Energy Methods

14

CHAPTER OBJECTIVES

In this chapter, we will show how to apply energy methods to solve problems involving deflection. The chapter begins with a discussion of work and strain energy, followed by a development of the principle of conservation of energy. Using this principle, the stress and deflection of a member are determined when the member is subjected to impact. The method of virtual work and Castigliano's theorem are then developed, and these methods are used to determine the displacement and slope at points on structural members and mechanical elements.

14.1 External Work and Strain Energy

The deflection of joints on a truss or points on a beam or shaft can be determined using energy methods. Before developing any of these methods, however, we will first define the work caused by an external force and couple moment and show how to express this work in terms of a body's strain energy. The formulations to be presented here and in the next section will provide the basis for applying the work and energy methods that follow throughout the chapter.

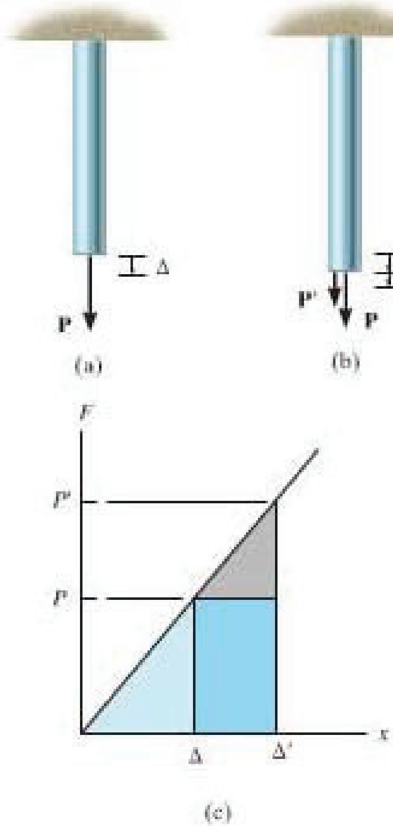

Fig. 14–1

Work of a Force. In mechanics, a force does *work* when it undergoes a displacement dx that is in the *same direction* as the force. The work done is a scalar, defined as $dU_e = F\,dx$. If the total displacement is Δ, the work becomes

$$U_e = \int_0^{\Delta} F\,dx \qquad (14\text{–}1)$$

To show how to apply this equation, we will calculate the work done by an axial force applied to the end of the bar shown in Fig. 14–1*a*. As the magnitude of the force is *gradually* increased from zero to some limiting value $F = P$, the final displacement of the end of the bar becomes Δ. If the material behaves in a linear-elastic manner, then the force will be directly proportional to the displacement; that is, $F = (P/\Delta)x$. Substituting into Eq. 14–1 and integrating from 0 to Δ, we get

$$U_e = \frac{1}{2}P\Delta \qquad (14\text{–}2)$$

Therefore, as the force is gradually applied to the bar, its magnitude builds from zero to some value P, and consequently, the work done is equal to the *average force magnitude*, $P/2$, times the total displacement Δ. We can represent this graphically as the light-blue shaded area of the triangle in Fig. 14–1*c*.

Suppose, however, that **P** is already applied to the bar and that *another force* **P**′ is now applied, so that the end of the bar is displaced *further* by an amount Δ', Fig. 14–1*b*. The work done by **P**′ is equal to the gray shaded triangular area, but now the work done by **P** when the bar undergoes this further displacement is

$$U_e' = P\Delta' \qquad (14\text{–}3)$$

Here the work represents the dark-blue shaded *rectangular area* in Fig. 14–1*c*. In this case **P** does not change its magnitude, since the bar's displacement Δ' is caused only by **P**′. Therefore, work here is simply the force magnitude P times the displacement Δ'.

Work of a Couple Moment. A couple moment **M** does work when it undergoes an angular displacement $d\theta$ along its line of action. The work is defined as $dU_e = M\,d\theta$, Fig. 14–2. If the total angular displacement is θ rad, the work becomes

$$U_e = \int_0^\theta M\,d\theta \tag{14–4}$$

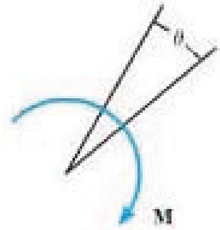

Fig. 14–2

As in the case of force, if the couple moment is applied to a *body* having linear elastic material behavior, such that its magnitude is increased gradually from zero at $\theta = 0$ to M at θ, then the work is

$$U_e = \frac{1}{2}M\theta \tag{14–5}$$

However, if the couple moment is already applied to the body and other loadings further rotate the body by an amount θ', then the work is

$$U'_e = M\theta'$$

Strain Energy. When loads are applied to a body, they will deform the material. Provided no energy is lost in the form of heat, the external work done by the loads will be converted into internal work called ***strain energy***. This energy, which is *always positive*, is stored in the body and is caused by the action of either normal or shear stress.

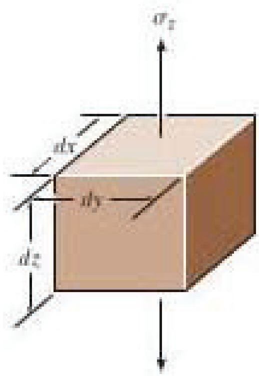

Fig. 14–3

Normal Stress. If the volume element shown in Fig. 14–3 is subjected to the normal stress σ_z, then the force created on the element's top and bottom faces is $dF_z = \sigma_z\,dA = \sigma_z\,dx\,dy$. If this force is applied gradually to the element, like the force **P** discussed previously, its magnitude is increased from zero to dF_z, while the element undergoes an elongation $d\Delta_z = \epsilon_z\,dz$. The work done by dF_z is therefore $dU_i = \frac{1}{2}dF_z\,d\Delta_z = \frac{1}{2}[\sigma_z\,dx\,dy]\epsilon_z\,dz$. Since the volume of the element is $dV = dx\,dy\,dz$, we have

$$dU_i = \frac{1}{2}\sigma_z\epsilon_z\,dV \tag{14–6}$$

Notice that dU_i is *always positive*, even if σ_z is compressive, since σ_z and ϵ_z will always be in the same direction.

In general then, if the body is subjected only to a uniaxial *normal stress* σ, the strain energy in the body is then

$$U_i = \int_V \frac{\sigma\epsilon}{2}\,dV \tag{14–7}$$

Also, if the material behaves in a linear-elastic manner, then Hooke's law applies, and we can express the strain energy in terms of the normal stress as

$$\boxed{U_i = \int_V \frac{\sigma^2}{2E}\,dV} \tag{14–8}$$

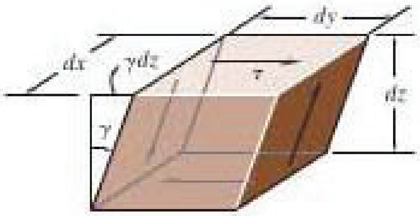

Fig. 14–4

Shear Stress. A strain-energy expression similar to that for normal stress can also be established for the material when it is subjected to shear stress. Consider the volume element shown in Fig. 14–4. Here the shear stress causes the element to deform such that only the shear force $dF = \tau(dx\,dy)$, acting on the top face of the element, is displaced $\gamma\,dz$ relative to the bottom face. The *vertical faces* only rotate, and therefore the shear forces on these faces do no work. Hence, the strain energy stored in the element is

$$dU_i = \frac{1}{2}[\tau(dx\,dy)]\gamma\,dz$$

or since $dV = dx\,dy\,dz$

$$dU_i = \frac{1}{2}\tau\gamma\,dV \tag{14–9}$$

The strain energy stored in the body is therefore

$$U_i = \int_V \frac{\tau\gamma}{2}\,dV \tag{14–10}$$

Like the case for normal strain energy, shear strain energy is always positive since τ and γ are always in the same direction. If the material is linear elastic, then, applying Hooke's law, $\gamma = \tau/G$, we can express the strain energy in terms of the shear stress as

$$\boxed{U_i = \int_V \frac{\tau^2}{2G}\,dV} \tag{14–11}$$

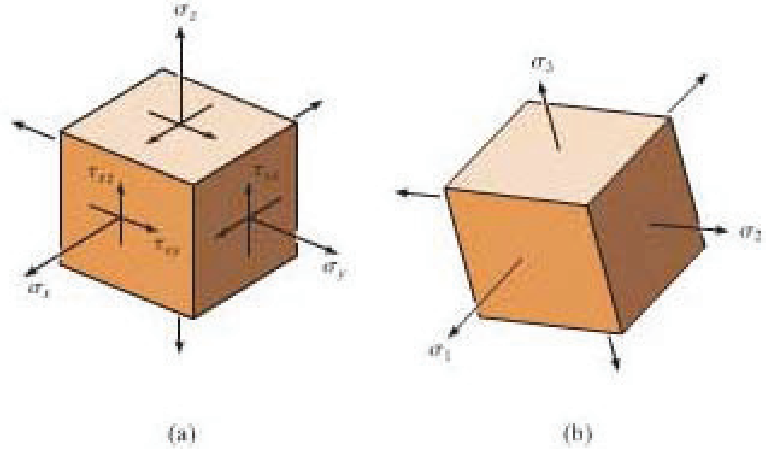

Fig. 14–5

In the next section, we will use Eqs. 14–8 and 14–11 to obtain formal expressions for the strain energy stored in members subjected to several types of loads. Once this is done we will then be able to develop the energy methods necessary to determine the displacement and slope at points on a body.

Multiaxial Stress. The previous development may be expanded to determine the strain energy in a body when it is subjected to a general state of stress, Fig. 14–5*a*. The strain energies associated with each of the normal and shear stress components can be obtained from Eqs. 14–6 and 14–9. Since energy is a scalar, the total strain energy in the body is therefore

$$U_i = \int_V \left[\frac{1}{2}\sigma_x\epsilon_x + \frac{1}{2}\sigma_y\epsilon_y + \frac{1}{2}\sigma_z\epsilon_z + \frac{1}{2}\tau_{xy}\gamma_{xy} + \frac{1}{2}\tau_{yz}\gamma_{yz} + \frac{1}{2}\tau_{xz}\gamma_{xz}\right] dV \qquad (14\text{–}12)$$

The strains can be eliminated by using the generalized form of Hooke's law given by Eqs. 10–18 and 10–19. After substituting and combining terms, we have

$$U_i = \int_V \left[\frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z) + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right)\right] dV \qquad (14\text{–}13)$$

If only the principal stresses σ_1, σ_2, σ_3 act on the element, Fig. 14–5*b*, this equation reduces to a simpler form, namely,

$$U_i = \int_V \left[\frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\nu}{E}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right] dV \qquad (14\text{–}14)$$

This equation was used in Sec. 10.7 as a basis for developing the maximum-distortion-energy theory.

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14.2 Elastic Strain Energy for Various Types of Loading

Using the equations for elastic strain energy developed in the previous section, we will now formulate the strain energy stored in a member when it is subjected to an axial load, bending moment, transverse shear, and torsional moment. Examples will be given to show how to calculate the strain energy in members subjected to each of these loadings.

Axial Load. Consider a bar of variable yet slightly tapered cross section, Fig. 14–6. The *internal axial force* at a section located a distance x from one end is N. If the cross-sectional area at this section is A, then the normal stress on the section is $\sigma = N/A$. Applying Eq. 14–8, we have

Fig. 14–6

$$U_i = \int_V \frac{\sigma_x^2}{2E} dV = \int_V \frac{N^2}{2EA^2} dV$$

If we choose an element or differential slice having a volume $dV = A\,dx$, the general formula for the strain energy in the bar is therefore

$$\boxed{U_i = \int_0^L \frac{N^2}{2AE} dx} \qquad (14\text{–}15)$$

For the more common case of a prismatic bar of constant cross-sectional area A, length L, and constant axial load N, Fig. 14–7, Eq. 14–15, when integrated, gives

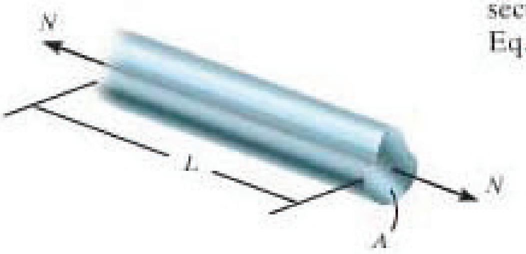

Fig. 14–7

$$\boxed{U_i = \frac{N^2 L}{2AE}} \qquad (14\text{–}16)$$

Notice that the bar's elastic strain energy will *increase* if the length of the bar is increased, or if the modulus of elasticity or cross-sectional area is decreased. For example, an aluminum rod $[E_{al} = 10(10^3)\text{ ksi}]$ will store approximately three times as much energy as a steel rod $[E_{st} = 29(10^3)\text{ ksi}]$ having the same size and subjected to the same load. However, doubling the cross-sectional area of a rod will decrease its ability to store energy by one-half. The following example illustrates this point numerically.

EXAMPLE 14.1

One of the two high-strength steel bolts A and B shown in Fig. 14–8 is to be chosen to support a sudden tensile loading. For the choice it is necessary to determine the greatest amount of elastic strain energy that each bolt can absorb. Bolt A has a diameter of 0.875 in. for 2 in. of its length and a root (or smallest) diameter of 0.731 in. within the 0.25-in. threaded region. Bolt B has "upset" threads, such that the diameter throughout its 2.25-in. length can be taken as 0.731 in. In both cases, neglect the extra material that makes up the threads. Take $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 44$ ksi.

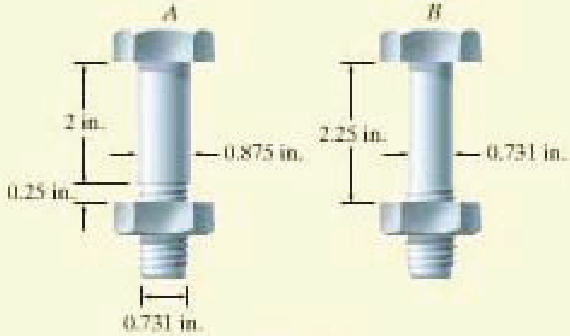

Fig. 14–8

SOLUTION

Bolt *A*. If the bolt is subjected to its maximum tension, the maximum stress of $\sigma_Y = 44$ ksi will occur within the 0.25-in. region. This tension force is

$$P_{max} = \sigma_Y A = 44 \text{ ksi}\left[\pi\left(\frac{0.731 \text{ in.}}{2}\right)^2\right] = 18.47 \text{ kip}$$

Applying Eq. 14–16 to each region of the bolt, we have

$$U_i = \sum \frac{N^2 L}{2AE}$$

$$= \frac{(18.47 \text{ kip})^2(2 \text{ in.})}{2[\pi(0.875 \text{ in.}/2)^2][29(10^3) \text{ ksi}]} + \frac{(18.47 \text{ kip})^2(0.25 \text{ in.})}{2[\pi(0.731 \text{ in.}/2)^2][29(10^3) \text{ ksi}]}$$

$$= 0.0231 \text{ in.} \cdot \text{kip} \qquad \textit{Ans.}$$

Bolt *B*. Here the bolt is assumed to have a uniform diameter of 0.731 in. throughout its 2.25-in. length. Also, from the calculation above, it can support a maximum tension force of $P_{max} = 18.47$ kip. Thus,

$$U_i = \frac{N^2 L}{2AE} = \frac{(18.47 \text{ kip})^2(2.25 \text{ in.})}{2[\pi(0.731 \text{ in.}/2)^2][29(10^3) \text{ ksi}]} = 0.0315 \text{ in.} \cdot \text{kip} \qquad \textit{Ans.}$$

NOTE: By comparison, bolt B can absorb 36% more elastic energy than bolt A, because it has a smaller cross section along its shank.

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Bending Moment. Since a bending moment applied to a straight prismatic member develops *normal stress* in the member, we can use Eq. 14–8 to determine the strain energy stored in the member due to bending. For example, consider the axisymmetric beam shown in Fig. 14–9. Here the internal moment is M, and the normal stress acting on the arbitrary element a distance y from the neutral axis is $\sigma = My/I$. If the volume of the element is $dV = dA\,dx$, where dA is the area of its exposed face and dx is its length, the elastic strain energy in the beam is

$$U_i = \int_V \frac{\sigma^2}{2E} dV = \int_V \frac{1}{2E}\left(\frac{My}{I}\right)^2 dA\,dx$$

or

$$U_i = \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2\,dA\right) dx$$

Realizing that the area integral represents the moment of inertia of the area about the neutral axis, the final result can be written as

$$\boxed{U_i = \int_0^L \frac{M^2\,dx}{2EI}} \qquad (14\text{–}17)$$

To evaluate the strain energy, therefore, we must first express the internal moment as a function of its position x along the beam, and then perform the integration over the beam's entire length.* The following examples illustrate this procedure.

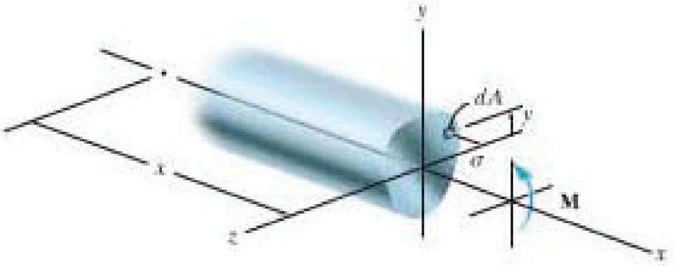
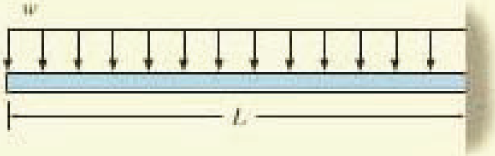


Fig. 14–9

*Recall that the flexure formula, as used here, can also be used with justifiable accuracy to determine the stress in slightly tapered beams. (See Sec. 6.4.) So in the general sense, I in Eq. 14–17 may also have to be expressed as a function of x.

EXAMPLE 14.2

Determine the elastic strain energy due to bending of the cantilevered beam in Fig. 14–10*a*. *EI* is constant.

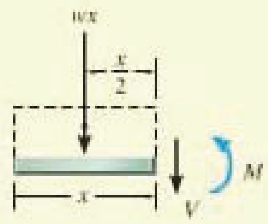


(a)

Fig. 14–10

SOLUTION

The internal moment in the beam is determined by establishing the x coordinate with origin at the left side. The left segment of the beam is shown in Fig. 14–10*b*. We have



(b)

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad M + wx\left(\frac{x}{2}\right) = 0$$

$$M = -w\left(\frac{x^2}{2}\right)$$

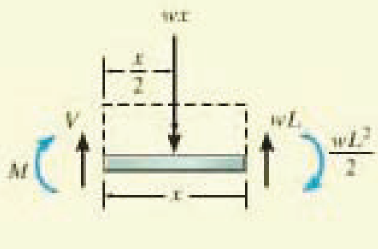
Applying Eq. 14–17 yields

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} = \int_0^L \frac{[-w(x^2/2)]^2\,dx}{2EI} = \frac{w^2}{8EI}\int_0^L x^4\,dx$$

or

$$U_i = \frac{w^2L^5}{40EI} \qquad \textit{Ans.}$$

We can also obtain the strain energy using an x coordinate having its origin at the right side of the beam and extending positive to the left, Fig. 14–10*c*. In this case,



(c)

$$\circlearrowleft + \Sigma M_{NA} = 0; \quad -M - wx\left(\frac{x}{2}\right) + wL(x) - \frac{wL^2}{2} = 0$$

$$M = -\frac{wL^2}{2} + wLx - w\left(\frac{x^2}{2}\right)$$

Applying Eq. 14–17, we obtain the same result as before; however, more calculations are involved in this case.

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EXAMPLE 14.3

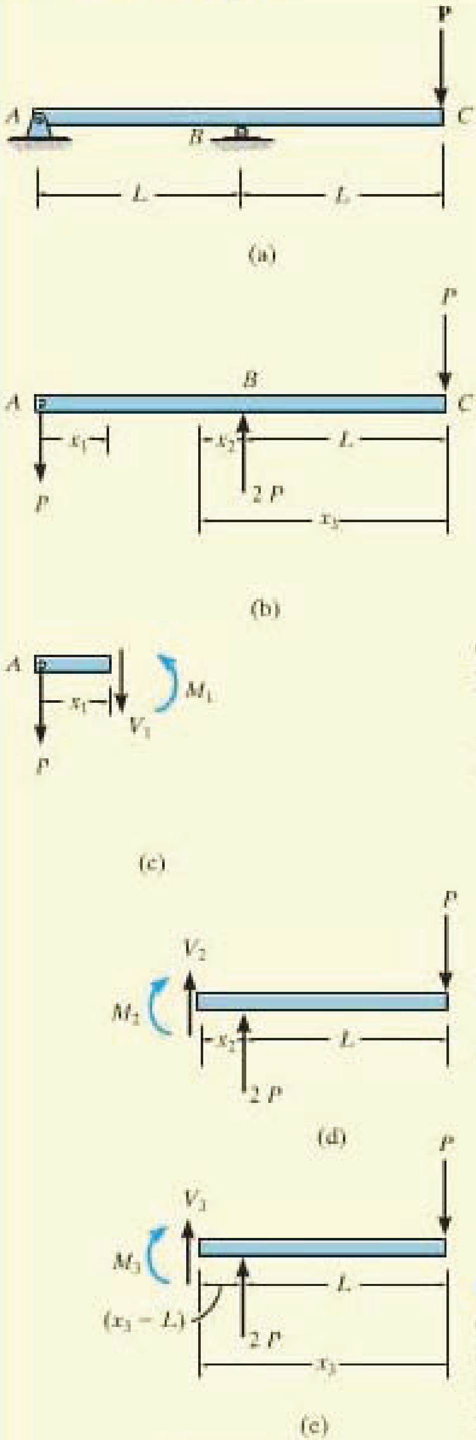

Fig. 14–11

Determine the bending strain energy in region AB of the beam shown in Fig. 14–11a. EI is constant.

SOLUTION

A free-body diagram of the beam is shown in Fig. 14–11b. To obtain the answer we can express the internal moment in terms of any one of the indicated three "x" coordinates and then apply Eq. 14–17. Each of these solutions will now be considered.

$0 \le x_1 \le L$. From the free-body diagram of the section in Fig. 14–11c, we have

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad M_1 + Px_1 = 0$$

$$M_1 = -Px_1$$

$$U_i = \int \frac{M^2\,dx}{2EI} = \int_0^L \frac{(-Px_1)^2\,dx_1}{2EI} = \frac{P^2L^3}{6EI} \qquad \textit{Ans.}$$

$0 \le x_2 \le L$. Using the free-body diagram of the section in Fig. 14–11d gives

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad -M_2 + 2P(x_2) - P(x_2 + L) = 0$$

$$M_2 = P(x_2 - L)$$

$$U_i = \int \frac{M^2\,dx}{2EI} = \int_0^L \frac{[P(x_2 - L)]^2\,dx_2}{2EI} = \frac{P^2L^3}{6EI} \qquad \textit{Ans.}$$

$L \le x_3 \le 2L$. From the free-body diagram in Fig. 14–11e, we have

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad -M_3 + 2P(x_3 - L) - P(x_3) = 0$$

$$M_3 = P(x_3 - 2L)$$

$$U_i = \int \frac{M^2\,dx}{2EI} = \int_L^{2L} \frac{[P(x_3 - 2L)]^2\,dx_3}{2EI} = \frac{P^2L^3}{6EI} \qquad \textit{Ans.}$$

NOTE: This and the previous example indicate that the strain energy for the beam can be found using *any* suitable x coordinate. It is only necessary to integrate over the range of the coordinate where the internal energy is to be determined. Here the choice of x_1 provides the simplest solution.

Transverse Shear. The strain energy due to shear stress in a beam element can be determined by applying Eq. 14–11. Here we will consider the beam to be prismatic and to have an axis of symmetry about the y axis as shown in Fig. 14–12. If the internal shear at the section x is V, then the shear stress acting on the volume element of material, having an area dA and length dx, is $\tau = VQ/It$. Substituting into Eq. 14–11, the strain energy for shear becomes

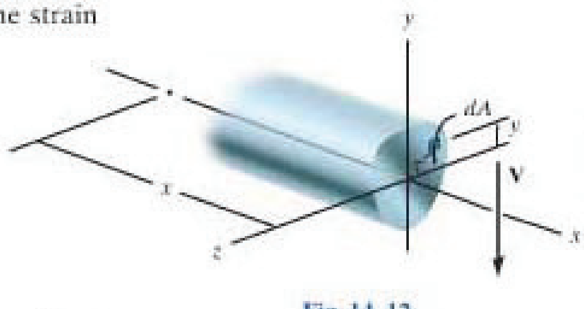

Fig. 14–12

$$U_i = \int_V \frac{\tau^2}{2G}dV = \int_V \frac{1}{2G}\left(\frac{VQ}{It}\right)^2 dA\,dx$$

$$U_i = \int_0^L \frac{V^2}{2GI^2}\left(\int_A \frac{Q^2}{t^2}dA\right)dx$$

The integral in parentheses can be simplified if we define the *form factor* for shear as

$$f_s = \frac{A}{I^2}\int_A \frac{Q^2}{t^2}dA \qquad (14\text{–}18)$$

Substituting into the above equation, we get

$$\boxed{U_i = \int_0^L \frac{f_s V^2\,dx}{2GA}} \qquad (14\text{–}19)$$

The form factor defined by Eq. 14–18 is a dimensionless number that is unique for each specific cross-sectional area. For example, if the beam has a rectangular cross section of width b and height h, Fig. 14–13, then

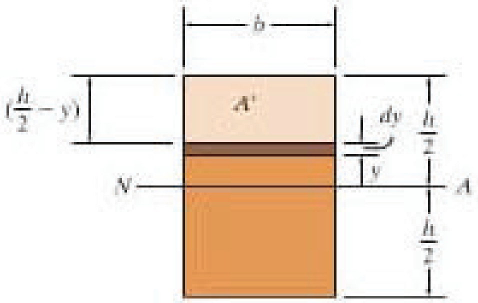

Fig. 14–13

$$t = b$$
$$dA = b\,dy$$
$$I = \frac{1}{12}bh^3$$
$$Q = \bar{y}'A' = \left(y + \frac{(h/2) - y}{2}\right)b\left(\frac{h}{2} - y\right) = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

Substituting these terms into Eq. 14–18, we get

$$f_s = \frac{bh}{\left(\frac{1}{12}bh^3\right)^2}\int_{-h/2}^{h/2} \frac{b^2}{4b^2}\left(\frac{h^2}{4} - y^2\right)^2 b\,dy = \frac{6}{5} \qquad (14\text{–}20)$$

The form factor for other sections can be determined in a similar manner. Once obtained, this factor is substituted into Eq. 14–19 and the strain energy for transverse shear can then be evaluated.

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EXAMPLE 14.4

Determine the strain energy in the cantilevered beam due to shear if the beam has a square cross section and is subjected to a uniform distributed load w, Fig. 14–14a. EI and G are constant.

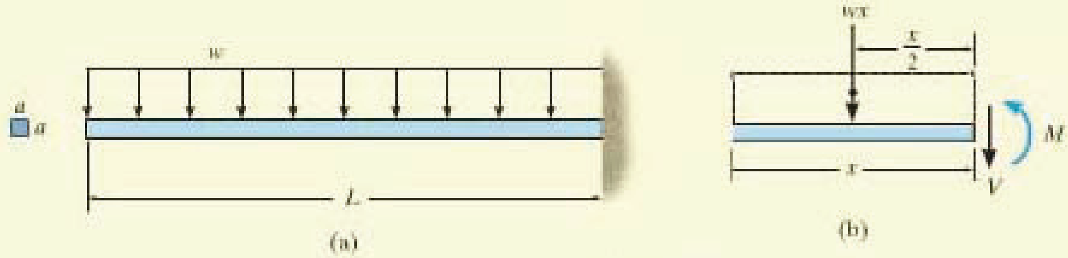


Fig. 14–14

SOLUTION

From the free-body diagram of an arbitrary section, Fig. 14–14b, we have

$+\uparrow \Sigma F_y = 0;$ $\qquad -V - wx = 0$

$$V = -wx$$

Since the cross section is square, the form factor $f_s = \frac{6}{5}$ (Eq. 14–20) and therefore Eq. 14–19 becomes

$$(U_i)_s = \int_0^L \frac{\frac{6}{5}(-wx)^2\,dx}{2GA} = \frac{3w^2}{5GA}\int_0^L x^2\,dx$$

or

$$(U_i)_s = \frac{w^2L^3}{5GA} \qquad \textit{Ans.}$$

NOTE: Using the results of Example 14.2, with $A = a^2$, $I = \frac{1}{12}a^4$, the ratio of shear to bending strain energy is

$$\frac{(U_i)_s}{(U_i)_b} = \frac{w^2L^3/5Ga^2}{w^2L^5/40E\left(\frac{1}{12}a^4\right)} = \frac{2}{3}\left(\frac{a}{L}\right)^2\frac{E}{G}$$

Since $G = E/2(1+\nu)$ and $\nu \le \frac{1}{2}$ (Sec. 10.6), then as an *upper* bound, $E = 3G$, so that

$$\frac{(U_i)_s}{(U_i)_b} = 2\left(\frac{a}{L}\right)^2$$

It can be seen that this ratio will increase as L decreases. However, even for very short beams, where, say, $L = 5a$, the contribution due to shear strain energy is only 8% of the bending strain energy. For this reason, the shear strain energy stored in beams is usually neglected in engineering analysis.

Torsional Moment. To determine the internal strain energy in a circular shaft or tube due to an applied torsional moment, we must apply Eq. 14–11. Consider the slightly tapered shaft in Fig. 14–15. A section of the shaft taken a distance x from one end is subjected to an internal torque T. The shear stress distribution that causes this torque varies linearly from the center of the shaft. On the arbitrary element of area dA and length dx, the stress is $\tau = T\rho/J$. The strain energy stored in the shaft is thus

$$U_i = \int_V \frac{\tau^2}{2G} dV = \int_V \frac{1}{2G}\left(\frac{T\rho}{J}\right)^2 dA\,dx = \int_0^L \frac{T^2}{2GJ^2}\left(\int_A \rho^2\,dA\right) dx$$

Since the area integral represents the polar moment of inertia J for the shaft at the section, the final result can be written as

$$\boxed{U_i = \int_0^L \frac{T^2}{2GJ} dx} \qquad (14\text{–}21)$$

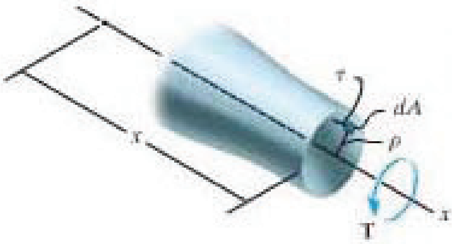


Fig. 14–15

The most common case occurs when the shaft (or tube) has a constant cross-sectional area and the applied torque is constant, Fig. 14–16. Integration of Eq. 14–21 then gives

$$\boxed{U_i = \frac{T^2 L}{2GJ}} \qquad (14\text{–}22)$$

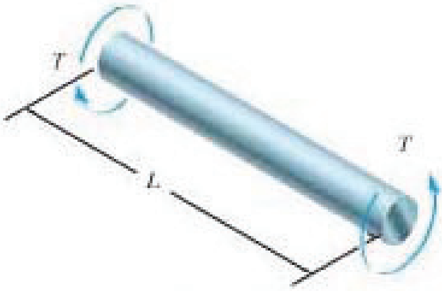


Fig. 14–16

From this equation we may conclude that, like an axially loaded member, the energy-absorbing capacity of a torsionally loaded shaft is *decreased* by increasing the diameter of the shaft, since this increases J.

Important Points

- A *force* does work when it moves through a *displacement*. When a force is applied to a body and its magnitude is increased gradually from zero to F, the work is $U = (F/2)\Delta$, whereas if the force is constant when the displacement occurs then $U = F\Delta$.
- A *couple moment* does work when it displaces through a *rotation*.
- *Strain energy* is caused by the internal work of the normal and shear stresses. It is always a *positive* quantity.
- The strain energy can be related to the resultant internal loadings N, V, M, and T.
- As the beam becomes longer, the strain energy due to bending becomes much larger than the strain energy due to shear. For this reason, the *shear strain energy* in beams can generally be *neglected*.

The following example illustrates how to determine the strain energy in a circular shaft due to a torsional loading.

EXAMPLE 14.5

The tubular shaft in Fig. 14–17*a* is fixed at the wall and subjected to two torques as shown. Determine the strain energy stored in the shaft due to this loading. $G = 75$ GPa.

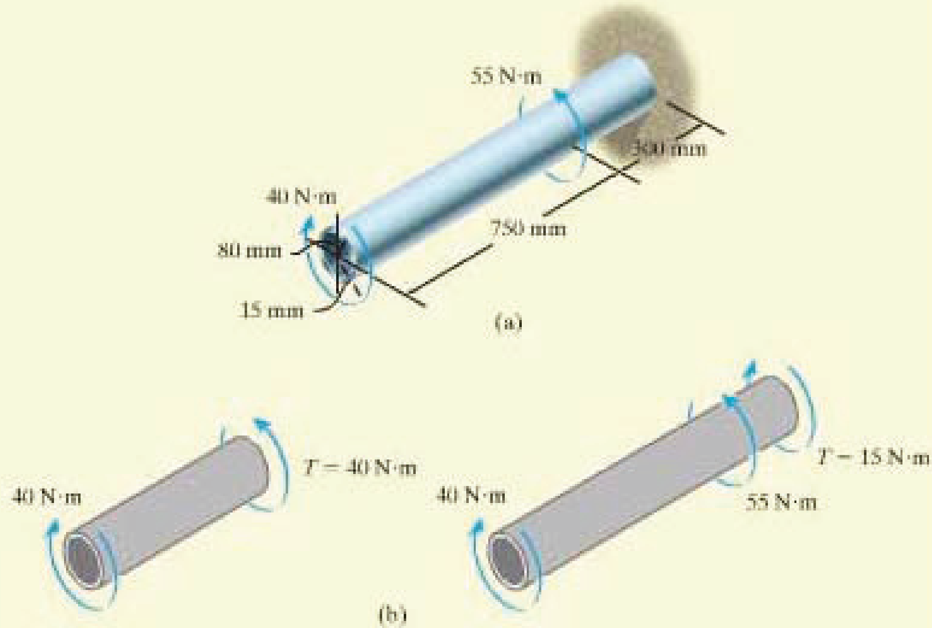


Fig. 14–17

SOLUTION

Using the method of sections, the internal torque is first determined within the two regions of the shaft where it is constant, Fig. 14–17*b*. Although these torques ($40\text{ N}\cdot\text{m}$ and $15\text{ N}\cdot\text{m}$) are in opposite directions, this will be of no consequence in determining the strain energy, since the torque is squared in Eq. 14–22. In other words, the strain energy is always positive. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}[(0.08\text{ m})^4 - (0.065\text{ m})^4] = 36.30(10^{-6})\text{ m}^4$$

Applying Eq. 14–22, we have

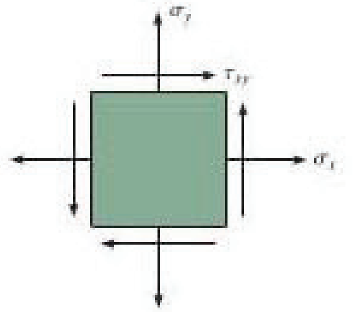
$$U_i = \sum \frac{T^2L}{2GJ}$$

$$= \frac{(40\text{ N}\cdot\text{m})^2(0.750\text{ m})}{2[75(10^9)\text{ N/m}^2]36.30(10^{-6})\text{ m}^4} + \frac{(15\text{ N}\cdot\text{m})^2(0.300\text{ m})}{2[75(10^9)\text{ N/m}^2]36.30(10^{-6})\text{ m}^4}$$

$$= 233\ \mu\text{J}$$ *Ans.*

PROBLEMS

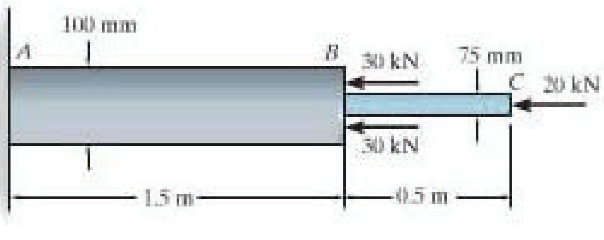
14–1. A material is subjected to a general state of plane stress. Express the strain energy density in terms of the elastic constants E, G, and ν and the stress components σ_x, σ_y, and τ_{xy}.



Prob. 14–1

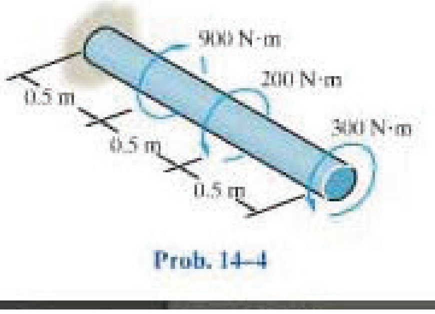
14–2. The strain-energy density must be the same whether the state of stress is represented by σ_x, σ_y, and τ_{xy}, or by the principal stresses σ_1 and σ_2. This being the case, equate the strain–energy expressions for each of these two cases and show that $G = E/[2(1 + \nu)]$.

14–3. Determine the strain energy in the stepped rod assembly. Portion AB is steel and BC is brass. $E_{br} = 101$ GPa, $E_{st} = 200$ GPa, $(\sigma_Y)_{br} = 410$ MPa, $(\sigma_Y)_{st} = 250$ MPa.



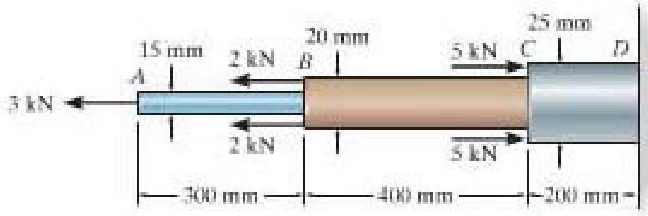
Prob. 14–3

***14–4.** Determine the torsional strain energy in the A-36 steel shaft. The shaft has a diameter of 40 mm.



Prob. 14–4

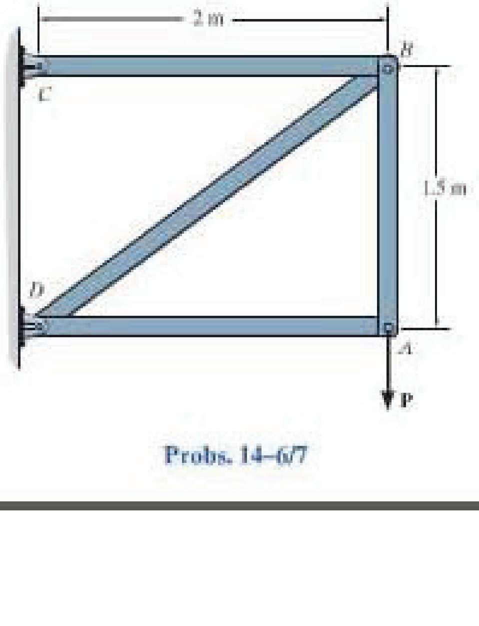
•14–5. Determine the strain energy in the rod assembly. Portion AB is steel, BC is brass, and CD is aluminum. $E_{st} = 200$ GPa, $E_{br} = 101$ GPa, and $E_{al} = 73.1$ GPa.



Prob. 14–5

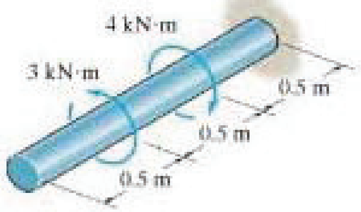
14–6. If $P = 60$ kN, determine the total strain energy stored in the truss. Each member has a cross-sectional area of $2.5(10^3)$ mm² and is made of A-36 steel.

14–7. Determine the maximum force **P** and the corresponding maximum total strain energy stored in the truss without causing any of the members to have permanent deformation. Each member has the cross-sectional area of $2.5(10^3)$ mm² and is made of A-36 steel.



Probs. 14–6/7

*14–8. Determine the torsional strain energy in the A-36 steel shaft. The shaft has a radius of 30 mm.

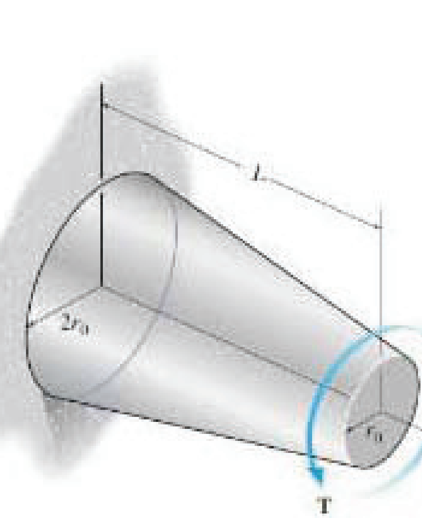


Prob. 14–8

•14–9. Determine the torsional strain energy in the A-36 steel shaft. The shaft has a radius of 40 mm.

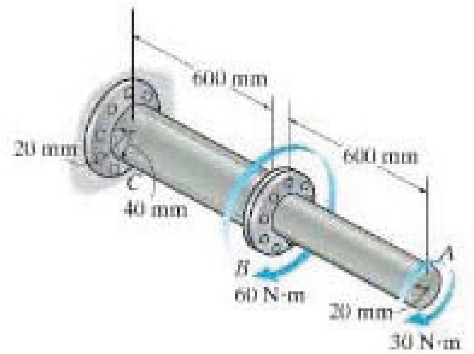
Prob. 14–9

14–10. Determine the torsional strain energy stored in the tapered rod when it is subjected to the torque **T**. The rod is made of material having a modulus of rigidity of G.



Prob. 14–10

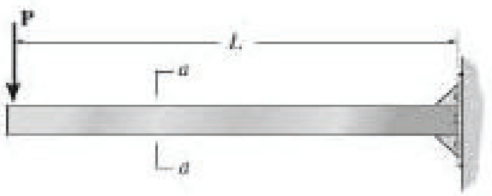
14–11. The shaft assembly is fixed at C. The hollow segment BC has an inner radius of 20 mm and outer radius of 40 mm, while the solid segment AB has a radius of 20 mm. Determine the torsional strain energy stored in the shaft. The shaft is made of 2014-T6 aluminum alloy. The coupling at B is rigid.

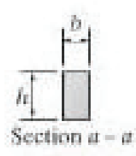


Prob. 14–11

*14–12. Consider the thin-walled tube of Fig. 5–28. Use the formula for shear stress, $\tau_{avg} = T/2tA_m$, Eq. 5–18, and the general equation of shear strain energy, Eq. 14–11, to show that the twist of the tube is given by Eq. 5–20. *Hint:* Equate the work done by the torque T to the strain energy in the tube, determined from integrating the strain energy for a differential element, Fig. 14–4, over the volume of material.

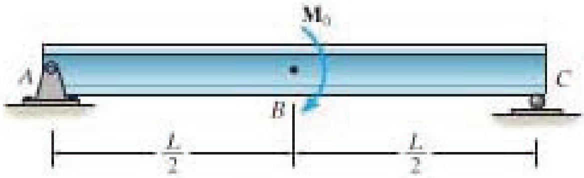
•14–13. Determine the ratio of shearing strain energy to bending strain energy for the rectangular cantilever beam when it is subjected to the loading shown. The beam is made of material having a modulus of elasticity of E and Poisson's ratio of ν.





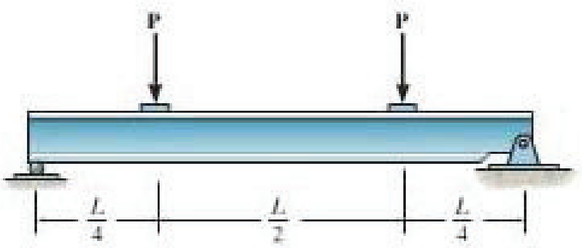
Prob. 14–13

14–14. Determine the bending strain-energy in the beam due to the loading shown. EI is constant.



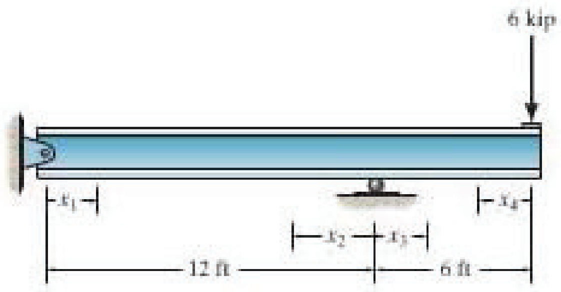
Prob. 14–14

14–15. Determine the bending strain energy in the beam. EI is constant.



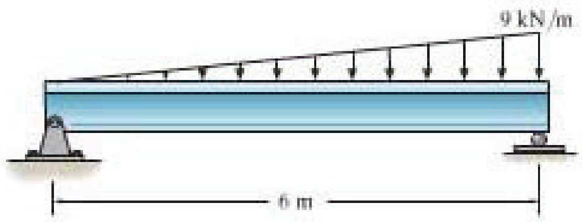
Prob. 14–15

***14–16.** Determine the bending strain energy in the A-36 structural steel W10 × 12 beam. Obtain the answer using the coordinates (a) x_1 and x_4, and (b) x_2 and x_3.



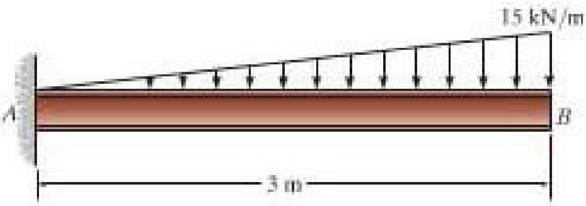
Prob. 14–16

•**14–17.** Determine the bending strain energy in the A-36 steel beam. $I = 99.2\ (10^6)\ \text{mm}^4$.



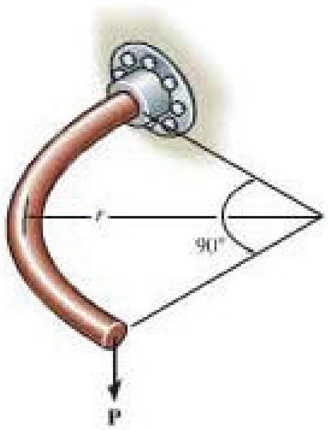
Prob. 14–17

14–18. Determine the bending strain energy in the A-36 steel beam due to the distributed load. $I = 122\ (10^6)\ \text{mm}^4$.



Prob. 14–18

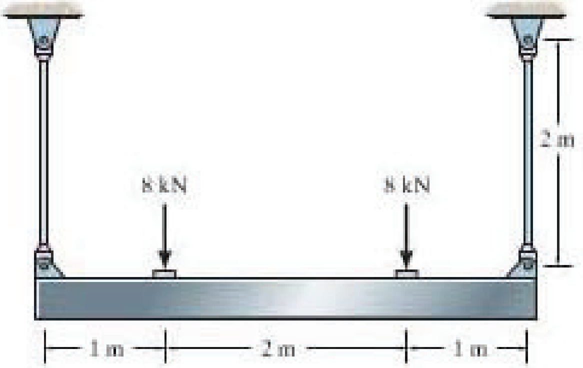
14–19. Determine the strain energy in the *horizontal* curved bar due to torsion. There is a *vertical* force **P** acting at its end. JG is constant.



Prob. 14–19

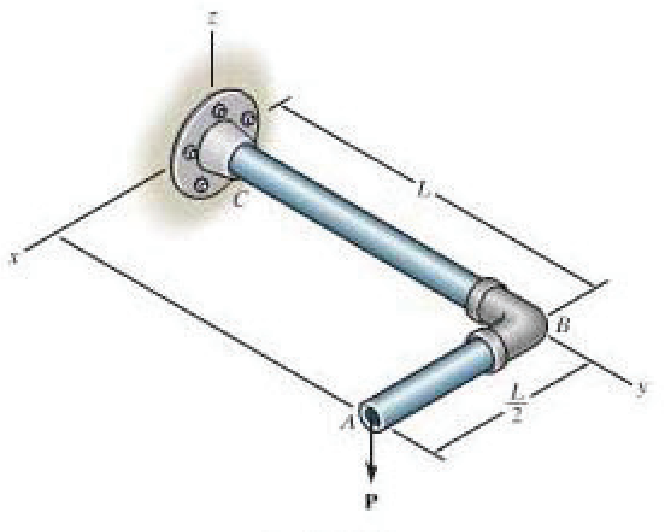
14

***14–20.** Determine the bending strain energy in the beam and the axial strain energy in each of the two rods. The beam is made of 2014-T6 aluminum and has a square cross section 50 mm by 50 mm. The rods are made of A-36 steel and have a circular cross section with a 20-mm diameter.



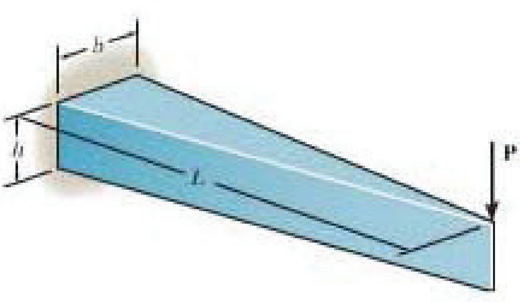
Prob. 14–20

•14–21. The pipe lies in the horizontal plane. If it is subjected to a vertical force **P** at its end, determine the strain energy due to bending and torsion. Express the results in terms of the cross-sectional properties I and J, and the material properties E and G.



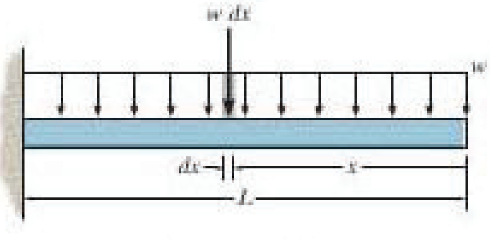
Prob. 14–21

14–22. The beam shown is tapered along its width. If a force **P** is applied to its end, determine the strain energy in the beam and compare this result with that of a beam that has a constant rectangular cross section of width b and height h.



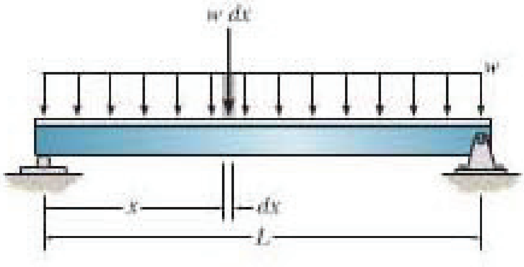
Prob. 14–22

14–23. Determine the bending strain energy in the cantilevered beam due to a uniform load w. Solve the problem two ways. (a) Apply Eq. 14–17. (b) The load $w\,dx$ acting on a segment dx of the beam is displaced a distance y, where $y = w(-x^4 + 4L^3x - 3L^4)/(24EI)$, the equation of the elastic curve. Hence the internal strain energy in the differential segment dx of the beam is equal to the external work, i.e., $dU_i = \frac{1}{2}(w\,dx)(-y)$. Integrate this equation to obtain the total strain energy in the beam. EI is constant.



Prob. 14–23

***14–24.** Determine the bending strain energy in the simply supported beam due to a uniform load w. Solve the problem two ways. (a) Apply Eq. 14–17. (b) The load $w\,dx$ acting on the segment dx of the beam is displaced a distance y, where $y = w(-x^4 + 2Lx^3 - L^3x)/(24EI)$, the equation of the elastic curve. Hence the internal strain energy in the differential segment dx of the beam is equal to the external work, i.e., $dU_i = \frac{1}{2}(w\,dx)(-y)$. Integrate this equation to obtain the total strain energy in the beam. EI is constant.



Prob. 14–24

14.3 Conservation of Energy

All energy methods used in mechanics are based on a balance of energy, often referred to as the conservation of energy. In this chapter, only mechanical energy will be considered in the energy balance; that is, the energy developed by heat, chemical reactions, and electromagnetic effects will be neglected. As a result, if a loading is applied *slowly* to a body, then physically the external loads tend to deform the body so that the loads do *external work* U_e as they are displaced. This external work on the body is transformed into *internal work* or strain energy U_i, which is stored in the body. Furthermore, when the loads are removed, the strain energy restores the body back to its original undeformed position, provided the material's elastic limit is not exceeded. The conservation of energy for the body can therefore be stated mathematically as

$$\boxed{U_e = U_i} \tag{14–23}$$

We will now show three examples of how this equation can be applied to determine the displacement of a point on a deformable member or structure. As the first example, consider the truss in Fig. 14–18 subjected to the load **P**. Provided **P** is applied gradually, the external work done by **P** is determined from Eq. 14–2, that is, $U_e = \frac{1}{2}P\Delta$, where Δ is the vertical displacement of the truss at the joint where **P** is applied. Assuming that **P** develops an axial force **N** in a particular member, the strain energy stored in this member is determined from Eq. 14–16, that is, $U_i = N^2L/2AE$. Summing the strain energies for all the members of the truss, we can write Eq. 14–23 as

$$\frac{1}{2}P\Delta = \sum \frac{N^2L}{2AE} \tag{14–24}$$

Once the internal forces (N) in all the members of the truss are determined and the terms on the right calculated, it is then possible to determine the unknown displacement Δ.

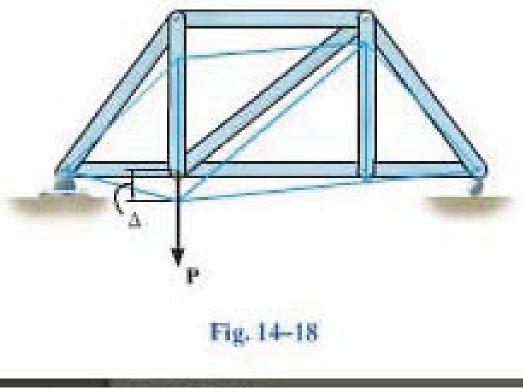


Fig. 14–18

14

As a second example, consider finding the vertical displacement Δ under the load **P** acting on the beam in Fig. 14–19. Again, the external work is $U_e = \frac{1}{2}P\Delta$. In this case the strain energy is the result of internal shear and moment loadings caused by **P**. In particular, the contribution of strain energy due to shear is generally *neglected* in most beam deflection problems unless the beam is short and supports a very large load. (See Example 14.4.) Consequently, the beam's strain energy will be determined only by the internal bending moment M, and therefore, using Eq. 14–17, Eq. 14–23 can be written symbolically as

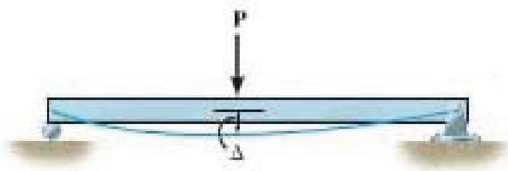


Fig. 14–19

$$\frac{1}{2}P\Delta = \int_0^L \frac{M^2}{2EI}dx \tag{14–25}$$

Once M is expressed as a function of position x and the integral is evaluated, Δ can then be determined.

As a last example, we will consider a beam loaded by a couple moment $\mathbf{M}_0$ as shown in Fig. 14–20. This moment causes the rotational displacement θ at the point of application of the couple moment. Since the couple moment only does work when it *rotates*, using Eq. 14–5, the external work is $U_e = \frac{1}{2}M_0\theta$. Therefore Eq. 14–23 becomes

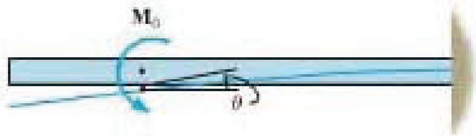


Fig. 14–20

$$\frac{1}{2}M_0\theta = \int_0^L \frac{M^2}{2EI}dx \tag{14–26}$$

Here the strain energy is the result of the internal bending moment M caused by application of the couple moment $\mathbf{M}_0$. Once M has been expressed as a function of x and the strain energy evaluated, then θ which measures the slope of the elastic curve can be determined.

In each of the above examples, it should be noted that application of Eq. 14–23 is *quite limited*, because only a *single* external force or couple moment must act on the member or structure. Also, the displacement can *only* be calculated at the point and in the direction of the external force or couple moment. If more than one external force or couple moment were applied, then the external work of each loading would involve its associated unknown displacement. As a result, *all* these unknown displacements could not be determined, since only the single Eq. 14–23 is available for the solution. Although application of the conservation of energy as described here has these restrictions, it does serve as an introduction to more general energy methods, which we will consider throughout the rest of this chapter.

EXAMPLE 14.6

The three-bar truss in Fig. 14–21*a* is subjected to a horizontal force of 5 kip. If the cross-sectional area of each member is 0.20 in^2, determine the horizontal displacement at point *B*. $E = 29(10^3)$ ksi.

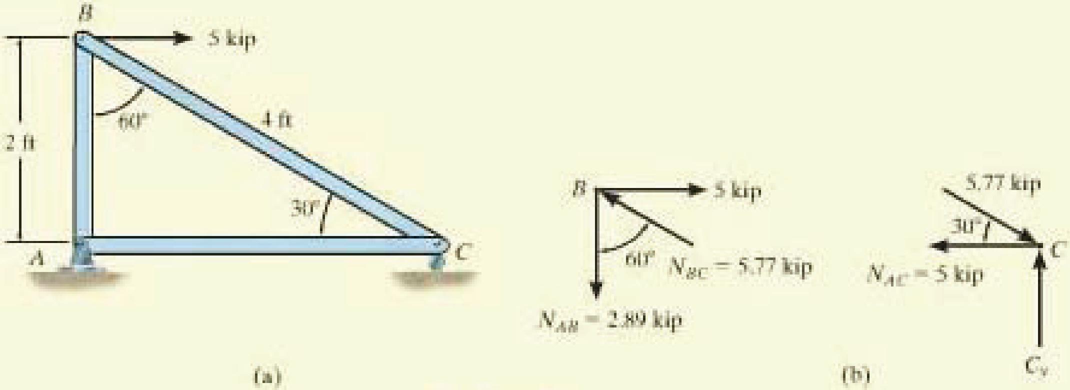


Fig. 14–21

SOLUTION

We can apply the conservation of energy to solve this problem because only a *single* external force acts on the truss and the required displacement happens to be in the *same direction* as the force. Furthermore, the reactive forces on the truss do no work since they are not displaced.

Using the method of joints, the force in each member is determined as shown on the free-body diagrams of the pins at *B* and *C*, Fig. 14–21*b*.

Applying Eq. 14–24, we have

$$\frac{1}{2}P\Delta = \sum \frac{N^2L}{2AE}$$

$$\frac{1}{2}(5\text{ kip})(\Delta_B)_h = \frac{(2.89\text{ kip})^2(2\text{ ft})}{2AE} + \frac{(-5.77\text{ kip})^2(4\text{ ft})}{2AE} + \frac{(5\text{ kip})^2(3.46\text{ ft})}{2AE}$$

$$(\Delta_B)_h = \frac{47.32\text{ kip}\cdot\text{ft}}{AE}$$

Notice that since *N* is squared, it does not matter if a particular member is in tension or compression. Substituting in the numerical data for *A* and *E* and solving, we get

$$(\Delta_B)_h = \frac{47.32\text{ kip}\cdot\text{ft}(12\text{ in./ft})}{(0.2\text{ in}^2)[29(10^3)\text{ kip/in}^2]}$$

$$= 0.0979\text{ in.} \rightarrow \qquad \textit{Ans.}$$

EXAMPLE 14.7

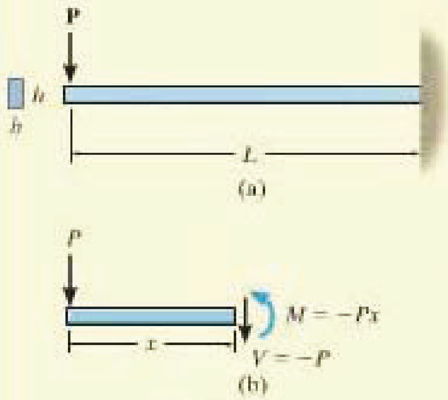


Fig. 14–22

The cantilevered beam in Fig. 14–22*a* has a rectangular cross section and is subjected to a load **P** at its end. Determine the displacement of the load. EI is constant.

SOLUTION

The internal shear and moment in the beam as a function of x are determined using the method of sections, Fig. 14–22*b*.

When applying Eq. 14–23 we will consider the strain energy due to both shear and bending. Using Eqs. 14–19 and 14–17, we have

$$\frac{1}{2}P\Delta = \int_0^L \frac{f_s V^2\,dx}{2GA} + \int_0^L \frac{M^2\,dx}{2EI}$$
$$= \int_0^L \frac{\left(\frac{6}{5}\right)(-P)^2\,dx}{2GA} + \int_0^L \frac{(-Px)^2\,dx}{2EI} = \frac{3P^2L}{5GA} + \frac{P^2L^3}{6EI} \quad (1)$$

The first term on the right side of this equation represents the strain energy due to shear, while the second is the strain energy due to bending. As stated in Example 14.4, for most beams the shear strain energy is much smaller than the bending strain energy. To show when this is the case for the beam in Figure 14–22*a*, we require

$$\frac{3}{5}\frac{P^2L}{GA} \ll \frac{P^2L^3}{6EI}$$
$$\frac{3}{5}\frac{P^2L}{G(bh)} \ll \frac{P^2L^3}{6E\left[\frac{1}{12}(bh^3)\right]}$$
$$\frac{3}{5G} \ll \frac{2L^2}{Eh^2}$$

Since $E \leq 3G$ (see Example 14.4), then

$$0.9 \ll \left(\frac{L}{h}\right)^2$$

Hence if L is relatively long compared with h, the beam becomes slender and the shear strain energy can be neglected. In other words, the *shear strain energy* becomes important *only* for *short, deep beams*. For example, beams for which $L = 5h$ have approximately 28 times more bending strain energy than shear strain energy, so neglecting the shear strain energy represents an error of about 3.6%. With this in mind, Eq. 1 can be simplified to

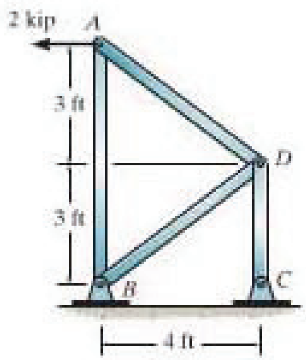
$$\frac{1}{2}P\Delta = \frac{P^2L^3}{6EI}$$

so that

$$\Delta = \frac{PL^3}{3EI} \qquad \textit{Ans.}$$

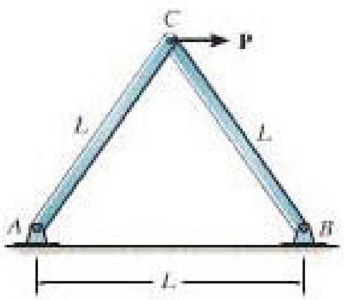
PROBLEMS

•**14–25.** Determine the horizontal displacement of joint *A*. Each bar is made of A-36 steel and has a cross-sectional area of 1.5 in^2.



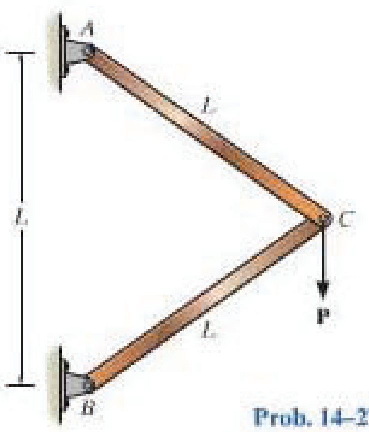
Prob. 14–25

14–26. Determine the horizontal displacement of joint *C*. *AE* is constant.



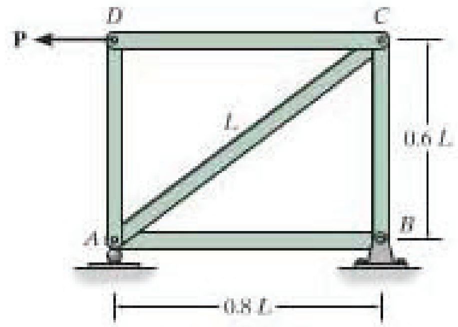
Prob. 14–26

14–27. Determine the vertical displacement of joint *C*. *AE* is constant.



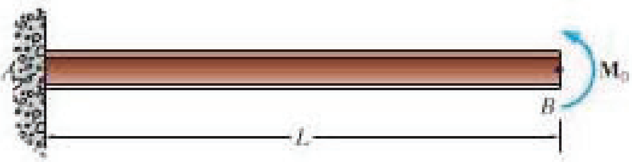
Prob. 14–27

*__14–28.__ Determine the horizontal displacement of joint *D*. *AE* is constant.



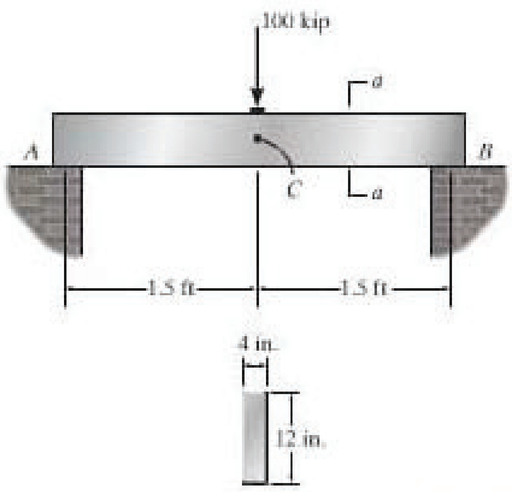
Prob. 14–28

•**14–29.** The cantilevered beam is subjected to a couple moment M_0 applied at its end. Determine the slope of the beam at *B*. *EI* is constant.



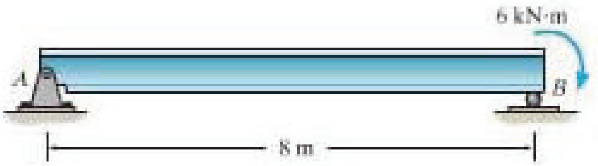
Prob. 14–29

14–30. Determine the vertical displacement of point *C* of the simply supported 6061-T6 aluminum beam. Consider both shearing and bending strain energy.



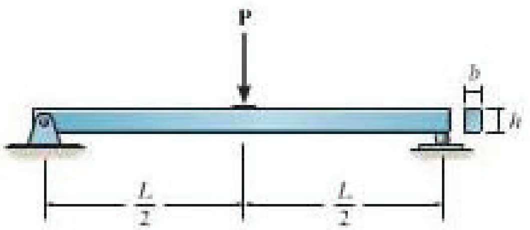
Prob. 14–30

14–31. Determine the slope at the end B of the A-36 steel beam. $I = 80(10^6)$ mm^4.



Prob. 14–31

*14–32. Determine the deflection of the beam at its center caused by shear. The shear modulus is G.

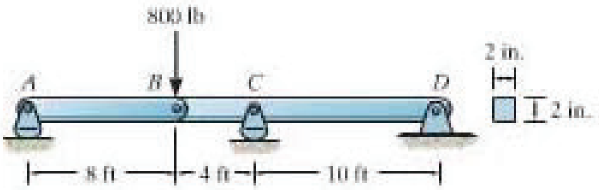


Prob. 14–32

•14–33. The A-36 steel bars are pin connected at B and C. If they each have a diameter of 30 mm, determine the slope at E.

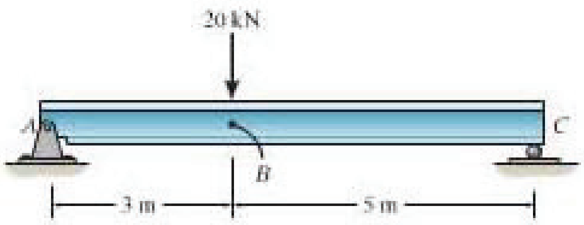
Prob. 14–33

14–34. The A-36 steel bars are pin connected at B. If each has a square cross section, determine the vertical displacement at B.



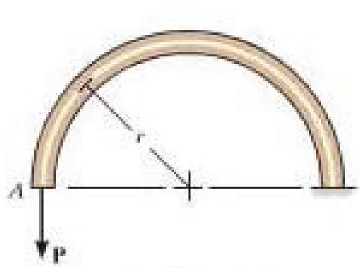
Prob. 14–34

14–35. Determine the displacement of point B on the A-36 steel beam. $I = 80(10^6)$ mm^4.



Prob. 14–35

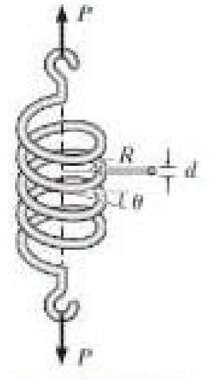
*14–36. The rod has a circular cross section with a moment of inertia I. If a vertical force **P** is applied at A, determine the vertical displacement at this point. Only consider the strain energy due to bending. The modulus of elasticity is E.



Prob. 14–36

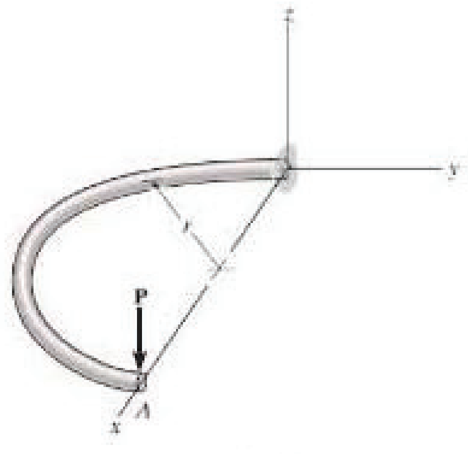
•14–37. The load **P** causes the open coils of the spring to make an angle θ with the horizontal when the spring is stretched. Show that for this position this causes a torque $T = PR\cos\theta$ and a bending moment $M = PR\sin\theta$ at the cross section. Use these results to determine the maximum normal stress in the material.

14–38. The coiled spring has n coils and is made from a material having a shear modulus G. Determine the stretch of the spring when it is subjected to the load **P**. Assume that the coils are close to each other so that $\theta \approx 0°$ and the deflection is caused entirely by the torsional stress in the coil.



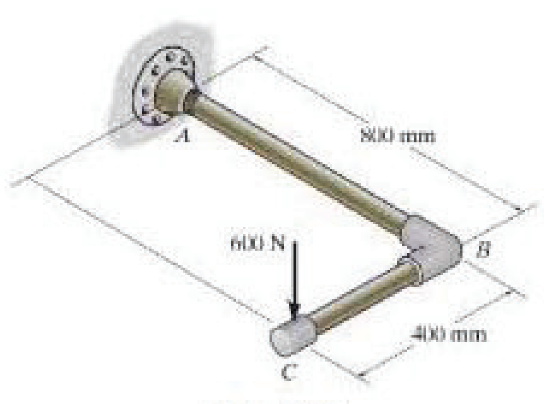
Probs. 14–37/38

*14–40. The rod has a circular cross section with a polar moment of inertia J and moment of inertia I. If a vertical force **P** is applied at A, determine the vertical displacement at this point. Consider the strain energy due to bending and torsion. The material constants are E and G.



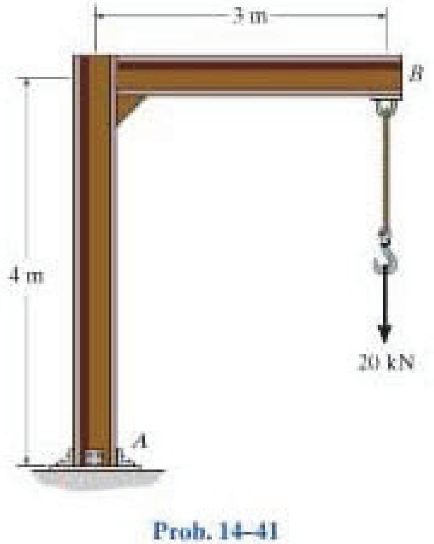
Prob. 14–40

14–39. The pipe assembly is fixed at A. Determine the vertical displacement of end C of the assembly. The pipe has an inner diameter of 40 mm and outer diameter of 60 mm and is made of A-36 steel. Neglect the shearing strain energy.



Prob. 14–39

•14–41. Determine the vertical displacement of end B of the frame. Consider only bending strain energy. The frame is made using two A-36 steel W460 × 68 wide-flange sections.



Prob. 14–41

14.4 Impact Loading

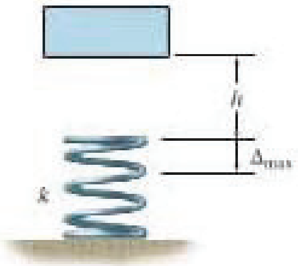


Fig. 14–23

Throughout this text we have considered all loadings to be applied to a body in a gradual manner, such that when they reach a maximum value the body remains static. Some loadings, however, are dynamic; that is, they vary with time. A typical example would be caused by the collision of objects. This is called an impact loading. Specifically, ***impact*** occurs when one object strikes another, such that large forces are developed between the objects during a very short period of time.

If we assume no energy is lost during impact, due to heat, sound or localized plastic deformations, then we can study the mechanics of impact using the conservation of energy. To show how this is done, we will first analyze the motion of a simple block-and-spring system as shown in Fig. 14–23. When the block is released from rest, it falls a distance h, striking the spring and compressing it a distance Δ_{max} before momentarily coming to rest. If we neglect the mass of the spring and assume that the spring responds *elastically*, then the conservation of energy requires that the energy of the falling block be transformed into stored (strain) energy in the spring; or in other words, the work done by the block's weight, falling $h + \Delta_{max}$, is equal to the work needed to displace the end of the spring by an amount Δ_{max}. Since the force in a spring is related to Δ_{max} by the equation $F = k\Delta_{max}$, where k is the spring stiffness, then applying the conservation of energy and Eq. 14–2, we have

$$U_e = U_i$$

$$W(h + \Delta_{max}) = \frac{1}{2}(k\Delta_{max})\,\Delta_{max}$$

$$W(h + \Delta_{max}) = \frac{1}{2}k\Delta_{max}^2 \qquad (14\text{–}27)$$

$$\Delta_{max}^2 - \frac{2W}{k}\Delta_{max} - 2\left(\frac{W}{k}\right)h = 0$$

This quadratic equation may be solved for Δ_{max}. The maximum root is

$$\Delta_{max} = \frac{W}{k} + \sqrt{\left(\frac{W}{k}\right)^2 + 2\left(\frac{W}{k}\right)h}$$

If the weight W is supported statically by the spring, then the top displacement of the spring is $\Delta_{st} = W/k$. Using this simplification, the above equation becomes

$$\Delta_{max} = \Delta_{st} + \sqrt{(\Delta_{st})^2 + 2\Delta_{st}h}$$

or

$$\Delta_{max} = \Delta_{st}\left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}\right] \qquad (14\text{–}28)$$

This crash barrier is designed to absorb the impact energy of moving vehicles.

Once Δ_{max} is calculated, the maximum force applied to the spring can be determined from

$$F_{max} = k\Delta_{max} \tag{14–29}$$

It should be realized, however, that this force and associated displacement occur only at an *instant*. Provided the block does not rebound off the spring, it will continue to vibrate until the motion dampens out and the block assumes the static position, Δ_{st}. Note also that if the block is held just above the spring, $h = 0$, and released, then, from Eq. 14–28, the maximum displacement of the block is

$$\Delta_{max} = 2\Delta_{st}$$

In other words, when the block is released from the top of the spring (a dynamic load), the displacement is *twice* what it would be if it were set on the spring (a static load).

Using a similar analysis, it is also possible to determine the maximum displacement of the end of the spring if the block is sliding on a smooth horizontal surface with a known velocity **v** just before it collides with the spring, Fig. 14–24. Here the block's kinetic energy,* $\frac{1}{2}(W/g)v^2$, will be transformed into stored energy in the spring. Hence,

$$U_e = U_i$$

$$\frac{1}{2}\left(\frac{W}{g}\right)v^2 = \frac{1}{2}k\Delta_{max}^2$$

$$\Delta_{max} = \sqrt{\frac{Wv^2}{gk}} \tag{14–30}$$

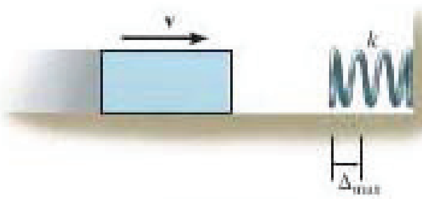


Fig. 14–24

Since the static displacement at the top of the spring caused by the weight W resting on it is $\Delta_{st} = W/k$, then

$$\Delta_{max} = \sqrt{\frac{\Delta_{st}v^2}{g}} \tag{14–31}$$

The results of this simplified analysis can be used to determine both the approximate deflection and the stress developed in a deformable member when it is subjected to impact. To do this we must make the necessary assumptions regarding the collision, so that the behavior of the colliding bodies is similar to the response of the block-and-spring models discussed above. Hence we will consider the moving body to be *rigid* like the block and the stationary body to be deformable like the spring. Also, it is assumed that the material behaves in a linear-elastic manner. When collision occurs, the bodies remain in contact until the elastic body reaches its maximum deformation, and during the motion the inertia or mass of the elastic body is neglected. Realize that each of these assumptions will lead to a *conservative* estimate of both the maximum stress and deflection of the elastic body. In other words, their values will be larger than those that actually occur.

*Recall from physics that kinetic energy is "energy of motion." For the translation of a body it is determined from $\frac{1}{2}mv^2$, where m is the body's mass, $m = W/g$.

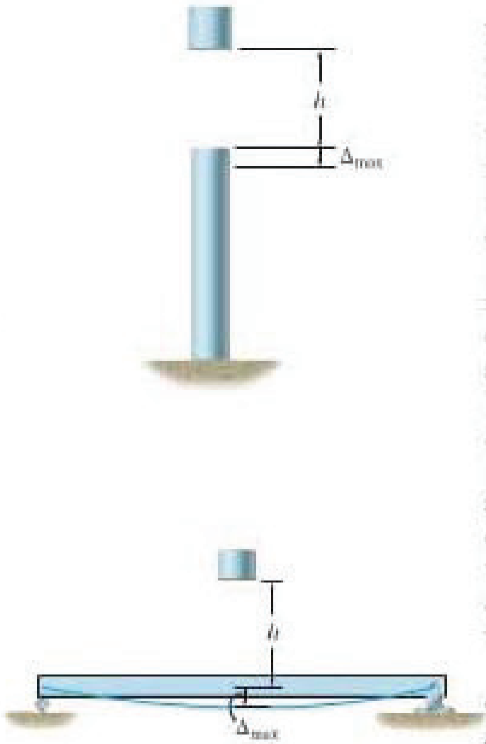


Fig. 14–25

A few examples of when this theory can be applied are shown in Fig. 14–25. Here a block of known weight is dropped onto a post and a beam, causing them to deform a maximum amount Δ_{max}. The energy of the falling block is transformed momentarily into axial strain energy in the post and bending strain energy in the beam.* In order to determine the deformation Δ_{max}, we could use the same approach as for the block–spring system, and that is to write the conservation-of-energy equation for the block and post or block and beam, and then solve for Δ_{max}. However, we can also solve these problems in a more direct manner by modeling the post and beam by an *equivalent spring*. For example, if a force **P** displaces the top of the post $\Delta = PL/AE$, then a spring having a stiffness $k = AE/L$ would be displaced the same amount by **P**, that is, $\Delta = P/k$. In a similar manner, from Appendix C, a force **P** applied to the center of a simply supported beam displaces the center $\Delta = PL^3/48EI$, and therefore an equivalent spring would have a stiffness of $k = 48EI/L^3$. It is not necessary, however, to actually find the equivalent spring stiffness to apply Eq. 14–28 or 14–30. All that is needed to determine the dynamic displacement, Δ_{max}, is to calculate the *static displacement*, Δ_{st}, due to the weight $P_{st} = W$ of the block resting on the member.

Once Δ_{max} is determined, the maximum dynamic force can then be calculated from $P_{max} = k\Delta_{max}$. If we consider P_{max} to be an *equivalent static load* then the maximum stress in the member can be determined using statics and the theory of mechanics of materials. Recall that this stress acts only for an *instant*. In reality, vibrational waves pass through the material, and the stress in the post or the beam, for example, does not remain constant.

The ratio of the equivalent static load P_{max} to the static load $P_{st} = W$ is called the *impact factor*, n. Since $P_{max} = k\Delta_{max}$ and $P_{st} = k\Delta_{st}$, then from Eq. 14–28, we can express it as

$$\boxed{n = 1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}} \qquad (14\text{–}32)$$

The members of this crash guard must be designed to resist a prescribed impact loading in order to arrest the motion of a rail car.

This factor represents the magnification of a statically applied load so that it can be treated dynamically. Using Eq. 14–32, n can be calculated for any member that has a linear relationship between load and deflection. For a complicated system of connected members, however, impact factors are determined from experience and experimental testing. Once n is determined, the dynamic stress and deflection at the point of impact are easily found from the static stress σ_{st} and static deflection Δ_{st} caused by the load W, that is, $\sigma_{max} = n\sigma_{st}$ and $\Delta_{max} = n\Delta_{st}$.

*Strain energy due to shear is neglected for reasons discussed in Example 14.4.

Important Points

- *Impact* occurs when a large force is developed between two objects which strike one another during a short period of time.
- We can analyze the effects of impact by assuming the moving body is rigid, the material of the stationary body is linear elastic, no energy is lost during collision, the bodies remain in contact during collision, and the inertia of the elastic body is neglected.
- The dynamic load on a body can be determined by multiplying the static load by an *impact factor*.

EXAMPLE 14.8

The aluminum pipe shown in Fig. 14–26 is used to support a load of 150 kip. Determine the maximum displacement at the top of the pipe if the load is (a) applied gradually, and (b) applied by suddenly releasing it from the top of the pipe when $h = 0$. Take $E_{al} = 10(10^3)$ ksi and assume that the aluminum behaves elastically.

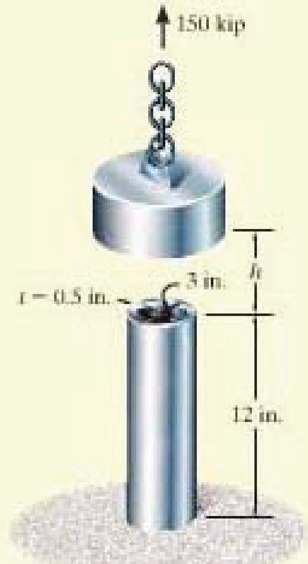


Fig. 14–26

SOLUTION

Part (a). When the load is applied gradually, the work done by the weight is transformed into elastic strain energy in the pipe. Applying the conservation of energy, we have

$$U_e = U_i$$

$$\frac{1}{2}W\Delta_{st} = \frac{W^2L}{2AE}$$

$$\Delta_{st} = \frac{WL}{AE} = \frac{150 \text{ kip}(12 \text{ in.})}{\pi[(3 \text{ in.})^2 - (2.5 \text{ in.})^2]10(10^3) \text{ kip/in}^2}$$

$$= 0.02083 \text{ in.} = 0.0208 \text{ in.} \qquad \textit{Ans.}$$

Part (b). Here Eq. 14–28 can be applied, with $h = 0$. Hence,

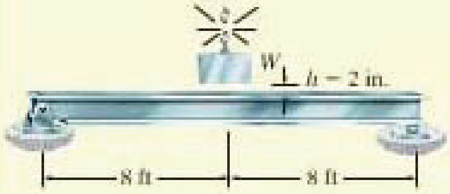
$$\Delta_{max} = \Delta_{st}\left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}\right]$$

$$= 2\Delta_{st} = 2(0.02083 \text{ in.})$$

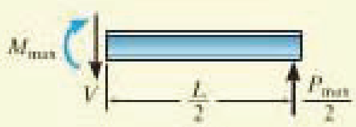
$$= 0.0417 \text{ in.} \qquad \textit{Ans.}$$

Hence, the displacement of the weight when applied dynamically is twice as great as when the load is applied statically. In other words, the impact factor is $n = 2$, Eq. 14–32.

EXAMPLE 14.9



(a)



(b)

Fig. 14–27

The A-36 steel beam shown in Fig. 14–27*a* is a W10 × 39. Determine the maximum bending stress in the beam and the beam's maximum deflection if the weight $W = 1.50$ kip is dropped from a height $h = 2$ in. onto the beam. $E_{st} = 29(10^3)$ ksi.

SOLUTION I

We will apply Eq. 14–28. First, however, we must calculate Δ_{st}. Using the table in Appendix C, and the data in Appendix B for the properties of a W10 × 39, we have

$$\Delta_{st} = \frac{WL^3}{48EI} = \frac{(1.50\text{ kip})(16\text{ ft})^3(12\text{ in./ft})^3}{48[29(10^3)\text{ ksi}](209\text{ in}^4)} = 0.03649\text{ in.}$$

$$\Delta_{max} = \Delta_{st}\left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}\right]$$

$$= 0.03649\text{ in.}\left[1 + \sqrt{1 + 2\left(\frac{2\text{ in.}}{0.03649\text{ in.}}\right)}\right] = 0.420\text{ in.} \quad \textit{Ans.}$$

The equivalent static load that causes this displacement is therefore

$$P_{max} = \frac{48EI}{L^3}\Delta_{max} = \frac{48(29(10^3)\text{ ksi})(209\text{ in}^4)}{(16\text{ ft})^3\ (12\text{ in./ft})^3}(0.420\text{ in.}) = 17.3\text{ kip}$$

The internal moment caused by this load is maximum at the center of the beam, such that by the method of sections, Fig. 14–27*b*, $M_{max} = P_{max}L/4$. Applying the flexure formula to determine the bending stress, we have

$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{P_{max}Lc}{4I} = \frac{12E\,\Delta_{max}c}{L^2}$$

$$= \frac{12[29(10^3)\text{ kip/in}^2](0.420\text{ in.})(9.92\text{ in./2})}{(16\text{ ft})^2(12\text{ in./ft})^2} = 19.7\text{ ksi} \quad \textit{Ans.}$$

SOLUTION II

It is also possible to obtain the dynamic or maximum deflection Δ_{max} from first principles. The external work of the falling weight W is $U_e = W(h + \Delta_{max})$. Since the beam deflects Δ_{max}, and $P_{max} = 48EI\Delta_{max}/L^3$, then

$$U_e = U_i$$

$$W(h + \Delta_{max}) = \frac{1}{2}\left(\frac{48EI\Delta_{max}}{L^3}\right)\Delta_{max}$$

$$(1.50\text{ kip})(2\text{ in.} + \Delta_{max}) = \frac{1}{2}\left[\frac{48[29(10^3)\text{ kip/in}^2]209\text{ in}^4}{(16\text{ ft})^3(12\text{ in./ft})^3}\right]\Delta_{max}^2$$

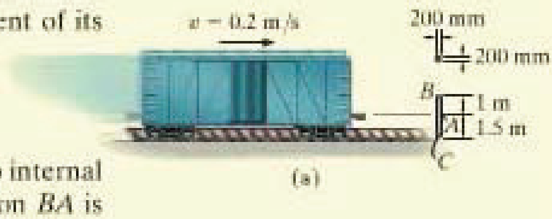
$$20.55\Delta_{max}^2 - 1.50\Delta_{max} - 3.00 = 0$$

Solving and choosing the positive root yields

$$\Delta_{max} = 0.420\text{ in.} \quad \textit{Ans.}$$

EXAMPLE 14.10

A railroad car that is assumed to be rigid and has a mass of 80 Mg is moving forward at a speed of $v = 0.2$ m/s when it strikes a steel 200-mm by 200-mm post at A, Fig. 14–28a. If the post is fixed to the ground at C, determine the maximum horizontal displacement of its top B due to the impact. Take $E_{st} = 200$ GPa.



14

SOLUTION

Here the kinetic energy of the railroad car is transformed into internal bending strain energy only for region AC of the post. (Region BA is not subjected to an internal loading.) Assuming that point A is displaced $(\Delta_A)_{max}$, then the force P_{max} that causes this displacement can be determined from the table in Appendix C. We have

$$P_{max} = \frac{3EI(\Delta_A)_{max}}{L_{AC}^3} \qquad (1)$$

$$U_e = U_i; \qquad \frac{1}{2}mv^2 = \frac{1}{2}P_{max}(\Delta_A)_{max}$$

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{3EI}{L_{AC}^3}(\Delta_A)_{max}^2; \qquad (\Delta_A)_{max} = \sqrt{\frac{mv^2L_{AC}^3}{3EI}}$$

Substituting in the numerical data yields

$$(\Delta_A)_{max} = \sqrt{\frac{80(10^3)\text{ kg}(0.2\text{ m/s})^2(1.5\text{ m})^3}{3[200(10^9)\text{ N/m}^2]\left[\frac{1}{12}(0.2\text{ m})^4\right]}} = 0.01162\text{ m} = 11.62\text{ mm}$$

Using Eq. 1, the force P_{max} is therefore

$$P_{max} = \frac{3[200(10^9)\text{ N/m}^2]\left[\frac{1}{12}(0.2\text{ m})^4\right](0.01162\text{ m})}{(1.5\text{ m})^3} = 275.4\text{ kN}$$

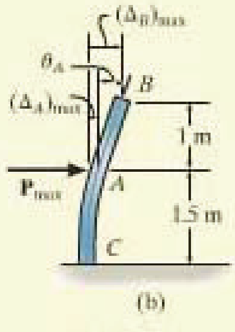


Fig. 14–28

With reference to Fig. 14–28b, segment AB of the post remains straight. To determine the maximum displacement at B, we must first determine the slope at A. Using the appropriate formula from the table in Appendix C to determine θ_A, we have

$$\theta_A = \frac{P_{max}L_{AC}^2}{2EI} = \frac{275.4(10^3)\text{ N }(1.5\text{ m})^2}{2[200(10^9)\text{ N/m}^2]\left[\frac{1}{12}(0.2\text{ m})^4\right]} = 0.01162\text{ rad}$$

The maximum displacement at B is thus

$$(\Delta_B)_{max} = (\Delta_A)_{max} + \theta_A L_{AB}$$
$$= 11.62\text{ mm} + (0.01162\text{ rad})\,1(10^3)\text{ mm} = 23.2\text{ mm} \qquad \textit{Ans.}$$

PROBLEMS

14–42. A bar is 4 m long and has a diameter of 30 mm. If it is to be used to absorb energy in tension from an impact loading, determine the total amount of elastic energy that it can absorb if (a) it is made of steel for which $E_{st} = 200$ GPa, $\sigma_Y = 800$ MPa, and (b) it is made from an aluminum alloy for which $E_{al} = 70$ GPa, $\sigma_Y = 405$ MPa.

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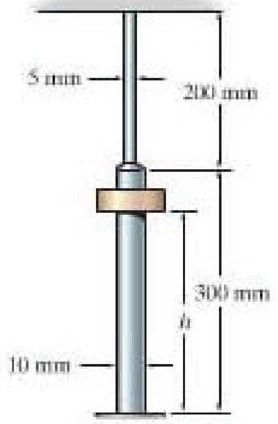
14–43. Determine the diameter of a red brass C83400 bar that is 8 ft long if it is to be used to absorb 800 ft·lb of energy in tension from an impact loading. No yielding occurs.

***14–44.** A steel cable having a diameter of 0.4 in. wraps over a drum and is used to lower an elevator having a weight of 800 lb. The elevator is 150 ft below the drum and is descending at the constant rate of 2 ft/s when the drum suddenly stops. Determine the maximum stress developed in the cable when this occurs. $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 50$ ksi.

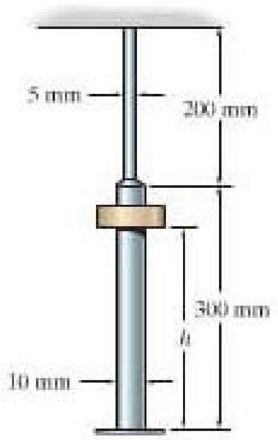
Prob. 14–44

•**14–45.** The composite aluminum bar is made from two segments having diameters of 5 mm and 10 mm. Determine the maximum axial stress developed in the bar if the 5-kg collar is dropped from a height of $h = 100$ mm. $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.



Prob. 14–45

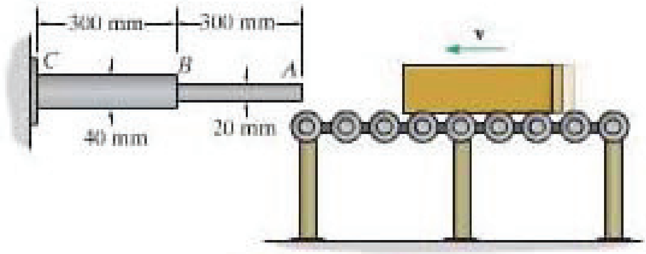
14–46. The composite aluminum bar is made from two segments having diameters of 5 mm and 10 mm. Determine the maximum height h from which the 5-kg collar should be dropped so that it produces a maximum axial stress in the bar of $\sigma_{max} = 300$ MPa, $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.



Prob. 14–46

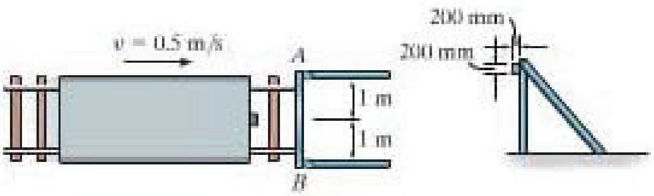
14–47. The 5-kg block is traveling with the speed of $v = 4$ m/s just before it strikes the 6061-T6 aluminum stepped cylinder. Determine the maximum normal stress developed in the cylinder.

*__14–48.__ Determine the maximum speed v of the 5-kg block without causing the 6061-T6 aluminum stepped cylinder to yield after it is struck by the block.



Probs. 14–47/48

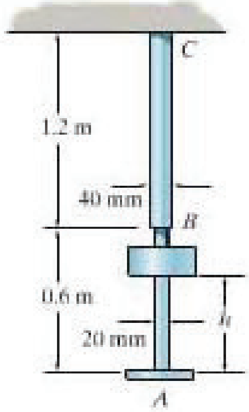
•**14–49.** The steel beam AB acts to stop the oncoming railroad car, which has a mass of 10 Mg and is coasting towards it at $v = 0.5$ m/s. Determine the maximum stress developed in the beam if it is struck at its center by the car. The beam is simply supported and only horizontal forces occur at A and B. Assume that the railroad car and the supporting framework for the beam remains rigid. Also, compute the maximum deflection of the beam. $E_{st} = 200$ GPa, $\sigma_Y = 250$ MPa.



Prob. 14–49

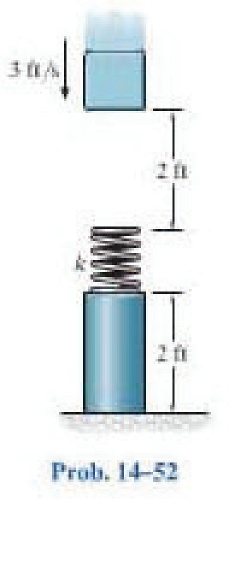
14–50. The aluminum bar assembly is made from two segments having diameters of 40 mm and 20 mm. Determine the maximum axial stress developed in the bar if the 10-kg collar is dropped from a height of $h = 150$ mm. Take $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.

14–51. The aluminum bar assembly is made from two segments having diameters of 40 mm and 20 mm. Determine the maximum height h from which the 60-kg collar can be dropped so that it will not cause the bar to yield. Take $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.



Probs. 14–50/51

*__14–52.__ The 50-lb weight is falling at 3 ft/s at the instant it is 2 ft above the spring and post assembly. Determine the maximum stress in the post if the spring has a stiffness of $k = 200$ kip/in. The post has a diameter of 3 in. and a modulus of elasticity of $E = 6.80(10^3)$ ksi. Assume the material will not yield.

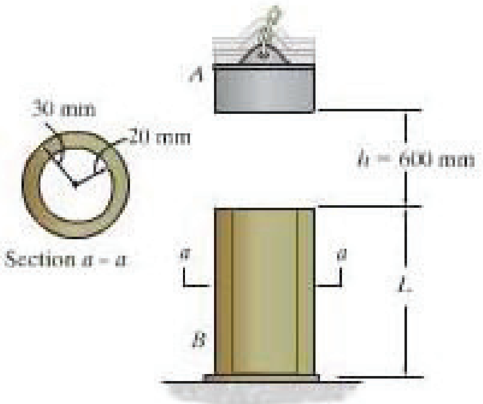


Prob. 14–52

14

•14–53. The 50-kg block is dropped from $h = 600$ mm onto the bronze C86100 tube. Determine the minimum length L the tube can have without causing the tube to yield.

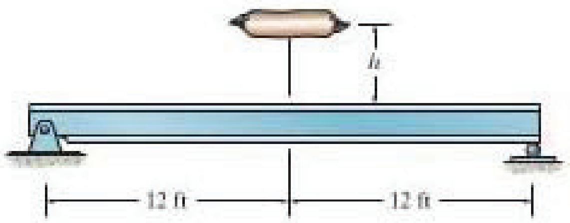
14–54. The 50-kg block is dropped from $h = 600$ mm onto the bronze C86100 tube. If $L = 900$ mm, determine the maximum normal stress developed in the tube.



Probs. 14–53/54

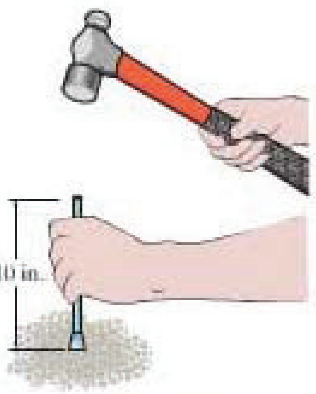
*14–56. The sack of cement has a weight of 90 lb. If it is dropped from rest at a height of $h = 4$ ft onto the center of the W10 × 39 structural steel A-36 beam, determine the maximum bending stress developed in the beam due to the impact. Also, what is the impact factor?

•14–57. The sack of cement has a weight of 90 lb. Determine the maximum height h from which it can be dropped from rest onto the center of the W10 × 39 structural steel A-36 beam so that the maximum bending stress due to impact does not exceed 30 ksi.



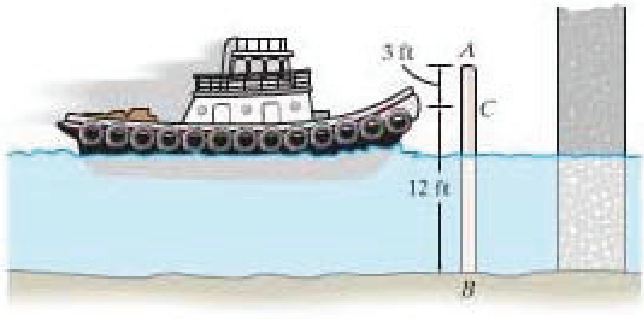
Probs. 14–56/57

14–55. The steel chisel has a diameter of 0.5 in. and a length of 10 in. It is struck by a hammer that weighs 3 lb, and at the instant of impact it is moving at 12 ft/s. Determine the maximum compressive stress in the chisel, assuming that 80% of the impacting energy goes into the chisel. $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 100$ ksi.



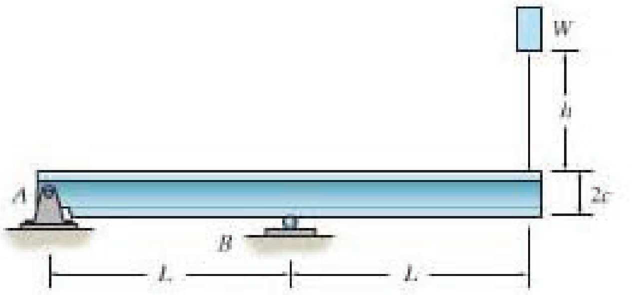
Prob. 14–55

14–58. The tugboat has a weight of 120 000 lb and is traveling forward at 2 ft/s when it strikes the 12-in.-diameter fender post AB used to protect a bridge pier. If the post is made from treated white spruce and is assumed fixed at the river bed, determine the maximum horizontal distance the top of the post will move due to the impact. Assume the tugboat is rigid and neglect the effect of the water.



Prob. 14–58

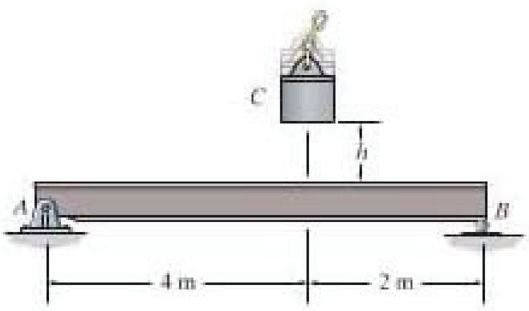
14–59. The wide-flange beam has a length of $2L$, a depth $2c$, and a constant EI. Determine the maximum height h at which a weight W can be dropped on its end without exceeding a maximum elastic stress σ_{max} in the beam.



Prob. 14–59

*14–60. The 50-kg block C is dropped from $h = 1.5$ m onto the simply supported beam. If the beam is an A-36 steel W250 × 45 wide-flange section, determine the maximum bending stress developed in the beam.

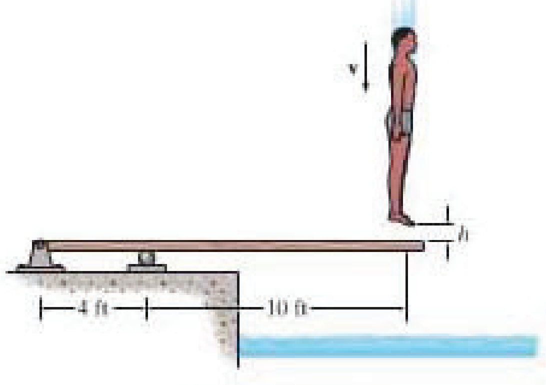
•14–61. Determine the maximum height h from which the 50-kg block C can be dropped without causing yielding in the A-36 steel W310 × 39 wide flange section when the block strikes the beam.



Probs. 14–60/61

14–62. The diver weighs 150 lb and, while holding himself rigid, strikes the end of a wooden diving board ($h = 0$) with a downward velocity of 4 ft/s. Determine the maximum bending stress developed in the board. The board has a thickness of 1.5 in. and width of 1.5 ft. $E_w = 1.8(10^3)$ ksi, $\sigma_Y = 8$ ksi.

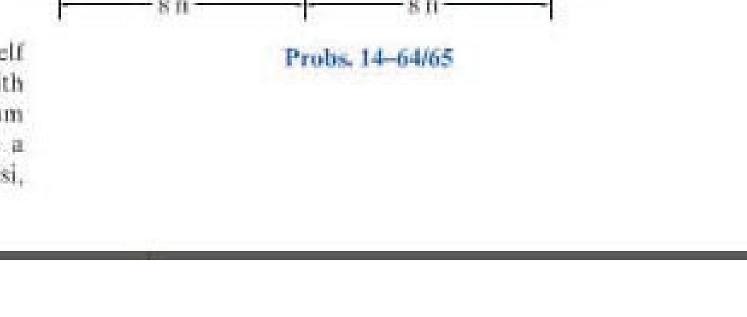
14–63. The diver weighs 150 lb and, while holding himself rigid, strikes the end of the wooden diving board. Determine the maximum height h from which he can jump onto the board so that the maximum bending stress in the wood does not exceed 6 ksi. The board has a thickness of 1.5 in. and width of 1.5 ft. $E_w = 1.8(10^3)$ ksi.



Probs. 14–62/63

*14–64. The weight of 175 lb is dropped from a height of 4 ft from the top of the A-36 steel beam. Determine the maximum deflection and maximum stress in the beam if the supporting springs at A and B each have a stiffness of $k = 500$ lb/in. The beam is 3 in. thick and 4 in. wide.

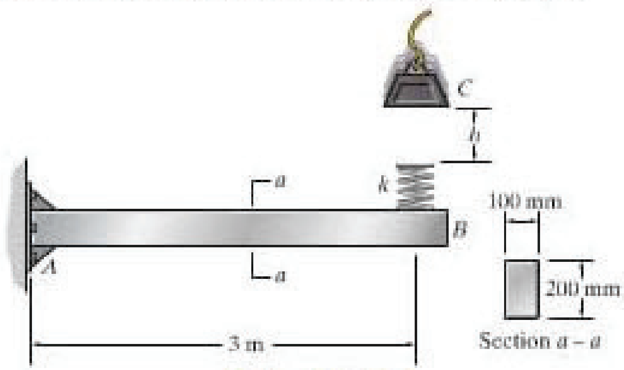
•14–65. The weight of 175 lb, is dropped from a height of 4 ft from the top of the A-36 steel beam. Determine the load factor n if the supporting springs at A and B each have a stiffness of $k = 300$ lb/in. The beam is 3 in. thick and 4 in. wide.



Probs. 14–64/65

14–66. Block C of mass 50 kg is dropped from height $h = 0.9$ m onto the spring of stiffness $k = 150$ kN/m mounted on the end B of the 6061-T6 aluminum cantilever beam. Determine the maximum bending stress developed in the beam.

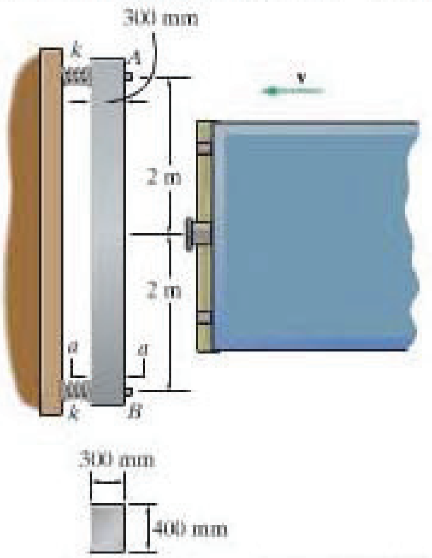
14–67. Determine the maximum height h from which 200-kg block C can be dropped without causing the 6061-T6 aluminum cantilever beam to yield. The spring mounted on the end B of the beam has a stiffness of $k = 150$ kN/m.



Probs. 14–66/67

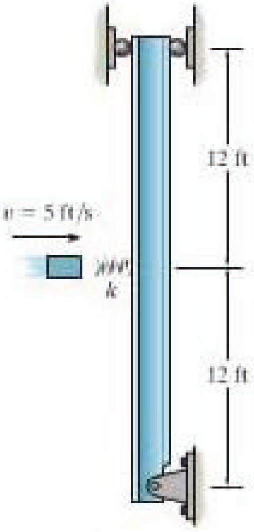
***14–68.** The 2014-T6 aluminum bar AB can slide freely along the guides mounted on the rigid crash barrier. If the railcar of mass 10 Mg is traveling with a speed of $v = 1.5$ m/s, determine the maximum bending stress developed in the bar. The springs at A and B have a stiffness of $k = 15$ MN/m.

•14–69. The 2014-T6 aluminum bar AB can slide freely along the guides mounted on the rigid crash barrier. Determine the maximum speed v the 10-Mg railcar without causing the bar to yield when it is struck by the railcar. The springs at A and B have a stiffness of $k = 15$ MN/m.



Probs. 14–68/69

14–70. The simply supported W10 × 15 structural A-36 steel beam lies in the horizontal plane and acts as a shock absorber for the 500-lb block which is traveling toward it at 5 ft/s. Determine the maximum deflection of the beam and the maximum stress in the beam during the impact. The spring has a stiffness of $k = 1000$ lb/in.



Prob. 14–70

14–71. The car bumper is made of polycarbonate-polybutylene terephthalate. If $E = 2.0$ GPa, determine the maximum deflection and maximum stress in the bumper if it strikes the rigid post when the car is coasting at $v = 0.75$ m/s. The car has a mass of 1.80 Mg, and the bumper can be considered simply supported on two spring supports connected to the rigid frame of the car. For the bumper take $I = 300(10^6)$ mm^4, $c = 75$ mm, $\sigma_Y = 30$ MPa and $k = 1.5$ MN/m.

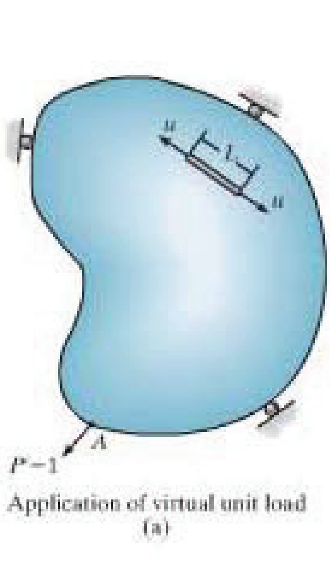
Prob. 14–71

*14.5 Principle of Virtual Work

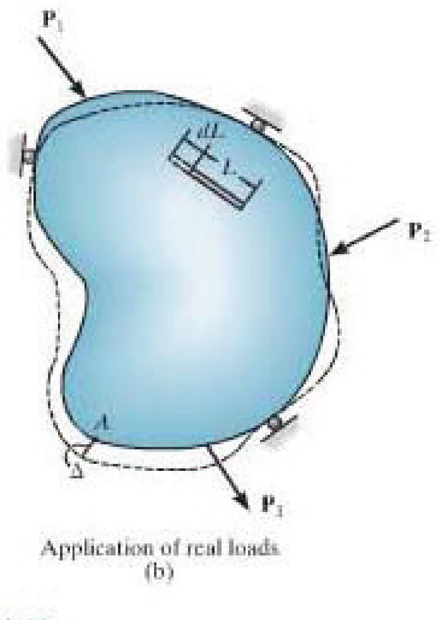
The principle of virtual work was developed by John Bernoulli in 1717, and like other energy methods of analysis, it is based on the conservation of energy. Although the principle of virtual work has many applications in mechanics, in this text we will use it to obtain the displacement and slope at a point on a deformable body.

To do this, we will consider the body to be of arbitrary shape as shown in Fig. 14–29*b*, and to be subjected to the "real loads" $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. It is assumed that these loads cause no movement of the supports; however, in general they can strain the material *beyond* the elastic limit. Suppose that it is necessary to determine the displacement Δ of point A on the body. Since there is no force acting at A, then Δ will *not* be included as an external "work term" in the equation when the conservation of energy principle is applied to the body. In order to get around this limitation, we will place an *imaginary* or "virtual" force $\mathbf{P}'$ on the body at point A, such that $\mathbf{P}'$ acts in the *same direction* as Δ. Furthermore, this load is applied to the body *before* the real loads are applied, Fig. 14–29*a*. For convenience, which will be made clear later, we will choose $\mathbf{P}'$ to have a "unit" magnitude; that is, $P' = 1$. It is to be emphasized that the term "***virtual***" is used to describe this load because it is *imaginary* and does not actually exist as part of the real loading.

This external virtual load, however, does create an internal virtual load $\mathbf{u}$ in a representative element or fiber of the body, as shown in Fig. 14–29*a*. As expected, P' and u can be related by the equations of equilibrium. Also, because of P' and u, the body and the element will each undergo a virtual (imaginary) displacement, although we will *not* be



Application of virtual unit load
(a)



Application of real loads
(b)

Fig. 14–29

concerned with their magnitudes. Once the virtual load is applied and *then* the body is subjected to the *real loads* $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, point A will be displaced a real amount Δ, which causes the element to be displaced dL, Fig. 14–29*b*. As a result, the external virtual force $\mathbf{P}'$ and internal virtual load $\mathbf{u}$ "ride along" or are displaced by Δ and dL, respectively. Consequently these loads perform *external virtual work* $1 \cdot \Delta$ on the body and *internal virtual work* $u \cdot dL$ on the element. Considering *only* the conservation of *virtual* energy, the external virtual work is then equal to the internal virtual work done on all the elements of the body. Therefore, we can write the virtual-work equation as

$$\overbrace{1 \cdot \underbrace{\Delta = \Sigma u}_{\text{real displacements}}}^{\text{virtual loadings}} \cdot\, dL \tag{14–34}$$

Here

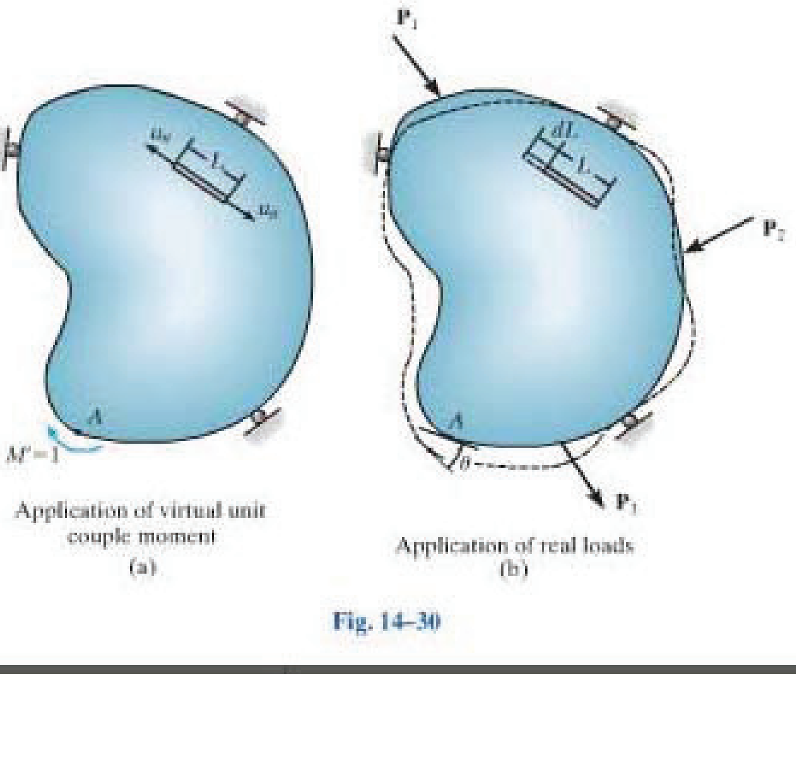
$P' = 1$ = external virtual unit load acting in the direction of Δ

u = internal virtual load acting on the element

Δ = external displacement caused by the real loads

dL = internal displacement of the element in the direction of $\mathbf{u}$, caused by the real loads

By choosing $P' = 1$, it can be seen that the solution for Δ follows directly, since $\Delta = \Sigma u\, dL$.



Application of virtual unit couple moment
(a)

Application of real loads
(b)

Fig. 14–30

In a similar manner, if the angular displacement or slope of the tangent at a point on the body is to be determined at A, Fig. 14–30*b*, then a virtual *couple moment* $\mathbf{M}'$, having a "unit" magnitude, is applied at the point, Fig. 14–30*a*. As a result, this couple moment causes a virtual load u_θ in one of the elements of the body. Assuming the real loads $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ deform the element an amount dL, the angular displacement θ can be found from the virtual-work equation

$$\overbrace{1 \cdot \underbrace{\theta = \Sigma u_\theta}_{}}^{\text{virtual loadings}} \underbrace{\,dL}_{\text{real displacements}} \tag{14–35}$$

Here

$M' = 1$ = external virtual unit couple moment acting in the direction of θ

u_θ = internal virtual load acting on an element

θ = external angular displacement in radians caused by the real loads

dL = internal displacement of the element in the direction of u_θ, caused by the real loads

This method for applying the principle of virtual work is often referred to as the *method of virtual forces*, since a *virtual force* is applied, resulting in a determination of an external *real displacement*. The equation of virtual work in this case represents a statement of *compatibility requirements* for the body. Although it is not important here, realize that we can also apply the principle of virtual work as a *method of virtual displacements*. In this case, *virtual displacements* are imposed on the body when the body is subjected to *real loadings*. This method can be used to determine the external reactive force on the body or an unknown internal loading. When it is used in this manner, the equation of virtual work is a statement of the *equilibrium requirements* for the body.*

Internal Virtual Work. The terms on the right side of Eqs. 14–34 and 14–35 represent the internal virtual work developed in the body. The real internal displacements dL in these terms can be produced in several different ways. For example, these displacements may result from geometric fabrication errors, from a change in temperature, or more commonly from stress. In particular, no restriction has been placed on the magnitude of the external loading, so the stress may be large enough to cause yielding or even strain hardening of the material.

*See *Engineering Mechanics: Statics*, 12th edition, R.C. Hibbeler, Prentice Hall, Inc., 2009.

14

TABLE 14–1		
Deformation caused by	**Strain energy**	**Internal virtual work**
Axial load N	$\int_0^L \frac{N^2}{2EA}dx$	$\int_0^L \frac{nN}{EA}dx$
Shear V	$\int_0^L \frac{f_sV^2}{2GA}dx$	$\int_0^L \frac{f_svV}{GA}dx$
Bending moment M	$\int_0^L \frac{M^2}{2EI}dx$	$\int_0^L \frac{mM}{EI}dx$
Torsional moment T	$\int_0^L \frac{T^2}{2GJ}dx$	$\int_0^L \frac{tT}{GJ}dx$

If we assume that the material behavior is linear elastic and the stress does not exceed the proportional limit, we can then formulate the expressions for internal virtual work caused by stress using the equations of elastic strain energy developed in Sec. 14.2. They are listed in the center column of Table 14–1. Recall that each of these expressions assumes that the internal loading **N**, **V**, **M**, or **T** was applied gradually from zero to its full value. As a result, the work done by these resultants is shown in these expressions as *one-half* the product of the internal loading and its displacement. In the case of the virtual-force method, however, the "full" virtual internal loading is applied *before* the real loads cause displacements, and therefore the work of the virtual loading is simply the product of the virtual load and its real displacement. Referring to these internal virtual loadings (u) by the corresponding lowercase symbols n, v, m, and t, the virtual work due to axial load, shear, bending moment, and torsional moment is listed in the right-hand column of Table 14–1. Using these results, the virtual-work equation for a body subjected to a general loading can therefore be written as

$$1 \cdot \Delta = \int \frac{nN}{AE}dx + \int \frac{mM}{EI}dx + \int \frac{f_s vV}{GA}dx + \int \frac{tT}{GJ}dx \quad (14\text{–}36)$$

In the following sections we will apply the above equation to problems involving the displacement of joints on trusses, and points on beams and mechanical elements. We will also include a discussion of how to handle the effects of fabrication errors and differential temperature. For application it is important that a consistent set of units be used for all the terms. For example, if the real loads are expressed in kilonewtons and the body's dimensions are in meters, a 1-kN virtual force or 1-kN · m virtual couple should be applied to the body. By doing so a calculated displacement Δ will be in meters, and a calculated slope will be in radians.

*14.6 Method of Virtual Forces Applied to Trusses

In this section, we will apply the method of virtual forces to determine the displacement of a truss joint. To illustrate the principles, the vertical displacement of joint A of the truss shown in Fig. 14–31b will be determined. To do this, we must place a virtual unit force at this joint, Fig. 14–31a, so that when the real loads $\mathbf{P}_1$ and $\mathbf{P}_2$ are applied to the truss, they cause the external virtual work $1 \cdot \Delta$. The internal virtual work in each member is $n\Delta L$. Since each member has a constant cross-sectional area A, and n and N are constant throughout the member's length, then from Table 14–1, the internal virtual work for each member is

$$\int_0^L \frac{nN}{AE}dx = \frac{nNL}{AE}$$

Therefore, the virtual-work equation for the entire truss is

$$\boxed{1 \cdot \Delta = \sum \frac{nNL}{AE}} \qquad (14\text{–}37)$$

Here

1 = external virtual unit load acting on the truss joint in the direction of Δ

Δ = joint displacement caused by the real loads on the truss

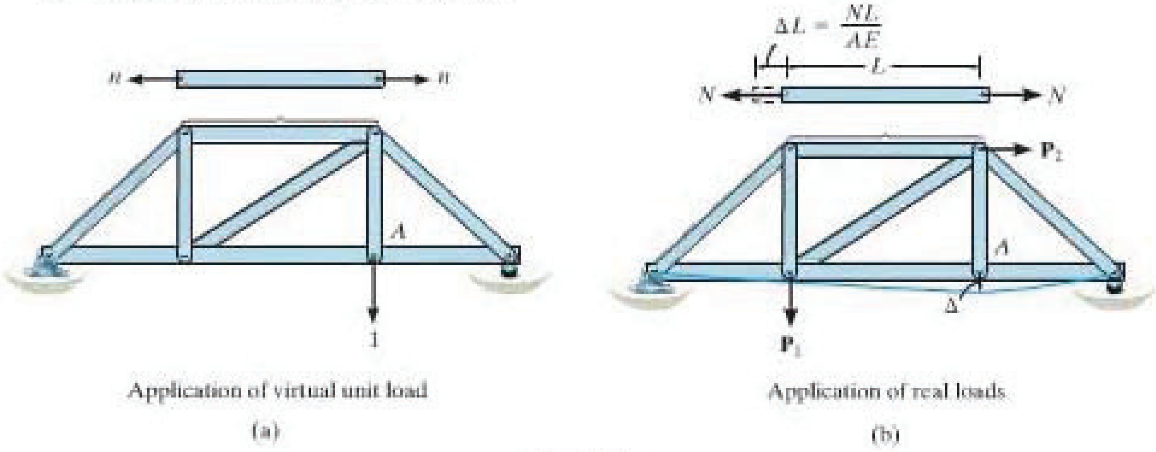
n = internal virtual force in a truss member caused by the external virtual unit load

N = internal force in a truss member caused by the real loads

L = length of a member

A = cross-sectional area of a member

E = modulus of elasticity of a member



Application of virtual unit load

(a)

Application of real loads

(b)

Fig. 14–31

Temperature Change. Truss members can change their length due to a change in temperature. If α is the coefficient of thermal expansion for a member and ΔT is the change in temperature, the change in length of a member is $\Delta L = \alpha\,\Delta T L$ (Eq. 4–4). Hence, we can determine the displacement of a selected truss joint due to this temperature change from Eq. 14–34, written as

$$\boxed{1 \cdot \Delta = \Sigma n \alpha\,\Delta T L} \qquad (14\text{–}38)$$

14

Here

1 = external virtual unit load acting on the truss joint in the direction of Δ

Δ = joint displacement caused by the temperature change

n = internal virtual force in a truss member caused by the external virtual unit load

α = coefficient of thermal expansion of material

ΔT = change in temperature of member

L = length of member

Fabrication Errors. Occasionally errors in fabricating the lengths of the members of a truss may occur. If this happens, the displacement Δ in a particular direction of a truss joint from its expected position can be determined from direct application of Eq. 14–34 written as

$$\boxed{1 \cdot \Delta = \Sigma n\,\Delta L} \qquad (14\text{–}39)$$

Here

1 = external virtual unit load acting on the truss joint in the direction of Δ

Δ = joint displacement caused by the fabrication errors

n = internal virtual force in a truss member caused by the external virtual unit load

ΔL = difference in length of the member from its intended length caused by a fabrication error

A combination of the right-hand sides of Eqs. 14–37 through 14–39 will be necessary if external loads act on the truss and some of the members undergo a temperature change or have been fabricated with the wrong dimensions.

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement of any joint on a truss using the method of virtual forces.

Virtual Forces n.

- Place the virtual unit load on the truss at the joint where the displacement is to be determined. The load should be directed along the line of action of the displacement.
- With the unit load so placed and all the real loads *removed* from the truss, calculate the internal n force in each truss member. Assume that tensile forces are positive and compressive forces are negative.

Real Forces N.

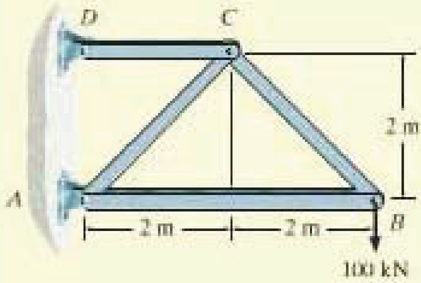
- Determine the N forces in each member. These forces are caused only by the real loads acting on the truss. Again, assume that tensile forces are positive and compressive forces are negative.

Virtual-Work Equation.

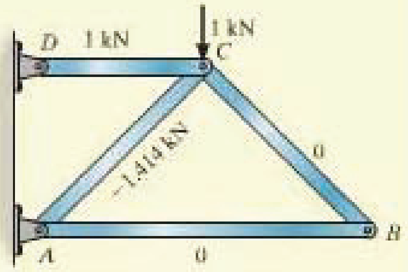
- Apply the equation of virtual work to determine the desired displacement. It is important to retain the algebraic sign for each of the corresponding n and N forces when substituting these terms into the equation.
- If the resultant sum $\Sigma nNL/AE$ is positive, the displacement Δ is in the same direction as the virtual unit load. If a negative value results, Δ is opposite to the virtual unit load.
- When applying $1 \cdot \Delta = \Sigma n\alpha\,\Delta TL$, an *increase* in temperature, ΔT, will be *positive*; whereas a *decrease* in temperature will be *negative*.
- For $1 \cdot \Delta = \Sigma n\,\Delta L$, when a fabrication error causes an increase in the length of a member, ΔL is *positive*, whereas a *decrease* in length is *negative*.
- When applying this method, attention should be paid to the units of each numerical quantity. Notice, however, that the virtual unit load can be assigned any arbitrary unit: pounds, kips, newtons, etc., since the n forces will have these *same* units, and as a result, the units for both the virtual unit load and the n forces will cancel from both sides of the equation.

EXAMPLE 14.11

Determine the vertical displacement of joint C of the steel truss shown in Fig. 14–32a. The cross-sectional area of each member is $A = 400 \text{ mm}^2$ and $E_{st} = 200$ GPa.



(a)

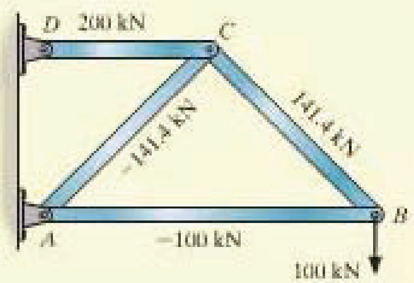


Virtual forces

(b)

SOLUTION

Virtual Forces *n*. Since the vertical displacement at joint C is to be determined, *only* a vertical 1-kN virtual load is placed at joint C; and the force in each member is calculated using the method of joints. The results of this analysis are shown in Fig. 14–32b. Using our sign convention, positive numbers indicate tensile forces and negative numbers indicate compressive forces.



Real forces

(c)

Fig. 14–32

Real Forces *N*. The applied load of 100 kN causes forces in the members that are also calculated using the method of joints. The results of this analysis are shown in Fig. 14–32c.

Virtual-Work Equation. Arranging the data in tabular form, we have

Member	n	N	L	nNL
AB	0	−100	4	0
BC	0	141.4	2.828	0
AC	−1.414	−141.4	2.828	565.7
CD	1	200	2	400
				Σ 965.7 kN²·m

Thus,

$$1 \text{ kN} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{965.7 \text{ kN}^2 \cdot \text{m}}{AE}$$

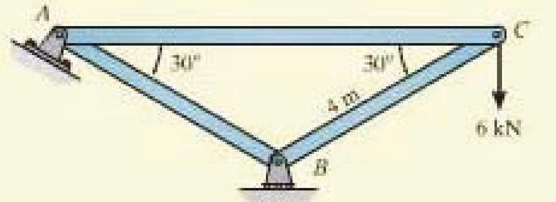
Substituting the numerical values for A and E, we have

$$1 \text{ kN} \cdot \Delta_{C_v} = \frac{965.7 \text{ kN}^2 \cdot \text{m}}{[400(10^{-6}) \text{ m}^2]\, 200(10^6) \text{ kN/m}^2}$$

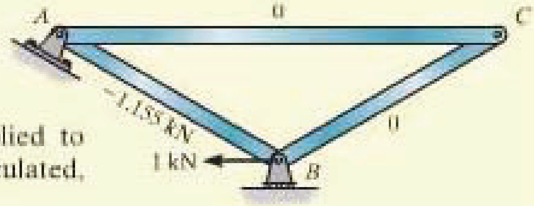
$$\Delta_{C_v} = 0.01207 \text{ m} = 12.1 \text{ mm}$$ *Ans.*

EXAMPLE 14.12

Determine the horizontal displacement of the roller at B of the truss shown in Fig. 14–33*a*. Due to radiant heating, member AB is subjected to an *increase* in temperature of $\Delta T = +60°C$, and this member has been fabricated 3 mm too short. The members are made of steel, for which $\alpha_{st} = 12(10^{-6})/°C$ and $E_{st} = 200$ GPa. The cross-sectional area of each member is 250 mm^2.



(a)



Virtual forces

(b)

SOLUTION

Virtual Forces *n*. A horizontal 1-kN virtual load is applied to the truss at joint B, and the forces in each member are calculated, Fig. 14–33*b*.

Real Forces *N*. Since the n forces in members AC and BC are *zero*, the N forces in these members do *not* have to be determined. Why? For completeness, though, the entire "real" force analysis is shown in Fig. 14–33*c*.

Virtual-Work Equation. The loads, temperature, and the fabrication error all affect the displacement of point B; therefore, Eqs. 14–37, 14–38, and 14–39 must be combined, which gives

$$1\text{ kN}\cdot\Delta_{B_h} = \sum\frac{nNL}{AE} + \Sigma n\alpha\,\Delta T L + \Sigma n\Delta L$$

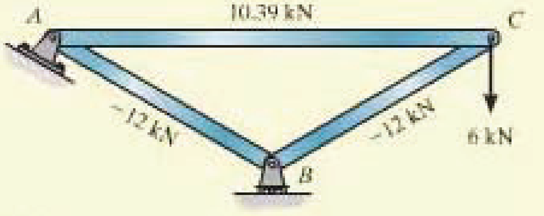
$$= 0 + 0 + \frac{(-1.155\text{ kN})(-12\text{ kN})(4\text{ m})}{[250(10^{-6})\text{ m}^2][200(10^6)\text{ kN/m}^2]}$$

$$+ 0 + 0 + (-1.155\text{ kN})[12(10^{-6})/°\text{C}](60°\text{C})(4\text{ m})$$

$$+ (-1.155\text{ kN})(-0.003\text{ m})$$

$$\Delta_{B_h} = 0.00125\text{ m}$$

$$= 1.25\text{ mm} \leftarrow \qquad \textit{Ans.}$$



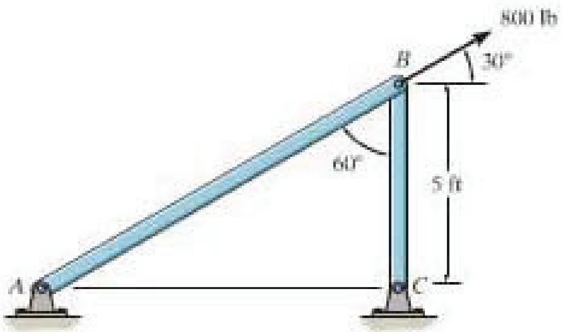
Real forces

(c)

Fig. 14–33

PROBLEMS

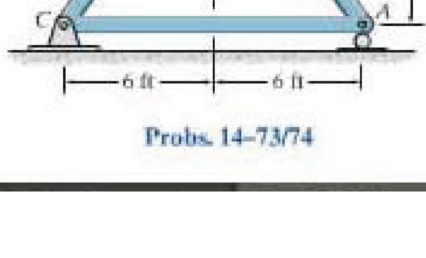
*14–72. Determine the horizontal displacement of joint B on the two-member frame. Each A-36 steel member has a cross-sectional area of $2\ \text{in}^2$.



Prob. 14–72

•14–73. Determine the horizontal displacement of point B. Each A-36 steel member has a cross-sectional area of $2\ \text{in}^2$.

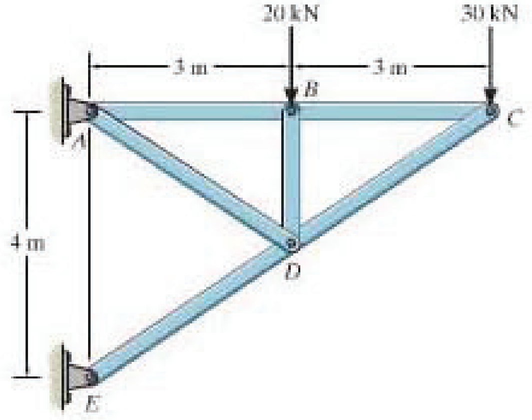
14–74. Determine the vertical displacement of point B. Each A-36 steel member has a cross-sectional area of $2\ \text{in}^2$.



Probs. 14–73/74

14–75. Determine the vertical displacement of joint C on the truss. Each A-36 steel member has a cross-sectional area of $A = 300\ \text{mm}^2$.

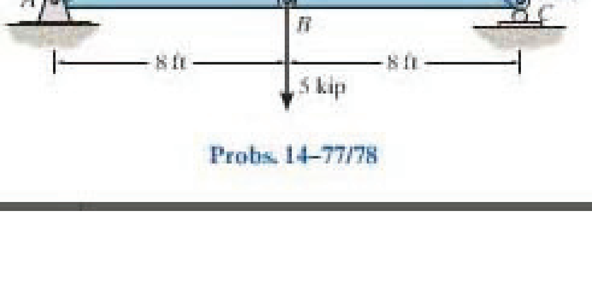
*14–76. Determine the vertical displacement of joint D on the truss. Each A-36 steel member has a cross-sectional area of $A = 300\ \text{mm}^2$.



Probs. 14–75/76

•14–77. Determine the vertical displacement of point B. Each A-36 steel member has a cross-sectional area of $4.5\ \text{in}^2$.

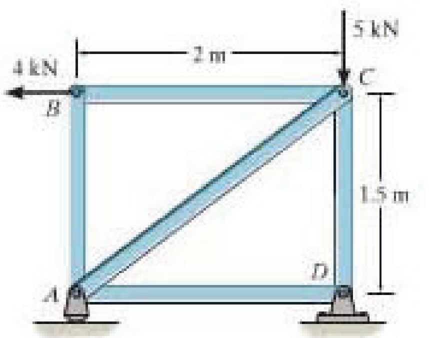
14–78. Determine the vertical displacement of point E. Each A-36 steel member has a cross-sectional area of $4.5\ \text{in}^2$.



Probs. 14–77/78

14–79. Determine the horizontal displacement of joint B of the truss. Each A-36 steel member has a cross-sectional area of 400 mm^2.

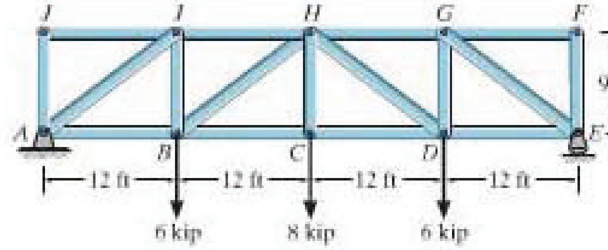
*14–80. Determine the vertical displacement of joint C of the truss. Each A-36 steel member has a cross-sectional area of 400 mm^2.



Probs. 14–79/80

14–83. Determine the vertical displacement of joint C. Each A-36 steel member has a cross-sectional area of 4.5 in^2.

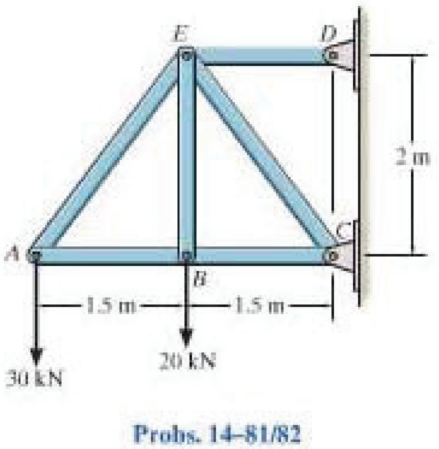
*14–84. Determine the vertical displacement of joint H. Each A-36 steel member has a cross-sectional area of 4.5 in^2.



Probs. 14–83/84

•14–81. Determine the vertical displacement of point A. Each A-36 steel member has a cross-sectional area of 400 mm^2.

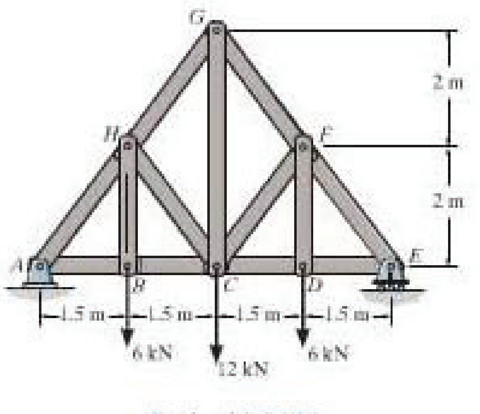
14–82. Determine the vertical displacement of point B. Each A-36 steel member has a cross-sectional area of 400 mm^2.



Probs. 14–81/82

•14–85. Determine the vertical displacement of joint C. The truss is made from A-36 steel bars having a cross- sectional area of 150 mm^2.

14–86. Determine the vertical displacement of joint G. The truss is made from A-36 steel bars having a cross-sectional area of 150 mm^2.



Probs. 14–85/86

14

*14.7 Method of Virtual Forces Applied to Beams

In this section we will apply the method of virtual forces to determine the displacement and slope at a point on a beam. To illustrate the principles, the vertical displacement Δ of point A on the beam shown in Fig. 14–34b will be determined. To do this we must place a vertical unit load at this point, Fig. 14–34a, so that when the "real" distributed load w is applied to the beam it will cause the internal virtual work $1 \cdot \Delta$. Because the load causes both a shear V and moment M within the beam, we must actually consider the internal virtual work due to both of these loadings. In Example 14.7, however, it was shown that beam deflections due to shear are negligible compared with those caused by bending, particularly if the beam is long and slender. Since this type of beam is most often used in practice, we will only consider the virtual strain energy due to bending, Table 14–1. Hence, the real load causes the element dx to deform so its sides rotate by an angle $d\theta = (M/EI)dx$, which causes internal virtual work $m\ d\theta$. Applying Eq. 14–34, the virtual-work equation for the entire beam, we have

$$\boxed{1 \cdot \Delta = \int_0^L \frac{mM}{EI}dx} \qquad (14\text{–}40)$$

Here

1 = external virtual unit load acting on the beam in the direction of Δ

Δ = displacement caused by the real loads acting on the beam

m = internal virtual moment in the beam, expressed as a function of x and caused by the external virtual unit load

M = internal moment in the beam, expressed as a function of x and caused by the real loads

E = modulus of elasticity of the material

I = moment of inertia of the cross-sectional area about the neutral axis

In a similar manner, if the slope θ of the tangent at a point on the beam's elastic curve is to be determined, a virtual unit couple moment must be applied at the point, and the corresponding internal virtual moment m_θ has to be determined. If we apply Eq. 14–35 for this case and neglect the effect of shear deformations, we have

$$\boxed{1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI}dx} \qquad (14\text{–}41)$$

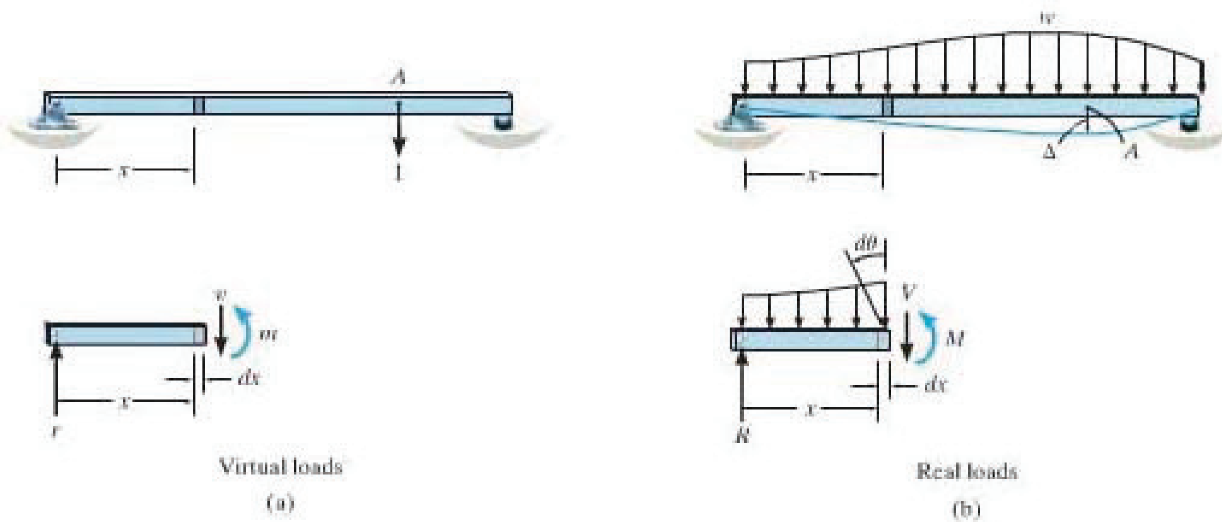


Fig. 14–34

When applying these equations, keep in mind that the integrals on the right side represent the amount of virtual bending strain energy that is *stored* in the beam. If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration *cannot* be performed across the beam's entire length. Instead, separate x coordinates must be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each x have the same origin; however, the x selected for determining the real moment M in a particular region must be the *same* x as that selected for determining the virtual moment m or m_θ within the same region. For example, consider the beam shown in Fig. 14–35. In order to determine the displacement at D, we can use x_1 to determine the strain energy in region AB, x_2 for region BC, x_3 for region DE, and x_4 for region DC. In any case, each x coordinate should be selected so that both M and m (or m_θ) can easily be formulated.

Unlike beams, as discussed here, some members may also be subjected to significant virtual strain energy caused by axial load, shear, and torsional moment. When this is the case, we must include in the above equations the energy terms for these loadings as formulated in Eq. 14–36.

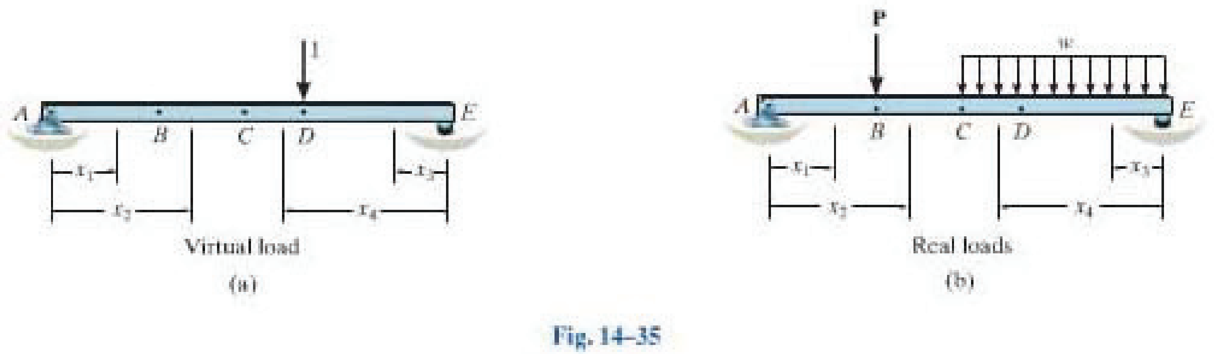


Fig. 14–35

14

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the method of virtual forces.

Virtual Moments m or m_θ.

- Place a *virtual unit load* on the beam at the point and directed along the line of action of the desired displacement.
- If the slope is to be determined, place a *virtual unit couple moment* at the point.
- Establish appropriate x coordinates that are valid within regions of the beam where there is no discontinuity of both real and virtual load.
- With the virtual load in place, and all the real loads *removed* from the beam, calculate the internal moment m or m_θ as a function of each x coordinate.
- Assume that m or m_θ acts in the positive direction according to the established beam sign convention for positive moment, Fig. 6–3.

Real Moments.

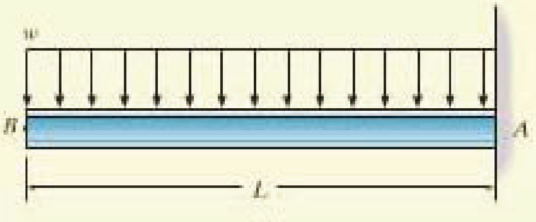
- Using the *same* x coordinates as those established for m or m_θ, determine the internal moments M caused by the real loads.
- Since positive m or m_θ was assumed to act in the conventional "positive direction," it is important that positive M acts in this *same direction*. This is necessary since positive or negative internal virtual work depends on the directional sense of *both* the virtual load, defined by $\pm m$ or $\pm m_\theta$, and displacement, caused by $\pm M$.

Virtual-Work Equation.

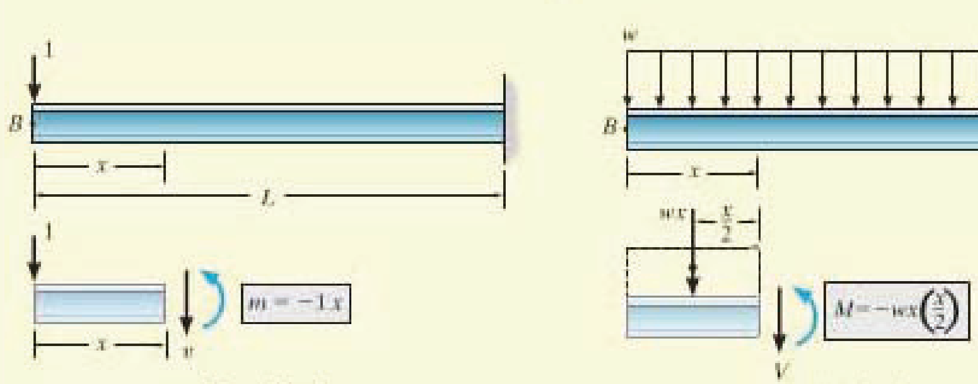
- Apply the equation of virtual work to determine the desired displacement Δ or slope θ. It is important to retain the algebraic sign of each integral calculated within its specified region.
- If the algebraic sum of all the integrals for the entire beam is positive, Δ or θ is in the same direction as the virtual unit load or virtual unit couple moment. If a negative value results, Δ or θ is opposite to the virtual unit load or couple moment.

EXAMPLE 14.13

Determine the displacement of point B on the beam shown in Fig. 14–36*a*. EI is constant.



(a)



Virtual loads

(b)

Real loads

(c)

Fig. 14–36

SOLUTION

Virtual Moment *m*. The vertical displacement of point B is obtained by placing a virtual unit load at B, Fig. 14–36*b*. By inspection, there are no discontinuities of loading on the beam for *both* the real and virtual loads. Thus, a *single* x coordinate can be used to determine the virtual strain energy. This coordinate will be selected with its origin at B, so that the reactions at A do not have to be determined in order to find the internal moments m and M. Using the method of sections, the internal moment m is shown in Fig. 14–36*b*.

Real Moment *M*. Using the *same* x coordinate, the internal moment M is shown in Fig. 14–36*c*.

Virtual-Work Equation. The vertical displacement at B is thus

$$1 \cdot \Delta_B = \int \frac{mM}{EI} dx = \int_0^L \frac{(-1x)(-wx^2/2)\,dx}{EI}$$

$$\Delta_B = \frac{wL^4}{8EI} \qquad \textit{Ans.}$$

14

EXAMPLE 14.14

Determine the slope at point B of the beam shown in Fig. 14–37a. EI is constant.

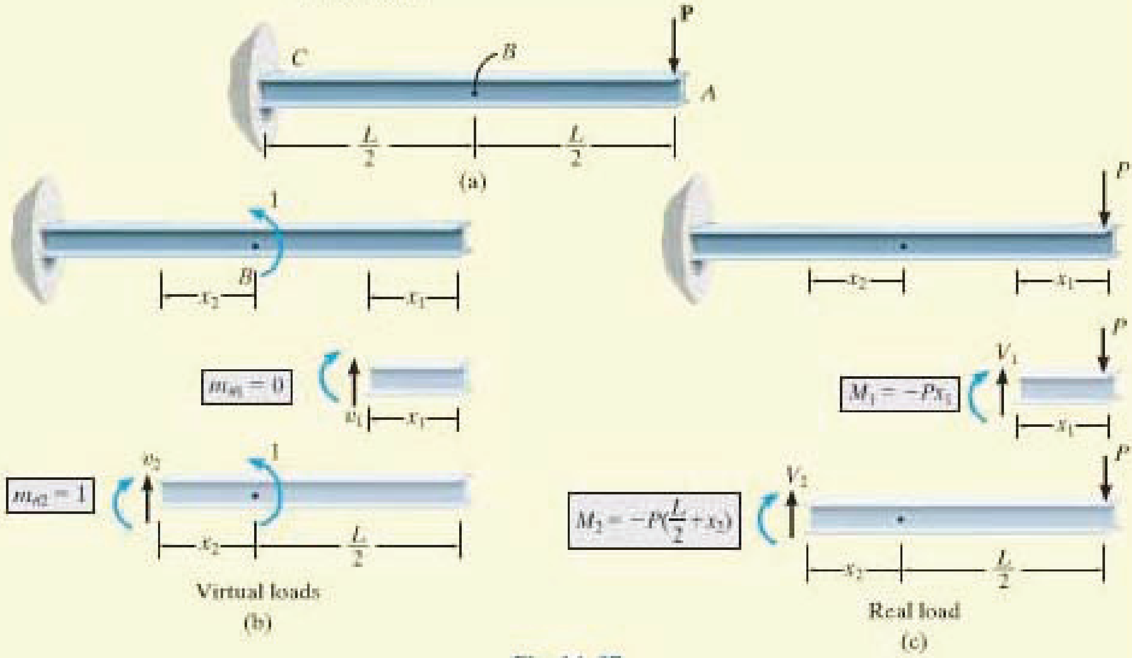


Fig. 14–37

SOLUTION

Virtual Moments m_θ. The slope at B is determined by placing a virtual unit couple moment at B, Fig. 14–37b. Two x coordinates must be selected in order to determine the total virtual strain energy in the beam. Coordinate x_1 accounts for the strain energy within segment AB, and coordinate x_2 accounts for the strain energy in segment BC. Using the method of sections the internal moments m_θ within each of these segments are shown in Fig. 14–37b.

Real Moments M. Using the *same coordinates* x_1 and x_2 (Why?), the internal moments M are shown in Fig. 14–37c.

Virtual-Work Equation. The slope at B is thus

$$1 \cdot \theta_B = \int \frac{m_\theta M}{EI} dx$$

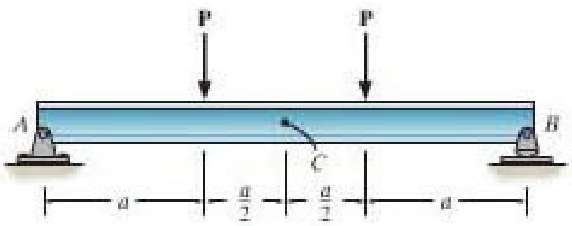
$$= \int_0^{L/2} \frac{0(-Px_1)\,dx_1}{EI} + \int_0^{L/2} \frac{1\{-P[(L/2) + x_2]\}\,dx_2}{EI}$$

$$\theta_B = -\frac{3PL^2}{8EI} \qquad \textit{Ans.}$$

The *negative sign* indicates that θ_B is *opposite* to the direction of the virtual couple moment shown in Fig. 14–37b.

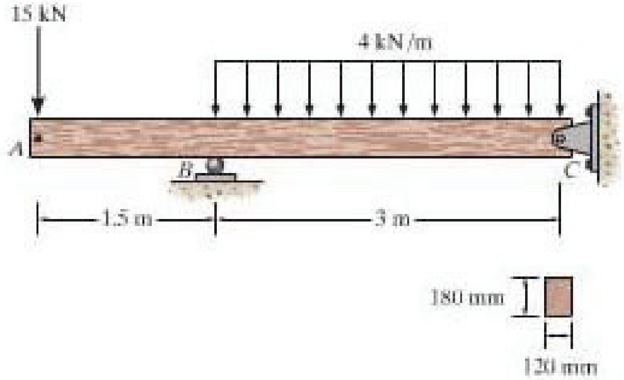
PROBLEMS

14–87. Determine the displacement at point C. EI is constant.



Prob. 14–87

***14–88.** The beam is made of southern pine for which $E_p = 13$ GPa. Determine the displacement at A.

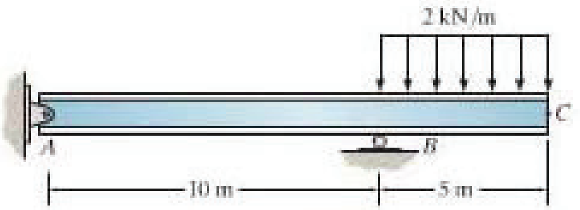


Prob. 14–88

•14–89. Determine the displacement at C of the A-36 steel beam. $I = 70(10^6)\ \text{mm}^4$.

14–90. Determine the slope at A of the A-36 steel beam. $I = 70(10^6)\ \text{mm}^4$.

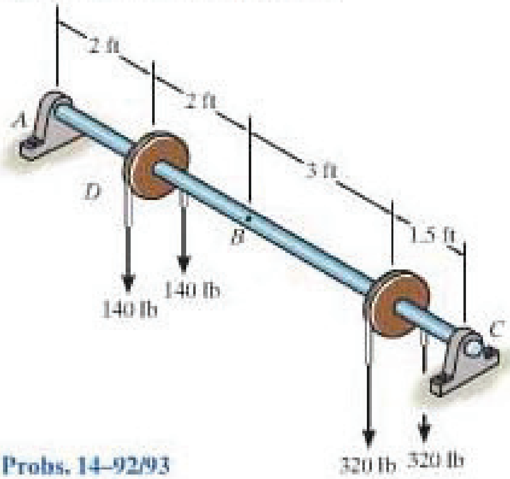
14–91. Determine the slope at B of the A-36 steel beam. $I = 70(10^6)\ \text{mm}^4$.



Probs. 14–89/90/91

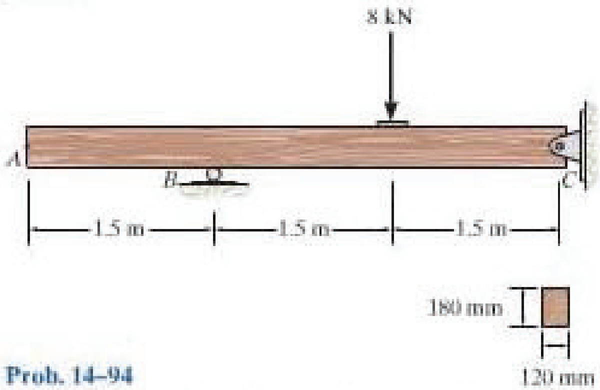
***14–92.** Determine the displacement at B of the 1.5-in-diameter A-36 steel shaft.

•14–93. Determine the slope of the 1.5-in-diameter A-36 steel shaft at the bearing support A.



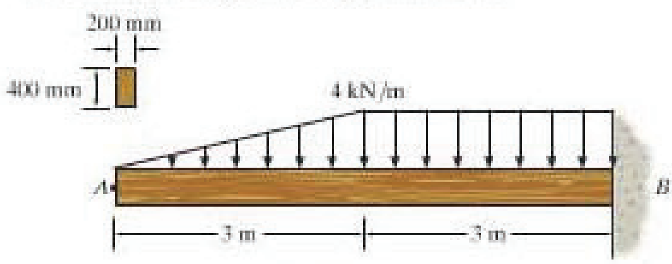
Probs. 14–92/93

14–94. The beam is made of Douglas fir. Determine the slope at C.



Prob. 14–94

14–95. The beam is made of oak, for which $E_o = 11$ GPa. Determine the slope and displacement at A.

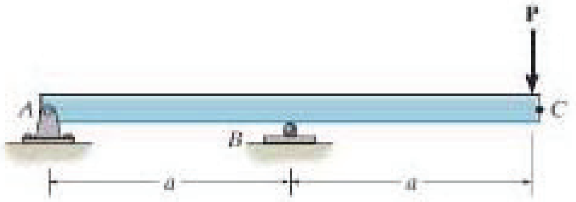


Prob. 14–95

*14–96. Determine the displacement at point C. EI is constant.

•14–97. Determine the slope at point C. EI is constant.

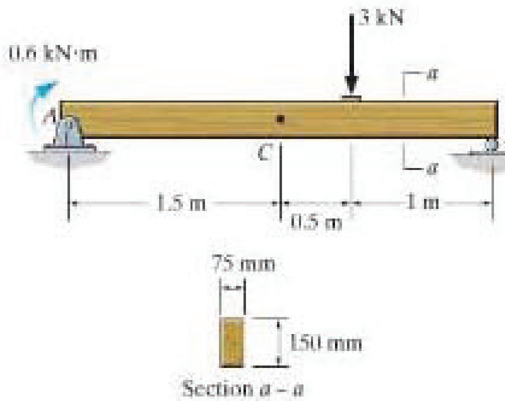
14–98. Determine the slope at point A. EI is constant.



Probs. 14–96/97/98

14–99. Determine the slope at point A of the simply supported Douglas fir beam.

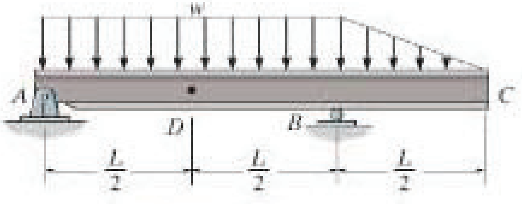
*14–100. Determine the displacement at C of the simply supported Douglas fir beam.



Probs. 14–99/100

•14–101. Determine the slope of end C of the overhang beam. EI is constant.

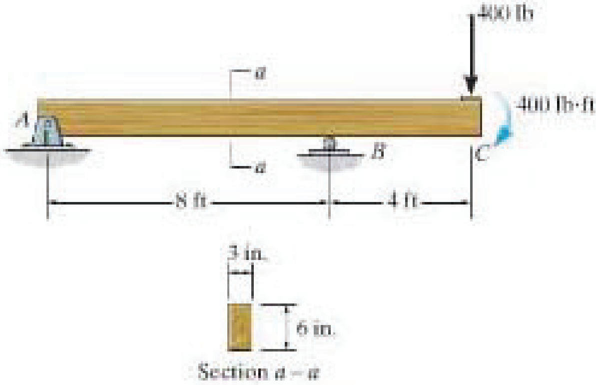
14–102. Determine the displacement of point D of the overhang beam. EI is constant.



Probs. 14–101/102

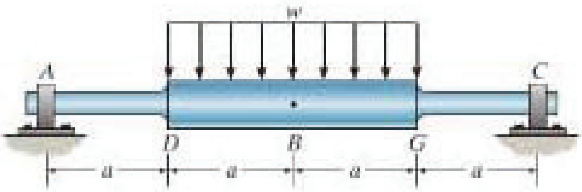
14–103. Determine the displacement of end C of the overhang Douglas fir beam.

*14–104. Determine the slope at A of the overhang white spruce beam.



Probs. 14–103/104

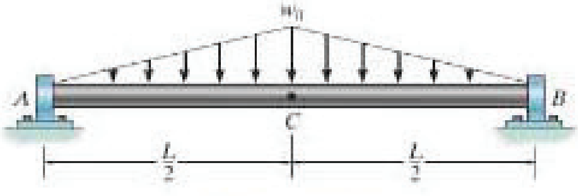
•14–105. Determine the displacement at point B. The moment of inertia of the center portion DG of the shaft is $2I$, whereas the end segments AD and GC have a moment of inertia I. The modulus of elasticity for the material is E.



Prob. 14–105

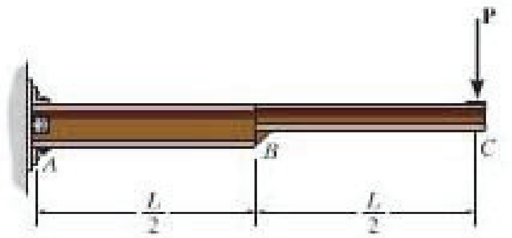
14–106. Determine the displacement of the shaft at C. EI is constant.

14–107. Determine the slope of the shaft at the bearing support A. EI is constant.



Probs. 14–106/107

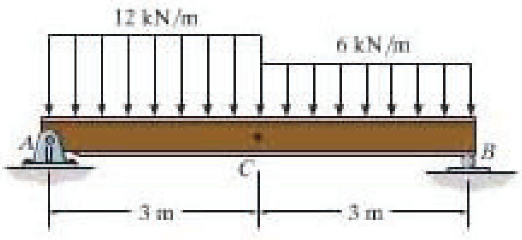
*14–108. Determine the slope and displacement of end C of the cantilevered beam. The beam is made of a material having a modulus of elasticity of E. The moments of inertia for segments AB and BC of the beam are $2I$ and I, respectively.



Prob. 14–108

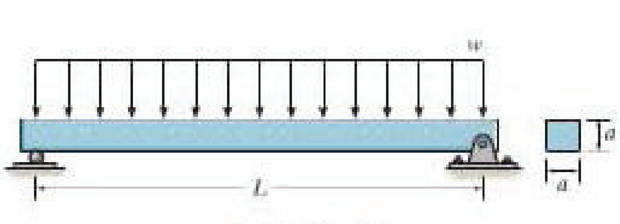
•14–109. Determine the slope at A of the A-36 steel W200 × 46 simply supported beam.

14–110. Determine the displacement at point C of the A-36 steel W200 × 46 simply supported beam.



Probs. 14–109/110

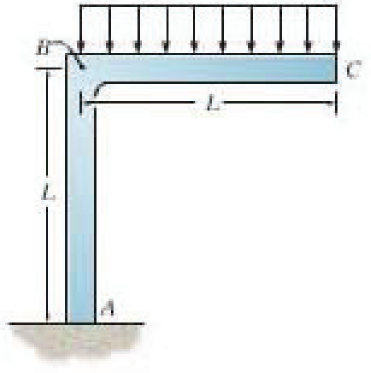
14–111. The simply supported beam having a square cross section is subjected to a uniform load w. Determine the maximum deflection of the beam caused only by bending, and caused by bending and shear. Take $E = 3G$.



Prob. 14–111

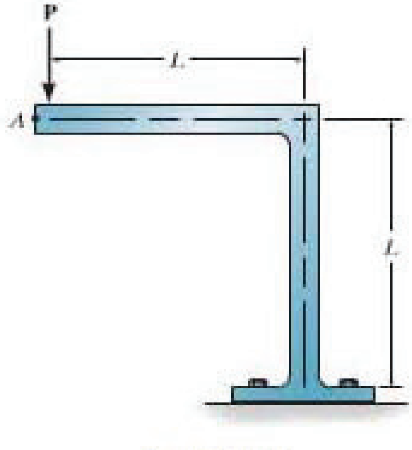
*14–112. The frame is made from two segments, each of length L and flexural stiffness EI. If it is subjected to the uniform distributed load determine the vertical displacement of point C. Consider only the effect of bending.

•14–113. The frame is made from two segments, each of length L and flexural stiffness EI. If it is subjected to the uniform distributed load, determine the horizontal displacement of point B. Consider only the effect of bending.



Probs. 14–112/113

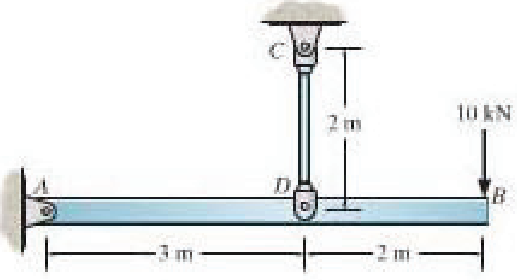
14–114. Determine the vertical displacement of point A on the angle bracket due to the concentrated force **P**. The bracket is fixed connected to its support. EI is constant. Consider only the effect of bending.



Prob. 14–114

14–115. Beam AB has a square cross section of 100 mm by 100 mm. Bar CD has a diameter of 10 mm. If both members are made of A-36 steel, determine the vertical displacement of point B due to the loading of 10 kN.

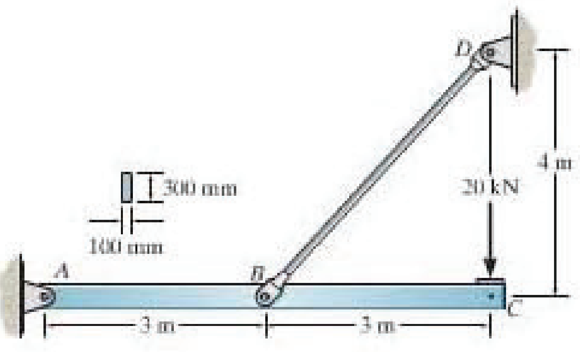
*__14–116.__ Beam AB has a square cross section of 100 mm by 100 mm. Bar CD has a diameter of 10 mm. If both members are made of A-36 steel, determine the slope at A due to the loading of 10 kN.



Probs. 14–115/116

14–117. Bar ABC has a rectangular cross section of 300 mm by 100 mm. Attached rod DB has a diameter of 20 mm. If both members are made of A-36 steel, determine the vertical displacement of point C due to the loading. Consider only the effect of bending in ABC and axial force in DB.

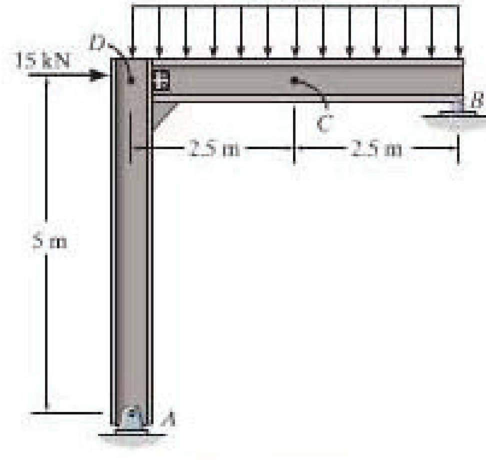
14–118. Bar ABC has a rectangular cross section of 300 mm by 100 mm. Attached rod DB has a diameter of 20 mm. If both members are made of A-36 steel, determine the slope at A due to the loading. Consider only the effect of bending in ABC and axial force in DB.



Probs. 14–117/118

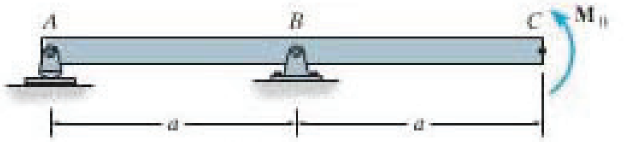
14–119. Determine the vertical displacement of point C. The frame is made using A-36 steel W250 × 45 members. Consider only the effects of bending.

*__14–120.__ Determine the horizontal displacement of end B. The frame is made using A-36 steel W250 × 45 members. Consider only the effects of bending.



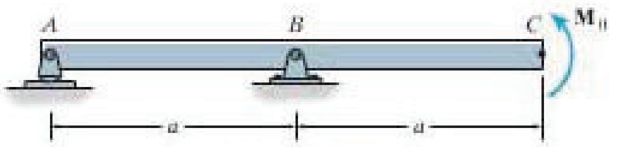
Probs. 14–119/120

•__14–121.__ Determine the displacement at point C. EI is constant.



Prob. 14–121

14–122. Determine the slope at B. EI is constant.



Prob. 14–122

*14.8 Castigliano's Theorem

In 1879, Alberto Castigliano, an Italian railroad engineer, published a book in which he outlined a method for determining the displacement and slope at a point in a body. This method, which is referred to as Castigliano's second theorem, applies only to bodies that have constant temperature and material with linear-elastic behavior. If the displacement at a point is to be determined, the theorem states that the displacement is equal to the first partial derivative of the strain energy in the body with respect to a force acting at the point and in the direction of displacement. In a similar manner, the slope of the tangent at a point in a body is equal to the first partial derivative of the strain energy in the body with respect to a couple moment acting at the point and in the direction of the slope angle.

To derive Castigliano's second theorem, consider a body of any arbitrary shape, which is subjected to a series of n forces $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, Fig. 14–38. According to the conservation of energy, the external work done by these forces is equal to the internal strain energy stored in the body. However, the external work is a function of the external loads, $U_e = \Sigma \int P\,dx$, Eq. 14–1, so the internal work is also a function of the external loads. Thus,

$$U_i = U_e = f(P_1, P_2, \ldots, P_n) \tag{14–42}$$

Now, if any one of the external forces, say P_j, is increased by a differential amount dP_j, the internal work will also be increased, such that the strain energy becomes

$$U_i + dU_i = U_i + \frac{\partial U_i}{\partial P_j} dP_j \tag{14–43}$$

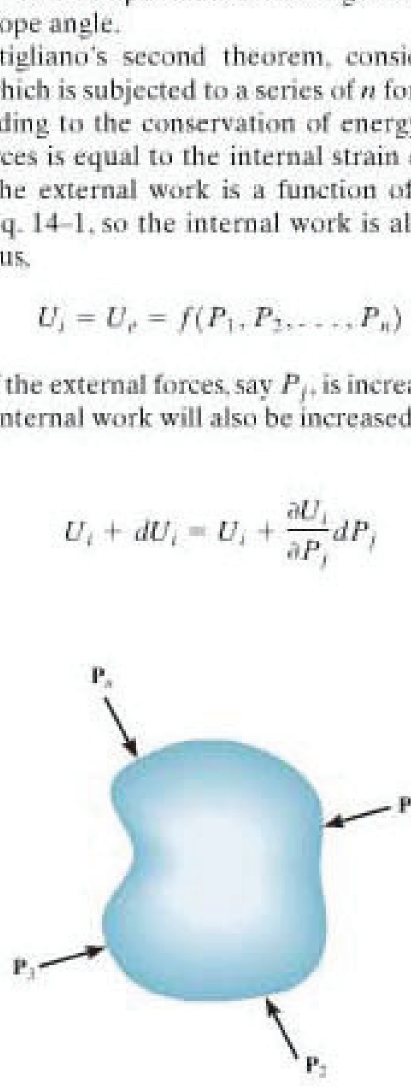


Fig. 14–38

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14

This value, however, will not depend on the sequence in which the n forces are applied to the body. For example, we could apply $d\mathbf{P}_j$ to the body *first*, then apply the loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$. In this case, $d\mathbf{P}_j$ would cause the body to displace a differential amount $d\Delta_j$ in the direction of $d\mathbf{P}_j$. By Eq. 14–2 ($U_e = \frac{1}{2}P_j\,\Delta_j$), the increment of strain energy would be $\frac{1}{2}dP_j\,d\Delta_j$. This quantity, however, is a second-order differential and may be neglected. Further application of the loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ causes $d\mathbf{P}_j$ to move through the displacement Δ_j so that now the strain energy becomes

$$U_i + dU_i = U_i + dP_j\,\Delta_j \tag{14–44}$$

Here, as above, U_i is the internal strain energy in the body, caused by the loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, and $dP_j\Delta_j$ is the *additional* strain energy caused by $d\mathbf{P}_j$.

In summary, Eq. 14–43 represents the strain energy in the body determined by first applying the loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, then $d\mathbf{P}_j$; Eq. 14–44 represents the strain energy determined by first applying $d\mathbf{P}_j$ and then the loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$. Since these two equations must be equal, we require

$$\Delta_j = \frac{\partial U_i}{\partial P_j} \tag{14–45}$$

which proves the theorem; i.e., the displacement Δ_j in the direction of $\mathbf{P}_j$ is equal to the first partial derivative of the strain energy with respect to $\mathbf{P}_j$.

Castigliano's second theorem, Eq. 14–45, is a statement regarding the body's *compatibility requirements*, since it is a condition related to displacement. Also, the above derivation requires that *only conservative forces* be considered for the analysis. These forces can be applied in any order, and furthermore, they do work that is independent of the path and therefore create no energy loss. As long as the material has linear-elastic behavior, the applied forces will be conservative and the theorem is valid. Castigliano's first theorem is similar to his second theorem; however, it relates the load P_j to the partial derivative of the strain energy with respect to the corresponding displacement, that is, $P_j = \partial U_i/\partial\Delta_j$. The proof is similar to that given above. This theorem is another way of expressing the *equilibrium requirements* for the body; however, it has limited application and therefore it will not be discussed here.

*14.9 Castigliano's Theorem Applied to Trusses

Since a truss member is only subjected to an axial load, the strain energy for the member is given by Eq. 14–16, $U_i = N^2L/2AE$. Substituting this equation into Eq. 14–45 and omitting the subscript i, we have

$$\Delta = \frac{\partial}{\partial P}\sum \frac{N^2L}{2AE}$$

It is generally easier to perform the differentiation prior to summation. Also, L, A, and E are constant for a given member, and therefore we can write

$$\boxed{\Delta = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE}} \qquad (14\text{–}46)$$

Here

Δ = displacement of the truss joint

P = an external force of *variable magnitude* applied to the truss joint in the direction of Δ

N = internal axial force in a member caused by *both* force **P** and the actual loads on the truss

L = length of a member

A = cross-sectional area of a member

E = modulus of elasticity of the material

By comparison, Eq. 14–46 is similar to that used for the method of virtual forces, Eq. 14–37 ($1 \cdot \Delta = \Sigma nNL/AE$), except that n is replaced by $\partial N/\partial P$. These terms, n and $\partial N/\partial P$, are the *same*, since they represent the change of the member's axial force with respect to the load P or, in other words, the axial force per unit load.

14

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement of any joint on a truss using Castigliano's second theorem.

External Force P.

- Place a force **P** on the truss at the joint where the displacement is to be determined. This force is assumed to have a *variable magnitude* and should be directed along the line of action of the displacement.

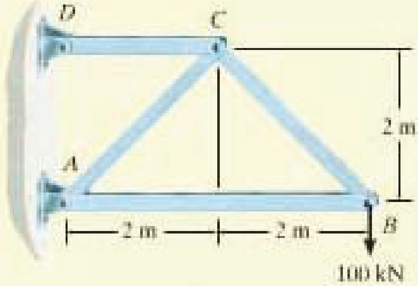
Internal Forces N.

- Determine the force N in each member in terms of both the actual (numerical) loads and the (variable) force P. Assume that tensile forces are positive and compressive forces are negative.
- Find the respective partial derivative $\partial N/\partial P$ for each member.
- *After* N and $\partial N/\partial P$ have been determined, assign P its numerical value if it has actually replaced a real force on the truss. Otherwise, set P equal to zero.

Castigliano's Second Theorem.

- Apply Castigliano's second theorem to determine the desired displacement Δ. It is important to retain the algebraic signs for corresponding values of N and $\partial N/\partial P$ when substituting these terms into the equation.
- If the resultant sum $\Sigma N(\partial N/\partial P)L/AE$ is positive, Δ is in the same direction as **P**. If a negative value results, Δ is opposite to **P**.

EXAMPLE 14.15



(a)

Fig. 14–39

Determine the vertical displacement of joint C of the steel truss shown in Fig. 14–39*a*. The cross-sectional area of each member is $A = 400\text{ mm}^2$, and $E_{st} = 200\text{ GPa}$.

SOLUTION

External Force P. A vertical force **P** is applied to the truss at joint C, since this is where the vertical displacement is to be determined, Fig. 14–39*b*.

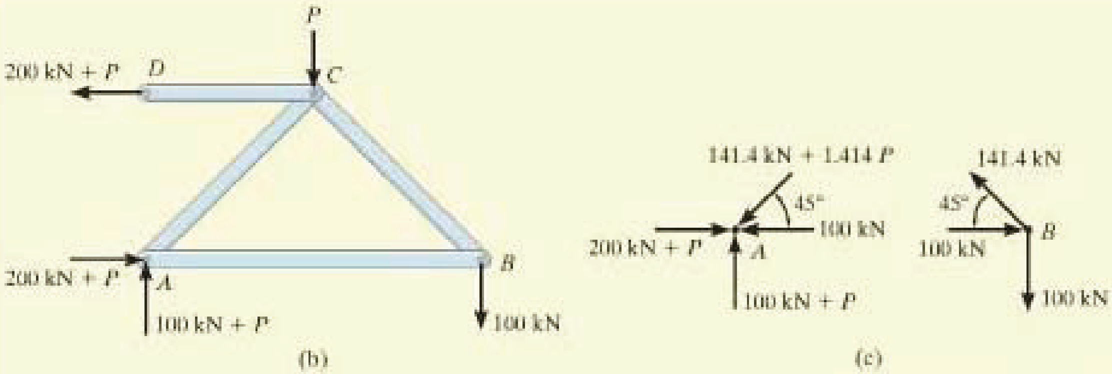


Fig. 14–39 (cont.)

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Internal Forces *N*. The reactions at the truss supports A and D are calculated and the results are shown in Fig. 14–39b. Using the method of joints, the N forces in each member are determined, Fig. 14–39c.* For convenience, these results along with their partial derivatives $\partial N/\partial P$ are listed in tabular form. Note that since **P** does not actually exist as a real load on the truss, we require $P = 0$.

Member	N	$\frac{\partial N}{\partial P}$	$N(P = 0)$	L	$N\left(\frac{\partial N}{\partial P}\right)L$
AB	-100	0	-100	4	0
BC	141.4	0	141.4	2.828	0
AC	$-(141.4 + 1.414P)$	-1.414	-141.4	2.828	565.7
CD	$200 + P$	1	200	2	400
					Σ 965.7 kN · m

Castigliano's Second Theorem. Applying Eq. 14–46, we have

$$\Delta_{C_v} = \Sigma N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{965.7 \text{ kN}\cdot\text{m}}{AE}$$

Substituting the numerical values for A and E, we get

$$\Delta_{C_v} = \frac{965.7 \text{ kN}\cdot\text{m}}{[400(10^{-6})\text{ m}^2]\,200(10^6)\text{ kN/m}^2}$$
$$= 0.01207 \text{ m} = 12.1 \text{ mm} \qquad \textit{Ans.}$$

This solution should be compared with that of Example 14.11, using the virtual-work method.

*It may be more convenient to analyze the truss with just the 100-kN load on it, then analyze the truss with the **P** load on it. The results can then be summed algebraically to give the N forces.

PROBLEMS

14–123. Solve Prob. 14–72 using Castigliano's theorem.

***14–124.** Solve Prob. 14–73 using Castigliano's theorem.

•14–125. Solve Prob. 14–75 using Castigliano's theorem.

14–126. Solve Prob. 14–76 using Castigliano's theorem.

14–127. Solve Prob. 14–77 using Castigliano's theorem.

***14–128.** Solve Prob. 14–78 using Castigliano's theorem.

•14–129. Solve Prob. 14–79 using Castigliano's theorem.

14–130. Solve Prob. 14–80 using Castigliano's theorem.

14–131. Solve Prob. 14–81 using Castigliano's theorem.

***14–132.** Solve Prob. 14–82 using Castigliano's theorem.

•14–133. Solve Prob. 14–83 using Castigliano's theorem.

14–134. Solve Prob. 14–84 using Castigliano's theorem.

*14.10 Castigliano's Theorem Applied to Beams

The internal strain energy for a beam is caused by both bending and shear. However, as pointed out in Example 14.7, if the beam is long and slender, the strain energy due to shear can be neglected compared with that of bending. Assuming this to be the case, the internal strain energy for a beam is given by $U_i = \int M^2\,dx/2EI$, Eq. 14–17. Omitting the subscript i, Castigliano's second theorem, $\Delta_i = \partial U_i/\partial P_i$, becomes

$$\Delta = \frac{\partial}{\partial P}\int_0^L \frac{M^2\,dx}{2EI}$$

Rather than squaring the expression for internal moment, integrating, and then taking the partial derivative, it is generally easier to differentiate prior to integration. Provided E and I are constant, we have

$$\boxed{\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}} \qquad (14\text{–}47)$$

Here

Δ = displacement of the point caused by the real loads acting on the beam

P = an external force of *variable magnitude* applied to the beam at the point and in the direction of Δ

M = internal moment in the beam, expressed as a function of x and caused by both the force P and the actual loads on the beam

E = modulus of elasticity of the material

I = moment of inertia of cross-sectional area about the neutral axis

If the slope of the tangent θ at a point on the elastic curve is to be determined, the partial derivative of the internal moment M with respect to an *external couple moment* M' acting at the point must be found. For this case,

$$\boxed{\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}} \tag{14-48}$$

The above equations are similar to those used for the method of virtual forces, Eqs. 14–40 and 14–41, except m and m_θ replace $\partial M/\partial P$ and $\partial M/\partial M'$, respectively.

In addition, if axial load, shear, and torsion cause significant strain energy within the member, then the effects of all these loadings should be included when applying Castigliano's theorem. To do this we must use the strain-energy functions developed in Sec. 14.2, along with their associated partial derivatives. The result is

$$\Delta = \Sigma N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} + \int_0^L f_s V\left(\frac{\partial V}{\partial P}\right)\frac{dx}{GA} + \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} + \int_0^L T\left(\frac{\partial T}{\partial P}\right)\frac{dx}{GJ} \tag{14-49}$$

The method of applying this general formulation is similar to that used to apply Eqs. 14–47 and 14–48.

14

Procedure for Analysis

The following procedure provides a method that may be used to apply Castigliano's second theorem.

External Force P or Couple Moment M'.

- Place a force **P** on the beam at the point and directed along the line of action of the desired displacement.
- If the slope of the tangent is to be determined at the point, place a couple moment **M'** at the point.
- Assume that both **P** and **M'** have a variable magnitude.

Internal Moments M.

- Establish appropriate x coordinates that are valid within regions of the beam where there is no discontinuity of force, distributed load, or couple moment.
- Determine the internal moments M as a function of x, the actual (numerical) loads, and P or M', and then find the partial derivatives $\partial M/\partial P$ or $\partial M/\partial M'$ for each coordinate x.
- *After* M and $\partial M/\partial P$ or $\partial M/\partial M'$ have been determined, assign P or M' its numerical value if it has actually replaced a real force or couple moment. Otherwise, set P or M' equal to zero.

Castigliano's Second Theorem.

- Apply Eq. 14–47 or 14–48 to determine the desired displacement Δ or θ. It is important to retain the algebraic signs for corresponding values of M and $\partial M/\partial P$ or $\partial M/\partial M'$.
- If the resultant sum of all the definite integrals is positive, Δ or θ is in the same direction as **P** or **M'**. If a negative value results, Δ or θ is opposite to **P** or **M'**.

EXAMPLE 14.16

Determine the displacement of point B on the beam shown in Fig. 14–40a. EI is constant.

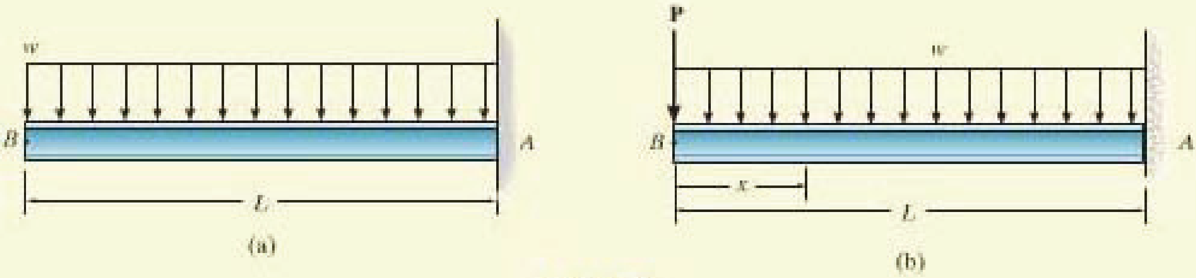
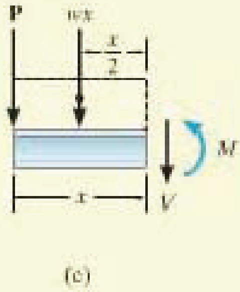


Fig. 14–40

SOLUTION



External Force P. A vertical force **P** is placed on the beam at B as shown in Fig. 14–40b.

Internal Moments M. A single x coordinate is needed for the solution, since there are no discontinuities of loading between A and B. Using the method of sections, Fig. 14–40c, the internal moment and its partial derivative are determined as follows:

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad M + wx\left(\frac{x}{2}\right) + P(x) = 0$$

$$M = -\frac{wx^2}{2} - Px$$

$$\frac{\partial M}{\partial P} = -x$$

Setting $P = 0$ gives

$$M = \frac{-wx^2}{2} \quad \text{and} \quad \frac{\partial M}{\partial P} = -x$$

Castigliano's Second Theorem. Applying Eq. 14–47, we have

$$\Delta_B = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} = \int_0^L \frac{(-wx^2/2)(-x)\,dx}{EI}$$

$$= \frac{wL^4}{8EI} \qquad \textit{Ans.}$$

The similarity between this solution and that of the virtual-work method, Example 14.13, should be noted.

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EXAMPLE 14.17

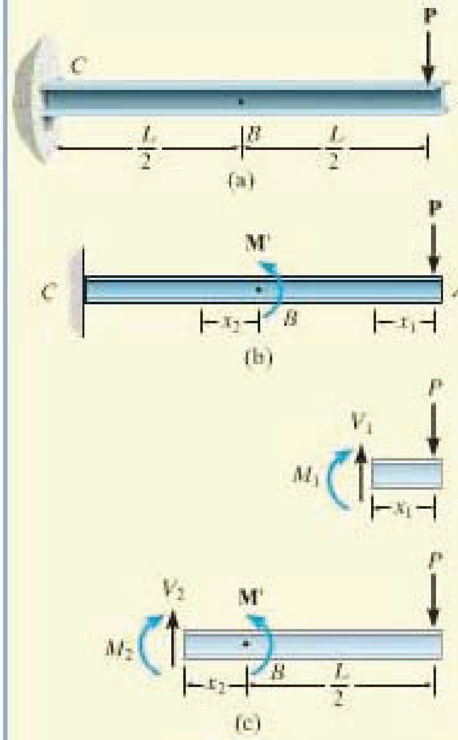


Fig. 14-41

Determine the slope at point B of the beam shown in Fig. 14–41a. EI is constant.

SOLUTION

External Couple Moment M'. Since the slope at point B is to be determined, an external couple moment $\mathbf{M}'$ is placed on the beam at this point, Fig. 14–41b.

Internal Moments M. Two coordinates, x_1 and x_2, must be used to completely describe the internal moments within the beam since there is a discontinuity, $\mathbf{M}'$, at B. As shown in Fig. 14–41b, x_1 ranges from A to B and x_2 ranges from B to C. Using the method of sections, Fig. 14–41c, the internal moments and the partial derivatives for x_1 and x_2 are determined as follows:

$$\circlearrowleft+\Sigma M_{NA} = 0; \qquad M_1 = -Px_1, \qquad \frac{\partial M_1}{\partial M'} = 0$$

$$\circlearrowleft+\Sigma M_{NA} = 0; \qquad M_2 = M' - P\left(\frac{L}{2} + x_2\right), \qquad \frac{\partial M_2}{\partial M'} = 1$$

Castigliano's Second Theorem. Setting $M' = 0$ and applying Eq. 14–48, we have

$$\theta_B = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI} = \int_0^{L/2}\frac{(-Px_1)(0)\,dx_1}{EI} + \int_0^{L/2}\frac{-P[(L/2) + x_2](1)\,dx_2}{EI} = -\frac{3PL^2}{8EI} \qquad \textit{Ans.}$$

Note the similarity between this solution and that of Example 14.14.

PROBLEMS

14–135. Solve Prob. 14–87 using Castigliano's theorem.

***14–136.** Solve Prob. 14–88 using Castigliano's theorem.

•14–137. Solve Prob. 14–90 using Castigliano's theorem.

14–138. Solve Prob. 14–92 using Castigliano's theorem.

14–139. Solve Prob. 14–93 using Castigliano's theorem.

***14–140.** Solve Prob. 14–96 using Castigliano's theorem.

14–141. Solve Prob. 14–97 using Castigliano's theorem.

14–142. Solve Prob. 14–98 using Castigliano's theorem.

14–143. Solve Prob. 14–112 using Castigliano's theorem.

***14–144.** Solve Prob. 14–114 using Castigliano's theorem.

•14–145. Solve Prob. 14–121 using Castigliano's theorem.

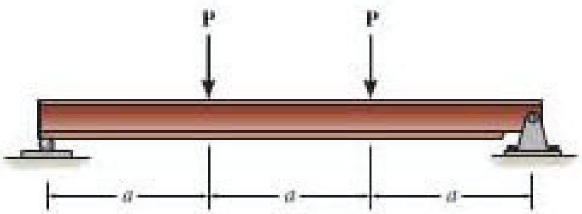
CHAPTER REVIEW

When a force (couple moment) acts on a deformable body it will do external work when it displaces (rotates). The internal stresses produced in the body also undergo displacement, thereby creating elastic strain energy that is stored in the material. The conservation of energy states that the external work done by the loading is equal to the internal elastic strain energy produced by the stresses in the body.	$U_e = U_i$
The conservation of energy can be used to solve problems involving elastic impact, which assumes the moving body is rigid and all the strain energy is stored in the stationary body. This leads to use of an impact factor n, which is a ratio of the dynamic load to the static load. It is used to determine the maximum stress and displacement of the body at the point of impact.	$n = 1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}$ $\sigma_{\max} = n\sigma_{st}$ $\Delta_{\max} = n\Delta_{st}$
The principle of virtual work can be used to determine the displacement of a joint on a truss or the slope and the displacement of points on a beam. It requires placing an external virtual unit force (virtual unit couple moment) at the point where the displacement (rotation) is to be determined. The external virtual work that is produced by the external loading is then equated to the internal virtual strain energy in the structure.	$1 \cdot \Delta = \sum \frac{nNL}{AE}$ $1 \cdot \Delta = \int_0^L \frac{mM}{EI}dx$ $1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI}dx$
Castigliano's second theorem can also be used to determine the displacement of a joint on a truss or the slope and the displacement at a point on a beam. Here a variable force P (couple moment M') is placed at the point where the displacement (slope) is to be determined. The internal loading is then determined as a function of P (M') and its partial derivative with respect to P (M') is determined. Castigliano's second theorem is then applied to obtain the desired displacement (rotation).	$\Delta = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE}$ $\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$ $\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$

14

REVIEW PROBLEMS

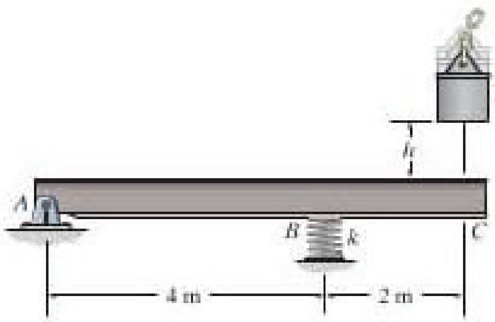
14–146. Determine the bending strain energy in the beam due to the loading shown. EI is constant.



Prob. 14–146

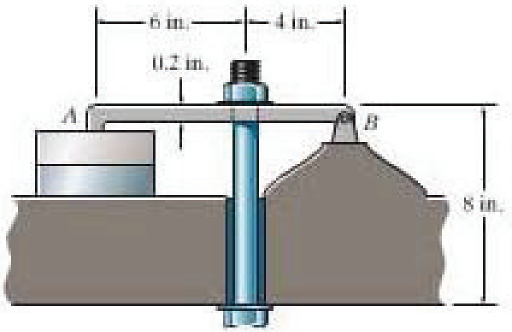
14–147. The 200-kg block D is dropped from a height $h = 1$ m onto end C of the A-36 steel W200 × 36 overhang beam. If the spring at B has a stiffness $k = 200$ kN/m, determine the maximum bending stress developed in the beam.

***14–148.** Determine the maximum height h from which the 200-kg block D can be dropped without causing the A-36 steel W200 × 36 overhang beam to yield. The spring at B has a stiffness $k = 200$ kN/m.



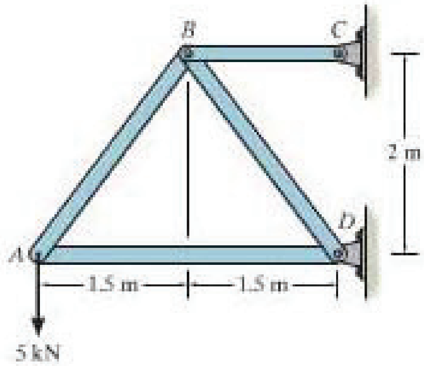
Probs. 14–147/148

•**14–149.** The L2 steel bolt has a diameter of 0.25 in., and the link AB has a rectangular cross section that is 0.5 in. wide by 0.2 in. thick. Determine the strain energy in the link AB due to bending, and in the bolt due to axial force. The bolt is tightened so that it has a tension of 350 lb. Neglect the hole in the link.



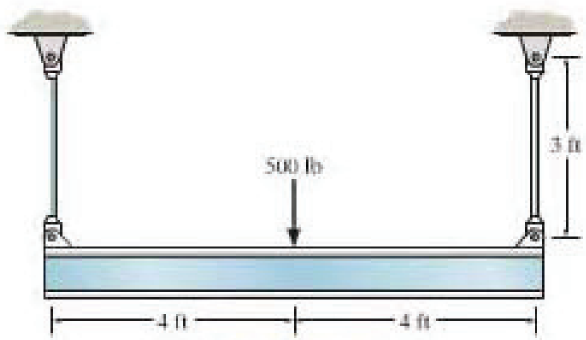
Prob. 14–149

14–150. Determine the vertical displacement of joint A. Each bar is made of A-36 steel and has a cross-sectional area of 600 mm². Use the conservation of energy.



Prob. 14–150

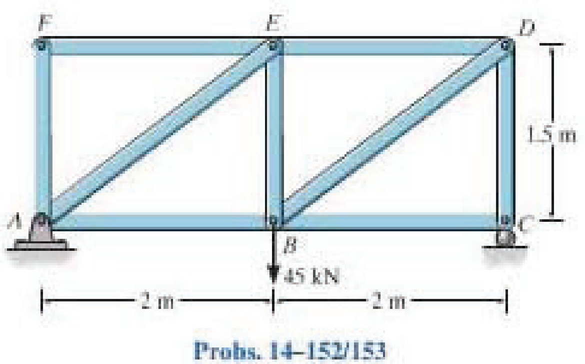
14–151. Determine the total strain energy in the A-36 steel assembly. Consider the axial strain energy in the two 0.5-in.-diameter rods and the bending strain energy in the beam for which $I = 43.4 \text{ in}^4$.



Prob. 14–151

*14–152. Determine the vertical displacement of joint E. For each member $A = 400 \text{ mm}^2$, $E = 200$ GPa. Use the method of virtual work.

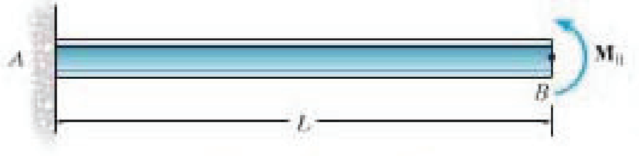
•14–153. Solve Prob. 14–152 using Castigliano's theorem.



Probs. 14–152/153

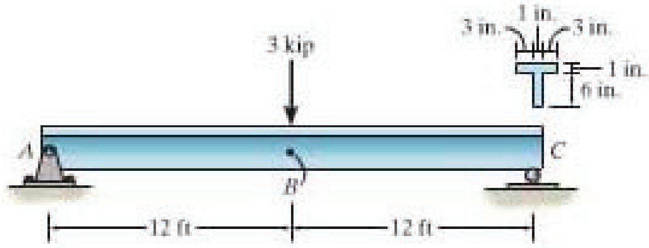
14–154. The cantilevered beam is subjected to a couple moment $\mathbf{M}_0$ applied at its end. Determine the slope of the beam at B. EI is constant. Use the method of virtual work.

14–155. Solve Prob. 14–154 using Castigliano's theorem.



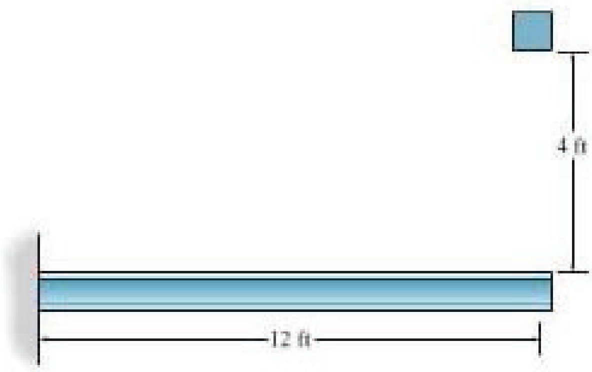
Probs. 14–154/155

*14–156. Determine the displacement of point B on the aluminum beam. $E_{al} = 10.6(10^3)$ ksi. Use the conservation of energy.



Prob. 14–156

14–157. A 20-lb weight is dropped from a height of 4 ft onto the end of a cantilevered A-36 steel beam. If the beam is a W12 × 50, determine the maximum stress developed in the beam.



Prob. 14–157