Chapter 8

8.1-1

(a) $P(\text{red card}) = (13+13)/52 = 0.5$

(b) $P(\text{black queen}) = (1+1)/52 = 1/26$

(c) $P(\text{picture card}) = 12/52 = 3/13$

(d) $P(7) = 4/52 = 1/13$

(e) $P(n \leq 5) = 20/52 = 5/13$

8.1-2

(a) $S = 4$ occurs from outcomes (1,1,2), (1,2,1), (2,1,1). There are a total of $6 \times 6 \times 6 = 216$ outcomes. Therefore, $P(S=4) = 3/216 = 1/72$

(b) $S = 9$ occurs from outcomes (1,2,6), (1,3,5), (1,4,4), (2,2,5), (2,3,4), (3,3,3), and their permutations. For example, (1,2,6) can also be (2,1,6), (6,2,1), (2,6,1), (6,1,2), (1,6,2). (1,2,6), (1,3,5), and (2,3,4) can be ordered in 6 different ways, whereas (1,4,4) and (2,2,5) can be ordered in 3 ways each. (3,3,3) has no permutation that is different. There are $3 \times 6 + 2 \times 3 + 1 = 25$ desired outcomes.

$$P(S=9) = \frac{25}{216}$$

(c) Similarly, $S = 15$ can occur as outcomes (3,6,6), (4,5,6), (5,5,5), and their distinct permutations. There are a total of $1 \times 3 + 1 \times 6 + 1 = 10$ different outcomes.

$$P(S=15) = \frac{10}{216}$$

8.1-3 The probability that the number i appears should be $k \times i$

$$1 = \sum_{i=1}^{6} k \times i = k \sum_{i=1}^{6} i = 21k \implies k = \frac{1}{21}$$

Therefore, $$P(i) = \frac{i}{21}, \qquad i = 1, 2, 3, 4, 5, 6$$

8.1-4 (a) We can draw 2 items out of 5 in 20 different ways: the 1st item can be any one of the 5; the second item can be any one of the remaining 4. All these outcomes are equally likely with probability 1/20.

(i) This event has 12 out of the 20 different ways. Hence, $P(E_1) = 12/20 = 0.6$

(ii) This event has combinations $E_2 = (P_1P_2) \cup (P_2P_1)$. $P(E_2) = 2/20 = 0.1$.

(iii) This event has combinations $E_3 = (O_1O_2) \cup (O_1O_3) \cup (O_2O_1) \cup (O_2O_3) \cup (O_3O_1) \cup (O_3O_2)$. $P(E_3) = 6/20 = 0.3$.

(iv) This event combines E_2 and E_3. E_2 and E_3 have no intersection. Hence,

$$P(E_2 \cup E_3) = P(E_2) + P(E_3) - P(E_2 \cap E_3) = P(E_2) + P(E_3) - P(\phi) = 0.1 + 0.3 - 0 = 0.4$$

8.1-5 $P(O_i, P_j) = P(O_i)P(P_j|O_i)$ and $P(P_j, O_i) = P(P_j)P(O_i|P_j)$ where

$$P(P_j|O_i) = \frac{2}{4},\ P(O_i|P_j) = \frac{3}{4},\ P(O_i) = \frac{3}{5},\ P(P_j) = \frac{2}{5}$$

(i) Hence, we have $P(P_j, O_i) + P(O_i, P_j) = \frac{3}{5}$, which is the same result obtained in Problem 8.1.4 (b) (i).
(ii) $P(P_i, P_j) = P(P_i|P_j)P(P_j) = \frac{1}{4}\frac{2}{5} = 0.1$
(iii) $P(O_i, O_j) = P(O_i|O_j)P(O_j) = \frac{2}{4}\frac{3}{5} = 0.3$
(iv) $P(O_i, O_j) + P(P_l, P_k) = 0.4$.

8.1-6

(a) $P(O|P) = \frac{3}{4}$

(b) $P(O|O) = \frac{2}{4}$

8.1-7

(a) We can have $\binom{10}{2}$ ways of getting two 1's and eight 0's in 10 digits. Hence

$$P(\text{two 1's and eight 0's}) = \binom{10}{2}(0.5)^2(0.5)^8 = 45(0.5)^{10} = 45/1024$$

(b)

$$\begin{aligned}
P(\text{at least four 0's}) &= 1 - P(\text{exactly three 0's}) - P(\text{exactly two 0's}) - P(\text{exactly one 0's}) - P(\text{exactly zero 0's}) \\
&= 1 - \binom{10}{3}(0.5)^{10} - \binom{10}{2}(0.5)^{10} - \binom{10}{1}(0.5)^{10} - \binom{10}{0}(0.5)^{10} \\
&= 1 - (\binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0})(0.5)^{10} \\
&= 1 - (120 + 45 + 10 + 1)(0.5)^{10} = 1 - \frac{176}{1024} = \frac{848}{1024} = \frac{53}{64}
\end{aligned}$$

8.1-8

(a) The total number of possible outcomes for drawing 6 balls out of 49 is

$$\binom{49}{6} = \frac{49!}{6!43!} = 13,983,816$$

Since there is only one outcome that will result in all 6 balls being matched, the probability of this event is given by

$$P(\text{6 balls match}) = \frac{1}{13,983,816}$$

(b) To match exactly 5 numbers means that we pick 5 of the chosen 6 numbers, and the last number can then be picked from the remaining 43 numbers. We can choose 5 number out of 6 in $\binom{6}{5} = 6$ ways, and we can choose one number out of 43 in $\binom{43}{1} = 43$ ways. Hence, we have 43×6 combinations in which exactly 5 numbers match. So the probability is given by

$$P(5\,\text{balls match}) = \frac{43 \times 6}{13,983,816} = 1.845 \times 10^{-5}$$

(c) To match exactly 4 balls we pick 4 out of the chosen 6 number in $\binom{6}{4} = 15$ ways and choose 2 out of the remaining 43 numbers $\binom{43}{2} = 903$. So the probability is given by

$$P(4\,\text{balls match}) = \frac{15 \times 903}{13,983,816} = 9.686 \times 10^{-4}$$

(d) Similarly, we can pick three numbers to match exactly in $\binom{6}{3}\binom{43}{3} = 246,820$ ways. Hence, the probability is given by

$$P(3\,\text{balls match}) = \frac{246820}{13983816} = 0.01765$$

8.1-9

(a) Let f denote system failure. Then

$$P(\bar{f}) = (1 - 0.01)^{100} = 0.90438$$

$$P(f) = 1 - (1 - 0.01)^{100} = 0.0956$$

(b) Now we have that $P(f) = 0.01$. $P(\bar{f}) = 0.99 = (1 - p)^{10}$. Hence, $p = 0.0010045$

8.1-10

(a) Let f represent system failure and let f_u and f_l represent the failure of the upper and the lower paths, respectively, in the network. Then

$$P(f) = P(f_u f_l) = P(f_u)P(f_l) = [P(f_u)]^2$$

The probability of one branch not failing, as calculated in problem **8.1.8(a)**, is 0.81. As shown in part **(a)** of Problem **8.1-8**, $P(f_u) = P_f = 0.183$. Hence, $P(f) = 0.183^2 = 0.0335$. Finally, the reliability of the network is given by $P_r = 1 - P(f) = 0.9665$.

(b) $P_r = 0.99$
$P(f) = 1 - 0.999 = 0.001$
$P(f_u) = \sqrt{0.001} = 0.0316$
$P(f_u) = (1 - p_f)^{10} = 1 - 0.0316$
$p_f = 0.003206$

8.1-11 Let p be the probability of failure of a link (s_1 or s_2).

For the system in Figure P.8.1-10a:
The system fails if both upper and lower branches fail simultaneously. The probability of any branch not failing is $(1-p)(1-p) = (1-p)^2$. So the probability of any branch failing is $1-(1-p)^2$. Clearly, P_f, the probability of the system failure is

$$P_f = [1-(1-p)^2][1-(1-p)^2] \approx 4p^2$$

where $p \ll 1$.

For the system in Figure P.8.1-10b:
We can look at this system as a cascade of two subsystems x_1 and x_2, where x_1 is the parallel combination of s_1 and s_1 and $x_2$is the parallel combination of s_2 and s_2. Let $P_f(x_i)$ be the probability of failure of x_i. Then $P_f(x_1) = P_f(x_2) = p^2$. The system functions if neither x_1 nor x_2 fails. Hence, the probability of not failing is $(1-p^2)(1-p^2)$. Therefore, the probability of system failure is

$$P_f = 1-(1-p^2)(1-p^2) \approx 2p^2$$

We can conclude, that the system in Figure P.8.1-10a is twice more likely to fail than the system in Figure P.8.1-10b.

8.1-12 There are $\binom{52}{5} = 2598960$ ways of getting 5 cards out of 52. The number of ways for drawing 5 cards of the same suit (of 13) is $\binom{13}{5} = 1287$. There are 4 suits. Hence, there are 4×1287 ways of getting a flush. Thus,

$$P(\text{flush}) = \frac{4 \times 1287}{2598960} = 0.0019808$$

8.1-13 Sum of 4 can be obtained as (1,3), (2,2), and (3,1). The two dice are independent. We denote x_1 as the regular die outcome and x_2 as the irregular die outcome.

$$P(x_1 = 1, x_2 = 3) = P(x_1 = 1)P(x_2 = 3) = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$$

$$P(x_1 = 2, x_2 = 2) = P(x_1 = 2)P(x_2 = 2) = \frac{1}{6} \times 0 = 0$$

$$P(x_1 = 3, x_2 = 1) = P(x_1 = 3)P(x_2 = 1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Thus, $P(\text{sum} = 4) = 1/18 + 1/36 = 1/12$.

Similarly, we can find

$$\begin{aligned} P(\text{sum} = 5) &= P(x_1 = 1, x_2 = 4) + P(x_1 = 2, x_2 = 3) + P(x_1 = 3, x_2 = 2) + P(x_1 = 4, x_2 = 1) \\ &= \frac{1}{6} \times 0 + \frac{1}{6} \times \frac{1}{3} + \frac{1}{6} \times \frac{1}{6} = \frac{1}{12} \end{aligned}$$

8.1-14

(a) $B = AB \cup A^C B$

$$P(B) = P(A)P(B|A) + P(A^C)P(B|A^C) = \left(\frac{1}{26}\right)\left(\frac{1}{51}\right) + \left(\frac{50}{52}\right)\left(\frac{2}{51}\right) = \frac{1}{26}$$

(b)

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{26}\frac{1}{51}}{\frac{1}{26}} = \frac{1}{51}$$

8.1-15

(a) Let N be a number of 1s that occur in the binary sequence. Then, the probability that there are exactly two 1s in an n binary sequence is given by

$$P(N=2)=\binom{n}{2}p(1)^2p(0)^{n-2}=\frac{n(n-1)}{2}0.8^2\cdot 0.1^{n-2}$$

(b)

$$P(N\geq 3)=1-\sum_{k=0}^{2}\binom{n}{k}p(1)^kp(0)^{n-k}=1-0.1^n-n\cdot 0.8\cdot 0.1^{n-1}-\frac{n(n-1)}{2}0.8^2\cdot 0.1^{n-2}$$

8.1-16 Let N be the number of errors. Then,

$$\begin{aligned}P(N\leq 4)&=\sum_{k=0}^{4}\binom{100}{k}{P_e}^k(1-P_e)^{100-k}\\&=(1-P_e)^{100}+100P_e(1-P_e)^{99}+4950P_e^2(1-P_e)^{98}\\&\quad+161700P_e^3(1-P_e)^{97}+3921225P_e^4(1-P_e)^{96}\end{aligned}$$

8.1-17

$$P_c=Prob(\text{correct detection over every link})=(1-P_1)(1-P_2)...(1-P_{15})$$

$$\begin{aligned}P_E=1-P_c&=1-(1-P_1)(1-P_2)...(1-P_{15})\\&=1-[1-(P_1+P_2+...+P_{15})+\text{higher order terms}]\\&\approx P_1+P_2+...+P_{15},\qquad \text{for } P_i\ll 1\end{aligned}$$

8.1-18

$$\begin{aligned}P(\varepsilon)&=\sum_{i=3}^{4}\binom{4}{3}P_e^k(1-P_e)^{4-k}+\frac{1}{2}\binom{4}{2}P_e^2(1-P_e)^2\\&=4P_e^3(1-P_e)+P_e^4+\frac{1}{2}6P_e^2(1-P_e)^2\end{aligned}$$

For $P_e\ll 1$, we have $P(\varepsilon)\approx 3P_e^2$. So the error probability is the same as in the case when we repeat 3 times; however, the rate of information transmission is reduced by a factor of 4 instead of 3.

8.1-19

(a) $P(\text{success in 1 trial})=\frac{1}{10}=0.1$

(b) $P(\text{success in 5 trials})=1-P(\text{failure in all 5 trials})=1-P(f_1)P(f_2)P(f_3)P(f_4)P(f_5)$.
$P(f_1)=\text{Prob(failure in the 1st trial)}=9/10$
$P(f_2)=\text{Prob(failure in the 2nd trial after failure in the 1st trial)}=8/9$
$P(f_3)=\text{Prob(failure in the 3rd trial after 2 failures)}=7/8$
$P(f_4)=\text{Prob(failure in the 4th trial after 3 failures)}=6/7$
$P(f_5)=\text{Prob(failure in the 5th trial after 4 failures)}=5/6$. Therefore,

$$P(\text{success in 5 trials})=1-\frac{9\times 8\times 7\times 6\times 5}{10\times 9\times 8\times 7\times 6}=1-\frac{5}{10}=0.5$$

8.1-20 Let x be the event of drawing the short straw and let $P_i(x)$ denote the even that the $i-$th person in the sequence draws the short straw.
First, $P_1(x) = 0.1$.
Next,

$$P_2(x) = \text{Prob(1st person did not draw but the 2nd person draws the short)} = (1 - 0.1) \times \frac{1}{9} = 0.1$$

Similarly,

$$P_3(x) = \text{Prob(1st and 2nd did not draw but the 3rd person draws the short)} = (1 - 0.1 - 0.1) \times \frac{1}{8} = 0.1$$

$$P_n(x) = \text{Prob(first } n-1 \text{ did not draw but the } n\text{-th person draws it)} = (1 - 0.1 \times n) \times \frac{1}{(10-n)} = 0.1$$

8.2-1

$$\begin{aligned} P_y(0) &= P_{xy}(1,0) + P_{xy}(0,0) \\ &= P_x(1)P_{y|x}(y|x) + P_x(0)P_{y|x}(0|0) \\ &= 0.6 \times 0.1 + 0.4[1 - P_{y|x}(1|0)] = 0.06 + 0.32 = 0.38 \\ P_y(1) &= 1 - P_y(0) = 0.62 \end{aligned}$$

8.2-2

(a) $P_{x|y}(1|1) = \dfrac{P_{y|x}(1,1)P_x(1)}{P_y(1)} = \dfrac{(1-P_e)q}{(1-Q)P_e + (1-P_e)Q}$

(b) $P_{x|y}(0|1) = 1 - P_{x|y}(1|1)$

8.2-3

(a)

$$P(\mathrm{x} \geq 1) = \int_1^\infty p_{\mathrm{x}}(x)dx == \frac{1}{2}\int_1^\infty xe^{-x}dx = e^{-1}$$

(b)

$$\begin{aligned} P(-1 < \mathrm{x} \leq 2) &= \int_{-1}^{2} p_{\mathrm{x}}(x)dx = \int_{-1}^{0} -xe^{x}dx \int_0^2 xe^{-x}dx \\ &= 1 - e^{-1} - 1.5e^{-2} \end{aligned}$$

(b)

$$\begin{aligned} P(\mathrm{x} \leq -2) &= \int_{-\infty}^{-2} p_{\mathrm{x}}(x)dx = \int_{-\infty}^{-2} -xe^{x}dx \\ &= 1.5e^{-2} \end{aligned}$$

8.2-4 Because

$$y = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

we have

$$F_{\mathrm{y}}(y) = P(\mathrm{y} \leq y) = P[\mathrm{x}u(\mathrm{x}) \leq y] = \begin{cases} 0, & y < 0 \\ P(\mathrm{x} < 0) = 0.5, & y = 0 \\ P(\mathrm{x} < y) = F_{\mathrm{x}}(y), & y > 0 \end{cases}$$

$$\begin{aligned} p_{\mathrm{y}}(y) &= \frac{d\,F_{\mathrm{y}}(y)}{dy} \\ &= \frac{1}{2}\delta(y) + \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) u(y) \end{aligned}$$

8.2-5

(a)

$$P(x \geq 2) = Q\left(\frac{2-4}{3}\right) = 1 - Q\left(\frac{2}{3}\right) \approx 1 - 0.2525 = 0.7475$$

(b)

$$P(x \leq -1) = 1 - P(x \geq -1) = 1 - Q(\frac{-1-4}{3}) = 1 - 1 + Q(\frac{5}{3}) = Q(\frac{5}{3}) = 0.0478$$

(c) $P(x \geq -2) = Q(\frac{-2-4}{3}) = 1 - Q(2) = 1 - 0.02275 = 0.9773$

8.2-6

(a) From Fig. S8.2-6, it is obvious that it is not Gaussian. However, it is a unilateral (rectified version of Gaussian pdf).

(b) We find **(i)** $P(x \geq 1) = 2P(y \geq 1) = 2Q(\frac{1}{4}) = 0.8026$
(ii) $P(-1 < x \leq 2) = 2P(-1 < y \leq 2) = 2[Q(-\frac{1}{4}) - Q(\frac{2}{4})] = 2[1 - Q(\frac{1}{4}) - Q(\frac{1}{2})] = 0.58033757391386$

(c) We can take the Gaussian random variable y and rectify y such that all negative values of y are multiplied by -1. The resulting variable is the desired random variable x.

8.2-7 The volume V under $p_{\mathrm{xy}}(x,y)$ must be unity.
$V = \frac{1}{2}(1 \times 1)A = \frac{A}{2} = 1$. Thus, we have $A = 2$.
$p_{\mathrm{x}}(x) = \int_y p_{\mathrm{xy}}(x,y)\,dy$. But $y = -x + 1$, so the limits on y are 0 to $1 - x$. Therefore,

$$p_{\mathrm{x}}(x) = \int_0^{1-x} 2dy = 2y \Big|_0^{1-x} = \begin{cases} 2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly,

$$p_{\mathrm{y}}(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$p_{\mathrm{x-y}}(x|y) = \frac{p_{\mathrm{xy}}(x,y)}{p_{\mathrm{y}}(y)} = \frac{2}{2(1-y)} = \begin{cases} 1/(1-y) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly,

$$p_{\mathrm{y-x}}(y|x) = \frac{p_{\mathrm{xy}}(x,y)}{p_{\mathrm{x}}(x)} = \begin{cases} 1/(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

From previous results it is easy to conclude that x and y are not independent.

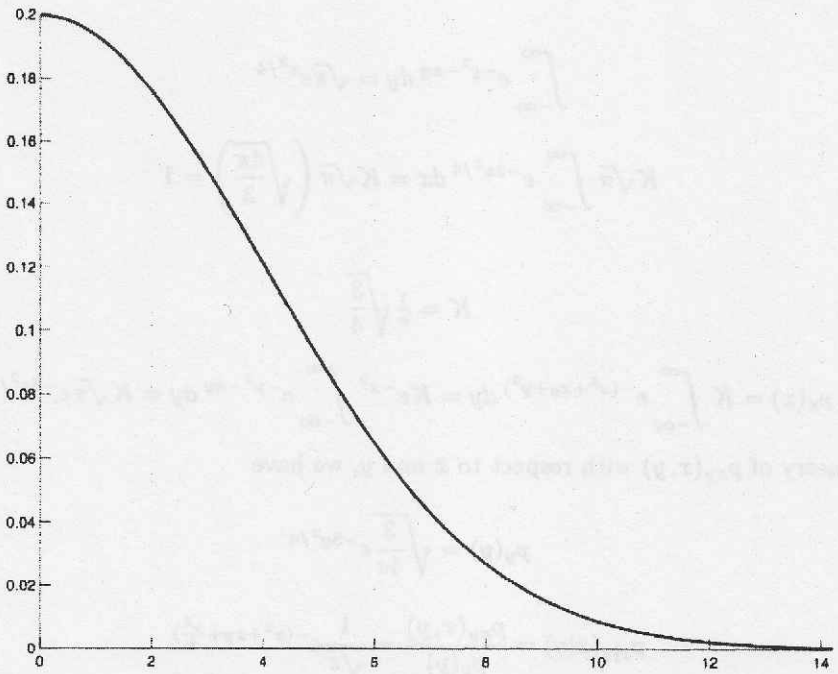

Fig. S8.2-6

8.2-8

(a) $p_x(x) = \int_0^{\infty} 2xye^{(-x^2)}e^{(-y^2/2)}u(x)\,dy = 2xe^{(-x^2)}u(x)$
$p_y(y) = \int_0^{\infty} 2xye^{(-x^2)}e^{(-y^2/2)}u(y)\,dx = ye^{(-y^2/2)}u(y)$
$p_x(x|y=y) = \frac{p_{xy}(x,y)}{p_y(y)} = 2xe^{(-x^2)}u(x)$
$p_y(y|x=x) = \frac{p_{xy}(x,y)}{x_x(x)} = ye^{(-y^2/2)}u(y)$

(b) From part (a), we can see that x and y are independent.

8.2-9

$$
\begin{aligned}
p_{\mathrm{x}}(x) &= \int_{-\infty}^{\infty} p_{\mathrm{xy}}(x,y)dy \\
&= \frac{1}{2\pi M}\int_{-\infty}^{\infty} e^{-(ax^2+by^2-2cxy)/2M}dy \\
&= \frac{1}{2\pi M}e^{-ax^2/2M}\int_{-\infty}^{\infty} e^{-(by^2-2cxy)/2M}dy = \frac{1}{\sqrt{2\pi b}}e^{-x^2/2b}
\end{aligned}
$$

Note that $b = \overline{\mathrm{x}^2}$. Similarly, we can show that $p_y(y) = \frac{1}{\sqrt{2\pi a}}e^{-y^2/2a}$ and that $a = \overline{\mathrm{y}^2}$. Therefore,

$$p_{\mathrm{x-y}}(x|y) = \frac{p_{xy}(x,y)}{p_y(y)} = \sqrt{\frac{a}{2\pi M}}e^{-a(x-\frac{c}{a}y)^2/2M}$$

$$p_{\mathrm{y\ x}}(y|x) = \frac{p_{xy}(x,y)}{p_x(x)} = \sqrt{\frac{a}{2\pi M}}e^{-a(x-\frac{c}{a}y)^2/2M}$$

8.2-10

$$K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}\,dx\,dy = K\int_{-\infty}^{\infty} e^{-x^2}[\int_{-\infty}^{\infty} e^{-y^2-xy}dy]dx = 1$$

But

$$\int_{-\infty}^{\infty} e^{-y^2-xy}\,dy = \sqrt{\pi}e^{x^2/4}$$

$$K\sqrt{\pi}\int_{-\infty}^{\infty} e^{-3x^2/4}\,dx = K\sqrt{\pi}\left(\sqrt{\frac{4\pi}{3}}\right) = 1$$

Hence,

$$K = \tfrac{1}{\pi}\sqrt{\frac{3}{4}}$$

$$p_{\mathrm{x}}(x) = K\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}\,dy = Ke^{-x^2}\int_{-\infty}^{\infty} e^{-y^2-xy}\,dy = K\sqrt{\pi}e^{-3x^2/4}$$

Because of the symmetry of $p_{xy}(x,y)$ with respect to x and y, we have

$$p_y(y) = \sqrt{\frac{3}{4\pi}}e^{-3y^2/4}$$

$$p_{x|y}(x|y) = \frac{p_{xy}(x,y)}{p_y(y)} = \frac{1}{\sqrt{\pi}}e^{-(x^2+xy+\frac{y^2}{4})}$$

and

$$p_{y|x}(y|x) = \frac{p_{xy}(x,y)}{p_x(x)} = \frac{1}{\sqrt{\pi}}e^{-(\frac{x^2}{4}+xy+y^2)}$$

Since $p_{xy}(x,y) \neq p_x(x)p_y(y)$, x and y are not independent.

8.2-11 $P_e = P(\varepsilon|1)P_x(1) + P(\varepsilon|0)P_x(0)$
If the optimum of the threshold is a, then

$$P(\varepsilon|1) = 1 - Q\left(\frac{a-A_p}{\sigma_n}\right) = Q\left(\frac{A_p-a}{\sigma_n}\right) \qquad P(\varepsilon|0) = Q\left(\frac{A_p+a}{\sigma_n}\right)$$

$$P_e = Q\left(\frac{A_p-a}{\sigma_n}\right)P_x(1) + Q\left(\frac{A_p+a}{\sigma_n}\right)P_x(0)$$

$$\frac{dP_e}{da} = \frac{1}{2\pi\sigma_n}\left[e^{-(A_p-a)^2/2\sigma_n^2}P_x(1) - e^{-(A_p+a)^2/2\sigma_n^2}P_x(0)\right] = 0$$

$$e^{-(A_p-a)^2/2\sigma_n^2}P_x(1) = e^{-(A_p+a)^2/2\sigma_n^2}P_x(0)$$

From here we get the optimum threshold to minimize P_e

$$a_{\text{optimum}} = \frac{\sigma_n^2}{2A_p}\ln\left[\frac{P_x(0)}{P_x(1)}\right]$$

8.3-1 Because $\mu_x = 2$, $\sigma_x = \sqrt{10}$, we have $p_x(x) = \frac{1}{\sqrt{20\pi}}e^{-(x-2)^2/20}$.

8.3-2 $p_{\mathrm{x}}(x) = \frac{1}{2}|x|e^{-|x|}$. Because of this symmetry, $\overline{\mathrm{x}} = 0$ and $p_{\mathrm{x}}(x) = \frac{1}{2}|x|e^{-|x|}$. Because of this symmetry,

$$\overline{\mathrm{x}^2} = 2\int_0^{\infty} x^2 p_{\mathrm{x}}(x)dx = 2\int_0^{\infty} x^2\frac{1}{2}xe^{-x}dx = 3! = 6$$

$$\sigma_{\mathrm{x}}^2 = \overline{\mathrm{x}^2} - (\overline{x})^2 = 6 - 0 = 6$$

8.3-3

$$p_{\mathrm{y}}(y) = \frac{1}{2}\delta(y) + \frac{1}{\sigma\sqrt{2\pi}}e^{-y^2/2\sigma^2}u(y)$$

Hence,

$$\overline{\mathrm{y}} = \int_{-\infty}^{\infty} y p_{\mathrm{y}}(y)dy = 0 + \int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} y e^{-y^2/2\sigma^2} dy = \frac{1}{\sqrt{2\pi}}\sigma$$

$$\begin{aligned}\overline{\mathrm{y}^2} &= \int_{-\infty}^{\infty} y^2 p_{\mathrm{y}}(y)dy = 0 + \int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} y^2 e^{-y^2/2\sigma^2} dy \\ &= \frac{1}{2\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y^2 e^{-y^2/2\sigma^2} dy \\ &= \frac{1}{2}\sigma^2\end{aligned}$$

$$\sigma_{\mathrm{y}}^2 = \frac{\sigma^2}{2}(1 - \frac{1}{\pi}).$$

8.3-4

$$\begin{aligned}\overline{\mathrm{x}^2} &= \int_0^{\infty} x^2 p_{\mathrm{x}}(x)dx = 2\int_0^{\infty} x^2 \frac{1}{4\sqrt{2\pi}} e^{-x^2/32}dx \\ &= \frac{4}{\sqrt{2\pi}}\int_0^{\infty} e^{-x^2/32}dx = 16.\end{aligned}$$

But

$$\begin{aligned}\overline{\mathrm{x}} &= \frac{1}{2\sqrt{2\pi}}\int_0^{\infty} x e^{-x^2/32}dx \\ &= \frac{8}{\sqrt{2\pi}}.\end{aligned}$$

Therefore,

$$\sigma_{\mathrm{y}}^2 = \overline{\mathrm{x}^2} - \overline{\mathrm{x}}^2 = 16 - 32/\pi.$$

8.3-5 The area of the triangle must be 1. Hence, $K = \frac{1}{2}$ and $p_{\mathrm{x}}(x) = \frac{1}{8}(x+1)$, $-1 \le x \le 3$

$$\overline{\mathrm{x}} = \int_{-1}^{3} x p_{\mathrm{x}}(x)\,dx = \frac{1}{8}\int_0^4 y(y-1)\,dy = \frac{1}{8}\left(\frac{y^3}{3} - \frac{y^2}{2}\right)\Big|_0^4 = \frac{5}{3}$$

$$\overline{\mathrm{x}^2} = \int_{-1}^{3} x^2(x+1)\,dx = \frac{1}{8}\left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_{-1}^{3} = \frac{11}{3}$$

$\sigma_x^2 = \overline{x^2} - (\overline{x})^2 = \frac{8}{9}$

8.3-6

$$\overline{\mathrm{x}} = \sum_{i=1}^{12} x_i P_{\mathrm{x}}(x_i) = \frac{1}{36}(2) + \frac{2}{36}(3) + \cdots + \frac{1}{36}(12) = \frac{256}{36} = 7\frac{1}{9}$$

$$\overline{\mathrm{x}^2} = \sum_{i=1}^{12} x_i^2 P_{\mathrm{x}}(x_i) = \frac{1}{36}(2)^2 + \frac{2}{36}(3)^2 + \cdots + \frac{1}{36}(12)^2 = 54.83$$

$\sigma_{\mathrm{x}}^2 = \overline{x^2} - (\overline{x})^2 = 5.83$

8.3-7

$$\overline{x^n} = \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-x^2/2\sigma_x^2}dx$$

For n odd, the integrand is an odd function of x. Therefore, $\overline{x^n} = 0$ when n is odd. For n even, we have

$$\begin{aligned}
\overline{x^n} &= \frac{2}{\sigma_x\sqrt{2\pi}}\int_{0}^{\infty} x^n e^{-x^2/2\sigma_x^2}dx \\
&= \frac{2}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-x^2/2\sigma_x^2}dx \\
&= 2(\sqrt{2}\sigma_x)^n \int_0^\infty y^n e^{-y^2}dy \\
&= 2(\sqrt{2}\sigma_x)^n \frac{(n-1)}{2}\int_0^\infty y^{n-2}e^{-y^2}dy \\
&= 2(\sqrt{2}\sigma_x)^n \frac{(n-1)(n-3)}{2^2}\int_0^\infty y^{n-4}e^{-y^2}dy \\
&= 2(\sqrt{2}\sigma_x)^n \frac{(n-1)(n-3)\cdots 1}{2(n/2)}
\end{aligned}$$

8.3-8 Let x_i be the outcome of the i−th die.

$$\overline{x_i} = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}, \qquad i = 1,2,3,\ldots,10.$$

$$\overline{x_i^2} = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \frac{91}{6}, \qquad i = 1,2,3,\ldots,10.$$

$$\sigma_{x_i}^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}, \qquad i = 1,2,3,\ldots,10.$$

Let x represent the sum, i.e. $x = x_1 + x_2 + x_3 \ldots + x_{10}$. Then,

$$\overline{x} = \overline{x_1} + \overline{x_2} + \ldots + \overline{x_{10}} = 35$$

$$\sigma_x^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \ldots + \overline{x_{10}}^2 = 35 \times 10/12 = \frac{175}{6}$$

$$\overline{x^2} = \overline{x}^2 + \sigma_x^2 = 1254.167$$

8.5-1 For any real a, $\overline{[(x-\overline{x}) - (y-\overline{y})]^2} \geq 0$, or $a^2\sigma_x^2 + \sigma_y^2 - 2a\sigma_{xy} \geq 0$.
Hence, the discriminant of this quadratic in a must be non-positive; that is

$$4\sigma_{xy}^2 - 4\sigma_x^2\sigma_y^2 \leq 0$$

In other words, that is

$$\left|\frac{\sigma_{xy}}{\sigma_x\sigma_y}\right| \leq 1$$

or $|\rho| \leq 1$.

8.5-2 When $y = k_1x + k_2$ we have $\overline{y} = k_1\overline{x} + k_2$,
$\sigma_y^2 = k_1^2\sigma_x^2$, and $\sigma_{xy}^2 = \overline{(x-\overline{x})(y-\overline{y})} = \overline{(x-\overline{x})(k_1x + k_2 - k_1\overline{x} - k)} = k_1\sigma_x^2$. Hence,

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x\sigma_y} = \frac{k_1\sigma_x^2}{|k_1|\sigma_x^2}$$

From here, we see that if k_1 is positive, $\rho_{xy} = 1$; however, if k_1 is negative, $\rho_{xy} = -1$.

8.5-3 $\overline{x} = \int_0^{2\pi} \cos\theta p(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta d\theta = 0$. Similarly, we have that $\overline{y} = 0$.
$\sigma_{xy} = \overline{xy} = \overline{\cos\theta \sin\theta} = \frac{1}{2}\overline{\sin 2\theta} = \frac{1}{2}\int_0^{2\pi} \sin 2\theta p(\theta)d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta d\theta = 0$.
Hence, $\sigma_{xy} = \overline{xy} = 0$, which means that x and y are uncorrelated. On the other hand, $x^2 + y^2 = 1$. Hence, x and y are not independent.

8.6-1 $p_x(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-3)$ and $p_n(n) = \frac{1}{2\sqrt{2\pi}}e^{-n^2/8}$
Since y=x+n,

$$\begin{aligned}
p_y(y) = p_x(x) * p_n(n) &= [\frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-3)] * \frac{1}{2\sqrt{2\pi}}e^{-n^2/8} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\left[\frac{1}{2\sqrt{2\pi}}e^{-(y-x)^2/8}\right]dx + \frac{1}{2}\int_{-\infty}^{\infty} \delta(x-3)\left[\frac{1}{2\sqrt{2\pi}}e^{-(y-x)^2/8}\right]dx \\
&= \frac{1}{4\sqrt{2\pi}}e^{-y^2/8} + \frac{1}{4\sqrt{2\pi}}e^{-(y-3)^2/8}
\end{aligned}$$

8.6-2 $p_x(x) = 0.4\delta(x) + 0.6\delta(x-3)$ and $p_n(n) = \frac{1}{2\sqrt{2pi}}e^{-n^2/8}$. We have

$$p_y(y) = \frac{1}{5\sqrt{2\pi}}e^{-y^2/8} + \frac{3}{10\sqrt{2\pi}}e^{-(y-3)^2/8}$$

8.6-3 Given $p_x(x) = Q\delta(x-1) + (1-Q)\delta(x+1)$ and $p_n(n) = P\delta(n-1) + (1-Q)\delta(n+1)$

$$\begin{aligned}
p_y(y) &= [q\delta(y-1) + (1-Q)\delta(y+1)] * [P\delta(y-1) + (1-P)\delta(y+1) \\
&= (P + Q - 2PQ)\delta(y) + PQ\delta(y-2) + (1-P)(1-Q)\delta(y+2)
\end{aligned}$$

8.6-4 Because $p_z(z) = p_x(x) * p_y(y)$, taking Fourier transform of both sides, we have $P_z(\omega) = P_x(w)P_y(w)$, where $P_x(\omega) = e^{-\sigma_x^2\omega^2}e^{-jw\overline{x}}$ and $P_y(y) = e^{-\sigma_y^2\omega^2}e^{-jw\overline{y}}$.
We have

$$P_z(z) = e^{-(\sigma_x^2+\sigma_y^2)\omega^2}e^{-j\omega(\overline{x}+\overline{y})}$$

Taking the inverse Fourier transform, we get

$$p_z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}}e^{-[z-(x+y)]^2/2(\sigma_x^2+\sigma_y^2)}$$

It is clear that $\overline{z} = \overline{x} + \overline{y}$ and $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$.

8.6-5 In this case
$R_{11} = R_{22} = R_{33} = \overline{m_k^2} = P_m$
$R_{12} = R_{21} = R_{23} = R_{32} = R_{01} = 0.825P_m$
$R_{13} = R_{31} = R_{02} = 0.562P_m$
$R_{03} = 0.308P_m$
Substituting these values in Equation (8.89) yields
$a_1 = 1.1025$, $a_2 = -0.2883$, and $a_3 = -0.0779$.
From Equation (8.90) we obtain

$$\overline{\epsilon^2 = [1 - (0.825a_1 + 0.562a_2 + 0.308a_3)]P_m = 0.2753P_m}$$

Hence, the SNR improvement is $10\log\left(\frac{P_m}{0.2753P_m}\right) = 5.63\,\text{dB}$