Art of Problem Solving
2017 Iran Team Selection Test

Iran Team Selection Test 2017

Test 1 Day 1

1 Let a,b,c,d be positive real numbers with $a+b+c+d=2$. Prove the inequality:

$$\frac{(a+c)^2}{ad+bc}+\frac{(b+d)^2}{ac+bd}+4\geq 4\left(\frac{a+b+1}{c+d+1}+\frac{c+d+1}{a+b+1}\right)$$

Proposed by Mohammad Jafari

2 In the country of *Sugarland*, there are 13 students in the IMO team selection camp. 6 team selection tests were taken and the results have came out. Assume that no students have the same score on the same test.To select the IMO team, the national committee of math Olympiad have decided to choose a permutation of these 6 tests and starting from the first test, the person with the highest score between the remaining students will become a member of the team.The committee is having a session to choose the permutation.
Is it possible that all 13 students have a chance of being a team member?

Proposed by Morteza Saghafian

3 In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A,I_a and intersects the extensions of sides AB,AC (extended from B,C) at X,Y respectively. Let S,T be points on segments I_aB,I_aC respectively such that $\angle AXI_a=\angle BTI_a$ and $\angle AYI_a=\angle CSI_a$.Lines BT,CS intersect at K. Lines KI_a,TS intersect at Z.
Prove that X,Y,Z are collinear.

Proposed by Hooman Fattahi

Test 1 Day 2

4 We arranged all the prime numbers in the ascending order: $p_1=2<p_2<p_3<\cdots$.
Also assume that $n_1<n_2<\cdots$ is a sequence of positive integers that for all $i=1,2,3,\cdots$ the equation $x^{n_i}\equiv 2 \pmod{p_i}$ has a solution for x.
Is there always a number x that satisfies all the equations?

Proposed by Mahyar Sefidgaran , Yahya Motevasel

5 In triangle ABC, arbitrary points P,Q lie on side BC such that $BP=CQ$ and P lies between B,Q.The circumcircle of triangle APQ intersects sides AB,AC at E,F respectively.The point T is the intersection of EP,FQ.Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X,Y respectively.
Prove that the circumcircle of triangle TXY and triangle APQ are tangent to each other.

Proposed by Iman Maghsoudi

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6 In the unit squares of a transparent 1×100 tape, numbers $1,2,\cdots,100$ are written in the ascending order.We fold this tape on it's lines with arbitrary order and arbitrary directions until we reach a 1×1 tape with 100 layers.A permutation of the numbers $1,2,\cdots,100$ can be seen on the tape, from the top to the bottom.
Prove that the number of possible permutations is between 2^{100} and 4^{100}.
(*e.g.* We can produce all permutations of numbers $1,2,3$ with a 1×3 tape)

Proposed by Morteza Saghafian

Test 2 Day 1

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1 $ABCD$ is a trapezoid with $AB \parallel CD$. The diagonals intersect at P. Let ω_1 be a circle passing through B and tangent to AC at A. Let ω_2 be a circle passing through C and tangent to BD at D. ω_3 is the circumcircle of triangle BPC.
Prove that the common chord of circles ω_1,ω_3 and the common chord of circles ω_2, ω_3 intersect each other on AD.

Proposed by Kasra Ahmadi

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2 Find the largest number n that for which there exists n positive integers such that non of them divides another one, but between every three of them, one divides the sum of the other two.

Proposed by Morteza Saghafian

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3 There are 27 cards, each has some amount of (1 or 2 or 3) shapes (a circle, a square or a triangle) with some color (white, grey or black) on them. We call a triple of cards a *match* such that all of them have the same amount of shapes or distinct amount of shapes, have the same shape or distinct shapes and have the same color or distinct colors. For instance, three cards shown in the figure are a *match* be cause they have distinct amount of shapes, distinct shapes but the same color of shapes.

What is the maximum number of cards that we can choose such that non of the triples make a *match*?

Proposed by Amin Bahjati

Test 2 Day 2

4 A $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ where $h_i(x_1, x_2, \cdots, x_n)$ are n variable polynomials with real coefficients is called *good* if the following condition holds:
For any n functions $f_1, f_2, \cdots, f_n : \mathbb{R} \to \mathbb{R}$ if for all $1 \le i \le n+1$, $P_i(x) = h_i(f_1(x), f_2(x), \cdots, f_n(x))$ is a polynomial with variable x, then $f_1(x), f_2(x), \cdots, f_n(x)$ are polynomials.

$a)$ Prove that for all positive integers n, there exists a *good* $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ such that the degree of all h_i is more than 1.

$b)$ Prove that there doesn't exist any integer $n > 1$ that for which there is a *good* $n+1$-tuple $(h_1, h_2, \cdots, h_{n+1})$ such that all h_i are symmetric polynomials.

Proposed by Alireza Shavali

5 k, n are two arbitrary positive integers. Prove that there exists at least $(k-1)(n-k+1)$ positive integers that can be produced by n number of k's and using only $+, -, \times, \div$ operations and adding parentheses between them, but cannot be produced using $n-1$ number of k's.

Proposed by Aryan Tajmir

6 Let $k > 1$ be an integer. The sequence $a_1, a_2, \cdots$ is defined as: $a_1 = 1, a_2 = k$ and for all $n > 1$ we have: $a_{n+1} - (k+1)a_n + a_{n-1} = 0$
Find all positive integers n such that a_n is a power of k.

Proposed by Amirhossein Pooya

Test 3 Day 1

1 Let $n > 1$ be an integer. Prove that there exists an integer $n-1 \ge m \ge \left\lfloor \frac{n}{2} \right\rfloor$ such that the following equation has integer solutions with $a_m > 0:$

$$\frac{a_m}{m+1} + \frac{a_{m+1}}{m+2} + \cdots + \frac{a_{n-1}}{n} = \frac{1}{\operatorname{lcm}(1, 2, \cdots, n)}$$

Proposed by Navid Safaei

2 Let P be a point in the interior of quadrilateral $ABCD$ such that:

$$\angle BPC = 2\angle BAC \ \ , \ \ \angle PCA = \angle PAD \ \ , \ \ \angle PDA = \angle PAC$$

Prove that:

$$\angle PBD = |\angle BCA - \angle PCA|$$

Proposed by Ali Zamani

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3 Find all functions $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions for all positive real numbers x, y, z :

$$f\left(f(x,y),z\right) = x^2y^2f(x,z)$$

$$f\left(x, 1+f(x,y)\right) \geq x^2 + xyf(x,x)$$

Proposed by Mojtaba Zare, Ali Daei Nabi

Test 3 Day 2

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4 There are 6 points on the plane such that no three of them are collinear. It's known that between every 4 points of them, there exists a point that it's power with respect to the circle passing through the other three points is a constant value k.(Power of a point in the interior of a circle has a negative value.)
Prove that $k=0$ and all 6 points lie on a circle.

Proposed by Morteza Saghafian

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5 Let $\{c_i\}_{i=0}^{\infty}$ be a sequence of non-negative real numbers with $c_{2017} > 0$. A sequence of polynomials if defined as:

$$P_{-1}(x) = 0 \ , \ P_0(x) = 1 \ , \ P_{n+1}(x) = xP_n(x) + c_nP_{n-1}(x)$$

Prove that there doesn't exist any integer $n > 2017$ and some real number c such that:

$$P_{2n}(x) = P_n(x^2+c)$$

Proposed by Navid Safaei

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6 In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E, F lie on AB, AC respectively such that $BE = PC$, $CF = PB$. Let K be the intersection point of AP, OH. Prove that $\angle EKF = 90^\circ$

Proposed by Iman Maghsoudi