پاسخ تمرین سری ۵

سوال اول)

a) $(\vec{A}\cdot\vec{\nabla})\varphi = \left(3xyz^2\frac{\partial}{\partial x} + 2xy^3\frac{\partial}{\partial y} - x^2yz\frac{\partial}{\partial z}\right)(2x^2yz^3)$

$= 3xyz^2(4xyz^3) + 2xy^3(2x^2z^3) - x^2yz(6x^2yz^2)$

$= 12x^2y^2z^5 + 4x^3y^3z^3 - 6x^4y^2z^4$

b) $\vec{A}\cdot\vec{\nabla}\varphi = (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k})\cdot\left(\frac{\partial}{\partial x}2x^2yz^3\hat{i} + \frac{\partial}{\partial y}2x^2yz^3\hat{j} + \frac{\partial}{\partial z}2x^2yz^3\hat{k}\right)$

$= (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k})\cdot(4xyz^3\hat{i} + 2x^2z^3\hat{j} + 6x^2yz^2\hat{k})$

$= 12x^2y^2z^5 + 4x^3y^3z^3 - 6x^4y^2z^3$

c) $(\vec{B}\cdot\vec{\nabla})\vec{A} = \left(x^2\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}\right)A = x^2\frac{\partial A}{\partial x} + yz\frac{\partial A}{\partial y} - xy\frac{\partial A}{\partial z}$

$= x^2(3yz^2\hat{i} + 2y^3\hat{j} - 2xyz\hat{k}) + yz(3xz^2\hat{i} + 6xy^2\hat{j} - x^2z\hat{k})$

$- xy(6xyz\hat{i} - x^2y\hat{k})$

$= (3x^2yz^2 + 3xyz^3 - 6x^2y^2z)\hat{i} + (2x^2y^3 + 6xy^3z)\hat{j} + (-2x^3yz - x^2yz^2 + x^3y^2)\hat{k}$

d) $\vec{\nabla}\cdot(\varphi\vec{A}) = \vec{\nabla}\cdot(6x^3y^2z^5\hat{i} + 4x^3y^4z^3\hat{j} - 2x^4y^2z^4\hat{k})$

$= 18x^2y^2z^5 + 16x^3y^3z^3 - 8x^4y^2z^3$

e) $\vec{\nabla}.(\vec{\nabla}\varphi) = \vec{\nabla}.(۴xyz^{۳}\hat{i} + ۲x^{۲}z^{۳}\hat{j} + ۶x^{۲}yz^{۲}\hat{k}) = ۴yz^{۳} + ۱۲x^{۲}yz$

f) $$\nabla \times B = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^{۲} & yz & -xy \end{vmatrix} = \left(\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(yz)\right)\hat{i} - \left(\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial z}(x^{۲})\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(x^{۲})\right)\hat{k}$$

$$= (-x-y)\hat{i} + y\hat{j}$$

سوال دوم) الف)

$$\varphi = axy^{۲} + byz + cz^{۲}x^{۲}$$

$$\vec{\nabla}\varphi = (ay^{۲} + ۲cz^{۲}x)\hat{i} + (۲axy + bz)\hat{j} + (by + ۲czx^{۲})\hat{k} \xrightarrow{(۱,۲,-۱)}$$

$$(۴a + ۲c)\hat{i} + (۴a - b)\hat{j} + (۲b - ۲c)\hat{k}$$

موازی با Z

$$\begin{cases} ۴a + ۲c = ۰ & \text{I} \\ ۴a - b = ۰ & \text{II} \end{cases}$$

$$|\vec{\nabla}\varphi| = ۶۴ = \sqrt{(۲b - ۲c)^{۲}} \Rightarrow ۲b - ۲c = ۶۴ \Rightarrow b - c = ۳۲ \quad \text{III}$$

$$\xrightarrow{\text{II},\text{III}} ۴a - c = ۳۲ \Rightarrow c = ۴a - ۳۲$$

$$\xrightarrow{\text{I}} ۴a + ۸a - ۶۴ = ۰$$

$$۱۲a = ۶۴$$

$$a = \frac{۱۶}{۳}$$

$$\Rightarrow b = \frac{۶۴}{۳}$$

$$c = -\frac{۳۲}{۳}$$

ب) می‌دانیم: اگر تعداد خطوطی که از محدوده خارج می‌شوند بیشتر از ورودی باشد: دیورژانس ⊕

" " " " " " " " " کمتر " " " : دیورژانس ⊖

" " " " " " " " " برابر با " " " : دیورژانس صفر

a) منفی　　　b) مثبت

c) صفر　　　d) منفی

سوال سوم) اول بدست آوریم که $\vec{\nabla} r^n$ چیست:

با فرض $n=1$: $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\Rightarrow \nabla r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{\vec{r}}{r} = \hat{r} \quad *$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial r^n}{\partial x} &= n r^{n-1}\frac{\partial r}{\partial x} \\ \frac{\partial r^n}{\partial y} &= n r^{n-1}\frac{\partial r}{\partial y} \\ \frac{\partial r^n}{\partial z} &= n r^{n-1}\frac{\partial r}{\partial z} \end{aligned}\right\} \Rightarrow \nabla r^n = n r^{n-1}\vec{\nabla} r \overset{*}{=} \boxed{n r^{n-1}\hat{r}}$$

$$\Rightarrow \nabla\cdot(n r^{n-1}\hat{r}) = n\,\nabla\cdot(r^{n-1}\hat{r}) = n\,\nabla\cdot(r^{n-2}(r\hat{r})) = n\,\nabla\cdot(r^{n-2}\vec{r})$$

$$= n\Big(r^{n-2}\underbrace{(\nabla\cdot\vec{r})}_{3} + \vec{r}\cdot(\nabla r^{n-2})\Big) \overset{**}{=} n\left(3r^{n-2} + (n-2)r^{n-2}\right) =$$

$$\boxed{\text{داریم}: (\nabla r^{n-2}) = (n-2)\,r^{n-3}\,\underbrace{\nabla r}_{\hat{r}} = (n-2)\,r^{n-3}\hat{r} = (n-2)\,r^{n-2}} \; **$$

$$n r^{n-2}(3 + n - 2) = n(n+1)r^{n-2}$$

سوال چهارم) الف)

$$\vec{E} = -\vec{\nabla}V = -\frac{V_0}{a}\Big[\Big(yz\, e^{-(x+y+z)/a} - xyz\left(\frac{1}{a}\right)e^{-(x+y+z)/a}\Big)\hat{i}$$

$$+\Big(xz\, e^{-(x+y+z)/a} - xyz\left(\frac{1}{a}\right)e^{-(x+y+z)/a}\Big)\hat{j}$$

$$+\Big(xy\, e^{-(x+y+z)/a} - xyz\left(\frac{1}{a}\right)e^{-(x+y+z)/a}\Big)\hat{k}\Big]$$

$$\Rightarrow \vec{E} = -\frac{V_0}{a}\, e^{-(x+y+z)/a}\Big[yz\left(1-\frac{x}{a}\right)\hat{i} + xz\left(1-\frac{y}{a}\right)\hat{j} + xy\left(1-\frac{z}{a}\right)\hat{k}\Big]$$

ب) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$$\frac{\partial E_x}{\partial x} = -\frac{V_0}{a}\Big[\frac{1}{a}\, e^{-(x+y+z)/a}\left(yz\left(1-\frac{x}{a}\right)\right) - \frac{yz}{a}\, e^{-(x+y+z)/a}\Big]$$

$$= +\frac{V_0}{a^2}\, e^{-(x+y+z)/a}\quad yz\left(2-\frac{x}{a}\right)$$

و به همین ترتیب $\frac{\partial E_y}{\partial y}$, $\frac{\partial E_z}{\partial z}$ رو بدست می‌آوریم.

$$\Rightarrow \rho = \frac{\varepsilon_0 V_0}{a^2}\, e^{-(x+y+z)/a}\Big(yz\left(2-\frac{x}{a}\right) + xz\left(2-\frac{y}{a}\right) + xy\left(2-\frac{z}{a}\right)\Big)$$

$$= \frac{\varepsilon_0 V_0}{a^2}\, e^{-(x+y+z)/a}\Big(2yz + 2xz + 2xy - 3\left(\frac{xyz}{a}\right)\Big)$$

سوال پنجم) برای این کار باید r' رو در دستگاه مختصات جدید بنویسیم که نتیجه r' را برحسب $\hat n$، $\vec r$، φ به دست آورده‌ایم و می‌توانیم توصیف کنیم.

① بردار $\vec r$ را به دو قسمت می‌نویسیم که یکی هم جهت با $\hat n$ و دیگری عمود بر آن است.

هم جهت: $(\vec r\cdot\hat n)\hat n$ ☑

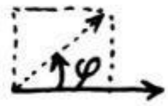
عمود: $(\vec r-(\vec r\cdot\hat n)\hat n)$

با دوران حول $\hat n$، دوران نمی‌کند پس حتما برای $\vec r'$ داریم: $\vec r' = (\vec r\cdot\hat n)\hat n + \,?$

تحت دوران φ داریم:

(۱) هم جهت با $(\vec r-(\vec r\cdot\hat n)\hat n) = (\vec r-(\vec r\cdot\hat n)\hat n)\cos\varphi$ ☑

(۲) عمود بر $(\vec r-(\vec r\cdot\hat n)\hat n) = (\vec r-(\vec r\cdot\hat n)\hat n)\sin\varphi$



$$(\vec r-\underbrace{(\vec r\cdot\hat n)\hat n)\sin\varphi \times \hat n}_{0} = 0$$

- چرا؟ چون با دوران، جهت و اندازه r نباید تغییر کند پس مؤلفه (۲) باید در راستای $(\vec r-(\vec r\cdot\hat n)\hat n)$ باشد!

پس مؤلفه‌ی بعدی $(\vec r\times\hat n)\sin\varphi$ ☑

$$\Rightarrow r' = \underline{(\vec r\cdot\hat n)\hat n} + \underbrace{(\vec r-(\vec r\cdot\hat n)\hat n)\cos\varphi}_{\vec r\cos\varphi - \underline{(\vec r\cdot\hat n)\hat n\cos\varphi}} + (\vec r\times\hat n)\sin\varphi$$

$$r' = \vec r\cos\varphi + (\vec r\cdot\hat n)\hat n(1-\cos\varphi) + (\vec r\times\hat n)\sin\varphi$$

ب) $\hat{n} = \hat{e}_z = \hat{k}$

$$\vec{r} \times \hat{n} = \begin{vmatrix} i & j & k \\ x & y & z \\ 0 & 0 & 1 \end{vmatrix} = y\hat{i} - x\hat{j}$$

$$\vec{r} \cdot \hat{n} = z$$

$$\vec{r}' = (x\hat{i} + y\hat{j} + z\hat{k})\cos\varphi + z\hat{k} - z\cos\varphi\,\hat{k} + (y\hat{i} - x\hat{j})\sin\varphi$$

$$= (x\cos\varphi + y\sin\varphi)\hat{i} + (y\cos\varphi - x\sin\varphi)\hat{j} + \underbrace{(\cancel{z\cos\varphi} + z - \cancel{z\cos\varphi})\hat{k}}_{z\hat{k}}$$

$$\Rightarrow \vec{r}' = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

ج) $\vec{r}' = \vec{r}\cos\varphi + (\vec{r} \times \vec{n})\sin\varphi + (\vec{r} \cdot \hat{n})\hat{n}\,(1 - \cos\varphi)$

0 0

- دلیل صفر شدن ضربشون چیه؟ برهم عمودند!

$$\Rightarrow r'^2 = r^2\cos^2\varphi + \underbrace{(\vec{r} \times \hat{n}) \cdot (\vec{r} \times \hat{n})}_{r^2 n^2 - (\vec{r} \cdot \hat{n})^2}\sin^2\varphi + (\vec{r} \cdot \hat{n})^2(1 - \cos\varphi)^2$$

$$+ 2(\vec{r} \cdot \hat{n}) \cdot (\vec{r} \cdot \hat{n})(1 - \cos\varphi)\cos\varphi$$

$$= \overbrace{r^2\cos^2\varphi + r^2\sin^2\varphi}^{r^2} - (\vec{r} \cdot \hat{n})^2\sin^2\varphi + (\vec{r} \cdot \hat{n})^2 + (r \cdot \hat{n})^2\cos^2\varphi - 2(\vec{r} \cdot \hat{n})^2\cos\varphi$$

$$+ 2(\vec{r} \cdot \hat{n})^2\cos\varphi - 2(\vec{r} \cdot \hat{n})^2\cos^2\varphi$$

$$= r^2 + (\vec{r} \cdot \hat{n})^2\left(-\sin^2\varphi + 1 + \cos^2\varphi - \cancel{2\cos\varphi} + \cancel{2\cos\varphi} - 2\cos^2\varphi\right) =$$

$$r^2 + (\vec{r} \cdot \hat{n})\left(\underbrace{-\cancel{(\sin^2\varphi + \cos^2\varphi)} + 1}_{0}\right) = r^2$$

$$\Rightarrow r'^2 = r^2 \quad \checkmark$$

سوال ششم)

$f(x,y)$ $\qquad \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$$S = \begin{bmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} \end{bmatrix}$$

$$S\vec{\nabla} f = \begin{bmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \\ \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} \end{bmatrix}$$

از راهنمایی :

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'}$$

$$\Rightarrow S\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x'} \\ \frac{\partial f}{\partial y'} \end{bmatrix}$$