PART III

12

A Unified Approach to Least Squares Adjustment

12.1. INTRODUCTION

In the different chapters of Part II of this book several techniques of least squares were introduced. In all these techniques a distinction was made between different groups of variables appearing in the mathematical formulation. Observations were designated as those variables for which covariance matrices are available a priori, and the parameters were the variables usually unknown in the adjustment. It was specified further that condition equations are those written as functions of both observations and parameters, and constraint equations are written only in terms of the parameters. Such classification of variables and equations follows what happens in reality and is therefore quite adequate for formulating many of the adjustment problems encountered in practice. However, recent rapid advances in the fields of geodesy, surveying, and photogrammetry have made it desirable to have an adjustment approach that can handle a mixture of input data in a more general and unified manner. As an example, computational orbital photogrammetry problems may involve regular photogrammetric data, orbital constraints, time measurements, auxiliary sensor information such as stellar

photomeasurements, laser altitude and doppler information, and gravity information. Although it would be possible to handle such complex adjustment problems by techniques developed in Part II, an approach in which varying types of data can be handled with minimum effort and specialization would certainly be more appropriate. It is the purpose of this chapter to introduce and discuss such an approach, which is being progressively incorporated in many of modern photogrammetric and geodetic computational systems.

The most important underlying factor to this approach is the assumption at the outset that *all variables involved in the mathematical formulation are observations.*

With this assumption no individual group classification becomes necessary as long as we introduce into the concept some kind of mechanism by which a differentiation can be made whenever necessary. Such a mechanism is conveniently given by the covariance, or weight, matrix of the *observations,* which in this case are all the variables in the model. To demonstrate the practicality of this approach, consider the following two extreme cases:

1. If an observation (in this case any variable in the model) is given an infinitely large variance, that is, its weight is $w = 0$, then it is allowed to vary freely in the adjustment and will therefore assume the role of an unknown parameter in the classical sense.
2. If on the other hand the observation is given a zero variance, or a weight that approaches infinity, $w \rightarrow \infty$, it is simply not allowed to change in the adjustment, with the consequence that its residual will be zero and it would assume the classical meaning of a constant.

Between the above two extremes lies a large set of possibilities within which actual observations (in the classical sense) fit. Of course, one of the most important points to watch is that when the covariances of a heterogeneous set of variables are all collected together, care must be exercised in regard to units and dimensions. This is particularly important if we choose to use cofactor matrices instead of covariance matrices. In such a case only one common factor must be chosen to relate the different covariance matrices together.

Having lifted up the distinction between observations and parameters, the differentiation between condition equations and constraint equations used in Part II no longer applies. This is because the parameters are now considered as observations, thus rendering the constraints as conditions. However, because the constraints usually refer to strict geometric or functional relations (such as points lying on a straight line, circle, and so on), their treatment is deferred until later. But first we shall consider the case of adjustment of observations and independent parameters treated in Chapter 6 (Part II).

12.2. FORMULATION

We begin by limiting consideration to linear functions. Let the classically designated observational vector be l with a priori cofactor matrix $\mathbf{Q}$ and a vector of residuals $\mathbf{v}$. The variables, usually referred to by parameters, will in the present case be treated as observations. For this purpose we use the vector $\mathbf{x}$ with a priori cofactor matrix $\mathbf{Q}_{xx}$ and a vector of "corrections" $\boldsymbol{\Delta}$. There are n elements (observations) in l (and of course in $\mathbf{v}$) and u elements (parameters) in $\mathbf{x}$ (and in $\boldsymbol{\Delta}$). In the classical approach of Part II the redundancy is $r = n - n_0$ and the total number of conditions is $c = r + u$, remembering that n_0 is the minimum number of variables necessary to specify the problem. In the unified concept all variables are observations, thus a total of $(n + u)$ observed values. This is equivalent to adjustment of observations only, given in Chapter 7, where the number of equations is equal to the redundancy. In the present case the redundancy is equal to the total number of observations $(n + u)$ minus n_0, or

$$(n + u - n_0) = r + u = c$$

Thus the new considerations under the unified approach lead to the same situation as would be the case when the classical procedure is followed.

Let us write the c originally linear equations in the form

$$\mathbf{A}(l + \mathbf{v}) + \mathbf{B}(\mathbf{x} + \boldsymbol{\Delta}) = \mathbf{d} \tag{12.1}$$

where

$\mathbf{A}$ is $c \times n$, rank $(\mathbf{A}) = c$

$\mathbf{B}$ is $c \times u$, rank $(\mathbf{B}) = u$

$\mathbf{d}$ is $c \times 1$

all of which are coefficient matrices. Rearranging equation (12.1), gives

$$\mathbf{A}\mathbf{v} + \mathbf{B}\boldsymbol{\Delta} = \mathbf{d} - \mathbf{A}l - \mathbf{B}\mathbf{x} = \bar{\mathbf{f}} \tag{12.2}$$

Equation (12.2) may be rewritten more concisely as

$$\bar{\mathbf{A}}\bar{\mathbf{v}} = \bar{\mathbf{f}} \tag{12.3}$$

where

$$\bar{\mathbf{A}} = [\mathbf{A} \quad \mathbf{B}] \tag{12.4a}$$

$$\bar{\mathbf{v}} = [\mathbf{v}^t \quad \boldsymbol{\Delta}^t]^t \tag{12.4b}$$

and the corresponding a priori cofactor matrix is

$$\bar{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{Q}_{xx} \end{bmatrix} \tag{12.4c}$$

where no correlation is assumed between the two vectors l and $\mathbf{x}$. The least squares solution for equation (12.3) is obtained by applying equations (7.4) and (7.5) as follows:

$$\bar{\mathbf{v}} = \bar{\mathbf{Q}}\bar{\mathbf{A}}^{t}(\bar{\mathbf{A}}\bar{\mathbf{Q}}\bar{\mathbf{A}}^{t})^{-1}\bar{\mathbf{f}}$$

$$= \bar{\mathbf{Q}}\bar{\mathbf{A}}^{t}(\mathbf{A}\mathbf{Q}\mathbf{A}^{t} + \mathbf{B}\mathbf{Q}_{xx}\,\mathbf{B}^{t})^{-1}\bar{\mathbf{f}}$$

or

$$\begin{bmatrix} \mathbf{v} \\ \Delta \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{xx} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{t} \\ \mathbf{B}^{t} \end{bmatrix} (\mathbf{Q}_{e} + \mathbf{B}\mathbf{Q}_{xx}\,\mathbf{B}^{t})^{-1}\bar{\mathbf{f}}$$

from which

$$\mathbf{v} = \mathbf{Q}\mathbf{A}^{t}(\mathbf{Q}_{e} + \mathbf{B}\mathbf{Q}_{xx}\,\mathbf{B}^{t})^{-1}\bar{\mathbf{f}} \tag{12.5a}$$

$$\Delta = \mathbf{Q}_{xx}\,\mathbf{B}^{t}(\mathbf{Q}_{e} + \mathbf{B}\mathbf{Q}_{xx}\,\mathbf{B}^{t})^{-1}\bar{\mathbf{f}} \tag{12.5b}$$

We first reduce equation (12.5b), by substituting for the inverse in it by its equivalent as given in equations (A68) and (A69) (Appendix A):

$$\Delta = \mathbf{Q}_{xx}\,\mathbf{B}^{t}\mathbf{Q}_{e}^{-1}[\mathbf{I} - \mathbf{B}(\mathbf{B}^{t}\mathbf{Q}_{e}^{-1}\mathbf{B} + \mathbf{Q}_{xx}^{-1})^{-1}\mathbf{B}^{t}\mathbf{Q}_{e}^{-1}]\bar{\mathbf{f}} \tag{12.6a}$$

Realizing that $\mathbf{W}_{e} = \mathbf{Q}_{e}^{-1}$ and referring to the auxiliaries used regularly,

$$\Delta = \mathbf{Q}_{xx}\,\mathbf{t} - \mathbf{Q}_{xx}\,\mathbf{N}(\mathbf{N} + \mathbf{W}_{xx})^{-1}\mathbf{t} \tag{12.6b}$$

Now applying equation (A70) (see Appendix A) to the inverse in equation (12.6b), then

$$\Delta = \mathbf{Q}_{xx}\,\mathbf{t} - \mathbf{Q}_{xx}\,\mathbf{N}\mathbf{N}^{-1}(\mathbf{N}^{-1} + \mathbf{Q}_{xx})^{-1}\mathbf{Q}_{xx}\,\mathbf{t}$$

$$= [\mathbf{Q}_{xx} - \mathbf{Q}_{xx}(\mathbf{N}^{-1} + \mathbf{Q}_{xx})^{-1}\mathbf{Q}_{xx}]\mathbf{t}$$

which from equations (A69) and (A68) again leads to

$$\Delta = (\mathbf{N} + \mathbf{Q}_{xx}^{-1})^{-1}\mathbf{t} = (\mathbf{N} + \mathbf{W}_{xx})^{-1}\mathbf{t} \tag{12.7}$$

Equation (12.7) shows that the only difference between the classical approach and the unified treatment is that the a priori weight matrix of the parameters $\mathbf{W}_{xx}$ is added to $\mathbf{N}$ and in computing $\mathbf{t}$ the vector $\bar{\mathbf{f}}$ of equation (12.2) is used instead of the usual $\mathbf{f}$ of equation (6.6). When $\mathbf{W}_{xx} = \mathbf{0}$ and a priori $\mathbf{x} = \mathbf{0}$, that is, when Δ represents strictly unknown variables (parameters), the unified approach immediately reduces to the "classical" one. Once Δ is computed from equation (12.7), the final estimate of the parameter vector would be $\hat{\mathbf{x}} = \mathbf{x} + \Delta$, where $\mathbf{x}$ is the a priori value.

Returning to equation (12.5a) we may reduce it in a manner similar to that used above. Thus

$$\mathbf{v} = \mathbf{Q}\mathbf{A}^{t}(\mathbf{Q}_{e} + \mathbf{B}\mathbf{Q}_{xx}\,\mathbf{B}^{t})^{-1}\bar{\mathbf{f}}$$

$$= \mathbf{Q}\mathbf{A}^{t}\mathbf{Q}_{e}^{-1}[\mathbf{I} - \mathbf{B}(\mathbf{B}^{t}\mathbf{Q}_{e}^{-1}\mathbf{B} + \mathbf{Q}_{xx}^{-1})^{-1}\mathbf{B}^{t}\mathbf{Q}_{e}^{-1}]\bar{\mathbf{f}}$$

$$= \mathbf{Q}\mathbf{A}^{t}\mathbf{W}_{e}[\bar{\mathbf{f}} - \mathbf{B}(\mathbf{N} + \mathbf{Q}_{xx}^{-1})^{-1}\mathbf{t}]$$

or

$$\mathbf{v} = \mathbf{QA}'\mathbf{W}_e(\bar{\mathbf{f}} - \mathbf{B\Delta}) \tag{12.8}$$

Equation (12.8) is similar to the result obtained in Chapter 6 (Part II) given by equations (6.13) and (6.17). Therefore the relation for computing $\mathbf{v}$ remains the same except for the fact that here $\bar{\mathbf{f}}$ and $\mathbf{\Delta}$ are computed from equations (12.2) and (12.7) instead of equation (6.24).

12.2.1. Alternative Derivation   In the foregoing development we treated the case of having the classically designated parameters considered directly as observations, and applied the results of adjustment of observations only to derive equations (12.7) and (12.8). The same results can be obtained by applying the minimum criterion of least squares to equation (12.2). Of basic importance here is the fact that $\mathbf{\Delta}$ in the present case plays a role equivalent to that of "residuals." In other words, although $\mathbf{v}$ is the vector of residuals for l, $\mathbf{\Delta}$ is the vector of "residuals" for the "observations" $\mathbf{x}$. Thus the scalar to be minimized in this case is

$$\phi = \mathbf{v}'\mathbf{W}\mathbf{v} + \mathbf{\Delta}'\mathbf{W}_{xx}\mathbf{\Delta} - 2\mathbf{k}'(\mathbf{Av} + \mathbf{B\Delta} - \bar{\mathbf{f}}) \tag{12.9}$$

and then,

$$\frac{\partial\phi}{\partial\mathbf{v}} = 2\mathbf{v}'\mathbf{W} - 2\mathbf{k}'\mathbf{A} = \mathbf{0}'$$

and

$$\frac{\partial\phi}{\partial\mathbf{\Delta}} = 2\mathbf{\Delta}'\mathbf{W}_{xx} - 2\mathbf{k}'\mathbf{B} = \mathbf{0}'$$

or

$$\mathbf{Wv} - \mathbf{A}'\mathbf{k} = 0 \tag{12.10a}$$

$$\mathbf{W}_{xx}\mathbf{\Delta} - \mathbf{B}'\mathbf{k} = 0 \tag{12.10b}$$

From equation (12.10a)

$$\mathbf{v} = \mathbf{QA}'\mathbf{k} \tag{12.10c}$$

and using it in equation (6.5) and solving for $\mathbf{k}$

$$\mathbf{k} = (\mathbf{AQA}')^{-1}(\bar{\mathbf{f}} - \mathbf{B\Delta}) = \mathbf{W}_e(\bar{\mathbf{f}} - \mathbf{B\Delta}) \tag{12.10d}$$

Substituting equation (12.10d) into (12.10b) leads to

$$\mathbf{W}_{xx}\mathbf{\Delta} - \mathbf{B}'\mathbf{W}_e(\bar{\mathbf{f}} - \mathbf{B\Delta}) = 0$$

or

$$(\mathbf{N} + \mathbf{W}_{xx})\mathbf{\Delta} = \mathbf{t} \tag{12.11}$$

The solution of equation (12.11) for Δ gives equation (12.7), thus proving that either of the two procedures given above leads to the same results. Equation (12.8) can also be obtained from combining equations (12.10c) and (12.10d).

12.2.2. Parameter/Observation Consideration The two methods given above dealt with the case when the parameters, as classically designated, are treated strictly as observations. At the end of each derivation we ended up with a relation for computing the vector Δ. In practice, even though the unified approach does not distinguish between observations and parameters, the groups of variables do in fact present themselves in specific categories, and the parameters are most often those variables that are required. On the basis of this premise it is possible to have a third approach to the unified concept.

We can begin by considering that l is an $n \times 1$ vector of observations with a cofactor matrix $\mathbf{Q}$ and a residual vector $\mathbf{v}$, and $\mathbf{x}$ is a $u \times 1$ vector of parameters with a correction vector Δ. Thus if $r = n - n_0$ is the redundancy, we may as usual write $c = r + u$ conditions in terms of $\mathbf{v}$, l, $\mathbf{x}$, and Δ. We may further consider that there are u "observations" $\mathbf{x}$ with a cofactor matrix $\mathbf{Q}_{xx}$ and a residual vector $\mathbf{v}_x$. This makes the total number of observations $(n + u)$, which with the given minimum number of required variables n_0 changes the redundancy to $(n + u - n_0)$ or $(r + u)$. Given u parameters in addition leads to a required total number of condition equations of $(r + 2u)$ or $(c + u)$.

The first set of c conditions is that usually available from the geometric or physical conditions of the problem and formulated in equation (12.2). Added to that we must have u more "conditions" to account for having a priori "observations" on the parameters. These may be formulated by recognizing that at the end of the adjustment, the value of the estimated parameters $\hat{\mathbf{x}}$ must be identical to the "estimated observations" for the same variables, or

$$\hat{\mathbf{x}} = \mathbf{x} + \Delta = \mathbf{x} + \mathbf{v}_x$$

or

$$\mathbf{v}_x - \Delta = 0 \tag{12.12}$$

Combining equations (12.2) and (12.12) gives

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v}_x \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{I} \end{bmatrix} \Delta = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{12.13a}$$

or

$$\dot{\mathbf{A}}\mathbf{v} + \dot{\mathbf{B}}\Delta = \dot{\mathbf{f}} \tag{12.13b}$$

for which the total cofactor matrix is $\bar{\mathbf{Q}}$ given by equation (12.4c). Least squares may now be applied directly to equation (12.13b). Thus

$$\dot{\mathbf{Q}}_e = \dot{\mathbf{A}}\bar{\mathbf{Q}}\dot{\mathbf{A}}^t = \begin{bmatrix} \mathbf{AQA}^t & 0 \\ 0 & \mathbf{Q}_{xx} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e & 0 \\ 0 & \mathbf{Q}_{xx} \end{bmatrix} \tag{12.14a}$$

$$\dot{\mathbf{N}} = \dot{\mathbf{B}}^t\dot{\mathbf{W}}_e \dot{\mathbf{B}} = [\mathbf{B}^t\mathbf{W}_e \mathbf{B} + \mathbf{W}_{xx}] = (\mathbf{N} + \mathbf{W}_{xx}) \tag{12.14b}$$

$$\dot{\mathbf{t}} = \dot{\mathbf{B}}^t\dot{\mathbf{W}}_e \dot{\mathbf{f}} = \mathbf{B}^t\mathbf{W}_e \bar{\mathbf{f}} = \mathbf{t} \tag{12.14c}$$

and thus

$$\boldsymbol{\Delta} = \dot{\mathbf{N}}^{-1}\dot{\mathbf{t}} = (\mathbf{N} + \mathbf{W}_{xx})^{-1}\mathbf{t} \tag{12.14d}$$

Equation (12.14d) is obviously identical to (12.7) and hence all three methods of derivation are identical. Again, $\hat{\mathbf{x}} = \mathbf{x} + \boldsymbol{\Delta}$.

In this linear case it is possible to assume that $\boldsymbol{\Delta}$ is the total parameter vector as has been the practice in Part II. Thus equation (12.12) becomes

$$\mathbf{v}_x - \boldsymbol{\Delta} = -\mathbf{x} \tag{12.15}$$

with $\mathbf{x}$ being the a priori "observational values" on the parameters, and $\dot{\mathbf{f}}$ changes to

$$\dot{\mathbf{f}} = \begin{bmatrix} \mathbf{f} \\ -\mathbf{x} \end{bmatrix} \tag{12.16a}$$

Hence,

$$\dot{\mathbf{t}} = \dot{\mathbf{B}}^t\dot{\mathbf{W}}_e \dot{\mathbf{f}} = [\mathbf{B}^t\mathbf{W}_e \mathbf{f} + \mathbf{W}_{xx} \mathbf{x}] = (\mathbf{t} + \mathbf{W}_{xx} \mathbf{x}) \tag{12.16b}$$

$$\boldsymbol{\Delta} = (\mathbf{N} + \mathbf{W}_{xx})^{-1}(\mathbf{t} + \mathbf{W}_{xx} \mathbf{x}) \tag{12.16c}$$

Note that the value of $\mathbf{t}$ in equation (12.16c) is computed on the basis of $\mathbf{f} = \mathbf{d} - \mathbf{A}l$ and not $\bar{\mathbf{f}}$ given in equation (12.2).

Example 12.1. Consider a very simple problem (Figure 12.1) of determining the elevation of a point P by measuring the difference in elevation between it and two reference points A, B, the elevations of which are zero. The observations are $l_1 = 1.10$ m and $l_2 = 1.12$ m with a covariance matrix $\Sigma = 10^{-4}\mathbf{I}_2$ m^2. Let the a priori value for the elevation of point P be 1.00 m with a variance of 0.01 m^2.
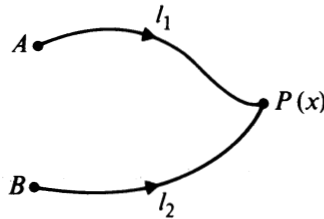

Figure 12.1

The first procedure of solving the problem is to consider having three observations 1.10, 1.12, and 1.00 m with the covariance matrix

$$\Sigma = \text{diag. } \{10^{-4}, 10^{-4}, 10^{-2}\} = 10^{-4} \text{ diag. } \{1, 1, 100\} = \sigma_0^2 \, \mathbf{Q}$$

Since there is only one elevation to be determined, we write two conditions

$$v_1 - v_x = (x - l_1) = -0.10 \text{ m}$$

$$v_2 - v_x = (x - l_2) = -0.12 \text{ m}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} -0.10 \\ -0.12 \end{bmatrix}$$

$$\mathbf{Q}_e = \begin{bmatrix} 101 & 100 \\ 100 & 101 \end{bmatrix} \qquad \mathbf{W}_e = \tfrac{1}{201} \begin{bmatrix} 101 & -100 \\ -100 & 101 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0.0094527 \\ -0.0105472 \end{bmatrix}$$

(the k's are computed to many decimal places for purposes of checking later). Next

$$[v_1 \quad v_2 \quad v_x] = [0.0094527 \quad -0.0105472 \quad 0.10945] \text{ m}$$

Thus

$$\hat{l}_1 = 1.109453 \text{ m} \qquad \hat{l}_2 = 1.109453 \text{ m} \qquad \text{and} \qquad \hat{x} = 1.10945 \text{ m}$$

which show the consistency of the results even to five decimal places. Practically these values should be rounded to two decimal places only.

A second procedure would be to apply equation (12.7). However, we first write the conditions

$$l_1 + v_1 - x - \Delta = 0 \qquad \text{and} \qquad l_2 + v_2 - x - \Delta = 0$$

$$\mathbf{v} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \Delta = \begin{bmatrix} x - l_1 \\ x - l_2 \end{bmatrix} = \begin{bmatrix} -0.10 \text{ m} \\ -0.12 \text{ m} \end{bmatrix} \qquad \text{or} \qquad \mathbf{v} + \mathbf{B}\Delta = \bar{\mathbf{f}}$$

Then with

$$\mathbf{Q} = \mathbf{W} = \mathbf{I} \qquad \text{and} \qquad \mathbf{W}_{xx} = (100)^{-1} = 0.01$$

$$N = 2 \qquad t = 0.22 \text{ m} \qquad \text{and} \qquad \Delta = (2 + 0.01)^{-1}(0.22) = 0.109453 \text{ m}$$

Hence

$$\hat{\mathbf{x}} = \mathbf{x} + \Delta = 1.109453 \text{ m}$$

The third technique allows for solving for the parameters directly rather than for a correction vector Δ since the equations are linear. In this case

$$\mathbf{f} = [-1.10 \qquad -1.12]^t$$

$$t = 2.22 \text{ m}$$

and

$$t = t + \mathbf{W}_{xx}\, \mathbf{x} = 2.22 + 0.01 = 2.23 \text{ m}$$

Applying equation (12.16c) yields

$$\Delta = (2 + 0.01)^{-1}(2.23) = 1.109453 \text{ m}$$

which is identical to those obtained before.

12.3. THE UNIFIED APPROACH AND NONLINEAR FUNCTIONS

In this section the following nonlinear conditions are considered

$$F(l, \mathbf{x}) = 0 \qquad (12.17)$$

Denoting by l^0 and $\mathbf{x}^0$ two approximation vectors for the observations and parameters, the linearization of equation (12.17) follows the procedure given in Section 11.1.2 or

$$\mathbf{A}\mathbf{v} + \mathbf{B}\Delta = \mathbf{f}^0 \qquad (11.8)$$

in which $\mathbf{f}^0$ is given by

$$\mathbf{f}^0 = -[F(l^0, \mathbf{x}^0) + \mathbf{A}(l - l^0)] \qquad (11.7)$$

In addition to the two approximation vectors, there are also two vectors of observations, l and $\mathbf{x}$, with a priori cofactor matrices $\mathbf{Q}$ and $\mathbf{Q}_{xx}$, respectively. The estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ is

$$\hat{\mathbf{x}} = \mathbf{x}^0 + \Delta = \mathbf{x} + \mathbf{v}_x$$

from which

$$\mathbf{v}_x - \Delta = \mathbf{x}^0 - \mathbf{x} = \mathbf{f}_x \qquad (12.18)$$

Equation (12.18) differs from equation (12.15) in that the latter does not contain $\mathbf{x}^0$, because no approximations are needed if the conditions are originally linear. Therefore if we replace $\mathbf{x}$ by $(-\mathbf{f}_x)$ in equation (12.16c), then the solution for equations (11.8) and (12.18) is

$$\Delta = (\mathbf{N} + \mathbf{W}_{xx})^{-1}(\mathbf{t} - \mathbf{W}_{xx}\,\mathbf{f}_x) \qquad (12.19)$$

In evaluating $\mathbf{t}$ in equation (12.19) the appropriate value of $\mathbf{f}^0$ must be used.

It is important to say a word about the value of $\mathbf{f}_x$ during different iterations of the nonlinear solution. At the beginning of the first iteration, it is quite practical and convenient to take $\mathbf{x}^0 = \mathbf{x}$, or to assume that the approximations for the parameters are equal to their a priori "observational" estimates. This will obviously lead to the first $\mathbf{f}_x$ being equal to zero. After the first iteration is completed a value Δ_1 is computed, which when added to $\mathbf{x}^0$ gives an updated value for the vector of approximations. In the meantime the "observational" values $\mathbf{x}$ do not change all through the iterative process. Consequently, for the second iteration, $\mathbf{f}_x$ is no longer zero but is equal to $\mathbf{x}^0 + \Delta_1 - \mathbf{x} = \Delta_1$, since $\mathbf{x}^0 = \mathbf{x}$. At the beginning of the third iteration, $\mathbf{f}_x = \Delta_1 + \Delta_2$, and so on.

The same discussion applies also for the term $\mathbf{A}(l - l^0)$ in the computation $\mathbf{f}^0$ from equation (11.7), when we iterate on the observations. Although it may be zero at the beginning, it acquires finite values in succeeding iterations.

Example 12.2. It is required to fit a straight line with the equation $y - mx = 0$ through the two points $(1.1, 2.1)$ and $(2.1, 4.0)$. The cofactor matrix for these coordinates is the identity matrix. An a priori estimate for the one parameter (the slope of the line) is $m = 1.0$, and its a priori cofactor is $q_m = 100$. Compute the final estimate m for the line slope from this data.

Solution: It would take one point with its two coordinates, in addition to the origin, to determine the line. The second point is intended to lie on the line and therefore *only one coordinate* is needed to fix its position on the line. Hence there is *one additional observation* more than the minimum necessary to determine the geometry of the problem, that is, 1 degree of freedom. In total, $n = 4$, $n_0 = 3$, and $r = 1$. With $u = 1$ for the parameter m, then $c = 2$ conditions, one line equation for each of the two given points,

$$y_i - mx_i = 0 \qquad \text{with } i = 1, 2$$

Denoting by m^0 the approximate value for the parameter, the linearized form of this equation becomes (we choose not to iterate on the observed coordinates in this problem)

$$-m^0 v_{x_i} + v_{y_i} - x_i \, \delta = m^0 x_i - y_i$$

or

$$[-m^0 \quad 1]v_i + [-x_i]\Delta = f$$

Using the given data and choosing $m^0 = m = 1$, the matrices for the two condition equations become

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1.1 \\ -2.1 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} -1.0 \\ -1.9 \end{bmatrix}$$

with

$$\mathbf{W} = \mathbf{I}, \qquad \mathbf{W}_e = \mathbf{AA}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathbf{N} = 2.81, \qquad \text{and} \qquad \mathbf{t} = 2.545$$

Because $m^0 = m$, the value of $\mathbf{f}_x$ for the first iteration is zero according to equation (12.18). Thus with $\mathbf{W}_{xx} = 0.01$ then

$$\Delta_1 = (2.81 + 0.01)^{-1}(2.545) = 0.902482$$

$$m_1^0 = m^0 + \Delta_1 = 1.902482$$

$$f_{x_1} = 0.902482$$

Using these values, the second iteration becomes

$$\Delta_2 = (1.216598 + 0.01)^{-1}(-0.005118) = -0.004172$$

For the third iteration,

$$m_2^0 = 1.898310, \qquad f_{x_2} = 0.898310, \qquad \text{and} \qquad \Delta_3 = 0.000025$$

which is considered to be sufficiently small to terminate the iterations. The final estimate of the line slope is

$$m = 1.898310 + 0.000025 = 1.898335$$

12.4. THE UNIFIED APPROACH AND PARAMETER CONSTRAINTS

Any constraint is in a real sense a function reflecting some geometric or physical relation between a number of variables. Therefore they can practically be viewed as "conditions" containing some type of "observations." For example, constraining a distance to a given value D can be done as closely as the known value of D. In the treatments of Chapter 9 (Part II), in which equations of the type $C\Delta = g$ were used, the distance D would be assumed known perfectly, or with infinite weight (zero variance). In the present situation under the unified concept, we would take D as an observed quantity and choose a value for its weight commensurate with the reliability of its known value. The higher the value of its weight, the closer we get to perfect satisfaction of the constraint.

Although the example of a distance can be easily visualized, there are many other instances in which constraints do not include variables that are amenable to being considered as observations. As an example we refer to the case of constraining a set of points to lie on a line or a circle. All variables in such functions are what have been conventionally considered parameters. The only other variable may simply be the numerical zero, which would obviously be a constant. It is that constant which we now convert to an "observation," regardless of its numerical value. Consequently, in each equation conventionally known as a "constraint," the constant is replaced by an observational variable that is denoted by l_c. (This would replace the distance D in the first example, and the "zero" in the second example.) In addition to the total vector of "constraint observations" l_c, the parameters $\mathbf{x}$ would be the only other variables in the set of equations replacing the constraints. These equations may in general be nonlinear and therefore take the form

$$F_c(l_c, \mathbf{x}) = \mathbf{0} \tag{12.20}$$

which is the same form as equation (12.17). Thus we can appreciate how the unified concept allows for only one general type of condition equations. In fact, with $\mathbf{x}$ also treated as a vector of observable quantities, we may combine equations (12.17) and (12.20) into one equation,

$$F_t(l_t) = \mathbf{0} \tag{12.21}$$

in which l_t represents the total vector variables. Equation (12.21) is essentially the same as that given by equation (11.10). The only difference is that in equation (11.10) the vector l denoted only the one set of variables conventionally known as observations, whereas in (12.21) the vector l_t includes several subvectors each referring to one class of conventional variables (that is, observations, parameters, and so on). It would be possible to develop a solution directly from equation (12.21). However, as has been shown on several occasions such a direct treatment invariably leads to undue complex-

ities in the derivation. Therefore for practical reasons as well as for convenience, it is prudent to partition the system of equation (12.21) and proceed with the development such that groups of variables usually encountered together in practice are treated as separate subsets.

Considering equations (12.17) and (12.20) together, the variables are

l $n \times 1$ vector of conventionally designated observations ($\mathbf{Q}$ is its a priori cofactor matrix.)

l_c $t \times 1$ vector of observations arising with the constraints ($\mathbf{Q}_{cc}$ is its a priori cofactor matrix.)

$\mathbf{x}$ $u \times 1$ vector of variables conventionally known as parameters ($\mathbf{Q}_{xx}$ is its a priori cofactor matrix.)

There are c equations in (12.17) and s equations in (12.20). If l^0, l_c^0, and $\mathbf{x}^0$ denote vectors of approximations, $\mathbf{v}$, $\mathbf{v}_c$, and $\mathbf{v}_x$ denote vectors of residuals, and Δ_l, Δ_c, and Δ represent vector of corrections to the approximate vectors, then

$$l + \mathbf{v} = l^0 + \Delta_l \tag{12.22a}$$

$$l_c + \mathbf{v}_c = l_c^0 + \Delta_c \tag{12.22b}$$

For the linearization of equations (12.17) and (12.20) we use the auxiliaries

$$\mathbf{A} = \frac{\partial F}{\partial l} \qquad \text{is } c \times n \text{ matrix of rank equal to } c$$

$$\mathbf{B} = \frac{\partial F}{\partial \mathbf{x}} \qquad \text{is } c \times u \text{ matrix of rank equal to } u$$

$$\mathbf{A}_c = \frac{\partial F_c}{\partial l_c} \qquad \text{is } s \times t \text{ matrix of rank equal to } s$$

$$\mathbf{C} = \frac{\partial F_c}{\partial \mathbf{x}} \qquad \text{is } s \times u \text{ matrix of rank equal to } s$$

Thus

$$\mathbf{A}\mathbf{v} + \mathbf{B}\Delta = \mathbf{f}^0 \tag{12.23a}$$

$$\mathbf{A}_c \mathbf{v}_c + \mathbf{C}\Delta = \mathbf{f}_c^0 \tag{12.23b}$$

in which

$$\mathbf{f}^0 = -[F(l^0, \mathbf{x}^0) - \mathbf{A}(l^0 - l)] \tag{12.24a}$$

$$\mathbf{f}_c^0 = -[F_c(l_c^0, \mathbf{x}^0) - \mathbf{A}_c(l_c^0 - l_c)] \tag{12.24b}$$

with

$$\mathbf{f}_x = \mathbf{x}^0 - \mathbf{x} \tag{12.24c}$$

then

$$
\begin{bmatrix} \mathbf{A} & 0 & 0 \\ 0 & \mathbf{A}_c & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v}_c \\ \mathbf{v}_x \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \\ -\mathbf{I} \end{bmatrix} \Delta = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}_c^0 \\ \mathbf{f}_x \end{bmatrix} \tag{12.25}
$$

Letting, as has been customary,

$$
\mathbf{W}_e = (\mathbf{A}\mathbf{Q}\mathbf{A}^t)^{-1} \tag{12.26a}
$$

$$
\mathbf{W}_{ec} = (\mathbf{A}_c \, \mathbf{Q}_{cc} \, \mathbf{A}_c^t)^{-1} \tag{12.26b}
$$

the normal equations for Δ from the conditions of equation (12.25) would be

$$
[\mathbf{B}^t \quad \mathbf{C}^t \quad -\mathbf{I}] \begin{bmatrix} \mathbf{W}_e & 0 & 0 \\ 0 & \mathbf{W}_{ec} & 0 \\ 0 & 0 & \mathbf{W}_{xx} \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \\ -\mathbf{I} \end{bmatrix} \Delta
$$

$$
= [\mathbf{B}^t \quad \mathbf{C}^t \quad -\mathbf{I}] \begin{bmatrix} \mathbf{W}_e & 0 & 0 \\ 0 & \mathbf{W}_{ec} & 0 \\ 0 & 0 & \mathbf{W}_{xx} \end{bmatrix} \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}_c^0 \\ \mathbf{f}_x \end{bmatrix}
$$

or

$$
[\mathbf{B}^t\mathbf{W}_e \, \mathbf{B} + \mathbf{C}^t\mathbf{W}_{ec} \, \mathbf{C} + \mathbf{W}_{xx}]\Delta
$$

$$
= [\mathbf{B}^t\mathbf{W}_e \, \mathbf{f}^0 + \mathbf{C}^t\mathbf{W}_{ec} \, \mathbf{f}_c^0 - \mathbf{W}_{xx} \, \mathbf{f}_x] \tag{12.27}
$$

Introducing new auxiliaries and repeating earlier ones for the sake of completeness,

$$
\mathbf{N} = \mathbf{B}^t\mathbf{W}_e \, \mathbf{B} \tag{12.28a}
$$

$$
\mathbf{t} = \mathbf{B}^t\mathbf{W}_e \, \mathbf{f}^0 \tag{12.28b}
$$

$$
\mathbf{N}_c = \mathbf{C}^t\mathbf{W}_{ec} \, \mathbf{C} \tag{12.28c}
$$

$$
\mathbf{t}_c = \mathbf{C}^t\mathbf{W}_{ec} \, \mathbf{f}_c^0 \tag{12.28d}
$$

equation (12.27) takes the shorter form,

$$
[\mathbf{N} + \mathbf{N}_c + \mathbf{W}_{xx}]\Delta = [\mathbf{t} + \mathbf{t}_c - \mathbf{W}_{xx} \, \mathbf{f}_x] \tag{12.29a}
$$

from which

$$
\Delta = (\mathbf{N} + \mathbf{N}_c + \mathbf{W}_{xx})^{-1}(\mathbf{t} + \mathbf{t}_c - \mathbf{W}_{xx} \, \mathbf{f}_x) \tag{12.29b}
$$

Equation (12.29b) gives the vector of estimates for corrections to parameter approximations when all variables have a priori values as well as a priori cofactor matrices. The terms $\mathbf{N}$ and $\mathbf{t}$ reflect the contribution of the conventionally known observations; the terms $\mathbf{N}_c$ and $\mathbf{t}_c$, the contribution of constraints; and the remaining terms, the contribution due to a priori knowledge of the parameters themselves.

In the forgoing development allowance was made for linearization and iteration on *all variables in the model* to make the analysis as general as possible. It should now be shown how the iterative process proceeds on this general case, and then the possible simplifications for special cases should be pointed out.

Let $x_{(0)}^0$, $l_{(0)}^0$, and $l_{c(0)}^0$ denote the initial approximation vectors, with, of course, x, l, and l_c being the a priori (observational) values that do not change during the adjustment. With these values the matrices and vectors A, B, A_c, C, f^0, f_c^0, and f_x are numerically evaluated and used to form N, t, N_c, and t_c, which in turn are substituted in equation (12.29b) to yield the first vector $\Delta_{(1)}$. This vector $\Delta_{(1)}$ is then added to $x_{(0)}^0$ to obtain an updated vector of approximation $x_{(1)}^0$. In order to get updated vectors of approximations $l_{(1)}^0$ and $l_{c(1)}^0$, we need to compute the two vectors v and v_c. From equation (12.8) we may easily write

$$v = QA'W_e(f^0 - B\Delta) \tag{12.29c}$$

and in a similar manner

$$v_c = Q_{cc} A_c^t W_{ec}(f_c^0 - C\Delta) \tag{12.30a}$$

finally,

$$v_x = x^0 - x + \Delta = f_x + \Delta \tag{12.30b}$$

Using the value $\Delta_{(1)}$ in equations (12.30a) and (12.30b), the first two values $v_{(1)}$ and $v_{c(1)}$ can be computed. These may be used in equations (12.22a) and (12.22b) to compute the correction vectors $\Delta_{l(1)}$ and $\Delta_{c(1)}$ for the corresponding approximation vectors $l_{(0)}^0$ and $l_{c(0)}^0$. These relations in recursive form are

$$\Delta_{l(i+1)} = l + v_{(i+1)} - l_{(i)}^0 \tag{12.30c}$$

$$\Delta_{c(i+1)} = l_c + v_{c(i+1)} - l_{c(i)}^0 \tag{12.30d}$$

in which we must note that l and l_c are the a priori "observations" that remain constant all through the iterations. Equations (12.30c) and (12.30d) may be rearranged to give a more convenient form from which the updated approximations l^0 and l_c^0 may be obtained directly.

$$l_{(i+1)}^0 = l_{(i)}^0 + \Delta_{l(i+1)} = l + v_{(i+1)} \tag{12.30e}$$

$$l_{c(i+1)}^0 = l_{c(i)}^0 + \Delta_{c(i+1)} = l_c + v_{c(i+1)} \tag{12.30f}$$

Thus with $v_{(1)}$ and $v_{c(1)}$ we can readily get, from equations (12.30e) and (12.30f) the vectors $l_{(1)}^0$ and $l_{c(1)}^0$, respectively. Given these, together with $x_{(1)}^0$ computed before, we may proceed to re-evaluate all matrices and vectors leading to computing a new vector $\Delta_{(2)}$. The procedure of obtaining a new $x_{(2)}^0$, $l_{(2)}^0$, and $l_{c(2)}^0$ is repeated in its entirety. At the end of each iteration a criterion, established a priori, is checked to see if the process is to be con-

tinued or stopped (see section 11.4). When such criterion is satisfied and the iterations terminated, we get the final estimates,

$$\hat{\mathbf{x}} = \mathbf{x}^0_{(0)} + \sum_j \mathbf{\Delta}_{(j)} \tag{12.31a}$$

$$\hat{l} = l + \mathbf{v}_{(f)} \tag{12.31b}$$

$$\hat{l}_c = l_c + \mathbf{v}_{c(f)} \tag{12.31c}$$

(f) denotes final value.

12.4.1. Special Cases   1. If all functions are originally *linear*, then

$$l^0 = l^0_c = \mathbf{x}^0 = 0 \tag{12.32a}$$

$$\mathbf{f}^0 = \mathbf{f} = \mathbf{d} - \mathbf{A}l \tag{12.32b}$$

$$\mathbf{f}^0_c = \mathbf{f}_c = \mathbf{d}_c - \mathbf{A}_c l_c \tag{12.32c}$$

$$\mathbf{f}_x = -\mathbf{x} \tag{12.32d}$$

$$\mathbf{\Delta} = (\mathbf{N} + \mathbf{N}_c + \mathbf{W}_{xx})^{-1}(\mathbf{t} + \mathbf{t}_c + \mathbf{W}_{xx}\,\mathbf{x}) \tag{12.32e}$$

noting that in computing $\mathbf{t}$ and $\mathbf{t}_c$, we use $\mathbf{f}^0$ and $\mathbf{f}^0_c$ as evaluated from equations (12.32b) and (12.32c).

2. If the functions are *linear* and there are no *constraints*, then equation (12.32e) reduces to equation (12.16c), or

$$\mathbf{\Delta} = (\mathbf{N} + \mathbf{W}_{xx})^{-1}(\mathbf{t} + \mathbf{W}_{xx}\,\mathbf{x}) \tag{12.16c}$$

by dropping the two terms $\mathbf{N}_c$ and $\mathbf{t}_c$.

3. For *nonlinear* functions *without constraints*, equation (12.33b) becomes the same as equation (12.19), or

$$\mathbf{\Delta} = (\mathbf{N} + \mathbf{W}_{xx})^{-1}(\mathbf{t} - \mathbf{W}_{xx}\,\mathbf{f}_x) \tag{12.19}$$

If we iterate on the observations, we compute $\mathbf{t}$ using $\mathbf{f}^0$ from equation (12.24a). If we do not iterate on the observations, a situation that occurs often in practice, then

$$\mathbf{f} = -\mathbf{F}(l, \mathbf{x}^0) \tag{11.9}$$

must be used in evaluating $\mathbf{t}$. The iterative process would involve computing successive values for $\mathbf{\Delta}$ only, whereas $\mathbf{v}$ may be computed only at the last iteration, unless it is needed for the criterion of termination of the iterations.

Example 12.3. Consider the problem of Example 9.5 and rework it to show how the unified approach accommodates parameter constraints by changing them to condition equations with appropriate weight matrices. For convenience first recall the data,

$$\mathbf{A}\mathbf{v} + \mathbf{B}\mathbf{\Delta} \qquad = \mathbf{f} \qquad \text{with } \mathbf{Q} = \mathbf{I}$$

$$\mathbf{D}_1\mathbf{\Delta} + \mathbf{D}_2\,\mathbf{\Delta}' = \mathbf{h}$$

where

$$A = I_3 \qquad \underset{3,\,2}{B} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \underset{3,\,1}{f} = \begin{bmatrix} -1.1 \\ 1.2 \\ 1.0 \end{bmatrix} \qquad h = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \Delta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \Delta' = x_3$$

Rewrite the equations in the general form

$$A v + [B \quad 0]\begin{bmatrix} \Delta \\ \Delta' \end{bmatrix} = f \qquad \text{or} \qquad A v + \tilde{B}\tilde{\Delta} = f$$

$$A_c v_c + [D_1 \quad D_2]\begin{bmatrix} \Delta \\ \Delta' \end{bmatrix} = h = f_c \qquad \text{or} \qquad A_c v_c + \tilde{C}\tilde{\Delta} = f_c$$

with obvious correspondence in terms. The numerical values of these matrices are

$$A_c = I$$

$$\underset{3,\,3}{\tilde{B}} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \underset{2,\,3}{\tilde{C}} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$$

and f and f_c $(= h)$ are as given above. Realizing that this is a linear case, the solution for $\tilde{\Delta} = [x_1 \quad x_2 \quad x_3]^t$ may be obtained by applying equation (12.32e), modified slightly to accommodate the fact that there were no a priori values for those three variables. Thus with $W_{xx} = 0$ and $x = 0$, we have

$$\Delta = (N + N_c)^{-1}(t + t_c)$$

which, because $A = A_c = I$, may be expanded to

$$\tilde{\Delta} = (\tilde{B}^t W \tilde{B} + \tilde{C}^t W_{cc} \tilde{C})^{-1}(\tilde{B}^t W f + \tilde{C}^t W_{cc} f_c)$$

with

$$W = Q^{-1} \qquad \text{and} \qquad W_{cc} = Q_{cc}^{-1}$$

The relative magnitudes of W and W_{cc} comprise the mechanism by which we can insure that the constraints are treated properly. Since $W = I$ as given in the data of the problem, we are free to choose a suitable value for W_{cc}. The ideal case is when W_{cc} is infinitely large relative to W. In order to determine which value of W_{cc} is "practically" infinite, consider several possibilities. Just for the sake of establishing a trend, begin by the rather unrealistic value of $W_{cc} = I$ as well. With these two values, the above equation gives

$$[x_1 \quad x_2 \quad x_3]_1 = [1.04486 \quad 1.07383 \quad -0.98785]$$

The answer is clearly different from the correct one obtained in Example 9.5, a fact that should have been expected. Next, use $\mathbf{W}_{cc} = 10\mathbf{I}$ and get

$$[x_1 \quad x_2 \quad x_3]_2 = [1.06000 \quad 1.08188 \quad -0.98038]$$

which is already closer to the correct answer. Further, with $\mathbf{W}_{cc} = 100\mathbf{I}$

$$[x_1 \quad x_2 \quad x_3]_3 = [1.06210 \quad 1.08299 \quad -0.97934]$$

which may be considered as essentially the same as the correct answer. If, however, a closer set of values is desired, we may increase $\mathbf{W}_{cc}$ to $1000\mathbf{I}$ or even more, depending on the accuracy of the required final answer.

12.5. PRECISION ESTIMATION

Applying the propagation principle to equation (12.29b), it can be readily shown that

$$\mathbf{Q}_{\Delta\Delta} = (\mathbf{N} + \mathbf{N}_c + \mathbf{W}_{xx})^{-1} \qquad (12.33a)$$

The fact that $\mathbf{Q}_{\Delta\Delta}$ is the inverse of the sum of three weight matrices should seem reasonable, since each matrix represents the contribution of one of the three sources of information. The matrix $\mathbf{W}_{xx}$ expresses what is already known about the parameters before the adjustment, and $\mathbf{N}$ and $\mathbf{N}_c$ express the contributions of the conditions and constraints, respectively.

Equation (12.33a) reduces to already known relationships when different restrictions are enforced. For example, when $\mathbf{W}_{xx} = \mathbf{N}_c = \mathbf{0}$, which means that there are no a priori parameter information and no constraints, then

$$\mathbf{Q}_{\Delta\Delta} = \mathbf{N}^{-1} \qquad (12.33b)$$

which is the same as equation (6.29).

If there are no constraints, that is, if $\mathbf{N}_c = \mathbf{0}$, but the a priori weight matrix for the parameters $\mathbf{W}_{xx}$ is given, then

$$\mathbf{Q}_{\Delta\Delta} = (\mathbf{N} + \mathbf{W}_{xx})^{-1} \qquad (12.33c)$$

A third interesting case is when $\mathbf{W}_{xx} = \mathbf{0}$ but $\mathbf{N}_c \neq 0$, for which

$$\mathbf{Q}_{\Delta\Delta} = (\mathbf{N} + \mathbf{N}_c)^{-1} \qquad (12.33d)$$

This equation can also be shown to reduce to an earlier form given the proper assumption. Rewriting equation (12.33d) in an expanded form in view of equation (12.28c), then

$$\mathbf{Q}_{\Delta\Delta} = (\mathbf{N} + \mathbf{C}^t \mathbf{W}_{ec} \mathbf{C})^{-1} \qquad (12.34a)$$

which may be expanded further, by applying equations (A68) and (A69) (Appendix A), to

$$\mathbf{Q}_{\Delta\Delta} = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{C}^t(\mathbf{Q}_{ec} + \mathbf{C}\mathbf{N}^{-1}\mathbf{C}^t)^{-1}\mathbf{C}\mathbf{N}^{-1} \qquad (12.34b)$$

In equation (12.34b), the matrix $\mathbf{Q}_{ec}$ is the inverse of $\mathbf{W}_{ec}$, which is from equation (12.26b)

$$\mathbf{Q}_{ec} = \mathbf{A}_c \, \mathbf{Q}_{cc} \, \mathbf{A}_c^t \tag{12.34c}$$

Conventional treatment of constraints requires those equations to be strictly satisfied. This is equivalent to saying that $\mathbf{Q}_{cc} = \mathbf{0}$ (or equivalently $\mathbf{W}_{cc} \to \infty$), which leads to having $\mathbf{Q}_{ec} = \mathbf{0}$ from equation (12.34c). With the vanishing of $\mathbf{Q}_{ec}$, equation (12.34b) may be rewritten as

$$\mathbf{Q}_{\Delta\Delta} = \mathbf{N}^{-1}[\mathbf{I} - \mathbf{C}^t(\mathbf{C}\mathbf{N}^{-1}\mathbf{C}^t)^{-1}\mathbf{C}\mathbf{N}^{-1}] \tag{12.34d}$$

which is identical to equation (9.15b) derived for the conventional case.

The next cofactor matrix to evaluate is that for *all* the observational residuals $\mathbf{v}_t$, which is denoted by $\mathbf{Q}_{v_t v_t}$. To derive this matrix, we first rewrite equation (12.25) in a more compact form

$$\mathbf{A}_t \, \mathbf{v}_t + \mathbf{B}_t \, \Delta = \mathbf{f}_t \tag{12.35a}$$

with associated total cofactor matrix,

$$\mathbf{Q}_{tt} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{cc} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_{xx} \end{bmatrix} \tag{12.35b}$$

Applying equation (6.30)

$$\mathbf{Q}_{v_t v_t} = \mathbf{Q}_{tt} \, \mathbf{A}_t^t \, \mathbf{W}_{et} \, \mathbf{A}_t \, \mathbf{Q}_{tt} - \mathbf{Q}_{tt} \, \mathbf{A}_t^t \, \mathbf{W}_{et} \, \mathbf{B}\mathbf{Q}_{\Delta\Delta} \, \mathbf{B}^t\mathbf{W}_{et} \, \mathbf{A}_t \, \mathbf{Q}_{tt} \tag{12.36}$$

in which

$$\mathbf{W}_{et} = (\mathbf{A}_t \, \mathbf{Q}_{tt} \, \mathbf{A}_t^t)^{-1} = \begin{bmatrix} (\mathbf{A}\mathbf{Q}\mathbf{A}^t)^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}_c \, \mathbf{Q}_{cc} \, \mathbf{A}_c^t)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_{xx}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{W} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{cc} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_{xx} \end{bmatrix} \tag{12.37}$$

Thus

$$\mathbf{Q}_{v_t v_t} = \begin{bmatrix} \mathbf{Q}_{vv} & \mathbf{Q}_{vv_c} & \mathbf{Q}_{vv_x} \\ & \mathbf{Q}_{v_c v_c} & \mathbf{Q}_{v_c v_x} \\ \text{symmetric} & & \mathbf{Q}_{v_x v_x} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{Q}\mathbf{A}^t\mathbf{W}_e\mathbf{A}\mathbf{Q} & [-\mathbf{Q}\mathbf{A}^t\mathbf{W}_e\mathbf{B}\mathbf{Q}_{\Delta\Delta}\mathbf{C}^t\mathbf{W}_{ec}\mathbf{A}_c\mathbf{Q}_{cc}] & [\mathbf{Q}\mathbf{A}^t\mathbf{W}_e\mathbf{B}\mathbf{Q}_{\Delta\Delta}] \\ -\mathbf{Q}\mathbf{A}^t\mathbf{W}_e\mathbf{B}\mathbf{Q}_{\Delta\Delta}\mathbf{B}^t\mathbf{W}_e\mathbf{A}\mathbf{Q}] & & \\ & [\mathbf{Q}_{cc}\mathbf{A}_c^t\mathbf{W}_{ec}\mathbf{A}_c\mathbf{Q}_{cc} & [\mathbf{Q}_{cc}\mathbf{A}_c^t\mathbf{W}_{ec}\mathbf{C}\mathbf{Q}_{\Delta\Delta}] \\ & -\mathbf{Q}_{cc}\mathbf{A}_c^t\mathbf{W}_{ec}\mathbf{C}\mathbf{Q}_{\Delta\Delta}\mathbf{C}^t\mathbf{W}_{ec}\mathbf{A}_c\mathbf{Q}_{cc}] & \\ \text{symmetric} & & [\mathbf{Q}_{xx} - \mathbf{Q}_{\Delta\Delta}] \end{bmatrix} \tag{12.38}$$

Equation (12.38) is rather general in as much as it gives not only autocofactor matrices for all three residual vectors, $\mathbf{v}$, $\mathbf{v}_c$, and $\mathbf{v}_x$, but also their crosscofactor matrices (for example, $\mathbf{Q}_{vv_c}$, $\mathbf{Q}_{vv_x}$, and so on). The total cofactor matrix of the estimated "observations" may readily be obtained from

$$
\hat{\mathbf{Q}}_{tt} = \mathbf{Q}_{tt} - \mathbf{Q}_{v_t v_t} = \begin{bmatrix} \mathbf{Q}_{ll} & \mathbf{Q}_{ll_c} & \mathbf{Q}_{l\hat{x}} \\ & \mathbf{Q}_{l_c l_c} & \mathbf{Q}_{l_c \hat{x}} \\ \text{symmetric} & & \mathbf{Q}_{\hat{x}\hat{x}} \end{bmatrix}
$$

$$
= \begin{bmatrix} (\mathbf{Q} - \mathbf{Q}_{vv}) & -\mathbf{Q}_{vv_c} & -\mathbf{Q}_{vv_x} \\ & (\mathbf{Q}_{cc} - \mathbf{Q}_{v_c v_c}) & -\mathbf{Q}_{v_c v_x} \\ \text{symmetric} & & (\mathbf{Q}_{xx} - \mathbf{Q}_{v_x v_x}) \end{bmatrix} \qquad (12.39)
$$

where the terms such as $\mathbf{Q}_{vv}$, $\mathbf{Q}_{vv_c}$, and so on, are obtainable from equation (12.38). Similar to equation (12.38), equation (12.39) is also of general nature such that various relationships derived earlier can be extracted from it. From example, the (1, 1) term (on the first row and first column) in equation (12.39) gives directly the familiar relation $\mathbf{Q}_{ll} = \mathbf{Q} - \mathbf{Q}_{vv}$. Another example is the crosscofactor $\mathbf{Q}_{l\hat{x}}$ in the (1, 3) term. Recognizing that $\hat{\mathbf{x}} = \Delta$ for the conventional treatment of linear functions given in Chapter 6 (Part II), then

$$
\mathbf{Q}_{l\Delta} = \mathbf{Q}_{l\hat{x}} = -\mathbf{Q}_{vv_x} = -\mathbf{Q}\mathbf{A}^t\mathbf{W}_e \, \mathbf{B}\mathbf{Q}_{\Delta\Delta}
$$

which, in view of equation (6.29) becomes

$$
\mathbf{Q}_{l\Delta} = -\mathbf{Q}\mathbf{A}^t\mathbf{W}_e \, \mathbf{B}\mathbf{N}^{-1} \qquad (12.40)
$$

Equation (12.40) gives a result that is identical to that given by equation (6.32).

Example 12.4. Compute the a posteriori cofactor matrices for the numerical Examples 12.1, 12.2, and 12.3.

Solution:

1. *The Problem of Example 12.1:* The cofactor of the elevation of point p is

$$
\mathbf{Q}_{\Delta\Delta} = (\mathbf{N} + \mathbf{W}_{xx})^{-1} = (2.01)^{-1} = 0.498
$$

2. *The Problem of Example 12.2:* The problem here was of fitting a straight line through the origin and two given points. A priori value for the one parameter, the slope of the line, was given. Thus this is a case of a nonlinear problem with a priori parameter data but no constraints. Given a priori $\mathbf{W}_{xx} = 0.01$, the solution was iterated three times in Example 12.2, for which we give the corresponding $\mathbf{W}_{\Delta\Delta}$,

$$\mathbf{W}_{\Delta\Delta(1)} = 2.81 + 0.01 \quad\ = 2.82 \qquad \text{after the first iteration}$$

$$\mathbf{W}_{\Delta\Delta(2)} = 1.216598 + 0.01 = 1.226598 \qquad \text{after the second iteration}$$

$$\mathbf{W}_{\Delta\Delta(3)} = 1.220788 + 0.01 = 1.230788 \qquad \text{after the third iteration}$$

We have given the computed values for all three iterations in order to illustrate a point. It is quite important when iterating a nonlinear problem to make sure that $Q_{\Delta\Delta}$ or $W_{\Delta\Delta}$ is computed from the last set of normal equations. Otherwise although the estimate of the vector itself, Δ, may be correct, the estimate of the associated cofactor or weight matrix can be considerably in error as shown by the above set of values.

3. *The Problem of Example 12.3* Here we had linear conditions, linear constraints, but no a priori W_{xx} for the parameters. According to the symbols in Example 12.3, then,

$$Q_{xx} = Q_{\tilde\Delta\tilde\Delta} = [\tilde{B}^t(W)\tilde{B} + \tilde{C}^t(W_{cc})\tilde{C}]^{-1}$$

with

$$\tilde{B} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & -2 \end{bmatrix} \quad \text{and} \quad W = I$$

The computed result will depend on the choice for a value of W_{cc}. If we write a general expression as $W_{cc} = \alpha I$, then

$$Q_{xx} = \left[\begin{bmatrix} 5 & -8 & 0 \\ -8 & 14 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 5 & -3 & -3 \\ -3 & 2 & 1 \\ -3 & 1 & 5 \end{bmatrix} \right]^{-1}$$

With an $\alpha = 10$,

$$Q_{xx(1)} = \begin{bmatrix} 505 & -308 & -300 \\ -308 & 214 & 100 \\ -300 & 100 & 500 \end{bmatrix}^{-1} = \begin{bmatrix} 0.125488 & 0.160414 & 0.043208 \\ & 0.210219 & 0.054204 \\ \text{symmetric} & & 0.017084 \end{bmatrix}$$

With an $\alpha = 100$,

$$Q_{xx(2)} = \begin{bmatrix} 5005 & -3008 & -3000 \\ & 2014 & 1000 \\ \text{symmetric} & & 5000 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0.117745 & 0.156300 & 0.039886 \\ & 0.208033 & 0.052173 \\ \text{symmetric} & & 0.013397 \end{bmatrix}$$

Finally, with an $\alpha = 1000$,

$$Q_{xx(3)} = \begin{bmatrix} 50005 & -30008 & -30000 \\ -30008 & 20014 & 10000 \\ -30000 & 10000 & 50000 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0.117133 & 0.156108 & 0.039058 \\ & 0.208107 & 0.052043 \\ \text{symmetric} & & 0.013046 \end{bmatrix}$$

The values in $Q_{xx(1)}$ agree with the proper values computed in Example 9.5 only to the first decimal place. As the value of α increases to bring W_{cc} closer to what it should be (infinity in the limit), the values computed from the present unified approach get closer and closer to the answer by conventional means.

12.6. THE REFERENCE VARIANCE

The computation of the a posteriori estimate of the reference variance is similar here to the other cases covered in Part II. We need to compute the total quadratic form $(v_t^t W_{tt} v_t)$ and divide it by the number of degrees of freedom in the adjustment. Since all the variables are considered as observations, the situation may be treated in the same way as the limited case of adjustment of observations only, covered in Chapter 7 (Part II). The quadratic form in the present case, however, is formed of several parts corresponding to the natural division of variables as they are presented in practice. For example, referring to equation (12.25) the quadratic form may be written as

$$\phi = v^t W v + v_c^t W_{cc} v_c + v_x^t W_{xx} v_x \qquad (12.41)$$

As regards to the number of degrees of freedom, this is often determined prior to the writing of the condition (and constraint) equations. A direct way to compute it is often to take the difference between the total number of equations and the number of unknowns.

Example 12.5. Consider the simple problem of the one-loop level net given in Example 12.1. If we assume for the moment that there was no a priori value for point P, the problem may be solved quickly by the method of adjustment of observations only:

$$[1 \quad -1]v = 0.02$$

from which

$$v = [0.01 \quad -0.01]^t$$

Then

$$\phi_1 = v^t W v = v^t v = 2 \times 10^{-4}$$

and with 1 degree of freedom

$$\hat{\sigma}_{01}^2 = \phi_1 = 2 \times 10^{-4} \text{ m}^2$$

From the unified approach solution

$$[v_1 \quad v_2 \quad v_x] = [0.0095 \quad -0.0105 \quad 0.1095] \text{ m}$$

and the quadratic form becomes

$$\phi_2 = v^t W v + v_x^t W_{xx} v_x = 2.005 \times 10^{-4} + 1.199 + 10^{-4} = 3.204 \times 10^{-4} \text{ m}^2$$

The degrees of freedom in the case of the unified treatment is equal to 2. Consequently, the estimate of the reference variance would be

$$\hat{\sigma}_{02}^2 = \frac{\phi_2}{2} = 1.602 \times 10^{-4} \text{ m}^2$$

The value of $\hat{\sigma}_{02}^2$ is not substantially different from $\hat{\sigma}_{01}^2$ because the a priori estimates of the elevation of point P and its variance were consistent. Therefore their

treatment as observations was quite realistic. If on the other hand the a priori estimates were selected in such a way as to allow for the convenience of applying the unified approach, then we must be careful in computing and interpreting the a posteriori reference variance. For example, if $W_{xx} = 10^{-6}$ were selected, then

$$\phi_2 = 2.005 \times 10^{-4} + 0.012 \times 10^{-6} = 2.00512 \times 10^{-4}$$

This value is almost identical to ϕ_1, but if we use 2 degrees of freedom, then

$$\hat{\sigma}_{02}^2 = 1.003 \times 10^{-4} \text{ m}^2$$

which is considerably different from $\hat{\sigma}_{01}^2$. Therefore it is advisable that we carefully analyze the given data, particularly when evaluating the degrees of freedom.

12.7. REVIEW EXAMPLES

In this section some of the problems used in the examples in Part II will be solved by applying the techniques of the unified approach developed in this chapter. In each of the examples to follow, several possibilities will be selected for the weight matrices of the parameters and the constraints, whenever applicable. The objective of these variations is to show how the solution vector behaves and when we can assume it to be the same as that obtained from the classical procedures.

Example 12.6. In this example, consider the problem of the three photos with two points given in Example 9.2, but without the distance constraint. Refer to Figure 9.1 and the data given, which consist of eight observations and their corresponding covariance matrix. The four parameters, representing the coordinates of the two given points A, B should have a priori observational values. Let these values be equal to the approximations used in Example 9.2 in order to be able to compare the results directly,

$$x_{1a} = 8.00 \text{ m} \qquad x_{2a} = 51.00 \qquad x_{1b} = 7.00 \text{ m} \qquad x_{2b} = 41.00 \text{ m}$$

Associated with these values we should have an a priori cofactor matrix Q_{xx}. We shall choose different values for this matrix and compare the results. These results are summarized in Table 12-1

Included in the table are four cases in which a priori Q_{xx} takes values from $10I$ to 10^4I, as well as the answers from the classical solution of Example 9.2. Note that with

TABLE 12-1

(m)	$Q_{xx} = 10I$	$Q_{xx} = 10^2I$	$Q_{xx} = 10^3I$	$Q_{xx} = 10^4I$	CLASSICAL $Q_{xx} = \infty I$
$\hat{x}_{1a}$	6.9954	6.9952	6.9952	6.9952	6.9952
$\hat{x}_{2a}$	49.7243	49.7181	49.7175	49.7174	49.7174
$\hat{x}_{1b}$	6.9820	6.9817	6.9816	6.9816	6.9816
$\hat{x}_{2b}$	41.9698	41.9698	41.9698	41.9698	41.9698

$Q_{xx} = 10I$ the estimates of the coordinates agree within less than 1 cm with those from the ultimate classical solution. This should be expected since a $Q_{xx} = 10I$ is already several order of magnitude larger than the a priori Q, and since the elements of the latter corresponding to measurements in the object space are 0.0025 m^2. This is a ratio of $10/0.0025$ or $4000 : 1$, which is certainly large enough to yield such close answers. As the magnitude of Q_{xx} is increased relative to Q, the answers get closer and closer to those from the classical procedure. It should be noted though that the gain in proximity of the results is not proportional to the increase in the value of Q_{xx}. Finally, the choice of $Q_{xx} = 10^4I$ gives identical results to within 0.1 mm in the object. The implication of this is that (10^4I) is practically a set of infinite values as far as this particular example is concerned.

As regards to the a posteriori estimate of the cofactor matrix of the parameters, that is, $Q_{\Delta\Delta}$, the values obtained behaved in much the same way as the vector of estimates $\hat{x}$ in Table 12-1, with $Q_{\Delta\Delta}$ being identical to $Q_{\Delta\Delta(f)}$ in Example 9.2 when $Q_{xx} = 10^4I$.

Example 12.7. In this example consider the same problem of the preceding example after introducing the distance constraint. This constraint seeks to enforce the distance between the two points A and B to be equal to 7.80 m. Here we need to select two a priori cofactor matrices, Q_{xx} for the four coordinates and Q_{cc} for the constraints. Running through five values of Q_{xx}, from $10I$ to 10^5I, a value $Q_{cc} = 0.0001$ associated with the linearized constraint equation was chosen. This means an a priori variance of 0.0001 m^2 (since a priori σ_0^2 is 1) to the given distance. The results are summarized in Table 12-2. These results show that the choice of $Q_{cc} = 0.0001$ m^2 is quite realistic since the answers by classical means are obtained to within less than 0.1 mm in the object space when Q_{xx} reached the value of 10^3I.

TABLE 12-2

	$Q_{cc} = 0.0001$					$Q_{cc} = 0$
$Q_{xx} =$	$10I$	10^2I	10^3I	10^4I	10^5I	CLASSICAL $Q_{xx} = \infty I$
$\hat{x}_{1a}$	6.99463	6.99437	6.99434	6.99434	6.99434	6.99434
$\hat{x}_{2a}$	49.75862	49.75716	49.75701	49.75700	49.75700	49.75700
$\hat{x}_{1b}$	6.98352	6.98334	6.98333	6.98332	6.98332	6.98332
$\hat{x}_{2b}$	41.95863	41.95717	41.95702	41.95700	41.95700	41.95700

A posteriori cofactor matrix of the parameters $Q_{\Delta\Delta}$ was also monitored in this example, and the conclusions obtained are essentially the same as those given in the preceding example.

Example 12.8. In this example we rework, using the unified approach, the problem of three photos containing four points given in Example 9.6, without applying the circle constraints. Figure 9.3 shows the schematic of the problem followed by the

pertinent data. For the a priori "observational" values for the eight parameters (the coordinates of the four points), select for simplicity the approximations used in Example 9.6, which are

$$x_{1a} = 8.00 \text{ m} \quad x_{2a} = 51.00 \text{ m} \quad x_{1b} = 7.00 \text{ m} \quad x_{2b} = 41.00 \text{ m}$$

$$x_{1c} = 4.00 \text{ m} \quad x_{2c} = 47.00 \text{ m} \quad x_{1d} = 4.00 \text{ m} \quad x_{2d} = 48.00 \text{ m}$$

The a priori cofactor matrix will be selected at different values in order to observe how closely the final estimates approach the correct answers obtained from the classical solution. The results of these different possibilities are summarized in Table 12-3.

TABLE 12-3

$\mathbf{Q}_{xx} =$ (m)	10I	10^2I	10^3I	CLASSICAL ∞I
$\hat{x}_{1a}$	6.9972	6.9970	6.9969	6.9969
$\hat{x}_{2a}$	49.7221	49.7152	49.7145	49.7144
$\hat{x}_{1b}$	6.9839	6.9834	6.9833	6.9833
$\hat{x}_{2b}$	41.9581	41.9675	41.9674	41.9674
$\hat{x}_{1c}$	2.9955	2.9955	2.9955	2.9955
$\hat{x}_{2c}$	46.1555	46.1511	46.1507	46.1506
$\hat{x}_{1d}$	4.0318	4.0814	4.0313	4.0313
$\hat{x}_{2d}$	48.6448	48.6441	48.6440	48.6440

Similar to the situation encountered in Example 12.6, with $\mathbf{Q}_{xx} = 10^3$I, we get the same answers (to within 0.1 mm in the object space) as those obtained classically in Example 9.6.

The a posteriori matrix $\mathbf{Q}_{\Delta\Delta}$ was also computed for several cases and the value obtained for the case of $\mathbf{Q}_{xx} = 10^3$I was practically the same as $\mathbf{Q}_{\Delta\Delta(f)}$ computed in Example 9.6.

Example 12.9. Continue in this example the problem of the preceding example by introducing the circle constraints and solve the problem by the unified approach. These constraint equations enforce the fact that each of the given four points lies on the circumference of a circle. The parameters of the circle, taken to be the two coordinates of its center and its radius, are not known but will be introduced as added parameters. In the unified solution we simply take them as observations and select the following values for their a priori estimates:

$$x_{10} = 7.00 \text{ m} \quad x_{20} = 46.00 \text{ m} \quad R = 4.00 \text{ m}$$

In the a priori $\mathbf{Q}_{xx}$ we shall also include the added parameters. Table 12-4 summarizes several cases of values of $\mathbf{Q}_{xx}$, and $\mathbf{Q}_{cc}$ is taken to be 0.0001I. This means that deviation between a point and the circumference of the constraint circle is allowed to an a priori variance (or cofactor, since $\sigma_0^2 = 1$) of 0.0001 m^2, which is an adequate and practical value. The results in Table 12-4 show that a value of $\mathbf{Q}_{xx} = 10^3$I gives answers within 0.1 mm from the classical answers.

Again as before, the cofactor matrix $\mathbf{Q}_{\Delta\Delta}$ was computed for each of the above cases with the conclusions corresponding to those given in the preceding three examples.

TABLE 12-4

$\mathbf{Q}_{xx} =$	10I	10^2I	10^3I	10^5I	CLASSICAL ∞I
$\hat{x}_{1a}$	6.99464	6.99424	6.99420	6.99419	6.99419
$\hat{x}_{2a}$	49.78021	49.77425	49.77365	49.77358	49.77358
$\hat{x}_{1b}$	6.98327	6.98270	6.98264	6.98263	6.98263
$\hat{x}_{2b}$	41.98046	41.97989	41.97982	41.97982	41.97981
$\hat{x}_{1c}$	2.99173	2.99157	2.99155	2.99155	2.99155
$\hat{x}_{2c}$	46.17063	46.16641	46.16598	46.16593	46.16593
$\hat{x}_{1d}$	4.04346	4.04338	4.04337	4.04337	4.04337
$\hat{x}_{2d}$	48.55662	48.55233	48.55189	48.55184	48.55184
$\hat{x}_{10}$	6.88226	6.87954	6.87926	6.87923	6.87923
$\hat{x}_{20}$	45.88049	45.87723	45.87690	45.87686	45.87686
$\hat{R}$	3.90134	3.89871	3.89844	3.89841	3.89841

12.8. SUMMARY OF EQUATIONS AND CONCLUDING REMARKS

12.8.1. Summary of Equations Given n observations l with a cofactor matrix $\mathbf{Q}$, u parameters with a priori estimates $\mathbf{x}$ and cofactor matrix $\mathbf{Q}_{xx}$, t observational elements in a vector l_c associated with constraints and a cofactor matrix $\mathbf{Q}_{cc}$, and the following nonlinear equations:

$$F(l, \mathbf{x}) = 0 \qquad (c \text{ equations}) \tag{12.17}$$

$$F_c(l_c, \mathbf{x}) = 0 \qquad (s \text{ equations}) \tag{12.20}$$

With l^0, $\mathbf{x}^0$, and l_c^0 as approximation vectors, the linearized form of these equations is

$$\mathbf{A}\mathbf{v} + \mathbf{B}\Delta = \mathbf{f}^0 \tag{12.23a}$$

$$\mathbf{A}_c \mathbf{v}_c + \mathbf{C}\Delta = \mathbf{f}_c^0 \tag{12.23b}$$

$\mathbf{A}$ is $c \times n$,   rank $(\mathbf{A}) = c$

$\mathbf{A}_c$ is $s \times t$,   rank $(\mathbf{A}_c) = s$

$\mathbf{B}$ is $c \times u$,   rank $(\mathbf{B}) = u$

$\mathbf{C}$ is $s \times u$,   rank $(\mathbf{C}) = s$

$$\mathbf{f}^0 = -[F(l^0, \mathbf{x}^0) - \mathbf{A}(l^0 - l)] \qquad \text{is } c \times 1 \tag{12.24a}$$

$$\mathbf{f}_c^0 = -[F_c(\mathbf{f}_c^0, \mathbf{x}^0) - \mathbf{A}_c(\mathbf{f}_c^0 - \mathbf{l}_c)] \qquad \text{is } s \times 1 \qquad (12.24b)$$

$$\mathbf{f}_x = \mathbf{x}^0 - \mathbf{x} \qquad (12.24c)$$

$$\mathbf{W}_e = \mathbf{Q}_e^{-1} = (\mathbf{A}\mathbf{Q}\mathbf{A}^t)^{-1} \qquad (12.26a)$$

$$\mathbf{W}_{ec} = (\mathbf{A}_c \, \mathbf{Q}_{cc} \, \mathbf{A}_c^t)^{-1} \qquad (12.26b)$$

$$\mathbf{N} = \mathbf{B}^t\mathbf{W}_e \, \mathbf{B} \qquad (12.28a)$$

$$\mathbf{t} = \mathbf{B}^t\mathbf{W}_e \, \mathbf{f}^0 \qquad (12.28b)$$

$$\mathbf{N}_c = \mathbf{C}^t\mathbf{W}_{ec} \, \mathbf{C} \qquad (12.28c)$$

$$\mathbf{t}_c = \mathbf{C}^t\mathbf{W}_{ec} \, \mathbf{f}_c^0 \qquad (12.28d)$$

$$\boldsymbol{\Delta} = (\mathbf{N} + \mathbf{N}_c + \mathbf{W}_{xx})^{-1}(\mathbf{t} + \mathbf{t}_c - \mathbf{W}_{xx} \, \mathbf{f}_x) \qquad (12.29b)$$

$$\mathbf{v} = \mathbf{Q}\mathbf{A}^t\mathbf{W}_e(\mathbf{f}^0 - \mathbf{B}\boldsymbol{\Delta}) \qquad (12.29c)$$

$$\mathbf{v}_c = \mathbf{Q}_{cc} \, \mathbf{A}_c^t \, \mathbf{W}_{ec}(\mathbf{f}_c^0 - \mathbf{C}\boldsymbol{\Delta}) \qquad (12.30a)$$

$$\mathbf{v}_x = \mathbf{f}_x + \boldsymbol{\Delta} \qquad (12.30b)$$

$$\mathbf{Q}_{\Delta\Delta} = (\mathbf{N} + \mathbf{N}_c + \mathbf{W}_{xx})^{-1} \qquad (12.33a)$$

$$\begin{bmatrix} \mathbf{Q}_{vv} & \mathbf{Q}_{vv_c} & \mathbf{Q}_{vv_x} \\ & \mathbf{Q}_{v_cv_c} & \mathbf{Q}_{v_cv_x} \\ \text{symmetric} & & \mathbf{Q}_{v_xv_x} \end{bmatrix} = \text{refer directly to equation (12.38)}$$

$$\begin{bmatrix} \mathbf{Q}_{ll} & \mathbf{Q}_{ll_c} & \mathbf{Q}_{l\hat{x}} \\ & \mathbf{Q}_{l_cl_c} & \mathbf{Q}_{l_c\hat{x}} \\ \text{symmetric} & & \mathbf{Q}_{\hat{x}\hat{x}} \end{bmatrix} = \begin{bmatrix} (\mathbf{Q} - \mathbf{Q}_{vv}) & -\mathbf{Q}_{v_cv_c} & -\mathbf{Q}_{vv_x} \\ & (\mathbf{Q}_{cc} - \mathbf{Q}_{v_cv_c}) & -\mathbf{Q}_{v_cv_x} \\ \text{symmetric} & & (\mathbf{Q}_{xx} - \mathbf{Q}_{v_xv_x}) \end{bmatrix}$$

$$\phi = \frac{(\mathbf{v}^t\mathbf{W}\mathbf{v} + \mathbf{v}_c^t \, \mathbf{W}_{cc} \, \mathbf{v}_c + \mathbf{v}_x^t \, \mathbf{W}_{xx} \, \mathbf{v}_x)}{r}$$

$$r = \text{redundancy or degrees of freedom} \qquad (12.41)$$

12.8.2. Concluding Remarks   The unified approach given in this chapter is the culmination of the developments recently undertaken in least squares adjustment. Allowing each variable to be treated as an observation and varying the a priori weight to account for practical considerations can result in quite a flexible method which is also in a standardized form. This alleviates having to make special decisions for different problems. In fact the procedure applies equally to extensive adjustment problems with varying inputs as well as relatively limited problems. While working with the unified approach, care must be exercised when setting up the equations and, more importantly, when determining the relative magnitudes of the a priori weights and values of the variables. Although the values and weights of the

actual observations are usually well specified, those for the conventionally termed parameters must often be estimated. Here, the chosen weights, in relation to the weights of the actual observations, must be consistent with the selected a priori values. After the adjustment, care must also be taken in interpreting the a posteriori estimate of the reference variance, particularly in regard to the number of degrees of freedom.

13

Sequential Data Processing

13.1. INTRODUCTION

We shall introduce our discussion of sequential data processing by refer-
ring to the problem of the two-parameter transformation as treated in
Example 8.7. Under the assumption that both the $\mathbf{A}$ and $\mathbf{Q}$ matrices are
block diagonal (by point), equations (8.14) and (8.15) show that both the
coefficient matrix $\mathbf{N}$ and the constant vector $\mathbf{t}$ are formed by summing the
contributions of each of the three given points. This summation process can
obviously be extended to any number of points m,

$$\mathbf{N} = \sum_{i=1}^{m} (\mathbf{B}^t \mathbf{W}_e \, \mathbf{B})_i \tag{13.1a}$$

$$\mathbf{t} = \sum_{i=1}^{m} (\mathbf{B}^t \mathbf{W}_e \, \mathbf{f})_i \tag{13.1b}$$

It should be emphasized that equations (13.1a) and (13.1b) are possible
only when the $\mathbf{A}$ *and* $\mathbf{Q}$ *matrices are block diagonal.* This means that the
conditions arising for the ith point contain observations that belong *only to*

those conditions (thus the block diagonality of **A**) and that are *uncorrelated with all other observations* in the model (hence the block diagonality of **Q**).

Stopping the summation at an intermediate step and denoting the partially formed (or incomplete) normal equations by

$$N_{(i-1)} = \sum_{j=1}^{i-1} (B^t W_e B)_j \tag{13.2a}$$

$$t_{(i-1)} = \sum_{j=1}^{i-1} (B^t W_e f)_j \tag{13.2b}$$

it then follows from the pair of equations in (13.1) that the succeeding step would be

$$N_{(i)} = N_{(i-1)} + (B^t W_e B)_i \tag{13.3a}$$

$$t_{(i)} = t_{(i-1)} + (B^t W_e f)_i \tag{13.3b}$$

An interesting possibility now arises if a solution is assumed possible from both pairs of equations (13.2) and (13.3), respectively, that is,

$$\Delta_{(i-1)} = N_{(i-1)}^{-1} t_{(i-1)} \tag{13.4a}$$

$$\Delta_{(i)} = N_{(i)}^{-1} t_{(i)} \tag{13.4b}$$

assuming that both inverses $N_{(i-1)}^{-1}$ and $N_{(i)}^{-1}$ exist. The possibility is to find the value of $\Delta_{(i)}$, not from $N_{(i)}$ and $t_{(i)}$, but from a previous solution vector $\Delta_{(i-1)}$. Such a scheme would lead to obtaining a least squares solution in a sequential or recursive manner. The following are examples of why and under what circumstances we would use such a scheme.

The first case is when an older adjustment has been performed and new information becomes available. One possibility would be to combine the new information with all the old information and perform a complete readjustment. This can be rather uneconomical, particularly if the new information is relatively limited compared to the old information. Instead, it would be better to perform a sort of an "add-on" adjustment if the *results* of the preceding larger adjustment can be used in combination with the *new observational data*. This is possible with the sequential techniques to be developed here.

Another possible use of sequential methods is in the operation of designing the actual observational experiment before the adjustment is performed. Given present-day modern computers with their large capacities and procedures of data simulation, the technique of sequential data reduction can be used to great advantage to assist in an optimum design. For example, in planning a trilateration net with a fixed set of points, various line lengths can be inserted or removed (or both) and numerous observational plans can all be tested in a computer program. The one configuration that would most economically meet selected accuracy requirements would be chosen for field

execution. Any computer simulation expense would be more than offset by savings accrued from selecting an optimum scheme.

In the above example of design of observational experiments it was implied that the sequential technique to be developed would be capable of handling both problems of adding information as well as deleting information. This is true, as the algorithm can be used with only minor sign changes to remove the effect of any designated condition equation(s) from the adjustment. Such a possibility offers another rather useful practical application, the automatic editing of data during the reduction process. Large and involved computer programs used in photogrammetric and geodetic net adjustments often incorporate techniques of data editing. The ability to eliminate the effect of undesirable information, or to reinstate the effect of that which has proven to be useful, sequentially and at the same time of the adjustment, can lead to computational savings.

Yet another possibility of using sequential techniques is in the field of activity of on-line systems of data acquisition. Rapid advances in computer technology are leading more and more to acquisition systems that use electronic computers interfaced with regular electro-optical-mechanical equipment of measurement. One such example is a dedicated computer attached to a multiple-plate comparator for the acquisition of photogrammetric plate measurements. Here the sequential nature of the measurement process and the availability of a computer that allows for real-time or near real-time computational capability leave no doubt as to the desirability of developing sequential data processing techniques.

13.2. MATHEMATICAL DERIVATION

In the example discussed in the introduction, the parameter vector Δ was the same during all the steps of adding more condition equations. This represents one class of problems in which the size of the partially reduced normal equations remains unaltered during both addition and deletion of condition equations. In a second class, the change in the number of condition equations causes a corresponding change in the size of the normal equations; that is, the number of parameters increases with the addition of conditions and decreases when conditions are deleted. We shall treat each case separately.

13.2.1. Fixed Number of Parameters   There are several procedures of deriving a sequential algorithm depending in part on the type of information assumed known in the preceding step. For example, given $N_{(i-1)}^{-1}$ and $t_{(i-1)}$, we would like to get $N_{(i)}^{-1}$ and $t_{(i)}$. Note that we assume that we know the inverse $N_{(i-1)}^{-1}$ itself. This is a most useful case since the inversion of such a matrix during each step can then be avoided. However, one inverse must be assumed possible at the beginning of the recursive scheme. (This assumption

will be eliminated later when the unified approach is considered.) In order to get $N_{(i)}^{-1}$ from $N_{(i-1)}^{-1}$, we simply apply the relations (A68) and (A69) (Appendix A) to equation (13.3a), or

$$\mathop{\mathbf{N}_{(i)}^{-1}}_{u,\,u} = \mathop{\mathbf{N}_{(i-1)}^{-1}}_{u,\,u}\left[\mathbf{I} \mp \mathop{\mathbf{B}_i^t}_{u,\,u}\left(\mathop{\mathbf{Q}_{ei}}_{r_ir_i} \pm \mathop{\mathbf{B}_i}_{r_i,\,u}\,\mathop{\mathbf{N}_{(i-1)}^{-1}}_{u,\,u}\,\mathop{\mathbf{B}_i^t}_{u,\,r_i}\right)^{-1}\,\mathop{\mathbf{B}_i}_{r_i,\,u}\,\mathop{\mathbf{N}_{(i-1)}^{-1}}_{u,\,u}\right]$$

(13.5a)

where u is the total number of parameters and r_i is the number of conditions involved in the ith step. Included in equation (13.5a) are cases of both addition and deletion of conditions: the *upper signs refer to condition addition* and the lower signs are for condition deletion. Although $N_{(i)}^{-1}$ is computed from equation (13.5a), the updated constant term vector can be directly written from equation (13.3b), or

$$\mathbf{t}_{(i)} = \mathbf{t}_{(i-1)} \pm \mathbf{B}_i^t\,\mathbf{W}_{ei}\,\mathbf{f}_i$$

(13.5b)

where the upper and lower signs in equation (13.5b) refer to condition addition and deletion, respectively. Once $N_{(i)}^{-1}$ and $\mathbf{t}_{(i)}$ are evaluated, Δ_i may be obtained, if desired, from equation (13.4b).

The second case to be considered is concerned with $N_{(i-1)}^{-1}$ and Δ_{i-1}. Obviously, $N_{(i)}^{-1}$ can be evaluated from equation (13.5a) for both situations of adding as well as deleting conditions. For developing Δ_i from Δ_{i-1} we shall take the case of adding conditions in the following derivation as an example.

From equation (13.4b) we write

$$\Delta_i = \mathbf{N}_{(i)}^{-1}\,\mathbf{t}_{(i)}$$

and using equations (13.5a) (with the upper signs) and (13.5b) leads to

$$\Delta_i = [\mathbf{N}_{(i-1)}^{-1} - \mathbf{N}_{(i-1)}^{-1}\,\mathbf{B}_i^t(\mathbf{Q}_{ei} + \mathbf{B}_i\,\mathbf{N}_{(i-1)}^{-1}\,\mathbf{B}_i^t)^{-1}\mathbf{B}_i\,\mathbf{N}_{(i-1)}^{-1}]$$
$$\times\,(\mathbf{t}_{(i-1)} + \mathbf{B}_i^t\,\mathbf{W}_{ei}\,\mathbf{f}_i)$$

For simplicity we temporarily drop the subscript e from $\mathbf{Q}_e$ and reinstate it at the end of the derivation. Introducing the auxiliary,

$$\mathbf{J} = \mathbf{B}_i\,\mathbf{N}_{i-1}^{-1}\,\mathbf{B}_i^t$$

(13.6a)

then

$$\Delta_i = \mathbf{N}_{i-1}^{-1}\,\mathbf{t}_{i-1}$$
$$\quad - \mathbf{N}_{i-1}^{-1}\,\mathbf{B}_i^t(\mathbf{Q}_i + \mathbf{J})^{-1}\mathbf{B}_i\,\mathbf{N}_{i-1}^{-1}\,\mathbf{t}_{i-1}$$
$$\quad + \mathbf{N}_{i-1}^{-1}\,\mathbf{B}_i^t\,\mathbf{W}_i\,\mathbf{f}_i$$
$$\quad - \mathbf{N}_{i-1}^{-1}\,\mathbf{B}_i^t(\mathbf{Q}_i + \mathbf{J})^{-1}(\mathbf{B}_i\,\mathbf{N}_{i-1}^{-1}\,\mathbf{B}_i^t)\mathbf{W}_i\,\mathbf{f}_i$$

(13.6b)

The last term in equation (13.6b) is

$$\mathbf{K} = -\mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t (\mathbf{Q}_i + \mathbf{J})^{-1} \mathbf{J} \mathbf{W}_i \mathbf{f}_i$$

which when using equation (A70) (Appendix A) gives

$$\mathbf{K} = -\mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t \mathbf{W}_i (\mathbf{W}_i + \mathbf{J}^{-1})^{-1} \mathbf{W}_i \mathbf{f}_i$$

and applying equation (A69), realizing that $\mathbf{U} = \mathbf{V} = \mathbf{I}$ in the present case, leads to

$$\mathbf{K} = -\mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t \mathbf{W}_i [\mathbf{Q}_i - \mathbf{Q}_i (\mathbf{J} + \mathbf{Q}_i)^{-1} \mathbf{Q}_i] \mathbf{W}_i \mathbf{f}_i$$

$$= -\mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t \mathbf{W}_i \mathbf{f}_i + \mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t (\mathbf{Q}_i + \mathbf{J})^{-1} \mathbf{f}_i$$

which when used in equation (13.6b) results in

$$\Delta_i = \Delta_{i-1} - \mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t (\mathbf{Q}_i + \mathbf{J})^{-1} \mathbf{B}_i \Delta_{i-1} + \mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t (\mathbf{Q}_i + \mathbf{J})^{-1} \mathbf{f}_i$$

Finally,

$$\Delta_i = \Delta_{i-1} + \mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t (\mathbf{Q}_i + \mathbf{B}_i \mathbf{N}_{i-1}^{-1} \mathbf{B}_i^t)^{-1} (\mathbf{f}_i - \mathbf{B}_i \Delta_{i-1}) \qquad (13.6c)$$

Restoring the subscript e and including the case of condition deletion yields (the reader is advised to work out the case of deletion for himself)

$$\Delta_i = \Delta_{i-1} \pm \mathbf{N}_{(i-1)}^{-1} \mathbf{B}_i^t (\mathbf{Q}_{ei} \pm \mathbf{B}_i \mathbf{N}_{(i-1)}^{-1} \mathbf{B}_i^t)^{-1} (\mathbf{f}_i - \mathbf{B}_i \Delta_{i-1}) \qquad (13.7)$$

Thus with $\mathbf{N}_{(i-1)}^{-1}$ and Δ_{i-1} known, the new matrices $\mathbf{N}_{(i)}^{-1}$ and Δ_i can be computed from the recursive formulas (13.5a) and (13.7).

All the preceding derivations were accomplished using only operations of matrix algebra. It should be recognized that the same relation must also be obtainable from applying the basic least squares criterion. As a demonstration we show the case of getting $\mathbf{N}_{(i)}^{-1}$ from $\mathbf{N}_{(i-1)}^{-1}$ for the addition of conditions and leave the other relations for the reader to derive as an exercise. Let

$$\mathbf{A}_{i-1} \mathbf{v}_{i-1} + \mathbf{B}_{i-1} \Delta = \mathbf{f}_{i-1}$$

represent the total conditions up to and including the $(i - 1)$ step; and

$$\mathbf{A}_i \mathbf{v}_i + \mathbf{B}_i \Delta = \mathbf{f}_i$$

denote the conditions to be added at the ith step. These two equations may be combined into the form

$$\begin{bmatrix} \mathbf{A}_{i-1} & 0 \\ 0 & \mathbf{A}_i \end{bmatrix} \begin{bmatrix} \mathbf{v}_{i-1} \\ \mathbf{v}_i \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{i-1} \\ \mathbf{B}_i \end{bmatrix} \Delta = \begin{bmatrix} \mathbf{f}_{i-1} \\ \mathbf{f}_i \end{bmatrix} \qquad (13.8)$$

With the total cofactor matrix (keeping the original assumption of no correlation)

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{(i-1)} & 0 \\ 0 & \mathbf{Q}_{(i)} \end{bmatrix}$$

then

$$\mathbf{Q}_e = \begin{bmatrix} (\mathbf{AQA}')_{(i-1)} & 0 \\ 0 & (\mathbf{AQA}')_{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{e(i-1)} & 0 \\ 0 & \mathbf{Q}_{e(i)} \end{bmatrix}$$

The scalar to be minimized is in this case

$$\phi = (\mathbf{v}'\mathbf{W}\mathbf{v})_{i-1} + (\mathbf{v}'\mathbf{W}\mathbf{v})_i - 2\mathbf{k}^t_{i-1}(\mathbf{A}_{i-1}\,\mathbf{v}_{i-1} + \mathbf{B}_{i-1}\,\Delta - \mathbf{f}_{i-1})$$
$$- 2\mathbf{k}^t_i(\mathbf{A}_i\,\mathbf{v}_i + \mathbf{B}_i\,\Delta - \mathbf{f}_i)$$

which must be differentiated with respect to $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, and Δ, and the resulting differentials equated to zero. These three equations combined with the pair in equation (13.8) make up five equations in five unknown vectors, $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, $\mathbf{k}_{i-1}$, $\mathbf{k}_i$, and Δ. After eliminating $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, and $\mathbf{k}_{i-1}$, the partially reduced normal equations take the form

$$\begin{bmatrix} -\mathbf{N}_{(i-1)} & \mathbf{B}^t_i \\ \mathbf{B}_i & \mathbf{W}_{e(i)} \end{bmatrix} \begin{bmatrix} \Delta \\ \mathbf{k}_i \end{bmatrix} = \begin{bmatrix} -\mathbf{t}_{(i-1)} \\ \mathbf{f}_i \end{bmatrix} \tag{13.9}$$

in which $\mathbf{N}_{(i-1)}$ and $\mathbf{t}_{(i-1)}$ are equivalent to those given by equations (13.2a) and (13.2b). The solution of equation (13.9) by partitioning can be carried out as follows:

$$\begin{bmatrix} -\mathbf{N}_{(i-1)} & \mathbf{B}^t_i \\ \mathbf{B}_i & \mathbf{W}_{e(i)} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{P} & \mathbf{R}^t \\ \mathbf{R} & \mathbf{S} \end{bmatrix}$$

and according to equation (A37a) (Appendix A)

$$-\mathbf{P} = (\mathbf{N}_{(i-1)} + \mathbf{B}^t_i\,\mathbf{W}_{e(i)}\,\mathbf{B}_i)^{-1}$$

which is equal to $\mathbf{N}_{(i)}^{-1}$ from equation (13.3a). Multiplying the coefficient matrix in equation (13.9) by its inverse and equating the result to the identity matrix gives

$$-\mathbf{P} = \mathbf{N}_{(i-1)}^{-1}(\mathbf{I} - \mathbf{B}^t_i\,\mathbf{R})$$
$$-\mathbf{B}_i = \mathbf{N}_{(i-1)}^{-1}(\mathbf{I} - \mathbf{B}^t_i\,\mathbf{R}) + \mathbf{Q}_{e(i)}\,\mathbf{R} = 0$$
$$\mathbf{R} = (\mathbf{Q}_{e(i)} + \mathbf{B}_i\,\mathbf{N}_{(i-1)}^{-1}\,\mathbf{B}^t_{i-1})^{-1}\mathbf{B}_i\,\mathbf{N}_{(i-1)}^{-1}$$

then

$$\mathbf{N}_{(i)}^{-1} = \mathbf{N}_{(i-1)}^{-1}[\mathbf{I} - \mathbf{B}^t_i(\mathbf{Q}_{e(i)} + \mathbf{B}_i\,\mathbf{N}_{(i-1)}^{-1}\,\mathbf{B}^t_i)^{-1}\mathbf{B}_i\,\mathbf{N}_{(i-1)}^{-1}]$$

which is identical to equation (13.5a) with the upper signs for addition of conditions.

The recursive formulas that have been derived exhibit an apparent advantage in avoiding the inversion of the normal equations coefficient matrix. On the other hand these sequential relationships, for example, equation (13.5a), involve an apparently large number of matrix multiplications and an inversion of a matrix of order r_i. As regards the inversion, r_i is, in general, much

smaller than u, since we would usually add or eliminate a few conditions at a time. Of course, if only one equation is involved, then the inversion degenerates into computing a reciprocal of a scalar. Concerning the number of matrix multiplications, the question becomes a matter of computational efficiency of the sequential scheme compared to the conventional "batch" least squares. As an example, 90 linear equations in 60 unknowns, of the form

$$\underset{90,\,1}{\mathbf{v}} + \underset{90,\,60}{\mathbf{B}} \underset{60,\,1}{\mathbf{\Delta}} = \underset{90,\,1}{\mathbf{f}}$$

were generated. As a start, a 60×60 normal equations coefficient matrix from all 90 condition equations was formed. The inverse of this matrix, which is used as a standard, consumed approximately 8 seconds on the computer used (UNIVAC 1108). Next, several cases of deleting a progressively larger number of condition equations (up to the limit of 30 equations) were computed using both methods of conventional batch least squares and by the sequential scheme developed here. To emphasize the difference between these two methods we can consider as an example the case of deleting 5 equations from the given 90 equations. In the batch process we would use the remaining 85 equations, form a 60×60 normal equations matrix and invert. In this case, the time used for these operations is a little less than 7 seconds. By contrast, in the sequential procedure the standard inverse (that is, that computed from all 90 condition equations) is used together with the 5 equations to be deleted to compute the inverse that would correspond to the remaining 85 equations. This sequential operation, which leads to the same inverse, consumed less than 1 second of computer time. The same process of taking off equations was performed several times by both methods; each time a larger number of equations was deleted, and the computing time was noted. The results are presented in Figure 13.1 in a form of two graphs, one for the batch solution and the other for the sequential solution.

Examining the graph for the batch solution first, we can see that the difference between the time required for getting an inverse from 90 equations and that required for getting it from 60 equations is rather small, being only slightly over 2 seconds. This indicates that the batch method is not strongly dependent either on the total number of condition equations or the number of equations to be deleted, but rather on the size of the reduced normal equation coefficient matrix under the assumption that all the condition equations are already available; that is, the time for their formulation is not considered here. By contrast, the sequential method is more strongly dependent on the number of condition equations, particularly those to be deleted. The time increases relatively rapidly as the number of such equations increases, until it reaches a cutoff point where the two graphs intersect and where the use of sequential solution offers no advantage over the conventional batch method. Therefore it would be advisable to limit the number of
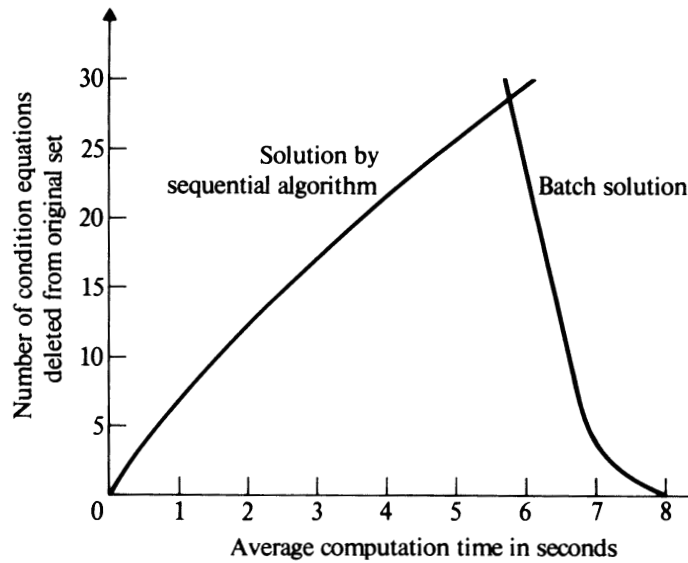
Figure 13.1. Example on Computational Time for Batch and Sequential Solutions (A 60-Parameter Problem)

condition equations by which the problem is changed in relation to the number of normal equations.

It should be emphasized that these results are only a demonstration of the possibilities and hold only for the data used (60 normal equations, inversion by Gauss elimination algorithm, UNIVAC 1108 computer, and so on). Different data may produce different results, but it is felt that the general characteristics of Figure 13.1 will remain. It is suspected further that if the condition equations were to be formed within the adjustment process, the sequential solution would offer additional computational time savings.

Example 13.1. Consider the following set of four condition equations:

$$v_1 - 2x_1 + x_2 = 1$$

$$v_2 - x_1 - 3x_2 = -9$$

$$v_3 + x_1 + 2x_2 = 8$$

$$\text{-------------------------}$$

$$v_4 - 4x_1 - x_2 = -7$$

or in matrix form

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ --- \\ v_4 \end{bmatrix} + \begin{bmatrix} -2 & +1 \\ -1 & -3 \\ +1 & +2 \\ ---- & ---- \\ -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 8 \\ --- \\ -7 \end{bmatrix}$$

which corresponds to

$$\mathbf{v} + \mathbf{B} \ \Delta = \mathbf{f}$$
$$\underset{4,\,1}{} \quad \underset{4,\,2}{} \underset{2,\,1}{} \quad \underset{4,\,1}{}$$

Assume for simplicity that $\mathbf{Q} = \mathbf{I}_4$. With this data we seek to demonstrate the concepts developed in the preceding section for both the addition and deletion of conditions. First consider *the case of condition addition*, by solving with all four conditions, then with three plus one added sequentially, and then comparing the final results.

For all four conditions,

$$(\mathbf{B}^t\mathbf{WB})\Delta = (\mathbf{B}^t\mathbf{Wf}) \qquad \text{or} \qquad \mathbf{N}\Delta = \mathbf{t}$$

or

$$\begin{bmatrix} 22 & 7 \\ 7 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 43 \\ 51 \end{bmatrix}$$

With

$$\mathbf{N}^{-1} = \frac{1}{281} \begin{bmatrix} 15 & -7 \\ -7 & 22 \end{bmatrix} \qquad \text{and} \qquad \mathbf{t} = \begin{bmatrix} 43 \\ 51 \end{bmatrix}$$

then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{281} \begin{bmatrix} 288 \\ 821 \end{bmatrix}$$

For the first three equations,

$$\mathbf{B}_0 = \begin{bmatrix} -2 & 1 \\ -1 & -3 \\ 1 & +2 \end{bmatrix}, \qquad \mathbf{f}_0 = \begin{bmatrix} 1 \\ -9 \\ 8 \end{bmatrix}, \qquad \mathbf{N}_0 = \begin{bmatrix} 6 & 3 \\ 3 & 14 \end{bmatrix}, \qquad \mathbf{t}_0 = \begin{bmatrix} 15 \\ 44 \end{bmatrix}$$

$$\mathbf{N}_0^{-1} = \frac{1}{75} \begin{bmatrix} 14 & -3 \\ -3 & 6 \end{bmatrix}$$

thus

$$\Delta_0 = \frac{1}{75} \begin{bmatrix} 78 \\ 219 \end{bmatrix}$$

In order to effect the addition of the fourth condition equation, its matrices are

$$\mathbf{B}_1 = [-4 \quad -1] \qquad \mathbf{f}_1 = [-7]$$

with the subscript 1 referring to the first (and only, in this case) sequential addition. First, use equation (13.5a) (with upper signs) to compute $\mathbf{N}_1^{-1}$, as follows. Because of having $\mathbf{Q} = \mathbf{I}$, the recursive equation becomes

$$\mathbf{N}_1^{-1} = \mathbf{N}_0^{-1}[\mathbf{I} - \mathbf{B}_1^t(1 + \mathbf{B}_1 \mathbf{N}_0^{-1}\mathbf{B}_1^t)^{-1}\mathbf{B}_1 \mathbf{N}_0^{-1}]$$

noting that the parenthetical expression to be inverted is only a scalar equal to

$$1 + \frac{1}{75}[-4 \quad -1] \begin{bmatrix} 14 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \frac{281}{75}$$

Performing the arithmetic operations, yields N_1^{-1} which is identical to N^{-1} obtained directly from the four equations. As for the constant term vector, apply equation (13.5b)

$$\mathbf{t}_1 = \mathbf{t}_0 + \mathbf{B}_1^t \ \mathbf{W} \mathbf{f}_1 = [43 \quad 51]^t$$

which is also identical to $\mathbf{t}$ computed from the four condition equations. Hence the estimates for x_1, x_2 will be identical from this sequential scheme as from the batch solution.

The next step is to apply equation (13.7) (with upper signs), which in this example becomes

$$\Delta_1 = \Delta_0 + N_0^{-1} \mathbf{B}_1^t (1 + \mathbf{B}_1 \ N_0^{-1} \mathbf{B}_1^t)^{-1} (\mathbf{f}_1 - \mathbf{B}_1 \ \Delta_0)$$

First,

$$(\mathbf{f}_1 - \mathbf{B}_1 \ \Delta_0) = \frac{2}{25}$$

and

$$N_0^{-1} \mathbf{B}_1^t (\mathbf{I} + \mathbf{B}_1 N_0^{-1} \mathbf{B}_1^t)^{-1} (\mathbf{f}_1 - \mathbf{B}_1 \ \Delta_0) = \frac{1}{75 \times 281} [-318 \quad 36]^t$$

then

$$\Delta_1 = \frac{1}{281} [288 \quad 821]^t$$

which is identical to the solution obtained before. Consequently, both schemes of sequential solution for addition of condition equations are valid. Now consider the same data for the *case of condition deletion*. The solution for all four conditions is modified by eliminating the contribution of the fourth equation, and the results obtained are compared to those directly available from solving the first three equations together. Thus

$$N_0^{-1} = \frac{1}{281} \begin{bmatrix} 15 & -7 \\ -7 & 22 \end{bmatrix} \qquad \mathbf{t}_0 = \begin{bmatrix} 43 \\ 51 \end{bmatrix} \qquad \Delta_0 = \frac{1}{281} \begin{bmatrix} 288 \\ 821 \end{bmatrix}$$

and

$$\mathbf{B}_1 = [-4 \quad -1] \qquad \Delta_1 = [-7] \qquad \mathbf{Q}_1 = 1$$

First apply equation (13.5a) with the lower signs to compute N_1^{-1} from N_0^{-1}.

$$N_1^{-1} = \frac{1}{75} \begin{bmatrix} 14 & -3 \\ -3 & 6 \end{bmatrix}$$

then from equation (13.5b) compute

$$\mathbf{t}_1 = [15 \quad 44]^t$$

Both N_1^{-1} and $\mathbf{t}_1$ are identical to N_0^{-1} and $\mathbf{t}_0$ for the case of addition for the batch solution of the first three conditions. Finally, we can also demonstrate the applicability of equation (13.7) with lower signs for the case of condition deletion, which is left as an exercise.

13.2.2. Variable Number of Parameters   In many problems, such as aerial triangulation in photogrammetry and net design with variable number of control points in surveying, the increase or decrease in the number of condition equations may change the number of parameters in the adjustment accordingly. For example, if the collinearity type of condition equations is used in a photogrammetric triangulation adjustment, each time a pass point is added three new parameters corresponding to its coordinates will also be added. In the following development first the basic foundation is given then each of the two situations of condition addition and deletion will be treated separately.

Consider the two sets of conditions

$$\mathbf{A}_{i-1}\,\mathbf{v}_{i-1} + \mathbf{B}_{i-1}\,\Delta = \mathbf{f}_{i-1} \tag{13.10a}$$

$$\mathbf{A}_i\,\mathbf{v}_i + \mathbf{B}_i\,\Delta + \mathbf{b}_i\,\delta \;=\; \mathbf{f}_i \tag{13.10b}$$

in which δ is a subvector of parameters that appears in the set of conditions to be added or subtracted sequentially. This pair of equations may be combined into

$$\begin{bmatrix} \mathbf{A}_{i-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_i \end{bmatrix} \begin{bmatrix} \mathbf{v}_{i-1} \\ \mathbf{v}_i \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{i-1} & \mathbf{0} \\ \mathbf{B}_i & \mathbf{b}_i \end{bmatrix} \begin{bmatrix} \Delta \\ \delta \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{i-1} \\ \mathbf{f}_i \end{bmatrix} \tag{13.11}$$

for which the reduced normal equations may be written directly as a coefficient matrix $\mathbf{N}_i$ and constant vector $\mathbf{t}_i$

$$\mathbf{N}_i = \begin{bmatrix} (\mathbf{B}^t\mathbf{W}_e\,\mathbf{B})_{i-1} + (\mathbf{B}^t\mathbf{W}_e\,\mathbf{B})_i & (\mathbf{B}^t\mathbf{W}_e\,\mathbf{b})_i \\ (\mathbf{b}^t\mathbf{W}_e\,\mathbf{B})_i & (\mathbf{b}^t\mathbf{W}_e\,\mathbf{b})_i \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{N}_{i-1} + \delta\mathbf{N}_i & \bar{\mathbf{n}}_i \\ \bar{\mathbf{n}}_i^t & \mathbf{n}_i \end{bmatrix} \tag{13.12a}$$

$$\mathbf{t}_i = \begin{bmatrix} (\mathbf{B}^t\mathbf{W}_e\,\mathbf{f})_{i-1} + (\mathbf{B}^t\mathbf{W}_e\,\mathbf{f})_i \\ (\mathbf{b}^t\mathbf{W}_e\,\mathbf{f})_i \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{i-1} + \delta\mathbf{t}_i \\ \bar{\mathbf{t}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{t}_i' \\ \bar{\mathbf{t}}_i \end{bmatrix} \tag{13.12b}$$

The inverse of $\mathbf{N}_i$ may be evaluated by partitioning as

$$\mathbf{N}_i^{-1} = \begin{bmatrix} \bar{\mathbf{N}}_i & \bar{\mathbf{n}}_i \\ \bar{\mathbf{n}}_i^t & \mathbf{n}_i \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{F}_i & \mathbf{G}_i \\ \mathbf{G}_i^t & \mathbf{H}_i \end{bmatrix} \tag{13.13}$$

with [see equations (A37), Appendix A]

$$\mathbf{F}_i = (\bar{\mathbf{N}}_i - \bar{\mathbf{n}}_i\,\mathbf{n}_i^{-1}\bar{\mathbf{n}}_i^t)^{-1} \tag{13.14a}$$

$$\mathbf{G}_i = -\mathbf{F}_i\,\bar{\mathbf{n}}_i\,\mathbf{n}_i^{-1} \tag{13.14b}$$

$$\mathbf{H}_i = \mathbf{n}_i^{-1} - \mathbf{n}_i^{-1}\bar{\mathbf{n}}_i^t\,\mathbf{G}_i \tag{13.14c}$$

If $\mathbf{G}_i$ and $\mathbf{F}_i$ are replaced by their expressions, $\mathbf{H}_i$ would become

$$\mathbf{H}_i = (\mathbf{n}_i - \bar{\mathbf{n}}_i^t\,\bar{\mathbf{N}}_i^{-1}\bar{\mathbf{n}}_i)^{-1} \tag{13.14d}$$

which is an alternate form. Equation (A69) may be applied to equation (13.14a) to give

$$\mathbf{F}_i = \bar{\mathbf{N}}_i^{-1} + \bar{\mathbf{N}}_i^{-1}\bar{\mathbf{n}}_i(\mathbf{n}_i - \bar{\mathbf{n}}_i^t\,\bar{\mathbf{N}}_i^{-1}\bar{\mathbf{n}}_i)^{-1}\bar{\mathbf{n}}_i^t\,\bar{\mathbf{N}}_i^{-1} \qquad (13.15a)$$

which in view of equation (13.14d) becomes

$$\mathbf{F}_i = \bar{\mathbf{N}}_i^{-1} + \bar{\mathbf{N}}_i^{-1}\bar{\mathbf{n}}_i\,\mathbf{H}_i\,\bar{\mathbf{n}}_i^t\,\bar{\mathbf{N}}_i^{-1} \qquad (13.15b)$$

and since from equations (13.12a) and (13.13)

$$\bar{\mathbf{N}}_i = \mathbf{N}_{i-1} + \delta\mathbf{N}_i = \mathbf{N}_{i-1} + \mathbf{B}_i^t\,\mathbf{W}_{ei}\,\mathbf{B}_i \qquad (13.16)$$

the application of equation (A69) to equation (13.16) leads to

$$\bar{\mathbf{N}}_i^{-1} = \mathbf{N}_{i-1}^{-1} - \mathbf{N}_{i-1}^{-1}\,\mathbf{B}_i^t(\mathbf{Q}_{ei} + \mathbf{B}_i\,\mathbf{N}_{i-1}^{-1}\,\mathbf{B}_i^t)^{-1}\mathbf{B}_i\,\mathbf{N}_{i-1}^{-1} \qquad (13.17)$$

The foregoing development establishes all the fundamental relationships necessary to handle this case of variable size of parameters. Because the case of addition increases the order of the normal equations coefficient matrix, whereas the case of deletion effects the opposite, each case must be handled separately.

ADDING CONDITION EQUATIONS   At the $(i - 1)$ step of the sequential processing we would have $\mathbf{N}_{i-1}^{-1}$ and $\mathbf{t}_{i-1}$, and we would want to add the new conditions given by equation (13.10b). These conditions include δ_i which increases the size of the parameters from Δ_{i-1} which is $u_{i-1} \times 1$ to

$$\Delta_i = \begin{bmatrix} \Delta_{i-1} \\ \delta_i \end{bmatrix}$$

which is $u_i \times 1$, where u_i is $u_{i-1} + w_i$ and w_i is the number of parameters in δ_i. It is important to note that the new observations in equation (13.10b) are assumed to be uncorrelated with all the original observations. This allows for constructing the total cofactor matrix as a block diagonal matrix. After the new conditions of equation (13.10b) are added, we would have $\mathbf{N}_i^{-1}$ and $\mathbf{t}_i$, which are obtained as follows.

First, equation (13.17) is used to compute $\bar{\mathbf{N}}_i^{-1}$ from $\mathbf{N}_{i-1}^{-1}$ and the matrices of the added conditions of equation (13.10b), noting that $\mathbf{Q}_{ei} = \mathbf{A}_i\,\mathbf{Q}_i\,\mathbf{A}_i^t$. Second, according to equation (13.12a) the auxiliary matrices $\bar{\mathbf{n}}_i$ and $\mathbf{n}_i$ are evaluated from

$$\bar{\mathbf{n}}_i = \mathbf{B}_i^t\,\mathbf{W}_{ei}\,\mathbf{b}_i \qquad (13.18a)$$

$$\mathbf{n}_i = \mathbf{b}_i^t\,\mathbf{W}_{ei}\,\mathbf{b}_i \qquad (13.18b)$$

Next, from $\bar{\mathbf{N}}_i^{-1}$ and these auxiliary matrices, $\mathbf{F}_i$, $\mathbf{G}_i$, and $\mathbf{H}_i$ are computed from equations (13.15), (13.14b), and (13.14c), respectively. These three matrices are then collected together to give $\mathbf{N}_i^{-1}$ according to equation

(13.13). As regards to the constant term vector, by reference to equation (13.12b) we compute

$$t'_i = t_{i-1} + B_i W_{ei} f_i \tag{13.19a}$$

$$\bar{t}_i = b^t_i W_{ei} f_i \tag{13.19b}$$

and collect them to form t_i as shown in equation (13.12b). This completes the ith step of the sequential solution.

DELETING CONDITION EQUATIONS   Here, in order to use the same basic relations developed at the beginning of this section, we shall for the moment go from the ith step to the $(i - 1)$ step, and then at the end change the subscript variable to give the more conventional relations. Thus at the ith step we would have N_i^{-1} and t_i and would wish to eliminate the effect of the conditions of equation (13.10b) to get N_{i-1}^{-1} and t_{i-1}. Since this elimination will reduce the parameters by δ, then both N_{i-1}^{-1} and t_{i-1} will be correspondingly of smaller dimensions than N_i^{-1} and t_i, respectively. Consequently, from the partitioning of N_i^{-1} in equation (13.13) only the submatrix F_i would be of interest, and the remaining matrices are simply discarded. Inverting equation (13.14a) yields

$$F_i^{-1} = \bar{N}_i - \bar{n}_i n_i^{-1} \bar{n}^t_i$$

or

$$\bar{N}_i = F^{-1} + \bar{n}_i n_i^{-1} \bar{n}^t_i \tag{13.20a}$$

and from equation (A69) the inverse of $\bar{N}_i$ would be

$$\bar{N}_i^{-1} = F_i - F_i \bar{n}_i (n_i + \bar{n}^t_i F_i \bar{n}_i)^{-1} \bar{n}^t_i F_i \tag{13.20b}$$

Next, from equation (13.16)

$$N_{i-1} = \bar{N}_i - B^t_i W_{ei} B_i \tag{13.21a}$$

which again may be inverted according to equation (A69) to give

$$N_{i-1}^{-1} = \bar{N}_i^{-1} + \bar{N}_i^{-1} B^t_i (Q_{ei} - B_i \bar{N}_i^{-1} B^t_i)^{-1} B_i \bar{N}_i^{-1} \tag{13.21b}$$

In a similar manner only the subvector t'_i of t_i is pertinent here, from which t_{i-1} may be directly obtained from equation (13.19a), or

$$t_{i-1} = t'_i - B_i W_{ei} f_i \tag{13.21c}$$

This completes showing how N_{i-1}^{-1} and t_{i-1} may be obtained from N_i and t_i. However, in a sequential procedure we are conventionally accustomed to ascending subscripts instead of this case of decreasing subscripts. Consequently, we recapitulate the above development while changing the subscript to correspond to starting with N_{j-1}^{-1} and t_{j-1} and seeking N_j^{-1} and t_j due to eliminating the conditions,

$$A_j v_j + B_j \Delta + b_j \delta = f_j \tag{13.22}$$

The sequence is summarized as follows:

1. Partition N_{j-1}^{-1} and take the upper left matrix F_{j-1} of order corresponding to the reduced number of parameters.
2. Compute the auxiliary matrices $\bar{n}_j$ and n_j using the matrices of equation (13.22) in equations (13.18a) and (13.18b), and noting that $W_{ej} = (A_j Q_j A_j')^{-1}$.
3. Evaluate $\bar{N}_{j-1}^{-1}$ from [see equation (13.20b)],

$$\bar{N}_{j-1}^{-1} = F_{j-1} - F_{j-1}\, \bar{n}_j(n_j + \bar{n}_j'\, F_{j-1}\, \bar{n}_j)^{-1}\bar{n}_j'\, F_{j-1} \tag{13.23}$$

4. Compute N_j^{-1} from [see equation (13.21b)],

$$N_j^{-1} = \bar{N}_{j-1}^{-1} + \bar{N}_{j-1}^{-1}B_j'(Q_{ej} - B_j\bar{N}_{j-1}^{-1}B_j')^{-1}B_j\bar{N}_{j-1}^{-1} \tag{13.24}$$

5. Compute t_j from [see equation (13.21c)],

$$t_j = t_{j-1}' - B_j'\, W_{ej}\, f_j \tag{13.25}$$

in which t_{j-1}' is the upper subvector of t_{j-1} with as many elements as the number of remaining parameters.

This concludes the steps of sequentially eliminating conditions that reduce the number of parameters.

Example 13.2. To demonstrate the applicability of the derivations in this section, take a numerical example and consider both *cases of adding and deleting information*. Assuming for simplicity that A and Q are identity matrices for all conditions and observations, let the total system of condition equations be denoted by [see equation (13.11)]

$$\begin{bmatrix} \dot{v} \\ \ddot{v} \end{bmatrix} + \begin{bmatrix} \dot{B} & 0 \\ \ddot{B} & b \end{bmatrix}\begin{bmatrix} \Delta \\ \delta \end{bmatrix} = \begin{bmatrix} \dot{f} \\ \ddot{f} \end{bmatrix} \qquad \text{or} \qquad v + B\Delta' = f$$

where the total coefficient matrix is

$$B = \begin{bmatrix} 0.44 & 0.76 & 0 \\ 1.97 & 3.18 & 0 \\ 8.35 & 3.52 & 0 \\ \hline 5.40 & 7.24 & 2.70 \\ 9.99 & 2.69 & 2.69 \\ 5.39 & 8.08 & 3.47 \\ 1.55 & 5.03 & 6.59 \end{bmatrix}$$

For simplicity we shall not consider the constant vector, as its modification is relatively straightforward. For batch processing, with $Q = I_7$, we get

$$\underset{3,\,3}{N^{-1}} = (B'B)^{-1} = \begin{bmatrix} 0.01036 & -0.01066 & 0.00301 \\ -0.01066 & 0.02681 & -0.02303 \\ 0.00301 & -0.02303 & 0.04025 \end{bmatrix}$$

For sequential processing, we originally have three condition equations with the coefficient matrix

$$\dot{\mathbf{B}}_{3,\,2} = \begin{bmatrix} 0.44 & 0.76 \\ 1.97 & 3.18 \\ 8.35 & 3.52 \end{bmatrix}$$

from which the inverse of the normal equations coefficient matrix is

$$\dot{\mathbf{N}}^{-1} = (\dot{\mathbf{B}}'\dot{\mathbf{B}})^{-1} = \begin{bmatrix} 0.05658 & 0.08823 \\ 0.08823 & 0.18091 \end{bmatrix}$$

The new set of equations to be added sequentially (as one group) has the following coefficient matrices:

$$\ddot{\mathbf{B}} = \begin{bmatrix} 5.40 & 7.24 \\ 9.99 & 2.69 \\ 5.39 & 8.08 \\ 1.55 & 5.03 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2.7 \\ 2.69 \\ 3.47 \\ 6.59 \end{bmatrix}$$

In order to construct the total augmented matrix $\mathbf{N}^{-1}$ [see equation (13.13)], first compute $\bar{\mathbf{N}}^{-1}$ from equation (13.17) using $\dot{\mathbf{N}}^{-1}$ (representing $\mathbf{N}_{i-1}^{-1}$) and $\ddot{\mathbf{B}}$, thus

$$\bar{\mathbf{N}}^{-1} = \begin{bmatrix} 0.01013 & -0.00894 \\ -0.00894 & 0.01363 \end{bmatrix}$$

Next, from equations (13.18a) and (13.18b) compute the following auxiliary matrices

$$\bar{\mathbf{n}} = \ddot{\mathbf{B}}_{2,\,4}\,\mathbf{b}_{4,\,1} = \begin{bmatrix} 70.37090 \\ 88.10440 \end{bmatrix} \qquad \mathbf{n} = \mathbf{b}'_{1,\,4}\,\mathbf{b}_{4,\,1} = [69.99510]$$

and use them and $\bar{\mathbf{N}}^{-1}$ above to compute $\mathbf{F}$ from equation (13.15)

$$\mathbf{F} = \begin{bmatrix} 0.01036 & -0.01066 \\ -0.01066 & 0.92681 \end{bmatrix}$$

which is the first submatrix of the required inverse. We finish up by using equations (13.14b) and (13.14c) to compute the remaining submatrices

$$\mathbf{G} = -\mathbf{F}\bar{\mathbf{n}}\mathbf{n}^{-1} = \begin{bmatrix} 0.00301 \\ -0.02303 \end{bmatrix}$$

$$\mathbf{H} = \mathbf{n}^{-1} - \mathbf{n}^{-1}\bar{\mathbf{n}}'\mathbf{G} = [0.04025]$$

Thus the total updated inverse is

$$\mathbf{N}^{-1}_{3,\,3} = \begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{G}' & \mathbf{H} \end{bmatrix} = \begin{bmatrix} 0.01036 & -0.01066 & 0.00301 \\ -0.01066 & 0.02681 & -0.02303 \\ 0.00301 & -0.02303 & 0.04025 \end{bmatrix}$$

which is identical to $\mathbf{N}^{-1}$ computed directly from all seven condition equations. The same example may be reworked to demonstrate *the case of deleting information*. Having $\mathbf{N}^{-1}$ which corresponds to all seven equations involving three parameters,

we wish to obtain the inverse $\dot{\mathbf{N}}^{-1}_{2,2}$ for only the top three condition equations and two parameters. Consequently, from $\mathbf{N}^{-1}$ only the top left 2×2 block diagonal submatrix, or $\mathbf{F}$ is relevant. Referring to equation (13.23) first compute $\bar{\mathbf{N}}^{-1}$, using $\bar{\mathbf{n}}$ and $\mathbf{n}$ already evaluated

$$\mathbf{F}\bar{\mathbf{n}} = \begin{bmatrix} -0.21015 \\ 1.61193 \end{bmatrix}, \qquad \bar{\mathbf{n}}'\mathbf{F}\bar{\mathbf{n}} = 127.22924, \qquad \mathbf{n}\bar{\mathbf{n}}'\mathbf{F}\bar{\mathbf{n}} = 197.22434$$

$$\mathbf{F}\bar{\mathbf{n}}(\mathbf{n} + \bar{\mathbf{n}}'\mathbf{F}\bar{\mathbf{n}})^{-1}\bar{\mathbf{n}}'\mathbf{F} = \begin{bmatrix} 0.000224 & -0.001718 \\ -0.001718 & 0.013174 \end{bmatrix}$$

and

$$\bar{\mathbf{N}}^{-1} = \begin{bmatrix} 0.01013 & -0.00894 \\ -0.00894 & 0.01363 \end{bmatrix}$$

which is the same as computed before. Finally, using equation (13.24), compute

$$\dot{\mathbf{N}}^{-1} = \begin{bmatrix} 0.05658 & 0.08823 \\ 0.08823 & 0.18091 \end{bmatrix}$$

which is identical to $\dot{\mathbf{N}}^{-1}$ computed directly from the first three condition equations.

13.3. SEQUENTIAL PROCESSING AND THE UNIFIED APPROACH

The concept of sequential data reduction may be extended to the case of having a priori estimates of functionally independent parameters and their weights. The question of parameter constraints will be dealt with later. Cases of both fixed number of parameters and of variable number of parameters are addressed separately.

13.3.1. Fixed Number of Parameters
In the unified approach with functionally independent parameters, we have the observations l and their cofactor matrix $\mathbf{Q}$, and a priori estimates $\mathbf{x}$ for the parameters and an a priori cofactor matrix $\mathbf{Q}_{xx}$. With an approximation vector $\mathbf{x}^0$, a constant vector $\mathbf{f}_x$ was defined by equation (12.24c) in the preceding chapter as

$$\mathbf{f}_x = \mathbf{x}^0 - \mathbf{x} \tag{12.24c}$$

The effect of these additional data is that $\mathbf{W}_{xx}(= \mathbf{Q}_{xx}^{-1})$ is added to the coefficient matrix, and the vector $(\mathbf{W}_{xx}\mathbf{f}_x)$ is subtracted from the constant term vector of the reduced normal equations. Thus if these matrices are redefined as

$$\mathbf{N} = \mathbf{B}'\mathbf{W}_e\,\mathbf{B} + \mathbf{W}_{xx} \tag{13.26a}$$

$$\mathbf{t} = \mathbf{B}'\mathbf{W}_e\,\mathbf{f} - \mathbf{W}_{xx}\,\mathbf{f}_x \tag{13.26b}$$

then all the relations in Section 13.2.1 apply directly without any change. For example, equation (13.5a) is used to update the inverse of the coefficient

matrix, equation (13.5b) for updating the constant term vector, and equation (13.7) for directly updating the value of the parameter vector. In the earlier case when no a priori information was available, the sequential algorithm was possible only if a set of conditions, which is *more than or equal to the minimum necessary for a unique solution*, was used to evaluate the initial values N_0^{-1}, t_0^{-1} or Δ_0. If such was not the case, neither N_0^{-1} nor Δ_0 could be computed and the sequential process could not be started.

By contrast, in the presence of a priori information this restriction need not be imposed. In fact, we can begin the sequential procedure *with zero conditions*. In this case

$$N_0^{-1} = Q_{xx}, \qquad t_0 = -W_{xx} f_x$$

with $x^0 = 0$ the first value of the parameters, Δ_0 would simply be equal to x, their a priori estimates. The following rather simple example demonstrates this useful concept.

Example 13.3. Consider a much simplified problem of seeking the elevation of a point C (Figure 13.2) by measuring two differences in elevation from two known points A and B, whose elevations are 5.00 m and 4.00 m, respectively. The observations are $l_1 = 1.74$ m and $l_2 = 2.76$ m with a covariance matrix $\Sigma = 10^{-4} I_2$ m^2. Assume further that an a priori value for the elevation of point C (that is, the unknown parameter) is $x = 6.70$ m with a variance of 0.01 m^2. Compute the elevation x of point C.
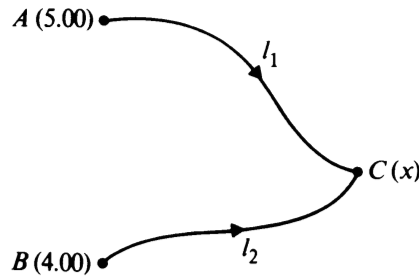
Figure 13.2

Solution: Several procedures are considered.

1. The simplest and most direct way is to begin by writing two condition equations, one for each observation, without regard to the given a priori information about point C. (Note that here $r = 1$, $u = 1$; thus $c = 2 = n$). Hence

$$5 + l_1 - x = 0 \qquad 4 + l_2 - x = 0$$

or

$$v + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x = \begin{bmatrix} -l_1 & -5 \\ -l_2 & -4 \end{bmatrix} = \begin{bmatrix} -6.74 \\ -6.76 \end{bmatrix}$$

Let $\sigma_0^2 = 10^{-4}$; thus $Q = I$ and $q_x = 100$ or $W_{xx} = 0.01$. Thus the normal equation is $2x = 13.50$ or $x = 6.75$ m.

2. Next, consider the given a priori information and write the normal equation

$$(\mathbf{N} + \mathbf{W}_{xx})\boldsymbol{\Delta} = (\mathbf{t} - \mathbf{W}_{xx}\,\mathbf{f}_x)$$

where, according to equation (13.24) and with $x^0 = 0$, $\mathbf{f}_x = -6.7$. Thus

$$(2 + 0.01)x = 13.5 - (0.01)(-6.7) = 13.567$$

and $x \cong 6.75$ m which is the same as in (1) above to two decimal places.

3. In another approach we may perform the solution *in two steps*: the first, using only the first condition equation, with the a priori information, and in the second step computing the contribution of the second condition equation. The first condition equation is

$$v_1 + [-1]x = -6.74$$

which with

$$W_0 = 1, \qquad W_{xx} = 0.01, \qquad \text{and} \qquad f_x = -6.7$$

yields

$$\mathbf{N}_0 = (\mathbf{B}_0^t\,\mathbf{W}_0\,\mathbf{B}_0 + \mathbf{W}_{xx}) = (-1)(1)(-1) + (0.01) = 1.01$$

$$\mathbf{t}_0 = (\mathbf{B}_0^t\,\mathbf{W}_0\,\mathbf{f}_0 - \mathbf{W}_{xx}\,\mathbf{f}_x) = (-1)(1)(-6.74) - (0.01)(-6.7) = 6.807$$

$$\boldsymbol{\Delta}_0 = \mathbf{N}_0^{-1}\mathbf{t}_0 = (1.01)^{-1}(6.807) = 6.7396$$

The second condition equation is

$$v_2 + [-1]x = -6.76$$

with $W_1 = 1$, for which the second term of equation (13.7) becomes

$$\delta\boldsymbol{\Delta} = \mathbf{N}_0^{-1}\mathbf{B}_1^t(\mathbf{A}_1\mathbf{Q}_1\mathbf{A}_1^t + \mathbf{B}_1\mathbf{N}_0^{-1}\mathbf{B}_1^t)^{-1}(\mathbf{f}_1 - \mathbf{B}_1\boldsymbol{\Delta}^0) = 0.01015$$

Hence

$$x = \boldsymbol{\Delta}_0 + \delta\boldsymbol{\Delta} = 6.74975 \cong 6.75 \text{ m}$$

which checks with the above answers to two decimal places.

4. The solution in (3) above was based on using a minimum set of conditions, which is one condition in this simple example, to obtain $\boldsymbol{\Delta}_0$. Here we shall begin with no conditions, then add one condition equation at a time. Therefore for $\boldsymbol{\Delta}_0$

$$\mathbf{B}_0 = \mathbf{0}\mathbf{f}_0 = \mathbf{0}$$

$$\mathbf{N}_0 = (\mathbf{0} + \mathbf{W}_{xx}) = 0.01$$

$$\mathbf{t}_0 = (\mathbf{0} - \mathbf{W}_{xx}\,\mathbf{f}_x) = 0.067$$

$$\boldsymbol{\Delta}_0 = \mathbf{N}_0^{-1}\mathbf{t}_0 = (0.01)^{-1}(0.067) = 6.7 \text{ m}$$

(which was said to be so in the text). Now, with $\mathbf{B}_1 = -1$ and $\mathbf{f}_1 = -6.74$, equation (13.7) yields $\delta\boldsymbol{\Delta}_1 = 0.0396$.

$$\boldsymbol{\Delta}_1 = \boldsymbol{\Delta}_0 + \delta\boldsymbol{\Delta}_1 = 6.7396 \text{ m}$$

[the same as $\boldsymbol{\Delta}_0$ in the two-step solution of (3)].

In order to compute the contribution of the second condition equation, evaluate first N_1^{-1} from N_0^{-1}, using equation (13.5a) with upper signs,

$$N_1^{-1} = N_0^{-1} - N_0^{-1}B_1^t(A_1QA_1^t + B_1N_0^{-1}B_1^t)^{-1}B_1N_0^{-1} = (1.01)^{-1}$$

The second condition equation has $B_2 = -1$ and $f_2 = -6.76$, for which equation (13.7) gives $\delta\Delta_2 = 0.01015$. Hence

$$x = \Delta_1 + \delta\Delta_2 = 6.74975 \cong 6.75 \text{ m}$$

For the sake of completeness, four different techniques were given, all leading to essentially the same solution. A casual consideration for a choice would almost certainly pick up the first technique as it is simple, short, and most direct. This is undeniably true for this nearly trivial example. Further consideration, however, is necessary when large size adjustment problems are involved. Furthermore, when a priori information is available and is realistic, we must use it, which would exclude the first technique.

If the observational data are in fact generated in a sequential manner, it is advisable, and even necessary in certain cases, to perform the adjustment sequentially. In this case several options are available. We may wait until enough observations are accumulated for a minimum unique solution, or we may simply begin with the a priori estimates that are given. The choice may not be simple since the overriding factor will be computational efficiency. Therefore for a given problem and its corresponding computational configuration, both hardware and software, we must analyze the different steps involved to determine which procedure would be more appropriate. As regards the redundant data, the choice of the number of equations to be added (or deleted) at a time would also influence the time of computation, as may be implied from Figure 13.1. Therefore, in a similar manner we ought to look into this aspect as well when designing the sequential scheme to be used for a particular adjustment application in order to arrive at an optimum configuration.

13.3.2. Variable Number of Parameters   In the case of a variable number of parameters the condition equations are of the form given in equations (13.10a) and (13.10b). The a priori information would be x and W_{xx} for Δ, and in addition x_δ and $W_{\delta\delta}$ for the new parameters δ. Consequently, besides redefining N and t as given in equations (13.26a) and (13.26b), we also need to redefine n and t appearing in equations (13.12a) and (13.12b). If x_δ^0 denotes an approximation vector, then

$$f_\delta = x_\delta^0 - x_\delta \tag{13.27a}$$

and

$$n_i = b_i^t W_{ei} b_i + W_{\delta\delta} \tag{13.27b}$$

$$\bar{t}_i = b_i^t W_{ei} f_i - W_{\delta\delta} f_\delta \tag{13.27c}$$

With all the new definitions, every relation developed in Section 13.2.2 can be used directly and without alteration for the situation when a priori parameter information is available.

Example 13.4. Suppose that we have the following six linear condition equations:

$$v_1 + x_1 \qquad\qquad = 1.1$$

$$v_2 + 2x_1 \qquad\qquad = 2.0$$

$$v_3 + x_1 + x_2 \qquad\quad = 2.1$$

$$v_4 + 2x_1 + x_2 \qquad\quad = 3.0$$

$$v_5 + 2x_1 + 2x_2 + x_3 = 5.0$$

$$v_6 + x_1 + x_2 + 2x_3 = 4.1$$

with

$$\mathbf{Q}_{6,6} = \mathbf{I}_6 \qquad \text{and} \qquad \mathbf{Q}_{xx}_{3,3} = 100\mathbf{I}_3$$

and the a priori estimates of the three parameters are all equal to 1.

Required: For the parameters (1) compute the estimates in one step using the a priori information, and (2) compute the estimates sequentially adding one equation at a time.

Solution

1. The matrix **B** and vector **f** are

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}^t \qquad \mathbf{f} = \begin{bmatrix} 1.1 & 2 & 2.1 & 3 & 5 & 4.1 \end{bmatrix}^t$$

$$\mathbf{N} = \mathbf{B}^t\mathbf{B} = \begin{bmatrix} 15 & 8 & 4 \\ 8 & 7 & 4 \\ 4 & 4 & 5 \end{bmatrix} \qquad \mathbf{t} = \mathbf{B}^t\mathbf{f} = \begin{bmatrix} 27.3 \\ 19.2 \\ 13.2 \end{bmatrix}$$

Because of the a priori estimates of the parameters and the fact that the conditions are linear, $\mathbf{x}^0 = \mathbf{0}$, from equation (13.24c)

$$\mathbf{f}_x = -\mathbf{x} = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^t$$

The corresponding normal equations would therefore be

$$(\mathbf{B}^t\mathbf{B} + \mathbf{W}_{xx})\mathbf{x} = (\mathbf{B}^t\mathbf{f} - \mathbf{W}_{xx}\,\mathbf{f}_x)$$

$$\begin{bmatrix} 15.01 & 8 & 4 \\ 8 & 7.01 & 4 \\ 4 & 4 & 5.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27.31 \\ 19.21 \\ 13.21 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1.015537 & 0.990998 & 1.034703 \end{bmatrix}.$$

2. We begin here by having $W_{xx} = 0.01$ for the normal equations coefficient matrix and $-W_{xx}f_x = 0.01$ for the constant vector, then we add the first equation for which

$$B_1 = 1, \qquad f_1 = 1.1, \qquad Q_1 = 1,$$

therefore

$$N_1^{-1} = (1.01)^{-1} \qquad \text{and} \qquad t_1 = 1.11$$

Next,

$$B_2 = 2, \qquad f_2 = 2, \qquad Q_2 = 1, \qquad N_2^{-1} = (5.01)^{-1}, \qquad \text{and} \qquad t_2 = 5.11$$

At this point the parameter number will increase from one to two. Thus with a priori $W_{\delta_2\delta_2} = 0.01$ the matrix N_2^{-1} should now be expanded to

$$\bar{N}_2^{-1} = \begin{bmatrix} (5.01)^{-1} & 0 \\ 0 & 100 \end{bmatrix} = \begin{bmatrix} 0.1996 & 0 \\ 0 & 100 \end{bmatrix}$$

$$B_3 = [1 \quad 1] \qquad f_3 = 2.1 \qquad Q_3 = 1 \qquad f_{\delta_3} = -1 \qquad W_{\delta_3\delta_3} = 0.01$$

$$N_3^{-1} = \bar{N}_2^{-1}[I_2 - B_3'(I + B_3\bar{N}_2^{-1}B_3')^{-1}B_3\bar{N}_2^{-1}] = \begin{bmatrix} 0.199206 & -0.197225 \\ -0.197225 & 1.19 \end{bmatrix}$$

The constant term vector also needs to be expanded to

$$\bar{t}_2 = \begin{bmatrix} 5.11 \\ -W_{\delta_3\delta_3}f_{\delta_3} \end{bmatrix} = \begin{bmatrix} 5.11 \\ 0.01 \end{bmatrix} \qquad t_3 = \bar{t}_2 + B_3'f_3 = \begin{bmatrix} 7.21 \\ 2.11 \end{bmatrix}$$

$$B_4 = [2 \quad 1] \qquad f_4 = 3.0 \qquad Q_3 = 1$$

$$N_4^{-1} = \begin{bmatrix} 0.18079 & -0.270045 \\ -0.270045 & 0.902047 \end{bmatrix} \qquad \text{and} \qquad t_4 = \begin{bmatrix} 13.21 \\ 5.11 \end{bmatrix}$$

$$B_5 = [2 \quad 2 \quad 1] \qquad f_5 = 5.0 \qquad Q_5 = 1 \qquad f_{\delta_5} = -1$$

$$\bar{N}_4^{-1} = \begin{bmatrix} 0.18079 & -0.270045 & 0 \\ & 0.902047 & 0 \\ \text{symmetric} & & 100 \end{bmatrix} \qquad \text{and} \qquad \bar{t}_4 = \begin{bmatrix} 13.21 \\ 5.11 \\ 0.01 \end{bmatrix}$$

$$N_5^{-1} = \begin{bmatrix} 0.180446 & -0.267606 & 0.172593 \\ & 0.884764 & -1.222095 \\ \text{symmetric} & & 3.068322 \end{bmatrix} \qquad t_5 = \begin{bmatrix} 23.21 \\ 15.11 \\ 5.01 \end{bmatrix}$$

$$B_6 = [1 \quad 1 \quad 2] \qquad f_6 = 4.1 \qquad Q_6 = 1$$

$$N_6^{-1} = \begin{bmatrix} 0.173515 & -0.218526 & 0.035937 \\ & 0.537242 & -0.254463 \\ \text{symmetric} & & 0.374073 \end{bmatrix} \qquad t_6 = \begin{bmatrix} 27.31 \\ 19.21 \\ 13.21 \end{bmatrix}$$

and

$$[x_1 \quad x_2 \quad x_3] = [1.015538 \quad 0.991017 \quad 1.034710]$$

which agree well with the values obtained previously, the differences being due to round-off errors.

13.3.3. An Alternative Direct Scheme The solution in the second part of Example 7.4 suggests a general scheme that may be applied equally well to cases of both a fixed and variable number of parameters, and for addition as well as deletion of condition equations. The only requisite for this scheme is that *all the parameters involved must have a priori estimates and corresponding a priori weight matrices*. Because of the allowance for varying the number of parameters, the case of adding conditions differs somewhat from that of deleting them.

THE DIRECT SCHEME FOR ADDING CONDITIONS The number of parameters is assumed to be variable since the case of fixed parameters is a special case (see also Section 13.3.1 and Example 13.3). Thus when a group of parameters are to be included in the adjustment an a priori estimate of their values and their weight matrix, $\mathbf{W}_{xx}$, must be given. If two subvectors of parameters Δ_i and Δ_j enter the adjustment sequentially ($j = i + 1$), the assumption will be made that only their respective a priori weight matrices $\mathbf{W}_{x_i x_i}$ and $\mathbf{W}_{x_j x_j}$ are given. This implies that they are not correlated, which is a totally practical assumption. At the end of the $(i - 1)$ cycle we have $\mathbf{N}_{i-1}^{-1}$ with dimensions $(u_{i-1} \times u_{i-1})$ and $\mathbf{t}_{i-1}$ for a computation of Δ_{i-1} of size u_{i-1}. For the ith cycle, we add the conditions

$$\underset{s_i,\, t_i\ \ t_i,\, 1}{\mathbf{A}_i\ \ \mathbf{v}_i} + \underset{s_i,\, u_i\ \ u_i,\, 1}{\mathbf{B}_i\ \ \Delta_i} = \underset{s_i,\, 1}{\mathbf{f}_i} \tag{13.28}$$

where Δ_i is increased over Δ_{i-1} by $\mathbf{w}_i = u_i - u_{i-1}$. Note that s_i must be larger than w_i in order that equation (13.28) may contribute to the adjustment. The added subvector of parameters δ_i would have an a priori estimate. Furthermore, we must also have the a priori weight matrix $\mathbf{W}_{\delta_i \delta_i}$ which is $(w_i \times w_i)$. Thus before computing $\mathbf{N}_i^{-1}$ from $\mathbf{N}_{i-1}^{-1}$, the latter must be expanded from a size of $u_{i-1} \times u_{i-1}$ to the larger size of $u_i \times u_i$, as follows:

$$\underset{u_i,\, u_i}{\bar{\mathbf{N}}_{i-1}^{-1}} = \begin{bmatrix} \mathbf{N}_{i-1}^{-1} & 0 \\ 0 & \mathbf{W}_{\delta_i \delta_i}^{-1} \end{bmatrix} \tag{13.29}$$

Then

$$\mathbf{N}_i^{-1} = \bar{\mathbf{N}}_{i-1}^{-1}[\mathbf{I} - \mathbf{B}_i^t(\mathbf{A}_i\, \mathbf{Q}_i\, \mathbf{A}_i^t + \mathbf{B}_i\, \bar{\mathbf{N}}_{i-1}^{-1}\mathbf{B}_i^t)^{-1}\mathbf{B}_i\, \bar{\mathbf{N}}_{i-1}^{-1}] \tag{13.30}$$

realizing, of course, that $\mathbf{Q}_i$ is the cofactor matrix of the new set of observations which is assumed by necessity to be uncorrelated with the preceding set already used. If the a priori estimate corresponding to δ_i is $\mathbf{x}_{\delta_i}$ and $\mathbf{x}_{\delta_i}^0$ is an approximate vector, then $\mathbf{f}_{\delta_i}$ is computed from equation (13.27a). The constant term vector $\mathbf{t}_{i-1}$ must also be expanded from a $u_{i-1} \times 1$ to a $u_i \times 1$ vector, or

$$\bar{\mathbf{t}}_{i-1} = \begin{bmatrix} \mathbf{t}_{i-1} \\ -\mathbf{W}_{\delta_i \delta_i}\, \mathbf{f}_{\delta_i} \end{bmatrix} \tag{13.31}$$

and then

$$\mathbf{t}_i = \bar{\mathbf{t}}_{i-1} + \mathbf{B}_i^t (\mathbf{A}_i \, \mathbf{Q}_i \, \mathbf{A}_i^t)^{-1} \mathbf{f}_i$$

which completes the computation of $\mathbf{N}_i$ and $\mathbf{t}_i$ necessary for evaluating $\boldsymbol{\Delta}_i$, the answer at the ith step.

THE DIRECT SCHEME FOR DELETING CONDITIONS   At the ith step of editing, for example, we may delete the effect of the condition equations (13.28) from the solution available as given by $\mathbf{N}_{i-1}^{-1}$ and $\mathbf{t}_{i-1}$, both of size u_{i-1}. For the sake of generality, we assume that u_{i-1} is larger than u_i, that is, the number of parameters decreases, by say w_i, when the effect of equation (13.28) is removed from the solution. Obviously, then, $u_i = u_{i-1} - w_i$. The matrix $\mathbf{N}_{i-1}^{-1}$ is first partitioned such that

$$\mathbf{N}_{i-1}^{-1} \atop {}_{u_{i-1}, u_{i-1}} = \begin{bmatrix} \bar{\mathbf{N}}_{i-1} & \bar{\mathbf{n}}_{i-1} \\ {}_{u_i, u_i} & {}_{u_i, w_i} \\ \bar{\mathbf{n}}_{i-1}^t & \mathbf{n}_{i-1} \\ {}_{w_i, u_i} & {}_{w_i, w_i} \end{bmatrix} \tag{13.32}$$

then the new inverse is obtained from submatrix $\bar{\mathbf{N}}_{i-1}$

$$\mathbf{N}_i^{-1} = \bar{\mathbf{N}}_{i-1}[\mathbf{I} + \mathbf{B}_i^t(\mathbf{A}_i \, \mathbf{Q}_i \, \mathbf{A}_i^t - \mathbf{B}_i \, \bar{\mathbf{N}}_{i-1}\mathbf{B}_i^t)^{-1}\mathbf{B}_i \, \bar{\mathbf{N}}_{i-1}] \tag{13.33}$$

in which $\mathbf{Q}_i$ is the cofactor matrix of the observations associated with $\mathbf{v}_i$. The remaining submatrices $\bar{\mathbf{n}}_{i-1}$ and $\mathbf{n}_{i-1}$ are not used, as they become irrelevant with regard to the new set of parameters $\boldsymbol{\Delta}_i$. It should also be pointed out that if no change in parameters is warranted in the ith step (because the conditions to be deleted do not reduce the number of parameters), $\bar{\mathbf{N}}_{i-1}$ will be the total matrix $\mathbf{N}_{i-1}^{-1}$ which must then be used directly in equation (13.33) to compute $\mathbf{N}_i^{-1}$. In a similar fashion $\mathbf{t}_{i-1}$ is partitioned

$$\mathbf{t}_{i-1} = \begin{bmatrix} \bar{\mathbf{t}}_{i-1} \\ \mathbf{t}_{i-1}' \end{bmatrix} \tag{13.34}$$

and the new constant term vector computed from

$$\mathbf{t}_i = \bar{\mathbf{t}}_{i-1} - \mathbf{B}_i^t(\mathbf{A}_i \, \mathbf{Q}_i \, \mathbf{A}_i^t)^{-1}\mathbf{f}_i \tag{13.35}$$

The same remark about the case of no change in the number of parameters also applies here, thus $\bar{\mathbf{t}}_{i-1}$ would be equal to $\mathbf{t}_{i-1}$.

Example 13.5.   Given the following four linear condition equations, $\mathbf{Q} = \mathbf{I}_4$, $\mathbf{Q}_{xx} = 100\mathbf{I}_2$, and the a priori estimates are both equal to 1.

$$v_1 + x_1 \qquad\quad = 1.1$$
$$v_2 + 2x_1 \qquad\ = 2.0$$
$$v_3 + x_1 + x_2 = 2.1$$
$$v_4 + 2x_1 + x_2 = 3.0$$

Compute first the estimates of x_1 and x_2 from all four conditions; then removing sequentially the last two equations verify that the value of x_1 thus obtained is the same as that obtained from solving the first two conditions directly by least squares.

Solution

$$\mathbf{B_0} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}^t \qquad \mathbf{f_0} = [1.1 \quad 2.0 \quad 2.1 \quad 3.0]^t$$

$$\mathbf{N_0} = \mathbf{B_0^t}\,\mathbf{B_0} + \mathbf{Q_{xx}^{-1}} = \begin{bmatrix} 10.01 & 3. \\ 3. & 2.01 \end{bmatrix}$$

$$\mathbf{N_0^{-1}} = \frac{1}{11.1201} \begin{bmatrix} 2.01 & -3 \\ -3 & 10.01 \end{bmatrix} \qquad \mathbf{f_x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{t_0} = \mathbf{B_0^t}\,\mathbf{f_0} - \mathbf{W_{xx}}\,\mathbf{f_x} = \begin{bmatrix} 13.21 \\ 5.11 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.009 \\ 1.036 \end{bmatrix}$$

Now to remove the effect of the fourth equation

$$\mathbf{A_1} = 1 \qquad \mathbf{B_1} = [2 \quad 1] \qquad \mathbf{f_1} = 3.0 \qquad \mathbf{Q_1} = 1$$

Because the number of parameters remains unaltered in this step, then $\bar{\mathbf{N}}_0 = \mathbf{N_0^{-1}}$ and we apply equation (13.33) to compute $\mathbf{N_1^{-1}}$

$$\mathbf{N_1^{-1}} = \begin{bmatrix} 0.19920711 & -0.19723476 \\ \text{symmetric} & 1.18538095 \end{bmatrix} \qquad \mathbf{t_1} = \begin{bmatrix} 7.21 \\ 2.11 \end{bmatrix}$$

To remove the effect of the third equation

$$\mathbf{A_2} = 1 \qquad \mathbf{B_2} = [1 \quad 1] \qquad \mathbf{f_2} = 2.1 \qquad \mathbf{Q_2} = 1$$

Since the number of parameters still remains unchanged we need not modify the solution from the above step but simply take $\bar{\mathbf{N}}_1 = \mathbf{N_1^{-1}}$ and $\bar{\mathbf{t}}_1 = \mathbf{t_1}$. Equation (13.33) thus yields

$$\mathbf{N_2^{-1}} = \begin{bmatrix} 0.19960079 & 0.00000027 \\ \text{symmetric} & 100.00001894 \end{bmatrix}, \qquad \mathbf{t_2} = \begin{bmatrix} 5.11 \\ 0.01 \end{bmatrix} \qquad \text{and}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_2 = \begin{bmatrix} 1.019960 \\ 1.000002 \end{bmatrix}$$

Returning to the original set of conditions, the first *two equations* yield

$$\mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} 1.1 \\ 2.0 \end{bmatrix}, \qquad \mathbf{W} = \mathbf{I_2}, \qquad \mathbf{W_{x_1 x_1}} = 0.01, \qquad \text{and} \qquad \mathbf{f_{x_1}} = -1$$

$$\mathbf{N} = \mathbf{B^t B} + \mathbf{W_{x_1 x_1}} = 5.01 \qquad \mathbf{N^{-1}} = 0.19960079$$

$$\mathbf{t} = \mathbf{B^t f} - \mathbf{W_{x_1 x_1}}\mathbf{f_{x_1}} = 5.11 \qquad x_1 = 1.019961$$

Note that $\mathbf{N^{-1}}$ is exactly the same value as the (1, 1) element of $\mathbf{N_2^{-1}}$ and x_1 differs only in the sixth decimal place. This shows that aside from rounding off error the direct scheme is both rigorous and general. Its generality is demonstrated by the fact

that x_2 computed after deleting the third and fourth equations is nothing but its a priori estimate of 1.00, except for round-off error.

To demonstrate the reduction in the number of parameters, we extend the problem one more step by subtracting the effect of the second equation for which $B_3 = 2$, $f_3 = 2.0$, and $Q_3 = 1$. Here only a submatrix of appropriate dimensions (in this case a scalar) from N_2^{-1} is needed. Thus $\bar{N}_2 = 0.19960079$ and equation (13.33) leads to $N_3^{-1} = 0.99009885$. Similarly, a subvector of t_2, $\bar{t}_2 = 5.11$, is relevant, hence $t_3 = 1.11$. From the first equation directly

$$N = B'B + W_{xx} = 1.01 \qquad N^{-1} = (1.01)^{-1} = 0.99009900$$

$$t = B'B - W_{xx} f_x = 1.11$$

which agree well with those computed by the sequential scheme.

13.4. SEQUENTIAL REDUCTION, THE UNIFIED APPROACH, AND PARAMETER CONSTRAINTS

The presence of parameter constraints poses no particular difficulties as long as the unified approach is applied. If the classical methods were used, there would be several special cases depending on whether the constraints were available at the beginning or arose with the new conditions. Since in the unified approach no distinction is made between conditions and constraints, the present case can be considered as essentially the same as that given in Section 13.3. Therefore no special derivation is necessary here and a numerical example is given instead to illustrate the points of treatment that are applicable to the present case.

Example 13.6. Given are two conditions and one "constraint,"

$$\mathbf{v} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 0.1 \end{bmatrix}$$

of the form

$$\mathbf{v} + \mathbf{B}\Delta = \mathbf{f}$$

and

$$\mathbf{v}_c + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1.0$$

of the form

$$\mathbf{v}_c + \mathbf{C}\Delta = \mathbf{g}$$

The a priori weight matrix associated with the first two equations is to be taken as $\mathbf{W} = \mathbf{I}_2$, and the weight associated with the third equation is $w_c = 100$; w_c must be

large relative to $\mathbf{W}$ in order to effect a practical enforcement of the third equation as a parameter constraint. The a priori estimate of the parameters is

$$\mathbf{x} = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$$

with the a priori weight matrix $\mathbf{W}_{xx} = 0.01\mathbf{I}_2$.

With the above data a solution is assumed to be found at the start. Then that solution is to be modified due to the addition of another equation,

$$v_3 + \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2.2$$

with $w_3 = 1.0$.

Solution: For the first part of the data we compute the normal equations as

$$\mathbf{N}_1 = \mathbf{B}^t\mathbf{W}\mathbf{B} + \mathbf{C}^t\mathbf{W}_c\,\mathbf{C} + \mathbf{W}_{xx} = \begin{bmatrix} 102.01 & 0 \\ 0 & 2.01 \end{bmatrix}$$

$$\mathbf{N}_1^{-1} = \begin{bmatrix} (102.01)^{-1} & 0 \\ 0 & (2.01)^{-1} \end{bmatrix}, \qquad \mathbf{f}_x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{t}_1 = \mathbf{B}^t\mathbf{W}\mathbf{f} + \mathbf{C}^t\mathbf{W}_c\,\mathbf{B} - \mathbf{W}_{xx}\,\mathbf{f}_x = \begin{bmatrix} 102.21 \\ 2.01 \end{bmatrix}$$

The answer for the first part is therefore

$$\mathbf{\Delta}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{N}_1^{-1}\mathbf{t}_1 = \begin{bmatrix} 1.002 \\ 1.000 \end{bmatrix}$$

For the second part

$$\mathbf{N}_2^{-1} = \frac{1}{309.06}\begin{bmatrix} 3.01 & -1 \\ -1 & 103.01 \end{bmatrix}, \qquad \mathbf{t}_2 = \begin{bmatrix} 104.41 \\ 4.21 \end{bmatrix}$$

thus

$$\mathbf{\Delta}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.003 \\ 1.065 \end{bmatrix}$$

In Example 13.6 the number of parameters remained fixed through the sequential treatment. As an illustration of how to apply the unified approach sequentially with constraints in the case of variable number of parameters, we shall take another numerical example.

Example 13.7. The original data include two conditions and one constraint:

$$\mathbf{v} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 = \begin{bmatrix} 1.1 \\ 2.0 \end{bmatrix}$$

of the form

$$\mathbf{v} + \mathbf{B}\mathbf{\Delta} = \mathbf{f}$$

and

$$\mathbf{v}_c + x_1 = 1$$

of the form

$$\mathbf{v}_c + \mathbf{C}\Delta = \mathbf{g}$$

for which

$$\mathbf{W} = \mathbf{I}_2, \qquad w_c = 100, \qquad x_1 = 1.0, \qquad \text{and} \qquad w_{xx} = 0.01$$

After obtaining the solution for this data the following conditions are to be added:

$$\mathbf{v}_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 2.1 \\ 3.0 \end{bmatrix}$$

of the form

$$\mathbf{v}_1 + \mathbf{B}_1 \Delta + \mathbf{b}\delta = \mathbf{f}_1$$

with

$$\mathbf{W} = \mathbf{I}_2, \qquad x_2 = 1.0, \qquad \text{and} \qquad w = 0.01$$

Solution: For the first part,

$$\mathbf{f}_{x_1} = -x_1 = -1.0, \qquad \text{and} \qquad \mathbf{f}_\delta = 1.0$$

thus

$$\mathbf{N} = \mathbf{B}^t\mathbf{W}\mathbf{B} + \mathbf{C}^t\mathbf{W}_c\,\mathbf{C} + \mathbf{W}_{xx} = 105.01$$

$$\mathbf{t} = \mathbf{B}^t\mathbf{W}\mathbf{f} + \mathbf{C}^t\mathbf{W}_c\,\mathbf{g} - \mathbf{W}_{xx}\,\mathbf{f}_{x_1} = 105.11$$

$$\Delta = 1.0010$$

With $\mathbf{A}_1 = \mathbf{I}$, $(\mathbf{AQA}^t) = 1$, thus

$$\mathbf{n}_1^{-1} = (\mathbf{b}^t(\mathbf{AQA}^t)_1^{-1}\mathbf{b} + \mathbf{W}_{\delta\delta})^{-1} = (2.01)^{-1}$$

$$\bar{\mathbf{n}}_1 = \mathbf{B}_1^t(\mathbf{AQA}^t)_1^{-1}\mathbf{b} = 3, \qquad \mathbf{B}_1^t(\mathbf{AQA}^t)_1^{-1}\mathbf{f}_1 = 8.1$$

$$\bar{\mathbf{t}} = \mathbf{b}^t(\mathbf{AQA}^t)_1^{-1}\mathbf{f}_1 - \mathbf{W}_{\delta\delta}\,\mathbf{f}_\delta = 5.11$$

Now equation (13.17) yields $\bar{\mathbf{N}}^{-1} = (110.01)^{-1}$.
To compute $\mathbf{F}$ use equation (13.15),

$$(\mathbf{n}_1 - \bar{\mathbf{n}}_1^t\,\bar{\mathbf{N}}_1^{-1}\bar{\mathbf{n}}_1)^{-1} = \frac{110.01}{212.1201}$$

$$\bar{\mathbf{n}}_1(\mathbf{n}_1 - \bar{\mathbf{n}}_1^t\bar{\mathbf{N}}_1^{-1}\bar{\mathbf{n}}_1)^{-1}\bar{\mathbf{n}}_1^t\mathbf{n}_1^{-1} = \frac{9}{212.1201}$$

$$\mathbf{F}_1 = (110.01)^{-1}\frac{221.1201}{212.1201}$$

From equation (13.14b)

$$G_1 = -F_1 \bar{n}_1 n_1^{-1} = -\frac{3}{313.1201}$$

$$H_1 = n_1^{-1} - n_1^{-1} \bar{n}_1' G_1 = (2.01)^{-1} \frac{221.1201}{212.1201}$$

$$N_1^{-1} = \begin{bmatrix} F & G \\ G' & H \end{bmatrix} = \frac{1}{212.1201} \begin{bmatrix} 2.01 & -3 \\ -3 & 110.01 \end{bmatrix}$$

Finally, according to equation (13.12b),

$$t = \begin{bmatrix} t + B_1'(AQA')_1^{-1} f_1 \\ t \end{bmatrix} = \begin{bmatrix} 113.21 \\ 5.11 \end{bmatrix}, \qquad \Delta = \begin{bmatrix} 1.0005 \\ 1.0490 \end{bmatrix}$$

In both Examples 13.6 and Example 13.7 the constraints were given at the start. The cases in which constraints arise later pose no new points for consideration since both types of conditions are the same. Consequently, no particular treatment of such cases is necessary. Finally, cases of information deletion are sufficiently straightforward and need no further elaboration.

13.5. PRECISION ESTIMATION

The question of a posteriori cofactor matrices is not much different from the regular practice since at the end of a sequential treatment we end up with the same information as would be obtained from the direct batch least squares. For fixed number of parameters, $Q_{\Delta\Delta}$ will simply be equal to the new inverse of the normal equations matrix after either addition or deletion of information. Thus in this case equation (13.5a) may be used for evaluating $Q_{\Delta\Delta}$. In case we are working with a variable number of parameters, equation (13.13) may be used for the addition of information and equation (13.21b) for deleting information. All we need to remember is to use the *updated* inverse of the normal equations coefficient matrix as the required cofactor matrix $Q_{\Delta\Delta}$.

With regard to Section 13.3 and the use of the unified approach, the same equations referred to above also apply. The only difference is that a priori W_{xx} is included here in N^{-1}.

Precision estimation is not much different for the situation when parameter constraints are present as given in Section 13.4. Since the unified approach is applied, no modification to the above statements is needed.

13.6. SEQUENTIAL PROCESSING AND NONLINEAR EQUATIONS

All derivations in this chapter concerned linear functions. For nonlinear problems, in addition to linearization by series expansion, further discussion is needed when considering sequential or recursive procedures. As an illus-

tration we shall consider the case of addition of information when the number of parameters is fixed. Let a set of nonlinear conditions be

$$F(l, \mathbf{x}) = 0 \tag{13.36}$$

and its linearized form

$$\underset{c,\,n}{\mathbf{A}} \underset{n,\,1}{\mathbf{v}} + \underset{c,\,u}{\mathbf{B}} \underset{u,\,1}{\mathbf{\Delta}} = \underset{c,\,1}{\mathbf{f}^0} \tag{13.37}$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{f}^0$ are evaluated at approximate vector $\mathbf{x}_0^0$ for the parameters (disregarding the aspect of iterating on the observations, for the sake of simplicity). From equation (13.37) a system of u normal equations may be formed and solved for the first vector of parameter corrections $\mathbf{\Delta}_1$. Updated approximations to be used in the second iteration would be

$$\mathbf{x}_1 = \mathbf{x}_0^0 + \mathbf{\Delta}_1$$

The process may be repeated many (m) times leading to the eventual answer

$$\hat{\mathbf{x}} = \mathbf{x}_0^0 + \sum_{k=1}^{m} \mathbf{\Delta}_k \tag{13.38}$$

Let an additional set of conditions be

$$\mathbf{F}_a(l, \mathbf{x}) = 0 \tag{13.39}$$

The combination of equations (13.36) and (13.39) gives a larger set of condition equations from which the final estimate $\mathbf{x}_f$ of the u parameters will obviously be different from that given by equation (13.38). The most direct process (which may not necessarily be the most efficient) is to linearize equation (13.39) at an approximation vector $\mathbf{x}_f^0$ equal to $\hat{\mathbf{x}}$ obtained from equation (13.38) to give

$$\underset{p,\,n}{\mathbf{A}_a} \underset{n,\,1}{\mathbf{v}} + \underset{p,\,u}{\mathbf{B}_a} \underset{u,\,1}{\mathbf{\Delta}_t} = \underset{p,\,1}{\mathbf{f}^0} \tag{13.40}$$

Next, equation (13.37) is re-evaluated at the same value $\mathbf{x}_f^0$ and combined with equation (13.40) to form a total set of $(c + p)$ condition equations. The corresponding normal equations are formed and solved to obtain a correction, $\mathbf{\Delta}_t$, and the solution is repeated, if necessary, until convergence is achieved. It should be noted that in this process the normal equation matrix of size u will have to be inverted every time a correction vector is computed.

Another possibility occurs when the equations in (13.37) are stored and kept until all the additional equations in (13.39) become available. In this case, equation (13.39) is linearized at the approximation vector $\mathbf{x}_0^0$ used for equation (13.36) and the set of linear equations is combined with equation (13.37). Normal equations are formed and solved and relinearizations carried out until the solution converges to a preset tolerance.

A number of possibilities arise when sequential procedures are used for nonlinear problems. Some of these possibilities entail exact solutions whereas others involve certain approximations. To begin with, the sequential algorithm is exact when used inside each linearization. For example, suppose that Δ_{t_1} is obtained from the first cycle of the direct solution for equations (13.36) and (13.39) together (both linearized at the same approximation vector x_0^0). If, then, the inverse of the normal equation matrix and constant term vector arising from the set of equations (13.36) (linearized at x_0^0) are modified by the sequential algorithm to include equation (13.39) (also linearized at x_0^0), the product of the resulting inverse and constant term vector will give Δ_{t_1} exactly. For the nonlinear case the set of equations in (13.37) is solved by the linearization process as shown in equation (13.38). The new set of equations in (13.40) is then added by the sequential algorithm after being linearized with the best estimates of the parameters from the original set of conditions, that is, $\hat{x}$ from equation (13.38). In this case the resulting estimated parameter Δ_t will not necessarily be precisely the same Δ_{t_1} estimated directly from linear least squares solution for all condition equations considered together. Obviously, the amount of difference between Δ_t from the sequential solution and Δ_{t_1} from the batch solution will depend, among other things, on the quality of the first vector of approximations, x_0^0.

To obtain an appreciation of this method of sequential reduction in the case of nonlinear models, the solution of different cases of a relative orientation problem of a pair of aerial photographs is given in Table 13-1. First, six points were used in a regular batch least squares and the solution iterated until convergence. The answers—that is, the values of the five relative orientation parameters—from this case are given in the first line. Adding three more points, the second line gives the results from a batch solution in which all nine points are used directly, and the fifth line gives the corresponding

TABLE 13-1 Comparison of Batch and Sequential Solution of the Nonlinear Model of Relative Orientation

TYPE OF SOLUTION		ESTIMATED PARAMETERS				
BATCH POINTS	SEQUENTIAL POINTS	Y(mm)	Z(mm)	ω	ϕ	κ
6	0	0.179	151.766	1′ 42.851″	36′ 0.070″	6′ 17.920″
9	0	0.168	151.770	1′ 56.564″	35′ 59.249″	6′ 19.278″
15	0	0.158	151.768	2′ 07.057″	36′ 33.944″	6′ 12.202″
40	0	0.152	151.773	2′ 12.462″	36′ 38.931″	6′ 12.551″
6	3	0.168	151.770	1′ 56.573″	35′ 59.238″	6′ 19.277″
6	9	0.158	151.768	2′ 07.097″	36′ 33.939″	6′ 12.207″
6	34	0.152	151.773	2′ 12.462″	36′ 38.931″	6′ 12.551″

results when the three points are added sequentially. In the third and sixth lines the results of a total of 15 points are given, and those in the fourth and seventh lines are for 40 points. The reader will note that when extensive redundancy exists, such as the case of 40 points, the two solutions are identical. For other cases there is some difference between the two methods of solution reflecting the already mentioned fact that the sequential algorithm is not exact for nonlinear cases. However, such differences, at least for the example given, are so small they can be neglected. Of course, it must be emphasized that this is only one example that shows tendencies, and other examples from different applications would demonstrate the concept.

13.7. SUMMARY OF EQUATIONS AND CONCLUDING REMARKS

13.7.1. Summary of Equations

For a case of fixed number of parameters, given $N_{(i-1)}^{-1}$, $t_{(i-1)}$, and a new set of conditions $A_i v_i + B_i \Delta = f_i$ with Q_i, then

$$Q_{ei} = (AQA^t)_i \qquad W_{ei} = Q_{ei}^{-1}$$

$$N_i^{-1} = N_{(i-1)}^{-1}[I \mp B_i^t(Q_{ei} \pm B_i N_{(i-1)}^{-1} B_i^t)^{-1} B_i N_{(i-1)}^{-1}] \qquad (13.5a)$$

$$t_i = t_{(i-1)} \pm B_i^t W_{ei} f_i \qquad (13.5b)$$

with upper signs for addition of conditions whereas the lower signs are for subtraction of conditions. Alternatively, given $N_{(i-1)}^{-1}$ and Δ_{i-1}, the effect of the new conditions would be

$$\Delta_i = \Delta_{i-1} \pm N_{(i-1)}^{-1} B_i^t(Q_{ei} \pm B_i N_{(i-1)}^{-1} B_i^t)^{-1}(f_i - B_i \Delta_{i-1}) \qquad (13.7)$$

again with upper signs for condition addition and vice versa.

For a case of variable number of parameters, given N_{i-1}^{-1}, t_{i-1}, and the new conditions $A_i v_i + B_i \Delta + b_i \delta = f_i$ with Q_i, and where δ is an added set of parameters, then for condition addition,

$$\bar{N}_i^{-1} = N_{i-1}^{-1}[I - B_i^t(Q_{ei} + B_i N_{i-1}^{-1} B_i^t)^{-1} B_i N_{i-1}^{-1}] \qquad (13.17)$$

$$\bar{n}_i = (B^t W_e b)_i \qquad (13.18a)$$

$$n_i = (b^t W_e b)_i \qquad (13.18b)$$

$$F_i = \bar{N}_i^{-1}[I + \bar{n}_i(n_i - \bar{n}_i^t \bar{N}_i^{-1} \bar{n}_i)^{-1} \bar{n}_i^t \bar{N}_i^{-1}] \qquad (13.15a)$$

$$G_i = -F_i \bar{n}_i n_i^{-1} \qquad (13.14b)$$

$$H_i = n_i^{-1} - n_i^{-1} \bar{n}_i^t G_i = (n_i - \bar{n}_i^t \bar{N}_i^{-1} \bar{n}_i)^{-1} \qquad (13.14c),$$

$$(13.14d)$$

$$N_i^{-1} = \begin{bmatrix} F_i & G_i \\ G_i^t & H_i \end{bmatrix} \qquad (13.13)$$

$$t'_i = t_{i-1} + B_i W_{ei} f_i \tag{13.19a}$$

$$\bar{t}_i = b'_i W_{ei} f_i \tag{13.19b}$$

$$t_i = \begin{bmatrix} t'_i \\ \bar{t}_i \end{bmatrix} \tag{13.12b}$$

$$\begin{bmatrix} \Delta_i \\ \delta \end{bmatrix} = N_i^{-1} t_i$$

For deletion of the conditions $A_j v_j + B_j \Delta + b_j \delta = f_j$ from the solution given by N_{j-1}^{-1} and t_{j-1}, then partition N_{j-1}^{-1} and t_{j-1},

$$N_{j-1}^{-1} = \begin{bmatrix} F_{j-1} & G_{j-1} \\ G'_{j-1} & H_{j-1} \end{bmatrix}, \qquad t_{j-1} = \begin{bmatrix} t'_{j-1} \\ \bar{t}_{j-1} \end{bmatrix}$$

where H and t are of order equal to that of δ.

Compute $\bar{n}_j$ and n_j from equations (13.18a) and (13.18b) above,

$$\bar{N}_{j-1}^{-1} = F_{j-1}[I - \bar{n}_j(n_j + \bar{n}'_j F_{j-1} \bar{n}_j)^{-1} \bar{n}'_j F_{j-1}] \tag{13.23}$$

$$N_j^{-1} = \bar{N}_{j-1}^{-1}[I + B'_j(Q_{ej} - B_j \bar{N}_{j-1}^{-1} B'_j)^{-1} B_j \bar{N}_{j-1}^{-1}] \tag{13.24}$$

$$t_j = t'_{j-1} - B'_j W_{ej} f_j \tag{13.25}$$

When unified approach is used, apply all formulas above but redefining

$$N = B'W_e B + W_{xx} \qquad\qquad t = B'W_e f - W_{xx} f_x \tag{13.26a, 13.26b}$$

$$n = b'W_e b + W_{\delta\delta} \qquad\qquad \bar{t} = b'W_e f - W_{\delta\delta} f_\delta \tag{13.27b, 13.27c}$$

where $f_x = x^0 - x$ and $f_\delta = x_\delta^0 - x_\delta$; x^0 and x_δ^0 are vectors of parameter approximation; and x, x_δ and W_{xx}, $W_{\delta\delta}$ are a priori estimates of the parameters and their weight matrices.

13.7.2. Concluding Remarks We introduced in this chapter several modern concepts that may be regarded as an extension of regularly practiced least squares adjustment procedures. Although these concepts may appear to be somewhat theoretical or academic, they may in fact take a place of prominence in adjustment methodology because of current rapid advances in on-line modes of data acquisition and the ever-increasing role played by electronic computers.

Because the sequential methods are relatively recent, we have by no means exhausted the subject. Actually, this chapter is mostly an introductory exposition. One point that must be emphasized is that the entire concept of derivation as given was built on the lack of correlation between the new set of observations and those preceding it. This assumption allowed for the block diagonality of the total cofactor matrices and made possible the derivation of the algorithms as given. Obviously, further development is

needed to deal with the effect on these algorithms of having correlated observations that need to be sequentially processed.

Another area that requires further elaboration concerns the nonlinearity of the model, and how it can best be handled by sequential procedures. Numerical factors and computational efficiency, further considerations of cases of variable number of parameters, and the unified approach combined with model nonlinearity all need more development. Therefore the material given here is intended to present the reader with enough basic concepts to prepare him for further study of more advanced work.

14

An Introduction to Least Squares Interpolation, Filtering, and Collocation

14.1. INTRODUCTION

The method of least squares has been used extensively in the major portions of this book for solving *adjustment* problems involving random variables and specified mathematical models. In such problems the selected mathematical model is treated as superior to the given sample, and the least squares criterion is used to estimate another set of random variables that is consistent with the model. Therefore *adjustment* procedures may be considered as one type of statistical estimation in cases in which a mathematical model is given a priori.

There are other estimation procedures that may also apply the least squares criterion and which rely on the basic concepts of stochastic processes. We may speak of " least squares prediction " when " time " is the variable involved, and therefore we estimate (or predict) what would happen in the future on the basis of past occurrences (history). On the other hand many of the applications in photogrammetry, geodesy, and surveying often involve location instead of time, and hence either least squares interpolation or extrapolation is used. The task of interpolation is to estimate (interpolate)

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in the future on the basis of past occurrences (history). On the other hand many of the applications in photogrammetry, geodesy, and surveying often involve location instead of time, and hence either least squares interpolation or extrapolation is used. The task of interpolation is to estimate (interpolate)

at locations other than those for which (observed) data are given. Because this situation is more often in practice, the term "interpolation" will be selected for use hereafter. At data points, measuring errors may also occur. If the estimation procedure "filters" these errors when estimating at new points, we speak of "least squares filtering."

When interpolating at new locations from data at given points, nonlinear relationships could, in general, be assumed. However, for practical reasons nonlinear interpolation is not often applied and instead "linear interpolation" is more commonly used. For this reason consideration will be given only to the linear case in the developments given in this chapter.

Least squares adjustment has been applied to problems in photogrammetry, geodesy, and surveying for a long time. On the other hand least squares interpolation and filtering have found their way into these fields only recently. In addition to applying interpolation to geodetic problems a general least squares technique combining classical adjustment with interpolation and filtering was devised. This technique has been given the name "least squares collocation."

Instead of commencing with the relatively advanced method of least squares collocation, least squares interpolation is introduced first. And in order to facilitate its introduction several basic concepts and definitions are concisely presented, without proofs.

14.1.1. Random or Stochastic Functions

The reader will recall that a *random variable* is defined as a variable that assumes various values depending on the result of an experiment (or observation, or trial) when such an experiment is repeated many times under similar conditions. The random variable has therefore a *probability distribution* associated with it, and it is considered specified when its *distribution function* is known. When the outcome of the experiment is a set of quantities (or numbers), we may consider them collectively in a vector as a single *multidimensional random vector*. In this case such random variables are specified by a multidimensional distribution function. Random variables, whether one dimensional or multidimensional, were sufficient for the treatment of different techniques of least squares adjustment. In order to explain least squares interpolation and filtering, however, the concept of random functions is required. A *random function* (also called "stochastic" function) is a function that is specified by the outcome of an experiment, and which may have different realizations as the experiment is repeated. It is composed of a set of random variables. An example of a random function is the diameter of the cross section of a thin wire (see Yaglom, 1962, in the Bibliography). Due to many factors in manufacturing a wire, its cross section is never a constant but is instead a *random function* along the wire. Another example concerns the position of image points (reseau marks) on an aerial photograph.

A *realization* of a random function (or a *sample function*) is one given set

of values for the function. A given segment of a thin wire is one realization since the diameters of various cross sections along this specified segment are known values in the sense that they can be measured. Another segment (of the same length) would be another realization which may, in general, have different values for the diameters of the wire cross sections. Similarly, a given photograph is a realization of a random function where the position of each specific point (reseau mark) is determined. The random variation is therefore from one realization to another.

The general term "random function" is used to designate either a *random sequence* or a *random process*. In the case of random sequences, the variable(s) in the function (for example, the position of a reseau point on an aerial photograph) assume only discrete values (when properly scaled, they become integer values). Random processes (often termed "stochastic" processes) on the other hand consider all values of such variable(s) (for example, the position of a wire cross section). Random functions are specified by the moments, particularly the first two, of their probability distributions. The first moment is the mean whereas the second moment would represent the covariance if the mean is zero. If $s(u)$ represents a random function, then the mean $\mu_s(u_1)$ at a particular value u_1 is defined as

$$\mu_s(u_1) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} s_k(u_1) \tag{14.1}$$

Similarly, the joint moment (which is called *autocorrelation function*) $C_s(u_1, u_1 + \Delta)$ at two different values u_1 and $(u_1 + \Delta)$ is given by

$$C_s(u_1, u_1 + \Delta) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} s_k(u_1)s_k(u_1 + \Delta) \tag{14.2}$$

For practical applications, μ_s and C_s are often sufficient for specifying the random function in question (in equations (14.1) and (14.2) the summation is over realizations).

14.1.2. Stationarity, Nonstationarity, and Ergodicity

If the conditions of the experiment, on whose outcome the values of the random function depend, remain unaltered (within one experiment and from one experiment to the next), then the function is called "stationary"; otherwise it is called "nonstationary." Consequently, if the random function $s(x)$ is nonstationary, then both μ_s and C_s of equations (14.1) and (14.2) would vary as u_1 varies. On the other hand for the special case of a stationary function, μ_s and C_s do not change in value as u_1 varies. It follows that for stationary functions the mean is constant (often taken as zero) and the autocorrelation function depends only on the increment Δ. An example of stationary function is when the manufacturing environment of the thin wire does not change with time. In this case the mean wire diameter is constant and the autocorrelation

function depends only on the distance between points along the wire. Any two different segments of the wire would have the same autocorrelation function.

For the purposes of the current presentation, consideration will be given mostly to stationary random functions. It is therefore convenient to take the mean as zero and speak of the autocorrelation function as the *autocovariance function*. If $\mu_s \neq 0$, a new random function $s'(u) = s(u) - \mu_s$, which has a zero mean, would replace $s(u)$.

The mean and autocorrelation would require a large number of realizations of the function s_k in order to be estimated from the data. This is often not practical and an alternative is desirable. In many cases in practice it is possible to describe the characteristics of a *stationary* function from *only one realization*. Here the mean and autocorrelation are given by

$$\mu_s(k) = \lim_{U \to \infty} \frac{1}{U} \int_0^U s_k(u) \, du \tag{14.3}$$

$$C_s(\Delta, k) = \lim_{U \to \infty} \frac{1}{U} \int_0^U s_k(u)s_k(u + \Delta) \, du \tag{14.4}$$

If the mean and autocorrelation as given by equations (14.3) and (14.4) turn out to be the same no matter which realization k is used, the stationary random function is called *ergodic*. The property of ergodicity applies only to those functions that are stationary.

A characteristic that is useful in practice pertains to *normal stationary* random functions. When the function mean is *zero*, the ergodic property will be satisfied when

$$\lim_{\Delta \to \infty} C_s(\Delta) = 0 \tag{14.5}$$

In other words, when the function is stationary and has a normal probability function† (which occurs frequently in practice) with zero mean and a covariance function that converges to zero as the interval Δ increases indefinitely, then the function is ergodic.

If the example of aerial photography is recalled, when ergodicity applies we may use data at different (reseau) points of the same photograph (one realization) to derive properties pertaining to the multitude of aerial photographs. In photogrammetry this is frequently carried out in tests on photographic quality without having to obtain an excessive number of photographic frames. In such tests ergodicity is assumed.

14.1.3. Multidimensional Random (Stochastic) Functions—Random Fields

So far, consideration has been given to single random functions such as

† A random function $s(x)$ has a normal distribution if the multidimensional joint distribution function $f[s(u_1), s(u_2), \ldots, s(u_n)]$ is normal for any arbitrary set of $u_1, u_2, \ldots, u_n$.

$s(u)$ with only one variable u which may or may not be time. Random vectors have been introduced before, however, and in a like manner a *random vector function* (or vector process) may be introduced, again of a single independent variable u. For example,

$$\mathbf{s}(u) = [s_1(u) \quad s_2(u) \cdots s_n(u)]^t \tag{14.6}$$

is an n-dimensional vector process that is a function of the one variable u. In this case we would have a mean n-vector $\boldsymbol{\mu}_s$, but the concept of correlation function must be expanded. Here, in addition to the autocorrelation (autocovariance) functions for each $s_i(u)$, there will be crosscorrelation (crosscovariance) functions for each pair $s_i(u)$ and $s_j(u)$, $i \neq j$. Consequently, the covariance function $C(u)$ is replaced by a covariance matrix $\mathbf{C}(u)$. This matrix is *symmetric* for *stationary* random vectors (of real functions). Each element $s_i(u)$ is a stationary random function, and each pair, $s_i(u)$ and $s_j(u)$, $i \neq j$, is only *stationarily correlated*, that is, the crosscovariance function depends only on the difference $(u_i - u_j)$.

Although the random vector function consists of several random functions all of which are in terms of only one variable (u), a *random field* consists of only *one* random function *but* in terms of more than one independent variable. For example, $s(\mathbf{u})$ is a random field that is a function of a vector of independent variables $\mathbf{u}$. Wind speed is an example of a four-dimensional random field, where $\mathbf{u}$ includes four independent variables—that is, time and the three dimensions of Euclidean space. For fields, the concept of stationarity is replaced by the concepts of "homogeneity" and "isotropy". A field is called "homogeneous" if its mean is constant and its autocovariances depend only on the vector separation (both length and direction) between different points of the field, not on their absolute location. If, in addition, autocovariances are invariant under rotation—that is, they depend only on the length of the vector separation—then the field is not only homogeneous, it is also *isotropic*.

A last kind of multidimensional random function combines both of the types that have been presented so far. A *random vector field* consists of several random functions (or processes), each of which is a function of a vector of variables $\mathbf{u}$, or

$$\mathbf{s}(\mathbf{u}) = [s_1(\mathbf{u}) \quad s_2(\mathbf{u}) \cdots s_n(\mathbf{u})]^t \tag{14.7}$$

The properties of such a relatively complex field would be expressed by averages, autocorrelations, and crosscorrelations in a manner similar to those presented previously.

14.1.4. Linear Interpolation and Filtering

In many branches of applied science and engineering, problems often occur in which an estimate of a random quantity is required from certain available data. The random quantity sought could, in general, be written as a nonlinear function of the var-

iables representing the data, but the solution in such a case would be difficult analytically. Instead, linear functions are often used to simplify the problem, hence the term "linear interpolation and filtering." For an explanation of interpolation and filtering, we define three random functions $l(u)$, $s(u)$, and $r(u)$ such that

$$l(u) = s(u) + r(u) \tag{14.8}$$

The observable function is $l(u)$, whereas $r(u)$ represents the measuring error or "noise," thus $s(u)$ is the so-called "signal" or response that is to be estimated. Interpolation and filtering, therefore, are the problems of finding an estimate $\hat{s}(u_0)$ of the random function $s(u_0)$, at $u = u_0$, when a discrete set of function values $l(u_1)$, $l(u_2)$, ..., $l(u_n)$ from a given realization $l(u)$ are available. The estimate is sought such that $\hat{s}(u_0)$ is a linear combination of $l(u_i)$, the function values, or

$$\hat{s}_0 = \hat{s}(u_0) = \mathbf{a}^t l \tag{14.9}$$

where

$$\mathbf{a}^t = [a_1 \quad a_2 \quad \cdots \quad a_n] \tag{14.10a}$$

is a vector of coefficients, and

$$l = [l(u_1) \quad l(u_2) \cdots l(u_n)]^t \tag{14.10b}$$

is the vector of given data values. Special cases of this general formulation include

1. If $r(u) = 0$, then $l(u) = s(u)$ implies that either no measuring errors occurred or if they did, they are of sufficiently small magnitude to be neglected. This is the problem of interpolation, where $s(u_0)$ is also $l(u_0)$.
2. If the noise $r(u) \neq 0$ and we are interested in estimating $s(u_0)$ at $u = u_0$ from the data values $l(u_i)$, then this is the filtering problem.

If the independent variable u is time, then the estimation process is called "prediction." In this case the data represent the "past" of the random function, and signal values at "future" times are to be estimated. In all cases the type of estimation considered is linear inasmuch as equation (14.9) represents the linear operator used for estimation.

Before dwelling on the details of interpolation and filtering, the concept of correlation (covariance) functions should first be introduced. Dealing with stationary random functions, it is convenient to assume zero means, or

$$\mu_s = E(s) = 0 \tag{14.11a}$$

$$\mu_r = E(r) = 0 \tag{14.11b}$$

$$\mu_l = E(l) = 0 \tag{14.11c}$$

Furthermore, stationarity means that autocorrelation C is only a function of the separation Δ. The functional expression relating C and Δ is called the

"correlation" or "covariance" function. There are in general three correlation functions—$C_l(\Delta)$, $C_s(\Delta)$, and $C_r(\Delta)$—which, under the assumption of zero means and that r and s are stationarily uncorrelated, are related by

$$C_l(\Delta) = C_s(\Delta) + C_r(\Delta) \qquad (14.12a)$$

and

$$C_{ls}(\Delta) = C_s(\Delta) \qquad (14.12b)$$

Figure 14.1(a) depicts graphically the relation of equation (14.12a) where the shaded area represents the correlation function of the noise.
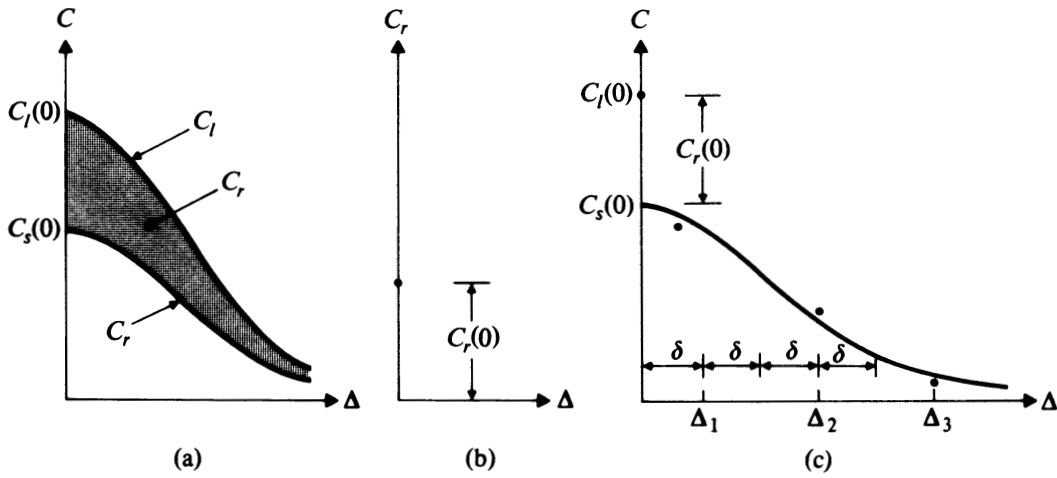
Figure 14.1

In practice, however, the noise $r(u)$ is more realistically assumed to be uncorrelated and therefore $C_r(\Delta)$ would be a single point (representing the variance of the noise) as depicted in Figure 14.1(b). Consequently, the three covariance functions in this case would be as shown in Figure 14.1(c), and for $\Delta \to 0$ the relation between variances is

$$C_l(0) = C_s(0) + C_r(0) \qquad (14.12c)$$

Covariance functions are either theoretically known, or they may be derived from the experimental data as will be explained in Section 14.5. Once the functions are established, the covariance $C(\Delta)$ may be evaluated for any two points separated by a distance Δ.

14.2. LINEAR INTERPOLATION OF STATIONARY FUNCTIONS

As given under the special case (1) of the preceding section the data pertain to $s(u)$, and hence equation (14.9) may be used replacing l by s. Let the deviation between the estimate $\hat{s}$ and s be designated by

$$v = s(u_0) - \hat{s}(u_0) = s_0 - \hat{s}_0 = s_0 - \mathbf{a}^t \mathbf{s}(u)$$

or

$$v = [1 \quad -\mathbf{a}^t]\begin{bmatrix} s_0 \\ \mathbf{s}(u) \end{bmatrix} \tag{14.13}$$

for which the variance is (applying the technique of variance propagation, see Chapter 4, Part I)

$$\sigma_v^2 = \mathbf{J}\Sigma\mathbf{J}^t \tag{14.14}$$

with the Jacobian, $\mathbf{J} = [1 \quad -\mathbf{a}^t]$ and the covariance matrix of the random variables involved given by

$$\Sigma = \begin{bmatrix} \sigma_{s0}^2 & \boldsymbol{\sigma}_{s0\,s} \\ \boldsymbol{\sigma}_{ss0} & \Sigma_{ss} \end{bmatrix} \tag{14.15}$$

This is an $(n + 1)$ square symmetric matrix of autocovariances and crossco-variances, with σ_{s0}^2 the variance of s_0 (a scalar) which is equal to $C_s(0)$ [see Figure 14.1(a)]; $\boldsymbol{\sigma}_{ss0} = \boldsymbol{\sigma}_{s0\,s}^t$, the $n \times 1$ vector of crosscovariances between s_0 and each element $s(u_i)$ of $\mathbf{s}$ (each value is obtained from the covariance function $C_s(\Delta)$ for each distance $\Delta_i = u_0 u_i$); and Σ_{ss} the $n \times n$ symmetric autocovariance matrix of $\mathbf{s}$. Its diagonal elements are all equal to $C_s(0)$, and each off-diagonal element ij is obtained from the covariance function using $\Delta_{ij} = u_i u_j$ as arguments.

Using equation (14.13) equation (14.12) may be expanded to

$$\sigma_v^2 = \sigma_{s0}^2 - 2\mathbf{a}^t\boldsymbol{\sigma}_{ss0} + \mathbf{a}^t\Sigma_{ss}\,\mathbf{a} \tag{14.16}$$

which represents the mean square interpolation error. For a minimum interpolation error, we select a vector $\mathbf{a}$ such that

$$\frac{\partial\sigma_v^2}{\partial\mathbf{a}} = -2\boldsymbol{\sigma}_{ss0}^t + 2\mathbf{a}^t\Sigma_{ss} = 0$$

or, after transposing and rearranging

$$\mathbf{a} = \Sigma_{ss}^{-1}\boldsymbol{\sigma}_{ss0} \tag{14.17}$$

This leads to the estimate of the interpolated value [see equation (14.9)]

$$\hat{s}_0 = \boldsymbol{\sigma}_{s0\,s}\Sigma_{ss}^{-1}\mathbf{s} \tag{14.18}$$

The estimate of the mean square error of estimation is

$$\sigma_{\hat{s}0}^2 = \sigma_v^2 = \sigma_{s0}^2 - 2\mathbf{a}^t\boldsymbol{\sigma}_{ss0} + \mathbf{a}^t\Sigma_{ss}\,\mathbf{a}$$

or in view of equation (14.17) it becomes

$$\sigma_{\hat{s}0}^2 = \sigma_{s0}^2 - \mathbf{a}^t\boldsymbol{\sigma}_{ss0} = C_s(0) - \mathbf{a}^t\boldsymbol{\sigma}_{ss0} = C_s(0) - \boldsymbol{\sigma}_{ss0}^t\Sigma_{ss}^{-1}\boldsymbol{\sigma}_{ss0} \tag{14.19a}$$

and alternatively

$$\sigma_{\hat{s}0}^2 = \sigma_{s0}^2 - \mathbf{a}^t\Sigma_{ss}\,\mathbf{a} = C_s(0) - \mathbf{a}^t\Sigma_{ss}\,\mathbf{a} \tag{14.19b}$$

14.3. LINEAR FILTERING OF STATIONARY FUNCTIONS

Linear filtering is actually a more interesting problem since it is of more practical usefulness than interpolation. In a manner similar to the derivation for interpolation, the estimate $\hat{s}_0$ for filtering may be obtained as follows:

$$v = s_0 - \mathbf{a}^t l = \begin{bmatrix} 1 & -\mathbf{a}^t \end{bmatrix} \begin{bmatrix} s_0 \\ l \end{bmatrix}$$

$$\sigma_v^2 = \begin{bmatrix} 1 & -\mathbf{a}^t \end{bmatrix} \Sigma \begin{bmatrix} 1 \\ -\mathbf{a} \end{bmatrix}$$

Here, the covariance matrix Σ is constructed as

$$\Sigma = \begin{bmatrix} \sigma_{so}^2 & \sigma_{sol} \\ \sigma_{lso} & \Sigma_{ll} \end{bmatrix}$$

The variance σ_{so}^2, in Σ, is equal to $C_s(0)$. Next, the vector σ_{sol} contains crosscovariances $C_{sl}(\Delta)$ which are equal to $C_s(\Delta)$ according to equation (14.12b). Finally, Σ_{ll} is the covariance matrix for the given data. Its diagonal elements are all equal to $C_l(0)$. Its off-diagonal elements are the same as those for Σ_{ss} because $C_l(\Delta) = C_s(\Delta)$ for $\Delta \neq 0$ [see Figure 14.1(c)].

Expanding σ_v^2 and differentiating to get a minimum, we get an equation similar to equation (14.17)

$$\mathbf{a} = \Sigma_{ll}^{-1} \sigma_{lso} \tag{14.20}$$

Finally, as in the case of interpolation

$$\hat{s}_0 = \sigma_{sol} \Sigma_{ll}^{-1} l \tag{14.21}$$

$$\sigma_{\hat{s}_0}^2 = \sigma_{so}^2 - \mathbf{a}^t \sigma_{lso}$$

or

$$\sigma_{\hat{s}_0}^2 = C_s(0) - \mathbf{a}^t \sigma_{lso} = C_s(0) - \sigma_{lso}^t \Sigma_{ll}^{-1} \sigma_{lso} \tag{14.22a}$$

and

$$\sigma_{\hat{s}_0}^2 = C_s(0) - \mathbf{a}^t \Sigma_{ll} \mathbf{a} \tag{14.22b}$$

This concludes discussion of filtering.

In the treatments of both interpolation and filtering, attention was limited to estimating one random function only. In practice, two- and three-dimensional random vectors may occur. Therefore both interpolation and filtering will be extended in the next section for multidimensional cases.

14.4. EXTENSION TO MULTIDIMENSIONAL CASES

Suppose that at each point more than one random variable are defined. For practical reasons only two- and three-dimensional random vectors will be considered. We will denote the components of these vectors by $x(u)$, $y(u)$, and $z(u)$. Thus for the two-dimensional case,

$$\begin{bmatrix} \hat{x}_0 \\ \hat{y}_0 \end{bmatrix} = \begin{bmatrix} \sigma_{xox} & \sigma_{xoy} \\ {}_{1,n} & {}_{1,n} \\ \sigma_{yox} & \sigma_{yoy} \\ {}_{1,n} & {}_{1,n} \end{bmatrix} \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ {}_{n,n} & {}_{n,n} \\ \Sigma_{yx} & \Sigma_{yy} \\ {}_{n,n} & {}_{n,n} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(u) \\ \mathbf{y}(u) \end{bmatrix} \tag{14.23}$$

and for the three-dimensional case

$$\begin{bmatrix} \hat{x}_0 \\ \hat{y}_0 \\ \hat{z}_0 \end{bmatrix} = \begin{bmatrix} \sigma_{xox} & \sigma_{xoy} & \sigma_{xoz} \\ \sigma_{yox} & \sigma_{yoy} & \sigma_{yoz} \\ \sigma_{zox} & \sigma_{zoy} & \sigma_{zoz} \end{bmatrix} \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zx} & \Sigma_{zy} & \Sigma_{zz} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(u) \\ \mathbf{y}(u) \\ \mathbf{z}(u) \end{bmatrix} \tag{14.24}$$

$\hat{x}_0$, $\hat{y}_0$, $\hat{z}_0$ are the estimates of the three components of the signal at $u = u_0$.

σ_{xox}, σ_{xoy}, σ_{xoz} are each a $1 \times n$ row vector of covariances between $x(u_0)$ and the data values $x(u_i)$, $y(u_i)$, and $z(u_i)$, respectively.

σ_{yox}, σ_{yoy}, ..., σ_{zoz} are vectors similarly defined.

Σ_{xx}, Σ_{yy}, Σ_{zz} are each an $n \times n$ square symmetric matrix of autocovariances of the given data.

Σ_{xy}, Σ_{xz}, Σ_{yz} are each an $n \times n$ square matrix of crosscovariances of the given data.

Equations (14.23) and (14.24) may be written more concisely as

$$\mathbf{s}_0 = \Sigma_{sos} \Sigma_{ss}^{-1} \mathbf{s} \qquad \text{(interpolation)} \tag{14.25}$$
$$\scriptstyle m,1 \qquad m,mn \quad mn,mn \quad mn,1$$

$$\mathbf{s}_0 = \Sigma_{sol} \Sigma_{ll}^{-1} \mathbf{l} \qquad \text{(filtering)} \tag{14.26}$$
$$\scriptstyle m,1 \qquad m,mn \quad mn,mn \quad mn,1$$

and the corresponding covariance matrices

$$\Sigma_{s_0 s_0} = \Sigma_{sos o} - \mathbf{a}^t \Sigma_{ss_0}$$
$$\scriptstyle m,m \qquad m,m \qquad m,mn \ mn,m$$

$$= \Sigma_{sos o} - \Sigma_{sos} \Sigma_{ss}^{-1} \Sigma_{sso} \tag{14.27}$$
$$\scriptstyle m,m \qquad m,mn \quad mn,mn \ mn,m$$

$$\Sigma_{s_0 s_0} = \Sigma_{sos o} - \mathbf{a}^t \Sigma_{lso} = \Sigma_{sos o} - \Sigma_{sol} \Sigma_{ll}^{-1} \Sigma_{lso} \tag{14.28}$$

where $m = 2$ for the two-dimensional case, and $m = 3$ for the three-dimensional problem. The covariance matrix $\Sigma_{s_0 s_0}$ is the computed matrix for the estimated signals $x(u_0)$, $y(u_0)$, and so on. The matrix $\Sigma_{sos o}$ is a covar-

iance matrix whose elements are the zero values $C_s(0)$ of the auto- and crosscovariance functions, or, for three dimensions,

$$
\Sigma_{s0s0} = \begin{bmatrix} C_{xx}(0) & C_{xy}(0) & C_{xz}(0) \\ C_{yx}(0) & C_{yy}(0) & C_{yz}(0) \\ C_{zx}(0) & C_{zy}(0) & C_{zz}(0) \end{bmatrix} \tag{14.29}
$$

In case the multidimensionality pertain to the independent variable, that is, in case of *random fields*, the relations of equations (14.18), (14.19) and equations (14.21), (14.22) for interpolation and filtering, respectively, would directly apply if the fields are assumed to be homogeneous and isotropic. Similarly, for homogeneous isotropic random vector fields, all equations in this section can be directly used. The increment Δ would be the scalar distance between the points in the field.

14.5. TREND SURFACES, COVARIANCE FUNCTIONS, AND EXPERIMENTAL RESULTS

Stationary random functions are characterized by a constant mean and the fact that the correlation function $C(\Delta)$ approaches the mean as $\Delta \to \infty$. For simplicity, the presented formulation of interpolation and filtering assumed zero mean; thus $C(\Delta) \to 0$ as $\Delta \to \infty$. In order to ascertain that these properties hold for given observational data, the *trend* must be removed first. A general definition of "trend" is that it is that component of a random phenomenon which has a period larger than the recorded data sample.

14.5.1. Trend Surfaces The trend is often represented by linear or low-order polynomial terms. The given data must therefore be preprocessed to remove the trend by transforming the data to another surface called "reference" or "trend" surface as shown in Figure 14.2. In this figure the signal surface (to be interpolated) would be referred to the trend surface instead of to the original datum. Consequently, the given values l, at the data or reference points, would be measured with respect to the trend surface. Both the signal surface and data points vary randomly about the trend surface such that their means are zero, one of the requirements for removing the trend.

Trend removal is an important operation. If the trend is not properly removed, distortions which cause erroneous results will occur. The type of function selected to represent the trend surface is essentially determined from the nature of the given data. In fact, caution should be exercised not to remove trend arbitrarily when there is no reason to believe that the data

Point to be interpolated   Reference or data points

————— Surface to be interpolated
— — — Reference or trend surface
— — — — Interpolated surface
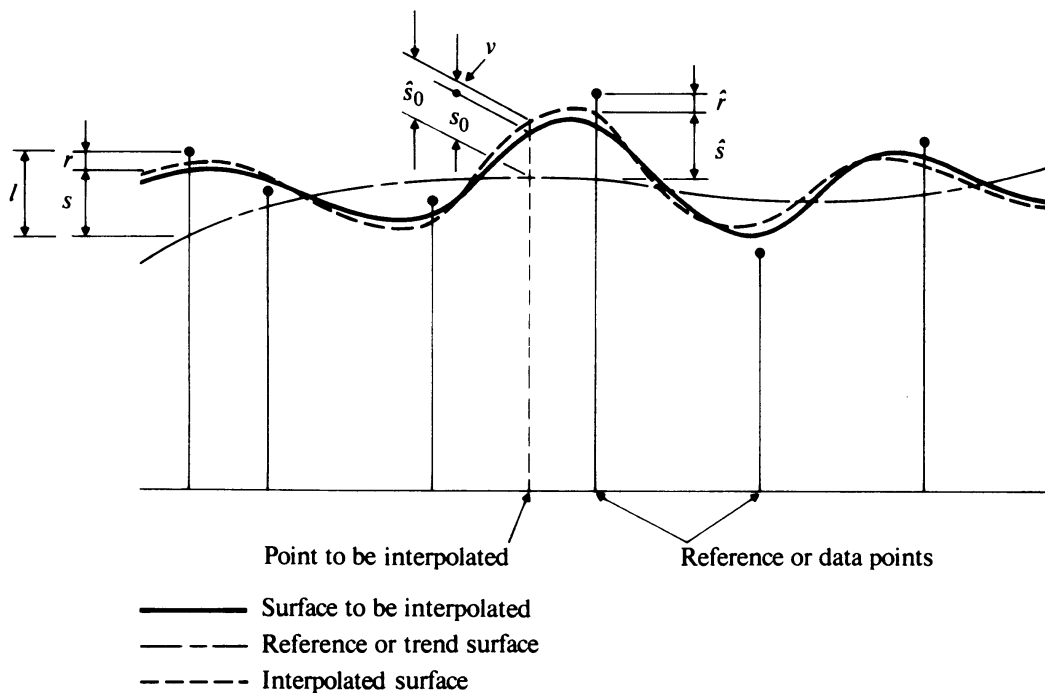
Figure 14.2

include a trend. On the other hand once it is recognized that a particular deformation exists in the data, it should be accounted for in the trend removal. For example, residual film deformation often causes affine distortion of a triangulated photogrammetric block. Consequently, we would choose an affine transformation to remove the trend when we are interested in external block adjustment using the filtering method presented here.

Having selected an appropriate expression for the trend surface, the transformation of the data is usually accomplished using the method of least squares adjustment. This is advisable because the number of reference (data) points is usually so large as to yield considerable redundancy. Therefore a least squares surface (curve) fitting would be the most suitable procedure. Once the fitting is accomplished, the data values are then referenced to the trend surface and become ready for the computation of the covariance function. Trend computation may be combined with interpolation or filtering in one operation. This operation is the so-called collocation, which will be given in detail in Section 14.7.

14.5.2. Covariance Functions   Although in some special instances it is possible to have a theoretically determined covariance function, it is often necessary to derive such a function from the given data. Since consideration is limited here to stationary random functions (or fields), the covariance function would usually take the general form shown in Figure 14.1. Addressing

first the one-dimensional case, we may compute the variances and covariances from the given data l_i as follows:

Variance:

$$C_l(0) = \frac{1}{n} \sum_{k=1}^{n} l_k^2 \qquad (14.30)$$

Covariance:

$$C_l(\Delta_k) = \frac{1}{n_k} \sum_{i<j} l_i l_j \qquad (14.31)$$

Equation (14.30) shows that the variance is computed by simply summing up the squares of all given data values l_k and dividing by the total number of reference points, n. If there is no noise, or measuring errors, in the data, the computed $C_l(0)$ will be identical to $C_s(0)$, the variance of the signal to be interpolated. Otherwise these two variances will be different by the amount $C_r(0)$, the error variance, as shown in Figure 14.1(c).

With respect to the covariance, reference is made to equation (14.12a) which shows that the covariance computed from the data $C_l(\Delta)$ is composed of two covariances $C_s(\Delta)$ and $C_r(\Delta)$, when s and r are uncorrelated. In practice the measuring errors are usually uncorrelated, thus $C_r(\Delta) = 0$ for $\Delta \neq 0$, and $C_l(\Delta) = C_s(\Delta)$ for all $\Delta > 0$, as shown in Figure 14.1(c). In the figure, Δ is the distance between the two points whose correlation is considered.

To compute the values $C_l(\Delta)$ of the covariance function from the data, Δ is divided into equal intervals. For example, as shown in Figure 14.1(c). the first is $\Delta_1 = \delta$, the second is $\Delta_2 = 3\delta$, and so on, with the interval being 2δ. Therefore the mean of all possible n_k products $l_i l_j$, at points whose separation Δ_{ij} is within an interval $(\Delta_k \pm \delta)$, is computed and regarded as the covariance $C_l(\Delta_k)$ at Δ_k according to equation (14.31). The increment δ is rather arbitrary and its value can usually be chosen on the basis of the knowledge of the given data. Once the number of covariance points is thus evaluated from the data, the corresponding covariance function may be derived. In order to determine an appropriate expression for the function, it is advisable to plot these covariance points as shown in Figure 14.1(c). When studying the disposition of the plotted points, and from knowledge of the conditions of the problem, we can usually elect the suitable function to fit the covariance points. Of the functions commonly used for autocovariance we may use a constant, sinusoidal, Gaussian, exponential, exponential cosine, and exponential sine and cosine. The function depicted in Figure 14.1(c) is the Gaussian with the equation

$$C(\Delta) = C(0) \exp\left(-k^2\Delta^2\right) \qquad (14.32)$$

which has been used frequently in applications to photogrammetric problems.

Usually the number of covariance points exceeds the number of coefficients in the covariance function, which are to be determined numer-

ically. Therefore a least squares curve fitting is applied. For example, in the case of the Gaussian curve of Figure 14.1(c), both $C(0)$ and k would be determined using all covariance points but not the point $C_l(0)$ along the covariance axis if filtering is to be effected. It is worthwhile to mention that linearization of equation (14.32) for least squares adjustment requires good approximations, otherwise difficulties with convergence may be encountered. When interpolation is to be carried out without filtering, we would use $C_l(0)$ and have the covariance function curve pass through it. This is accomplished by using a constraint equation.

Once $C(0)$ and k (as an example) are determined, then the covariance for any value of Δ is computed using equation (14.32). Consequently, we can construct the autocovariance matrices Σ for both interpolation and filtering. In case of interpolation only, the diagonal elements of Σ_{ss} [see equation (14.18)] would all be equal to $C_s(0)$—which is also equal to $C_l(0)$ in this case. Next, each off diagonal element in Σ_{ss} is obtained by evaluating Δ for each corresponding pair of points, and using it to compute the corresponding covariance using equation (14.32). For instance, considering points 3 and 5, we compute Δ_{35} and use it as an argument in equation (14.32) to get the element σ_{35}. The same procedure is repeated for all other off diagonal elements of Σ_{ss}.

In case of filtering, the diagonal elements of Σ_{ll} [see equation (14.21)] are equal to $C_l(0)$ and not $C_s(0)$. This is the major difference that leads to filtering. Other than that, the remaining off diagonal elements of Σ_{ll} would be obtained in the same manner as for Σ_{ss}.

To have an appreciation of the importance of the covariance function a simulated problem of one-dimensional filtering of a profile was set up. Figure 14.3 shows the empirical covariance functions used, limiting consideration to Gaussian and exponential functions only. Figure 14.4 shows several interpolated profiles corresponding to different covariance functions with different ratios between $C_s(0)$ and $C_l(0)$. Points to note here follow.
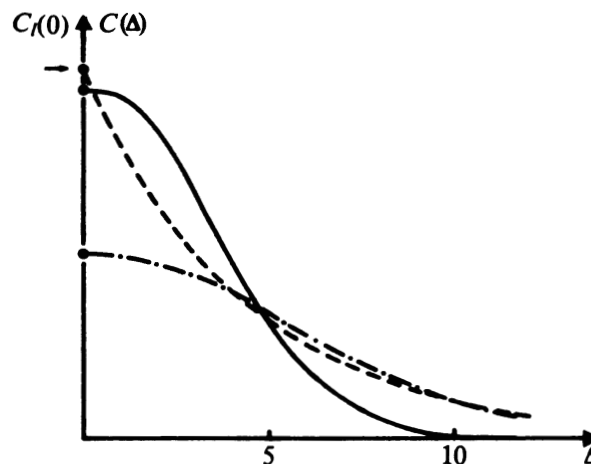
Figure 14.3. Experimentation with Different Covariance Functions
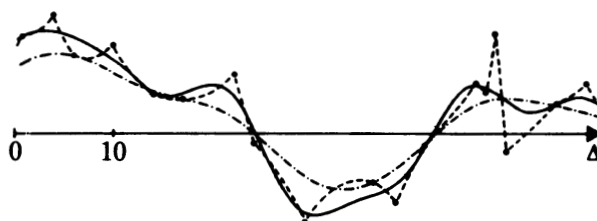
Figure 14.4. Interpolated Profiles Corresponding to the Different Covariance Functions in Figure 14.3

1. The Gaussian function produces estimated profiles with continuous first derivative, whereas the exponential function leads to profiles that exhibit discontinuous slope changes (that is, breaks or sharp peaks) at the locations of the reference points. This is one reason for using exponential covariance functions for interpolation problems in the digital terrain model, for example.

2. When $C_s(0) = C_i(0)$, the estimated profiles pass through the reference points (that is, no filtering). In practice, however, it is always better to filter even with $C_s(0) = 0.99C_i(0)$ in order to avoid potential numerical problems.

3. As the value of $C_s(0)$ gets smaller relative to $C_i(0)$ the profiles in general get farther from the reference points (that is, more filtering).

14.5.3. Multidimensional Cases

So far discussion has been limited to covariance functions for one-dimensional random functions. Extension to random fields, that is, when the argument of the one random function is multidimensional, is not problematic if the field is homogeneous and isotropic. In such a case the increment Δ becomes the distance between the points. If the field is not isotropic, then we must consider both distance and azimuth in constructing the covariance function—a situation that is not quite so common in practice. When considering random vector functions (or isotropic random vector fields), we must evaluate both autocovariance as well as crosscovariance functions. For each element of the random vector, an autocovariance function may be evaluated in a manner similar to the development given in the preceding section. In addition, for each pair of elements, a crosscovariance function also needs to be evaluated. If all such crosscovariances, when evaluated from the data, turned out to be small enough not to be significant, then an n-dimensional random vector would break down to n one-dimensional functions. Consequently, both interpolation and filtering of such random vectors can be performed as n separate operations each of one-dimensional random function. This turns out to be the case in many applications in practice.

When crosscorrelation is not insignificant, the evaluation of a crosscovariance function is done in a similar manner as the autocovariance function. As an example, consider the case of a two-dimensional random vector with

components $x(u)$ and $y(u)$. The autocovariance functions $C_{xx}(\Delta)$ and $C_{yy}(\Delta)$ can be derived from the data using equation (14.31), or

$$C_{xx}(\Delta_k) = \frac{1}{n_k} \sum_{i<j} x_i x_j \tag{14.33}$$

$$C_{yy}(\Delta_k) = \frac{1}{n_k} \sum_{i<j} y_i y_j \tag{14.34}$$

The crosscovariance functions $C_{xy}(\Delta)$ and $C_{yx}(\Delta)$ may be computed from the data using

$$C_{yx}(\Delta_k) = \frac{1}{n_k} \sum_{i<j} x_i y_i \tag{14.35}$$

$$C_{yx}(\Delta_k) = \frac{1}{n_k} \sum_{i<j} y_i x_j \tag{14.36}$$

Once the autocovariance and crosscovariance functions are evaluated, the elements of the covariance matrices Σ_{xx}, Σ_{xy}, and so on, may be computed from the functions in a straightforward manner.

14.5.4. Experimentation and Evaluation   The filtering method presented in Section 14.3 can be practically evaluated when the possibility of filtering at each of the given n reference points is considered. Hence, for the ith point, the estimated value according to equation (14.21) is given by

$$\hat{s}_i = \sigma_{li} \Sigma_{ll}^{-1} l \tag{14.37}$$

Referring to the discussion in Section 14.5.2 on covariance functions, the vector σ_{li} is the same as the ith column of Σ_{ll} except for the diagonal element σ_{ii}. For completeness we give both,

$$\sigma_{li} = \begin{bmatrix} \sigma_{i1} \\ \sigma_{i2} \\ \vdots \\ \sigma_{ii} = C_s(0) \\ \vdots \\ \sigma_{in} \end{bmatrix} \quad \text{and} \quad \Sigma_{ll} = \begin{bmatrix} C_l(0) & \sigma_{12} & \cdots & \sigma_{1n} \\ & C_l(0) & \cdots & \sigma_{2n} \\ & & \cdots & \\ \text{symmetric} & & & C_l(0) \end{bmatrix} \tag{14.38}$$

The vector σ_{li} may be considered as a column of a square symmetric matrix $\bar{\Sigma}_{ll}$,

$$\bar{\Sigma}_{ll} = \begin{bmatrix} C_s(0) & \sigma_{12} & \cdots & \sigma_{1n} \\ & C_s(0) & \cdots & \sigma_{2n} \\ & & \cdots & \\ \text{symmetric} & & & C_s(0) \end{bmatrix} \tag{14.39}$$

and the total vector of estimated signals at the n reference points becomes

$$\hat{\mathbf{s}} = \bar{\Sigma}_{ll}\Sigma_{ll}^{-1}l \tag{14.40}$$

Recalling that $s + r = l$ [see equation (14.8)], an estimate of the error vector $\mathbf{r}$ would be

$$\hat{\mathbf{r}} = l - \hat{\mathbf{s}} = (\mathbf{I} - \bar{\Sigma}_{ll}\Sigma_{ll}^{-1})l \tag{14.41}$$

$$\hat{\mathbf{r}} = (\Sigma_{ll} - \bar{\Sigma}_{ll})\Sigma_{ll}^{-1}l$$

The matrices Σ_{ll} and $\bar{\Sigma}_{ll}$ differ only in the diagonal elements [see equations (14.38) and (14.39)], and from equation (14.12c) and Figure 14.1(c), equation (14.41) reduces to

$$\hat{\mathbf{r}} = C_r(0)\Sigma_{ll}^{-1}l \tag{14.42}$$

This is the vector of the components filtered out at the reference points. If the assumption of zero mean is adhered to, an a posteriori estimate of $C_r(0)$ may be computed from

$$\hat{C}_r(0) = \frac{1}{n}\hat{\mathbf{r}}^t\hat{\mathbf{r}} \tag{14.43}$$

Since $C_r(0)$ is often obtained a priori from the data (when establishing the covariance function) whereas $\hat{C}_r(0)$ is an estimate after filtering, their relative magnitudes may be considered as an indication of how well the filtering was performed. As an example with a one-dimensional problem, the a priori variance was $C_r(0) = 0.0026$ and the estimate was $\hat{C}_r(0) = 0.0019$. In terms of standard deviation, 86% of r was filtered out. This type of evaluation has the same practical usefulness as the reference variance (σ_0^2) in least squares adjustment. Just as σ_0^2 may be used to judge both the model and the given data, the test on $C_r(0)$ helps evaluate the overall adequacy of the method, particularly the amount of filtering.

14.6. EXAMPLES OF FILTERING APPLICATIONS

14.6.1. Compensation for Aerial Film Deformation The mathematical model in computational photogrammetry relies on the collinearity condition that the camera position, the image point, and the object point all lie on a straight line. This is, in fact, a simplification of the actual physical shape of the light ray from the object point to the image point, which is usually a curve. Instead of formulating much too complex an expression for that curved path, the collinear form is introduced together with shifting the image position in the plane of the photograph to make its application realistic. In addition to the image shift to compensate for the curvature of the projection ray, there is also an image shift to account for the deformation of

the emulsion and film base between the moment of exposure and the time of image coordinate measurement.

To account for image deformation, reference points with calibrated positions, such as fiducial marks or reseau points (or both) are used. Although procedures using polynomials and other functions have been applied for some time, only recently has linear filtering been used. Figure 14.5 shows the
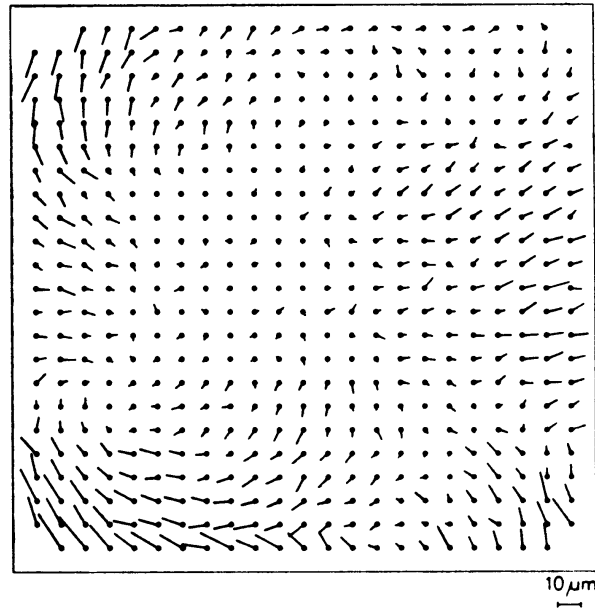
Figure 14.5. Residuals at 524 Points

field of deviations between measured coordinates and calibrated positions at 524 reseau points after the trend has been removed using a similarity transformation. These vectors represent the given data l. Each l_i can be considered in two components l_x and l_y. The computed correlation between these two components was small and therefore each component was treated independently as a one-dimensional filtering problem. From the given data, Gaussian covariance functions [see equation (14.32)] were evaluated for both the x and y components. Figure 14.6 shows the covariance function for the y component, as an example, for which $C_l(0) = 18.4 \ \mu\text{m}^2$, $C_s(0) = 12.2 \ \mu\text{m}^2$, and $k = 0.0173 \ \text{mm}^{-2}$. In fact, the experiment involved three photographs, the results of which are summarized in Table 14.1, with

$$\sigma_s = \sqrt{C_s(0)} \quad \text{and} \quad \sigma_r = \sqrt{C_r(0)} = \sqrt{C_l(0) - C_s(0)}$$

Continuing with the results on one photograph, Figure 14.8 shows the computed $\hat{s}$ component at all 524 reseau points when 144 reference points are used. The vectors depicted in the figure are drawn as the resultants of the separately computed $\hat{s}_x$ and $\hat{s}_y$. Comparison of Figure 14.7 to Figure 14.5
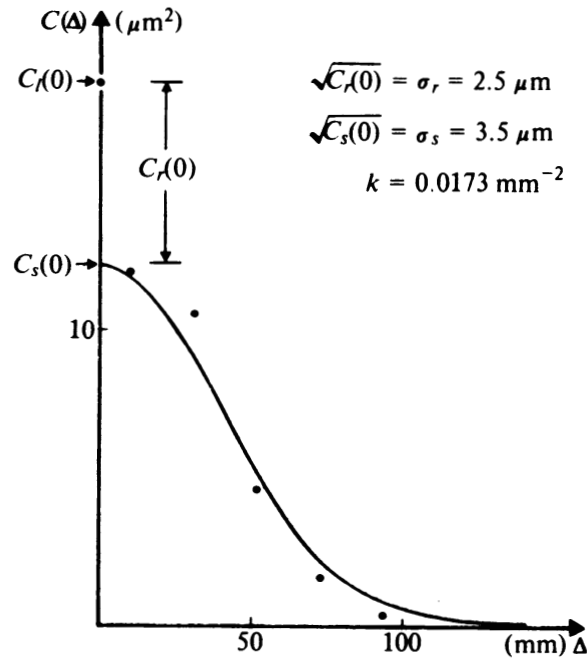
Figure 14.6. Covariance Function for Y Component of Film Deformation

TABLE 14.1 Uncorrelated Random Component of Film Deformation—Standard Deviation σ_r. Correlated Random Component for Film Deformation—Standard Deviation σ_s. Constant k of the Covariance Function

PLATE	x COMPONENT			y COMPONENT		
	σ_r (μm)	σ_s (μm)	k (mm^{-2})	σ_r (μm)	σ_s (μm)	k (mm^{-2})
302	1.7	2.8	0.017	2.8	3.9	0.017
358	1.8	3.3	0.014	2.5	3.5	0.017
412	2.1	3.9	0.015	3.0	4.9	0.021

shows that the original field is approximated quite well. The filtered uncorrelated random components, $\hat{r}$, are plotted in Figure 14.8 and show how effective filtering is for this type of problem.

14.6.2. Radial Lens Distortion Similar to film deformation, lens distortion is another factor for which measured image coordinates are treated before utilizing them in an analytical photogrammetry application. Limiting consideration to radial distortion, Figure 14.9 shows the distortion curves along
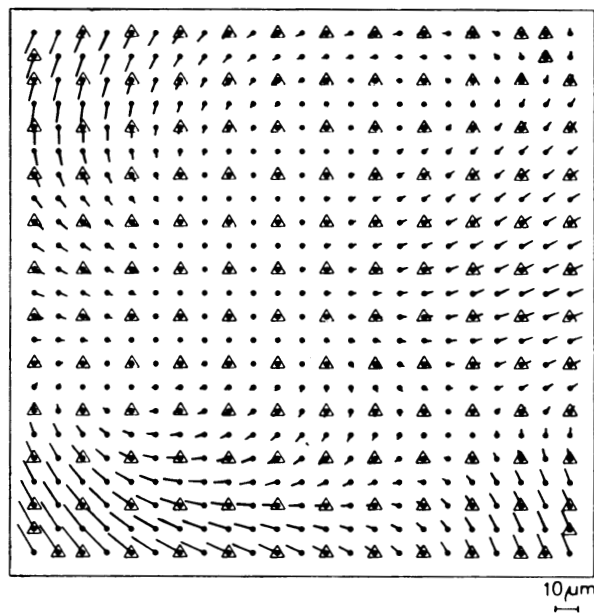
Figure 14.7. Computed $\hat{s}$ Component of Film Deformation Using 144 Reference Points
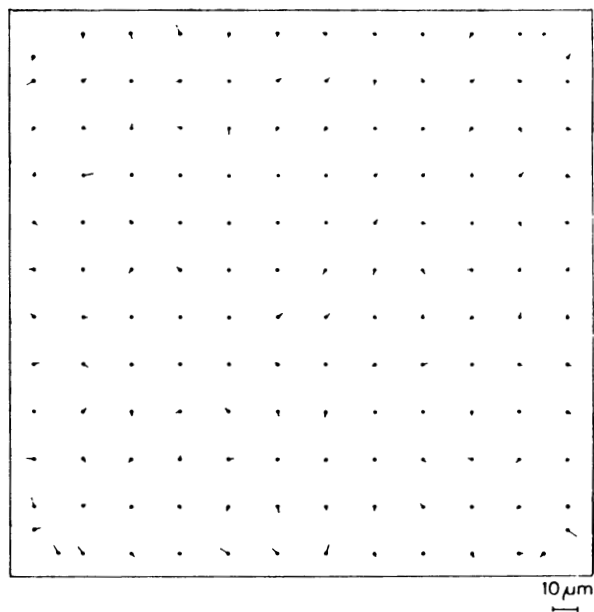
Figure 14.8. Filtered $\hat{r}$ Component of Film Deformation Using 144 Reference Points

the four radii of the format. It is not uncommon in practice simply to take the mean of all four values (see Figure 14.9) and apply a uniform radial correction to each point depending on its radial distance from the center of the format. This is obviously inadequate and better results can be obtained by a more effective method that operates directly with the given distortions along the different radii.
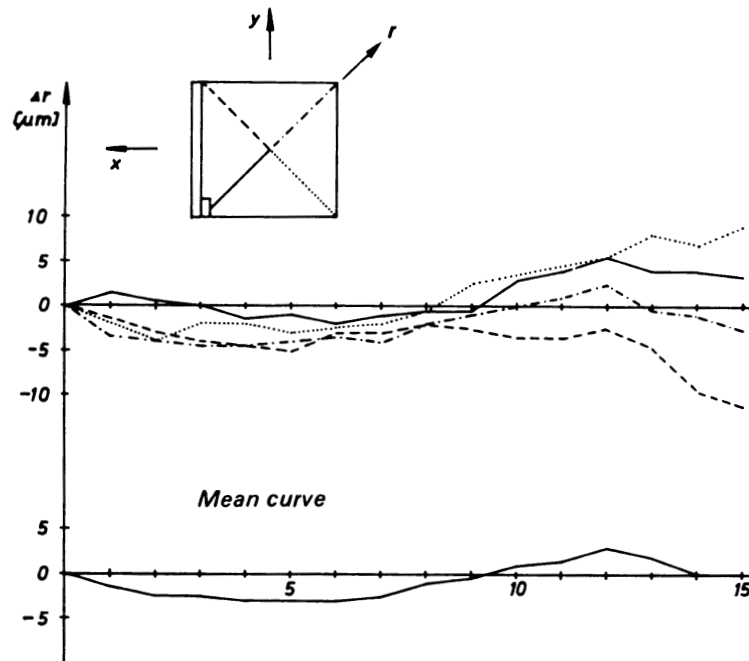
Figure 14.9. Radial Lens Distortion Curves

The first step is to seek an appropriate trend surface that was found to be the polynomial (see Vlcek, 1969, in the Bibliography),

$$d_r = b_1 r + b_2 r^3 + b_3 r^5 + a_1 r^2 \cos \phi + a_2 r^2 \sin \phi \qquad (14.44)$$

where d_r is the radial distortion, r is the radial distance, and ϕ is the direction angle.

After removing the trend using this polynomial, the remaining field of residuals is shown, for 20-mm points, in Figure 14.10. As usual an appropriate covariance function, in this case a Gaussian, was found for both the x and y directions as shown in Figure 14.11. Using least squares filtering the resulting correlated $\hat{s}$ components are added to those from the polynomial and the combined effects are plotted for the 2-cm grid of points in Figure 14.12. Finally, the uncorrelated random $\hat{r}$ components filtered out during estimation are shown in Figure 14.13.

14.6.3. External Photogrammetric Block Adjustment After block triangulation, particularly by relatively simplified methods such as Anblock, residual vectors remain at the check control points (or at used control points if they were not assigned infinite weights during triangulation). Residuals at different points invariably include correlated components (s components) and usually uncorrelated random errors (r components). In order to account for these correlated residuals and to compute corresponding quantities at
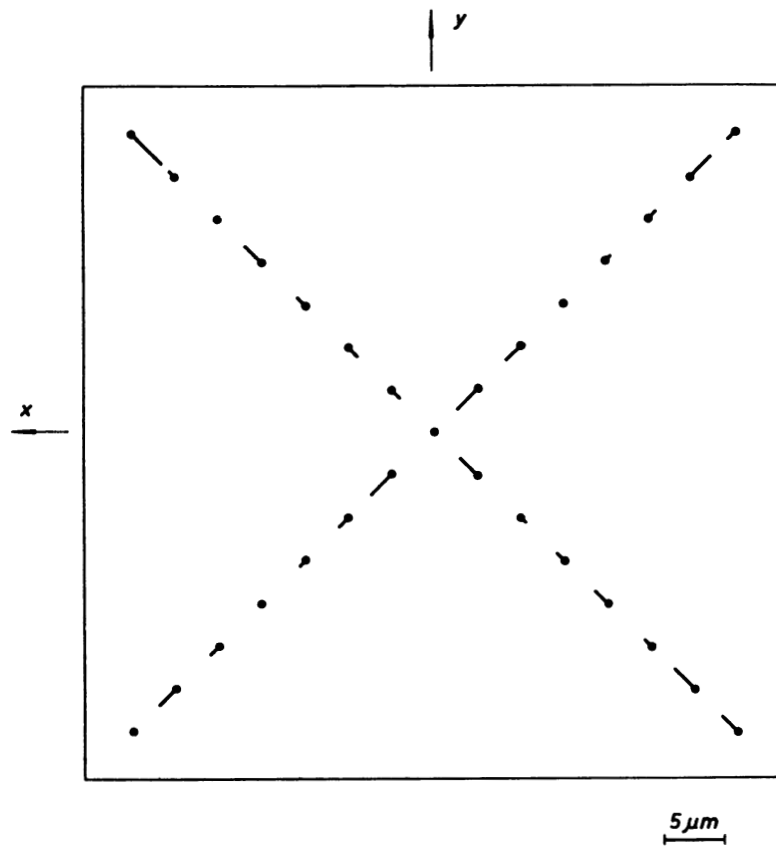
Figure 14.10. Lens Distortion Residuals after Trend Removal
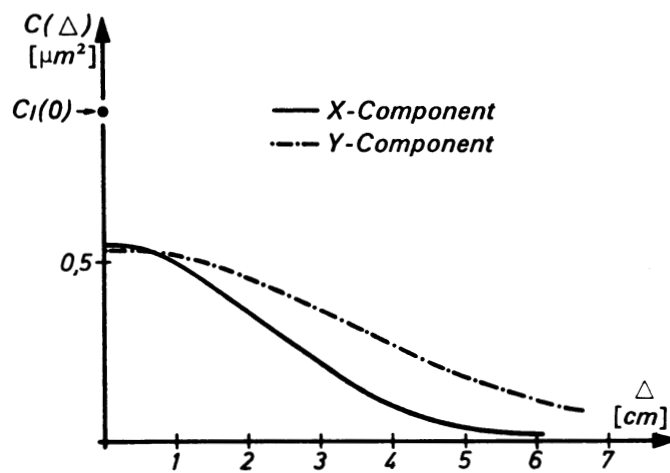
Figure 14.11. Gaussian Covariance Functions for Lens Distortion
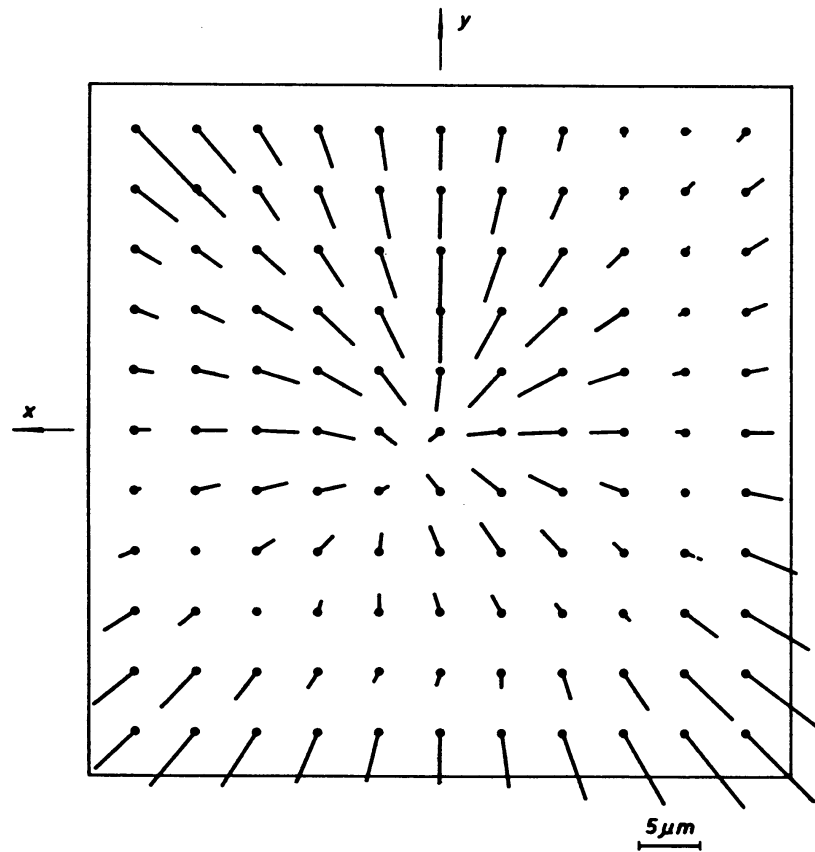
Figure 14.12. Combined Trend and $\hat{s}$ Component (Total Signal) for Lens Distortion
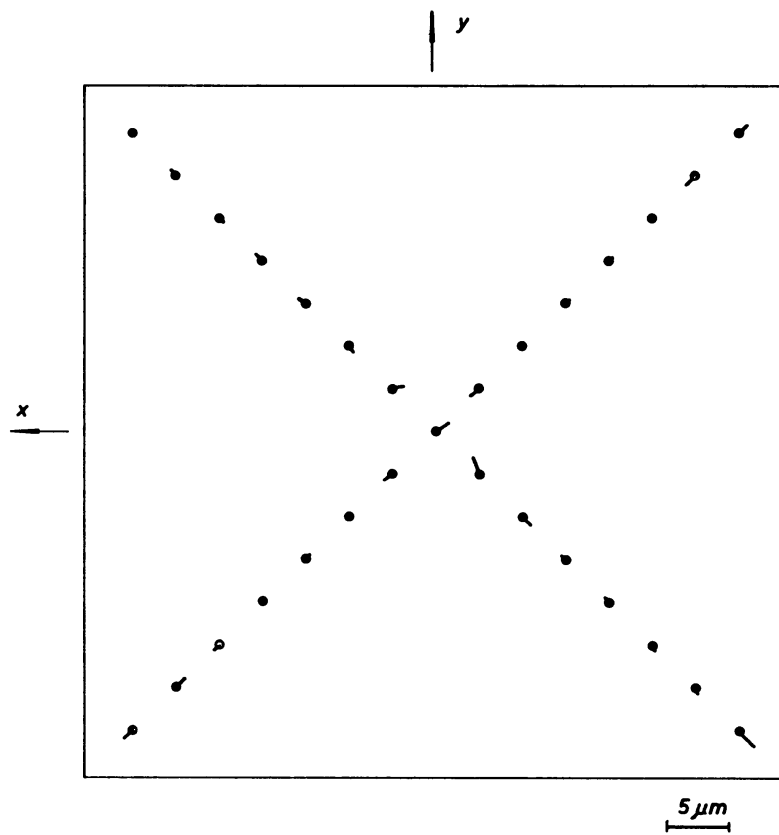
Figure 14.13. Filtered Uncorrelated $\hat{r}$ Component of Lens Distortion

415
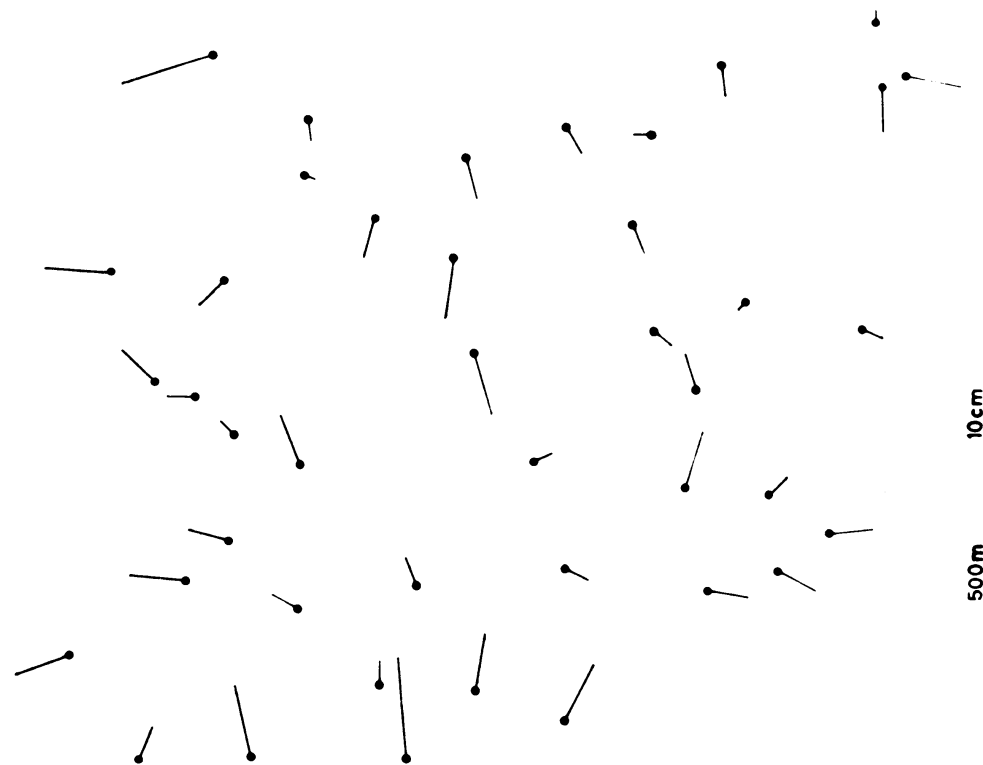
Figure 14.14. External Block Adjustment (Residuals at $n = 42$ Horizontal Check Control Points)
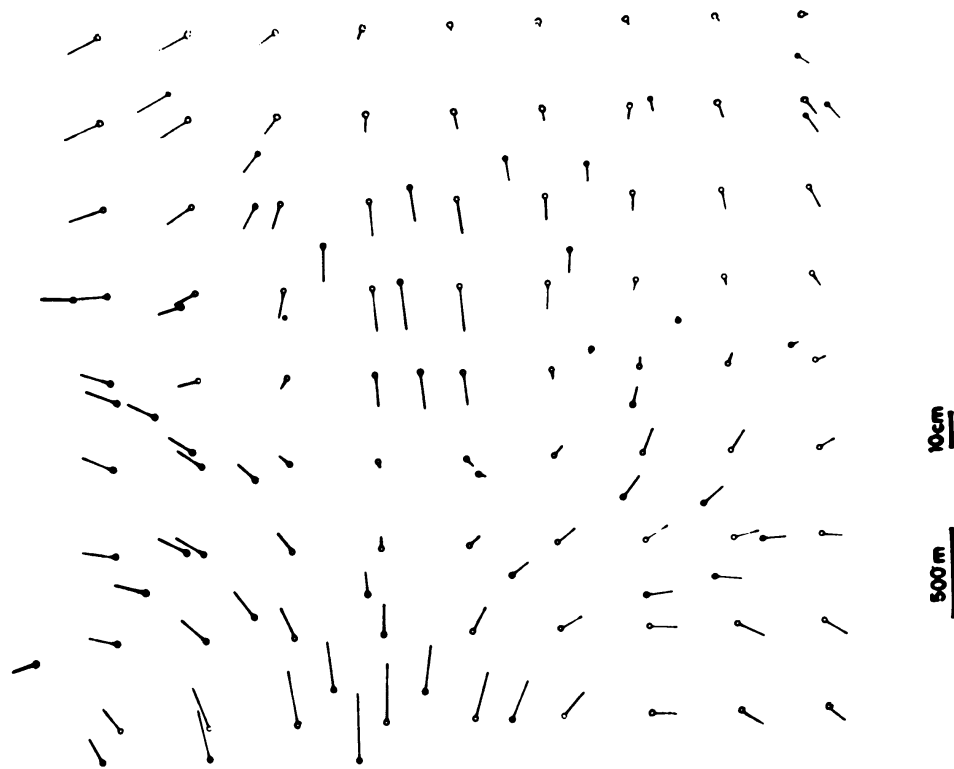
Figure 14.15. Estimated $\hat{s}$ Components at Control Points (·) and Pass Points (○)

416
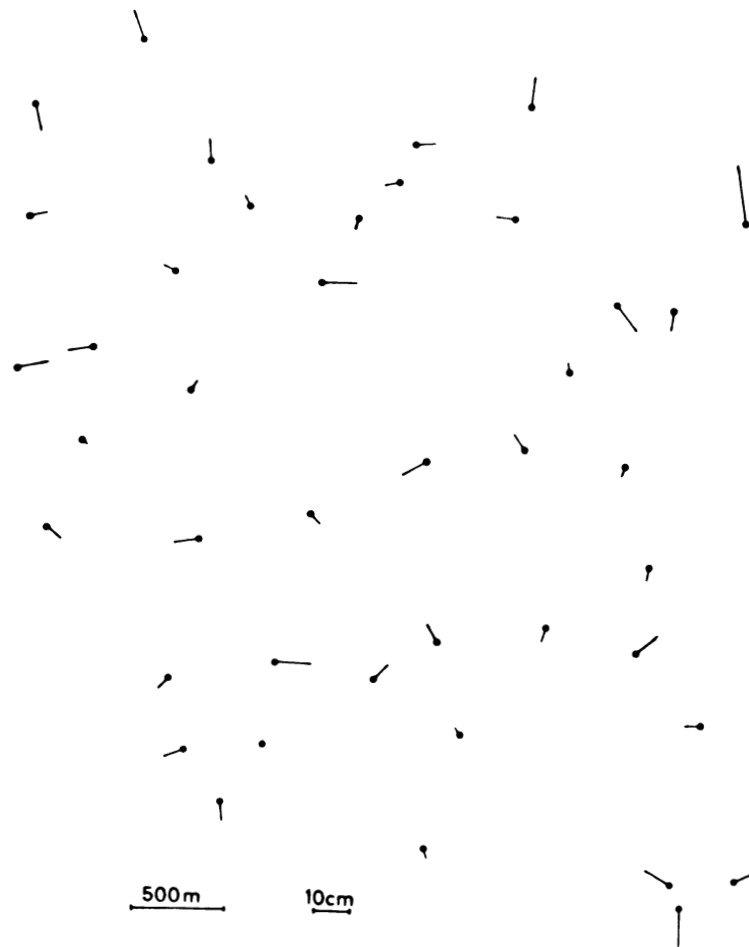
Figure 14.16. Estimated $\hat{r}$ Components

the pass points (s components) without the uncorrelated random part, least squares filtering proved to be an excellent procedure. As an example for horizontal adjustment, Figure 14.14 shows a photogrammetric block with the residuals after triangulation at 42 horizontal check control (reference) points. Using a Gaussian covariance function the values $C_s(0) = 0.72$, $C_l(0) = 1.00$, and $k = 0.00086$ m^2 were estimated from the data. Both X and Y components were estimated independently and separately, but then combined to plot total horizontal vectors.

Figure 14.15 shows the estimated $\hat{s}$ for both the check control (reference) points (designated by solid circles) and the interpolated pass points (hollow circles $\circ$). Figure 14.16 shows the remaining $\hat{r} = l - \hat{s}$ component of the residuals at the reference points. It can be ascertained by comparing Figures 14.15 and 14.16 how the $\hat{s}$ and $\hat{r}$ components vary in appearance and behavior. Although the $\hat{s}$ components are quite correlated, the r components obviously are not.

14.7. LEAST SQUARES COLLOCATION

It has been mentioned earlier in this chapter that least squares collocation combines the well-established techniques of adjustment with those of inter-polation and filtering. The presentation by Moritz (1972, see Bibliography) considers the technique of adjustment of indirect observation and interpola-tion. Although we shall rely mainly on this work, we shall extend it slightly to include the more general least squares technique of adjustment of obser-vations and functionally independent parameters, for which the conditions are

$$\underset{c,\,n}{\mathbf{A}}\,(l+\mathbf{v})+\underset{c,\,u\ \ u,\,1}{\mathbf{B}}\ \underset{}{\Delta}=\underset{c,\,1}{\mathbf{d}} \tag{14.45}$$

with Σ as the covariance matrix of the original observations. A new set of equivalent observations l_e may be introduced so that

$$l_e+\mathbf{v}_e=(-\mathbf{A}l+\mathbf{d})+(\mathbf{A}\mathbf{v})$$

Thus

$$\underset{c,\,1}{l_e}=-\mathbf{A}l+\mathbf{d} \tag{14.46a}$$

$$\underset{c,\,1}{\mathbf{v}_e}=\mathbf{A}\mathbf{v} \tag{14.46b}$$

and from propagation techniques

$$\underset{c,\,c}{\Sigma_e}=\mathbf{A}\Sigma\mathbf{A}' \tag{14.46c}$$

is the covariance matrix of l_e. Equation (14.45) may now be rewritten as

$$\mathbf{v}_e=-\mathbf{B}\Delta+l_e \tag{14.47}$$

which implies a transformation from a technique of combined observations and parameters to that of indirect observations.

In equation (14.47) the residuals $\mathbf{v}_e$ are decomposed into a correlated component (signal) $\mathbf{s}$ and noise $\mathbf{r}$, or $\mathbf{v}_e=\mathbf{s}+\mathbf{r}$, hence

$$\mathbf{r}+\mathbf{s}=l_e-\mathbf{B}\Delta \tag{14.48}$$

Comparing equations (14.48) to (14.8) shows that subtracting $\mathbf{B}\Delta$ from l_e amounts to trend computation. Thus collocation means combining estimat-ing the trend simultaneously with interpolation and filtering.

Since both $\mathbf{r}$ and $\mathbf{s}$ are random vectors whose expectations are zero vectors, or

$$\mu_r=E(\mathbf{r})=0 \tag{14.49}$$

$$\mu_s=E(\mathbf{s})=0 \tag{14.50}$$

then $\mu_{v_e} = 0$, and $\mathbf{B}\Delta$ represents the mean vector of l_e or $E(l_e)$. Consequently,

$$\Sigma_{v_e} = E(l_e - \mathbf{B}\Delta)(l_e - \mathbf{B}\Delta)^t$$
$${}_{c,\,c}$$

$$= E(l_e - E(l_e))(l_e - E(l_e))^t$$

which is by definition the covariance matrix of l_e, or

$$\Sigma_{v_e v_e} = \Sigma_e \tag{14.51}$$

The vector of observations is assumed to be known at n points (the reference or data points). In addition to estimating Δ, we may also wish to interpolate and filter at another set of m points for which the vector $\mathbf{s}_0$ will be used. In other words $\mathbf{s}_0$ is the vector of estimated signals at m points that may be different from the data points. Combining the two random vectors

$$\mathbf{s}_0 \qquad \text{and} \qquad \mathbf{v}_e$$
$${}_{m,\,1} \qquad\qquad\qquad {}_{c,\,1}$$

into one, let

$$\dot{\mathbf{v}} = \begin{bmatrix} \mathbf{s}_0 \\ \mathbf{v}_e \end{bmatrix} \qquad \text{with } p = m + c \tag{14.52}$$
$${}_{p,\,1}$$

and so equation (14.47) becomes

$$[\mathbf{0} \quad \mathbf{I}] \begin{bmatrix} \mathbf{s}_0 \\ \mathbf{v}_e \end{bmatrix} + \mathbf{B}\Delta = l_e = \mathbf{f} \tag{14.53}$$

or

$$\dot{\mathbf{A}}\dot{\mathbf{v}} + \mathbf{B}\Delta = \mathbf{f} \tag{14.54}$$

Taking the a priori reference variance equal to one, the covariance and cofactor matrices associated with equation (14.54) may be constructed as

$$\Sigma = \begin{bmatrix} \Sigma_{s_0 s_0} & \Sigma_{s_0 e} \\ \Sigma_{s_0 e}^t & \Sigma_e \end{bmatrix} = \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{s_0 s_0} & \mathbf{Q}_{s_0 e} \\ \mathbf{Q}_{s_0 e}^t & \mathbf{Q}_e \end{bmatrix} \tag{14.55}$$

According to equation (6.20), the least squares estimate of Δ from equation (14.54) is

$$\Delta = [\mathbf{B}^t(\dot{\mathbf{A}}\mathbf{Q}\dot{\mathbf{A}}^t)^{-1}\mathbf{B}]^{-1}[\mathbf{B}^t(\dot{\mathbf{A}}\mathbf{Q}\dot{\mathbf{A}}^t)^{-1}\mathbf{f}] \tag{14.56}$$

From equations (14.53), (14.54), and (14.55)

$$(\dot{\mathbf{A}}\mathbf{Q}\dot{\mathbf{A}}^t) = [\mathbf{0} \quad \mathbf{I}] \begin{bmatrix} \mathbf{Q}_{s_0 s_0} & \mathbf{Q}_{s_0 e} \\ \mathbf{Q}_{s_0 e}^t & \mathbf{Q}_e \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} = \mathbf{Q}_e$$

hence equation (14.56) becomes

$$\Delta = (\mathbf{B}^t \mathbf{W}_e \mathbf{B})^{-1}(\mathbf{B}^t \mathbf{W}_e \mathbf{f}) \tag{14.57}$$

Equation (14.57) is identical to the results in equations (6.21), (6.22), and (6.24), which shows that the extension to least squares collocation still yields the proper adjustment estimate for the parameters. It can also be shown that the a posteriori cofactor matrix $\mathbf{Q}_{\Delta\Delta}$ is in fact $(\mathbf{B}^t\mathbf{W}_e\mathbf{B})^{-1}$, which is consistent with the results in Chapter 6 (Part II).

To continue with filtering as a part of collocation, we may write

$$\dot{\mathbf{v}} = \mathbf{Q}\dot{\mathbf{A}}^t\mathbf{W}_e(l_e - \mathbf{B}\Delta)$$

or

$$\begin{bmatrix} \hat{\mathbf{s}}_0 \\ \mathbf{v}_e \end{bmatrix} = \begin{bmatrix} \Sigma_{s_0s_0} & \Sigma_{s_0e} \\ \Sigma_{s_0e}^t & \Sigma_e \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \Sigma_e^{-1}(l_e - \mathbf{B}\Delta)$$

Thus the estimate $\hat{\mathbf{s}}_0$ of the filtered signal at the m points is

$$\hat{\mathbf{s}}_0 = \Sigma_{s_0e}\Sigma_e^{-1}(l_e - \mathbf{B}\Delta) \tag{14.58}$$

If the original least squares adjustment were of the indirect observation technique, then l_e in equation (14.58) would be replaced by the observations themselves (data) l, thus

$$\hat{\mathbf{s}}_0 = \Sigma_{s_0l}\Sigma_{ll}^{-1}(l - \mathbf{B}\Delta) \tag{14.59}$$

in which $\hat{\mathbf{s}}_0$ is the $m \times 1$ vector of interpolated or filtered values at the m new points; $\Sigma_{s_0l} = \Sigma_{ls_0}^t$ is the $m \times n$ cross covariance matrix between the signal and the given data; Σ_{ll} is the $n \times n$ symmetric autocovariance matrix of the observations (data); l is the vector of observations ($= \mathbf{s}$ for interpolation); and $\mathbf{B}\Delta$ is the mean of the observations (representing the trend surface) as the product of the $m \times u$ coefficient matrix and the $u \times 1$ vector of parameters. If the trend has been removed before interpolation and filtering, $\mathbf{B}\Delta$ would be missing and equation (14.59) becomes identical to equation (14.26).

As a further demonstration if only one value s_0 is to be estimated, the least squares collocation reduces to

$$\hat{s}_0 = \underset{1,\,n}{\sigma_{s_0l}}\Sigma_{ll}^{-1}(l - \mathbf{B}\Delta) \tag{14.60a}$$

in which σ_{s_0l} is a row vector. Furthermore, when the vector of parameters is zero due to a priori trend removal, least squares collocation reduces to interpolation and filtering, and equation (14.60a) becomes

$$\hat{s}_0 = \sigma_{s_0l}\Sigma_{ll}^{-1}l \tag{14.60b}$$

which is identical to equation (14.21). Consequently, it is obvious that both least squares adjustment [see equation (14.57)] and least squares interpolation and filtering are special cases of collocation.

By way of summary then, least squares collocation makes possible the estimation of parameters, Δ, as well as estimating filtered values for the

variables representing the observations (signals) not only at the data points, but also at other points. These filtered (or interpolated) values will by necessity be referred to the trend surface $\mathbf{B}\Delta$. Consequently, if we are interested in the total estimate of the signal $\mathbf{t}$ with respect to the original datum of the data, then

$$\hat{\mathbf{t}}_0 = \hat{\mathbf{s}}_0 + \mathbf{D}\Delta \tag{14.61}$$

where $\mathbf{D}$ is the appropriate coefficient matrix necessary for computing values of the function at the m new filtered points. Obviously, if filtering is done at the original n reference or data points, the matrix $\mathbf{D}$ would be identical to the original matrix $\mathbf{B}$. The computation of the total signal is demonstrated by the example succeeding the following section.

A last remark concerns the fact that all the functions given in the preceding derivations were linear. This is in keeping with the premise that least squares is practical mainly in linear estimation. Naturally, if the original functional model is nonlinear, as it often is in practice, a linearized form must be derived and used in the developed algorithms.

14.7.1. Precision Estimation for Least Squares Collocation

The a posteriori cofactor (covariance) matrix of the parameters is

$$\mathbf{Q}_{\Delta\Delta} = (\mathbf{B}^t\mathbf{W}_e\,\mathbf{B})^{-1} = [\mathbf{B}^t(\mathbf{A}\mathbf{Q}_{ll}\,\mathbf{A}^t)^{-1}\mathbf{B}]^{-1} \tag{14.62}$$

In order to evaluate the a posteriori cofactor matrix of the estimated signals $\hat{\mathbf{s}}_0$, we should recall that in general the a posteriori cofactor matrix of the estimated observations is equal to the a priori cofactor matrix minus the a posteriori cofactor matrix of the residuals. According to equation (6.30) the cofactor matrix $\mathbf{Q}_{\dot{v}\dot{v}}$ may be evaluated

$$\mathbf{Q}_{\dot{v}\dot{v}} = \mathbf{Q}\dot{\mathbf{A}}^t(\mathbf{W}_e - \mathbf{W}_e\,\mathbf{B}\mathbf{Q}_{\Delta\Delta}\,\mathbf{B}^t\mathbf{W}_e)\dot{\mathbf{A}}\mathbf{Q} \tag{14.63}$$

From the structure of $\dot{\mathbf{A}}$ and $\mathbf{Q}$, see equations (14.53), (14.54), and (14.55), the cofactor submatrix relating to $\hat{\mathbf{s}}_0$ is

$$\mathbf{Q}_{v_s v_s} = \mathbf{Q}_{soe}\,\mathbf{W}_e\,\mathbf{Q}_{soe}^t - \mathbf{Q}_{soe}\,\mathbf{W}_e\,\mathbf{B}\mathbf{Q}_{\Delta\Delta}\,\mathbf{B}^t\mathbf{W}_e\,\mathbf{Q}_{soe}^t \tag{14.64}$$

Therefore the estimated cofactor matrix of $\hat{\mathbf{s}}_0$ is

$$\mathbf{Q}_{\hat{s}_0\hat{s}_0} = \mathbf{Q}_{soso} - \mathbf{Q}_{v_s v_s}$$

or

$$\mathbf{Q}_{\hat{s}_0\hat{s}_0} = \mathbf{Q}_{soso} - \mathbf{Q}_{soe}\,\mathbf{Q}_e^{-1}\mathbf{Q}_{soe}^t + \mathbf{Q}_{soe}\,\mathbf{Q}_e^{-1}\,\mathbf{B}\mathbf{Q}_{\Delta\Delta}\,\mathbf{B}^t\mathbf{Q}_e^{-1}\mathbf{Q}_{soe}^t \tag{14.65}$$

with the last term in equation (14.65) representing the contribution of the errors in the parameters Δ. In the absence of parameters, equation (14.65) in terms of covariance matrices becomes

$$\Sigma_{\hat{s}_0\hat{s}_0} = \Sigma_{soso} - \Sigma_{soe}\Sigma_e^{-1}\Sigma_{soe}^t \tag{14.66}$$

which is very similar to equation (14.28) except that here l_e replaces the l used in the earlier treatment. Finally, if only one value is estimated, and the conditions are of the indirect observation type, then (using the original symbols for covariances)

$$\sigma_{\hat{s}_0}^2 = C_s(0) - \sigma_{ls_0}^t \Sigma_{ll}^{-1} \sigma_{ls_0} \tag{14.67}$$

which is identical to equation (14.22a). This shows that the results from least squares collocation will reduce to the results of interpolation and filtering once the parameters are not considered, which means that the trend has been removed a priori.

A crosscofactor matrix $\mathbf{Q}_{\Delta s_0}$ may also be evaluated as follows. Let the vector of observational variables associated with equation (14.54) be denoted by l. The estimated observations are therefore

$$\hat{l} = l + \dot{v} = \begin{bmatrix} 0 \\ l_e \end{bmatrix} + \begin{bmatrix} s_0 \\ v_e \end{bmatrix} = \begin{bmatrix} s_0 \\ \hat{l}_e \end{bmatrix} \tag{14.68}$$

Referring to equation (6.33), the crossfactor matrix between the parameters Δ and $\hat{l}$ is

$$\mathbf{Q}_{\Delta l} = -\mathbf{N}^{-1}\mathbf{B}^t\mathbf{W}_e\dot{\mathbf{A}}\mathbf{Q} = -\mathbf{Q}_{\Delta\Delta}\ \mathbf{B}^t\mathbf{W}_e[0 \quad \mathbf{I}]\begin{bmatrix} \mathbf{Q}_{s_0 s_0} & \mathbf{Q}_{s_0 e} \\ \mathbf{Q}_{s_0 e}^t & \mathbf{Q}_e \end{bmatrix}$$

or

$$[\mathbf{Q}_{\Delta\hat{s}_0}\mathbf{Q}_{\Delta l_e}] = -\mathbf{Q}_{\Delta\Delta}\ \mathbf{B}^t\mathbf{W}_e[\mathbf{Q}_{s_0 e}^t \quad \mathbf{Q}_e]$$

from which

$$\mathbf{Q}_{\Delta\hat{s}_0} = -\mathbf{Q}_{\Delta\Delta}\ \mathbf{B}^t\mathbf{W}_e\ \mathbf{Q}_{s_0 e}^t \tag{14.69}$$

or in terms of covariance matrices

$$\Sigma_{\Delta\hat{s}_0} = -\Sigma_{\Delta\Delta}\ \mathbf{B}^t\mathbf{W}_e\Sigma_{s_0 e} \tag{14.70}$$

Finally, an a posteriori cofactor matrix is needed for the final estimate $\hat{t}$ of the total signal given by equation (14.61). Applying the propagation principle

$$\mathbf{Q}_{tt} = \mathbf{Q}_{s_0 s_0} + \mathbf{D}\mathbf{Q}_{\Delta\Delta}\ \mathbf{D}^t + \mathbf{Q}_{\hat{s}_0\Delta}\ \mathbf{D}^t + \mathbf{D}\mathbf{Q}_{\Delta\hat{s}_0}$$

which according to equations (14.65) and (14.69) becomes

$$\mathbf{Q}_{\hat{t}\hat{t}} = \mathbf{Q}_{s_0 s_0} - \mathbf{Q}_{s_0 e}\ \mathbf{Q}_e^{-1}\mathbf{Q}_{s_0 e}^t + \mathbf{Q}_{s_0 e}\ \mathbf{Q}_e^{-1}\ \mathbf{B}\mathbf{Q}_{\Delta\Delta}\ \mathbf{B}^t\mathbf{Q}_e^{-1}\mathbf{Q}_{s_0 e}^t$$

$$+ \mathbf{D}\mathbf{Q}_{\Delta\Delta}\ \mathbf{D}^t - \mathbf{Q}_{s_0 e}\mathbf{Q}_e^{-1}\mathbf{B}\mathbf{Q}_{\Delta\Delta}\ \mathbf{D}^t - \mathbf{D}\mathbf{Q}_{\Delta\Delta}\ \mathbf{B}^t\mathbf{Q}_e^{-1}\mathbf{Q}_{s_0 e}^t$$

$$= \mathbf{Q}_{s_0 s_0} - \mathbf{Q}_{s_0 e}\ \mathbf{Q}_e^{-1}\mathbf{Q}_{s_0 e}^t + (\mathbf{Q}_{s_0 e}\ \mathbf{Q}_e^{-1}\mathbf{B} - \mathbf{D})\mathbf{Q}_{\Delta\Delta}$$

$$\times\ (\mathbf{B}^t\mathbf{Q}_e^{-1}\mathbf{Q}_{s_0 e}^t - \mathbf{D}^t)$$

or

$$\mathbf{Q}_{ii} = \mathbf{Q}_{s_0 s_0} - \mathbf{Q}_{soe} \, \mathbf{Q}_e^{-1} \mathbf{Q}_{soe}^t + (\mathbf{Q}_{soe} \, \mathbf{Q}_e^{-1} \mathbf{B} - \mathbf{D}) \mathbf{Q}_{\Delta\Delta}$$
$$\times (\mathbf{Q}_{se} \, \mathbf{Q}_e^{-1} \mathbf{B} - \mathbf{D})^t \tag{14.71}$$

in which $\mathbf{Q}_{\Delta\Delta}$ is as given by equation (14.62). In terms of covariance matrices, equation (14.71) becomes

$$\Sigma_{ii} = \Sigma_{s_0 s_0} - \Sigma_{soe}\Sigma_e^{-1}\Sigma_{soe}^t + (\Sigma_{soe}\Sigma_e^{-1}\mathbf{B} - \mathbf{D})\Sigma_{\Delta\Delta}$$
$$\times (\Sigma_{soe}\Sigma_e^{-1}\mathbf{B} - \mathbf{D})^t \tag{14.72}$$

14.7.2. Example on Least Squares Collocation It was pointed out previously that the removal of the trend in the data is an important prerequisite to the application of interpolation and filtering. The operation of trend removal has usually been performed prior to the process of interpolation or filtering. Least squares collocation as presented here allows the determination of the trend surface, simultaneous with filtering. The trend surface therefore becomes the functional part of the model.

This example considers a one-dimensional filtering problem with the trend surface limited to a straight line. It is taken from the report by Moritz (1972, see Bibliography) in which Dr. K. P. Schwarz is acknowledged as having performed the computations. Figure 14.17 is a schematic of the problem.
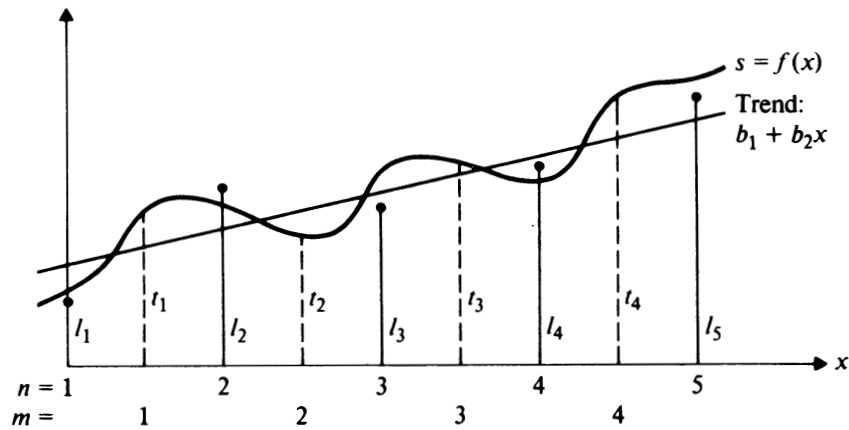
Figure 14.17

The data, or observations, are given at $n = 5$ points in Table 14.2. Assuming stationarity and ergodicity, a covariance function may be written in terms of distances between points, d. The expression

$$C_l(d) = C_l(0)e^{-a^2 d^2} \tag{14.73}$$

is used for such a covariance function with the constants

$$C_l(0) = 0.1260 \qquad a = 0.6$$

TABLE 14-2

x	l
0.000	0.6108
1.445	1.0863
2.890	2.9034
4.335	4.5925
5.780	6.2714

Taking the covariance function for the uncorrelated random component r to be a constant equal to 0.01, then

$$C_r(0) = 0.01 \qquad C_r(d) = 0 \qquad \text{for all } d \neq 0$$

Consequently, the a priori covariance matrices are

$$\Sigma_{ll} = \begin{bmatrix} 0.1360 & 0.0594 & 0.0062 & 0.0002 & 0.0000 \\ & 0.1360 & 0.0594 & 0.0062 & 0.0002 \\ & & 0.1360 & 0.0594 & 0.0062 \\ & & & 0.1360 & 0.0594 \\ \text{symmetric} & & & & 0.1360 \end{bmatrix}$$

$$\Sigma_{ss} = \begin{bmatrix} 0.1260 & 0.0594 & 0.0062 & 0.0002 \\ & 0.1260 & 0.0594 & 0.0062 \\ & & 0.1260 & 0.0594 \\ \text{symmetric} & & & 0.1260 \end{bmatrix}$$

The reader should note that the diagonal elements of Σ_{ll} are equal and each is the sum of one diagonal element of Σ_{ss} plus 0.01. This satisfies the fact that $C_l(0) = C_s(0) + C_r(0)$ [see equation (14.12c)],

$$\Sigma_{sol} = \begin{bmatrix} 0.1044 & 0.1044 & 0.0232 & 0.0012 & 0.0000 \\ 0.0232 & 0.1044 & 0.1044 & 0.0232 & 0.0012 \\ 0.0012 & 0.0232 & 0.1044 & 0.1044 & 0.0232 \\ 0.0000 & 0.0012 & 0.0232 & 0.1044 & 0.1044 \end{bmatrix}$$

Before presenting the results, the appropriate equations are first given. The equation for the trend surface is

$$y = b_1 + b_2 x$$

Thus with $\Delta = [b_1 \quad b_2]^t$, the functional model is expressed by the equation [see equation (14.48)] for each data point

$$s_i + r_i = l_i - [1 \quad x_i]\Delta = l_i - \mathbf{B}_i \Delta$$

with $\mathbf{B}_i = [1 \quad x_i]$. The total coefficient matrix $\mathbf{B}_{5,2}$ is

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^t$$

and the estimate of the parameters is [see equation (14.57)]

$$\Delta = (\mathbf{B}^t\mathbf{\Sigma}_{ll}^{-1}\mathbf{B})(\mathbf{B}^t\mathbf{\Sigma}_{ll}^{-1}l) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.3252 \\ 0.9891 \end{bmatrix}$$

Note that since the model is that of indirect observations, $\mathbf{W}_e$ becomes $\mathbf{\Sigma}_{ll}^{-1}$. Next, the filtered values are obtained at the $m = 4$ points shown in Figure 14.17, from [see equation (14.59)]

$$\hat{\mathbf{s}}_0 = \mathbf{\Sigma}_{s_0 l}\mathbf{\Sigma}_{ll}^{-1}(l - \mathbf{B}\Delta) = [-0.221 \quad -0.5590 \quad -0.1052 \quad 0.1082]^t$$

The points at which filtering is performed are taken as halfway between data points. Their coordinates are needed to compute the values of the total signal at them. For example $\hat{t}_1$ at the point $x_1 = 0.7225$ is computed as [see equation (14.61)]

$$\hat{t}_1 = 0.2221 + [1 \quad 0.7225]\Delta = 0.8177$$

Collecting all values for computing the total signal we get the data in Table 14-3.

TABLE 14-3

x	s_0	t
0.7225	-0.2221	0.8177
2.1675	-0.5590	1.9101
3.6125	-0.1052	3.7932
5.0575	0.1082	5.4359

Finally, the a posteriori covariance matrices are

$$\mathbf{\Sigma}_{\Delta\Delta} = (\mathbf{B}^t\mathbf{\Sigma}_{ll}^{-1}\mathbf{B})^{-1} = \begin{bmatrix} 0.1136 & -0.0234 \\ -0.0234 & 0.0081 \end{bmatrix}$$

$$\mathbf{\Sigma}_{s_0 s_0} = \begin{bmatrix} 0.1034 & 0.0532 & 0.0343 & 0.0025 \\ & 0.0564 & 0.0369 & 0.0343 \\ & & 0.0564 & 0.0532 \\ \text{symmetric} & & & 0.1034 \end{bmatrix}$$

$$\mathbf{\Sigma}_{ii} = \begin{bmatrix} 0.0131 & -0.0024 & 0.0015 & -0.0009 \\ & 0.0121 & -0.0019 & 0.0015 \\ & & 0.0121 & -0.0024 \\ \text{symmetric} & & & 0.0131 \end{bmatrix}$$

The value of $\mathbf{D}$ used to compute Σ_{ii} is

$$\mathbf{D}^t = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.7225 & 2.1675 & 3.6125 & 5.0575 \end{bmatrix}$$

$$\Sigma_{\Delta s_0} = \begin{bmatrix} 0.1007 & 0.0578 & 0.0298 & -0.0080 \\ -0.0188 & -0.0048 & 0.0048 & 0.0188 \end{bmatrix}$$

Problems for Part III

1. Using the unified approach rework Problem 16 of Part II taking the a priori coordinates of point B as $x_1 = 500.00$ m and $x_2 = 866.00$ m with a priori covariance matrix $\Sigma_{xx} = 25I_2 (m^2)$.

2. Using the unified approach rework Problem 10 of Part II taking the a priori elevations as follows:

Station	1	2	3	4
Elevation (m)	542.0	529.5	487.2	495.7

Assume that the cofactor for each of these a priori elevations is 20 times the largest cofactor for the given observed differences in elevation.

3. Rework Problem 20, Part II, using a priori value for x as 60 degrees with a priori weight $w_{xx} = 10$ times the weight of the observations.

4. Refer to Problem 39, Part II. Let $x_1 = 10$, $x_2 = 4$, $y_2 = 2$ be a priori values for the parameters, with a priori $\mathbf{W}_{xx} = 100I_3$. Using the unified approach compute the estimates of x_1, x_2, y_2 and their a posteriori cofactor matrix.

5. For the traverse in Problem 36, Part II, the coordinates of the control points B and E are assumed to have a priori covariance matrix $\Sigma = 4 \times 10^{-6} I_4$ m^2. In order

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to apply the unified approach totally to the problem, a priori values for the coordinates of points C and D are needed. Compute these using a minimum subset of the given observations, and take the estimate of their a priori covariance matrix as $0.04I_4$ m^2.

6. The figure shows the geometry of lines and points in a plane. The observations are $l_1 = 7.07$ m; $l_2 = 7.08$ m; $l_3 = 10.00$ m; $l_4 = 10.02$ m; with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.01$ m; $l_5 = 30°$; $\sigma_5 = 0.01°$. Points A and B are control points with *perfectly known*
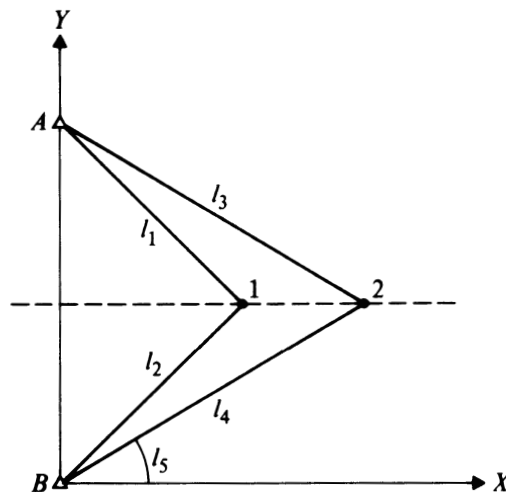
coordinates $X_A = 0$, $Y_A = 10$ m, $X_B = Y_B = 0$. For *all the parts to follow* we are interested in the least squares estimates of the coordinates of points 1 and 2.

a. Write the elements of the *functional* and *stochastic* models. (Give the equations, but do not linearize if nonlinear. Instead write symbolically the linearized form and give the dimensions of the matrices).

b. If, in addition to the information above, points 1 and 2 are on a line parallel to the X axis, give the elements of the *functional* model (again no linearization—only matrices' dimensions).

c. If, instead of assuming the coordinates of points A and B to be perfectly known, the given coordinates (that is, $X_A = X_B = Y_B = 0$ and $Y_B = 10.0$ m) *each* has a standard deviation of 0.001 m, give the elements of the *functional* and *stochastic* models. (No linearization—only matrices' dimensions).

d. In addition to the information in (b) and (c) a priori coordinates for points 1 and 2 are $X_1 = Y_1 = 5.0$ m, $X_2 = 8.7$ m, $Y_2 = 5.0$ m, *each* with a standard deviation of 0.2 m. Taking the approximations for all the observations equal to their a priori values, and the approximations for the parameters to be $X_1^0 = Y_1^0 = 5.0$ m, $X_2^0 = 8.0$ m, $Y_2^0 = 5.0$ m, give the *linearized* form of the functional model, as well as the stochastic model. Show how you would construct the reduced normal equations, giving the numerical values of the contributions from the a priori information on the parameters X_1, Y_1, X_2, Y_2.

7. For Problem 30, Part II, use enough points to determine uniquely the parameters, then add the contribution of the remaining points sequentially.

8. For Problem 7 above use the parameter values from a unique solution as a priori estimates and assume a suitable a priori weight matrix for these estimates. Add

the contribution of all five points sequentially using two points, then two points, then one point.

9. For Problem 38, Part II, find the least squares solution for the first two points, then add the contribution of the third point sequentially.

10. Rework Problem 31, Part II, beginning with a minimum solution and adding the remaining data sequentially.

11. Using the a priori data in Problem 3 above, add *sequentially* the conditions arising from the given observational data to get the least squares estimate of the angle *x*.

12. The figure shows a schematic of a simulated level net involving a bench mark (elevation 100.000 m) and five points 1, 2, 3, 4, 5. There is a total of 13 observed
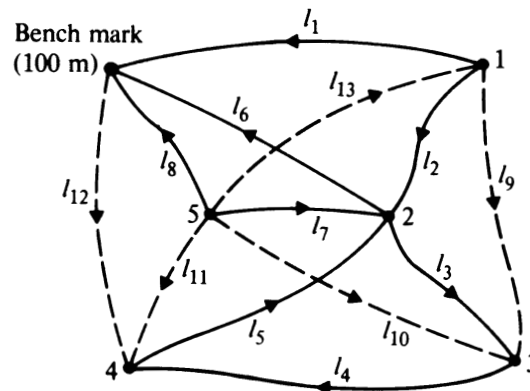
differences in elevation, the values of which as well as corresponding lengths of the lines are given in the following table:

LINE	l_1	l_2	l_3	l_4	l_5	l_6	
Observations (m)	5.013	2.985	6.987	10.014	17.013	1.982	
Length (km)	1.4	1.03	1.2	1.71	1.42	1.27	

LINE	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}
Observations (m)	7.982	10.014	10.014	15.016	25.017	14.982	4.981
Length (km)	0.6	0.95	1.82	1.66	1.14	2.0	1.42

The variance of each observation in square metres is taken proportional to the corresponding line length in kilometres. Use the first eight observations, shown in solid lines in the figure, to perform a batch least squares for estimating the elevations of the five points. Then sequentially add one observation at a time, l_9 through l_{13} (dotted lines in the figure), and compute new estimates for the five elevations as well

as the reference variance after each observation. Use an F test to determine at which observation, if any, is the change in the estimate of the reference variance, $\hat{\sigma}_0^2$, significant at $\alpha = 0.05$.

13. With respect to the figure the coordinates are as follows:

POINT	X	Y
A	1	1
1	2	1.2
2	3	1.3
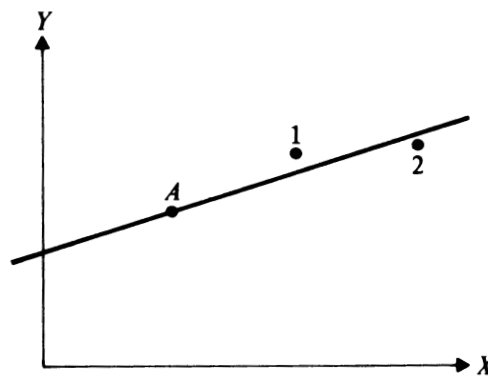
A straight line is to pass through point A *exactly*. Points 1 and 2, whose Y coordinates are taken as the observations, are to be used to determine the straight line. Get a solution from point 1 first, then sequentially add the contribution of the data from point 2. Assume $\mathbf{Q} = \mathbf{I}$.

APPENDIX

A

Review of Matrix Theory

A1. DEFINITIONS

A *matrix* is an array of numbers of some algebraic system, which in general is written in the geometric form of a rectangle of specified number of rows and columns. Thus

$$
\underset{m,\,n}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
$$

is a rectangular matrix of order $m \times n$. The first subscript always refers to the number of the row and the second refers to the column number. The capital boldface letter refers to the whole matrix, whereas the lowercase letter refers to one element only. It follows then that a_{ij} lies at the intersection of the ith row and the jth column. Another way of expressing the matrix $\mathbf{A}$ is by typical element such as

$$
\mathbf{A} = \{a_{ij}\} i = 1, 2, \ldots, m; \qquad j = 1, 2, \ldots, n
$$

If the matrix is of the order 1×1, it is called a "scalar."

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A2. TYPES OF MATRICES

A2.1. Square Matrix   A square matrix is the matrix with an equal number of rows and columns. In this case $A_{m, m}$ is a square matrix of order j. The principal (or main, or leading) diagonal is that composed of the elements a_{ij} for $i = j$. Two special cases of the square matrix are the *symmetric* and *skew-symmetric* ones. These will be introduced later.

A2.2. Row Matrix or Row Vector   A row matrix or row vector is the matrix composed of only one row. It is designated by a lowercase boldface roman letter, for example,

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \qquad \text{and} \qquad \underset{1,\,3}{\mathbf{a}} = [1 \quad 2 \quad 4]$$

A2.3. Column Matrix, or Column Vector   A column matrix or column vector is composed of only one column, and is denoted by

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \qquad \text{for example,} \quad \underset{2,\,1}{\mathbf{b}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

A2.4. Diagonal Matrix   A diagonal matrix is the square matrix with all elements off the main diagonal being zero:

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & d_{mm} \end{bmatrix}$$

where $d_{ij} = 0$ for all $i \neq j$, and $d_{ij} \neq 0$ for some or all $i = j$.

Example

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

We will also denote a diagonal matrix by the form

$$\mathbf{B} = \text{diag.} \{b_1, b_2, b_3, \ldots, b_n\}.$$

A2.5. Scalar Matrix A scalar matrix is a diagonal matrix whose elements are *all* equal to the same scalar:

$$\mathbf{A} = \begin{bmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{bmatrix} \qquad \begin{array}{ll} a_{ij} = 0 & \text{for } all \ i \neq j \\ a_{ij} = a & \text{for } all \ i = j \end{array}$$

Thus

$$\mathbf{H} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is a scalar matrix.

A2.6. Unit or Identity Matrix A unit or identity matrix is a diagonal matrix whose elements are all equal to 1. It is always referred to by

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \begin{array}{ll} a_{ij} = 0 & \text{for } all \ i \neq j \\ a_{ij} = 1 & \text{for } all \ i = j \end{array}$$

A2.7. Null or Zero Matrix A null or zero matrix is a matrix whose elements are all zero. It is denoted by a boldface zero, **0**.

A2.8. Triangular Matrix A triangular matrix is a square matrix whose elements above (or below), but not including, the main diagonal are all zero:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix} \begin{array}{l} = \text{upper triangular} \\ \qquad a_{ij} = 0, \ i > j \end{array}$$

Example

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{array}{l} = \text{lower triangular} \\ \qquad a_{ij} = 0, \ i < j \end{array}$$

$$\mathbf{B} = \begin{bmatrix} 8 & 0 \\ 2 & -1 \end{bmatrix}$$

A3. MATRIX OPERATIONS

A3.1. Equality   Two matrices **A** and **B** of the *same order* are *equal* if $a_{ij} = b_{ij}$ for all i and j. Matrices of different orders cannot be equated. The laws of equality that hold are

$$\mathbf{A} = \mathbf{A} \quad \text{for all } \mathbf{A} \quad \text{(reflexive law)} \tag{A1}$$

$$\text{If } \mathbf{A} = \mathbf{B}, \text{ then } \mathbf{B} = \mathbf{A} \quad \text{for all } \mathbf{A} \text{ and } \mathbf{B} \quad \text{(symmetric law)} \tag{A2}$$

$$\text{If } \mathbf{A} = \mathbf{B}, \mathbf{B} = \mathbf{C} \quad \text{then } \mathbf{A} = \mathbf{C} \quad \text{for all } \mathbf{A}, \mathbf{B}, \mathbf{C} \tag{A3}$$

A3.2. Sums   The *sum* of two matrices **A** and **B**, of the *same order*, is a matrix **C** of that order whose elements are $c_{ij} = a_{ij} + b_{ij}$ for all i and j. Matrices of different orders cannot be added. The laws of addition that hold are

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(commutative law)} \tag{A4}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + \mathbf{B} + \mathbf{C}$$
$$\text{(associative law)} \tag{A5}$$

With the null matrix **0**, we have

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A} \tag{A6}$$

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0} \tag{A7}$$

where $(-\mathbf{A})$ is the matrix composed of $(-a_{ij})$ as elements.

A3.3. Scalar Multiplication   Scalar multiplication of a matrix **A** by a scalar α is another matrix **B** whose elements are $b_{ij} = \alpha a_{ij}$, for all i and j. (We will in general denote a scalar by a lowercase Greek letter.) Thus

$$\mathbf{B} = \alpha \mathbf{A} \quad \text{and} \quad \{b_{ij}\} = \{\alpha a_{ij}\}$$

Example. For

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$\mathbf{B} = 2\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 6 & -8 \end{bmatrix}$$

The following laws hold true:

$$\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B} \tag{A8}$$

$$(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A} \tag{A9}$$

$$\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B}) \tag{A10}$$

$$\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A} \tag{A11}$$

A3.4. Matrix Multiplication The product of two matrices **A** and **B**, *in that order*, whose dimensions are $m \times k$ and $k \times n$, respectively, is another matrix **C** of order $m \times n$ and elements,

$$c_{ij} = \sum_{r=1}^{r=k} a_{ir} b_{rj} \qquad \text{for all } i \text{ and } j \tag{A12}$$

or

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj}$$

This process is shown schematically in the following relation:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & & & & b_{2n} \\ & \cdots & b_{2j} & \cdots & \\ \vdots & & \vdots & & \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & & & & \vdots \\ c_{i1} & \cdots & c_{ij} = \sum_{r=1}^{r=k} a_{ir} b_{rj} & \cdots & c_i \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

It is important to note that for matrix multiplication to be defined the *number of columns of the first matrix must be equal to the number of rows of the second matrix.*

Example

$$\mathop{\mathbf{A}}_{3,2} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathop{\mathbf{B}}_{2,2} = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\mathop{\mathbf{C}}_{3,2} = \mathop{\mathbf{A}}_{3,2} \mathop{\mathbf{B}}_{2,2} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} [1(5) + 2(2)] & [1(1) + 2(3)] \\ [3(5) + 0(2)] & [3(1) + 0(3)] \\ [1(5) + 1(2)] & [1(1) + 1(3)] \end{bmatrix} = \begin{bmatrix} 9 & 7 \\ 15 & 3 \\ 7 & 4 \end{bmatrix}$$

The following relationships regarding matrix multiplication hold:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \qquad \text{with } \mathbf{I} = \text{identity matrix} \tag{A13}$$

$$\mathbf{A(BC)} = \mathbf{(AB)C} = \mathbf{ABC} \qquad \text{(associative law)} \tag{A14}$$

$$\mathbf{A(B + C)} = \mathbf{AB} + \mathbf{AC} \qquad \text{(distributive law)} \tag{A15}$$

$$\mathbf{(A + B)C} = \mathbf{AC} + \mathbf{BC} \qquad \text{(distributive law)} \tag{A16}$$

In all these relations the sequence of the matrices is strictly preserved. Note that in general the commutative law does not hold even if multiplication is defined in both orders, i.e.,

$$\mathbf{AB} \neq \mathbf{BA} \qquad \text{in general} \tag{A17}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix} \qquad \text{and} \qquad \mathbf{BA} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

Of course if orders are different (that is, component matrices are not square) the equality is not even defined, for example,

$$\underset{m,\,k}{\mathbf{A}} \ \underset{k,\,m}{\mathbf{B}} = \underset{m,\,m}{\mathbf{C}}$$

and

$$\underset{k,\,m}{\mathbf{B}} \ \underset{m,\,k}{\mathbf{A}} = \underset{k,\,k}{\mathbf{D}}$$

It is important to note that the product of two matrices can be the null matrix without either matrix being the zero matrix.

Examples

$$1. \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2. \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 2 & -3 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$3. \quad \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

It is useful to note some properties of diagonal matrices as regards multiplication. If $\mathbf{A}$ is a square matrix and $\mathbf{D}$ is a diagonal matrix, then

1. **DA** causes each *row* $\mathbf{A}_i$ of $\mathbf{A}$ to be multiplied by the corresponding element d_{ii} of $\mathbf{D}$.
2. **AD** causes each *column* $\mathbf{A}_j$ of $\mathbf{A}$ to be multiplied by the corresponding element d_{jj} of $\mathbf{D}$.

Example. For

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$\mathbf{DA} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \beta a_{21} & \beta a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{AD} = \begin{bmatrix} \alpha a_{11} & \beta a_{12} \\ \alpha a_{21} & \beta a_{22} \end{bmatrix}$$

The concept of the *fractional power of a diagonal matrix* applies to diagonal matrices with nonnegative elements. Thus if $d_{ii} \geq 0$ and given a scalar $\gamma > 0$, then the matrix $\mathbf{D}^\gamma$ is

$$\mathbf{D}^\gamma_{n,n} = \begin{bmatrix} d^\gamma_{11} & 0 & \cdots & 0 \\ 0 & d^\gamma_{22} & \cdots & 0 \\ 0 & 0 & \cdots & d^\gamma_{nn} \end{bmatrix}$$

Obviously, if $\alpha > 0$ and $\beta > 0$, then

$$\mathbf{D}^\alpha \cdot \mathbf{D}^\beta = D^{(\alpha + \beta)}$$

and in particular

$$\mathbf{D}^{1/2}\mathbf{D}^{1/2} = \mathbf{D}$$

A3.5. Matrix Transpose   The transpose of a matrix $\mathbf{A}$ of order $m \times n$ is an $n \times m$ matrix formed from $\mathbf{A}$ by interchanging rows and columns such that row i of $\mathbf{A}$ becomes column i of the transposed matrix. We denote the transpose by $\mathbf{A}^t$. If $\mathbf{B} = \mathbf{A}^t$, it follows that $b_{ij} = a_{ji}$ for all i and j.

Examples

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -6 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^t = \begin{bmatrix} 1 & -6 \\ 3 & 5 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -2 & 7 \\ 0 & 8 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^t = \begin{bmatrix} -2 & 0 & 1 \\ 7 & 8 & 4 \end{bmatrix}$$

The following relationships hold true

$$(\mathbf{A} + \mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t \tag{A18}$$

$$(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t \tag{A19}$$

$$(\alpha\mathbf{A})^t = \alpha\mathbf{A}^t \tag{A20}$$

$$(\mathbf{A}^t)^t = \mathbf{A} \tag{A21}$$

If $\mathbf{D}$ is a diagonal matrix, then $\mathbf{D}^t = \mathbf{D}$. Similarly, for the scalar matrix $\mathbf{H}$ and the identity matrix $\mathbf{I}$ we have $\mathbf{H}^t = \mathbf{H}$ and $\mathbf{I}^t = \mathbf{I}$.

If x is a column vector, then $x^t x$ is a nonnegative scalar that is equal to the sum of squares of the vector's components. On the other hand, xx^t is a *square symmetric* matrix of the same order as the vector x.

A3.6. Symmetry and Skew Symmetry   A matrix is symmetric if $A^t = A$ and it is skew symmetric if $A^t = -A$. These matrices must always be square. Thus

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

is symmetric and

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

is skew symmetric.

There are at most $(n^2 + n)/2$ distinct elements in a symmetric matrix of order n, although a skew-symmetric matrix of the same order would have at most $(n^2 - n)/2$ non-zero elements. The only matrix that is both symmetric and skew symmetric is the null matrix 0. Every square matrix can be uniquely decomposed into the sum of a symmetric and a skew symmetric matrix:

$$A = A + \tfrac{1}{2}A^t - \tfrac{1}{2}A^t = \tfrac{1}{2}(A + A^t) + \tfrac{1}{2}(A - A^t) \qquad (A22)$$

The matrix

$$A_s = \tfrac{1}{2}(A + A^t)$$

is symmetric because

$$A_s^t = \tfrac{1}{2}(A + A^t)^t = \tfrac{1}{2}(A + A^t) = A_s$$

and the matrix

$$A_a = \tfrac{1}{2}(A - A^t)$$

is skew symmetric since

$$A_a^t = \tfrac{1}{2}(A - A^t)^t = -\tfrac{1}{2}(A - A^t) = -A_a$$

Thus we have expressed the matrix A as the sum of the symmetric matrix A_s and the skew symmetric matrix A_a.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A^t = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$B = \tfrac{1}{2}(A + A^t) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{symmetric}$$

$$C = \tfrac{1}{2}(A - A^t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{skew symmetric}$$

and

$$\mathbf{B} + \mathbf{C} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \mathbf{A}$$

For *any matrix* $\mathbf{A}$ and any *symmetric matrix* $\mathbf{B}$, the matrices $\mathbf{AA}^t$, $\mathbf{A}^t\mathbf{A}$, $\mathbf{ABA}^t$, and $\mathbf{A}^t\mathbf{BA}$ are all symmetric. The same applies for skew symmetry if $\mathbf{B}$ were skew symmetric. Note that the sum and difference of symmetric (or skew symmetric) matrices are also symmetric (or skew symmetric). On the other hand the product of symmetric matrices is symmetric only if the matrices commute.

A3.7. Partitioning of Matrices   We often wish to study a subset of elements from a given matrix $\mathbf{A}$. The matrix formed by that subset of elements is called a *submatrix* of $\mathbf{A}$. For example, the matrix

$$\begin{bmatrix} a_{11} & a_{13} & a_{15} \\ a_{31} & a_{33} & a_{35} \end{bmatrix}$$

is a 2×3 submatrix obtained by deleting the second row and second and fourth columns of the 3×5 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

The scheme called *partitioning* allows for the writing of a matrix not in terms of its original elements but in terms of submatrices. The submatrices will be denoted by boldface uppercase (or lowercase in the event that they are vectors) letters with appropriate subscripts. Thus the matrix

$$\mathbf{A} = \left[\begin{array}{ccc:cccc} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & a_{mn} \end{array} \right]$$

may be partitioned several ways. Considering the vertical dotted line only, we write

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix}$$

where $\mathbf{A}_1$ is an $m \times 3$ submatrix and $\mathbf{A}_2$ an $m \times (n - 3)$ submatrix. Similarly, considering the horizontal dotted line only,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$$

where in this case A_1 is a $2 \times n$ submatrix and A_2 an $(m - 2) \times n$ submatrix. Considering both horizontal and vertical partition lines

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is 2×3, A_{12} is $2 \times (n - 3)$, A_{21} is $(m - 2) \times 3$, and A_{22} is $(m - 2) \times (n - 3)$.

All matrix operations given in the preceding sections can be performed on the submatrices as if they are elements, provided necessary precautions are exercised regarding dimensions. Thus, for example,

$$\mathop{\mathbf{B}}_{n,\,m} = \mathbf{A}^t = \begin{bmatrix} \mathop{A^t_{11}}\limits_{3,\,2} & \mathop{A^t_{21}}\limits_{3,\,(m-2)} \\ \mathop{A^t_{12}}\limits_{(n-3),\,2} & \mathop{A^t_{22}}\limits_{(n-3),\,(m-2)} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Example. Let

$$\mathop{\mathbf{A}}_{3,\,4} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ \hline 2 & 1 & 3 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathop{\mathbf{B}}_{4,\,2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ \hline 2 & 1 \\ 1 & 3 \end{bmatrix}$$

then

$$\mathop{\mathbf{A}}_{3,\,4} \mathop{\mathbf{B}}_{4,\,2} = \mathop{\mathbf{C}}_{3,\,2}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

then

$$AB = \begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21}) \\ (A_{21}B_{11} + A_{22}B_{21}) \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$$

where

$$\mathop{A_{11}}_{2,\,2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \qquad \mathop{A_{12}}_{2,\,2} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}, \qquad \mathop{A_{21}}_{1,\,2} = [2 \quad 1], \qquad \mathop{A_{22}}_{1,\,2} = [3 \quad 4]$$

$$\mathop{B_{11}}_{2,\,2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathop{B_{21}}_{2,\,2} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Thus

$$A_{11}B_{11} = \begin{bmatrix} 7 & 10 \\ 13 & 20 \end{bmatrix}, \qquad A_{12}B_{21} = \begin{bmatrix} 10 & 15 \\ 5 & 5 \end{bmatrix},$$

$$A_{21}B_{11} = [5 \quad 8], \qquad A_{22}B_{21} = [10 \quad 15]$$

leading to

$$
AB = \begin{bmatrix} \begin{bmatrix} 7 & 10 \\ 13 & 20 \end{bmatrix} + \begin{bmatrix} 10 & 15 \\ 5 & 5 \end{bmatrix} \\ [\,5 \quad 8] + [10 \quad 15] \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 17 & 25 \\ 18 & 25 \end{bmatrix} \\ [15 \quad 23] \end{bmatrix} = \begin{bmatrix} 17 & 25 \\ 18 & 25 \\ 15 & 23 \end{bmatrix}
$$

which is the same answer as that obtained directly from

$$
AB = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 25 \\ 18 & 25 \\ 15 & 23 \end{bmatrix}
$$

A3.8. Matrix Inverse† Unlike scalars, *division* of matrices is *not defined*. In fact, we may have $AB = AC$ without having $B = C$. This implies that the operation of "dividing" by A, even if $A \neq 0$, is not possible. As an example let

$$
A = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
$$

where obviously $B \neq C$, but computing we get

$$
AB = \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} = AC
$$

In place of division, we introduce the concept of *inverse*, A^{-1}, of a matrix A (similar to the reciprocal α^{-1} of the scalar α).

The *inverse* of a *square* matrix A, *if it exists*, is the unique matrix A^{-1} with the property

$$
AA^{-1} = A^{-1}A = I \tag{A23}
$$

The following rules hold:

$$
(AB)^{-1} = B^{-1}A^{-1} \tag{A24}
$$

$$
(A^{-1})^{-1} = A \tag{A25}
$$

$$
(A^t)^{-1} = (A^{-1})^t \tag{A26}
$$

$$
(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1} \tag{A27}
$$

A square matrix that has an inverse is called *nonsingular*; one that does not have an inverse is called *singular*.

† Consideration in this Appendix is limited to Cayley matrix algebra where an inverse is only defined for a square matrix. Although a square matrix may not have an inverse in this algebra, it has a generalized inverse. However, generalized matrix algebra is not utilized in this book.

We showed in Section A3.4 that $\mathbf{AB} = \mathbf{0}$ did not necessarily imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. If, however, either $\mathbf{A}$ or $\mathbf{B}$ is nonsingular then the other matrix must be a null matrix. Hence, the product of two nonsingular matrices cannot be a null matrix.

To appreciate a method of inverting matrices we need to introduce the concept of determinants of square matrices. We associate with each square matrix $\mathbf{A}$ a uniquely defined scalar that is called the *determinant* of $\mathbf{A}$ and is denoted by either "det $\mathbf{A}$" or $|\mathbf{A}|$. The determinant of order n (for an $n \times n$ matrix) will be defined in terms of determinants of order $(n - 1)$. In order to apply this procedure we need to define the determinant of the least-order matrix, namely, that of a 1×1 matrix. For a matrix consisting of a single element,

$$\mathbf{A}_{1,1} = [a_{11}]$$

the determinant is defined as

$$|\mathbf{A}| = \det \mathbf{A} = a_{11}$$

An $(n - 1) \times (n - 1)$ determinant obtained from the $n \times n$ matrix is called a *minor*. The minor m_{ij} of an element a_{ij} of a square matrix $\mathbf{A}$ is the determinant of the square submatrix formed from $\mathbf{A}$ by deleting the ith row and jth column.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad m_{12} \text{ (for } a_{12}) = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

The *cofactor* c_{ij} of an element a_{ij} in a matrix $\mathbf{A}$ is given by

$$c_{ij} = (-1)^{i+j} m_{ij}$$

where m_{ij} is the minor of the element a_{ij}. We may now define a determinant as

$$|\mathbf{A}| = \det \mathbf{A} = \sum_{j=1}^{n} a_{1j} c_{1j}$$

or that $|\mathbf{A}|$ is the sum of the products of the elements of the first row times their respective cofactors. For the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the cofactors for the elements a_{11} and a_{12} are

$$c_{11} = |a_{22}| = a_{22} \qquad \text{and} \qquad c_{12} = -|a_{21}| = -a_{21}$$

Thus

$$|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$$

Similarly, for

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

There are several properties of determinants that make their evaluation relatively less complex.

1. The determinant of a triangular matrix is the product of its diagonal elements (same for a diagonal matrix).
2. $|A_1 \cdot A_2 \cdot \cdots \cdot A_k| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|$
3. The determinant of a matrix with a zero vector is zero.
4. The determinant of a matrix with two rows or two columns that are identical is zero.
5. The determinant of a matrix with one row (or column) a multiple of another row (or column) is zero.
6. $|A| = |A^t|$
7. Interchanging two rows or two columns changes the sign of the determinant.
8. If a row or a column is multiplied by a constant, the value of the determinant is multiplied by that constant.
9. The value of a determinant does not change if a multiple of a row (or column) is added to another row (or column).
10. If a matrix A can be partitioned to form a triangular matrix with square matrices $A_{11}, A_{22}, \ldots, A_{kk}$ on main diagonal, then

$$|A| = |A_{11}| \cdot |A_{22}| \cdot |A_{33}| \cdot \cdots \cdot |A_{kk}|$$

Having introduced the concept and some properties of determinants, we may now give some of the more direct methods of evaluating an inverse of a matrix.

1. The *cofactor matrix* C of a matrix A is the square matrix of the same order as A and in which each element a_{ij} is replaced by its cofactor c_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

and the cofactor matrix

$$
C = \begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1m} \\
c_{21} & c_{22} & \cdots & c_{2m} \\
\vdots & & & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mm}
\end{bmatrix}
$$

2. The *adjoint* matrix of a matrix A is the transpose of its cofactor matrix, that is, C^t, and is denoted by adj A.

$$
\text{adj } A = C^t = \begin{bmatrix}
c_{11} & c_{21} & \cdots & c_{m1} \\
c_{12} & c_{22} & \cdots & c_{m2} \\
\vdots & \vdots & & \vdots \\
c_{1m} & c_{2m} & \cdots & c_{mm}
\end{bmatrix}
$$

It can be shown that

$$
A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I
$$

Example. Let

$$
A = \begin{bmatrix}
2 & 1 & 0 \\
1 & 3 & 2 \\
1 & 2 & 1
\end{bmatrix}
$$

$$
|A| = 2(-1) - 1(-1) + 0(-1) = -1
$$

The elements of the cofactor matrix are

$$
\begin{array}{lll}
c_{11} = +(-1) & c_{12} = -(-1) & c_{13} = +(-1) \\
c_{21} = -(1) & c_{22} = +(2) & c_{23} = -(3) \\
c_{31} = +(2) & c_{32} = -(4) & c_{33} = +(5)
\end{array}
$$

Thus

$$
C = \begin{bmatrix}
-1 & 1 & -1 \\
-1 & 2 & -3 \\
2 & -4 & 5
\end{bmatrix}
$$

and

$$
\text{adj } A = C^t = \begin{bmatrix}
-1 & -1 & 2 \\
1 & 2 & -4 \\
-1 & -3 & 5
\end{bmatrix}
$$

$$
A(\text{adj } A) = \begin{bmatrix}
2 & 1 & 0 \\
1 & 3 & 2 \\
1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & -1 & 2 \\
1 & 2 & -4 \\
-1 & -3 & 5
\end{bmatrix}
=
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
$$

$$
= (-1)I = |A| \cdot I
$$

The reader can ascertain that (adj A) A is also equal $|A| \cdot I$. From the preceding relation between A, adj A, and $|A|$, and the basic definition of a matrix inverse given in equation (A23), it follows readily that the inverse of a square matrix A is computed from

$$A^{-1} = \frac{\text{adj } A}{|A|} \tag{A28}$$

It is apparent from equation (A28) that $|A|$ *must not vanish* in order that A^{-1} may exist.

Example. Given the simultaneous pair of linear equations

$$2X_1 + 3X_2 = 8$$

$$3X_1 - X_2 = 1$$

find a unique solution.

First the two equations are written in matrix form as

$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix} \quad \text{or} \quad Ax = b$$

Evaluating the cofactors,

$$c_{11} = (-1)^2(-1) = -1$$

$$c_{12} = (-1)^3(3) = -3$$

$$c_{21} = (-1)^3(3) = -3$$

$$c_{22} = (-1)^4(2) = 2$$

and

$$|A| = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} = -2 - 9 = -11$$

$$C = \begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix} \quad \text{adj } A = C^t = \begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = -\tfrac{1}{11}\begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{11} & \tfrac{3}{11} \\ \tfrac{3}{11} & -\tfrac{2}{11} \end{bmatrix}$$

From $Ax = b$ we may write

$$A^{-1}Ax = x = A^{-1}b = \begin{bmatrix} \tfrac{1}{11} & \tfrac{3}{11} \\ \tfrac{3}{11} & -\tfrac{2}{11} \end{bmatrix}\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence $X_1 = 1$ and $X_2 = 2$.

A very important matrix which is encountered frequently in practice is the *orthogonal* matrix. A square matrix M is orthogonal if

$$M^t M = M M^t = I \tag{A29}$$

which implies that

$$\mathbf{M}^{-1} = \mathbf{M}^t \tag{A30}$$

The columns of an orthogonal matrix are mutually orthogonal vectors of unit length. Also,

$$|\mathbf{M}| = \pm 1 \tag{A31}$$

When $|\mathbf{M}| = +1$, then $\mathbf{M}$ is called proper orthogonal, otherwise it is termed "improper."

The product of two orthogonal matrices is also an orthogonal matrix.

MATRIX INVERSE BY PARTITIONING   Let $\mathbf{A}$ be an $n \times n$ square nonsingular matrix whose inverse is to be evaluated. We partition $\mathbf{A}$ in the form,

$$\mathbf{A} = \begin{matrix} & s & m \\ & \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} & \begin{matrix} s \\ m \end{matrix} \end{matrix}$$

where $\mathbf{A}_{11}$ is $s \times s$, $\mathbf{A}_{12}$ is $s \times m$, $\mathbf{A}_{21}$ is $m \times s$, $\mathbf{A}_{22}$ is $m \times m$, and $m + s = n$. The inverse $\mathbf{A}^{-1}$ exists and we shall denote it, in the correspondingly partitioned form, by

$$\mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

From the basic definition of an inverse we have $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}\mathbf{B} = \mathbf{I}$, or in the partitioned form

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_s & 0 \\ 0 & \mathbf{I}_m \end{bmatrix}$$

which leads to

$$\mathbf{A}_{11} \mathbf{B}_{11} + \mathbf{A}_{12} \mathbf{B}_{21} = \mathbf{I}_s \tag{A32a}$$

$$\mathbf{A}_{11} \mathbf{B}_{12} + \mathbf{A}_{12} \mathbf{B}_{22} = 0 \tag{A32b}$$

$$\mathbf{A}_{21} \mathbf{B}_{11} + \mathbf{A}_{22} \mathbf{B}_{21} = 0 \tag{A32c}$$

$$\mathbf{A}_{21} \mathbf{B}_{12} + \mathbf{A}_{22} \mathbf{B}_{22} = \mathbf{I}_m \tag{A32d}$$

From equation (A32c) we have

$$\mathbf{B}_{21} = -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \mathbf{B}_{11} \tag{A33}$$

Substituting into equation (A32a), we obtain the value of $\mathbf{B}_{11}$ as

$$\mathbf{B}_{11} = [\mathbf{A}_{11} - \mathbf{A}_{12} \mathbf{A}_{22}^{-1}\mathbf{A}_{21}]^{-1} \tag{A34}$$

From equation (A32d) we have

$$B_{22} = A_{22}^{-1} - A_{22}^{-1}A_{21} \, B_{12} \tag{A35}$$

which, with equation (A32b) and taking equation (A34) into account, gives

$$B_{12} = -B_{11} \, A_{12} \, A_{22}^{-1} \tag{A36}$$

Thus all the submatrices have been evaluated and their formulas may be collected as

$$B_{11} = [A_{11} - A_{12} \, A_{22}^{-1}A_{21}]^{-1} \tag{A37a}$$

$$B_{12} = -B_{11} \, A_{12} \, A_{22}^{-1} \tag{A37b}$$

$$B_{21} = -A_{22}^{-1}A_{21} \, B_{11} \tag{A37c}$$

$$B_{22} = A_{22}^{-1} - A_{22}^{-1}A_{21} \, B_{12} \tag{A37d}$$

If A is originally a symmetric matrix, then $A_{21} = A'_{12}$ and correspondingly $B_{21} = B'_{12}$.

In inverting by partitioning we end up computing directly thé inverse of a matrix A_{22} of a smaller order $(m < n)$ than the original matrix. In fact, if m is taken equal to 1, then A_{22}^{-1} is simply the reciprocal of a scalar. Inversion by partitioning can, of course, be performed in more than one step.

A4. RANK OF A MATRIX

The rank of a matrix is the order of the largest nonzero determinant that can be formed from the elements of the matrix by appropriate deletion of rows or columns (or both). Thus a matrix is said to be of *rank* m if and only if it has *at least one nonsingular submatrix of order* m, but has no nonsingular submatrix of order more than m. A nonsingular matrix of order n has a rank n. A matrix with zero rank has elements that must all be zero.

A4.1 A rectangular matrix has a rank that is less than or equal to the smallest dimension. Therefore if A is an $n \times k$ matrix and $k < n$, then rank $(A) \le k$.

A4.2 The rank of the product of a number of matrices does not exceed the *least* rank of individual multiplicands, or

$$\text{rank} \, (A_1 \, A_2 \, A_3 \, \cdots \, A_k) \le \min \, [\text{rank} \, (A_1), \, \text{rank} \, (A_2), \, \ldots, \, \text{rank} \, (A_k)] \tag{A38}$$

A4.3 Multiplication of a matrix by a nonsingular matrix does not change its rank.

A4.4 If A and B are $m \times k$ and $k \times n$, respectively, and each is of rank k, then AB is of rank k.

Example. Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad r(A) = 2, \qquad r(B) = 2$$

then

$$C = A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad r(C) = 2$$

and

$$D = AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r(D) = 1 < 2$$

An interesting property for the product of two matrices is that if

$$r_c = \operatorname*{rank\,(C)}_{m,\,s}, \qquad r_d = \operatorname*{rank\,(D)}_{s,\,n}, \qquad \text{and} \qquad CD = 0$$

then

$$r_c + r_d \leq s \tag{A39}$$

This relation can be verified by the three examples given in Section A3.4. As an illustration, let us consider the third example, where $CD = 0$, or

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}_{1,\,3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix}_{3,\,3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

The rank of the first matrix is $r_c = 1$, and for the second matrix, $r_d = 2$, leading to $r_c + r_d = 3 = s$.

A4.5 An important fact is that AA' and $A'A$ have the same rank as A. It follows that if A is of order $n \times k$ and $n > k$ and is of (full) rank k, then $A'A$ is nonsingular, whereas AA' is singular with rank k. Furthermore, $A'BA$ is nonsingular if B is also nonsingular.

A4.6 The rank of the sum of two matrices cannot exceed the sum of their ranks.

The determination of the *rank* of a matrix by direct application of the definition is not practical, particularly when we are dealing with other than simple small matrices. Instead, we obtain a much faster answer if we realize

that *primary row or column operations* do not change either the order or the rank of the matrix. These operations are as follows:

1. The interchange of any two rows (or columns).
2. The multiplication of all the elements of any row (or column) by the same nonzero constant.
3. The addition to any row (column) of an arbitrary multiple of any other row (column).

Example. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$$

Subtracting twice the first row from the second and adding the first row to the third yields

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

Subtracting the new second row from the new third row gives

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Primary row operations may be stopped at this point since

$$\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} \neq 0$$

leading to rank = 2. However, the primary operations may be continued until

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

leading directly to the rank being 2.

Two matrices are said to be *equivalent* if one can be transformed to the other by elementary operations. Hence they must be of equal order and rank.

The primary row (or column) operations are also useful in several methods of solving sets of *linear equations*. We shall briefly give here two of the more common procedures.

THE GAUSS METHOD   In the Gauss method, elementary row operations are used to reduce the matrix to an upper triangular form or canonical form. This is classically called the *forward solution*, which is followed by a *back-*

ward solution to determine the values of the unknowns. Let us consider as an example the two equations solved by the method of inverse in Section A3.8.

$$\mathbf{Ax} = \mathbf{b}$$

with

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

First we set the matrices

$$\mathbf{A} \vdots \mathbf{b} \vdots \mathbf{I}$$

side by side

$$\begin{bmatrix} 2 & 3 & \vdots & 8 & \vdots & 1 & 0 \\ 3 & -1 & \vdots & 1 & \vdots & 0 & 1 \end{bmatrix}$$

Augmentation by $\mathbf{I}$ is useful for the following method. Divide the first row by 2 and multiply by -1 $(R_1' = -\frac{1}{2}R_1)$; multiply R_1' by 3 and add to negative of second row $(R_2' = -R_2 + 3R_1')$

$$\begin{bmatrix} 1 & \frac{3}{2} & \vdots & 4 & \vdots & \frac{1}{2} & 0 \\ 0 & \frac{11}{2} & \vdots & 11 & \vdots & \frac{3}{2} & -1 \end{bmatrix}$$

At this point we can readily write from the second row that

$$\tfrac{11}{2}x_2 = 11 \quad \text{or} \quad x_2 = 2$$

and substituting backward

$$x_1 = 4 - \tfrac{3}{2}x_2 = 1$$

We could also continue until the upper triangular matrix (in $\mathbf{A}$) has ones along the diagonal by

$$R_2'' = \tfrac{2}{11}R_2'$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \vdots & 4 & \vdots & \frac{1}{2} & 0 \\ 0 & 1 & \vdots & 2 & \vdots & \frac{3}{11} & -\frac{2}{11} \end{bmatrix}$$

From which $x_2 = 2$ directly and x_1 computed as above.

THE GAUSS-JORDAN METHOD   In the Gauss-Jordan method, the elementary row operations are continued until $\mathbf{A}$ becomes a unit matrix, $\mathbf{b}$ thus becoming the answer vector for $\mathbf{x}$.

$$R_2'' = \tfrac{2}{11}R_2'$$

$$R_1'' = R_1' - \tfrac{3}{2}R_2''$$

$$\begin{bmatrix} 1 & 0 & \vdots & 1 & \vdots & \frac{1}{11} & \frac{3}{11} \\ 0 & 1 & \vdots & 2 & \vdots & \frac{3}{11} & \frac{1}{11} \end{bmatrix}$$

giving $x_1 = 1$ and $x_2 = 2$ directly. It is also quite interesting to note that although **A** reduced to **I**, the augmented identity matrix **I** became the inverse A^{-1}. This is useful when, in addition to solving the equations, the inverse is also needed.

A5. TRACE OF A MATRIX

If **A** is an $n \times n$ *square* matrix, then the sum of its diagonal elements is a scalar called the *trace* of **A** and is denoted by tr (**A**). (Some textbooks on matrices use the German word *Spur* instead of the English translation "trace.")

$$\text{tr }(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn}$$

The definition leads directly to the fact that tr $(\mathbf{A}^t)$ = tr (**A**). If **A** and **B** are both of order m, then

$$\text{tr }(\mathbf{A} + \mathbf{B}) = \text{tr }(\mathbf{A}) + \text{tr }(\mathbf{B}) \tag{A40}$$

Also, if **E** is of order $p \times q$ and **F** of order $q \times p$, then

$$\text{tr }(\mathbf{EF}) = \text{tr }(\mathbf{FE}) \tag{A41}$$

Finally, if **F** is nonsingular, that is, if

$$|\mathbf{F}| \neq 0 \quad \text{and} \quad \mathbf{B} = \mathbf{FAF}^{-1}$$

then

$$\text{tr }(\mathbf{B}) = \text{tr }(\mathbf{A}) \tag{A42}$$

A6. THE EIGENVALUE PROBLEM

For a square matrix **A** of order n, we seek a nonzero vector **x** and a scalar λ such that

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{A43}$$

This is called the "eigenvalue problem." A solution λ_0 and $\mathbf{x}_0$ to this problem is called an *eigenvalue* (proper value, characteristic value) and the corresponding *eigenvector* (proper vector, characteristic vector) of the matrix **A**. An eigenvector, if one exists, can be determined only to a scalar multiplication, for if λ_0, $\mathbf{x}_0$ satisfy equation (A43), then λ_0, $\alpha\mathbf{x}_0$, where α is an arbitrary scalar, will also do.

A6.1. The Characteristic Equation   Equation (A43) can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{A44}$$

which represents a set of homogenous linear equations. For a nontrivial solution to this set the following must be satisfied,

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{A45}$$

Equation (A45) represents a real polynomial equation of degree n:

$$b_n(-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + \cdots + b_0 = 0 \tag{A46}$$

where

$$b_n \quad = 1$$

$$b_{n-1} = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii} = \text{tr } (\mathbf{A}) = \text{trace of } \mathbf{A}$$

$$\vdots$$

$$b_{n-r} = \text{sum of all principal minors of order } r \text{ of } \mathbf{A}$$

$$\vdots$$

$$b_0 \quad = |\mathbf{A}| = \text{determinant of } \mathbf{A} \tag{A47}$$

Equation (A46) is called the *characteristic equation* of $\mathbf{A}$, or the *eigenvalue equation*. The matrix $(\mathbf{A} - \lambda\mathbf{I})$ is called the *characteristic matrix*. There are n roots for equation (A46), counting multiplicity. These are the n eigenvalues of $\mathbf{A}$, $\lambda_1, \lambda_2, \ldots, \lambda_n$. For an eigenvalue λ_i, we solve the set of (homogeneous) linear equations $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{x} = \mathbf{0}$ to determine the components of the corresponding eigenvector $\mathbf{x}_i$. In general, λ_i and $\mathbf{x}_i$ are either real or complex numbers and vectors, respectively.

If the matrix $\mathbf{A}$ is *symmetric*, then

1. The eigenvalues are real.
2. The eigenvectors are all mutually orthogonal, that is,

$$\mathbf{x}_i^t \mathbf{x}_j = \mathbf{x}_j^t \mathbf{x}_i = 0$$

Theorem: For every symmetric matrix $\mathbf{A}$ there exists a *rotation (orthogonal)* matrix $\mathbf{R}$ such that $(\mathbf{R}^t\mathbf{A}\mathbf{R})$ is a *diagonal* matrix. The *columns* of $\mathbf{R}$ are the *normalized* eigenvectors of $\mathbf{A}$ and the elements of the diagonal matrix are the corresponding eigenvalues of $\mathbf{A}$ ($\mathbf{R}$ is not unique). We demonstrate the validity of this fact for the case in which all eigenvalues λ_i are distinct. Here the corresponding eigenvectors

$$\mathbf{x}_i = [x_{i1} \quad x_{i2} \quad \cdots \quad x_{in}]$$

are also distinct.

Since, as we mentioned earlier, these vectors are orthogonal, if they are *normalized* so that $\mathbf{x}_i^t \mathbf{x}_i = 1$, then the matrix $\mathbf{R}$ whose columns are these normalized vectors, is an *orthogonal* matrix. Thus

$$\mathbf{R} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_2] = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} \tag{A48}$$

has the property

$$\mathbf{R}^t = \mathbf{R}^{-1}$$

and

$$\mathbf{AR} = [\mathbf{Ax}_1 \quad \mathbf{Ax}_2 \quad \cdots \quad \mathbf{Ax}_n] = [\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2 \quad \cdots \quad \lambda_n\mathbf{x}_n]$$

$$= \begin{bmatrix} \lambda_1 x_{11} & \cdots & \lambda_n x_{n1} \\ \vdots & & \vdots \\ \lambda_1 x_{1n} & \cdots & \lambda_n x_{nn} \end{bmatrix} = \mathbf{R} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \lambda_n \end{bmatrix} \tag{A49}$$

Premultiplying equation (A49) by $\mathbf{R}^{-1} = \mathbf{R}^t$,

$$\mathbf{R}^{-1}\mathbf{AR} = \mathbf{R}^t\mathbf{AR} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \mathbf{D} = \text{diagonal matrix}$$

$$\tag{A50}$$

Example. Find the eigenvalues and eigenvectors for the matrix,

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

From equation (A47), we have

$$b_3 = 1, \qquad b_2 = 1 + 2 + 1 = 4$$

$$b_1 = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 2 + 2 + 0 = 4$$

$$b_0 = 2 - 2 = 0$$

Thus the characteristic polynomial is

$$-\lambda^3 + 4\lambda^2 - 4\lambda = 0 \qquad \text{or} \qquad \lambda(\lambda^2 - 4\lambda + 4) = 0$$

Hence

$$\lambda_1 = 0, \qquad \lambda_2 = 2, \qquad \lambda_3 = 2$$

are the three eigenvalues. To compute the eigenvectors, we have for $\lambda_1 = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

from which

$$x_2 = 0, \qquad x_1 = -x_3$$

hence $[x_1, \quad 0, \quad -x_1]$ is an eigenvector.

For λ_2 and $\lambda_3 = 2$,

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$x_2 = 0, \qquad x_1 = x_3$$

and $[x_1, \quad 0, \quad x_1]$ is an eigenvector.

Example. Find the eigenvalues and eigenvectors of the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Show that $\mathbf{A}$ is orthogonally similar to a diagonal matrix whose elements are the eigenvalues of $\mathbf{A}$, with the transforming orthogonal matrix composed of the normalized eigenvectors of $\mathbf{A}$ as columns.

The characteristic polynomial is

$$\begin{vmatrix} (1 - \lambda) & 2 \\ 2 & (1 - \lambda) \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

from which

$$\lambda_1 = -1 \qquad \text{and} \qquad \lambda_2 = 3$$

are the eigenvalues.

For $\lambda_1 = -1$, we write

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

or $\mathbf{x}_1 = (1, -1)$ is an eigenvector.

For $\lambda_2 = 3$, we have

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

or $\mathbf{x}_2 = (1, 1)$ is an eigenvector.

These two vectors are orthogonal since

$$\mathbf{x}_1^t \, \mathbf{x}_2 = 1 - 1 = 0$$

The normalized vector $\mathbf{x}_1$ is

$$\mathbf{y}_1 = \left(\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}} \right)$$

and for $\mathbf{x}_2$

$$\mathbf{y}_2 = \left(\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}} \right).$$

Then the orthogonal matrix is

$$R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

for which

$$R^t AR = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

which demonstrates equation (A50).

A7. BILINEAR AND QUADRATIC FORMS

If A is a square matrix of order n and x and y are two arbitrary n vectors, then the scalar

$$u = x^t Ay \tag{A51}$$

is called a *bilinear form*. If, however, the matrix A is also *symmetric*, then

$$v = x^t Ax \tag{A52}$$

is called a *quadratic form* with the *kernal* A.

The matrix A is called *positive definite* if $v > 0$ for all $x \neq 0$, and we write $A > 0$. If $v \geq 0$ for all x and there exists a nonzero vector x for which equality holds, we say A is *positive semidefinite* (or *nonnegative definite*) and write $A \geq 0$. There are corresponding definitions for *negative definite* (or *nonpositive definite*). If there exist vectors x_1 and x_2 such that $x_1^t Ax_1 > 0$ and $x_2^t Ax_2 < 0$, we say A is *indefinite*.

For a positive definite matrix A it is necessary and sufficient that

$$a_{11} > 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \qquad \ldots, \qquad |A| > 0 \tag{A53}$$

Thus the matrix

$$B = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

is positive definite, because $3 > 0$; and

$$\begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = 9 - 4 = 5 > 0$$

and

$$|B| = 3(11) + 2(-9) + 1(-5) = 10 > 0$$

A quadratic form represents, in general, a conic section of some kind. Considering the two-dimensional case for simplicity we write

$$x^t Ax = b \qquad \text{with } A \text{ symmetric}$$

or

$$a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2 = b$$

Figure A1 represents an equation of an ellipse when the quadratic form is positive definite, with axes oblique to the $x_1 x_2$ axis system of coordinates. If we construct the orthogonal matrix $\mathbf{R}$ whose columns are the normalized eigenvectors of $\mathbf{A}$, then perform the transformation,

$$\mathbf{x} = \mathbf{Ry}$$
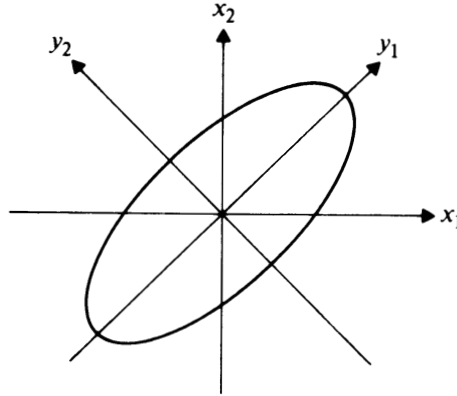
We get

$$(\mathbf{Ry})^t \mathbf{A}(\mathbf{Ry}) = \mathbf{y}^t (\mathbf{R}^t \mathbf{AR})\mathbf{y} = \mathbf{y}^t \mathbf{Dy}$$

where

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{with } \lambda_i \text{ the eigenvalues of } \mathbf{A}$$

Thus the quadratic form becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = b$$

or

$$\frac{y_1^2}{(b/\lambda_1)} + \frac{y_2^2}{(b/\lambda_2)} = 1$$

which is the equation of the ellipse with respect to the $y_1 y_2$ system as shown in Figure A1. It is seen that $\mathbf{R}$ is a rotation matrix, and that the semimajor and semiminor axes are $\sqrt{b/\lambda_1}$ and $\sqrt{b/\lambda_2}$, respectively.

A8. MATRIX DIFFERENTIATION

1. If we are given a vector $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^t$ whose elements are functions of another variable u, then the derivative dx/du is given by

$$\frac{d\mathbf{x}}{du}_{n,1} = \begin{bmatrix} \dfrac{dx_1}{du} \\ \vdots \\ \dfrac{dx_n}{du} \end{bmatrix} \tag{A54}$$

The differential change dx of the vector $\underset{n,1}{\mathbf{x}}$ is defined as

$$d\mathbf{x}_{n,1} = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} \tag{A55}$$

2. If the elements of the matrix $A_{m,n}$ are functions of the (scalar) variable u, then the derivative dA/du is given by

$$\frac{d\mathbf{A}}{du}_{m,n} = \begin{bmatrix} \dfrac{da_{11}}{du} & \cdots & \dfrac{da_{1n}}{du} \\ \vdots & & \vdots \\ \dfrac{da_{m1}}{du} & \cdots & \dfrac{da_{mn}}{du} \end{bmatrix} \tag{A56}$$

3. If a vector $\underset{m,1}{\mathbf{y}}$ represents m functions of some or all of the elements of a variable vector $\underset{n,1}{\mathbf{x}}$, then the total differential of y is given by

$$d\mathbf{y} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} d\mathbf{x} \tag{A57}$$

where the total differentials dx and dy follow the definition given in the first

item above. By necessity the partial derivative $\partial \mathbf{y}/\partial \mathbf{x}$ is an $m \times n$ matrix, which is called the *Jacobian matrix*, and is written in detail as

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \dfrac{\partial y_m}{\partial x_2} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{bmatrix} \tag{A58}$$

4. For the expression

$$\underset{m,\,1}{\mathbf{v}} = \underset{m,\,n}{\mathbf{A}} \underset{n,\,1}{\mathbf{y}}$$

where $\mathbf{A}$ is independent of $\mathbf{y}$,

$$\underset{m,\,1}{d\mathbf{v}} = \underset{m,\,n}{\frac{\partial \mathbf{v}}{\partial \mathbf{y}}} \underset{n,\,1}{d\mathbf{y}} = \underset{m,\,n}{\frac{\partial (\mathbf{Ay})}{\partial \mathbf{y}}} \underset{n,\,1}{d\mathbf{y}}$$

or

$$\frac{\partial (\mathbf{Ay})}{\partial \mathbf{y}} = \underset{m,\,n}{\mathbf{A}} \tag{A59}$$

5. For

$$\underset{1,\,q}{\mathbf{w}} = \underset{1,\,p}{\mathbf{x}^t} \underset{p,\,q}{\mathbf{B}}$$

where $\underset{p,\,q}{\mathbf{B}}$ is independent of $\mathbf{x}$

$$\underset{1,\,q}{d\mathbf{w}} = d\left(\underset{q,\,p}{\mathbf{B}^t} \underset{p,\,1}{\mathbf{x}} \right)^t = \left(\underset{q,\,p}{\frac{\partial (\mathbf{B}^t \mathbf{x})}{\partial \mathbf{x}}} \underset{p,\,1}{d\mathbf{x}} \right)^t$$

$$= (\mathbf{B}^t d\mathbf{x})^t$$

$$= \underset{1,\,p}{d\mathbf{x}^t} \underset{p,\,q}{\mathbf{B}}$$

6. For the bilinear form,

$$u = \underset{1,\,n}{\mathbf{x}^t} \underset{n,\,n}{\mathbf{A}} \underset{n,\,1}{\mathbf{y}}$$

where $\mathbf{A}$ is independent of both $\mathbf{x}$ and $\mathbf{y}$,

$$du = \underset{\substack{1,\,n}}{\frac{\partial u}{\partial \mathbf{y}}} \underset{n,\,1}{d\mathbf{y}} \qquad \text{or} \qquad \frac{\partial u}{\partial \mathbf{y}} = \mathbf{x}^t \mathbf{A} \tag{A60}$$

and

$$du = \left(\frac{\partial u}{\partial x}_{n,1} dx\right)^t \quad \text{or} \quad \frac{\partial u}{\partial x}_{1,n} = \mathbf{y}^t_{1,n} \mathbf{A}^t_{n,n} \tag{A61}$$

7. For the quadratic form,

$$u = \mathbf{x}^t \mathbf{A} \mathbf{x}$$

where **A** is independent of **x**,

$$\frac{\partial u}{\partial x}_{1,n} = \mathbf{x}^t_{1,n} \mathbf{A} + \mathbf{x}^t_{1,n} \mathbf{A}^t = \mathbf{x}^t_{1,n}\left(\mathbf{A}_{n,n} + \mathbf{A}^t\right) = 2\mathbf{x}^t_{1,n}\mathbf{A} \tag{A62}$$

when $\mathbf{A}_{n,n}$ is symmetric.

8. The derivative of the inverse $\mathbf{A}^{-1}$, assuming that it exists, is obtained from

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\frac{d}{dx}(\mathbf{A}\mathbf{A}^{-1}) = \frac{d\mathbf{I}}{dx} = \mathbf{0}$$

$$\mathbf{A}\frac{d\mathbf{A}^{-1}}{dx} + \frac{d\mathbf{A}}{dx}\mathbf{A}^{-1} = \mathbf{0}$$

hence

$$\frac{d\mathbf{A}^{-1}}{dx} = -\mathbf{A}^{-1}\frac{d\mathbf{A}}{dx}\mathbf{A}^{-1} \tag{A63}$$

9. If the elements of a determinant $r = |\mathbf{R}|$ are functions of a variable x, then

$$\frac{dr}{dx} = \frac{d}{dx}|\mathbf{R}|$$

$$= \begin{vmatrix} \dfrac{dr_{11}}{dx} & r_{12} & \cdots & r_{1n} \\ \dfrac{dr_{21}}{dx} & r_{22} & \cdots & r_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{dr_{n1}}{dx} & r_{n2} & \cdots & r_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} r_{11} & r_{12} & \cdots & \dfrac{dr_{1n}}{dx} \\ r_{21} & r_{22} & \cdots & \dfrac{dr_{2n}}{dx} \\ \cdot & \cdot & \cdots & \cdot \\ r_{n1} & 2_{n2} & \cdots & \dfrac{dr_{nn}}{dx} \end{vmatrix} \tag{A64}$$

or

$$\frac{dr}{dx} = \begin{vmatrix} \dfrac{dr_{11}}{dx} & \dfrac{dr_{12}}{dx} & \cdots & \dfrac{dr_{1n}}{dx} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{dr_{n1}}{dx} & \dfrac{dr_{n2}}{dx} & \cdots & \dfrac{dr_{nn}}{dx} \end{vmatrix}$$

(A65)

Note that the derivative of a determinant of order n is the sum of n determinants.

A9. SOME USEFUL MATRIX RELATIONSHIPS

1. For the bilinear and quadratic forms, we may write

$$\mathbf{x}^t \mathbf{A} \mathbf{y} = \text{tr } (\mathbf{x} \mathbf{y}^t \mathbf{A}^t) \tag{A66}$$

$$\mathbf{x}^t \mathbf{A} \mathbf{x} = \text{tr } (\mathbf{x} \mathbf{x}^t \mathbf{A}) \qquad (\text{A symmetric}) \tag{A67}$$

2. Given the matrix expression

$$\underset{n,\,n}{\mathbf{X}} = \underset{n,\,n}{\mathbf{Y}} \pm \underset{n,\,p}{\mathbf{U}} \; \underset{p,\,p}{\mathbf{Z}} \; \underset{p,\,n}{\mathbf{V}} \tag{A68}$$

then

$$\mathbf{x}^{-1} = \mathbf{Y}^{-1} \mp \mathbf{Y}^{-1}\mathbf{U}(\mathbf{Z}^{-1} \pm \mathbf{V}\mathbf{Y}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{Y}^{-1} \tag{A69}$$

provided that all inverses in equation (A69) do exist.

3.

$$(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1}(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1}\mathbf{B}^{-1} \tag{A70}$$

$$= \mathbf{B}^{-1}(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1}\mathbf{A}^{-1} \tag{A71}$$

provided that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist.

4. In general

$$(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) \neq \mathbf{A}^2 - \mathbf{B}^2 \tag{A72}$$

and

$$\frac{d}{dx}\mathbf{A}^2 \neq 2\mathbf{A}\frac{d\mathbf{A}}{dx}$$

5. An idempotent matrix is a matrix that is equal to its square, or

$$\mathbf{H} = \mathbf{H}^2 \tag{A73}$$

For instance,

$$C = I - B(B'B)^{-1}B'$$

is idempotent, since

$$C^2 = I - 2B(B'B)^{-1}B' + B(B'B)^{-1}B'B(B'B)^{-1}B'$$

$$= I - B(B'B)^{-1}B'$$

$$= C$$

6. A block diagonal matrix is a square matrix with square submatrices along the main diagonal and zero submatrices elsewhere. It may be written symbolically as

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & A_{kk} \end{bmatrix}$$

as an example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 4 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 & 0 \\ 0 & -5 & 0 & 7 & & \\ 0 & 0 & 0 & 0 & 11 & 1 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix}$$

is a block diagonal matrix.

The inverse of a nonsingular block diagonal matrix is another block diagonal matrix with corresponding diagonal submatrices being inverses of the respective submatrices in the original matrix, or

$$\begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & A_{kk}^{-1} \end{bmatrix}$$

B

Linearization by Series Expansion

When the original condition (and constraint) equations are nonlinear, they are usually linearized using the zero and first-order terms of Taylor series. In the following sections a number of cases are given.

B1. ONE FUNCTION OF ONE VARIABLE

For $y = f(x)$ the series expansion is

$$y = f(x^0) + \frac{dy}{dx}\bigg|_{x_0} \Delta x + \frac{1}{2!}\frac{d^2y}{dx^2}\bigg|_{x_0} (\Delta x)^2 + \cdots \tag{B1}$$

Any term beyond the second on the right hand side of equation (B1) is again nonlinear. Therefore, for purposes of adjustment, only the first two terms (zero and first order) are used to form a linear expression approximating the original nonlinear form. Thus

$$y \cong f(x^0) + \frac{dy}{dx}\bigg|_{x^0} \cdot \Delta x = y^0 + j\,\Delta x \tag{B2}$$

is the linearized form to be used.
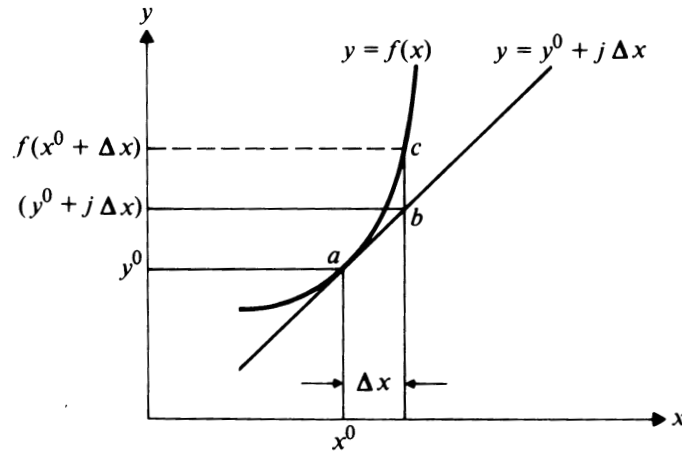
The technique of linearization is demonstrated in Figure B1. The curve represents the original nonlinear function $f(x)$, whereas the straight line represents the linearized form of equation (B2). That line is tangent to the curve at the given point a, (x^0, y^0). When Δx is given (or evaluated), the value of the function would be approximated by point b whose ordinate is $(y^0 + j \cdot \Delta x)$, and the exact value from the nonlinear function is point c, with ordinate $\underline{f}(x^0 + \Delta x)$. The error arising from using the linear form is the line segment $\overline{bc}$.

Example Given $y = x^2 + x^3$, evaluate the linearized form at $x^0 = 1$ and the value of the function and its linearized form at $x = 1.1$

$$j = \frac{dy}{dx}\bigg|_{x^0} = 2x + 3x^2 \bigg|_{x^0 = 1} = 5$$

$$y^0 = f(x^0) = 2 \quad \text{and} \quad \Delta x = 0.1$$

$$y = y^0 + j \, \Delta x = 2 + 0.5 = 2.5$$

$$f(x^0 + \Delta x) = f(1.1) = 1.21 + 1.331 = 2.541$$

B2. ONE FUNCTION OF TWO VARIABLES

$$y = f(x_1, x_2)$$

$$= f(x_1^0, x_2^0) + \frac{\partial y}{\partial x_1}\bigg|_{x_1^0, x_2^0} \cdot \Delta x_1 + \frac{\partial y}{\partial x_2}\bigg|_{x_1^0, x_2^0} \cdot \Delta x_2$$

$$+ \frac{1}{2!}\frac{\partial^2 y}{\partial x_1^2}\bigg|_{x_1^0, x_2^0} \cdot (\Delta x_1)^2 + \frac{1}{2!}\frac{\partial^2 y}{\partial x_2^2}\bigg|_{x_1^0, x_2^0} (\Delta x_2)^2$$

$$+ \frac{\partial y}{\partial x_1}\bigg|_{x_1^0, x_2^0} \cdot \frac{\partial y}{\partial x_2}\bigg|_{x_1^0, x_2^0} (\Delta x_1)(\Delta x_2) + \cdots$$

$$\tag{B3}$$

For linearized form, equation (B3) is truncated to

$$y = y^0 + j_1 \, \Delta x_1 + j_2 \, \Delta x_2 \tag{B4}$$

where

$$y^0 = f(x_1^0, \, x_2^0)$$

$$j_1 = \frac{\partial y}{\partial x_1}\bigg|_{x_1^0, \, x_2^0}$$

$$j_2 = \frac{\partial y}{\partial x_2}\bigg|_{x_1^0, \, x_2^0}$$

Equation (B4) can be rewritten introducing vectors, or

$$y = y^0 + [j_1 \quad j_2]\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

or

$$y = y^0 + \mathbf{J}_{yx} \, \Delta \mathbf{x} \tag{B5}$$

where

$$\mathbf{J}_{yx} = \frac{\partial y}{\partial \mathbf{x}}$$

is the Jacobian of y with respect to $\mathbf{x}$.

B3. TWO FUNCTIONS OF ONE VARIABLE

$$\begin{aligned} y_1 &= f_1(x) \cong y_1^0 + j_1 \, \Delta x \\ y_2 &= f_2(x) \cong y_2^0 + j_2 \, \Delta x \end{aligned} \tag{B6}$$

with

$$y_1 = f_1(x)^0$$

$$y_2 = f_2(x)^0$$

$$\mathbf{J}_{yx} = [j_1 \quad j_2] = \begin{bmatrix} \dfrac{dy_1}{dx}\bigg|_{x^0} & \dfrac{dy_2}{dx}\bigg|_{x^0} \end{bmatrix}$$

B4. TWO FUNCTIONS OF TWO VARIABLES EACH

$$\begin{aligned} y_1 &= f_1(x_1, x_2) \cong y_1^0 + j_{11} \, \Delta x_1 + j_{12} \, \Delta x_2 \\ y_2 &= f_2(x_1, x_2) \cong y_2^0 + j_{21} \, \Delta x_1 + j_{22} \, \Delta x_2 \end{aligned} \tag{B7a}$$

or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \cong \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} + \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} \tag{B7b}$$

or

$$y = y^0 + J_{yx} \, \Delta x \tag{B7c}$$

where

$$y^0 = \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} = \begin{bmatrix} f_1(x_1^0, \, x_2^0) \\ f_2(x_1^0, \, x_2^0) \end{bmatrix}$$

and

$$J_{yx} = \frac{\partial y}{\partial x} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\[2ex] \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix}$$

evaluated at $x_1^0, \, x_2^0$.

B5. GENERAL CASE OF m FUNCTIONS OF n VARIABLES

$$y_1 = f_1(x_1, \, x_2, \, \ldots, \, x_n)$$
$$y_2 = f_2(x_1, \, x_2, \, \ldots, \, x_n)$$
$$\text{-----------------------------}$$
$$y_m = f_m(x_1, \, x_2, \, \ldots, \, x_n) \tag{B8}$$

With the auxiliaries,

$$y^0 = \begin{bmatrix} y_1^0 \\ y_2^0 \\ \vdots \\ y_m^0 \end{bmatrix} = \begin{bmatrix} f_1(x_1^0, \, x_2^0, \, \ldots, \, x_n^0) \\ f_2(x_2^0, \, x_2^0, \, \ldots, \, x_n^0) \\ \text{-----------------------} \\ f_m(x_1^0, \, x_2^0, \, \ldots, \, x_n^0) \end{bmatrix}$$

$$J_{yx} = \frac{\partial y}{\partial x} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \text{------------------------} \\ \dfrac{\partial y_m}{\partial x_1} & \dfrac{\partial y_m}{\partial x_2} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{bmatrix} \quad \text{evaluated at } x^0$$

$$\Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

the linearized form of equation (B8) becomes

$$y \cong y^0 + J_{yx} \, \Delta x$$

Example Given

$$y_1 = x_1^2 - x_3$$

$$y_2 = \frac{1}{x_2} + x_3^2$$

write the linearized form, in matrix notation, at the point $x^0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^t$

$$\mathbf{y}^0 = \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\mathbf{J}_{yx} = \begin{bmatrix} 2x_1^0 & 0 & -1 \\ 0 & -x^{0-2} & 2x_3^0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Thus

$$\mathbf{y} \cong \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix}$$

C

Ranks of a Posteriori Cofactor (Covariance) Matrices

After a least squares adjustment is performed, the residuals v and estimated observations $\hat{l}$ are both functionally dependent variables. Consequently, their cofactor matrices, Q_{vv} and Q_{ll}, are singular. In the text the fact was mentioned that the rank of Q_{vv} is r (the redundancy) and the rank of Q_{ll} is $(n - r)$, but without proof. It is the purpose of this appendix to give the proofs of these facts for different techniques of least squares.

C1. ADJUSTMENT OF OBSERVATIONS ONLY

The conditions are:

$$\underset{r,\,n}{A}\ \underset{n,\,1}{v} = \underset{r,\,1}{f}$$

with cofactor matrix $\underset{n,\,n}{Q}$ and

$$n > r, \qquad \text{rank } (A) = r, \qquad \text{rank } (Q) = n$$

$$v = QA^tW_e\,f$$

$$\hat{l} = l + v$$

$$Q_{vv} = QA^tW_e\,AQ$$

$$Q_{ll} = Q - Q_{vv}$$

It is required to show that:

$$\text{rank} \ (\mathbf{Q}_{vv}) = r \quad \text{and} \quad \text{rank} \ (\mathbf{Q}_{ll}) = n - r$$

Let

$$\underset{n,\,n}{\mathbf{J}} = \mathbf{A}^t \mathbf{W}_e \, \mathbf{A} \mathbf{Q} \tag{C1}$$

which has the following properties:

1. $\mathbf{J}^2 = \mathbf{A}^t \mathbf{W}_e (\mathbf{A} \mathbf{Q} \mathbf{A}^t) \mathbf{W}_e \, \mathbf{A} \mathbf{Q} = \mathbf{J}$ (idempotent)
2. Rank $(\mathbf{J}) = r$ because

   rank $(\mathbf{J}) \leq$ least rank of constituent matrices in equation (C1)

   $$\leq r$$

since

$$\mathbf{J} \mathbf{A}^t = \mathbf{A}^t \quad \text{and} \quad \text{rank} \ (\mathbf{A}^t) = r$$

then

$$r = \text{rank} \ (\mathbf{A}^t) \leq \min \ (\text{rank} \ (\mathbf{J}), r)$$

$$\text{rank} \ (\mathbf{J}) \geq r$$

Thus

$$\text{rank} \ (\mathbf{J}) = r$$

3. There exists an orthogonal matrix $\mathbf{F}$ that diagonalizes $\mathbf{J}$,

$$\mathbf{F}^t \mathbf{J} \mathbf{F} = \mathbf{D}$$

$$\mathbf{D}^2 = \mathbf{F}^t \mathbf{J} \mathbf{F} \mathbf{F}^t \mathbf{J} \mathbf{F} = \mathbf{F}^t \mathbf{J}^2 \mathbf{F} = \mathbf{D}$$

Thus the diagonal elements of $\mathbf{D}$ (eigenvalues) are either 0 or 1. The number of elements for which $d_i = 1$ is the same as the rank of $\mathbf{J}$, or is r.

4. Because $\mathbf{J} \mathbf{A}^t = \mathbf{A}^t$, the row vectors of $\mathbf{A}$ (column vectors of $\mathbf{A}^t$) are the eigenvectors of $\mathbf{J}$

Let

$$\underset{n,\,n}{\mathbf{K}} = \mathbf{I} - \mathbf{J} \tag{C2}$$

which has the properties

1. $\mathbf{K}^2 = \mathbf{I} - 2\mathbf{J} + \mathbf{J}^2 = \mathbf{I} - \mathbf{J} = \mathbf{K}$ (idempotent)
2. Rank $(\mathbf{K}) = (n - r)$, because applying the orthogonal transformation

$$\mathbf{F}^t \mathbf{K} \mathbf{F} = \mathbf{I} - \mathbf{F}^t \mathbf{J} \mathbf{F} = \mathbf{I} - \mathbf{D}$$

thus

$$\text{rank } (\mathbf{K}) = \text{rank } (\mathbf{I} - \mathbf{D})$$

$$= \text{ the number of elements for which } d_i = 0$$

$$= (n - r)$$

3. If eigenvalues of $\mathbf{J}$ are λ, then for any vector $\mathbf{x}$

$$\mathbf{Jx} = \lambda\mathbf{x}$$

and thus

$$\mathbf{Kx} = \mathbf{x} - \mathbf{Jx} = \mathbf{x} - \lambda\mathbf{x} = (\mathbf{I} - \lambda)\mathbf{x}$$

Thus the eigenvalues of $\mathbf{J}$ are eigenvalues of $\mathbf{K}$ except that those of value 1 for $\mathbf{J}$ are of value 0 for $\mathbf{K}$ and vice versa.

From the above preliminary notes the ranks of $\mathbf{Q}_{vv}$ and $\mathbf{Q}_{ll}$ may be derived.

C1.1. Rank of $\mathbf{Q}_{vv}$

$$\mathbf{Q}_{vv} = \mathbf{QJ}$$

$$\text{rank } (\mathbf{Q}_{vv}) \le \min [n, r] \le r \qquad (\text{because } r < n)$$

and

$$\mathbf{WQ}_{vv} = \mathbf{J}$$

$$\text{rank } (\mathbf{J}) = r \le \min [n, \text{rank } (\mathbf{Q}_{vv})]$$

$$\text{rank } (\mathbf{Q}_{vv}) \ge r$$

Thus

$$\text{rank } (\mathbf{Q}_{vv}) = r \qquad\qquad\qquad\qquad\qquad\qquad (\text{C3})$$

which is the first required result.

C1.2. Rank of $\mathbf{Q}_{ll}$

$$\mathbf{Q}_{ll} = \mathbf{Q} - \mathbf{Q}_{vv} = \mathbf{Q} - \mathbf{QJ} = \mathbf{Q}(\mathbf{I} - \mathbf{J}) = \mathbf{Qk}$$

$$\text{rank } (\mathbf{Q}_{ll}) \le \min [n, (n - r)]$$

$$\le (n - r)$$

$$\mathbf{WQ}_{ll} = \mathbf{K}$$

$$\text{rank } (\mathbf{K}) = (n - r) \le \min [n, \text{rank } (\mathbf{Q}_{ll})]$$

$$\text{rank } (\mathbf{Q}_{ll}) \ge (n - r)$$

Thus

$$\text{rank } (Q_{ll}) = (n - r) \qquad\qquad (C4)$$

which is the second required result.

C2. ADJUSTMENT OF INDIRECT OBSERVATIONS

The conditions are

$$\underset{n,\,1}{v} + \underset{n,\,u}{B} \underset{u,\,1}{\Delta} = \underset{n,\,1}{f} \qquad \text{with } \underset{n,\,n}{Q}$$

and

$$n > u, \qquad \text{rank } (B) = u, \qquad \text{rank } (Q) = n$$

$$\Delta = (B^t WB)^{-1}(B^t Wf)$$

$$v = f - B\Delta$$

$$Q_{\Delta\Delta} = (B^t WB)^{-1} = N^{-1}$$

$$Q_{vv} = Q - BN^{-1}B^t$$

$$Q_{ll} = BN^{-1}B^t$$

$$= Q - Q_{vv}$$

It is required to show that

$$\text{rank } (Q_{vv}) = (n - u) = r \qquad \text{or degrees of freedom}$$

$$\text{rank } (Q_{ll}) = u$$

which is the minimum number of variables (observations) necessary for a unique solution.

In this case

$$\underset{n,\,n}{J} = BN^{-1}B^t W$$

with the properties:
1. $J^2 = J$
2. Rank $(J) = u$ because

$$\text{rank } (J) \le \text{min rank of constituent matrices}$$

$$< u$$

$$JB = B$$

$$u \le \text{min } [\text{rank } (J), u]$$

$$\text{rank } (J) \ge u$$

or rank (J) must equal u.

3. Orthogonal transformation

$$\mathbf{F}^t\mathbf{JF} = \mathbf{D}$$

where $\mathbf{D}$ is a diagonal matrix with u of $d_i = 1$ and $(n - u)$ of $d_i = 0$, since $\mathbf{D}^2 = \mathbf{D}$.

4. $\mathbf{JB} = \mathbf{B}$, or columns of $\mathbf{B}$ are eigenvectors of $\mathbf{J}$. Next,

$$\mathbf{K} = \mathbf{I} - \mathbf{J}$$

with

$$\mathbf{K}^2 = \mathbf{K}$$

$$\text{rank } (\mathbf{K}) = (n - u)$$

and eigenvalues of $\mathbf{J}$ are also eigenvalues for $\mathbf{K}$ as in Section C1.

C2.1. Rank of $\mathbf{Q}_{\tilde{l}}$

$$\mathbf{Q}_{ll} = \mathbf{JQ}$$

$$\text{rank } (\mathbf{Q}_{ll}) \leq u$$

$$\mathbf{Q}_{ll}\mathbf{W} = \mathbf{J}$$

$$\text{rank } (\mathbf{Q}_{ll}) \geq u$$

Thus

$$\text{rank } (\mathbf{Q}_{ll}) = u = (n - r) \tag{C5}$$

C2.2. Rank of $\mathbf{Q}_{vv}$

$$\mathbf{Q}_{vv} = \mathbf{Q} - \mathbf{Q}_{ll} = \mathbf{Q} - \mathbf{JQ} = \mathbf{KQ}$$

$$\text{rank } (\mathbf{Q}_{vv}) \leq (n - u)$$

$$\mathbf{Q}_{vv}\mathbf{W} = \mathbf{K}$$

$$\text{rank } (\mathbf{Q}_{vv}) \geq (n - u)$$

Hence

$$\text{rank } (\mathbf{Q}_{vv}) = (n - u) = r \tag{C6}$$

C3. ADJUSTMENT OF COMBINED OBSERVATIONS AND PARAMETERS

The conditions are

$$\underset{c,\,n\ \ n,\,1}{\mathbf{A}\quad\mathbf{v}} + \underset{c,\,u\ \ u,\,1}{\mathbf{B}\quad\Delta} = \underset{c,\,1}{\mathbf{f}} \qquad \text{with } \underset{n,\,n}{\mathbf{Q}}$$

and

$$c < n \qquad c > u \qquad c = u + r \qquad r = \text{redundancy}$$

$$\text{rank } (\mathbf{A}) = c \qquad \text{rank } (\mathbf{B}) = u \qquad \text{rank } (\mathbf{Q}) = n$$

$$\Delta = \left(\underset{u,c}{\mathbf{B}^t} \; \underset{c,c}{\mathbf{W}_e} \; \underset{c,u}{\mathbf{B}} \right)^{-1} (\mathbf{B}^t \mathbf{W}_e \, \mathbf{f}) = \mathbf{N}^{-1} \mathbf{t}$$

$$\mathbf{v} = \mathbf{Q} \mathbf{A}^t \mathbf{W}_e (-\mathbf{B}\Delta + \mathbf{f})$$

$$\mathbf{Q}_{\Delta\Delta} = \mathbf{N}^{-1}$$

$$\underset{n,n}{\mathbf{Q}_{vv}} = \underset{n,n}{\mathbf{Q}} \; \underset{n,c}{\mathbf{A}^t} [\mathbf{W}_e - \mathbf{W}_e \, \mathbf{B} \mathbf{N}^{-1} \mathbf{B}^t \mathbf{W}_e] \mathbf{A} \mathbf{Q}$$

$$\mathbf{Q}_{ll} = \mathbf{Q} - \mathbf{Q}_{vv}$$

It is required to show that

$$\text{rank } (\mathbf{Q}_{vv}) = c - u = r$$

$$\text{rank } (\mathbf{Q}_{ll}) = n - r$$

C3.1. Rank of $\mathbf{Q}_{vv}$

$$\mathbf{Q} \mathbf{A}^t \mathbf{W}_e (\mathbf{I} - \mathbf{B} \mathbf{N}^{-1} \mathbf{B}^t \mathbf{W}_e) \mathbf{A} \mathbf{Q}$$

Let

$$\underset{c,c}{\bar{\mathbf{J}}} = \mathbf{B} \mathbf{N}^{-1} \mathbf{B}^t \mathbf{W}_e$$

$$\bar{\mathbf{J}}^2 = \mathbf{B} \mathbf{N}^{-1} (\mathbf{B}^t \mathbf{W}_e \, \mathbf{B}) \mathbf{N}^{-1} \mathbf{B}^t \mathbf{W}_e = \mathbf{B} \mathbf{N}^{-1} \mathbf{B}^t \mathbf{W}_e = \bar{\mathbf{J}} \qquad \text{(idempotent)}$$

rank $(\bar{\mathbf{J}}) \leq$ min rank of constituent matrices

$$\leq u$$

$$\bar{\mathbf{J}} \mathbf{B} = \mathbf{B}$$

rank $(\bar{\mathbf{J}}) \geq u$

Thus

rank $(\bar{\mathbf{J}}) = u$

Let

$$\bar{\mathbf{K}} = \mathbf{I} - \bar{\mathbf{J}}$$

$$\bar{\mathbf{K}}^2 = \bar{\mathbf{K}} \qquad \text{(idempotent)}$$

and following the scheme of orthogonal transformation

$$\bar{\mathbf{F}}^t \bar{\mathbf{K}} \bar{\mathbf{F}} = \mathbf{I} - \bar{\mathbf{D}}$$

we can get

$$\text{rank } (\bar{\mathbf{K}}) = c - u = r$$

Then

$$\mathbf{Q}_{vv} = \mathbf{Q}\mathbf{A}'\mathbf{W}_e \bar{\mathbf{K}}\mathbf{A}\mathbf{Q}$$

$$\text{rank } (\mathbf{Q}_{vv}) \leq \text{min rank of constituent matrices}$$

$$\leq r$$

Premultiplying both sides by $\mathbf{A}$, and postmultiplying by $\mathbf{A}'$, then

$$\mathbf{A}\mathbf{Q}_{vv} \mathbf{A}^t = \bar{\mathbf{K}}\mathbf{Q}_e$$

or

$$\mathbf{A}\mathbf{Q}_{vv} \mathbf{A}'\mathbf{W}_e = \bar{\mathbf{K}}$$

$$r = \text{rank } (\bar{\mathbf{K}}) \leq \min (c, \text{rank } (\mathbf{Q}_{vv}))$$

$$\text{rank } (\mathbf{Q}_{vv}) \geq r$$

Hence

$$\text{rank } (\mathbf{Q}_{vv}) = r \tag{C7}$$

Next,

$$\mathbf{Q}_{ll} = \mathbf{Q} - \mathbf{Q}_{vv} = \mathbf{Q} - \mathbf{Q}\mathbf{A}'\mathbf{W}_e \bar{\mathbf{K}}\mathbf{A}\mathbf{Q}$$

$$= \mathbf{Q}(\mathbf{I} - \mathbf{A}'\mathbf{W}_e \bar{\mathbf{K}}\mathbf{A}\mathbf{Q})$$

Let

$$\dot{\mathbf{J}} = \mathbf{A}'\mathbf{W}_e \bar{\mathbf{K}}\mathbf{A}\mathbf{Q}$$

$$\dot{\mathbf{J}}^2 = \mathbf{A}'\mathbf{W}_e \bar{\mathbf{K}}(\mathbf{A}\mathbf{Q}\mathbf{A}')\mathbf{W}_e \bar{\mathbf{K}}\mathbf{A}\mathbf{Q} = \mathbf{A}'\mathbf{W}_e \bar{\mathbf{K}}^2\mathbf{A}\mathbf{Q} = \dot{\mathbf{J}}$$

$$\text{rank } (\dot{\mathbf{J}}) \leq \text{min rank of constituent matrices}$$

$$\leq r$$

Premultiplying $\dot{\mathbf{J}}$ by $\mathbf{A}\mathbf{Q}$ and postmultiplying by $\mathbf{A}^t$

$$\mathbf{A}\mathbf{Q}\dot{\mathbf{J}}\mathbf{A}^t = (\mathbf{A}\mathbf{Q}\mathbf{A}')\mathbf{W}_e \bar{\mathbf{K}}(\mathbf{A}\mathbf{Q}\mathbf{A}')$$

$$= \bar{\mathbf{K}}\mathbf{Q}_e$$

$$\mathbf{A}\mathbf{Q}\dot{\mathbf{J}}\mathbf{A}'\mathbf{W}_e = \bar{\mathbf{K}}$$

$$r = \text{rank } (\bar{\mathbf{K}}) \leq \min (\text{rank } (\dot{\mathbf{J}}), c)$$

$$\text{rank } (\dot{\mathbf{J}}) \geq r$$

Let

$$\dot{\mathbf{K}} = \mathbf{I} - \dot{\mathbf{J}}$$

$$\dot{\mathbf{K}}^2 = \dot{\mathbf{K}}$$

and following the scheme of orthogonal transformation

$$\mathbf{F}^t\dot{\mathbf{K}}\mathbf{F} = \mathbf{I} - \mathbf{D}$$

as used previously, we conclude

$$\text{rank}\ (\dot{\mathbf{K}}) = (n - r)$$

Finally,

$$\mathbf{Q}_{ll} = \mathbf{Q}\dot{\mathbf{K}}$$

$$\text{rank}\ (\mathbf{Q}_{ll}) \leq (n - r)$$

$$\mathbf{W}\mathbf{Q}_{ll} = \dot{\mathbf{K}}$$

$$\text{rank}\ (\mathbf{Q}_{ll}) \geq (n - r)$$

therefore

$$\text{rank}\ (\mathbf{Q}_{ll}) = n - r \tag{C8}$$

D

Pertinent Statistical Tables

TABLE D1   Area under the Standard Normal Density Function

$$\alpha = \int_{z_\alpha}^{\infty} f(z)\, dz = P(z > z_\alpha) = 1 - \int_{-\infty}^{z_\alpha} f(z)\, dz$$
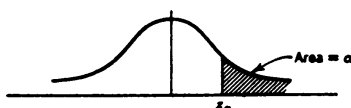
| $z_\alpha \rightarrow$ $\downarrow$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0076	0.0073	0.0071	0.0070	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0042	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0014	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002

TABLE D2 Area Under the Student t Density Function

$$t_{\alpha, m} \text{ such that } P(t_m > t_{\alpha, m}) = \alpha = \int_{t_{\alpha,m}}^{\infty} f(t)\, dt = 1 - \int_{-\infty}^{t_{\alpha,m}} f(t)\, dt$$

Area = α

$t_{\alpha, m}$
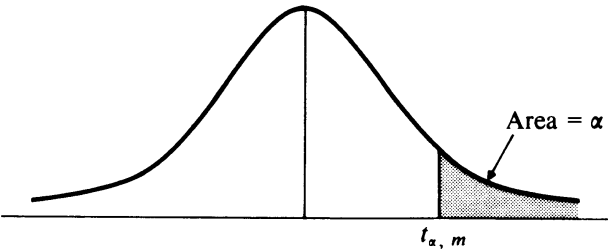
m	$\alpha = 0.25$	0.20	0.15	0.10	0.050	0.025	0.010	0.005	0.0005
1	1.001	1.376	1.963	3.078	6.314	12.706	31.821	63.657	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	31.598
3	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	12.941
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	6.859
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	5.405
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	4.073
16	0.690	0.866	1.071	1.337	1.746	2.120	2.583	2.921	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.762
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.551
60	0.679	0.848	1.046	1.296	1.671	2.000	2.390	2.660	3.460
120	0.677	0.844	1.042	1.289	1.658	1.980	2.358	2.617	3.380

## TABLE D3 Area Under the Chi-Square Density Function

$$\chi^2_{\alpha, m} \text{ such that } P(\chi^2_m > \chi^2_{\alpha, m}) = \alpha = \int_{\chi^2_{\alpha, m}}^{\infty} f(\chi^2) \, d\chi^2 = 1 - \int_0^{\chi^2_{\alpha, m}} f(\chi^2) \, d\chi^2$$
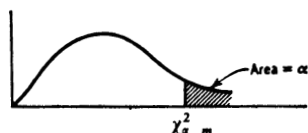
m	$\alpha = 0.995$	0.990	0.975	0.950	0.900	0.500	0.10	0.05	0.025	0.010	0.005
1	0.00	0.00	0.00	0.00	0.02	0.46	2.71	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	0.21	1.39	4.61	5.99	7.38	9.21	10.60
3	0.07	0.12	0.22	0.35	0.58	2.37	6.25	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	1.06	3.36	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	4.35	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	5.35	10.64	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	6.35	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	7.34	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	9.34	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	9.34	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	10.34	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	11.34	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	12.34	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	6.79	13.34	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	14.34	22.31	25.00	27.49	40.58	32.80
16	5.14	5.81	6.91	7'96	9.31	15.34	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.08	16.34	24.77	27.59	20.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	17.34	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	14.34	24.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.88	41.92	45.64	49.29
27	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77
60	35.53	37.48	40.48	43.19	46.64	59.33	74.40	79.08	83.30	88.38	91.95
120	83.85	86.92	91.58	95.70	100.62	119.33	140.23	146.57	152.21	158.95	163.65

TABLE D4(a) Area Under the F Density Function

$F_{0.05, m_1, m_2}$ such that $P(F_{m_1, m_2} > F_{0.05, m_1, m_2}) = 0.05$

$$= \int_{F_{0.05, m_1, m_2}}^{\infty} f(F)\, dF = 1 - \int_{0}^{F_{0.05, m_1, m_2}} f(F)\, dF$$

$F_{0.5, m_1, m_2}$
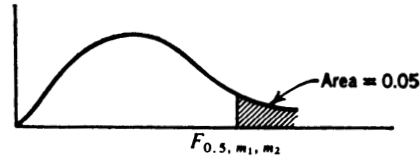
m_2 \ m_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16
1	161	200	216	225	230	234	237	239	241	242	243	244	245	245	246
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.69
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.84
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.73	4.70	4.68	4.66	4.64	4.60
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.92
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.49
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.20
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	2.99
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.83
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.70
12	4.75	3.89	3.49	3.25	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.60
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.51
14	4.60	3.74	3.35	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.44
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.33
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.25
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.18
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.13
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18	2.15	2.13	2.09
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.05
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.02
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	1.99
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00	1.97	1.95	1.90
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99	1.95	1.92	1.89	1.85
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92	1.89	1.86	1.82
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91	1.88	1.84	1.82	1.77
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89	1.85	1.82	1.79	1.75
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80	1.77	1.74	1.69
500	3.86	3.01	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.81	1.77	1.74	1.71	1.66
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75	1.72	1.69	1.64

TABLE D4(a)—(*Continued*)

$$F_{0.05, m_1, m_2} \text{ such that } P(F_{m_1, m_2} > F_{0.05, m_1, m_2}) = 0.05$$

$$= \int_{F_{0.05,m,m}}^{\infty} f(F)\, dF = 1 - \int_0^{F_{0.05,m,m}} f(F)\, dF$$

18	20	22	24	26	28	30	40	50	60	80	100	200	500	∞	m_1 / m_2
247	248	249	249	249	250	250	251	252	252	252	253	254	254	254	1
19.4	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	2
8.67	8.66	8.65	8.64	8.63	8.62	8.62	8.59	8.59	8.57	8.56	8.55	8.54	8.53	8.53	3
5.82	5.80	5.79	5.77	5.76	5.75	5.75	5.72	5.70	5.69	5.67	5.66	5.65	5.64	5.63	4
4.58	3.56	4.54	4.53	4.52	4.50	4.50	4.46	4.44	4.43	4.41	4.41	4.39	4.37	4.37	5
3.90	3.87	3.86	3.84	3.83	3.82	3.81	3.77	3.75	3.74	3.72	3.71	3.69	3.68	3.67	6
3.47	3.44	3.43	3.41	3.40	3.39	3.38	3.34	3.32	3.30	3.29	3.27	3.25	3.24	3.23	7
3.17	3.15	3.13	3.12	3.10	3.09	3.08	3.04	3.02	3.01	2.99	2.97	2.95	2.94	2.93	8
2.96	2.94	2.92	2.90	2.89	2.87	2.86	2.83	2.80	2.79	2.77	2.76	2.73	2.72	2.71	9
2.80	2.77	2.75	2.74	2.72	2.71	2.70	2.66	2.64	2.62	2.60	2.59	2.56	2.55	2.54	10
2.67	2.65	2.63	2.61	2.59	2.58	2.57	2.53	2.51	2.49	2.47	2.46	2.43	2.42	2.40	11
2.57	2.54	2.52	2.51	2.49	2.48	2.47	2.43	2.40	2.38	2.36	2.35	2.32	2.31	2.30	12
2.48	2.46	2.44	2.42	2.41	2.39	2.38	2.34	2.31	2.30	2.27	2.26	2.23	2.22	2.21	13
2.41	2.38	2.37	2.35	2.33	2.32	2.31	2.27	2.24	2.22	2.20	2.19	2.16	2.14	2.13	14
2.30	2.28	2.25	2.24	2.22	2.21	2.19	2.15	2.12	2.11	2.08	2.07	2.04	2.02	2.01	16
2.22	2.19	2.17	2.15	2.13	2.12	2.11	2.06	2.04	2.02	1.99	1.98	1.95	1.93	1.92	18
2.15	2.12	2.10	2.08	2.07	2.05	2.04	1.99	1.97	1.95	1.92	1.91	1.88	1.86	1.84	20
2.10	2.07	2.05	2.03	2.01	2.00	1.98	1.94	1.91	1.89	1.86	1.85	1.82	1.80	1.78	22
2.05	2.03	2.00	1.98	1.97	1.95	1.94	1.89	1.86	1.84	1.82	1.80	1.77	1.75	1.73	24
2.02	1.99	1.97	1.95	1.93	1.91	1.90	1.84	1.82	1.80	1.78	1.76	1.73	1.71	1.69	26
1.99	1.96	1.93	1.91	1.90	1.88	1.87	1.82	1.79	1.77	1.74	1.73	1.69	1.67	1.65	28
1.96	1.93	1.91	1.89	1.87	1.85	1.84	1.79	1.76	1.74	1.71	1.70	1.66	1.64	1.62	30
1.87	1.84	1.81	1.79	1.77	1.76	1.74	1.69	1.66	1.64	1.61	1.59	1.55	1.53	1.51	40
1.81	1.78	1.76	1.74	1.72	1.70	1.69	1.63	1.60	1.58	1.54	1.52	1.48	1.46	1.44	50
1.78	1.75	1.72	1.70	1.68	1.66	1.65	1.59	1.56	1.53	1.50	1.48	1.44	1.41	1.39	60
1.73	1.70	1.68	1.65	1.63	1.62	1.60	1.54	1.51	1.48	1.45	1.43	1.38	1.35	1.32	80
1.71	1.68	1.65	1.63	1.61	1.59	1.57	1.52	1.48	1.45	1.41	1.39	1.34	1.31	1.28	100
1.66	1.62	1.60	1.57	1.55	1.53	1.52	1.46	1.41	1.39	1.35	1.32	1.26	1.22	1.19	200
1.62	1.59	1.56	1.54	1.52	1.50	1.48	1.42	1.38	1.34	1.30	1.28	1.21	1.16	1.11	500
1.60	1.57	1.54	1.52	1.50	1.48	1.46	1.39	1.35	1.32	1.27	1.24	1.17	1.11	1.00	∞

TABLE D4(b)

$F_{0.025, m_1, m_2}$ such that $P(F_{m_1, m_2} > F_{0.025, m_1, m_2}) = 0.025$

$$= \int_{F_{0.025, m, m}}^{\infty} f(F) \, dF = 1 - \int_{0}^{F_{0.025, m, m}} f(F) \, dF$$

Area = 0.025

$F_{0.25, m_1, m_2}$
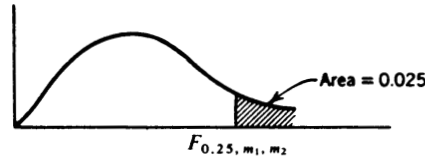
m_2 \ m_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16
1	648	800	864	900	922	937	948	957	963	969	973	977	980	983	987
2	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4	39.4
3	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.4	14.3	14.3	14.3	14.2
4	12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.79	8.75	8.72	8.69	8.64
5	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.57	6.52	6.49	6.46	6.41
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.41	5.37	5.33	5.30	5.25
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	4.63	4.60	4.54
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	4.16	4.13	4.08
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	3.83	3.80	3.74
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.66	3.62	3.58	3.55	3.50
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.47	3.43	3.39	3.36	3.30
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.32	3.28	3.24	3.21	3.15
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.20	3.15	3.12	3.08	3.03
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.09	3.05	3.01	2.98	2.92
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.93	2.89	2.85	2.82	2.76
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.81	2.77	2.73	2.70	2.64
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.72	2.68	2.64	2.60	2.55
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.65	2.60	2.56	2.53	2.47
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.59	2.54	2.50	2.47	2.41
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.54	2.49	2.45	2.42	2.36
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.49	2.45	2.41	2.37	2.32
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.46	2.41	2.37	2.34	2.28
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.33	2.29	2.25	2.21	2.15
50	5.34	3.98	3.39	3.06	2.83	2.67	2.55	2.46	2.38	2.32	2.26	2.22	2.18	2.14	2.08
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.22	2.17	2.13	2.09	2.03
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.36	2.38	2.21	2.16	2.11	2.07	2.03	1.97
100	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	2.12	2.08	2.04	2.00	1.94
200	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18	2.11	2.06	2.01	1.97	1.93	1.87
500	5.05	3.72	3.14	2.81	2.59	2.43	2.31	2.22	2.14	2.07	2.02	1.97	1.93	1.89	1.83
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.99	1.94	1.90	1.87	1.80

TABLE D4(b)—*(Continued)*

$$F_{0.025, m_1, m_2} \text{ such that } P(F_{m_1, m_2} > F_{0.025, m_1, m_2}) = 0.025$$

$$= \int_{F_{0.025,m\ .m}}^{\infty} f(F)\, dF = 1 - \int_{0}^{F_{0.025,m\ .m}} f(F)\, dF$$

18	20	22	24	26	28	30	40	50	60	80	100	200	500	∞	m_1 / m_2
990	993	995	997	999	1000	1001	1006	1008	1010	1012	1013	1016	1017	1018	1
39.4	39.4	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	39.5	2
14.2	14.2	14.1	14.1	14.1	14.1	14.1	14.0	14.0	14.0	14.0	14.0	13.9	13.9	13.9	3
8.60	8.56	8.53	8.51	8.49	8.48	8.46	8.41	8.38	8.36	8.33	8.32	8.29	8.27	8.26	4
6.37	6.33	6.30	6.28	6.26	6.24	6.23	6.18	6.14	6.12	6.10	6.08	6.05	6.03	6.01	5
5.21	5.17	5.14	5.12	5.10	5.08	5.07	5.01	4.98	4.96	4.93	4.92	4.88	4.86	4.85	6
4.50	4.47	4.44	4.42	4.39	4.38	4.36	4.31	4.28	4.25	4.23	4.21	4.18	4.16	4.14	7
4.03	4.00	3.97	3.95	3.93	3.91	3.89	3.84	3.81	3.78	3.76	3.74	3.70	3.68	3.67	8
3.70	3.67	3.64	3.61	3.59	3.58	3.56	3.51	3.47	3.45	3.42	3.40	3.37	3.35	3.33	9
3.45	3.42	3.39	3.37	3.34	3.33	3.31	3.26	3.22	3.20	3.17	3.15	3.12	3.09	3.08	10
3.26	3.23	3.20	3.17	3.15	3.13	3.12	3.06	3.03	3.00	2.97	2.96	2.92	2.90	2.88	11
3.11	3.07	3.04	3.02	3.00	2.98	2.96	2.91	2.87	2.85	2.82	2.80	2.76	2.74	2.72	12
2.98	2.95	2.92	2.89	2.87	2.85	2.84	2.78	2.74	2.72	2.69	2.67	2.63	2.61	2.60	13
2.88	2.84	2.81	2.79	2.77	2.75	2.73	2.67	2.64	2.61	2.58	2.56	2.53	2.50	2.49	14
2.72	2.68	2.65	2.63	2.60	2.58	2.57	2.51	2.47	2.45	2.42	2.40	2.36	2.33	2.32	16
2.60	2.56	2.53	2.50	2.48	2.46	2.44	2.38	2.35	2.32	2.29	2.27	2.23	2.20	2.19	18
2.50	2.46	2.43	2.41	2.39	2.37	2.35	2.29	2.25	2.22	2.19	2.17	2.13	2.10	2.09	20
2.43	2.39	2.36	2.33	2.31	2.29	2.27	2.21	2.17	2.14	2.11	2.09	2.05	2.02	2.00	22
2.36	2.33	2.30	2.27	2.25	2.23	2.21	2.15	2.11	2.08	2.05	2.02	1.98	1.95	1.94	24
2.31	2.28	2.24	2.22	2.19	2.17	2.16	2.09	2.05	2.03	1.99	1.97	1.92	1.90	1.88	26
2.27	2.23	2.20	2.17	2.15	2.13	2.11	2.05	2.01	1.98	1.94	1.92	1.88	1.85	1.83	28
2.23	2.20	2.16	2.14	2.11	2.09	2.07	2.01	1.97	1.94	1.90	1.88	1.84	1.81	1.79	30
2.11	2.07	2.03	2.01	1.98	1.96	1.94	1.88	1.83	1.80	1.76	1.74	1.69	1.66	1.64	40
2.03	1.99	1.96	1.93	1.91	1.88	1.87	1.80	1.75	1.72	1.68	1.66	1.60	1.57	1.55	50
1.98	1.94	1.91	1.88	1.86	1.83	1.82	1.74	1.70	1.67	1.62	1.60	1.54	1.51	1.48	60
1.93	1.88	1.85	1.82	1.79	1.77	1.75	1.68	1.63	1.60	1.55	1.53	1.47	1.43	1.40	80
1.89	1.85	1.81	1.78	1.76	1.74	1.71	1.64	1.59	1.56	1.51	1.48	1.42	1.38	1.35	100
1.82	1.78	1.74	1.71	1.68	1.66	1.64	1.56	1.51	1.47	1.42	1.39	1.32	1.27	1.23	200
1.78	1.74	1.70	1.67	1.64	1.62	1.60	1.51	1.46	1.42	1.37	1.34	1.25	1.19	1.14	500
1.75	1.71	1.67	1.64	1.61	1.59	1.57	1.48	1.43	1.39	1.33	1.30	1.21	1.13	1.00	∞

TABLE D4(c)

$$F_{0.01, m_1, m_2} \text{ such that } P(F_{m_1, m_2} > F_{0.01, m_1, m_2}) = 0.01$$

$$= \int_{F_{0.01,m \,,m}}^{\infty} f(F) \, dF = 1 - \int_0^{F_{0.01,m \,,m}} f(F) \, dF$$

$F_{.01, m_1, m_2}$ — Area = 0.01
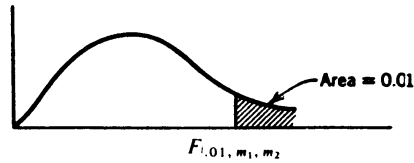
m_2 \ m_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16
*1	405	500	540	563	576	586	593	598	602	606	608	611	613	614	617
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1	27.0	26.9	26.8
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.4	14.3	14.2	14.2
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.96	9.89	9.82	9.77	9.68
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72	7.66	7.60	7.52
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	6.41	6.36	6.27
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	5.61	5.56	5.48
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	5.05	5.00	4.92
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71	4.65	4.60	4.52
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40	4.34	4.29	4.21
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16	4.10	4.05	3.97
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96	3.91	3.86	3.78
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.86	3.80	3.75	3.70	3.62
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55	3.50	3.45	3.37
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37	3.32	3.27	3.19
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23	3.18	3.13	3.05
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12	3.07	3.02	2.94
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03	2.98	2.93	2.85
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96	2.90	2.86	2.78
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90	2·84	2.79	2.72
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84	2.79	2.74	7.66
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66	2.61	2.56	2.48
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.79	2.70	2.63	2.56	2.51	2.46	2.38
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50	2.44	2.39	2.31
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.48	2.42	2.36	2.31	2.23
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.43	2.37	2.31	2.26	2.19
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2 41	2.34	2.27	2.22	2.17	2.09
500	6.69	4.65	3.82	3.36	3.05	2.84	2.68	2.55	2.44	2.36	2.28	2.22	2.17	2.12	2.04
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18	2.13	2.08	2.00

* Multiply the numbers of the first row ($m_2 = 1$) by 10.

TABLE D4(c)—(*Continued*)

$$F_{0.01,\,m_1,\,m_2} \text{ such that } P(F_{m_1,\,m_2} > F_{0.01,\,m_1,\,m_2}) = 0.01$$

$$= \int_{F_{0.01,m\,,m}}^{\infty} f(F)\,dF = 1 - \int_{0}^{F_{0.01,m\,,m}} f(F)\,dF$$

18	20	22	24	26	28	30	40	50	60	80	100	200	500	∞	m_1/m_2
619	621	622	623	624	625	626	629	630	631	633	633	635	636	637	1
99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	2
26.8	26.7	26.6	26.6	26.6	26.5	26.5	26.4	26.4	26.3	26.3	26.2	26.2	26.1	26.1	3
14.1	14.0	14.0	13.9	13.9	13.9	13.8	13.7	13.7	13.7	13.6	13.6	13.5	13.5	13.5	4
9.61	9.55	9.51	9.47	9.43	9.40	9.38	9.29	9.24	9.20	9.16	9.13	9.08	9.04	9.02	5
7.45	7.40	7.35	7.31	7.28	7.25	7.23	7.14	7.09	7.06	7.01	6.99	6.93	6.90	6.88	6
6.21	6.16	6.11	6.07	6.04	6.02	5.99	5.91	5.86	5.82	5.78	5.75	5.70	5.67	5.65	7
5.41	5.36	5.32	5.28	5.25	5.22	5.20	5.12	5.07	5.03	4.99	4.96	4.91	4.88	4.85	8
4.86	4.81	4.77	4.73	4.70	4.67	4.65	4.57	4.52	4.48	4.44	4.42	4.36	4.33	4.31	9
4.46	4.41	4.36	4.33	4.30	4.27	4.25	4.17	4.12	4.08	4.04	4.01	3.96	3.93	3.91	10
4.15	4.10	4.06	4.02	3.99	3.96	3.94	3.86	3.81	3.78	3.73	3.71	3.66	3.62	3.60	11
3.91	3.86	3.82	3.78	3.75	3.72	3.70	3.62	3.57	3.54	3.49	3.47	3.41	3.38	3.36	12
3.72	3.66	3.62	3.59	3.56	3.53	3.51	3.43	3.38	3.34	3.30	3.27	3.22	3.19	3.16	13
3.56	3.51	3.46	3.43	3.40	3.37	3.35	3.27	3.22	3.18	3.14	3.11	3.06	3.03	3.00	14
3.31	3.26	3.22	3.18	3.15	3.12	3.10	3.02	2.97	2.93	2.89	2.86	2.81	2.78	2.75	16
3.13	3.08	3.03	3.00	2.97	2.94	2.92	2.84	2.78	2.75	2.70	2.68	2.62	2.59	2.57	18
2.99	2.94	2.90	2.86	2.83	2.80	2.78	2.69	2.64	2.61	2.56	2.54	2.48	2.44	2.42	20
2.88	2.83	2.78	2.75	2.72	2.69	2.67	2.58	2.53	2.50	2.45	2.42	2.36	2.33	2.31	22
2.79	2.74	2.70	2.66	2.63	2.60	2.58	2.49	2.44	2.40	2.36	2.33	2.27	2.24	2.21	24
2.72	2.66	2.62	2.58	2.55	2.53	2.50	2.42	2.36	2.33	2.28	2.25	2.19	2.16	2.13	26
2.65	2.60	2.56	2.52	2.49	2.46	2.44	2.35	2.30	2.26	2.22	2.19	2.13	2.09	2.06	28
2.60	2.55	2.51	2.47	2.44	2.41	2.39	2.30	2.25	2.21	2.16	2.13	2.07	2.03	2.01	30
2.42	2.37	2.33	2.29	2.26	2.23	2.20	2.11	2.06	2.02	1.97	1.94	1.87	1.83	1.80	40
2.32	2.27	2.22	2.18	2.15	2.12	2.10	2.01	1.95	1.91	1.86	1.82	1.76	1.71	1.68	50
2.25	2.20	2.15	2.12	2.08	2.05	2.03	1.94	1.88	1.84	1.78	1.75	1.68	1.63	1.60	60
2.17	2.12	2.07	2.03	2.00	1.97	1.94	1.85	1.79	1.75	1.69	1.66	1.58	1.53	1.49	80
2.12	2.07	2.02	1.98	1.94	1.92	1.89	1.80	1.73	1.69	1.63	1.60	1.52	1.47	1.43	100
2.02	1.97	1.93	1.89	1.85	1.82	1.79	1.69	1.63	1.58	1.52	1.48	1.39	1.33	1.28	200
1.97	1.92	1.87	1.83	1.79	1.76	1.74	1.63	1.56	1.52	1.45	1.41	1.31	1.23	1.16	500
1.93	1.88	1.83	1.79	1.76	1.72	1.70	1.59	1.52	1.47	1.40	1.36	1.25	1.15	1.00	∞

* Multiply the number of the first row ($m_2 = 1$) by 10.

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