2009 Durban Invitational World Youth Mathematics Intercity Competition

10th IWYMIC

Solutions of Team Contest

Time limit: 60 minutes   8th July 2009   Durban, South Africa

1. The cards 1 to 15 are arranged in a deck, not in numerical order. The top card is placed on the table and the next card is transferred to the bottom of the deck. Now the new top card is placed on top of the card on the table and the next card is transferred to the bottom of the remaining deck. This process is repeated until all 15 cards are on the table. If the cards on the table are now in their natural order, 1 to 15, from top to bottom, what was the fourth card from the bottom in the original deck?

***Solution* 1**

Let us reconstruct the original deck horizontally. Clearly the top card in the original deck, being the bottom card on the table, must be 15. The third card from the top must be 14. Proceed.

Top	15	“	14	??	13		12		11		10		9		8	Bottom

[10 points]

The eight indicated cards are no longer in the deck, so the card marked “ is at the top of the deck after the eight has been placed on the table. This card goes to the bottom of the deck so ?? is 7. Continuing in this manner, we can fill the rest of the blanks to get

Top	15	4	14	7	13	2	12	6	11	3	10	5	9	1	8	Bottom

[10 points]

so the original deck is

The fourth card from the bottom is 5. [20 points]

***Solution* 2**

The 4th card from the bottom call it card-4, *is the 12th from the top. This card is transferred to the bottom of the original deck at the stage when the original deck has only 9 cards left. There are then 6 cards on the table.* [20 points]

When card-4 *again appears at the top of the original deck it is placed on the table and becomes the 11th card from the bottom, which is the 5th from the top.* [20 points]

2. Find the smallest positive integer with at least one factor ending in each of the digits 0 to 9 i.e. at least one factor ends in 0, at least one factor ends in 1, … , at least one factor ends in 9.

Solution

To get a factor ending in 0, the number must have a 2 and a 5 in its prime factorization. Thus this number also has a factor ending in 2 and in 5. Of course, it has a factor ending in 1, namely 1 itself. The construction may be completed by including 3^3 in the prime factorization, yielding $2\times 3^3\times 5=270$. Among its factors are 3, $2\times 3^3=54$, $2\times 3=6$, $3^3=27$, $2\times 3^2=18$ and $3^2=9$. To show that there is no smaller number, the factor ending in 9 must be 9 or 19 and the factor ending in 7 must be 7 or 17. Since these numbers are relatively prime, the smallest value is $9\times 7\times 2\times 5>270$.

3. Place the digits 1 to 6 in each of the rows and columns as well as the two diagonals such that no digit is repeated in a row, column or diagonal.

2			1		
					4
	2				
				6	
		5			1
3					

Solution

Label the rows 1 to 6 from bottom to top and the columns *a* to *f* from left to right. The squares *f*1 and *f*6 must contain the figures 5 and 6, so that the square *f*4 must contain 3 and the square *f*3 must contain 2. In row 2, the figure 2 must be in *d*2 and the figure 3 must be in *e*2. The figure 3 in column *b* must be in *b*3. Hence the one in row 6 is in *c*6 and the one in row 5 is in *d*5. The figure 2 in the up-diagonal must be in *e*5, Hence the remaining one must be in *c*1. This takes us to the diagram below on the left, where all the figures 2 and 3 are in place.

6	2		3	1		
5				3	2	4
4		2				3
3		3			6	2
2			5	2	3	1
1	3		2			
	a	*b*	*c*	*d*	*e*	*f*

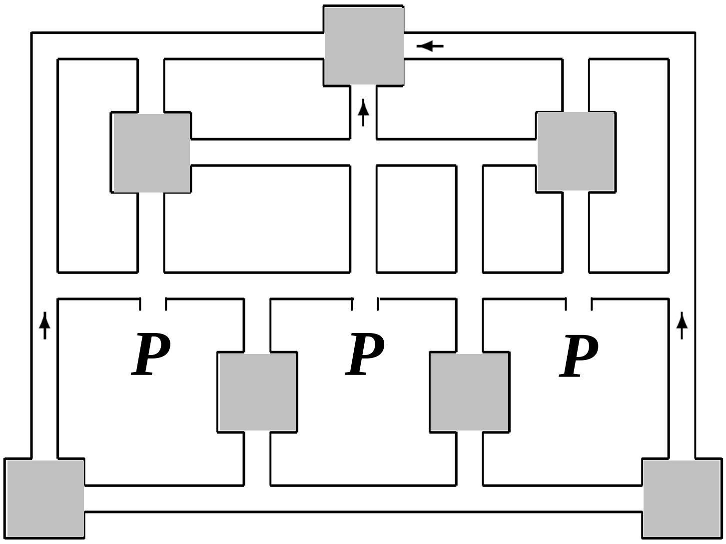
6	2	6	3	1	4	5
5	5	1	6	3	2	4
4	1	2	4	6	5	3
3	4	3	1	5	6	2
2	6	4	5	2	3	1
1	3	5	2	4	1	6
	a	*b*	*c*	*d*	*e*	*F*

We now test whether the figure 6 in column *f* is in *f*1 or *f*6. Putting it in *f*6 leads to an impossible situation. Hence it is in *f*1, and the remaining squares can be filled readily, taking us to the completed diagram above on the right.

Alternate Solution: First place the remaining four 2's in their unique positions. Then do the same for the 1's, the 3's, the 4's, the 6's and the 5's in that order.

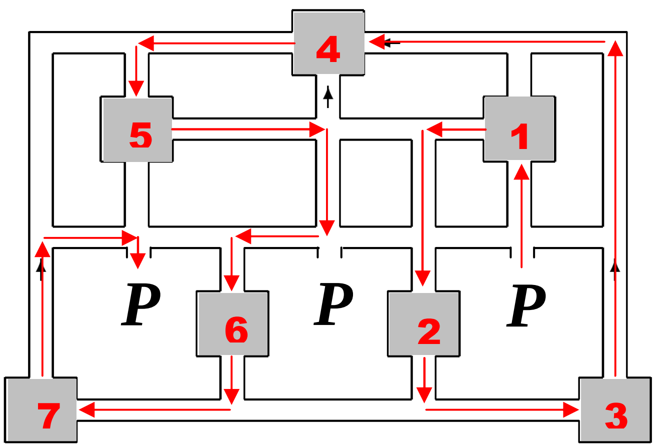
If a digit is repeated in any row, column or diagonal, student gets 0 points. If the square is partially filled in, give 1 point for every correct cell, and -1 for every wrong cell.

4. We have indicated the positions of three parking areas (indicated by the letter *P*) and seven squares (the shaded areas) on the map of this small town centre. Some of the streets only allow one-way traffic. This is shown by arrows which indicate the direction of traffic up to the first side street. Can you find a route that begins at one of the parking areas, passes through all the squares and ends at another parking area? Make sure that you do not visit any point, including intersection points, on your route more than once.



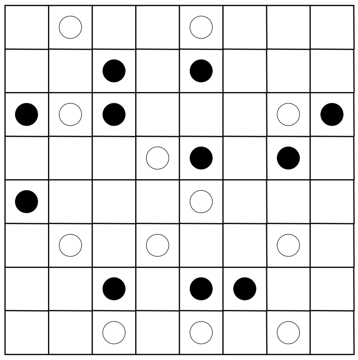
Solution

We may begin from the parking space in the east, visit the shops in numerical order along the unique route indicated by the arrows in the diagram below, and end at the parking space in the west. After visiting shops 4 and 7, the route cannot collide at the northwest corner. Hence shop 4 must be followed by shop 5, and shop 7 should be followed by the west parking space, so that we must exit shop 5 towards the east. This means that we must enter shop 4 from the east, and it must follow shop 3. The rest of the route is easy to determine.



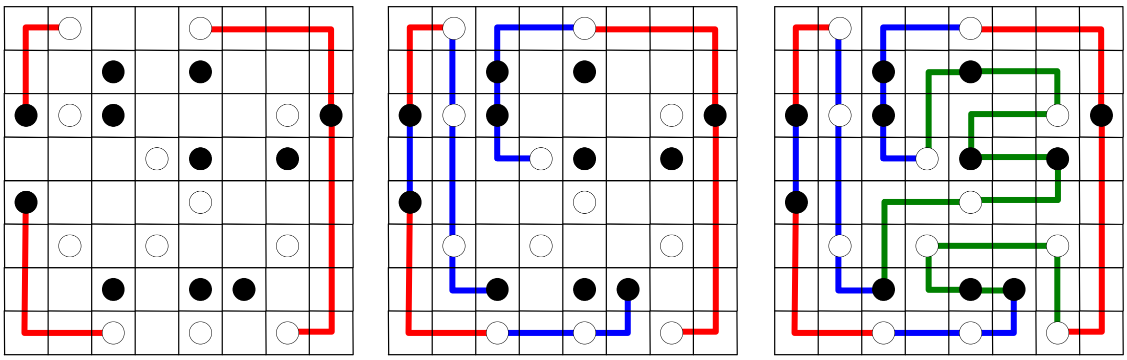
If a point is visited more than once, or a route goes in a wrong direction in one of the one-way streets, student gets 0 points. If the route is partially correct (and none of the preceding apply), give 2 points for every correct segment, and -2 points for every incorrect one.

5. In the diagram below, draw a continuous path that begins and ends at the same place and runs through every square exactly once without crossing itself, so that between two consecutive circles on the path, if those circles are the same colour, then they must be joined by one straight line segment and if they are different colours, then they must be joined by two straight line segments which form a right angle. (You may only move horizontally or vertically.)



Solution

We start from the border of the figure and note that the path must pass through corner square. To avoid forming a closed path without passing through the inner squares, we must look for places where the path either must enter the interior or may do so. Working layer by layer from the outside yields the paths shown in the diagrams below.



STEP 1　　STEP 2　　STEP 3

If any rule is violated (e.g., two consecutive circles of same colour are connected by segments forming a right angle, or circles of different colour are connected by a straight segment, or the path crosses itself), then student gets 0 points. If the path is partially connect (and none of the preceding apply), give 1 point for every correct connection, and -1 point for every incorrect one.

6. Let $a_n = \dfrac{2^n}{2^{2n+1} - 2^{n+1} - 2^n + 1}$ for all positive integers n.

Prove that $a_1 + a_2 + \cdots + a_{2009} < 1$.

Solution

Note that $a_n = \dfrac{2^n}{(2^{n+1}-1)(2^n-1)} = \dfrac{1}{2^n - 1} - \dfrac{1}{2^{n+1}-1}$. [20 points]

Hence

$$a_1 + a_2 + \cdots + a_{2009} = \frac{1}{2-1} - \frac{1}{2^2-1} + \frac{1}{2^2-1} - \frac{1}{2^3-1} + \cdots + \frac{1}{2^{2009}-1} - \frac{1}{2^{2010}-1}$$

$$= 1 - \frac{1}{2^{2010}-1}$$

$$< 1$$

[20 points]

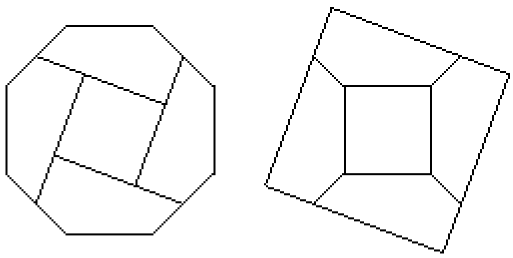
7. Find all the possible ways of splitting the positive integers into cold numbers and hot numbers such that the sum of a hot number and a cold number is hot and their product is cold.

Solution

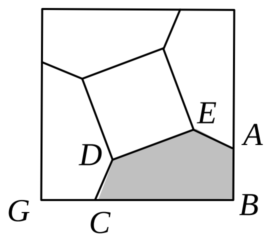
Let m and n be cold and let k be hot. Then $m+k$ is hot so that $(m+k)n$ is cold. On the other hand, kn is cold so that $(m+k)n=mn+kn$ implies mn is cold. Let m be the smallest cold number. Then all the multiply of m are cold. [20 points]

Suppose n is cold and $n=qm+r$ with $0<r<m-1$. If r is hot, then $qm+r$ is hot, a contradiction. [20 points]

8. The diagram below shows how a regular octagon may be cut into a 1×1 square and four congruent pentagons which may be reassembled to form a square. Determine the perimeter of one of those pentagons.



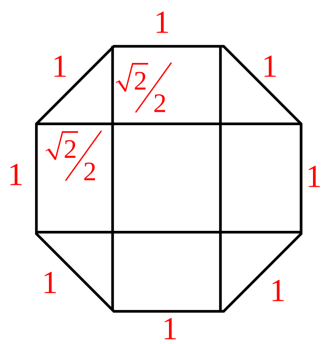
Solution



Let the length of the side of the square be a.

Then $AB + BC = CG + BC = a$ [5 points]

We know that the length of the side of the regular octagon is 1. And we can divide the regular octagon as the right diagram. Hence the area of the regular octagon is



$$1+1\times\frac{\sqrt{2}}{2}\times4+\frac{1}{2}\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}\times4=2+2\sqrt{2}.$$

Thus

$$a^2=2+2\sqrt{2}$$
$$a=\sqrt{2+2\sqrt{2}}$$ [20 points]

i.e. $AB+BC=CG+BC=\sqrt{2+2\sqrt{2}}$ [5 points]

And $AE+DC$ is equal to the side of the regular octagon, i.e. $AE + DC = 1$. [5 points]

Hence $AB+BC+CD+DE+EA=2+\sqrt{2+2\sqrt{2}}$. [5 points]

9. A game of cards involves 4 players. In a contest, the total number of games played is equal to the total number of players entered in the contest. Every two players are together in at least one game. Determine the maximum number of players that can enter the contest.

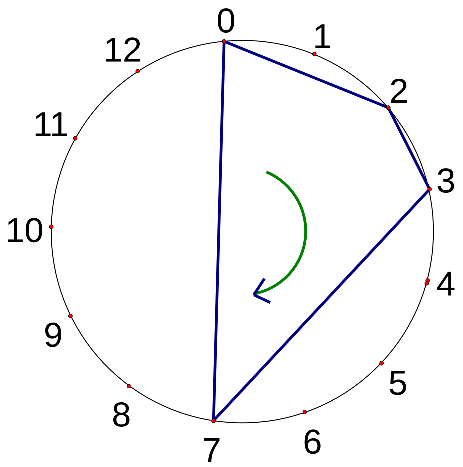
Solution

Let n be the number of players. Each game involves 6 pairs of players. Hence $\binom{n}{2}\le 6n$ which yields $n\le 13$.

We may have n=13. [20 points]

Number the players 0 to 12 and let the 13 games be represented by the quartet of players involved in it:

(0, 2, 3, 7), (1, 3, 4, 8), (2, 4, 5, 9), (3, 5, 6, 10), (4, 6, 7, 11), (5, 7, 8, 12), (6, 8, 9, 0), (7, 9, 10, 1), (8, 10, 11, 2), (9, 11, 12, 3), (10, 12, 0, 4), (11, 0, 1, 5) and (12, 1, 2, 6). [20 points]



10. Which of the numbers 2008, 2009 and 2010 may be expressed in the form $x^3+y^3+z^3-3xyz$, where x, y and z are positive integers?

Solution

We have

$$\begin{aligned}&x^3+y^3+z^3-3xyz\\&=(x+y+z)(x^2+y^2+z^2-yz-zx-xy)\\&=(x+y+z)\left[x(x-y)+y(y-z)+z(z-x)\right]\end{aligned}.$$

Letting x = 670 and y = z = 669, we have 2008; [10 points]

Letting x = y = 670 and z = 669, we have 2009. [10 points]

Now 2010 is a multiple of 3. If $3 \mid (x+y+z)$, then $3 \mid (x+y+z)^2$ so that

$3 \mid (x^2+y^2+z^2-yz-zx-xy) = (x+y+z)^2 - 3(xy+yz+zx)$. However 9 does not divides 2010. [10 points]

A similar contradiction is obtained if we assume that

$3 \mid (x^2+y^2+z^2-yz-zx-xy)$. [10 points]