Seventh Edition

CHAPTER

19

VECTOR MECHANICS FOR ENGINEERS:

DYNAMICS

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Mechanical Vibrations

Contents

Introduction

- *Mechanical vibration* is the motion of a particle or body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable due to increased stresses and energy losses.

- Time interval required for a system to complete a full cycle of the motion is the *period* of the vibration.

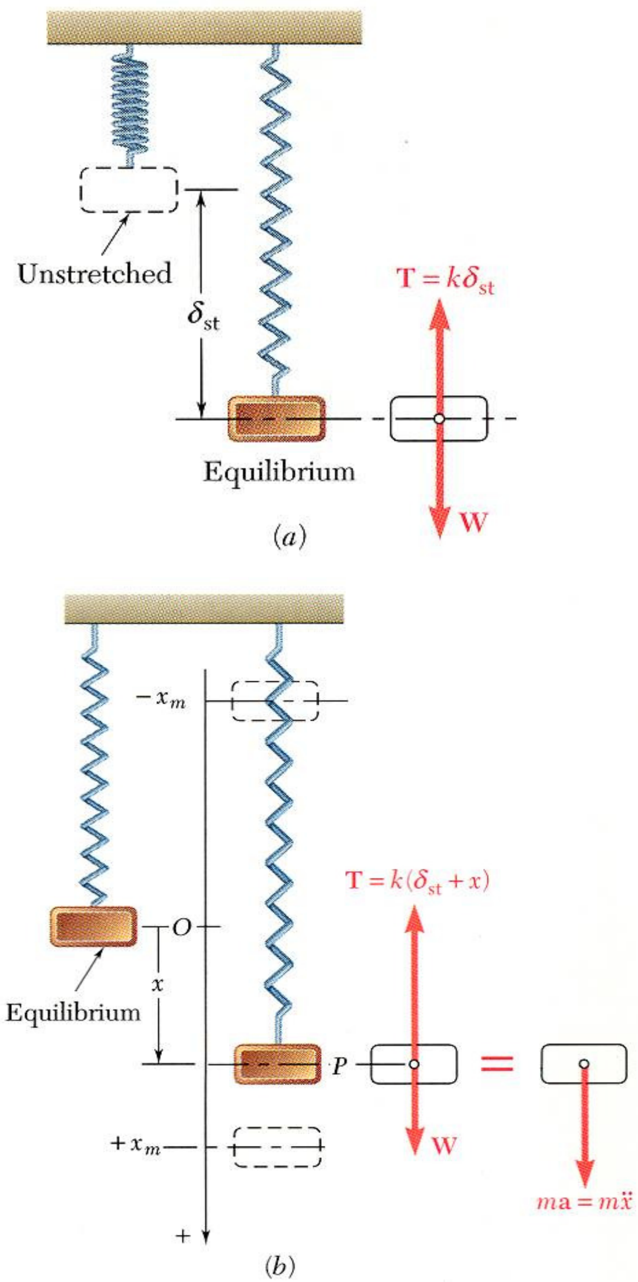
- Number of cycles per unit time defines the *frequency* of the vibrations.

- Maximum displacement of the system from the equilibrium position is the *amplitude* of the vibration.

- When the motion is maintained by the restoring forces only, the vibration is described as *free vibration*. When a periodic force is applied to the system, the motion is described as *forced vibration*.

- When the frictional dissipation of energy is neglected, the motion is said to be *undamped*. Actually, all vibrations are *damped* to some degree.

Free Vibrations of Particles. Simple Harmonic Motion



- If a particle is displaced through a distance x_m from its equilibrium position and released with no velocity, the particle will undergo *simple harmonic motion*,

$$ma = F = W - k(\delta_{st} + x) = -kx$$

$$m\ddot{x} + kx = 0$$

- General solution is the sum of two *particular solutions*,

$$x = C_1 \sin\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$= C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$$

- x is a *periodic function* and ω_n is the *natural circular frequency* of the motion.

- C_1 and C_2 are determined by the initial conditions:

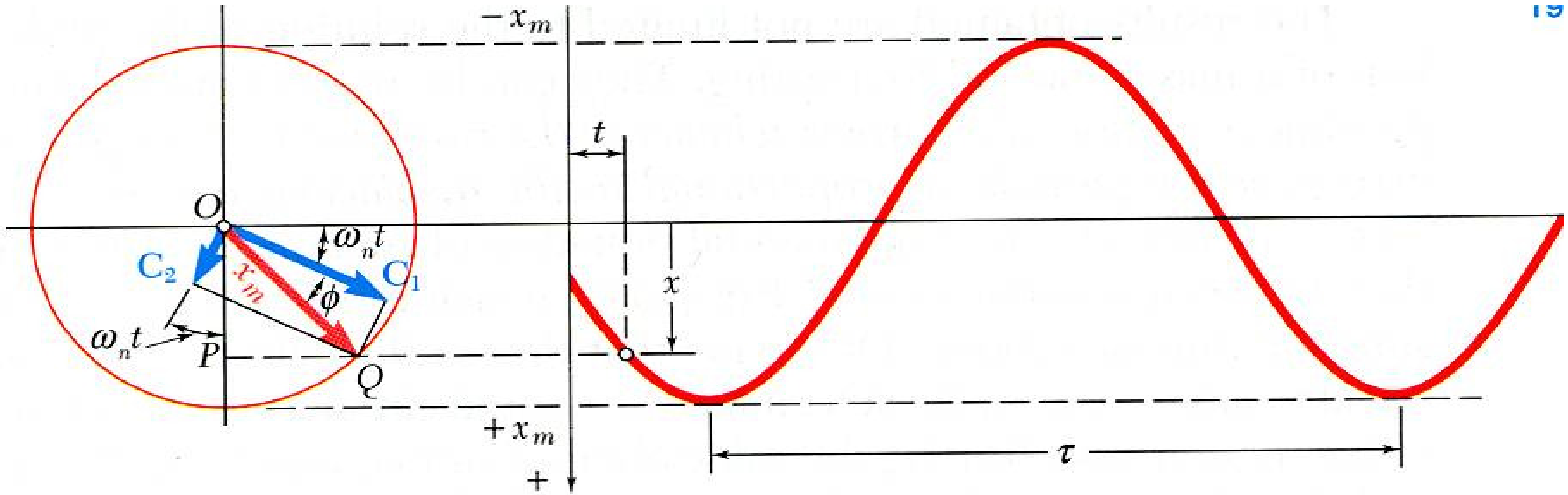
$$x = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t) \qquad C_2 = x_0$$

$$v = \dot{x} = C_1\omega_n \cos(\omega_n t) - C_2\omega_n \sin(\omega_n t) \qquad C_1 = v_0/\omega_n$$

Free Vibrations of Particles. Simple Harmonic Motion

$$C_1 = \frac{v_0}{\omega_n}$$

$$C_2 = x_0$$



- Displacement is equivalent to the x component of the sum of two vectors $\vec{C}_1 + \vec{C}_2$ which rotate with constant angular velocity ω_n.

$$x = x_m \sin(\omega_n t + \phi)$$

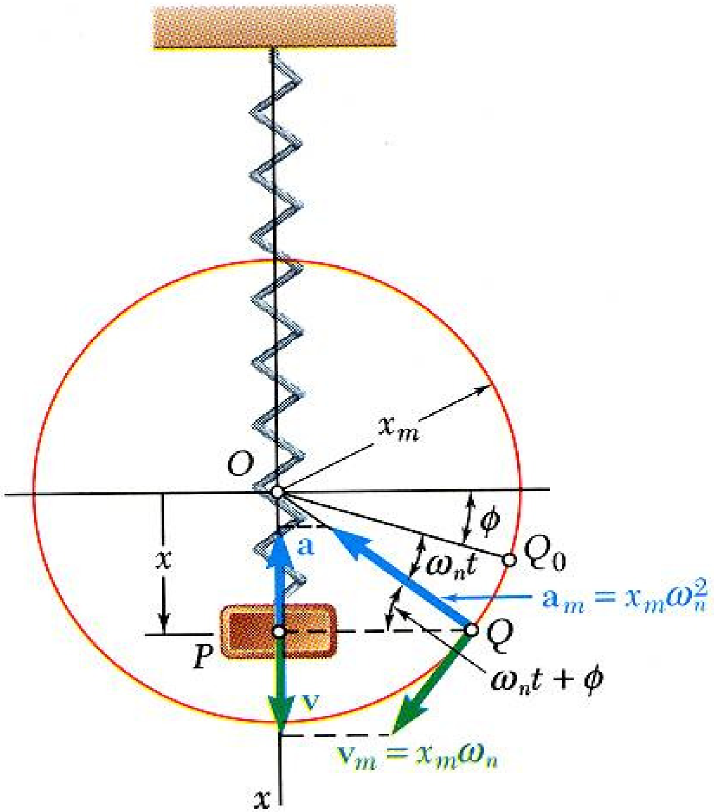
$$x_m = \sqrt{(v_0/\omega_n)^2 + x_0^2} = \textit{amplitude}$$

$$\phi = \tan^{-1}(v_0/x_0\omega_n) = \textit{phase angle}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \textit{period}$$

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} = \textit{natural frequency}$$

Free Vibrations of Particles. Simple Harmonic Motion



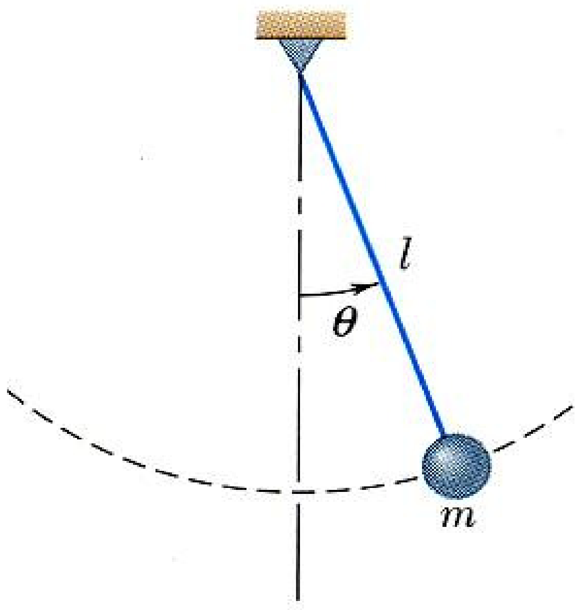
- Velocity-time and acceleration-time curves can be represented by sine curves of the same period as the displacement-time curve but different phase angles.

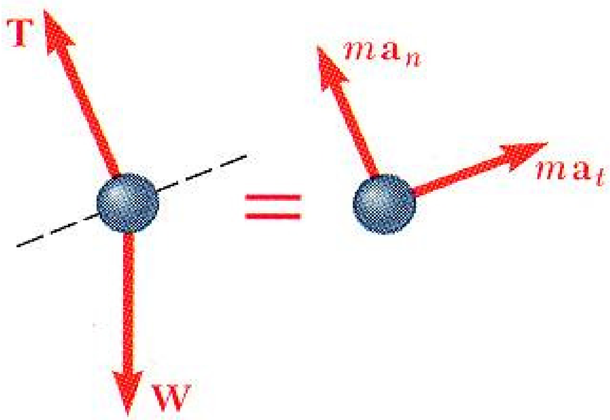
$$x = x_m \sin(\omega_n t + \phi)$$

$$\begin{aligned} v &= \dot{x} \\ &= x_m \omega_n \cos(\omega_n t + \phi) \\ &= x_m \omega_n \sin(\omega_n t + \phi + \pi/2) \end{aligned}$$

$$\begin{aligned} a &= \ddot{x} \\ &= -x_m \omega_n^2 \sin(\omega_n t + \phi) \\ &= x_m \omega_n^2 \sin(\omega_n t + \phi + \pi) \end{aligned}$$

Simple Pendulum (Approximate Solution)





- Results obtained for the spring-mass system can be applied whenever the resultant force on a particle is proportional to the displacement and directed towards the equilibrium position.
- Consider tangential components of acceleration and force for a simple pendulum,

$$\sum F_t = ma_t : \quad -W \sin\theta = ml\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

for small angles,

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}$$

Simple Pendulum (Exact Solution)

An exact solution for $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

leads to $$\tau_n = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2(\theta_m/2)\sin^2\phi}}$$

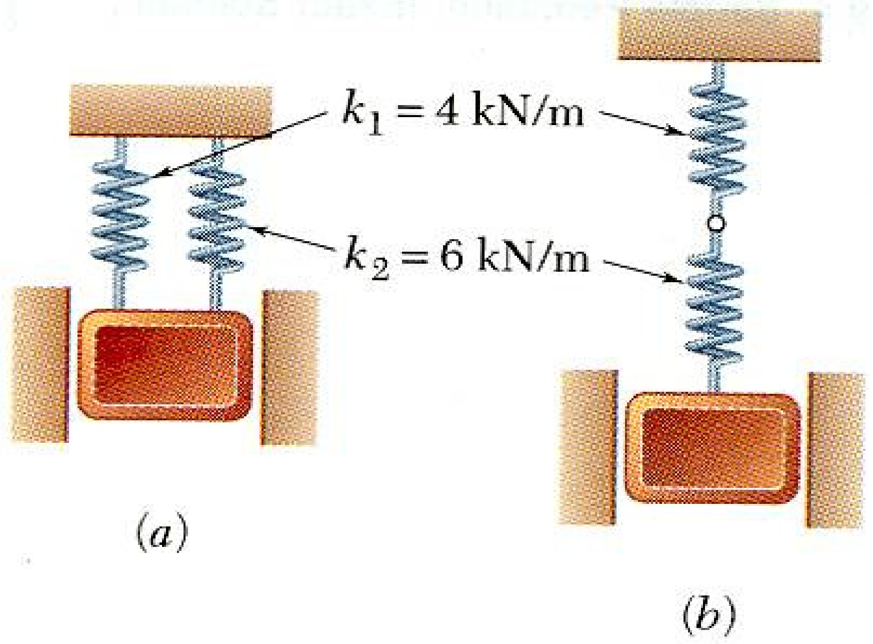
which requires numerical solution.

$$\tau_n = \frac{2K}{\pi}\left(2\pi\sqrt{\frac{l}{g}}\right)$$

Table 19.1. **Correction Factor for the Period of a Simple Pendulum**

θ_m	0°	10°	20°	30°	60°	90°	120°	150°	180°
K	1.571	1.574	1.583	1.598	1.686	1.854	2.157	2.768	∞
$2K/\pi$	1.000	1.002	1.008	1.017	1.073	1.180	1.373	1.762	∞

Sample Problem 19.1



A 50-kg block moves between vertical guides as shown. The block is pulled 40mm down from its equilibrium position and released.

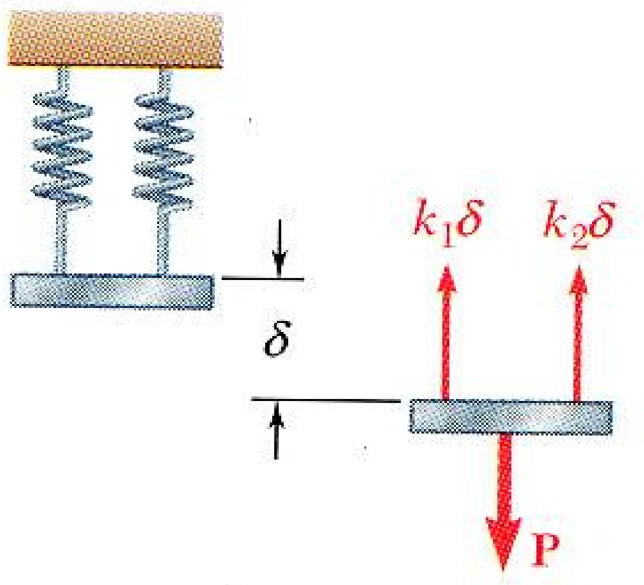
For each spring arrangement, determine *a*) the period of the vibration, *b*) the maximum velocity of the block, and *c*) the maximum acceleration of the block.

SOLUTION:

- For each spring arrangement, determine the spring constant for a single equivalent spring.
- Apply the approximate relations for the harmonic motion of a spring-mass system.

Sample Problem 19.1

$k_1 = 4\,\text{kN/m} \quad k_2 = 6\,\text{kN/m}$



$$P = k_1\delta + k_2\delta$$

$$k = \frac{P}{\delta} = k_1 + k_2$$

$$= 10\,\text{kN/m} = 10^4\,\text{N/m}$$

SOLUTION:

- Springs in parallel:
  - determine the spring constant for equivalent spring
  - apply the approximate relations for the harmonic motion of a spring-mass system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4\,\text{N/m}}{20\,\text{kg}}} = 14.14\,\text{rad/s}$$

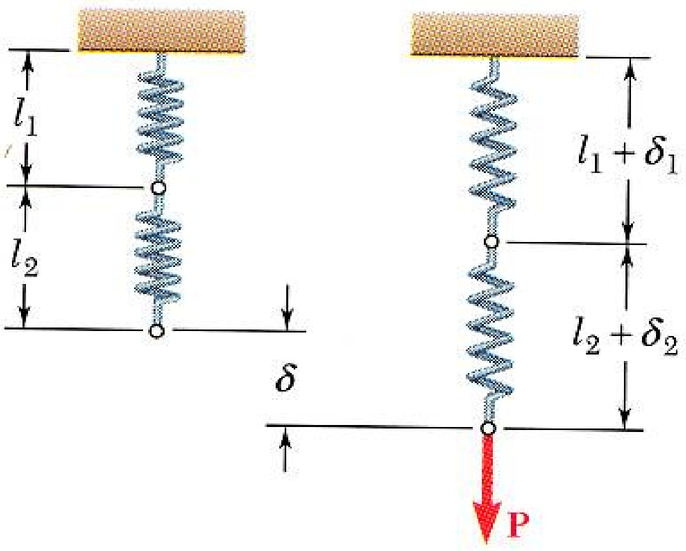
$$\tau_n = \frac{2\pi}{\omega_n} \qquad \boxed{\tau_n = 0.444\ \text{s}}$$

$$v_m = x_m\omega_n$$
$$= (0.040\ \text{m})(14.14\ \text{rad/s}) \qquad \boxed{v_m = 0.566\,\text{m/s}}$$

$$a_m = x_m a_n^2$$
$$= (0.040\ \text{m})(14.14\ \text{rad/s})^2 \qquad \boxed{a_m = 8.00\,\text{m/s}^2}$$

Sample Problem 19.1

$$k_1 = 4\,\mathrm{kN/m} \quad k_2 = 6\,\mathrm{kN/m}$$



$$P = k_1\delta + k_2\delta$$

$$k = \frac{P}{\delta} = k_1 + k_2$$

$$= 10\,\mathrm{kN/m} = 10^4\ \mathrm{N/m}$$

- Springs in series:
  - determine the spring constant for equivalent spring
  - apply the approximate relations for the harmonic motion of a spring-mass system

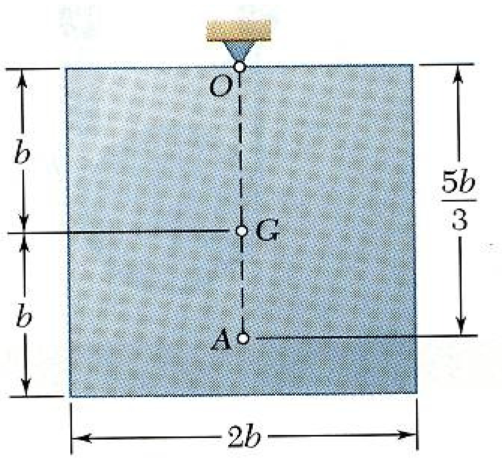
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2400\mathrm{N/m}}{20\,\mathrm{kg}}} = 6.93\,\mathrm{rad/s}$$

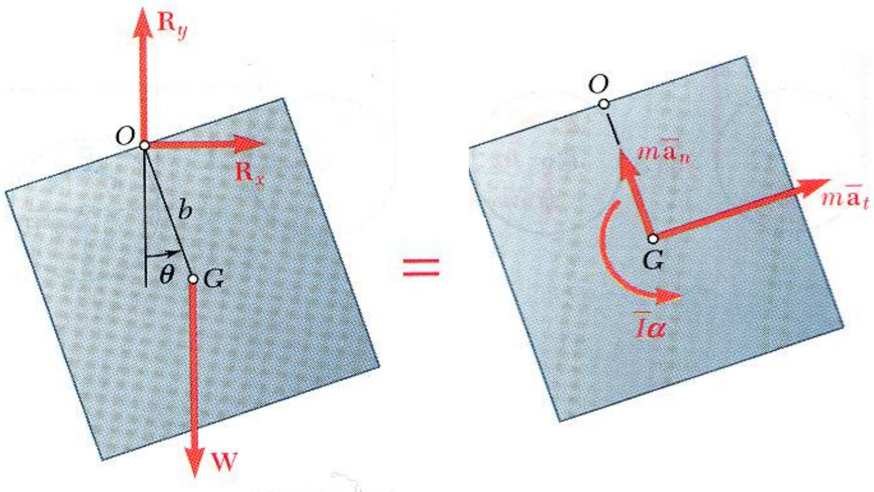
$$\tau_n = \frac{2\pi}{\omega_n} \qquad \boxed{\tau_n = 0.907\ \mathrm{s}}$$

$$v_m = x_m\omega_n$$
$$= (0.040\ \mathrm{m})(6.93\ \mathrm{rad/s}) \qquad \boxed{v_m = 0.277\ \mathrm{m/s}}$$

$$a_m = x_m a_n^2$$
$$= (0.040\ \mathrm{m})(6.93\ \mathrm{rad/s})^2 \qquad \boxed{a_m = 1.920\,\mathrm{m/s^2}}$$

Free Vibrations of Rigid Bodies





- If an equation of motion takes the form

$$\ddot{x} + \omega_n^2 x = 0 \quad \text{or} \quad \ddot{\theta} + \omega_n^2 \theta = 0$$

the corresponding motion may be considered as simple harmonic motion.

- Analysis objective is to determine ω_n.

- Consider the oscillations of a square plate

$$+\circlearrowleft \; -W(b\sin\theta) = (mb\ddot{\theta}) + \bar{I}\ddot{\theta}$$

but $\bar{I} = \frac{1}{12} m\left[(2b)^2 + (2b)^2\right] = \frac{2}{3} mb^2, \quad W = mg$

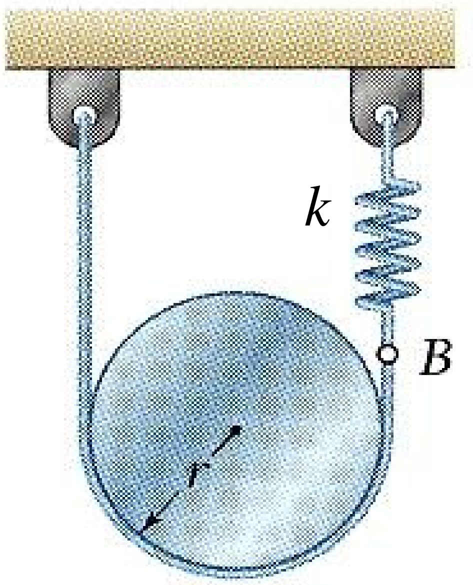
$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\sin\theta \cong \ddot{\theta} + \frac{3}{5}\frac{g}{b}\theta = 0$$

then $\omega_n = \sqrt{\frac{3g}{5b}}, \quad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{5b}{3g}}$

- For an equivalent simple pendulum,

$$l = 5b/3$$

Sample Problem 19.2



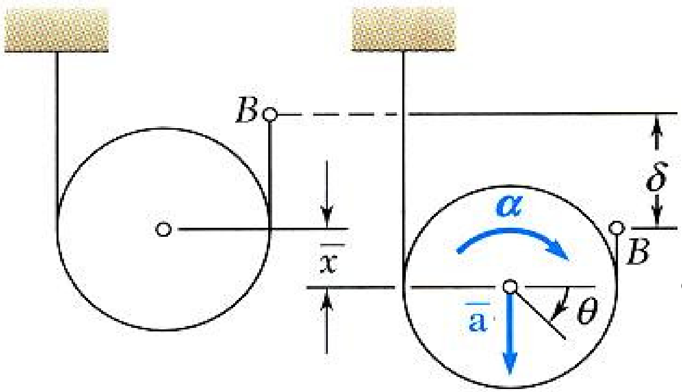
A cylinder of weight W is suspended as shown.

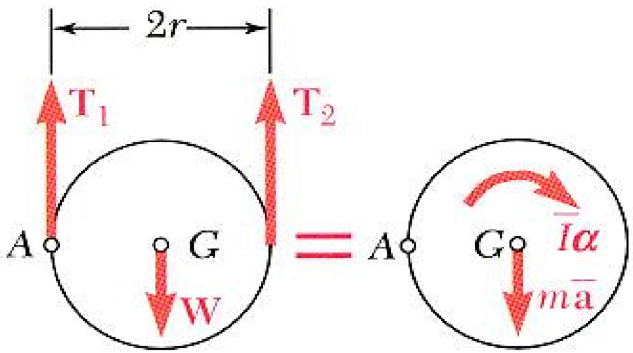
Determine the period and natural frequency of vibrations of the cylinder.

SOLUTION:

- From the kinematics of the system, relate the linear displacement and acceleration to the rotation of the cylinder.
- Based on a free-body-diagram equation for the equivalence of the external and effective forces, write the equation of motion.
- Substitute the kinematic relations to arrive at an equation involving only the angular displacement and acceleration.

Sample Problem 19.2





SOLUTION:

- From the kinematics of the system, relate the linear displacement and acceleration to the rotation of the cylinder.

$$\bar{x} = r\theta \qquad \delta = 2\bar{x} = 2r\theta$$

$$\vec{\alpha} = \ddot{\theta} \;+\circlearrowright \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \vec{\bar{a}} = r\ddot{\theta} \;+\!\downarrow$$

- Based on a free-body-diagram equation for the equivalence of the external and effective forces, write the equation of motion.

$$+\circlearrowright \sum M_A = \sum (M_A)_{eff} : \qquad Wr - T_2(2r) = m\bar{a}r + \bar{I}\alpha$$

but $\quad T_2 = T_0 + k\delta = \frac{1}{2}W + k(2r\theta)$

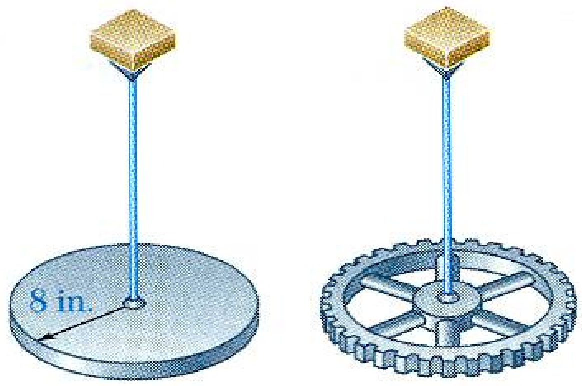
- Substitute the kinematic relations to arrive at an equation involving only the angular displacement and acceleration.

$$Wr - \left(\tfrac{1}{2}W + 2kr\theta\right)(2r) = m(r\ddot{\theta})r + \tfrac{1}{2}mr^2\theta$$

$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

$$\boxed{\omega_n = \sqrt{\frac{8k}{3m}}} \qquad \boxed{\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3m}{8k}}} \qquad \boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{8k}{3m}}}$$

Sample Problem 19.3



$W = 20$ lb

$\tau_n = 1.13$ s $\tau_n = 1.93$ s

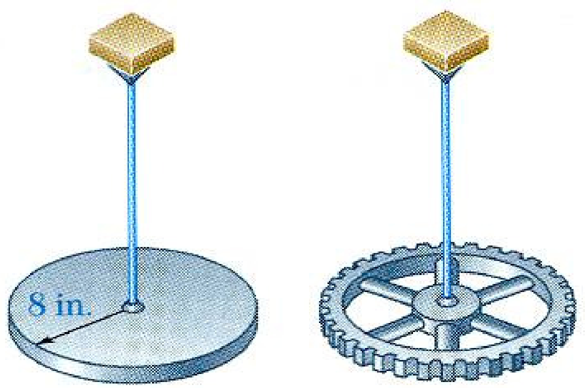
The disk and gear undergo torsional vibration with the periods shown. Assume that the moment exerted by the wire is proportional to the twist angle.

Determine *a*) the wire torsional spring constant, *b*) the centroidal moment of inertia of the gear, and *c*) the maximum angular velocity of the gear if rotated through 90° and released.

SOLUTION:

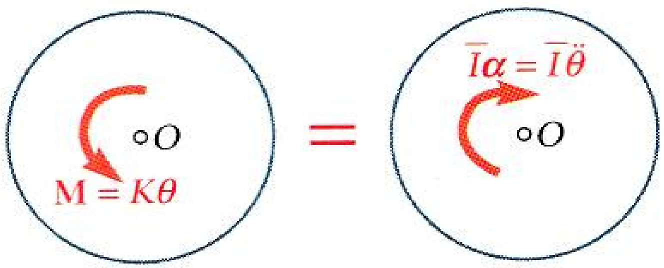
- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.
- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.
- With natural frequency and spring constant known, calculate the moment of inertia for the gear.
- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.

Sample Problem 19.3



$W = 20$ lb

$\tau_n = 1.13$ s $\qquad \tau_n = 1.93$ s



SOLUTION:

- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.

$$+\circlearrowright \sum M_O = \sum (M_O)_{eff}: \qquad +K\theta = -\bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

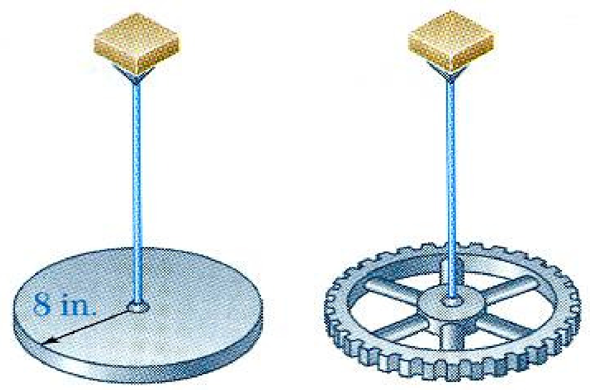
$$\omega_n = \sqrt{\frac{K}{\bar{I}}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{I}}{K}}$$

- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.

$$\bar{I} = \tfrac{1}{2}mr^2 = \frac{1}{2}\left(\frac{20}{32.2}\right)\left(\frac{8}{12}\right)^2 = 0.138 \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

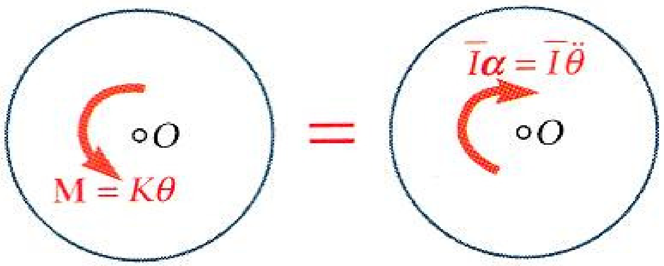
$$1.13 = 2\pi\sqrt{\frac{0.138}{K}} \qquad \boxed{K = 4.27 \text{ lb}\cdot\text{ft/rad}}$$

Sample Problem 19.3



$W = 20$ lb

$\tau_n = 1.13\,\text{s}$ $\tau_n = 1.93\ \text{s}$



$$\omega_n = \sqrt{\frac{K}{\bar{I}}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{I}}{K}}$$

$$K = 4.27\ \text{lb}\cdot\text{ft/rad}$$

- With natural frequency and spring constant known, calculate the moment of inertia for the gear.

$$1.93 = 2\pi\sqrt{\frac{\bar{I}}{4.27}}$$

$$\boxed{\bar{I} = 0.403\,\text{lb}\cdot\text{ft}\cdot\text{s}^2}$$

- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.

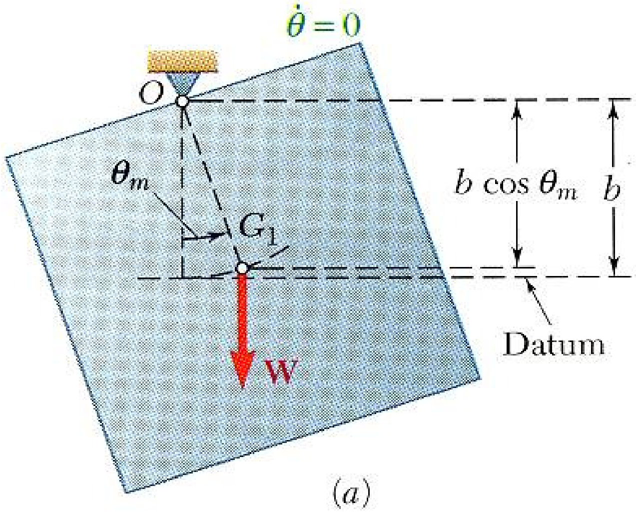
$$\theta = \theta_m \sin\omega_n t \qquad \omega = \theta_m\omega_n \sin\omega_n t \qquad \omega_m = \theta_m\omega_n$$

$$\theta_m = 90° = 1.571\ \text{rad}$$

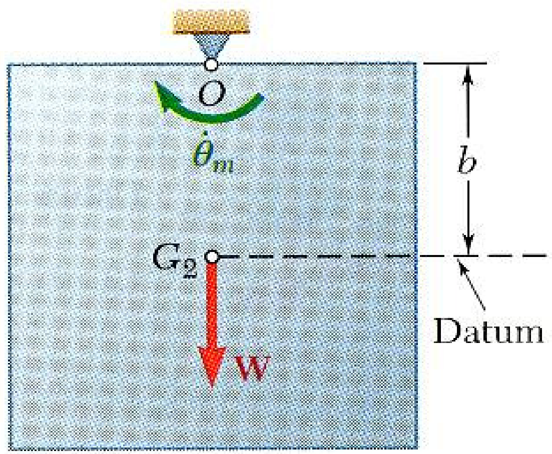
$$\omega_m = \theta_m\left(\frac{2\pi}{\tau_n}\right) = (1.571\ \text{rad})\left(\frac{2\pi}{1.93\ \text{s}}\right)$$

$$\boxed{\omega_m = 5.11\,\text{rad/s}}$$

Principle of Conservation of Energy



(a)



(b)

- Resultant force on a mass in simple harmonic motion is conservative - total energy is conserved.

$$T + V = \text{constant} \qquad \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 = \text{constant}$$

$$\dot{x}^2 + \omega_n^2 x^2 =$$

- Consider simple harmonic motion of the square plate,

$$T_1 = 0 \qquad V_1 = Wb(1 - \cos\theta) = Wb\left[2\sin^2(\theta_m/2)\right]$$

$$\cong \tfrac{1}{2} Wb\theta_m^2$$

$$T_2 = \tfrac{1}{2} m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 \qquad V_2 = 0$$

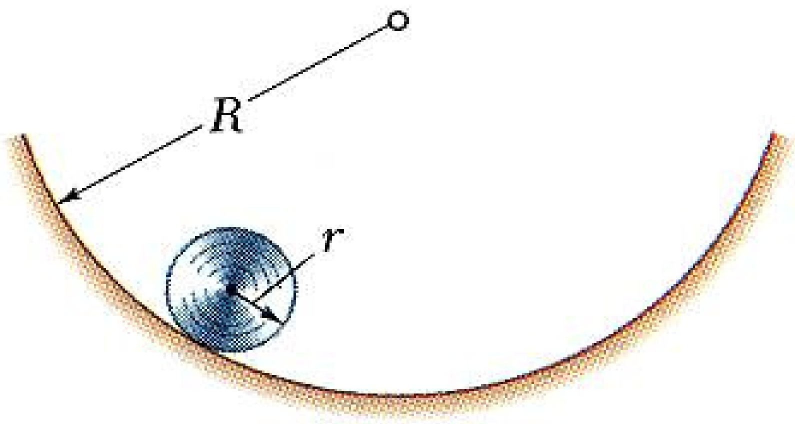
$$= \tfrac{1}{2} m\left(b\dot{\theta}_m\right)^2 + \tfrac{1}{2}\left(\tfrac{2}{3} mb^2\right)\omega_m^2$$

$$= \tfrac{1}{2}\left(\tfrac{5}{3} mb^2\right)\dot{\theta}_m^2$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + \tfrac{1}{2} Wb\theta_m^2 = \tfrac{1}{2}\left(\tfrac{5}{3} mb^2\right)\theta_m^2\omega_n^2 + 0 \qquad \omega_n = \sqrt{3g/5b}$$

Sample Problem 19.4

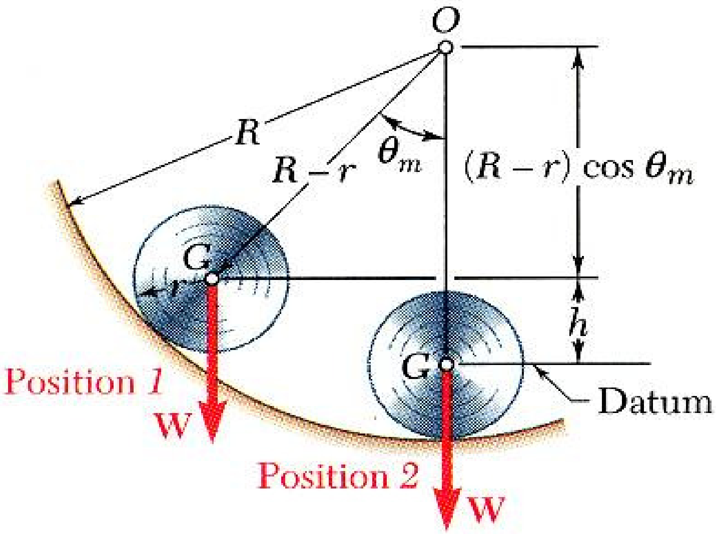


Determine the period of small oscillations of a cylinder which rolls without slipping inside a curved surface.

SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.
- Solve the energy equation for the natural frequency of the oscillations.

Sample Problem 19.4

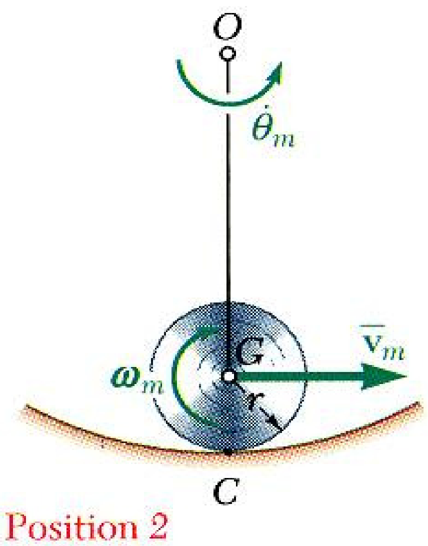


SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.

$$T_1 + V_1 = T_2 + V_2$$

$$T_1 = 0 \qquad V_1 = Wh = W(R-r)(1-\cos\theta) \cong W(R-r)\left(\theta_m^2/2\right)$$

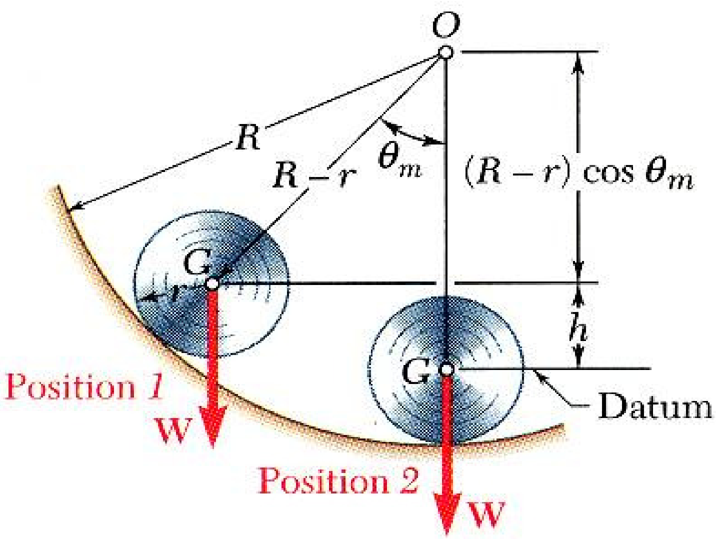


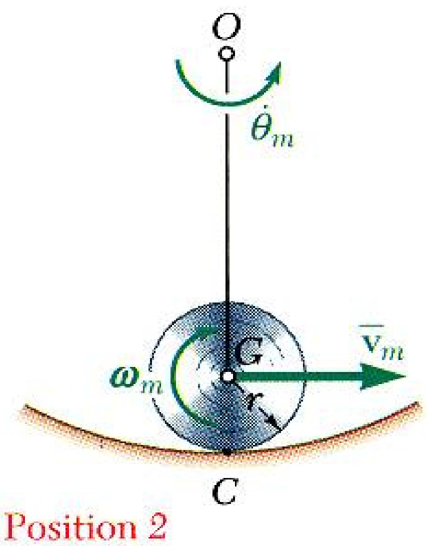
$$T_2 = \tfrac{1}{2} m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 \qquad V_2 = 0$$

$$= \tfrac{1}{2} m(R-r)\dot{\theta}_m^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\left(\frac{R-r}{r}\right)^2\dot{\theta}_m^2$$

$$= \tfrac{3}{4} m(R-r)^2\dot{\theta}_m^2$$

Sample Problem 19.4





- Solve the energy equation for the natural frequency of the oscillations.

$$T_1 = 0 \qquad V_1 \cong W(R-r)\left(\theta_m^2/2\right)$$

$$T_2 = \tfrac{3}{4}m(R-r)^2\dot{\theta}_m^2 \qquad V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2$$

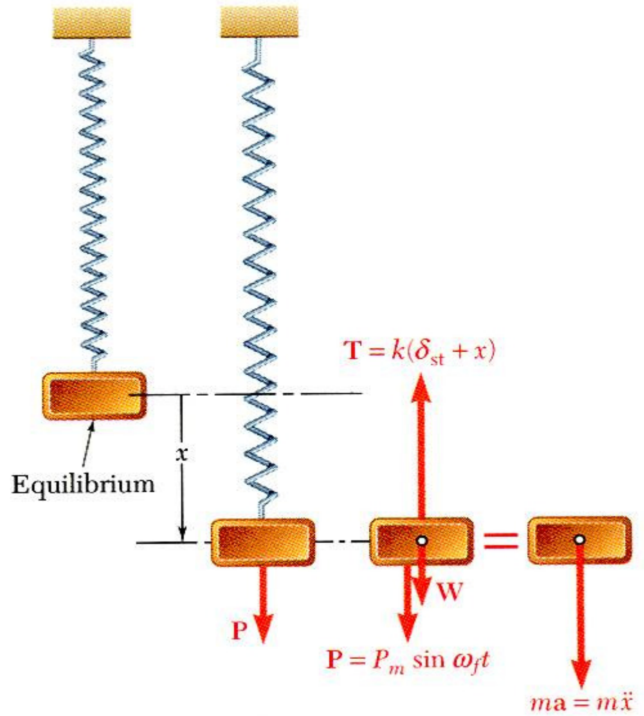
$$0 + W(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R-r)^2\dot{\theta}_m^2 + 0$$

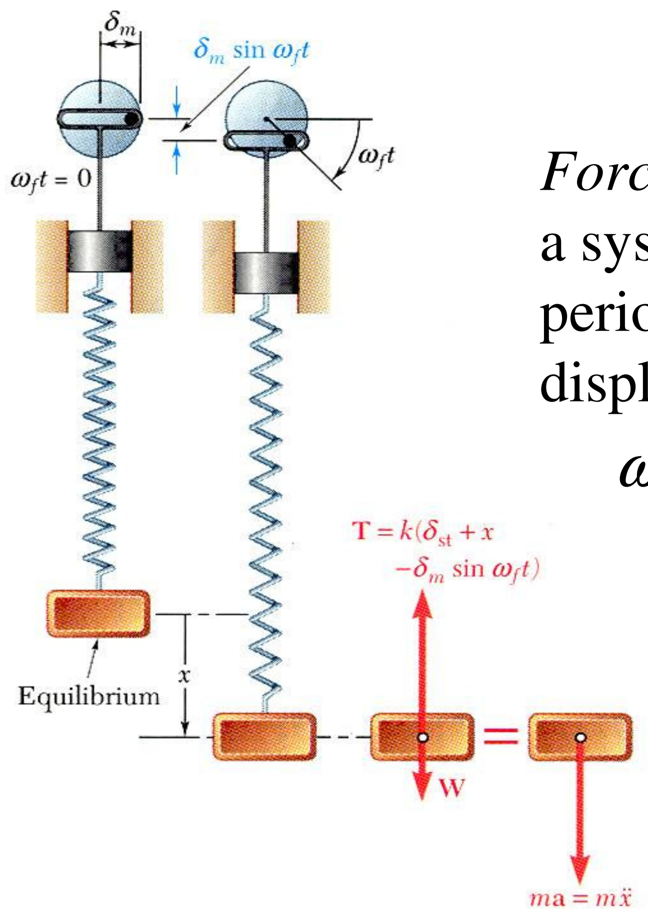
$$(mg)(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R-r)^2(\theta_m\omega_n)_m^2$$

$$\omega_n^2 = \frac{2}{3}\frac{g}{R-r}$$

$$\boxed{\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3}{2}\frac{R-r}{g}}}$$

Forced Vibrations





Forced vibrations - Occur when a system is subjected to a periodic force or a periodic displacement of a support.

$$\omega_f = \textit{forced frequency}$$

$$+\downarrow \sum F = ma :$$

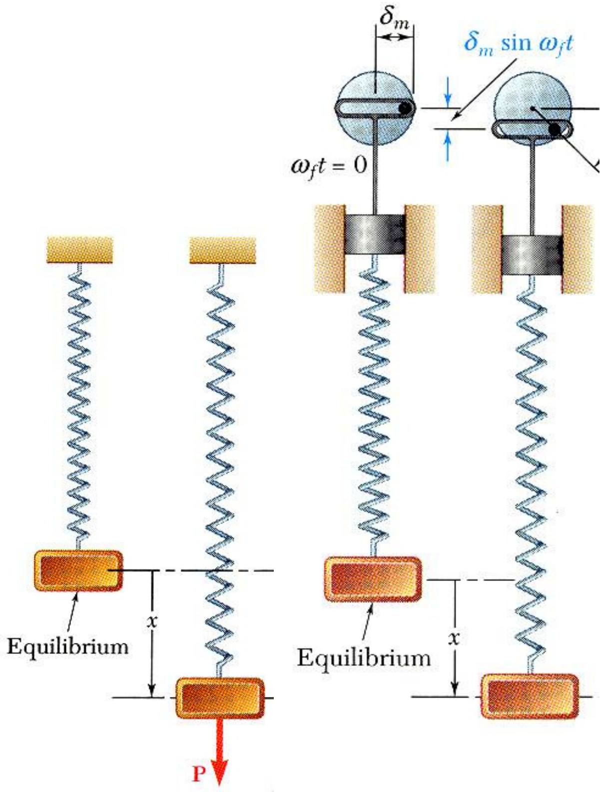
$$P_m \sin\omega_f t + W - k(\delta_{st} + x) = m\ddot{x}$$

$$m\ddot{x} + kx = P_m \sin\omega_f t$$

$$W - k(\delta_{st} + x - \delta_m \sin\omega_f t) = m\ddot{x}$$

$$m\ddot{x} + kx = k\delta_m \sin\omega_f t$$

Forced Vibrations



$$m\ddot{x} + kx = P_m \sin \omega_f t$$

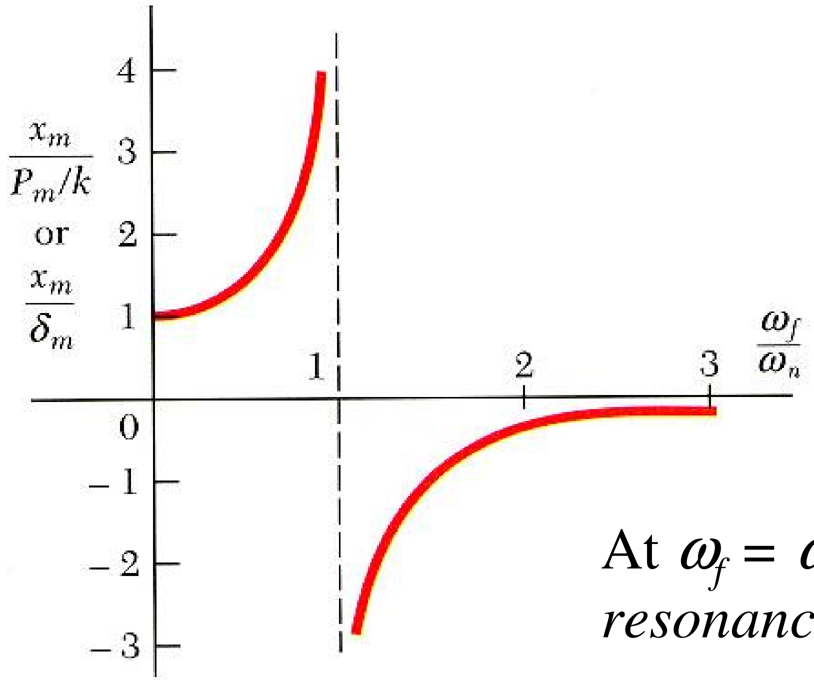
$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

$$x = x_{complementary} + x_{particular}$$
$$= \left[C_1 \sin \omega_n t + C_2 \cos \omega_n t\right] + x_m \sin \omega_f t$$

Substituting particular solution into governing equation,

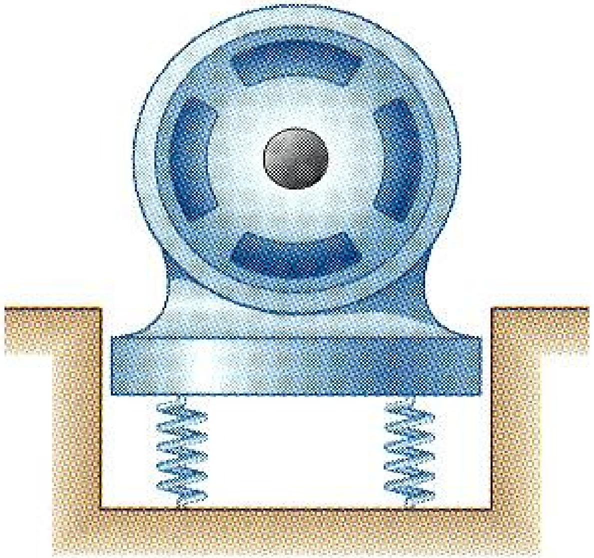
$$-m\omega_f^2 x_m \sin \omega_f t + kx_m \sin \omega_f t = P_m \sin \omega_f t$$

$$x_m = \frac{P_m}{k - m\omega_f^2} = \frac{P_m/k}{1 - (\omega_f/\omega_n)^2} = \frac{\delta_m}{1 - (\omega_f/\omega_n)^2}$$



At $\omega_f = \omega_n$, forcing input is in *resonance* with the system.

Sample Problem 19.5



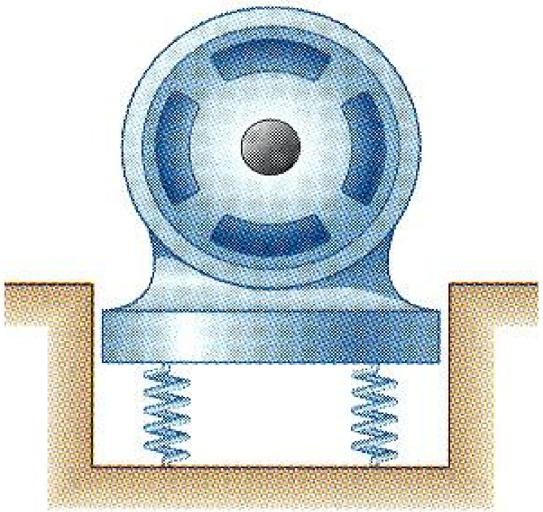
A motor weighing 350 lb is supported by four springs, each having a constant 750 lb/in. The unbalance of the motor is equivalent to a weight of 1 oz located 6 in. from the axis of rotation.

Determine *a*) speed in rpm at which resonance will occur, and *b*) amplitude of the vibration at 1200 rpm.

SOLUTION:

- The resonant frequency is equal to the natural frequency of the system.
- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.

Sample Problem 19.5



$W = 350$ lb

$k = 4(350$ lb/in)

SOLUTION:

- The resonant frequency is equal to the natural frequency of the system.

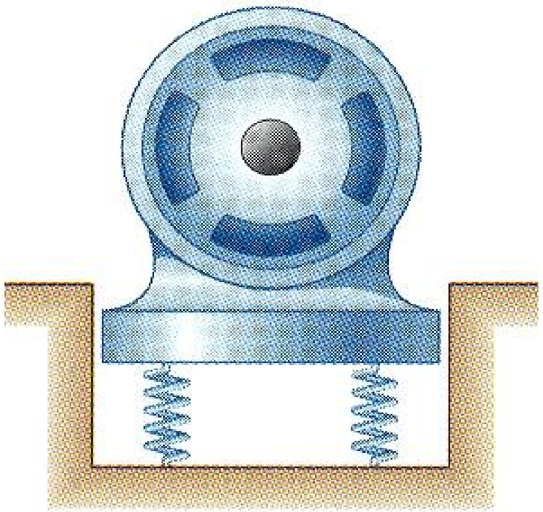
$$m = \frac{350}{32.2} = 10.87\,\text{lb}\cdot\text{s}^2/\text{ft}$$

$$k = 4(750) = 3000\,\text{lb/in}$$
$$= 36{,}000\,\text{lb/ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{36{,}000}{10.87}}$$
$$= 57.5\ \text{rad/s} = 549\ \text{rpm}$$

$$\boxed{\text{Resonance speed} = 549\ \text{rpm}}$$

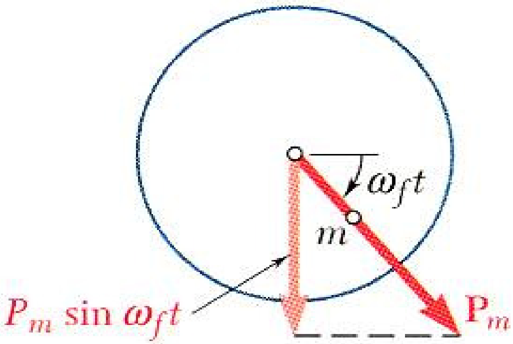
Sample Problem 19.5



$W = 350 \text{ lb}$

$k = 4(350 \text{ lb/in})$

$\omega_n = 57.5 \text{ rad/s}$



- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.

$$\omega_f = \omega = 1200 \text{ rpm} = 125.7 \text{ rad/s}$$

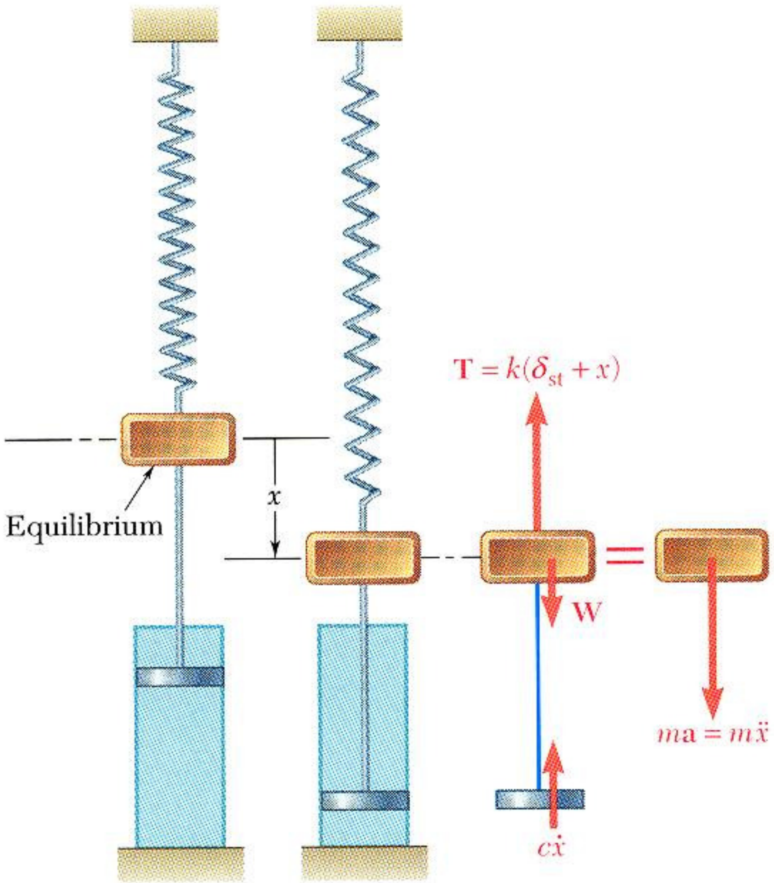
$$m = (1 \text{ oz})\left(\frac{1 \text{ lb}}{16 \text{ oz}}\right)\left(\frac{1}{32.2 \text{ ft}/\text{s}^2}\right) = 0.001941 \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$P_m = ma_n = mr\omega^2$$
$$= (0.001941)\left(\tfrac{6}{12}\right)(125.7)^2 = 15.33 \text{ lb}$$

$$x_m = \frac{P_m/k}{1-\left(\omega_f/\omega_n\right)^2} = \frac{15.33/3000}{1-(125.7/57.5)^2}$$
$$= -0.001352 \text{ in}$$

$$\boxed{x_m = 0.001352 \text{ in. (out of phase)}}$$

Damped Free Vibrations



- All vibrations are damped to some degree by forces due to *dry friction*, *fluid friction*, or *internal friction*.
- With *viscous damping* due to fluid friction,

$$+\downarrow \sum F = ma: \qquad W - k(\delta_{st} + x) - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

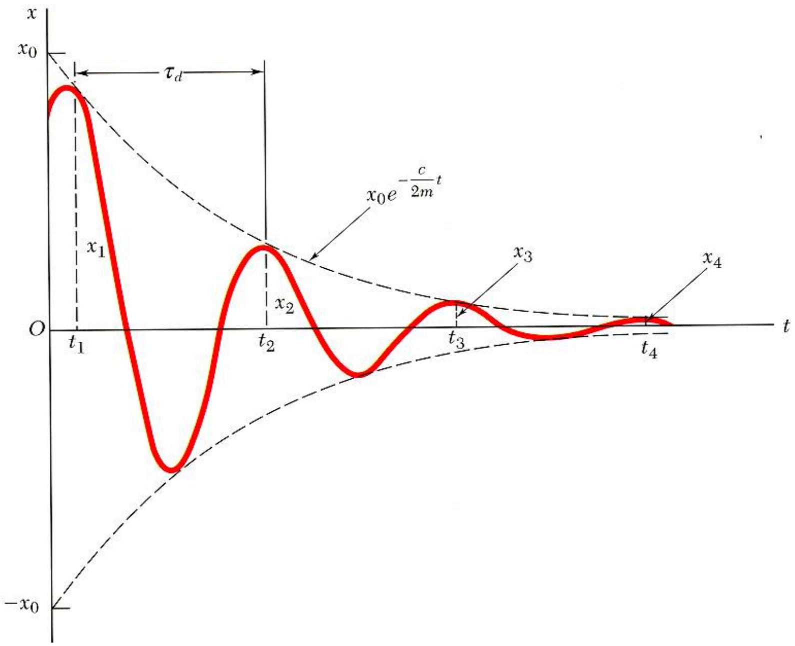
- Substituting $x = e^{\lambda t}$ and dividing through by $e^{\lambda t}$ yields the *characteristic equation*,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- Define the critical damping coefficient such that

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

Damped Free Vibrations



- Characteristic equation,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$c_c = 2m\omega_n = \text{critical damping coefficient}$$

- *Heavy damping*: $c > c_c$

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

  - negative roots
  - nonvibratory motion

- *Critical damping*: $c = c_c$

$$x = (C_1 + C_2 t)e^{-\omega_n t}$$

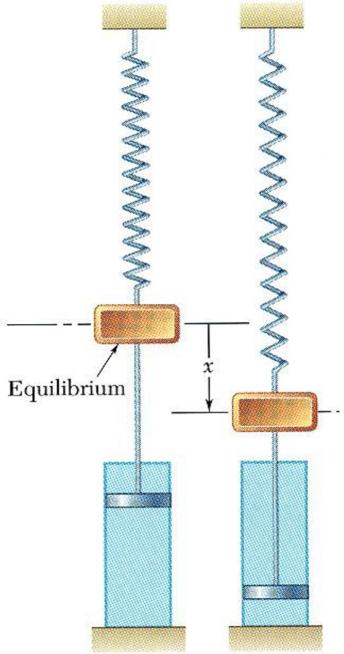
  - double roots
  - nonvibratory motion

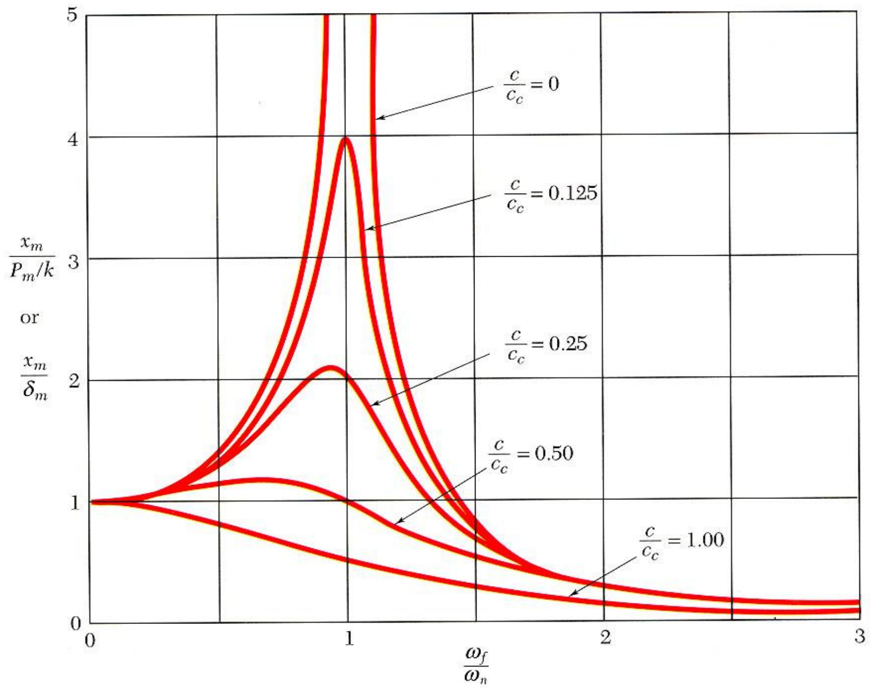
- *Light damping*: $c < c_c$

$$x = e^{-(c/2m)t}\left(C_1 \sin \omega_d t + C_2 \cos \omega_d t\right)$$

$$\omega_d = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2} = \text{damped frequency}$$

Damped Forced Vibrations



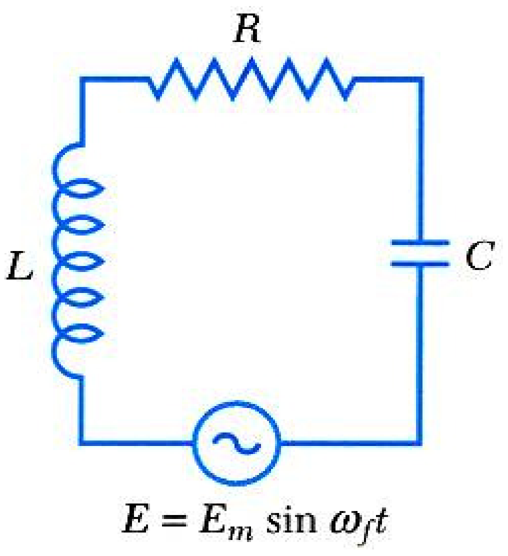


$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \qquad x = x_{complementary} + x_{particular}$$

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta} = \frac{1}{\sqrt{\left[1-\left(\omega_f/\omega_n\right)^2\right]^2 + \left[2\left(c/c_c\right)\left(\omega_f/\omega_n\right)\right]^2}} = \text{magnification factor}$$

$$\tan\phi = \frac{2\left(c/c_c\right)\left(\omega_f/\omega_n\right)}{1-\left(\omega_f/\omega_n\right)^2} = \text{phase difference between forcing and steady state response}$$

Electrical Analogues



- Consider an electrical circuit consisting of an inductor, resistor and capacitor with a source of alternating voltage

$$E_m \sin \omega_f t - L\frac{di}{dt} - Ri - \frac{q}{C} = 0$$

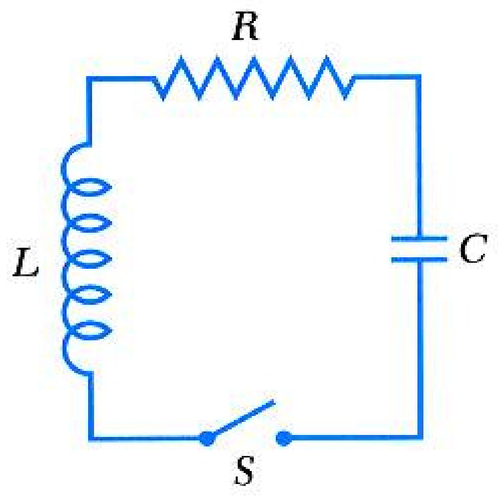
$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_m \sin \omega_f t$$

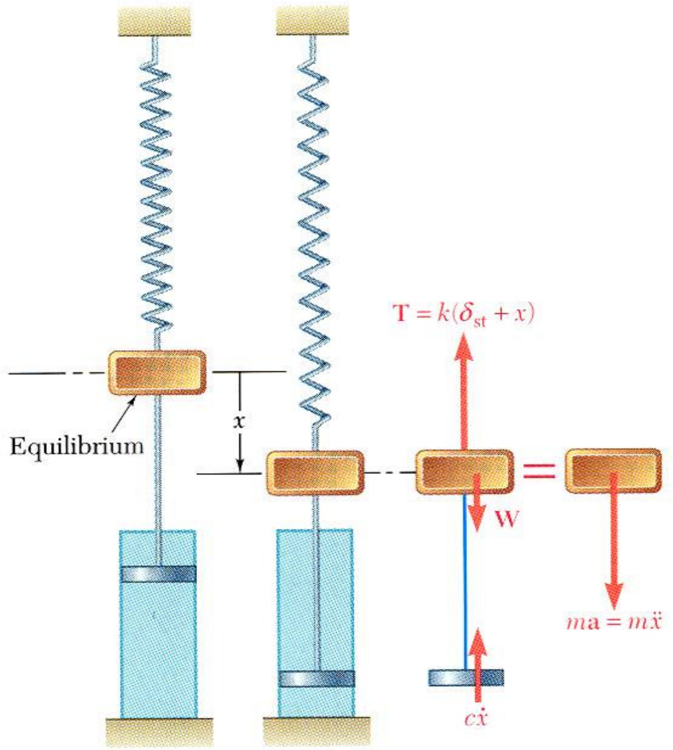
- Oscillations of the electrical system are analogous to damped forced vibrations of a mechanical system.

Table 19.2. Characteristics of a Mechanical System and of Its Electrical Analogue

Mechanical System		Electrical Circuit	
m	Mass	L	Inductance
c	Coefficient of viscous damping	R	Resistance
k	Spring constant	$1/C$	Reciprocal of capacitance
x	Displacement	q	Charge
v	Velocity	i	Current
P	Applied force	E	Applied voltage

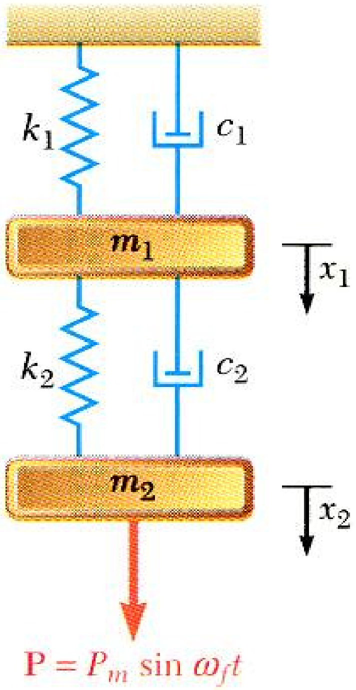
Electrical Analogues

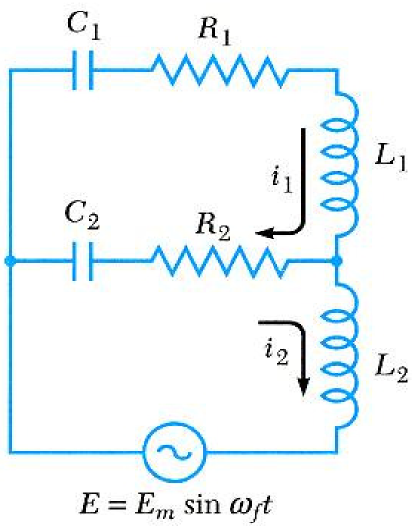




- The analogy between electrical and mechanical systems also applies to transient as well as steady-state oscillations.
- With a charge $q = q_0$ on the capacitor, closing the switch is analogous to releasing the mass of the mechanical system with no initial velocity at $x = x_0$.
- If the circuit includes a battery with constant voltage E, closing the switch is analogous to suddenly applying a force of constant magnitude P to the mass of the mechanical system.

Electrical Analogues





- The electrical system analogy provides a means of experimentally determining the characteristics of a given mechanical system.

- For the mechanical system,

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + c_2 (\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_2 (x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) = P_m \sin \omega_f t$$

- For the electrical system,

$$L_1 \ddot{q}_1 + R_1 (\dot{q}_1 - \dot{q}_2) + \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_2} = 0$$

$$L_2 \ddot{q}_2 + R_2 (\dot{q}_2 - \dot{q}_1) + \frac{q_2 - q_1}{C_2} = E_m \sin \omega_f t$$

- The governing equations are equivalent. The characteristics of the vibrations of the mechanical system may be inferred from the oscillations of the electrical system.