پاسخ تمرین «دلتای دیراک»

سوال اول - الف)

$$1)\int_{-\infty}^{\infty} f(x)\,\delta(g(x))\,dx \overset{*}{=} \sum_{i=1}^{n}\int_{x_{i}-\varepsilon}^{x_{i}+\varepsilon} f(x)\,\delta\big((x-x_{i})\,g'(x_{i})\big)dx$$

$$^{*}\,g(x)=g(x_0)+(x-x_0)\,g'(x_0)+(x-x_0)^2 g''(x_0)/2!+\cdots$$

با توجه به خواص دیگر داریم:

$$① \quad \delta(ax)=\frac{1}{a}\,\delta(x)$$

$$② \quad \int_{-\infty}^{\infty}\delta(x-x_0)\,f(x)\,dx=f(x_0)$$

$$\sum_{i=1}^{n}\int_{x_{i}-\varepsilon}^{x_{i}+\varepsilon} f(x)\,\delta\big((x-x_i)\,g'(x_i)\big)dx \overset{①}{=} \sum_{i=1}^{n} f(x_i)\int_{x_i-\varepsilon}^{x_i+\varepsilon}\delta\big((x-x_i)\,g'(x_i)\big)dx \overset{②}{=}$$

$$\sum_{i=1}^{n} f(x_i)\,\frac{1}{|g'(x_i)|}$$

با استفاده از انتگرال جزء به جزء:

$$2)\int_{-\infty}^{+\infty}\underbrace{f(x)}_{u}\,\underbrace{\delta'(x-x_0)\,dx}_{dv}=$$

$$\underbrace{\cancel{f(x)\,\delta(x-x_0)}}_{0}\Big|_{-\infty}^{+\infty}-\int f'(x)\,\delta(x-x_0)\,dx \overset{②}{=} -f'(x_0)$$

ب)

$$1)\int_{-\infty}^{+\infty}\underbrace{(x^2+5x)}_{f(x)}\,\delta(x-2)\,dx=f(2)=2^2+5(2)=14$$

$$2)\int_{-\infty}^{+\infty}\underbrace{\sin(x)\,(x^2+1)}_{f(x)}\,\delta(x-\pi)\,dx=f(\pi)=\sin(\pi)(\pi^2+1)=0$$

۳) $\int_{-\infty}^{+\infty} \ln(1+x)\sin\left(\frac{\pi x}{2}\right)\delta'(x^3-1)\,dx =$

$$\int_{-\infty}^{+\infty} \underbrace{\ln(1+x)\,\sin\left(\frac{\pi x}{2}\right)\Big/3x^2}_{f(x)}\; 3x^2\,\delta'(x^3-1)\,dx = \int_{-\infty}^{+\infty} \underbrace{f(x)}_{u}\;\underbrace{d\big(\delta(x^3-1)\big)}_{dv} =$$

$$f(x)\,\delta(x^3-1)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(x)\,\delta(x^3-1)\,dx = -\frac{f'(x)}{3x^2}\Big|_{x=1} = \boxed{-\frac{1-2\ln 2}{9}}$$

$$f'(x) = \left(\frac{1}{x}\right)\sin\left(\frac{\pi}{2}x\right)\left(\frac{1}{3}x^{-2}\right) + \ln(1+x)\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\right)\left(\frac{1}{3}x^{-2}\right) + \ln(1+x)\sin\left(\frac{\pi}{2}x\right)\left(-\frac{2}{3}x^{-3}\right)$$

$$x=1 \Rightarrow f'(x) = \frac{1}{3} - \frac{2}{3}\ln 2$$

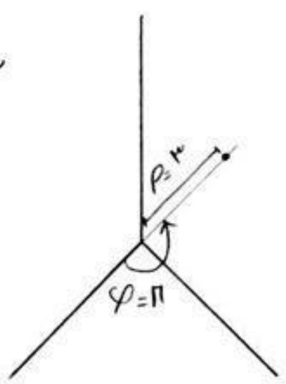
سوال دوم)

الف)

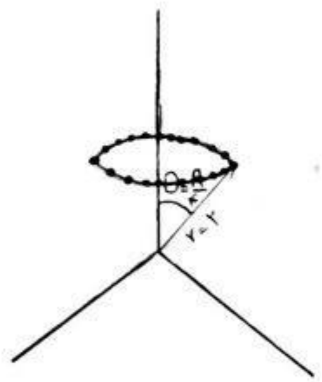
$\rho = 3$

$\varphi = \pi$

$z = 0$



ب)



$r = 2$

$\theta = \frac{\pi}{4}$

$\varphi = \frac{\pi}{10}, \frac{2\pi}{10}, \dots, \frac{20\pi}{10}$

سوال سوم)

الف) $\rho(r) = q\,\delta(r-r')$, $\int_{-\infty}^{+\infty} q\,\delta(r-r')\,dr = q$

ب) $\rho(x) = -q\,\delta(x) + q\,\delta(x-a)$

ج) $\rho(r) = Q\,\delta(r-R)$

د) $\rho(r,z) = Q\,\delta(r-a)\,\delta(z)$

سوال چهارم)

$$\int_a^b \underbrace{e^x \sin\frac{\pi x}{2}}_{f(x)} \delta(\underbrace{x^2-1}_{g(x)})\,dx = \sum \frac{f(x_i)}{|g'(x_i)|}$$

$$x^2-1=0 \Rightarrow x=\pm 1 \qquad g'(x)=2x$$

$$\begin{cases} a=0,\ b=+\infty \quad \Rightarrow \displaystyle\int_0^{+\infty} f(x)\,\delta(g(x)) = \frac{f(1)}{|g'(1)|} = \frac{e}{2} \\[2ex] a=-2,\ b=2 \quad \Rightarrow \displaystyle\int_{-2}^{2} f(x)\,\delta(g(x)) = \frac{f(-1)}{|g'(-1)|} + \frac{f(1)}{|g'(1)|} = \frac{1}{2e} + \frac{e}{2} \end{cases}$$

سوال پنجم)

$$F=\frac{dP}{dt} \Rightarrow dp = F\,dt$$

$$\Delta P = \int F\,dt$$

$$m\Delta V = \int F\,dt$$

$$m\Delta V = \int_0^{t_0} a_0\,\delta(t-t_0)\,dt + \int_{t_0}^{t} a_1\,\delta(t-t_1)\,dt = a_0 + a_1$$

$$\Delta V = \frac{a_0+a_1}{m}$$

$$V = \frac{a_0+a_1}{m} + V_0$$

سوال ششم)

$$\frac{dL}{dt} = T \qquad dL = T\,dt$$

$$\Delta L = \int T\,dt$$

$$I\Delta\omega = \int T\,dt = \int_{-\infty}^{+\infty} a(t^2+t)\,\delta(t-2) = \left. 6a \right|_{t=2}$$

$$I(0-\omega_0) = 6a \Rightarrow a = \frac{-I\omega_0}{6}$$