Seventh Edition

CHAPTER

14

VECTOR MECHANICS FOR ENGINEERS:

DYNAMICS

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Lecture Notes:

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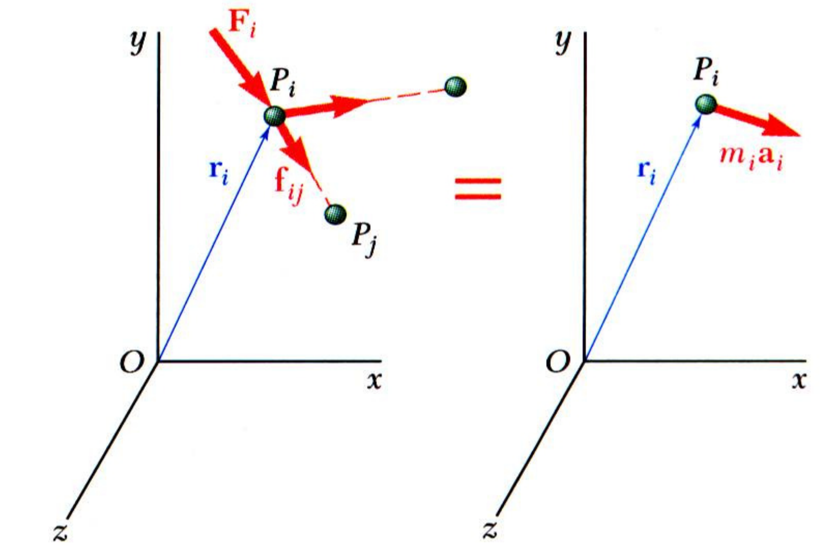
Systems of Particles

Contents

Introduction

- In the current chapter, you will study the motion of *systems of particles*.
- The *effective force* of a particle is defined as the product of it mass and acceleration. It will be shown that the *system of external forces* acting on a system of particles is *equipollent* with the *system of effective forces* of the system.
- The *mass center* of a system of particles will be defined and its motion described.
- Application of the *work-energy principle* and the *impulse-momentum principle* to a system of particles will be described. Result obtained are also applicable to a system of rigidly connected particles, i.e., a *rigid body*.
- Analysis methods will be presented for *variable systems of particles*, i.e., systems in which the particles included in the system change.

Application of Newton's Laws. Effective Forces



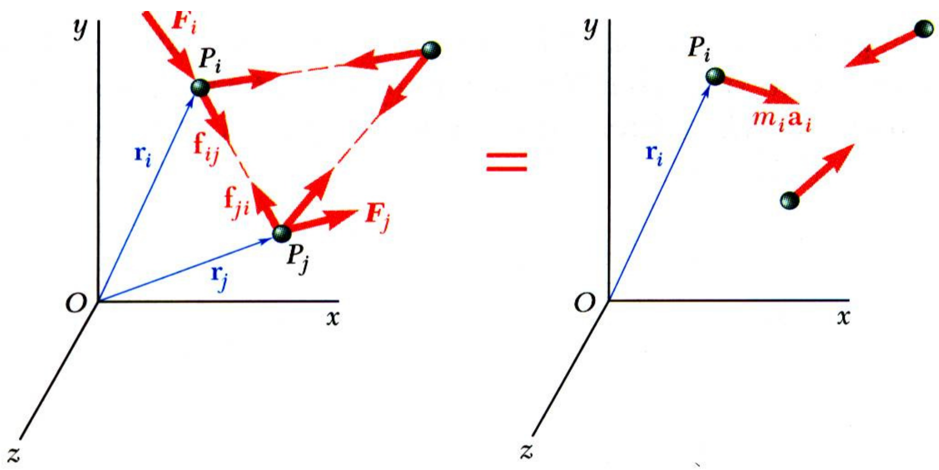
- Newton's second law for each particle P_i in a system of n particles,

$$\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i \vec{a}_i$$

$$\vec{r}_i \times \vec{F}_i + \sum_{j=1}^{n} \left(\vec{r}_i \times \vec{f}_{ij} \right) = \vec{r}_i \times m_i \vec{a}_i$$

$\vec{F}_i$ = external force $\qquad$ $\vec{f}_{ij}$ = internal forces

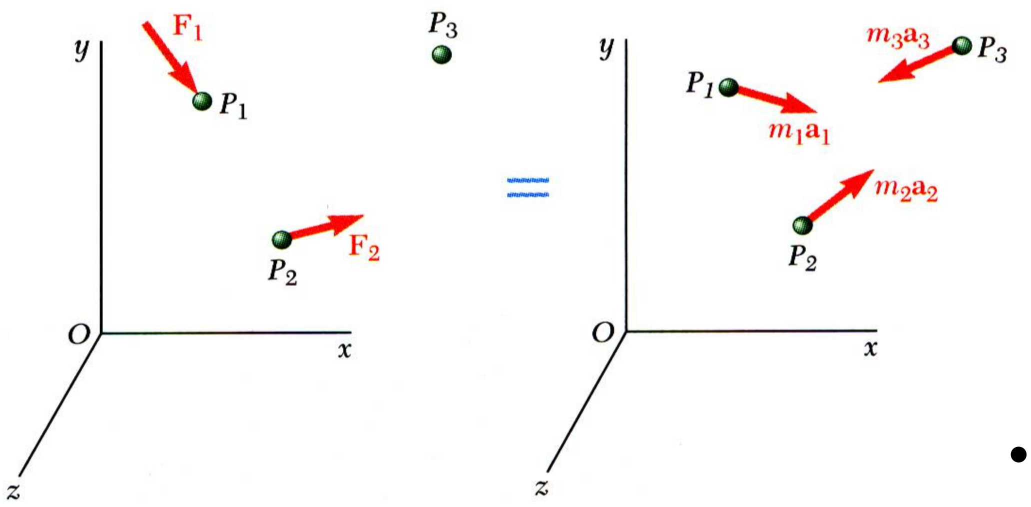
$m_i \vec{a}_i$ = effective force

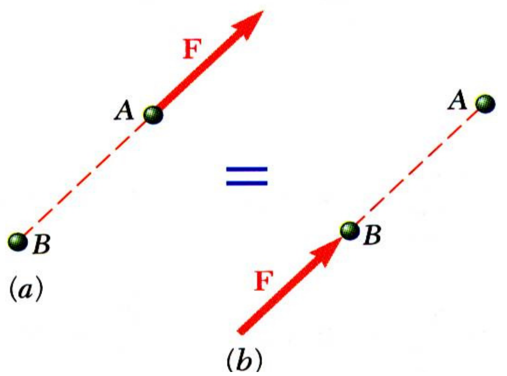


- The system of external and internal forces on a particle is *equivalent* to the effective force of the particle.

- The system of external and internal forces acting on the entire system of particles is *equivalent* to the system of effective forces.

Application of Newton's Laws. Effective Forces





- Summing over all the elements,

$$\sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \vec{f}_{ij} = \sum_{i=1}^{n} m_i \vec{a}_i$$

$$\sum_{i=1}^{n} \left(\vec{r}_i \times \vec{F}_i\right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\vec{r}_i \times \vec{f}_{ij}\right) = \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

- Since the internal forces occur in equal and opposite collinear pairs, the resultant force and couple due to the internal forces are zero,

$$\sum \vec{F}_i = \sum m_i \vec{a}_i$$
$$\sum \left(\vec{r}_i \times \vec{F}_i\right) = \sum \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

- The system of external forces and the system of effective forces are *equipollent* by not *equivalent*.

Linear & Angular Momentum

- Linear momentum of the system of particles,

$$\vec{L} = \sum_{i=1}^{n} m_i \vec{v}_i$$

$$\dot{\vec{L}} = \sum_{i=1}^{n} m_i \dot{\vec{v}}_i = \sum_{i=1}^{n} m_i \vec{a}_i$$

- Resultant of the external forces is equal to rate of change of linear momentum of the system of particles,

$$\sum \vec{F} = \dot{\vec{L}}$$

- Angular momentum about fixed point O of system of particles,

$$\vec{H}_O = \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \vec{v}_i \right)$$

$$\dot{\vec{H}}_O = \sum_{i=1}^{n} \left(\dot{\vec{r}}_i \times m_i \vec{v}_i \right) + \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \dot{\vec{v}}_i \right)$$

$$= \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \vec{a}_i \right)$$

- Moment resultant about fixed point O of the external forces is equal to the rate of change of angular momentum of the system of particles,

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$

Motion of the Mass Center of a System of Particles

- Mass center G of system of particles is defined by position vector $\vec{r}_G$ which satisfies

$$m\vec{r}_G = \sum_{i=1}^{n} m_i \vec{r}_i$$

- Differentiating twice,

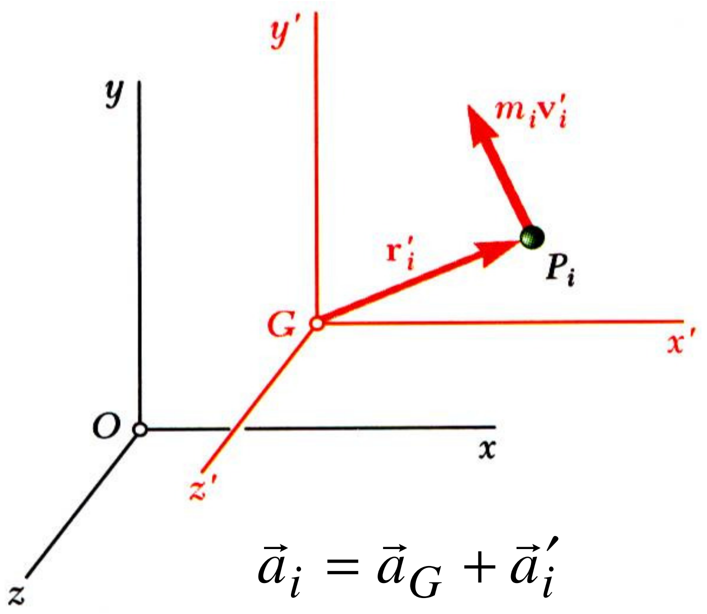
$$m\dot{\vec{r}}_G = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i$$

$$m\vec{v}_G = \sum_{i=1}^{n} m_i \vec{v}_i = \vec{L}$$

$$m\vec{a}_G = \dot{\vec{L}} = \sum \vec{F}$$

- The mass center moves as if the entire mass and all of the external forces were concentrated at that point.

Angular Momentum About the Mass Center



$$\vec{a}_i = \vec{a}_G + \vec{a}'_i$$

- Consider the centroidal frame of reference *Gx'y'z'*, which translates with respect to the Newtonian frame *Oxyz*.
- The centroidal frame is not, in general, a Newtonian frame.

- The angular momentum of the system of particles about the mass center,

$$\vec{H}'_G = \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i \vec{v}'_i\right)$$

$$\dot{\vec{H}}'_G = \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i \vec{a}'_i\right) = \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i\left(\vec{a}_i - \vec{a}_G\right)\right)$$

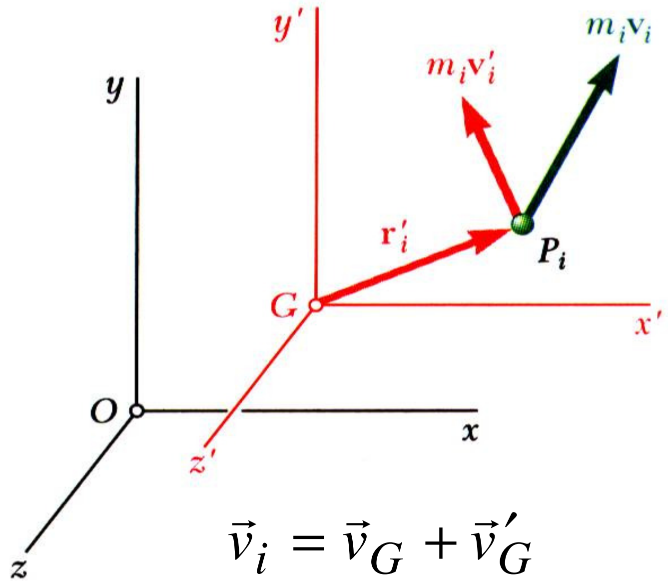
$$= \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i \vec{a}_i\right) - \left(\sum_{i=1}^{n} m_i \vec{r}'\right) \times \vec{a}_G$$

$$= \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i \vec{a}_i\right) = \sum_{i=1}^{n}\left(\vec{r}'_i \times \vec{F}_i\right)$$

$$= \sum \vec{M}_G$$

- The moment resultant about *G* of the external forces is equal to the rate of change of angular momentum about *G* of the system of particles.

Angular Momentum About the Mass Center



$$\vec{v}_i = \vec{v}_G + \vec{v}'_G$$

- Angular momentum about *G* of the particles in their motion relative to the centroidal *Gx'y'z'* frame of reference,

$$\vec{H}'_G = \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i\vec{v}'_i\right)$$

- Angular momentum about *G* of particles in their absolute motion relative to the Newtonian *Oxyz* frame of reference.

$$\vec{H}_G = \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i\vec{v}_i\right)$$

$$= \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i\left(\vec{v}_G + \vec{v}'_i\right)\right)$$

$$= \left(\sum_{i=1}^{n} m_i\vec{r}'_i\right) \times \vec{v}_G + \sum_{i=1}^{n}\left(\vec{r}'_i \times m_i\vec{v}_i\right)$$

$$\vec{H}_G = \vec{H}'_G = \sum \vec{M}_G$$

- Angular momentum about *G* of the particle momenta can be calculated with respect to either the Newtonian or centroidal frames of reference.

Conservation of Momentum

- If no external forces act on the particles of a system, then the linear momentum and angular momentum about the fixed point *O* are conserved.

$$\dot{\vec{L}} = \sum \vec{F} = 0 \qquad \dot{\vec{H}}_O = \sum \vec{M}_O = 0$$
$$\vec{L} = \text{constant} \qquad \vec{H}_O = \text{constant}$$

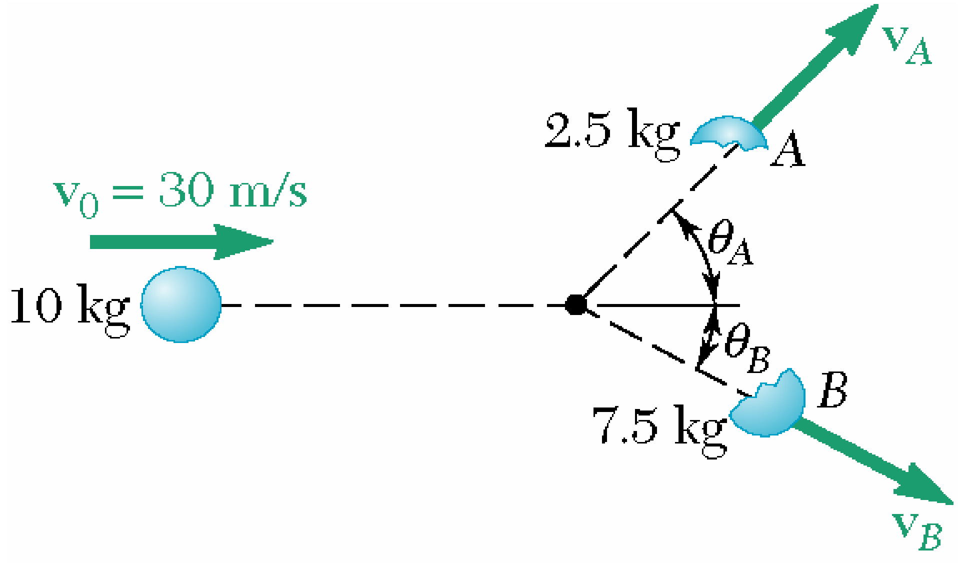
- In some applications, such as problems involving central forces,

$$\dot{\vec{L}} = \sum \vec{F} \neq 0 \qquad \dot{\vec{H}}_O = \sum \vec{M}_O = 0$$
$$\vec{L} \neq \text{constant} \qquad \vec{H}_O = \text{constant}$$

- Concept of conservation of momentum also applies to the analysis of the mass center motion,

$$\dot{\vec{L}} = \sum \vec{F} = 0 \qquad \dot{\vec{H}}_G = \sum \vec{M}_G = 0$$
$$\vec{L} = m\vec{v}_G = \text{constant}$$
$$\vec{v}_G = \text{constant} \qquad \vec{H}_G = \text{constant}$$

Sample Problem 14.2



A 10 kg projectile is moving with a velocity of 30 m/s when it explodes into 2.5 and 7.5 kg fragments. Immediately after the explosion, the fragments travel in the directions $\theta_A = 45^o$ and $\theta_B = 30^o$.

Determine the velocity of each fragment.

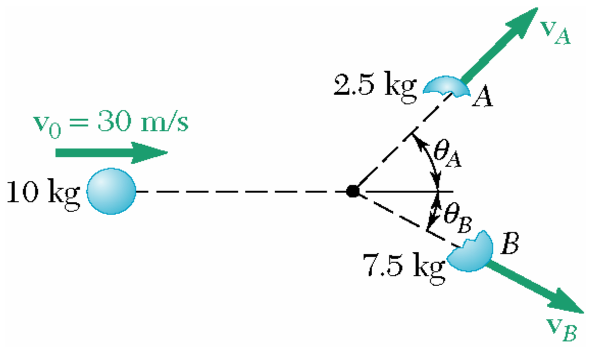
SOLUTION:

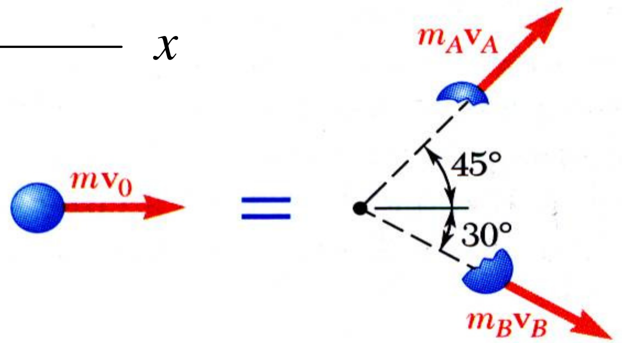
- Since there are no external forces, the linear momentum of the system is conserved.
- Write separate component equations for the conservation of linear momentum.
- Solve the equations simultaneously for the fragment velocities.

Sample Problem 14.2

SOLUTION:

- Since there are no external forces, the linear momentum of the system is conserved.





- Write separate component equations for the conservation of linear momentum.

$$m_A \vec{v}_A + m_B \vec{v}_B = m\vec{v}_0$$

$$(2.5)\vec{v}_A + (7.5)\vec{v}_B = (10)\vec{v}_0$$

x components:

$$2.5 v_A \cos 45° + 7.5 v_B \cos 30° = 10(30)$$

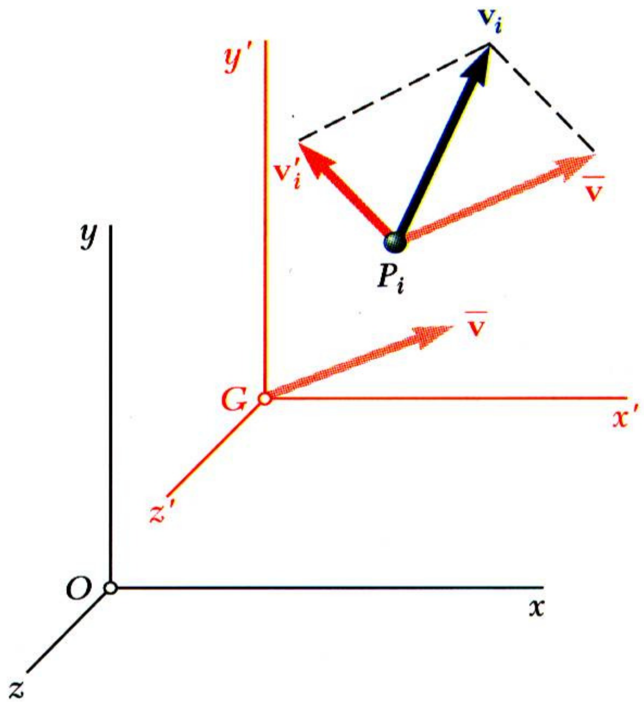
y components:

$$2.5 v_A \sin 45° - 7.5 v_B \sin 30° = 0$$

- Solve the equations simultaneously for the fragment velocities.

$$v_A = 62.1\,\text{m/s} \qquad v_B = 29.3\,\text{m/s}$$

Kinetic Energy



$$\vec{v}_i = \vec{v}_G + \vec{v}_i'$$

- Kinetic energy of a system of particles,

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\vec{v}_i \bullet \vec{v}_i\right) = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2$$

- Expressing the velocity in terms of the centroidal reference frame,

$$T = \frac{1}{2}\sum_{i=1}^{n}\left[m_i\left(\vec{v}_G + \vec{v}_i'\right)\bullet\left(\vec{v}_G + \vec{v}_i'\right)\right]$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right) v_G^2 + \vec{v}_G \bullet \sum_{i=1}^{n} m_i \vec{v}_i' + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2$$

$$= \frac{1}{2} m \vec{v}_G^2 + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2$$

- Kinetic energy is equal to kinetic energy of mass center plus kinetic energy relative to the centroidal frame.

Work-Energy Principle. Conservation of Energy

- Principle of work and energy can be applied to each particle P_i,

$$T_1 + U_{1\to 2} = T_2$$

where $U_{1\to 2}$ represents the work done by the internal forces $\vec{f}_{ij}$ and the resultant external force $\vec{F}_i$ acting on P_i.

- Principle of work and energy can be applied to the entire system by adding the kinetic energies of all particles and considering the work done by all external and internal forces.

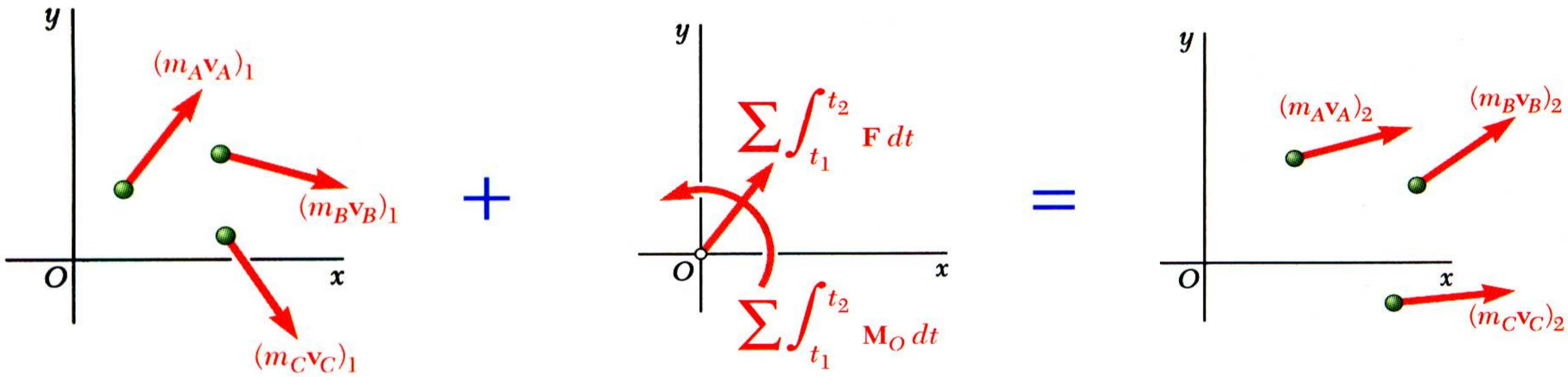
- Although $\vec{f}_{ij}$ and $\vec{f}_{ji}$ are equal and opposite, the work of these forces will not, in general, cancel out.

- If the forces acting on the particles are conservative, the work is equal to the change in potential energy and

$$T_1 + V_1 = T_2 + V_2$$

which expresses the principle of conservation of energy for the system of particles.

Principle of Impulse and Momentum



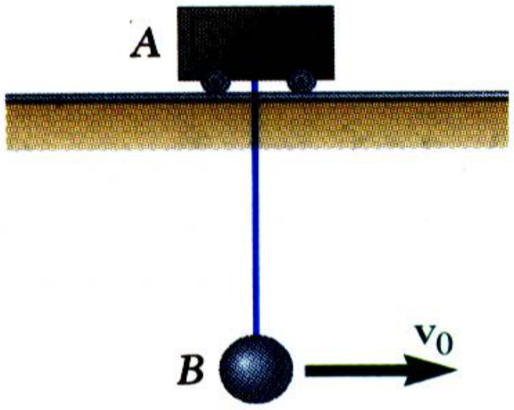
$$\sum \vec{F} = \dot{\vec{L}} \qquad \sum \vec{M}_O = \dot{\vec{H}}_O$$

$$\sum \int_{t_1}^{t_2} \vec{F}dt = \vec{L}_2 - \vec{L}_1 \qquad \sum \int_{t_1}^{t_2} \vec{M}_O dt = \vec{H}_2 - \vec{H}_1$$

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F}dt = \vec{L}_2 \qquad \vec{H}_1 + \sum \int_{t_1}^{t_2} \vec{M}_O dt = \vec{H}_2$$

- The momenta of the particles at time t_1 and the impulse of the forces from t_1 to t_2 form a system of vectors *equipollent* to the system of momenta of the particles at time t_2 .

Sample Problem 14.4



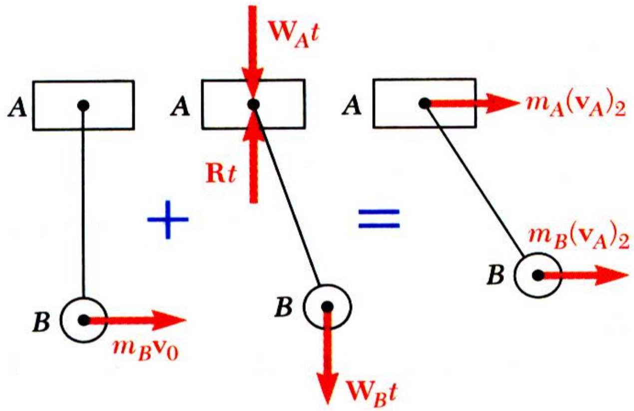
Ball *B*, of mass m_B,is suspended from a cord, of length *l*, attached to cart *A*, of mass m_A, which can roll freely on a frictionless horizontal tract. While the cart is at rest, the ball is given an initial velocity $v_0 = \sqrt{2gl}$.

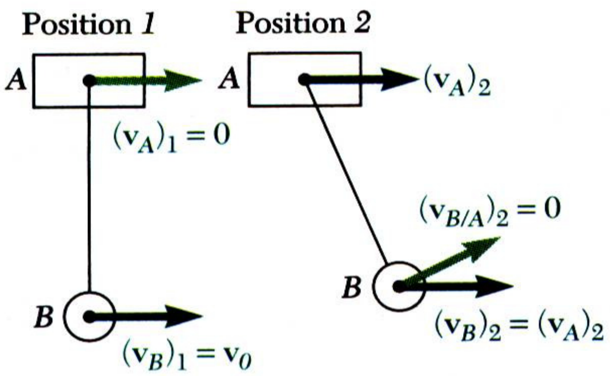
Determine *(a)* the velocity of *B* as it reaches it maximum elevation, and *(b)* the maximum vertical distance *h* through which *B* will rise.

SOLUTION:

- With no external horizontal forces, it follows from the impulse-momentum principle that the horizontal component of momentum is conserved. This relation can be solved for the velocity of *B* at its maximum elevation.
- The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy. The maximum vertical distance is determined from this relation.

Sample Problem 14.4





SOLUTION:

- With no external horizontal forces, it follows from the impulse-momentum principle that the horizontal component of momentum is conserved. This relation can be solved for the velocity of *B* at its maximum elevation.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2$$

x component equation:

$$m_A v_{A,1} + m_B v_{B,1} = m_A v_{A,2} + m_B v_{B,2}$$

Velocities at positions 1 and 2 are

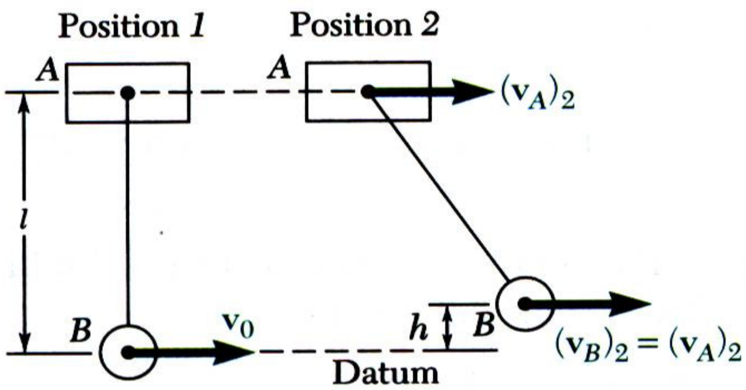
$$v_{A,1} = 0 \qquad v_{B,1} = v_0$$

$$v_{B,2} = v_{A,2} + v_{B/A,2} = v_{A,2}$$ (velocity of *B* relative to *A* is zero at position 2)

$$m_B v_0 = (m_A + m_B) v_{A,2}$$

$$v_{A,2} = v_{B,2} = \frac{m_B}{m_A + m_B} v_0$$

Sample Problem 14.4



- The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy.

$$T_1 + V_1 = T_2 + V_2$$

Position 1 - Potential Energy: $V_1 = m_A g l$

Kinetic Energy: $T_1 = \frac{1}{2} m_B v_0^2$

Position 2 - Potential Energy: $V_2 = m_A g l + m_B g h$

Kinetic Energy: $T_2 = \frac{1}{2}(m_A + m_B) v_{A,2}^2$

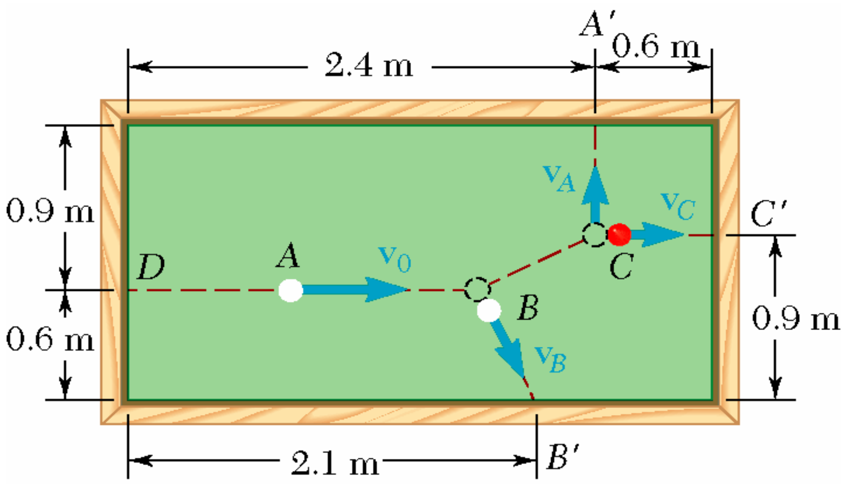
$$\tfrac{1}{2} m_B v_0^2 + m_A g l = \tfrac{1}{2}(m_A + m_B) v_{A,2}^2 + m_A g l + m_B g h$$

$$h = \frac{v_0^2}{2g} - \frac{m_A + m_B}{m_B}\frac{v_{A,2}^2}{2g} = \frac{v_0^2}{2g} - \frac{m_A + m_B}{2g\, m_B}\left(\frac{m_B}{m_A + m_B} v_0\right)^2$$

$$h = \frac{v_0^2}{2g} - \frac{m_B}{m_A + m_B}\frac{v_0^2}{2g}$$

$$h = \frac{m_A}{m_A + m_B}\frac{v_0^2}{2g}$$

Sample Problem 14.5



Ball *A* has initial velocity $v_0 = 3$ m/s parallel to the axis of the table. It hits ball *B* and then ball *C* which are both at rest. Balls *A* and *C* hit the sides of the table squarely at *A'* and *C'* and ball *B* hits obliquely at *B'*.

Assuming perfectly elastic collisions, determine velocities v_A, v_B, and v_C with which the balls hit the sides of the table.

SOLUTION:

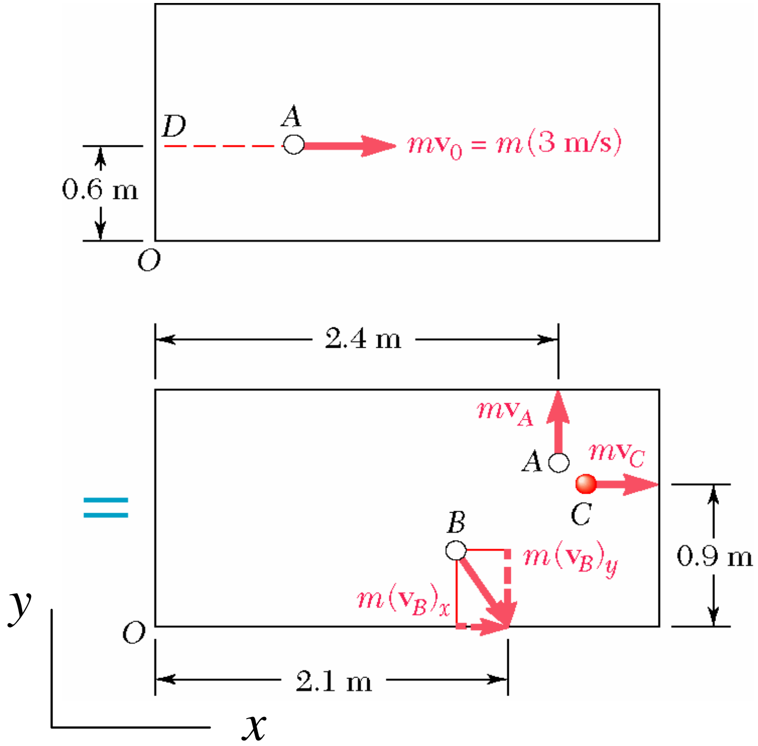
- There are four unknowns: v_A, $v_{B,x}$, $v_{B,y}$, and v_C.
- Solution requires four equations: conservation principles for linear momentum (two component equations), angular momentum, and energy.
- Write the conservation equations in terms of the unknown velocities and solve simultaneously.

Sample Problem 14.5

SOLUTION:

- There are four unknowns: v_A, $v_{B,x}$, $v_{B,y}$, and v_C.

$$\vec{v}_A = v_A \vec{j}$$
$$\vec{v}_B = v_{B,x}\vec{i} + v_{B,y}\vec{j}$$
$$\vec{v}_C = v_C \vec{i}$$



- The conservation of momentum and energy equations,

$$\vec{L}_1 + \sum \int \vec{F}dt = \vec{L}_2$$

$$mv_0 = mv_{B,x} + mv_C \qquad 0 = mv_A - mv_{B,y}$$

$$\vec{H}_{O,1} + \sum \int \vec{M}_O dt = \vec{H}_{O,2}$$

$$-(0.6\,\text{m})mv_0 = (2.4\,\text{m})mv_A - (2.1\,\text{m})mv_{B,y} - (0.9\,\text{m})mv_C$$

$$T_1 + V_1 = T_2 + V_2$$

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_A^2 + \tfrac{1}{2}m\left(v_{B,x}^2 + v_{B,y}^2\right) + \tfrac{1}{2}mv_C^2$$

Solving the first three equations in terms of v_C,

$$v_A = v_{B,y} = 3v_C - 6 \qquad v_{B,x} = 3 - v_C$$

Substituting into the energy equation,

$$2(3v_C - 6)^2 + (3 - v_C)^2 + v_C^2 = 9$$

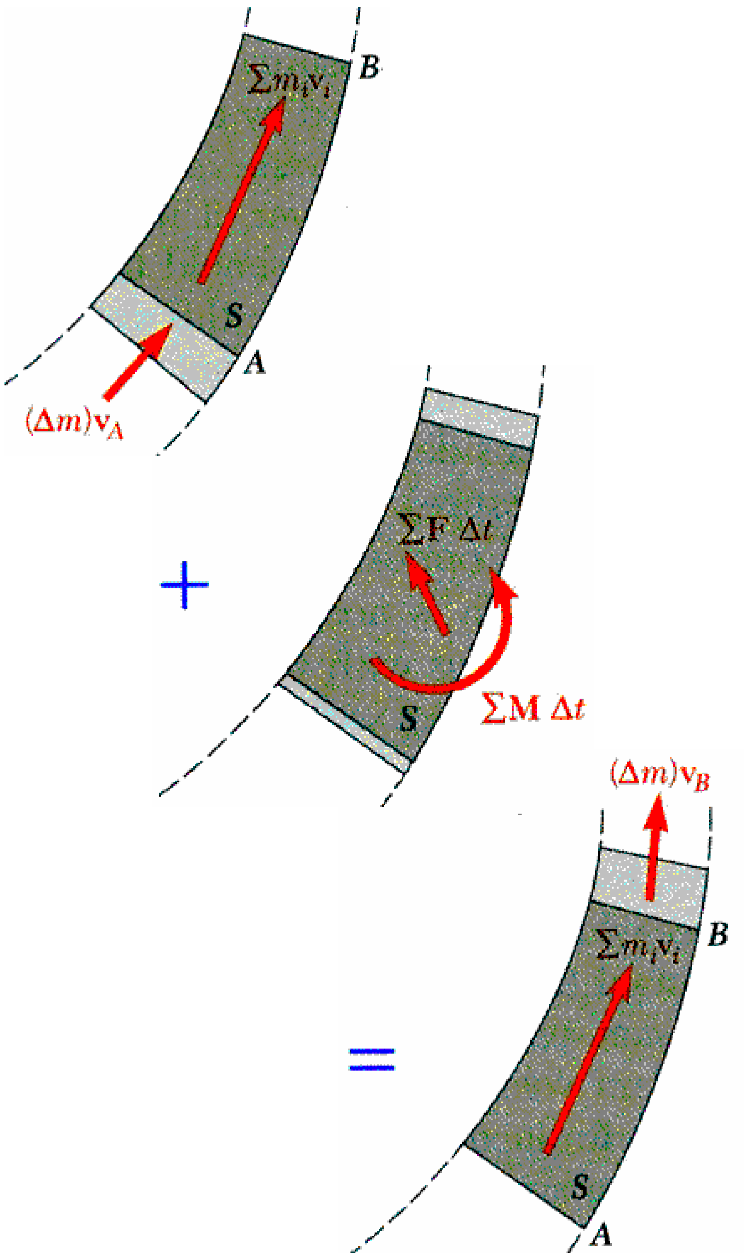
$$20v_C^2 - 78v_C + 72 = 0$$

$$v_A = 1.2\,\text{m/s} \qquad v_C = 2.4\,\text{m/s}$$
$$\vec{v}_B = \left(2\vec{i} - 4\vec{j}\right)\text{m/s} \qquad v_B = 1.34\,\text{m/s}$$

Variable Systems of Particles

- Kinetics principles established so far were derived for constant systems of particles, i.e., systems which neither gain nor lose particles.
- A large number of engineering applications require the consideration of variable systems of particles, e.g., hydraulic turbine, rocket engine, etc.
- For analyses, consider auxiliary systems which consist of the particles instantaneously within the system plus the particles that enter or leave the system during a short time interval. The auxiliary systems, thus defined, are constant systems of particles.

Steady Stream of Particles



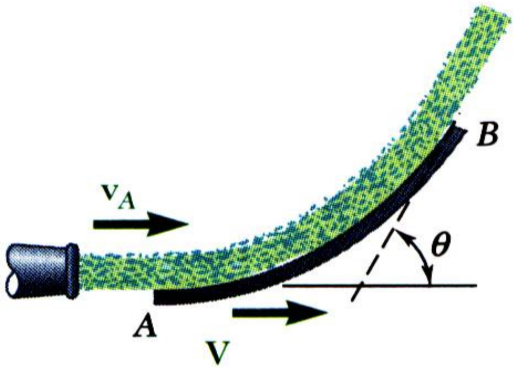
- System consists of a steady stream of particles against a vane or through a duct.
- Define auxiliary system which includes particles which flow in and out over Δt.
- The auxiliary system is a constant system of particles over Δt.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2$$

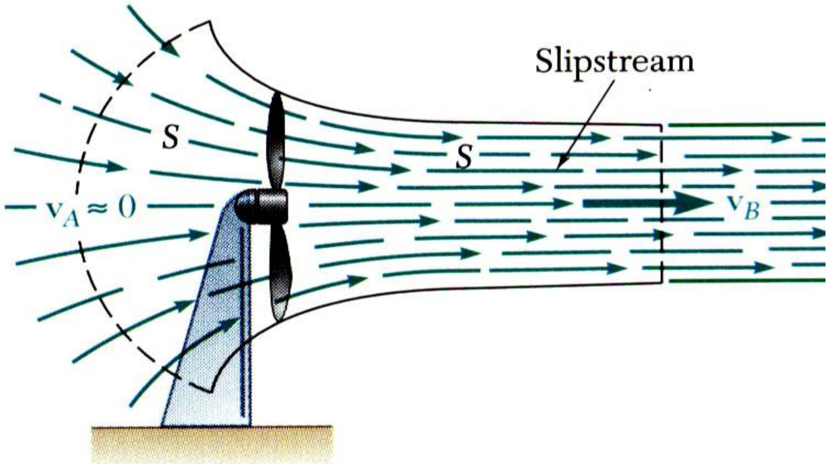
$$\left[\sum m_i \vec{v}_i + (\Delta m)\vec{v}_A\right] + \sum \vec{F}\, \Delta t = \left[\sum m_i \vec{v}_i + (\Delta m)\vec{v}_B\right]$$

$$\sum \vec{F} = \frac{dm}{dt}\left(\vec{v}_B - \vec{v}_A\right)$$

Steady Stream of Particles. Applications

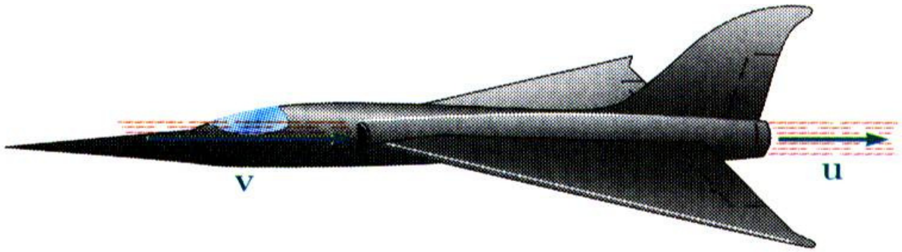


- Fluid Stream Diverted by Vane or Duct

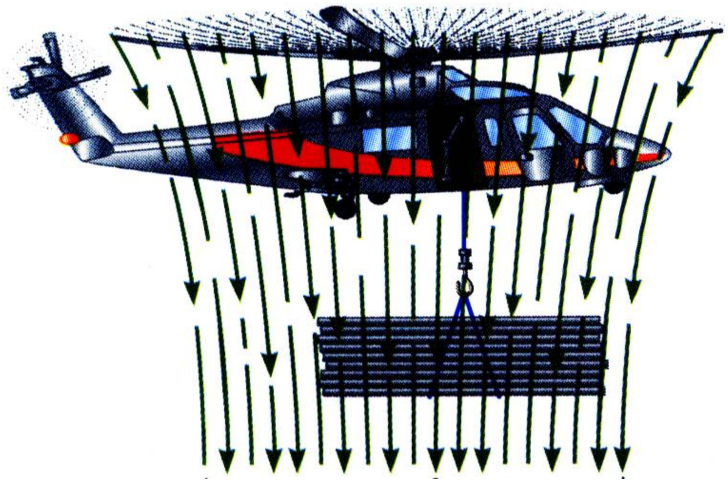


- Fan

- Fluid Flowing Through a Pipe

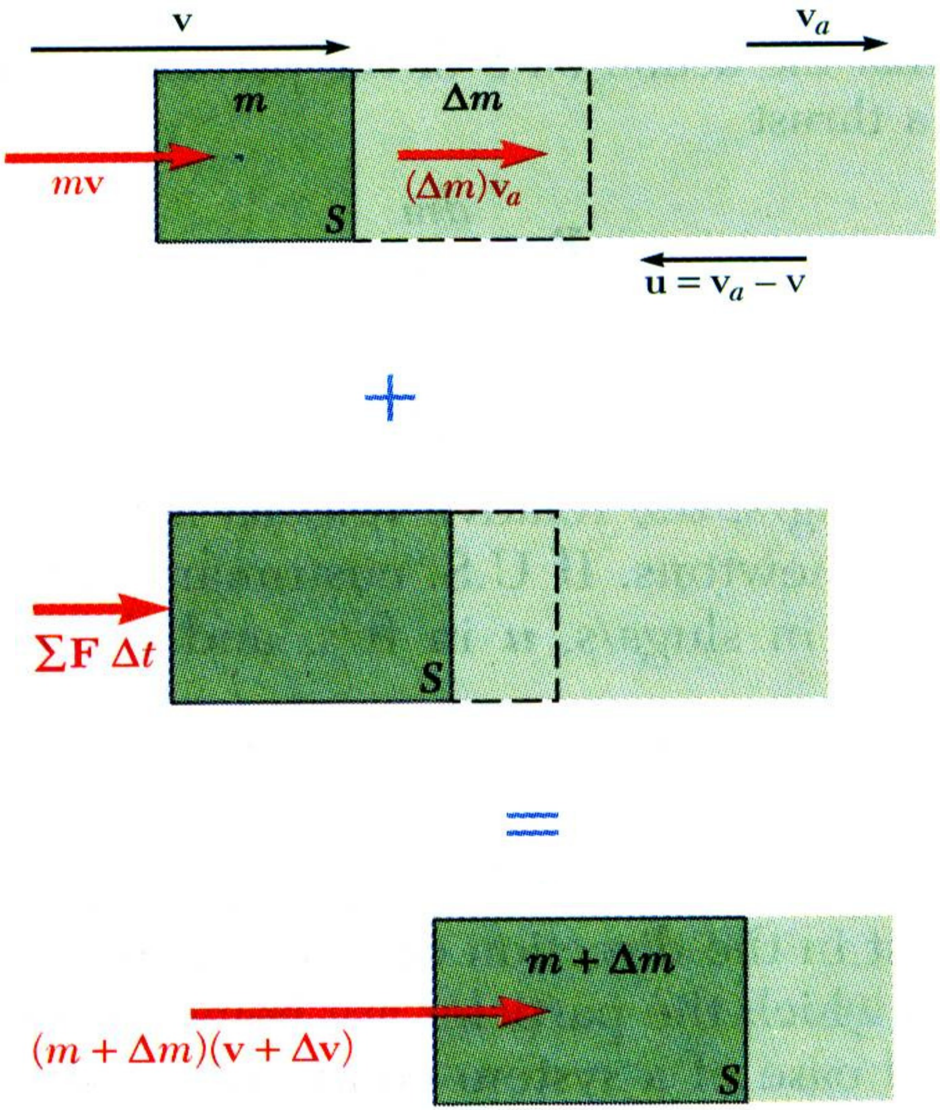


- Jet Engine



- Helicopter

Streams Gaining or Losing Mass



- Define auxiliary system to include particles of mass m within system at time t plus the particles of mass Δm which enter the system over time interval Δt.
- The auxiliary system is a constant system of particles.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2$$

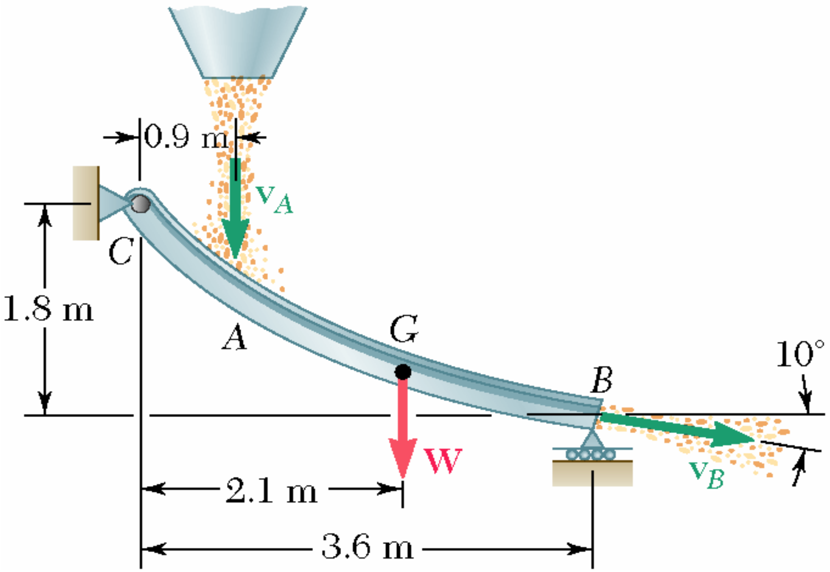
$$\left[m\vec{v} + (\Delta m)\vec{v}_a\right] + \sum \vec{F}\,\Delta t = (m + \Delta m)(\vec{v} + \Delta\vec{v})$$

$$\sum \vec{F}\,\Delta t = m\Delta\vec{v} + \Delta m(\vec{v} - \vec{v}_a) + (\Delta m)\Delta\vec{v}$$

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{u}$$

$$m\vec{a} = \sum \vec{F} - \frac{dm}{dt}\vec{u}$$

Sample Problem 14.6



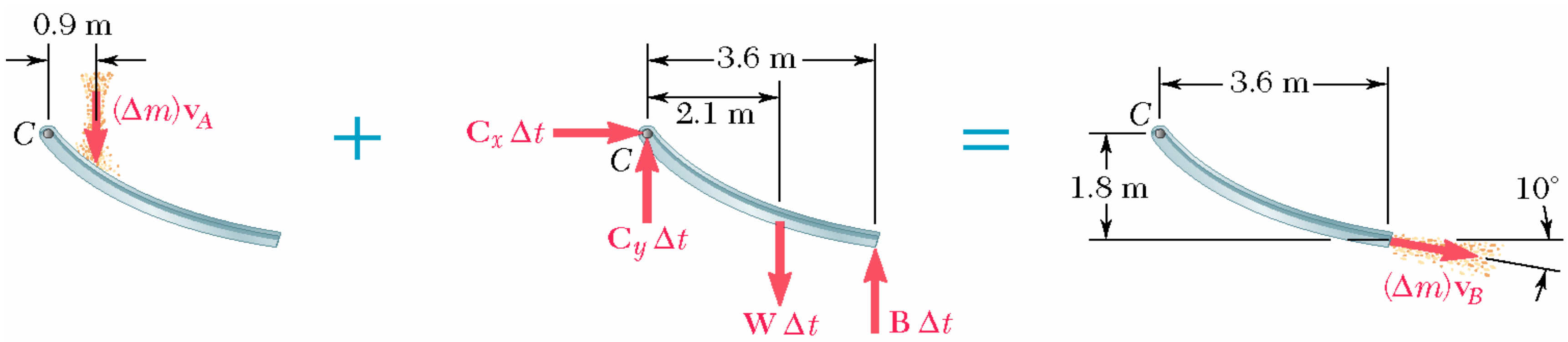
Grain falls onto a chute at the rate of 120 kg/s. It hits the chute with a velocity of 6 m/s and leaves with a velocity of 4.5 m/s. The combined weight of the chute and the grain it carries is 3 kN with the center of gravity at *G*.

Determine the reactions at *C* and *B*.

SOLUTION:

- Define a system consisting of the mass of grain on the chute plus the mass that is added and removed during the time interval Δt.
- Apply the principles of conservation of linear and angular momentum for three equations for the three unknown reactions.

Sample Problem 14.6



SOLUTION:

- Define a system consisting of the mass of grain on the chute plus the mass that is added and removed during the time interval Δt.
- Apply the principles of conservation of linear and angular momentum for three equations for the three unknown reactions.

$$\vec{L}_1 + \sum \int \vec{F} dt = \vec{L}_2$$

$$C_x \Delta t = (\Delta m) v_B \cos 10°$$

$$-(\Delta m) v_A + (C_y - W + B)\Delta t = -(\Delta m) v_B \sin 10°$$

$$\vec{H}_{C,1} + \sum \int \vec{M}_C dt = \vec{H}_{C,2}$$

$$-0.9(\Delta m) v_A + (-2.1W + 3.6B)\Delta t$$
$$= 1.8(\Delta m) v_B \cos 10° - 3.6(\Delta m) v_B \sin 10°$$

Solve for C_x, C_y, and B with

$$\frac{\Delta m}{\Delta t} = \frac{120\,\text{kg/s}}{32.2\,\text{ft/s}^2} = 7.45\,\text{slug/s}$$

$$\boxed{B = 2102\ \text{N} \quad \vec{C} = (110.1\vec{i} + 307\vec{j})\ \text{N}}$$