ریاضیات مهندسی پیشرفته

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پاییز 1392

سرفصل مطاالب جلسه سوم

معادلات همگن مرتبه اول

$$\frac{dy}{dt} + \alpha y(t) = 0$$

$$y' + \alpha y = 0 \Rightarrow \frac{y'}{y} = \alpha \xrightarrow{\int} \int \frac{y'}{y} = \int -\alpha \, dt$$

$$\Rightarrow \ln y(t) = -\alpha t + C$$

هر دو طرف را به توان e می‌رسانیم:

$$e^{\ln y(t)} = e^{-\alpha t + C} \Rightarrow y(t) = e^{C} e^{-\alpha t} = C_1 e^{-\alpha t}$$

$$\boxed{y(t) = C_1 e^{-\alpha t}}$$ جواب عمومی معادله همگن:

مثال 1)

$$y(0) = 10$$

$$y(0) = C_1 e^{-\alpha \times 0} = \boxed{C_1 = 10} \Rightarrow y(t) = 10 e^{-\alpha t}$$

ضریب ارتعاش در سیم صوتی

معادله موج یک بعدی – در سیم

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad u(x,t)$$

معادله موج دو بعدی – در صفحه

$$c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2} \qquad u(x,y,t)$$

معادله حرارت یک بعدی – در سیم

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,y)$$

یکی از دسته جوابها به صورت زیر می‌باشد:

$$u(x,y) = x^2 + y^2$$

$$u(x,y) = e^x \cos y \Rightarrow \frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial^2 u}{\partial x^2} = \boxed{e^x \cos y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^x \cdot -\sin y, \quad \frac{\partial^2 u}{\partial y^2} = \boxed{-e^x \cos y} \Rightarrow = 0$$

حل معادلات دیفرانسیل با روش تفکیک پذیری

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad u(x,t) = ?$$

شرایط کرانی

$$u(0,t) = u(l,t) = 0$$
$$u(x,0) = f(x)$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

$$u(x,t) = F(x) \cdot G(t) = F \cdot G$$

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \Rightarrow c^2 \frac{\partial^2}{\partial x^2}\left(F_{(x)} \cdot G_{(t)}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}(F \cdot G) \Rightarrow c^2 F'' G = F G'' \xrightarrow{\text{تقسیم بر } c^2FG} \frac{F''}{F} = \frac{G''}{c^2 G} = -\lambda^2$$

$$\begin{cases} F'' + \lambda^2 F = 0 \\ G'' + c^2 \lambda^2 G = 0 \end{cases} \qquad \boxed{F = A e^{sx}}$$

جواب عمومی تمام معادلات به صورت روبرو می‌باشد

مثال 2)

$$F'' + \lambda^2 F = 0$$
$$\Rightarrow s^2 F + \lambda^2 F = 0 \Rightarrow F(s^2 + \lambda^2) = 0$$
$$F \neq 0 \ , \ (s^2 + \lambda^2) = 0 \Rightarrow \boxed{s = \pm i\lambda}$$

$$\Rightarrow F = A_1 e^{s_1 x} + A_2 e^{s_2 x}$$
$$\Rightarrow F = A_1 e^{i\lambda x} + A_2 e^{-i\lambda x}$$

$$A e^{i\theta} = A \cos\theta + i A \sin\theta$$

$$F = A_1(\cos\lambda x + i \sin\lambda x) + A_2(\cos\lambda x - i \sin\lambda x)$$ تبدیل فرم مختلط به کارتزین

$$= \cos\lambda x \underbrace{(A_1 + A_2)}_{A} + \sin\lambda x \underbrace{(iA_1 - iA_2)}_{B}$$

$$\begin{cases} F'' + \lambda^2 F = 0 \Rightarrow F(x) = A\cos\lambda x + B\sin\lambda x \\ G'' + c^2\lambda^2 G = 0 \Rightarrow G(t) = D\cos c\lambda t + E\sin c\lambda t \end{cases}$$

$$u(0,t) = 0 \Rightarrow F(0)\,\underset{\neq 0}{G(t)} = 0 \Rightarrow F(0) = 0$$

$$\Rightarrow A\cos(\cancel{\lambda \times 0}) + B\sin(\cancel{\lambda \times 0}) = 0 \Rightarrow \boxed{A = 0}$$

$$u(l,t) = 0 \Rightarrow F(l)\,\underset{\neq 0}{G(t)} = 0 \Rightarrow F(l) = B\sin$$

$$\Rightarrow \underset{\neq 0}{B}\sin\lambda l = 0 \Rightarrow \sin\lambda l = 0 \quad \lambda l = k\pi \quad \boxed{\lambda = \frac{k\pi}{l}}$$

$$F_k(x) = B_k \sin\frac{k\pi x}{l}$$

$$G_k(t) = D_k \cos\frac{k\pi c t}{l} + E_k \sin\frac{k\pi c t}{l}$$

$$u_k(x,t) = F_k(x) \cdot G_k(t)$$

$$= B_k \sin\frac{k\pi}{l} x \left(D_k \cos\frac{k\pi c t}{l} + E_k \sin\frac{k\pi c t}{l} \right)$$

نامگذاری: $B_k D_k = b_k$ و $B_k E_k = a_k$

$$= b_k \sin\frac{k\pi}{l} x \cos\frac{k\pi c t}{l} + a_k \sin\frac{k\pi x}{\lambda} \sin\frac{k\pi c t}{l}$$

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{l} x \cos\frac{k\pi c t}{l} + a_k \sin\frac{k\pi x}{l} \sin\frac{k\pi c t}{l}$$

$$\Rightarrow u(x,0) = f(x)$$

$$\sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{l} \times 1 + a_k \sin\frac{k\pi x}{l} \times 0$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty} b \sin\frac{k\pi x}{l} = f(x) \qquad 0 < x < l}$$ معادله سیم مرتعش

b_k بدست آمده در بالا بسط نیم دامنه فرد سینوسی تابع $f(x)$ می‌باشد.

بنابراین طبق فرمول بسط نیم دامنه فرد:

$$\boxed{b_k = \frac{2}{l}\int_0^l f(x) \sin\frac{k\pi x}{l}\, dx}$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = g(x)$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{l} \cdot \frac{-k\pi c}{l}\underbrace{\cancel{\sin\frac{k\pi ct}{l}}}_{\sin 0 = 0} + a_k \sin\frac{k\pi x}{l} \cdot \frac{k\pi c}{l} \underbrace{\cancel{\cos\frac{k\pi ct}{l}}}_{\cos 0 = 1}$$

$$\Rightarrow g(x) = \sum_{k=1}^{\infty} a_k \frac{k\pi c}{l} \sin\frac{k\pi x}{l} = g(x)$$ بسط نیم دامنه فرد

$$a_k \frac{k\pi c}{l} = \frac{2}{l}\int_0^l g(x) \sin\frac{k\pi x}{l}\, dx$$

$$a_k = \frac{2}{k\pi c}\int_0^l g(x) \sin\frac{k\pi x}{l}\, dx$$

$$b_k = \frac{2}{l}\int_0^l f(x) \sin\frac{k\pi x}{l}\, dx$$

$$u(x,t) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l} \sin\frac{k\pi ct}{l} + b_k \sin\frac{k\pi x}{l} \cos\frac{k\pi ct}{l}$$

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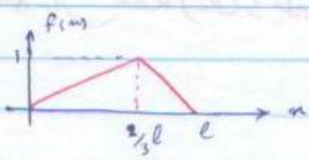
تمرین 1) معادله دیفرانسیل موج یک بعدی را برای مقادیر مقابل بدست آورید.

الف) $l = \pi \;\&\; c^2 = 1$

$f(x) = k \sin 2x \;\&\; g(x) = x$

ب)



$g(x) = x \;,\; l = \pi$

تمرین 2) معادلات تفکیک شده معادله دیفرانسیل با مشتقات جزیی را بدست آورید.

الف) $\dfrac{x\,\partial u}{\partial x} = y\,\dfrac{\partial u}{\partial y}$

ب) $\dfrac{x^2\,\partial^2 u}{\partial x\,\partial y} + 3y^2 u = 0$

تمرین 3) حل معادله موج یک بعدی به ازای k>0

جواب : غیرممکن $u(x,t) = 0$

ادامه حل مثال صفحه قبل:

مثال قبل برای $-\lambda^2$ حل شد

$$\frac{F''}{F} = \frac{G''}{c^2 G} = k = \begin{cases} -\lambda^2 & \checkmark \\ 0 & \text{غ ق ق} \\ \lambda^2 & ? \end{cases}$$

حالا برای $k=0$ (مثال را حل می‌کنیم)

$F'' = 0 \Rightarrow F(t) = Ax + B$

$G'' = 0 \Rightarrow G(t) = Ct + D$

$u(0,t) = 0 \rightarrow F(0)\,G(t) = 0$

$\Rightarrow F(0) = 0 \Rightarrow A \times 0 + B = 0 \Rightarrow \boxed{B = 0}$

$u(\ell,t) = 0 \rightarrow F(\ell) = 0 \Rightarrow A\ell = 0 \rightarrow \boxed{A = 0}$

$F(x) = Ax + B = 0 \Rightarrow u(x,t) = 0$ بنابراین غیر ممکن است

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(0,t) = \rho_1 \quad u(\ell,t) = \rho_2$$

$$u(x,0) = f(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

حل:

$$u(x,t) = V(x) + w(x,t)$$

$$u(0,t) = \rho_1 \Rightarrow \rho_1 = \underbrace{V(0)}_{\rho_1} + \underbrace{w(0,t)}_{0}$$

$$u(\ell,t) = \rho_2 \Rightarrow \rho_2 = \underbrace{V(\ell)}_{\rho_2} + \underbrace{w(\ell,t)}_{0}$$

$$\frac{c^2 \partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \Rightarrow c^2\left(\frac{\partial^2 V(x)}{dx^2} + \frac{d^2 w(x,t)}{dx^2}\right)$$

$$= \cancel{\frac{\partial^2 V(x)}{\partial t^2}} + \frac{\partial^2 w(x,t)}{\partial t^2}$$

$$= c^2 V''(x) + c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\left\{\begin{array}{l} \dfrac{c^2 \partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{\partial t^2} \\ w(0,t) = 0 \\ w(\ell,t) = 0 \end{array}\right.$$

معادله موج می‌شود با شرایط مرزی صفر!

$$\Rightarrow w(x,t)=\sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l}\sin\frac{k\pi ct}{l}+b_k\sin\frac{k\pi x}{l}\cos\frac{k\pi ct}{l}$$

$$u(x,t)=\frac{P_2-P_1}{l}x+P_1+\sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l}\sin\frac{k\pi ct}{l}+b_k\sin\frac{k\pi x}{l}\cos\frac{k\pi ct}{l}$$

$u(x,0)=f(x)$

تمرین 4) محاسبه ضرایب فوریه از رابطه فوق. ak - bk

$v''(x)=0$

$v(0)=P_1$

$v(l)=P_2$

$v''(x)=0 \Rightarrow v(x)=Ax+B$

$v(0)=P_1 \Rightarrow A\times 0+B=P_1 \Rightarrow \boxed{B=P_1}$

$v(l)=P_2 \Rightarrow Al+P_1=P_2 \Rightarrow A=\frac{P_2-P_1}{l}$

$$\boxed{v(x)=\frac{P_2-P_1}{l}x+P_1}$$

$$u(x,t)=\sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{\ell}\sin\frac{k\pi ct}{\ell}+b_k\sin\frac{k\pi x}{\ell}\cos\frac{k\pi ct}{\ell}$$

$$a_k=\frac{2}{k\pi c}\int^{\ell} g(x)\sin\frac{k\pi x}{\ell}\,dx$$

فرم سری فوریه

با فرض $g(x)=0$

$$\Rightarrow a_k=0$$

$$u(x,t)=\sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{\ell}\cos\frac{k\pi ct}{\ell}$$

یاد آوری

$$\sin(a\pm b)=\sin a\cos b\pm\sin b\cos a$$

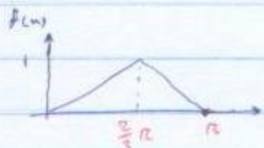
$$\sin a\cos b=\frac{1}{2}\left(\sin(a+b)+\sin(a-b)\right)$$

طبق فرمول $\Rightarrow \frac{1}{2}\sum_{k=1}^{\infty} b_k\sin\frac{k\pi}{\ell}(x+ct)+b_k\sin\frac{k\pi}{\ell}(x-ct)$

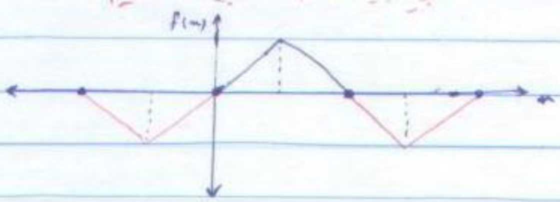
$$=\underbrace{\frac{1}{2}\sum_{k=1}^{\infty} b_k\sin\frac{k\pi}{\ell}(x+ct)}_{f(x+ct)}+\underbrace{\frac{1}{2}\sum_{k=1}^{\infty} b_k\sin\frac{k\pi}{\ell}(x-ct)}_{f(x-ct)}$$

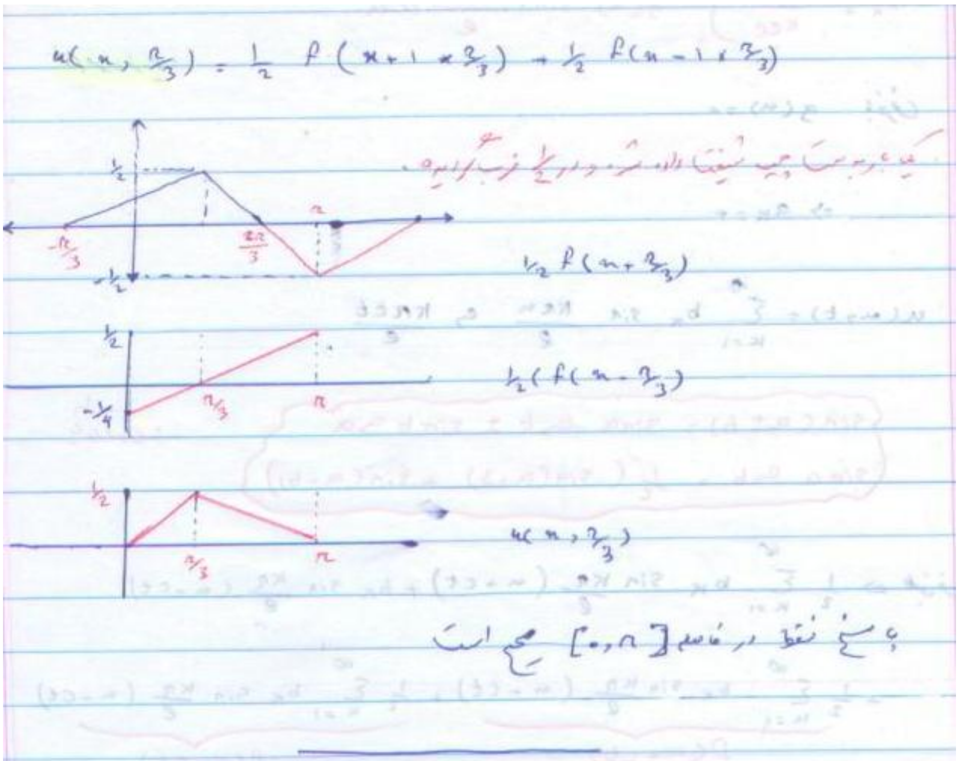
$$=\frac{1}{2}f(x+ct)+\frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)$$

مثال) وضعیت میله را در زمان $\pi/3$ بدست آورید.



چون بسط نیم دامنه فرد است نمودار بقیه شکل را بصورت تابع فرد ادامه میدهیم





تمرین 5) مثال فوق را در $t=\pi/2$ و $t=2\pi/3$ بدست آورید.

حل معادله موج یک بعدی با استفاده از روش دالامبر:

$$\xi = x + ct$$
$$\eta = x - ct$$

$$c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\frac{\partial \eta}{\partial x} = 1 \quad , \quad \frac{\partial \xi}{\partial x} = 1$$

$$\frac{\partial \eta}{\partial t} = -c \quad , \quad \frac{\partial \xi}{\partial t} = c$$

$$\frac{\partial u(\xi,\eta)}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta} = c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2}\cancel{\frac{\partial \xi}{\partial x}} + \frac{\partial^2 u}{\partial \eta \partial \xi}\cancel{\frac{\partial \eta}{\partial x}} + \frac{\partial^2 u}{\partial \xi \partial \eta}\cancel{\frac{\partial \xi}{\partial x}} + \frac{\partial^2 u}{\partial \eta^2}\cancel{\frac{\partial \eta}{\partial x}}$$

$$= \frac{\partial^2 u}{\partial \xi^2} + \underbrace{\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \xi \partial \eta}}_{2\frac{\partial^2 u}{\partial \xi \partial \eta}} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\, c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$$= c\left(\frac{\partial^2 u}{\partial \xi^2}\underbrace{\cancel{\frac{\partial \xi}{\partial t}}}_{c} + \frac{\partial^2 u}{\partial \eta \partial \xi}\underbrace{\cancel{\frac{\partial \eta}{\partial t}}}_{-c} - \frac{\partial^2 u}{\partial \xi \partial \eta}\underbrace{\frac{\partial \xi}{\partial t}}_{c} - \cancel{\frac{\partial^2 u}{\partial \eta^2}}\underbrace{\cancel{\frac{\partial \eta}{\partial t}}}_{-c}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial \xi^2} - \frac{2\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}\right)$$

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \Rightarrow \cancel{c^2}\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} + \frac{2\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right)$$

$$\Rightarrow \cancel{c^2}\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} - \frac{2\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right) \Rightarrow \boxed{\frac{\partial^2 u}{\partial \xi \partial \eta} = 0}$$

$$u(\xi, \eta) = \varphi(\xi) + \psi(\eta)$$

$$\frac{\partial u}{\partial \xi} = \varphi'(\xi) + 0$$

$$\frac{\partial^2 u}{\partial \eta \partial \xi} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = \frac{\partial}{\partial \eta}\left(\varphi'(\xi)\right) = 0$$

$$u(x,t)\Big|_{t=0} = f(x) \Rightarrow \varphi(x+ct) + \psi(x-ct)\Big|_{t=0} = f(x)$$

$$\Rightarrow f(x) = \varphi(x) + \psi(x)$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = g(x) \Rightarrow c\left(\frac{\partial u}{\partial \rho} - \frac{\partial u}{\partial \zeta}\right)\Big|_{t=0} = g(x)$$

$$\Rightarrow c\left(\varphi'(x+ct) - \psi'(x-ct)\right)\Big|_{t=0} = g(x)$$

$$\Rightarrow g(x) = c\left(\varphi'(x) - \psi'(x)\right)$$

$$\varphi(x) - \psi(x) = f(x) \qquad \varphi(x) + \psi(x) = f(x)$$

$$\int_0^x \varphi'(x) - \psi'(x) = \int \frac{1}{c}\, g(x) \Rightarrow \varphi(x) - \psi(x) = \frac{1}{c}\int_0^x g(s)\,ds$$

$$\Rightarrow \begin{cases} \psi(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(s)\,ds\right) \\ \varphi(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(s)\,ds\right) \end{cases}$$

$$u(x,t) = u(\rho,\zeta) = \varphi(\rho) + \psi(\zeta)$$

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(s)\,ds\right) + \frac{1}{2}\left(f(x-ct) - \frac{1}{c}\int_0^{x-ct} g(s)\,ds\right)$$

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$