Poularikas A. D. “Probability and Stochastic Processes”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC,1999

34

Probability and Stochastic Processes

34.1 Axioms of Probability

34.1.1 Axioms of Probability

I. $P(A) \geq 0$,II. $P(S) = 1$, III. If $AB = 0$ then $P(A + B) = P(A) + P(B)$ [. S = a set of elements of outcomes $\{\zeta_i\}$ of an experiment (certain event), 0 = empty set (impossible event). $\{\zeta_i\}$ = elementary event if $\{\zeta_i\}$ consists of a single element. A + B = *union* of events, AB = *intersection of events*, event = a subset of S, P(A) = probability of event A.

34.1.2 Corollaries of Probability

$$P(0),\ P(A) = 1 - P(\overline{A}) \le 1,\ (\overline{A} = \text{ complement set of } A)$$

$$P(A+B) \ne P(A) + P(B), \quad P(A+B) = P(A) + P(B) - P(AB) \le P(A) + P(B)$$

Example

S = {*hh,ht,th,tt*} (tossing a coin twice), *A* = {heads at first tossing} = {*hh,ht*}, *B* = {only one head came up} = {*ht,th*}, *G* = {heads came up at least once} = {*hh,ht,th*}, *D* = {tails at second tossing} = {*ht, tt*}

34.2 Conditional Probabilities—Independent Events

34.2.1 Conditional Probabilities

$$P(A|M) = \frac{P(AM)}{P(M)} = \frac{\text{probability of event } AM}{\text{probabilty of event } M} = \text{conditional probaqbility of } A \text{ given } M.$$

1. $P(A|M) = 0 \quad \text{if} \quad AM = 0$
2. $P(A|M) = \dfrac{P(A)}{P(M)} \ge P(A) \quad \text{if} \quad AM = A\ (A \subset M)$
3. $P(A|M) = \dfrac{P(M)}{P(M)} = 1 \quad \text{if} \quad M \subset A$
4. $P(A+B|M) = P(A|M) + P(B|M) \quad \text{if} \quad AB = 0$

Example

$P(f_i) = 1/6, \quad i = 1,\cdots 6.\ M = \{\text{odd}\} = \{f_1, f_3, f_5\},\ A = \{f_1\},\ AM = \{f_1\},\ P(M) = 3/6,\ P(AM) = 1/6$, then $P(f_1|\text{even}) = P(AM)/P(M) = 1/3$

34.2.2 Total Probability

$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)$ arbitrary event, $A_iA_j = 0 \quad i \ne j = 1,2,\cdots n,\ A_1 + \cdots + A_n$ = S = certain event.

34.2.3 Baye's Theorem

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$$

$$A_iA_j = 0,\ i \ne j = 1,2,\cdots n, \quad A_1 + A_2 + \cdots A_n = S = \text{certain event, B=arbitrary}$$

34.2.4 Independent Events

$P(AB) = P(A)P(B)$ implies A and B are independent events.

34.2.5 Properties

1. $P(A|B) = P(A)$

2. $P(B|A) = P(B)$

3. $P(A_1A_2\cdots A_n) = P(A_1)\cdots P(A_n)$, $A_i =$ independent events

4. $P(A+B) = P(A)+P(B)-P(A)P(B)$

5. $\overline{A}\overline{B} = (\overline{A+B})$, $P(\overline{A+B}) = 1-P(A+B)$, $P = (\overline{A}\overline{B}) = P(\overline{A})P(\overline{B})$

   If A nad B are independent. Overbar means complement set.

6. If A_i, A_2, A_3 are independent and A_1 is independent of A_2A_3 then

$$P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3) = P(A_1)P(A_2A_3). \quad \text{Also}$$
$$P[A_1(A_2+A_3)] = P(A_1A_2)+P(A_1A_3)-P(A_1A_2A_3) = P(A_1)$$
$$[P(A_2)+P(A_3)-P(A_2)P(A_3)] = P(A_1)P(A_2+A_3)$$

34.2.6 $P(A+B+C) = P(A)+P(B)+P(C)-P(AB)-P(AC)-P(BC)+P(ABC)$

34.3 Compound (Combined, Experiments

34.3.1 $S = S_1 \times S_2$ = *Cartesian product*

Example

S_1={1,2,3}, S_2={heads, tails}, $S = S_1 \times S_2$ = {(1 heads),(1 tails),(2 heads),(2 tails),(3 heads),(3 tails)}

34.3.2 If $A_1 \subset S_1$, $A_2 \subset S_2$ then $A_1 \times A_2 = (A_1 \times S_2)(A_2 \times S_1)$ (see Figure 34.1)

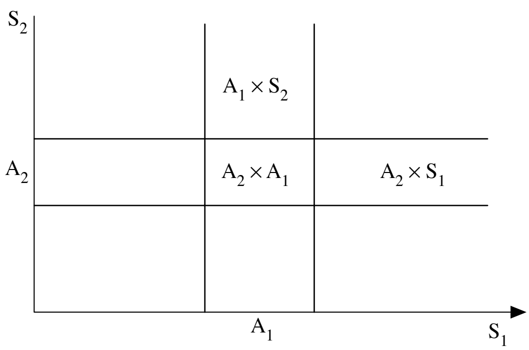

FIGURE 34.1

34.3.3 Probability in Compound Experiments

$$P(A_1) = P(A_1 \times S_2) \text{ where } \zeta_1 \in A_1 \text{ and } \zeta_2 \in A_2$$

34.3.4 Independent Compound Experiments

$P(A_1 \times A_2) = P_1(A_1)P_2(A_2)$

Example

P(heads) = p, P(tails) = q, $p+q=1$, E = experiment tossing the coin twice = $E_1 \times E_2$ (E_1 = experiment of first tossing), E_2 = experiment of second tossing), $S_1 = \{h,t\}$ $P_1\{h\}=p$ $P_2\{t\}=q$, $E_2 = E_1$ = experiment of the second tossing, $S = S_1 \times S_2 = [hh,ht,th,tt\}$, $P\{hh\} = P_1\{h\}P_2\{h\} = p^2 =$ assume independence, $P\{ht\} = pq$, $P\{th\} = qp$, $P[t,t\} = q^2$. For heads at the first tossing, $H_1 = \{hh,ht\}$ or $P(H_1) = P\{hh\} + P\{ht\} = p^2 + pq = p$

34.3.5 Sum of more Spaces

$S = S_1 + S_2$, $S_1 =$ outcomes of experiment E_1 and S_2 = outcomes of experiment E_2. $S =$ space of the experiment $E = E_1 + E_2$; $A = A_1 + A_2$ where A_1 and A_2 are events of E_1 and E_2: $A_1 \subset S_1$, $A_2 \subset S_2$; $P(A) = P(A_1) + P(A_2)$.

34.3.6 Bernoulli Trials

$P(A) =$ probability of event A, E×E×E×...×E = perform experiment n times = combined experiment.

$p_n(k) = \binom{n}{k} p^k q^{n-k}$ = probability that events occurs k times in any order $P(A) = p,\ P(\overline{A}) = q,\ p + q = 1$

Example

A fair die was rolled 5 times. $p_5(2) = \frac{5!}{(5-2)!2!}\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{5-2}$ = probability that "four" will come up twice.

Example

Two fair dice are tossed 10 times. What is the probability that the dice total seven points exactly four times?

Solution

Event $B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}, P(B) = 6 \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{6} = p, P(\bar{8}) = 1 - p = \frac{5}{6}$. The probability of B occuring four times and $\overline{B}$ six times is $\binom{10}{4}\left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 = 0.0543$.

34.3.7

$P\{k_1 \le k \le k_2\} =$ probability of success of A (event) will lie between k_1 and k_2

$$P\{k_1 \le k \le k_2\} = \sum_{k=k_1}^{k_2} p_n(k) = \sum_{k=k_1}^{k} \binom{n}{k} p^k q^{n-k}$$

1. Approximate value: $\sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k} \cong \frac{1}{\sqrt{2\pi npq}} \sum_{k=k_1}^{k_2} e^{-(k-np)^2/2npq}, npq >> 1$

34.3.8 DeMoivre-Laplace Theorem

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \cong \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)/2npq}, npq >> 1$$

34.3.9 Poisson Theorem

$$\frac{n!}{k!(n-k)!} p^k q^{n-k} \cong e^{-np} \frac{(np)^k}{k!} = e^{-a} \frac{a^k}{k!}, n \to \infty, p \to 0, np \to a$$

34.3.10 Random Points in Time

$P\{k \text{ in } t_a\} \cong e^{-nt_a/T} \dfrac{(nt/T)^k}{k!} = e^{-\lambda t_a} \dfrac{(\lambda t_a)^k}{k!}, t_2 - t_1 = t_a << T,$ n random points in (0,T) , $\lambda = n/T$. If $n \to \infty, \quad T \to \infty, \quad n/T \to \lambda$ the approximation becomes equality.

1. $P\{\text{one in } t_a\} \cong e^{-\lambda t_a} \lambda t_a \cong \lambda t_a$
2. $\lim\limits_{t_a \to 0} \dfrac{P\{\text{one in } t_a\}}{t_a} = \lambda$
3. $P(k \text{ in } (t_1, t_2)\} = e^{-\int_{t_1}^{t_2} \lambda(t)dt} \dfrac{\left[\int_{t_1}^{t_2} \lambda(t)dt\right]^k}{k!}, \quad p = \int_{t_1}^{t_2} \alpha(t)dt, \quad n\alpha(t) = \lambda(t)$

34.4 Random Variables

34.4.1 Random Variable

To every outcome ζ of any experiment we assign a number $X(\zeta) = x$. The function X, whose domain in the space S of all outcomes and its range is a set of numbers, is called a random variable (r.v.).

34.4.2 Distribution Function

$F_x(x) = P\{X \le x\}$ defined on any number $-\infty < x < \infty$. $\{X \le x\}$ is an event for any real number x.

34.4.3 Properties of Distribution Function

1. $F(-\infty) = 0, \quad F(+\infty) = 1$
2. $F(x_1) \le F(x_2)$ for $x_1 < x_2$
3. $F(x+) = F(x)$ continuous from the right

34.4.4 Density Function (or Frequency Function)

$f(x) = \dfrac{dF(x)}{dx}; \quad f(x) = \lim\limits_{\Delta x \to o} \dfrac{P\{x \le X \le x + \Delta x\}}{\Delta x}; \quad P\{X = x\} = 0$ for continuous distribution function; $f(x) = \sum_i p_i \delta(x - x_i) =$ density of discrete type, $p_i = F(x_i) - F(x_i -)$.

Example

Poisson distribution: $P\{X = k\} = e^{-\lambda} \dfrac{\lambda^k}{k!}, \quad k = 0, 1, \cdots, \lambda > 0$. Then $f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \dfrac{\lambda^k}{k!} \delta(x - k)$.

Example

If X is normally distributed $\left(f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x - m_x)^2/2\sigma^2} \right)$ with m_x=1000 and σ=50, then the probability that X is between 900 and 1,050 is $P\{x_1 \le X \le x_2\} = \int_{-\infty}^{x_2} f9y)d(y) - \int_{-\infty}^{x_1} f(y)d(y) = \dfrac{1}{2} + \text{erf} \dfrac{x_2 - m_x}{\sigma}$ $- \dfrac{1}{2} - \text{erf} \dfrac{x_1 - m_x}{\sigma} = \text{erf}1 + \text{erf}2 = 0.819$ where error function of $x = \text{erf } x = \dfrac{1}{\sqrt{2\pi}} \int_0^x \exp(-y^2/2)dy$

34.4.5 Tables of Distribution Functions (see Table 34.1)

TABLE 34.1 Distribution and Related Quantities

Definitions

1. Distribution function (or cumulative distribution function [c.d.f.]): $F(x) =$ probability that the variate takes values less than or equal to $x = P\{X \le x\} = \int_{-\infty}^{x} f(u)du$
2. Probability density function (p.d.f.):

$$f(x);\ P\{x_l < X \le x_u\} = \int_{x_l}^{x_u} f(x)dx;\ \ f(x) = \frac{dF(x)}{dx}$$

3. Probability function (discrete variates) $f(x) =$ probability that the variate takes the value $x = P\{X = x\}$
4. Probability generating function (discrete variates):

$$P(t) = \sum_{x=0}^{\infty} t^x f(x), \quad f(x) = (1/x!)\left(\frac{\partial^x P(t)}{\partial t^x}\right)_{t=0}, \quad x = 0,1,2,\cdots, X > 0$$

5. Moment generating function (m.f.g):

$$M(t) = \int_{-\infty}^{\infty} t^x f(x)dx. \quad M(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots + \frac{\mu_r' t^r}{r!} + \cdots,$$

$$\mu_r' = r^{th} \text{ moment about the origin } = \int_{-\infty}^{\infty} x^r f(x)dx = \left.\frac{\partial^r M(t)}{\partial t^r}\right|_{t=0}$$

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

6. Laplace transform of p.d.f.: $f^L(s) = \int_0^{\infty} e^{-sx} f(x)dx,\ \ X \ge 0$
7. Characteristic function : $\Phi(t) = \int_{-\infty}^{\infty} e^{jtx} f(x)dx, \quad \Phi_{X+Y}(t) = \Phi_X(t)\Phi_Y(t)$
8. Cumulant function: $K(t) = \log \Phi(t), \quad K_{X+Y}(t) = K_X(t) + K_Y(t)$
9. r^{th} cumulant: the coefficient of $(jt)^r / r!$ in the expansion of K(t)
10. r^{th} moment about the origin:

$$\mu_r^{©} = \int_{-\infty}^{\infty} x^r f(x)dx = \left.\left(\frac{\partial^r M(t)}{\partial t^r}\right)\right|_{t=0} = (-j)^r \left.\left(\frac{\partial^r \Phi(t)}{\partial t^r}\right)\right|_{t=0}$$

11. Mean: $\mu =$ first moment about the origin $= \int_{-\infty}^{\infty} x f(x)dx = \mu_1'$
12. r^{th} moment about the mean: $\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)dx$
13. Variance: σ^2 second moment about the mean $= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \mu_2$
14. Standard deviation: $\sigma = \sqrt{\sigma^2}$
15. Mean derivation: $\int_{-\infty}^{\infty} |x-\mu| f(x)dx$
16. Mode: A fractile (value of r.v.) for which the p.d.f is a local maximum
17. Median: m = the fractile which is exceeded with probability 1/2.

18. Standardized r[th] moment about the mean: $\eta_r = \int_{-\infty}^{\infty}\left(\frac{x-\mu}{\sigma}\right)^r f(x)dx = \frac{\mu_r}{\sigma^r}$

19. Coefficient of skewness: $\eta_3 = \mu_3 / \sigma^3$

20. Coefficint of kurtois: $\eta_4 = \mu_4 / \sigma^4$

21. Coefficient of variation: (standard deviation) / mean $= \sigma / \mu$

22. Information content: $I = -\int_{-\infty}^{\infty} f(x)\log_2(f(x))dx$

23. r[th] factorial moment about the origin (discrete case):

$$\mu^{\copyright}_{(r)} = \sum_{x=0}^{\infty} f(x)x(x-1)\cdots(x-r+1), \quad X \geq 0, \quad \mu^{\copyright}_{(r)} = \left(\frac{\partial^r P(t)}{\partial t^r}\right)_{t=1}$$

24. r[th] factorial moment moment about the mean (discrete case):

$$\mu_{(r)} = \sum_{x=0}^{\infty} f(x-\mu)(x-\mu)(x-\mu-1)\cdots(x-\mu-r+1), \; X \geq 0$$

25. Relationships between moments:

$$\mu'_r = \sum_{i=0}^{r}\binom{r}{i}\mu_{r-i}(\mu'_1); \quad \mu_r = \sum_{i=0}^{r}\binom{r}{i}\mu'_{r-i}(-\mu'_1)^i, \quad \mu_0 = \mu'_0 = 1, \quad \mu_1 = 0$$

26. log is the natural logarithm

Distributions

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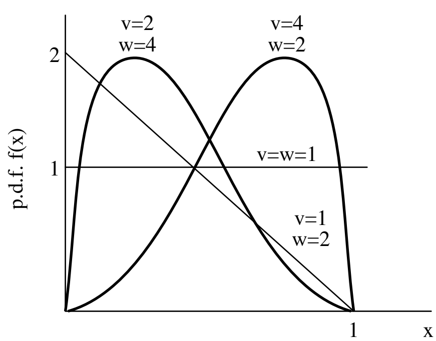
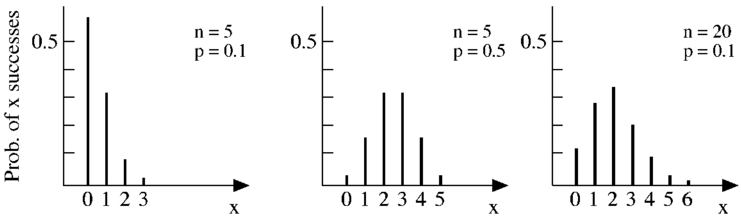
1. Beta: p.d.f $= f(x) = x^{v-1}(1-x)^{w-1} / B(v,w) \quad 0 \leq x \leq 1, B(v,w) =$ beta function $= \int_0^1 u^{v-1}(1-u)^{w-1}du$;r[th] moment about the origin $\prod_{i=0}^{r-1}(v+i)(v+w+i)$; mean $= v/(v+w)$; variance $= vw/(v+w)^2(v+w+1)$; mode $= (v-1)/(v+w+2), \quad v>1$, w>1; coefficient of skewness: $[2(w-v)(v+w+1)^{1/2}]/[(v+w+2)(vw)^{1/2}]$; coefficient of kurtois: $([3(v+w)(v+w+1)$ $(v+1)(2w-v)]/\{vw(v+w+2)(v+w+3)]) + [v(v-w)]/(v+w)$; coefficient of variation: $[w/[v(v+w+1)]]^{1/2}$; p.d.f. $=$ $f(x) = [(v+w-1)!x^{v-1}(1-x)^{w-1}]/[(v-1)!(w-1)!]$, v and w integers; $B(v,w) = \Gamma(v)\Gamma(w)/\Gamma(v+w) = B(w,v), \quad \Gamma(c) =$ $(c-1)\Gamma(c-1)$



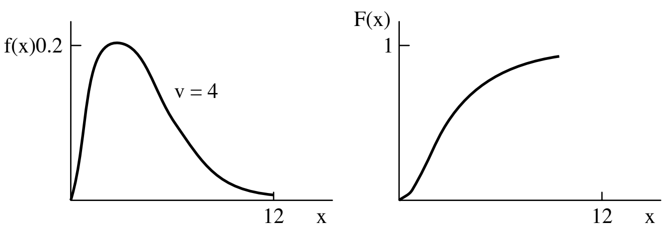
2. Binominal: n, p is the number of successes in n independent Bemoulli trials where the probability of success at each trial is p and the probability of failure is $q = 1-p, n =$ positive integer $0 < p < 1$. c.d.f $= F(x) = \sum_{i=0}^{x}\binom{n}{i}p^i q^{n-i}$, $x =$

integer; p.d.f. $= f(x) = \binom{n}{x} p^x q^{n-x}$, $x =$ integer; moment generating function: $[p\exp(t)+q]^n$; probability generating function: $(pt+q)^n$; characteristic function : $\Phi(t) = [p\exp(jt)+q]^n$. moments about the origin: mean=np, second = *np(np + q)*, third $= np[(n-1)(n-2)p^2 + 3p(n-1)+1]$; moment about the mean: variance = npq, third = npq(q - p), fourth = $npq[1+3pq(n-2)]$ standard deviation : $(npq)^{1/2}$; mode: $p(n+1)-1 \le x \le p(n+1)$; coefficient of skewness: $(q-p)/(npq)^{1/2}$; coefficient of kurtois: 3-(6/*n*)+(1/*npq);* factorial moments about the mean: second = npq, third = $-2npq(1+q)$; coefficient of variation = $(q/np)^{1/2}$



3. Cauchy: p.d.f $= f(x) = 1/[\pi b[(x-a)/b]^2 + 1]]$, $\alpha =$ shift parameter, b= ,scale parameter, $-\infty < x < \infty$; mode = a median = a

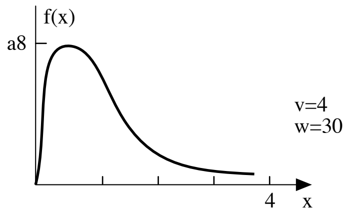
4. Chi-Squared: p.d.f. $f(x) = [x^{(v-2)/2}\exp(-x/2)]/[2^{v/2}\Gamma(v/2)]$, v (shape parameter) = degrees of freedom, $0 \le x < \infty$; moment generating function : $(1-2t)^{-v/2}, t > 1/2$; characteristic function: $\Phi(t) = (1-2jt)^{-v/2}$; cumulant function: $(-v/2)\log(1-2jt)$; r^{th} cumulant; $2^{r-1}v[(r-1)!]$; r^{th} moment about the origin: $2^r \prod_{i=0}^{r-1}[i+(v/2)]$; mean = v; variance: $2v$; standard deviation $(2v)^{1/2}$; Laplace transform of the p.d.f: $(1+2s)^{-v/2}$

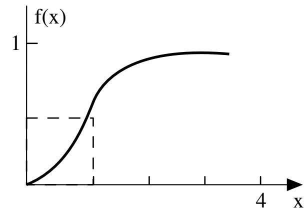


5. Discrete uniform: $a \le x \le a+b-1$, $x =$ integer, a = lower limit of the range, b = scale parameter; c.d.f = *F(x)* = $(x-a+1)/b$; p.d.f.= $f(x) = 1/b$;probability generating function: $(t^a - t^{a=b})/(1-t)$;characteristic function: exp[*j*(*a*-1)*t*] $\sinh(jtb/2)\sinh(jt/2)/b$; mean: $a+(b-1)/2$; variance: $(b^2-1)/12$; coefficient of skewness 0; information content: $\log_2 b$.

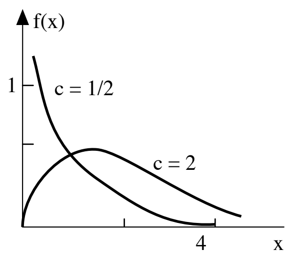
6. Exponential: $0 \le x < \infty$, b = scale parameter = mean, $\lambda = 1/b =$ alternative parameter; c.d.f = $F(x) = 1 - \exp(-x/b)$; p.d.f = $f(x) = (1/b)\exp(-x/b)$; moment generating function: $1/(1-bt), t > (1/b)$; Laplace transform of the p.d.f: $1/(1+bs)$; characteristic function: $1/(1-jbt)$; cumulant function: $-\log(1-jbt)$; r^{th} cumulant: $(r-1)!b^r$; r^{th} moment about the origin: $r!b^r$; mean: b : variance: b^2; standard deviation: b; mean deviation: 2 b/e (e base and natural log); mode: 0; median: b log 2; coefficient of skewness: 2; coefficient of kurtosis 9; coefficient of variation: 1; information content: $\log_2(eb)$ **.**

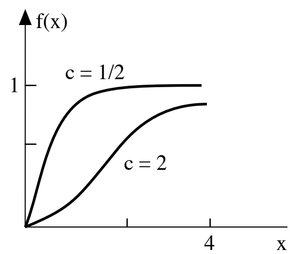
7. F-distribution: $0 \le x < \infty$, v and $w \equiv$ positive integers $\equiv$ degrees of freedom: p.d.f = $f(x) = [\Gamma[\frac{1}{2}(v+w)](v/w)^{v/2} x^{(v-2)/2}]/[\Gamma(\frac{1}{2}v)\Gamma(\frac{1}{2}w)(1+xv/w)^{(v+w)/2}]$; r^{th} moment about the origin: $[(w/v)^r\Gamma(\frac{1}{2}v+r)\Gamma(\frac{1}{2}w-r)/[\Gamma(\frac{1}{2}v)\Gamma(\frac{1}{2}w)]$, $w > 2r$; mean: $w/(w-2), w > 2$; variance: $[2w^2(v+w-2)]/[v(w-2)^2(w-4)]$, $w > 4$; mode $[w(v-2)]/[v(w+2)]$, $v > 1$;coefficient of skewness: $[(2v+w-2)[8(w-4)]^{1/2}]/[(w-6)(v+w-2)^{1/2}]$, $w > 6$;coefficient of variation: $[[2(v+w-2)/[v9w-4)]]^{1/2}, w > 4$.



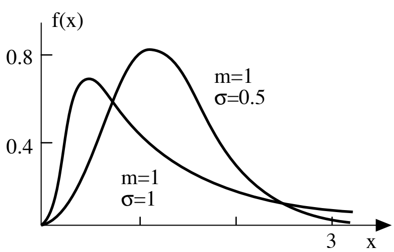


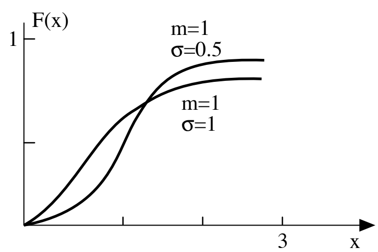
8. Gamma: $0 \le x < \infty$, b = scal e parameter > 0 (or $\lambda = 1/b$), c>0 shaper parameter; p.d.f = $f(x) = (x/b)^{c-1}$ $[\exp(-x/b)]/[b\Gamma(c)], \Gamma(c) = \int_0^\infty \exp(-u)u^{c-1}du$; moment generating function: $(1-bt)^{-c}$, $t > 1/b$; Laplace fransform of the p.d.f.: $(1+bs)^{-c}$; characteristic function: $(1-jbt)^{-c}$; cumulant function: $-c\log 91 - jbt)$; r^th^ cumulant: $(r-1)!cb^r$; r^th^ moment about the origin: $b^r \prod_{i=0}^{r-1}(c+i)$; mean: bc; variance: b^2c ; standard deviation: $b\sqrt{c}$; mode: $b(c-1), c \ge 1$; coefficient of skewness: $2c^{-1/2}$; coefficient of kurtosis: 3 + 6/c: coefficient of variation: $c^{-1/2}$





9. Logonormal: $0 \le x < \infty$, m = scale parameter = median > 0, μ = mean of $\log X > 0$, $m = \exp\mu$, $\mu = \log m$, σ =shape parameter= standard deviation of $\log X$, $w = \exp(\sigma^2)$; p.d.f = $f(x) = [1/x\sigma(2\pi)^{1/2}]\exp[-[\log(x/m)]^2/2\sigma^2]$ r^th^ moment about the origin: $m^r \exp(r^2\sigma^2/2)$; mean: $m\exp(\sigma^2/2)$; variance: $m^2w(w-1)$: standard deviation: $m(w^2-w)^{1/2}$; mode *m/w*; median; m; coefficient of skewness: $(w+2)(w-1)^{1/2}$; coefficient of kurtosis: $w^4+2w^3+3w^2-3$; coefficient of variation: $(w-1)^{1/2}$.

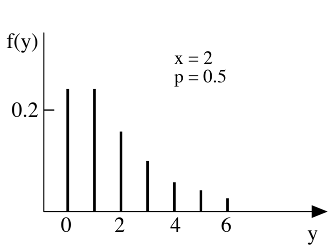


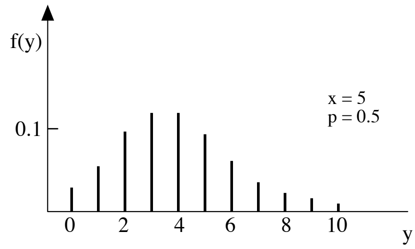


10. Negative bionomial: y = number of failures (integer), x = number of failures before x^{th} success in a sequence of Bernoulli trials; p = probability of success at each trial, $q = 1-p$, $0 \le y < \infty$, $0 < p < 1$; c.d.f. = $F(y) = \sum_{i=1}^{y}\binom{x+i-1}{i}p^x q^i$;

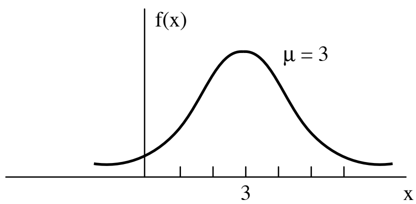
p.d.f. = $f(y) = \binom{x+y-1}{y}p^x q^y$; moment generating function: $p^x(1-q\exp t)^{-x}$; probability generating function: $p^x(1-qt)^{-x}$; ; characteristic function: $p^x[1-q\exp(jt)]^{-x}$; cumulant function: $x\log(p) - x\log(1-q\exp t)$; Cumulants: first = xq/p, second = xq/p^2, third = $xq(1+q)/p^3$, fourth = $xq(6q+p^2)/p^4$;mean: xq/p; Moments about the mean: variance = xq/p^2 , third = $xq(1+q)/p^3$, fourth = $(xq/p^4)(3xq+6q+p^2)$; standard deviation: $(xq)^{1/2}/p$; coefficient of skewness:

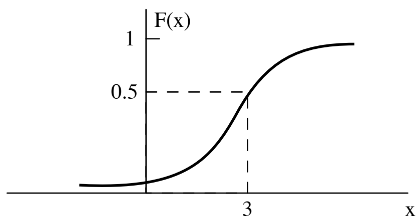
$(1+q)(xq)^{-1/2}$; coefficient of kurtosis: $3+\frac{6}{x}+\frac{p^2}{xq}$; factorial moment generating function: $(1-q^t/p)^{-x}$ r^th factorial moment about the origin: $(q/p)^r(x+r-1)^r$; coefficient of variaton: $(xq)^{-1/2}$.



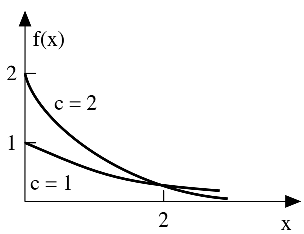


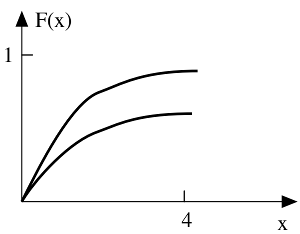
11. Normal: $-8 < x < \infty$, μ = mean = location parameter, σ = standard deviation = scale parameter, $\sigma > 0$; p.d.f. = $f(x) = [1/\sigma(2\pi)^{1/2}]\exp[-(x-\mu)^2/2\sigma^2]$; moment generating function: $\exp(\mu t+\frac{1}{2}\sigma^2 t^2)$; characteristic function: $\exp(j\mu t-\frac{1}{2}\sigma^2 t^2)$; cumulant function: $j\mu t-\frac{1}{2}\sigma^2 t^2$; r^{th} cumulant: $K_2=\sigma^2$, $K_r=0$, $r>2$; mean: μ r^{th} moment about the mean: $\mu_r=0$ for r odd, $\mu_r=(\sigma^r r!)/[2^{r/2}[(r/2)!]]$ for r even; variance: σ^2 ; standard deviation: σ ; mean deviation: $\sigma(2/\pi)^{1/2}$; mode: μ; median: μ; coefficient of skewness: 0; coefficient of kurtosis: 3; information content: $\log_2[\sigma(2\pi e)^{1/2}]$





12. Pareto: $1 \le x < \infty$, $c =$ shape parameter; c.d.f.= $F(x)=1-x^{-c}$; p.d.f.= $f(x)=cx^{-c-1}$; r^{th} moment about the origin: $c/(c-r), c>r$; mean : $c/(c-1), c>1$; variance: $[c/(c-2)]-[c/(c-1)]^2, c>2$; coefficient of variation: $(c-1)/[c(c-1)]^{1/2}$, $c>2$.

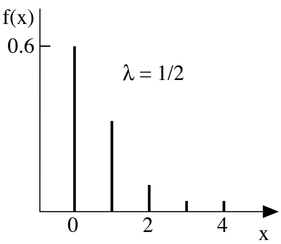


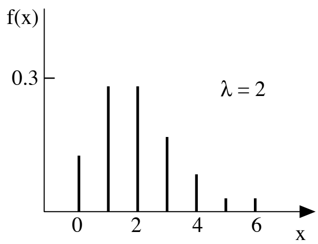


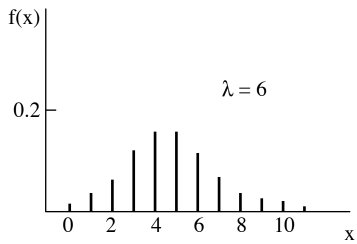
13. Pascal: n = number of teals, $n \ge 1$, x = the Bernoulli success parameter = the number of trials up to and including the x^th success, p = probability of success at each trial, $0<p<1, q=1-p$; p.d.f. = $f(n)=\binom{n-1}{n-x}p^x q^{n-x}$; moment generating function: $p^x\exp(tx)/(1-q\exp t)^x$ probability generating function: $(pt)^x/(1-qt)^x$; characteristic function: $p^x\exp(jtx)/(1-q\exp(jt)^x$; mean: x/p; variance: xq/p^2; standard deviation: $(xq)^{1/2}/p$; coefficient of variation: $(q/x)^{1/2}$.

14. Poisson: $0 \le x < \infty, \lambda =$ mean (a parameter); c.d.f. = $F(x)=\sum_{i=0}^{x}\lambda^i\exp(-\lambda)/i!$; p.d.f. = $f(x)=\lambda^x\exp(-\lambda)/x!$; moment generating function: $\exp[\lambda[\exp(t)-]]$; probability generating function: $\exp[-\lambda(1-t)]$; characteristic function:

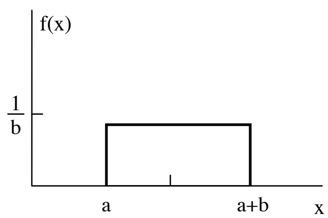
$\exp[\lambda[\exp(jt)-1]]$;cumulant function: $\lambda[\exp(t)-1] = \sum_{i=0}^{\infty} t^i / i!$; r^{th} cumulant: λ;moment about the origin: mean=λ,second= $\lambda + \lambda^2$; third= $\lambda[(\lambda+1)^2 + \lambda]$,fourth = $\lambda(\lambda^3 + 6\lambda^2 + 7\lambda + 1)$; r^{th} moment about the mean, $\mu_i : \lambda \sum_{i=0}^{r-2} \binom{r-1}{i} \mu_i, r > 1, \mu_0 = 1.$ Moments about the mean: variance = λ, third=λ, fourth = $\lambda(1+3\lambda)$, fifth = $\lambda(1+10\lambda)$, sixth = $\lambda(1+25\lambda+15\lambda^2)$; standard deviation = $\lambda^{1/2}$;coefficient of skewness: $\lambda^{-1/2}$;coefficient of kurtosis: $3 + 1/\lambda$;factorial moments about the mean: second = λ, third = -2λ, fourth = 3λ(λ+2); coefficient of variation: $\lambda^{-1/2}$.

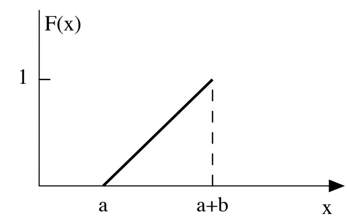




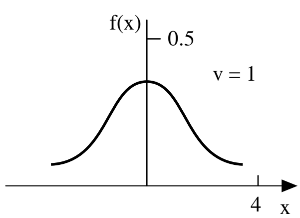


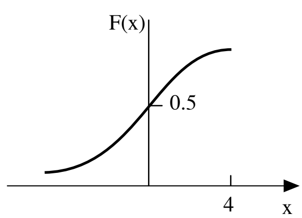
15. Rectangular: $a \le x \le a+b$, $x =$ range, a = lower limit,b=scale parameter; c.d.f = $F(x) = (x-a)/b$; p.d.f. = $f(x) = 1/b$; moment generating function: $\exp(at)[\exp(bt)-1]/bt$; Laplace transform of the p.d.f: $\exp(-as)[1-\exp(-bs)]/bs$; characteristic function: $\exp(jat)[\exp(jbt)-1]/jbt$; mean: $a + b/2$; r^{th} moment about the mean: $\mu_r = 0$ for r odd, $\mu_r = (b/2)^r/(r+1)$ for r even;variance: $b^2/12$;standard deviation: $b/\sqrt{12}$;mean deviation $b/4$;median $a + b/2$;standardized r^{th} moment about the mean: $\mu_r = 0$ for r odd, $\mu_r = 3^{r/2}/(r+1)$ for r even; coefficient of skewness: 0; coefficient of kurtosis: 915; coefficient of variation: $b/[3^{1/2}(2a+b)]$; information content: $\log_2 b$.



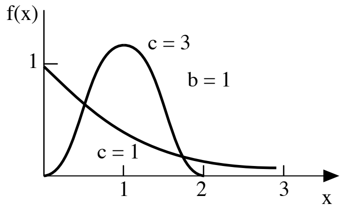


16. Student's: $-\infty < x < \infty$, $v =$ shape parameter (degrees of freedom), $v \equiv$ positive integer; p.d.f. = $f(x) = [\Gamma[(v+1)/2][1+(x^2/v)]^{-(v+1)/2}]/[(\pi v)^{1/2}\Gamma(v/2)]$; mean: 0; r^{th} moment about the mean: $\mu_r = 0$ for r odd, $\mu_r = [1 \cdot 3 \cdot 5 \cdots (r-1) v^{r/2}]/[(v-2)(v-4)\cdots(v-r)]$ for r even, $r<v$: variance: $v/(v-2), v > 2$; mean deviation: $v^{1/2}\Gamma(\frac{1}{2}(v-1)/\pi^{1/2}\Gamma(\frac{1}{2}v)$; mode: 0; coefficient of skewness and kurtosis: 0





17. Weibull: $0 \le x < \infty, b > 0$ scale parameter, c = shape aprameter c>0; c.d.f. = $F(x) = 1 - \exp[-(x/b)$; p.d.f. = $f(x) = (cx^{c-1}/b^c)\exp[-(x/b)^c]$; r^{th} moment about the origin: $b^r\Gamma[(c+r)/c]$; mean: $b\Gamma[(c+1)/c]$.



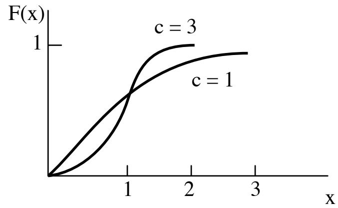


TABLE 34.2 Normal Distribution Tables.

$f(x) =$ distribution density $= (1/\sqrt{2\pi})e^{-x^2/2}$, $F(x) =$ cumulative distribution function $= \int_{-\infty}^{x} (1/\sqrt{2\pi})e^{-\gamma^2/2}d\tau$,

$f'(x) = -xf(x),\ f''(x) = (x^2 - 1)f(x), \quad F(-x) = 1 - F(x), \quad P\{-x < X < x\} = 2F(x) - 1$

x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$	x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$
.00	.5000	.3989	-.0000	-.3989	.50	.6915	.3521	-.1760	-.2641
.01	.5040	.3989	-.0000	-.3989	.51	.6950	.3503	-.1787	-.2592
.02	.5080	.3989	-.0080	-.3987	.52	.6985	.3485	-.1812	-.2543
.03	.5120	.3988	-.0120	-.3984	.53	.7019	.3467	-.1837	-.2493
.04	.5160	.3986	-.0159	-.3980	.54	.7054	.3448	-.1862	-.2443
.05	.5199	.3984	-.0199	-.3975	.55	.7088	.3429	-.1886	-.2392
.06	.5239	.3982	-.0239	-.3968	.56	.7123	.3410	-.1920	-.2341
.07	.5279	.3980	-.0279	-.3960	.57	.7157	.3391	-.1933	-.2289
.08	.5319	.3977	-.0318	-.3951	.58	.7190	.3372	-.1956	-.2238
.09	.5359	.3973	-.0358	-.3941	.59	.7224	.3352	-.1978	-.2185
.10	.5398	.3970	-.0199	-.3975	.60	.7257	.3332	-.1999	-.2133
.11	.5438	.3965	-.0239	-.3968	.61	.7291	.3312	-.2020	-.2080
.12	.5478	.3961	-.0279	-.3960	.62	.7324	.3292	-.2041	-.2027
.13	.5517	.3956	-.0318	-.3951	.63	.7357	.3271	-.2061	-.1973
.14	.5557	.3951	-.0358	-.3941	.64	.7389	.3251	-.2080	-.1919
.15	.5596	.3945	-.0592	-.3856	.65	.7422	.3230	-.2099	-.1865
.16	.5636	.3939	-.0630	-.3838	.66	.7454	.3209	-.2118	-.1811
.17	.5675	.3932	-.0668	-.3819	.67	.7486	.3187	-.2136	-.1757
.18	.5714	.3925	-.0707	-.3798	.68	.7517	.3166	-.2153	-.1702
.19	.5753	.3918	-.0744	-.3777	.69	.7549	.3144	-.2170	-.1647
.20	.5793	.3910	-.0782	-.3754	.70	.7580	.3132	-.2186	-.1593
.21	.5832	.3902	-.0820	-.3730	.71	.7611	.3101	-.2201	-.1538
.22	.5871	.3894	-.0857	-.3706	.72	.7642	.3079	-.2217	-.1483
.23	.5910	.3885	-.0894	-.3680	.73	.7673	.3056	-.2231	-.1428
.24	.5948	.3876	-.0930	-.3653	.74	.7704	.3034	-.2245	-.1373
.25	.5987	.3867	-.0967	-.3625	.75	.7734	.3011	-.2259	-.1318
.26	.6026	.3857	-.1003	-.3596	.76	.7764	.2989	-.2271	-.1262
.27	.6064	.3847	-.1039	-.3566	.77	.7794	.2966	-.2284	-.1207
.28	.6103	.3836	-.1074	-.3535	.78	.7823	.2943	-.2296	-.1153
.29	.6141	.3825	-.1109	-.3504	.79	.7852	.2920	-.2307	-.1098
.30	.6179	.3814	-.1144	-.3471	.80	.7881	.2897	-.2318	-.1043
.31	.6217	.3802	-.1179	-.3437	.81	.7910	.2874	-.2328	-.0988
.32	.6255	.3791	-.1213	-.3402	.82	.7939	.2850	-.2337	-.0934
.33	.6293	.3778	-.1247	.3367	.83	.7967	.2827	-.2346	-.0880
.34	.6331	.3765	.-.1280	-.3330	.84	.7995	.2803	-.2355	-.0825

TABLE 34.2 Normal Distribution Tables. (continued)

$f(x) =$ distribution density $= (1/\sqrt{2\pi})e^{-x^2/2}$, $F(x) =$ cumulative distribution function $= \int_{-\infty}^{x}(1/\sqrt{2\pi})e^{-\gamma^2/2}d\tau$,

$f'(x) = -xf(x)$, $f''(x) = (x^2 - 1)f(x)$, $F(-x) = 1 - F(x)$, $P\{-x < X < x\} = 2F(x) - 1$

x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$	x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$
.35	.6368	.3752	-.1313	-.3293	.85	.8023	.2780	-.2363	-.0771
.36	.6406	.3739	-.1346	-.3255	.86	.8051	.2756	-.2370	-.0718
.37	.6443	.3725	-.1378	-.3216	.87	.8078	.2732	-.2377	-.0664
.38	.6480	.3712	-.1410	-.3176	.88	.8106	.2709	-.2384	-.0611
.39	.6517	.3697	-.1442	-.3135	.89	.8133	.2685	-.2389	-.0558
.40	.6554	.3683	-.1473	-.3094	.90	.8159	.2661	-.2395	-.0506
.41	.6591	.3668	-.1504	-.3015	.91	.8186	.2637	-.2400	-.0453
.42	.6628	.3653	-.1534	-.3008	.92	.8212	.2613	-.2404	-.0401
.43	.6664	.3637	-.1564	-.2965	.93	.8238	.2589	-.2408	-.0350
.44	.6700	.3621	-.1593	-.2920	.94	.8264	.2565	-.2411	-.0299
.45	.6736	.3605	-.1622	-.2875	.95	.8289	3.2541	-.2414	-.0248
.46	.6772	.3589	-.1651	-.2830	.96	.8315	.2516	-.2416	-.0197
.47	.6808	.3572	-.1679	-.2783	.97	.8340	.2492	-.2417	-.0147
.48	.6844	.3555	-.1707	-.2736	.98	.8365	.2468	-.2419	-.0098
.49	.6879	.3538	-.1734	-.2689	.99	.8389	.2444	-.2420	-.0049
.50	.6915	.3521	-.1760	-.2641	1.00	.8413	.2420	-.2420	-.0000
1.00	.8413	.2420	-.2420	.0000	1.50	.9332	.1295	-.1943	.1619
1.01	.8438	.2396	-.2420	.0048	1.51	.9345	.1276	-.1927	.1633
1.02	.8461	.2371	-.2419	.0096	1.52	.9357	.1257	-.1910	.1647
1.03	.8485	.2371	-.2418	.0143	1.53	.9370	.1238	-.1894	.1660
1.04	.8508	.2323	-.2416	.0190	1.54	.9382	.1219	-.1877	.1672
1.05	.8531	.2299	-.2414	.0236	1.55	.9394	.1200	-.1860	.1683
1.06	.8554	.2275	-.2411	.0281	1.56	.9406	.1182	-.1843	.1694
1.07	.8577	.2251	-.2408	.0326	1.57	.9418	.1163	-.1826	.1704
1.08	.8599	.2227	-.2405	.0371	1.58	.9429	.1145	-.1809	.1714
1.09	.8621	.2203	-.2401	.0414	1.59	.9441	.1127	-.1792	.1722
1.10	.8643	.2176	-.2396	.0458	1.60	.9452	.1109	-.1775	.1730
1.11	.8665	.2155	-.2392	.0500	1.61	.9463	.1092	-.1757	.1738
1.12	.8686	.2131	-.2386	.0542	1.62	.9474	.1074	-.1740	.1745
1.13	.8708	.2107	-.2381	.0583	1.63	.9484	.1057	-.1723	.1751
1.14	.8729	.1083	-.2375	.0624	1.64	.9495	.1040	-.1705	.1757
1.15	.8749	.2059	-.2368	.0664	1.65	.9505	.1023	-.1687	.1762
1.16	.8770	.2036	-.2361	.0704	1.66	.9515	.1006	-.1670	.1766
1.17	.8790	.2012	-.1354	.0742	1.67	.9525	.0989	-.1652	.1770
1.18	.8810	.1989	-.2347	.0780	1.68	.9535	.0973	-.1634	.1773
1.19	.8830	.1965	-.2339	.0818	1.69	.9545	.0957	-.1617	.1776
1.20	.8849	.1942	-.2330	.0854	1.70	.9554	.0940	-.1599	.1778
1.21	.8869	.1919	-.2322	.0890	1.71	.9564	.0925	-.1581	.1779
1.22	.8888	.1895	-.2312	.0926	1.72	.9573	.0909	-.1563	.1780
1.23	.8907	.1872	-.2303	.0960	1.73	.9582	.0893	-.1546	.1780
1.24	.8925	.1849	-.2293	.0994	1.74	.9591	.0878	-.1528	.1780
1.25	.8944	.1826	-.2283	.1027	1.75	.9599	.0863	-.1510	.1780
1.26	.8962	.1804	-.2273	.1060	1.76	.9608	.0848	-.1492	.1778
1.27	.8980	.1781	-.2262	.1092	1.77	.9616	.0833	-.1474	.1777

TABLE 34.2 Normal Distribution Tables. (continued)

$f(x) =$ distribution density $= (1/\sqrt{2\pi})e^{-x^2/2}$, $F(x) =$ cumulative distribution function $= \int_{-\infty}^{x}(1/\sqrt{2\pi})e^{-\gamma^2/2}d\tau$,

$f'(x) = -xf(x),\ f''(x) = (x^2 - 1)f(x),\quad F(-x) = 1 - F(x),\quad P\{-x < X < x\} = 2F(x) - 1$

x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$	x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$
1.28	.8997	.1758	-.2251	.1123	1.78	.9625	.0818	-.1457	.1774
1.29	.9015	.1736	-.2240	.1153	1.79	.9633	.0804	-.1439	.1772
1.30	.9032	.1714	-.2228	.1182	1.80	.9641	.0790	-.1421	.1769
1.31	.9049	.1691	-.2204	.1211	1.81	.9649	.0775	-.1403	.1765
1.32	.9066	.1669	-.2204	.1239	1.82	.9556	.0761	-.1386	.1761
1.33	.9082	.1647	-.2191	.1267	1.83	.9664	.0748	-.1368	.1756
1.34	.9099	.1626	-.2178	.1293	1.84	.9671	.0734	-.1351	.1751
1.35	.9115	.1604	-.2165	.1319	1.85	.9678	.0721	-.1333	.1746
1.36	.9131	.1582	-.2152	.1344	1.86	.9686	.0707	-.1316	.1740
1.37	.9147	.1561	-.2138	.1369	1.87	.9693	.0694	-.1298	.1734
1.38	.9162	.1539	-.2125	.1392	1.88	.9699	.0681	-.1281	.1727
1.39	.9177	.1518	-.2110	.1415	1.89	.9706	.0689	-.1264	.1720
1.40	.9192	.1479	-.2096	.1437	1.90	.9713	.0656	-.1247	.1713
1.41	.9207	.1476	-.2082	.1459	1.91	.9719	.0644	-.1230	.1705
1.42	.9222	.1456	-.2067	.1480	1.92	.9726	.0632	-.1213	.1697
1.43	.9236	.1435	-.2052	.1500	1.93	.9732	.0620	-.1196	.1688
1.44	.9251	.1415	-.2037	.1519	1.94	.9738	.0608	-.1179	.1679
1.45	.9265	.1394	-.2022	.1537	1.95	.9744	.0596	-.1162	.1670
1.46	.9279	.1374	-.2006	.1555	1.96	.9750	.0584	-.1145	.1661
1.47	.9306	.1354	-.1991	.1572	1.97	.9756	.0573	-.1129	.1651
1.48	.9306	.1334	-.1975	-.1588	1.98	.9761	0.562	-.1112	.1641
1.49	.9319	.1315	-.1959	.1604	1.99	.9767	.0551	-.1096	.1630
1.50	.9332	.1295	-.1943	.1619	2.00	.9772	.0540	-.1080	.1622
2.00	.9773	.0540	-.1080	.1620	2.50	.9938	.0175	-.0438	.0920
2.01	.9778	.0529	-.1064	.1609	2.51	.9940	.0171	-.0429	.0906
2.02	.9783	.0519	-.1048	.1598	2.52	.9941	.0167	-.0420	.0892
2.03	.9788	.0508	-.1032	.1586	2.53	.9943	.0163	-.0411	.0868
2.04	.9793	.0498	-.1016	.1575	2.54	.9945	.0158	-.0403	.0878
2.05	.9798	.0488	-.1000	.1563	2.55	.9946	.0155	-.0394	.0850
2.06	.9803	.0478	-.0985	.1550	2.56	.9948	.0151	-.0386	.0836
2.07	.9809	.0468	-.0969	.1538	2.57	.9949	.0147	-.0377	.0823
2.08	.9812	.0459	-.0954	.1526	2.58	.9951	.0143	-.0369	.0809
2.09	.9817	.0449	-.0939	.1513	2.59	.9952	.0139	-.0361	.0796
2.10	.9821	.0440	-.0924	.1500	2.60	.9953	.0136	-.0353	.0782
2.11	.9826	.0431	-.0909	.1487	2.61	.9955	.0132	.0345	.0769
2.12	.9830	.0422	-.0894	.1474	2.62	.9956	.0129	-.0338	.0756
2.13	.9834	.0413	-.0879	.1460	2.63	.9957	.0126	-.0330	.0743
2.14	.9838	.0404	-.0865	.1446	2.64	.9959	.0122	-.0323	.0730
2.15	.9842	.0396	-.0850	.1433	2.65	.9960	.0119	-.0316	.0717
2.16	.9846	.0387	-.0836	.1419	2.66	.9961	.0116	-.0309	.0705
2.17	.9850	.0379	-.0822	.1405	2.67	.9962	.0113	-.0302	.0692
2.18	.9854	.0371	-.0808	.1391	2.68	.9963	.0110	-.0295	.0680
2.19	.9857	.0363	-.0794	.1377	2.69	.9964	.0107	-.0288	.0668

TABLE 34.2 Normal Distribution Tables. (continued)

$f(x) =$ distribution density $= (1/\sqrt{2\pi})e^{-x^2/2}$, $F(x) =$ cumulative distribution function $= \int_{-\infty}^{x}(1/\sqrt{2\pi})e^{-\gamma^2/2}d\tau$,

$f'(x) = -xf(x),\ f''(x) = (x^2-1)f(x),\quad F(-x) = 1-F(x),\quad P\{-x < X < x\} = 2F(x)-1$

x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$	x	$F(x)$	$f(x)$	$f'(x)$	$f''(x)$
2.20	.9861	.0355	-.0780	.1362	2.70	.9965	.0104	-.0281	.0656
2.21	.9864	.0347	-.0767	.1348	2.71	.9966	.0101	-.0275	.0644
2.22	.9868	.0339	-.0754	.1333	2.72	.9967	.0099	-.0269	.0632
2.23	.9871	.0332	-.0740	.1319	2.73	.9968	.0096	-.0262	.0620
2.24	.9875	.0325	-.0727	.1304	2.74	.9969	.0093	-.0256	.0608
2.25	.9868	.0317	-.0714	.1289	2.75	.9970	.0091	-.0250	.0597
2.26	.9881	.0310	-.0701	.1275	2.76	.9971	.0088	-.0244	.0585
2.27	.9884	.0303	-.0689	.1260	2.77	.9972	.0086	-.0238	.0574
2.28	.9887	.0297	-.0676	.1245	2.78	.9973	.0084	-.0233	.0563
2.29	.9890	.0290	-.0664	.1230	2.79	.9974	.0081	-.0227	.0562
2.30	.9893	.0283	-.0652	.1215	2.80	.9974	.0079	-.0222	.0541
2.31	.9896	.0277	-.0639	.1200	2.81	.9975	.0077	-.0216	.0531
2.32	.9898	.0270	-.0628	.1185	2.82	.9976	.0075	-.0211	.0520
2.33	.9901	.0264	-.0616	.1170	2.83	.9977	.0073	-.0206	.0510
2.34	.9904	.2058	-.0604	.1155	2.84	.9977	.0071	-.0201	.0500
2.35	.9906	.0252	-.0593	.1141	2.85	.9978	.0069	-.0196	.0490
2.36	.9909	.0246	-.0581	.1126	2.86	.9979	.0067	-.0191	.0480
2.37	.9911	.0241	-.0570	.1111	2.87	.9979	.0065	-.0186	.0470
2.38	.9913	.0235	-.0559	.1096	2.88	.9980	.0063	-.0182	.0460
2.39	.9916	.0229	-.0548	.1081	2.89	.9981	.0061	-.0177	.0451
2.40	.9918	.0224	-.0538	.1066	2.90	.9981	.0060	-.0173	.0441
2.41	.9920	.0219	-.0527	.1051	2.91	.9982	.0058	-.0168	.0432
2.42	.9922	.0213	-.0516	.1036	2.92	.9982	.0056	-.0164	.0423
2.43	.9925	.0208	-.0506	.1022	2.93	.9983	.0055	-.0160	.0414
2.44	.9927	.0203	-.0496	.1007	2.94	.9984	.0053	-.0156	.0405
2.45	.9929	.0198	-.0486	.0992	2.95	.9984	.0051	-.0152	.0396
2.46	.9931	.0194	-.0476	.0978	2.96	.9985	.0050	-.0148	.0388
2.47	.9932	.0189	-.0467	.0963	2.97	.9985	.0048	-.0144	.0379
2.48	.9934	.0184	-.0457	.0949	2.98	.9986	.0047	-.0140	.0371
2.49	.9936	.0180	-.0448	.0935	2.99	.9986	.0046	-.0137	.0363
2.50	.9938	.0175	-.0438	.0920	3.00	.9987	.0044	-.0133	.0355
3.00	.9987	.0044	-.0133	.0355	3.50	.9998	.0009	-.0031	.0098
3.05	.9989	.0038	-.0116	.0316	3.55	.9998	.0007	-.0026	.0085
3.10	.9990	.0033	-.0101	.0281	3.60	.9998	.0006	-.0022	.0073
3.15	.9992	.0028	-.0088	.0249	3.65	.9999	.0005	-.0019	.0063
3.20	.9993	.0024	-.0076	.0220	3.70	.9999	.0004	-.0016	.0054
3.25	.9994	.0020	-.0066	.0194	3.75	.9999	.0004	-.0013	.0046
3.30	.9995	.0017	-.0057	.0170	3.80	.9999	.0003	-.0011	.0039
3.35	.9996	.0015	-.0049	.0149	3.85	.9999	.0002	-.0009	.0033
3.40	.9997	.0012	-.0042	.0130	3.90	1.0000	.0002	-.0008	.0028
3.45	.9997	.0010	-.0036	.0113	3.95	1.0000	.0002	-.0006	.0024
3.50	.9998	.0009	-.0031	.0098	4.00	1.0000	.0001	-.0005	.0020

TABLE 34.3 Student t-Distribution Table

$$f(x)=\int_{-\infty}^{x}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma 9n/2)}\left(1+\frac{y^2}{n}\right)^{-(n+1)/2}dy$$

n= number of degrees of freedom, numbers give x of distribution, e.g., for n=6 and F=0.975, x=2.447, $F(-x)=1-F(x)$

n \ F	.60	.75	.90	.95	.975	.99	.995	.9995
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	.289	.816	1.886	2.920	4.303	6.965	9.925	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	8.610
5	.267	.727	1.476	2.015	.2.571	3.365	4.032	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.555	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.646
40	.255	.681	1.303	1.684	2.201	2.423	2.704	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	3.291

34.4.6 Conditional Distribution

$$F_X(x|M)=P\{X\le x|M\}=\frac{P\{X\le x,M\}}{P\{M\}},$$

$\{X\le x,M\}=$ event of all outcomes ζ such that $X(\zeta)\le x$ and $\zeta\in M$

1. $F(\infty|M)=1, F(-\infty|M)=0$
2. $F(x_2|M)-F(x_1|M)=P\{x_1<X\le x_2|M\}=\dfrac{P\{x_1<X\le x_2,M\}}{P\{M\}}$

TABLE 34.4 The Chi-Squared Distribution

$$F(x) = \int_0^x \frac{1}{2^{n/2} F(n/2)} y^{(n-2)/2} e^{-y/2} dy$$

n = number of degrees of freedom

n\F	.005	,010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.990	.995
1	.0000393	.000157	.000982	.00393	..0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	6.25	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	.584	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.06	1.92	3.36	5.39	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	2.20	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14,8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	2.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.94	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

TABLE 34.5 The F-Distribution

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma(r_1 + r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1+(r_1 x/r_2)]^{(r_1+r_2)/2}} dx$$

$$P\{F \le f\} = 0.95$$

$r_2 \backslash r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	262.22
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.74	5.72	5.69
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.71	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38
13	4.67	3.81	3.41	3.18	3.03	2.92	1.83	2.77	2.71	2.67	2.60	2.53	2.46	2,41	2.38	2.34	2.30
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.37	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.34	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.32	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.30	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84

25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79
28	1.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma[(r_1+r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1+(r_1x/r_2)]^{(r_1+r_2)/2}} dx \equiv \text{F distribution}$$

$P\{F \le f\} = 0.975$

$r_2 \backslash r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.6	968.6	976.7	984.9	993.1	997.2	1001	1006	1010
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.85	8.66	8.56	8.51	8.46	8.41	8.36
5	1.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96
7	807	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78
9	5.21	2.71	2.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45
10	6.94	5.46	4.83	4.47	4.24	2.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45
17	6.04	4.62	4.01	3.66	3.44	3.28	3.26	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27

TABLE 34.5 The F-Distribution

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma(r_1 + r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1 + (r_1 x/r_2)]^{(r_1+r_2)/2}} dx$$

$P\{F \le f\} = 0.95$

$r_2 \backslash r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14
23	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	.234	2.23	2.17	2.11	2.05	1.98
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1 + (r_1 x/r_2)]^{(r_1+r_2)/2}} dx$$

$P\{F \le f\} = 0.99$

$r_2 \backslash r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022	6056	6106	6157	6209	6235	6261	6287	6313
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82

8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.225	4.10	4.10	4.02	3.94	3.86
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.96	3.23	3.08	3.00	2.92	2.84	2.75
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.32	3.43	3.80	3.15	3.00	2.92	2.824	2.76	2.67
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55
22	7.95	5.72	4.82	4.31	3.99	3.76	2.59	3.45	3.35	3.26	2.12	2.98	2.83	2.75	2.67	2.58	2.83
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.75
24	7.82	5.61	4.72	4.22	3.90	3.67	3.77	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.67
25	7.77	5.57	4.68	4.28	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	2.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	1.52	2.44	2.35	2.26
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	.3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma[(r_1+r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1+(r_1x/r_2)]^{(r_1+r_2)/2}} dx$$

$P\{F \le f\} = 0.995$

$r_2 \backslash r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	16211	20000	21615	22500	23056	23437	23715	23925	24091	24224	24426	24630	24836	24920	25044	25148	25253

TABLE 34.5 The F-Distribution (continued)

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma(r_1 + r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1+(r_1 x/r_2)]^{(r_1+r_2)/2}} dx$$

$P\{F \le f\} = 0.995$

$r_2 \backslash r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15
4	31.33	26.28	24.26	23.15	22.46	21.87	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.65	7.53	7.42	7.31
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.44
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12
13	11.37	8.198	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3,61	3.51	3.41	3.31	3.21
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.72
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66

25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.37	3.20	3.01	2.92	2.82	2.72	2.61
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.87	2.77	2.67	2.84
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.77
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.71
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	6.48	3.48	3.21	3.04	2.86	2.76	2.66	2.56	2.66
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96

TABLE 34.6 The Poisson function

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

x \ λ	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0	3.5	4	5	6	7	8	9	10
0	0.9048	0.6703	0.5488	0.4493	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0067	0.0025	0.0009	0.0003	0.0001	0.0000
1	0.0905	0.2681	0.3293	0.3595	0.3679	0.3347	0.2707	0.2052	0.1494	0.1057	0.0733	0.0337	0.0149	0.0064	0.0027	0.0011	0.0005
2	0.0045	0.0536	0.0988	0.1438	0.1839	0.2510	0.2707	0.2565	0.2240	0.1850	0.1465	0.0842	0.0446	0.0223	0.0107	0.0050	0.0023
3	0.0002	0.0072	0.0198	0.0383	0.0613	0.1255	0.1804	0.2138	0.2240	0.2158	0.1954	0.1404	0.0892	0.0521	0.0286	0.0150	0.0076
4		0.0007	0.0030	0.0077	0.0153	0.0471	0.0902	0.1336	0.1680	0.1888	0.1954	0.1755	0.1339	0.0912	0.0573	0.0337	0.0189
5		0.0001	0.0004	0.0012	0.0031	0.0141	0.0361	0.0668	0.1008	0.1322	0.1563	0.1755	0.1606	0.1277	0.0916	0.0607	0.0378
6				0.0002	0.0005	0.0035	0.0120	0.0278	0.0504	0.0771	0.1042	0.1462	0.1606	0.1490	0.1221	0.0911	0.0631
7					0.0001	0.0008	0.0034	0.0099	0.0216	0.0385	0.0595	0.1044	0.0137	0.1490	0.1396	0.1171	0.0901
8						0.0001	0.0009	0.0031	0.0081	0.0169	0.0298	0.0653	0.1033	0.1304	0.1396	0.1318	0.1126
9							0.0002	0.0009	0.0027	0.0066	0.0132	0.0363	0.0688	0.1014	0.1241	0.1318	0.1251
10								0.0002	0.0008	0.0023	0.0053	0.0181	0.0413	0.0710	0.0993	0.1186	0.1251
11									0.0002	0.0007	0.0019	0.0082	0.0225	0.0452	0.0722	0.0970	0.1137
12									0.0001	0.0002	0.0006	0.0034	0.0113	0.0264	0.0481	0.0728	0.0948
13										0.0001	0.0002	0.0013	0.0052	0.0142	0.0296	0.0504	0.0729
14											0.0001	0.0005	0.0022	0.0071	0.0169	0.0324	0.0521
15												0.0002	0.0009	0.0033	0.0090	0.0194	0.0347
16													0.0003	0.0014	0.0045	0.0109	0.0217
17													0.0001	0.0006	0.0021	0.0058	0.0128
18														0.0002	0.0009	0.0029	0.0071
19														0.0001	0.0004	0.0014	0.0037
20															0.0002	0.0006	0.0019
21															0.0001	0.0003	0.0009
22																0.0001	0.0004
23																	0.0002
24																	0.0001

TABLE 34.7 The Poisson Distribution

$$F(x) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}$$

x \ λ	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.5	3.0	3.5	4.0
0	0.8187	0.6703	0.5488	0.4493	0.3679	0.3012	0.2466	0.2019	0.1653	0.1353	0.0821	0.0489	0.0302	0.0183
1	0.9825	0.9384	0.8781	0.8088	0.7358	0.6626	0.5918	0.5249	0.4268	0.4060	0.2873	0.1991	0.1359	0.0916
2	0.9989	0.9921	0.9769	0.9526	0.9197	0.8795	0.8335	0.7834	0.7306	0.6767	0.5438	0.4232	0.3208	0.2381
3	0.9999	0.9992	0.9966	0.9909	0.9810	0.9662	0.9463	0.9212	0.8913	0.8571	0.7576	0.6472	0.5366	0.4335
4	1.0000	0.9999	0.9996	0.9986	0.9963	0.9923	0.9857	0.9763	0.9636	0.9473	0.8912	0.8153	0.7254	0.6288
5		1.0000	1.0000	0.9998	0.9994	0.9985	0.9968	0.9940	0.9896	0.9834	0.9580	0.9161	0.8576	0.7851
6				1.0000	0.9999	0.9997	0.9994	0.9987	0.9974	0.9955	0.9858	0.9665	0.9347	0.8893
7					1.0000	1.0000	0.9999	0.9997	0.9994	0.9989	0.9958	0.9881	0.9733	0.9489
8							1.0000	1.0000	0.9999	0.9998	0.9989	0.9962	0.9901	0.9786
9									1.0000	1.0000	0.9997	0.9989	0.9967	0.9919
10											0.9999	0.9997	0.9990	0.9972
11											1.0000	0.9999	0.9997	0.9991
12												1.0000	0.9999	0.9997
13													1.0000	0.9999
14														1.0000

34.4.7 Conditional Density

$$f(x|M) = \frac{dF(x|M)}{dx} = \lim_{\Delta x \to 0} \frac{P\{x \le X \le x + \Delta x|M\}}{\Delta x}$$

$\int_{-\infty}^{\infty} f(x|M)dx = F(\infty|M) = 1$

Example

$X(f_i) = 10i, \quad i = 1, \cdots 6$ where f_i = face of a die. $M = \{f_2, f_4, f_6]$ = even event. For $x \ge 60, \{X \le x, M\}$ $= \{f_2, f_4, f_6\}, \quad f(x|M) = \frac{F\{f_2, f_4, f_6\}}{P\{M\}} = 1$; for 40≤x<60, $\{X \le x, M\} = \{f_2, f_4\}, F(x|M) = P\{f_2, f_4\}/$ $P\{M\} = (2/6)/(3/6) = 2/3$; for $20 \le x < 40, \{X \le x, M\} = \{f_2\}, F(x|M) = P\{f_2\}/P\{M\}$ =(1/6)(3/6) = 1/3; for x< 20, $\{X \le x, M\} = 0$ and $F(x|M) = 0$.

34.4.8 Total Probability

$F(x) = F(x|A_1)P(A_1) + F(x|A_2)P(A_2) + \cdots + F(x|A_n)P(A_n)P(A_n)$, A_i's are mutually exclusive and their sum is equal to the certain event **S**.

34.5 Function of One Random Variable (r.v.)

34.5.1 Random Variable (Definition)

To every experimental outcome ζ we assign a number $X(\zeta)$. The domain of X is the space S, and its range is the set I_X of the real numbers $X(\zeta)$..

34.5.2 Function of r.v.

$Y = g(X) = g[X(\zeta)]$

34.5.3 Distribution Function of Y (see 34.5.2)

$F_y(y) = P\{Y \le y\} = P\{g(X) \le y\} = P\{X \in I_y\}$

Note: To find $F_y(y)$ for a given y we must find that set I_y and the probability that X is in I_y .Refer to Figure 34.2:If $y \ge k$ then $g(x) \le y$ for any x. Hence $\{Y \le y\}$ = certain event and $F_y(y) = P\{Y \le y\} =$ 1.If $y = y_1$, then $g(x) \le y_1$ for $x \le x_1$ and,hence, $F_y(y_1) = P\{Y \le y_1\} = P\{X \le x_1\} = F_x(x_1)$ (x_1 depends on y_1), If $y = y_2$, then $g(x) = y_2$ has three solutions $x_2', x_2'', x_2''': g(x_2') = g(x_2'') = g(x_2''') = y_2$ and from Figure 34.2 $g(x) \le y_2$ if $x \le x_2'$ or $x_2'' \le x \le x_2'''$ and hence, $F_Y(y_2) = P\{X \le x_2'\} + P\{x_2'' \le x \le x_2'''\} =$ $F_X(x_2') + F_X(x_2''') - F_X(x_2'')$. If $y < \ell$ no value of x produces $g(x) \le y$ and the event $\{Y \le y\}$ has zero probability: $F_y(y) = 0$.

Example

$Y = 1/X^2$. If $y > 0$, there are two solutions: $x_1 = -\sqrt{y}, x_2 = 1/\sqrt{y}. g(x) \le y$ if $x \le x_1$ or $x \ge x_2$ and thus

$$F_y(y) = P\{Y \le y\} = P\{X \le -1/\sqrt{y}\} + P\{X \ge 1/\sqrt{y}\} = F_x(-1/\sqrt{y}) + 1 - F_x(1/\sqrt{y}).$$

if y < 0, no x will produce $g(x) \le y$ and, hence, $F_y(y) = 0$.

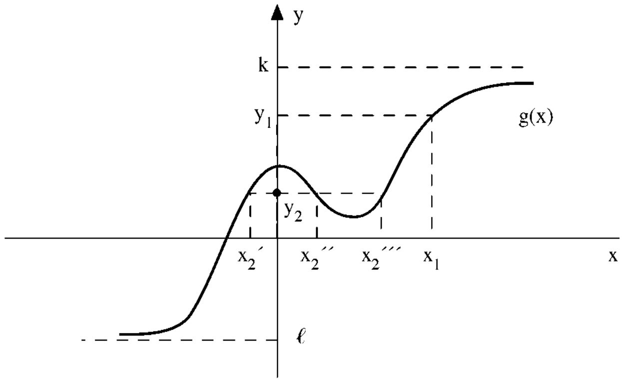


FIGURE 34.2

34.5.4 Density Function of Y=g(X) in Terms of $f_X(x)$ of X

1) Solve *y=g(x)* for x in terms of y. If $x_1, x_2, \cdots, x_n$ are all its real roots, then $y = g(x_1) = \cdots = g(x_n) = \cdots$, then $f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|} + \cdots$, $g'(x) = dg(x)/dx$. If *y=g(x)* has no real roots then $f_Y(y) = 0$.

Example 1

$g(x) = aX + b$ and $x = (y-b)/a$ for every *y*. $g'(x) = a$ and hence $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

Example 2

$g(X) = aX^2$ with the r.v. $y = ax^2, a > 0$. If $y < 0$ roots are imaginary and $f_Y(y) = 0$. If $y > 0$ then $x_1 = \sqrt{y/a}$ and $x_2 = -\sqrt{y/a}$ .Since $g'(x_1) = 2ax_1 = 2\sqrt{ay}$ and $g'(x_2) = 2ax_2 = 2\sqrt{ay}$,then f_Y (y)= $\frac{1}{2\sqrt{ay}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right] u(y)$, $u(y) =$ unit step function.

Example 3

$Y = a\sin(X+\theta), a > 0$. If $|y| < a$ then $y = a\sin(x+\theta)$ has infinitely many solutions $x_n = \sin^{-1}\frac{y}{a} - \theta$, $n = \cdots, -1, 0, 1, \cdots . dg(x_n)/dx = a\cos(x_n + \theta) = \sqrt{a^2 - y^2}$ and from 34.5.4 $f_Y(y) = \left(1/\sqrt{a^2-y^2}\right)\sum_{n=-\infty}^{\infty} f_X(x_n)$, $|y| < a$. For $|y| > 0$ there exist no solutions, and $f_Y(y) = 0$.;

Example 4

$Y = be^{-aX}u(X), a > 0, b > 0$. If $y < 0$ or $y > b$ then the equation $y = b\exp(-ax)u(x)$ has no solution, and hence $f_Y(y) = 0$. If $0 < y < b$, then $x = -(1/a)\ln(y/b).g@x) = -abe^{-ax} = -ay$ and $f_Y(y) = f_X(-(1/a)\ln(y/b))/ay, 0 < y < b$. .

34.5.5 Conditional Density of Y=g(x)

$$f_Y(y|M)=\frac{f_X(x_1|M)}{|g'(x_1)|}+\cdots+\frac{f_X(x_n|M)}{|g'(x_n)|}+\cdots$$

Example

$Y=aX^2, a>0, X\ge 0, f_x(x|X\ge 0)=\dfrac{f_X(x)}{1-F_X(0)}u(x)$ (see 34.5.4 Example 2), and hence $f_Y(y|X\ge 0)=$ $[1/(2\sqrt{ay})]\dfrac{f_X(\sqrt{y/a})}{1-F_X(0)}u(x)$.

$$\left[f(x|X\ge t)=f(x)/\{1-F(t)\}=\Big/\int_t^\infty f(x)dx, x\ge t\right]$$

34.5.6 Expected Value

$$E\{X\}=\int_{-\infty}^{\infty}xf(x)dx \text{ continuous r.v.}$$

$$E\{X\}=\sum_n x_nP\{X=x_n\}=\sum_n x_np_n \text{ discrete r.v.}$$

34.5.7 Expected Value of a Function g(X)

$$E\{Y=g(X)\}=\int_{-\infty}^{\infty}yf_Y(y)dy=\int_{-\infty}^{\infty}g(x)f_x(x)dx \text{ continuous r.v.}$$

$$E\{g(X)\}=\sum_k g(x_k)P\{X=x_k\} \text{ discrete type of r.v.}$$

34.5.8 Conditional Expected Value

$$E\{X|M\}=\int_{-\infty}^{\infty}xf(x|M)dx \text{ continuous r.v.}$$

$$E\{X|M\}=\sum_n x_nP\{X=x_n|M\} \text{ discrete r.v.}$$

34.5.9 Variance

$$\sigma^2=E\{(X-\mu)^2=\int_{-\infty}^{\infty}(x-\mu)^2f(x)dx \text{ continuous r.v.}$$

$$\sigma^2=\sum_n(x_n-\mu)^2P\{X=x_n\} \text{ discrete r.v.}$$

$$\sigma^2 = E\{X^2\} - E^2\{X\}$$

Example

$P\{X = k) = e^{-\lambda}\dfrac{\lambda^k}{k!}, k = 0,1,\cdots =$ Poisson distribution.

$$E\{X) = \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!}.$$

but

$$\frac{d}{d\lambda}e^{\lambda} = \frac{d}{d\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^{k-1}}{k!} = \frac{1}{\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!} = e^{\lambda}$$

or

$$\lambda = e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!}$$

and hence, $E\{X\} = \lambda.$.

34.5.10 Moments About the Origin

$\mu_k' = E\{X^k\} = \int_{-\infty}^{\infty} x^k f(x)dx = \sum_{r=0}^{k}\binom{k}{r}\mu^r \mu_{k-r}, \mu_1' = \mu = E\{X\}, \mu_0' = 1$

34.5.11 Central Moments

$$\mu_k = E\{X^k\} = \int_{-\infty}^{\infty}(x-\mu)^k f(x)dx = E\left\{\sum_{r=0}^{k}\binom{k}{r}(-1)^r \mu^r X^{k-r}\right\} = \sum_{r=0}^{k}\binom{k}{r}(-1)^r \mu^r \mu_{k-r}'$$

$\mu_0 = \mu_0' = 1, \mu_1 = \mu_1' - \mu = 0,\ \mu_2 = \mu_2' - 2\mu\mu_1' + \mu^2 = \mu_2' - \mu^2, \mu_3 = \mu_3' - 3\mu\mu_2' + 3\mu^2\mu_1' - \mu^3 = \mu_3' - 3\mu\mu_2' + 2\mu^3$

34.5.12 Absolute Moments

$M_k = E\{|X|^k\} = \int_{-\infty}^{\infty}|x| f(x)dx$

34.5.13 Generalized Moments

${}_a\mu_k' = E\{(X-a)^k\}, \quad {}_aM_k' = E\{|X-a|^k\}$

Example 1

$$E\{X^{2n}\} = \frac{1}{2a}\int_{-a}^{a} x^{2n} dx = \frac{a^{2n}}{2n+1}, \sigma^2 = E\{x^2\} = \frac{a^2}{3}$$

for X uniformly distributed in (-a,a).

Example 2

$$E\{X^n\} = \frac{a^{b+1}}{\Gamma(b+1)}\int_0^{\infty} x^n x^b e^{-ax} dx = \frac{a^{b+1}\Gamma(b+n+1)}{a^{b+n+1}\Gamma(b+1)}$$

for a gamma density $f(x) = \{a^{b+1} / \Gamma(b+1)]x^b e^{-ax} u(x), u(x) =$ unit step function.

34.5.14 Tchebycheff Inequality

$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}, \mu = E\{X\}$. Regardless of the shape of $f(x)$, $P\{\mu - \varepsilon < X < \mu + \varepsilon\} \geq 1 - \frac{\sigma^2}{\varepsilon^2}$

Generalizations:

1. If $f_y(y) = 0$ then $P\{Y \geq \alpha\} \leq \frac{E\{Y\}}{\alpha}, \alpha > 0$
2. $P\{|X-\alpha|^n \geq \varepsilon^n\} \leq \frac{E\{|X-\alpha|^n\}}{\varepsilon^n}$

34.5.15 Characteristic Function

$$\Phi(\omega) = E\{e^{j\omega x}\} = \int_{-\infty}^{\infty} f(x)dx \text{ for continuous r.v.}$$

$$\Phi(\omega) = \sum_k e^{j\omega x_k} P\{X = x_k\} \text{ for discrete type r.v.}$$

$$\Phi(0) = 1, \quad |\Phi(\omega)| \leq 1$$

Example 1

$$\Phi(\omega) = E\{e^{j\omega Y}\} = E\{e^{j\omega(aX+b)}\} = e^{j\omega b} E\{e^{j\omega aX}\}, \text{ if } Y = aX + b$$

Example 2

$$P\{X = k\} = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0,1,\cdots \text{ Poisson distribution } \Phi(\omega) = e^{-\lambda}\sum_{k=0}^{\infty} e^{j\omega k}\frac{\lambda^k}{k!} = e^{\lambda}(e^{j\omega} - 1)$$

34.5.16 Second Characteristic Function

$\Psi(\omega) = \ln \Phi(\omega)$

34.5.17 Inverse of the Characteristic Function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega) e^{-j\omega x} d\omega$$

34.5.18 Moment Theorem and Characteristic Function

$$\frac{d^n \Phi(0)}{d\omega^n} = j^n \mu'_{n\copyright} \quad E\{X^n\}$$

34.5.19 Convolution and Characteristic Function

$\Phi(\omega) = \Phi_1(\omega)\Phi_2(\omega)$, where $\Phi_1(\omega)$ and $\Phi_2(\omega)$ are the characteristic functions of the density functions $f_1(x)$ and $f_2(x)$. $\Phi(\omega) = E\{e^{j\omega(X_1+X_2)}\}$ and $f(x) = f_1(x) * f_2(x)$ where * indicates convolution.

34.5.20 Characteristic Function of Normal r.v.

$\Phi(\omega) = \exp(j\mu\omega - \frac{1}{2}\sigma^2\omega^2)$

34.6 Two Random Variables

34.6.1 Joint Distribution Function

$$F_{xy}(xy) = P\{X \le x, Y \le y\}, \quad F_{xy}(x,\infty) = F_x(x), F_{xy}(\infty, y) = F_y(y),$$
$$F_{xy}(\infty,\infty) = 1, F_{xy}(-\infty, y) = 0, F_{xy}(x, -\infty) = 0$$

34.6.2 Joint Density Function

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}, f(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad f_y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

34.6.3 Conditional Distribution Function

$$F_y(y|M) = P\{Y \le y|M\} = \frac{P\{Y \le y, M\}}{P\{M\}}, F\{y|X \le x\} = \frac{P\{X \le x, Y \le y\}}{P\{X \le x\}} = \frac{F_{xy}(x,y)}{F_x(x)}$$

$$F_y(y|X \le a, Y \le b) = \frac{P\{X \le a, Y \le b, Y \le y\}}{P\{X \le a, Y \le b\}} = \begin{cases} 1 & y \ge b \\ F_{xy}(a,y)/F_{xy}(a,b) & y < b \end{cases}$$

34.6.4 Conditional Density Function

$$f_y(y|X \le x) = \frac{\partial F_{xy}(x,y)/\partial y}{F_x(x)} = \frac{\int_{-\infty}^{x} f_{xy}(\xi,y)d\xi}{\int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{xy}(\xi,y)d\xi dy}, \quad f_y(y|x < X \le x_2 = \frac{\int_{x_1}^{x_2} f_{xy}(x,y)dx}{\Gamma_x(x_2) - F_x(x_1)},$$

$$f_y(y|X = x) = \frac{f_{xy}(x,y)}{f_x(x)}$$

34.6.5 Baye's Theorem

$$f_y(y|X = x) = \frac{f_x(x|Y = y) f_y(y)}{f_x(x)}$$

34.6.6 Joint Conditional Distribution

$$F_{xy}(x,y|a < X \le b) = \frac{P\{X \le x, Y \le y, a < Y \le b\}}{P\{a < X \le b\}} = \begin{cases} \dfrac{F_{xy}(b,y) - F_{xy}(a,y)}{F_x(b) - F_x(a)} & x > b \\ \dfrac{F_{xy}(x,y) - F_{xy}(a,y)}{F_x(b) - F_x(a)} & a < x \le b \\ 0 & x \le a \end{cases}$$

34.6.7 Conditional Expected Value

$$E\{g(Y)|X = x\} = \int_{-\infty}^{\infty} g(y) f_y(y|X = x)dy = \frac{\int_{-\infty}^{\infty} g(y) f_{xy}(x,y)dy}{\int_{-\infty}^{\infty} f_{xy}(x,y)dy}, \quad E\{E\{Y|X\}\} = E\{Y\}$$

34.6.8 Independent r.v.

$$F_{xy}(x,y) = F_x(x)F_y(y);\ f_{xy}(x,y) = f(x)f(y);\ f_y(y|x) = f_y(y);\ f_x(x|y) = f_x(x)$$

34.6.9 Jointly Normal r.v.

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2r(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right]$$

$E\{X\} = \mu_1$, $E\{Y\} = \mu_2$, $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$. If $r = 0$, $f(x,y) = f_x(x) f_y(y) \equiv$ independent. $|r| < 1$, $r =$ correlation coeffifient.

Conditional Densities

$$f_y(y|X=x)=\frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}}\exp\left[-\frac{1}{2\sigma^2(1-r^2)}\left[y-\mu_2-\frac{r\sigma_2}{\sigma_1}(x-\mu_1)\right]^2\right]$$

$$E\{Y|X=x\}=\mu_2+\frac{r\sigma_2}{\sigma_1}(x-\mu_1),\quad \sigma_{y|x=x}=\sigma_2\sqrt{1-r^2}$$

If $\mu_1=\mu_2=0$ then $E\{Y^2|X=x\}=\sigma_2^2(1-r^2)+\dfrac{r^2\sigma_2^2}{\sigma_1^2}x^2$

34.7 Functions of Two Random Variables

34.7.1 Definitions

$Z=g(X,Y)=g[X(\zeta),Y(\zeta)]$, $F_z(z)=P\{Z\le z\}$, $D_z=$ region of xy-plane such that $g(x,y)\le z$, $\{Z\le z\}$ $=\{(X,Y)\in D_z\}$

34.7.2 Distribution Function

$$f_z(z)dz=P\{z<Z\le z+dz\}=\iint_{\Delta D_z} f_{xy}(x,y)dxdy$$

34.7.3 Density Function

$$f_z(z)dz=P\{z<Z\le z+dz\}=\iint_{\Delta D_z} f_{xy}(x,y)dxdy$$

Example 1

$Z=X+Y, x+y\le z$, $F_z(z)=\int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{xy}(x,y)dxdy, \dfrac{dF_z(z)}{dz}=f_z(z)=\int_{-\infty}^{\infty} f_{xy}(z-y,y)dy$. If the r.v. are independent then $f_{xy}(x,y)=f_x(x)f_y(y)$ and hence $f_z(z)=\int_{-\infty}^{\infty} f_x(z-y)f_y(y)dy=\int_{-\infty}^{\infty} f_x(x)f_y(z-x)dx=$ $f_x(z)*f_y(z)=$ convolution of densities.

Example 2

$Z=X^2+Y^2$, if $z>0$ so then $x^2+y^2\le z=$ circle with radius $\sqrt{z}, F_z(z)=\iint_{x^2+y^2\le z} f(x,y)dxdy$, if $z<0$, $F_z(z)=0$. $f_{xy}(x,y)=(1/2\pi\sigma^2)\exp[-(x^2+y^2)/2\sigma^2$ then $F_z(z)=\dfrac{1}{2\pi\sigma^2}\int_0^z 2\pi re^{-r^2/2\sigma^2}dr=1-e^{-z/2\sigma^2}$, $z>0$ and $f_z(z)=\dfrac{1}{2\sigma^2}e^{-z/2\sigma^2}, z\ge 0$

Example 3

$f_{xy}(x,y) = (1/2\pi\sigma^2)\exp[-(x^2+y^2)/2\sigma^2]$, $Z = +\sqrt{X^2+Y^2}$, $F_z(z) = \frac{1}{2\pi\sigma^2}\int_0^z 2\pi r e^{-r^2/2\sigma^2} dr = 1 - e^{-z^2/2\sigma^2}$, $z > 0$, $f_z(z) = (z/\sigma^2)\exp(-z^2/2\sigma^2 0, z > 0 \equiv$ Rayleigh distributed, $E\{Z\} = \sigma\sqrt{\pi/2}$, $E\{Z^2\} = 2\sigma^2, \sigma_z^2$ $= (2-(\pi/2))\sigma^2$

Example 4

If $f_{xy}(x,y) = f_{xy}(-x,-y)$ then $F_z(z) = 2\int_0^\infty \int_{-\infty}^{yz} f_{xy}(x,y)dxdy$, $f_z(z) = 2\int_0^\infty y f_{xy}(zy,y)dy$. Then for $f_{xy}(x,y)$ $= (1/[2\pi\sigma_1\sigma_2\sqrt{1-r^2}])\exp\left[-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_1^2}\right)\right]$ then $f_z(z)$ of $Z = X/Y$ is

$$f_z(z) = [2/(2\pi\sigma_1\sigma_2\sqrt{1-r^2})]\int_0^\infty y\exp\left[-\frac{y^2}{2(1-r^2)}\left[\frac{z^2}{\sigma_1^2} - \frac{2rz}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right]\right]dy.$$

But $\int_0^\infty y\exp[-y^2/2a^2]dy = a^2\int_0^\infty e^{-w}dw = a^2$ and hence

$$f_z(z) = [(\sqrt{1-r^2}\sigma_1\sigma_2/\pi]/[\sigma_2^2(z - r\sigma_1/\sigma_2)^2 + \sigma_1^2(1-r^2)].$$

If $\mu_2 = \mu_2 = 0$ then $f_z(z)$ is Cauchy density.

34.8 Two Functions of Two Random Variables

34.8.1 Definitions

$Z = g(X,Y)$, $W = h(X,Y)$, D_{zw} region of the xy plane such that $g(x,y) \le z$ and $h(x,y) \le w$, $\{Z \le z, W \le w\} = \{(X,Y) \in D_{xy}\}$, $F_{zw}(z,w) = \iint_{D_{zw}} f_{xy}(x,y)dxdy$

34.8.2 Density Function $f_{zw}(z,w)$

$f_{zw}(z,w) = \frac{f_{xy}(x_1,y_1)}{|J(x_1,y_1)|} + \cdots + \frac{f_{xy}(x_n,y_{n1})}{|J(x_n,y_n)|} + \cdots, z = g(x_i,y_i), w = h(x_i,y_i)$ where (x_i,y_i) are solutions. if there are no real solutions for certain values of (z,w) then $f_{zw}(z,w) = 0$.

Jacobian of transformation

$$J(x,y) = \begin{vmatrix} \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \\ \frac{\partial h(x,y)}{\partial x} & \frac{\partial h(x,y)}{\partial y} \end{vmatrix}$$

Example 1

If $z = ax + by,\ w = cx + dy$ then $x = a_1 z + b_1 y,\ y = c_1 z + d_1 w,$ where a_1, b_1, c_1 and d_1 are functions of a,b,c, and d.

$$J(x,y) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,\ f_{zw} = (z,w) = 1/[|ad - bc|]\, f_{xy}(a_1 z + b_1 w, c_1 z + d_1 w)$$

Example 2

$z = +\sqrt{x^2 + y^2},\ w = x/y.$ If $z > 0$ then the system has two solutions: $x_1 = zw/\sqrt{1+w^2},\ y_1 = z/\sqrt{1+w^2}$ and $x_2 = -x_1, y_2 = -y_1$ for any w.

$$J(x,y) = \begin{vmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ 1/y & -x/y^2 \end{vmatrix} = (1+w^2)/(-z)$$

and from 34.8.2

$$f_{zw}(z,w) = [z/(1+w^2)]\left[f_{xy}\left(\frac{zw}{\sqrt{1+w^2}}, \frac{z}{\sqrt{1+w^2}} \right) + f_{xy}\left(\frac{-zw}{\sqrt{1+w^2}}, \frac{-z}{\sqrt{1+w^2}} \right) \right].$$

If $z < 0,\ f_{zw}(z,w) = 0.$

34.8.3 Auxiliary Variable

If $z = g(x,y)$ we can introduce an auxiliary function w = x or w = y. $f_z(z) = \int_{-\infty}^{\infty} f_{zw}(z,w)\,dw.$

Example

If $z = xy$ set auxiliary function w = x. The system has solutions

$$x = w,\ y = z/w. \quad J(x,y) = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -w$$

and, hence,

$$f_{zw}(z,w) = (1/|w|)\, f_{xy}(w, z/w) \quad \text{and} \quad f_z(z) = \int_{-\infty}^{\infty} (1/|w|)\, f_{xy}(w, z/w)\,dw.$$

34.8.4 Functions of Independent r.v.'s

If X and Y are independent then $Z = g(X)$ and $W = h(Y)$ are independent and

$$f_{zw}(z,w) = \frac{f_x(x_1)}{|g'(x_1)|}\ \frac{f_y(y_1)}{|h'(y_1)|}$$

since

$$J(x,y)=\begin{vmatrix} g'(x) & 0 \\ 0 & h'(x) \end{vmatrix} = g'(x)h'(x)$$

34.9 Expected Value, Moments, and Characteristic Function of Two Random Variables

34.9.1 Expected Value

$$E\{g(X,Y)\} = \iint_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dxdy;$$

$$E\{z\} = \int_{-\infty}^{\infty} z f_z(z) dz \text{ if } z = g(x,y);\; E\{g(X,Y)\} = \sum_{k,n} g(x_k, y_n) p_{kn},$$

$$P\{X = x_k,\; Y = y_n\} = p_{kn} \text{ discrete case r.v.}$$

34.9.2 Conditional Expected Values

$$E\{g(X,Y|M)\} = \iint_{-\infty}^{\infty} g(x,y)\, f(x,y|M)\, dxdy;$$

$$E\{g(X,Y|X = x\} = \int_{-\infty}^{\infty} g(x,y)\, f(x,y) dy / f_x(x) = \int_{-\infty}^{\infty} g(x,y)\, f(y|X = x) dy$$

34.9.3 Moments

$$\mu'_{kr} = E\{X^k Y^r\} = \iint_{-\infty}^{\infty} x^k y^r f(x,y) dxdy, \mu'_{11} = R_{xy} = E\{XY\}$$

$$\mu_{kr} = E\{(X-\mu_x)^k (Y-\mu_y)^r\} = \iint_{-\infty}^{\infty} (x-\mu_x)^k (y-\mu_y)^r f(x,y) dxdy$$

$$\mu_{20} = \sigma_x^2,\; \mu_{02} = \sigma_y^2,\; \mu = \mu'_1$$

34.9.4 Covariance

$$\mu_{11} = E\{(X-\mu_x)(Y-\mu_y)\} = E\{XY\} - \mu_x E\{Y\} - \mu_y E\{X\} + \mu_x \mu_y$$

34.9.5 Correlation Coefficient

$$r = E\{(X-\mu_x)(Y-\mu_y)\} / \sqrt{E\{(X-\mu_x)^2 E\{(Y-\mu_y)^2\}} = \mu'_{11} / \sigma_x \sigma_y$$

$$\mu_{11}'^2 \le \mu_{20}'\mu_{02}', \mu_{11}'^2 \le \mu_{20}\mu_{02}, \ |r| = \left|\mu_{11}'^2\right| / \sqrt{\mu_{20}\mu_{02}} \le 1$$

34.9.6 Uncorrelated r.v.'s

$E\{XY\} = E\{X\}E\{Y\}$

34.9.7 Orthogonal r.v.'s

$E\{XY\} = 0$

34.9.8 Independent r.v.'s

$f(x,y) = f_x(x) f_y(y)$

Note:

1. If X and Y are independent, *g(X)* and *h(Y)* are independent or

$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$$

2. If X and Y are uncorrelated, then

   a. $E\{(X-\mu_x)(Y-\mu_y)\} = 0, \ r = 0$
   b. $\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$
   c. $E\{(X+Y)^2\} = E\{X^2\} + E\{Y^2\}$
   d. $E\{g(X)h(Y)\} \ne E\{g(X)\}E\{h(Y)\}$ in general

34.9.9 Joint Characteristic Function

$$\Phi_{xy}(\omega_1,\omega_2) = E\{e^{j(\omega_1 x+\omega_2 y)} = \int_{-\infty}^{\infty}\int f_{xy}(x,y)e^{j(\omega_1 x+\omega_2 y)}dxdy, \Psi_{xy}(\omega_1,\omega_2) = \ln \Phi_{xy}(\omega_1,\omega_2)$$

$$f_{xy}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int e^{-j(\omega_1 x+\omega_2 y)}\Psi_{xy}(\omega_1\omega_2)d\omega_1 d\omega_2$$

$$\Phi_x(\omega) = E\{e^{j\omega X}\} = \Phi_{xy}(\omega,0), \ \Phi_y(\omega) = \Phi_{xy}(0,\omega)$$

Example

$$\Phi_z(\omega) = E\{e^{j\omega z}\} = E\{e^{j(a\omega X+b\omega Y)}\} = \Phi_{xy}(a\omega,b\omega) \text{ if } Z = aX+bY.$$

$$\Phi_{xy}(\omega_1,\omega_2) = \Phi_x(\omega_1)\Phi_y(\omega_2)$$

if X and Y are independent.

34.9.10 Moment Theorem

$$\frac{\partial^k \partial^r \Phi(0,0)}{\partial\omega_1^k \partial\omega_2^r} = j^{(k+r)} \mu'_{kr}$$

34.9.11 Series Expansion of $\Phi(\omega_1,\omega_2)$

$$\Phi(\omega_1,\omega_2) = 1 + jE\{X\}\omega_1 + jE\{Y\}\omega_2 - \tfrac{1}{2}\{X^2\}\omega_1^2$$

$$-\tfrac{1}{2}E\{Y^2\}\omega_2^2 - E\{XY\}\omega_1\omega_2 + \cdots + \frac{1}{4!}\binom{4}{2}E\{X^2Y^2\}\omega_1^2\omega_2^2 + \cdots,$$

$$\Psi(\omega_1,\omega_2) = \ln\Phi(\omega_1,\omega_2) = j\mu_x\omega_1 + j\mu_y\omega_2 - \tfrac{1}{2}\sigma_x^2\omega_1^2 - r\sigma_x\sigma_y\omega_1\omega_2 - \tfrac{1}{2}\sigma_y^2\omega_2^2 + \cdots$$

34.10 Mean Square Estimation of R.V.'s

34.10.1 Mean Square Estimation of r.v.'s

a. a minimizes $E\{X-a)^2\}$ if $a = E\{X\} = \mu_x$

b. The function $g(X) = E\{Y|X\}$ = regression curve minimizes

$$E\{[Y-g(X)]^2\} = \int_{-\infty}^{\infty}\int [y-g(x)]^2 f(x,y)dxdy$$

c. $a = \dfrac{r\sigma_y}{\sigma_x}$ and $b = E\{Y\} - aE\{X\}$ minimize the m.s. error

$$e = E\{[Y-(aX+b)^2]\} = \int_{-\infty}^{\infty}\int [y-(ax-b)^2] f(x,y)dxdy$$

$e_m =$ minimum error $= \sigma_y^2(1-r^3)$, $r =$ correlation coefficient of X and Y.

d. If $E\{X\} = E\{Y\} = 0$ the constant a that minimizes the m.s. error $e = E\{(y-ax)^2\}$ is such that $E\{(Y-aX)X\} = 0$ (orthogonality principle) and the minimum m.s. error is: $e_m = E\{(Y-aX)Y\}$

$a = E\{XY\}/E\{X^2\}$ and hence $e_m = E\{Y^2\} - \dfrac{E^2\{XY\}}{E\{X^2\}}$, also $e_m = E\{Y^2\} - E\{(aX)^2\}$

$e_m \ge E\{[Y - E\{Y|X\}]^2\}$

34.11 Normal Random Variables

34.11.1 Jointly Normal

If $e\{X\} = E\{Y\} = 0$ the normal joint density is:

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma\sqrt{1-r^2}}\exp\left[-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)\right],\quad E\{X^2\}=\sigma_1^2,\quad E\{Y^2\}=\sigma_2^2$$

34.11.2 Conditional Density

$$f(y|x)=\frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}}\exp\left[-\frac{1}{2\sigma_2^2(1-r^2)}\left(y-\frac{r\sigma_2}{\sigma_1}x\right)^2\right],$$

$$E\{Y|X\}=\frac{r\sigma_2}{\sigma_1}X,\quad E\{Y^2\big|X\}=\sigma_2^2(1-r^2)+\frac{r^2\sigma_2^2}{\sigma_1^2}x^2$$

$$E\{XY\}=r\sigma_1\sigma_2,\quad E\{X^2Y^2]=\sigma_1^2\sigma_2^2+2r^2\sigma_1^2\sigma_2^2$$

34.11.3 Mean Value

$$E\{(X-\mu_x)(Y-\mu_y)\}=r\sigma_1\sigma_2$$

34.11.4 Linear Transformations

$$Z=aX+bY,\quad W=cX+dY.$$

If X and Y are jointly normal with zero mean then

$$\sigma_z^2=E\{Z^2\}=E\{(aX+bY)^2\}=a^2\sigma_x^2+b^2\sigma_y^2+2abr_{xy}\sigma_x\sigma_y$$

$$\sigma_w^2=E\{W^2\}=c^2\sigma_x^2+d^2\sigma_y^2+2cdr_{xy}\sigma_x\sigma_y,$$

$$r_{zw}\sigma_z\sigma_w=E\{ZW\}=ac\sigma_x^2+bd\sigma_y^2+(ad+bc)r_{xy}\sigma_x\sigma_y$$

34.12 Characteristic Functions of Two Normal Random Variables

34.12.1 Characteristic Function

$\Phi(\omega_1,\omega_2)=E\{\exp[j(\omega_1X+\omega_2Y]\}=\exp[-\tfrac{1}{2}(\sigma_1^2\omega_1^2+2r\sigma_1\sigma_2\omega_1\omega_2+\sigma_2^2\omega_2^2)]$ for $E\{X\}=E\{Y\}=0$, and X and Y jointly normal.

34.12.2 Characteristic Function with Means

$\Phi(\omega_1,\omega_2)=\exp[j(\omega_1\mu_x+\omega_2\mu_y)]\exp\{-\tfrac{1}{2}\mu_{20}\omega_1^2+2\mu_{11}\omega_1\omega_2+\mu_{02}\omega_2^2]$, $\mu_{ij}=$ joint moments about the means.

34.13 Price Theorem for Two R.V's

34.13.1 Price Theorem

If X and Y are jointly normal with

$$\mu_{11} = E\{(X-\mu_x)(Y-\mu_y)\} = E\{XY\} - E\{X\}E\{Y\},$$

then,

$$E\{g(X,Y)\} = \iint_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dxdy$$

a. If $\mu_{11} = 0$ (r.v.'s independent) $E\{X^k Y^r\} = E\{X^k\}E\{Y^r\}$

b. $E\{X^k Y^r\} = kr\int_0^{\mu_{11}} E\{X^{k-1}Y^{r-1}\}d\mu_{11} + E\{X^k\}E\{Y^r\}$

c. $E\{X^2Y^2\} = 4\int_0^{\mu_{11}} E\{XY\}d\mu_{11} + E\{X^2\}E\{Y^2\} = 4\int_0^{\mu_{11}}(\mu_{11} + E\{X\}E\{Y\})d\mu_{11}$
$+E\{X^2\}E\{Y^2\} = 2\mu_{11}^2 + 4\mu_{11}E\{X\}E\{Y\} + E\{X^2\}E\{Y^2\}$

34.14 Sequences of Random Variables

34.14.1 Definitions

31.14.1.1 Definitions

n real r.v. $X_1, X_2, \cdots, X_n; F(x_1, x_2, \cdots, x_n) = P\{X_1 \le x_1, \cdots, X_n \le x_n\} =$ distribution function; $f(x_1, \cdots, x_n)$ $= \partial^n F / \partial x_1, \cdots, \partial x_n =$ density function.

31.14.1.2 Marginal Densities

$F(x_1, x_3) = F(x_1, \infty, x_3, \infty) =$ marginal distribution for a sequence of four r.v. ; $f(x_1, x_3) = \iint_{-\infty}^{\infty}$ $f(x_1, x_2, x_3, x_4)dx_2dx_4$ marginal density.

31.14.1.3 Functions of r.v.'s

$Y_1 = g_1(X_1, \cdots, X_n), \cdots, \quad Y_n = g_n(X_1, \cdots, X_n),$

$$f_{y_1,\cdots y_n}(y_1, \cdots, y_n) = f(x_1, x_2, \cdots, x_n)/\left|J(x_1, \cdots, x_n)\right|, \quad J(x_1, \cdots, x_n) = \begin{vmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \dfrac{\partial g_n}{\partial x_1} & \cdots & \dfrac{\partial g_n}{\partial x_n} \end{vmatrix}$$

31.14.1.4 Conditional Densities

$$f(x_1, \cdots, x_k \mid x_{k+1}, \cdots, x_n) = f(x_1, \cdots, x_k, \cdots, x_n)/ f(x_{k+1}, \cdots, x_n).$$

Example

$$f(x_1 \mid x_2, x_3) = f(x_1, x_2, x_3)/ f(x_2, x_3), \quad F(x_1 \mid x_2, x_3) = \int_{-\infty}^{x_1} f(\xi_1, x_2, x_3)d\xi_1 / f(x_2, x_3)$$

34.14.1.5 Chain Rule

$f(x_1,\cdots,x_n) = f(x_n|x_{n-1},\cdots,x_1)\cdots f(x_2|x_1)f(x_1)$

34.14.1.6 Removal Rule

$$f(x_1|x_3) = \int_{-\infty}^{\infty} f(x_1,x_2|x_3)dx_2,$$

$$f(x_1|x_4) = \iint_{-\infty}^{\infty} f(x_1|x_2,x_3,x_4)f(x_2,x_3|x_4)dx_2dx_3, \quad f(x_1|x_3) = \int_{-\infty}^{\infty} f(x_1|x_2,x_3)f(x_2|x_3)dx_2$$

34.14.1.7 Independent r.v.

$$F(x_1,\cdots,x_n) = F(x_1)\cdots F(x_n);\ f(x_1,\cdots,x_n) = f(x_1)\cdots f(x_n)$$
$$f(x_1,\cdots,x_k,x_{k+1},\cdots,x_n) = f(x_1,\cdots,x_k)f(x_{k+1},\cdots,x_n)$$

if $X_1,\cdots,X_k$ are independent of $X_{k+1},\cdots,X_n$

34.14.2 Mean, Moments, Characteristic Function

34.14.2.1 Expected Value

$$E\{g(X_1,\cdots,X_n)\} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\cdots,x_n)f(x_1,\cdots,x_n)dx_1\cdots dx_n$$

34.14.2.2 Conditional Expected Values

$$E\{X_1|x_2,\cdots,x_n\} = \int_{-\infty}^{\infty} x_1 f(x_1|x_2,\cdots,x_n)dx_1 = \int_{-\infty}^{\infty} x_1 f(x_1,\cdots,x_n)dx_1 / f(x_2,\cdots,x_n)$$

a. $E\{E\{X_1|X_2,\cdots,X_n\}\} = E\{X_1\}$

b. $E\{X_1X_2|X_3\} = E\{E\{X_1X_2|X_2,X_3\} = E\{X_2E\{X_1|X_2,X_3\}|X_3\}$

c. $E\{X_1|x_2,\cdots,x_n\} = E\{X_1\}$ if X_1 is independent from the remaining r.v.'s

34.14.2.3 Uncorrelated r.v.'s

$X_1,\cdots,X_n$ are uncorrelated if the covariance of any two ofthem is zero, $E\{X_iX_j\} = E\{X_i\}E\{X_j\}$ for $i \neq j$

34.14.2.4 Orthogonal r.v.'s

$E\{X_iX_j\} = 0$ for any $i \neq j$

34.14.2.5 Variance of Uncorrelated r.v.'s

$\sigma^2_{x_1+\cdots+x_n} = \sigma^2_{x_1} + \cdots + \sigma^2_{x_n}$, $\sigma_z^2 = E\{|Z - E\{Z\}|^2\}$ if $Z = X + jY \equiv$ complex r.v. , $E\{Z_iZ_j^*\} = E\{Z_i\}E\{Z_j^*\}$ $\equiv$ uncorrelated r.v.'s $i \neq j$, $E\{Z_iZ_j^*\} = 0 \equiv$ orthogonal,

$$f(x_1,y_1,x_2,y_2) = f(x_1,y_1)f(x_2,y_2) \text{ if } Z_1 = X_1 + jY_1 \text{ and } Z_2 = X_2 + jY_2$$

are independent,

$E\{Z_i Z_2^*\} = E\{Z_1\}E\{Z_2^*\} = \int\int_{-\infty}^{\infty} z_1 f(x_1,y_1)dx_1 dy_1 \int\int_{-\infty}^{\infty} z_2^* f(x_2,y_2)dx_2 dy_2 =$ uncorrelated if they are independent.

34.14.2.6 Characteristic Functions

$\Phi(\omega_1,\cdots,\omega_n) = E\{e^{j(\omega_1 X + \cdots + \omega_n X_n)}\}$,

$$\frac{\partial^r \Phi(0,0)}{\Phi\omega_1^{k_1},\cdots,\partial\omega_n^{k_n}} = j^r \mu'_{k_1\cdots k_n},\ \mu'_{k_1\cdots k_n} = E\{X_1^{k_1}\cdots X_n^{k_n}\},\ r = k_1 + k_2 + \cdots + k_n$$

for dependent variables

34.14.2.7 Characteristic Functions for Independent Variables

$E\{e^{j\omega_1 X_1}\cdots e^{j\omega_n X_n}\} = E\{e^{j\omega_1 X_1}\}\cdots E\{e^{j\omega_n X_n}\}$

Example

$Z = X_1 + X_2 + X_3,\ \Phi_z(\omega) = E\{e^{j\omega(X_1+X_2+X_3)}\} = \Phi_1(\omega)\Phi_2(\omega)\Phi_3(\omega)$. Hence,

$$f_z(z) = \text{density function } = f_1(z) * f_2(z) * f_3(z)$$

34.14.2.8 Sample Mean

$\overline{X} = (X_1 + \cdots + X_n)/n,\ \overline{X} =$

34.14.2.9 Sample Variance

$$\overline{S} = [(X_1 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2]/n = \sum_{i=1}^{n} \frac{X_i^2}{n} - \overline{X}^2,\ \overline{S} = \text{random variable}$$

34.14.2.10 Statistic

A function of one or more variables that does not depend upon any unknown parameter.

Example

$Y = \sum_{i=1}^{n} X_k \equiv$ is a statistic; $Y = (X_1 - \mu)/\sigma \equiv$ is not a statistic unless μ and σ are known: the sample mean is a statistic.

34.14.2.11 Random Sums

$Y = \sum_{k=1}^{n} X_k =$ random sum; $E\{Y\} = \mu E\{n\}$ if n is an r.v. of discrete type and X_i 's are independent of n and $E\{X_k\} = \mu$; $E\{Y^2\} = \mu^2 E\{n^2\} + \sigma^2 E(n)$ if the r.v. X_k are uncorrelated with the same variance σ^2.

34.14.3 Normal Random Variables

34.14.3.1 Density Function

$f(x_1,\cdots,x_n) = (1/[(2\pi)^{n/2}\sigma^n])\exp[-(x_1^2 + \cdots + x_n^2)/2\sigma^2]$ where the r.v. X_i are normal, independent with same variance σ^2.

34.14.3.2 Density Eunction of the Sample Mean

$f_{\bar{x}}(x) = (1/\sqrt{2\pi\sigma^2/n})\exp(-nx^2/2\sigma^2)$ (see also 34.14.3.1)

34.14.3.3 Density Function of χ =[X$_1$+ … +X$_n$]$^{1/2}$ (see also 34.14.3.1)

$f_x(\chi) = \dfrac{2}{2^{n/2}\sigma^n\Gamma(n/2)}\chi^{n-1}\exp(-\chi^2/2\sigma^2)u(\chi)$, $n = n$ degrees of freedpm, $u(\chi)$ = unit step function.

34.14.3.4 Density Function of $Y = \chi^2 = X_1^2 + \cdots + X_n^2$ (see also 34.14.3.1)

$f_y(y) = \dfrac{1}{2^{n/2}\sigma^n\Gamma(n/2)}y^{(n-2)/2}\exp(-ns/2\sigma^2)u(y)$, $u(y) =$ unit step function

34.14.3.5 Density Function of the Variance $\bar{\mathrm{S}}$ (see also 34.14.2.9 and 34.14.3.1)

$f_{\bar{s}}(s) = \dfrac{1}{2^{(n-2)/2}(\sigma/\sqrt{n})^{n-1}\Gamma[(n-1)/2]}s^{(n-3)/2}\exp(-ns/2\sigma^2)u(s)$

34.14.3.6 Characteristic Function

$\Phi(\omega_1,\omega_2) = E\{e^{j(\omega_1 X_1+\omega_2 X_2)}\} = \exp[-\tfrac{1}{2}(\sigma_1^2\omega_1^2 + r\sigma_1\sigma_2\omega_1\omega_2 + r\sigma_1\sigma_2\omega_2\omega_1 + \sigma_2^2\omega_2^2)]$

$\sigma_1^2 = \mu'_{11},\ \sigma_2^2 = \mu'_{22},\ r\sigma_1\sigma_2 = \mu'_{12}$

34.14.3.7 Matrix Form of Density Function

$f(x_1,\cdots,x_2) = \exp(-\tfrac{1}{2}x^T\mu'^{-1}x)/\sqrt{(2\pi)^n|\mu'|}$

$\mu' = \begin{bmatrix} \mu'_{11} & \mu'_{12} & \cdots & \mu'_{1n} \\ \vdots & & & \vdots \\ \mu'_{n1} & \mu'_{n2} & \cdots & \mu'_{nn} \end{bmatrix}$, $\mu'^{-1=}$ inverse of μ', $x = [x_1,\cdots,x_n]^T, |\mu'|$ = determinant

34.14.3.8 Characteristic Function in Matrix Form

$\Phi(\omega_1,\cdots,\omega_n) = \exp(-\tfrac{1}{2}\omega^T\mu'\omega)$, $\omega = [\omega_1,\cdots,\omega_n]^T$, $\mu' \equiv$ see 34.14.3.7

34.14.4 Convergence Concepts, Central Limit Theorem

34.14.4.1 Chebyshev Inequality (see 34.5.14)

$P\{\mu - \varepsilon < X < \mu + \varepsilon\} \ge 1 - \dfrac{\sigma^2}{\varepsilon^2} \cong 1$ for $\sigma << \varepsilon$ .Hence the probability of the event $\{|X-\mu| < \varepsilon\}$ is close to 1 If the observed value of X is $X(\zeta)$ of an experiment, then $\mu - \varepsilon < X(\zeta) < \mu + \varepsilon$ or $X(\zeta) - \varepsilon < \mu$ $< X(\zeta) + \varepsilon$ which estimates the mean.

34.14.4.2 Limiting Distribution

If $\lim\limits_{n\to\infty} F_n(y) = F(y)$ for every point y at which $F(y)$ is continuous, then the r.v. Y_n is said to have a limiting distribution $F(y)$.

34.14.4.3 Stochastic Convergence

The r.v. Y_n converges stochastically to the constant c if and only if,for every $\varepsilon > 0$, $\lim\limits_{n\to\infty} P\{|Y_n - c| < \varepsilon\} = 1$.

Example

$$P\{|\overline{X}_n - \mu| \geq \varepsilon\} = P\left\{|\overline{X}_n - \mu| \geq \frac{k\sigma}{\sqrt{n}}\right\} \leq \frac{\sigma^2}{n\varepsilon^2} \quad (k = \varepsilon\sqrt{n}/\sigma)$$

the last inequality is due to Chebyshev inequality, $\overline{X}_n$ = mean of random sample of size n from a distribution with mean μ and variance σ^2, the mean and variance of $\overline{X}_n$ are μ and σ^2/n. Hence, $\lim_{n\to\infty} P\{|\overline{X}_n - \mu| \geq \varepsilon\} \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$ which implies that $\overline{X}_n$ converges stochastically to μ if σ^2 is finite.

34.14.4.4 Convergence in Probability

$\lim_{n\to\infty} P\{|Y_n - c| < \varepsilon] = 1$ implies Y_n converges to c in probability (same as stochastic convergence).

34.14.4.5 Convergence with Probability One

$P\{\lim_{n\to\infty} Y_n = c\} = 1$ implies Y_n converges to c with probability one.

34.14.4.6 Limiting Distribution and Moment-Generating Function

If $\lim_{n\to\infty} M(t;n) = M(t)$ then Y_n has a limiting distribution with distribution function $F(Y)$. $M(t;n) =$ moment generating function of Y_n in $-h < t < h$ for all $n; M(t) =$ m.g.f. of Y with d.f. $F(Y)$ in $|t| \leq h_1 < h$ (see Table 34.1).

Example

Y_n's have binomial distribution $f(n,p)$. Let $\mu = np$ be the same for every n, that is, $p = \mu/n$. $M(t;n) = E\{e^{tY_n}\} = [(1-p) + pe^t]^n = \left[1 + \frac{\mu(e^t - 1)}{n}\right]^n$ for all real values of t. Hence, $\lim_{n\to\infty} M(t;n) = \exp[\mu(e^t - 1)]$ for all t.

Note: From advanced calculus

$$\lim_{n\to\infty}\left[1 + \frac{b}{n} + \frac{\psi(n)}{n}\right]^{cn} = \lim_{n\to\infty}\left(1 + \frac{b}{n}\right)^{cn} = e^{bc}, \ \lim_{n\to\infty}\psi(n) = 0,$$

c and b do not depend upon n. Hence, Y_n has a limiting Poisson distribution with mean μ since the moment generating function of Poisson distribution is $\exp[\mu(e^t - 1)]$.

34.14.4.7 Central Limit Theorem

$Y_n = \left(\sum_{n=1}^{n} X_i\right)\Big/\sqrt{n}\sigma$ has a limiting distribution that is normal with mean zero and variance 1. $X_1, X_2, \cdots, X_n$ is a random sample with mean μ and variance σ^2.

Example

Let $X_1, X_2, \cdots, X_n$ are r.v. from binomial distribution with $\mu = p$ and $\sigma^2 = p(1-p)$. to find the $P\{Y = 48, 49, 50, 51, 52\}$ is equivalent to finding the $P\{47.5 < Y < 52.5\}$. But $(Y_n - np)/\sqrt{np(1-p)}$ where $Y_n = \overline{X}_n = X_1 + X_2 + \cdots + X_n$ has a limiting distribution that is normal with mean zero and variance

1. If $n = 100$ and $p = 1/2$ then $np = 50$ and $\sqrt{np(1-p)} = 5$ and hence $P_r\{47.5 < Y < 52.5\} = P\left\{\frac{47.5-50}{5} < \frac{Y-50}{5} < \frac{52.5-50}{5}\right\} = P\left\{-0.5 < \frac{Y-50}{5} < 0.5\right\}$. But $(Y-50)/5$ has a limiting distribution that is normal with mean zero and variance one. From tables (see 34.2), we find that the probability $(Y-50)/5$ between -0.5 and 0.5 is 0.383.

34.14.4.8 Limiting Distribution of W_n=U_n/V_n

If $\lim_{n\to\infty} F_n(u) = F(u)$ and if V_n converges stochastically (see 34.14.4.3) to l, then the limiting distribution of the r.v. $W_n = U_n / V_n$ is the same as that of U_n (the distribution $F(w)$).

Example

Let the distribution be $N(\mu,\sigma^2)$ (normal with mean μ and variance σ^2)and a sample from it with mean $\overline{X}_n$ and variance S_n^2. $\overline{X}_n$ converges stochastically to μ and S_n^2 converges stochastically to σ^2. Since S_n/σ converges stochastically to l (see 34.14.4.9), then $W_n = \overline{X}_n /(S_n/\sigma)$ has the same distribution as the limiting distribution of $\overline{X}_n$.

34.14.4.9 LiIniting Distribution of U_n/c

If $\lim_{n\to\infty} F_n(u) = F(u)$ and U_n and U_n converges stochastically (see 34.14.4.3) to $c \neq 0$ then U_n / c converges stochastically to l.

34.15 General Concepts of Stochastic Processes

34.15.1 Introduction

34.15.1.1 The X(t) Real Function (see Figure 34.3)

$X(t)$ $(X(t,\zeta))$ represents four different things:

1. A family of time functions (t variable, ζ variable)
2. A single time function (t variable, ζ fixed)
3. An r.v. (t fixed, ζ variable)
4. A single number (t, ζ fixed)

($\zeta \equiv$ outcomes of an experiment forming the space S, certain subsets of S are events, and probably probability of these events.)

34.15.1.2 Distribution Function (first order) F(x;t)

$F(x;t) = P\{X(t) \leq x\}$. The event $\{X(t) \leq x\}$ consists of all outcomes ζ such that at specified t, the functions $X(t)$ of the process do not exceed the given number x.

34.15.1.3 Density Function

$$f(x,t) = \frac{\partial F(x,t)}{\partial x}$$

34.15.1.4 Distribution Function (second order)

$$F(x_1,x_2;t_1,t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

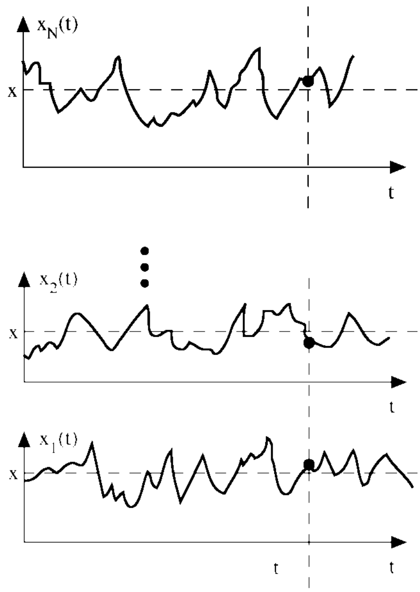


FIGURE 34.3

34.15.1.5 Density Function (second order)

$$f(x_1,x_2;t_1,t_2)=\frac{\partial F(x_1,x_2;t_1,t_2)}{\partial x_1\partial x_2}$$

Note: $F(x_1,\infty;t_1,t_2)=F(x_1;t_1),\ f(x_1;t_1)=\int_{-\infty}^{\infty} f(x_1,x_2;t_1,t_2)dx_2$

34.15.1.6 Conditional Density

$$f(x_1,t_1\big|X_2(t_2)=x_2)=f(x_1,x_2;t_1,t_2)/f(x_2;t_2)$$

34.15.1.7 Mean

$$\mu(t)=\mu'(t)=E\{X(t)\}=\int_{-\infty}^{\infty} xf(x;t)dx$$

34.15.1.8 Autocorrelation

$$R(t_1,t_2)=E\{X(t_1)X(t_2)\}\int_{-\infty}^{\infty} x_1x_2 f(x_1,x_2;t_1,t_2)dx_1dx_2$$

34.15.1.9 Autocovariance

$$C(t_1,t_2)=E\{[X(t_1)-\mu(t_1)][X(t_2)-\mu(t_2)\}R(t_1.t_2)-\mu(t_1)\mu(t_2)$$

34.15.1.10 Variance

$$\sigma^2_{x(t)}=C(t,t)=R(t,t)-\mu^2(t)$$

34.15.1.11 Distribution Function (n^th order)

$$F(x_1,\cdots,x_n;t_1,\cdots,t_n)=P\{X(t_1)\le x_1,\cdots,X(t_n)\le x_n\}$$

34.15.1.12 Density Function (n^th order)

$$f(x_1,\cdots,x_n;t_1,\cdots,t_n)=\frac{\partial F(x_1,\cdots,x_n;t_1,\cdots,t_n)}{\partial x_1,\partial x_2,\cdots,\partial x_n}$$

34.15.1.13 Complex Process

$Z(t)=X(t)+jY(t)$ is a family of complex functions and is statistically determined in terms of the two dimensional processes $X(t)$ and $Y(t)$.

34.15.1.14 Mean of Complex Process

$\mu_x(t)=E\{X(t)\},\ X(t)\equiv$ real or complex

34.15.1.15 Autocorrelation of Complex Process

$R(t_1,t_2)=E\{X(t_1)X^*(t_2)\},\quad *$ indicates conjugation.

34.15.1.16 Autocovariance of Complex Process

$C(t_1,t_2)=E\{[X(t_1)-\mu(t_1)][X^*(t_2)-\mu^*(t_2)]\}=R(t_1,t_2)-\mu(t_1)\mu^*(t_2)$

34.15.1.17 Crosscorrelation of Complex Processes

$R_{xy}(t_1,t_2)=E\{X(t_1)Y^*(t_2)\}$

34.15.1.18 Crosscovariance of Complex Processes

$C_{xy}(t_1,t_2)=R_{xy}(t_1,t_2)-\mu_x(t_1)\mu_y^*(t_2)$

34.15.1.19 Uncorrelated Complex Processes

$R_{xy}(t_1,t_2)=\mu_x(t_1)\mu_y^*(t_2),\quad C_{xy}(t_1,t_2)=0$

34.15.1.20 Orthogonal

$R_{xy}(t_1,t_2)=0$

34.15.1.21 Independent Processes X(t) and Y(t)

The processes are independent if the group $X(t_1),\cdots,X(t_n)$ is independent of the group $Y(t_1'),\cdots,Y(t_n')$ for any $t_1,\cdots,t_y,t_1',\cdots,t_n'$

34.15.1.22 Normal Processes

A process *X(t)* is normal if r.v. $X(t_1),\cdots,X(t_n)$ are jointly normal for any n.

Example

$X(t)=A\cos at+B\sin at,$ A and B = independent normal r.v. with $E\{A\}=E\{B\}=0$ and $E\{A^2\}=E\{B^2\}=\sigma^2$ and $a=$ constant. Hence $E\{X(t)\}=E\{A\}\cos at+E\{B\}\sin at=0,$ $R(t_1,t_2)=E\{(A\cos at_1+B\sin at_1)(A\cos at_2+B\sin at_2)\}=E\{A^2\cos at_1\cos at_2+E\{B^2\}\sin at_1\sin at_2=\sigma^2\cos\omega(t_2-t_1)$ which implies that *X(t)* has mean value zero and variance σ^2 and,hence $f(x,t)=(1/\sigma\sqrt{2\pi})\exp(-x^2/2\sigma^2)$. But $r=R(t_1,t_2)/\sqrt{R(t_1,t_1)R(t_2,t_2)}=\cos a(t_1-t_2)$ and from (34.11.1) $f(x_1,x_2;t_1,t_2)=[1/(2\pi\sigma^2\sqrt{1-\cos^2 a\tau})]\exp[-(x_1^2-2x_1x_2\cos a\tau+x_2^2)/(2\sigma^2(1-\cos^2 a\tau))]$ where $\tau=t_1-t_2$.

Note: $f(x;t)$ is independent of *t* and $f(x_1,x_2;t_1,t_2)$ depends only on $t_1-t_2=$ time difference.

34.16 Stationary Processes

34.16.1 Strict and Wide Sense Stationary

34.16.1.1 Strict Sense

$X(t)$ and $X(t+\varepsilon)$ have the same statistics for all shifts ε.

34.16.1.2 Jointly Stationary

$X(t), Y(t)$ the same joint statistics as $X(t+\varepsilon)$, $Y(t+\varepsilon)$ for all shifts ε.

34.16.1.3 Complex Process

$Z(t) = X(t) + jY(t)$ is stationary if $X(t)$ and $Y(t)$ are jointly stationary.

34.16.1.4 Density Function of the n[th] Order

$f(x_1,\cdots,x_n;t_1,\cdots,t_n) = f(x_1,\cdots,x_2;t_1+\varepsilon,\cdots,t_n+\varepsilon)$ for all shifts ε.

1. $f(x,t) = f(x) =$ independent of time,
2. $E\{X(t)\} = \mu =$
3. $f(x_1,x_2;t_1,t_2) = f(x_1,x_2;\tau)$, $\tau = t_1 - t_2 \equiv$ joint density of $X(t+\tau)$ and $X(t)$
4. $R(\tau) = E\{X(t+\tau)X(t)\} = R(-\tau)$,
5. $R_{xy}(\tau) = E\{X(t+\tau)Y(t)\}$

34.16.1.5 Wide Sense Stationary

$X(t)$ is wide sense stationary if $E\{X(t)\} = \mu =$ constant and $E\{X(t+\tau)X(t)\} = R(\tau)$.

Note: A wide sense process may not be a strict one. The converse is always true. A strict sense normal process is also a wide one.

Example

$X(t) = \cos(at+\varphi)$, φ is an r.v. and a is a constant. Then $E\{X(t)\} = E\{\cos(at+\varphi)\} = E\{\cos at \cos\varphi)\}$ $-\{\sin at \sin\varphi)\} = \cos at\, E\{\cos\varphi\} - \sin at E\{\sin\varphi\}$ and to have a stationary process $X(t)$ we must have the mean independent of t or equivalently

$$E\{\cos\varphi\} = E\{\sin\varphi\} = 0.\ R(t+\tau,t) = E\{\cos(a(t+\tau)+\varphi)\cos(at+\varphi)\}$$
$$= \tfrac{1}{2}\cos a\tau + \tfrac{1}{2}E\{\cos(2at + a\tau + 2\varphi)\}.$$

For the autocorrelation to be independent of t, we must have $E\{\cos 2\varphi\} = E\{\sin 2\varphi\} = 0$. Hence $R(\tau)$ $= \tfrac{1}{2}\cos a\tau$ and the process is wide stationary.

34.17 Stochastic Processes and Linear Deterministic Systems

34.17.1 Memoryless and Time-Invariant System

34.17.1.1 Output System

$Y(t) = g[X(t)]$

34.17.1.2 Mean of Output

$$E\{Y(t)\} = \int_{-\infty}^{\infty} g(x) f_x(x;t)\,dx$$

34.17.1.3 Autocorrelation of Output

$E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{\infty}\int g(x_1)g(x_2)f_x(x_1x_2;t_1,t_2)dx_1dx_2$, x_i is the r.v. of the process $X(t)$ at $t = t_i$.

34.17.1.4 Output Density Function

$f_y(y_1,\cdots,y_k;t_1,\cdots,t_k) = f_x(x_1,\cdots,x_k;t_1,\cdots,t_k)/[|g'(x_1)|\cdots|g'(x_k)|]$
$y_1 = g(x_1),\cdots,y_k = g(y_k)$, prime indicates derivative.

Example

If

$$Y(t) = \begin{cases} 1 & X(t) \le x \\ 0 & X(t) > x \end{cases}$$

then $P\{Y(t)=1\} = P\{X(t) \le x\}$, $P\{Y(t)=0\} = P\{X(t) \ge x\}$ and hence, $E\{Y(t)\} = 1\cdot P\{X(t) \le x\} + 0\cdot P\{X(t) \ge x\} = F_x(x)$

34.17.2 Linear System

34.17.2.1 Linear Operator

$Y(t) = L[X(t)]$, $L[a_1X_1(t) + a_2X_2(t)] = a_1L[X_1(t)] + a_2L[X_2(t)]$, $L[AX(t)] = AL[X(t)]$, $A \equiv$ random variable, $Y(t+\tau) = L[X(t+\tau)]$ implies L (system) is time invariant, L = transformation (system).

34.17.2.2 Linear Transformation

$E\{Y(t)\} = L[E\{X(t)\}]$

34.17.2.3 Autocorrelation

$R_{xy}(t_1,t_2) = L_{t_2}[R_{xx}(t_1,t_2)] = L_{t_2}[E\{X(t_1)X(t_2)\}]$; $R_{yy}(t_1,t_2) = L_{t_1}[R_{xy}(t_1,t_2)] = L_{t_1}[E\{X(t_1)Y(t_2)\}]$;
$E\{Y(t_1)Y(t_2)Y(t_3)] = L_{t_1}L_{t_2}L_{t_3}[E\{X(t_1)X(t_2)X(t_3)\}]$ which means to evaluate the left-hand side we must know the mean of the right-hand side for every t_1,t_2,t_3.

34.17.3 Stochastic Continuity and Differentiation

34.17.3.1 Continuity in the m.s. Sense

If $E\{[X(t+\tau) - X(T)]^2\} \to 0$ as $\tau \to 0$ the $X(t)$ is continuous in the mean square (m.s.) sense.

34.17.3.2 Continuity with Probability 1

$\lim X(t+\varepsilon) = X(t)$, $\varepsilon \to 0$ for all outcomes

34.17.3.3 Continuity in the m.s.

$E\{[X(t+\tau) - X(t)]^2\} \to 0$ with $\tau \to 0$

34.17.3.4 Continuity in the Mean

$E\{X(t+\tau)\} \to E\{X(t)\}$, $\tau \to 0$

34.17.3.5 Continuity of Stationary Process

$E\{[X(t+\tau) - X(t)]^2\} = 2[R(0) - R(\tau)]$ The process is continuous if its autocorrelation function is con tinuous at $\tau = 0$.

34.17.3.6 Differentiable Stationary Process

A stationary process $X(t)$ is differentiable in the m.s. sense if its autocorrelation $R(\tau)$ has derivatives of order up to two.

34.17.3.7 Autocorrelation of Derivatives

$$R_{xx'}(\tau) = -\frac{dR_{xx}(\tau)}{d\tau}, \; R_{x'x'}(\tau) = \frac{dR_{xx'}(\tau)}{d\tau},$$

$$R_{x'x'}(\tau) = -\frac{d^2 R_{xx}(\tau)}{d\tau^2}, \; E\{[X'(t)]^2\} = R_{x'x'}(0) = -\frac{d^2 R_{xx}(0)}{d\tau^2}.$$

Primes indicate differentiation.

34.17.3.8 Taylor Series

If

$$R(\tau) = \sum_{n=0}^{\infty} R^{(n)}(0)\frac{\tau^n}{n!}$$

then $X^{(n)}(t)$ exists in the m.s. sense and

$$X(t+\tau) = \sum_{n=0}^{\infty} X^{(n)}(t)\frac{\tau^n}{n!}$$

where (n) in exponent means nth derivative and $R(\tau)$ must have derivatives of any order.

34.17.4 Stochastic Integrals, Averages, and Ergoticity

34.17.4.1 Integral of a Process

$S = \int_a^b X(t)dt \equiv$ and defines a number $S(\zeta)$ for each outcome.

34.17.4.2 M.S. Limit of a Sum

$$\lim E\left\{\left[S - \sum_{i=1}^{n} X(t_i)\Delta t_i\right]^2\right\} = o \quad \text{for} \quad \Delta t_i \to 0$$

34.17.4.3 Mean Value of Integrals

$E\{S\} = \int_a^b E\{X(t)\}dt = \int_a^b \mu(t)dt$ (see 34.17.4.1) since the integral can be equated to a sum.

34.17.4.4 The Mean of S²

$$E\{S^2\} = \int_a^b \int_a^b E\{X(t_1)X(t_2)\}dt_1 dt_2 = \int_a^b \int_a^b r(t_1,t_2)dt_1 dt_2$$

34.17.4.5 Variance of S

$$\sigma_s^2 = \int_a^b \int_a^b [R(t_1,t_2) - \mu(t_1)\mu(t_2)dt_1 dt_2 = \int_a^b \int_a^b C(t_1,t_2)dt_1 dt_2;$$

$$\sigma_s^2 = \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C(\tau)d\tau$$

for $a=-T$ and $b=T$; if $X(t)$ is real $C(\tau)$ is even and

$$\sigma_s^2 = \frac{1}{T}\int_0^{2T}(1-\frac{\tau}{2T})C(\tau)d\tau = \frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)[R(\tau)-\mu^2]d\tau$$

34.17.4.6 Ergotic Process

A process $X(t)$ is ergotic if all its statistics can be determined from a single function (realization) $X(t,\zeta)$ of the process.

34.17.4.7 Ergoticity of the Mean

$\lim_{T\to\infty}\overline{X}_T(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)dt = E\{X(t)\} = \mu =$ constant (stationary process) iff

$$\lim_{T\to\infty}\frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)[R(\tau)-\mu^2]d\tau = 0$$

Example

$E\{X(t)\},\ R(\tau)=\exp(-\alpha|t|),\quad E\{S\} = \frac{1}{2T}\int_{-T}^{T}E\{(t)\}dt0,$

$$\sigma_s^2 = \frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)\exp(-\alpha\tau)d\tau = \frac{1}{\alpha T}-\frac{1-e^{-2\alpha T}}{2\alpha^2T^2}$$

which approaches zero as $T\to\infty$ and hence $X(t)$ is ergotic in the mean.

34.17.4.8 Ergoticity of the Autocorrelation

$\lim_{T\to\infty}R_T(\lambda) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t+\lambda)X(t)dt = E\{X(t+\lambda)X(t)\} = R(\lambda)$ iff

$$\lim_{T\to\infty}\frac{1}{T}\int_0^{2T}\left(1-\frac{\tau}{2T}\right)[R_{yy}(\tau)-R^2(\lambda)]d\tau = 0$$

where $Y(t)=X(t+\lambda)X(t),\ E\{Y(t)\}=E\{X(t+\lambda)X(t)\}=R(\lambda)$

34.18 Correlation and Power Spectrum of Stationary Processes

34.18.1 Correlation and Covariance

34.18.1.1 Correlation

$R(\tau) = E\{X(t+\tau)X^*(t)\} = R_x(\tau)$, $*$ indicates conjugation; $R(-\tau) = R^*(\tau)$

34.18.1.2 Crosscorrelation

$R_{xy}(\tau) = E\{X(t+\tau)Y^*(t)\} = R^*_{yx}(-\tau)$

34.18.1.3 Auto and Crosscovariance

$C(\tau) = E\{[X(t+\tau) - \mu^*]\} = R(\tau) - |\mu| \equiv$ covariance;

$C_{xy}(\tau) = E\{[X(t+\tau) - \mu_x][Y^*(t) - \mu^*_y]\} = R_{xy}(\tau) - \mu_x\mu^*_y \equiv$ crosscovariance

34.18.1.4 Properties of Correlation

1. $R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$ if $Z(t) = X(t) + Y(t)$
2. $R_{ww}(\tau) = R_{xx}(\tau)R_{yy}(\tau)$ if $W(t) = X(t)Y(t)$ and $X(t)$ is independent of $Y(t)$
3. $R(0) = E\{|X(t)|^2 \geq 0$
4. $E\{[X(t+\tau) \pm X(t)]^2\} = 2[R(0) \pm R(\tau)]$
5. $|R(\tau)| \leq R(0)$, $R(\tau)$ is maximum at the origin,
6. $E\{[X(t+\tau) + aY^*(t)]^2 = R_{xx}(0) + 2aR_{xy}(\tau) + a^2R_{yy}(0)$, $X(t)$ and $Y(t)$ are real processes.
7. $2|R_{xy}(\tau)| \leq R_{xx}(0) + R_{yy}(0)$

34.18.2 Power Spectrum of Stationary Processes

34.18.2.1 Power Spectrum (spectral density; see Table 34.8.)

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \equiv \text{ real, since } R(-\tau) = R^*(\tau);$$

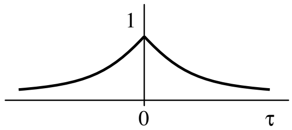
$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega = \int_{-\infty}^{\infty} S(f)e^{j2\pi f\tau}df; R(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega = E\{|X(t)|^2\};$$

if $X(t)$ is real implies $R(\tau)$ is real and even and, hence, $S(-\omega) = S(\omega) \equiv$ even, $R(\tau)$ correlation

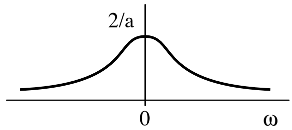
TABLE 34.8

$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega$	$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$

$e^{-a|t|}$

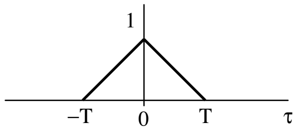


$\frac{2_a}{a^2+\omega^2}$

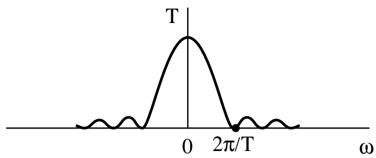


$1-\frac{|\tau|}{T},\ 0 \le \tau \le T$

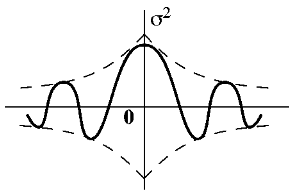
$0\ ,\ |t| > 0$



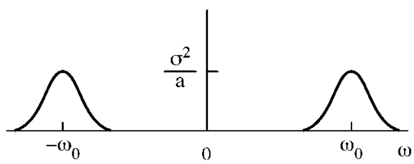
$\frac{4\sin^2(\omega T/2)}{T\omega^2}$

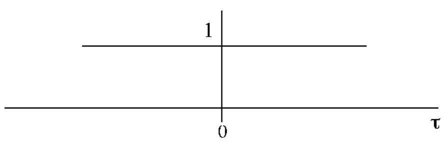


$\sigma^2 e^{-a|t|}\cos\omega_0\tau$

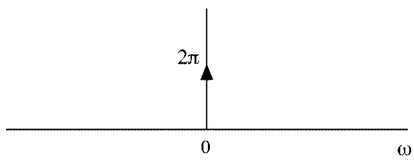


$\sigma^2\left[\frac{a}{(\omega-\omega_0)^2+a^2}+\frac{a}{(\omega+\omega_0)^2+a^2}\right]$

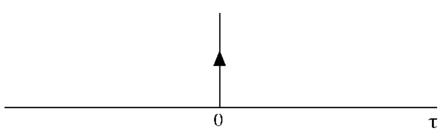


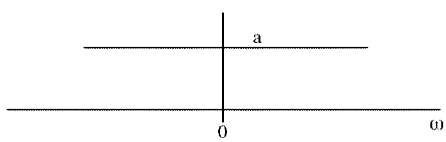


$2\pi\delta(\omega)$

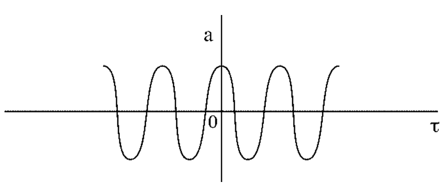


$a\delta(\tau)$





$a\cos\omega_0\tau$



$a\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$

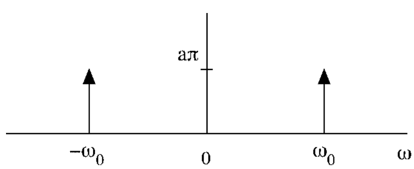
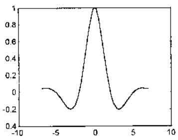
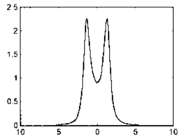
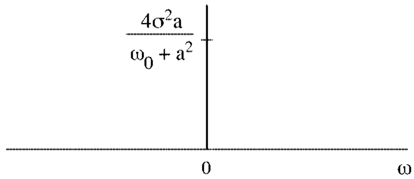
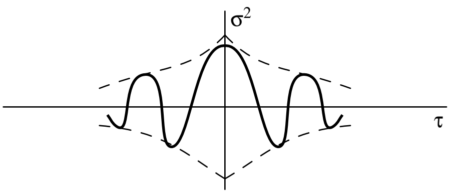
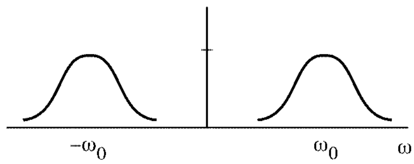


TABLE 34.8 (continued)

$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega$	$S(\omega)=\int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$
$R(\tau)=\sigma^2 e^{-a\lvert\tau\rvert}\left(\cos\omega_0\tau+\frac{a}{\omega_0}\sin\omega_0\lvert\tau\rvert\right)$	$4\sigma^2 a\frac{a^2+\omega_0^2}{(\omega^2-\omega_0^2-a^2)^2+4a^2\omega^2}$
 	
$\sigma^2 e^{-a^2 e^{-a^2 t^2}} \cos\omega_0\tau$	$\frac{\sigma^2\sqrt{\pi}}{2a}[e^{\frac{(\omega+\omega_0)^2}{4a^2}}+e^{\frac{(\omega-\omega_0)^2}{4a^2}}]$
	

34.18.2.2 Cross-power Spectrum

$$S_{xy}(\omega)=\int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j\omega\tau}d\tau=S_{yx}^*(\omega),\ R_{xy}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{j\omega\tau}d\omega;$$

$$R_{xy}(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)d\omega=E\{X(t)Y^*(t)\}$$

34.18.2.3 Orthogonal Processes

$R_{xy}(\tau)=0,\ S_{xy}(\omega)=0,\ R_{x+y}(\tau)=R_x(\tau)+R_y(\tau),\ S_{x+y}(\omega)=S_x(\omega)+S_y(\omega)$ (see 34.14.2.4)

34.18.2.4 Relationships Between Processes (see Table 34.9)

TABLE 34.9

$X(t)$	$R(\tau)=E\{X(t+\tau)X(t)\}$	$S(\omega)=\int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$
$aX(t)$	$\lvert a\rvert^2 R(\tau)$	$\lvert a\rvert^2 S(\omega)$
$\frac{dX(t)}{dt}$	$-\frac{d^2R(\tau)}{d\tau^2}$	$\omega^2 S(\omega)$
$\frac{d^nX(t)}{dt^n}$	$(-1)^n\frac{d^{2n}R(\tau)}{d\tau^{2n}}$	$\omega^{2n}S(\omega)$

TABLE 34.9 (continued)

$X(t)$	$R(\tau)=E\{X(t+\tau)X(t)\}$	$S(\omega)=\int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$
$X(t)e^{\pm j\omega_0 t}$	$e^{\pm j\omega_0 t}R(\tau)$	$S(\omega\mp\omega_0)$
$b_0\dfrac{d^m X(t)}{dt^m}+b_1\dfrac{d^{m-1}X(t)}{dt^{m-1}}+\cdots+b_m X(t)$	$\dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\left\|b_0(j\omega)^m+b_1(j\omega)^{m-1}+\cdots+b_m\right\|^2 S(\omega)d\omega$	$\left\|b_0(j\omega)^m+b_1(j\omega)^{n-1}+\cdots+b_m\right\|^2 S(\omega)$
$b_0\dfrac{d^2X(t)}{dt^2}+b_1\dfrac{dX(t)}{dt}+b_2X(t)$	$\dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\left\|b_0(j\omega)^2+b_1(j\omega)+b_2\right\|S(\omega)d\omega$	$\left\|b_0(j\omega)+b_1(j\omega)+b_2\right\|S(\omega)$

34.18.2.5 Power Spectrum as Time Average (ergoticity)

$$\lim_{T\to\infty} E\{S_T(\omega)\}=S(\omega)=\int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \quad\text{where}\quad S_T(\omega)=\frac{1}{2T}\left|\int_{-T}^{T} X(t)e^{-j\omega t}dt\right|^2 \quad\text{and}\quad \int_{-\infty}^{\infty}|\tau R(\tau)|d\tau<\infty.$$

Example

$R(\tau)=e^{-2\lambda|t|}\equiv$ autocorrelation of random telegraph signal,

$$S(\omega)=\int_{-\infty}^{\infty} e^{-2\lambda|t|}e^{-j\omega t}d\tau=\int_{-\infty}^{0} e^{2\lambda\tau}e^{-j\omega t}d\tau+\int_{0}^{\infty} e^{-2\lambda\tau}e^{-j\omega t}d\tau=\frac{4\lambda}{4\lambda^2+\omega^2}$$

Example

If $X(t)=\sum_{i=1}^{n} A_i e^{j\omega_i t}$ then $R(\tau)=\sum_{i=1}^{n}\sigma_i^2 e^{j\omega_i\tau}$ where $E\{A_i^2\}=\sigma_i^2$ and, hence,

$$S(\omega)=\sum_{i=1}^{n}\sigma_i^2\int_{-\infty}^{\infty} e^{-j(\omega-\omega_i)}d\tau=2\pi\sum_{i=1}^{n}\sigma_i^2\delta(\omega-\omega_i)$$

34.18.3 Linear Systems and Stationary Processes

34.18.3.1 Transfer Function

$H(\omega)=\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt;$ $h(t)=$ impulse response of a linear and time invariant system. $h(t)$ is produced if delta function is the input to the system.

34.18.3.2 Mean

$E\{Y(t)\}=\mu_y=\int_{-\infty}^{\infty} E\{X(t-\alpha)h(\alpha)d\alpha=\mu_x\int_{-\infty}^{\infty} h(\alpha)d\alpha=\mu_x H(0),$ $Y(t)\equiv$ output of the system, $X(t)\equiv$ input of the system, $h(t)$ =impulse response of the system

34.18.3.3 Cross-Correlation (outputs—inputs)

$$E\{Y(t)X^*(t-\tau)\} = R_{yx}(\tau) = \int_{-\infty}^{\infty} E\{X(t-\alpha)X^*(t-\tau)\}h(\alpha)d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau-\alpha)h(\alpha)d\alpha = R_{xx}(\tau) * h(\tau)$$

34.18.3.4 Autocorrelation of Input

$E\{X(t-\alpha)X^*(t-\tau)\} = R_{xx}[t-\alpha-(t-\tau)] = R_{xx}(\tau-\alpha)$

34.18.3.5 Autocorrelation of Output

$$R_{yy}(\tau) = E\{Y(t+\tau)Y^*(t)\} = \int_{-\infty}^{\infty} E\{Y(t+\tau)X^*(t-\alpha)\}h^*(\alpha)d\alpha = R_{yx}(\tau) * h^*(-\tau);$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau),\;\; R_{yy}(\tau) = R_{xy}(\tau) * h(\tau),\;\; R_{yy}(\tau) = R_{xx}(\tau) * h^*(-\tau) * h(\tau)$$

(see 34.18.3.2 and 34.18.3.3)

34.18.3.6 White Noise Input

$X(t) =$ white noise, $R_{xx}(\tau) = \delta(\tau)$, $h(t) = 0$ for $t < 0$ (casual system), then $R_{xy}(\tau) = h(-\tau) = 0$ for $\tau > 0$ which implies $Y(t)$ and $X(t)$ are orthogonal.

34.18.3.7 Power Spectrum

$S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$ and $S_{yy}(\omega) = S_{xy}(\omega)H(\omega)$ and hence, $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$ $\mathcal{F}\{h^*(-t)\} = H^*(\omega)$, $\mathcal{F}$ stands for Fourier transform)

34.18.3.8 Positiveness of Power Spectrum

$S(\omega) \geq 0$

Example

$Y(t) = dX(t)/dt$, $H(\omega) = j\omega \equiv$ transfer function of a differentiation, and hence, 34.18.3.7 gives $S_{xx'}(\omega) = S_{xx}(\omega)(-j\omega)$ and $S_{x'x'}(\omega) = \omega^2 S_{xx}(\omega)$. Thus $R_{xx'}(\tau) = \mathcal{F}^{-1}\{S_{xx}(\omega)(-j\omega)\} = -\dfrac{dR_{xx}(\tau)}{d\tau}$ and $R_{x'x'}(\tau) = \mathcal{F}^{-1}\{\omega^2 S_{xx}(\omega)\} = -\dfrac{d^2R_{XX}(\tau)}{d\tau^2}$ where $\mathcal{F}^{-1}$ where F-1 stands for inverse Fourier transform.

34.18.3.9 Multiple Terminals Spectra (see Figures 34.4 and 34.5)

$$R_{y_1y_2}(\tau) = \int_{-\infty}^{\infty} R_{x_1y_2}(t-\alpha)h_1(\alpha)d\alpha = R_{x_1y_2}(\tau) * h_1(\tau) = \int_{-\infty}^{\infty} R_{x_1x_2}(\tau+\beta)h_2^*(\beta)d\beta$$

$$= R_{x_1x_2}(\tau) * h_2^*(-\tau); S_{y_1y_2}(\omega) = S_{y_1y_2}(\omega)H_1(\omega);\; S_{x_1y_2}(\omega) = S_{x_1x_2}(\omega)H_2^*(\omega);$$

$$S_{y_1y_2}(\omega) + S_{x_1x_2}(\omega)H_1(\omega)H_2^*(\omega)$$

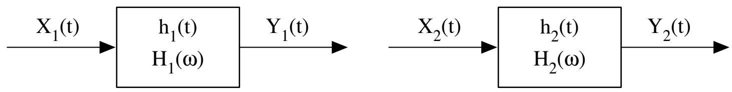


FIGURE 34.4

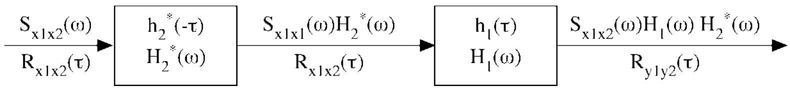


FIGURE 34.5

Note: If $H_1(\omega)H_2(\omega) = 0 \equiv$ disjoint systems,, $S_{y_1y_2}(\omega) = 0$; if also $E\{X(t)\} = 0$, $Y_1(t)$ and $Y_2(t)$ are uncorerelated.

Example (see Figure 34.6)

$H_1(\omega) = \dfrac{a + jw}{2a + jw}$, $H_2(\omega) = \dfrac{a}{2a + j\omega}$, $R_x(\tau) = e^{-2\lambda|\tau|} \equiv$ random telegraph signal, $S_x(\omega) = \dfrac{4\lambda}{4\lambda^2 + \omega^2}$,

$S_{v_1}(\omega) = \dfrac{4\lambda}{4\lambda^2 + \omega^2}\dfrac{a^2 + \omega^2}{4a^2 + \omega^2}$©$S_{v_2}(\omega) = \dfrac{4\lambda}{4\lambda^2 + \omega^2}\dfrac{a^2}{4a^2 + \omega^2}$, $S_{v_1v_2}(\omega) = \dfrac{4\lambda}{4\lambda^2 + \omega^2}\dfrac{a + j\omega}{2a + j\omega}\dfrac{a}{2a - j\omega}$

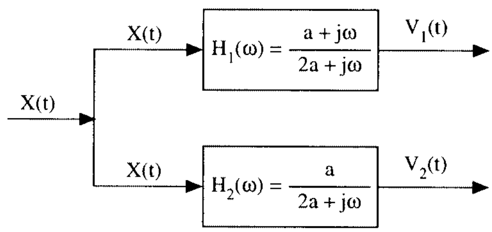


FIGURE 34.6

34.18.3.10 Stochastic Differential Equations

1. $a_nY^{(n)}(t) + \cdots + a_0Y(t) = X(t)$, $E\{Y(t)\} = \frac{1}{2}E\{X(t)\}$ (see (34.18.3.2)

$$H(\omega) = \frac{1}{a_n(j\omega)^n + \cdots + a_0}, \quad S_y(\omega) = S_x(\omega)|H(\omega)|^2$$

2. $a_nY^{(n)}(t) + \cdots + a_0Y(t) = b_mX^{(m)} + \cdots + b_0X(t)$

$$S_y(\omega) = \frac{|b_m(j\omega)^m + \cdots + b_0|^2}{|a_n(j\omega)^n + \cdots + a_0|}S_x(\omega)$$

Exponent in parenthesis indicates derivatives.

Example

$Y^{(2)}(t) + 2hY^{(1)}(t) + k^2Y(t) = X(t) \equiv$ white noise process, then $R_x(t) = \sigma^2\delta(\tau)$ and $S_x(\omega) = \sigma^2$,

$$S_y(\omega) = \frac{1}{\left|(j\omega)^2 + 2hj\omega + k^2\right|^2} S_x(\omega) = \frac{\sigma^2}{(\omega^2 - k^2)^2 + 4h^2\omega^2} = \frac{4\gamma^2\alpha(\alpha^2+\beta^2)}{[\omega^2 - (\beta^2+\alpha^2)]^2 + 4\alpha^2\omega^2}$$

where $\alpha = h,\ \beta = \sqrt{k^2 - h^2}$ and $\gamma^2 = \sigma^2/(2hk^2)$. From Table 34.8 $R_y(\tau) = \gamma^2 e^{-a|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|\right)$

34.18.3.11 Hilbert Transform

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -j\,\mathrm{sgn}(\omega),\ \hat{X}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X(t)}{t-\tau}d\tau = X(t) * \frac{1}{\pi t} \equiv \text{Hilbert transform},\ \mathcal{F}\{\hat{X}(t)\} = -j\,\mathrm{sgn}(\omega)\mathcal{F}\{X(t)\}$$

$$= H(\omega)\mathcal{F}\{X(\omega)\} S_{\hat{x}}(\omega) = S_x(\omega)|H(\omega)|^2 = S_x(\omega),\ S_{\hat{x}x}(\omega) = S_x(\omega)H(\omega) = \begin{cases} -jS_x(\omega) & \omega > 0 \\ jS_x(\omega) & \omega < 0 \end{cases}$$

$$\text{(see 34.18.3.7)},\quad R_{\hat{x}}(\tau) = R_x(\tau),\quad R_{\hat{x}x}(\tau) = R_{xx}(\tau) * h(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_x(\xi)}{\tau - \xi}d\xi = \hat{R}_x(\tau), R_{\hat{x}x}(-\tau) = -R_{\hat{x}x}(\tau),$$

$$R_{\hat{x}x}(0) = 0,\ R_{x\hat{x}}(\tau) = R_{\hat{x}x}(-\tau) - R_{\hat{x}x}(\tau)$$

34.18.3.12 Analytic Signal

$$Z(t) = X(t) + j\hat{X}(t),\ R_z(\tau) = 2[R_x(\tau) + jR_{\hat{x}x}(\tau)] = 2[R_x(\tau) + j\hat{R}_x(\tau)],$$

$$S_z(\omega) = 2[S_x(\omega) + jS_{\hat{x}x}(\omega)] = \begin{cases} 4S_x(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases},\ X(t) = \mathrm{Re}[Z(t)],\ R_x(\tau) = \tfrac{1}{2}\mathrm{Re}\{R_z(\tau)\}$$

34.18.3.13 Periodic Stochastic Process

$R(\tau + T) = R(\tau)$ then $X(t)$ is periodic in the m.s. sense.

$$R(\tau) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0\tau},\ \omega_0 = 2\pi/T,\ \alpha_n = \frac{1}{T}\int_0^T R(\tau)e^{-jn\omega_0\tau}d\tau,\ S(\omega) = \mathcal{F}\{R(\tau)\}$$

$$= 2\pi\sum_{n=-\infty}^{\infty}\alpha_n\delta(\omega - n\omega_0)$$

34.18.3.14 Periodic Processes in Linear System (see 34.18.3.13)

$$S_y(\omega) = 2\pi|H(\omega)|^2\sum_{n=-\infty}^{\infty}\alpha_n\delta(\omega - n\omega_0) = 2\pi\sum_{n=-\infty}^{\infty}\alpha_n|H(n\omega_0)|^2\delta(\omega - n\omega_0) \equiv \text{ output spectra density}$$

$$R_{yy}(\tau) = \mathcal{F}^{-1}\{S_y(\omega)\} = \sum_{n=-\infty}^{\infty}\alpha_n|H(n\omega_0)|^2 e^{jn\omega_0\tau} \equiv \text{ periodic}$$

34.18.3.15 Fourier Series

$$X(t)=\sum_{n=-\infty}^{\infty}A_n e^{jn\omega_o},\ A_n=\frac{1}{T}\int_0^T X(u)e^{-jn\omega_0 u}du \equiv \text{ uncorrelated (and orthogonal) r.v.,}$$

$$E\{an\}=\begin{cases}E[X(t)\} & n=0\\ 0 & n\neq 0\end{cases} E\{A_k A_n^*\}=\begin{cases}\alpha_n & k=n\\ 0 & k\neq n\end{cases},\ \alpha_n \equiv \text{ coefficient of } R(\tau)$$

(see 34.18.3.13)

34.18.4 Band-Limited Processes

34.18.4.1 Lowpass Process

$$S_x(\omega)=0,\ |\omega|>\omega_N,\ R_x^{(n)}(\tau)=\frac{1}{2\pi}\int_{-\omega_N}^{\omega_N}(j\omega)^n S_x(\omega)e^{j\omega\tau}d\omega<\infty,$$

$$X(t+\tau)=\sum_{n=0}^{\infty}X^{(n)}(t)\frac{\tau^n}{n!}\ \text{(see 17.3.8).}$$

34.18.4.2 Sampling Theorem

$$R_x(\tau)=\sum_{n=-\infty}^{\infty}R(nT)\frac{\sin(\omega_N\tau-n\pi)}{\omega_N\tau-n\pi},\ \omega_N=\frac{\pi}{T},\ \text{if } S_x(\omega)$$

is bandlimited (see 34.18.4.1);

$$R_x(\tau-a)=\sum_{n=-\infty}^{\infty}R_x(nT-a)\frac{\sin(\omega_N\tau-n\pi)}{\omega_N\tau-n\pi}$$

or

$$R_x(\tau)=\sum_{n=-\infty}^{\infty}R_x(nT-a)\frac{\sin(\omega_N(\tau+a)-n\pi)}{\omega_N(\tau+a)-n\pi};\ X(t)=\sum_{n=-\infty}^{\infty}X(nT)\frac{\sin(\omega_N t-n\pi)}{\omega_N t-n\pi},\ \omega_N=\pi/T$$

If $X(t)$ is bandlimited.

34.18.4.3 Bandpass Process

The narrow-band (quasi-monochromatic $\omega_N << \omega_0$) process $W(t)=X(t)\cos\omega_0 t+Y(t)\sin\omega_0 t$ is wide-sense stationary iff $E\{X(t)\}=E\{Y(t)\}=0$ and $R_x(\tau)=R_y(\tau)$ and $R_{xy}(\tau)=-R_{yx}(\tau)$ then $R_w(\tau)=R_x(\tau)\cos\omega_0\tau+R_{yx}(\tau)\sin\omega_0\tau$.

34.19 Linear Mean-Square Estimation

34.19.1 Definitions

34.19.1.1 Estimate

$\hat{G}(t) \equiv$ estimate of $G(t)$

34.19.1.2 Mean Square (m.s.) Error

$e = E\{\left|G(t) - \hat{G}(t)\right|^2\}$

34.19.1.3 Minimum m.s. Error, Linear Operation on Data

$\hat{G}(t) = L\{X(t)\}$, $L =$ linear operation

Example

Find a linear operator L such that $e = E\{[G(t) - L\{X(\xi)\}]^2\}$, $\xi \in 1$ is minimum. It turns out that $G(t) - L\{X(\xi)\}$ is orthogonal to $X(\xi)$ for every ξ in I in I and if there is an optimum solution, then L is this optimum. Hence, if $E\{[G(t) - L\{X(\xi)\}]X(\xi_i)\} = 0$, $\xi_i \in I$, then the m.s. is minimum and is given by $e_m = E\{[G(t) - L\{X(\xi)\}]G(t)\}$.

34.19.2 Orthogonality in Linear m.s. Estimation

34.19.2.1 Orthogonality

$E\{X_i X_j^*\} = 0$ implies X_i and X_j are orthogonal r.v. If $E\{X_i Z^*\} = 0$ for $i = 1,2,\cdots,n$ then $E\{a_1 X_1 + \cdots + a_n X_n\}Z^*\} = 0$, a_i 's are constatns.

34.19.2.2 m.s. Error Estimation

The m.s. error $e = E\{\left|S_0 - (a_1 S_1 + \cdots + a_n S_n\right|^2\}$ is minimum if we find the constants a_i such that the error $S_0 - (a_1 S_1 + \cdots + a_n S_n$ is orthogonal to the data $S_1, \cdots, S_n$ or $E\{[S_0 - (a_1 S_1 + \cdots + a_n S_n)]S_i^*\} = 0$, $i = 1,2,\cdots,n$ or equivalently $E\{[S_0^* - (a_1^* S_1^* + \cdots + a_n^* S_n^*)]S_i\} = 0$,

Example

To estimate $S(t+\tau)$ given $S(t)$ we use 34.19.2.2 $E\{[S(t+\tau) - aS(t)]S(t)\} = 0$ or $R(\tau) - aR(0) = 0$ or $a = R(\tau)/R(0)$ where $S_0 \equiv S(t+\tau)$, $S_1 = S(t)$ end the error is $S(t+\tau) - aS(t)$.

Mean Square error (34.19.2.2)
To estimate

$$e = E\{[S(t+\tau) - aS(t)]^2\} = E\{[S(t+\tau) - aS(t)][S(t+\tau) - aS(t)]\}$$
$$= E\{[S(t+\tau) - aS(t)]S(t+\tau)\} = R(0) - aR(\tau) = R(0) - R^2(\tau) - R^2(\tau)/R(0)$$

since $S \equiv$ data is orthogonal to the error <picture>). (This is a prediction problem.)

Example

To estimate $S(t)$ in terms of $X(t)$ we obtain (34.19.2.2) $E\{[S(t) - aX(t)]X(t)\} = 0$ or $R_{sx}(0) - aR_x(0) = 0$ or $a = R_{sx}(0)/R_x(0)$. Mean square error: $e = E\{[S(t) - aX(t)]S(t)\} = R_s(0) - aR_{sx}(0) = R_s(0) - R_{sx}^2(0)/$. If $X(t) = S(t) + n(t)$ and $n(t) \equiv$ noise is orthogonal $S(t)$, then $R_{sx}(\tau) = R_s(\tau)$, $R_x(\tau) = R_s(\tau) + R_n(\tau)$ and $a = R_s(0)/[R_s(0) + R_n(0)]$ and $e = R_s(0)R_n(0)/[R_s(0) + R_n(0)]$. . (This is a filtering problem.)

Examples

To estimate $S(t)$ in terms of $S(0)$ and $S(t_0)$ we set the error to be orthogonal to the data $S(0)$ and $S(t_0)$. Hence, $E\{[S(t) - a_1 S(0) - a_2 S(t_0)S(0)\} = 0$ and $E\{[S(t) - a_1 S(0) - a_2 S(t_0)]S(t_0)\} = 0$. Therefore, we obtain the systems $R(t) = a_1 R(0) + a_2 R(t_0)$ and $R(t_0 - t) = a_1 R(t_0) + a_2 R(0)$ and solve it for the unknown a_1 and a_2 The m.s. error is $e = E\{[S(t) - a_1 S(0) - a_2 S(t_0)]S(t)\} = R(0) - [a_1 R(t) + a_2 R(t_0 - t)]$. (This is an interpolation problem.)

34.20 The Filtering Problem for Stationary Processes

34.20.1 Wiener Theory

34.20.1.1 Wiener Integral Equation

$$R_{gx}(t-\xi) = \int_a^b R_x(\alpha - \xi) h(t,\alpha) d\alpha, \; a \le \xi \le b.$$

If $h(t,\xi)$ is found to satisfy the integral equation for the process G(t) and X(t), then the m.s. error is given by (orthogonality principle):

$$e = E\left\{\left[G(t) - \int_a^b X(\alpha) h(t,\alpha) d\alpha\right] G(t)\right\} = R_g(0) - \int_a^b R_{gx}(t-\alpha) h(t,\alpha) d\alpha$$

34.20.1.2 Wiener Integral Equation for Stationary Processes

$$R_{gx}(\tau) = \int_a^b R_x(\tau - \alpha) h(\alpha) d\alpha$$

34.20.1.3 Solution of 34.20.1.2

$S_{gx}(\omega) = S_x(\omega) H(\omega)$ or $H(\omega) = S_{gx}(\omega) / S_x(\omega)$

Example

$X(t) = S(t) + N(t)$, $X(t) \equiv$ given in $-\infty < t < \infty$, $N(t) \equiv$ noise, $G(t) = S(t)$ and

$$R_{sx}(\tau) = \int_{-\infty}^{\infty} R_x(\tau - \alpha) h(\alpha) d\alpha, \; S_{sx}(\omega) = S_x(\omega) H(\omega), e = \text{m.s. error}$$

$$= E\{[S(t) - \int_{-\infty}^{\infty} X(t-\alpha) h(\alpha) d\alpha] S(t)\} = R_s(0) - \int_{-\infty}^{\infty} R_{sx}(\alpha) h(\alpha) d\alpha.$$

If $f(\tau) = R_s(\tau) - R_{sx}(-\tau) * h(\tau)$ then $e = f(0)$. But

$$F(\omega) = S_s(\omega) - S_{sx}(-\omega) H(\omega) = S_s(\omega) - \frac{S_{sx}(-\omega) S_{sx}(\omega)}{S_x(\omega)}$$

and from inverse formula with $\omega = 0$

$$e = f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[S_s(\omega) - \frac{S_{sx}(-\omega)S_{sx}(\omega)}{S_x(\omega)}]d\omega.$$

34.20.1.4 Signal and Noise Uncorrelated

$S_{sn}(\omega) = 0,\ S_x(\omega) = S_s(\omega) + S_n(\omega),\ S_{sx}(\omega) = S_s(\omega),\ H(\omega) = S_s(\omega)/[S_s(\omega) + S_n(\omega)],$

$$e = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_s(\omega)S_n(\omega)}{S_s(\omega) + S_n(\omega)}d\omega$$

Example

If $R_s(\tau) = Ae^{-|\tau|}\cos\omega_0\tau,\ \omega_0 >> 1$ and $S_n(\omega) = N,\ S_s(\omega) = A/[1 + (\omega - \omega_0)^2]$ for $\omega > 0,\ H(\omega) = A/[A + N + N(\omega - \omega_0)^2],\ \omega > 0,$

$$e = \frac{1}{\pi}\int_{-\infty}^{\infty} ANd\omega/[A + N + N(\omega - \omega_0)^2] = A/[1 + A/N]^{1/2},\ E\{S^2(t)\} = R_s(0) = A.$$

If $N>>A$ the filtering does not improve the estimation of $S(t)$, $e = A$. . If $A = 3N$ then $e = R_s(0)/2$.

34.20.1.5 Wiener-Hopf Equation

$$R(\tau + \lambda) = \int_0^{\infty} R(\tau - \alpha)h(\alpha)d\alpha,\ \tau \ge 0$$

$$e = \text{ m.s. error} = E\{[S(\tau + \lambda) - \int_0^{\infty} S(\tau - \alpha)h(\alpha)d\alpha]S(\tau + \lambda)\} = R(0) - \int_0^{\infty} R(-\lambda - \alpha)h(\alpha)d\alpha$$

Solution to Wiener-Hopf Equation

$S(\omega) =$ is a rational function of ω.

Step 1

$$S(\omega)\Big|_{\omega = p/j} = \frac{A(p^2)}{B(p^2)} = F(p)F(-p),\ F(p) = \frac{C(p)}{D(p)},\ F(-p) = \frac{C(-p_i)}{D(-p)}\Bigg|,$$

C and D contain all the roots of $A(p^2)$ and $B(p^2)$

Step 2

$$D(p_i) = 0,\ \operatorname{Re} p_i \le 0,\ F(p) = \frac{C(p)}{D(p)} = \frac{a_1}{p - p_1} + \cdots + \frac{a_n}{p - p_n},\ a_i = \frac{C(p_i)}{D'(p_i)}$$

Step 3

$$F_1(p) = \frac{a_1 e^{p_1\lambda}}{p - p_1} + \cdots + \frac{a_n e^{P_n\lambda}}{p - p_n} = \frac{C_1(p)}{D(p)}$$

$\lambda =$ the constant in Wiener-Hopf equation $R(\tau+\lambda),\ C_1(p_i) = C(p_i)e^{p_i\lambda}$.

Step 4

$$H(p) = \frac{F_1(p)}{F(p)} = \frac{C_1(p)}{C(p)}$$

Example

$R(\tau) = Ae^{-|t|},\ S(\omega) = 2A/(1+\omega^2 0.$

Step 1

$$S(p/j) = \frac{2A}{1-p^2} = \frac{\sqrt{2A}}{1+p}\frac{\sqrt{2A}}{1-p}$$

Step 2

$$F(p) = \frac{C(p)}{D(p)} = \frac{\sqrt{2A}}{1+p},\ p_1 = -1.$$

Step 3

$$F_1(p) = \frac{\sqrt{2A}}{1+p}e^{-\lambda}$$

Step 4

$$H(p) = \frac{F_1(p)}{F(p)} = e^{-\lambda}h(t) = e^{-\lambda}\delta(t).$$

Estimate $S(t+\lambda) \sim e^{-\lambda}\ \ S(t) \equiv$ prediction.

$$e = E\{[S(t+\lambda) - e^{-\lambda}S(t)]S(t+\lambda)\} = R(0) - e^{-\lambda}R(\lambda) = A(1-e^{-2\lambda}0.$$

34.21 Harmonic Analysis

34.21.1 Series Expansion

34.21.1.1 $R(\tau) =$ periodic $(R(\tau+T) = R(\tau))$:

$$X(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_0 t},\ \omega_0 = 2\pi/T,$$

$$A_n = \frac{1}{T}\int_{-T/2}^{T/2} X(t)e^{-jn\omega_0 t}dt,\ E\{A_n A_n^*\} = 0,\ n \neq 0 \text{ orthogonal},$$

$$R(\tau)=\sum_{n=-\infty}^{\infty}\alpha_n e^{jn\omega_0\tau},\ S(\omega)=2\pi\sum_{n=-\infty}^{\infty}\alpha_n\delta(\omega-n\omega_0),\ E\{|A_n|^2\}=\alpha_n$$

34.21.1.2 Karhunen-Loeve Expansion

If the expansion

$$X(t)\cong\sum_{n=1}^{\infty}B_n\varphi_n(t),\ |t|<\tfrac{T}{2}$$

has orthogonal coefficients $E\{B_nB_m^*\}=0,\ n\neq m$, then φ_n must satisfy the integral equation

$$\int_{-T/2}^{T/2}r(t_1,t_2)\varphi(t_2)dt_2=\lambda\varphi(t_1),\ |t_1|<\tfrac{T}{2}$$

for term $\lambda=\lambda_n$ of λ and the variance of B_n must be equal to $\lambda_n : E\{|B_n|^2\}=\lambda_n$

34.21.1.3 If 34.21.1.2 is satisfied $E\{|X(t)-\hat{X}(t)|^2\}=R(t,t)-\sum_{n=1}^{\infty}\lambda_n|\varphi_n(t)|^2,\ |t|<\tfrac{T}{2}$

34.21.2 Fourier Transforms

34.21.2.1 Fourier Transforms

$$X(\omega)=\int_{-\infty}^{\infty}X(t)e^{-j\omega t}dt,\ \int_{-\infty}^{\infty}|X(t)|^2dt<\infty,$$

$$X(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega,\ E\{X(\omega)\}=\int_{-\infty}^{\infty}E\{X(t)\}e^{-j\omega t}dt$$

34.21.2.2 Autocorrelation of X(ω)

$$R(t_1t_2)=E\{X(t_1)X^*(t_2)\}$$

$$\Gamma(\omega_1,\omega_2)=\iint_{-\infty}^{\infty}R(t_1,t_2)e^{-j(\omega_1t_1-\omega_2t_2)}dt_1dt_2=E\{X(\omega_1)X^*(\omega_2)\};$$

$$R(t_1,t_2)\xleftrightarrow{\mathcal{F}_{t_1}}E\{X(\omega_1)X^*(t_2)\}\xleftrightarrow{\mathcal{F}_{t_2}}E\{X(\omega_1)X^*(\omega_2)\}$$

34.21.2.3 Linear Systems

$Y(\omega)=$ output $=X(\omega)H(\omega);\ E\{Y_1(\omega)Y_2^*(\omega)\}=E\{X_1(\omega)H_1(\omega)X_2(\omega)H_2(\omega)\}=$

$E\{X_1(\omega_1)X_2^*(\omega_2)H_1(\omega_1)H_2^*(\omega_2);\ \Gamma_{y_1y_2}(\omega_1,\omega_2)=\Gamma_{x_1x_2}(\omega_1,\omega_2)H_1(\omega_1)H_2^*(\omega_2)$

34.22 Markoff Sequences and Processes

34.22.1 Definitions

34.22.1.1 Definition

$F(x_n|x_{n-1},\cdots,x_1) = F(x_n|x_{n-1})$. For continuous r. v. type $f(x_n|x_{n-1},\cdots,x_1) = f(x_n|x_{n-1})$. Also

$$f(x_1,x_2,\cdots,x_n) = f(x_n|x_{n-1}) = f(x_{n-1}|x_{n-2})\cdots f(x_2|x_1)f(x_1)$$

$$f(x_n|x_{n-1},\cdots,x) = f(x_1,x_2,\cdots,x_n)/f(x_1,x_2,\cdots,x_{n-1}) = f(x_n|x_{n-1})$$

34.22.1.2 Homogeneous

$f(x_n|x_{n-1}) \equiv$ independent of n

34.22.1.3 Stationary

$f_{x_n}(x) \equiv$ the sequence is homogeneous and the r.v. X_n have the same density.

34.22.1.4 Chapman-Kalmogoroff Equation

$f(x_n|x_s) = \int_{-\infty}^{\infty} f(x_n|x_r)f(x_r|x_s)dx_r$ where $n > r > s$ are any integers.

34.22.2 Markoff Chains

34.22.2.1 Markoff Chains

$P\{X_n = a_{i_n}\Big|X_{n-1} = a_{i_{n-1}},\cdots,X_1 = a_{i_1}] = P\{X_n a_{i_n}\Big|X_{n-1} = a_{i_{n-1}}\}X_n \equiv$ Markoff chain

34.22.2.2 Conditional Densities

$p_i(n) = P\{X_n = a_i\}$; $P_{ij}(n,s) = P\{X_n = a_i|X_s = a_j\}, n > s;$ $\quad p_i(n) = \sum_j P_{ij}(n,s)p_j(s); \sum_i p_i(n) = 1,$

$\sum_j p_{ij}(n,s) = 1$; $P_{ij}(n,s) = \sum_k P_{ik}(n,r)P_{kj}(r,s)$, n > r >s discrete case Chapman-Kolmogoroff equation..

34.22.2.3 Matrices

$P(n,s) =$ square matrix with $P_{ij}(n,s)$ elements; $p(n)$ column matrix with elements $p_i(n)$; $P(n) = P(n,s)p(s)$; $P(n,s) = P(n,r)P(r,s)$

34.22.2.4 Homogeneous Chains

The conditional probabilities $P_{ij}(n,s)$ depend only on the difference n - s, $P_{ij}(n-s)$. Matrix $P(n-s)$ with $P(1) = \Pi$. WIth $r = s+1$, $n = s+2$, $P(2) = P(1)P(1) = \Pi^2$ and $P(n) = \Pi^n$. Also, $p(n+k) = \Pi^n p(k)$.

34.22.3 Markoff Processes

34.22.3.1 Markoff Process

$P\{X(t_n) \le x_n|X(t_{n-1}),\cdots,X(t_1)\} = P\{X(t_n) \le x_n|X(t_{n-1})\}$ for every n and $t_1 < t_2 \cdots < t_n$.

Note:

1. For $t_1 < t_2$, $P\{X(t_1) \le x_1|X(t)$ for all $t \ge t_2\} = P\{X(t_1) \le x_1|X(t_2)\}$
2. If $X(t_2) - X(t_1)$ is independent of $X(t)$ for every $t \le t_1$ $(t_1 < t_2)$ the process $X(t)$ is Markoff.

3. $E\{X(t_n)|X(t_{n-1}),\cdots,X(t_1)\}=E\{X(t_n)|X(t_{n-1})\}$

4. $R(t_3,t_2)R(t_2,t_1)=R(t_3,t_1)r(t_2,t_2)$ for every $t_3>t_2>t_1$ and if $X(t)$ is normal Markoff with zero mean.

34.22.3.2 Continuous Process (Chapman-Kolmogoroff equation)

$p(x,t;x_0,t_0)=\int_{-\infty}^{\infty} p(x,t;x_1,t_1)p(x_1,t_1;x_0,t_0)dx_1$ if $t>t_1>t_0$. $p(x,t;x_0,t_0)=f_{x(t)}(x|X(t_0)=x_0)$ for $t\ge t_0\equiv$ conditional density of $X(t)$

34.22.3.3 Conditional Mean and Variance

$E\{X(t)|X(t_0)=x_0=a(x_0,t,t_0)$; $E[[X(t)-a(x_0,t,t_0)]^2|X(t_0)=x_0\}=b(x_0,t,t_0)$; $a(x_0,t,t_0)=\int_{-\infty}^{\infty} xp(x,t;x_0,t_0)dx$, $b(x_0,t,t_0)=\int_{-\infty}^{\infty}(x-a)^2 p(x,t;x_0,t_0)dx$; $a(x_0,t_0,t_0)=x_0$, $b(x_0,t_0,t_0)=0$

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