Closed-Form Solutions for Black-Litterman Model with Conditional Value at Risk

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Abstract

We consider a portfolio optimization problem of Black-Litterman type, using the conditional value-at-risk (CVaR) as the risk measure. Further, we use the multi-variate elliptical distributions, instead of the multi-variate normal distribution, to model the financial asset returns. We propose an approximation algorithm and establish the convergence results. Based on the approximation algorithm, we derive the closed-form solution of the portfolio optimization problem with CVaR.

Keywords: Black-Litterman Model, Portfolio Optimization, Robust Optimization, CVaR, Elliptical Distribution

1 Introduction

In the classical Markowitz portfolio optimization model, the historical mean vector and covariance matrix of the risky assets are used to obtain the optimal portfolio allocation while normal distributions are assumed (Markowitz [17]). Mean variance optimization considers only the first two moments of the return distribution. This restriction was consistent with reality if asset returns followed a normal distribution (Fabozzi [6] and Meucci [18]). But we know that asset return's coskewness and cokurtosis values differ from normality(See more at Harvey et. al. [10] and Jondeau and Rockinger [16]). Moreover, historical data tells us that the covariance matrix is

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a random variable by itself(Hull [12]). Additionally, historical returns are not good estimates of the future returns and they are very difficult to estimate. Furthermore, the mapping between expected returns and portfolio weights are complicated. In addition to that optimal portfolio weights are highly sensitive to the parameters of the optimization problem(see Meucci [18] and Fabozzi et. al. [6]). Hence, Markowitz's optimal allocation vectors are lack of diversification and/or has corner solutions.

Black and Litterman [5] proposes a portfolio optimization technique in which the investor's view can be integrated with the historical performance to obtain the optimal portfolio. The Black-Litterman Model (BLM) combines the intuitions of the investors about the selected assets with their historical information to update their mean vector and covariance matrix using Bayesian framework. The BLM assumes that the expected returns are random variables themselves which normally distributed and centered at the CAPM equilibrium returns with historical covariance matrix. There are two different types of BLMs in the literature: the original model(canonical model) and the alternate model, which is proposed later (see Walters [26] and Meucci [18]).

Most of the current literatures on the BLM are still using normal distributions with variance as the risk measure. In this paper, we consider elliptical distributions with conditional value-at-risk as the risk measure. Note that CAPM holds as long as return distributions are elliptical(see Meucci [18] and references therein). Derivation of the posterior distribution for this case is given at Xiao and Valdez [27](see Proposition 3). On the other hand, CVaR has become more and more popular as a coherent risk measure in the financial industry. Derivation of the optimal solution analytically is extremely difficult. Hence, we propose an efficient approximation algorithm for optimization problems under CVaR risk measure. Then, based on the approximation algorithm, we derive the closed-form solutions for the BLM with elliptical distributions and CVaR. To our best knowledge, no closed-form solutions for BLM with CVaR have been derived before.

The rest of the paper is organized as follows. Definitions of the portfolio allocation problems (PAP) are given in Section 2. The BLM with CVaR as the risk measure is reviewed in Section 3. We propose an efficient approximation algorithm for optimization problems under CVaR risk measure in Section 4. The closed-form solution of the optimization is given in Section 5. Finally, we conclude the paper in Section 6.

2 Portfolio Allocation Problem (PAP)

We consider a market with n risk assets. Risky asset returns are denoted by the random vector $\mathbf{r} \in \mathbb{R}^n$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$. Vectors are defined as column vectors unless otherwise stated. The mean vector and the covariance matrix of risky asset returns are denoted by $\boldsymbol{\mu} = \mathbb{E}[\mathbf{r}]$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ respectively. The risk free rate of return is denoted as $r_f \in \mathbb{R}_+ \cup 0$. Moreover, $\mathbf{x} \in \mathbb{R}^n$ is the portfolio weight vector of risky assets and $(1 - \mathbf{e}'\mathbf{x})$ is the allocation on the risk-free asset, where $\mathbf{e} = (1, 1, \cdots, 1)'$ is a vector of ones in $\mathbb{R}^n$.

First we define the space of portfolio returns for a given number of available risky assets and a risk-free asset.

Definition 1 (Space of Portfolio Returns)**.**

$$\mathcal{V} = \{\tilde{v} \in \mathbb{R} : \exists (r_f, \mathbf{x}) \ s.t. \ \tilde{v} = (1 - \mathbf{e}'\mathbf{x})r_f + \mathbf{r}'\mathbf{x}\}. \tag{1}$$

Proposition 1. *Each portfolio return can be represented as a combination of return with certainty and return with uncertainity.*

Proof. Consider $\tilde{v} \in \mathcal{V}$ such that $\tilde{v} = v_0 + \mathbf{r}'\mathbf{x}$ where $v_0 = (1 - \mathbf{e}'\mathbf{x})r_f$ then we can represent $\tilde{v}$ as follows

$$\begin{aligned}\tilde{v} &= v_0 + \mathbf{r}'\mathbf{x} = v_0 + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x} + \boldsymbol{\mu}'\mathbf{x} \\ &= \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}.\end{aligned}$$

Note that $\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f$ is the return with certainty. On the other hand, we have uncertainty on $(\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}$. □

Let us continue with the definition of the constrained Markowitz's portfolio allocation problem (PAP) (Markowitz [17]):

Definition 2 (Markowitz's PAP)**.**

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq L\}, \tag{2}$$

where $L \in \mathbb{R}_+$ is a predefined risk tolerance level of the investor.

In Markowitz's PAP, the variance is used as the risk measure. There are other risk measures that are widely used, such as value at risk (VaR) and conditional value at risk (CVaR).

Definition 3 (VaR)**.** *Given $\alpha \in (0, 1)$ and a random variable Y, the VaR of the r.v. Y with α quantile is,*

$$VaR_\alpha(Y) = \inf\{y \in \mathbb{R} : \mathbb{P}(y + Y \leq 0) \leq 1 - \alpha\}.$$

Definition 4 (CVaR)**.** *Given $\alpha \in (0, 1)$ and a random variable (i.e. Y) the conditional value at risk of the random variable with confidence level α is,*

$$CVaR_\alpha(Y) = -\mathbb{E}[Y | Y \leq -VaR_\alpha(Y)].$$

VaR_α is not a coherent risk measure (see Artzner et. al. [1]). In particular, diversification benefit may not present under VaR. On the other hand, CVaR is coherent (The properties of CVaR can be found in Rockafellar and Uryasev [21] and [22].). It is the main reason that CVaR is now a very popular risk measure.

Here we consider the PAP with CVaR:

Definition 5 (PAP using CVaR)**.**

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f : CVaR_\alpha((\mathbf{r}-r_f\mathbf{e})'\mathbf{x}+r_f) \leq L\}. \tag{3}$$

More generally, we can also use a generic coherent risk measure by $\rho(\tilde{v})$ instead of CVaR. We give the general portfolio allocation problem(GPAP) using generic coherent risk measure next.

Definition 6 (GPAP with coherent risk measure ρ)**.**

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}^{'}\mathbf{x} + (1-\mathbf{e}^{'}\mathbf{x})r_f : \rho((\mathbf{r}-r_f\mathbf{e})^{'}\mathbf{x}+r_f) \leq L\} \tag{4}$$

On the other hand, we can use robust programming(with elliptical uncertainty sets to be specified below) to model the GPAP. In addition, if we use a generic coherent risk measure, we can convert it to robust optimization under some certain conditions and vice versa(see Natarajan et. al. [20]).

We use uncertainty sets in order to manage the uncertainty part of the portfolio return (i.e. $(\mathbf{r}-\boldsymbol{\mu})$). In particular, we use elliptical uncertainty sets:

$$\mathcal{U}_\beta = \{\mathbf{r} : \|\boldsymbol{\Sigma}^{-1/2}(\mathbf{r}-\boldsymbol{\mu})\|_2 \leq \beta\}$$

where β is the scaling parameter which models the risk averseness of the investor from the deviation of the realized returns from their forecasted values.

The reasons of using elliptical uncertainty sets is twofold: First, when uncertainty set is elliptical then the robust programming can be converted into conic programming (Ben Tal and Nemirovski [2]); Second, elliptical uncertainty sets can be used for leptokurtic behavior of asset returns.

Once we define the uncertainty set then we can find the robust counterpart of the problem on hand. The robust counterpart is just a deterministic problem, but it keeps the random structure of the optimization problem. Now, we are ready to state the GPAP in the sense of robust optimization.

Definition 7 (GPAP with robust optimization)**.**

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}^{'}\mathbf{x} + (1-\mathbf{e}^{'}\mathbf{x})r_f : (\mathbf{r}-r_f\mathbf{e})^{'}\mathbf{x} \geq -L \ \forall \mathbf{r} \in \mathcal{U}_\beta \ \}$$

The process of finding the robust counterpart is straight forward. We first start with defining the robust counterpart risk measure, $\eta_{\mathcal{U}_\beta}(\tilde{v})$, given by Natarajan et. al. [20]. Then we find the dual of the robust optimization problem via second order conic programming. Luckily, $((1-\mathbf{e}'\mathbf{x})r_f+\mathbf{r}'\mathbf{x})$ is an affine function. The closed-form solution of minimizing an affine function over a single ellipsoidal constraint is given by Ben Tal and Nemirovski[3].

We give the connection between PAP with generic risk measure and PAP in the sense of robust optimization in the following proposition (Note that we use CVaR as our coherent risk measure).

Proposition 2 (CVaR and Uncertanity Sets)**.** *Consider the following uncertainty set:*

$$\mathcal{U}_\beta = \{\mathbf{r} : (\mathbf{r}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{r}-\boldsymbol{\mu}) \leq \beta^2\}. \tag{5}$$

Then for a portfolio return $\tilde{v}$ in a space given by (1)*, the closed-form solution of CVaR is:*

$$CVaR_\alpha(\tilde{v}) = -(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f) + \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}. \tag{6}$$

Proof.

$$\begin{aligned}
\rho(v_0+\mathbf{r}'\mathbf{x}) &= \eta_{\mathcal{U}_\beta}(v_0+\mathbf{r}'\mathbf{x}) \\
&\quad \text{(by Theorem 4 of [20])} \\
&= \eta_{\mathcal{U}_\beta}\left(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}\right) \\
&\quad \text{(by Proposition 1)} \\
&= -\min_{\mathbf{r}\in\mathcal{U}_\beta}\left(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}\right) \\
&\quad \text{(by Natarajan et. al. [20])} \\
&= -\left(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f\right) + \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \\
&\quad \text{(by Ben Tal and Nemirovski [3]} \\
&= CVaR_\alpha((\mathbf{r}-r_f\mathbf{e})'\mathbf{x}+r_f). \\
&\quad \text{(by Lemma 4.1 in Xiao and Valdez [27])}
\end{aligned}$$

□

Next, let us look at some examples of Proposition 2.

Example 1. *Consider the uncertainty set below:*

$$\mathcal{U} = \{\mathbf{r} : (\mathbf{r}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{r}-\boldsymbol{\mu}) \leq \beta^2\}. \tag{7}$$

We consider three cases.

(i) If $\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *and* $\beta = f(z_\alpha)/(1-\alpha)$*, where* $f(\cdot)$ *is the standard normal density function and* z_α *is the z-score, we get the PAP with CVaR for normal distributions.*

(ii) If $\mathbf{r}$ *follows a multivariate Student-t distribution i.e.* $\mathbf{r} \sim t(\boldsymbol{\mu}, \boldsymbol{\Sigma}, m)$ *where* m *stands for the degree of freedom, and* $\beta = \frac{c_2 m}{(1-\alpha)(m-1)}\left(1+\frac{y_\alpha^2}{m}\right)^{\frac{1-m}{2}}$*, where* $c_2 = \frac{(\pi m)^{-1/2}\Gamma((m+1)/2)}{\Gamma(m/2)}$*, then we get the PAP with CVaR for Student-t distributions.*

(iii) If $\mathbf{r}$ *follows a multivariate Logistics distribution i.e.* $\mathbf{r} \sim ML(\boldsymbol{\mu}, \boldsymbol{\Sigma})$*, and* $\beta = \frac{c_3}{2(1-\alpha)}\left(1-\frac{1}{1+e^{-y_\alpha^2}}\right)$ *where* $c_3 = \frac{\Gamma(n/2)}{\pi^{1/2}}\left(\int_0^\infty u^{n/2-1}\frac{e^{-u}}{(1+e^{-u})^2}du\right)^{-1}$*, then we get the PAP with CVaR for multivariate Logistics distributions.*

We can use the definition of the elliptical uncertainty set and Theorem 4 of Natarjan et. al. [20] together with Lemma 4.1 of Xiao and Valdez [27] to derive the results above. Note that one can also use multivariate elliptical distributions directly and come up with the corresponding β values(see Landsman and Valdez [14]).

3 BLM with CVaR

We continue with the BLM under elliptical distributions ($ED_n(\cdot)$) given in Xiao and Valdez [27]. Let $\mathbf{r} \sim ED_n(\boldsymbol{\mu}, \mathbf{D}, g_n)$ be an n dimensional vector, denotes the market factors, where $\boldsymbol{\mu}$, $\mathbf{D}$ and g_n are the location parameter, dispersion matrix and the density generator function respectively(For more on elliptical distributions please see Fang et. al. [7]). Furthermore, conditional random view vector is: $\mathbf{v}|\mathbf{r} \sim ED_k(\mathbf{P}\boldsymbol{\mu}, \boldsymbol{\Omega}, g_k(\cdot; p(\mathbf{r})))$ where $p(\mathbf{r}) = (\mathbf{r}-\boldsymbol{\Pi})'\boldsymbol{\Sigma}'(\mathbf{r}-\boldsymbol{\Pi})$. The posterior distribution is given by the following proposition.

Proposition 3. *The posterior distribution is*

$$
\begin{aligned}
&\mathbf{r}|\mathbf{v} \sim ED_k(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}, g_n(\cdot; q(\mathbf{v}))), \textit{ where}\\
&\boldsymbol{\mu}_{BL} = \boldsymbol{\Pi} + \mathbf{D}\mathbf{P}'(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\Pi})\\
&\mathbf{D}_{BL} = \boldsymbol{\Sigma} - \mathbf{D}\mathbf{P}'(\boldsymbol{\Omega} + \mathbf{P}\mathbf{D}\mathbf{P}')^{-1}\mathbf{P}\mathbf{D}, \textit{ and}\\
&q(\mathbf{v}) = (\mathbf{v} - \mathbf{P}\boldsymbol{\Pi})'(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\Pi})\\
&\boldsymbol{\Sigma}_{BL} = \mathbf{D}_{BL} C_k(q(\mathbf{v})/2)
\end{aligned}
$$

where C_k *is a distribution specific function from* $\mathbb{R}$ *to* $\mathbb{R}$*.*

The key assumption of BLM is that every player in the market solves the Markowitz's problem. In other words, BLM takes CAPM equilibrium as prior for the excess return distribution. However, in our case, investors have views under the CVaR risk measure. There are other generalization fo the BLM with CAPM equilibrium. For example, Silva et. al. [24] gives the BLM under active management, Giacometti et. al. [9] proposes a model where asset returns follow stable distributions with different types of risk measures. Unlike models in those papers, here we consider elliptical distributions for asset returns and solve the constrained model.

Consider the constrained portfolio optimization problem with CVaR(i.e. Definition 5) with uncertainty set given by (5). Then by using Proposition 2 we get it's Lagrangian function:

$$\begin{aligned}\mathcal{L}(\mathbf{x},\lambda) =&\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f - \lambda CVaR_\alpha(\tilde{v})\\ =&\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f -\\ &\left(-\mathbf{x}'(\boldsymbol{\mu}-\mathbf{e}r_f) - r_f + \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}\right)\lambda\\ =&(1+\lambda)[\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f] - \lambda\beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}.\end{aligned}$$

Now, take the partial derivative with respect to $\mathbf{x}$ to get the first order necessary condition and use inverse optimization to find an estimate of expected excess return vector:

$$\boldsymbol{\Pi} = \frac{\lambda\beta}{1+\lambda}\left((\mathbf{x}'_{mkt}\boldsymbol{\Sigma}\mathbf{x}_{mkt})^{-1/2}\boldsymbol{\Sigma}\mathbf{x}_{mkt}\right), \tag{8}$$

where we take $\mathbf{x} = \mathbf{x}_{mkt}$ as the market weights.

He and Litterman [11] considers the unconstrained mean variance optimization problem. Therefore, CAPM equilibrium is centered at $\boldsymbol{\Pi} = 2\delta\boldsymbol{\Sigma}\mathbf{x}_{mkt}$ where they fixed the value of δ as 1.25. Note that in (8), if we take

$$\frac{\lambda\beta}{1+\lambda}(\mathbf{x}'_{mkt}\boldsymbol{\Sigma}\mathbf{x}_{mkt})^{-1/2} = 2.5$$

then we will get the same $\boldsymbol{\Pi}$ value. Hence, our model is a generalization of He and Litterman [11]. We get the return distribution for the updated excess return vector by using Proposition 3. We also need to determine the investor risk aversion parameter. In our problem, investor risk aversion is reflected in two parameters. The first one is the choice of the parameter β when the investor picks an elliptical distribution (see Example 1) for the asset returns. The second parameter is λ, which is related with the risk reward trade off. Now, we can solve the portfolio optimization problem under CVaR (i.e. definition (5)) with the new(updated) excess return vector and dispersion matrix.

4 CVaR Approximation

Consider the constrained portfolio optimization problem with CVaR(i.e. Definition 5) with uncertainty set given by (5). Then by using Proposition 2 we get it's Lagrangian function:

$$\begin{aligned}\mathcal{L}(\mathbf{x},\lambda) =&\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f - \\ &\left(CVaR_\alpha\left((\mathbf{r}-r_f\mathbf{e})'\mathbf{x} - r_f\right) - L\right)\delta \\ =&\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f - \\ &\left(-(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f) + \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} - L\right)\delta\end{aligned}$$

Now, take the partial derivative with respect to $\mathbf{x}$ to get the first order necessary condition.

$$(1+\delta)\left(\boldsymbol{\mu} - \mathbf{e}r_f\right) - \delta\left(\beta(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{-1/2}\boldsymbol{\Sigma}\mathbf{x}\right) = 0$$

Ladnsman [13] derives the closed-form solution of the problem of minimizing the root of a quadratic functional subject to some affine constraints. In addition to that, we can find the numerical solution using convex (or semidefinite) programming. But there is no explicit solution solution for constrained PAP with CVaR yet. Hence, we propose an algorithm to find the closed-form of the optimal solution for that problem.

The asset return distribution is assumed to be elliptical with the parameters defined in Proposition 3. Furthermore, historical mean vector and covariance matrix are taken as the mean of the asset returns and covariance matrix, respectively. We can rewrite the constraint as (using Proposition 2)

$$\begin{aligned}&CVaR_\alpha((\mathbf{r}-r_f\mathbf{e})'\mathbf{x} + r_f) \leq L && (9)\\ \Leftrightarrow &-(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f) + \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq L && (10)\\ \Leftrightarrow &\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \frac{(L + \boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f)}{\beta} \equiv \tilde{L}(\mathbf{x}). && (11)\end{aligned}$$

For a give $\mathbf{x}_n$, we define $\mathbf{x}_{n+1}$ as the solution of the following PAP

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f\} \quad \text{s.t} \quad \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \tilde{L}(\mathbf{x}_n). \tag{12}$$

As we can see, the above PAP is of Markowitz mean-variance type, and it is very easy to solve.

If we start with an initial $\mathbf{x}_0$ such that (9) holds(For example, we can take $\mathbf{x}_0 = \mathbf{0}$), then we can show that the sequence $\tilde{L}(\mathbf{x}_n)$ converges. Using the limit of $\tilde{L}(\mathbf{x}_n)$, we can derive the optimal solution of the PAP with CVaR.

We have the following results.

Lemma 1. *For any given $\mathbf{x}_n \in \mathbb{R}^n$, $\tilde{L}(\mathbf{x}_n)$ is bounded above and below.*

Proof. Since $\boldsymbol{\mu}$ is a given vector, by the definition of $\tilde{L}(\cdot)$, it is easy to show that $\tilde{L}(\mathbf{x}_n)$ is bounded. □

Lemma 2. *If the initial vector* $\mathbf{x}_0$ *satisfies* (9)*, then* $\{\tilde{L}(\mathbf{x}_n)\}$ *is a non-decreasing sequence.*

Proof. We prove the result by induction. Since $\mathbf{x}_0$ satisfies (9) , we have that $\sqrt{\mathbf{x}_0'\boldsymbol{\Sigma}\mathbf{x}_0} \leq \tilde{L}(\mathbf{x}_0)$. By virtue of the definition of $\mathbf{x}_1$, we can get that

$$\boldsymbol{\mu}'\mathbf{x}_1 + (1 - \mathbf{e}'\mathbf{x}_1)r_f \geq \boldsymbol{\mu}'\mathbf{x}_0 + (1 - \mathbf{e}'\mathbf{x}_0)r_f.$$

Then, by the definition of $\tilde{L}(\mathbf{x})$ (11), we have

$$\tilde{L}(\mathbf{x}_1) \geq \tilde{L}(\mathbf{x}_0).$$

Further, we have that

$$\sqrt{\mathbf{x}_1'\boldsymbol{\Sigma}\mathbf{x}_1} \leq \tilde{L}(\mathbf{x}_0) \leq \tilde{L}(\mathbf{x}_1).$$

Now we can assume that

$$\tilde{L}(\mathbf{x}_n) \geq \tilde{L}(\mathbf{x}_{n-1}), \quad \sqrt{\mathbf{x}_n'\boldsymbol{\Sigma}\mathbf{x}_n} \leq \tilde{L}(\mathbf{x}_n).$$

Then using the same arguments as we used for $\mathbf{x}_0$ and $\mathbf{x}_1$, we can show that

$$\tilde{L}(\mathbf{x}_{n+1}) \geq \tilde{L}(\mathbf{x}_n), \quad \sqrt{\mathbf{x}_{n+1}'\boldsymbol{\Sigma}\mathbf{x}_{n+1}} \leq \tilde{L}(\mathbf{x}_{n+1}).$$

Therefore, $\{\tilde{L}(\mathbf{x}_n)\}$ is a non-decreasing sequence. □

Theorem 1 (Conv. of $\tilde{L}(\mathbf{x}_n)$)**.** $\tilde{L}(\mathbf{x}_n) \xrightarrow{n\to\infty} \tilde{L}(\mathbf{x}^*)$ *where* $\mathbf{x}^*$ *is the optimal solution to the PAP with CVaR (i.e. problem given by Definition 5).*

Proof. From Lemma 1 and 2, we can see that $\tilde{L}(\mathbf{x}_n)$ is a bounded and non-decreasing sequence in $\mathbb{R}$. Thus, the sequence $\{\tilde{L}(\mathbf{x}_n)\}$ must converge to a real number. We denote the limit as $\tilde{L}^*$. We will show that $\tilde{L}^* = \tilde{L}(\mathbf{x}^*)$.

Define $\hat{\mathbf{x}}$ as the solution of

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f\} \quad \text{s.t} \quad \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \tilde{L}^*. \tag{13}$$

We now claim that

$$\tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*. \tag{14}$$

First let us assume that $\tilde{L}(\hat{\mathbf{x}}) > \tilde{L}^*$. Now we have

$$\sqrt{\hat{\mathbf{x}}'\boldsymbol{\Sigma}\hat{\mathbf{x}}} = \tilde{L}^* < \tilde{L}(\hat{\mathbf{x}}).$$

By the definition of $\tilde{L}$, the above inequality implies that

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) < L.$$

Taking $\mathbf{x}_0 = \hat{\mathbf{x}}$ in Lemma 2, we can get that $\tilde{L}(\hat{\mathbf{x}}) \le \tilde{L}^*$, which is a contradiction with the assumption $\tilde{L}(\hat{\mathbf{x}}) > \tilde{L}^*$.

Now let us assume that $\tilde{L}(\hat{\mathbf{x}}) < \tilde{L}^*$. Since $\tilde{L}^*$ is the limit of the non-decreasing sequence $\{\tilde{L}(\mathbf{x}_n)\}$, there must be an integer n such that $\tilde{L}(\mathbf{x}_k) > \tilde{L}(\hat{\mathbf{x}})$, $\forall k \ge n$. Let $\mathbf{x}_{n+1}$ be the solution of

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f\} \quad \text{s.t} \quad \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \le \tilde{L}(\mathbf{x}_n). \tag{15}$$

Then we must have $\tilde{L}(\mathbf{x}_{n+1}) \le \tilde{L}^*$. However, as $\hat{\mathbf{x}}$ and $\mathbf{x}_{n+1}$ are the optimal solutions of (13) and (15), respectively and $\tilde{L}(\mathbf{x}_n) \le \tilde{L}^*$, we must have

$$\boldsymbol{\mu}'\mathbf{x}_{n+1} + (1 - \mathbf{e}'\mathbf{x}_{n+1})r_f \le \boldsymbol{\mu}'\hat{\mathbf{x}} + (1 - \mathbf{e}'\hat{\mathbf{x}})r_f.$$

By the definition of $\tilde{L}(\cdot)$ (see (11)), we can get $\tilde{L}(\mathbf{x}_{n+1}) \le \tilde{L}(\hat{\mathbf{x}})$, which is a contradiction, too. So (14) must hold.

On the other hand, by definition, we have

$$\mathbf{x}^* \in \operatorname{argmax}\{(\boldsymbol{\mu} - \mathbf{e})'\mathbf{x} + r_f : CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \le L\}.$$

Further, the constraint should be binding for the optimal solution, so we can get $CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x}^* + r_f) = L$. By the definition of $\tilde{L}$ (see (11)), we can get

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x}^* + r_f) = L \Leftrightarrow \sqrt{(\mathbf{x}^*)'\boldsymbol{\Sigma}\mathbf{x}^*} = \tilde{L}(\mathbf{x}^*)$$

From (14) and (11), we can get

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\hat{\mathbf{x}} + r_f) \le L.$$

Therefore, $\hat{\mathbf{x}}$ is a feasible solution to the PAP with CVaR. Because $\mathbf{x}^*$ is the optimal solution, we can get that

$$\begin{aligned}
&(\boldsymbol{\mu} - \mathbf{e})'\hat{\mathbf{x}} + r_f \le (\boldsymbol{\mu} - \mathbf{e})'\mathbf{x}^* + r_f \\
\Rightarrow &\frac{((\boldsymbol{\mu} - \mathbf{e})'\hat{\mathbf{x}} + r_f + L)}{\beta} \le \frac{((\boldsymbol{\mu} - \mathbf{e})'\mathbf{x}^* + r_f + L)}{\beta} \\
\Rightarrow &\tilde{L}(\hat{\mathbf{x}}) \le \tilde{L}(\mathbf{x}^*).
\end{aligned} \tag{16}$$

Now, taking $\mathbf{x}_0$ as $\mathbf{x}^*$ and using Lemma 2, we can get that

$$\tilde{L}(\mathbf{x}^*) \le \tilde{L}^* = \tilde{L}(\hat{\mathbf{x}}).$$

Therefore, we must have $\tilde{L}(\mathbf{x}^*) = \tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*$. This completes the proof. $\square$

5 Closed-From Solutions of BLM with CVaR

Now, we are ready to give the closed-form solution for PAP with elliptical distributions under CVaR for the BLM. Under the BLM with CVaR, we have an updated mean vector $\boldsymbol{\mu}_{BL}$ and covariance matrix $\boldsymbol{\Sigma}_{BL}$ which are given by Proposition 3 using the new $\boldsymbol{\Pi}$ vector (i.e. equation (8)).

Theorem 2. *Let* $\mathbf{x}^*$ *be the optimal solution for PAP under CVaR then the closed-form solution is as follows:*

$$\mathbf{x}^* = (L + r_f)\left(d\beta\mathbf{I} - \mathbf{V}\right)^{-1}\boldsymbol{\Sigma}_{BL}^{-1}\left(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f\right) \tag{17}$$

where

$$d = \sqrt{(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}, \tag{18}$$

$$\mathbf{V} = \boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'. \tag{19}$$

Proof. From Theorem 1 we know that the sequence $\tilde{L}(\mathbf{x}_n)$ converges. We now consider the sequence of $\{\mathbf{x}_n\}$ and will find the explicit optimal solution of the PAP with CVaR.

Let $\mathbf{x}_{n+1}$ be the optimal solution of the optimization problem defined by the problem parameters (i.e. $\boldsymbol{\mu}_{BL}$, r_f and $\boldsymbol{\Sigma}_{BL}$) and $\tilde{L}(\mathbf{x}_n)$. The closed-form solution of this problem is well-known:

$$\mathbf{x}_{n+1} = \frac{\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}{2\delta}, \tag{20}$$

where δ is the Lagrange multiplier. Since $\mathbf{x}_{n+1}$ is the optimal solution, the value of δ is given by:

$$\delta = \frac{d}{2\tilde{L}(\mathbf{x}_n)},$$

where d is defined by (18). If we plug in the value of δ to (20) and use the definition of $\tilde{L}(\cdot)$, we can get

$$\mathbf{x}_{n+1} = \frac{\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\tilde{L}(\mathbf{x}_n)}{\beta d}.$$

So we have

$$\tilde{L}(\mathbf{x}_{n+1}) = \tilde{L}\left(\frac{\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\tilde{L}(\mathbf{x}_n)}{\beta d}\right).$$

By virtue of Theorem 1, as $n \to \infty$, we can get

$$\tilde{L}(\mathbf{x}^*) = \tilde{L}\left(\frac{\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\tilde{L}(\mathbf{x}^*)}{\beta d}\right).$$

By the definition of $\tilde{L}$, the above equation is equivalent to

$$(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)' \Big[(\beta d\mathbf{I} - \mathbf{V})\mathbf{x}^* \\ - (L + r_f)\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f) \Big] = 0.$$

Since it is true for any $\boldsymbol{\mu}_{BL}$, we must have

$$(\beta d\mathbf{I} - \mathbf{V})\mathbf{x}^* = (L + r_f)\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f).$$

Solve it and we can get

$$\mathbf{x}^* = (L + r_f)(d\beta\mathbf{I} - \mathbf{V})^{-1}\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f).$$

This completes the proof. □

We continue with the relationship between the constrained models and unconstrained model of PAP. First, let us start with the unconstrained model of PAP:

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \frac{\lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}{2} \right\}, \tag{21}$$

where λ denotes the investor's risk(variance in this case) reward trade-off. The optimal solution of this problem is:

$$\mathbf{x}^* = \lambda^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f).$$

We continue with the optimal allocation vector for the constrained model PAP as given by Definition 2. Using the same arguments given above, we can get the optimal solution

$$\mathbf{x}^* = \left(\frac{\sqrt{(\boldsymbol{\mu} - \mathbf{e}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}}{L} \right)^{-1} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f).$$

We can see that when the investor has an upper limit for her portfolio then she is doing nothing but changing her risk-reward coefficient as if she is solving an unconstrained model with an updated risk-reward trade-off coefficient (see also Steinbach [25]). Lastly, we give the solution of the BLM with CVaR (see Theorem 2):

$$\mathbf{x}^* = \left(\frac{d\beta\mathbf{I} - \mathbf{V}}{(L + r_f)} \right)^{-1} \boldsymbol{\Sigma}_{BL}^{-1} \left(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f \right).$$

Here, we can see that when the investor uses CVaR in an constrained setting then her risk-reward coefficient turns into a matrix. This is interesting and one can perturb the new risk-reward trade-off matrix in order to understand the behavior of the optimal portfolio vector under different problem settings.

6 Conclusion

Black and Litterman [5] proposed the BLM in order to overcome the Markowitz Model's drawbacks. Their model uses the Bayesian framework to combine the intuitions and/or inside information about the selected assets or market parameters with the historical information of the market to update the mean vector and covariance matrix. In our work, we use CVaR as a risk measure, instead of the variance risk measure proposed in the original model. In addition, elliptical uncertainty sets are used to model the uncertainty of asset returns in order to capture the non-normal behavior of the asset returns. For constrained problem, deriving the closed-form optimal solutions analytically is extremely difficult. Hence, we propose an efficient approximation algorithm for the BLM type optimization problems under CVaR and establish the convergence results. Based on the approximation, we derived the closed-form solution of the BLM with CVaR.

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