## Casting a polyhedron

# Computational Geometry

## Lecture 5: Casting a polyhedron

# CAD/CAM systems

CAD/CAM systems allow you to design objects and test how they can be constructed

Many objects are constructed used a mold

# Casting

# Casting

A general question: Given an object, can it be made with a particular design process?

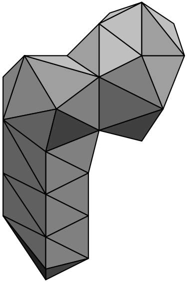
For casting, can the object be removed from its mold without breaking the cast?

# Casting

Objects to be made are 3D polyhedra

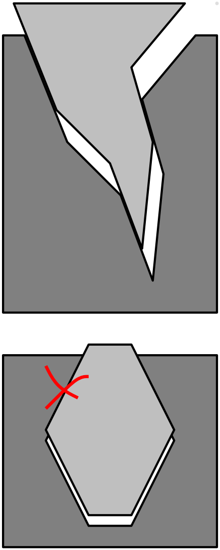
Its boundary is like a planar graph, but the coordinates of vertices are 3D

We can use a doubly-connected edge list with three coordinates in each vertex object

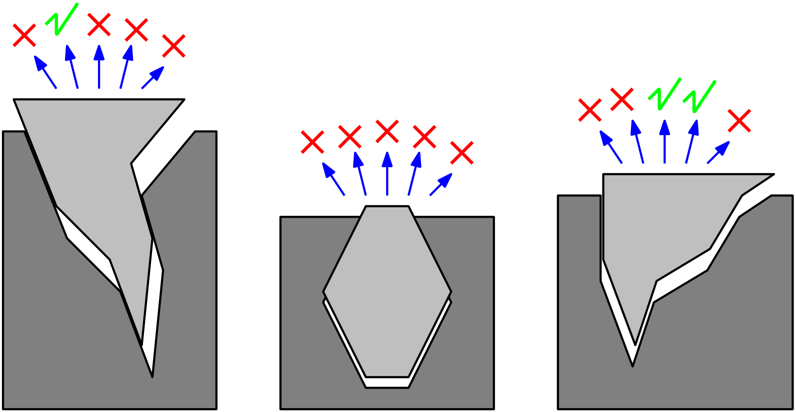


# Casting in 2D

First the 2D version: can we remove a 2D polygon from a mold?



# Casting in 2D



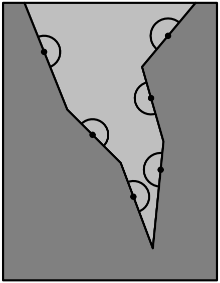
Certain removal directions may be good while others are not

# Casting in 2D

What top facet should we use?

When can we even begin to move the object out?

What kind of movements do we allow?

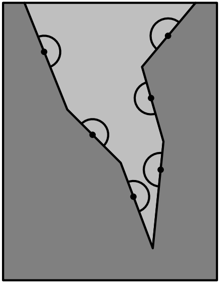


# Casting in 2D

Assume the top facet is fixed; we can try all

Let us consider *translations* only

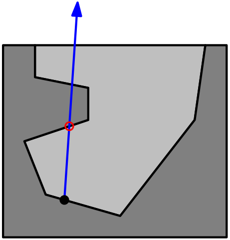
An edge of the polygon should not *directly* run into the coinciding mold edge



# Casting in 2D

**Observe:** For a given top facet, if the object can be translated over some (small) distance, then it can be translated all the way out

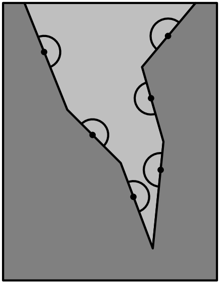
Consider a point $p$ that at first translates away from its mold side, but later runs into the mold …



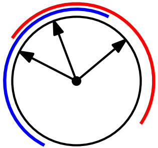
# Casting in 2D

A polygon can be removed from its cast *by a single translation* if and only if there is a direction so that every polygon edge does not cross the adjacent mold edge

Sequences of translations do not help; we would not be able to construct more shapes than by a single translation

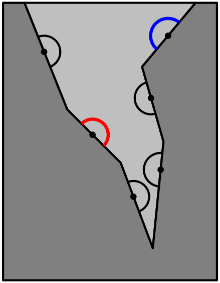


# Circle of directions



We need a representation of directions in 2D

Every polygon edge requires the removal direction to be in a semi-circle



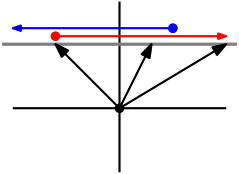
⇒ compute the common intersection of a set of circular intervals (semi-circles)

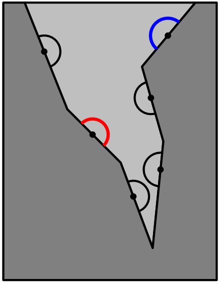
# Line of directions

We only need to represent upward directions: we can use points on the line $y = 1$

Every polygon edge requires the removal direction to be in a half-line

$\Rightarrow$ compute the common intersection of a set of half-lines in 1D

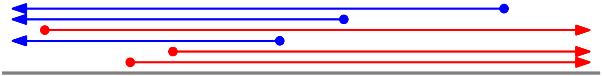




# Common intersection of half-lines

The common intersection of a set of half-lines in 1D:

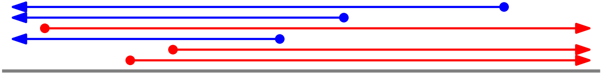
- Determine the endpoint $p_l$ of the rightmost left-bounded half-line
- Determine the endpoint $p_r$ of the leftmost right-bounded half-line
- The common intersection is $[p_l, p_r]$ (can be empty)



# Common intersection of half-lines

The algorithm takes only $O(n)$ time for $n$ half-lines

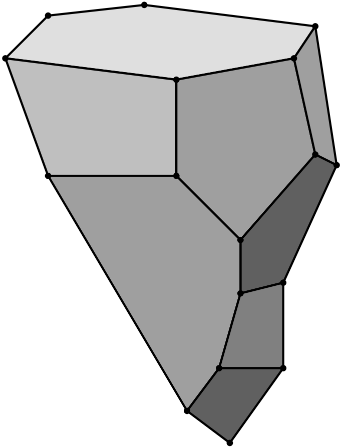
Note: we need not sort the endpoints



# Casting in 3D

Can we do something similar in 3D?

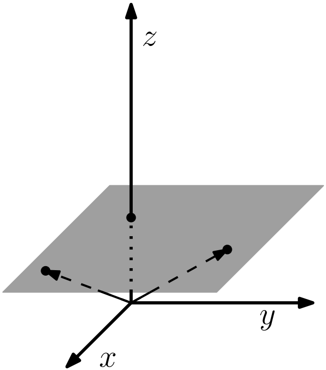
Again each facet must not move into the corresponding mold facet



# Representing directions in 3D

The circle of directions for 2D becomes a sphere of directions for 3D; the line of directions for 2D becomes a plane of directions for 3D: take $z = 1$

Which directions represented in the plane does a facet rule out as removal directions?

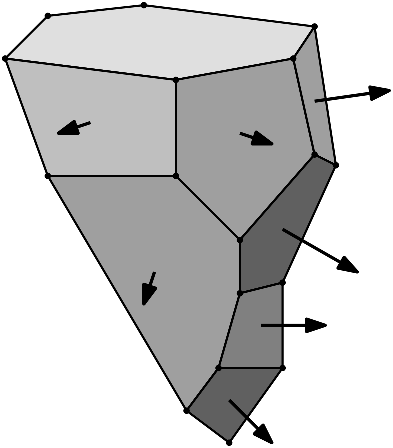


# Directions in 3D

Consider the outward normal vectors of all facets

An allowed removal direction must make an angle of at least $\pi/2$ with every facet (except the topmost one)

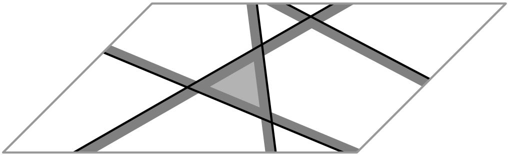
$\Rightarrow$ every facet in 3D makes a half-plane in $z = 1$ invalid



# Common intersection of half-planes

We get: common intersection of half-planes in the plane

The problem of deciding castability of a polyhedron with $n$ facets, with a given top facet, where the polyhedron must be removed from the cast by a single translation, can be solved by computing the common intersection of $n-1$ half-planes



# Common intersection of half-planes

Half-planes in the plane:

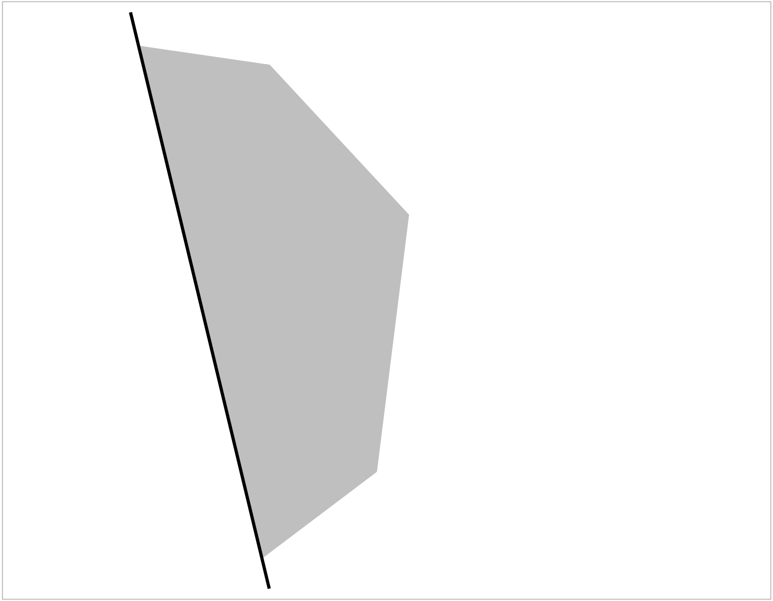
- $y \geq m \cdot x + c$
- $y \leq m \cdot x + c$
- $x \geq c$
- $x \leq c$

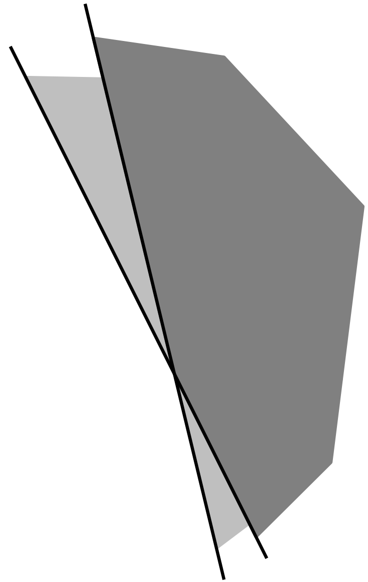
# An approach

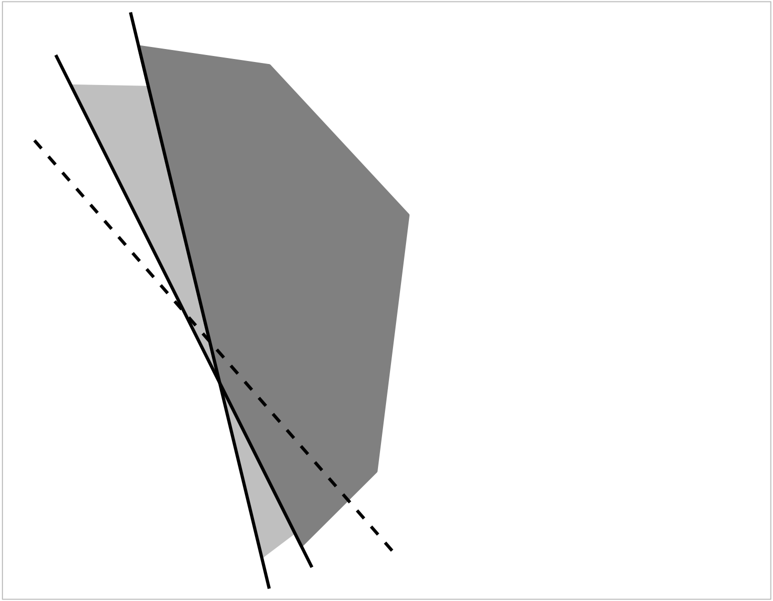
Take the first set:

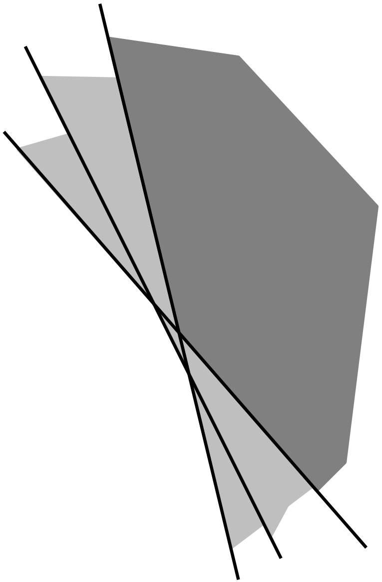
- $y \geq m \cdot x + c$

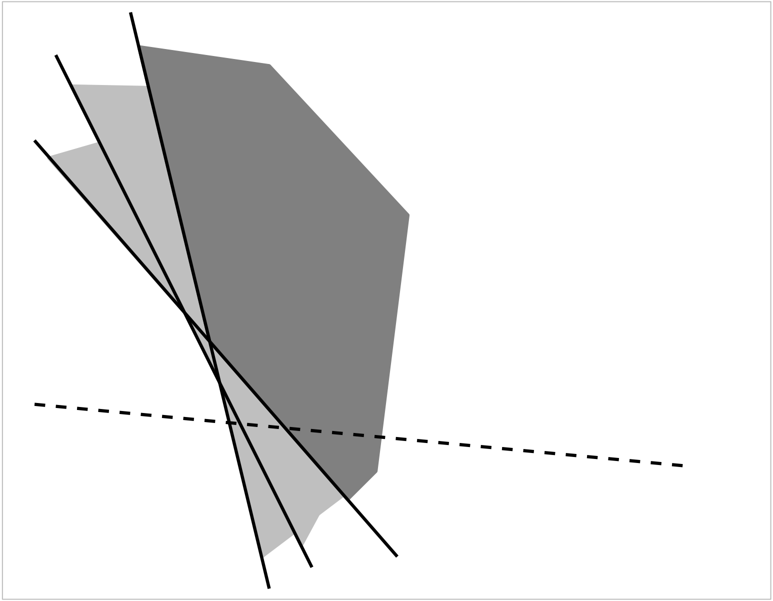
Sort by angle, and add incrementally

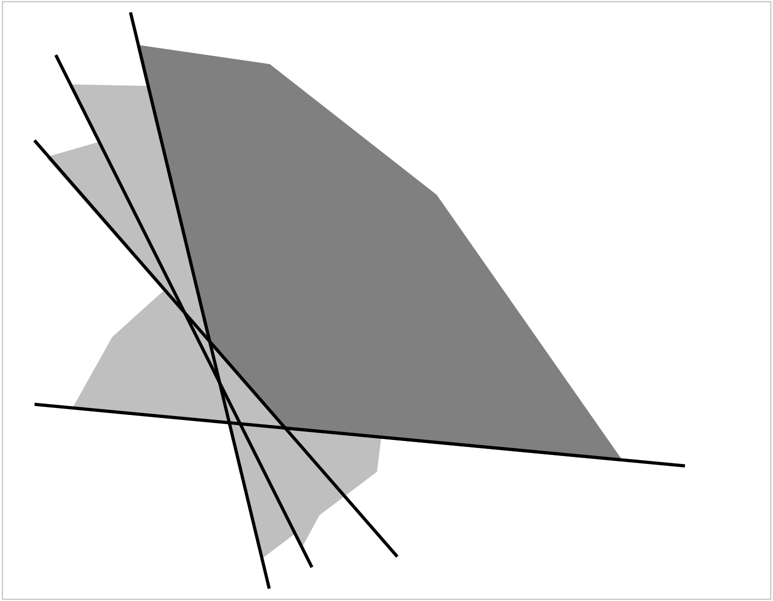


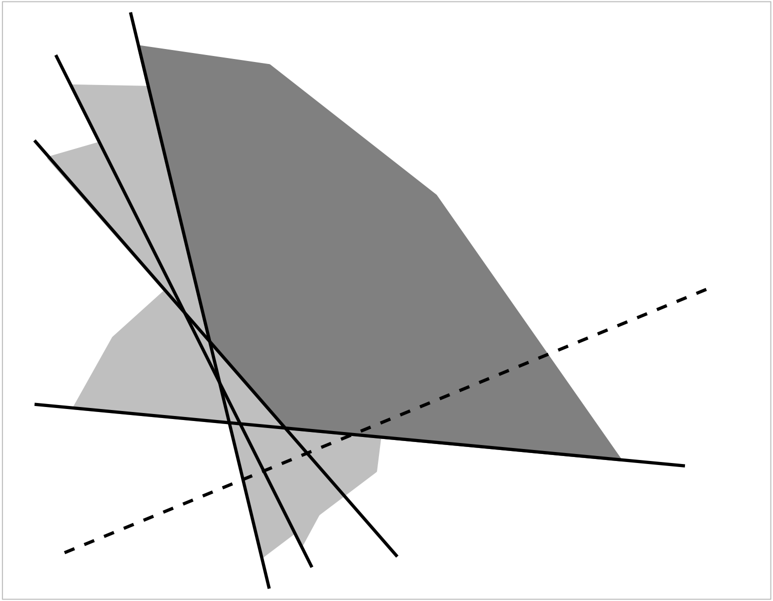


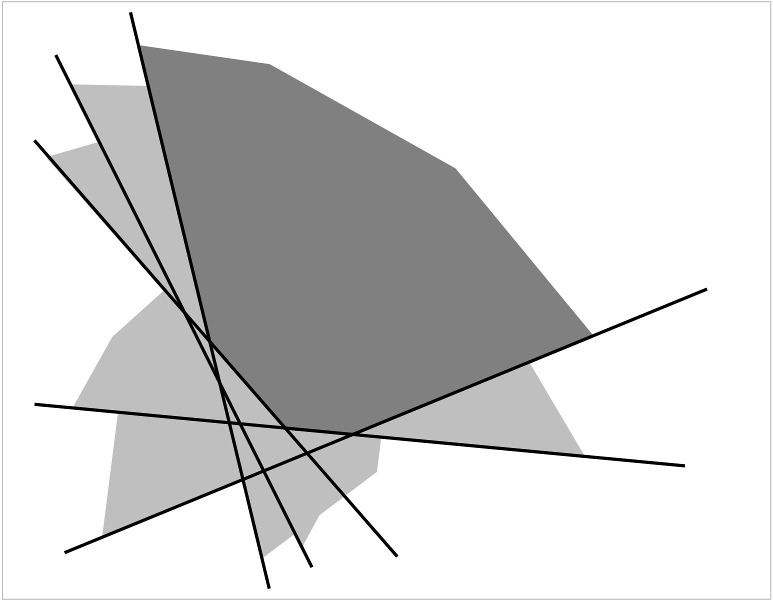


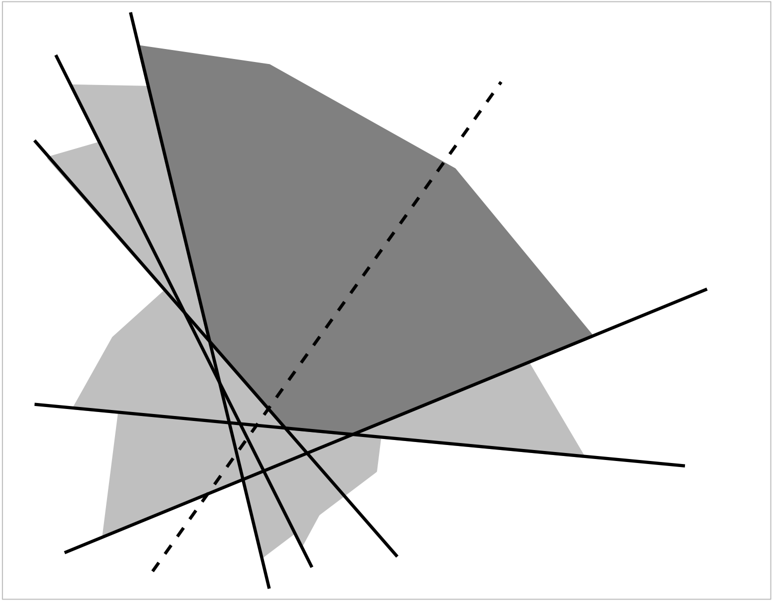








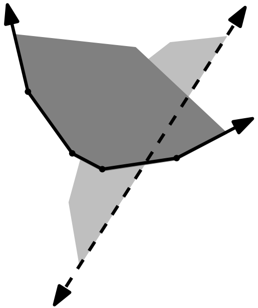




# Incremental common intersection

The boundary of the valid region is a polygonal convex chain that is unbounded at both sides

The next half-plane has a steeper bounding line and will always contribute to the next valid region

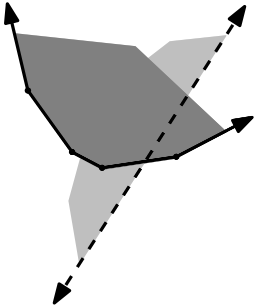


# Incremental common intersection

Maintain the contributing bounding lines in increasing angular order

For the new half-plane, remove any no longer contributing bounding lines from the end

Then add the line bounding the new half-plane



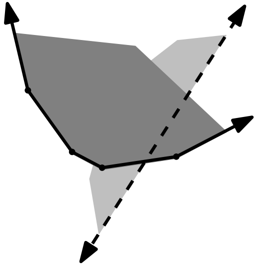
# Incremental common intersection

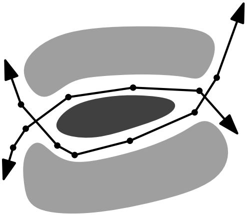
After sorting on angle, this takes only $O(n)$ time

**Question:** Why?

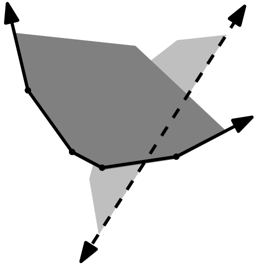
The half-planes bounded from above give a similar chain

Intersecting the two chains is simple with a left-to-right scan



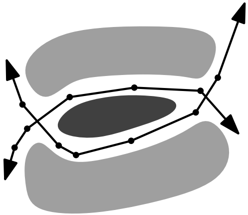


# Incremental common intersection



Half-planes with vertical bounding lines can be added by restricting the region even more

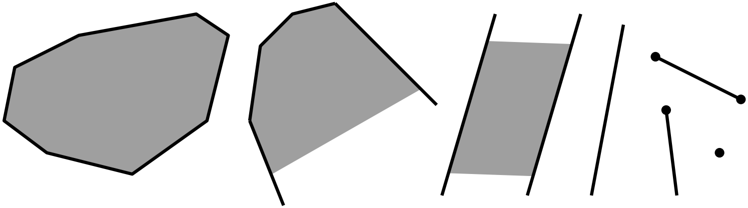
This can also be done in linear time



# Result

**Theorem:** The common intersection of $n$ half-planes in the plane can be computed in $O(n \log n)$ time

The common intersection may be empty, or a convex polygon that can be bounded or unbounded



# Back to casting

The common intersection of half-planes cannot be computed faster (we are sorting the lines along the boundary)

The region we compute represents *all mold removal directions*

...

... but to determine castability, we only need one!

# Linear programming

We will find the *lowest* point in the common intersection

Notice that half-planes are linear constraints

**Minimize** $y$

**Subject to**

$$
\begin{aligned}
y &\geq m_1 \cdot x + c_1 \\
y &\geq m_2 \cdot x + c_2 \\
&\vdots \\
y &\geq m_i \cdot x + c_i \\
y &\leq m_{i+1} \cdot x + c_{i+1} \\
&\vdots \\
y &\leq m_n \cdot x + c_n
\end{aligned}
$$

# Linear programming

**Minimize** $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$

**Subject to**

$$\begin{array}{c} a_{1,1} \cdot x_1 + \cdots + a_{k,1} \cdot x_k \leq b_1 \\ a_{1,2} \cdot x_1 + \cdots + a_{k,2} \cdot x_k \leq b_2 \\ \vdots \\ a_{1,n} \cdot x_1 + \cdots + a_{k,n} \cdot x_k \leq b_n \end{array}$$

where $a_{1,1}, \ldots, a_{k,n}$, $b_1, \ldots, b_n$, $c_1, \ldots, c_k$ are given coefficients

This is LP with $k$ unknowns (dimensions) and $n$ inequalities

**Question:** Where are the $\geq$ inequalities?

# Terminology

LP with $k$ unknowns (dimensions) and $n$ inequalities: $k$-dimensional linear programming

The subspace that is the common intersection is the feasible region. If it is empty, the LP is infeasible

The vector $(c_1, \ldots, c_k)^T$ is the objective vector or cost vector

If the LP has solutions with arbitrarily low cost, then the LP is unbounded

Note: The feasible region may be unbounded while the LP is bounded

# LP for casting

It is 2-dimensional linear programming with $n$ constraints

We only want to decide feasibility, so we can choose any objective function

We will make it ourselves easy

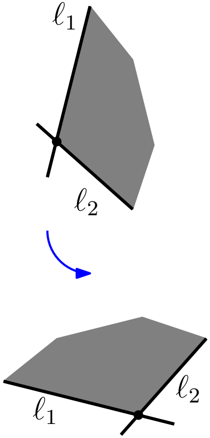
# Incremental LP

Let $h_1, \ldots, h_n$ be the constraints and $\ell_1, \ldots, \ell_n$ their bounding lines

Find any two constraints $h_1$ and $h_2$ where $\ell_1$ and $\ell_2$ are non-parallel

Rotate $h_1$ and $h_2$ over an angle $\alpha$ around the origin to make $\ell_1 \cap \ell_2$ the optimal solution for the objective function that minimizes $y$

Rotate all other constraints over $\alpha$ too

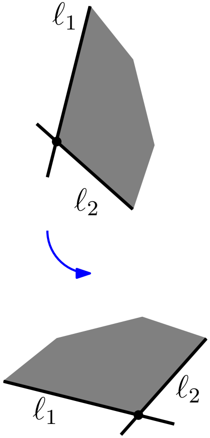


# Incremental LP

Solve the LP with the rotated constraints

If the rotated LP is infeasible, then so is the unrotated version

If the rotated LP gives an optimal solution $(p_x, p_y)$, then rotate if over an angle $-\alpha$ around the origin to get the removal direction for the original position of the polyhedron



# Incremental LP

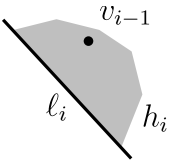
The algorithm adds the constraints $h_3, \ldots, h_n$ incrementally and maintains the optimum so far

Let $H_i = \{h_1, \ldots, h_i\}$

Let $v_i$ be the optimum for $H_i$ (unless we already have infeasibility)

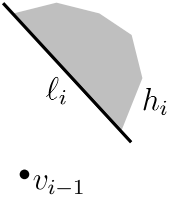
# LP for casting

The incremental step: suppose we know $v_{i-1}$ and want to add $h_i$

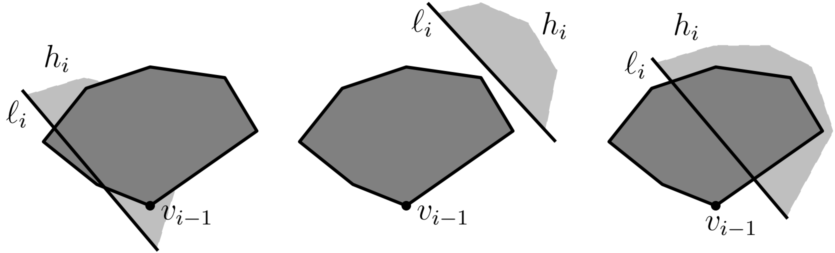



There are two possibilities:

- If $v_{i-1} \in h_i$, then $v_i = v_{i-1}$
- If $v_{i-1} \notin h_i$, then either the LP is infeasible, or $v_i$ lies on $\ell_i$



# Incremental LP



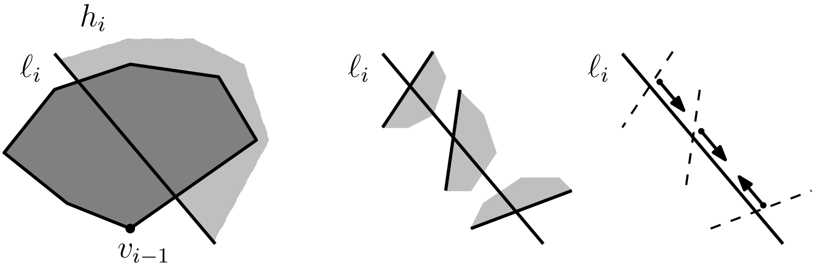
# LP for casting

**Algorithm** $\text{LPFORCASTING}(H)$

1. Let $h_1$, $h_2$, and $v_2$ be as chosen
2. **for** $i \leftarrow 3$ **to** $n$
3.     **do if** $v_{i-1} \in h_i$
4.         **then** $v_i \leftarrow v_{i-1}$
5.         **else** $v_i \leftarrow$the point $p$ on $\ell_i$ that minimizes $y$, subject to the constraints in $H_{i-1}$.
6.             **if** $p$ does not exist
7.                 **then** Report that the LP is infeasible, and quit.
8. **return** $v_n$

# LP for casting

If $v_{i-1} \notin h_i$, how do we find the point $p$ on $\ell_i$?

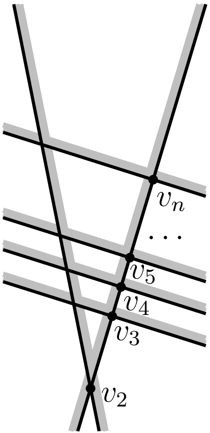


# Efficiency

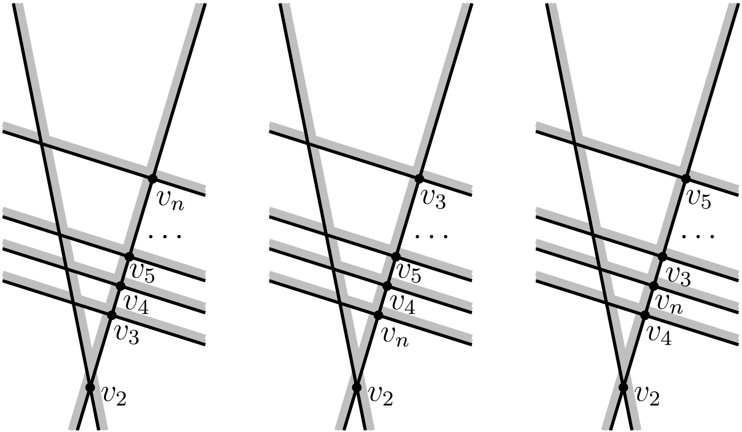
If $v_{i-1} \in h_i$, then the incremental step takes only $O(1)$ time

If $v_{i-1} \notin h_i$, then the incremental step takes $O(i)$ time

The LP-for-casting algorithm takes $O(n^2)$ time in the worst case



# Efficiency



# Randomized algorithm

**Algorithm** $\textsc{RandomizedLPforCasting}(H)$

Let $h_1$, $h_2$, and $v_2$ be as chosen
Let $h_3, h_4, \ldots, h_n$ be in a random order
**for** $i \leftarrow 3$ **to** $n$
  **do if** $v_{i-1} \in h_i$
    **then** $v_i \leftarrow v_{i-1}$
    **else** $v_i \leftarrow$the point $p$ on $\ell_i$ that minimizes $y$, subject to the constraints in $H_{i-1}$.
      **if** $p$ does not exist
        **then** Report that the LP is infeasible, and quit.
**return** $v_n$

# Putting in random order

The constraints may be given in any order, the algorithm will just reorder them

- Let $j$ be a random integer in $[3, n]$
- Swap $h_j$ and $h_n$
- Recursively shuffle $h_3, \ldots, h_{n-1}$

Putting in random order takes $O(n)$ time

# Expected running time

Every one of the $(n-2)!$ orders is equally likely

The expected time taken by the algorithm is the *average* time over all orders

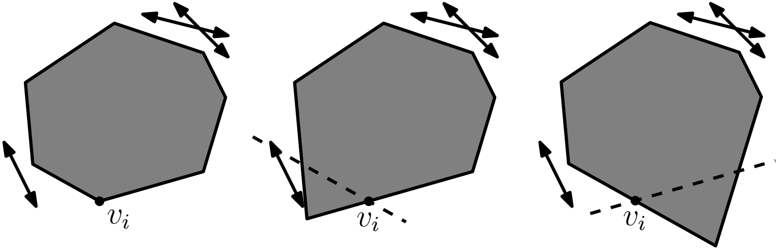
$$\frac{1}{(n-2)!} \cdot \sum_{\Pi \text{ permutation}} \text{time if the random order is } \Pi$$

# Expected running time

If the order of the constraints $h_3,\ldots,h_n$ is random, what is the probability that $v_{i-1} \in h_i$ ?

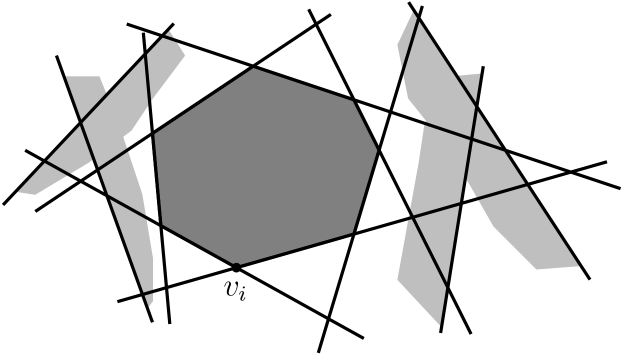
We use backwards analysis: consider the situation *after $h_i$ is inserted*, and $v_i$ is computed (either by $v_i = v_{i-1}$, or somewhere on $\ell_i$)

# Expected running time




Only if one of the dashed lines was $\ell_i$, the last step where $h_i$ was added was expensive and took $\Theta(i)$ time

# Expected running time




If $h_i$ does not bound the feasible region, or not at $v_i$, then the addition step was cheap and took $\Theta(1)$ time

# Expected running time

There are $i-2$ half-planes that could have been one of the lines defining $v_i$

Since the order was random, each of the $i-2$ half-planes has the same probability to be the last one added, and only $2$ of these caused the expensive step

- $2$ out of $i-2$ cases: expensive step; $\Theta(i)$ time for $i$-th addition
- $i-4$ out of $i-2$ cases: cheap step; $\Theta(1)$ time for $i$-th addition

# Expected running time

Expected time for $i$-th addition:

$$\frac{i-4}{i-2} \cdot \Theta(1) + \frac{2}{i-2} \cdot \Theta(i) = \Theta(1)$$

Total running time:

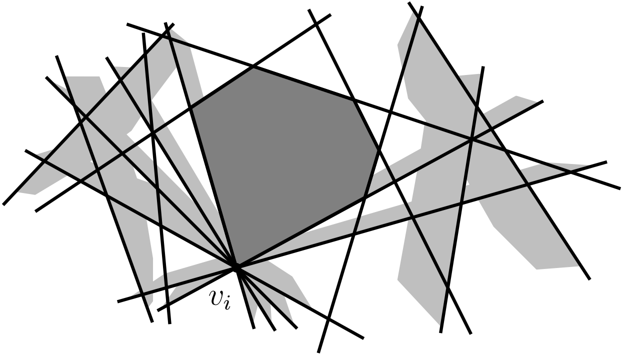
$$\Theta(n) + \sum_{i=3}^{n} \Theta(1) = \Theta(n) \text{ expected time}$$

# Degenerate cases

The optimal solution may not be unique, if the feasible region is bounded from below by a horizontal line. How to solve it?

There may be many lines from $\ell_3, \ldots, \ell_i$ passing through $v_i$; how does this affect the probability of an expensive step?

# Degenerate cases



# Degenerate cases

In degenerate cases, the probability that the last addition was expensive is even smaller: $1/(i-2)$, or $0$

Without any adaptations, the running time holds

# Result

**Theorem:** Castability of a simple polyhedron with $n$ facets, given a top facet, can be decided in $O(n)$ expected time

**Theorem:** 2-dimensional linear programming with $n$ constraints can be solved in $O(n)$ expected time

**Question:** What does "expected time" mean? Expectation over what?

# Higher dimensions?

**Question:** Can you imagine whether we can also solve 3-dimensional linear programming efficiently?