بسم الله الرحمن الرحیم

Theory of Computer Science

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What we are going to discuss?

Theory of Computation

Complexity Theory

another topic …

- Mining Massive Datasets
- Computational Learning Theory
- Parameterized Algorithms

Theory of Computation

- Church-Turing thesis, Turing machine & its variations
- exploring the limits of algorithmic solvability
- reducibility as a key method to prove unsolvability
- recursive/partial recursive functions
- decidability in terms of recursion
- arriving at Turing machines
- decidability of logical theories
- Turing reducibilities

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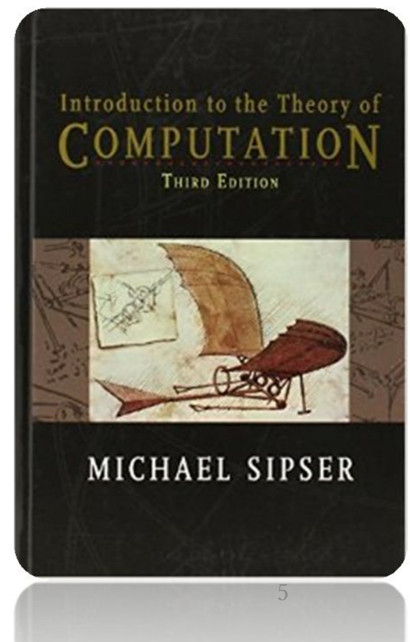
[S12] Sipser, Michael. **Introduction to the Theory of Computation**, 3rd edition. Cengage Learning, 2012. (Chapters 3 to 6)



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- Church-Turing thesis, Turing machine & its variations
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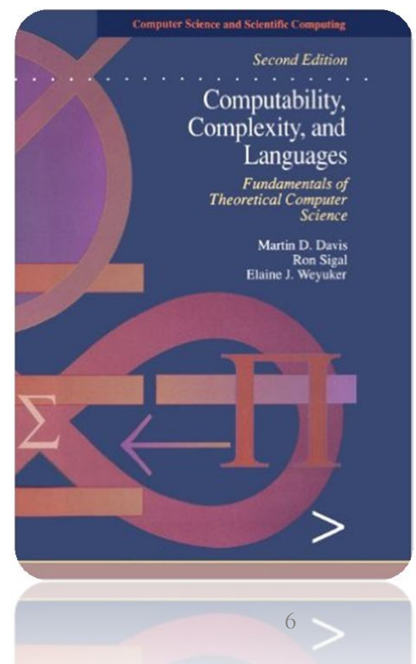

[DSW94] Davis, Martin, Ron Sigal, and Elaine J. Weyuker. **Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science**. Newnes, 1994. (Chapters 2 to 6)

Theory of Computation

Course note based on *Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science*, 2nd edition, authored by Martin Davis, Ron Sigal, and Elaine J. Weyuker.

course note prepared by

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Week 1, Spring 2010

Textbook

Martin Davis, Ron Sigal, and Elaine J. Weyuker. *Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science*, 2nd edition. February 1994, Morgan Kaufmann. ISBN: 0122063821.

- Written for people who may know programming, but from a mathematical view of the subjects. Enjoyably readable but very rigorous.
- "It is our purpose . . . to provide an introduction to the various aspects of theoretical computer science for undergraduate and graduate students that is sufficiently comprehensive that . . . research papers will become accessible to our readers." (the authors)
- We will cover just one half of the materials in the book.

Outline of Today's Lecture

- Review some preliminary materials.
- Define an abstract programming language $\mathscr{S}$ that is extremely simple.
- Write some programs in $\mathscr{S}$.

Cartesian Product

- If $S_1, S_2, \ldots, S_n$ are given sets, then we write $S_1 \times S_2, \times \cdots \times S_n$ for the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_1 \in S_1, a_2 \in S_2, \ldots, a_n \in S_n$.
- $S_1 \times S_2, \times \cdots \times S_n$ is called the *Cartesian product* of $S_1, S_2, \ldots, S_n$.
- In case $S_1 = S_2 = \cdots = S_n = S$ we write S^n for the Cartesian product $S_1 \times S_2, \times \cdots \times S_n$.

Functions

- A function f is a set whose members are ordered pairs (i.e., 2-tuples) and has the special property

  $$(a, b) \in f \text{ and } (a, c) \in f \quad \text{implies} \quad b = c.$$

  We write $f(a) = b$ to mean that $(a, b) \in f$.

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- A *partial function* on a set S is a function whose domain is a subset of S. If a partial function on S has the domain S, then it is called a *total function*.

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- A *partial function* on a set S is a function whose domain is a subset of S. If a partial function on S has the domain S, then it is called a *total function*.
- We write $f(a)\downarrow$ and say that $f(a)$ is *defined* if a is in the domain of f; if a is not in the domain of f, we write $f(a)\uparrow$ and say that $f(a)$ is *undefined*.

Examples of Functions

- Let f be the set of ordered pairs (n, n^2) for $n \in N$. Then, for each $n \in N$, $f(n) = n^2$. The domain of f is N. The range of f is the set of perfect squares. f is a total function.

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- For a partial function f on a Cartesian product $S_1 \times S_2, \times \cdots \times S_n$, we write $f(a_1, \ldots, a_n)$ rather than $f((a_1, \ldots, a_n))$.

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- A partial function f on a set S^n is called an *n-ary* partial function on S, or a function of n variables on S. We use *unary* and *binary* for 1-ary and 2-ary, respectively.

Predicate

A *predicate*, or a *Boolean-valued function*, on a set S is a *total* function P on S such that for each $a \in S$, either

$$P(a) = \text{TRUE} \quad \text{or} \quad P(a) = \text{FALSE}$$

We also identify the truth value TRUE with number 1 and the truth value FALSE with number 0.

Logic Connectives

The three *logic connectives*, or *propositional connectives*, $\sim, \vee, \&$ are defined by the two tables below.

p	$\sim p$
0	1
1	0

p	q	$p \,\&\, q$	$p \vee q$
1	1	1	1
0	1	0	1
1	0	0	1
0	0	0	0

Characteristic Function

Given a predicate P on a set S, there is a corresponding subset R of S consisting of all elements $a \in S$ for which $P(a) = 1$. We write

$$R = \{a \in S \mid P(a)\}.$$

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Conversely, given a subset R of a given set S, the expression $x \in R$ defines a predicate P on S:

$$P(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{if } x \notin R. \end{cases}$$

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The predicate P is called the *characteristic function* of the set R. Note the easy translations between the two notations:

$$\begin{aligned} \{x \in S \mid P(x) \,\&\, Q(x)\} &= \{x \in S \mid P(x)\} \cap \{x \in S \mid Q(x)\}, \\ \{x \in S \mid P(x) \vee Q(x)\} &= \{x \in S \mid P(x)\} \cup \{x \in S \mid Q(x)\}, \\ \{x \in S \mid \sim P(x)\} &= S - \{x \in S \mid P(x)\}. \end{aligned}$$

Bounded Existential Quantifier

Let $P(t, x_1, \ldots, x_n)$ be a $(n+1)$-ary predicate. Let predicate $Q(y, x_1, \ldots, x_n)$ be defined by

$$\begin{array}{rcl} Q(y, x_1, \ldots, x_n) & = & P(0, x_1, \ldots, x_n) \\ & \vee & P(1, x_1, \ldots, x_n) \\ & \vee & \ldots \\ & \vee & P(y, x_1, \ldots, x_n) \end{array}$$

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That is, $Q(y, x_1, \ldots, x_n)$ is true if there is a value $t \leq y$ such that $P(t, x_1, \ldots, x_n)$ is true. We write this predicate Q as

$$(\exists t)_{\leq y} P(t, x_1, \ldots, x_n)$$

"$(\exists t)_{\leq y}$" is called a *bounded existential quantifier*.

Bounded Universal Quantifier

Let $P(t, x_1, \ldots, x_n)$ be a $(n+1)$-ary predicate. Let predicate $Q(y, x_1, \ldots, x_n)$ be defined by

$$\begin{aligned} Q(y, x_1, \ldots, x_n) &= P(0, x_1, \ldots, x_n) \\ &\;\& \;\; P(1, x_1, \ldots, x_n) \\ &\;\& \;\; \ldots \\ &\;\& \;\; P(y, x_1, \ldots, x_n) \end{aligned}$$

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That is, $Q(y, x_1, \ldots, x_n)$ is true if for all value $t \leq y$ such that $P(t, x_1, \ldots, x_n)$ is true. We write this predicate Q as

$$(\forall t)_{\leq y} P(t, x_1, \ldots, x_n)$$

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- Labels: $A_1, B_1, C_1, D_1, E_1, A_2, B_2, C_2, D_2, E_2, A_3, \ldots$
- Exit label: E.

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- Exit label: E.
- All variables and labels are in the global scope.

Programming in $\mathscr{S}$

- A program is a list (i.e., a finite sequence) of instructions.
- *The output variable* Y *and the local variables* Z_i *initially have the value* 0.
- A program halts when there is no more instruction to execute.
- A program also halts if an instruction labeled L is to be executed, but there is no instruction with that label.

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- A program halts when there is no more instruction to execute.
- A program also halts if an instruction labeled L is to be executed, but there is no instruction with that label.
- What does this program do?

$$
\begin{array}{ll}
[A] & X \leftarrow X - 1 \\
 & Y \leftarrow Y + 1 \\
 & \text{IF } X \neq 0 \text{ GOTO } A
\end{array}
$$

A Bug?

- What does this program do?

$$
\begin{array}{ll}
[A] & X \leftarrow X - 1 \\
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- The above program *computes* the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{otherwise.} \end{cases}$$

A Program That Computes $f(x) = x$

$$
\begin{array}{ll}
[A] & \text{IF } X \neq 0 \text{ GOTO } B \\
 & Z \leftarrow Z + 1 \\
 & \text{IF } Z \neq 0 \text{ GOTO } E \\
[B] & X \leftarrow X - 1 \\
 & Y \leftarrow Y + 1 \\
 & Z \leftarrow Z + 1 \\
 & \text{IF } Z \neq 0 \text{ GOTO } A
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- What does Z actually do?

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\end{array}
$$

- What does Z actually do?
- What does the following do?

$$
\begin{array}{l}
Z \leftarrow Z + 1 \\
\text{IF } Z \neq 0 \text{ GOTO } L
\end{array}
$$

A Program That Computes $f(x) = x$

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[A] & \text{IF } X \neq 0 \text{ GOTO } B \\
 & Z \leftarrow Z + 1 \\
 & \text{IF } Z \neq 0 \text{ GOTO } E \\
[B] & X \leftarrow X - 1 \\
 & Y \leftarrow Y + 1 \\
 & Z \leftarrow Z + 1 \\
 & \text{IF } Z \neq 0 \text{ GOTO } A
\end{array}
$$

- What does Z actually do?
- What does the following do?

$$
\begin{array}{l}
Z \leftarrow Z + 1 \\
\text{IF } Z \neq 0 \text{ GOTO } L
\end{array}
$$

- That is an unconditional goto!

$$\text{GOTO } L$$

A *Macro* for Unconditional GOTO

- Before macro expansion:

$$
\begin{array}{ll}
[A] & \text{IF } X \neq 0 \text{ GOTO } B \\
 & \text{GOTO } E \\
[B] & X \leftarrow X - 1 \\
 & Y \leftarrow Y + 1 \\
 & \text{GOTO } A
\end{array}
$$

A *Macro* for Unconditional GOTO

- Before macro expansion:

$[A]$	IF $X \neq 0$ GOTO B
	GOTO E
$[B]$	$X \leftarrow X - 1$
	$Y \leftarrow Y + 1$
	GOTO A

- After macro expansion:

$[A]$	IF $X \neq 0$ GOTO B
	$Z_1 \leftarrow Z_1 + 1$
	IF $Z_1 \neq 0$ GOTO E
$[B]$	$X \leftarrow X - 1$
	$Y \leftarrow Y + 1$
	$Z_2 \leftarrow Z_2 + 1$
	IF $Z_2 \neq 0$ GOTO A

A *Macro* for Unconditional GOTO

- Before macro expansion:

  $[A]$ IF $X \neq 0$ GOTO B
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  $[B]$ $X \leftarrow X - 1$
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- After macro expansion:

  $[A]$ IF $X \neq 0$ GOTO B
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  IF $Z_1 \neq 0$ GOTO E
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z_2 \leftarrow Z_2 + 1$
  IF $Z_2 \neq 0$ GOTO A

- *Fresh local variables are always used during macro expansions.*

Copy The Value of Variable X to Variable Y

- $[A]$ IF $X \neq 0$ GOTO B
  GOTO E
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
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Copy The Value of Variable X to Variable Y

- $[A]$ IF $X \neq 0$ GOTO B
  GOTO E
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  GOTO A
- Anything wrong?

Copy The Value of Variable X to Variable Y

- $$\begin{array}{ll} [A] & \text{IF } X \neq 0 \text{ GOTO } B \\ & \text{GOTO } E \\ [B] & X \leftarrow X - 1 \\ & Y \leftarrow Y + 1 \\ & \text{GOTO } A \end{array}$$
- Anything wrong?
- The value of X is "destroyed" while copied to Y!

Copy The Value of Variable X to Variable Y, Continued

- $[A]$ IF $X \neq 0$ GOTO B
  GOTO C
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z \leftarrow Z + 1$
  GOTO A
  $[C]$ IF $Z \neq 0$ GOTO D
  GOTO E
  $[D]$ $Z \leftarrow Z - 1$
  $X \leftarrow X + 1$
  GOTO C

Copy The Value of Variable X to Variable Y, Continued

- $[A]$ IF $X \neq 0$ GOTO B
  GOTO C
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z \leftarrow Z + 1$
  GOTO A
  $[C]$ IF $Z \neq 0$ GOTO D
  GOTO E
  $[D]$ $Z \leftarrow Z - 1$
  $X \leftarrow X + 1$
  GOTO C
- Anything wrong?

Copy The Value of Variable X to Variable Y, Continued

- [A] IF $X \neq 0$ GOTO B
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  $Y \leftarrow Y + 1$
  $Z \leftarrow Z + 1$
  GOTO A
  [C] IF $Z \neq 0$ GOTO D
  GOTO E
  [D] $Z \leftarrow Z - 1$
  $X \leftarrow X + 1$
  GOTO C
- Anything wrong?
- This program is correct only when Y and Z are initialized to the value 0. It cannot be used as a macro.

A Macro for $V \leftarrow V'$

- $$\begin{array}{ll}
 & V \leftarrow 0 \\
[A] & \text{IF } V' \neq 0 \text{ GOTO } B \\
 & \text{GOTO } C \\
[B] & V \leftarrow V' - 1 \\
 & V \leftarrow V + 1 \\
 & Z \leftarrow Z + 1 \\
 & \text{GOTO } A \\
[C] & \text{IF } Z \neq 0 \text{ GOTO } D \\
 & \text{GOTO } E \\
[D] & Z \leftarrow Z - 1 \\
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 & V \leftarrow V + 1 \\
 & Z \leftarrow Z + 1 \\
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[C] & \text{IF } Z \neq 0 \text{ GOTO } D \\
 & \text{GOTO } E \\
[D] & Z \leftarrow Z - 1 \\
 & V' \leftarrow V' + 1 \\
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$$
- Anything wrong?

A Macro for $V \leftarrow V'$

- $$\begin{array}{ll} & V \leftarrow 0 \\ [A] & \text{IF } V' \neq 0 \text{ GOTO } B \\ & \text{GOTO } C \\ [B] & V \leftarrow V' - 1 \\ & V \leftarrow V + 1 \\ & Z \leftarrow Z + 1 \\ & \text{GOTO } A \\ [C] & \text{IF } Z \neq 0 \text{ GOTO } D \\ & \text{GOTO } E \\ [D] & Z \leftarrow Z - 1 \\ & V' \leftarrow V' + 1 \\ & \text{GOTO } C \end{array}$$
- Anything wrong?
- $V \leftarrow 0$ is not an instruction in $\mathscr{S}$.

A Macro for $V \leftarrow 0$

A Macro for $V \leftarrow 0$

$$
\begin{array}{ll}
[L] & V \leftarrow V - 1 \\
 & \text{IF } V \neq 0 \text{ GOTO } L
\end{array}
$$

A Program That Computes $f(x_1, x_2) = x_1 + x_2$

$$
\begin{array}{ll}
 & Y \leftarrow X_1 \\
 & Z \leftarrow X_2 \\
[B] & \text{IF } Z \neq 0 \text{ GOTO } A \\
 & \text{GOTO } E \\
[A] & Z \leftarrow Z - 1 \\
 & Y \leftarrow Y + 1 \\
 & \text{GOTO } B
\end{array}
$$

Note that Z is used to preserve the value of X_2 so that it will not be destroyed during the computation.