

2005 Kaohsiung Invitational World Youth Mathematics Intercity Competition

Team Contest

2005/8/3 Kaohsiung

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1. The positive integers a , b and c are such that $a + b + c = 20 = ab + bc - ca - b^2$. Determine all possible values of abc .

We have $(a-b)(b-c)=20$. Without loss of generality, we may assume that $a \geq b \geq c$. We summarize various cases in the chart below:

$b-c$	$a-b$	c	b	a	$a + b + c = 20$	abc
1	20	1	2	22	No	
2	10	2	4	14	Yes	112
4	5	2	6	11	No	
		3	7	12	No	
5	4	2	7	11	Yes	154
10	2	1	11	13	No	
20	1	1	21	22	No	

Hence the possible values of abc are 112 and 154.

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2. The sum of 49 positive integers is 624. Prove that three of them are equal to one another.

If each positive integer appears at most twice, then the sum of 49 of them is at least $2(1+2+\dots+24)+25=625>624$. Hence at least one of them must appear thrice.

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3. The list 2, 3, 5, 6, 7, 10, ... consists of all positive integers which are neither squares nor cubes in increasing order. What is the 2005th number in this list?

Let us see when 2005 appears. Since $44^2 < 2005 < 45^2$ and $2005 - 44 = 1961$, it moves up temporarily to the 1961-st place. Moreover, $12^3 < 2005 < 13^3$. So it moves up another 12 places to the 1949-th. However, we have counted the six-th powers as both squares and cubes. Since $3^6 < 2005 < 4^6$, 2005 drops back 3 places to the 1952-nd. Since $2005 - 1952 = 53$, we expect the 2005th number to be $2005 + 53 = 2058$. However, we have now pass over another square, namely, $45^2 = 2025$. So the 2005-th number is 2059.

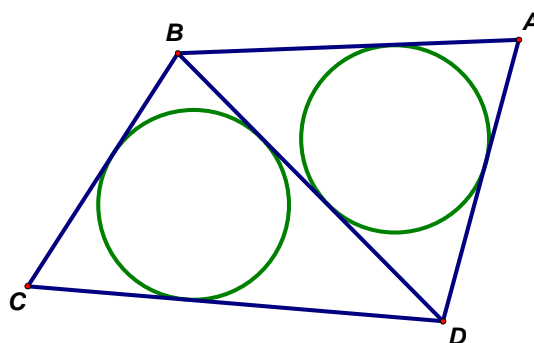
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4. $ABCD$ is a convex quadrilateral such that the incircles of triangles BAD and BCD are tangent to each other. Prove that $ABCD$ has an incircle.



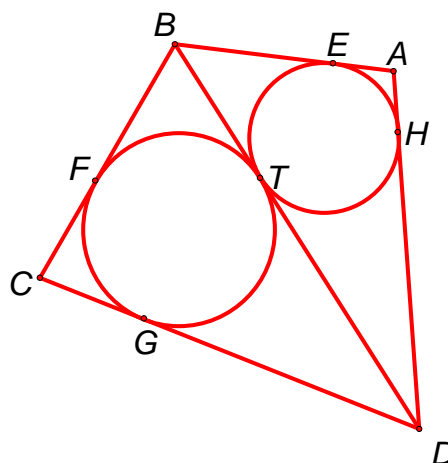
The incircles of triangles BAD and BCD must fall at the different side of BD , it tangent to each other only at the same point on BD .

Let the incircle of triangle BAD touch AB , BD and DA at E , T and H , respectively. Let the incircle of triangle BCD touch CB , BD and DC at F , T and G , respectively. Then $AB+CD=AE+BE+CG+DG=AE+BT+CG+DT=AH+BF+CF+DH=AD+BC$.

If $ABCD$ is a convex quadrilateral with $AB+CD=AD+BC$, we can easily prove that $ABCD$ has an incircle.

Let circle O tangent to AB , BC and DC at P , Q and R . Drawing a tangent line from A intersect CD at D' . Suppose $D \neq D'$.

Hence $AB+CD=AD+BC$ and $AB+CD'=AD'+BC$. $AD-AD'=CD-CD'=DD'$, it implies that $AD=DD'+AD'$, it's contradiction, and this implies that $ABCD$ has an incircle.



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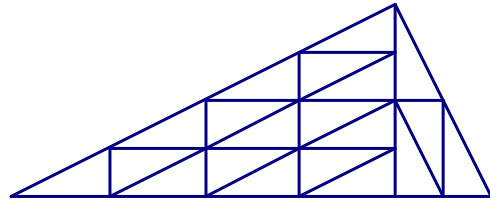
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5. Find a dissection of a triangle into 20 congruent triangles.

A right triangle with legs in the ratio 2:1 may be dissected into twenty congruent triangles of the same shape as shown in the diagram below.



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6. You are gambling with the Devil with 3 dollars in your pocket. The Devil will play five games with you. In each game, you give the Devil an integral number of dollars, from 0 up to what you have at the time. If you win, you get back from the Devil double the amount of what you pay. If you lose, the Devil just keeps what you pay. The Devil guarantees that you will only lose once, but the Devil decides which game you will lose, after receiving the amount you pay. What is the highest amount of money you can guarantee to get after the five games?

Bet 2 dollars on the first game. If you lose, bet everything you have on the remaining games and get 16 dollars. If you win the first game and have 5 dollars, bet 3 dollars on the second game. If you lose, bet everything you have on the remaining games and get 16 dollars. If you win the second game and have 8 dollars, bet 4 dollars on the third game. If you lose, bet everything you have on the remaining games and get 16 dollars. If you win the third game and have 12 dollars, bet 4 dollars on the fourth game. If you lose, bet everything you have on the fifth game and get 16 dollars. If you win the fourth game and have 16 dollars, quit. You cannot do any better. On each game before you lose, if you bet less than the suggested amount, the Devil will let you win and you will get less than 16 dollars at the end. If you bet more than the suggested amount, the Devil will win and you will not have enough left to get 16 dollars at the end.

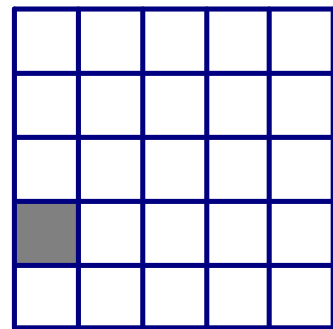
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7. A frog is sitting on a square adjacent to a corner square of a 5×5 board. It hops from square to adjacent square, horizontally or vertically but not diagonally. Prove that it cannot visit each square exactly once.



Paint the squares of the board alternately black and white in checkerboard fashion, with the corner squares black. Then there are 13 black squares and 12 white squares. In each hop, the frog goes between adjacent squares which have opposite colours. In order to visit each square exactly once, it must start on a black square and ends on another black square. Since its starting square is white, the task is impossible.

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8. Determine all integers n such that $n^4 - 4n^3 + 15n^2 - 30n + 27$ is a prime number.

Since $n^4 - 4n^3 + 15n^2 - 30n + 27 = (n^2 - 3n + 3)(n^2 - n + 9)$ is a prime number.

Due to $(n^2 - n + 9)$ always larger than 8, hence $(n^2 - 3n + 3)$ must equal 1.

It follows that $n^2 - 3n + 2 = 0$ so that $n = 1$ or 2 . For $n = 1$, $n^2 - n + 9 = 9$ is a multiple of 3.

Hence $n^4 - 4n^3 + 15n^2 - 30n + 27 = 11$ only for $n = 2$.

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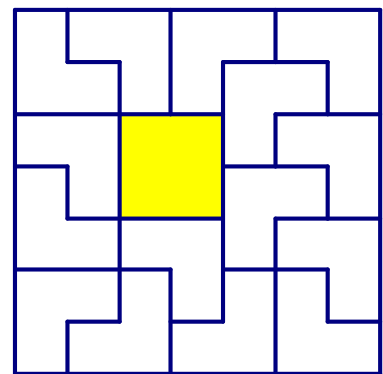
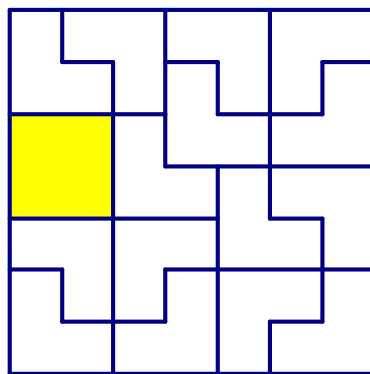
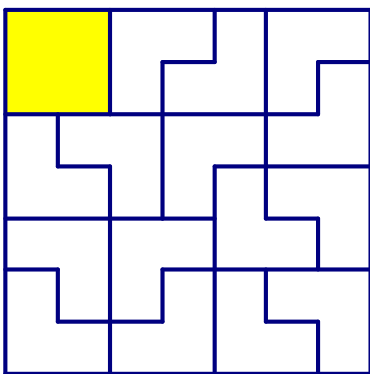
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9. A V-shaped tile consists of a 2×2 square with one corner square missing. Show that no matter which square is omitted from a 7×7 board, the remaining part of the board can be covered by 16 tiles.



V-shaped

There are ten non-equivalent positions of a square in a 7×7 board. By rotation and reflection, we can arrange so that it appears in the 2×2 subboard in one of the three diagrams below. In each diagram, the rest of the 7×7 board is tiled, and an additional tile can be placed on the 2×2 subboard so as to leave any of its four squares uncovered.



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10. Let $a_0, a_1, a_2, \dots, a_n$ be positive integers and $a_0 > a_1 > a_2 > \dots > a_n > 1$ such that

$$\left(1 - \frac{1}{a_1}\right) + \left(1 - \frac{1}{a_2}\right) + \dots + \left(1 - \frac{1}{a_n}\right) = 2\left(1 - \frac{1}{a_0}\right).$$

Find all possible solutions for $(a_0, a_1, a_2, \dots, a_n)$.

It is obvious that $n \neq 1$.

If $a > 1$, then $1 > 1 - \frac{1}{a} > \frac{1}{2}$. We have

$$2 > 2\left(1 - \frac{1}{a_0}\right) = \left(1 - \frac{1}{a_1}\right) + \left(1 - \frac{1}{a_2}\right) + \dots + \left(1 - \frac{1}{a_n}\right) > \frac{n}{2}.$$

It implies that $3 \geq n$. Hence we only need to consider two cases (i) $n=2$, (ii) $n=3$.

(i) When $n=2$. We need to solve the equation :

$$\frac{2}{a_0} = \frac{1}{a_1} + \frac{1}{a_2}. \quad \text{But it is impossible since } \frac{1}{a_1} > \frac{1}{a_0} \text{ and } \frac{1}{a_2} > \frac{1}{a_0}.$$

(ii) When $n=3$. We need to solve the equation :

$$1 + \frac{2}{a_0} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}. \quad \text{If } (a_2, a_3) \neq (3, 2), \text{ Then}$$

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} < \frac{1}{5} + \frac{1}{4} + \frac{1}{3} < 1. \quad \text{It is contradiction.}$$

$$\text{Hence we have } a_2=3, a_3=2. \quad \text{that is } \frac{1}{6} + \frac{2}{a_0} = \frac{1}{a_1}.$$

Therefore we got $(a_0, a_1) = (24, 4)$ or $(60, 5)$.

So, all possible solutions for $(a_0, a_1, a_2, \dots, a_n)$ are $(24, 4, 3, 2)$ and $(60, 5, 3, 2)$.