An Interactive Introduction to Mathematical Analysis: Instructor's Manual

This printable form of the instructor's manual is a companion to the on-screen instructors manual for my book. To install the on-screen instructor's manual, you need to convert your installation of the book from a student version to an instructor's version. After installing the book, you should run the executable file

lewin-analysis-book-instructors-manual.exe

which can be found at the address

http://science.kennesaw.edu/~jlewin/analysis/instructors-manual/lewin-analysis-book-instructors-manual.exe

http://science.kennesaw.edu/~jlewin/CUP/lewin-analysis-instructor's-manual.exe

The running of this executable file requires a password that can be obtained by bona fide instructors at

solutions@cambridge.org

In both the student version and the instructor's version of the text, each group of exercises comes with a link to a solutions document. In the instructors version, the link takes you to a solutions document that shows the solutions provided to students in blue and shows the solutions provided only to instructors in green. If you make a monochrome print of this printable form of the manual, then the solutions provided to students will appear in a bold italics font and the solutions provided only to instructors will appear in an upright sans serif font.

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2 Mathematical Grammar

Exercises on Use of Quantifiers

Except in Exercise 2, decide whether the the sentence that appears in the exercise is meaningful or meaningless. If the sentence is meaningful, say whether what it says is true or false.

1. a. $\sqrt{x^2} = x$.

   Solution: ***This statement is meaningless because*** x ***is unquantified.***

b. For every real number x we have $\sqrt{x^2} = x$.

Solution: ***This statement is meaningful but false because the equation $\sqrt{x^2} = x$ is false whenever x is negative.***

c. For every positive number x we have $\sqrt{x^2} = x$.

Solution: ***This statement is true.***

2. a. Point at the expression $\sqrt{x^2}$ and click on the Evaluate button .

   b. Point at the expression $\sqrt{x^2}$ and click on the Simplify button .

   c. Point at the equation $\sqrt{x^2} = x$, open the Maple menu and click on Check Equality.

   d. Point at the equation $x = -2$ and click on the button to supply the definition $x = -2$ to *Scientific Notebook*. Then try a Check Equality on the equation $\sqrt{x^2} = x$.

3. For every number x and every number y there is a number z such that $z = x + y$.

   Solution: ***This statement is true.***

4. For every number x there is a number z such that for every number y we have $z = x + y$.

   Solution: ***This statement is false.***

5. For every number x and every number z there is a number y such that $z = x + y$.

   Solution: ***This statement is true.***

6. $\sin^2 x + \cos^2 x = 1$.

   Solution: ***This statement is meaningless because x is unquantified.***

7. For every number x we have $\sin^2 x + \cos^2 x = 1$.

   Solution: ***This statement is true.***

8. For every integer $n > 1$, if $n^2 \leq 3$ then the number 57 is prime.

   Solution: ***Since it is impossible to find an integer $n > 1$ such that $n^2 \leq 3$, the assertion that 57 is prime for every such integer is*** **true.** ***The fact that the number 57 happens not to be prime has no bearing on this exercise.***

Exercises on Order of Appearance of Unknowns

For each of the following pairs of statements, decide whether or not the statements are saying the same thing. Except in the first two exercises, say whether or not the given statements are true.

1. a. Every person in this room has seen a good movie that has started playing this week.

   b. A good movie that has started playing this week has been seen by every person in this room.

   Solution: ***The second statement asserts that there is one particular good movie that everyone in the room has seen. The first statement says less. It asserts that everyone has seen a good movie but leaves open the possibility that different people may have seen different movies.***

2. a. Only men wearing top hats may enter this hall.

b. Only men may enter this hall wearing top hats.

c. Men wearing top hats only may enter this hall.

d. Men wearing only top hats may enter this hall.

e. Men wearing top hats may enter this hall only.

Hint: ***These five statements are all different from one another.*** The first statement is ambiguous. It could mean that the only people who may enter this hall are men who are wearing top hats. However, with a different voice inflection it could mean that top hats are permitted only to men and that women will not be permitted entry if they are wearing top hats, leaving open the question of whether a woman who is not wearing a top hat may enter the hall. In other words, a change in voice inflection could make statements a. and b. say the same thing.

3. a. For every nonzero number x there is a number y such that $xy = 1$.

   Solution: *This statement is true because of x is any nonzero number then we have $x(1/x) = 1$.*

   b. There is a number y for which the equation $xy = 1$ is true for every nonzero number x.

   Solution: ***This statement is false.***

4. a. For every number $x \in [0,1)$ there exists a number $y \in [0,1)$ such that $x < y$.

   Solution: ***True***

   b. There is a number $y \in [0,1)$ satisfying $x < y$ for every number $x \in [0,1)$.

   Solution: ***False***

5. a. For every number $x \in [0,1]$ there exists a number $y \in [0,1]$ such that $x < y$.

   b. There is a number $y \in [0,1]$ satisfying $x < y$ for every number $x \in [0,1]$.

Solution: ***These two statements are not saying the same thing but who cares! Both statements are false.***

6. a. For every number $x \in [0,1)$ there exists a number $y \in [0,1)$ such that $x \leq y$.

   This statement is true. Given any $x \in [0,1]$, if we define $y = x$ then we have found a number $y \in [0,1]$ such that $x \leq y$.

   b. There is a number $y \in [0,1)$ satisfying $x \leq y$ for every number $x \in [0,1)$.

   This statement is false because it asserts that the interval $[0,1)$ has a largest member.

7. a. For every number $x \in [0,1]$ there exists a number $y \in [0,1]$ such that $x \leq y$.

   b. There is a number $y \in [0,1]$ satisfying $x \leq y$ for every number $x \in [0,1]$.

This time, both of the statements are true.

8. a. For every odd integer m it is possible to find an integer n such that mn is even.

   b. It is possible to find an integer n such that for every odd integer m the number mn is even.

Solution: ***These two statements do not say the same thing but they are both true.***

9. a. For every number x it is possible to find a number y such that $xy = 0$.

   b. It is possible to find a number y such that for every number x we have $xy = 0$.

Solution: ***These two statements do not say the same thing but they are both true.***

10. a. For every number x it is possible to find a number y such that $xy \neq 0$.

b. It is possible to find a number y such that for every number x we have $xy \neq 0$.

Solution: ***These two statements do not say the same thing but they are both false.***

11. a. For every number a and every number b there exists a number c such that $ab = c$.

Solution: ***True***

b. For every number a there exists a number c such that for every number b we have $ab = c$.

Solution: ***False***

12. a. For every number a and every number c there exists a number b such that $ab = c$.

b. For every number a there exists a number b such that for every number c we have $ab = c$.

Solution: ***These two statements do not say the same thing but they are both false.***

13. a. For every nonzero number a and every number c there exists a number b such that $ab = c$.

Solution: ***True***

b. For every nonzero number a there exists a number b such that for every number c we have $ab = c$.

Solution: ***False***

Some Exercises on Negations and the Quantifiers

Write a negation for each of the following statements:

1. All roses are red.

Solution: ***The negation could say: Not all roses are red. Alternatively it could say: There is at least one rose that isn't red. However it would be wrong to write the negation as: All roses are not red.***

2. In Sam's flower shop there is at least one rose that isn't red.

Solution: ***All the roses in Sam's flower shop are red.***

3. In every flower shop there is at least one rose that isn't red.

Solution: ***There is at least one flower shop in which all the roses are red.***

4. I believe that all roses are red.

Solution: ***I do not believe that all roses are red.***

5. There is at least one person in this room who thinks that all roses are red.

Solution: ***No one in this room thinks that all roses are red.***

6. Every person in this room believes that all roses are red.

There is at least one person in this room who does not believe that all roses are red.

7. At least half of the people in this room believe that all roses are red.

Solution: ***More than half of the people in this room do not believe that all roses are red.***

8. Every man believes that all women believe that all roses are red.

Solution: ***There are men who do not believe that all women believe that all roses are red.***

9. You were at least an hour late for work every day last week.

Solution: ***There was at least one day last week on which you began work less than an hour late.***

10. It has never rained on a day on which you have remembered to take your umbrella.

Solution: ***There have been days on which it has rained and on which you have remembered to take your umbrella.***

11. You told me that it has never rained on a day on which you have remembered to take your umbrella.

Solution: ***You have not told me that it has never rained on a day on which you have remembered to take your umbrella.***

12. You lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.

Solution: ***You did not lie when you told me that it has never rained on a day on which you have remembered to take your umbrella.***

13. I was joking when I said that you lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.

Solution: ***I was not joking when I said that you lied when you told me that it has never rained on a day on which you have remembered to take your umbrella.***

14. This, Watson, if I mistake not, is our client now.

Solution: ***This statement is meaningless because it refers to itself. We should have been able to expect better logical precision from Sherlock Holmes.***

15. a. For every real number x there exists a real number y such that

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement is true?

Solution: ***Yes this statement is true.***

Negation: ***There exists a real number*** x ***such that for every real number*** y***, the equation***

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}$$

is either false or meaningless.

b. There exists a real number x such that for every real number y we have

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement is true?

Solution: ***This statement is false because the equation shown her can't be true when*** $y = x$***.***

Negation: ***For every real number*** x ***there exists a real number*** y ***for which the equation***

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}$$

is either false or meaningless.

c. For every real number x and every real number $y \neq x$ we have

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}.$$

Is this statement is true?

Solution: ***This statement is true.***

Negation: ***There exists a real number*** x ***and there exists a real number*** $y \neq x$ ***such that the equation***

$$\frac{2x^2+xy-y^2}{x^3-y^3} = \frac{2}{3(x-y)} + \frac{5y+4x}{3(x^2+xy+y^2)}$$

is either false or meaningless.

Some Exercises on The Use of Conditionals

In the exercises that follow you should assume that P, Q, R and S are given statements that may be either true or false.

1. Write down the denial, the converse and the contrapositive form of each of the following statements:

   a. *All cats scratch.*

   The converse: ***All things that scratch are cats.***

   The contrapositive: ***All things that do not scratch are not cats.***

   The denial: ***There is at least one cat that does not scratch.***

   b. *If what you said yesterday is correct, then Jim has red hair.*

   The converse: ***If Jim has red hair then what you said yesterday is correct.***

   The contrapositive: ***If Jim does not have red hair then what you said yesterday is false.***

   The denial: ***What you said yesterday is correct and Jim does not have red hair.***

   c. *If a triangle* $\triangle ABC$ *has a right angle at* C *then*

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   The converse: ***Every triangle*** $\triangle ABC$ ***satisfying the the condition***

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   must have a right angle at C***.***

   The contrapositive: ***Every triangle*** $\triangle ABC$ ***that fails to satisfy the the condition***

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   will not have a right angle at C***.***

   The denial: ***There is at least one triangle*** $\triangle ABC$ ***that has a right angle at*** C ***and for which the condition***

   $$(AB)^2 = (AC)^2 + (BC)^2.$$

   does not hold.

   d. *If some cats scratch, then all dogs bite.*

   The converse: ***If all dogs bite then some cats scratch.***

   The contrapositive: ***If there exists a dog that does not bite then no cats scratch.***

   The denial: ***Some cats scratch but not all dogs bite.***

   e. *It is with regret that I inform you that someone in this room is smoking.*

   The converse: ***This statement doesn't have a converse.***

   The contrapositive: ***This statement doesn't have a contrapositive form.***

   The denial: ***It is without regret that I inform you that someone in this room is smoking.***

f. *If a function is differentiable at a given number then it must be continuous at that number.*

The converse: ***If a function is continuous at a given number then it must be differentiable at that number.***

The contrapositive: ***If a function is not continuous at a given number then it cannot be differentiable at that number.***

The denial: ***There exists a function and a real number such that the function is differentiable at the number but fails to be continuous at that number.***

g. *Every boy or girl alive is either a little liberal or else a conservative.*

The converse: ***Every little liberal and every conservative is a living boy or girl***

The contrapositive: ***Any individual who fails to be either a little liberal or a conservative cannot be either a living boy or a living girl.***

The denial: ***There is at least one living boy or girl who is neither a little liberal nor a conservative.***

2. In each of the following exercises, write down a denial of the given statement.

a. *All cats scratch and some dogs bite.*

The denial: ***Either there is a cat that does not scratch or no dogs bite.***

b. *Either some cats scratch or if all dogs bite then some birds sing.*

The denial: ***No cats scratch and all dogs bite and no birds sing.***

c. *He walked into my office this morning, told me a pack of lies and punched me on the nose.*

The denial: ***Either he did not walk into my office this morning or he did not tell me a pack of lies or he did not punch me on the nose.***

d. *No one has ever seen an Englishman who is not carrying an umbrella.*

The denial: ***At least one person has seen an Englishman who is not carrying an umbrella.***

e. *For every number x there exists a number y such that $y > x$.*

***The denial: There exists a number** x **such that for every number** y **we have** $y \le x$.*

3. In each of the following exercises we assume that f and g are given functions. Write down a denial of each of the following statements:

a. *Whenever $x > 50$, we have $f(x) = g(x)$.*

The denial: ***There exists a number** $x > 50$ **such that** $f(x) \neq g(x)$.*

b. *There exists a number w such that $f(x) = g(x)$ for all numbers $x > w$.*

The denial: ***For every number** w **there exists a number** $x > w$ **such that** $f(x) \neq g(x)$.*

c. *For every number x there exists a number $\delta > 0$ such that for every number t satisfying the condition $|x - t| < \delta$, we have $|f(x) - f(t)| < 1$.*

The denial: ***There exists a number** x **such that for every number** $\delta > 0$ **there is at least one number** t **satisfying the condition** $|x - t| < \delta$ **such that** $|f(x) - f(t)| \ge 1$.*

d. *There exists a number $\delta > 0$ such that for every pair of numbers x and t satisfying the condition $|x - t| < \delta$, we have $|f(x) - f(t)| < 1$.*

The denial: ***For every number** $\delta > 0$ **it is possible to find a pair of numbers** x **and** t **satisfying the condition** $|x - t| < \delta$ **and for which** $|f(x) - f(t)| \ge 1$.*

e. *For every number $\varepsilon > 0$ and for every number x, there exists a number $\delta > 0$ such that for every number t satisfying $|x - t| < \delta$, we have $|f(t) - f(x)| < \varepsilon$.*

The denial: ***There exists a number** $\varepsilon > 0$ **and a number** x **such that for every number** $\delta > 0$ **it is possible to find a number** t **such that** $|x - t| < \delta$ **and** $|f(x) - f(t)| \geq \varepsilon$.*

f. *For every positive number ε there exists a positive number δ such that for every pair of numbers x and t satisfying the condition $|x - t| < \delta$, we have $|f(x) - f(t)| < \varepsilon$.*

The denial: ***There exists a number** $\varepsilon > 0$ **such that for every number** $\delta > 0$ **it is possible to find a pair of numbers** x **and** t **satisfying the condition** $|x - t| < \delta$ **for which** $|f(x) - f(t)| \geq \varepsilon$.*

4. Explain why the statement $\neg(P \Rightarrow Q)$ is equivalent to the statement $P \wedge (\neg Q)$.

 The assertion $P \Rightarrow Q$ says that if P is true then Q must also be true. This assertion says nothing at all about Q in the event that P is false. The only way in which the assertion $P \Rightarrow Q$ can be false is that P is true and Q is not. In other words, the denial of the condition $P \Rightarrow Q$ says that $P \wedge (\neg Q)$.

5. Explain why the statement $\neg(P \Leftrightarrow Q)$ is equivalent to the assertion that either (P is true and Q is false) or (P is false and Q is true).

 The assertion $P \Leftrightarrow Q$ says that P and Q have the same truth value. So the assertion $\neg(P \Leftrightarrow Q)$ says that they don't, which means that one of them is true and the other is false.

6. Explain why the statement $\neg(P \vee Q)$ is equivalent to the statement $(\neg P) \wedge (\neg Q)$.

 The assertion $P \vee Q$ says that at least one of the statements P and Q is true. So the denial of the the assertion $P \vee Q$ says that they are both false.

7. Explain why the statement $\neg(P \Rightarrow (Q \vee R))$ is equivalent to the assertion that P is true and that both of the statements Q and R are false.

 The denial of the assertion $P \Rightarrow (Q \vee R)$ says that P is true but that the assertion $Q \vee R$ is false. In other words, it says that P is true and that both of the statements Q and R are false.

8. Explain why the converse of the statement $P \Rightarrow (Q \vee R)$ is equivalent to the condition $(R \Rightarrow P) \wedge (Q \Rightarrow P)$.

 The converse of the statement $P \Rightarrow (Q \vee R)$ says that $(Q \vee R) \Rightarrow P$ and this says that P must be true if at least one of the statments Q and R are true. In other words, the satement $(Q \vee R) \Rightarrow P$ says that $Q \Rightarrow P$ and also that $R \Rightarrow P$.

9. Write the assertion $P \Rightarrow (Q \vee R)$ as simply as you can in its contrapositive form.

 The contrapositive form of the assertion $P \Rightarrow (Q \vee R)$ says that $(\neg(Q \vee R)) \Rightarrow (\neg P)$ and this says that if both of the statements Q and R are false then P is false. In other words, this contrapositive form says that $((\neg Q) \wedge (\neg R)) \Rightarrow (\neg P)$.

10. Write the assertion $(P \wedge Q) \Rightarrow (R \vee S)$ as simply as you can in its contrapositive form.

 The contrapositive form of the assertion $(P \wedge Q) \Rightarrow (R \vee S)$ says that

$$\neg(R \vee S) \Rightarrow \neg(P \wedge Q)$$

 which we can write as

$$(\neg R) \wedge (\neg S) \Rightarrow (\neg P) \vee (\neg Q).$$

 The latter statement says that if both of the statements R and S are false then at least one of the statements P and Q is false.

3 Strategies for Writing Proofs

Some Exercises on Statements Containing “And”

1. Prove the following assertions:

   a. If a, b, c, x and y are positive numbers and $a^x = b$ and $b^y = c$, then $a^{xy} = c$.

   The point of this exercise is to emphasize that you don’t alrady have numbers a, b, c, x and y quantified for you in the statement of the exercise.

   Solution: ***You need to begin: Suppose that*** a, b, c, x ***and*** y ***are positive numbers and that*** $a^x = b$ ***and*** $b^y = c$. ***Then you can argue that***

   $$a^{xy} = (a^x)^y = b^y = c.$$

   b. If two integers m and n are both even then mn has a factor 4.

   Solution: ***You need to begin: Suppose that*** m ***and*** n ***are even integers. Since*** m ***is even we know that*** $m/2$ ***is an integer and since*** n ***is even we know that*** $n/2$ ***is an integer. Since***

   $$mn = 4\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)$$

   we conclude that mn ***has a factor*** 4.

   c. If an integer m is even and an integer n has a factor 3 then mn has a factor 6.

   You need to begin: Suppose that m and n are integers and that m is even and that n has a factor 3. We know that the numbers $m/2$ and $n/3$ are both integers. Since

   $$mn = 6\left(\frac{m}{2}\right)\left(\frac{n}{3}\right)$$

   we deduce that mn has a factor 6.

2. “Ladies and gentlemen of the jury” said the prosecutor, “We shall demonstrate beyond a shadow of doubt that on the night of June 13, 1997, the accused, Slippery Sam Carlisle, did willfully, unlawfully and maliciously kill and murder the deceased, Archibald Bott by striking him on the head with a blunt instrument”. Outline a strategy that the prosecutor might use in order to prove this charge. How many separate assertions must the prosecutor prove in order to carry out his promise to the jury?

   Of course this problem isn’t serious in suggesting that Slippery Sam should be found not guilty if any one of the actions decribed by the prosecutor turns out to be untrue. But, had the prosecutor been a mathematician he would have agreed that his obligation, in order to convict Slippery Sam, is to show that Sam’s action was wilful and also that it was unlawful and also that it was malicious and also that it involved killing Mr. Bott and also that it involved murdering Mr. Bott and also that the act was performed by striking Mr. Bott on the head and also that the murder weapon was blunt. If even one of these conditions is found to be false then a mathematician might be inclined to find Slippery Sam not guilty.

3. One of the basic laws of arithmetic tells us that if a and b are any two numbers satisfying the condition $a < b$ and if $x > 0$ then $ax < bx$. Show how this law may be used to show that if $0 < u < 1$ and $0 < v < 1$ then $0 < uv < 1$.

   Solution: ***We begin the proof by quantifying*** u ***and*** v***: Suppose that*** u ***and*** v ***are numbers satisfying the inequalities*** $0 < u < 1$ ***and*** $0 < v < 1$. ***Since*** $u < 1$ ***and*** $v > 0$ ***we know that*** $uv < 1v$ ***which tells us that*** $uv < v$. ***Since*** $uv < v$ ***and*** $v < 1$ ***we conclude that*** $uv < 1$.
   Now since $0 < v$ ***and*** $u > 0$ ***we know that*** $0u < uv$ ***which tells us that*** $0 < uv$. ***Therefore*** $0 < uv < 1$.

4. In this exercise, if we are given three nonnegative integers a, b and c then the integer that consists of a hundreds, b tens and c units will be written as $\ulcorner a,b,c \urcorner$. Given nonnegative integers a, b and c, prove the assertion $P \wedge Q \wedge R \wedge S$ where P, Q, R and S are, respectively, the following assertions

   P. If the number $\ulcorner a,b,c \urcorner$ is divisible by 3 then the number $a + b + c$ is also divisible by 3.

Q. If the number $a+b+c$ is divisible by 3 then the number $\lceil a,b,c \rceil$ is also divisible by 3.

R. If the number $\lceil a,b,c \rceil$ is divisible by 9 then the number $a+b+c$ is also divisible by 9.

S. If the number $a+b+c$ is divisible by 9 then the number $\lceil a,b,c \rceil$ is also divisible by 9.

Hint: ***In this exercise we are actually given three nonnegative integers a, b and c and so there is no need to quantify them. In order to prove the assertion $P \wedge Q \wedge R \wedge S$ we have to show that all of the statements P, Q, R, and S are true. We therefore have four separate proofs to write. We show the proof of statement Q here and leave the rest to you:***

Suppose that the number $a+b+c$ is divisible by 3. We need to show that the number

$$\lceil a,b,c \rceil = 100a + 10b + c$$

is also divisible by 3. Now since the number $\frac{a+b+c}{3}$ is an integer and since

$$\lceil a,b,c \rceil = 100a + 10b + c = 99a + 9b + a + b + c$$
$$= 3\left(33a + 3b + \frac{a+b+c}{3}\right)$$

we know that $\lceil a,b,c \rceil$ has a factor 3.

Some Exercises on Statements Containing "Or"

1. Given that m and n are integers and that the number mn is not divisible by 4, prove that either m is odd or n is odd.

   Solution: ***The required statement can be proved by showing that if both m and n are even then the number mn has a factor 4. Suppose that both m and n are even. We know that the numbers $\frac{m}{2}$ and $\frac{n}{2}$ are integers and that***

   $$mn = 4\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)$$

   from which it follows that mn has a factor 4.
   Instead of making a new problem out of this exercise we can interpret it as asking us to prove that if both m and n are even then mn has a factor 4. We did this problem in the 3.2.4.

2. Given that m and n are integers, that neither m nor n is divisible by 4 and that at least one of the numbers m and n is odd, prove that the number mn is not divisible by 4.

   Solution: ***We can interpret this exercise as asking us to prove the following three assertions:***

   a. ***If m and n are even then mn must have a factor 4.***

   b. ***If m and n are integers and m has a factor 4 then mn has a factor 4.***

   c. ***If m and n are integers and n has a factor 4 then mn has a factor 4.***

   The first of these was handled in the previous exercise and also in an earlier exercise***. The other two are easy and, of course, they are analogs of one another so we need not do both of them.***

3. Prove that if $x = -\cos 40^\circ$ or $x = -\cos 80^\circ$ then

   $$8x^3 - 6x - 1 = 0.$$

   Solution: ***We have two jobs to do:***

   - ***We need to show that if $x = -\cos 40^\circ$ then $8x^3 - 6x - 1 = 0$.***
   - ***We need to show that if $x = -\cos 80^\circ$ then $8x^3 - 6x - 1 = 0$.***

   To prove the first of these assertions, we assume that $x = -\cos 40^\circ$. In order to show that $8x^3 - 6x - 1 = 0$

we shall use the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

that holds for every number θ. We now observe that

$$\begin{aligned} 8x^3 - 6x - 1 &= 8(-\cos 40^\circ)^3 - 6(-\cos 40^\circ) - 1 \\ &= -2(4\cos^3 40^\circ - 3\cos 40^\circ) - 1 \\ &= -2\cos(3(40^\circ)) - 1 \\ &= -2\cos 120^\circ - 1 \\ &= -2\left(-\frac{1}{2}\right) - 1 = 0 \end{aligned}$$

4. A theorem in elementary calculus, known as Fermat's theorem, says that if a function f defined on an interval has either a maximum or a minimum value at a number c inside that interval then either $f'(c) = 0$ or $f'(c)$ does not exist. Give a brief outline of a strategy for approaching the proof of this theorem.

Solution:

Case 1: ***Suppose that f is a function defined on an interval and that has a maximum value a number c inside that interval, and suppose that $f'(c)$ exists. We need to show that $f'(c) = 0$.***

Case 2: ***Suppose that f is a function defined on an interval and that has a minimum value a number c inside that interval, and suppose that $f'(c)$ exists. We need to show that $f'(c) = 0$.***

5. A well known theorem on differential calculus that is known as **L'Hôpital's rule** may be stated as follows:

Suppose that f and g are given functions, that c is a given number, that

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L,$$

and that one or other of the following two conditions holds

a. *Both $f(x)$ and $g(x)$ approach 0 as $x \to c$.*

b. *$g(x) \to \infty$ as $x \to c$.*

Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

Describe how the proof of L'Hôpital's rule can be broken down into two parts. For each part of the proof, say what is being assumed and what is being proved.

The purpose of this exercise is not to dig into the underlying ideas of L'Hôpital's rule. Nor do we need to concern ourselves with the question as to whether it is actually necessary to break the proof into two parts. Instead, we assume that the two parts are needed. The student should be able to see that, approached this way, the theorem requires two separate proofs. In one proof we assume that both $f(x)$ and $g(x)$ approach 0 as $x \to c$ and in the other proof we assume that $g(x) \to \infty$ as $x \to c$.

Exercises on the Symbol $\Rightarrow$

1. a. Outline a strategy for proving an assertion that has the form $P \Rightarrow (Q \wedge R)$.

   Solution: ***Assume that P is true. Then write a proof that Q is true. Then write a proof that R is true.***

   b. Write down the assertion $P \Rightarrow (Q \wedge R)$ in its contrapositive form and outline a strategy for proving it in this form.

Solution: ***The contrapositive form says that $(\neg Q) \vee (\neg R) \Rightarrow \neg P$. Once again we have two jobs to do. We assume that the statement Q is false and prove that P is false. Then we assume that the statement R is false and again prove that P is false.***

2. a. Outline a strategy for proving an assertion that has the form $P \Rightarrow (Q \vee R)$.

   We need to show that if P is true than at least one of the assertions Q and R is true. We could begin by writing: "Suppose that P is true." Then we could try to prove that at least one of the assertions Q and R is true. One way to do this is to assume that Q is false and then show that R must be true.

   b. Write down the assertion $P \Rightarrow (Q \vee R)$ in its contrapositive form and outline a strategy for proving it in this form.

   The contrapositive form of the assertion $P \Rightarrow (Q \vee R)$ says that

$$(\neg Q) \wedge (\neg R) \Rightarrow \neg P$$

   To prove the assertion this way we should assume that both Q and R are false and then use this information to show that P is false.

3. a. Outline a strategy for proving an assertion that has the form $(P \wedge Q) \Rightarrow R$.

   Solution: ***Assume that both of the statements P and Q are true and write a proof that R is true.***

   b. Write down the assertion $(P \wedge Q) \Rightarrow R$ in its contrapositive form and outline a strategy for proving it in this form.

   Solution: ***The contrapositive form says that $\neg R \Rightarrow (\neg P) \vee (\neg Q)$. Assume that the statement R is false and show that at least one of the statements P and Q must be false.***

4. a. Outline a strategy for proving an assertion that has the form $(P \vee Q) \Rightarrow R$.

   We need to write two proofs. First we need to show that if we assume P then R must be true. Then we need to show that if we assume Q then R must be true.

   b. Write down the assertion $(P \vee Q) \Rightarrow R$ in its contrapositive form and outline a strategy for proving it in this form.

   The contrapositive form of the assertion $(P \vee Q) \Rightarrow R$ says that $(\neg R) \Rightarrow (\neg P) \wedge (\neg Q)$ and the strategy for proving this form of the assertion was given in Exercise 1a.

5. a. Outline a strategy for proving an assertion that has the form $(P \vee (Q \Rightarrow P)) \Rightarrow R$.

   Solution: ***Assume that the statement P is true and write a proof that R must be true. Then assume that the statement Q is false and write a proof that R must be true.***

   b. Write down the assertion $(P \vee (Q \Rightarrow P)) \Rightarrow R$ in its contrapositive form and outline a strategy for proving it in this form.

   Solution: ***The contrapositive form says that $\neg R \Rightarrow (\neg P) \wedge Q$. Assume that R is false and write a proof that P must be false. Then assume that R is false and write a proof that Q must be true.***

Exercises on Statements Containing Quantifiers

1. Physicist's proof that all odd natural numbers are prime: 1 is prime. 3 is prime. 5 is prime. 7 is prime. 9 is experimental error. 11 is prime. 13 is prime. We have now taken sufficiently many readings to verify the hypothesis. Comment!

2. You know that there are 1000 people in a hall. Upon inspection you determine that 999 of these people are men. What can you conclude about the 1000'th person?

   Solution: ***You can't make any conclusion at all about her. Don't even try.***

3. The *product rule for differentiation* says that for every number x and all functions f and g that are differentiable at $x,$ we have

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Write down the opening line of a proof of the product rule. Your opening line should start: *Suppose that ...*

Solution: ***Suppose that** x **is a real number and that** f **and** g **are functions that are differentiable at the number** x**.***

4. Given that $P(x)$ and $Q(x)$ are statements that contain an unknown x and that S is a set, outline a strategy for the proving the assertion $P(x) \Rightarrow Q(x)$ for every $x \in S.$ Write down the opening line of your proof.

Solution: ***Suppose that** $x \in S$ **and that the condition** $P(x)$ **is true. Then write a proof that** $Q(x)$ **must be true.***

5. Given that $P(x)$ is a statement that contains an unknown x and that S is a set, write down an opening line of a proof of the assertion that $P(x)$ is true for every $x \in S.$

Solution: ***Suppose that** $x \in S$**.***

6. You are given that $P(x)$ and $Q(x)$ are statements that contain an unknown $x,$ that S is a set, that $P(x)$ is true for every $x \in S$ and that $P(x) \Rightarrow Q(x).$ Is it possible to deduce that $Q(x)$ is true for every member x of the set S?

Solution: ***Yes it is possible. For each** x **we know that** $P(x)$ **is true and that** $P(x) \Rightarrow Q(x)$ **and so we know that** $Q(x)$ **is true.***

7. You are given that $P(x)$ and $Q(x)$ are statements that contain an unknown $x,$ that S is a set, that $P(x)$ is true for every $x \in S$ and that $P(x) \Rightarrow Q(x).$ Is it possible to deduce that $Q(x)$ is true for at least one member x of the set S?

Solution: ***No it isn't possible to deduce that** $Q(x)$ **is true for at least one member** x **of the set** S **because we have no information to the effect that the set** S **is nonempty.***

8. Write down the contrapositive form of the statement that for every member x of a given set S we have $P(x) \Rightarrow Q(x).$

Solution: ***For every member** x **of the set** S **we have** $(\neg Q(x)) \Rightarrow (\neg P(x))$**.***

9. Write down the denial of the statement that for every x we have $P(x) \Rightarrow Q(x).$

Solution: ***There is at least one** x **for which** $P(x)$ **is true and** $Q(x)$ **is false.***

10. Prove that for every number x in the interval $[-2,2],$ if we define

$$u = \sqrt[3]{\frac{3\sqrt{3}\,x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}}$$

and

$$v = \sqrt[3]{\frac{3\sqrt{3}\,x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}}$$

then

$$u^2 + v^2 = 1 + uv.$$

Hint: ***With an eye on the proof** shown earlier**, show that** $u^3 + v^3 = u + v.$*

That earlier proof shows that if x is any number in the interval $[-2,2]$ then

$$\sqrt[3]{\frac{3\sqrt{3}\,x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} + \sqrt[3]{\frac{3\sqrt{3}\,x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}} = x$$

which we can express as $u + v = 1$. On the other hand,

$$u^3 + v^3 = \frac{3\sqrt{3}\,x + (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}} + \frac{3\sqrt{3}\,x - (2x^2+1)\sqrt{4-x^2}}{6\sqrt{3}}$$

and we conclude that $u^3 + v^3 = x$. Thus

$$u^3 + v^3 = u + v$$

and, by factorizing the left side, we obtain the desired result

$$u^2 + v^2 = 1 + uv$$

without difficulty.

Some Exercises on Solution of Cubic Equations

1. Solve the following equations:

a. $x^3 - 6x + 6 = 0$

Solution: ***We look for two numbers r and s such that***

$$r^3 + s^3 = 6$$

and

$$3rs = 6$$

Thus

$$r^3 + \left(\frac{2}{r}\right)^3 = 6$$

and so

$$r^6 - 6r^3 + 8 = 0$$

Which gives us

$$(r^3 - 2)(r^3 - 4) = 0$$

So there are two possibilities: Either $r = \sqrt[3]{2}$, in which case $s = \sqrt[3]{4}$ or $r = \sqrt[3]{4}$, in which case $s = \sqrt[3]{2}$. The equation

$$x^3 - 6x + 6 = 0$$

becomes

$$x^3 + r^3 + s^3 - 3rsx = 0$$

which we can write as

$$(x + r + s)(x^2 + r^2 + s^2 - xr - xs - rs) = 0.$$

and so either

$$x = -\sqrt[3]{2} - \sqrt[3]{4}$$

or

$$x = \frac{\sqrt[3]{2} + \sqrt[3]{4} \pm \sqrt{3}\,i\left(\sqrt[3]{2} - \sqrt[3]{4}\right)}{2}.$$

b. $x^3 - 9x + 12 = 0$

We look for numbers r and s such that

$$r^3 + s^3 = 12$$

$$3rs = 9$$

which gives us

$$r^3 + \left(\frac{3}{r}\right)^3 = 12$$

and so

$$(r^3-3)(r^3-9)=0$$

and so we have the possibilities $r=\sqrt[3]{3}$ or $r=\sqrt[3]{9}$ that boil down to the one possibility that $r=\sqrt[3]{3}$ and $s=\sqrt[3]{9}$. So the equation $x^3-9x+12=0$ becomes

$$(x+r+s)(x^2+r^2+s^2-xr-xs-rs)=0.$$

which gives us the real solution $x=-\sqrt[3]{3}-\sqrt[3]{9}$ and also the possibility

$$x=\frac{\sqrt[3]{3}+\sqrt[3]{9}\pm\sqrt{3}\,i\left(\sqrt[3]{3}-\sqrt[3]{9}\right)}{2}=\frac{\sqrt[3]{3}+\sqrt[3]{9}\pm i\left(\sqrt[3]{9}-3\right)}{2}$$

c. $x^3-3x+2=0$

One might notice that $x=1$ is an obvious solution of this equation. That observation leads to

$$(x-1)(x^2+x-2)=0$$

giving us the solutions $x=1$ or $x=1$ or $x=-2$.

On the other hand, if we look for r and s to make

$$r^3+s^3=2$$
$$3rs=3$$

then we obtain

$$r^3+\left(\frac{1}{r}\right)^3=2$$

which becomes

$$r^6-2r^3+1=0$$

which leads to the equation $r=1$. Thus $r=s=1$ and we obtain the solutions

$$x=-r-s=-2$$

or

$$x=\frac{r+s\pm\sqrt{3}\,i(r-s)}{2}=\frac{2\pm0}{2}=1.$$

d. $x^3-6x+10=0$

We look for numbers r and s such that

$$r^3+s^3=10$$
$$3rs=6$$

which gives us

$$r^3+\left(\frac{2}{r}\right)^3=10$$

in other words,

$$r^6-10r^3+8=0$$

and we obtain

$$r^3=5\pm\sqrt{17}$$

and we may assume that $r=\sqrt[3]{5+\sqrt{17}}$ and $s=\sqrt[3]{5-\sqrt{17}}$. Thus we have the possibilities

$$x=-r-s=\sqrt[3]{\sqrt{17}-5}-\sqrt[3]{\sqrt{17}+5}$$

or

$$x=\frac{r+s\pm\sqrt{3}\,i(r-s)}{2}=\frac{\sqrt[3]{5+\sqrt{17}}+\sqrt[3]{5-\sqrt{17}}\pm\sqrt{3}\,i\left(\sqrt[3]{5+\sqrt{17}}-\sqrt[3]{5-\sqrt{17}}\right)}{2}.$$

e. $x^3-6x^2+6x+14=0$. We begin by making the substitution $u=x-c$ and the equation $x^3-6x^2+6x+14=0$ becomes

$$(u+c)^3 - 6(u+c)^2 + 6(u+c) + 14 = 0$$

giving us

$$u^3 + (3c-6)u^2 + (3c^2 - 12c + 6)u + c^3 - 6c^2 + 6c + 14 = 0$$

and now we see that the definition $c = 2$ will convert the equation into a form with no quadratic term. We obtain

$$u^3 - 6u + 10 = 0$$

We now copy the solution obtained in part d and obtain

$$u = \sqrt[3]{\sqrt{17} - 5} - \sqrt[3]{\sqrt{17} + 5}$$

or

$$u = \frac{\sqrt[3]{5 + \sqrt{17}} + \sqrt[3]{5 - \sqrt{17}} \pm \sqrt{3}\, i\left(\sqrt[3]{5 + \sqrt{17}} - \sqrt[3]{5 - \sqrt{17}}\right)}{2}$$

and so

$$x = 2 + \sqrt[3]{\sqrt{17} - 5} - \sqrt[3]{\sqrt{17} + 5}$$

or

$$x = 2 + \frac{\sqrt[3]{5 + \sqrt{17}} + \sqrt[3]{5 - \sqrt{17}} \pm \sqrt{3}\, i\left(\sqrt[3]{5 + \sqrt{17}} - \sqrt[3]{5 - \sqrt{17}}\right)}{2}.$$

2. Show that if $w^3 < k^2$ then the cubic equation

$$x^3 - 3wx + 2k = 0$$

has one real solution

$$x = \sqrt[3]{\sqrt{k^2 - w^3} - k} - \sqrt[3]{\sqrt{k^2 - w^3} + k}$$

We look for numbers r and s such that

$$r^3 + s^3 = 2k$$
$$3rs = 3w$$

which gives us

$$r = \sqrt[3]{k \pm \sqrt{k^2 - w^3}}\,.$$

In other words, we can assume that

$$r = \sqrt[3]{k + \sqrt{k^2 - w^3}}$$

and

$$s = \sqrt[3]{k - \sqrt{k^2 - w^3}}$$

and we conclude that the only real solution of the equation

$$x^3 - 3wx + 2k = 0$$

is

$$x = \sqrt[3]{\sqrt{k^2 - w^3} - k} - \sqrt[3]{\sqrt{k^2 - w^3} + k}\,.$$

3. By considering the solution of the equation

$$(x-1)(x^2 + x + 3) = 0$$

(or otherwise) deduce that

$$\sqrt[3]{\frac{9\sqrt{3} + 5\sqrt{11}}{6\sqrt{3}}} + \sqrt[3]{\frac{9\sqrt{3} - 5\sqrt{11}}{6\sqrt{3}}} = 1.$$

We see at once that the only real solution of the equation

$$(x-1)(x^2+x+3)=0$$

is $x=1$. On the other hand, the equation can be written as

$$x^3+2x-3=0$$

and we can solve by looking for numbers r and s satisfying

$$r^3+s^3=-3$$

$$3rs=-2$$

From which we obtain

$$r^3+\left(\frac{-2}{3r}\right)^3=-3$$

which gives us

$$r^3=\frac{-9\sqrt{3}\pm 5\sqrt{11}}{6\sqrt{3}}$$

from which we can select

$$r=\sqrt[3]{\frac{-9\sqrt{3}+5\sqrt{11}}{6\sqrt{3}}}$$

and

$$r=\sqrt[3]{\frac{-9\sqrt{3}-5\sqrt{11}}{6\sqrt{3}}}$$

which gives us the real solution of the equation

$$(x-1)(x^2+x+3)=0$$

as

$$x=-r-s=\sqrt[3]{\frac{9\sqrt{3}+5\sqrt{11}}{6\sqrt{3}}}+\sqrt[3]{\frac{9\sqrt{3}-5\sqrt{11}}{6\sqrt{3}}}$$

and we conclude that

$$\sqrt[3]{\frac{9\sqrt{3}+5\sqrt{11}}{6\sqrt{3}}}+\sqrt[3]{\frac{9\sqrt{3}-5\sqrt{11}}{6\sqrt{3}}}=1.$$

4. a. Show that if α and β are given numbers and if the equation

$$(x-\alpha)(x^2+\alpha x+\beta)=0$$

is written in the form

$$x^3+ax+b=0$$

then $a=\beta-\alpha^2$ and $b=-\alpha\beta$.
Since

$$(x-\alpha)(x^2+\alpha x+\beta)=x^3+(\beta-\alpha^2)x-\alpha\beta$$

the desired result follows at once.

b. Given that α and β are real numbers and that $\alpha^2<4\beta$, explain why the equation

$$(x-\alpha)(x^2+\alpha x+\beta)=0$$

has only one real solution. By considering this equation, or otherwise, deduce that

$$\sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta+(2\alpha^2+\beta)\sqrt{4\beta-\alpha^2}}{6\sqrt{3}}}+\sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta-(2\alpha^2+\beta)\sqrt{4\beta-\alpha^2}}{6\sqrt{3}}}=\alpha.$$

With $a=\beta-\alpha^2$ and $b=-\alpha\beta$ the inequality

$$27b^2 + 4a^3 > 0$$

says that

$$(4\beta - \alpha^2)(\beta + 2\alpha^2)^2 > 0$$

which holds if and only if $\alpha^2 < 4\beta$. Now suppose that $\alpha^2 < 4\beta$. We look for two numbers r and s such that

$$r^3 + s^3 = -\alpha\beta$$

$$3rs = -(\beta - \alpha^2)$$

giving us

$$r^3 + \left(\frac{\alpha^2 - \beta}{3r}\right)^3 = -\alpha\beta$$

$$27r^6 + 27\alpha\beta r^3 + (\alpha^2 - \beta)^3 = 0$$

and we can take

$$r = \sqrt[3]{\frac{-3\sqrt{3}\,\alpha\beta + (\beta + 2\alpha^2)\sqrt{(4\beta - \alpha^2)}}{6\sqrt{3}\,\alpha\beta}}$$

and

$$s = \sqrt[3]{\frac{-3\sqrt{3}\,\alpha\beta - (\beta + 2\alpha^2)\sqrt{(4\beta - \alpha^2)}}{6\sqrt{3}\,\alpha\beta}}$$

then the real solution of the equation

$$(x - \alpha)(x^2 + \alpha x + \beta) = 0$$

appears as

$$-r - s = \sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta + (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}} + \sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta - (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}}.$$

Since that real solution has to be α we obtain

$$\sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta + (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}} + \sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta - (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}} = \alpha.$$

c. Given that α and β are real numbers and that $\alpha^2 < 4\beta$, consider the two complex roots of the equation

$$(x - \alpha)(x^2 + \alpha x + \beta) = 0$$

and deduce that

$$\sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta + (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}} = \frac{\alpha}{2} + \frac{1}{2}\sqrt{\frac{4\beta - \alpha^2}{3}}$$

and

$$\sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta - (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}} = \frac{\alpha}{2} - \frac{1}{2}\sqrt{\frac{4\beta - \alpha^2}{3}}$$

The complex roots are

$$\frac{r + s \pm \sqrt{3}\,i(r - s)}{2}$$

and since these two complex roots are also

$$\frac{-\alpha \pm i\sqrt{4\beta - \alpha^2}}{2}$$

we deduce that

$$r + s = -\alpha$$

and

$$r - s = \pm\frac{\sqrt{4\beta - \alpha^2}}{\sqrt{3}}$$

By looking at the values of r and s we see that $r - s > 0$ and so the preceding equation is actually

$$r - s = \frac{\sqrt{4\beta - \alpha^2}}{\sqrt{3}}.$$

Adding we obtain

$$2r = -\alpha + \frac{\sqrt{4\beta - \alpha^2}}{\sqrt{3}}$$

which gives us

$$\sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta - (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}} = \frac{\alpha}{2} - \frac{1}{2}\sqrt{\frac{4\beta - \alpha^2}{3}}$$

and by subtracting we obtain

$$\sqrt[3]{\frac{3\sqrt{3}\,\alpha\beta + (2\alpha^2 + \beta)\sqrt{4\beta - \alpha^2}}{6\sqrt{3}}} = \frac{\alpha}{2} + \frac{1}{2}\sqrt{\frac{4\beta - \alpha^2}{3}}$$

Some Exercises on the Trigonometric Method

1. By considering the equation

$$(x-1)(x-2)(x+3) = 0$$

show that

$$x^3 - 7x + 6 = 0$$

$$\frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right)\right) = 2$$

and

$$\frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) + \frac{2\pi}{3}\right) = -3$$

and

$$\frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) - \frac{2\pi}{3}\right) = 1$$

Solution: ***We observe first that***

$$(x-1)(x-2)(x+3) = x^3 - 7x + 6.$$

To solve the equation

$$x^3 - 7x + 6 = 0$$

we look for a solution of the form $x = k\cos\theta$ ***and express the equation as***

$$k^3\cos^3\theta - 7k\cos\theta = -6$$

and we choose k ***in such a way that***

$$\frac{k^3}{7k} = \frac{4}{3}$$

which will hold when

$$k = \frac{2\sqrt{7}}{\sqrt{3}}.$$

The equation

$$x^3 - 7x + 6 = 0$$

now becomes

$$\left(\frac{2\sqrt{7}}{\sqrt{3}}\right)^3 \cos^3\theta - 7\left(\frac{2\sqrt{7}}{\sqrt{3}}\right)\cos\theta = -6$$

which gives us

$$4\cos^3\theta - 3\cos\theta = -\frac{9\sqrt{3}}{7\sqrt{7}}$$

in other words,

$$\cos 3\theta = -\frac{9\sqrt{3}}{7\sqrt{7}}$$

and we deduce that either

$$x = \frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right)\right)$$

or

$$x = \frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) + \frac{2\pi}{3}\right)$$

or

$$x = \frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) - \frac{2\pi}{3}\right)$$

Finally we observe that since

$$0 < \frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) < \frac{\pi}{3}$$

we have

$$\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right)\right) > \frac{1}{2}$$

and

$$\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) + \frac{2\pi}{3}\right) < 0$$

and

$$\left|\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) - \frac{2\pi}{3}\right)\right| < \frac{1}{2}$$

Thus the largest of the solutions of the equation is

$$\frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right)\right)$$

and the negative solution is

$$\frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right) + \frac{2\pi}{3}\right)$$

and the remaining solution is

$$\frac{2\sqrt{7}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{9\sqrt{3}}{7\sqrt{7}}\right)-\frac{2\pi}{3}\right)$$

2. Given real numbers α, and β, prove that the number

$$\frac{2\sqrt{\alpha\beta+\alpha^2+\beta^2}}{\sqrt{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{3\sqrt{3}\,(\alpha^2\beta+\alpha\beta^2)}{2(\alpha\beta+\alpha^2+\beta^2)\sqrt{\alpha\beta+\alpha^2+\beta^2}}\right)\right)$$

is the largest of the three numbers α and β and $-(\alpha+\beta)$.

Hint: ***Apply the method of Exercise 1 to the equation***

$$(x-\alpha)(x-\beta)(x+\alpha+\beta)=0.$$

Exercises on Proof by Contradiction

1. Prove that the following numbers are irrational:

a. $\log_{10}5$

To obtain a contradiction, assume that $\log_{10}5$ is rational and choose positive integers m and n such that

$$\log_{10}5=\frac{m}{n}$$

Since

$$10^{m/n}=5$$

we have

$$10^m=5^n$$

which is impossible because 10^m is even and 5^n is odd.

b. $\log_{12}24$

Solution: ***To obtain a contradiction, assume that the number*** $\log_{12}24$ ***is rational and choose positive integers*** m ***and*** n ***such that***

$$\log_{12}24=\frac{m}{n}.$$

Since

$$12^{m/n}=24$$

we have

$$12^m=24^n$$

which we can express as

$$12^{m-n}=2^n$$

Since $2^n>1$ ***we see that*** $m>n$ ***and so the number*** 12^{m-n} ***is an integer that has a factor*** 3***. But the integer*** 2^n ***cannot have a factor*** 3 ***and we have contradicted the fact that***

$$12^{m-n}=2^n.$$

c. $\sqrt[3]{4}$

To obtain a contradiction, suppose that the number $\sqrt[3]{4}$ is rational. Choose positive integers m and n that have no common factor such that

$$\sqrt[3]{4}=\frac{m}{n}.$$

Since

$$4n^3 = m^3$$

and since n is a factor of the number $4n^3$ we conclude that n is a factor of m^3. Therefore, since m and n have no common factor we must have $n = 1$ and the equation $4n^3 = m^3$ becomes

$$m^3 = 4$$

which is impossible since there is no integer whose cube is 4.

d. Any solution of the equation $8x^3 - 6x - 1 = 0$.

Solution: ***Suppose that*** x ***is a solution of the equation*** $8x^3 - 6x - 1 = 0$ ***and, to obtain a contradiction, assume that*** x ***is rational. Choose integers*** m ***and*** n ***such that*** $n > 0$ ***and*** m ***and*** n ***have no common factor and*** $x = \frac{m}{n}$***. Since***

$$8\left(\frac{m}{n}\right)^3 - 6\left(\frac{m}{n}\right) - 1 = 0$$

we have

$$8m^3 - 6mn^2 - n^3 = 0.$$

From the fact that

$$8m^3 = n(6mn + n^2)$$

we deduce that n ***is a factor of*** $8m^3$***. Therefore, since*** m ***and*** n ***have no common factor, the number*** n ***must be*** ± 1 ***or*** ± 2 ***or*** ± 4 ***or*** ± 8***.***

Returning to the equation

$$8m^3 - 6mn^2 - n^3 = 0.$$

we deduce that

$$n^3 = m(8m^2 - 6n^2)$$

and so n ***is a factor of*** n^3***. Therefore, since*** m ***and*** n ***have no common factor we must have*** $m = \pm 1$***. We deduce that*** x ***must be one of the numbers*** ± 1***,*** $\pm\frac{1}{2}$***,*** $\pm\frac{1}{4}$ ***or*** $\pm\frac{1}{8}$ ***and, by trying each of these numbers in the equation*** $8x^3 - 6x - 1 = 0$***, we can verify that none of them are solutions. Therefore no rational number can be a solution of this equation.***

Note: ***A quick way to verify that none of the numers*** ± 1***,*** $\pm\frac{1}{2}$***,*** $\pm\frac{1}{4}$ ***or*** $\pm\frac{1}{8}$ ***can be solutions of the equation*** $8x^3 - 6x - 1 = 0$ ***is to supply the equation***

$$f(x) = 8x^3 - 6x - 1$$

as a definition to Scientific Notebook by pointing at this equation and clicking on the new definition button ***and then evaluating the expression***

$$f\begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ -1 \\ -\frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{8} \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -\frac{19}{8} \\ -\frac{111}{64} \\ -3 \\ 1 \\ \frac{3}{8} \\ -\frac{17}{64} \end{pmatrix}$$

which shows us at once that $f(x)$ ***is not zero when*** x ***is one of the eight rational numbers that might have been solutions of the equation*** $8x^3 - 6x - 1 = 0$***.***

2. Given that m and n are integers and that mn does not have a factor 3, prove that neither m nor n can have a factor 3.

   What we want to prove is that if at least one of the integers m and n has a factor 3 then mn must have a factor 3. Strictly speaking, there are two cases to consider because the statement that at least one of the integers m and n has a factor 3 means that either m has a factor 3 or n has a factor 3. However, these two cases are analogous. We can turn one into the other by renaming the numbers. We can therefore restrict ourselves safely to just one of the two cases. Our proof can begin:
   Without loss of generality, we assume that m has a factor 3. Since $m/3$ is an integer and since
   $$mn = 3(m/3)n$$
   we see that mn has a factor 3, which is what we wanted to prove.

3. Suppose that we know that $x^2 - 2x < 0$ and that we wish to prove that $0 < x < 2$. Write down the first line of a proof of this assertion that uses the method of proof by contradiction. Do so in such a way that your proof splits into two cases and compete the proof in each of these cases.

   To obtain a contradiction we assume that the condition $0 < x < 2$ is false. That means that either $x \leq 0$ or $x \geq 2$ and so we must consider two cases.
   The case $x \leq 0$: In this case, since $x^2 - 2x = x(x-2)$ and since $x \leq 0$ and $x - 2 < 0$ we have $x^2 - x \geq 0$, contradicting the assumption that $x^2 - x < 0$.
   The case $x \geq 2$: In this case, since $x^2 - 2x = x(x-2)$ and since $x > 0$ and $x - 2 \geq 0$ we have $x^2 - x \geq 0$, contradicting the assumption that $x^2 - x < 0$.
   Since each of the two cases leads to a contradiction our proof is complete.

4. Suppose that f is a given function defined on the interval $[0,1]$ and suppose that we wish to prove that this function f has the property that there exists a number $\delta > 0$ such that whenever t and x belong to the interval $[0,1]$ and $|t-x| < \delta$, we have $|f(t) - f(x)| < 1$. Write down the first line of a proof of this assertion that uses the method of proof by contradiction.

   Solution: ***To obtain a contradiction, suppose that it is impossible to find a number*** $\delta > 0$ ***such that whenever*** t ***and*** x ***belong to the interval*** $[0,1]$ ***and*** $|t-x| < \delta$ ***we have*** $|f(t) - f(x)| < 1$***.***

   Alternative Solution: ***To obtain a contradiction, suppose that for every number*** $\delta > 0$ ***there exist numbers*** t ***and*** x ***in the interval*** $[0,1]$ ***such that*** $|t-x| < 1$ ***and*** $|f(t) - f(x)| \geq 1$***.***

5. Suppose that $\{x_1, x_2, \cdots x_n\}$ is a subset of a vector space V and that we wish to prove that the set $\{x_1, x_2, \cdots x_n\}$ is linearly independent. Write down the first line of a proof of this assertion that uses the method of proof by contradiction. (Try to be specific. Don't just suppose that the set is linearly dependent.)
   The assertion that the set $\{x_1, x_2, \cdots x_n\}$ is linearly dependent says that there exist numbers $c_1, c_2, \cdots, c_n$ that are not all zero such that
   $$\sum_{j=1}^{n} c_j x_j = \mathbf{0}$$
   where $\mathbf{0}$ is the "origin" in the vector space. A proof by contradiction that $\{x_1, x_2, \cdots x_n\}$ is linearly independed could begin as follows:
   To obtain a contradiction, assume that $\{x_1, x_2, \cdots x_n\}$ is linearly dependent. Choose numbers $c_1, c_2, \cdots, c_n$ that are not all zero such that
   $$\sum_{j=1}^{n} c_j x_j = \mathbf{0}$$
   and, using the fact that not all of the numbers $c_1, c_2, \cdots, c_n$ are zero, choose j such that $c_j \neq 0$.

Some Additional Exercises

In each of the following exercises, decide whether the statement is true or false and then write a carefully worded proof to justify your assertion.

1. For every number $x \in [0,1]$ there exists a positive integer N such that for every number $\varepsilon > 0$ and every integer $n \geq N$ we have
$$\frac{nx}{1+n^2x^2} < \varepsilon.$$
Solution: ***This statement is false. The denial of the statement says that there exists a number*** $x \in [0,1]$ ***such that for every positive integer*** N ***there exists a number*** $\varepsilon > 0$ ***and an integer*** $n \geq N$ ***such that***
$$\frac{nx}{1+n^2x^2} \geq \varepsilon.$$
To prove this denial we shall give an example of a number $x \in [0,1]$ ***such that for every positive integer*** N ***there exists a number*** $\varepsilon > 0$ ***and an integer*** $n \geq N$ ***such that***
$$\frac{nx}{1+n^2x^2} \geq \varepsilon.$$
We define $x = \frac{1}{2}$***.***

We now want to prove that something happens for every positive integer N ***and so we continue: Suppose that*** N ***is a positive integer.***

Now we want to prove that there exists a number $\varepsilon > 0$ ***and an integer*** $n \geq N$ ***such that***
$$\frac{nx}{1+n^2x^2} \geq \varepsilon.$$
and we shall do so by giving an example of such a number ε***.***

We define
$$\varepsilon = \frac{N\left(\frac{1}{2}\right)}{1+N^2\left(\frac{1}{2}\right)^2}.$$
To prove that there exists an integer $n \geq N$ ***such that***
$$\frac{nx}{1+n^2x^2} \geq \varepsilon$$
we make the observation that
$$\frac{Nx}{1+N^2x^2} \geq \varepsilon.$$

2. For every number $x \in [0,1]$ and every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number $t \in [0,1]$ satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

Solution: ***We shall prove that this statement is true. Since we want to prove that something happens for every number*** $x \in [0,1]$***, we begin as follows:***

Suppose that $x \in [0,1]$***.***

Now we want to prove that something happens for every number $\varepsilon > 0$ ***and so we continue:***

Suppose that $\varepsilon > 0$***.***

We are now seeking a number $\delta > 0$ ***such that the inequality*** $|t^2 - x^2| < \varepsilon$ ***will hold whenever*** $t \in [0,1]$ ***and*** $|t-x| < \delta$***. We observe first that whenever*** $t \in [0,1]$ ***we have***
$$|t^2 - x^2| = |t-x||t+x| \leq |t-x|(|t|+|x|) \leq 2|t-x|.$$
With this inequality in mind we define $\delta = \varepsilon/2$***.***

We see at once that whenever $t \in [0,1]$ ***and*** $|t-x| < \delta$ ***we have***
$$|t^2 - x^2| \leq 2|t-x| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

3. For every number $\varepsilon > 0$ and every number $x \in [0,1]$ there exists a number $\delta > 0$ such that for every number $t \in [0,1]$ satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

Hint: ***The statement that appears here is identical to the one in Exercise 2.***

4. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number $x \in [0,1]$ and every number $t \in [0,1]$ satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

Solution: ***We shall prove that this statement is true. Since we want to prove that something happens for every number*** $\varepsilon > 0$ ***we begin:***

Suppose that $\varepsilon > 0$.

We are now seeking a number $\delta > 0$ ***such that the intequality*** $|t^2 - x^2| < \varepsilon$ ***will hold for all numbers*** t ***and*** x ***in the interval*** $[0,1]$ ***satisfying the condition*** $|t-x| < \delta$. ***To give an example of such a number*** δ ***we continue:***

Define $\delta = \varepsilon/2$.

We see at once that whenever $t \in [0,1]$ ***and*** $x \in [0,1]$ ***and*** $|t-x| < \delta$ ***we have***

$$|t^2 - x^2| \le 2|t-x| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

5. For every number $\varepsilon > 0$ and every number x there exists a number $\delta > 0$ such that for every number t satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

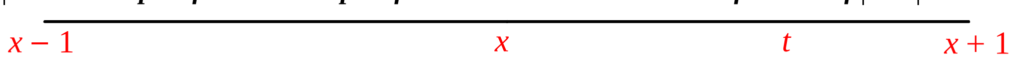
Solution: ***The statement made in this exercise says a little more than the one in Exercise 2 but it is still true. Since we want to prove that something happens for every number*** $\varepsilon > 0$ ***we begin:***

Suppose that $\varepsilon > 0$.

Now we want to prove that something happens for every number x ***and so we continue:***

Suppose that x ***is a real number.***

We are now seeking a number $\delta > 0$ ***such that for every number*** t ***satisfying*** $|t-x| < \delta$ ***we have*** $|t^2 - x^2| < \varepsilon$. ***To help us find an example of such a number*** δ ***we observe first that if*** $|t-x| < 1$



then, since

$$|t| = |t-x+x| \le |t-x| + |x| < 1 + |x|$$

we have

$$|t^2 - x^2| = |t-x||t+x| \le |t-x|(|t| + |x|) \le |t-x|(1 + 2|x|)$$

With this inequality in mind we define δ ***to be the smaller of the two numbers*** 1 ***and*** $\varepsilon/(1+2|x|)$.

We observe that whenever a number t ***satisfies the inequality*** $|t-x| < \delta$ ***we have***

$$|t^2 - x^2| \le |t-x|(1+2|x|) < \frac{\varepsilon}{1+2|x|}(1+2|x|) = \varepsilon.$$

6. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number x and every number t satisfying $|t-x| < \delta$ we have $|t^2 - x^2| < \varepsilon$.

Solution: ***This statement is stronger than the statement in Exercise 4 but it is too strong to be true. The denial of this statement says that there exists a number*** $\varepsilon > 0$ ***such that for every number*** $\delta > 0$ ***it is possible to find two numbers*** t ***and*** x ***such that*** $|t-x| < \delta$ ***and*** $|t^2 - x^2| \ge \varepsilon$. ***We shall prove that this denial is true. In order to prove the existence of a number*** ε ***as described in this assertion, we shall give an example.***

We define $\varepsilon = 1$.

Now we want to prove that something happens for every number $\delta > 0$ ***and so we continue:***

Suppose that $\delta > 0$.

We are now seeking two number t and x such that $|t-x| < \delta$ and $|t^2 - x^2| \geq 1$. To help ourselves find such numbers we make the observation that if $x > 0$ then

$$\left(x + \frac{1}{x}\right)^2 - x^2 = \frac{2x^2+1}{x^2} \geq 2$$

We choose a positive number x such that

$$\frac{1}{x} < \delta.$$

Then we define

$$t = x + \frac{1}{x}$$

and we observe that

$$|t-x| = \frac{1}{x} < \delta$$

and

$$|t^2 - x^2| \geq 2 > 1.$$

7. For every number $x \in (0,1]$ there exists a number $\delta > 0$ such that for every number $t \in (0,1]$ satisfying $|t-x| < \delta$ we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| < 1.$$

Hint: ***This statement is true. Write out a proof!***

Suppose that $x \in (0,1]$. We shall prove the existence of the required number δ by giving an example of one.
We begin by observing that for each $t \in (0,1]$ we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| = \frac{|x-t|}{tx}$$

In order to make the latter expression less than 1 we must prevent the denominator from being too small. In fact, looking at the following figure

$0 \qquad \frac{x}{2} \qquad x \qquad \frac{3x}{2}$

we see that whenever t is close enough to x we must have $t > \frac{x}{2}$. In fact, whenever $|t-x| < \frac{x}{2}$ we have $t > \frac{x}{2}$ and for such numbers t we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| = \frac{|x-t|}{tx} \leq \frac{1}{x^2}|x-t|.$$

The latter number will be lesss than 1 as long as $|x-t| < x^2$.
These observations tell us how to write our proof: We define δ to be the smaller of the two numbers x^2 and $\frac{x}{2}$. Then whenever $t \in (0,1]$ and $|x-t| < \delta$ we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| = \frac{|x-t|}{tx} \leq \frac{1}{x^2}|x-t| < \frac{1}{x^2}x^2 = 1.$$

8. There exists a number $\delta > 0$ such that for every number $x \in (0,1]$ and every number $t \in (0,1]$ satisfying $|t-x| < \delta$ we have

$$\left|\frac{1}{t} - \frac{1}{x}\right| < 1.$$

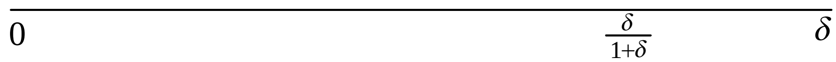
Hint: ***This statement is false. Say why!***

Suppose that $\delta > 0$. We shall find two numbers t and x in the interval $(0,1]$ such that $|x-t| < \delta$ and such that

$$\left|\frac{1}{t} - \frac{1}{x}\right| \geq 1.$$

For this purpose we define

$$x = \delta \quad \text{and} \quad t = \frac{\delta}{1+\delta}$$



We see at once that $|x-t|<\delta$ and that

$$\left|\frac{1}{x}-\frac{1}{t}\right| = 1.$$

This approach is actually too slick. A more motivated approach is to take $x=\delta$ and to look for a number t between 0 and x for which

$$\frac{1}{t}-\frac{1}{x} = 1.$$

The latter equation tells us that

$$\frac{1}{t}-\frac{1}{\delta} = 1$$

and we see at once that the value of t that we want is $\frac{\delta}{1+\delta}$.

9. For every number $p>0$ there exists a number $\delta>0$ such that for every number $x\in[p,1]$ and every number $t\in[p,1]$ satisfying $|t-x|<\delta$ we have

$$\left|\frac{1}{t}-\frac{1}{x}\right| < 1.$$

Hint: ***This statement is true. Write out a proof!***

In the event that $p>1$ the interval $[p,1]$ is empty and every positive number δ has the desired properties. We now assume that $0<p\le 1$.

The key to the proof we are seeking is the fact that if t and x are any numbers in the interval $[p,1]$ then

$$\left|\frac{1}{t}-\frac{1}{x}\right| = \frac{|x-t|}{tx} \le \frac{1}{p^2}|x-t|$$

The latter expression will be less than 1 when $|x-t|<p^2$. With this observation we define $\delta=p^2$ and we have found a number δ with the required properties.

10. a. If either $0<\theta<\pi$ or $\pi<\theta<2\pi$ then

$$\arctan(\tan(\theta/2)) + \arctan(\tan(\pi/2-\theta)) = \frac{\pi-\theta}{2}.$$

We have two cases to consider:

Case 1: Suppose that $0<\theta<\pi$. In this case we have $0<\theta<\frac{\pi}{2}$ and also

$$0<\frac{\pi}{2}-\theta<\frac{\pi}{2}.$$

Therefore

$$\arctan(\tan(\theta/2)) + \arctan(\tan(\pi/2-\theta)) = \frac{\theta}{2}+\frac{\pi}{2}-\theta = \frac{\pi-\theta}{2}.$$

Case 2: Suppose that $\pi<\theta<2\pi$. Since

$$\frac{\pi}{2}<\frac{\theta}{2}<\pi$$

we see that

$$\arctan\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}-\pi$$

and since

$$-\frac{3\pi}{2}<\frac{\pi}{2}-\theta<-\frac{\pi}{2}$$

and so

$$\arctan(\tan(\pi/2-\theta)) = \frac{\pi}{2}-\theta+\pi$$

Therefore

$$\arctan(\tan(\theta/2)) + \arctan(\tan(\pi/2-\theta)) = \frac{\theta}{2}-\pi+\frac{\pi}{2}-\theta+\pi = \frac{\pi-\theta}{2}.$$

b. Ask *Scientific Notebook* to solve the equation

$$\arctan(\tan(\theta/2)) + \arctan(\tan(\pi/2 - \theta)) = \frac{\pi - \theta}{2}.$$

Are you satisfied with the answer that it gives?

11. If x is any rational number then

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}(\cos(n!\pi x))^m\right) = 1.$$

We suppose that x is rational and we express x in the form $\frac{p}{q}$ where p and q are integers and $q > 0$. Whenever an integer n is greater than q, the number $(n!)x$ is an even integer and for such integers n we have

$$\lim_{m\to\infty}(\cos(n!\pi x))^m = \lim_{m\to\infty} 1 = 1$$

Since $\lim_{m\to\infty}(\cos(n!\pi x))^m = 1$ whenever n is sufficiently large we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}(\cos(n!\pi x))^m\right) = 1.$$

12. If x is any irrational number then

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}(\cos(n!\pi x))^m\right) = 0.$$

We suppose that x is irrational. Given any integer n we know from the fact that $(n!)x$ is not an integer that

$$|(\cos(n!\pi x))| < 1$$

and therefore that

$$\lim_{m\to\infty}(\cos(n!\pi x))^m = 0.$$

Therefore

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}(\cos(n!\pi x))^m\right) = 0.$$

4 Elements of Set Theory

Exercises on Set Notation

1. Given objects a, b and y and given that $\{a,b\} = \{a,y\}$, prove that $b = y$.

Solution: ***Since** $y \in \{a,y\}$ **and** $\{a,b\} = \{a,y\}$**, we know that** $y \in \{a,b\}$**. We know, therefore, that either** $y = a$ **or** $y = b$**. However, if** $y = a$ **then the equation** $\{a,b\} = \{a,y\}$ **becomes** $\{a,b\} = \{a\}$ **and we deduce from the fact that** $b \in \{a\}$ **that** $b = a$**. So in this case too we have** $y = b$**.***

2. Prove that if a, b, x and y are any given objects and if $\{a,b\} = \{x,y\}$ then either $a = x$ and $b = y$, or $a = y$ and $b = x$.

Solution: ***Since** $a \in \{a,b\}$ **and** $\{a,b\} = \{x,y\}$ **we know that** $a \in \{x,y\}$**. Therefore either** $a = x$ **or** $a = y$**. In the event that** $a = x$ **we have** $\{a,b\} = \{a,y\}$ **and it follows from Exercise 1 that** $b = y$**. Similarly, if** $a = y$ **then** $b = x$**.***

3. Prove that if a, b, x and y are any given objects and if

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\},$$

then $a = x$ and $b = y$.

The preceding exercises guarantee that if

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\},$$

then either $\{a\} = \{x\}$ or $\{a,b\} = \{x,y\}$ from which we can see at once that $a = x$ and $b = y$.

4. Describe the set $p(\emptyset)$.

Solution: ***Since the only subset of the set $\emptyset$ is $\emptyset$ itself we have $p(\emptyset) = \{\emptyset\}$.***

5. Describe the set $p(p(\emptyset))$.

Hint: ***You should be able to prove that $p(p(\emptyset)) = \{\emptyset,\{\emptyset\}\}$.***

6. Given that $A = \{a,b,c,d\}$, list all of the members of the set $p(A)$.

Solution: ***You should be able to show that $p(\{a,b,c,d\})$ is***

$$\left\{\emptyset,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\right\}$$

7. Given that $A = \{a,b\}$, list all of the members of the set $p(p(A))$.

Since $p(A) = \left\{\emptyset,\{a\},\{b\},\{a,b\}\right\}$ the set $p(p(A))$ is

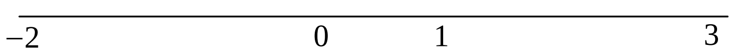
$$\left\{\begin{array}{c}\emptyset,\left\{\emptyset\right\},\left\{\{a\}\right\},\left\{\{b\}\right\},\left\{\{a,b\}\right\},\left\{\emptyset,\{a\}\right\},\left\{\emptyset,\{b\}\right\},\left\{\emptyset,\{a,b\}\right\},\left\{\{a\},\{b\}\right\},\left\{\{a\},\{a,b\}\right\},\\ \left\{\{b\},\{a,b\}\right\},\left\{\emptyset,\{a\},\{b\}\right\},\left\{\emptyset,\{a\},\{a,b\}\right\},\left\{\emptyset,\{b\},\{a,b\}\right\},\left\{\{a\},\{b\},\{a,b\}\right\},\left\{\emptyset,\{a\},\{b\},\{a,b\}\right\}\end{array}\right\}$$

8. Use the Evaluate operation in *Scientific Notebook* to evaluate the sets $\{1,2,3\} \cup \{2,3,4\}$ and $\{1,2,3\} \cap \{2,3,4\}$.

Exercises on Set Operations

1. Express the set $[-2,3] \setminus (0,1]$ as the union of two intervals.

Draw a figure.



We see that

$$[-2,3] \setminus (0,1] = [-2,0] \cup (1,3].$$

2. Given two sets A and B prove that the condition $A \subseteq B$ is equivalent to the condition $A \cup B = B$.

The right way to approach this sort of problem will depend upon the background and strength of the students. Ideally, one should be able to say that, in any case, $B \subseteq A \cup B$ and that the condition $A \cup B \subseteq B$ holds when $A \subseteq B$. The alternative is to write two careful proofs, one starting with the assumption $A \subseteq B$ and the other starting with the assumption $A \cup B = B$.

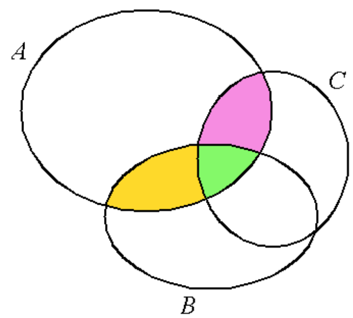
3. Given two sets A and B prove that the condition $A \subseteq B$ is equivalent to the condition $A \cap B = A$.

4. Given two sets A and B prove that the condition $A \subseteq B$ is equivalent to the condition $A \setminus B = \emptyset$.

5. Illustrate the identity

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

by drawing a figure. Then write out a detailed proof.



Solution: ***We obtain the identity***

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

by showing first that

$$A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

and then showing that

$$A \setminus (B \cup C) \supseteq (A \setminus B) \cap (A \setminus C).$$

To obtain the identity

$$A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

we suppose that $x \in A \setminus (B \cup C)$***. We know that*** $x \in A$ ***and that*** x ***does not belong to the set*** $B \cup C$***. Thus*** $x \in A$ ***and*** x ***cannot belong to either of the sets*** B ***and*** C***. In other words,*** $x \in A \setminus B$ ***and*** $x \in A \setminus C$ ***which tells us that*** $x \in (A \setminus B) \cap (A \setminus C)$***. The proof of the assertion***

$$A \setminus (B \cup C) \supseteq (A \setminus B) \cap (A \setminus C)$$

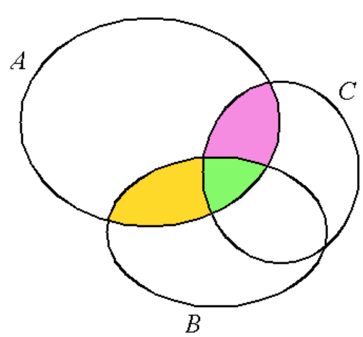
is similar and will be left to the reader.

A Shorter Solution: ***A given object*** x ***will belong to the set*** $A \setminus (B \cup C)$ ***when*** $x \in A$ ***and*** x ***belongs to neither of the sets*** B ***and*** C***. The latter condition says that*** $x \in A$ ***and*** $x \notin B$ ***and also that*** $x \in A$ ***and*** $x \notin C$ ***which says that*** $x \in (A \setminus B) \cap (A \setminus C)$***.***

6. Illustrate the identity

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

by drawing a figure. Then write out a detailed proof.
Use the same figure



that was used in Exercise 5. As before, we can write out the complete solution or the quick version. The quick version follows:

Solution: A given object x will belong to the set $A \setminus (B \cap C)$ when $x \in A$ and x fails to belong to the set $B \cap C$ which means that $x \in A$ and x fails to belong to at least one of the sets B and C. The latter condition says that either $x \in A$ and $x \notin B$ or $x \in A$ and $x \notin C$ which says that $x \in (A \setminus B) \cup (A \setminus C)$**.**

7. Given that $A,$ B and C are subsets of a set $X,$ prove that the condition $A \cap B \cap C = \emptyset$ holds if and only if
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) = X.$$
We know that
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) \subseteq X.$$
Now if we assume that $A \cap B \cap C = \emptyset$ then every member of X must fail to belong to at least one of the sets A and B and C, which tells us that
$$X \subseteq (X \setminus A) \cup (X \setminus B) \cup (X \setminus C).$$
Thus if we are given that $A \cap B \cap C = \emptyset$ then the equation
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) = X$$
must hold.
We now assume that the equation
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) = X$$
holds. This equation tells us that every member of X must belong to at least one of the sets $X \setminus A$ and $X \setminus B$ and $X \setminus C$ which means that every member of X must fail to belong to at least one of the sets A and B and C. Thus no member of X can belong to $A \cap B \cap C$. Since the sets A and B and C are subsets of X we also know that no object x lying outside the set X can belong to $A \cap B \cap C$. Therefore $A \cap B \cap C = \emptyset$.

8. Given sets $A,$ B and $C,$ determine which of the following identities are true.
Hint: ***The truth values are given here. Explain carefully why they are correct.***

a. $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$ ***True***

b. $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$ ***False***

c. $A \cup (B \setminus C) = (A \cup B) \cap (A \setminus C)$ ***False***

d. $A \cup (B \setminus C) = (A \cup B) \setminus (A \cap C)$ ***False***

e. $A \setminus (B \setminus C) = (A \setminus B) \setminus C$ ***False***

f. $A \setminus (B \setminus C) = (A \setminus B) \setminus (A \setminus C)$ ***False***

g. $A \setminus (B \setminus C) = (A \setminus B) \cap (A \setminus C)$ ***False***

h. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C)$ ***False***

i. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ ***True***

j. $A \setminus (B \setminus C) = (A \setminus B) \cap (A \cup C)$ ***False***

k. $A = (A \cap B) \cup (A \setminus B)$ ***True***

l. $p(A \cup B) = p(A) \cup p(B)$ ***False***

m. $p(A \cap B) = p(A) \cap p(B)$ ***True***

n. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ ***True***

o. $A \times (B \cap C) = (A \times B) \cap (A \times C)$ ***True***

p. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ ***True***

9. Is it true that if A and B are sets and $A = A \setminus B$ then the sets A and B are disjoint from each other?
Hint: ***Yes***

10. Given that A is a set with ten members, B is a set with seven members and that the set $A \cap B$ has four members, how many members does the set $A \cup B$ have?

11. Give an example of a set A that contains at least three members and that satisfies the condition $A \subseteq p(A)$.
Hint: ***Define***

$$A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$$

12. For which sets A do we have $A \in p(A)$?
Solution: ***This happens for all sets. Every set is a subset of itself.***

Exercises on Functions

1. Given that $f(x) = x^2$ for every real number x, simplify the following expressions:

   a. $f[[0,3]]$
   We have
   $$f[[0,3]] = [0,9].$$

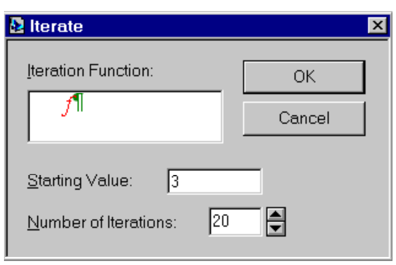
   b. $f[(-2,3]]$
   We have
   $$f[[-2,3]] = [0,9].$$

   c. $f^{-1}[[-3,4]]$
   We have
   $$f^{-1}[[-3,4]] = [-2,2].$$

2. Point at the equation $f(x) = x^2$ and then click on the button in your computing toolbar. Then work out the the expressions in parts a and b of the preceding exercise by pointing at them and clicking on the evaluate button.

3. Supply each of the definitions $f(x) = x^2$ and $g(x) = 2 - 3x$ to *Scientific Notebook* and then ask *Scientific Notebook* to solve the equation
$$(f \circ g)(x) = (g \circ f)(x).$$

4. Supply the definition
$$f(x) = \frac{x-2}{1-2x}$$
to *Scientific Notebook*. In this exercise we shall see how to evaluate the composition of the function f with itself up to 20 times starting at a variety of numbers. Open the Maple menu, click on Calculus and move to the right and select Iterate. In the iterate dialogue box



fill in the function as f, the starting value as 3 and the number of iterations as 20. Repeat this process with different starting values. Can you draw a conclusion from what you see?

5. Given that $f(x) = x^2$ for all $x \in \mathbf{R}$ and $g(x) = 1 + x$ for all $x \in \mathbf{R}$, simplify the following expressions:

   a. $(f \circ g)[[0,1]]$
   We have
   $$(f \circ g)[[0,1]] = f[g[[0,1]]] = f[[1,2]] = [1,4].$$

   b. $(g \circ f)[[0,1]]$
   We have
   $$(g \circ f)[[0,1]] = g[f[[0,1]]] = g[[0,1]] = [1,2].$$

   c. $(g \circ g)[[0,1]]$
   We have
   $$(g \circ g)[[0,1]] = g[g[[0,1]]] = g[[1,2]] = [2,3].$$

6. a. Given that $f(x) = (3x-2)/(x+1)$ for all $x \in \mathbf{R} \setminus \{-1\}$, determine whether or not f is one-one and find its range.
   Given any number y, the equation
   $$y = \frac{3x-2}{x+1}$$
   holds when
   $$(x+1)y = 3x-2$$
   which can be expressed as
   $$x(3-y) = 2+y.$$
   In the event that $y = 3$ the lattter equation says that $0 = 5$ and is therefore impossible. If $y \neq 3$ then the equation
   $$x(3-y) = 2+y$$
   says that
   $$x = \frac{2+y}{3-y}.$$
   We conclude that the range of f is $\mathbf{R} \setminus \{3\}$ and that for every number $y \in \mathbf{R} \setminus \{3\}$ there is a unique number x for which $y = f(x)$. Therefore the function f is one-one.

   b. Point at the equation
   $$y = \frac{3x-2}{x+1}$$
   and ask *Scientific Notebook* to solve for x. How many values of x are given? Is this result consistent with the answer you gave in part a of the question?

7. Suppose that $f : A \to B$ and that $E \subseteq A$. Is it true that $E = f^{-1}[f[E]]$? What if f is one-one? What if f is onto B?
   We certainly have $E \subseteq f^{-1}[f[E]]$ but the inclusion can be strict. For example, if we define $f(x) = x^2$ for every number x then
   $$f^{-1}[f[[0,1]]] \neq [0,1].$$
   Now suppose that f is a one-one function from a set A to a set B and that $E \subseteq A$. We shall prove that $f^{-1}[f[E]] \subseteq E$ and, for this purpose, we suppose that $x \in f^{-1}[f[E]]$. We know that $f(x) \in f[E]$ and, using this fact, we choose a member t of the set E such that $f(x) = f(t)$. Since f is one-one we have $x = t$ and so $x \in E$ which is what we needed to show.

8. Suppose that $f : A \to B$ and that $E \subseteq B$. Is it true that $E = f[f^{-1}[E]]$? What if f is one-one? What if f is onto B?
   No it isn't true. There is no reason to suppose that every member of E has to be in the range of f. For example, if we define $f(x) = x^2$ for every number x and $E = [-1,1]$ Then $E \neq f[f^{-1}[E]]$. In the

event that f is onto the set B the equation $E = f[f^{-1}[E]]$ will hold.

9. Suppose that $f : A \to B$ and that P and Q are subsets of B. Prove the identities

$$f^{-1}[P \cup Q] = f^{-1}[P] \cup f^{-1}[Q],$$

Solution: ***Given any member x of the set A, the condition $x \in f^{-1}[P \cup Q]$ says that $f(x) \in P \cup Q$ which says that either $f(x) \in P$ or $f(x) \in Q$ which says that either $x \in f^{-1}[P]$ or $x \in f^{-1}[Q]$.***

$$f^{-1}[P \cap Q] = f^{-1}[P] \cap f^{-1}[Q],$$

$$f^{-1}[P \setminus Q] = f^{-1}[P] \setminus f^{-1}[Q],$$

10. Suppose that $f : A \to B$ and that P and Q are subsets of A. Which of the following statements are true? What if f is one-one? What if f is onto B?

$$f[P \cup Q] = f[P] \cup f[Q]$$

Hint: ***This statement is true.***

$$f[P \cap Q] = f[P] \cap f[Q]$$

This statement is false. Give an example. Then prove that the statement is true if f is one-one.

$$f[P \setminus Q] = f[P] \setminus f[Q]$$

This statement is true when f is one-one.

11. Given that f is a one-one function from A to B and that g is a one-one function from B to C, prove that the function $g \circ f$ is one-one from A to C.

Solution: ***We need to prove that whenever t and x are members of the set A and $t \neq x$ we have***

$$(g \circ f)(t) \neq (g \circ f)(x).$$

Suppose that t and x are members of the set A and that $t \neq x$. Since f is one-one we have $f(t) \neq f(x)$. Therefore, since g is one-one we have $g(f(t)) \neq g(f(x))$ and we have shown that $(g \circ f)(t) \neq (g \circ f)(x)$.

12. Given that f is a function from A onto B and that g is a function from B onto C, prove that the function $g \circ f$ is a function from A onto C.
Suppose that $z \in C$. Using the fact that the function g is onto the set C, choose a member y of the set B such that $z = g(y)$. Now, using the fact that the function f is onto the set B we choose a member x of the set A such that $y = f(x)$. We have found a member x of A such that $z = (f \circ g)(x)$. Therefore C is the range of the function $f \circ g$.

13. Given that $f : A \to B$ and that $g : B \to C$ and that the function $g \circ f$ is one-one, prove that f must be one-one. Give an example to show that the function g does not have to be one-one.

Solution: ***To prove that f is one-one, suppose that x_1 and x_2 are members of the set A and that $x_1 \neq x_2$. Since the function $g \circ f$ is one-one we know that $g(f(x_1)) \neq g(f(x_2))$ and we see at once that $f(x_1) \neq f(x_2)$. Now we construct an example to show that the function g does not have to be one-one. We define $f(x) = x$ for every $x \in [0,1]$ and we define***

$$g(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2 & \text{if } 1 < x \leq 5 \end{cases}$$

14. Given that f is a function from A onto B and that $g : B \to C$ and that the function $g \circ f$ is one-one, prove that both of the functions f and g have to be one-one.

Solution: ***To see that f is one-one, suppose that x and t are members of the set A and that $t \neq x$. Since $g(f(t)) \neq g(f(x))$ we see at once that $f(t) \neq f(x)$.***

Now to see that g is one-one, suppose that u and y are members of the set B and that $u \neq y$. Using the fact

that the function f is onto the set B we choose members t and x of A such that $u = f(t)$ and $y = f(x)$. We see at once that $t \neq x$ and therefore

$$g(u) = g(f(t)) \neq g(f(x)) = g(y).$$

15. Given any set S, the **identity function** i_S on S is defined by $i_S(x) = x$ for every $x \in S$. Prove that if f is a one-one function from a set A onto a set B then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.
   There is really nothing to prove. The fact that
   $$f^{-1}(f(x)) = x = i_A(x)$$
   for every $x \in A$ follows at once from the definition of the function f^{-1}. The equation $f \circ f^{-1} = i_B$ follows in a similar manner.

16. Suppose that $f : A \to B$.
   a. Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$, what can be said about the functions f and g?
      The function f must be one-one because if t and x belong to A and $f(t) = f(x)$ then we have
      $$t = g(f(t)) = g(f(x)) = x.$$
      The function g must be onto the set A because if x is any member of A we have
      $$x = g(f(x)).$$
   b. Given that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f and h?
      The function f must be onto the set B and the function h must be one-one.
   c. Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$ and that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f, g and h?
      From parts a and b we see that all three functions are one-one and that f is onto B and that the functions g and h are onto A.

17. As in a previous example, we define
   $$f_a(x) = \frac{x-a}{1-ax}$$
   whenever $a \in (-1,1)$ and $x \in [-1,1]$.
   a. Prove that if a and b belong to $(-1,1)$ then so does the number
      $$c = \frac{a+b}{1+ab}.$$
      Hint: An quick way to do this exercise is to observe that $c = f_{-b}(a)$.
      There really isn't much more to say here. The earlier material showed that f_{-b} is a one-one function from $[0,1]$ onto $[0,1]$ and we see at once that $f_{-b}(-1) = -1$ and $f_{-b}(1) = 1$.
   b. Given a and b in $(-1,1)$ and
      $$c = \frac{a+b}{1+ab},$$
      prove that $f_b \circ f_a = f_c$.
      Given any number $x \in [0,1]$ we have
      $$\begin{aligned} f_b \circ f_a(x) &= f_b(f_a(x)) = f_b\left(\frac{x-a}{1-ax}\right) = \frac{\frac{x-a}{1-ax} - b}{1 - b\left(\frac{x-a}{1-ax}\right)} \\ &= \frac{x-a-b(1-ax)}{1-ax-b(x-a)} = \frac{x(1+ab)-a-b}{1+ab-(a+b)x} \\ &= \frac{x - \frac{a+b}{1+ab}}{\frac{1+ab}{a+b} - x} = f_c(x) \end{aligned}$$

Alternative 4: A More Detailed Presentation of Set Theory

Some Elementary Exercises on Sets

1. Given objects a, b and y and given that $\{a,b\} = \{a,y\}$, prove that $b = y$.

   Solution: ***From the fact that*** $\{a,b\} = \{a,y\}$ ***and the fact that*** $b \in \{a,b\}$ ***we deduce that*** $b \in \{a,y\}$. ***Therefore there are two possibilities; either*** $b = a$ ***or*** $b = y$. ***In the event that*** $b = a$ ***then the equation*** $\{a,b\} = \{a,y\}$ ***becomes*** $\{a\} = \{a,y\}$ ***and we conclude that*** $y \in \{a\}$ ***which tells us that*** $y = a$. ***So in this case we have*** $a = b = y$. ***So in either case we know that*** $b = y$.

2. Prove that if a, b, x and y are any given objects and if $\{a,b\} = \{x,y\}$ then either $a = x$ and $b = y$, or $a = y$ and $b = x$.
   Since $a \in \{a,b\}$ and $\{a,b\} = \{x,y\}$ we know that $a \in \{x,y\}$. Therefore either $a = x$ or $a = y$. In the event that $a = x$ we have $\{a,b\} = \{a,y\}$ and it follows from Exercise 1 that $b = y$. Similarly, if $a = y$ then $b = x$.

3. Prove that if a, b, x and y are any given objects and if
   $$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\},$$
   then $a = x$ and $b = y$.
   The preceding exercises guarantee that if
   $$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\},$$
   then either $\{a\} = \{x\}$ or $\{a,b\} = \{x,y\}$ from which we can see at once that $a = x$ and $b = y$.

4. Describe the set $p(\emptyset)$.
   Since the only subset of the set $\emptyset$ is $\emptyset$ itself we have $p(\emptyset) = \{\emptyset\}$.

5. Describe the set $p(p(\emptyset))$.
   You should be able to prove that $p(p(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.

6. Given that $A = \{a,b,c,d\}$, list all of the subsets of A. Describe the set $p(A)$.
   You should be able to show that $p(\{a,b,c,d\})$ is
   $$\left\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\right\}$$

7. Given that $A = \{a,b\}$, list all of the subsets of A. Describe the set $p(p(A))$.
   Since $p(A) = \left\{\emptyset, \{a\}, \{b\}, \{a,b\}\right\}$ the set $p(p(A))$ is
   $$\left\{\begin{array}{c} \emptyset, \left\{\emptyset\right\}, \left\{\{a\}\right\}, \left\{\{b\}\right\}, \left\{\{a,b\}\right\}, \left\{\emptyset, \{a\}\right\}, \left\{\emptyset, \{b\}\right\}, \left\{\emptyset, \{a,b\}\right\}, \left\{\{a\}, \{b\}\right\}, \left\{\{a\}, \{a,b\}\right\}, \\ \left\{\{b\}, \{a,b\}\right\}, \left\{\emptyset, \{a\}, \{b\}\right\}, \left\{\emptyset, \{a\}, \{a,b\}\right\}, \left\{\emptyset, \{b\}, \{a,b\}\right\}, \left\{\{a\}, \{b\}, \{a,b\}\right\}, \left\{\emptyset, \{a\}, \{b\}, \{a,b\}\right\} \end{array}\right\}$$

8. Given any set S, the **successor** of S is the set S^+ defined by
   $$S^+ = S \cup \{S\}.$$

   a. Describe the sets $\emptyset^+$, $\emptyset^{++}$, $\emptyset^{+++}$ and $\emptyset^{++++}$.

$$\begin{aligned}
\emptyset^{+} &= \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
\emptyset^{++} &= \emptyset^{+} \cup \{\emptyset^{+}\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} \\
\emptyset^{+++} &= \emptyset^{++} \cup \{\emptyset^{++}\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
\emptyset^{++++} &= \emptyset^{+++} \cup \{\emptyset^{+++}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}
\end{aligned}$$

b. Is it true that if $A \subseteq B$ then $A^{+} \subseteq B^{+}$?
This assertion is false. Note that $\emptyset \subseteq \{\{\emptyset\}\}$ but the set $\emptyset^{+} = \{\emptyset\}$ is not a subset of the set $\{\{\emptyset\}\}^{+} = \{\{\emptyset\}, \{\{\emptyset\}\}\}$.

c. Given that A and B are sets and that at least one of the conditions $A \in B$ and $B \in A$ is false, and given that $A^{+} = B^{+}$, prove that $A = B$.

Solution: ***We are given that*** $A \cup \{A\} = B \cup \{B\}$***. We shall assume that the condition*** $A \in B$ ***is false. The other case*** $B \notin A$ ***is analogous to this one. Now since*** $A \in A \cup \{A\} = B \cup \{B\}$ ***we know that either*** $A \in B$ ***or*** $A \in \{B\}$***. We have assumed that*** A ***does not belong to*** B ***and so the condition*** $A \in \{B\}$ ***must hold; and we conclude that*** $A = B$***.***

9. A set A is said to be **transitive** if every member of A is a subset of A.

a. Is it true that if A and B are transitive then the set $A \cup B$ is transitive?
This statement is true. Any member of A has to be a subset of A and is certainly a subset of $A \cup B$. The same argument can be applied to any member of B.

b. Is it true that if A and B are transitive then the set $A \cap B$ is transitive?
This statement is true. A member of $A \cap B$, being a member of A, must be a subset of A and, being a member of B, must be a subset of B. Therefore every member of $A \cap B$ must be a subset of $A \cup B$.

c. Is it true that if A and B are transitive then the set $A \setminus B$ is transitive?
This statement is false. Look at the set $\{\{\emptyset\}, \emptyset\} \setminus \{\emptyset\} = \{\{\emptyset\}\}$.

d. Is it true that if A is transitive then the successor A^{+} of A (as defined in the preceding exercise) is transitive?
This statement is true. Suppose that A is transitive. Every member of A, being a subset of A, must be a subset of $A \cup \{A\}$. Furthermore, the set A is also a subset of $A \cup \{A\}$.

Exercises on Set Operations

1. Express the set $[-2, 3] \setminus (0, 1]$ as the union of two intervals.
Draw a figure.

-2 $\qquad$ 0 $\qquad$ 1 $\qquad$ 3

We see that

$$[-2, 3] \setminus (0, 1] = [-2, 0] \cup (1, 3].$$

2. Given two sets A and B prove that the condition $A \subseteq B$ is equivalent to the condition $A \cup B = B$.
The right way to approach this sort of problem will depend upon the background and strength of the students. Ideally, one should be able to say that, in any case, $B \subseteq A \cup B$ and that the condition $A \cup B \subseteq B$ holds when $A \subseteq B$. The alternative is to write two careful proofs, one starting with the assumption $A \subseteq B$ and the other starting with the assumption $A \cup B = B$.

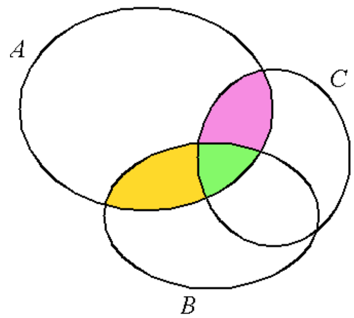
3. Given two sets A and B prove that the condition $A \subseteq B$ is equivalent to the condition $A \cap B = A$.

4. Given two sets A and B prove that the condition $A \subseteq B$ is equivalent to the condition $A \setminus B = \emptyset$.

5. Illustrate the identity

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

by drawing a figure. Then write out a detailed proof.



Solution: ***We obtain the identity***

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

by showing first that

$$A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

and then showing that

$$A \setminus (B \cup C) \supseteq (A \setminus B) \cap (A \setminus C).$$

To obtain the identity

$$A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$$

we suppose that $x \in A \setminus (B \cup C)$***. We know that*** $x \in A$ ***and that*** x ***does not belong to the set*** $B \cup C$***. Thus*** $x \in A$ ***and*** x ***cannot belong to either of the sets*** B ***and*** C***. In other words,*** $x \in A \setminus B$ ***and*** $x \in A \setminus C$ ***which tells us that*** $x \in (A \setminus B) \cap (A \setminus C)$***. The proof of the assertion***

$$A \setminus (B \cup C) \supseteq (A \setminus B) \cap (A \setminus C)$$

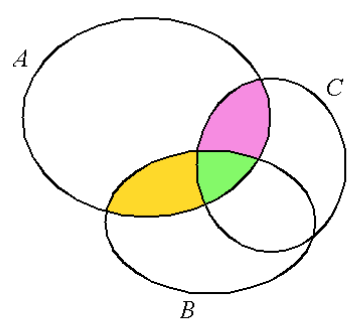
is similar and will be left to the reader.

A Shorter Solution: ***A given object*** x ***will belong to the set*** $A \setminus (B \cup C)$ ***when*** $x \in A$ ***and*** x ***belongs to neither of the sets*** B ***and*** C***. The latter condition says that*** $x \in A$ ***and*** $x \notin B$ ***and also that*** $x \in A$ ***and*** $x \notin C$ ***which says that*** $x \in (A \setminus B) \cap (A \setminus C)$***.***

6. Illustrate the identity

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

by drawing a figure. Then write out a detailed proof.
Use the same figure



that was used in Exercise 5. As before, we can write out the complete solution or the quick version. The quick version follows:

Solution: A given object x will belong to the set $A \setminus (B \cap C)$ when $x \in A$ and x fails to belong to the set $B \cap C$ which means that $x \in A$ and x fails to belong to at least one of the sets B and C. The latter condition says that either $x \in A$ and $x \notin B$ or $x \in A$ and $x \notin C$ which says that $x \in (A \setminus B) \cup (A \setminus C)$.

7. Given that A, B and C are subsets of a set X, prove that the condition $A \cap B \cap C = \emptyset$ holds if and only if
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) = X.$$
We know that
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) \subseteq X.$$
Now if we assume that $A \cap B \cap C = \emptyset$ then every member of X must fail to belong to at least one of the sets A and B and C, which tells us that
$$X \subseteq (X \setminus A) \cup (X \setminus B) \cup (X \setminus C).$$
Thus if we are given that $A \cap B \cap C = \emptyset$ then the equation
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) = X$$
must hold.
We now assume that the equation
$$(X \setminus A) \cup (X \setminus B) \cup (X \setminus C) = X$$
holds. This equation tells us that every member of X must belong to at least one of the sets $X \setminus A$ and $X \setminus B$ and $X \setminus C$ which means that every member of X must fail to belong to at least one of the sets A and B and C. Thus no member of X can belong to $A \cap B \cap C$. Since the sets A and B and C are subsets of X we also know that no object x lying outside the set X can belong to $A \cap B \cap C$. Therefore $A \cap B \cap C = \emptyset$.

8. Given sets A, B and C, determine which of the following identities are true.
Hint: ***The truth values are given here. Explain carefully why they are correct.***

   a. $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$ ***True***

   b. $A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$ ***False***

   c. $A \cup (B \setminus C) = (A \cup B) \cap (A \setminus C)$ ***False***

   d. $A \cup (B \setminus C) = (A \cup B) \setminus (A \cap C)$ ***False***

   e. $A \setminus (B \setminus C) = (A \setminus B) \setminus C$ ***False***

   f. $A \setminus (B \setminus C) = (A \setminus B) \setminus (A \setminus C)$ ***False***

   g. $A \setminus (B \setminus C) = (A \setminus B) \cap (A \setminus C)$ ***False***

   h. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C)$ ***False***

   i. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ ***True***

   j. $A \setminus (B \setminus C) = (A \setminus B) \cap (A \cup C)$ ***False***

   k. $A = (A \cap B) \cup (A \setminus B)$ ***True***

   l. $p(A \cup B) = p(A) \cup p(B)$ ***False***

   m. $p(A \cap B) = p(A) \cap p(B)$ ***True***

   n. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ ***True***

   o. $A \times (B \cap C) = (A \times B) \cap (A \times C)$ ***True***

   p. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ ***True***

9. Is it true that if A and B are sets and $A = A \setminus B$ then the sets A and B are disjoint from each other?
Hint: ***Yes***

10. Given that A is a set with ten members, B is a set with seven members and that the set $A \cap B$ has four

members, how many members does the set $A \cup B$ have?

11. Give an example of a set A that contains at least three members and that satisfies the condition $A \subseteq p(A)$.
Hint: ***Define***

$$A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$$

12. For which sets A do we have $A \in p(A)$?
Solution: ***This happens for all sets. Every set is a subset of itself.***

Exercises on Families of Sets

For each of the following families $\mathfrak{I}$ of subsets of $\mathbf{R}$, describe the sets $\cup\mathfrak{I}$ and $\cap\mathfrak{I}$:

1. a. $\mathfrak{I} = \left\{[-\frac{1}{n}, \frac{1}{n}] \mid n \text{ is a positive integer}\right\}$.
   If n is any positive integer then $[-\frac{1}{n}, \frac{1}{n}] \subseteq [-1, 1]$ and we see at once that $\cup\mathfrak{I} = [-1, 1]$. We can also see that $\cap\mathfrak{I} = \{0\}$ because if x is any nonzero number then the condition $x \in [-\frac{1}{n}, \frac{1}{n}]$ will fail whenever $\frac{1}{n} < |x|$ which holds when $n > 1/|x|$.

   b. $\mathfrak{I} = \left\{(1 + \frac{1}{n}, 3 - \frac{1}{n}) \mid n \text{ is a positive integer}\right\}$.
   Using methods similar to part a we can see that $\cup\mathfrak{I} = (1, 3)$ and $\cap\mathfrak{I} = \emptyset$.

   c. $\mathfrak{I}$ is the family of all those subsets of $\mathbf{R}$ that have no more than two members.
   We have $\cup\mathfrak{I} = \mathbf{R}$ and $\cap\mathfrak{I} = \emptyset$.

   d. $\mathfrak{I}$ is the family of all those subsets A of $\mathbf{R}$ for which the set $\mathbf{R} \setminus A$ has no more than two members.
   We have $\cup\mathfrak{I} = \mathbf{R}$ and $\cap\mathfrak{I} = \emptyset$.

2. Write out careful proofs for the three DeMorgan laws for families of sets that were not proved above.
We show just one more proof, the proof that if $\mathfrak{I}$ is a nonempty family of sets then

$$S \setminus \left(\bigcup \mathfrak{I}\right) = \bigcap \{S \setminus A \mid A \in \mathfrak{I}\}.$$

Suppose that $\mathfrak{I}$ is a nonempty family of sets. Given any $x \in S \setminus (\bigcup \mathfrak{I})$ we know that $x \in S$ and that x fails to be in $\bigcup \mathfrak{I}$. Thus if A is any member of $\mathfrak{I}$ then we know that $x \in S$ and $x \notin A$, and we conclude that $x \in S \setminus A$ for every $A \in \mathfrak{I}$. We have therefore shown that

$$S \setminus \left(\bigcup \mathfrak{I}\right) \subseteq \bigcap \{S \setminus A \mid A \in \mathfrak{I}\}.$$

Now suppose that $x \in \bigcap \{S \setminus A \mid A \in \mathfrak{I}\}$. Using the fact that $\mathfrak{I} \neq \emptyset$ we choose a member B of $\mathfrak{I}$. Since $x \in S \setminus B$ we know that $x \in S$. Finally, if A is any member of $\mathfrak{I}$ then it follows from the condition $x \in S \setminus A$ that $x \notin A$ and we conclude that $x \notin \bigcup \mathfrak{I}$. We have therefore shown that

$$\bigcap \{S \setminus A \mid A \in \mathfrak{I}\} \subseteq S \setminus \left(\bigcup \mathfrak{I}\right).$$

3. A family $\mathfrak{I}$ of sets is said to be **nested** if for any two members A and B of $\mathfrak{I}$ we have either $A \subseteq B$ or $B \subseteq A$. Given that $\mathfrak{I}$ is a nested family of sets and that $x \in \cup\mathfrak{I}$ and $y \in \cup\mathfrak{I}$, prove that there exists a member A of $\mathfrak{I}$ such that both x and y belong to A.
Using the fact that x and y belong to $\cup\mathfrak{I}$ we choose a member A of $\mathfrak{I}$ such that $x \in A$ and a member B of $\mathfrak{I}$ such that $y \in B$. Since $\mathfrak{I}$ is nested we know that either $A \subseteq B$ or $B \subseteq A$ and in either event we have found a member of $\mathfrak{I}$ to which both x and y belong.

4. A family $\mathfrak{I}$ of subsets of a set X is said to be **field** of subsets of X if $\emptyset \in \mathfrak{I}$ and for all sets A and B that belong to $\mathfrak{I}$ we have $X \setminus A \in \mathfrak{I}$ and $A \cup B \in \mathfrak{I}$.

   a. Prove that if $\mathfrak{I}$ is a field of subsets of a set X and A and B are sets that belong to $\mathfrak{I}$, then the set $A \cap B$ belongs to $\mathfrak{I}$.

   Hint: ***Given any two subsets*** A ***and*** B ***of*** X ***we have***

$$A \cap B = X \setminus \big((X \setminus A) \cup (X \setminus B)\big).$$

b. Prove that if $\mathfrak{I}$ is a field of subsets of a set X and A and B are sets that belong to $\mathfrak{I}$, then the set $A \setminus B$ belongs to $\mathfrak{I}$.
Given two subsets A and B of X we have
$$A \setminus B = A \cap (X \setminus B).$$

c. Prove that if $\mathfrak{C}$ is a collection of fields of subsets of a set X then the family $\cap\mathfrak{C}$ is also a field of subsets of X.
Since every member of $\mathfrak{C}$ contains $\emptyset$ we know that the family $\cup\mathfrak{C}$ also contains $\emptyset$. Given any member A of $\cap\mathfrak{C}$ we know that A belongs to every member of $\mathfrak{C}$ and therefore, since every member of $\mathfrak{C}$ is a field, $X \setminus A$ belongs to every member of $\mathfrak{C}$, and therefore belongs to $\cap\mathfrak{C}$. Finally, if A and B belong to $\cap\mathfrak{C}$ then, since both A and B belong to every member of $\mathfrak{C}$, so does $A \cup B$, which tells us that $A \cup B \in \cap\mathfrak{C}$.

5. Suppose that $\mathfrak{I}$ is a family of subsets of a set X and that Γ is defined by
$$\Gamma = \{X \setminus A \mid A \in \mathfrak{I}\}.$$

a. Prove that if the intersection of any two members of $\mathfrak{I}$ belongs to $\mathfrak{I}$ then the union of any two members of Γ belongs to Γ.
Suppose that the intersection of any two members of $\mathfrak{I}$ must belong to $\mathfrak{I}$. Suppose that A and B belong to the family Γ. Since both $X \setminus A$ and $X \setminus B$ belong to $\mathfrak{I}$ we know that $(X \setminus A) \cap (X \setminus B) \in \mathfrak{I}$ and therefore
$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \in \mathfrak{I}$$
from which we conclude that $A \cup B \in \Gamma$.

b. Prove that if $\cap\mathfrak{I} = \emptyset$ then $\cup\Gamma = X$.
We assume that $\cap\mathfrak{I} = \emptyset$. To prove that $\cup\Gamma = X$, suppose that $x \in X$. Using the fact that $x \notin \cap\mathfrak{I}$ we choose a member A of $\mathfrak{I}$ such that $x \notin A$. Since
$$x \in X \setminus A \in \Gamma$$
we know that $x \in \cup\Gamma$.

c. Given that for every subfamily $\mathfrak{I}_1$ of $\mathfrak{I}$ we have $\cup\mathfrak{I}_1 \in \mathfrak{I}$, and given that Γ_1 is a subfamily of Γ, prove that $\cap\Gamma_1 \in \Gamma$.

Solution: ***Suppose that*** Γ_1 ***is a subfamily of*** Γ***. We define***
$$\mathfrak{I}_1 = \{X \setminus A \mid A \in \Gamma_1\}$$
and we observe that, since $\mathfrak{I}_1 \subseteq \mathfrak{I}$***, we have*** $\cup\mathfrak{I}_1 \in \mathfrak{I}$***. Therefore***
$$\cap\,\Gamma_1 = X \setminus \cup\mathfrak{I}_1 \in \Gamma.$$

6. This exercise will explore some of the properties of the family $\mathfrak{I}$ that was defined in an earlier example. Recall that $\mathfrak{I}$ is defined to be the family of all those subsets A of $\mathbf{R}$ for which it is possible to find a real number a such that
$$A = \{x \in \mathbf{R} \mid x - a \text{ is a rational number}\}.$$

a. Suppose that A is a member of the family $\mathfrak{I}$ and that $v \in A$. Prove that a real number u will belong to A if and only if $u - v$ is rational.

Solution: ***Choose a number*** a ***such that***
$$A = \{x \in \mathbf{R} \mid x - a \text{ is a rational number}\}.$$
Since $v \in A$ ***we know that the number*** $v - a$ ***is rational. Now given any number*** u ***we see that***
$$u - a = (u - v) + (v - a)$$

and we deduce that the number $u - a$ ***will be rational if and only if the number*** $u - v$ ***is rational. In other words, the condition*** $u \in A$ ***holds if and only if*** $u - v$ ***is rational.***

b. Prove that $\cup\Im = \mathbf{R}$.
Suppose that x is any real number. We define

$$A = \left\{t \in \mathbf{R} \mid t - x \text{ is rational}\right\}$$

and observe that $x \in A \in \Im$.

c. Prove that if A and B are any two different members of $\Im$ then $A \cap B = \emptyset$.
Suppose that A and B are members of $\Im$ and that $A \neq B$. We may assume, without loss of generality that $A \setminus B \neq \emptyset$. Choose $x \in A \setminus B$. Given any member Y of the set B we know from part a of this exercise and from the fact that $x \notin B$ that the number $x - y$ can't be rational; and therefore we know that $y - x$ can't be rational. Thus no member y of the set B can belong to A and we have shown that $A \cap B = \emptyset$.

d. Suppose that we have selected exactly one number in each member A of $\Im$ and have then collected these numbers together to form a set E. Suppose that for each rational number r we have defined

$$E_r = \{r + x \mid x \in E\}.$$

Prove that

$$\cup\{E_r \mid r \in \mathbf{Q}\} = \mathbf{R}$$

and that whenever r and s are different rational numbers then $E_r \cap E_s = \emptyset$.
Suppose that u is any real number. Using the fact that $\cup\Im = \mathbf{R}$ we choose a member A of $\Im$ such that $u \in A$. We know that the set E has exactly one member that belongs the set A and we call this member x. Since

$$u = x + (u - x)$$

and since $u - x$ is rational we know that $u \in E_{u-x}$.
Finally suppose that r and s are rational numbers and that $E_r \cap E_s \neq \emptyset$. We shall show that $r = s$. Choose a number $u \in E_r \cap E_s$. Choose x and y in the set E such that

$$u = r + x = s + y.$$

Since $x - y = s - r \in \mathbf{Q}$ we know that x and y must belong to the same member of the family $\Im$ and therefore, from the way in which the set E was specified we conclude that $x = y$. The equation $r + x = s + y$ now guarantees that $r = s$, as promised.

Exercises on Relations

1. Under what conditions do we have $(x,y) = (y,x)$?
The condition $(x,y) = (y,x)$ implies that $x = y$ and is obviously true in this case.

2. For each of the relations defined earlier, find its domain and the range.

a. We write P for the set of all people and define

$$r = \left\{(x,y) \in P \times P \mid x \text{ is a brother of } y\right\}.$$

The domain of r is the set of all those males who have at least one sibling. The range of r is the set of all those people who have at least one male sibling.

b. We write P for the set of all people and define

$$r = \left\{(x,y) \in P \times P \mid x \text{ is a blood relation of } y\right\}.$$

c. We suppose that J is a bag (set) of jelly beans and define

$$r = \left\{(x,y) \in J \times J \mid x \text{ and } y \text{ have the same same color}\right\}.$$

Both the domain and range of r are the entire set J.

d. Writing $\mathbf{R}$ for the set of all real numbers, we define

$$r = \{(x,y) \in \mathbf{R}\times\mathbf{R} \mid x \le y\}.$$

Both the domain and range of r are the entire set $\mathbf{R}$.

e. Writing $\mathbf{Z}$ for the set of all integers, we define

$$r = \left\{(x,y) \in \mathbf{Z}\times\mathbf{Z} \mid \exists n \in \mathbf{Z} \text{ such that } y = nx\right\}.$$

Both the domain and range of r are the entire set $\mathbf{Z}$.

f. We define

$$r = \left\{(x,y) \in \mathbf{R}\times\mathbf{R} \mid x - y \text{ is rational}\right\}.$$

Both the domain and range of r are the entire set $\mathbf{R}$.

g. We define

$$r = \{(0,1),(0,2),(4,1),(3,2)\}.$$

Note that r is a subset of $\{0,4,3\}\times\{1,2\}$.
The domain of r is $\{0,4,3\}$ and the range of r is $\{1,2\}$.

h. We define

$$r = \{(x,y) \in \mathbf{R}\times\mathbf{R} \mid x^2 + y^2 = 1\}.$$

where $\mathbf{R}$ is the set of all real numbers.
Given any numbers x and y, the equation $x^2 + y^2 = 1$ requires that $-1 \le x \le 1$ and $-1 \le y \le 1$.
Both the domain and range of r are equal to $[-1,1]$.

i. Suppose that S is any set and define

$$r = \{(x,y) \in S\times S \mid y = x\}.$$

Both the domain and range of r are the entire set S.

3. Suppose that S is the set of all integers n such that $2 \le n \le 20$ and that r is the relation in S that consists of all pairs (x,y) for which x is a factor of y but $x \ne y$. What are the domain and range of r?

4. Suppose that r is the relation in $\mathbf{R}$ that contains all the ordered pairs (x,y) satisfying $x^2 + 4y^2 = 1$. What are the domain and range of r?

Solution: ***A real number x is in the domain of the relation r defined in this exercise if and only if it is possible to find a real number y such that $x^2 + 4y^2 = 1$. Such a real number y will exist if and only if $1 - x^2 \ge 0$ and to the domain of the relation r is the interval $[-1,1]$. In a similar way we can see that the range of the relation r is the interval $[-4,4]$.***

5. Suppose that r is the relation in $\mathbf{R}$ that contains all the ordered pairs (x,y) satisfying $x^2 + 4y^2 \ge 1$. What are the domain and range of r?

6. Suppose that r is the relation in $\mathbf{R}$ that contains all the ordered pairs (x,y) satisfying $x^2 - 4y^2 = 1$. What are the domain and range of r?

7. Is it true that if r_1 and r_2 are relations in a set S then the domain of the relation $r_1 \cup r_2$ is the union of the domains of r_1 and r_2?
The assertion is obviously true.

8. Suppose that

$$S = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}.$$

Show that the relations $\subseteq$ and $\in$ are the same in S?

9. Suppose that $\mathfrak{I}$ is a collection of families of sets and that $\mathfrak{I}$ is nested. Suppose that for every member $\mathfrak{C}$ of the collection $\mathfrak{I}$ the relations $\subseteq$ and $\in$ are the same in $\mathfrak{C}$. Prove that the relations $\subseteq$ and $\in$ are the same in the family $\cup\mathfrak{I}$.
Suppose that A and B belong to $\cup\mathfrak{I}$ and choose a member $\mathfrak{C}$ of $\mathfrak{I}$ to which both A and B belong.
Since the relations $\in$ and $\subseteq$ are the same in $\mathfrak{C}$, the condition $A \in B$ is equivalent to the condition

$A \subseteq B$.

Exercises on Equivalence Relations

1. For each of the relations given in the list of examples, determine whether or not the relation is reflexive, whether or not it is symmetric and whether or not it is transitive.

2. Give an example of a relation that is reflexive and symmetric but not transitive.
   We define
   $$r = \{(x,y) \in \mathbf{R} \times \mathbf{R} \mid |x - y| \leq 1\}.$$

3. Give an example of a relation that is reflexive and transitive but not symmetric.
   The relation $\subseteq$ in the power set $p(\mathbf{R})$ of $\mathbf{R}$ is reflexive and transitive but not symmetric.

4. Give an example of a relation that is symmetric and transitive but not reflexive.
   Take the relation
   $$r = \left\{(x,y) \in \mathbf{R} \times \mathbf{R} \mid x \neq 1 \text{ and } y \neq 1\right\}.$$

5. Suppose that r is a relation in a set S and that r satisfies the following three conditions:

   a. For every $x \in S$ there exists a member $y \in S$ such that $x\, r\, y$.

   b. The relation r is symmetric

   c. The relation r is transitive.

   Prove that r is an equivalence relation.
   All we have to do is prove that r is reflexive. Suppose that $x \in S$. Using condition a, choose a member y of S such that $x\, r\, y$. Since r is symmetric we know that $y\, r\, x$ and therefore, since r is transitive we have $x\, r\, x$.

6. Suppose that $\sim$ is an equivalence relation in a set S and that E is a subset of S that contains precisely one member of each equivalence class of the relation $\sim$. Prove that for every $x \in S$ there exists one and only one member y of the set E such that $x \sim y$.

   Solution: ***Suppose that $x \in S$. We define C to be the equivalence class of the ralation $\sim$ that contains the member x and we define y to be the member of E that lies in the equivalence class C. Since x and y belong to the same equivalence class we know that $x \sim y$. Now, to show that this member y of E is the only member of the set E that can be related by r to x, suppose that $z \in E$ and that $x \sim z$. From the fact that $x \sim y$ and $x \sim z$ we deduce that $y \sim z$. Therefore y and z must belong to the same equivalence class of the relation $\sim$ and, since E never contains more than one member in any one equivalence class of $\sim$ we conclude that $y = z$.***

7. Suppose that $\Im$ is a family of sets and that for any two members A and B of $\Im$ we define $A\, r\, B$ to mean that either $A \subseteq B$ or $B \subseteq A$. Is r an equivalence relation in $\Im$?
   This relation need not be transitive. For example, if $B \subseteq A$ and $B \subseteq C$ then there is no reason to expect that one of the sets A and C should be included in the other.

8. Which of the following relations in $\mathbf{R}$ is an equivalence relation?

   a. $r = \left\{(x,y) \in \mathbf{R}^2 \mid x - y \text{ is an integer}\right\}$.
      Yes

   b. $r = \left\{(x,y) \in \mathbf{R}^2 \mid x - y \text{ is a positive integer}\right\}$.
      No

   c. $r = \left\{(x,y) \in \mathbf{R}^2 \mid x - y \text{ is an even integer}\right\}$.
      Yes

d. $r = \left\{(x,y) \in \mathbf{R^2} \mid x - y \text{ is an odd integer}\right\}$.
No

e. $r = \left\{(x,y) \in \mathbf{R^2} \mid x - y \text{ is rational}\right\}$.
Yes

f. $r = \left\{(x,y) \in \mathbf{R^2} \mid x - y \text{ is irrrational}\right\}$.
No

9. Given a subset G of $\mathbf{R}$, prove that the following two conditions are equivalent:

a. The relation r defined by

$$r = \{(x,y) \in \mathbf{R^2} \mid x - y \in G\}$$

is an equivalence relation in $\mathbf{R}$.

b. The set G is nonempty and for all numbers x and y in the set G the numbers $-x$ and $x + y$ must also belong to G.

Proof that a implies b: We assume that r is an equivalence relation in $\mathbf{R}$. Since r is reflexive we know that $2\ r\ 2$ which says that $2 - 2 \in G$. Therefore G is nonempty.
Given any member x of the set G we know from the fact that $x - 0 \in G$ that $x\ r\ 0$ and it follows from the fact that r is symmetric that $0\ r\ x$ which says that $-x \in G$.
Finally, suppose that x and y belong to G. Since $0 - x = -x \in G$ we have $0\ r\ x$ and since $x - (x + y) = -y \in G$ we have $x\ r\ x + y$. Therefore, since r is transitive we have $0\ r\ x + y$ which tells us that $x + y = x + y - 0 \in G$.

Proof that b implies a: We assume that condition b holds. Using the fact that $G \neq \emptyset$ we choose a member u of G. We know that $-u$ also belongs to G and that $u + (-u) \in G$. In other words, $0 \in G$.
Given any number x we deduce from the fact that $x - x = 0 \in G$ that $x\ r\ x$ and so r is reflexive.
Given any numbers x and y, if $x - y \in G$ then $y - x \in G$ and so the condition $x\ r\ y$ implies that $y\ r\ x$ and we conclude that r is symmetric.
Finally, suppose that x and y and z are given numbers and that $x\ r\ y$ and $y\ r\ z$. Since $x - y \in G$ and $y - z \in G$ we have $x - y + y - z \in G$ which tells us that $x\ r\ z$ and we conclude that r is transitive.

10. Suppose that $G \subseteq \mathbf{R}$ and that the relation

$$r = \{(x,y) \in \mathbf{R^2} \mid x - y \in G\}$$

is an equivalence relation in $\mathbf{R}$. Suppose that in every equivalence class C of r we have chosen a number and named it x_C. Prove that every real number x can be written in one and only one way in the form $x_C + y$ where C is an equivalence class of $\sim$ and $y \in G$.

Solution: ***Suppose that** x **is any real number. We define** C **to be the equivalence class of the relation** r **such that** $x \in C$**. Since** $x_C \in C$ **we know that the ordered pair** (x, x_C) **must belong to the relation** r **and so** $x - x_C \in G$**. Since***

$$x = x_C + (x - x_C)$$

we have shown that there exists an equivalence class** C **of** r **and a member** y **of the set** G **such that** $x = x_C + y$**. Now we need to show that this decomposition of** x **into the form** $x_C + y$ **is unique. Suppose that** C **and** D **are both equivalence classes of** r **and that** y **and** z **are both members of the set** G **and that

$$x_C + y = x_D + z.$$

Since

$$x_C - x_D = z - y \in G$$

we see that the ordered pair** (x_C, x_D) **belongs to the relation** r **and we conclude that** $C = D$**. Therefore

$$x = x_C + y = x_C + z$$

and it follows that** $y = z$**.

11. Given that S is the set of all nonzero real numbers and that $G \subseteq S$ and that

$$r = \left\{(x,y) \in \mathbf{S}\times\mathbf{S} \mid \frac{x}{y} \in G\right\},$$

what properties must the set G have in order to make r an equivalence relation in S?
This exercise is similar to Exercise 9. The set G must be nonempty and whenever x and y belong to G, the numbers x^{-1} and xy must belong to G.

12. Suppose that P is the set of all positive numbers that are unequal to 1 and that $\mathbf{Q}$ is (as usual) the set of all rational numbers.

   a. Prove that the relation r defined by

$$r = \{(x,y) \in P\times P \mid \log_x y \in \mathbf{Q}\}$$

   is an equivalence relation in P.
   Given any number $x \in P$ we have $\log_x x = 1 \in \mathbf{Q}$ and so $x\,r\,x$. Therefore r is reflexive.
   Given numbers x and y in P it follows from the equation

$$\log_x y = \frac{1}{\log_y x}$$

   that $\log_x y \in \mathbf{Q}$ if and only if $\log_y x \in \mathbf{Q}$. Therefore r is symmetric.
   Given x, y and z in P, if the numbers $\log_x y$ and $\log_y z$ are rational then since

$$\log_x z = \frac{\log_y z}{\log_y x} = \left(\log_y z\right)(\log_x y) \in \mathbf{Q}$$

   we conclude that r is transitive.

   b. Suppose that E is a subset of P that contains precisely one member of each equivalence class of the relation r. Prove that if x is any positive number unequal to 1 then x can be expressed in one and only one way in the form $x = y^q$ where $y \in E$ and q is a rational number.
   We deduce from Exercise 6 that whenever $x \in P$ there is exactly one member y of the set E such that $\log_y x$ is rational.

13. Suppose that G is a set of nonzero real numbers and that P is the set of all positive numbers that are unequal to 1. Prove that the following two conditions are equivalent:

   a. The relation r defined by

$$r = \{(x,y) \in P\times P \mid \log_x y \in G\}$$

   is an equivalence relation in $\mathbf{P}$.

   b. The set G is nonempty and for all numbers a and b in the set G the number $\frac{a}{b}$ belongs to G.

This exercise can be completed by the methods that were used in Exercises 9 and 11.
An alternative is to use the conclusion of Exericise 11 as follows: We define

$$W = \{\log_2 x \mid x \in P\}$$

and we define the relation s in W by saying that if u and v belong to W then $u\,s\,v$ means that $\frac{u}{v} \in G$. Note that W is the set of nonzero real numbers. Since the condition $u\,s\,v$ holds exactly when $2^u\,r\,2^v$ we see at once that r is an equivalence relation in P if and only if s is an equivalence relation in the set W, the desired result follows at once from Exercise 11.

14. Suppose that S is a linearly independent subset of a vector space V. Prove that the relation r defined by

$$r = \left\{(x,y) \in V\times V \mid S\cup\{x-y\} \text{ is linearly dependent}\right\}$$

is an equivalence relation in V.
Given $x \in S$ we know from the fact that the set $S\cup\{x-x\}$ contains the additive identity $\mathbf{0}$ that $S\cup\{x-x\}$ is linearly dependent. Therefore the relation r is reflexive.
Given x and y in S, the set $S\cup\{x-y\}$ is linearly dependent if and only if the set $S\cup\{y-x\}$ is linearly dependent. Therefore r is symmetric.
Finally suppose that x, y and z belong to S and that $x\,r\,y$ and that $y\,r\,z$. The sets $S\cup\{x-y\}$ and $S\cup\{y-z\}$ are linearly dependent. Since both $x-y$ and $y-z$ must lie in the span $\langle S\rangle$ of S we know

that $x - y + y - z \in \langle S \rangle$ and so the set $S \cup \{x - z\}$ is linearly dependent.

15. What can be said in the above exercise if the set S is linearly dependent?
If S is linearly dependent then the condition $x\, r\, y$ holds for all x and y in S, in other words, $r = S \times S$ which is an equivalence relation in S.

Exercises on Order Relations

1. Suppose that $\preceq$ is the relation in $\mathbf{R}^2$ defined by $(a,b) \preceq (x,y)$ if and only if $a \le x$ and $b \le y$, where $\le$ is the usual order in $\mathbf{R}$. Prove that $\preceq$ is a partial order in $\mathbf{R}^2$ but is not a total order.
It is clear that the relation $\preceq$ is a partial order. Since neither of the points $(0,1)$ and $(1,0)$ precedes the other.

2. Suppose that $\preceq$ is the relation in $\mathbf{R}^2$ defined by $(a,b) \preceq (x,y)$ if and only if $a \le x$ and $b \ge y$, where $\le$ is the usual order in $\mathbf{R}$. Prove that $\preceq$ is a partial order in $\mathbf{R}^2$ but is not a total order.
This exercise is just like Exercise 1.

3. Suppose that $\preceq$ is the relation in $\mathbf{R}^2$ defined by $(a,b) \preceq (x,y)$ if and only if either $a \le x$ or $b \le y$, where $\le$ is the usual order in $\mathbf{R}$. Is the relation $\preceq$ a partial order in $\mathbf{R}^2$?
This relation fails to be transitive. Note that $(0,4) \preceq (1,0) \preceq (-1,2)$.

4. Suppose that $\preceq$ is the relation in $\mathbf{R}^2$ defined by $(a,b) \preceq (x,y)$ if and only if either $a < x$ or $\left(a = x \text{ and } b \le y\right)$. Prove that $\preceq$ is a total order in $\mathbf{R}^2$.
We see at once that the relation $\preceq$ is reflexive and it is also clear that if $(a,b) \preceq (x,y)$ and $(a,b) \preceq (x,y)$ then $a = x$ and $b = y$. To see that $\preceq$ is transitive, suppose that $(a,b) \preceq (x,y) \preceq (u,v)$. If either $a < x$ or $x < u$ then we have $a < u$ and the condition $(a,b) \preceq (u,v)$ is assured. Otherwise $a = x = u$ and it follows from the fact that $b \le y \le v$ that $b \le v$. So in this case we again have $(a,b) \preceq (u,v)$.
Finally suppose that (a,b) and (x,y) are any points in $\mathbf{R}^2$. If $a < x$ then we have $(a,b) \preceq (x,y)$ and if $x < a$ then we have $(x,y) \preceq (a,b)$. In the event that $a = x$ then the condition $(a,b) \preceq (x,y)$ holds when $b \le y$ and the condition $(x,y) \preceq (a,b)$ holds when $y \le b$.
We conclude that $\preceq$ is a total order in $\mathbf{R}^2$.

5. Suppose that $\mathfrak{I}$ is a family of sets and that r is the relation in $\mathfrak{I}$ that consists of all pairs (A,B) such that either $A = B$ or $A \setminus B \neq \emptyset$. Is r always a partial order?
This relation fails to be transitive. Look at the sets $\{1\}$ and $\{2\}$ and $\{1,3\}$.

6. Given a set S, is $S \times S$ a partial order of S?
No it isn't. If x and y are any two different members of S then, although both (x,y) and (y,x) belong to $S \times S$ we do not have $x = y$.

7. Prove that if r_1 and r_2 are partial orders in a set S then $r_1 \cap r_2$ is also a partial order in S. What about $r_1 \cup r_2$?
It is easy to see that if r_1 and r_2 are partial orders of S then so is $r_1 \cap r_2$. However $r_1 \cup r_2$ can fail. For example, both $\le$ and $\ge$ are partial orders of $\mathbf{R}$ but the union of these two relations is $\mathbf{R} \times \mathbf{R}$ which is not a partial order of $\mathbf{R}$.

Exercises on Functions

1. Given that $f(x) = x^2$ for every real number x, simplify the following expressions:

    a. $f[[0,3]]$
    We have $f[[0,3]] = [0,9]$.

    b. $f[(-2,3]]$
    We have $f[[-2,3]] = [0,9]$.

    c. $f^{-1}[[-3,4]]$

We have $f^{-1}[[-3,4]] = [0,16]$.

2. Point at the equation $f(x) = x^2$ and then click on the button in your computing toolbar. Then work out the expressions in parts (a) and (b) of the preceding exercise by pointing at them and clicking on the evaluate button.

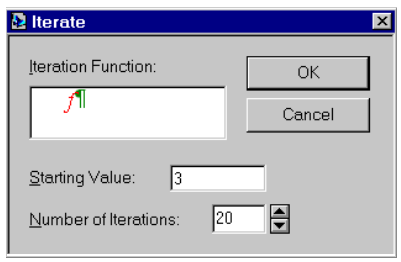
3. Supply each of the definitions $f(x) = x^2$ and $g(x) = 2 - 3x$ to *Scientific Notebook* and then ask *Scientific Notebook* to solve the equation

$$(f \circ g)(x) = (g \circ f)(x).$$

4. Supply the definition

$$f(x) = \frac{x-2}{1-2x}$$

to *Scientific Notebook*. In this exercise we shall see how to evaluate the composition of the function f with itself up to 20 times starting at a variety of numbers. Open the Compute menu, click on Calculus and move to the right and select Iterate. In the iterate dialogue box



fill in the function as f, the starting value as 3 and the number of iterations as 20. Repeat this process with different starting values. Can you draw a conclusion from what you see?

5. Given that $f(x) = x^2$ for all $x \in \mathbf{R}$ and $g(x) = 1 + x$ for all $x \in \mathbf{R}$, simplify the following expressions:

   a. $(f \circ g)[[0,1]]$
   We have $(f \circ g)[[0,1]] = f[g[[0,1]]] = f[[1,2]] = [1,4]$.

   b. $(g \circ f)[[0,1]]$
   We have $(g \circ f)[[0,1]] = g[f[[0,1]]] = g[[0,1]] = [1,2]$.

   c. $(g \circ g)[[0,1]]$
   We have $(g \circ g)[[0,1]] = g[g[0,1]] = g[[1,2]] = [2,3]$.

6. a. Given that $f(x) = (3x-2)/(x+1)$ for all $x \in \mathbf{R} \setminus \{-1\}$, determine whether or not f is one-one and find its range.
   Given any numbers x and y, the equation $y = f(x)$ says that

$$y = \frac{3x-2}{x+1}$$

   which gives us

$$xy + y = 3x - 2$$

   and finally

$$x = \frac{2+y}{3-y}.$$

   Since, for each number y, there is only one number x that makes the equation $y = \frac{3x-2}{x+1}$ hold, the function f is one-one.
   Given any number y, the existence of a number x that makes the equation $y = \frac{3x-2}{x+1}$ hold will be

assured as long as $y \neq 3$ and so the range of the function f is $\mathbf{R} \setminus \{3\}$.

b. Point at the equation

$$y = \frac{3x-2}{x+1}$$

and ask *Scientific Notebook* to solve for x. How many values of x are given? Is this result consistent with the answer you gave in part a of the question?

7. Suppose that $f : A \to B$ and that $E \subseteq A$. Is it true that $E = f^{-1}[f[E]]$? What if f is one-one? What if f is onto B?

Since

$$f^{-1}[f[E]] = \{x \in A \mid f(x) \in f[E]\}$$

we see at once that $E \subseteq f^{-1}[f[E]]$. However, the equation $E = f^{-1}[f[E]]$ does not have to hold, even if f is onto the set B.

For example, if $f(x) = x^2$ for $-3 \leq x \leq 3$ then f is a function from $[-3,3]$ onto $[0,9]$ and

$$f^{-1}[f[0,1]] = \{x \in [-3,3] \mid x^2 \in [0,1]\} = [-1,1] \neq [0,1].$$

On the other hand, if f is one-one then the equation $E = f^{-1}[f[E]]$ must hold. Suppose that f is a one-one function from A to B and that $E \subseteq A$. To see that $f^{-1}[f[E]] \subseteq E$, suppose that $x \in f^{-1}[f[E]]$. Using the fact that $f(x) \in f[E]$, choose $t \in E$ such that $f(x) = f(t)$. Since f is one-one we have $x = t \in E$.

8. Suppose that $f : A \to B$ and that $E \subseteq B$. Is it true that $E = f[f^{-1}[E]]$? What if f is one-one? What if f is onto B?

This exercise is similar to Exercise 7. No it isn't true that $E = f[f^{-1}[E]]$ and it doesn't help to assume that f is one-one. We do have the inclusion $f[f^{-1}[E]] \subseteq E$ and the equation $E = f[f^{-1}[E]]$ will be assured if f is onto the set B (or even if the range of f is known to include E).

9. Suppose that $f : A \to B$ and that P and Q are subsets of B. Prove the identities

$$f^{-1}[P \cup Q] = f^{-1}[P] \cup f^{-1}[Q],$$

Solution: ***Given any member x of the set A, the condition $x \in f^{-1}[P \cup Q]$ says that $f(x) \in P \cup Q$ which says that either $f(x) \in P$ or $f(x) \in Q$ which says that either $x \in f^{-1}[P]$ or $x \in f^{-1}[Q]$.***

$$f^{-1}[P \cap Q] = f^{-1}[P] \cap f^{-1}[Q],$$

$$f^{-1}[P \setminus Q] = f^{-1}[P] \setminus f^{-1}[Q],$$

10. Suppose that $f : A \to B$ and that P and Q are subsets of A. Which of the following statements are true? What if f is one-one? What if f is onto B?

$$f[P \cup Q] = f[P] \cup f[Q]$$

Hint: ***This statement is true.***

$$f[P \cap Q] = f[P] \cap f[Q]$$

This statement is false. Give an example. Then prove that the statement is true if f is one-one.

If we define $f(x) = x^2$ for every number x then

$$f[\{-1\} \cap \{1\}] \neq f[\{-1\}] \cap f[\{1\}].$$

In general it is clear that $f[P \cap Q] \subseteq f[P] \cap f[Q]$. Now suppose that f is one-one and that $y \in f[P] \cap f[Q]$. Using the fact that $y \in f[P]$ we choose $x \in P$ such that $y = f(x)$ and, using the fact that $y \in f[Q]$ we choose $t \in Q$ such that $y = f(t)$. Since $f(x) = f(t)$ and f is one-one we have $x = t$ and so $x \in P \cap Q$ and we conclude that $y \in f[P \cap Q]$.

$$f[P \setminus Q] = f[P] \setminus f[Q]$$

This statement is true when f is one-one.

11. Given that f is a one-one function from A to B and that g is a one-one function from B to C, prove that the

function $g \circ f$ is one-one from A to C.

Solution: ***We need to prove that whenever t and x are members of the set A and $t \neq x$ we have***

$$(g \circ f)(t) \neq (g \circ f)(x).$$

Suppose that t and x are members of the set A and that $t \neq x$. Since f is one-one we have $f(t) \neq f(x)$. Therefore, since g is one-one we have $g(f(t)) \neq g(f(x))$ and we have shown that $(g \circ f)(t) \neq (g \circ f)(x)$.

12. Given that f is a function from A onto B and that g is a function from B onto C, prove that the function $g \circ f$ is a function from A onto C.
Suppose that $z \in C$. Using the fact that g is onto the set C, choose $y \in B$ such that $g(y) = z$. Now we use the fact that f is onto the set B to choose $x \in A$ such that $f(x) = y$. We observe that $(g \circ f)(x) = z$. In this way we have shown that every member of the set C belongs to the range of the function $g \circ f$.

13. Given that $f : A \to B$ and that $g : B \to C$ and that the function $g \circ f$ is one-one, prove that f must be one-one. Give an example to show that the function g does not have to be one-one.

Solution: ***To prove that f is one-one, suppose that x_1 and x_2 are members of the set A and that $x_1 \neq x_2$. Since the function $g \circ f$ is one-one we know that $g(f(x_1)) \neq g(f(x_2))$ and we see at once that $f(x_1) \neq f(x_2)$. Now we construct an example to show that the function g does not have to be one-one. We define $f(x) = x$ for every $x \in [0,1]$ and we define***

$$g(x) = \begin{cases} x & \text{if} \quad x \in [0,1] \\ 2 & \text{if} \quad 1 < x \leq 5 \end{cases}$$

14. Given that f is a function from A onto B and that $g : B \to C$ and that the function $g \circ f$ is one-one, prove that both of the functions f and g have to be one-one.

Solution: ***To see that f is one-one, suppose that x and t are members of the set A and that $t \neq x$. Since $g(f(t)) \neq g(f(x))$ we see at once that $f(t) \neq f(x)$.***

Now to see that g is one-one, suppose that u and y are members of the set B and that $u \neq y$. Using the fact that the function f is onto the set B we choose members t and x of A such that $u = f(t)$ and $y = f(x)$. We see at once that $t \neq x$ and therefore

$$g(u) = g(f(t)) \neq g(f(x)) = g(y).$$

15. Suppose that $f : A \to B$ and that r is the relation defined by

$$r = \{(x,y) \in A \times A \mid f(x) = f(y)\}.$$

   a. Prove that r is an equivalence relation in the set A.
   Since $f(x) = f(x)$ whenever $x \in A$ the relation r is reflexive. Since the condition $f(x) = f(y)$ is the same as the condition $f(y) = f(x)$ the relation r is symmetric. A similar argument shows that r is transitive.

   b. Prove that if E is a subset of A that contains precisely one member of each equivalence class of r then the restriction of f to E is a one-one function from E into B.
   Suppose that E is a subset of A containing precisely one member of each equivalence class of r. Given x and t different members of E we know from the fact that x and t do not lie in the same equivalence class of r that $f(x) \neq f(t)$. Therefore the restriction of f to E is one-one.

16. Suppose that S is a given set and that $\mathfrak{I}$ is a set of functions from S to $\mathbf{R}$. Prove that if for any two members f and g of $\mathfrak{I}$ we define the condition $f \leq g$ to mean that $f(x) \leq g(x)$ for every $x \in S$, then $\leq$ is a partial order of the set $\mathfrak{I}$.
Given any function $f \in \mathfrak{I}$ we see from the fact that $f(x) \leq f(x)$ for every $x \in S$ that $f \leq f$. Therefore the relation $\leq$ in $\mathfrak{I}$ is reflexive and similar arguments can be used to show that $\leq$ is also symmetric and transitive. We omit the details.

17. Suppose that S is a given set and that $\Im$ is the set of all functions from S to $\mathbf{R}$. Suppose that the partial order $\leq$ is defined in the set $\Im$ above. Given two members f and g of $\Im$, prove that there exists a member u of $\Im$ such that the following two conditions hold:

a. $f \leq u$ and $g \leq u$

b. Whenever a member h of $\Im$ satisfies $f \leq h$ and $g \leq h$, we have $u \leq h$.

Suppose that f and g belong to the family $\Im$. We define

$$u(x) = \begin{cases} f(x) & \text{if } g(x) \leq f(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

It is easy to see that this function u has the desired properties.

18. Given any set S, the **identity function** i_S on S is defined by $i_S(x) = x$ for every $x \in S$. Prove that if f is a one-one function from a set A onto a set B then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.
The equation $f^{-1} \circ f = i_A$ says that $f^{-1}(f(x)) = x$ for every $x \in A$ and this assertion is exactly the definition of the function f^{-1}. The equation $f \circ f^{-1} = i_B$ follows similarly.

19. Suppose that $f : A \to B$.

a. Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$, what can be said about the functions f and g?
If x and t are any members of A for which $f(x) = f(t)$ then it follows that

$$x = g(f(x)) = g(f(t)) = t$$

and so the function f is one-one.
Given any $x \in A$ we see from the fact that $x = g(f(x))$ that the function g is onto the set A.

b. Given that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f and h?
This is just part a again. The function f must be onto the set B and the function h must be one-one.

c. Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$ and that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f, g and h?
Now the function f must be one-one and onto the set B and the functions g and h are one-one from B onto A and, in fact $g = h$.

20. As in a previous example, we define

$$f_a(x) = \frac{x - a}{1 - ax}$$

whenever $a \in (-1, 1)$ and $x \in [-1, 1]$.

a. Prove that if a and b belong to $(-1, 1)$ then so does the number

$$c = \frac{a + b}{1 + ab}.$$

Hint: An quick way to do this exercise is to observe that $c = f_{-b}(a)$.
We saw in that earlier example that whenever $-1 < a < 1$, the function f_a is a one-one function from $[-1, 1]$ onto $[-1, 1]$. and that $f(-1) = -1$ and $f(1) = 1$. Therefore

$$c = f_{-b}(a) \in (-1, 1).$$

b. Given a and b in $(-1, 1)$ and

$$c = \frac{a + b}{1 + ab},$$

prove that $f_b \circ f_a = f_c$.
Given any $x \in [-1, 1]$ we have

$$f_b(f_a(x)) = f_b\left(\frac{x-a}{1-ax}\right) = \frac{\frac{x-a}{1-ax} - b}{1 - \left(\frac{x-a}{1-ax}\right)b}$$
$$= \frac{\left(\frac{x-a}{1-ax} - b\right)(1-ax)}{\left(1 - \left(\frac{x-a}{1-ax}\right)b\right)(1-ax)} = \frac{(1+ab)x - (a+b)}{1+ba-(a+b)x}$$
$$= \frac{\left(\frac{1+ab}{a+b}\right)x - 1}{\frac{1+ba}{a+b} - x} = f_c(x).$$

Some Elementary Exercises on Set Equivalence

1. Prove that $\mathbf{Z}^+ \sim \mathbf{Z}$.

Solution: ***The function*** $f : \mathbf{Z} \to \mathbf{Z}^+$ ***defined by the equation***

$$f(n) = \begin{cases} 2n+1 & \text{if } n \geq 0 \\ -2n & \text{if } n < 0 \end{cases}$$

is a one-one function from $\mathbf{Z}$ ***onto*** $\mathbf{Z}^+$.

2. Prove that $[0,1) \sim (0,1)$.

Solution: ***The function defined by the equation***

$$f(x) = \begin{cases} \frac{1}{n+1} & \textbf{if } n \textbf{ is an integer and } n \geq 2 \textbf{ and } x = \frac{1}{n} \\ \frac{1}{2} & \textbf{if } x = 0 \\ x & \textbf{if } x \in [0,1) \setminus \left(\{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbf{Z} \textbf{ and } n \geq 2\right\}\right) \end{cases}$$

is a one-one function from $[0,1)$ ***onto*** $(0,1)$.

3. Prove that if $a < b$ then any two of the four intervals $(a,b), (a,b], [a,b)$ and $[a,b]$ are equivalent.

Solution: ***At this stage we already know that***

$$[0,1] \sim [0,1) \sim (0,1] \sim (0,1).$$

If we define

$$f(x) = a + (b-a)x$$

whenever $x \in [0,1]$ ***then we see that*** f ***is a one-one function from*** $[0,1]$ ***onto*** $[a,b]$ ***and, in a similar way, we can see that*** $[0,1) \sim [a,b)$ ***and*** $(0,1] \sim (a,b]$ ***and*** $(0,1) \sim (a,b)$.

4. Given that A and B are sets that are disjoint from each other and that $A \sim \mathbf{Z}^+$ and $B \sim \mathbf{Z}^+$, prove that $A \cup B \sim \mathbf{Z}^+$.

Solution: ***Choose a one-one function*** f ***from*** A ***onto*** $\mathbf{Z}^+$ ***and a one-one function*** g ***from*** B ***onto*** $\mathbf{Z}^+$. ***We now define a function*** h ***on the set*** $A \cup B$ ***as follows:***

$$h(x) = \begin{cases} 2f(x) - 1 & \textbf{if } x \in A \\ 2g(x) & \textbf{if } x \in B \end{cases}$$

and we observe that h ***is a on-one function from*** $A \cup B$ ***onto*** $\mathbf{Z}^+$.

5. Given that $A \sim B$, prove that $A \times A \sim B \times B$.
Using the fact that $A \sim B$ we choose a one-one function f from A onto B. We now define

$$g((x,y)) = (f(x), f(y))$$

for every point $(x,y) \in A \times B$.
To see that g is one-one, suppose that (x,y) and (u,v) belong to $A \times A$ and that $g((x,y)) = g((u,v))$. We have $(f(x),f(y)) = (f(u),f(v))$ and so $f(x) = f(u)$ and $f(y) = f(v)$. Since f is one-one we have $x = u$ and $y = v$ which tells us that $(x,y) = (u,v)$.
Now to see that g is onto the set $B \times B$, suppose that $(s,t) \in B \times B$. Using the fact that f is onto B, choose x and y in A such that $f(x) = s$ and $f(y) = t$. We see that $g((x,y)) = (s,t)$.

Exercises on Binary Operations

1. Prove that if we define $x * y = x + y - xy$ for all numbers x and y unequal to 1 then $*$ is an associative commutative binary operation in the set $\mathbf{R} \setminus \{1\}$.
   It is clear that the operation $*$ is commutative. Now suppose that x, y and z are numbers unequal to 1.
$$\begin{aligned}(x * y) * z &= x * y + z - (x * y)z \\ &= x + y - xy + z - (x + y - xy)z \\ &= x + y + z - xy - xz - yz + xyz\end{aligned}$$
   and we see similarly that
$$x * (y * z) = x + y + z - xy - xz - yz + xyz.$$
   An alternative way of looking at this exercise is to observe that if x and y are any numbers unequal to 1 then
$$x * y = 1 - (1 - x)(1 - y)$$
   and so
$$\begin{aligned}(x * y) * z &= 1 - (1 - x * y)(1 - z) \\ &= 1 - (1 - x)(1 - y)(1 - z).\end{aligned}$$

2. Prove that if we define
$$x * y = \frac{x + y}{1 + xy}$$
   for all numbers x and y in the interval $(-1,1)$ then $*$ is an associative commutative binary operation in $(-1,1)$.
   We have already seen in an earlier exercise that $*$ is a binary operation in the set $(-1,1)$. It is clear that the operation $*$ is commutative. To see that $*$ is associative, suppose that x, y and z belong to $(-1,1)$. We observe that
$$x * (y * z) = \frac{x + \frac{y+z}{1+yz}}{1 + x\left(\frac{y+z}{1+yz}\right)} = \frac{x + y + z + xyz}{1 + yz + xy + xz}$$
   and we can see similarly that
$$(x * y) * z = \frac{x + y + z + xyz}{1 + yz + xy + xz}$$

3. Prove that if we define
$$(x_1,y_1) * (x_2,y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$
   for all members (x_1,y_1) and (x_2,y_2) of $\mathbf{R}^2$, then $*$ is an associative commutative binary operation in $\mathbf{R}^2$.

4. We can see the properties of this operation $*$ just as we did in the two preceding exercises. Alternatively, we can jump ahead to the material on complex numbers where this exercise appears again as part of the narrative.

5. (This exercise requires a little linear algebra.) Prove that if S is the set of all 2×2 matrices with real entries then matrix multiplication is an associative but not a commutative binary operation in S.
   The assertions made here are part of standard linear algebra.

6. Prove that if S is the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where a and b are real numbers, then matrix multiplication is an associative commutative binary operation in S.

We observe that if x_1,y_1,x_2 and y_2 then

$$\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}\begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + y_1x_2 \\ -y_1x_2 - x_1y_2 & x_1x_2 - y_1y_2 \end{bmatrix}$$

and so, by relating ordered pairs (x_1, y_1) and matrices $\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$ we can see that this exercise is really Exercise 3 again in disguise.

7. Prove that if S is the set of all 2×2 matrices of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

where θ is any real number, then matrix multiplication is an associative commutative binary operation in S.

Once we have shown that the product of any two members of S must lie in S we shall know that this exercise is a special case of Exercises 3 and 5. We can think of the members of S as being the complex numbers whose distance from 0 is 1.

Exercises on Finite Sets

Many of the results you are asked to prove in the following exercises can be proved quite easily by mathematical induction.

1. Given that n is a positive integer and that S is a proper subset of the set $\{1, 2, \cdots, n\}$, prove that $\text{card}(S) < n$.

Solution: ***The assertion is obvious if*** $n = 1$***. We shall now assume that*** $n > 1$ ***and we shall write*** n ***in the form*** $m + 1$ ***where*** m ***is a positive integer. Suppose that*** S ***is a proper subset of the set*** $\{1, 2, \cdots, m + 1\}$***. In the event that*** $m + 1 \notin S$ ***we know that*** $S \subseteq \{1, 2, \cdots, m\}$ ***and so*** $\text{card}(S) \leq m < m + 1$***. Suppose now that*** $m + 1 \in S$***. Using the fact that that*** S ***is a proper subset of the set*** $\{1, 2, \cdots, m + 1\}$ ***we choose an integer*** $k \in \{1, 2, \cdots, m\} \setminus S$***. Since***

$$S \sim S \cup \{k\} \setminus \{m + 1\}$$

and since

$$\text{card}\left(S \cup \{k\} \setminus \{m + 1\}\right) < m + 1$$

we see that $\text{card}(S) < m + 1$***.***

2. Prove that if A is a finite set and B is a proper subset of A then we do not have $A \sim B$.

Using the fact that A is a nonempty finite set we choose a positive integer n such that $A \sim \{1, 2, \cdots, n\}$. Choose a one-one function f from A onto $\{1, 2, \cdots, n\}$. Since B is a proper subset of A the set $f[B]$ must be a proper subset of $\{1, 2, \cdots, n\}$. If we had $A \sim B$ then we would have

$$f[B] \sim B \sim A \sim \{1, 2, \cdots, n\}$$

which is impossible by Exercise 1. Therefore we can't have $A \sim B$.

3. Prove that if A and B are finite sets and $A \cap B = \emptyset$ then $\text{card}(A \cup B) = \text{card}(A) + \text{card}(B)$.

Solution: ***We use mathematical induction. Suppose that*** A ***is a finite set and define*** $m = \text{card}(A)$***. For each nonnegative integer*** n ***we take*** p_n ***to be the assertion that if*** B ***is any finite set disjoint from*** A ***and satisfying*** $\text{card}(B) = n$ ***then the set*** $A \cup B$ ***is finite and*** $\text{card}(A \cup B) = m + n$***.***

The assertion* p_0 *is obvious. We shall now prove that* p_1 *is true. Suppose that* B *is any set disjoint from* A *and satisfying* $\text{card}(B) = 1$*. We can express* B *in the form* $\{u\}$*. Choose a one-one function* f *from* A *onto* $\{1, 2, \cdots, m\}$ *and extend the function* f *to* $A \cup B$ *by defining* $f(u) = m + 1$*. Since this extension of* f *is a one-one function from* $A \cup B$ *onto* $\{1, 2, \cdots, m, m+1\}$*, we have* $\text{card}(A \cup B) = m + 1$*.

Now suppose that* n *is any positive integer for which the assertion* p_n *is true and that* B *is a set disjoint from* A *and satisfying* $\text{card}(B) = n + 1$*. Choose a member* u *of the set* B*. Since the assertion* p_1 *is true and* $\{u\}$ *is disjoint from the finite set* $B \setminus \{u\}$ *we have

$$\text{card}(B) = \text{card}(B \setminus \{u\}) + 1$$

which tells us that* $\text{card}(B \setminus \{u\}) = n$*. Therefore, since the assertion* p_n *is true we have

$$\text{card}(A \cup (B \setminus \{u\})) = m + n$$

and since* p_1 *is true and since* $\{u\}$ *is disjoint from* $A \cup (B \setminus \{u\})$ *we have

$$\text{card}(A \cup B) = \text{card}\Big(\big(A \cup (B \setminus \{u\})\big) \cup \{u\}\Big) = m + n + 1$$

4. Given that A and B are finite sets and that $\text{card}(A) = m$ and $\text{card}(B) = n$, prove that $A \cup B$ is finite and that $\text{card}(A \cup B) \le m + n$.

   The point is that $A \cup B = A \cup (B \setminus A)$ and that since $B \setminus A$ is a subset of B we have $\text{card}(B \setminus A) \le \text{card}(B)$.

5. Given that A and B are finite sets and that $\text{card}(A) = m$ and $\text{card}(B) = n$, prove that $A \times B$ is finite and that $\text{card}(A \times B) = mn$.

   Solution: ***We use mathematical induction. Suppose that* A *is a finite set and that* $\text{card}(A) = m$*. For each nonnegative integer* n *we take* p_n *to be the assertion that whenever* B *is a finite set and* $\text{card}(B) = n$ *then the set* $A \times B$ *is finite and* $\text{card}(A \times B) = mn$*.***

   The assertion* p_0 *is obvious because, if* B *is empty then* $A \times B$ *is empty.

   The assertion* p_1 *is obvious because, if* $B = \{u\}$ *then* $A \times B \sim A$*.

   Now suppose that* n *is any positive integer for which the assertion* p_n *is true and suppose that* $\text{card}(B) = n + 1$*. Choose a member* u *of the set* B*. Since* $\text{card}(B \setminus \{u\}) = n$ *we have

$$\text{card}(A \times (B \setminus \{u\})) = mn$$

   and since the assertion* p_1 *gives us* $\text{card}(A \times \{u\}) = m$ *and since the set* $A \times \{u\}$ *is disjoint from* $A \times (B \setminus \{u\})$*, Exercise 3 guarantees that

$$\text{card}(A \times B) = \text{card}\Big(\big(A \times (B \setminus \{u\})\big) \cup \big(A \times \{u\}\big)\Big) = mn + m = m(n+1)$$

6. Given that S is a finite set and $\text{card}(S) = n$, prove that the power set $p(S)$ of S is a finite set and that $\text{card}(p(S)) = 2^n$.

   Solution: ***We use mathematical induction. For each nonnegative integer* n *we take* p_n *to be the assertion that whenever* S *is a finite set satisfying* $\text{card}(S) = n$ *then* $p(S)$ *is a finite set and* $\text{card}(p(S)) = 2^n$*.***

   If* $S = \emptyset$ *then* $p(S) = \{\emptyset\}$ *and so

$$\text{card}(p(S)) = 1 = 2^0 = 2^{\text{card}(S)}.$$

   Therefore the assertion* p_0 *is true.

   To see that the assertion* p_1 *is true, suppose that* $\text{card}(S) = 1$*. We write* S *in the form* $S = \{u\}$*. Since

$$p(S) = \{\emptyset, \{u\}\}$$

   we see that

$$\text{card}(p(S)) = 2 = 2^1 = 2^{\text{card}(S)}.$$

   Now suppose that* n *is any positive integer for which the assertion* p_n *is true and suppose that* $\text{card}(S) = n + 1$*. Choose a member* u *of the set* S*. Since* $\text{card}(S \setminus \{u\}) = n$ *we know that

$$\text{card}\big(p(S \setminus \{u\})\big) = 2^n.$$

   We now define

$$W = \left\{A \cup \{u\} \ \mid \ A \in p(S \setminus \{u\})\right\}$$

and we observe that $W \sim p(S \setminus \{u\})$***. (To see this, define*** $f(A) = A \cup \{u\}$ ***for every*** $A \in p(S \setminus \{u\})$***.) Since*** W ***is disjoint from*** $p(S \setminus \{u\})$ ***and*** $\mathrm{card}(W) = n$ ***we deduce from Exercise 3 that***

$$\mathrm{card}(p(S)) = \mathrm{card}(p(S \setminus \{u\}) \cup W) = 2^n + 2^n = 2^{n+1}.$$

7. Prove that if S is a nonempty finite set of real numbers then S has a largest member.
   We use mathematical induction. For each positive integer n we take p_n to be the assertion that whenever S is a finite set satisfying $\mathrm{card}(S) = n$ then S has a largest member. The assertion p_1 is obvious.
   Now suppose that n is any positive integer for which the assertion p_n is true and suppose that S is a finite set satisfying $\mathrm{card}(S) = n+1$. We choose a member u of the set S. Since $\mathrm{card}(S \setminus \{u\}) = n$ we know that the set $S \setminus \{u\}$ has a largest member that we shall call v. In the event that $u \leq v$ we see that v is the largest member of S and in the event that $v < u$ we see that u is the largest member of S. In either event the set S must have a largest ,member.

8. Prove that none of the sets the set $\mathbf{Z}^+, \mathbf{Z}, \mathbf{Q},$ and $\mathbf{R}$ are finite.
   All of these sets are nonempty and none of them has a largest member.

9. Prove that a subset S of $\mathbf{Z}^+$ is finite if and only if it is possible to find an integer n such that the inequality $m \leq n$ holds for every member m of S.
   We already know that if n is any positive integer then every subset of the set $\{1, 2, \cdots, n\}$ is finite. On the other hand, if S is a finite subset of $\mathbf{Z}^+$ then either S is empty, in which case $m \leq 1$ for every $m \in S$, or S has a largest member n, in which case $m \leq n$ for every $m \in S$.

10. Prove that if m and n are positive integers then the number of possible functions from the set $\{1, \cdots, n\}$ into the set $\{1, \cdots, m\}$ is m^n.
    Solution: ***We want to know that if*** $\mathfrak{I}$ ***is the set of all functions from*** $\{1, 2, \cdots, n\}$ ***into*** $\{1, 2, \cdots, m\}$ ***then the set*** $\mathfrak{I}$ ***is finite and*** $\mathrm{card}(\mathfrak{I}) = m^n$***. We suppose that*** m ***is a positive integer and for each positive integer*** n ***we define*** $\mathfrak{I}_n$ ***to be the set of all functions from*** $\{1, 2, \cdots, n\}$ ***into*** $\{1, 2, \cdots, m\}$***. For each positive integer*** n ***we take*** p_n ***to be the assertion that*** $\mathrm{card}(\mathfrak{I}_n) = m^n$***.***
    To see that the assertion p_1 ***is true we observe that if we define*** $\phi(j)$ ***for each*** $j \in \{1, 2, \cdots, m\}$ ***to be the function from*** $\{1\}$ ***to*** $\{1, 2, \cdots, m\}$ ***whose value at*** 1 ***is*** j ***then*** ϕ ***is a one-one function from*** $\{1, 2, \cdots, m\}$ ***onto*** $\mathfrak{I}_1$***. Therefore***

$$\mathrm{card}(\mathfrak{I}_1) = m = m^1.$$

Now suppose that n ***is any positive integer for which the assertion*** p_n ***is true. In order to prove that*** $\mathrm{card}(\mathfrak{I}_{n+1}) = m^{n+1}$ ***we shall show that***

$$\mathfrak{I}_n \times \{1, 2, \cdots m\} \sim \mathfrak{I}_{n+1}$$

and for this purpose we shall define a function

$$\phi : \mathfrak{I}_n \times \{1, 2, \cdots, m\} \to \mathfrak{I}_{n+1}$$

as follows: For each member (f, j) ***of the set*** $\mathfrak{I}_n \times \{1, 2, \cdots, m\}$ ***we define***

$$\phi(f,j)(k) = \begin{cases} f(k) & \textbf{\textit{if}} \quad k \in \{1, 2, \cdots, n\} \\ j & \textbf{\textit{if}} \quad k = n+1 \end{cases}.$$

The proof will be complete when we have seen that ϕ ***is one-one and is onto the set*** $\mathfrak{I}_{n+1}$***.***
To see that ϕ ***is one-one we suppose that*** (f_1, j_1) ***and*** (f_2, j_2) ***belong to*** $\mathfrak{I}_n \times \{1, 2, \cdots, m\}$ ***and that*** $\phi(f_1, j_1) = \phi(f_2, j_2)$***. For each*** $k \in \{1, 2, \cdots, n\}$ ***we have***

$$f_1(k) = \phi(f_1, j_1)(k) = \phi(f_2, j_2)(k) = f_2(k)$$

and so $f_1 = f_2$***. We see also that***

$$j_1 = \phi(f_1, j_1)(n+1) = \phi(f_2, j_2)(n+1) = j_2.$$

Finally, to see that ϕ ***is onto the set*** $\mathfrak{I}_{n+1}$***, suppose that*** $g \in \mathfrak{I}_{n+1}$ ***and define*** $j = g(n+1)$ ***and define*** $f(k) = g(k)$ ***for every*** $k \in \{1, 2, \cdots, n\}$***. We see at once that*** $g = \phi(f, j)$***.***

11. a. Prove that if for each positive integer n we define F_n to be the set of all one-one functions from $\{1,2,\cdots,n\}$ onto $\{1,2,\cdots,n\}$ then for each n we have

$$F_{n+1} \sim F_n \times \{1,2,\cdots,n,n+1\}.$$

Solution: *To motivate the solution of this exercise we should consider that a one-one function from the set $\{1,2,\cdots,n\}$ onto itself is a way or arranging the members of the set $\{1,2,\cdots,n\}$ in a sequence $(x_1,\cdots,x_n)$. For each such arrangement we can obtain an arrangement of the members of the larger set $\{1,2,\cdots,n,n+1\}$ by placing the number $n+1$ in any of the $n+1$ positions shown*

$$\begin{gathered} n+1,x_1,x_2,\cdots,x_n \\ x_1,n+1,x_2,\cdots,x_n \\ \vdots \\ x_1,x_2,\cdots,x_n,n+1 \end{gathered}$$

Thus the number of ways of arranging the members of $\{1,2,\cdots,n,n+1\}$ should be $n+1$ times the number of ways of arranging the members of $\{1,2,\cdots,n\}$ in a sequence.

Now we begin: We define a function ϕ from $F_n \times \{1,2,\cdots,n,n+1\} \to F_{n+1}$ as follows: Given any member (f,m) of the set $F_n \times \{1,2,\cdots,n,n+1\}$ we define

$$\phi(f,m)(j) = \begin{cases} f(j) & \textbf{\textit{if}} \quad j < m \\ n+1 & \textbf{\textit{if}} \quad j = m \\ f(j-1) & \textbf{\textit{if}} \quad j > m \end{cases}.$$

Each such function $\phi(f,m)$ is a one-one function from $\{1,2,\cdots,n,n+1\}$ onto $\{1,2,\cdots,n,n+1\}$. To see that the function ϕ is one-one, suppose that (f,m) and (g,k) belong to the set $F_n \times \{1,2,\cdots,n,n+1\}$ and that $\phi(f,m) = \phi(g,k)$. Since the function $\phi(g,k)$ is one-one and since

$$\phi(g,k)(m) = \phi(f,m)(m) = n+1 = \phi(g,k)(k)$$

we deduce that $k = m$.
Given any $j < m$ we have

$$f(j) = \phi(f,m)(j) = \phi(g,m)(j) = g(j)$$

and given any $j \geq m$ we have

$$f(j) = \phi(f,m)(j+1) = \phi(g,m)(j+1) = g(j)$$

and we conclude that $f = g$. Therefore the function ϕ is one-one.
Finally, to show that ϕ is onto the set F_{n+1} we assume that $h \in F_{n+1}$. In other words, h is a one-one function from the set $\{1,2,\cdots,n,n+1\}$onto $\{1,2,\cdots,n,n+1\}$. We see easily that if $m = h^{-1}(n+1)$ and if we define $f : \{1,2,\cdots,n\} \to \{1,2,\cdots,n\}$ by the equation

$$f(j) = \begin{cases} h(j) & \textbf{\textit{if}} \quad j < m \\ h(j+1) & \textbf{\textit{if}} \quad j \geq m \end{cases}$$

then $f \in F_n$ and $h = \phi(f,m)$.

b. Prove that for each positive integer n, the number of ways of ordering the numbers in the set $\{1,\cdots,n\}$ into a finite one-one sequence is $n!$

Solution: ***We use mathematical induction. The exercise is asking us to prove that if, for each positive integer n we write F_n for the set of all one-one functions from the set $\{1,2,\cdots,n\}$ onto $\{1,2,\cdots,n\}$ then for each n we have $\operatorname{card}(F_n) = n!$.***
For each positive integer n we take p_n to be the assertion that $\operatorname{card}(F_n) = n!$ The assertion p_1 is obvious.
Now suppose that n is any positive integer for which the assertion p_n is true. From part a and from Exercise 5 we see that

$$\begin{aligned}\operatorname{card}(F_{n+1}) &= \left(\operatorname{card}(F_n)\right)\left(\operatorname{card}\{1,2,\cdots,n,n+1\}\right)\\ &= (n!)(n+1) = (n+1)!\end{aligned}$$

12. Given that r and n are integers and that $0 \le r \le n$, the **binomial coefficient** $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

a. Prove that if n and r are integers and $1 \le r \le n$ then

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

The identity follows at once when we combine the terms on the left side.

b. Suppose that r is a nonnegative integer. Prove that for every integer $n \ge r$ and every finite set S that has n members, the number of subsets of S that have r members is $\binom{n}{r}$.

Solution: ***For each nonnegative integer*** n ***we define*** p_n ***to be the assertion that, whenever*** r ***is an integer and*** $0 \le r \le n$***, the number of subsets with exactly*** r ***members of any given set that has exactly*** n ***members is*** $\binom{n}{r}$***. The statement*** p_0 ***is obviously true. Now suppose that*** n ***is a nonnegative integer for which the statement*** p_n ***is true, suppose that*** r ***is an integer, that*** $0 \le r \le n+1$ ***and that*** S ***is a set with exactly*** $n+1$ ***members. Choose a member*** q ***of the set*** S***. We observe that the set*** $S \setminus \{q\}$ ***has exactly*** n ***members.***

Now we define A ***to be the family of all subsets of the set*** $S \setminus \{q\}$ ***and having exactly*** r ***members and*** B ***to be the family of all sets of the form*** $\{q\} \cup E$ ***where*** E ***is a subset of*** $S \setminus \{q\}$ ***that has exactly*** $r-1$ ***members. (In the event that*** $r = 0$***, this definition makes the set*** B ***empty.) We see that***

$$\operatorname{card}(A) = \binom{n}{r}$$

and

$$\operatorname{card}(B) = \binom{n}{r-1}$$

and therefore, by Exercise 3 we deduce that the number of subsets of S ***that have exactly*** r ***members is***

$$\operatorname{card}(A \cup B) = \operatorname{card}(A) + \operatorname{card}(B) = \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

The desired assertion therefore follows by mathematical induction.

c. Prove that if n is a positive integer then

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

An intuitive approach to this question is to conclude from part b that the number $\sum_{j=0}^{n}\binom{n}{j}$ is the number of subsets of $\{1,2,\cdots,n\}$ which we know from Exercise 6 to be 2^n. We could write a more careful proof using mathematical induction but that proof would be the special case of Exercise 13 obtained when $a = b = 1$.

13. Prove that if n is a nonnegative integer and a and b are any real numbers then

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}a^0 b^n.$$

This equation, as you may know, is known as the binomial theorem.
The solution of this exercise would be the standard proof by mathematical induction of the binomial theorem. Of course the calculus part of the text shows much more elegant ways of deducing the binomial theorem.

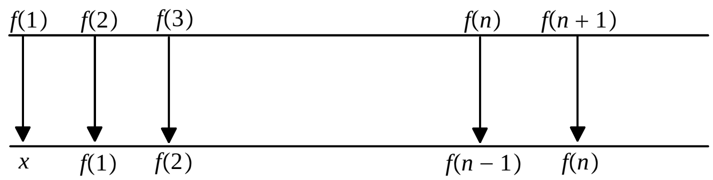
Exercises on Infinite Sets

1. Prove that a subset E of $\mathbf{Z}^+$ is equivalent to $\mathbf{Z}^+$ if and only if for every integer n there exists a member m of E such that $m > n$.

From an earlier exercise we know that a subset E of $\mathbf{Z}^+$ is infinite if and only if for every integer n there exists a member m of E such that $m > n$. We also know that a subset of $\mathbf{Z}^+$ is equivalent to $\mathbf{Z}^+$ if and only if it is infinite.

2. Given that S is an infinite set and that $x \in S$, prove that $S \sim S \setminus \{x\}$.
   Since $S = (S \setminus \{x\}) \cup \{x\}$ and S is infinite, the set $S \setminus \{x\}$ must be infinite. Choose a one-one function f from $\mathbf{Z}^+$ into $S \setminus \{x\}$. We now define

$$g(u) = \begin{cases} x & \text{if } u = f(1) \\ f(n) & \text{if } u = f(n+1) \text{ and } n \text{ is a positive integer} \\ u & \text{if } u \in S \setminus (\{x\} \cup f[\mathbf{Z}^+]) \end{cases}$$



   This function g is a one-one function from $S \setminus \{x\}$ onto S.

3. Combine the preceding exercise and an earlier exercise to conclude that if S is a given set and $x \in S$ then S is infinite if and only if $S \sim S \setminus \{x\}$.
   The desired statement follows at once.

4. Suppose that S is any set and that $\mathfrak{I}$ is the family of subsets E of S for which the set $S \setminus E$ is finite. Prove that the intersection of any two members of $\mathfrak{I}$ must belong to $\mathfrak{I}$.
   Suppose that A and B belong to $\mathfrak{I}$. Since the sets $S \setminus A$ and $S \setminus B$ are finite, so is the set

$$(S \setminus A) \cup (S \setminus B) = S \setminus (A \cap B)$$

   and we conclude that $A \cap B \in \mathfrak{I}$.

Exercises on Countability

1. Prove that if E is the set of irrational numbers then $\mathbf{R} \sim E$.
   Since the set $\mathbf{Q}$ of rational numbers is countable, the desired result follows at once from the theorem on removal of a countable subset from an uncountable set.

2. Suppose that A is a countable set and n is a positive integer. Suppose that B is the set of finite sequences $(x_1, x_2, \cdots, x_n)$ for which $x_j \in A$ for every $j \in \{1, \cdots, n\}$. Prove that the set B is countable.
   Solution: ***For each positive integer*** n ***we define*** B_n ***to be he set of finite sequences*** $(x_1, x_2, \cdots, x_n)$ ***for which*** $x_j \in A$ ***for every*** $j \in \{1, \cdots, n\}$***. We shall prove by mathematical induction that the set*** B_n ***is countable for every positive integer*** n***. Since*** $B_1 \sim A$ ***we know that*** B_1 ***is countable. Now given any positive integer*** n ***for which the set*** B_n ***is countable we deduce from the fact that***

$$B_{n+1} \sim B_n \times A$$

   and a an earlier theorem ***that*** B_{n+1} ***is countable.***

3. Justify the claim that was made in the proof of that a countable union of countable sets is countable that the function h defined there is one-one.
   For each positive integer n, we choose a one-one function from A_n into $\mathbf{Z}^+$ and we call this function f_n. Now given any

$$x \in \bigcup_{n=1}^{\infty} A_n,$$

   if n is the least positive integer for which $x \in A_n$, then we define

$$h(x) = (n, f_n(x)).$$

We are being asked to show that the function h defined in this way is one-one.
Suppose that x and t belong to $\bigcup_{n=1}^{\infty} A_n$ and that $h(t) = h(x)$. We define m to be the least positive integer for which $t \in A_m$ and x to be the least positive integer for which $x \in A_n$. We see that

$$(m, f_m(t)) = h(t) = h(x) = (n, f_n(x))$$

and so $m = n$ and $f_m(t) = f_m(x)$. Since the function f_m is one-one we conclude that $t = x$. Therefore h is one-one.

4. Suppose that n is a positive integer and that P_n is the set of all polynomials that have rational coefficients and whose degrees do not exceed n. Prove that the set P_n is countable.

   Hint: ***This exercise follows very simply from Exercise 2.***

5. Prove that if P is the set of all polynomials with rational coefficients then P is countable.

   Hint: ***Use the fact that if*** P_n ***is defined as in Exercise 4 for each*** n ***then***

$$\mathrm{P} = \bigcup_{n=1}^{\infty} \mathrm{P}_n.$$

6. As indicated earlier, a number x is said to be **algebraic** if there exists a nonzero polynomial f with integer coefficients such that $f(x) = 0$. Prove that the set of algebraic numbers is countable.

   Solution: ***The set*** F ***of polynomials with integer coefficients is countable by Exercise 5. Now for every member*** f ***of the set*** F ***we know that the set of solutions of the equation*** $f(x) = 0$ ***is finite; and therefore countable. Therefore the set***

$$\bigcup_{f \in F} \{x \in \mathbf{R} \mid f(x) = 0\}$$

   is countable.

7. A real number that is not algebraic is said to be **transcendental**. Complete the details of Cantor's proof outlined earlier that there exist transcendental numbers.
   The work has been done. The set of algebraic numbers is countable and the set $\mathbf{R}$ of all real numbers is uncountable. Therefore there must exist numbers that are not algebraic. In fact, we can do better. If T is the set of transcendental numbers then, since $\mathbf{R} \setminus T$ is countable we have $\mathbf{R} \sim T$.

8. Prove that the set of all functions from $\mathbf{Z}^+$ into $\{0,1\}$ is uncountable.
   The proof given here is modelled on the diagonal method given earlier, We define F to be the set of all functions from $\mathbf{Z}^+$ into $\{0,1\}$ and, to obtain a contradiction suppose that the members of F can be arranged in a sequence $f_1, f_2, f_3 \cdots$. For each n we define

$$g(n) = 1 - f_n(n).$$

   We see that g is a function from $\mathbf{Z}^+$ into $\{0,1\}$ and that g is not equal to any of the functions f_n. This is the desired contradiction.

9. Prove that if S is the set of all functions from $\mathbf{Z}^+$ into $\{0,1\}$ and $p(\mathbf{Z}^+)$ is the family of all subsets of $\mathbf{Z}^+$, then $S \sim p(\mathbf{Z}^+)$. Deduce that $p(\mathbf{Z}^+)$ is uncountable.

   Hint: ***For every subset*** E ***of*** Z^+ ***we define a function*** $\phi(E)$ ***from*** Z^+ ***to*** $\{0,1\}$ ***by the equation***

$$\phi(E)(x) = \begin{cases} 1 & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \in \mathbf{Z}^+ \setminus E \end{cases}$$

   Now show that the function ϕ ***defined in this way is a one-one function from*** $p(\mathbf{Z}^+)$ ***onto*** S.

Exercises on Subequivalence

1. Given sets A, B and C satisfying $A \sim\subseteq B$ and $B \sim\subseteq C$, prove that $A \sim\subseteq C$.
   Choose a one-one function f from A into B and a one-one function g from B into C. From Exercise 11 of the exercises on functions we see that $g \circ f$ is a one-one function from A into C.

2. Given that A is strictly subequivalent to B and that $B \sim\subseteq C$, explain why A must be strictly subequivalent to C.

   Solution: ***The assertion we want to prove is obvious if*** A ***is empty. Suppose now that*** $A \neq \emptyset$***. Using the fact that that*** $B \sim\subseteq C$ ***we choose a function*** f ***from*** C ***onto*** B***. Now to obtain a contradiction, suppose that*** A ***fails to be strictly subequivalent to*** C ***and choose a function*** g ***from*** A ***onto*** C***. The composition*** $f \circ g$ ***is a function from*** A ***onto*** B***, contradicting our assumption that*** A ***is strictly subequivalent to*** B***.***

3. Given that a set A is countable and that B is uncountable, explain why A is strictly subequivalent to B.
   Since B is an infinite set we have $\mathbf{Z}^+ \sim\subseteq B$. Since $A \sim\subseteq \mathbf{Z}^+ \sim\subseteq B$ we have $A \sim\subseteq B$. Since A is countable and B is uncountable it is impossible to have $A \sim B$.

4. Given that a set S is finite and that $x \in S$, explain why the set $S \setminus \{x\}$ must be strictly subequivalent to S.
   Since $S \setminus \{x\} \subseteq S$ we know that $S \setminus \{x\}$ is subequivalent to S. We also know from Exercise 2 of the exercises on finite sets that the sets S and $S \setminus \{x\}$ are not equivalent to each other.

5. Given that $A \sim\subseteq B$ prove that $p(A) \sim\subseteq p(B)$.

   Hint: ***Using the fact that*** $A \sim\subseteq B$ ***we choose a one-one function*** f ***from*** A ***to*** B***. We now define a function*** ϕ ***from*** $p(A)$ ***to*** $p(B)$ ***as follows:***
   $$\phi(E) = f[E]$$
   for every subset E ***of*** A***. Show that the function*** ϕ ***defined in this way is one-one.***
   To see that ϕ is one-one, suppose that E and F are subsets of A and that $\phi(E) = \phi(F)$, in other words $f[E] = f[F]$. Given any member x of the set E we know that $f(x) \in f[F]$. Choose $u \in F$ such that $f(x) = f(u)$. Since f is one-one we have $x = u$ and so $x \in F$. Therefore $E \subseteq F$ and we see similarly that $F \subseteq E$. We conclude that $E = F$.

6. Prove that $\mathbf{R} \sim\subseteq p(\mathbf{Q})$.

   Hint: ***Consider the function*** $f : R \to p(\mathbf{Q})$ ***defined by***
   $$f(x) = \{r \in \mathbf{Q} \mid r < x\}$$
   for every $x \in \mathbf{R}$.

7. Give an example of two nonempty sets A and B such that the set $A \cup B$ fails to be subequivalent to the set $A \times B$.
   The set $\{0\} \cup \{1\}$ fails to be subequivalent to $\{0\} \times \{1\}$.

8. Given that each of two given sets A and B has more than one member, prove that $A \cup B \sim\subseteq A \times B$.
   Choose two different members x and y of A and choose two different members u and v of B. We define
   $$A_1 = A \times \{u\}$$
   and
   $$B_1 = (\{x\} \times B) \cup \{(y,v)\} \setminus \{(x,u)\}$$
   Since $A \sim A_1$ and $B \sim B_1$ and since $A_1 \cap B_1 = \emptyset$ it is easy to see that $A \cup B \sim\subseteq A_1 \cup B_1$. As a matter of fact, if we define $f(t,u) = t$ for all $t \in A$ and define $f(x,t) = t$ for all $t \in B \setminus \{u\}$ and define $f(y,v) = u$ then f is a function from $A_1 \cup B_1$ onto $A \cup B$.
   Since $A_1 \cup B_1 \subseteq A \times B$ we conclude that $A \cup B \sim\subseteq A \times B$.

Exercises on The Equivalence Theorem

The exercises in this subsection explore some of the harder properties of set equivalence that can be deduced with the help of the equivalence theorem. We also need a little notation: Given any two sets A and B, the set A^B is defined to be the set of all functions from B to A. Thus

$$A^B = \{f \mid f : B \to A\}.$$

1. Prove that for any set A we have

$$\{0,1\}^A \sim p(A).$$

Hint: ***We define a function ϕ from $\{0,1\}^A$ to $p(A)$ as follows:***

$$\phi(f) = \{x \in A \mid f(x) = 0\}$$

for every member f of the set $\{0,1\}^A$. Prove that the function ϕ is one-one and onto the set $p(A)$.
To see that ϕ is one-one, suppose that f and g belong to $\{0,1\}^A$ and that $\phi(f) = \phi(g)$. In other words

$$\{x \in A \mid f(x) = 0\} = \{x \in A \mid g(x) = 0\}.$$

Given any $x \in A$, if $f(x) = 0$ then $g(x) = 0 = f(x)$ and if $f(x) \neq 0$ then $f(x) = 1$ and $g(x) \neq 0$ which gives us $g(x) = 1 = f(x)$. Therefore $f = g$.
Finally, to see that ϕ is onto the set $p(A)$, suppose that B is any subset of A. We define

$$f(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \in A \setminus B \end{cases}$$

and observe that $\phi(f) = B$.

2. Prove that if S is the set defined in the proof of this theorem then $S \sim \mathbf{R}$.

Solution: ***Since $S \subseteq \mathbf{R}$ we know that $S \sim\subseteq \mathbf{R}$. On the other hand,***

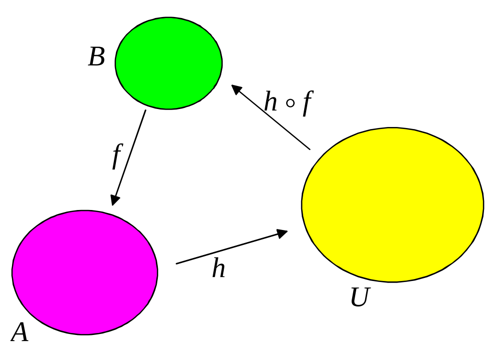
$$S \sim \{0,1\}^{\mathbf{Z}^+} \sim p(\mathbf{Z}^+) \sim \mathbf{R}$$

and so the result follows from the equivalence theorem.

3. Prove that if A and B are finite sets and A and m members and B has n members, then the set A^B has m^n members. (See also this exercise.)
This exercise is identical to the earlier one and the solution was given to the earlier one in the instructor's manual.

4. Given $A \sim\subseteq U$ and any set B, prove that $A^B \sim\subseteq U^B$.

Solution: ***Using the fact that $A \sim\subseteq U$ we choose a one-one function h from A to U. We now define a function ϕ from A^B to U^B as follows:***



For each member f of the set A^B we define $\phi(f)$ to be the function $h \circ f : B \to U$. In other words,

$$\phi(f)(x) = h(f(x))$$

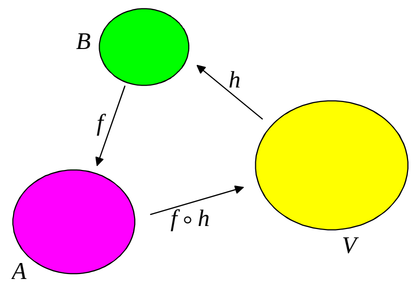
whenever $x \in B$. To see that ϕ is one-one, suppose that f_1 and f_2 are different members of the st A^B. Choose $x \in B$ such that $f_1(x) \neq f_2(x)$. Since h is one-one we have

$$h(f_1(x)) \neq h(f_2(x)),$$

in other words, $\phi(f_1)(x) \neq \phi(f_2)(x)$ ***and therefore the functions*** $\phi(f_1)$ ***and*** $\phi(f_2)$ ***are unequal to each other.***

5. Given that A and B are nonempty sets and that $B \sim\subseteq V$, prove that $A^B \sim\subseteq A^V$.

Solution: ***Using the fact that*** B ***is nonempty and*** $B \sim\subseteq V$ ***we choose a function*** h ***from*** V ***onto*** B***. We now define a function from*** A^B ***into*** A^V ***as follows:***



For each member f ***of the set*** A^B ***we define*** $\phi(f)$ ***to be the function*** $f \circ h$***. In other words***

$$\phi(f)(x) = f(h(x))$$

whenever $x \in V$***. To see that the function*** ϕ ***is one-one, suppose that*** f_1 ***and*** f_2 ***are different members of the set*** A^B ***and choose a member*** y ***of*** B ***such that*** $f_1(y) \neq f_2(y)$***. Using the fact that the function*** h ***is onto the set*** B***, choose a member*** x ***of*** V ***such that*** $h(x) = y$***. Thus***

$$\phi(f_1(x)) = f_1(h(x)) = f_1(y) \neq f_2(y) = f_2(h(x)) = \phi(f_2)(x)$$

and so the functions $\phi(f_1)$ ***and*** $\phi(f_2)$ ***are different from each other.***

6. Combine this exercise and this exercise to show that if $A \neq \emptyset$, $A \sim\subseteq U$ and $B \sim\subseteq V$, then $A^B \sim\subseteq U^V$.

7. Use this exercise and the equivalence theorem to give a quick proof that if $A \neq \emptyset$, $A \sim U$ and $B \sim V$, then $A^B \sim U^V$.

8. Given any sets A, B and C, prove that

$$\left(A^B\right)^C \sim A^{(B\times C)}.$$

Solution: ***We begin by observing that a typical member of the set*** $\left(A^B\right)^C$ ***is a function*** f ***from*** C ***to*** A^B***. For such a function*** f ***we know that, whenever*** $x \in C$***, the function*** $f(x)$ ***is a function from*** B ***to*** A***. Thus if*** $f \in \left(A^B\right)^C$ ***then for every member*** x ***of the set*** C ***and every member*** y ***of the set*** B ***we have***

$$f(x)(y) \in A.$$

Now we take a similar look at the set $A^{(B\times C)}$***. A typical member of this set is a function*** g ***from*** $B \times C$ ***into*** A***. Thus if*** $g \in A^{(B\times C)}$ ***then whenever*** $x \in C$ ***and*** $y \in B$ ***we have***

$$g(y,x) \in A.$$

With these thoughts in mind we define a function ϕ ***from*** $\left(A^B\right)^C$ ***to*** $A^{(B\times C)}$ ***as follows: Given any member*** f ***of the set*** $\left(A^B\right)^C$***, the function*** $\phi(f)$ ***is the function from*** $B \times C$ ***to*** A ***defined by***

$$\phi(f)(y,x) = f(x)(y)$$

whenever $x \in C$ ***and*** $y \in B$***.***
We need to show that the function ϕ ***is one-one and that the range of*** ϕ ***is the entire set*** $A^{(B\times C)}$***.***
To see that ϕ ***is one-one, suppose that*** f_1 ***and*** f_2 ***are different members of the set*** $\left(A^B\right)^C$***. Choose a member*** x ***of*** C ***such that*** $f_1(x) \neq f_2(x)$***. Using the fact that*** $f_1(x)$ ***and*** $f_2(x)$ ***are different functions from*** B ***to*** A ***we now choose a member*** y ***of*** B ***such that***

$$f_1(x)(y) \neq f_2(x)(y)$$

and we observe that

$$\phi(f_1)(y,x) \neq \phi(f_2)(y,x)$$

which tells us that $\phi(f_1) \neq \phi(f_2)$***.***
Finally, to see that ϕ ***is onto the set*** $A^{(B\times C)}$***, suppose that*** $g \in A^{(B\times C)}$***. We need to find a member*** f ***of the set*** $\left(A^B\right)^C$ ***such that*** $g = \phi(f)$***. Given any member*** x ***of the set*** C ***we define*** $f(x)$ ***to be the function from*** B ***to*** A

whose value at each member y of the set B is $g(y,x)$. The function f from C to A^B that we have defined clearly satisfies the condition $g = \phi(f)$.

9. Show that $\mathbf{R} \sim \{0,1\}^{\mathbf{Z}^+}$ and then use this exercise to show that $\mathbf{R}^{\mathbf{Z}^+} \sim \mathbf{R}$.

   Solution: ***We already know that***

$$\mathbf{R} \sim p(\mathbf{Z}^+) \sim \{0,1\}^{\mathbf{Z}^+}.$$

   Therefore

$$\mathbf{R}^{\mathbf{Z}^+} \sim \left(\{0,1\}^{\mathbf{Z}^+}\right)^{\mathbf{Z}^+} \sim \{0,1\}^{\mathbf{Z}^+\times\mathbf{Z}^+} \sim \{0,1\}^{\mathbf{Z}^+} \sim \mathbf{R},$$

10. Prove that $\mathbf{R} \times \mathbf{R} \sim \mathbf{R}$.

   Solution: ***We know that $\mathbf{R} \sim\subseteq \mathbf{R} \times \mathbf{R}$. On the other hand, if we define***

$$f(x,y) = (x,y,0,0,0,\cdots)$$

   whenever $(x,y) \in R \times R$ then we can see that $\mathbf{R} \times \mathbf{R} \sim\subseteq \mathbf{R}^{\mathbf{Z}^+} \sim \mathbf{R}$ and so it follows from the equivalence theorem that $R \times R \sim\subseteq R$.

11. Prove that $\mathbf{R}^3 \sim [0,1]$.

   We know that $\mathbf{R} \sim\subseteq \mathbf{R}^3$. On the other hand, if we define

$$f(x,y,z) = (x,y,z,0,0,\cdots)$$

   whenever $(x,y,z) \in \mathbf{R}^3$ then we can see that $\mathbf{R}^3 \sim\subseteq \mathbf{R}^{\mathbf{Z}^+} \sim \mathbf{R}$ and so it follows from the equivalence theorem that $\mathbf{R}^3 \sim\subseteq \mathbf{R}$.

12. Prove that $(p(\mathbf{R}))^{\mathbf{Z}^+} \sim (p(\mathbf{R}))^{\mathbf{R}} \sim p(\mathbf{R})$.

   Solution: ***On the one hand we know that $p(\mathbf{R}) \sim\subseteq (p(\mathbf{R}))^{\mathbf{Z}^+}$ and on the other other hand we know that***

$$(p(\mathbf{R}))^{\mathbf{Z}^+} \sim\subseteq (p(\mathbf{R}))^{\mathbf{R}} \sim \left(\{0,1\}^{\mathbf{R}}\right)^{\mathbf{R}} \sim \{0,1\}^{\mathbf{R}\times\mathbf{R}} \sim \{0,1\}^{\mathbf{R}} \sim p(\mathbf{R})$$

   and so the result follows from the equivalence theorem.

13. Prove that $\mathbf{R}^{\mathbf{R}} \sim p(\mathbf{R})$.

   On the one hand, $p(\mathbf{R}) \sim \{0,1\}^{\mathbf{R}} \sim\subseteq \mathbf{R}^{\mathbf{R}}$ and on the other hand

$$\mathbf{R}^{\mathbf{R}} \sim \left(\{0,1\}^{\mathbf{Z}^+}\right)^{\mathbf{R}} \sim \{0,1\}^{\mathbf{Z}^+\times\mathbf{R}} \sim\subseteq \{0,1\}^{\mathbf{R}\times\mathbf{R}} \sim \{0,1\}^{\mathbf{R}} \sim p(\mathbf{R})$$

   and so the result follows from the equivalence theorem.

14. Prove that if $A \sim\subseteq \mathbf{R}$ and A has more than one member then $A^{\mathbf{R}} \sim p(\mathbf{R})$.

   Since A has more than one member we have

$$p(\mathbf{R}) \sim \{0,1\}^{\mathbf{R}} \sim\subseteq A^{\mathbf{R}}.$$

   On the other hand,

$$A^{\mathbf{R}} \sim\subseteq \mathbf{R}^{\mathbf{R}} \sim p(\mathbf{R})$$

   and so the result follows from the equivalence theorem.

15. Given that S is a set with more than one member and that $S \times S \sim S$, prove that $S^S \sim (p(S))^S \sim p(S)$. We shall show in a later theorem that the condition $S \times S \sim S$ is satisfied by every infinite set.

   Since S has more than one member we have

$$p(S) \sim \{0,1\}^S \sim\subseteq S^S \sim\subseteq (p(S))^S.$$

   On the other hand,

$$(p(S))^S \sim \left(\{0,1\}^S\right)^S \sim \{0,1\}^{S\times S} \sim \{0,1\}^S \sim p(S)$$

   and so the result follows from the equivalence theorem.

16. Given that $S \times S \sim S$, that $A \sim\subseteq S$ and that A has more than one member, prove that $A^S \sim p(S)$.

   Since A has more than one member we have

$$p(S) \sim \{0,1\}^S \sim\subseteq A^S.$$

On the other hand,

$$A^S \sim\subseteq S^S \sim p(S)$$

by Exercise 15. So the result follows from the equivalence theorem.

17. The preceding exercises show that when $A \sim\subseteq B$ and the set A has more than one member and the set B satisfies the condition $B \times B \sim B$, then we have $A^B \sim p(B)$. Now what happens when B is the smaller set? In other words, what happens when B is strictly subequivalent to A? In this case, Exercises this one, this one, this one and this one seem to suggest that we should have $A^B \sim A$. In this exercise we see that this statement is false even if A is uncountable and B is countable:

    a. Suppose that (S_n) is a sequence of sets, that

    $$S = \bigcup_{n \in \mathbf{Z}^+} S_n$$

    and that, for each n, the set S_n is strictly subequivalent to S. Prove that the sets S and $S^{\mathbf{Z}^+}$ are not equivalent to each other. Show, in fact, that there is no function from S onto $S^{\mathbf{Z}^+}$.

    Solution: ***Suppose that f is any function from S to the set $S^{\mathbf{Z}^+}$. Given any member x of S we know that $f(x)$ is a function from $\mathbf{Z}^+$ to S which means that $f(x)$ is a sequence of members of S. For each positive integer n and each $x \in S$ we define $f_n(x)$ to be the nth member of the sequence $f(x)$. In other words, if $x \in S$ then $f(x)$ is the sequence***

    $$(f_1(x), f_2(x), f_3(x), \cdots).$$

    We need to show that the range of f must be a proper subset of $S^{\mathbf{Z}^+}$ and for this purpose we shall find a sequence $(x_1, x_2, x_3, \cdots)$ of members of S such that $(x_1, x_2, x_3, \cdots)$ is not in the range of f. For each positive integer n we use the fact that

    $$S \setminus f_n[S_n] \neq \emptyset$$

    to choose a member x_n of S such that

    $$x_n \in S \setminus f_n[S_n]$$

    Now given any member x of the set S it follows from the fact that

    $$S = \bigcup_{n \in \mathbf{Z}^+} S_n$$

    that for some n we have $x \in S_n$. For any such n we have $f_n(x) \neq x_n$ and therefore

    $$f(x) \neq (x_1, x_2, x_3, \cdots)$$

    and so the sequence $(x_1, x_2, x_3, \cdots)$ has the desired properties.

    b. Prove that if S is the gigantic set defined earlier then the sets S and $S^{\mathbf{Z}^+}$ are not equivalent.

18. Suppose that to each member i of a given set I there is associated a set S_i that is strictly subequivalent to a given set U. Prove that $\bigcup_{i \in I} S_i$ is strictly subequivalent to the set U^I.

    Solution: ***Suppose that f is a function from $\bigcup_{i \in I} S_i$ to U^I and for each $i \in I$ we define a function $f_i : \bigcup_{i \in I} S_i \to U$ by the equation***

    $$f_i(x) = f(x)(i)$$

    for every $x \in \bigcup_{i \in I} S_i$. For each i we use the fact that $f_i[S_i] \neq U$ to choose a member that we shall call $g(i)$ of the set U such that $g(i) \in U \setminus f[S_i]$. In this way we have defined a member g of U^I. To see that g does not lie in the range of f we observe that if $x \in \bigcup_{i \in I} S_i$ then for some i we have $x \in S_i$ and for any such i we have

    $$f(x)(i) \neq g(i).$$

Exercises on The Axiom of Choice

1. One of the assertions of an earlier theorem was that if A and B are given sets and if there exists a function g from B onto A then there must exist a one-one function from A into B. Rewrite the proof of this part of the theorem and show how and where the axiom of choice is used.

Solution: ***We assume that g is a function from B onto A. For every member x of the set A we define***

$$E_x = \{y \in B \mid g(y) = x\}.$$

Since the function g is onto A we know that all of the sets E_x are nonempty. The axiom of choice therefore guarantees the existence of a function f defined on A such that $f(x) \in E_x$ for every $x \in A$. We observe that for every member x of the set A we have $g(f(x)) = x$.
To see that the function f is one-one, suppose that x_1 and x_2 belong to A and that $f(x_1) = f(x_2)$. We see that

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2.$$

2. Suppose that I is a given set and that to each member $i \in I$ there is associated a nonempty set A_i of natural numbers. Explain why the axiom of choice does not have to be used to produce a choice function relative to this association.

Solution: ***We can provide a specific definition of a choice function in this example by defining $f(i)$ to be the least member of the set A_i for each $i \in I$.***

3. Suppose that to each member i of a given set I there is associated a nonempty finite set A_i. Do you think that the axiom of choice is needed to produce a choice function relative to this association?

Solution: ***Yes, the axiom of choice is needed here.***

4. Suppose that to each member i of a given set I there is associated a nonempty well ordered set A_i. Do you think that the axiom of choice is needed to produce a choice function relative to this association?

Solution: ***No, the axiom of choice is not needed. We can give a specific definition of a choice function f by defining $f(i)$ to be the least member of the set A_i for each $i \in I$.***

5. Given a finite set A, it is easy to give A a well order. Thus, if to each member i of a given set I there is associated a nonempty finite set A_i, then we can assign a well order to A_i for each i and then define $f(i)$ to be the least member of A_i for each i. In view of this fact, do you want to change your mind about the answer you gave for the above exercise?

Solution: ***No you do not want to change your mind. The fact that each set A_i can be given a well order is not the same as having each set A_i provided with a specific well order. In order to make use of the fact that each set A_i can be given a well order it is necessary to choose a well order of A_i for each i and this process requires the axiom of choice.***

6. Suppose that I is a given set and that for each $i \in I$ we have $A_i = [0,1]$. Do you think that the axiom of choice is needed to produce a choice function relative to this association?

7. Suppose that I is a given set and that for each $i \in I$ we have $A_i = \mathbf{R}$. Do you think that the axiom of choice is needed to produce a choice function relative to this association?

Solution: ***No the axiom of choice is not needed. We can obtain a choice function f very simply by defining $f(i) = 0$ for each $i \in I$.***

8. Given that A and B are nonempty sets, do we need the axiom of choice to guarantee that the set $A \times B$ is nonempty?

Solution: ***No, the axiom of choice is not needed. To show that $A \times B$ is nonempty, choose $x \in A$ and then choose $y \in B$. The ordered pair (x,y) must belong to $A \times B$.***

9. Using the axiom of choice, prove that there exists a subset S of $\mathbf{R}$ such that the following two conditions hold:

    a. Whenever x and y belong to S and $x \neq y$, the number $x - y$ is irrational.

    b. For every real number x there exists a member y of S such that the number $x - y$ is rational.

Solution: ***We define a relation ~ in R by saying that if x and y are any real numbers then the condition $x \sim y$ means that the number $x - y$ is rational. The relation ~ is an equivalence relation in R. We express the family of equivalence classes of this relation as $\{E_i \mid i \in I\}$. Using the axiom of choice we choose a choice function f relative to the association of i to E_i. In other words, $f(i) \in E_i$ for each i. We define $S = \{f(i) \mid i \in I\}$. Since every number must lie in precisely one equivalence class of the relation ~ we know that if $x \in \mathbf{R}$ then there is precisely one member $i \in I$ for which $x - f(i)$ is rational. Furthermore, if $i \neq j$ then the fact that $f(i)$ and $f(j)$ lie in different equivalence classes tells us that the number $f(i) - f(j)$ is irrational.***

10. Prove that if ~ is an equivalence relation in a set X then there exists a subset S of X such that the following two conditions hold:

    a. Whenever x and y belong to S and $x \neq y$, we have $\neg(x \sim y)$.

    b. For every $x \in X$ there exists a unique member y of S such that $x \sim y$.
    We use the axiom of choice to choose a set S that contains precisely one member of every equivalence class of the relation ~.

11. Given any real numbers x and y we define the statement $x \sim y$ to mean that there exists an integer n such that

$$x - y = n\sqrt{2}.$$

    a. Prove that ~ is an equivalence relation in **R**.
    The fact that ~ is an equivalence relation should be clear by now. There are many exercises just like this in the exercises on equivalance relations.

    b. Without using the axiom of choice, find a subset S of **R** the properties described in the above exercise.
    The interval $\left[0, \sqrt{2}\right)$ has the desired properties.

12. This exercise introduces the concept of the **product** of a family of sets. Suppose that I is a given set and that to each member i of I there is associated a given nonempty set A_i. The set of choice functions relative to this association is called the **product** of the indexed family of set A_i and is written as

$$\prod_{i \in I} A_i.$$

In other words, the set $\prod_{i \in I} A_i$ is the set of all functions h with domain I that satisfy the condition $h(i) \in A_i$ for every member i of the set I. The axiom of choice tells us that if I is a given set and that if to each member i of I there is associated a given nonempty set A_i then the product $\prod_{i \in I} A_i$ must be nonempty.
The purpose of this exercise is to provide an even stronger statement about the size of the product $\prod_{i \in I} A_i$.
This stronger statement is known as **König's inequality.**

    a. (König's inequality) Suppose that I is a given set and that to each member i of the set I are associated two sets A_i and B_i and suppose that for each $i \in I$ the set A_i is strictly subequivalent to the set B_i. Prove that the set $\bigcup_{i \in I} A_i$ is strictly subequivalent to the set $\prod_{i \in I} B_i$.

Solution: ***We need to show that there is no function from the set $\bigcup_{i \in I} A_i$ onto the set $\prod_{i \in I} B_i$. Suppose that***

$$f : \bigcup_{i \in I} A_i \to \prod_{i \in I} B_i.$$

We recall that if g is any member of the product then g is a function whose domain is I and for each $i \in I$ we have $g(i) \in B_i$. Given any member j of the set I we define the function π_j from $\prod_{i \in I} B_i$ to B_j by the equation

$$\pi_j(g) = g(j)$$

for every member g of the product $\prod_{i \in I} B_i$. We observe that whenever $j \in I$ the function $\pi_j \circ f$ is a function from $\bigcup_{i \in I} A_i$ into B_j. In fact, if $x \in \bigcup_{i \in I} A_i$ and $j \in I$ then

$$(\pi_j \circ f)(x) = \pi_j(f(x)) = f(x)(j) \in B_j.$$

For each j, since the set A_j is strictly subequivalent to B_j we know that the set $B_j \setminus (\pi_j \circ f)[A_j] \neq \emptyset$ and, using the axiom of choice, we choose a choice function ϕ whose domain is I and that satisfies the condition

$$\phi(j) \in B_j \setminus (\pi_j \circ f)[A_j]$$

for every member j of I. This function ϕ belongs to the product $\prod_{i \in I} B_i$. However, whenever $x \in \bigcup_{i \in I} A_i$ there must exist a member j of the set I such that $x \in A_j$ and for such j we have

$$f(x)(j) = (\pi_j \circ f)(x) \in (\pi_j \circ f)[A_j]$$

and the fact that

$$\phi(j) \in B_j \setminus (\pi_j \circ f)[A_j]$$

guarantees that $\phi \neq f(x)$. Therefore ϕ cannot belong to the range of f and we have shown that the function f is not onto the product $\prod_{i \in I} B_i$.

b. Explain why part a is a stronger form of this earlier exercise.
Suppose that S_i is strictly subequivalent to U for each $i \in I$. König's inequality tells us that $\bigcup_{i \in I} S_i$ is strictly subequivalent to the product $\prod_{i \in I} U$. Now all we have to observe is that

$$\prod_{i \in I} U = U^I.$$

c. Suppose that S is a given set and that to each member x of the set S are associated the sets $A_x = \{x\}$ and $B_x = \{0,1\}$. Explain how König's inequality can be applied to this association to show that S is strictly subequivalent to the set $\{0,1\}^S$ and deduce that Cantor's inequality is a special case of König's inequality.
Since

$$\bigcup_{x \in S} \{x\} = S$$

and since $\{x\}$ is strictly subequivalent to $\{0,1\}$ whenever $x \in S$ we deduce from König's inequality that S is strictly subequivalent to $\{0,1\}^S$ and we already know that $\{0,1\}^S \sim p(S)$.

Exercises on Well Orders

1. Prove that the lexicographic order $\preceq$ in $\{1,2\} \times \mathbf{Z}^+$ that was defined earlier is a well order of $\{1,2\} \times \mathbf{Z}^+$.
The desired result follows at once from Exercise 3.

2. Prove that if $\preceq$ is lexicographic order from the left in the set $\{1,2,3\} \times \mathbf{Z}^+$ then $\preceq$ is a well order of $\{1,2,3\} \times \mathbf{Z}^+$.
The desired result follows at once from Exercise 3.

3. Prove that if $\leq$ is a well order in a set S and $\preceq$ is lexicographic order from the left in the set $S \times S$, then $\preceq$ is a well order of S.
Solution: ***We leave it as an exercise to show that the relation $\preceq$ is a total order in $S \times S$. We shall now show that every nonempty subset of $S \times S$ has a member that is least with respect to the order $\preceq$. Suppose that E is a nonempty subset of $S \times S$. We define***

$$P = \left\{x \in S \mid \text{There is at least one member } y \text{ of } S \text{ such that } (x,y) \in E\right\}.$$

Since the set P is nonempty, it must have a least member that we shall call u. We now define v to be the least member of the set

$$\{y \in S \mid (u,y) \in E\}.$$

Now write a simple but careful explanation of why the pair (u,v) must be the member of $S \times S$ that is least with respect to the order $\preceq$.

4. Given that a and b belong to a well ordered set S and that $a < b$, prove that $a^+ < b^+$.
The desired result follows at once from the fact that $a^+ \leq b$ and $b < b^+$.

5. Given that a and b belong to a well ordered set S and that $a^+ = b^+$, prove that $a = b$.
From Exercise 4 we know that neither of the conditions $a < b$ or $b < a$ can hold.

6. Prove that if a belongs to a well ordered set then $a = (a^+)^-$.
We know that $a < a^+$ and that no member of the set can lie between a and a^+. Therefore a is the predecessor of a^+.

7. Prove that if a member a of a well ordered set has a predecessor then $a = (a^-)^+$.
Suppose that a has a predecessor. We know that $a^- < a$ and that no member of the set can lie between a^- and a. Therefore a is the successor of a^-.

8. Prove that if a belongs to a well ordered set S then a fails to be a successor if and only if
$$\forall x \in S(x < a \Rightarrow x^+ < a).$$
Suppose that a belongs to a well ordered set S.
Suppose that a is not a successor. We know that whenever $x < a$ we have $x^+ \leq a$ and since the equation $x^+ = a$ does not occur we must have $x^+ < a$ whenever $x < a$.
Now suppose that $x^+ < a$ whenever $x < a$. This condition tells us that there can't exist a member x of S for which $x^+ = a$ and so a fails to be a successor.

9. If S is a well ordered set and $a \in S$, then we saw that a is a **limit member** of S if a is not the least member of S and a has no predecessor. Prove that if a well ordered set S has a limit member then S has a nonempty subset E that has no limit member and no largest member.
Suppose that a well ordered set S has a limit member. We define q to be the least limit member of S and we define $E = \{x \in S \mid x < q\}$. We see at once that E is nonempty and has no limit member. Now given any member x of E, it follows from Exercise 8 and the fact that $x < q$ that $x^+ \in E$ and so E has no largest member.

10. Give an example of a totally ordered set S that is not well ordered even though every member of S has a successor.
The system $\mathbf{Z}$ of integers has the desired properties.

Exercises on Zorn's Lemma

1. a. A set S of real numbers is said to be **linearly independent** over the set $\mathbf{Q}$ of rational numbers if for every positive integer n and every choice of members $x_1, \cdots, x_n$ of the set S and rational numbers $r_1, \cdots, r_n$, the condition
$$r_1x_1 + \cdots + r_nx_n = 0$$
cannot hold unless all of the numbers $r_1, \cdots, r_n$ are zero. Prove that if $\mathfrak{I}$ is the family of subsets of $\mathbf{R}$ that are linearly independent over $\mathbf{Q}$ and $\mathfrak{I}$, is partially ordered by the relation $\subseteq$, then $\mathfrak{I}$ has a maximal member.
We need to show that every chain in the partially ordered set $\mathfrak{I}$ has an upper bound. Suppose that $\mathfrak{C}$ is a chain in $\mathfrak{I}$. Each member of $\mathfrak{C}$ is linearly independent over the set $\mathbf{Q}$ of rational numbers and if A and B are any two members of $\mathfrak{C}$ then either $A \subseteq B$ or $B \subseteq A$. We define $U = \bigcup\mathfrak{I}$ and we observe that U is a set of numbers and $A \subseteq U$ whenever $A \in \mathfrak{C}$. All we need to do is show that U is linearly independent.
Suppose that n is a positive integer and that $x_1, x_2, \cdots, x_n$ belong to $\mathfrak{C}$. We see easily that some member A of $\mathfrak{C}$ contains all of the numbers $x_1, x_2, \cdots, x_n$. Since A is linearly independent, no linear combination
$$r_1x_1 + \cdots + r_nx_n$$
with rational coefficients can be zero unless all of the coefficients are zero. This completes the proof that $U \in \mathfrak{I}$ and so the chain $\mathfrak{C}$ has an upper bound in $\mathfrak{I}$.

b. Prove that if H is a maximal member of the family $\Im$ that was defined in part a then every real number can be expressed uniquely in the form $r_1 x_1 + \cdots + r_n x_n$.

2. An **additive subgroup** of the system $\mathbf{R}$ of real numbers is defined to be a nonempty subset G of $\mathbf{R}$ with the property that whenever x and y belong to G, then so do the numbers $x+y$ and $x-y$.

a. Which of the following subsets of $\mathbf{R}$ are additive subgroups of $\mathbf{R}$?
These simple exercises have appeared before.

i. The set $\{0\}$.

ii. The set $\{1\}$.

iii. The set $\{-1, 0, 1\}$.

iv. The set $\mathbf{Z}$ of all integers.

v. The set $\mathbf{Q}$ of all rational numbers.

vi. The set $\mathbf{R} \setminus \mathbf{Q}$ of all irrational numbers.

vii. The empty set $\emptyset$.

b. Given that G is an additive subgroup of $\mathbf{R}$ and that $c \in \mathbf{R} \setminus G$, prove that there exists an additive subgroup S of $\mathbf{R}$ such that $G \subseteq S$ and $c \in \mathbf{R} \setminus S$ and such that every additive subgroup of $\mathbf{R}$ that includes S must also contain the number c.
We look at the family $\Im$ of all additive subgroups of $\mathbf{R}$ that fail to contain the number c and give it the partial order $\subseteq$. Since the union of all the groups in any chain in $\Im$ must belong to $\Im$, each chain in $\Im$ has an upper bound in $\Im$.

c. Given that S is an additive subgroup of $\mathbf{R}$ with the properties just described and that $y \in \mathbf{R} \setminus S$ then it possible to find a member x of S and an integer n such that $ny = x + c$.
The set $\{ny + x \mid n \in \mathbf{Z} \text{ and } x \in S\}$ is clearly a group that properly includes S and therefore

$$c \in \{ny + x \mid n \in \mathbf{Z} \text{ and } x \in S\}$$

3. A nonempty family $\Im$ of nonempty subsets of a set S is said to be a **filter** in S if the intersection of any two members of $\Im$ belongs to $\Im$ and any subset of S that includes a member of $\Im$ must belong to $\Im$. A filter that is not properly included in any other filter is called an **ultrafilter**.

a. Prove that every filter in a set S is included in an ultrafilter.
If we give the family of all filters in S the partial order $\subseteq$ then it is clear that the union of any chain of filters is a filter and so every chain has an upper bound.

b. Given that $\mathfrak{C}$ is a family of subsets of a set S and that the intersection of any finite number of members of $\mathfrak{C}$ is nonempty, prove that $\mathfrak{C}$ is included in an ultrafilter in S.
We define $\Im$ to be the family of all those subsets of S that include the intersection of finitely many members of $\mathfrak{C}$. Since $\mathfrak{C} \subseteq \Im$ and $\Im$ is a filter, the desired result follow from part a.

c. Suppose that $\Im$ is a filter in a set S and that for every subset E of S, either $E \in \Im$ or $S \setminus E \in \Im$. Prove that $\Im$ must be an ultrafilter in S.
Whenever a subset E of S fails to belong to $\Im$ we know that E is disjoint from the member $S \setminus E$ of $\Im$ and that, consequently, E can't belong to any filter that includes $\Im$.

d. Prove that if $\Im$ is an ultrafilter in a set S and $E \subseteq S$, then either $E \in \Im$ or $S \setminus E \in \Im$.
Suppose that $\Im$ is a filter in a set S and that $E \subseteq S$ and that neither of the sets E and $S \setminus E$ belong to $\Im$. Then, since neither of the sets E and $S \setminus E$ can include a member of $\Im$, neither of these sets can be disjoint from a member of $\Im$ and it is clear that the family of all those subsets of S that include an intersection $E \cap F$ with $F \in \Im$ must be a filter that properly includes $\Im$. Therefore if $\Im$ is an ultrafilter, one of the sets E and $S \setminus E$ must belong to $\Im$.

e. Prove that there is an ultrafilter $\Im$ in $\mathbf{Z}^+$ such that for every positive integer n we have

$$\{m \in \mathbf{Z}^+ \mid m \geq n\} \in \mathfrak{I}.$$

Prove that if for each $E \subseteq \mathbf{Z}^+$ we define

$$\varphi(E) = \begin{cases} 1 & \text{if } E \in \mathfrak{I} \\ 0 & \text{if } E \notin \mathfrak{I} \end{cases}$$

then $\varphi(\mathbf{Z}^+) = 1,$ $\varphi(E) = 0$ whenever E is a finite set, and

$$\varphi(E_1 \cup E_2) = \varphi(E_1) + \varphi(E_2)$$

whenever the sets E_1 and E_2 are disjoint from each other.
The existence of an ultrafilter that contains every set of the form $\{m \in \mathbf{Z}^+ \mid m \geq n\}$ with n a positive integer follows from part b.
The desired properties of the function ϕ follow at once because, if E_1 and E_2 are subsets of $\mathbf{Z}^+$ and $E_1 \cap E_2 = \emptyset$ then the condition $E_1 \cup E_2 \in \mathfrak{I}$ will hold if and only if exactly one of the sets E_1 and E_2 belongs to $\mathfrak{I}$.

Exercises on Choice Dependent Properties of Sets

1. Given that A and B are well ordered sets and that every initial segment of A is strictly subequivalent to B and every initial segment of B is strictly subequivalent to $A,$ prove that A and B are order isomorphic.
 We know that if A and B are not order isomorphic to one another then either A is order isomorphic to an initial segment of B or B is order isomorphic to an initial segment of A. But both of these options are impossible because no initial segment of B can be equivalent to A and so initial segment of A can be equivalent to B.

2. Given that S is an infinite set and that A is strictly subequivalent to $S,$ prove that $S \setminus A \sim S.$
 We observe first that if $S \setminus A$ were subequivalent to A then it would follow from the preceding theorem on unions that $S = A \cup (S \setminus A) \sim A,$ which we know to be false. It therefore follows from the theorem on comparision of sets that $A \sim\subseteq S \setminus A$ and we can apply the union theorem again to yield

$$S = A \cup (S \setminus A) \sim S \setminus A.$$

3. Suppose that (S_n) is a sequence of sets, that for each $n,$ the set S_n is strictly subequivalent to $S_{n+1},$ and that

$$S = \bigcup_{n \in \mathbf{Z}^+} S_n.$$

 Prove that there does not exist a set E such that $p(E) \sim S.$
 This exercise is related to an earlier exercise on applications of the equivalence theorem. If we had $S \sim p(E)$ then we would have

$$S^{\mathbf{Z}^+} \sim \left(\{0,1\}^E\right)^{\mathbf{Z}^+} \sim \{0,1\}^{E \times \mathbf{Z}^+}.$$

 Now since the set S is infinite, so is the set E and so $\mathbf{Z}^+ \sim\subseteq E$ and the theorem on products gives us

$$S^{\mathbf{Z}^+} \sim \left(\{0,1\}^E\right)^{\mathbf{Z}^+} \sim \{0,1\}^{E \times \mathbf{Z}^+} \sim \{0,1\}^E \sim S$$

 which we know to be false.

4. Prove that there is a well ordered set S such that $\aleph_1$ is strictly subequivalent to S and such that for every $x \in S,$ we have $P(S,x) \sim\subseteq \aleph_1.$ Prove that any set of this type is cardinally ordered and that any two sets of this type are order isomorphic.
 Choose a set T such that $\aleph_1$is strictly subequivalent to T. For example, we chould take $T = p(\aleph_1)$. We now assign a well order to the set T. If for every member x of T we have $P(S,x) \sim\subseteq \aleph_1$ then we take $S = T$. Otherwise we define y to be the least member of T for which the segment $P(S,y)$ fails to be subequivalent to $\aleph_1$ and we define $S = P(S,y)$. It is clear that S has the desired properties.
 Finally, if S_1 and S_2 are two sets with the specified properties then, neither of the sets can be order isomorphic with an initial segment of the other, it follows from the uniqueness theorem for well ordered sets that S_1 and S_2 are order isomorphic to one another.

5. Prove that if we choose any one set of the type described in the preceding exercise and call it $\aleph_2$ then for any set S, the set $\aleph_1$ will be strictly subequivalent to S if and only if $\aleph_2 \sim\subseteq S$.
It is clear that if $\aleph_2 \sim\subseteq S$ then $\aleph_1$ must be strictly subequivalent to S.
Now suppose that $\aleph_1$ is strictly subequivalent to S. We assign a well order to S. Since S cannot be order isomorphic to any initial segment in $\aleph_2$ we conclude that either $\aleph_2$ is order isomorphic to S or $\aleph_2$ is order isomorphic to an initial segment of S. In either case we have $\aleph_2 \sim\subseteq S$.

6. Assuming the continuum hypothesis, prove that there is a subset S of the unit square $[0,1] \times [0,1]$ such that for every horizontal line L in the square, the set $L \cap S$ is countable and for every vertical line L in the square, the set $L \setminus S$ is countable.
We assign a well order $\preceq$ to the set $[0,1]$ in such a way that $[0,1]$ is order isomorphic to $\aleph_1$. We observe that every initial segment of $[0,1]$ relative to the order $\preceq$ must be countable. We now define
$$S = \{(x,y) \in [0,1] \times [0,1] \mid x \preceq y\}$$
and observe that S has the desired properties.

7. Suppose that A and B are uncountable sets and that B is strictly subequivalent to A. By a *horizontal line* in $A \times B$ we mean a set of the form $\{(x,y) \mid x \in A\}$ where y is any member of B. Similarly, a *vertical line* in $A \times B$ is a set of the form $\{(x,y) \mid y \in B\}$ where x is any member of A. Prove that if S is any subset of $A \times B$ whose intersection with every horizontal line is subequivalent to B then there is a vertical line L such that $S \cap L = \emptyset$.
For each $y \in B$ we define
$$S_y = \{x \in A \mid (x,y) \in S\}.$$
Since each set S_y is subequivalent to B we deduce from the theorem on unions that
$$\bigcup_{y \in B} S_y \sim\subseteq B$$
and therefore
$$A \setminus \bigcup_{y \in B} S_y \neq \emptyset.$$
If we choose $x \in A \setminus \bigcup_{y \in B} S_y$ then the vertical line $\{(x,y) \mid y \in B\}$ is disjoint from S.

8. Suppose that A is an infinite set. Prove that there is a subset S of $A \times A$ such that for every horizontal line L in $A \times A$ the set $L \cap S$ is strictly subequivalent to A and for every vertical line L in $A \times A$, the set $L \cap S$ is equivalent to A.
We assign a cardinal order $\leq$ to the set A and we define
$$S = \{(x,y) \in A \times A \mid x \leq y\}.$$
It is clear that if L is any horizontal line than $L \cap S$ is strictly subequivalent to A and it follows from Exercise 2 that if L is any vertical line then $L \cap S \sim A$.

9. A subset E of $\aleph_1$ is said to be *closed* if for every $x \in \aleph_1 \setminus E$, there is a member y of $\aleph_1$ such that $y < x$ and no member of E lies between y and x.

   a. Prove that if A and B are closed uncountable subsets of $\aleph_1$ then $A \cap B$ is a closed uncountable subset of $\aleph_1$.
   As we know, a subset of $\aleph_1$ is uncountable if and only if it has an upper bound. Suppose that A and B are closed uncountable subsets of $\aleph_1$. In order to prove that $A \cap B$ is uncountable we shall show that this set has no upper bound. Suppose that $y \in \aleph_1$. We shall find a member of $A \cap B$ such that $y < x$.
   We begin by defining x_1 be the least member of A that is greater than y. More precisely, we define x_1 to be the least member of the set $\{t \in A \mid y < t\}$. In order to make this definition we have made use of the fact that the set $\{t \in A \mid y < t\}$ is nonempty; which we know because A has no upper bound.
   We define x_2 to be the least member of B that is greater than x_1. We define x_3 to be the least member of A greater than x_2 and we continue in this fashion.
   For each positive integer n we have $x_n < x_{n+1}$ and $x_n \in A$ when n is odd and $x_n \in B$ when n is

even. Since the set $\{x_n \mid n \in \mathbf{Z}^+\}$ is countable, it has upper bounds. We define x to be the least upper bound of the set $\{x_n \mid n \in \mathbf{Z}^+\}$. Given any member u of $\aleph_1$ such that $u < x$ we know that u fails to be an upper bound of the set $\{x_n \mid n \in \mathbf{Z}^+\}$ and so the inequality $u < x_n$ holds for some n. Therefore, whenever $u < x$ there are members of A and also members of B between u and x and it follows from the fact that A and B are closed that $x \in A \cap B$.
Finally we need to observe that $A \cap B$ is closed. Suppose that $x \in \aleph_1 \setminus (A \cap B)$. We know that either $x \in \aleph_1 \setminus A$ or $x \in \aleph_1 \setminus B$. We assume, without loss of generality that $x \in \aleph_1 \setminus A$. Choose $u < x$ such that no member of A lies between u and x. Of course, no member of $A \cap B$ can lie between u and x.

b. Prove that if A_n is a closed uncountable subset of $\aleph_1$ for every positive integer n, then the set $\bigcap_{n=1}^{\infty} A_n$ is a closed uncountable subset of $\aleph_1$.
The proof of this stronger assertion is similar to the one given in part a. However, in order to show that $\bigcap_{n=1}^{\infty} A_n$ has no upper bound we start with an arbitrary member y of $\aleph_1$ and we define the members x_n as follows:
x_1 is the least member of A_1 greater than y.
x_2 is the least member of A_2 greater than x_1.
x_3 is the least member of A_1 greater than x_2.
x_4 is the least member of A_2 greater than x_3.
x_5 is the least member of A_3 greater than x_4.
x_6 is the least member of A_1 greater than x_5.
x_7 is the least member of A_2 greater than x_8.
x_8 is the least member of A_3 greater than x_7.
x_9 is the least member of A_4 greater than x_8.
x_{10} is the least member of A_1 greater than x_9.
The sets A_n appear in this sequence as

$$A_1, A_2, A_1, A_2, A_3, A_1, A_2, A_3, A_4, A_1, A_2, A_3, A_4, A_5, A_1, A_2, A_3, A_4, A_5, A_6, A_1, \cdots$$

The purpose of this arrangement of the sets is that, for each positive integer k, the condition $x_n \in A_k$ holds for infinitely many values of n.
A less intuitive but more constructive way of arranging the sets A_k is make $x_n \in A_k$ whenever n can be expressed in the form $2^k 3^m$ for some positive integer m and making $x_n \in A_1$ whenever n can't be expressed in the form $2^k 3^m$.

5 The Real Number System

Some Exercises on Surds

1. Given that a, b and c are surds, that $a \neq 0$ and that $b^2 - 4ac \geq 0$, explain why the solutions of the equation

$$ax^2 + bx + c = 0$$

are surds.
This fact follows at once from the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the solutions of the equation $ax^2 + bx + c = 0$.

2. A detailed discussion of the material of this exercise can be found in the document on cubic equations. Please refer to that discussion for a detailed solution to this exercise.

a. Verify by direct multiplication that for any numbers u, v and x,

$$(u + v + x)(u^2 + v^2 + x^2 - uv - ux - vx) = u^3 + v^3 + x^3 - 3uvx.$$

b. Given numbers x, a and b, show that the expression $x^3 + ax + b$ can be written in the form

$$x^3 + ax + b = x^3 + u^3 + v^3 - 3uvx$$

by solving the simultaneous equations

$$3uv = -a$$
$$u^3 + v^3 = b.$$

Now show that as long as $27b^2 + 4a^3 \geq 0$ these equations can be solved giving

$$u = \sqrt[3]{\frac{3\sqrt{3}\,b + \sqrt{27b^2 + 4a^3}}{6\sqrt{3}}}$$

and

$$v = \sqrt[3]{\frac{3\sqrt{3}\,b - \sqrt{27b^2 + 4a^3}}{6\sqrt{3}}}.$$

c. Deduce that if a and b are surds and $27b^2 + 4a^3 \geq 0$ then the solutions of the cubic equation

$$x^3 + ax + b = 0$$

are also surds. Why doesn't this fact contradict the claim made earlier that the solutions of the equation

$$8x^3 - 6x - 1 = 0$$

are not surds?

d. How many real solutions does the equation

$$x^3 + ax + b = 0$$

have in the case $27b^2 + 4a^3 > 0$? What if $27b^2 + 4a^3 = 0$?

3. Use *Scientific Notebook* to find the exact form of the solutions of the equation

$$125x^3 - 300x^2 + 195x - 28 + 4\sqrt{7} = 0$$

and show graphically that only one of these solutions is real.

Exercises on Inequalities

1. Prove that if x and y are positive real numbers then their product xy is positive. We assume that $x > 0$ and $y > 0$. Since $0 < x$ and $y > 0$ it follows from the order axiom for the real number system that

$$0y < xy$$

and the fact that $0y = 0$ allows us to deduce that $0 < xy$.

2. Prove that if x and y are negative real numbers then their product xy is positive.
If x and y are negative then $-x$ and $-y$ are positive and since

$$xy = (-x)(-y)$$

the fact that xy is positive follows from Exercise 1.

3. Given real numbers a and b, prove that

$$|a| - |b| \leq |a - b|.$$

Since

$$a = b + (a - b)$$

it follows from the triangle inequality that

$$
\begin{aligned}
|a| &= |b + (a - b)| \\
&\le |b| + |a - b|
\end{aligned}
$$

and therefore

$$|a| - |b| \le |a - b|.$$

4. Given real numbers a and b, prove that

$$||a| - |b|| \le |a - b|.$$

From Exercise 3 we know that

$$|a| - |b| \le |a - b|.$$

and

$$|b| - |a| \le |b - a| = |a - b|$$

Since $||a| - |b||$ is one of the numbers $|a| - |b|$ and $|b| - |a|$ the inequality

$$||a| - |b|| \le |a - b|$$

follows at once.

5. In each of the following cases, find the numbers x for which the given inequality is true. Compare your answers with the answers given by *Scientific Notebook*

a. $|2x - 3| \le |6 - x|$.

Method 1: We separate the problem into three cases as illustrated in the figure:

$\frac{3}{2}$ $\qquad\qquad$ 6

Case 1: When $x \le \frac{3}{2}$ the inequality $|2x - 3| \le |6 - x|$ says that

$$3 - 2x \le 6 - x$$

which tells us that $x \ge -3$.

Case 2: When $\frac{3}{2} < x \le 6$ the inequality $|2x - 3| \le |6 - x|$ says that

$$2x - 3 \le 6 - x$$

which tells us that $x \le 3$.

Case 3: When $x > 6$ the inequality $|2x - 3| \le |6 - x|$ says that

$$2x - 3 \le x - 6$$

which tells us that $x \le -3$ (which is impossible). The set of numbers x for which the inequality $|2x - 3| \le |6 - x|$ holds is therefore

$$\left[-3, \frac{3}{2}\right] \cup \left(\frac{3}{2}, 3\right] = [-3, 3].$$

Method 2: We look first at the equation

$$|2x - 3| = |6 - x|$$

which says that either $2x - 3 = 6 - x$ or $2x - 3 = x - 6$. The latter condition says that either $x = 3$ or $x = -3$ and so we separate the problem into the cases indicated by the next figure

-3 $\qquad\qquad$ 3

This method leads us even more quickly to the solutions set $[-3, 3]$. We omit the details.

b. $||x| - 5| < |x - 6|$.

We provide one solution here. Another approach is suggested by the solution provided below for part c. We begin by looking at the equation

$$||x| - 5| = |x - 6|$$

which says that either $|x| - 5 = x - 6$ or $|x| - 5 = 6 - x$. When $x \ge 0$ these equations say that either $x - 5 = x - 6$ (which is impossible) or $x - 5 = 6 - x$, which tells us that $x = \frac{11}{2}$.

When $x < 0$ the equations say that either $-x-5 = x-6$ or $-x-5 = 6-x$ which are both impossible. Therefore the only value of x at which the inequality $||x|-5| < |x-6|$ can switch from true to false or from false to true is $\frac{11}{2}$ and by looking at a specimen value of x less than $\frac{11}{2}$ and a specimen value greater then $\frac{11}{2}$ we see that the inequality holds if and only if $x < \frac{11}{2}$.

c. $|2|x|-5| \leq |4-|x-1||$.

Solution: ***The inequality***

$$|2|x|-5| \leq |4-|x-1||$$

can be written as

$$-(4-|x-1|) \leq 2|x|-5 \leq 4-|x-1|$$

which says that

$$1+|x-1| \leq 2|x| \leq 9-|x-1|$$

0 1

In order to express this inequality without any absolute value signes we shall look separately at the cases $x < 0$ ***and*** $x \geq 0$ ***and also the cases*** $x < 1$ ***and*** $x \geq 1$***.***

When $x < 0$***, the inequality***

$$1+|x-1| \leq 2|x| \leq 9-|x-1|$$

becomes

$$1+1-x \leq -2x \leq 9-(1-x)$$

which says that $x \leq -2$ ***and*** $-8/3 \leq x$***. In other words, when*** $x < 0$***, we must have***

$$-\frac{8}{3} \leq x \leq -2.$$

When $0 \leq x \leq 1$***, the inequality***

$$1+|x-1| \leq 2|x| \leq 9-|x-1|$$

becomes

$$1+1-x \leq 2x \leq 9-(1-x)$$

which says that $x \geq 2/3$ ***and*** $x \leq 8$***. Therefore the required inequality holds when***

$$\frac{2}{3} \leq x \leq 1.$$

When $x \geq 1$***, the inequality***

$$1+|x-1| \leq 2|x| \leq 9-|x-1|$$

becomes

$$1+x-1 \leq 2x \leq 9-(x-1)$$

which says that $x \geq 0$ ***and*** $x \leq 10/3$***. In other words***

$$1 \leq x \leq \frac{10}{3}.$$

Thus the solution of the required inequality is the set

$$\left[-\frac{8}{3},-2\right] \cup \left[\frac{2}{3},1\right] \cup \left[1,\frac{10}{3}\right] = \left[-\frac{8}{3},-2\right] \cup \left[\frac{2}{3},\frac{10}{3}\right]$$

6. Prove that if a,b,c,x,y and z are any real numbers then

$$(ax+by+cz)^2 \leq (a^2+b^2+c^2)(x^2+y^2+z^2).$$

One way to produce this inequality is to oberve that

$$\begin{aligned}&(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2\\ &= a^2y^2+a^2z^2+b^2x^2+b^2z^2+c^2x^2+c^2y^2-2axby-2axcz-2bycz\\ &= (ay-bx)^2+(az-cx)^2+(bz-cy)^2 \geq 0\end{aligned}$$

There are several other possible approaches.

7. Given that a, b and c are positive numbers and that $c < a+b$, prove that
$$\frac{c}{1+c} < \frac{a}{1+a} + \frac{b}{1+b}.$$
We observe that
$$\begin{aligned}\frac{a}{1+a}+\frac{b}{1+b}-\frac{c}{1+c} &= \frac{a(1+b)(1+c)+b(1+a)(1+c)-c(1+a)(1+b)}{(1+a)(1+b)(1+c)}\\ &= \frac{2ab+acb+a+b-c}{(1+a)(1+b)(1+c)} > \frac{2ab+acb+0}{(1+a)(1+b)(1+c)} > 0.\end{aligned}$$

Exercises on Integers and Rational Numbers

1. a. Explain why the numbers $\frac{3}{2}+\frac{2}{7}$ and $\frac{3}{2}\times\frac{2}{7}$ are both rational.
These facts become obvious when we simplify:
$$\frac{3}{2}+\frac{2}{7} = \frac{25}{14} \quad \text{and} \quad \frac{3}{2}\times\frac{2}{7} = \frac{3}{7}.$$

b. Explain why the number 2.345 is rational.
We need only observe that
$$2.345 = \frac{2345}{1000}.$$

c. Explain why the sum, difference, product and quotient of rational numbers must always be rational.

2. For each of the following statements, say whether the statement is true or false and justify your assertion:

a. If x is rational and y is irrational then $x+y$ is irrational.
This statement is true. To see why we should observe that an equivalent way of making this assertion is to say that if x is rational and $x+y$ is rational then y must be rational; a statement that follows at once from the equation
$$y = (x+y)-x.$$

b. If x is rational and y is irrational then xy is irrational.
As long as $x \neq 0$ we can use the approach used in part a to prove this assertion from the equation
$$y = \left(\frac{y}{x}\right)x.$$
However, if $x=0$ then $xy=0$ for every y and so it would be false to say that y must be rational.

c. If x is irrational and y is irrational then $x+y$ is irrational.
This statement is false. Observe that
$$\left(3+\sqrt{2}\right)+\left(2-\sqrt{2}\right) = 5.$$

Exercises on Upper And Lower Bounds

1. Suppose that A is a nonempty bounded set of real numbers that has no largest member and that $a \in A$. Explain why the sets A and $A \setminus \{a\}$ have exactly the same upper bounds.

Solution: ***Since the set $A \setminus \{a\}$ is a subset of A, it is clear that any upper bound of A must be an upper bound of $A \setminus \{a\}$. Now, to show that any upper bound of $A \setminus \{a\}$ must be an upper bound of A, suppose that x is an upper bound of $A \setminus \{a\}$. Using the fact that a is not the largest member of A, choose a member b of A such that $a < b$. Since $x \geq b$ we see at once that $x > a$ and so x is an upper bound of the set A.***

Note that the assumption that A has no largest member isn't really needed here. All we need to know is that the specific number a is not the largest member of A.

2. a. Give an example of a set A that has a largest member a such that the sets A and $A \setminus \{a\}$ have exactly the same upper bounds.
   We can look at the interval $[0, 1]$.

   b. Give an example of a set A that has a largest member a such that the sets A and $A \setminus \{a\}$ do not have exactly the same upper bounds.
   We can look at the set $[0, 1] \cup \{3\}$.

3. a. Given that S is a subset of a given interval $[a, b]$ explain why, for every member x of the set S we have
$$|x| \leq |a| + |b - a|.$$

   Solution: ***Suppose that $x \in S$. We see that***
$$|x| = |x - a + a| \leq |x - a| + |a|.$$
   Now since $a \leq x \leq b$ we see that
$$|x - a| = x - a \leq b - a$$
   and so
$$|x| \leq |a| + |b - a|.$$

   b. Given that a set S of numbers is bounded and that
$$T = \{|x| \mid x \in S\},$$
   prove that the set T must also be bounded.
   Choose a lower bound a of S and an upper bound b of S and then observe from part a that the number $|a| + |b - a|$ is an upper bound of T.

4. Given that A is a set of real numbers and that $\sup A \in A$, explain why $\sup A$ must be the largest member of A.

   Solution: This assertion is obvious. We are given that $\inf A$ is a member of A and we know that no member of A can be less than $\inf A$, and so $\inf A$ is the least member of A.

5. Given that A is a set of real numbers and that $\inf A \in A$, explain why $\inf A$ must be the smallest member of A.
   This assertion is obvious. We are given that $\sup A$ is a member of A and we know that no member of A can be larger than $\sup A$, and so $\sup A$ is the largest member of A.

6. Is it possible for a set of numbers to have a supremum even though it has no largest member?

   Solution: ***You bet it's possible! That's the whole point of this chapter!***

7. Given that α is an upper bound of a set A and that $\alpha \in A$, explain why $\alpha = \sup A$.
   We are given that α is an upper bound of A. Now if w is any number less than α then, because $\alpha \in A$, the number w can't be an upper bound of A. Therefore α must be the least upper bound of A.

8. Explain why the empty set does not have a supremum.
   Since every number is an upper bound of the empty set, there can't be a least upper bound of the empty set.

9. Explain why the set $[1, \infty)$ does not have a supremum.
   The interval $[1, \infty)$ is unbounded above. This set doesn't have any upper bounds and so it can't

have a least one.

10. Given that two sets A and B are bounded above, explain why their union $A \cup B$ is bounded above.
Using the fact that A is bounded above, choose an upper bound u of A. Now, using the fact that B is bounded above, choose an upper bound v of B. We now define w to be the larger of the two numbers u and v. Given any member x of the set $A \cup B$, there are two possibilities: Either $x \in A$, in which case $x \leq u \leq w$ or $x \in B$, in which case $x \leq v \leq w$. Therefore no member of the set $A \cup B$ can be greater than w and we have shown that $A \cup B$ is bounded above.

11. a. If a man says truthfully that he sells more BMWs than anyone in the Southeast, what can you deduce about him?

Solution: ***He is not in the Southeast.***
Next problem: "I sell diamonds for less than anyone in the industry." No wonder he can sell them for less. He isn't in the industry. The diamonds are hot.

b. Given that $\alpha = \sup A$ and that $x < \alpha$, what conclusions can you draw about the number x?

Solution: ***The number x cannot be an upper bound of A. In other words, there must exist a member of A that is larger than x.***

c. Given that $\alpha = \inf A$ and that $x > \alpha$, what conclusions can you draw about the number x?
The number x cannot be a lower bound of A. In other words, there must exist a member of A that is less than x.

12. If A and B are sets of real numbers then the sets $A + B$ and $A - B$ are defined by

$$A + B = \{x \mid \exists a \in A \text{ and } \exists b \in B \text{ such that } x = a + b\}$$

and

$$A - B = \{x \mid \exists a \in A \text{ and } \exists b \in B \text{ such that } x = a - b\}$$

a. Work out $A + B$ and $A - B$ in each of the following cases:

i. $A = [0,1]$ and $B = [-1,0]$.
We have $A + B = [-1,1]$. Certainly, the most that $a + b$ can be if $a \in A$ and $b \in B$ is $1 + 0 = 1$ and the least that $a + b$ can be if $a \in A$ and $b \in B$ is $0 + (-1) = -1$. Now if $x \in [-1,1]$ then there are two possibilities:
Case 1: $-1 \leq x \leq 0$. In this case, the fact that $0 \in A$ and $x \in B$ and $x = 0 + x$ shows that $x \in A + B$.
Case 2: $0 < x \leq 1$. In this case, the fact that $x \in A$ and $0 \in B$ and $x = x + 0$ shows that $x \in A + B$.

ii. $A = [0,1]$ and $B = \{1,2,3\}$.
One can use an argument similar to that used for part i to show that $[0,1] + \{1\} = [1,2]$ or, perhaps, one could declare this fact to be obvious. Similar remarks apply to $[0,1] + \{2\}$ and to $[0,1] + \{3\}$. Finally,

$$A + B = [0,1] + \{1,2,3\} = [1,2] \cup [2,3] \cup [3,4] = [1,4].$$

iii. $A = (0,1)$ and $B = \{1,2,3\}$.
We have

$$A + B = (1,2) \cup (2,3) \cup (3,4).$$

b. Prove that if two sets A and B are bounded then so are $A + B$ and $A - B$.
Suppose that A and B are bounded. Choose lower bounds u_1 and u_2 of A and B respectively and choose upper bounds v_1 and v_2 of A and B respectively. To show that $v_1 + v_2$ is an upper bound of $A + B$, suppose that $x \in A + B$. Choose $a \in A$ and $b \in B$ such that $x = a + b$. Since $a \leq u_1$ and $b \leq u_2$ we have $x = a + b \leq u_1 + u_2$. Thus $u_1 + u_2$ is an upper bound of $A + B$ and similar argument show that $u_1 - v_2$ is an upper bound of $A - B$ and $v_1 + v_2$ is a lower bound of

$A+B$ and $v_1 - u_2$ is a lower bound of $A-B$.

Exercises on Supremum And Infimum

1. Suppose that A is a nonempty bounded set of real numbers that has no largest member and that $a \in A$. Prove that $\sup A = \sup(A \setminus \{a\})$.

   You saw in an earlier exercise that the sets A and $A \setminus \{a\}$ have exactly the same upper bounds.

2. Given that A and B are sets of numbers, that A is nonempty, that B is bounded above and that $A \subseteq B$, explain why $\sup A$ and $\sup B$ exist and why $\sup A \leq \sup B$.
   Since A is nonempty and $A \subseteq B$ we know that B is nonempty. Since $A \subseteq B$ we know that any upper bound of B must be an upper bound of A. Therefore, since B is bounded above we know that that A is bounded above. Finally, since $\sup B$, being an upper bound of B, must be an upper bound of A, and since $\sup A$ is the **least** upper bound of A we have $\sup A \leq \sup B$.

3. Given that A is a nonempty bounded set of numbers, explain why $\inf A \leq \sup A$.
   Using the fact that A is nonempty we choose a member a of A. We know that
   $$\inf A \leq a \leq \sup A.$$

4. It is given that A and B are nonempty bounded sets of real numbers, that for every $x \in A$ there exists $y \in B$ such that $x < y$ and for every $y \in B$ there exists $x \in A$ such that $y < x$. Prove that $\sup A = \sup B$.

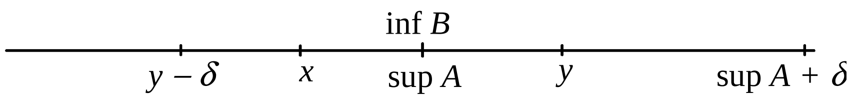
   Solution: ***We need to show that the two sets A and B have exactly the same upper bounds and for this purpose we shall show that a number fails to be an upper bound of A if and only if it fails to be an upper bound of B.***

   Suppose that u fails to be an upper bound of A. Choose a member x of A such that $u < x$. Using the given property of A and B we now choose a member y of the set B such that $x < y$ and, since $u < x < y$, we conclude that u can't be an upper bound of B. We can show similarly that a number that fails to be an upper bound of B must also fail to be an upper bound of A.

5. Suppose that A and B are nonempty sets of real numbers and that for every $x \in A$ and every $y \in B$ we have $x < y$. Prove that $\sup A \leq \inf B$. Give an example of sets A and B satisfying these conditions for which $\sup A = \inf B$.
   Given any member y of the set B it follows from the fact that $x < y$ for every $x \in A$ that y must be an upper bound of A. In other words, every member of B is an upper bound of A and the fact that B is nonempty shows that A is bounded above. A similar argument shows that every member of A is a lower bound of B and the fact that A is nonempty guarantees that B is bounded below. Thus $\sup A$ and $\inf B$ exist.
   Given any member y of B, the fact that y is an upper bound of A and $\sup A$ is the **least** upper bound of A tells us that $\sup A \leq y$. In other words, $\sup A$ is a lower bound of B. Since $\inf B$ is the **greatest** lower bound of B we deduce that $\sup A \leq \inf B$.

6. Suppose that A and B are nonempty sets of real numbers and that $\sup A = \inf B$. Prove that for every number $\delta > 0$ it is possible to find a member x of A and a member y of B such that $x + \delta > y$.

   Solution: ***Suppose that $\delta > 0$. Using the fact that***
   $$\sup A + \delta = \inf B + \delta > \inf B$$
   and that $\sup A + \delta$ is therefore not a lower bound of B, choose a member y of the set B such that
   $$\sup A + \delta > y.$$
   From the fact that
   $$y - \delta < \sup A$$
   we deduce that $y - \delta$ is not an upper bound of A and, using this fact, we choose a member x of A such that $y - \delta < x$. In this way we have found $x \in A$ and $Y \in B$ such that $x + \delta > y$.



7. Suppose that A and B are nonempty sets of real numbers, that $\sup A \leq \inf B$ and that for every number $\delta > 0$ it is possible to find a member x of A and a member y of B such that $x + \delta > y$. Prove that $\sup A = \inf B$.

Solution: ***To obtain a contradiction, assume that*** $\sup A < \inf B$ ***and define***

$$\delta = \inf B - \sup A.$$

We observe that $\delta > 0$***. Now for all*** $x \in A$ ***and*** $y \in B$***, it follows from the facts that*** $x \leq \sup A$ ***and*** $\inf B \leq y$ ***that***

$$x \leq \sup A = \inf B - \delta \leq y - \delta$$

which tells us that $x + \delta \leq y$***. Therefore it is impossible to find*** $x \in A$ ***and*** $y \in B$ ***such that*** $x + \delta > y$ ***and we have reached the desired contradiction.***

8. Suppose that A is a nonempty bounded set of real numbers, that A has no largest member and that $x < \sup A$. Prove that there are at least two different members of A lying between x and $\sup A$.
Since x is less than the least upper bound of A we know that x can't be an upper bound of A. Using the fact that x isn't an upper bound of A we choose a member a of A such that $x < a$. Since A has no largest member it must have a member larger than the member a. We choose a member b of A such that $a < b$. Since A has members greater than b we deduce that

$$x < a < b < \sup A$$

and we have found two different members of A between x and $\sup A$.

9. Suppose that A is a nonempty bounded set of real numbers, that $\delta > 0$ and that for any two different members x and y of A we have $|x - y| \geq \delta$. Prove that A has a largest member. You can find a hint to the solution of this exercise in a forthcoming theorem.

10. Suppose that S is a nonempty bounded set of real numbers, that $\alpha = \inf S$ and $\beta = \sup S$, and that every number that lies between two members of S must also belong to S. Prove that S must be one of the four intervals $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, (α, β). See a coming theorem for a solution of this exercise.

11. Suppose that A is a nonempty bounded set of real numbers, that $\alpha = \inf A$ and that $\beta = \sup A$. Suppose that

$$S = \left\{x - y \mid x \in A \text{ and } y \in A\right\}.$$

Prove that $\sup S = \beta - \alpha$. You will find a solution to this exercise in the next section.

12. Suppose that A is a set of numbers and that A is nonempty and bounded above. Suppose that q is a given number and that the set C is defined as follows:

$$C = \{q + x \mid x \in A\}.$$

Prove that the set C is nonempty and bounded above and that

$$\sup C = q + \sup A$$

This exercise is a special case of the next exercise because $C = A + \{q\}$.

13. Suppose that A and B are nonempty bounded sets of numbers and that the sets $A + B$ and $A - B$ are defined as above. Prove that

$$\sup(A + B) = \sup A + \sup B$$

and

$$\sup(A - B) = \sup A - \inf B$$

Solution: ***We shall prove that***

$$\sup(A + B) = \sup A + \sup B$$

Step 1: We want to show that the number $\sup A + \sup B$ ***is an upper bound of the set*** $A + B$***. Suppose that***

$x \in A + B$. ***Using the definition of*** $A + B$ ***we choose a number*** $a \in A$ ***and a number*** $b \in B$ ***such that*** $x = a + b$. ***Since*** $a \leq \sup A$ ***and*** $b \leq \sup B$ ***we see that***

$$x = a + b \leq \sup A + \sup B.$$

Step 2: We want to show that the number $\sup A + \sup B$ ***is actually the least upper bound of the set*** $A + B$. ***Suppose that*** u ***is any upper bound of the set*** $A + B$. ***Given any number*** $a \in A$ ***and*** $b \in B$ ***we have*** $a + b \leq u$. ***Therefore, whenever*** $b \in B$ ***we know that the inequality***

$$a \leq u - b$$

holds for all $a \in A$. ***Therefore, whenever*** $b \in B$, ***the number*** $u - b$ ***is an upper bound of*** A ***and must satisfy the condition***

$$\sup A \leq u - b$$

which we can also write as

$$b \leq u - \sup A.$$

We conclude that $u - \sup A$ ***is an upper bound of*** B ***and so***

$$\sup B \leq u - \sup A$$

which we can write as

$$\sup B + \sup A \leq u.$$

Thus $\sup A + \sup B$ ***is the least upper bound of*** $A + B$, ***as promised.***

Exercises on the Archimedean Property of the System R

1. Prove that if A is the set of all rational numbers in the interval $[0, 1]$ then $\sup A = 1$.
   Quite simply, the largest member of the set A is 1.

2. Suppose that

   $$A = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots\right\} = \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\}.$$

   Prove that $\inf A = 0$.
   The number 0 is clearly a lower bound of A. Now given any number $x > 0$ we know that there exists a positive integer n such that $\frac{1}{n} < x$ and, consequently, that x cannot be a lower bound of A. Therefore 0 is the greatest lower bound of A.

3. A nonempty set G of real numbers is said to be a **subgroup** of $\mathbf{R}$ if whenever x and y belong to G then so do the numbers $x + y$ and $x - y$.

   a. Determine which of the following sets are subgroups of $\mathbf{R}$.

$\emptyset$	$\{1, 0, -1\}$	$\mathbf{Q}$	$\mathbf{Z}$
$\mathbf{Z}^+$	$\mathbf{Q}^+$	$\{2n \mid n \in \mathbf{Z}\}$	$\{2n \mid n \in \mathbf{Z}^+\}$
$\{0\}$	$\mathbf{R}$	$\mathbf{R} \setminus \mathbf{Q}$	$\{m + n\sqrt{2} \mid m \in \mathbf{Z} \text{ and } n \in \mathbf{Z}\}$

   The set $\emptyset$ is not a subgroup of $\mathbf{R}$ because the definition requires a subgroup to be nonempty.
   The set $\{1, 0, -1\}$ is not a subgroup because $1 + 1$ does not belong to this set.
   The set $\mathbf{Q}$ is a subgroup of $\mathbf{R}$.
   The set $\mathbf{Z}$ is a subgroup of $\mathbf{R}$.
   The set $\mathbf{Z}^+$ is not a subgroup of $\mathbf{R}$ because $2 - 4$ does not belong to $\mathbf{Z}^+$. The same argument applies to the sets $\mathbf{Q}^+$ and $\{2n \mid n \in \mathbf{Z}^+\}$.
   The set $\{2n \mid n \in \mathbf{Z}\}$ is a subgroup of $\mathbf{R}$.
   The set $\{0\}$ is a subgroup of $\mathbf{R}$.
   The set $\mathbf{R}$ is a subgroup of $\mathbf{R}$.
   The set $\mathbf{R} \setminus \mathbf{Q}$ is not a subgroup of $\mathbf{R}$ because the number $\sqrt{2} - \sqrt{2}$ does not belong to $\mathbf{R} \setminus \mathbf{Q}$.

The set $\left\{m+n\sqrt{2} \;\mid\; m \in \mathbf{Z} \text{ and } n \in \mathbf{Z}\right\}$ is a subgroup of $\mathbf{R}$.

b. Explain why every subgroup of $\mathbf{R}$ must contain the number 0. Show that if G is any subgroup of $\mathbf{R}$ other than $\{0\}$ then G must contain infinitely many positive numbers.
Suppose that G is a subgroup of $\mathbf{R}$. Using the fact that G is not empty we choose a member x of G. Since $0 = x - x$ we conclude that $0 \in G$.
Now suppose that the subgroup G is not $\{0\}$. Choose a member x of G such that $x \neq 0$. Since the infinite set $\{nx \mid n \in \mathbf{Z}\}$ is an infinite subset of G we know that G is an infinite set.

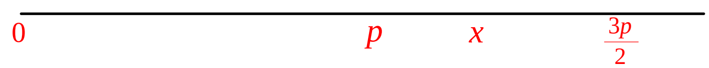
c. Suppose that G is a subgroup of $\mathbf{R}$ other than $\{0\}$, that

$$p = \inf\{x \in G \mid x > 0\}$$

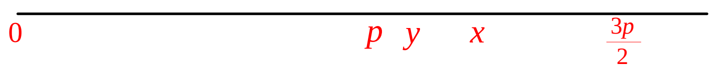
and that the number p is positive. Prove that either $p \in G$ or the set G must contain at least two different members between p and $3p/2$.

Solution: ***We assume that*** p ***does not belong to the group*** G***. Now since*** p ***is the greatest lower bound of*** G ***and*** $p < 3p/2$ ***we know that*** $3p/2$ ***fails to be a lower bound of*** G***. Using this fact we choose a member*** x ***of*** G ***such that***

$$x < \frac{3p}{2}.$$



Since p ***does not lie in*** G ***we know that*** $p < x$ ***and so*** x ***is not a lower bound of*** G***. Choose a member*** y ***of*** G ***such that*** $y < x$***. In this way we have found two members of*** G ***lying between*** p ***and*** $3p/2$***.***



d. Suppose that G is a subgroup of $\mathbf{R}$ other than $\{0\}$, that

$$p = \inf\{x \in G \mid x > 0\}$$

and that the number p is positive. Prove that p is the smallest positive member of G and that

$$G = \{np \mid n \in \mathbf{Z}\}.$$

Solution: ***Given any members*** x ***and*** y ***of the set*** G ***we know that if*** $x < y$ ***then*** $y - x \geq p$***. Therefore no two different members of*** G ***can both lie between*** p ***and*** $3p/2$ ***and we conclude from part (c) that*** $p \in G$***. We therefore know that***

$$G \supseteq \{np \mid n \in \mathbf{Z}\}.$$

Now suppose that x ***is any member of*** G***. We define*** n ***to be the largest integer that does not exceed*** x/p ***and we observe that, since***

$$pn \leq x < p(n+1)$$

we have

$$0 \leq x - pn < p.$$

Since the set G ***has no positive members that are less than*** p ***we conclude that*** $x - pn = 0$ ***and we have shown that every member of*** G ***is an integer multiple of*** p***. Therefore***

$$G = \{np \mid n \in \mathbf{Z}\}.$$

e. Suppose that G is a subgroup of $\mathbf{R}$ other than $\{0\}$, and that

$$0 = \inf\{x \in G \mid x > 0\}.$$

Prove that if a and b are any real numbers satisfying $a < b$ then it is possible to find a member x of the set G such that $a < x < b$.

Solution: ***Suppose that*** a ***and*** b ***are real numbers and that*** $a < b$***. Using the fact that*** $b - a > 0$

and that $\inf G = 0$***, choose a member*** q ***of*** G ***such that***

$$0 < q < b - a.$$

We now define

$$S = \{n \in \mathbf{Z} \mid nq > a\}.$$

Since the inequality $nq > a$ ***is the same as saying that*** $n > a/q$ ***and since the set*** Z ***of integers is unbounded above we know that*** S ***is nonempty. Furthermore, since the number*** a/q ***is a lower bound of*** S ***we know that*** S ***is bounded below. We deduce from*** *an order property* ***of the system of integers that the set*** S ***has a least member that we shall call*** m***. Thus*** $mq > a$ ***but*** $(m-1)q \leq a$***.***

We observe that

$$a < mq = (m-1)q + q < a + b - a = b$$

and so the member mq ***of*** G ***must lie between*** a ***and*** b***.***

4. This exercise invites you to explore the so-called **division algorithm** which describes the process by which an integer b can be divided into an integer a to yield a quotient q and a remainder r.
   Suppose that a and b are positive integers and that

$$S = \{n \in \mathbf{Z} \mid nb \leq a\}.$$

   a. Prove that $S \neq \emptyset$ and that S is bounded above.
      The fact that S is bounded above is clear since the number $\frac{a}{b}$ must be an upper bound of S.

   b. Prove that if the largest member of S is called q and we define $r = a - qb$ then $a = qb + r$ and $0 \leq r < b$.
      We need to recall that every nonempty set of integers wil,if it is bounded above, have a largest member. If we define q to be the largest member of S then, since $qb \leq a$ we have $a - qb \geq 0$. On the other hand, since q is the largest member of S we have

$$(q+1)b > a$$

      which gives us $a - qb < b$.

Some Exercises on Suprema and Infima of Functions

1. Given that $f(x) = 1/x$ whenever $x > 0$, prove that the function f is unbounded above. Prove that for every number $\delta > 0$, the restriction of f to the interval $[\delta, \infty)$ is bounded.
   To show that f is unbounded above, suppose that w is any positive number. We observe that

$$f\left(\frac{1}{1+w}\right) = 1 + w > w$$

   and therefore w is not an upper bound of the function f.
   Now suppose that $\delta > 0$. Given any number $x \geq \delta$ we have $f(x) \leq \frac{1}{\delta}$ and so $\frac{1}{\delta}$ is an upper bound of the restriction of f to the interval $[\delta, \infty)$.

2. Give an example of a bounded function on the interval $[0,1]$ that has a minimum value but does not have a maximum value.
   We could define

$$f(x) = \begin{cases} x & \text{if} \quad 0 \leq x < 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$$

3. A function f is said to be **increasing** on a set S if the inequality $f(t) \leq f(x)$ holds whenever t and x belong to S and $t \leq x$. Prove that every increasing function on the interval $[0,1]$ must have both a maximum and a minimum value.
   The definition of an increasing function makes it clear that if f is an increasing function on theinterval $[0,1]$ then $f(0)$ and $f(1)$ are the minimum and maximum values, respectively, of the function f.

4. Prove that if f and g are bounded above on a nonempty set S then

$$\sup(f+g) \leq \sup f + \sup g.$$

Solution: ***Given any member** x **of the set** S**, since** $f(x) \leq \sup f$ **and** $g(x) \leq \sup g$**, we have***

$$(f+g)(x) = f(x) + g(x) \leq \sup f + \sup g.$$

Thus the number** $\sup f + \sup g$ **is an upper bound of the function** $f+g$ **and so

$$\sup(f+g) \leq \sup f + \sup g.$$

5. Give an example of two bounded functions f and g on the interval $[0,1]$ such that

$$\sup(f+g) < \sup f + \sup g.$$

6. Given that f is a bounded function on a nonempty set S and that c is a real number prove that

$$\sup(cf) = \begin{cases} c\sup f & \text{if} \quad c > 0 \\ c\inf f & \text{if} \quad c < 0 \end{cases}$$

Solution: ***We give the solution here for the case** $c < 0$ **and leave it to you to handle the case** $c > 0$**.***
Suppose that** $c < 0$**.
Given any member** x **of the set** S**, it follows from the fact that** $\inf f \leq f(x)$ **that

$$cf(x) \leq c\inf f$$

and we conclude that** $c\inf f$ **is an upper bound of the function** cf**. To show that** $c\inf f$ **is the least upper bound of** cf **we need to show that no number less than** $c\inf f$ **can be an upper bound of** cf**. Suppose that** $p < c\inf f$**.
Since** $p/c > \inf f$ **we know that** p/c **fails to be a lower bound of** f**. Using this fact, we choose a member** x **of** S **such that

$$\frac{p}{c} > f(x).$$

Since** $p < cf(x)$ **we have shown, as promised, that** p **fails to be an upper bound of the function** cf**.

7. Prove that if f is a bounded function on a nonempty set S then

$$|\sup f| \leq \sup|f|.$$

Solution: ***For every** $x \in S$ **we have***

$$f(x) \leq |f(x)| \leq \sup|f|$$

and so

$$\sup f \leq \sup|f|.$$

Futhermore, for every** $x \in S$ **we have

$$f(x) \geq -|f(x)| \geq -\sup|f|.$$

Using the fact that the set** S **is nonempty, choose a member** x **of the set** S**. We see that

$$-\sup|f| \leq f(x) \leq \sup f$$

and so

$$-\sup|f| \leq \sup f \leq \sup|f|$$

and we have shown that

$$|\sup f| \leq \sup|f|.$$

Exercises on Sequences of Sets

1. Evaluate

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right].$$

We observe first that if n is a positive integer then $[\frac{1}{n},1] \subseteq (0,1]$. Now given any number $x \in (0,1]$ there exist positive integers n for which $\frac{1}{n} < x$ and for such n we have $x \in [\frac{1}{n},1]$. Therefore

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right] = (0,1].$$

2. Evaluate

$$\bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},5-\frac{2}{n}\right].$$

Using an argument similar to the one used in Exercise 1 we can see that

$$\bigcup_{n=1}^{\infty}\left[1+\frac{1}{n},5-\frac{2}{n}\right] = (1,5).$$

3. Evaluate

$$\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n},2+\frac{1}{n}\right).$$

Using an argument similar to the one used in Exercise 1 we can see that

$$\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n},2+\frac{1}{n}\right) = [1,2].$$

4. Explain why if (A_n) is an expanding sequence of subsets of $\mathbf{R}$ then the sequence $(\mathbf{R}\setminus A_n)$ is a contracting sequence.

Suppose that (A_n) is an expanding sequence of subsets of $\mathbf{R}$. For each n it follows at once from the fact that $A_n \subseteq A_{n+1}$ that

$$\mathbf{R}\setminus A_{n+1} \subseteq \mathbf{R}\setminus A_n.$$

5. Suppose that (A_n) is a sequence of sets.

   a. Prove that if we define

$$B_n = \bigcup_{j=n}^{\infty} A_j$$

   for each n then the sequence (B_n) is a contracting sequence of sets.

   Given any positive integer n, the inequality

$$\bigcup_{j=n+1}^{\infty} A_j \subseteq \bigcup_{j=n}^{\infty} A_j$$

   follows at once from the observation that if $x \in \bigcup_{j=n+1}^{\infty} A_j$ then there is an integer $j \geq n+1$ for which $x \in A_j$ and so there must certainly exist an integer $j \geq n$ for which $x \in A_j$.

   b. Prove that if we define

$$B_n = \bigcap_{j=n}^{\infty} A_j$$

   for each n then the sequence (B_n) is an expanding sequence of sets.

   Given any positive integer n, the inequality

$$\bigcap_{j=n}^{\infty} A_j \subseteq \bigcap_{j=n+1}^{\infty} A_j$$

follows at once from the observation that if $x \in \bigcap_{j=n}^{\infty} A_j$ then x belongs to A_j for every integer $j \geq n$ and so x certainly belongs to A_j for every integer $j \geq n+1$.

c. Prove that if we define

$$B_n = \bigcap_{j=1}^{n} A_j$$

for each n then the sequence (B_n) is an contracting sequence of sets.
Given any positive integer n, the inequality

$$\bigcap_{j=1}^{n+1} A_j \subseteq \bigcap_{j=1}^{n} A_j$$

follows by the same sort of argument that was given for parts a and b of this exercise.

d. Prove that if we define

$$B_n = \bigcup_{j=1}^{n} A_j$$

for each n then the sequence (B_n) is an expanding sequence of sets.
Given any positive integer n, the inequality

$$\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n+1} A_j$$

follows by the same sort of argument that was given for parts a and b of this exercise.

6. Prove that if (A_n) is a sequence of subsets of $\mathbf{R}$ then

$$\mathbf{R} \setminus \bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} (\mathbf{R} \setminus A_n).$$

Hint: ***Given any number x that belongs to the set***

$$\mathbf{R} \setminus \bigcup_{n=1}^{\infty} A_n$$

we know from the fact that x does not belong to the union

$$\bigcup_{n=1}^{\infty} A_n$$

that, for each n, x must fail to belong to the set A_n. Thus, whenever

$$x \in \mathbf{R} \setminus \bigcup_{n=1}^{\infty} A_n$$

we know that

$$x \in \mathbf{R} \setminus A_n$$

for every n and so

$$x \in \bigcap_{n=1}^{\infty} (\mathbf{R} \setminus A_n).$$

You can use an almost identical argument to show that whenever a number x belongs to the set

$$\bigcap_{n=1}^{\infty} (\mathbf{R} \setminus A_n)$$

we must have

$$x \in \mathbf{R} \setminus \bigcup_{n=1}^{\infty} A_n$$

and we have therefore shown that

$$\mathbf{R} \setminus \bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} (\mathbf{R} \setminus A_n).$$

Exercises on Mathematical Induction

1. Given that $f(x) = \log(1+x)$ for all $x > -1$, prove that the identity

$$(1+x)f^{(n+1)}(x) + nf^{(n)}(x) = 0$$

holds whenever n is a positive integer and $x > -1$.
For each positive integer n we define P_n to be the assertion that the equation

$$(1+x)f^{(n+1)}(x) + nf^{(n)}(x) = 0$$

holds. The assertion P_1 says that

$$(1+x)\left(\frac{-1}{(1+x)^2}\right) + 1\left(\frac{1}{1+x}\right) = 0$$

and this assertion is obviously true. Now suppose that n is any positive integer for which the statement P_n is true. From the equation

$$(1+x)f^{(n+1)}(x) + nf^{(n)}(x) = 0$$

we have

$$1f^{(n+1)}(x) + (1+x)f^{(n+2)}(x) + nf^{(n+1)}(x) = 0$$

which we can write as

$$(1+x)f^{(n+2)}(x) + (n+1)f^{(n+1)}(x) = 0.$$

Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

2. Given that $f(x) = \arctan(1+x)$ for every number x, prove that the identity

$$(x^2+2x+2)f^{(n+1)}(x) + 2n(x+1)f^{(n)}(x) + n(n-1)f^{(n-1)}(x) = 0$$

holds whenever n is a positive integer and x is a real number.
For each positive integer n we define P_n to be the assertion that the equation

$$(x^2+2x+2)f^{(n+1)}(x) + 2n(x+1)f^{(n)}(x) + n(n-1)f^{(n-1)}(x) = 0$$

holds. The assertion P_1 says that

$$(x^2+2x+2)\left(\frac{-2x-2}{(x^2+2x+2)^2}\right) + 2(x+1)\left(\frac{1}{2x+x^2+2}\right) = 0$$

and this assertion is obviously true. Now suppose that n is any positive integer for which the statement P_n is true. From the equation

$$(x^2+2x+2)f^{(n+1)}(x) + 2n(x+1)f^{(n)}(x) + n(n-1)f^{(n-1)}(x) = 0$$

we have

$$(2x+2)f^{(n+1)}(x) + (x^2+2x+2)f^{(n+3)}(x) + 2nf^{(n)}(x) + 2n(x+1)f^{(n+1)}(x) + n(n-1)f^{(n)}(x) = 0$$

which we can write as

$$(x^2+2x+2)f^{(n+3)}(x) + 2nf^{(n)}(x) + 2(n+1)(x+1)f^{(n+1)}(x) + (n+1)nf^{(n)}(x) = 0$$

Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

3. Given that $x_1 = \sqrt{2}$ and that the equation

$$x_{n+1} = \sqrt{2 + x_n}$$

holds for every natural number n, prove that $x_n < 2$ for every natural number n.
For each positive integer n we define P_n to be the assertion that $x_n < 2$. The assertion P_1 says that $\sqrt{2} < 2$ which is obviously true. Now suppose that n is any positive integer for which the statement P_n is true. We see that

$$x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + 2} = 2.$$

Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

4. Given that $x_1 = \sqrt{2}$ and that the equation

$$x_{n+1} = \sqrt{2 + x_n}$$

holds for every natural number n, prove that $x_n < x_{n+1}$ for every natural number n.
For each positive integer n we define P_n to be the assertion that $x_n < x_{n+1}$. The assertion P_1 says that

$$\sqrt{2} < \sqrt{2 + \sqrt{2}}$$

which is obviously true. Now suppose that n is any positive integer for which the statement P_n is true. We see that

$$x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + x_{n+1}} = x_{n+2}.$$

Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

5. Given that $x_1 = 0$ and that the equation

$$8x_{n+1}^3 = 6x_n + 1$$

holds for every natural number n, prove the following two assertions:

a. For every natural number n we have $x_n < 1$.
For each positive integer n we define P_n to be the assertion that $x_n < 1$. The assertion P_1 says that $0 < 1$ which is true. Now suppose that n is any positive integer for which the statement P_n is true. We see that

$$x_{n+1} = \sqrt[3]{\frac{6x_n + 1}{8}} < \sqrt[3]{\frac{6 + 1}{8}} < 1.$$

Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

b. For every natural number n we have $x_n < x_{n+1}$.
For each positive integer n we define P_n to be the assertion that $x_n < x_{n+1}$. The assertion P_1 says that

$$0 < \frac{1}{2}$$

which is obviously true. Now suppose that n is any positive integer for which the statement P_n is true. We see that

$$x_{n+1} = \sqrt[3]{\frac{6x_n + 1}{8}} < \sqrt[3]{\frac{6x_{n+1} + 1}{8}} = x_{n+2}.$$

Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

6. Prove that every nonempty finite set of real numbers has a largest member.
For each positive integer n we define P_n to be the assertion that any set of real numbers that has exactly n members must have a largest member. Since the only member of any singleton is clearly the largest member of that singleton we know that the assertion P_1 is true. Now suppose that n is any positive integer for which the assertion P_n is true. To prove that the assertion P_{n+1} must also be true we suppose that S is a set of real numbers that has exactly $n + 1$ members. Choose a member

a of the set S. Since the set $S \setminus \{a\}$ has exactly n members, it follows from the assertion P_n that $S \setminus \{a\}$ has a largest member that we shall call b. The larger of the two numbers a and b is clearly the largest member of the set S.
Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

7. Given that $S \subseteq \mathbf{Z}^+$, prove that the following three conditions are equivalent:

   a. The set S is infinite.

   b. The set S has no largest member.

   c. The set S is unbounded above.

   It it probably easier to explain why the denials of these three statements are equivalent to one another. We shall show that the following three satements are equivalent:

   a. The set S is finite.

   b. The set S has a largest member.

   c. The set S is bounded above.

   We already know that every finite set of numbers must have a largest member. Therefore condition a implies condition b. If a set of numbers has a largest member then that largest member is clearly an upper bound of the set. Therefore condition b implies condition c. Finally, we know that every nonempty set of integers that is bounded above must have a largest member. If the largest member of a set of postive integers is n then that set cannot have more than n members and must therefore be finite. Therefore condition c implies condition a.

8. Prove that for every positive integer n, if n horses run in a race and no two horses tie then there are exactly $n!$ possible outcomes.
   For each positive integer n we define P_n to be the assertion that if n norses run in a race and no two horses tie then there are exactly $n!$ possible outcomes.
   The assertion P_1 is obviously true. Now suppose that n is any positive integer for which P_n is true. To show that the assertion P_{n+1} is also true, suppose that $n+1$ horses run in a race tha that no two horses tie. Choose one of the horses whom we shall call Dobbin and, for the moment let's focus our attention on the other n horses. Those other horses present us with $n!$ possible outcomes. For each of those outcomes, if we add in Dobbin then there are $n+1$ positions in which Dobbin can place among the other horses. Therefore the number of outcomes, including Dobbin is $(n+1)(n!) = (n+1)!$.
   Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

9. Given nonnegative integers n and r satisfying $r \le n$, the **binomial coefficient** $\binom{n}{r}$ is defined by the equation
$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

   a. Prove that if n and r are nonnegative integers and $r \le n$ then
$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$
   Hint: No special techniques are needed for this proof. Just combine the expressions on the left side.

   b. Prove that if n and r are nonnegative integers and $r \le n$ then every set which has n members must have exactly $\binom{n}{r}$ subsets that have r members.
   For each positive integer n we define P_n to be the assertion that if a set has exactly n members and if r is a nonnegative integer and $r \le n$ then the set has exactly $\binom{n}{r}$ subsets with r members.
   The assertion P_1 is obviously true. Now suppose that n is a positive integer for which the assertion P_n is true and suppose that S is a set with $n+1$ members, that r is a nonnegative integer and that $r \le n$.

Choose a member of S that we shall call w.
Now given any subset E of S, if $w \in E$ then E is the union of $\{w\}$ and a subset of $S \setminus \{w\}$ that contains $r-1$ members. Since the assertion P_n is true, there are $\binom{n}{r-1}$ such subsets.
Furthermore, if E is a subset of S and E has r members and w does not belong to E then E is a subset containing r members of the set $S \setminus \{w\}$. Since the assertion P_n is true there are $\binom{n}{r}$ such subsets.
The total number of subsets of S with n members is therefore

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

Since P_1 is true and since the condition $P_n \Rightarrow P_{n+1}$ holds for every positive integer n it follows from the principle of mathematical induction that P_n is true for every positive integer n.

10. Suppose that for every positive integer n we define $p(n)$ to be the assertion that if in any crowd of men, at least one of them has red hair then all of them have red hair. What is wrong with the following proof by mathematical induction that the assertion $p(n)$ is true for every n?

    a. The statement $p(1)$ is obviously true.

    b. Now suppose that n is any positive integer for which the statement $p(n)$ happens to be true and suppose that S is a crowd containing $n+1$ men and that at least one of these $n+1$ men has red hair. Choose such a man and let's call him Harry. Now, in the crowd of $n+1$ men, choose any man and call him Joe and ask him to step away from the others. There are n men left and, since at least one of them has red hair, they all have. Now ask Harry to step away and ask Joe to come back. Again we are looking at a crowd of n men and, since Joe is the only man about whom we have any doubt, at least one man in this crowd has red hair; and therefore they all have red hair. Now ask Harry to come back and we see that all $n+1$ men have red hair.

    The argument given above has to refer to three different men. It refers to Harry and to Joe and also to at least one other red-haired man. The argument therefore cannot be used to show that $P_2 \Rightarrow P_3$.

Some Exercises on the Extended Real Number System

1. Thinking of the rule for sums of limits

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

that you saw in elementary calculus, give some examples to show why the expression $\infty + (-\infty)$ should not be defined.

Solution: ***If we define*** $f(x) = x$ and $g(x) = 1 - x$ ***for every number*** x ***then***

$$\lim_{x\to\infty} f(x) = \infty$$

and

$$\lim_{x\to\infty} g(x) = -\infty$$

$$\lim_{x\to\infty}(f(x)+g(x)) = 1.$$

With this example in mind we might conclude that if $\infty + (-\infty)$ ***is to be defined, it ought to be equal to*** 1***. However, if we define*** $f(x) = x$ ***and*** $g(x) = 2 - x$ ***for every*** x ***then we should conclude that*** $\infty + (-\infty)$ ***ought to be*** 2***. If we define*** $f(x) = x + \sin x$ ***and*** $g(x) = -x$ ***for every*** x ***then we ought to conclude that*** $\infty + (-\infty)$ ***doesn't exist. If we define*** $f(x) = x^2$ ***and*** $g(x) = -x$ ***for all*** x ***then we ought to conclude that*** $\infty + (-\infty)$ ***ought to be*** ∞***.***
It is clear that no single definition of the symbol $\infty + (-\infty)$ ***would be useful to us.***

2. Thinking of the rule for products of limits

$$\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

that you saw in elementary calculus, give some examples to show why the expression $\infty \times 0$ should not be defined.
If we define $f(x) = x$ and $g(x) = \frac{1}{x}$ for all $x > 0$ then
$$\lim_{x\to\infty} f(x) = \infty$$
and
$$\lim_{x\to\infty} g(x) = 0$$
and
$$\lim_{x\to\infty} f(x)g(x) = 1.$$
This example suggests that if $\infty \times 0$ ought to be defined then its value should be zero. However, if we were to define $f(x) = x^2$ and $g(x) = \frac{1}{x}$ for all $x > 0$ we would again have
$$\lim_{x\to\infty} f(x) = \infty$$
and
$$\lim_{x\to\infty} g(x) = 0$$
but, this time, we would have
$$\lim_{x\to\infty} f(x)g(x) = \infty.$$
Alternatively we could define $f(x) = x$ and $g(x) = \frac{1}{x^2}$ for all $x > 0$ and obtain
$$\lim_{x\to\infty} f(x)g(x) = 0.$$
These examples show that there is no way to define $\infty \times 0$ that will make the rule for products of limits work in the case in which one limit is zero and the other is ∞.

3. Thinking of the rule for quotients of limits
$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$
that you saw in elementary calculus, give some examples to show why the expression $\frac{\infty}{\infty}$ should not be defined.

4. Given that A and B are intervals and that $A \cap B \neq \emptyset$, prove that the set $A \cup B$ is an interval.
Choose a number $w \in A \cap B$.
We define $u = \inf(A \cup B)$ and $v = \sup(A \cup B)$. We see at once that u is the smaller of the two numbers $\inf A$ and $\inf B$ and that v is the larger of the two numbers $\sup A$ and $\sup B$.
Case 1: Assume that $u = \inf A$ and $v = \sup A$: In this case it is clear that
$$u = \inf A \leq \inf B \leq \sup B \leq \sup A = v$$
and $A \cup B$ is one of the intervals $[u,v]$, $[u,v)$, $(u,v]$ and (u,v).
Case 2: Assume that $u = \inf B$ and $v = \sup B$. This case is similar to case 1.
Case 3: Assume that $u = \inf A$ and $v = \sup B$. In this case
$$u = \inf A \leq \inf B \leq w \leq \sup A \leq \sup B = v$$
Once again we can see that $A \cup B$ must be one of the intervals $[u,v]$, $[u,v)$, $(u,v]$ and (u,v). To justify this assertion we need to see why every number x between u and v must belong to $A \cup B$. But if $u < x < v$ then either $u < x \leq w$ or $w < x < v$. The inequality $u < x \leq w$ guarantees that $x \in A$ and the inequality $w < x < v$ guarantees that $x \in B$.
Case 4: Assume that $u = \inf B$ and $v = \sup A$. This case is similar to case 3.

5. Given that A, B and C are intervals and that the sets $A \cap B$ and $B \cap C$ are nonempty, prove that $A \cup B \cup C$ is an interval.
We deduce from Exercise 4 that $A \cup B$ is an interval. Since
$$(A \cup B) \cap C \neq \emptyset$$
we deduce from Exercise 4 again that $(A \cup B) \cup C$ is an interval.

Exercises on the Complex Number System

1. Find two complex numbers $z = x + iy$ for which $z^2 = 3 + 4i$.

Solution: ***The equation*** $(x+iy)^2 = 3+4i$ ***gives us the system of two equations***

$$x^2 - y^2 = 3$$
$$2xy = 4$$

which implies that $x^4 - 3x^2 - 4 = 0$ ***from which we deduce that*** $x = \pm 2$***. We see at once that***

$$x + iy = \pm(2+i).$$

2. Solve the quadratic equation $z^2 + 2z + 4 = 0$. Why must the solutions of this equation be cube roots of 8?
The equation $z^2 + 2z + 4 = 0$ says that $(z+1)^2 = -3$ which yields $z = -1 \pm \sqrt{3}\, i$. Since the equation also implies that $(z-2)(z^2+2z+4) = 0$ which says that $z^3 - 8 = 0$, the solutions of the equation must be cube roots of 8.

3. Given two complex numbers z and w with $w \neq 0$, prove that the complex conjugate of z/w is $\bar{z}/\bar{w}$. Establish a similar identity for absolute value.

Hint: ***Once the corresponding assertion for multiplication has been obtained we can obtain the assertion for division very quickly using the fact that***

$$z = \left(\frac{z}{w}\right) w.$$

4. Work out $(1+i)^n$ for a few positive integers n. What is $(1+i)^4$?

5. Prove that if n is an even natural number that is not a multiple of 4 then the real part of $(1+i)^n$ is zero. *Hint*: Use mathematical induction.
We want to prove that whenever n is a positive integer, the real part of the number $(1+i)^{4n-2}$ is zero. This fact is obvious if $n = 1$. Now given any positive integer n for which the real part of the number $(1+i)^{4n-2}$ is zero we have

$$(1+i)^{4(n+1)-2} = (1+i)^{4n-2}(1+i)^4 = -4(1+i)^{4n-2}$$

and therefore this number has a zero real part. It follows from the principle of mathematical induction that $(1+i)^{4n-2}$ has a zero real part for every positive integer n.

6. Prove that if n is a natural number then $(1+i)^n$ is real and positive if and only if n is a multiple of 8.
If n is any positive integer then, since

$$(1+i)^{4n} = (1+i)^{4n-2}(1+i)^2 = 2i(1+i)^{4n-2}$$

and since the number $(1+i)^{4n-2}$ is imaginary, we deduce that $(1+i)^{4n}$ is real. We deduce that if n is any even positive integer then the number $(1+i)^n$ must either be real or imaginary and it follows that the number

$$(1+i)^{n+1} = (1+i)^n + i(1+i)^n$$

cannot be real. We conclude that if n is an odd positive integer then $(1+i)^n$ cannot be real.
Finally, if n is a positive integer then since

$$(1+i)^{4n} = \left((1+i)^4\right)^n = (-4)^n$$

we see that the number $(1+i)^{4n}$ is positive if and only if n is even. Therefore the number $(1+i)^n$ is real and positive if and only if n is a multiple of 8.

7. Assuming the standard identities from trigonometry, prove that if α and β are any real numbers then we have

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Then prove by induction (or otherwise) that if n is any natural number then

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

Does the same identity hold if n is a negative integer?
The first equation follows simply:

$$\begin{aligned}(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha)\\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta).\end{aligned}$$

The equation

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

is obvious when $n = 1$. Now given any positive integer n for which the equation

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

holds we have

$$\begin{aligned}(\cos\alpha + i\sin\alpha)^{n+1} &= (\cos\alpha + i\sin\alpha)^n(\cos\alpha + i\sin\alpha)\\ &= (\cos n\alpha + i\sin n\alpha)(\cos\alpha + i\sin\alpha)\\ &= \cos(n\alpha+\alpha) + i\sin(n\alpha+\alpha)\end{aligned}$$

and so it follows from the principle of mathematical induction that the equation

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

holds for every positive integer n. Since

$$\begin{aligned}(\cos\alpha + i\sin\alpha)^{-1} &= \frac{1}{\cos\alpha + i\sin\alpha} = \frac{\cos\alpha - i\sin\alpha}{(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}\\ &= \frac{\cos\alpha - i\sin\alpha}{\cos^2\alpha + \sin^2\alpha} = \cos(-\alpha) + i\sin(-\alpha)\end{aligned}$$

we see that the equation

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

also holds when $n = -1$. Finally, if n is any positive integer then we have

$$\begin{aligned}(\cos\alpha + i\sin\alpha)^{-n} &= \left((\cos\alpha + i\sin\alpha)^{-1}\right)^n = (\cos(-\alpha) + i\sin(-\alpha))^n\\ &= \cos(n(-\alpha)) + i\sin(n(-\alpha)) = \cos(-n\alpha) + i\sin(-n\alpha)\end{aligned}$$

and so the equation

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

does also hold when n is negative. Incidentally, this equation also holds when $n = 0$.

8. Making use of Exercise 7, prove that

$$\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}$$

is a cube root of

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^3 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

9. Suppose that n is a natural number. Making use of Exercise 7, find a complex number w such that $w^n = 1$ but $w^k \neq 1$ whenever $k = 1, 2, \cdots, n-1$. A number with these properties is called a **primitive** nth root of 1.
The number $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ will work.

10. Prove that if n is a natural number, $w^n = 1$, and $w \neq 1$, then

$$1 + w + w^2 + w^3 + \cdots + w^{n-1} = 0.$$

This fact follows at once from the fact that

$$1 + w + w^2 + w^3 + \cdots + w^{n-1} = \frac{1 - w^n}{1 - w}.$$

11. Prove that if w is a cube root of 1 and $w \neq 1$, then for any numbers a, b, and c we have

$$(a + b + c)(a + wb + w^2c)(a + w^2b + w^4c) = a^3 + b^3 + c^3 - 3abc.$$

If you know something about determinants, prove that these expressions are also equal to

$$\det\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

Can you generalize this result to higher order determinants?
This does generalise nicely to a general n by n determinant

$$\det\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}.$$

If w is an nth root of 1 and $w \neq 1$ then

$$\det\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix} = \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} w^j a_j\right)\left(\sum_{j=1}^{n} w^{2j} a_j\right)\cdots\left(\sum_{j=1}^{n} w^{(n-1)j} a_j\right)$$

It's easy to find all of these factors by performing elementary row operations on the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$$

6 Elementary Topology of the Real Line

Exercises on Neighborhoods

1. Complete the following sentence *"A set U fails to be a neighborhood of a number x when for every number* $\delta > 0$, *..."*
   A set U fails to be a neighborhood of a number x when for every number $\delta > 0$ the interval $(x-\delta, x+\delta)$ contains at least one number that does not belong to U.

2. Explain carefully why the assertion

   $$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

   that was made above is true. You will need to make use of an earlier result.
   Since the number 0 belongs to the interval $(-\frac{1}{n}, \frac{1}{n})$ for every positive integer n we have

   $$0 \in \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right).$$

Now we need to show that 0 is the only number in the set $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$. Suppose that $x \neq 0$. Using the fact that $|x| > 0$ and that earlier result, choose a positive integer n such that

$$\frac{1}{n} < |x|.$$

$$-|x| \_\_\_\_\_\_\_\_ -\frac{1}{n} \_\_\_\_\_\_ 0 \_\_\_\_\_\_ \frac{1}{n} \_\_\_\_\_\_\_\_ |x|$$

Since we have found a value of n for which x fails to belong to the interval $\left(-\frac{1}{n}, \frac{1}{n}\right)$ we know that

$$x \notin \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right).$$

Thus 0 is the only number in the set $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$, which is what we wanted to prove.

3. Given that x is an interior point of U and that $U \subseteq V$, explain why x must be an interior point of V.
   Using the fact that x is an interior point of U, choose $\delta > 0$ such that
   $$(x-\delta, x+\delta) \subseteq U.$$
   Since $U \subseteq V$ we therefore have
   $$(x-\delta, x+\delta) \subseteq V.$$
   Therefore x is an interior point of V.

4. Suppose that x is a real number and that $U \subseteq \mathbf{R}$. Prove that the following two conditions are equivalent:
   a. The set U is a neighborhood of the number x.
   b. It is possible to find two numbers a and b such that
   $$x \in (a,b) \subseteq U.$$

   Solution: ***To prove that condition a implies condition b we assume that*** U ***is a neighborhood of the number*** x***. Choose*** $\delta > 0$ ***such that***
   $$(x-\delta, x+\delta) \subseteq U.$$
   By defining $a = x - \delta$ ***and*** $b = x + \delta$ ***we obtain two numbers*** a ***and*** b ***such that*** $a < b$ ***and such that***
   $$x \in (a,b) \subseteq U.$$

   Now to prove that condition b implies condition a we assume that condition b holds. Choose numbers a ***and*** b ***such that*** $a < b$ ***and such that***
   $$x \in (a,b) \subseteq U.$$
   Using the fact that the interval (a,b) ***is a neighborhood of*** x***, choose*** $\delta > 0$ ***such that***
   $$(x-\delta, x+\delta) \subseteq (a,b).$$
   Since $(x-\delta, x+\delta) \subseteq U$ ***we have shown that*** U ***is a neighborhood of*** x***.***

5. Suppose that x and y are two different real numbers. Prove that it is possible to find a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.
   Solution: ***We may assume, without loss of generality, that*** $x < y$***. Choose a number*** c ***between*** x ***and*** y***. The intervals*** $(-\infty, c)$ ***and*** (c, ∞) ***are, respectively, neighborhoods of*** x ***and*** y ***and the intersection of these two intervals is empty.***

6. Given that S is a set of real numbers and that x is an upper bound of S, explain why S cannot be a neighborhood of x.
   Solution: ***If*** δ ***is any positive number then, since all of the numbers between*** x ***and*** $x+\delta$ ***must lie in*** $R \setminus S$***, the interval*** $(x-\delta, x+\delta)$ ***cannot be included in*** S***.***

7. Given that a set S of real numbers is nonempty and bounded above, explain why neither S nor $\mathbf{R} \setminus S$ can be a neighborhood of $\sup S$.
   Solution: ***We see from Exercise 6 that*** S ***is not a neighborhood of*** $\sup S$***. Now we observe that,***

whenever $\delta > 0$, since the number $x - \delta$ fails to be an upper bound of S, there must be members of S in the interval $(x - \delta, x]$. Therefore, whenever $\delta > 0$, the interval $(x - \delta, x + \delta)$ fails to be included in the set $R \setminus S$ and so $R \setminus S$ must also fail to be a neighborhood of x.

8. Suppose that A and B are sets of real numbers and that x is an interior point of the set $A \cup B$. Is it true that x must either be an interior point of A or an interior point of B?

   Hint: ***The answer is no. Give an example to show what can go wrong.***

9. Suppose that A and B are sets of real numbers and that x is an interior point both of A and of B. Is it true that x must be an interior point of the set $A \cap B$?

   Yes it is true. Suppose that x is an interior point of both of the sets A and B. Choose a number $\delta_1 > 0$ such that

$$(x - \delta_1, x + \delta_1) \subseteq A$$

   and choose a number $\delta_2 > 0$ such that

$$(x - \delta_2, x + \delta_2) \subseteq B.$$

   We now define δ to be the smaller of the two numbers δ_1 and δ_2 and we observe that

$$(x - \delta, x + \delta) \subseteq A \cap B.$$

10. Suppose that x and y are real numbers and that U is a neighborhood of y. Prove that the set V defined by

$$V = \{x + u \mid u \in U\}$$

   is a neighborhood of the number $x + y$.

   Solution: ***We need to find a number $\delta > 0$ such that the interval $(x + y - \delta, x + y + \delta)$ is included in V. Using the fact that U is a neighborhood of y, we choose $\delta > 0$ such that***

$$(y - \delta, y + \delta) \subseteq U.$$

   Now given any number t in the interval $(x + y - \delta, x + y + \delta)$ we deduce from the fact that

$$x + y - \delta < t < x + y + \delta$$

   that

$$y - \delta < t - x < y + \delta$$

   and therefore we know that $t - x \in U$. Therefore, since

$$t = x + t - x$$

   we know that $t \in V$ and we have shown that

$$(x + y - \delta, x + y + \delta) \subseteq V.$$

Exercises on Open Sets and Closed Sets

1. Explain why if U is open and H is closed then the set $U \setminus H$ must be open.

   Solution: ***The set $U \setminus H$ can be expressed as $U \cap (\mathbf{R} \setminus H)$ which is the intersection of two open sets.***

2. Explain why if U is open and H is closed then the set $H \setminus U$ must be closed.

   We observe that

$$H \setminus U = H \cap (\mathbf{R} \setminus U)$$

   which is the intersection of two closed sets. As we know, the intersection of closed sets is always closed.

3. Give an example of an infinite family of open sets whose intersection fails to be open.

   We saw earlier that

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

and, of course, the set $\{0\}$ isn't open.

4. Give an example of an infinite family of closed sets whose union fails to be closed.

$$\bigcup_{n=1}^{\infty}\left[0, 1-\frac{1}{n}\right] = [0,1)$$

which is not closed.

5. Give an example of two sets A and B neither of which is open but for which the set $A \cup B$ is open.

Solution: ***Look at the union*** $(0,1] \cup [1,2)$**.**

6. Given a set H of real numbers, prove that the following conditions are equivalent:

   a. The set H is closed.

   b. For every number $x \in \mathbf{R} \setminus H$ it is possible to find a number $\delta > 0$ such that $(x-\delta, x+\delta) \cap H = \emptyset$.

   Condition b can be expressed by saying for every number $x \in \mathbf{R} \setminus H$ it is possible to find a number $\delta > 0$ such that

$$(x-\delta, x+\delta) \subseteq \mathbf{R} \setminus H$$

   and this is exactly the condition that the set $\mathbf{R} \setminus H$ is open.

7. a. Given any number x, prove that the singleton $\{x\}$ is closed.
   The fact that $\{x\}$ is closed follows from the fact that the set $\mathbf{R} \setminus \{x\}$ is the union of the two open intervals $(-\infty, x)$ and (x, ∞), the fact that an open interval is an open set and the fact that the union of open sets is always open.

   b. Use part a. and the fact that every finite set is a finite union of singletons to deduce that every finite set is closed.
   There isn't really anything to add. The union of finitely many closed sets is always closed.

8. Given that $\mathbf{Q} \subseteq H$ and that H is closed, prove that $H = \mathbf{R}$.

Solution: ***The set*** $\mathbf{R} \setminus H$ ***is open and contains no rational number. To prove that*** $\mathbf{R} \setminus H$ ***must be empty we shall observe that every nonempty open set must contain a rational number. Suppose that*** U ***is open and nonempty. Choose*** $x \in U$ ***and choose*** $\delta > 0$ ***such that*** $(x-\delta, x+\delta) \subseteq U$***. Since the interval*** $(x-\delta, x+\delta)$ ***must contain rational numbers, so must the set*** U***.***

9. Given that H is closed, nonempty and bounded below, prove that H must have a least member.
   We need to show that $\inf H$ belongs to H. Since H is closed we can show that $\inf H$ belongs to H by showing that $\inf H$ is close to the set H. Suppose that $\delta > 0$. Since $\inf H + \delta > \inf H$ and since $\inf H$ is the *greatest* lower bound of H we know that $\inf H + \delta$ must fail to be a lower bound of H. Using this fact we choose a member x of H such that $x < \inf H + \delta$. We observe that

$$\inf H - \delta < \inf H \leq x < \inf H + \delta$$

   and so

$$(\inf H - \delta, \inf H + \delta) \cap H \neq \emptyset$$

   and we have shown that $\inf H$ is close to H.

10. Prove that no open set can have a largest member.
   Suppose that U is open and that $x \in U$. We need to show that x can't be the largest member of U. Using the fact that x must be an interior point of U we choose $\delta > 0$ such that

$$(x-\delta, x+\delta) \subseteq U.$$

   The interval $(x, x+\delta)$ is non empty and all its members must belong to U. Therefore x is not the largest member of U.

11. Given a real number x and a set S of real numbers, prove that the following two conditions are equivalent:

   a. The number x is an interior point of S.

b. It is possible to find an open set U such that $x \in U \subseteq S$.

Proof that condition a implies condition b: We assume that x is an interior point of S. Choose a number $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq S$ and define $U = (x-\delta, x+\delta)$. In this way we have found an open set U such that $x \in U \subseteq S$.
Proof that condition b implies condition a. We assume that condition b holds. Choose an open set U such that $x \in U \subseteq S$. Using the fact that x is an interior point of U we choose $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$. Since $(x-\delta, x+\delta) \subseteq S$ we deduce that x is an interior point of S.

12. Prove that if S is any set of real numbers then the set of all interior points of S must be open.
Suppose that S is a set of real numbers and define U to be the set of interior points of S. To show that U is open, suppose that $x \in U$. Choose $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq S$. We deduce from the results obtained in Exercise 11 that every number in the interval must be an interior point of S. In other words, $(x-\delta, x+\delta) \subseteq U$. Therefore, since every member of U is an interior point of U the set U must be open.

13. This exercise refers to the sum of two sets as it was defined in this exercise. Prove that if A is any set of real numbers and U is an open set then the set $A+U$ must be open.

Solution: ***We need to show that every member of*** $A+U$ ***is an interior point of*** $A+U$***. Suppose that*** $x \in A+U$***. Choose a member*** a ***of*** A ***and a member*** u ***of*** U ***such that*** $x = a+u$***. Using the fact that*** $u \in U$***, choose*** $\delta > 0$ ***such that*** $(u-\delta, u+\delta) \subseteq U$***. We shall show that*** x ***is an interior point of*** $A+U$ ***by showing that***

$$(x-\delta, x+\delta) \subseteq A+U.$$

Suppose that $t \in (x-\delta, x+\delta)$***. Thus***

$$|t-(a+u)| < \delta$$

which gives us

$$|(t-a)-u| < \delta$$

and so

$$t-a \in (u-\delta, u+\delta) \subseteq U.$$

Since

$$t = a+(t-a)$$

we see that $t \in A+U$ ***and so***

$$(x-\delta, x+\delta) \subseteq A+U$$

as promised.

Exercises on Closure

1. Suppose that

$$S = [0,1) \cup (1,2).$$

a. What is the set of interior points of S?
The set of interior points of S is $(0,1) \cup (1,2)$.

b. Given that U is the set of interior points of S, evaluate $\overline{U}$.

$$\overline{(0,1) \cup (1,2)} = [0,1] \cup [1,2] = \overline{S}.$$

The purpose of parts a and b is to exhibit a set S such that, if U is the set of interior points of S then $\overline{U} = \overline{S}$.

c. Give an example of a set S of real numbers such that if U is the set of interior points of S then $\overline{U} \neq \overline{S}$.
We could take S to be a singleton like $\{3\}$ or it could be the set of all integers. It could also be the set of all rational numbers between 0 and 1.

d. Give an example of a subset S of the interval $[0,1]$ such that $\overline{S} = [0,1]$ but if U is the set of interior points of S then $\overline{U} \neq [0,1]$.
Once again, take the set of all rational numbers between 0 and 1.

2. Given that

$$S = \left\{ \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\},$$

evaluate $\overline{S}$.

Hint: ***Show that***

$$\overline{S} = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\}.$$

First show that $0 \in \overline{S}$***. Then observe that every negative number belongs to the set*** $(-\infty, 0)$ ***and that if*** x ***is any positive number then*** x ***belongs to the interval***

$$\left(\frac{1}{n+1}, \frac{1}{n} \right)$$

for some positive integer n***.***

3. Given that S is a set of real numbers, that H is a closed set and that $S \subseteq H$, prove that $\overline{S} \subseteq H$.
We could argue that $\overline{S} \subseteq \overline{H}$ and that, because H is closed, $\overline{H} = H$.

4. Given two sets A and B of real numbers, prove that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Solution: ***Since*** $A \subseteq \overline{A}$ ***and*** $B \subseteq \overline{B}$ ***we have***

$$A \cup B \subseteq \overline{A} \cup \overline{B}.$$

and therefore, since the union of the two closed sets $\overline{A}$ ***and*** $\overline{B}$ ***is closed we have***

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

On the other hand, since A ***is included in the closed set*** $\overline{A \cup B}$ ***we have***

$$\overline{A} \subseteq \overline{A \cup B}$$

and, similarly we can see that

$$\overline{B} \subseteq \overline{A \cup B}$$

and so

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

Therefore

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

5. Given two sets A and B of real numbers, prove that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Do the two sides of this inclusion have to be equal? What if A and B are open? What if they are closed?
Since $A \cap B \subseteq A$ we have $\overline{A \cap B} \subseteq \overline{A}$ and similarly that $\overline{A \cap B} \subseteq \overline{B}$. Thus

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Now observe that if $A = (0,1)$ and $B = (1,2)$ then

$$\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$$

and

$$\overline{A \cap B} = \overline{\emptyset} = \emptyset.$$

Of course, we could give more spectacular examples like $A = \mathbf{Q}$ and $B = \mathbf{R} \setminus \mathbf{Q}$.

6. Prove that if S is any set of real numbers then the set $\mathbf{R} \setminus \overline{S}$ is the set of interior points of the set $\mathbf{R} \setminus S$.
Most students should be encouraged to write two separate arguments here. The first task is to show that every member of the set $\mathbf{R} \setminus \overline{S}$ must be an interior point of $\mathbf{R} \setminus S$. Then one should show that every interior point of $\mathbf{R} \setminus S$ must belong to $\mathbf{R} \setminus \overline{S}$.

On the other hand, a strong student could be permitted to observe that if x is any given number then the statement that x does not belong to $\overline{S}$ is the statement that there exists a number $\delta > 0$ such that $(x-\delta, x+\delta) \cap S = \emptyset$, and that the latter equation is just the condition that $(x-\delta, x+\delta) \subseteq \mathbf{R} \setminus S$.

7. Given that α is an upper bound of a given set S of real numbers, prove that the following two conditions are equivalent:

   a. We have $\alpha = \sup S$.

   b. We have $\alpha \in \overline{S}$.

   To prove that condition a implies condition b we assume that $\alpha = \sup S$. We need to show that $\alpha \in \overline{S}$. Suppose that $\delta > 0$. Using the fact that α is the **least** upper bound of S and that $\alpha - \delta < \alpha$ we choose a member x of S such that $\alpha - \delta < x$. Since $x \in (\alpha-\delta, \alpha+\delta) \cap S$ we have $(\alpha-\delta, \alpha+\delta) \cap S \neq \emptyset$.
   To prove that condition b implies condition a we assume that $\alpha \in \overline{S}$. We need to show that α is the least upper bound of S. Suppose that $p < \alpha$. Since the set (p, ∞) is a neighborhood of α we have $(p, \infty) \cap S \neq \emptyset$. Thus, since α is an upper bound of S and since no number $p < \alpha$ can be an upper bound of S we conclude that α is the least upper bound of S.

8. Is it true that if A and B are sets of real numbers and
$$\overline{A} = \overline{B} = \mathbf{R}$$
   then $\overline{A \cap B} = \mathbf{R}$?
   The answer is no. Look at $A = \mathbf{Q}$ and $B = \mathbf{R} \setminus \mathbf{Q}$.

9. Prove that if A and B are open sets and
$$\overline{A} = \overline{B} = \mathbf{R}$$
   then $\overline{A \cap B} = \mathbf{R}$. What if only one of the sets A and B is open?
   Solution: ***All we need to know is that at least one of the sets*** A ***and*** B ***is open. Suppose that*** A ***and*** B ***are sets of real numbers, that***
$$\overline{A} = \overline{B} = \mathbf{R}$$
   and that the set A ***is open.***
   To prove that
$$\overline{A \cap B} = \mathbf{R},$$
   suppose that x ***is any real number and that*** $\delta > 0$***. Since*** $x \in \overline{A}$ ***we know that the set***
$$(x-\delta, x+\delta) \cap A$$
   is nonempty and we also know that this set is open. Therefore, since $\overline{B} = \mathbf{R}$ ***we know that***
$$(x-\delta, x+\delta) \cap A \cap B \neq \emptyset.$$
   We have therefore shown that every real number must belong to $\overline{A \cap B}$***.***

10. Two sets A and B are said to be **separated** from each other if
$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$
   Which of the following pairs of sets are separated from each other?

   a. $[0,1]$ and $[2,3]$. Yes.

   b. $(0,1)$ and $(1,2)$. Yes.

   c. $(0,1]$ and $(1,2)$. No because $(0,1] \cap \overline{(1,2)} = \{1\} \neq \emptyset$.

   d. $\mathbf{Q}$ and $\mathbf{R} \setminus \mathbf{Q}$. No.

11. Prove that if A and B are closed and disjoint from one another then A and B are separated from each other.
   Suppose that A and B are closed and disjoint from one another. Since $A = \overline{A}$ and $B = \overline{B}$, the fact that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ follows at once from the fact that $A \cap B = \emptyset$.

12. Prove that if A and B are open and disjoint from one another then A and B are separated from each other.
Suppose that A and B are open and disjoint from one another. Given any number $x \in A$, we deduce from the fact that A is a neighborhood of x and $A \cap B = \emptyset$ that x is not close to B. Therefore $A \cap \overline{B} = \emptyset$ and we see similarly that $\overline{A} \cap B = \emptyset$.

13. Suppose that S is a set of real numbers. Prove that the two sets S and $\mathbf{R} \setminus S$ will be separated from each other if and only if the set S is both open and closed. What then do we know about the sets S for which S and $\mathbf{R} \setminus S$ are separated from each other?
Suppose that S and $\mathbf{R} \setminus S$ are separated from each other. To show that S is open, suppose that $x \in S$. Since $S \cap \left(\overline{\mathbf{R} \setminus S}\right) = \emptyset$ we know that x is not close to $\mathbf{R} \setminus S$. Choose $\delta > 0$ such that

$$(x-\delta, x+\delta) \cap (\mathbf{R} \setminus S) = \emptyset$$

and observe that $(x-\delta, x+\delta) \subseteq S$. Thus S is open and a similar argument shows that $\mathbf{R} \setminus S$ is also open. We therefore know that if the sets S and $\mathbf{R} \setminus S$ are separated from one another then S is both open and closed.
Now suppose that S is both open and closed. Since the two set S and $\mathbf{R} \setminus S$ are closed and disjoint from one other they are separated from one another.

14. This exercise refers to the notion of a subgroup of $\mathbf{R}$ that was introduced in an earlier exercise. That exercise should be completed before you start this one.

a. Given that H and K are subgroups of $\mathbf{R}$, prove that the set $H+K$ defined in the sense of an earlier exercise is also a subgroup of $\mathbf{R}$.
To prove that $H+K$ is a subgroup of $\mathbf{R}$ we need to show that $H+K$ is nonempty and that the sum and difference of any members of $H+K$ must always belong to $H+K$.
To show that $H+K$ is nonempty we use the fact that H and K are nonempty to choose $x \in H$ and $y \in K$. Since $x+y \in H+K$ we have $H+K \neq \emptyset$.
Now suppose that w_1 and w_2 are any members of the set $H+K$. Choose members x_1 and x_2 of H and members y_1 and y_2 of K such that $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$. Since the numbers $x_1 + x_1$ and $x_1 - x_2$ belong to H and the numbers $y_1 + y_2$ and $y_1 - y_2$ belong to K, and since

$$w_1 + w_2 = (x_1 + x_2) + (y_1 + y_2)$$

and

$$w_1 - w_2 = (x_1 - x_2) + (y_1 - y_2)$$

we see at once that $w_1 + w_2$ and $w_1 - w_2$ belong to $H+K$.

b. Prove that if a, b and c are integers and if

$$a\sqrt{2} = b\sqrt{3} + c$$

then $a = b = c = 0$.

Solution: ***From the equation***

$$a\sqrt{2} = b\sqrt{3} + c$$

we see that

$$2a^2 = 3b^2 + 2bc\sqrt{3} + c^2.$$

Therefore, unless $bc = 0$ ***we have***

$$\sqrt{3} = \frac{2a^2 - 3b^2 - c^2}{2bc}$$

which contradicts the fact that the number $\sqrt{3}$ ***is irrational. Therefore at least one of the number*** b ***and*** c ***must be zero.***
In the event that $c = 0$***, the equation***

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = b\sqrt{3}$$

and, unless $a = 0$***, the latter equation gives us***

$$\sqrt{\frac{2}{3}} = \frac{b}{a}$$

which contradicts the fact that $\sqrt{\frac{2}{3}}$ ***is irrational. So in the case*** $c = 0$ ***we also have*** $a = 0$ ***and we see at once that*** $b = 0$ ***as well.***

In the event that $b = 0$***, the equation***

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = c$$

and, unless $a = 0$***, the latter equation gives us***

$$\sqrt{2} = \frac{c}{a}$$

which contradicts the irrationality of $\sqrt{2}$***. So, once again,*** $a = 0$ ***and we see at once that*** $c = 0$ ***as well.***

c. Prove that if m, n, p and q are integers then it is impossible to have

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

and deduce that if α is any real number and if $H = \{n\alpha \mid n \in \mathbf{Z}\}$ then the subgroup $H + \mathbf{Z}$ cannot contain both of the numbers $\sqrt{2}$ and $\sqrt{3}$.

Solution: ***The equation***

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

implies that

$$q\sqrt{2} = n\sqrt{3} - np + mq$$

which, by part b, tells us that

$$0 = q = n = mq - np$$

which is clearly impossible since n ***and*** q ***appear denominators of the fractions in the equation***

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}.$$

Now, to obtain a contradiction, suppose that the subgroup $H + Z$ ***contains both of the numbers*** $\sqrt{2}$ ***and*** $\sqrt{3}$***. Choose integers*** m ***and*** n ***such that***

$$\sqrt{2} = m + n\alpha$$

and choose integers p ***and*** q ***such that***

$$\sqrt{3} = p + q\alpha.$$

Since $\sqrt{2}$ ***is irrational, we know that*** $\sqrt{2} \neq m$ ***and so*** $n \neq 0$***; and we know similarly that*** $q \neq 0$***. Thus***

$$\frac{\sqrt{2} - m}{n} = \alpha = \frac{\sqrt{3} - p}{q}$$

which we know to be impossible.

d. Suppose that G is a subgroup of $\mathbf{R}$ other than $\{0\}$, that

$$p = \inf\{x \in G \mid x > 0\}$$

and that the number p is positive. Prove that the set G is closed.

Solution: ***We know from*** *an earlier exercise* ***that***

$$G = \{np \mid n \in \mathbf{Z}\}.$$

e. Prove that if G is a subgroup of $\mathbf{R}$ other than $\{0\}$ and that G has no least positive member then $\overline{G} = \mathbf{R}$.

Solution: ***This fact was established in*** *an earlier exercise.*

f. Suppose that α is an irrational number, that

$$H = \{n\alpha \mid n \in \mathbf{Z}\}$$

and that $G = H + \mathbf{Z}$ (in the sense of this exercise). Prove that although the sets H and $\mathbf{Z}$ are closed subgroups of $\mathbf{R}$ and although the set G is also a subgroup of $\mathbf{R}$, the set G is not closed.

Solution: ***Since G cannot contain both of the numbers $\sqrt{2}$ and $\sqrt{3}$ we know that $G \neq R$. To show that G is not closed we shall make the observation that $\overline{G} = R$ and, for this purpose, all we have to show is that if***

$$p = \inf\{x \in G \mid x > 0\}$$

then $p = 0$. Suppose that p is defined in this way and, to obtain a contradiction, suppose that $p > 0$. We know that

$$G = \{np \mid n \in \mathbf{Z}\}$$

and, using the fact that both of the numbers 1 and α belong to G, we choose integers m and n such that

$$1 = mp$$

and

$$\alpha = np.$$

From the fact that $p = 1/m$ we see that p is rational but from the fact that $p = \alpha/n$ we see that p must be irrational. Thus we have arrived at the promised contradiction.

Exercises on Limit Points

1. Prove that $\mathbf{L}(\mathbf{Z}) = \emptyset$.
   Given any number x, the interval $(x-1, x+1)$ can contain at most two integers. We know that a neighborhood of a limit point of a set must always contain infinitely many members of that set and so we conclude that no number x can be a limit point of the set $\mathbf{Z}$ of integers.

2. Prove that $\mathbf{L}(\mathbf{Q}) = \mathbf{R}$.
   Suppose that x is any real number. To show that x is a limit point of $\mathbf{Q}$, suppose that $\delta > 0$. Since there are rational numbers in the interval $(x, x+\delta)$ we know that the set $(x-\delta, x+\delta) \cap \mathbf{Q} \setminus \{x\} \neq \emptyset$.

3. Prove that $\mathbf{L}(\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}) = \{0\}$.
   If x is any negative number then the interval $(-\infty, 0)$ is a neighborhood of x that fails to contain any members of the set $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$. Thus a negative number can not be a limit point of $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$.
   If x is any positive number then the interval $\left(\frac{x}{2}, \infty\right)$ is a neighborhood of x which fails to contain infinitely many members of the set $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$. To see why, note that if n is a positive integer then the condition

   $$\frac{1}{n} \in \left(\frac{x}{2}, \infty\right)$$

   can hold only if $n < \frac{2}{x}$. Therefore no positive number can be a limit point of $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$.
   Finally, we need to explain why 0 must be a limit point of $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$. Suppose that $\delta > 0$. Choose an integer $k > \frac{1}{\delta}$ and observe that

   $$\frac{1}{k} \in (0-\delta, 0+\delta) \cap \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\} \setminus \{0\}$$

   from which we deduce that the set $(0-\delta, 0+\delta) \cap \{\frac{1}{n} \mid n \in \mathbf{Z}^+\} \setminus \{0\}$ must be nonempty.

4. a. Give an example of an infinite set that has no limit point.
      As we saw in Exercise 1, the infinite set $\mathbf{Z}$ has no limit point.

   b. Give an example of a bounded set that has no limit point.
      A finite set like $\{2\}$ will not have any limit points. We could also look at the empty set $\emptyset$.

   c. Give an example of an unbounded set that has no limit point.

As we saw in Exercise 1, the infinite set $\mathbf{Z}$ has no limit point.

d. Give an example of an unbounded set that has exactly one limit point.
The unbounded set $\mathbf{Z} \cup \{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$ has only the limit point 0.

e. Give an example of an unbounded set that has exactly two limit points.
The set
$$\mathbf{Z} \cup \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\} \cup \left\{1 + \frac{1}{n} \mid n \in \mathbf{Z}^+\right\}$$
has the two limit points 0 and 1. We can see this directly or we can use the assertion proved in Exercise 6 below.

5. Prove that if A and B are sets of real numbers and if $A \subseteq B$ then $\mathbf{L}(A) \subseteq \mathbf{L}(B)$.
Suppose that A and B are sets of real numbers and that $A \subseteq B$. Suppose that x is a limit point of A. We need to explain why x has to be a limit point of B. Suppose that $\delta > 0$. Since the set $(x - \delta, x + \delta) \cap A \setminus \{x\}$ is nonempty and since
$$(x - \delta, x + \delta) \cap A \setminus \{x\} \subseteq (x - \delta, x + \delta) \cap B \setminus \{x\}$$
we deduce that the set $(x - \delta, x + \delta) \cap B \setminus \{x\}$ is nonempty.

6. Prove that if A and B are sets of real numbers then
$$\mathbf{L}(A \cup B) = \mathbf{L}(A) \cup \mathbf{L}(B).$$

Solution: ***Since*** $A \subseteq A \cup B$ ***we know that*** $\mathbf{L}(A) \subseteq \mathbf{L}(A \cup B)$ ***and similarly we know that*** $L(B) \subseteq L(A \cup B)$***. Thus***
$$\mathbf{L}(A) \cup \mathbf{L}(B) \subseteq \mathbf{L}(A \cup B).$$
Now suppose that a number x ***fails to belong to the set*** $\mathbf{L}(A) \cup \mathbf{L}(B)$***. Choose a number*** $\delta_1 > 0$ ***such that the interval*** $(x - \delta_1, x + \delta_1)$ ***contains only finitely many members of the set*** A***. Choose a number*** $\delta_2 > 0$ ***such that the interval*** $(x - \delta_2, x + \delta_2)$ ***contains only finitely many members of the set*** B***. We now define*** δ ***to be the smaller of the two numbers*** δ_1 ***and*** δ_2 ***and we observe that, although*** $\delta > 0$***, the interval*** $(x - \delta, x + \delta)$ ***contains only finitely many members of the set*** $A \cup B$***. Therefore no number that lies outside the set*** $\mathbf{L}(A) \cup \mathbf{L}(B)$ ***can be a limit point of*** $A \cup B$ ***and we conclude that***
$$\mathbf{L}(A \cup B) = \mathbf{L}(A) \cup \mathbf{L}(B).$$

7. Is it true that if A and B are sets of real numbers then
$$\mathbf{L}(A \cap B) = \mathbf{L}(A) \cap \mathbf{L}(B)?$$
What if A and B are closed? What if A and B are open? What if A and B are intervals?
The answers are no, no, no and no. Look at the following example:
$$A = [0,1] \text{ and } B = [1,2]$$
These two sets are closed and
$$\mathbf{L}(A \cap B) = \mathbf{L}(\{1\}) = \emptyset$$
while
$$\mathbf{L}(A) \cap \mathbf{L}(B) = [0,1] \cap [1,2] = \{1\}.$$
Now look at the following example:
$$A = (0,1) \qquad \text{and} \qquad B = (1,2).$$
In this case
$$\mathbf{L}(A \cap B) = \mathbf{L}(\emptyset) = \emptyset$$
and
$$\mathbf{L}(A) \cap \mathbf{L}(B) = [0,1] \cap [1,2] = \{1\}.$$

8. Is it true that if $\overline{D} = \mathbf{R}$ then $\mathbf{L}(D) = \mathbf{R}$?

Hint: ***Yes.***

Suppose that $\overline{D} = \mathbf{R}$. We know that whenever a and b are real numbers and $a < b$ there must be members of D lying between a and b. Now suppose that x is a real number. To show that x is a limit point of D, suppose that $\delta > 0$. Since there must be members of D in the interval $(x, x+\delta)$ we conclude that the set $(x-\delta, x+\delta) \cap D \setminus \{x\}$ is nonempty.

9. Given that a set S of real numbers is nonempty and bounded above but that S does not have a largest member, prove that $\sup S$ must be a limit point of S. State and prove a similar result about $\inf S$.
To show that $\sup S$ is a limit point of S, suppose that $\delta > 0$. Since $\sup S - \delta < \sup S$ and since $\sup S$ is the **least** upper bound of S the number $\sup S - \delta$ fails to be an upper bound of S. Choose a member x of S such that $\sup S - \delta < x$. Since $x \leq \sup S$ and since $\sup S$ does not belong to S we have $x < \sup S$. We conclude that
$$(\sup S - \delta, \sup S + \delta) \cap S \setminus \{\sup S\} \neq \emptyset.$$

10. Given any set S of real numbers, prove that the set $\mathbf{L}(S)$ must be closed.
Solution: ***We shall show that any number that fails to belong to*** $\mathbf{L}(S)$ ***must fail to belong to*** $\overline{\mathbf{L}(S)}$***. Suppose that*** $x \in R \setminus L(S)$***. Choose a number*** $\delta > 0$ ***such that the interval*** $(x-\delta, x+\delta)$ ***contains only finitely many members of S. Given any number*** t ***in the interval*** $(x-\delta, x+\delta)$***, it follows from the fact that*** $(x-\delta, x+\delta)$ ***is a neighborhood of*** t ***and the fact that*** $(x-\delta, x+\delta)$ ***contains only finitely many members of*** S ***that*** t ***is not a limit point of S. Thus***
$$(x-\delta, x+\delta) \cap \mathbf{L}(S) = \emptyset$$
and we have shown, as promised, that x ***does not belong to*** $\overline{\mathbf{L}(S)}$***.***

11. Prove that if a set U is open then $\mathbf{L}(U) = \overline{U}$.
Of course $\mathbf{L}(U) \subseteq \overline{U}$. Now suppose that $x \in \overline{U}$. To show that $x \in \mathbf{L}(U)$, suppose that $\delta > 0$. Using the fact that $x \in \overline{U}$, choose a number y in the set $U \cap (x-\delta, x+\delta)$. Using the fact that the set $U \cap (x-\delta, x+\delta)$ is open, choose $\varepsilon > 0$ such that
$$(y-\varepsilon, y+\varepsilon) \subseteq U \cap (x-\delta, x+\delta).$$
We have now found more than one member of U that belongs to the interval $(x-\delta, x+\delta)$ and so we know that
$$U \cap (x-\delta, x+\delta) \setminus \{x\} \neq \emptyset.$$

12. Suppose that S is a set of real numbers, that $\mathbf{L}(S) \neq \emptyset$, and that $\delta > 0$. Prove that there exist two different numbers x and y in S such that $|x-y| < \delta$.
Solution: ***Choose a limit point*** t ***of the set S. Using the fact that the interval*** $(t-\delta/2, t+\delta/2)$ ***contains infinitely many members of S, choose two different members*** x ***and*** y ***of*** S ***that lie in the interval*** $(t-\delta/2, t+\delta/2)$***. We observe that*** $|x-y| < \delta$***.***

Alternative 6: The Topology of Metric Spaces

Some Exercises on Euclidean Spaces

1. Two points $\mathbf{x}$ and $\mathbf{y}$ of the space $\mathbf{R}^k$ are said to be **orthogonal** to one another when $\mathbf{x} \cdot \mathbf{y} = 0$. Prove that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to one another then
$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$
Is the converse of this statement true?
Since

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})$$
$$= \mathbf{x}\cdot\mathbf{x}+\mathbf{y}\cdot\mathbf{x}+\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y} = \mathbf{x}\cdot\mathbf{x}+2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y},$$

the equation $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2+\|\mathbf{y}\|^2$ holds if and only if $\mathbf{x}\cdot\mathbf{y} = 0$.

2. Given any points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^k$, prove that
$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2.$$
This identity is known as the **parallelogram law**. Is there also a parallelogram law for the ∞-norm?
The parallelogram law follows at once when we expand the left side. The parallelogram law does not hold for the ∞-norm. For example
$$\|(1,0)-(0,1)\|_\infty^2+\|(1,0)+(0,1)\|_\infty^2 \neq 2\|(1,0)\|_\infty^2+2\|(0,1)\|_\infty^2.$$

3. a. Prove that if $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbf{R}^k$ and
$$\|\mathbf{x}\| = \|\mathbf{y}\| = \mathbf{x}\cdot\mathbf{y} = 1$$
then $\mathbf{x} = \mathbf{y}$. Hint: Look at the expression $\|\mathbf{x}-\mathbf{y}\|^2$.
The hint says it all:
$$\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})$$
$$= \mathbf{x}\cdot\mathbf{x}-\mathbf{y}\cdot\mathbf{x}-\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y}$$
$$= 1-1-1+1 = 0.$$
It may be worth asking students to interpret this fact in terms of the angle between $\mathbf{x}$ and $\mathbf{y}$ as it appears in Exercise 4.

   b. Prove that if $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbf{R}^k\setminus\{\mathbf{O}\}$ and
$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|$$
then there exists a positive number t such that $\mathbf{x} = t\mathbf{y}$.
We define
$$t = \frac{\|\mathbf{x}\|}{\|\mathbf{y}\|}$$
and we observe that
$$\left\|\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| = \left\|\frac{t\mathbf{y}}{\|\mathbf{x}\|}\right\| = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\cdot\left(\frac{t\mathbf{y}}{\|\mathbf{x}\|}\right) = 1$$
and we deduce from part a that
$$\frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{t\mathbf{y}}{\|\mathbf{x}\|}$$
which gives us $\mathbf{x} = t\mathbf{y}$.

4. If $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbf{R}^k\setminus\{\mathbf{O}\}$ then we define the **angle** between $\mathbf{x}$ and $\mathbf{y}$ to be
$$\arccos\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right).$$

   a. Why is the Cauchy-Schwarz inequality needed to make this definition make sense?
The Cauchy-Schwarz inequality guarantees that
$$\left|\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right| \leq 1.$$

   b. Prove that if θ is the angle between two points $\mathbf{x}$ and $\mathbf{y}$ then
$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta.$$
This equation follows at once from the definition of θ.

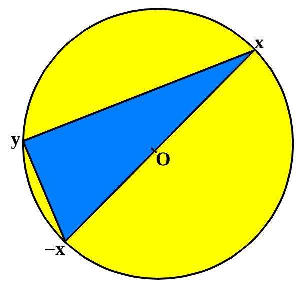
   c. What is the angle between two points $\mathbf{x}$ and $\mathbf{y}$ if these points are orthogonal to one another?
When $\mathbf{x}\cdot\mathbf{y} = 0$ then the angle is $\arccos 0 = \frac{\pi}{2}$.

5. Prove that if $\mathbf{x}$ and $\mathbf{y}$ are any points in $\mathbf{R}^k$ then the points $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are orthogonal to one another if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$. Can you interpret this statement geometrically?

The result follows at once from the equation

$$(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2.$$

A common interpretation of this exercise is that the diagonals of a parallelogram will be perpendicular to each other if and only if the parallelogram is a rhombus. Here is another interpretation:



The angle subtended at any point on a circle by a diameter of the circle is a right angle.

6. In this exercise we suppose that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are points of $\mathbf{R}^k$ and that

$$\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\|.$$

Suppose that $\mathbf{x} = \mathbf{a}+\mathbf{b}+\mathbf{c}$. Prove that the points $\mathbf{x}-\mathbf{a}$ and $\mathbf{b}-\mathbf{c}$ are orthogonal to one another. Deduce two more similar statements and make a geometric interpretation of these statements that concerns the three altitudes of a triangle with vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

We see at once that

$$(\mathbf{x}-\mathbf{a}) \cdot (\mathbf{b}-\mathbf{c}) = (\mathbf{a}+\mathbf{b}+\mathbf{c}-\mathbf{a}) \cdot (\mathbf{b}-\mathbf{c}) = (\mathbf{b}+\mathbf{c}) \cdot (\mathbf{b}-\mathbf{c}) = 0.$$

The point $\mathbf{x}$ is shown to be the common point of intrersection of the three altitudes of the triangle.

7. The **cross product** $\mathbf{a}\times\mathbf{b}$ of two points $\mathbf{a} = (a_1,a_2,a_3)$ and $\mathbf{b} = (b_1,b_2,b_3)$ in $\mathbf{R}^3$ is defined by the equation

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

a. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are points in $\mathbf{R}^3$ then

$$\mathbf{a} \cdot (\mathbf{a}\times\mathbf{b}) = \mathbf{b} \cdot (\mathbf{a}\times\mathbf{b}) = 0.$$

The desired result follows at once when we work out the left side.

b. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are points in $\mathbf{R}^3$ and t is a real number then

$$(t\mathbf{a})\times\mathbf{b} = t(\mathbf{a}\times\mathbf{b}).$$

This result follows directly from the definition.

c. Prove that if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are points in $\mathbf{R}^3$ then

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$$

This result follows directly from the definition.

d. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are points in $\mathbf{R}^3$ then

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2.$$

If you don't want to work with messy computations in this exercise, let Maple do the work for you.

The working steps needed for this result are somewhat messy but they are routine.

e. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are points in $\mathbf{R}^k$ then

$$\|\mathbf{a}\times\mathbf{b}\| \le \|\mathbf{a}\|\|\mathbf{b}\|.$$

This result follows at once from part d.

f. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are points in $\mathbf{R}^k \setminus \{\mathbf{O}\}$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta.$$

Suppose that θ is the angle between **a** and **b**. Since $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ we have

$$\begin{aligned}\|\mathbf{a} \times \mathbf{b}\|^2 &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2 \cos^2\theta \\ &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1 - \cos^2\theta) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 \sin^2\theta.\end{aligned}$$

Since $0 \le \theta \le \pi$ we know that $\sin\theta \ge 0$ and so

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta.$$

g. Prove that if **a**, **b** and **c** are points in $\mathbf{R}^3$ then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

and

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

If you don't want to work with messy computations in this exercise, let Scientific Notebook do the work for you.

Hint: ***The identity***

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

is easy to verify for an arbitrary choice of* b *and* c *in the event that* $\mathbf{a} = (1, 0, 0)$ *and a similar argument shows that the identity holds when* $\mathbf{a} = (0, 1, 0)$ *or* $\mathbf{a} = (0, 0, 1)$. *We now define* G *to be the set of all members* x *of* $\mathbf{R}^3$ *for which the identity

$$\mathbf{x} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{x} \cdot \mathbf{c})\mathbf{b} - (\mathbf{x} \cdot \mathbf{b})\mathbf{c}$$

holds for all points* b *and* c *of* $\mathbf{R}^3$. *It is easy to verify that the sum and difference of any to members of* G *must belong to* G *and that the product of any real number with a member of* G *must belong to* G.

Exercises on Convex Sets

1. Given two distinct points **a** and **b** in $\mathbf{R}^k$, prove that there is exactly one point **x** on the line segment from **a** to **b** such that

$$\|\mathbf{x} - \mathbf{a}\| = \|\mathbf{x} - \mathbf{b}\|.$$

Solution: ***We are being asked to show that there is just one number*** $t \in [0, 1]$ such that

$$\|(1 - t)\mathbf{a} + t\mathbf{b} - \mathbf{a}\| = \|(1 - t)\mathbf{a} + t\mathbf{b} - \mathbf{b}\|$$

This equation says that

$$\|t(\mathbf{b} - \mathbf{a})\| = \|(1 - t)(\mathbf{a} - \mathbf{b})\|$$

which we can express as

$$t\|\mathbf{a} - \mathbf{b}\| = (1 - t)\|\mathbf{a} - \mathbf{b}\|$$

and so we conclude that $t = \frac{1}{2}$.

2. Prove that the intersection of any two convex subsets of $\mathbf{R}^k$ is convex.

Suppose that A and B are convex subsets of $\mathbf{R}^k$ and that **a** and **b** belong to $A \cap B$ and that $0 \le t \le 1$. Since A is convex we know that $(1 - t)\mathbf{a} + t\mathbf{b} \in A$ and since B is convex we know that $(1 - t)\mathbf{a} + t\mathbf{b} \in B$. Therefore $(1 - t)\mathbf{a} + t\mathbf{b} \in \mathbf{A} \cap \mathbf{B}$.

3. Suppose that **a**, **b** and **c** are points in $\mathbf{R}^k$ and that H is the set of all points of the form $r\mathbf{a} + s\mathbf{b} + t\mathbf{c}$ where r, s and t are nonnegative numbers and $r + s + t = 1$. Prove that the set H is convex. Interpret this fact geometrically.

Suppose that r_1, r_2, s_1, s_2, t_1 and t_2 are nonnegative numbers and that $r_1 + s_1 + t_1 = r_2 + s_2 + t_2 = 1$, and suppose that $0 \le u \le 1$. We see that

$$(1-u)(r_1\mathbf{a}+s_1\mathbf{b}+t_1\mathbf{c})+u(r_2\mathbf{a}+s_2\mathbf{b}+t_2\mathbf{c}) = ((1-u)r_1+ur_2)\mathbf{a}+((1-u)s_1+us_2)\mathbf{b} + ((1-u)t_1+ut_2)\mathbf{c}.$$

Since the numbers $(1-u)r_1+ur_2$ and $(1-u)s_1+us_2$ and $(1-u)t_1+ut_2$ are all nonnegative and since

$$\begin{aligned}(1-u)r_1+ur_2+(1-u)s_1+us_2+(1-u)t_1+ut_2 &= (1-u)(r_1+s_1+t_1)+u(r_2+s_2+t_2)\\ &= 1-u+u = 1\end{aligned}$$

we conclude that

$$(1-u)(r_1\mathbf{a}+s_1\mathbf{b}+t_1\mathbf{c})+u(r_2\mathbf{a}+s_2\mathbf{b}+t_2\mathbf{c}) \in H.$$

4. In an earlier exercise we saw the definition of the sum $A+B$ of two sets A and B of real numbers. A similar definition can be given if A and B are subsets of $\mathbf{R}^k$:

$$A+B = \left\{\mathbf{x} \mid \exists\mathbf{a}\in A \text{ and } \exists\mathbf{b}\in B \text{ such that } \mathbf{x} = \mathbf{a}+\mathbf{b}\right\}$$

which we can write more briefly in the form

$$A+B = \left\{\mathbf{a}+\mathbf{b} \mid \mathbf{a}\in A \text{ and } \mathbf{b}\in B\right\}.$$

Prove that if A and B are convex subsets of $\mathbf{R}^k$ then the set $A+B$ is also convex.
Suppose that A and B are convex. Suppose that $\mathbf{a}_1$ and $\mathbf{a}_2$ belong to A and $\mathbf{b}_1$ and $\mathbf{b}_2$ belong to B, and suppose that $0\le t\le 1$. We have

$$(1-t)(\mathbf{a}_1+\mathbf{b}_1)+t(\mathbf{a}_2+\mathbf{b}_2) = ((1-t)\mathbf{a}_1+t\mathbf{a}_2)+((1-t)\mathbf{b}_1+t\mathbf{b}_2)$$

which is the sum of a member of A and a member of B.

5. If A is a subset of $\mathbf{R}^k$ then a **convex combination** of members of A is defined to be any sum of the form

$$r_1\mathbf{a}_1+r_2\mathbf{a}_2+\cdots+r_n\mathbf{a}_n = \sum_{j=1}^{n} r_j\mathbf{a}_j$$

where n is a positive integer and each of the points $\mathbf{a}_1,\ \mathbf{a}_2,\cdots,\mathbf{a}_n$ belongs to the set A and each of the coefficients $r_1,r_2,\cdots,r_n$ is nonnegative and

$$\sum_{j=1}^{n} r_j = 1.$$

Prove that if A is convex then every convex combination of members of A must belong to A.

Solution: ***Suppose that A is a convex set. We prove the assertion by mathematical induction. For each integer $n\ge 2$ we define $p(n)$ to be the assertion that whenever $r_1,r_2,\cdots,r_n$ are nonnegative numbers satisfying the condition***

$$\sum_{j=1}^{n} r_j = 1$$

and whenever $a_1,a_2,\cdots,a_n$ are points of the set A, we have

$$\sum_{j=1}^{n} r_j\mathbf{a}_j \in A.$$

It follows at once from the convexity of the set A that the assertion $p(2)$ is true. Now suppose that n is any integer such that $n\ge 2$ and such that the assertion $p(n)$ is true. Suppose that $r_1,r_2,\cdots,r_n,r_{n+1}$ are nonnegative number, that

$$\sum_{j=1}^{n+1} r_j = 1$$

and that $a_1,a_2,\cdots,a_n,a_{n+1}$ are all points of A. In the event that

$$\sum_{j=1}^{n} r_j = 0$$

we have $r_{n+1} = 1$ ***and***

$$\sum_{j=1}^{n} r_j \mathbf{a}_j = a_{n+1} \in A.$$

We suppose now that

$$\sum_{j=1}^{n} r_j \neq 0$$

and we define

$$t = \sum_{j=1}^{n} r_j.$$

We observe that $r_{n+1} = 1 - t$ ***and that***

$$\begin{aligned}\sum_{j=1}^{n+1} r_j \mathbf{a}_j &= \sum_{j=1}^{n} r_j \mathbf{a}_j + r_{n+1} \mathbf{a}_{n+1} \\ &= t\left(\sum_{j=1}^{n} \frac{r_j}{t} \mathbf{a}_j\right) + (1-t)\mathbf{a}_{n+1}.\end{aligned}$$

Since the assertion $p(n)$ ***is true we know that***

$$\sum_{j=1}^{n} \frac{r_j}{t} \mathbf{a}_j \in A$$

and it follows from the convexity of A ***that***

$$\sum_{j=1}^{n+1} r_j \mathbf{a}_j \in A.$$

6. Given a nonempty subset A of $\mathbf{R}^k$, the **convex hull** $\operatorname{co}(A)$ of A is defined to be the set of all possible convex combinations of points of A.

   a. Prove that if A is any nonempty subset of $\mathbf{R}^k$ then $A \subseteq \operatorname{co}(A)$.
   This assertion is obvious.

   b. Prove that if A and B are nonempty subsets of $\mathbf{R}^k$ and $A \subseteq B$ then $\operatorname{co}(A) \subseteq \operatorname{co}(B)$.
   This assertion is obvious.

   c. Prove that if A is any nonempty subset of $\mathbf{R}^k$ then $\operatorname{co}(A)$ is convex.
   Suppose n is a positive integer, that $\mathbf{a}_j$ and $\mathbf{b}_j$ belong to A and that r_j and s_j are nonnegative numbers for $j = 1, \cdots, n$ and suppose that

$$\sum_{j=1}^{n} r_j = \sum_{j=1}^{n} s_j = 1.$$

   Suppose finally that $0 \leq t \leq 1$. Then we have

$$(1-t)\sum_{j=1}^{n} r_j \mathbf{a}_j + t\sum_{j=1}^{n} s_j \mathbf{b}_j = \sum_{j=1}^{n}\Big((1-t) r_j \mathbf{a}_j + t s_j \mathbf{b}_j\Big).$$

   Since

$$\sum_{j=1}^{n}\Big((1-t) r_j + t s_j\Big) = 1$$

the expression $\sum_{j=1}^{n}\left((1-t)r_j\mathbf{a}_j + ts_j\mathbf{b}_j\right)$ is also a convex combination of points of A.

d. Prove that if A is a nonempty subset of $\mathbf{R}^k$ then A is convex if and only if $A = \mathrm{co}(A)$.
The result follows at once from part c and Exercise 5.

e. Prove that if A and B are nonempty subsets of $\mathbf{R}^k$ and $A \subseteq B$ and B is convex then $\mathrm{co}(A) \subseteq B$.
The result follows at once from parts b and d.

7. a. Prove that if A and B are nonempty subsets of $\mathbf{R}^k$ then

$$\mathrm{co}(A+B) \subseteq \mathrm{co}(A) + \mathrm{co}(B).$$

We shall prove in part c that this inequality can be replaced by an equation.
Suppose that n is a positive integer, that $\mathbf{a}_j \in A$ and $\mathbf{b}_j \in B$ and $r_j \geq 0$ for $j = 1,\cdots,n$ and that $\sum_{j=1}^{n} r_j = 1$. Then

$$\sum_{j=1}^{n} r_j(\mathbf{a}_j + \mathbf{b}_j) = \sum_{j=1}^{n} r_j\mathbf{a}_j + \sum_{j=1}^{n} r_j\mathbf{b}_j \in \mathrm{co}(A) + \mathrm{co}(B).$$

b. Prove that if A and B are nonempty subsets of $\mathbf{R}^k$ and $\mathbf{x} \in \mathrm{co}(A)$ and $\mathbf{y} \in B$ then $\mathbf{x}+\mathbf{y} \in \mathrm{co}(A+B)$.
We assume that $\mathbf{x} \in \mathrm{co}(A)$ and that $\mathbf{y} \in B$.
Choose a positive integer n and members $\mathbf{a}_j$ of the set A and numbers $r_j \geq 0$ for $j = 1,\cdots,n$ such that $\sum_{j=1}^{n} r_j = 1$ and such that $\mathbf{x} = \sum_{j=1}^{n} r_j\mathbf{a}_j$. We see that

$$\mathbf{x}+\mathbf{y} = \sum_{j=1}^{n} r_j\mathbf{a}_j + \sum_{j=1}^{n} r_j\mathbf{y} = \sum_{j=1}^{n} r_j(\mathbf{a}_j + \mathbf{y}) \in \mathrm{co}(A+B).$$

c. Prove that if A and B are nonempty subsets of $\mathbf{R}^k$ and $\mathbf{x} \in \mathrm{co}(A)$ and $\mathbf{y} \in \mathrm{co}(B)$ then

$$\mathbf{x}+\mathbf{y} \in \mathrm{co}(A) + \mathrm{co}(B).$$

Choose a positive integer n and nonnegative numbers r_j and members $\mathbf{a}_j$ of the set A for $j = 1,\cdots,n$ such that $\mathbf{x} = \sum_{j=1}^{n} r_j\mathbf{a}_j$. Then

$$\mathbf{x}+\mathbf{y} = \sum_{j=1}^{n} r_j\mathbf{a}_j + \sum_{j=1}^{n} r_j\mathbf{y} = \sum_{j=1}^{n} r_j(\mathbf{a}_j + \mathbf{y}).$$

From part b we know that each number $\mathbf{a}_j + \mathbf{y}$ must belong to $\mathrm{co}(A+B)$ and since $\mathrm{co}(A+B)$ is convex we have

$$\mathbf{x}+\mathbf{y} = \sum_{j=1}^{n} r_j(\mathbf{a}_j + \mathbf{y}) \in \mathrm{co}(A+B).$$

Exercises on Metric Spaces

1. Suppose that (X,d) is a metric space and that we have defined

$$\delta(x,y) = \begin{cases} d(x,y) & \text{if } d(x,y) \leq 1 \\ 1 & \text{if } d(x,y) > 1 \end{cases}.$$

Prove that δ is also a metric on the set X.
The equation $\delta(x,y) = \delta(y,x)$ either says that $d(x,y) = d(y,x)$ or it says that $1 = 1$. In either case, the equation $\delta(x,y) = \delta(y,x)$ is true.
The equation $\delta(x,y) = 0$ holds if and only if $d(x,y) = 0$ which holds if and only if $x = y$.
Finally suppose that x, y and z belong to X. If either of the numbers $\delta(x,y)$ or $\delta(y,z)$ is equal to 1 then the inequality

$$\delta(x,z) \leq \delta(x,y) + \delta(y,z)$$

is assured. Otherwise

$$\delta(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \delta(x,y) + \delta(y,z)$$

and once again the inequality $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$ holds.

2. Prove that if we define

$$d(\mathbf{a},\mathbf{b}) = |x_1 - x_2| + |y_1 - y_2|$$

whenever $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ are points in $\mathbf{R}^2$ then d is a metric on $\mathbf{R}^2$.

The fact that $d(\mathbf{a},\mathbf{b}) = 0$ if and only if $\mathbf{a} = \mathbf{b}$ is obvious and so is the equation $d(\mathbf{a},\mathbf{b}) = d(\mathbf{a},\mathbf{b})$ for all $\mathbf{a}$ and $\mathbf{b}$. Now suppose that $\mathbf{a} = (x_1, y_1)$, $\mathbf{b} = (x_2, y_2)$ and $\mathbf{c} = (x_3, y_3)$. We observe that

$$\begin{aligned} d(\mathbf{a},\mathbf{c}) &= |x_1 - x_3| + |y_1 - y_3| \\ &= |x_1 - x_2 + x_2 - x_3| + |y_1 - y_2 + y_2 - y_3| \\ &\leq |x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3| \\ &= d(\mathbf{a},\mathbf{b}) + d(\mathbf{b},\mathbf{c}). \end{aligned}$$

3. Prove that if we define

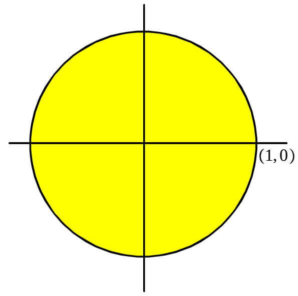
$$d(\mathbf{a},\mathbf{b}) = 2|x_1 - x_2| + 3|y_1 - y_2|$$

whenever $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ are points in $\mathbf{R}^2$ then d is a metric on $\mathbf{R}^2$.

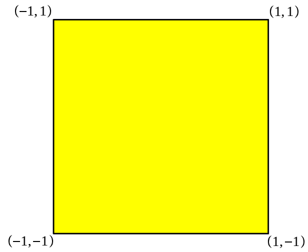
The solution of this exercise is virtually identical to that of Exercise 2.

4. Sketch the ball $B[\mathbf{O},1]$ in $\mathbf{R}^2$ with the Euclidean metric, with the ∞-metric and with each of the metrics defined in Exercises 2 and 3.

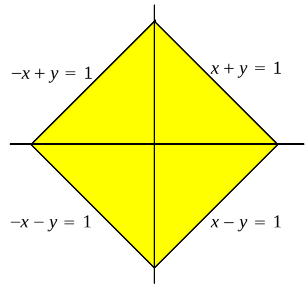
Of course, with the Euclidean metric, the ball is the disk $x^2 + y^2 \leq 1$.



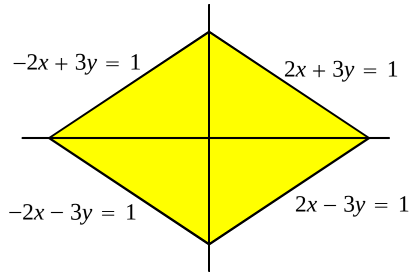
With the ∞-metric the condition for a point (x,y) to lie in the ball is that the larger of $|x|$ and $|y|$ does not exceed 1. In other words, $(x,y) \in B[\mathbf{O},1]$ if and only if $|x| \leq 1$ and $|y| \leq 1$. So the ball is a square.



With the metric of Exercise 2, the condition $(x,y) \in B[\mathbf{O},1]$ says that $|x| + |y| \leq 1$.



Finally, with the metric of Exercise 3, the ball is the set of points (x, y) for which $2|x| + 3|y| \leq 1$



5. Prove that a metric space X is bounded if and only if it is possible to find a positive number r and a member c of the space X such that $X = B(c, r)$.
   If $X = B(c, r)$ and x and y belong to X then
   $$d(x, y) \leq d(x, c) + d(c, y) \leq r + r = 2r$$
   which tells us that $\operatorname{diam}(X) \leq 2r$.
   On the other hand, if we know that X is bounded and choose $r > 0$ such that $d(x, y) < r$ whenever x and y belong to X then, given any point $c \in X$ we have $X = B(c, r)$.

6. Prove that a metric space X is bounded if and only if it is possible to find a positive number r such that for every member c of the space X we have $X = B(c, r)$.
   The solution given to Exercise 5 satisfies the conditions stated in this exercise too.

7. Suppose that x and y are points in a metric space X and that $x \neq y$. Suppose that
   $$\delta = d(x, y).$$
   Prove that
   $$B\left(x, \frac{\delta}{2}\right) \cap B\left(y, \frac{\delta}{2}\right) = \emptyset.$$
   Solution: ***Given any member u of the ball $B\left(x, \frac{\delta}{2}\right)$ we have***
   $$d(x, y) \leq d(x, u) + d(u, y)$$
   and since $d(x, u) < \frac{\delta}{2}$ we obtain
   $$\delta \leq \frac{\delta}{2} + d(u, y)$$
   from which we conclude that $d(u, y) \geq \frac{\delta}{2}$. Therefore no member of the ball $B\left(x, \frac{\delta}{2}\right)$ can belong to the ball $B\left(y, \frac{\delta}{2}\right)$.

8. a. Prove that in the metric space $\mathbf{R}^k$ with the Euclidean metric the diameter of every ball $B(\mathbf{c}, r)$ and every ball $B[\mathbf{c}, r]$ is $2r$.
   On the one hand we know that whenever $\mathbf{x}$ and $\mathbf{y}$ belong to the ball $B[\mathbf{c}, r]$ we have
   $$\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{c}\| + \|\mathbf{c} - \mathbf{y}\| \leq r + r = 2r$$
   and we conclude that the diameter of $B[\mathbf{c}, r]$ does not exceed $2r$. Now, to show that the diameter of $B(\mathbf{c}, r)$ can't be less than $2r$, suppose that $0 < p < 2r$. We shall find two points $\mathbf{x}$ and $\mathbf{y}$ in the ball $B(\mathbf{c}, r)$ such that $\|\mathbf{x} - \mathbf{y}\| > 2r$. We begin by choosing a number q between p and $2r$. Then we define
   $$\mathbf{x} = \mathbf{c} + \frac{q}{2}\mathbf{e}$$
   and
   $$\mathbf{y} = \mathbf{c} - \frac{q}{2}\mathbf{e}$$
   and observe that $\mathbf{x}$ and $\mathbf{y}$ belong to $B(\mathbf{c}, r)$ and that $\|\mathbf{x} - \mathbf{y}\| = q > p$.

   b. Prove that in the metric space $\mathbf{R}^k$ with the ∞-metric the diameter of every cube $I(\mathbf{c}, r)$ and every cube $I[\mathbf{c}, r]$ is $2r$.
   Since the cubes are just the balls in $\mathbf{R}^k$ with the ∞-norm, the solution of this exercise is identical

to that of part a with the Euclidean norm replaced by the ∞-norm.

c. Prove that in the metric space $\mathbf{R}^k$ with the ∞-metric the diameter of every ball $B(\mathbf{c},r)$ and every ball $B[\mathbf{c},r]$ is $2r$.
Given any points $\mathbf{x}$ and $\mathbf{y}$ in $B[\mathbf{c},r]$ we have

$$\|\mathbf{x}-\mathbf{y}\|_\infty \le \|\mathbf{x}-\mathbf{y}\| < 2r.$$

on the other hand, if we follow the argument given in the solution of part a then we obtain

$$\|\mathbf{x}-\mathbf{y}\|_\infty = \left\|\left(\mathbf{c}+\frac{q}{2}\mathbf{e}\right)-\left(\mathbf{c}-\frac{q}{2}\mathbf{e}\right)\right\|_\infty = \|q\mathbf{e}\|_\infty = \|q\mathbf{e}\| = q$$

and so the argument given there applies here too.

d. Prove that in the metric space $\mathbf{R}^k$ with the Euclidean metric the diameter of every cube $I(\mathbf{c},r)$ and every cube $I[\mathbf{c},r]$ is $2r\sqrt{k}$.
Given any points $\mathbf{x}$ and $\mathbf{y}$ in the cube $I[\mathbf{c},r]$ we have

$$\|\mathbf{x}-\mathbf{y}\| \le \sqrt{k}\,\|\mathbf{x}-\mathbf{y}\|_\infty \le 2r\sqrt{k}.$$

Now suppose that $0 < p < 2r\sqrt{k}$. We define $\mathbf{u}$ to be the point $(u_1,u_2,\cdots,u_k)$ of $\mathbf{R}^k$ whose coordinates u_j are all equal to 1, we choose a number q between p and $2r\sqrt{k}$ and we define

$$\mathbf{x} = \mathbf{c}-\frac{q}{2\sqrt{k}}\mathbf{u}$$

and

$$\mathbf{y} = \mathbf{c}+\frac{q}{2\sqrt{k}}\mathbf{u}.$$

The points $\mathbf{x}$ and $\mathbf{y}$ lie in the cube $I(\mathbf{c},r)$ and

$$\|\mathbf{x}-\mathbf{y}\| = \left\|\left(\mathbf{c}+\frac{q}{2\sqrt{k}}\mathbf{u}\right)-\left(\mathbf{c}-\frac{q}{2\sqrt{k}}\mathbf{u}\right)\right\| = \frac{q}{\sqrt{k}}\|\mathbf{u}\| = q > p.$$

9. Prove that if d is the discrete metric on a set X and $c \in X$ then the diameter of the ball $B(c,1)$ is 0 and the diameter of the ball $B[c,1]$ is 1.
These assertions follow at once from the observation that $B(c,1) = \{c\}$ and $B[c,1] = X$. Strictly speaking, the exercise should have stipulated that the set X shoud contain more than one point.

10. Given that $\mathbf{c} \in \mathbf{R}^k$ and $r > 0$ prove that the sets $B(\mathbf{c},r)$, $B[\mathbf{c},r]$, $I(\mathbf{c},r)$ and $I[\mathbf{c},r]$ are all convex.
Suppose that $\mathbf{x}$ and $\mathbf{y}$ belong to $B(\mathbf{c},r)$ and that $0 \le t \le 1$. We see that

$$\begin{aligned}\|\mathbf{c}-((1-t)\mathbf{x}+t\mathbf{y})\| &= \|(1-t)(\mathbf{c}-\mathbf{x})+t(\mathbf{c}-\mathbf{y})\| \\ &\le \|(1-t)(\mathbf{c}-\mathbf{x})\|+\|t(\mathbf{c}-\mathbf{y})\| \\ &< (1-t)r+tr = r.\end{aligned}$$

Thus $B(\mathbf{c},r)$ is convex and we can argue in the same way that $B[\mathbf{c},r]$ is convex. By replacing the Euclidean norm by the ∞-norm we obtain the analogous results for $I(\mathbf{c},r)$ and $I[\mathbf{c},r]$.

11. Suppose that S is a nonempty subset of $\mathbf{R}^k$ and that H is the convex hull of S.

a. Prove that for every point $\mathbf{a} \in \mathbf{R}^k$ we have

$$\sup\{\|\mathbf{a}-\mathbf{x}\| \mid \mathbf{x} \in S\} = \sup\{\|\mathbf{a}-\mathbf{y}\| \mid \mathbf{y} \in H\}.$$

Since $S \subseteq H$ we have

$$\sup\{\|\mathbf{a}-\mathbf{x}\| \mid \mathbf{x} \in S\} \le \sup\{\|\mathbf{a}-\mathbf{y}\| \mid \mathbf{y} \in H\}.$$

on the other hand, if n is a positive integer and $\mathbf{x}_1,\cdots,\mathbf{x}_n$ belong to S and $r_1,\cdots,r_n$ are nonnegative numbers and $\sum_{j=1}^n r_j = 1$ then

$$\left\| \mathbf{a} - \sum_{j=1}^{n} r_j \mathbf{x}_j \right\| = \left\| \sum_{j=1}^{n} r_j \mathbf{a} - \sum_{j=1}^{n} r_j \mathbf{x}_j \right\| = \left\| \sum_{j=1}^{n} r_j (\mathbf{a} - \mathbf{x}_j) \right\|$$

$$\leq \sum_{j=1}^{n} r_j \|(\mathbf{a} - \mathbf{x}_j)\| \leq \sum_{j=1}^{n} r_j \sup\{\|\mathbf{a} - \mathbf{x}\| \mid \mathbf{x} \in S\}$$

$$= \sup\{\|\mathbf{a} - \mathbf{x}\| \mid \mathbf{x} \in S\}$$

b. Prove that

$$\operatorname{diam}(S) = \operatorname{diam}(H).$$

We see at once that $\operatorname{diam}(S) \leq \operatorname{diam}(H)$. Now given any point $\mathbf{x} \in S$ we deduce from part a that

$$\sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in H\} = \sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in S\} = \operatorname{diam}(S).$$

Using part a again we deduce that if $\mathbf{y} \in H$ then

$$\sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x} \in H\} = \sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x} \in S\} \leq \operatorname{diam}(S).$$

Some Exercises on Total Boundedness

1. Given that S is a subset of a metric space X, prove that S is totally bounded if and only if for every $\varepsilon > 0$ it is possible to find finitely many points $x_1, x_2, \cdots, x_n$ that *belong to the set* S such that

$$S \subseteq \bigcup_{j=1}^{n} B(x_j, \varepsilon).$$

Solution: ***The "if" part of this exercise is obvious. We assume that a subset S of a given metric space X is totally bounded and that $\varepsilon > 0$. and we need to find finitely many points $x_1, x_2, \cdots, x_n$ in S such that***

$$S \subseteq \bigcup_{j=1}^{n} B(x_j, \varepsilon).$$

Using the fact that S is totally bounded we choose a positive integer k and points $y_1, y_2, \cdots, y_k$ in X such that

$$S \subseteq \bigcup_{j=1}^{k} B\left(y_j, \frac{\varepsilon}{2}\right).$$

For each j, in the event that the set

$$S \cap B\left(y_j, \frac{\varepsilon}{2}\right)$$

is nonempty, we choose a member of this set and call it x_j. We can now show that S is included in the union of the balls $B(x_j, \varepsilon)$ where j runs through those integers for which x_j has been defined:

Suppose that $x \in S$ and, using the fact that

$$x \in \bigcup_{j=1}^{k} B\left(y_j, \frac{\varepsilon}{2}\right)$$

choose j such that

$$x \in B\left(y_j, \frac{\varepsilon}{2}\right).$$

Since

$$B\left(y_j, \frac{\varepsilon}{2}\right) \cap S \neq \emptyset$$

we know that x_j is defined and we have

$$d(x, x_j) \leq d(x, y_j) + d(y_j, x_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

2. Given that S is an infinite set, find a bounded subset E of the metric space $\ell^\infty(S)$ such that E fails to be totally bounded.
See the solution of Exercise 3.

3. True or false? If x is a point in a metric space X then there exists a number $\varepsilon > 0$ such that the ball $B(x,\varepsilon)$ is totally bounded.
We take an infinite set S and define $X = \ell^\infty(S)$. Suppose that $\varepsilon > 0$. For each $x \in S$ we define the function $f_x : S \to \mathbf{R}$ by the equation
$$f_x(t) = \begin{cases} 0 & \text{if } t \in S \setminus \{x\} \\ \frac{\varepsilon}{2} & \text{if } t = x \end{cases}.$$
The set E of all these functions f_x fails to be totally bounded because no ball with radius $\frac{\varepsilon}{2}$ can contain more than one member of E. Since E is included in the ball $B(\mathbf{O},\varepsilon)$ where $\mathbf{O}$ is the constant function 0 we conclude that the assertion made in Exercise 3 is false.

4. Prove that if S is an infinite totally bounded subset of a metric space X and if $\varepsilon > 0$ then it is possible to find two different members x and y of the set S such that $d(x,y) < \varepsilon$.
Suppose that S is an infinite totally bounded subset of a metric space X and that $\varepsilon > 0$. Choose finitely many sets with diameter less than ε whose union includes S. At least one of these sets must contain more than one member of S.

5. True or false? If X is a metric space then the following two conditions are equivalent:

   a. No infinite subset of X is totally bounded.

   b. There exists a number $\varepsilon > 0$ such that for every pair x and y of different points of X we have $d(x,y) \geq \varepsilon$.

   If you decide that these two conditions are equivalent, prove that each implies the other. If you decide that one of these conditions is sufficient but not necessary for the other, supply a proof and a counter example. If you decide that neither of these statements implies the other, supply two counter examples.
The fact that condition b implies condition a follows at once from Exercise 4.
Condition a does not imply condition b. We define
$$X = \mathbf{Z}^+ \cup \left\{ n + \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\}.$$
An infinite subset of this set X has to be unbounded but, in spite of this, condition b does not hold.

Exercises on Open Sets and Closed Sets

1. Given a subset H of a metric space X, prove that the following conditions are equivalent:

   a. The set H is closed.

   b. For every point $x \in X \setminus H$ there exists a number $\delta > 0$ such that
$$B(x,\delta) \cap H = \emptyset.$$
Condition b says that for every $x \in X \setminus H$ there exists a number $\delta > 0$ such that $B(x,\delta) \subseteq X \setminus H$ and this is the condition for the set $X \setminus H$ to be open.

2. Prove that if X is any metric space then the singleton $\{x\}$ is closed. Then use the fact that every finite set is a finite union of singletons to show that every finite subset of X must be closed.

   Solution: ***Suppose that*** x ***is a point in a metric space*** X***. We need to show that the set*** $X \setminus \{x\}$ ***is open. Suppose that*** $y \in X \setminus \{x\}$***. We define*** $\delta = d(x,y)$ ***and we observe that*** $\delta > 0$ ***and that***
$$B(y,\delta) \subseteq X \setminus \{x\}.$$

3. Give an example of a sequence (U_n) of open subsets of the metric space $\mathbf{R}$ such that the set

$$\bigcap_{n=1}^{\infty} U_n$$

fails to be open.

We have observed that

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right) = \{0\}.$$

4. Given that H is a closed subset of $\mathbf{R}$ and that every rational number belongs to H, prove that $H = \mathbf{R}$.

   Solution: ***The set $\mathbf{R} \setminus H$ is open in $\mathbf{R}$ and contains no rational number. To prove that $\mathbf{R} \setminus H$ must be empty we shall observe that every nonempty open set must contain a rational number. Suppose that U is open and nonempty. Choose $x \in U$ and choose $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$. Since the interval $(x-\delta, x+\delta)$ must contain rational numbers, so must the set U.***

5. Given that H is a closed subset of $\mathbf{R}^2$ and that $\mathbf{Q} \times \mathbf{Q} \subseteq H$, prove that $H = \mathbf{R}^2$.
   We need to show that the open set $\mathbf{R}^2 \setminus H$ is empty. To obtain a contradiction, suppose that $\mathbf{R}^2 \setminus H \neq \emptyset$ and choose a point $(a,b) \in \mathbf{R}^2 \setminus H$. Using the theorem on open subsets of $\mathbf{R}^k$ we now choose $\delta > 0$ such that whenever $a-\delta < x < a+\delta$ and $b-\delta < y < b+\delta$ we have $(x,y) \in \mathbf{R}^2 \setminus H$. Using the theorem on densesess of the set of rational numbers we choose rational numbers x and y such that $a-\delta < x < a+\delta$ and $b-\delta < y < b+\delta$ and we observe that $(x,y) \in \mathbf{Q} \times \mathbf{Q} \setminus H$, contradiction our assumption that $\mathbf{QQ} \subseteq H$.

6. Given that U is an open subset of a metric space X and that H is closed, prove that the set $U \setminus H$ is open and that the set $H \setminus U$ is closed.
   Since
   $$U \setminus H = U \cap (X \setminus H)$$
   which is the intersection of two open sets, the set $U \setminus H$ is open. Since
   $$H \setminus U = H \cap (X \setminus U)$$
   which is the intersection of two closed sets, the set $H \setminus U$ is closed.

7. Prove that every nonempty closed set of real numbers that is bounded below must have a least member.
   Suppose that H is a nonempty closed set of real numbers and that H is bounded below. We need to show that $\inf H \in H$. To obtain a contradiction, suppose that $\inf H$ belongs to the open set $\mathbf{R} \setminus H$. Choose $\delta > 0$ such that $(\inf H - \delta, \inf H + \delta) \subseteq \mathbf{R} \setminus H$. Since no member of H can be less than $\inf H$ and no member of H can lie in the interval $(\inf H - \delta, \inf H + \delta)$ we see that the number $\inf H + \delta$ is a lower bound of H, contradicting the fact that $\inf H$ is the **greatest** lower bound of H.

8. If A and B are subsets of the space $\mathbf{R}^k$ for a given positive integer k, then the **sum** $A + B$ of A and B is defined, by analogy with an earlier exercise. Prove that if A is any subset of $\mathbf{R}^k$ and U is an open subset of $\mathbf{R}^k$ then the set $A + U$ is open.

   Solution: ***We need to show that every member of $A + U$ is an interior point of $A + U$. Suppose that $x \in A + U$. Choose a member a of A and a member u of U such that $x = a + u$. Using the fact that $u \in U$, choose $\delta > 0$ such that $B(u,\delta) \subseteq U$. We shall show that x is an interior point of $A + U$ by showing that***
   $$B(x,\delta) \subseteq A + U.$$
   Suppose that $t \in B(x,\delta)$. Thus
   $$\|t - (a+u)\| < \delta$$
   which gives us
   $$\|(t-a) - u\| < \delta$$
   and so
   $$t - a \in B(u,\delta) \subseteq U.$$
   Since

$$t = a + (t - a)$$

we see that** $t \in A + U$ **and so

$$B(x, \delta) \subseteq A + U$$

as promised.

9. Prove that the interval $(1, 3]$ is not an open subset of the metric space $\mathbf{R}$ but that this interval is an open subset of the metric space $[0, 3]$ which is a subspace of $\mathbf{R}$.
   Given any positive number δ, the ball in the metric space $\mathbf{R}$ with center 3 and radius δ will contain all of the numbers between 3 and $3 + \delta$ and will therefore fail to be included in the interval $(1, 3]$. Thus, in the metric space $\mathbf{R}$, the point 3 fails to be an interior point of the interval $(1, 3]$ and therefore this interval fails to be open in the metric space $\mathbf{R}$. However, if $0 < \delta < 1$ then, in the metric space $[0, 3]$, the ball center 3 with radius δ is the interval $(3 - \delta, 3]$ which is included in the interval $(1, 3]$. Since all other members of this interval are obviously interior points, this interval is open in the metric space $[0, 3]$.

10. Given that S is a nonempty subset of a metric space (X, d) and that U is an open subset of the metric space (X, d) and that $U \subseteq S$, prove that U must be open in the metric space (S, d).
    Suppose that $x \in U$. Using the fact that U is open in the space (X, d), choose a number $\delta > 0$ such that $B(x, \delta) \subseteq U$. In the metric space (S, d), the ball center x with radius δ is $S \cap B(x, \delta)$ and since
    $$S \cap B(x, \delta) \subseteq B(x, \delta) \subseteq U$$
    we have shown that x is an interior point of the set U in the metric space (S, d).

11. Suppose that S is a nonempty subset of a metric space (X, d) and that $U \subseteq S$. Prove that the following two conditions are equivalent:
    a. The set U is open in the metric space (S, d).
    b. There exists an open subset V of the metric space (X, d) such that
    $$U = S \cap V.$$

Solution: ***First we prove that condition b implies condition a. Suppose that condition b holds and choose an open subset*** V ***of the metric space*** X ***such that*** $U = S \cap V$***. To prove that the set*** U ***is open in the subspace*** S***, suppose that*** $x \in U$***. Using the fact that*** $x \in V$ ***and the fact that*** V ***is open in the metric space*** X ***we now choose*** $\delta > 0$ ***such that***

$$B(x, \delta) \subseteq V$$

and we observe that

$$\{y \in S \mid d(x, y) < \delta\} \subseteq S \cap V.$$

Since the set $\{y \in S \mid d(x, y) < \delta\}$ ***is the ball center*** x ***with radius*** δ ***in the metric space*** S***, we have shown that the set*** U ***is open in the metric space*** S***.***

Now we want to prove that condition a implies conditio b. Suppose that condition a holds. In other words, suppose that the set U ***is open in the metric space*** S***. We know that for every point*** $x \in U$ ***it is possible to find a positive number*** δ ***such that***

$$B(x, \delta) \cap S \subseteq U.$$

We now define V ***to be the union of all the balls of the form*** $B(x, \delta)$ ***for which*** $x \in U$ ***and*** $\delta > 0$ ***and***

$$B(x, \delta) \cap S \subseteq U.$$

The set V***, being a union of balls in the space*** X***, must be open in*** X***. Finally, we need to explain why*** $U = V \cap S$***.***

Now given any point $x \in U$ ***we can choose*** $\delta > 0$ ***such that***

$$B(x, \delta) \cap S \subseteq U$$

and since $x \in B(x, \delta)$ ***we have*** $x \in V$***.***

Now suppose that $x \in V \cap S$***. Using the definition of the set*** V ***we choose*** $y \in U$ ***and*** $\delta > 0$ ***such that***

$$B(y,\delta) \cap S \subseteq U$$

and $x \in B(y,\delta)$***. Thus*** $x \in B(y,\delta) \cap S \subseteq U$***.***

12. Given that S is an open nonempty subset of a metric space (X,d) and that U is an open subset of the metric space (S,d), prove that U must be open in the metric space (X,d).
Using Exercise 11 we choose a set V that is open in the metric space (X,d) such that $U = V \cap S$. Since both of the sets S and V are open in the space (X,d), so is their intersection, which is U.

13. Given that S is a nonempty subset of a metric space (X,d) and that H is a closed subset of the metric space (X,d) and that $H \subseteq S$, prove that H must be closed in the metric space (S,d).
From Exercise 11 and the fact that the set $X \setminus H$ is open in the metric space (X,d) we conclude that the set $S \cap (X \setminus H)$ is open in the metric space (S,d). Since
$$S \cap (X \setminus H) = S \setminus H$$
we conclude that $S \setminus H$ is open in the space (S,d) which tells us that H is closed in the space (S,d).

14. Given that S is a closed nonempty subset of a metric space (X,d) and that H is a closed subset of the metric space (S,d), prove that H must be closed in the metric space (X,d).
Using Exercise 11 and the fact that $S \setminus H$ is open in the space (S,d) we choose a set V that is open in the space X such that
$$S \setminus H = S \cap V$$
and we observe that
$$H = S \cap (X \setminus V).$$
Thus H, being the intersection of two sets that are both closed in the space (X,d), must be closed in the space (X,d).

15. Skip this exercise if you are not familiar with the concept of an uncountable set. Prove that every nonempty open subset of $\mathbf{R}^k$ is uncountable.
The proof follows at once from an earlier theorem about equivalence to $\mathbf{R}$ of subsets of $\mathbf{R}$ that include intervals.

16. Suppose that (X,d) is a given metric space and that we have defined
$$\delta(x,y) = \begin{cases} d(x,y) & \text{if} \quad d(x,y) \leq 1 \\ 1 & \text{if} \quad d(x,y) > 1 \end{cases}.$$

    a. Prove that the function δ is also a metric on the set X.

    b. Prove that the metric spaces (X,d) and (X,δ) have exactly the same open sets.

We saw in an earlier exercise that the function δ is a metric. Part b follows from the fact that whenever $x \in X$ and $0 < r < 1$, the ball with center x and radius r n the metric space (X,d) is exactly the same as the ball with center x and radius r in the metric space (X,δ).

17. This exercise refers to the optional topic of a convex hull that appears in the reading material on convexity. Prove that if U is an open subset of $\mathbf{R}^k$ then the convex hull of U is open.
We assume that U is an open subset of $\mathbf{R}^k$. Suppose that $\mathbf{x}$ is any member of the convex hull $\mathrm{co}(U)$ of U and choose a positive integer n and numbers $r_j \in [0,1]$ for $j = 1,2,\cdots,n$ such that $\sum_{j=1}^{n} r_j = 1$ and
$$x = \sum_{j=1}^{n} r_j \mathbf{x}_j.$$
We may assume that the coefficient r_1 is not zero. Now using the fact that U is open and that $x_1 \in U$ we choose $\delta > 0$ such that $B(\mathbf{x}_1,\delta) \subseteq U$. We see that if $\mathbf{y}$ is any member of the ball $B(\mathbf{x}_1, r_1\delta)$, since

$$\mathbf{y} = \mathbf{y} - \mathbf{x} + \sum_{j=1}^{n} r_j \mathbf{x}_j$$
$$= r_1 \left(\frac{\mathbf{y} - \mathbf{x}}{r_1} + \mathbf{x}_1 \right) + \sum_{j=2}^{n} r_j \mathbf{x}_j$$

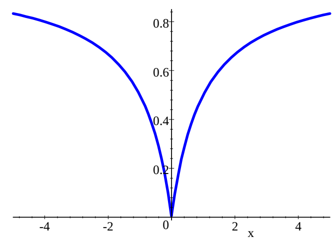
and since

$$\frac{\mathbf{y} - \mathbf{x}}{r_1} + \mathbf{x}_1 \in B(\mathbf{x}_1, \delta) \subseteq U$$

we see that $\mathbf{y} \in \mathrm{co}(U)$.

18. The purpose of this exercise is to exhibit an example of a closed subset H of $\mathbf{R}^2$ whose convex hull fails to be closed. We define

$$f(x) = \begin{cases} 1 - \frac{1}{1+x} & \text{if } x \geq 0 \\ 1 - \frac{1}{1-x} & \text{if } x < 0 \end{cases}$$



and we define

$$H = \{(x,y) \mid 0 \leq y \leq f(x)\}.$$

Prove that the set H has the desired properties.

Given any positive integer n and points $A_j = (x_j, y_j)$ in the set H for $j = 1, 2, \cdots, n$ and nonnegative numbers r_j for which $\sum_{j=1}^{n} r_j = 1$ we have

$$\sum_{j=1}^{n} r_j A_j = \sum_{j=1}^{n} r_j (x_j, y_j) = \left(\sum_{j=1}^{n} r_j x_j, \sum_{j=1}^{n} r_j y_j \right)$$

and since $\sum_{j=1}^{n} r_j y_j < 1$ we know that the point $(0,1)$ does not belong to $\mathrm{co}(H)$. On the other hand, whenever $p > 0$ the point

$$\left(0, \frac{p^2}{p^2 - 1} \right) = \frac{1}{2} \left(p, 1 - \frac{1}{1+p} \right) + \frac{1}{2} \left(-p, 1 - \frac{1}{1-p} \right)$$

must belong to $\mathrm{co}(H)$ and so the point $(0,1)$ is close to $\mathrm{co}(H)$. Therefore $\mathrm{co}(H)$ is not closed.

We can, of course, do much better. In fact

$$\mathrm{co}(H) = \{(0,0)\} \cup \{(x,y) \in \mathbf{R}^2 \mid 0 < y < 1\}.$$

Exercises on Closure

1. Suppose that

$$S = [0,1) \cup (1,2).$$

a. What is the set of interior points of S in the metric space $\mathbf{R}$?

The set of interior points of S is $(0,1) \cup (1,2)$.

b. Given that U is the set of interior points of S, evaluate $\overline{U}$.

$$\overline{(0,1) \cup (1,2)} = [0,2]$$

c. Give an example of a set S of real numbers such that if U is the set of interior points of S then $\overline{U} \neq \overline{S}$.
We could take S to be a singleton like $\{3\}$ or it could be the set of all integers. It could also be the set of all rational numbers between 0 and 1.

d. Give an example of a subset S of the interval $[0,1]$ such that $\overline{S} = [0,1]$ but if U is the set of interior points of S then $\overline{U} \neq [0,1]$.
Take S to be the set of rational numbers that lie in the interval $[0,1]$. The set of interior points of S is empty.

2. Given that

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbf{Z}^{+} \right\},$$

evaluate $\overline{S}$.

Hint: ***Show that***

$$\overline{S} = \{0\} \cup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbf{Z}^{+} \right\}.$$

First show that $0 \in \overline{S}$***. Then observe that every negative number belongs to the set*** $(-\infty, 0)$ ***and that if*** x ***is any positive number then*** x ***belongs to the interval***

$$\left(\frac{1}{n+1}, \frac{1}{n} \right)$$

for some positive integer n***.***

3. Given that S is a subset of a metric space, that H is a closed set and that $S \subseteq H$, prove that $\overline{S} \subseteq H$.
From the elementary properties of closure we know that $\overline{S} \subseteq \overline{H}$ and we know from the relationship between closure and the property of being closed that $\overline{H} = H$.

4. Given two subsets A and B of a metric space, prove that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Solution: ***Since*** $A \subseteq \overline{A}$ ***and*** $B \subseteq \overline{B}$ ***we have***

$$A \cup B = \overline{A} \cup \overline{B}.$$

and therefore, since the union of the two closed sets $\overline{A}$ ***and*** $\overline{B}$ ***is closed we have***

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

On the other hand, since A ***is included in the closed set*** $\overline{A \cup B}$ ***we have***

$$\overline{A} \subseteq \overline{A \cup B}$$

and, similarly we can see that

$$\overline{B} \subseteq \overline{A \cup B}$$

and so

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

Therefore

$$A \cup B = \overline{A} \cup \overline{B}.$$

5. Given two subsets A and B of a metric space, prove that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Do the two sides of this inclusion have to be equal? What if A and B are open? What if they are closed?
From the fact that $A \cap B \subseteq A$ we deduce that $\overline{A \cap B} \subseteq \overline{A}$ and we see similarly that $\overline{A \cap B} \subseteq \overline{B}$. Therefore $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
In the metric space $\mathbf{R}$, if $A = (0,1)$ and $B = (1,2)$ then A and B are open and $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.
On the other hand, if A and B are closed subsets of a metric space then

$$\overline{A \cap B} = A \cap B = \overline{A} \cap \overline{B}.$$

6. Prove that if S is any subset of a metric space X then the set $X \setminus \overline{S}$ is the set of interior points of the set $X \setminus S$.
   Given any point $x \in X$, the condition $x \in X \setminus \overline{S}$ means that there exists a number $\delta > 0$ such that $B(x,\delta) \cap S = \emptyset$ and this condition says that there exists a number $\delta > 0$ such that $B(x,\delta) \subseteq X \setminus S$. The latter assertion says that x is an interior point of the set $X \setminus S$.

7. Given that α is an upper bound of a given set S of real numbers, prove that the following two conditions are equivalent:

   a. We have $\alpha = \sup S$.

   b. We have $\alpha \in \overline{S}$.

   To prove that condition a implies condition b we assume that $\alpha = \sup S$. We need to show that $\alpha \in \overline{S}$. Suppose that $\delta > 0$. Using the fact that α is the **least** upper bound of S and that $\alpha - \delta < \alpha$ we choose a member x of S such that $\alpha - \delta < x$. Since $x \in (\alpha - \delta, \alpha + \delta) \cap S$ we have $(\alpha - \delta, \alpha + \delta) \cap S \neq \emptyset$.
   To prove that condition b implies condition a we assume that $\alpha \in \overline{S}$. We need to show that α is the least upper bound of S. Suppose that $p < \alpha$. Since the set (p,∞) is a neighborhood of α we have $(p,\infty) \cap S \neq \emptyset$. Thus, since α is an upper bound of S and since no number $p < \alpha$ can be an upper bound of S we conclude that α is the least upper bound of S.

8. Is it true that if A and B are sets of real numbers and
   $$\overline{A} = \overline{B} = \mathbf{R}$$
   then $\overline{A \cap B} = \mathbf{R}$?
   The answer is no. Look at $A = \mathbf{Q}$ and $B = \mathbf{R} \setminus \mathbf{Q}$.

9. Prove that a subset S of a metric space X is dense in X if and only if we have
   $$S \cap U \neq \emptyset$$
   whenever U is a nonempty open subset of X.
   Suppose that S is dense in X and that U is a nonempty open set. Choose $x \in U$. Since $x \in \overline{S}$ and U is a neighborhood of x we know that $U \cap S \neq \emptyset$.
   Now suppose that the condition $U \cap S \neq \emptyset$ holds for every nonempty open subset U of X. To show that $\overline{S} = X$, suppose that $x \in X$ and that $\varepsilon > 0$. Since the ball $B(x,\varepsilon)$ is a nonempty open set we must have $B(x,\varepsilon) \cap S \neq \emptyset$.

10. Prove that if U and V are open dense subsets of a metric space X then the set $U \cap V$ is also dense in X. What if only one of the two sets U and V is open?

    Solution: ***All we need to know is that at least one of the sets*** U ***and*** V ***is open. Suppose that*** A ***and*** B ***are sets of real numbers, that***
    $$\overline{U} = \overline{V} = X$$
    and that the set U ***is open.***
    To prove that
    $$\overline{U \cap V} = X,$$
    suppose that $x \in X$ ***and that*** $\delta > 0$***. Since*** $x \in \overline{U}$ ***we know that the set*** $B(x,\delta) \cap U$ ***is nonempty and we also know that this set is open. Therefore, since*** $\overline{V} = X$ ***we know that***
    $$B(x,\delta) \cap U \cap V \neq \emptyset.$$
    We have therefore shown that every point of the space X ***must belong to*** $\overline{U \cap V}$***.***

11. Skip this exercise if you are not familiar with the concept of a countable set. Find a sequence (U_n) of dense open subsets of the metric space $\mathbf{Q}$ such that
    $$\bigcap_{n=1}^{\infty} U_n = \emptyset.$$

Using the fact that the set $\mathbf{Q}$ of rational numbers is countable we express $\mathbf{Q}$ in the form

$$\mathbf{Q} = \{r_1, r_2, r_3, \cdots, r_n, \cdots\}.$$

We now define

$$U_n = \{r_j \mid j \geq n\}$$

for each positive integer n. Since the set $\mathbf{Q} \setminus U_n$ is finite for each n we know that each set U_n is open in the metric space $\mathbf{Q}$. Finally, since every neighborhood of a point in $\mathbf{Q}$ must contain infiitely many rational numbers, each neighborhood of a point of $\mathbf{Q}$ must intersect with each of the sets $\mathbf{Q} \setminus U_n$ and so each of the sets $\mathbf{Q} \setminus U_n$ must be dense.

12. Skip this exercise if you are not familiar with the concept of a countable set. Prove that if S is a countable subset of the metric space $\mathbf{R}$ then $\mathbf{R} \setminus S$ is dense in $\mathbf{R}$. Extend this fact to the metric space $\mathbf{R}^k$ for an arbitrary positive integer k.
Suppose that x is a real number and that $\delta > 0$. Since the interval $(x - \delta, x + \delta)$ is uncountable we know that $(x - \delta, x + \delta) \setminus S \neq \emptyset$, in other words

$$(x - \delta, x + \delta) \cap (\mathbf{R} \setminus S) \neq \emptyset.$$

Therefore $\mathbf{R} \setminus S$ is dense in the metric space $\mathbf{R}$.
Now we repeat the same argument in $\mathbf{R}^k$. We assume that k is a positive integer and that S is a countable subset of $\mathbf{R}^k$. Suppose that $\mathbf{x} \in \mathbf{R}^k$ and that $\delta > 0$. Since the ball $B(\mathbf{x}, \delta)$ is uncountable we know that $B(\mathbf{x}, \delta) \setminus S \neq \emptyset$, in other words

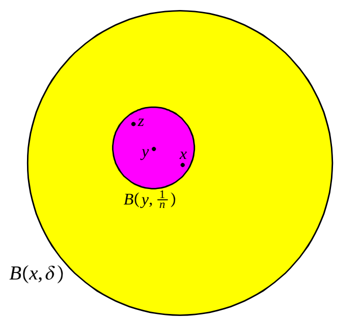
$$B(\mathbf{x}, \delta) \cap (\mathbf{R}^k \setminus S) \neq \emptyset.$$

Therefore $\mathbf{R}^k \setminus S$ is dense in the metric space $\mathbf{R}^k$.

13. Suppose that D is a dense subset of a metric space X and that U is a neighborhood of a point $x \in X$. Prove that there exists a point $y \in D$ and a rational number $r > 0$ such that

$$x \in B(y, r) \subseteq U.$$

Choose $\delta > 0$ such that $B(x, \delta) \subseteq U$. Using the fact that D is dense we now choose a member y of the set D such that $d(x, y) < \frac{\delta}{3}$. Finally we choose a positive integer $n > \frac{3}{\delta}$ which makes $\frac{1}{n} < \frac{\delta}{3}$.



We shall now observe that

$$B\left(y, \frac{1}{n}\right) \subseteq B(x, \delta).$$

Suppose that $z \in B(y, \frac{1}{n})$. We observe that

$$d(x, z) \leq d(x, y) + d(y, z) < \frac{\delta}{3} + \frac{1}{n} < \frac{\delta}{3} + \frac{\delta}{3} < \delta.$$

14. Given a point x in a metric space X and given $r > 0$, prove that

$$\overline{B(x, r)} \subseteq B[x, r].$$

Give an example to show that the latter inequality does not have to be an equation.
We know from an earlier theorem that the set $B[x, r]$ is a closed set that includes $B(x, r)$.

15. Prove that if $\mathbf{x} \in \mathbf{R}^k$ and $r > 0$ then

$$\overline{B(x, r)} = B[x, r]$$

and

$$\overline{I(x,r)} = I[x,r].$$

Both assertions follow automatically from Exercise 14.

16. Which of the following pairs of sets are separated from each other in the metric space $\mathbf{R}$?

  a. $[0,1]$ and $[2,3]$. Yes.

  b. $(0,1)$ and $(1,2)$. Yes.

  c. $(0,1]$ and $(1,2)$. No because $(0,1] \cap \overline{(1,2)} = \{1\} \neq \emptyset$.

  d. $\mathbf{Q}$ and $\mathbf{R} \setminus \mathbf{Q}$. No.

17. Prove that if A and B are closed in a metric space X and disjoint from one another then A and B are separated from each other.
Suppose that A and B are closed and disjoint from one another. Since $A = \overline{A}$ and $B = \overline{B}$, the fact that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ follows at once from the fact that $A \cap B = \emptyset$.

18. Prove that if A and B are open in a metric space X and disjoint from one another then A and B are separated from each other.
Suppose that A and B are open and disjoint from one another. Given any number $x \in A$, we deduce from the fact that A is a neighborhood of x and $A \cap B = \emptyset$ that x is not close to B. Therefore $A \cap \overline{B} = \emptyset$ and we see similarly that $\overline{A} \cap B = \emptyset$.

19. Suppose that S is a subset of a metric space X. Prove that the two sets S and $X \setminus S$ will be separated from each other if and only if the set S is both open and closed. What then do we know about the subset S of the metric space $\mathbf{R}$ for which S and $\mathbf{R} \setminus S$ are separated from each other?
Suppose that S and $X \setminus S$ are separated from each other. To show that S is open, suppose that $x \in S$. Since $S \cap \left(\overline{X \setminus S}\right) = \emptyset$ we know that x is not close to $X \setminus S$. Choose $\delta > 0$ such that
$$(x-\delta, x+\delta) \cap (X \setminus S) = \emptyset$$
and observe that $(x-\delta, x+\delta) \subseteq S$. Thus S is open and a similar argument shows that $X \setminus S$ is also open. We therefore know that if the sets S and $X \setminus S$ are separated from one another then S is both open and closed.
Now suppose that S is both open and closed. Since the two set S and $X \setminus S$ are closed and disjoint from one other they are separated from one another.

20. Prove that a metric space X is connected if and only if it cannot be written as the union of two nonempty sets that are separated from one another.
This assertion follows at once from Exercise 19.

21. Prove that if S is a connected subspace of a metric space X then the subspace $\overline{S}$ is also connected.
We want to show that if the space $\overline{S}$ fails to be connected then the space S must also fail to be connected.
Assume that the space $\overline{S}$ fails to be connected and choose a nonempty open closed subset E of the space $\overline{S}$ such that $E \neq \overline{S}$. In order to show that S fails to be connected we shall show that both $S \cap E$ and $S \setminus E$ are nonempty open subsets of the space S.
Using the fact that E is open in $\overline{S}$ and the theorem on open subsets of a subspace, choose a set V that is open in the metric space X such that $E = \overline{S} \cap V$. Since $S \cap E = S \cap V$ the set $S \cap E$ is open in the metric space S. Using the fact that $E \neq \emptyset$ choose a point $x \in E$. Since V is a neighborhood of x and $x \in \overline{S}$ we know that $S \cap V \neq \emptyset$. This shows that $S \cap E \neq \emptyset$.
The same argument applied to the set $\overline{S} \setminus E$ shows that $S \setminus E$ is a nonempty open subset of the metric space S. Therefore the space S fails to be connected, as we promised.

22. True or false? If S is a bounded subset of a metric space X then the set $\overline{S}$ is also bounded.
This statement is true. Suppose that S is a bounded subset of a metric space X and choose a number p such that the inequality $d(x,y) \leq p$ holds whenever x and y belong to S. We shall now observe that the ame inequality holds for all members x and y of $\overline{S}$. In other words, $\operatorname{diam}(\overline{S}) \leq p$.

To obtain a contradiction, suppose that x and y belong to $\overline{S}$ and that $d(x,y) > p$. We define

$$\delta = d(x,y) - p.$$

Using the fact that x and y are close to S we choose members u and v of S such that $u \in B\left(x, \frac{\delta}{2}\right)$ and $v \in B\left(y, \frac{\delta}{2}\right)$. We observe that

$$\begin{aligned} d(x,y) &\leq d(x,u) + d(u,v) + d(v,y) \\ &< \frac{\delta}{2} + d(u,v) + \frac{\delta}{2} = d(u,v) + \delta = d(u,v) + d(x,y) - p \end{aligned}$$

which implies that $d(u,v) > p$, contradicting the way in which p was chosen.

23. True or false? If S is a totally bounded subset of a metric space X then the set $\overline{S}$ is also totally bounded.
This statement is true. Suppose that S is a totally bounded subset of a metric space X and that $\varepsilon > 0$. Using the fact that S is totally bounded we choose a positive integer n and points $x_1, x_2, \cdots, x_n$ such that

$$S \subseteq \bigcup_{j=1}^{n} B\left(x_j, \frac{\varepsilon}{2}\right).$$

We shall now show that

$$\overline{S} \subseteq \bigcup_{j=1}^{n} B(x_j, \varepsilon).$$

Suppose that $y \in \overline{S}$. Choose a member x of S such that $d(x,y) < \frac{\varepsilon}{2}$. Choose j such that $x \in B\left(x_j, \frac{\varepsilon}{2}\right)$. Since

$$d(y,x_j) \leq d(y,x) + d(x,x_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

we see that $y \in B(x_j, \varepsilon)$.

24. Prove that the closure of a convex subset of $\mathbf{R}^k$ is also convex.

25. Suppose that S is a convex subset of $\mathbf{R}^k$, that $\mathbf{x}$ is an interior point of S, that $\mathbf{y} \in \overline{S}$ and that $0 < t < 1$. Prove that the point

$$(1-t)\mathbf{x} + \mathbf{y}$$

is an interior point of S.

26. This exercise refers to the notion of a subgroup of $\mathbf{R}$ that was introduced in an earlier exercise. That exercise should be completed before you start this one.

   a. Given that H and K are subgroups of $\mathbf{R}$, prove that the set $H+K$ defined in the sense of an earlier exercise is also a subgroup of $\mathbf{R}$.
   To prove that $H+K$ is a subgroup of $\mathbf{R}$ we need to show that $H+K$ is nonempty and that the sum and difference of any members of $H+K$ must always belong to $H+K$.
   To show that $H+K$ is nonempty we use the fact that H and K are nonempty to choose $x \in H$ and $y \in K$. Since $x+y \in H+K$ we have $H+K \neq \emptyset$.
   Now suppose that w_1 and w_2 are any members of the set $H+K$. Choose members x_1 and x_2 of H and members y_1 and y_2 of K such that $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$. Since the numbers $x_1 + x_1$ and $x_1 - x_2$ belong to H and the numbers $y_1 + y_2$ and $y_1 - y_2$ belong to K, and since

$$w_1 + w_2 = (x_1 + x_2) + (y_1 + y_2)$$

   and

$$w_1 - w_2 = (x_1 - x_2) + (y_1 - y_2)$$

   we see at once that $w_1 + w_2$ and $w_1 - w_2$ belong to $H+K$.

   b. Prove that if a, b and c are integers and if

$$a\sqrt{2} = b\sqrt{3} + c$$

   then $a = b = c = 0$.

Solution: ***From the equation***

$$a\sqrt{2} = b\sqrt{3} + c$$

we see that

$$2a^2 = 3b^2 + 2bc\sqrt{3} + c^2.$$

Therefore, unless $bc = 0$ ***we have***

$$\sqrt{3} = \frac{2a^2 - 3b^2 - c^2}{2bc}$$

which contradicts the fact that the number $\sqrt{3}$ ***is irrational. Therefore at least one of the number*** b ***and*** c ***must be zero.***
In the event that $c = 0$***, the equation***

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = b\sqrt{3}$$

and, unless $a = 0$***, the latter equation gives us***

$$\sqrt{\frac{2}{3}} = \frac{b}{a}$$

which contradicts the fact that $\sqrt{\frac{2}{3}}$ ***is irrational. So in the case*** $c = 0$ ***we also have*** $a = 0$ ***and we see at once that*** $b = 0$ ***as well.***
In the event that $b = 0$***, the equation***

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = c$$

and, unless $a = 0$***, the latter equation gives us***

$$\sqrt{2} = \frac{c}{a}$$

which contradicts the irrationality of $\sqrt{2}$***. So, once again,*** $a = 0$ ***and we see at once that*** $c = 0$ ***as well.***

c. Prove that if m, n, p and q are integers then it is impossible to have

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

and deduce that if α is any real number and if $H = \{n\alpha \mid n \in \mathbf{Z}\}$ then the subgroup $H + \mathbf{Z}$ cannot contain both of the numbers $\sqrt{2}$ and $\sqrt{3}$.

Solution: ***The equation***

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

implies that

$$q\sqrt{2} = n\sqrt{3} - np + mq$$

which, by part b, tells us that

$$0 = q = n = mq - np$$

which is clearly impossible since n ***and*** q ***appear denominators of the fractions in the equation***

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}.$$

Now, to obtain a contradiction, suppose that the subgroup $H + Z$ ***contains both of the numbers*** $\sqrt{2}$ ***and*** $\sqrt{3}$***. Choose integers*** m ***and*** n ***such that***

$$\sqrt{2} = m + n\alpha$$

and choose integers p ***and*** q ***such that***

$$\sqrt{3} = p + q\alpha.$$

Since $\sqrt{2}$ is irrational, we know that $\sqrt{2} \neq m$ and so $n \neq 0$; and we know similarly that $q \neq 0$. Thus

$$\frac{\sqrt{2} - m}{n} = \alpha = \frac{\sqrt{3} - p}{q}$$

which we know to be impossible.

d. Suppose that G is a subgroup of $\mathbf{R}$ other than $\{0\}$, that

$$p = \inf\{x \in G \mid x > 0\}$$

and that the number p is positive. Prove that the set G is closed.

Solution: ***We know from** an earlier exercise **that***

$$G = \{np \mid n \in \mathbf{Z}\}.$$

e. Prove that if G is a subgroup of $\mathbf{R}$ other than $\{0\}$ and that G has no least positive member then $\overline{G} = \mathbf{R}$.

Solution: ***This fact was established in** an earlier exercise.*

f. Suppose that α is an irrational number, that

$$H = \{n\alpha \mid n \in \mathbf{Z}\}$$

and that $G = H + \mathbf{Z}$. Prove that although the sets H and $\mathbf{Z}$ are closed subgroups of $\mathbf{R}$ and although the set G is also a subgroup of $\mathbf{R}$, the set G is not closed.

Solution: ***Since G cannot contain both of the numbers $\sqrt{2}$ and $\sqrt{3}$ we know that $G \neq R$. To show that G is not closed we shall make the observation that $\overline{G} = R$ and, for this purpose, all we have to show is that if***

$$p = \inf\{x \in G \mid x > 0\}$$

then $p = 0$. Suppose that p is defined in this way and, to obtain a contradiction, suppose that $p > 0$. We know that

$$G = \{np \mid n \in \mathbf{Z}\}$$

and, using the fact that both of the numbers 1 and α belong to G, we choose integers m and n such that

$$1 = mp$$

and

$$\alpha = np.$$

From the fact that $p = 1/m$ we see that p is rational but from the fact that $p = \alpha/n$ we see that p must be irrational. Thus we have arrived at the promised contradiction.

Exercises on Limit Points

1. Prove that in the metric space $\mathbf{R}$ we have $\mathbf{L}(\mathbf{Z}) = \emptyset$.
   Given any number x, the interval $(x-1, x+1)$ can contain at most two integers. We know that a neighborhood of a limit point of a set must always contain infinitely many members of that set and so we conclude that no number x can be a limit point of the set $\mathbf{Z}$ of integers.

2. Prove that in the metric space $\mathbf{R}$ we have $\mathbf{L}(\mathbf{Q}) = \mathbf{R}$.
   Suppose that x is any real number. To show that x is a limit point of $\mathbf{Q}$, suppose that $\delta > 0$. Since there are rational numbers in the interval $(x, x+\delta)$ we know that the set $(x-\delta, x+\delta) \cap \mathbf{Q} \setminus \{x\} \neq \emptyset$.

3. Prove that in the metric space $\mathbf{R}$ we have $\mathbf{L}(\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}) = \{0\}$.
   If x is any negative number then the interval $(-\infty, 0)$ is a neighborhood of x that fails to contain any members of the set $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$. Thus a negative number can not be a limit point of $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$.

If x is any positive number then the interval $\left(\frac{x}{2},\infty\right)$ is a neighborhood of x which fails to contain infinitely many members of the set $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$. To see why, note that if n is a positive integer then the condition

$$\frac{1}{n} \in \left(\frac{x}{2},\infty\right)$$

can hold only if $n < \frac{2}{x}$. Therefore no positive number can be a limit point of $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$. Finally, we need to explain why 0 must be a limit point of $\{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$. Suppose that $\delta > 0$. Choose an integer $k > \frac{1}{\delta}$ and observe that

$$\frac{1}{k} \in (0-\delta, 0+\delta) \cap \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\} \setminus \{0\}$$

from which we deduce that the set $(0-\delta, 0+\delta) \cap \{\frac{1}{n} \mid n \in \mathbf{Z}^+\} \setminus \{0\}$ must be nonempty.

4. a. Give an example of an infinite subset of $\mathbf{R}$ that has no limit point.
   As we saw in Exercise 1, the infinite set $\mathbf{Z}$ has no limit point.

   b. Give an example of a bounded subset of $\mathbf{R}$ that has no limit point.
   A finite set like $\{2\}$ will not have any limit points. We could also look at the empty set $\emptyset$.

   c. Give an example of an unbounded subset of $\mathbf{R}$ that has no limit point.
   As we saw in Exercise 1, the infinite set $\mathbf{Z}$ has no limit point.

   d. Give an example of an unbounded subset of $\mathbf{R}$ that has exactly one limit point.
   The unbounded set $\mathbf{Z} \cup \{\frac{1}{n} \mid n \in \mathbf{Z}^+\}$ has only the limit point 0.

   e. Give an example of an unbounded subset of $\mathbf{R}$ that has exactly two limit points.
   The set

$$\mathbf{Z} \cup \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\} \cup \left\{1+\frac{1}{n} \mid n \in \mathbf{Z}^+\right\}$$

   has the two limit points 0 and 1. We can see this directly or we can use the assertion proved in Exercise 6 below.

5. Prove that if A and B are subsets of a metric space X and if $A \subseteq B$ then $\mathbf{L}(A) \subseteq \mathbf{L}(B)$.
   Suppose that A and B are subsets of a metric space X and that $A \subseteq B$. Suppose that x is a limit point of A. We need to explain why x has to be a limit point of B. Suppose that $\delta > 0$. Since the set $B(x,\delta) \cap A \setminus \{x\}$ is nonempty and since

$$B(x,\delta) \cap A \setminus \{x\} \subseteq B(x,\delta) \cap B \setminus \{x\}$$

   we deduce that the set $B(x,\delta) \cap B \setminus \{x\}$ is nonempty.

6. Prove that if A and B are subsets of a metric space X then

$$\mathbf{L}(A \cup B) = \mathbf{L}(A) \cup \mathbf{L}(B).$$

   Solution: ***Since*** $A \subseteq A \cup B$ ***we know that*** $\mathbf{L}(A) \subseteq \mathbf{L}(A \cup B)$ ***and similarly we know that*** $\mathbf{L}(B) \subseteq \mathbf{L}(A \cup B)$***. Thus***

$$\mathbf{L}(A) \cup \mathbf{L}(B) \subseteq \mathbf{L}(A \cup B).$$

   Now suppose that a point x ***fails to belong to the set*** $L(A) \cup L(B)$***. Choose a number*** $\delta_1 > 0$ ***such that the ball*** $B(x,\delta_1)$ ***contains only finitely many members of the set*** A***. Choose a number*** $\delta_2 > 0$ ***such that the ball*** $B(x,\delta_2)$ ***contains only finitely many members of the set*** B***. We now define*** δ ***to be the smaller of the two numbers*** δ_1 ***and*** δ_2 ***and we observe that, although*** $\delta > 0$***, the interval*** $B(x,\delta)$ ***contains only finitely many members of the set*** $A \cup B$***. Therefore no number that lies outside the set*** $\mathbf{L}(A) \cup \mathbf{L}(B)$ ***can be a limit point of*** $A \cup B$ ***and we conclude that***

$$\mathbf{L}(A \cup B) = \mathbf{L}(A) \cup \mathbf{L}(B).$$

7. Is it true that if A and B are subsets of a metric space X then

$$\mathbf{L}(A \cap B) = \mathbf{L}(A) \cap \mathbf{L}(B)?$$

   What if A and B are closed? What if A and B are open? What if A and B are intervals in $\mathbf{R}$?
   The answers are no, no, no and no. Look at the following example:

$$A = [0,1] \text{ and } B = [1,2]$$

These two sets are closed in $\mathbf{R}$ and

$$\mathbf{L}(A \cap B) = \mathbf{L}(\{1\}) = \emptyset$$

while

$$\mathbf{L}(A) \cap \mathbf{L}(B) = [0,1] \cap [1,2] = \{1\}.$$

Now look at the following example:

$$A = (0,1) \qquad \text{and} \qquad B = (1,2).$$

In this case

$$\mathbf{L}(A \cap B) = \mathbf{L}(\emptyset) = \emptyset$$

and

$$\mathbf{L}(A) \cap \mathbf{L}(B) = [0,1] \cap [1,2] = \{1\}.$$

8. Is it true that if D is a dense subset of $\mathbf{R}$ then $\mathbf{L}(D) = \mathbf{R}$?
   The assertion is true. Suppose that $\overline{D} = \mathbf{R}$. We know that whenever a and b are real numbers and $a < b$ there must be members of D lying between a and b. Now suppose that x is a real number. To show that x is a limit point of D, suppose that $\delta > 0$. Since there must be members of D in the interval $(x, x+\delta)$ we conclude that the set $(x-\delta, x+\delta) \cap D \setminus \{x\}$ is nonempty.

9. Is it true that if D is a dense subset of a metric space X then $\mathbf{L}(D) = X$?
   No! Consider, for example, a metric space like $\{1,2,3\}$ that has no limit points at all. The set $\{1,2,3\}$ is dense in this space.

10. Is it true that if D is a dense subset of a connected metric space X then $\mathbf{L}(D) = X$?
    Yes, as long as the space X contains at least two different points then if X connected and D is dense in X we must have $\mathbf{L}(D) = X$. As a matter of fact, we can do better than this. The condition that X be connected says that X has no open closed subsets other than $\emptyset$ and X. All we actually need to know is that no singleton in X can be an open set.
    Suppose that X is a metric space in which no singleton is open. Suppose that D is a dense subset of X. Suppose that $x \in X$ and, to show that x is a limit point of D, suppose that $\delta > 0$. Since the ball $B(x,\delta)$ is open and $\{x\}$ is not open we know that $B(x,\delta) \neq \{x\}$, in other words, $B(x,\delta) \setminus \{x\} \neq \emptyset$. Furthermore, since

$$B(x,\delta) \setminus \{x\} = B(x,\delta) \cap (X \setminus \{x\})$$

    which is open, the set $B(x,\delta) \setminus \{x\}$, being a nonempty open subset of X, must intersect with the set D. Thus

$$B(x,\delta) \cap D \setminus \{x\} \neq \emptyset.$$

11. Is it true that if X is a metric space and $\mathbf{L}(X) = X$ then for every dense subset D of X we have $\mathbf{L}(D) = X$?
    The answer is yes. See the solution to Exercise 10.

12. Given that a set S of real numbers is nonempty and bounded above but that S does not have a largest member, prove that $\sup S$ must be a limit point of S. State and prove a similar result about $\inf S$.
    To show that $\sup S$ is a limit point of S, suppose that $\delta > 0$. Since $\sup S - \delta < \sup S$ and since $\sup S$ is the **least** upper bound of S the number $\sup S - \delta$ fails to be an upper bound of S. Choose a member x of S such that $\sup S - \delta < x$. Since $x \leq \sup S$ and since $\sup S$ does not belong to S we have $x < \sup S$. We conclude that

$$(\sup S - \delta, \sup S + \delta) \cap S \setminus \{\sup S\} \neq \emptyset.$$

13. Given that S is a closed subset of a metric space X and that every infinite subset of X has at least one limit point, prove that every infinite subset of S must have at least one limit point that belongs to S.
    Every infinite subset of S, being an infinite subset of X, must have a limit point somewhere in X. But, since S is closed, every limit point of a subset of S must belong to S.

14. Suppose that S is a bounded set of real numbers and that $\mathbf{L}(S) = \emptyset$. Prove that every nonempty subset of S

must have both a largest and a smallest member. Can such a set S be infinite?
The fact that every nonempty subset of S must have both a greatest and a least member follows at once from Exercise 12. To see that such a set S cannot be infinite we shall show that if S is an infinite set of real numbers and if every nonempty subset of S has a least member then S must have a subset that does not have a greatest member.
We assume that S is an infinite set of real numbers and that every nonempty subset of S has a least member. Using the fact that $S \neq \emptyset$ we define x_1 to be the least member of S.
Using the fact that the set $S \setminus \{x_1\}$ is nonempty we define x_2 to be the least member of the set $S \setminus \{x_1\}$.
Using the fact that the set $S \setminus \{x_1, x_2\}$ is nonempty we define x_3 to be the least member of the set $S \setminus \{x_1, x_2\}$.
Continuing in this way we obtain a strictly increasing sequence (x_n) in the set S and we see that the set $\{x_n \mid n \in \mathbf{Z}^+\}$ does not have a greatest member.

15. Given any subset S of a metric space X, prove that the set $\mathbf{L}(S)$ must be closed.

Solution: ***We shall show that any point that fails to belong to*** $\mathbf{L}(S)$ ***must fail to belong to*** $\overline{\mathbf{L}(S)}$***. Suppose that*** $x \in X \setminus L(S)$***. Choose a number*** $\delta > 0$ ***such that the ball*** $B(x,\delta)$ ***contains only finitely many members of*** S***. Given any point*** t ***in the ball*** $B(x,\delta)$***, it follows from the fact that*** $B(x,\delta)$ ***is a neighborhood of*** t ***and the fact that*** $B(x,\delta)$ ***contains only finitely many members of*** S ***that*** t ***is not a limit point of*** S***. Thus***

$$B(x,\delta) \cap \mathbf{L}(S) = \emptyset$$

and we have shown, as promised, that x ***does not belong to*** $\overline{\mathbf{L}(S)}$***.***

16. Prove that if U is an open subset of the metric space $\mathbf{R}^k$ then $\mathbf{L}(U) = \overline{U}$.
Of course $\mathbf{L}(U) \subseteq \overline{U}$. Now suppose that $\mathbf{x} \in \overline{U}$. To show that $\mathbf{x} \in \mathbf{L}(U)$, suppose that $\delta > 0$. Using the fact that $\mathbf{x} \in \overline{U}$, choose a point $\mathbf{y}$ in the set $U \cap B(\mathbf{x},\delta)$. Using the fact that the set $U \cap B(\mathbf{x},\delta)$ is open, choose $\varepsilon > 0$ such that

$$B(\mathbf{y},\varepsilon) \subseteq U \cap B(\mathbf{x},\delta).$$

We have now found more than one member of U that belongs to the ball $B(\mathbf{x},\delta)$ and so we know that

$$U \cap B(\mathbf{x},\delta) \setminus \{\mathbf{x}\} \neq \emptyset.$$

17. Suppose that S is a subset of a metric space, that $\mathbf{L}(S) \neq \emptyset$, and that $\delta > 0$. Prove that there exist two different members x and y of S such that $d(x,y) < \delta$.
Choose a limit point u of the set S. Using the fact that the ball $B\left(u, \frac{\delta}{2}\right)$ contains infinitely many points of S, choose two different points x and y in $S \cap B\left(u, \frac{\delta}{2}\right)$. We observe that

$$d(x,y) \leq d(x,u) + d(u,y) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

18. Prove that if S is a convex subset of $\mathbf{R}^k$ and S contains more than one point then every member of S is a limit point of S.
Suppose that S is a convex subset of $\mathbf{R}^k$ and that S contains two different points x and $\mathbf{y}$. In order to show that $\mathbf{x}$ is a limit point of S, suppose that $\delta > 0$. We observe that if t is a real number and if $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$ then

$$\|\mathbf{z} - \mathbf{x}\| = \|(1-t)\mathbf{x} + t\mathbf{y} - \mathbf{x}\| = |t|\|\mathbf{x} - \mathbf{y}\|$$

and so the inequality $\|\mathbf{z} - \mathbf{x}\| < \delta$ will hold as long as $|t|\|\mathbf{x} - \mathbf{y}\| < \delta$. We now choose a number t such that

$$0 < t < \frac{\delta}{\|\mathbf{x} - \mathbf{y}\|}$$

and observe that the point $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$ belongs to the set $B(\mathbf{x},\delta) \cap S \setminus \{\mathbf{x}\}$. Therefore $\mathbf{x}$ is a limit point of S.

19. For the purpose of this exercise we shall call a subset S of a metric space X **compressed** if for every number $\varepsilon > 0$ there exist two different points x and y in S such that $d(x,y) < \varepsilon$.

a. Prove that whenever a subset S of a metric space has a limit point, it must be compressed.
This is just Exercise 17 again.

b. Give an example of a compressed subset of the metric space $\mathbf{R}$ that has no limit point.
The set

$$S = \mathbf{Z}^{+} \cup \left\{ n + \frac{1}{n} \ \middle|\ n \in \mathbf{Z}^{+} \right\}$$

is compressed but, since no interval of length 1 can contain more then three members of S , the set S has no limit point.

c. Give an example of a compressed subset of the metric space $(0,1]$ that has no limit point in $(0,1]$.
The set $\{\frac{1}{n} \mid n \in \mathbf{Z}^{+}\}$ has the desired properties.

d. Prove that if a metric space X is compressed and $x \in X$ then the set $X \setminus \{x\}$ is also compressed.
To obtain a contradiction, suppose that the set $X \setminus \{x\}$ fails to be compressed and choose $\delta > 0$ such that whenever a and b are different points in the set $X \setminus \{x\}$ we have $d(a,b) \geq \delta$.
Since X is compressed and since the inequality $d(a,b) < \frac{\delta}{2}$ never holds when a and b are different points in the set $X \setminus \{x\}$ there must exist a point $a \in X \setminus \{x\}$ such that $d(a,x) < \frac{\delta}{2}$.
Choose such a point a.
Again, using the fact that X is compressed, choose a point $b \in X \setminus \{x\}$ such that $d(b,x) < d(a,x)$. We see at once that $b \neq a$. Moreover

$$d(a,b) \leq d(a,x) + d(x,b) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

contradicting the choice of δ.

e. Prove that if a metric space X is compressed and F is a finite subset of X then the set $X \setminus F$ is compressed.
The set $X \setminus F$ can be obtained by removing the points of F from X one at a time and applying part d each time.

f. Prove that a metric space X is totally bounded if and only if every infinite subset of X is compressed.
The desired result follows from an earlier stated property of total boundedness.

20. Given that S is a subset of a metric space X and that every infinite subset of S has a limit point, prove that every infinite subset of the set $\overline{S}$ must have a limit point.
Suppose that E is an infinite subset of $\overline{S}$. Using the fact that E is infinite we choose a sequence (x_n) of points of E such that whenever $m \neq n$ we have $x_m \neq x_n$.
For each positive integer n we choose a point $y_n \in S$ such that $d(x_n, y_n) < \frac{1}{n}$.
In the event that the set $\{y_n \mid n \in \mathbf{Z}^{+}\}$ is finite there must be a point $y \in S$ such that $y_n = y$ for infinitely many values of n; and we choose such a point y.
In the event that the set $\{y_n \mid n \in \mathbf{Z}^{+}\}$ is infinite, it must have a limit point and we choose such a limit point and call it y.
In either event we have found a point $y \in X$ such that whenever $\delta > 0$ there are infinitely many values of n for which $d(y, y_n) < \delta$. We complete the proof by showing that this point y must be a limit point of E. Suppose that $\delta > 0$.
There are infinitely many integers $n > \frac{2}{\delta}$ for which $d(y, y_n) < \frac{\delta}{2}$. For each of these integers n we have

$$d(x_n, y) \leq d(x_n, y_n) + d(y_n, y) < \frac{1}{n} + \frac{\delta}{2} < \delta$$

and since the sequence (x_n) is one-one we conclude that the ball $B(y, \delta)$ contains more than one point of the set E. Therefore y is a limit point of E.

7 Limits of Sequences

Some Exercises on Subsequences

Decide whether each of the following assertions is true or false. If it is true, prove it. If it is false, illustrate this fact by giving an example.

1. If a sequence (x_n) is eventually in a given set S then every subsequence of (x_n) is eventually in S.
   The assertion is true. Suppose that (x_n) is a sequence that is eventually in a given set S and that (x_{n_i}) is a subsequence of (x_n). Choose an integer N such that the condition $x_n \in S$ holds whenever $n \geq N$. Now choose an integer j such that the condition $n_i \geq N$ holds whenever $i \geq j$. We see that $x_{n_i} \in S$ whenever $i \geq j$.

2. If a sequence (x_n) is frequently in a given set S then every subsequence of (x_n) is frequently in S.
   The assertion is false. If we define $x_n = (-1)^n$ for every positive integer n then, although (x_n) is frequently in the set $\{-1\}$, the subsequence (x_{2n}) fails to be frequently in the set $\{-1\}$.

3. If every subsequence of a sequence is frequently in a given set S then (x_n) is frequently in S.
   The statement is obviously true because every sequence is a subsequence of itself.

4. If every subsequence of a sequence (x_n) is frequently in a given set S then (x_n) is eventually in S.
   The assertion is true. We shall show that if (x_n) fails to be eventually in S then (x_n) must have a subsequence that is not frequently in S. Suppose that (x_n) is a sequence that fails to be eventually in a given set S. We know that (x_n) is frequently in the set $\mathbf{R} \setminus S$ and from the preceding theorem we deduce that (x_n) has a subsequence in the set $\mathbf{R} \setminus S$. Such a subsequence cannot be frequently in S.

5. If (x_n) is a sequence of real numbers and $S \subseteq \mathbf{R}$, and if (x_n) is not eventually in S then (x_n) has a subsequence that is eventually in $\mathbf{R} \setminus S$.
   Since a sequence that is not eventually in S must be frequently in $\mathbf{R} \setminus S$, the desired result follows at once from a theorem on subsequences.

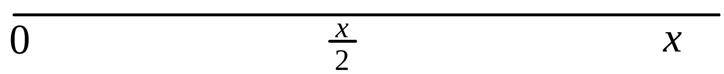
Some Exercises on Limits and Partial Limits

1. Given that
$$x_n = 3 + \frac{1}{n}$$
   for each positive integer n, prove that 3 is a limit of (x_n).
   We need to prove that for every number $\varepsilon > 0$ the sequence (x_n) will eventually be in the interval $(3 - \varepsilon, 3 + \varepsilon)$. So we start: Suppose that $\varepsilon > 0$.
   Before we go any further we need to ask ourselves what it means to say that $x_n \in (3 - \varepsilon, 3 + \varepsilon)$. We observe that the inequality $3 - \varepsilon < x_n < 3 + \varepsilon$ is equivalent to the assertion that
$$3 - \varepsilon < 3 + \frac{1}{n} < 3 + \varepsilon$$
   which holds when $1/n < \varepsilon$. This tells how how to continue:
   Using the fact that the number $1/\varepsilon$ is not an upper bound of the set $\mathbf{Z}$ of integers we choose an integer N such that $N > 1/\varepsilon$, in other words,
$$\frac{1}{N} < \varepsilon.$$
   Then, whenever $n \geq N$ we have
$$3 - \varepsilon < 3 < 3 + \frac{1}{n} \leq 3 + \frac{1}{N} < 3 + \varepsilon$$

and so we have shown that (x_n) is eventually in $(3-\varepsilon, 3+\varepsilon)$.

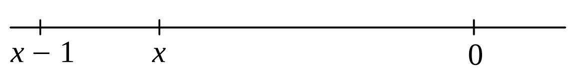
2. Given that
$$x_n = 3 + \frac{2}{n}$$
for each positive integer n, prove that 3 is a limit of (x_n).
We need to prove that for every number $\varepsilon > 0$ the sequence (x_n) will eventually be in the interval $(3-\varepsilon, 3+\varepsilon)$. So we start: Suppose that $\varepsilon > 0$. This time we make the observation that the inequality
$$3-\varepsilon < 3 + \frac{2}{n} < 3+\varepsilon$$
holds when $2/n < \varepsilon$.
Using the fact that the number $2/\varepsilon$ is not an upper bound of the set $\mathbf{Z}$ of integers we choose an integer N such that $N > 2/\varepsilon$, in other words,
$$\frac{2}{N} < \varepsilon.$$
Then, whenever $n \geq N$ we have
$$3-\varepsilon < 3 < 3 + \frac{2}{n} \leq 3 + \frac{2}{N} < 3+\varepsilon$$
and so we have shown that (x_n) is eventually in $(3-\varepsilon, 3+\varepsilon)$.

3. Given that $x_n = 1/n$ for each positive integer n and that $x \neq 0$, prove that x is not a partial limit of (x_n).
Solution: ***In the event that $x < 0$, the interval $(-\infty, 0)$ is a neighborhood of x and it is clear that (x_n) fails to be frequently in this neighborhood. Therefore no negative number can be a partial limit of (x_n). Suppose now that $x > 0$. The interval $(x/2, \infty)$ is a neighborhood of x***



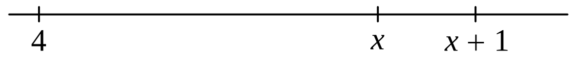
and the condition $x_n \in (x/2, \infty)$ must fail to hold whenever $n > 2/x$. Therefore the sequence (x_n) cannot be frequently in the interval $(x/2, \infty)$ and the number x cannot be a partial limit of (x_n).

4. Given that
$$x_n = \begin{cases} (-1)^n n^3 & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{if } n \text{ is one more than a multiple of 3} \\ 4 & \text{if } n \text{ is two more than a multiple of 3} \end{cases},$$
Prove that the partial limits of (x_n) are $-\infty$, ∞, 0 and 4.
Since (x_n) is unbounded both above and below, it follows from the discussion of infinite partial limits we saw earlier that both ∞ and $-\infty$ are partial limits of (x_n). Since the equation $x_n = 0$ holds for infinitely many values of n we know that (x_n) is frequently in every neighborhood of 0 and so 0 is a partial limit of (x_n). In the same way we can see that 4 is a partial limit of (x_n).
Now we need to explain why any real number other than 0 and 4 must fail to be a partial limit of (x_n). Suppose that $x \in \mathbf{R} \setminus \{0, 4\}$.
In the event that $0 < x < 4$, the fact that x is not a partial limit of (x_n) follows from the fact that (x_n) is not frequently (or ever) in the interval $(0, 4)$ which is a neighborhood of x.
Now suppose that $x < 0$.



In order to show that x is not a partial limit of (x_n) we shall make the observation that (x_n) is not frequently in the interval $(x-1, 0)$ which is a neighborhood of x. In fact, the inequality
$$x - 1 < x_n < 0$$
can hold only when n is odd and
$$x - 1 < -n^3$$
which is equivalent to saying that $n^3 < 1 - x$. Since there are only finitely many such positive integers n we conclude that (x_n) is not frequently in the interval $(x-1, 0)$

Finally we must consider the case $x > 4$.



In this case we observe that there can be only finitely many positive integers n for which

$$4 < n^3 < x+1$$

and so, once again, x can't be a partial limit of (x_n).

5. Give an example of a sequence of real numbers whose set of partial limits is the set $\{1\} \cup [4,5]$.

Hint: ***For each positive integer*** n***, if*** n ***can be written in the form***

$$n = 2^m 3^k$$

for some positive integers m ***and*** k ***and if***

$$4 \le \frac{m}{k} \le 5$$

then we define

$$x_n = \frac{m}{k}.$$

In all other cases we define $x_n = 1$***. Observe that the range of the sequence*** (x_n) ***is the set***

$$\{1\} \cup (\mathbf{Q} \cap [4,5])$$

and then show that the set of partial limits of (x_n) ***is*** $\{1\} \cup [4,5]$***.***

Since the equation $x_n = 1$ holds for infinitely many positive integers n the number 1 must be a partial limit of (x_n). To see that every number in the interval $[4,5]$ must be a partial limit of (x_n), suppose that $x \in [4,5]$ and suppose that $\varepsilon > 0$. Since the interval $(x-\delta, x+\delta)$ must contain infinitely many members of the set $\mathbf{Q} \cap [4,5]$ we know that the condition $x_n \in (x-\delta, x+\delta)$ must hold for infinitely many positive integers n and so x must be a partial limit of (x_n).

Finally we observe that if $x \in \mathbf{R} \setminus (\{1\} \cup [4,5])$ then the open set $\mathbf{R} \setminus (\{1\} \cup [4,5])$ is a neighborhood of x and that (x_n) fails to be frequently (or ever) in $\mathbf{R} \setminus (\{1\} \cup [4,5])$ and so x can't be a partial limit of (x_n).

6. Given that

$$x_n = \frac{3+2n}{5+n}$$

for every positive integer n, prove that $x_n \to 2$ as $n \to \infty$.

We begin by observing that if n is a positive integer then

$$\left|\frac{3+2n}{5+n} - 2\right| = \frac{7}{5+n}$$

Now suppose that $\varepsilon > 0$. The inequality

$$\left|\frac{3+2n}{5+n} - 2\right| < \varepsilon$$

says that

$$\frac{7}{5+n} < \varepsilon$$

which holds when

$$\frac{5+n}{7} > \frac{1}{\varepsilon}$$

in other words

$$n > \frac{7}{\varepsilon} - 5$$

With these inequalities in mind we choose an integer N such that

$$N > \frac{7}{\varepsilon} - 5$$

and we observe that whenever n is an integer and $n \ge N$ we have

$$\left|\frac{3+2n}{5+n} - 2\right| < \varepsilon.$$

7. Given that

$$x_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even} \\ \frac{1}{n^2+1} & \text{if } n \text{ is odd} \end{cases}$$

prove that $x_n \to 0$ as $n \to \infty$.
We observe that if n is a positive integer then

$$0 \le x_n \le \frac{1}{n}.$$

Now suppose that $\varepsilon > 0$. Choose an integer N such that $N > 1/\varepsilon$. We observe that whenever $n \ge N$,

$$|x_n - 0| = x_n < \frac{1}{n} \le \frac{1}{N} < \varepsilon.$$

8. Suppose that (x_n) is a sequence of real numbers and that $x \in \mathbf{R}$. Prove that the following conditions are equivalent:

a. $x_n \to x$ as $n \to \infty$.

b. For every number $\varepsilon > 0$ the sequence (x_n) is eventually in the interval $(x - 5\varepsilon, x + 5\varepsilon)$.

To prove that condition a implies condition b we assume that $x_n \to x$ as $n \to \infty$.
Suppose that $\varepsilon > 0$ and, using the fact that $x_n \to x$ as $n \to \infty$, choose N such that the condition

$$x_n \in (x - \varepsilon, x + \varepsilon)$$

will hold whenever $n \ge N$. Then, whenever $n \ge N$ we have

$$x_n \in (x - \varepsilon, x + \varepsilon) \subseteq (x - 5\varepsilon, x + 5\varepsilon).$$

Now to prove that condition b implies condition a we assume that condition b holds. To prove that condition a holds, suppose that $\varepsilon > 0$. Using the fact that $\varepsilon/5$ is a positive number, we choose an integer N such that the condition

$$x_n \subseteq \left(x - 5\left(\frac{\varepsilon}{5}\right), x + 5\left(\frac{\varepsilon}{5}\right)\right)$$

holds whenever $n \ge N$. Thus for every $n \ge N$ we have

$$x_n \subseteq \left(x - 5\left(\frac{\varepsilon}{5}\right), x + 5\left(\frac{\varepsilon}{5}\right)\right) = (x - \varepsilon, x + \varepsilon).$$

9. Prove that

$$\frac{n^2 + 3n + 1}{2n^2 + n + 4} \to \frac{1}{2}$$

as $n \to \infty$.
We begin by observing that if n is any positive integer then

$$\left| \frac{n^2 + 3n + 1}{2n^2 + n + 4} - \frac{1}{2} \right| = \left| \frac{5n - 2}{4n^2 + 2n + 8} \right| \le \frac{5n}{4n^2} = \frac{5}{4n}.$$

Now suppose that $\varepsilon > 0$. The inequality

$$\left| \frac{n^2 + 3n + 1}{2n^2 + n + 4} - \frac{1}{2} \right| < \varepsilon$$

will hold when

$$\frac{5}{4n} < \varepsilon$$

which requires that

$$n > \frac{5}{4\varepsilon}$$

Choose an integer N such that $N > 5/(4\varepsilon)$ and observe that, whenever $n \ge N$ we have

$$\left| \frac{n^2 + 3n + 1}{2n^2 + n + 4} - \frac{1}{2} \right| \le \frac{5}{4n} \le \frac{5}{4N} < \varepsilon.$$

10. For each positive integer n, if n can be written in the form

$$n = 2^m 3^k$$

where m and k are postive integers and $0 \leq \frac{m}{k} \leq 1$ then we define

$$x_n = \frac{m}{k}.$$

Otherwise we define $x_n = 0$. Prove that the set of partial limits of the sequence (x_n) is $[0,1]$.

Since the range of (x_n) is the set of all rational numbers in the interval $[0,1]$, every neighborhood of a number x in the interval $[0,1]$ must contain infinitely many members of the range of (x_n) and must, therefore, contain the number x_n for infinitely many integers n. Thus every member of the interval $[0,1]$ is a partial limit of (x_n).

If x is any number in the set $\mathbf{R} \setminus [0,1]$ then, since $\mathbf{R} \setminus [0,1]$ is a neighborhood of x and (x_n) is not frequently (or, indeed, ever) in the set $\mathbf{R} \setminus [0,1]$, the number x must fail to be a partial limit of (x_n).

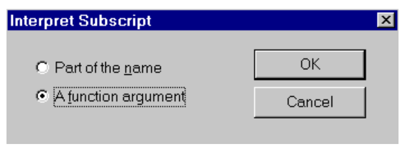
Exercises on the Elementary Properties of Limits

1. The purpose of this exercise is to use *Scientific Notebook* to gain an intuitive feel for the limit behaviour of a rather difficult sequence.

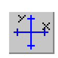
   a. Point at the equation

   $$x_n = \frac{n^n \sqrt{n}}{(n!)e^n}$$

   and then click on the button  to supply the definition to *Scientific Notebook*. When you see the screen

   make the selection “A function argument” so that *Scientific Notebook* knows that you are defining a sequence.

   b. Point at the expression x_n and click on the button  to display the sequence graphically. Revise your graph and set the domain interval as $[1,500]$. Double click into your graph to make the buttons

   appear in the top right corner and click on the bottom button to select it. Trace your graph with the mouse and show graphically that

   $$\lim_{n\to\infty} \frac{n^n \sqrt{n}}{(n!)e^n} \approx 0.3989.$$

   c. Point at the expression

   $$\lim_{n\to\infty} \frac{n^n \sqrt{n}}{(n!)e^n}$$

   and ask *Scientific Notebook* to evaluate it numerically. Compare the result with the limit value that you found graphically.

d. Point at the expression and ask *Scientific Notebook* to evaluate it exactly to show that the limit is $1/\sqrt{2\pi}$.

2. Prove that $5^n/n! \to 0$ as $n \to \infty$.

Solution: ***Whenever*** $n \geq 5$ ***we have***

$$0 \leq \frac{5^n}{n!} = \left(\frac{5}{1}\right)\left(\frac{5}{2}\right)\left(\frac{5}{3}\right)\left(\frac{5}{4}\right)\left(\frac{5}{5}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{5}{n}\right) \leq \left(\frac{5^4}{4!}\right)\left(\frac{5}{n}\right)$$

Now, to prove that $5^n/n! \to 0$ ***as*** $n \to \infty$***, suppose that*** $\varepsilon > 0$***.***
Choose an integer N ***such that***

$$N > \frac{5^5}{4!}\varepsilon.$$

Then whenever $n \geq N$ ***we have***

$$0 \leq \frac{5^n}{n!} \leq \left(\frac{5^4}{4!}\right)\left(\frac{5}{n}\right) \leq \left(\frac{5^4}{4!}\right)\left(\frac{5}{N}\right) < \varepsilon.$$

3. Prove that $n!/n^n \to 0$ as $n \to \infty$.

Hint: ***Make use of the fact that, for each*** n ***we have***

$$0 \leq \frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right) \leq \frac{1}{n}.$$

4. Given that (x_n) is a sequence of real numbers, that $x > 0$ and that $x_n \to x$ as $n \to \infty$, prove that there exists an integer N such that the inequality $x_n > 0$ holds for all integers $n \geq N$.

Hint: ***The interval*** $(0, \infty)$ ***is a neighborhood of the number*** x***.***

5. Given that $x_n \geq 0$ for every positive integer n and that x is a partial limit of the sequence (x_n), prove that $x \geq 0$.
We need to show that no negative number can be a partial limit of (x_n). Suppose that $y < 0$. Since the interval $(-\infty, 0)$ is a neighborhood of y and (x_n) is not frequently (or, indeed, ever) in the interval $(-\infty, 0)$, we deduce that y is not a partial limit of (x_n).

6. Suppose that (x_n) is a sequence of real numbers and that $x \in \mathbf{R}$. Prove that the following conditions are equivalent:

   a. $x_n \to x$ as $n \to \infty$.

   b. $|x_n - x| \to 0$ as $n \to \infty$.
   Condition a says that for every $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ we have $|x_n - x| < \varepsilon$.
   Condition b says that for every $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ we have $||x_n - x| - 0| < \varepsilon$.
   These two conditions clearly say the same thing.

7. Suppose that (x_n) is a sequence of real numbers, that $x \in \mathbf{R}$ and that $x_n \to x$ as $n \to \infty$. Prove that $|x_n| \to |x|$ as $n \to \infty$.

Hint: ***Make use of the fact that for each*** n ***we have***

$$0 \leq ||x_n| - |x|| \leq |x_n - x|.$$

8. Suppose that (x_n) is a sequence of real numbers, that $x \in \mathbf{R}$ and that $x_n \to x$ as $n \to \infty$. Suppose that p is an integer and that for every positive integer n we have

$$y_n = x_{n+p}.$$

Prove that $y_n \to x$ as $n \to \infty$.
Suppose that $\varepsilon > 0$. Using the fact that $x_n \to x$ as $n \to \infty$ we choose an integer N such that whenever $n \geq N$ we have $|x_n - x| < \varepsilon$. We observe that the inequality $|y_n - x| < \varepsilon$ will hold whenever

$n+p \geq N$ which is the same as saying that $n \geq N-p$.

9. Given that $a_n \leq b_n$ for every positive integer n and given that $a_n \to \infty$, prove that $b_n \to \infty$.
Suppose that w is a real number. Using the fact that $a_n \to \infty$ as $n \to \infty$ we choose an integer N such that the inequality $a_n > w$ holds whenever $n \geq N$. Then whenever $n \geq N$ we have $b_n \geq a_n > w$.

10. Suppose that (a_n) and (b_n) are sequences of real numbers and that $|a_n - b_n| \leq 1$ for every positive integer n and that ∞ is a partial limit of the sequence (a_n). Prove that ∞ is a partial limit of (b_n).
We know that $b_n \geq a_n - 1$ for each n. Suppose that w is a real number. Since ∞ is a partial limit of (a_n) there are infinitely many integers n for which $a_n > w+1$ and for all such integers we have $b_n > w$.

11. Two sequences (a_n) and (b_n) of real numbers X are said to be *eventually close* if for every number $\varepsilon > 0$ there exists an integer N such that the inequality $d(a_n, b_n) < \varepsilon$ holds for all integers $n \geq N$.

    a. Prove that if two sequences (a_n) and (b_n) are eventually close and if a number x is the limit of the sequence (a_n) then x is also the limit of the sequence (b_n).
    Suppose that (a_n) and (b_n) are eventually close and that $a_n \to x$ as $n \to \infty$. To show that $b_n \to x$ as $n \to \infty$, suppose that $\varepsilon > 0$.
    Choose N_1 such that the inequality $|a_n - x| < \frac{\varepsilon}{2}$ holds whenever $n \geq N_1$. Choose N_2 such that the inequality $|a_n - b_n| < \frac{\varepsilon}{2}$ holds whenever $n \geq N_2$. We define N to be the larger of N_1 and N_2 and we see that whenever $n \geq N$ we have
    $$|b_n - x| \leq |b_n - a_n| + |a_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

    b. Prove that if two sequences (a_n) and (b_n) are eventually close and if a number x is a partial limit of the sequence (a_n) then x is also a partial limit of the sequence (b_n).
    Suppose that (a_n) and (b_n) are eventually close and that x is a partial limit of (a_n). Suppose that $\varepsilon > 0$. Choose an integer N such that the inequality $|a_n - b_n| < \frac{\varepsilon}{2}$ holds whenever $n \geq N$. Since there are infinitely many integers n for which $|a_n - x| < \frac{\varepsilon}{2}$ there must be infinitely many integers $n \geq N$ for which $|a_n - x| < \frac{\varepsilon}{2}$. For every one of these integers n we have
    $$|b_n - x| \leq |b_n - a_n| + |a_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

12. Suppose that (a_n) and (b_n) are sequences of real numbers, that $a_n \to a$ and $b_n \to b$ as $n \to \infty$, and that $a < b$. Prove that there exists an integer N such that the inequality $a_n < b_n$ holds for all integers $n \geq N$.
Choose a number p between a and b.

a $\qquad\qquad$ p $\qquad\qquad\qquad\qquad$ b

Using the fact that $a_n \to a$ as $n \to \infty$ and the fact that the interval $(-\infty, p)$ is a neighborhood of a, choose an integer N_1 such that $a_n < p$ whenever $n \geq N_1$. Similarly, choose an integer N_2 such that $b_n > p$ whenever $n \geq N_2$. We define N to be the larger of N_1 and N_2 and we observe that $a_n < p < b_n$ whenever $n \geq N$.

13. Give an example of two sequences (a_n) and (b_n) of real numbers satisfying $a_n > b_n$ for every positive integer n even though the sequence (a_n) has a partial limit a that is less than a partial limit b of the sequence (b_n).
We define $a_n = 2 + (-1)^n$ and $b_n = 3 + (-1)^n$ for every positive integer n. We see that $a_n < b_n$ for each n and that 3 is a partial limit of (a_n) and that 2 is a partial limit of (b_n).

Some Exercises on the Algebraic Rules for Limits

1. Write out proofs of those cases of this theorem that were not proved above.
We shall show just two more cases here.
Proof of Part 3 when $x < 0$ **and** $y = \infty$. Since $x \times \infty = -\infty$, we need to show that $x_n y_n \to -\infty$. Now since $-x_n \to -x > 0$ and $y_n \to \infty$ as $n \to \infty$, it follows from a case we have already considered that $-x_n y_n \to \infty$ as $n \to \infty$. From this fact and part 2 of the theorem we conclude that $x_n y_n \to -\infty$.

Alternatively we can prove this case directly: We begin by choosing a number q such that $x < q < 0$. (For example, one may define $q = x/2$.)

x $\qquad\qquad$ q $\qquad\qquad$ 0

To show that $x_n y_n \to -\infty$, suppose that w is any real number. In view of the theorem of infinite limits we need to show that $x_n y_n < w$ for all sufficiently large integers n. Using the fact that the interval $(-\infty, q)$ is a neighborhood of x and the fact that $x_n \to x$ we choose an integer N_1 such that $x_n < q$ for all integers $n \geq N_1$. Now we choose an integer N_2 such that $y_n > |w/q|$ for every integer $n \geq N_2$ and we define N to be the larger of the two numbers N_1 and N_2. Then for every integer $n \geq N$ we have

$$x_n y_n < q y_n < q\left|\frac{w}{q}\right| = -|w| \leq w$$

and the proof is complete.

Proof of Part 4 when $x = \infty$ and y is a positive real number

Since $\infty/y = \infty$ we need to prove that $x_n/y_n \to \infty$ as $n \to \infty$. Since

$$\frac{x_n}{y_n} = x_n\left(\frac{1}{y_n}\right)$$

for each n and since $1/y_n \to 1/y$ as $n \to \infty$ the desired result follows from part 3 of the theorem. Alternatively, this part of the theorem can be proved directly.

2. Given that (x_n) and (y_n) are sequences of real numbers, that (x_n) converges to a number x and that a real number y is a partial limit of the sequence (y_n), prove that $x + y$ is a partial limit of the sequence $(x_n + y_n)$.

Suppose that $\varepsilon > 0$. Using the fact that (x_n) converges to the number x we choose an integer N such that the inequality

$$|x_n - x| < \frac{\varepsilon}{2}$$

holds whenever $n \geq N$. Since y is a partial limit of the sequence (y_n) and since there are only finitely many positive integers $n < N$ there must be infinitely many integers $n \geq N$ for which the inequality

$$|y_n - y| < \frac{\varepsilon}{2}$$

holds. For each of these infinitely many integers we have

$$\begin{aligned} |x_n + y_n - (x + y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

3. State and prove some analogues of this exercise for subtraction, multiplication and division.

Suppose that (x_n) and (y_n) are sequences of numbers, that (x_n) converges to a number x and that a real number y is a partial limit of (y_n). We shall prove that the number xy must be a partial limit of the sequence $(x_n y_n)$.

Using the fact that (x_n) is convergent, and therefore bounded, we choose a number p such that $|x_n| < p$ for every n. For each n we observe that

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &\leq p|y_n - y| + |y||x_n - x| \end{aligned}$$

Now, to show that xy is a partial limit of the sequence $(x_n y_n)$, suppose that $\varepsilon > 0$. Using the fact that $x_n \to x$ as $n \to \infty$ we choose an integer N such that the inequality

$$|x_n - x| < \frac{\varepsilon}{2|y| + 1}$$

holds whenever $n \geq N$. Since there are only finitely many positive integers $n < N$, and y is a partial limit of the sequence (y_n), there must be infinitely many integers $n \geq N$ for which the inequality

$$|y_n - y| < \frac{\varepsilon}{2p}$$

holds. For each of these infinitely many integers n we have

$$\begin{aligned}|x_ny_n - xy| &= |x_ny_n - x_ny + x_ny - xy| \\ &\le |x_ny_n - x_ny| + |x_ny - xy| \\ &\le p|y_n - y| + |y||x_n - x| \\ &< p\left(\frac{\varepsilon}{2p}\right) + |y|\left(\frac{\varepsilon}{2|y|+1}\right) < \varepsilon.\end{aligned}$$

4. Give an example of two sequences (x_n) and (y_n) and a partial limit x of (x_n) and a partial limit y of (y_n) such that $x + y$ fails to be a partial limit of the sequence $(x_n + y_n)$.
   Define $x_n = y_n = (-1)^n$ for each n. We observe that the numbers -1 and 1 are partial limits of (x_n) and (y_n) respectively but that $-1 + 1$ fails to be a partial limit of $(x_n + y_n)$.

5. Give an example of two divergent sequences (x_n) and (y_n) such that the sequence $(x_n + y_n)$ is convergent.
   We define $x_n = (-1)^n$ and $y_n = (-1)^{n-1}$ for each n. Note that the sequence $(x_n + y_n)$ is the sequence with constant value 0.

6. Give an example of two sequences (x_n) and (y_n) such that $x_n \to 0$ and $y_n \to \infty$ and

   a. $x_ny_n \to 0$
      We define $x_n = n$ and $y_n = 1/n^2$ for each n.

   b. $x_ny_n \to 6$
      We define $x_n = n$ and $y_n = 6/n$ for each n.

   c. $x_ny_n \to \infty$
      We define $x_n = n^2$ and $y_n = 1/n$ for each n.

   d. The sequence (x_ny_n) is bounded but has no limit.
      We define $x_n = n$ and $y_n = (-1)^n/n$ for each n.

7. Given two sequences (x_n) and (y_n) of real numbers such that both of the sequences (x_n) and $(x_n + y_n)$ are convergent, is it true that the sequence (y_n) must be convergent?
   Yes. Since
   $$y_n = (x_n + y_n) - x_n$$
   for each n it follows at once that
   $$\lim_{n\to\infty} y_n = \lim_{n\to\infty}(x_n + y_n) - \lim_{n\to\infty} x_n$$

8. Given that (x_n) is a sequence of real numbers and that $x_n \to 0$, prove that
   $$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \to 0.$$

Solution: ***Suppose that $\varepsilon > 0$. Using the fact that $x_n \to 0$ as $n \to \infty$, choose an integer N_1 such that the inequality $|x_n| < \varepsilon/2$ holds whenever $n \ge N_1$. Whenever $n \ge N_1$ we see that***

$$\begin{aligned}\left|\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right| &= \left|\frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{n} + \frac{x_{N_1} + x_{N_1+1} + \cdots + x_n}{n}\right| \\ &\le \left|\frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{n}\right| + \left|\frac{x_{N_1} + x_{N_1+1} + \cdots + x_n}{n}\right|.\end{aligned}$$

Now we choose an integer N_2 such that

$$N_2 \ge \frac{2|x_1 + x_2 + x_3 + \cdots + x_{N_1}|}{\varepsilon}$$

and we define N to be the larger of the two numbers N_1 and N_2. Then whenever $n \ge N$ we have

$$\begin{aligned}\left|\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} - 0\right| &\le \left|\frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{n}\right| + \left|\frac{x_{N_1} + x_{N_1+1} + \cdots + x_n}{n}\right| \\ &\le \left|\frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{N_2}\right| + \frac{|x_{N_1+1}| + |x_{N_1+2}| + \cdots + |x_n|}{n} \\ &< \frac{\varepsilon}{2} + \left(\frac{n - N_1}{n}\right)\frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

9. Given that (x_n) is a sequence of real numbers, that x is a real number and that $x_n \to x$, prove that
$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \to x.$$

Solution: ***Since*** $x_n - x \to 0$ ***as*** $n \to \infty$ ***we have***
$$\frac{(x_1 - x) + (x_2 - x) + (x_3 - x) + \cdots + (x_n - x)}{n} \to 0$$
as $n \to \infty$***. Therefore***
$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} - x = \frac{(x_1 - x) + (x_2 - x) + (x_3 - x) + \cdots + (x_n - x)}{n} \to 0$$
as $n \to \infty$***.***

10. Given that (x_n) and (y_n) are sequences of real numbers and that $x_n - y_n \to 0$, prove that (x_n) and (y_n) have the same set of partial limits.
Since $x_n = (x_n - y_n) + y_n$ and $y_n = x_n - (x_n - y_n)$ for each n the present result follows at once from Exercise 2.

11. Suppose that (x_n) and (y_n) are sequences of real numbers, that $x_n - y_n \to 0$ and that the number 0 fails to be a partial limit of at least one of the sequences (x_n) and (y_n). Prove that
$$\frac{x_n}{y_n} \to 1$$
as $n \to \infty$.

Solution: ***From Exercise 10 we know that*** (x_n) ***and*** (y_n) ***have the same sets of partial limits and we know, therefore, that*** 0 ***is not a partial limit of either of these two sequences.***
Using the fact that 0 ***is not a partial limit of*** (y_n)***, choose an integer*** N_1 ***and a number*** $\delta > 0$ ***such that the inequality***
$$|y_n| \geq \delta$$
holds whenever $n \geq N_1$***. For every*** $n \geq N_1$ ***we see that***
$$\left|\frac{x_n}{y_n} - 1\right| = \left|\frac{x_n - y_n}{y_n}\right| \leq \frac{|x_n - y_n|}{\delta}$$
and so the fact that
$$\frac{x_n}{y_n} \to 1$$
as $n \to \infty$ ***follows from the sandwich theorem.***

12. Give an example to show that the requirement in this exercise that 0 not be a partial limit of at least one of the two sequences is really needed.
We define $x_n = 2/n$ and $y_n = 1/n$ for each n and observe that, even though $x_n - y_n \to 0$ as $n \to \infty$, $x_n/y_n \to 2$ as $n \to \infty$.

13. Suppose that (x_n) and (y_n) are sequences of real numbers, that $x_n/y_n \to 1$ and that at least one of the sequences (x_n) and (y_n) is bounded. Prove that $x_n - y_n \to 0$. Give an example to show that the conclusion $x_n - y_n \to 0$ can fail if both (x_n) and (y_n) are unbounded.

Hint: ***Make the observation that both of the sequences*** (x_n) ***and*** (y_n) ***must be bounded. Explain carefully how you arrive at this conclusion. Now make use of the fact that***
$$|x_n - y_n| = |y_n|\left|\frac{x_n}{y_n} - 1\right|$$
for every n***.***

14. Suppose that (x_n) and (y_n) are sequences of real numbers, that $y_n \to 1$ and that for each n we have $z_n = x_n y_n$. Prove that the sequences (x_n) and (z_n) have the same set of partial limits.
It follows from the part of Exercise 3 for which a solution is provided above that if x is an partial limit of (x_n) then the number $x(1) = x$ is a partial limit of the sequence (z_n). At the same time, since $x_n = z_n/y_n$ for all sufficiently large n we know that if x is any partial limit of (z_n) then the number $x/1 = x$ is a partial limit of the sequence (x_n).

Exercises on Sequences and the Topology of R

1. Prove that a set S of real numbers is unbounded below if and only if there exists a sequence (x_n) in S such that $x_n \to -\infty$.

   Solution: ***We want to show that, for a given set S of real numbers, the following conditions are equivalent:***

   a. ***The set S is unbounded below.***

   b. ***There exists a sequence (x_n) in the set S such that $x_n \to -\infty$ as $n \to \infty$.***

   To show that condition b implies condition a, assume that condition b holds and choose a sequence (x_n) in S such that $x_n \to -\infty$ as $n \to \infty$. If w is any real number then it follows at once from the fact that the sequence (x_n) is eventually in the interval $(-\infty, w)$ that w is not a lower bound of S. Therefore S is not bounded above.

   Now to show that condition a implies condition b, assume that condition a holds. For each positive integer n we use the fact that the number $-n$ is not a lower bound of S to choose a number that we shall call x_n such that $x_n < -n$. The sequence (x_n) that we have made in this way is in the set S and it is clear that $x_n \to -\infty$.

2. Suppose that S is a nonempty set of real numbers and that α is an upper bound of S. Prove that the following conditions are equivalent:

   a. We have $\alpha = \sup S$.

   b. There exists a sequence (x_n) in S such that $x_n \to \alpha$ as $n \to \infty$.
      In Exercise 7 of the exercises on closure we saw that an upper bound α of set S is close to S if and only if α is equal to $\sup S$. We also saw in a recent theorem that a number α is close to a set S if and only if there exists a sequence in S that converges to α. The present exercise follows at once from these two facts. Of course, we can write this exercise out directly if we wish.

3. Given a sequence (x_n) that is frequently in a set S of real numbers, and given a partial limit x of the sequence (x_n), is it necessarily true that $x \in \overline{S}$?
   No, this statement need not be true. If we define $x_n = (-1)^n$ for each n then, although (x_n) is frequently in the set $\{-1\}$ it has the partial limit 1 that is not close to $\{-1\}$.

4. Prove that a set U of real numbers is open if and only if every sequence that converges to a member of U must be eventually in U.
   We know from a recent theorem that a set H is closed if and only if no sequence that is frequently in H can have a limit that doesn't belong to H. Therefore if U is a set of real numbers then the set $\mathbf{R} \setminus U$ is closed if and only if no sequence with a limit in U can fail to be eventually in U.
   Of course the exercise can also be done directly.

5. Given that S is a set of real numbers and that x is a real number, prove that the following conditions are equivalent:

   a. The number x is a limit point of the set S.

   b. There exists a sequence (x_n) in the set $S \setminus \{x\}$ such that $x_n \to x$.

   Solution: ***The assertion in this exercise follows at once from the corresponding theorem about limits of sequences and closure of a set, and from the fact that x is a limit point of S if and only if $x \in \overline{S \setminus \{x\}}$.***

6. Prove that if (x_n) is a sequence of real numbers then the set of all partial limits of (x_n) is closed.
   Suppose that (x_n) is a sequence of real numbers and write the set of partial limits of (x_n) as H. In order to show that H is closed we shall show that $\mathbf{R} \setminus H$ is open. Suppose that $x \in \mathbf{R} \setminus H$. In other words, suppose that x is a number that is not a partial limit of (x_n). Choose a number $\varepsilon > 0$ such that the condition

$$x_n \in (x - \varepsilon, x + \varepsilon)$$

   holds for at most finitely many integers n. Given any number $y \in (x - \varepsilon, x + \varepsilon)$, it follows from the fact that $(x - \varepsilon, x + \varepsilon)$ is a neighborhood of y and the fact that x_n belongs to this neighborhood of y for at most finitely many integers n that y is not a partial limit of (x_n). In other words,

$$(x - \varepsilon, x + \varepsilon) \subseteq \mathbf{R} \setminus H$$

   and we have shown, as promised, that the set $\mathbf{R} \setminus H$ is open.

7. Suppose that A and B are nonempty sets of real numbers and that for every number $x \in A$ and every number $y \in B$ we have $x < y$. Prove that the following conditions are equivalent:

   a. We have $\sup A = \inf B$.

   b. There exists a sequence (x_n) in the set A and a sequence (y_n) in the set B such that $y_n - x_n \to 0$ as $n \to \infty$.

   Solution: ***From the information given about A and B we know that $\sup A \le \inf B$. In the event that $\sup A < \inf B$ we know that for every sequence (x_n) in A and every sequence (y_n) in B we have***

$$y_n - x_n \to \inf B - \sup A > 0$$

   for every n and so we can't have $y_n - x_n \to 0$ as $n \to \infty$. Therefore if condition a is false then so is condition b.

   Suppose now that condition a is true; in other words, that $\sup A = \inf B$. For every positive integer n, we use the fact that

$$\sup A - \frac{1}{n}$$

   is not an upper bound of A to choose a member x_n of A such that

$$\sup A - \frac{1}{n} < x_n.$$

   For every positive integer n we use the fact that

$$\inf B + \frac{1}{n}$$

   is not a lower bound of B to choose a member y_n of B such that

$$y_n < \inf B + \frac{1}{n}.$$

   Thus for each n we have

$$\sup A - \frac{1}{n} < x_n \le y_n < \inf B + \frac{1}{n}$$

   and therefore

$$0 \le y_n - x_n < \frac{2}{n}$$

   from which it follows that $y_n - x_n \to 0$ as $n \to \infty$.

8. Suppose that S is a nonempty bounded set of real numbers. Prove that there exist two sequences (x_n) and (y_n) in the set S such that

$$y_n - x_n \to \sup S - \inf S$$

   as $n \to \infty$.
   We have already seen that there exists a sequence (x_n) in S such that $x_n \to \sup S$ as $n \to \infty$. In the same way we can see that there exists a sequence (y_n) in S such that $y_n \to \inf S$ as $n \to \infty$. We now have

$$y_n - x_n \to \sup S - \inf S$$

as $n \to \infty$.

Exercises on Monotone Sequences

1. Given that $c > 1$, use the following method to prove that $c^n \to \infty$:

   a. Write $\delta = c - 1$, so that $c = 1 + \delta$, and then use mathematical induction to prove that, if n is any positive integer then $c^n \geq 1 + n\delta$.
   For each positive integer n we take p_n to be the assertion that $c^n \geq 1 + n\delta$. Since $c = 1 + \delta$ we know that the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n is true. We observe that
   $$\begin{aligned} c^{n+1} &= cc^n \geq (1+\delta)(1+n\delta) \\ &= 1 + (n+1)\delta + n\delta^2 > 1 + (n+1)\delta \end{aligned}$$
   and so the assertion p_{n+1} is also true. We deduce from mathematical induction that the inequality $c^n \geq 1 + n\delta$ is true for every positive integer n.

   b. Explain why $1 + n\delta \to \infty$ and then use this exercise in this subsection to show that $c^n \to \infty$.
   To prove that $1 + n\delta \to \infty$ as $n \to \infty$, suppose that w is a real number. The inequality
   $$1 + n\delta > w$$
   will hold when
   $$n > \frac{w-1}{\delta}.$$
   We choose a positive integer N such that
   $$N > \frac{w-1}{\delta}$$
   and we observe that $1 + n\delta > w$ whenever $n \geq N$.

2. Given that $c > 1$, use the following method to prove that $c^n \to \infty$:

   a. Explain why the sequence (c^n) is increasing and deduce that it has a limit.
   The fact that the sequence (c^n) is increasing follows at once from the inequality
   $$c^{n+1} = cc^n > 1c^n.$$
   It now follows from the monotone sequences theorem that the sequence (c^n) has a limit.

   b. Call the limit x and show that if x is finite then the equation
   $$c^{n+1} = cc^n$$
   leads to the equation $x = cx$, which implies that $x = 0$. But x cannot be equal to zero? Why not?
   From the equation
   $$c^{n+1} = cc^n$$
   we obtain
   $$\lim_{n\to\infty} c^{n+1} = \lim_{n\to\infty}(cc^n)$$
   which gives us $x = cx$. Since $c^n > 1$ for every n we know that $x \geq 1$ and therefore $x \neq 0$.

3. Suppose that $|c| < 1$ and that, for every positive integer n,
   $$x_n = \sum_{i=1}^{n} c^{i-1}.$$
   Explain why
   $$x_n \to \frac{1}{1-c}.$$
   From the identity

$$(1-c)\sum_{i=1}^{n} c^{i-1} = 1-c^n$$

we deduce that

$$\sum_{i=1}^{n} c^{i-1} = \frac{1-c^n}{1-c}$$

Now since $1/|c| > 1$ we know that $1/|c|^n \to \infty$ as $n \to \infty$ and we conclude that $|c|^n \to 0$ as $n \to \infty$. Therefore $c^n \to 0$ as $n \to \infty$ and we conclude that

$$\lim_{n\to\infty} \frac{1-c^n}{1-c} = \frac{1}{1-c}.$$

4. Suppose that (x_n) is a given sequence of real numbers, that $x_1 = 0$ and that the equation

$$8x_{n+1}^3 = 6x_n + 1$$

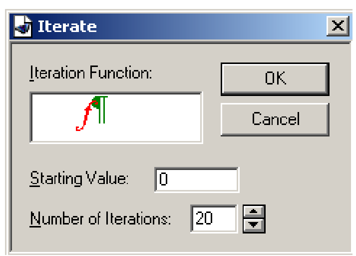
holds for every positive integer n.

a. Use *Scientific Notebook* to work out the first twenty terms in the sequence (x_n).

To use *Scientific Notebook* for this purpose, place your cursor in the equation

$$f(x) = \sqrt[3]{\frac{6x+1}{8}}$$

and click on the button in your computing toolbar to supply the definition of the function f to *Scientific Notebook*. Then open the *Compute* menu, scroll down to *Calculus* and move across to Iterate. Fill the dialogue box giving the name of the function as f, the number of iterations as 20 and the starting value as 0.



a. Prove that $x_n < 1$ for every positive integer n.

We use mathematical induction. For each positive integer n we take p_n to be the assertion that $x_n < 1$. Since $x_1 = 0$, the assertion p_1 is true. Now suppose that n is any positive integer for which the statement p_n is true. Then

$$8x_{n+1}^3 = 6x_n + 1 < 6(1) + 1$$

and so

$$x_{n+1} < \sqrt[3]{\frac{7}{8}} < 1$$

from which it follows that the assertion p_{n+1} is also true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

b. Prove that the sequence (x_n) is strictly increasing.

Once again we use mathematical induction. For each positive integer n we take p_n to be the assertion that $x_n < x_{n+1}$. Since $x_1 = 0 < \frac{1}{2} = x_2$, the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n is true. Since

$$x_{n+2} = \sqrt[3]{\frac{6x_{n+1}+1}{8}} > \sqrt[3]{\frac{6x_n+1}{8}} = x_{n+1}$$

we see that the assertion p_{n+1} must also be true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

c. Deduce that the sequence (x_n) is convergent and discuss its limit. Assuming an unofficial knowledge of the trigonometric functions, prove that the limit of the sequence (x_n) is $\cos\frac{\pi}{9}$.

Solution to part d: ***We write the limit of this sequence as** x**. from the fact that***

$$8x_{n+1}^3 = 6x_n + 1$$

for each** n **we obtain

$$8x^3 - 6x - 1 = 0.$$

This equation has one positive solution and two negative solutions and from the fact that

$$\begin{aligned} 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1 &= 2\left(4\cos^3\frac{\pi}{9} - 3\cos\frac{\pi}{9}\right) - 1 \\ &= 2\left(\cos\left(3\left(\frac{\pi}{9}\right)\right)\right) - 1 = 2\cos\frac{\pi}{3} - 1 = 0 \end{aligned}$$

we see that the positive root is** $\cos\frac{\pi}{9}$**.

5. In this exercise we study the sequence (x_n) defined by the equation

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

for every integer $n \geq 1$. You will probably want to make use of the binomial theorem when you do this exercise.

a. Ask *Scientific Notebook* to make a 2D plot of the graph of the function f defined by the equation

$$f(n) = \left(1 + \frac{1}{n}\right)^n$$

for $1 \leq n \leq 100$.

b. Prove that $x_n < 3$ for every n.

Solution: ***For each** $n \geq 2$ **we see that***

$$\begin{aligned} x_n &= \left(1 + \frac{1}{n}\right)^n = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j = 1 + 1 + \sum_{j=2}^{n} \frac{n(n-1)\cdots(n-j+1)}{(j!)n^j} \\ &= 1 + 1 + \sum_{j=2}^{n} (1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{j-1}{n}\right)\frac{1}{j!} \\ &< 2 + \sum_{j=2}^{n} \frac{1}{j!} < 2 + \sum_{j=2}^{n} \frac{1}{2^{j-1}} < 3 \end{aligned}$$

c. Prove that the sequence (x_n) is increasing.

Solution: ***For each** $n \geq 2$ **we see that***

$$\begin{aligned} x_n &= 1 + 1 + \sum_{j=2}^{n} (1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{j-1}{n}\right)\frac{1}{j!} \\ &< 1 + 1 + \sum_{j=2}^{n} (1)\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{j-1}{n+1}\right)\frac{1}{j!} = x_{n+1}. \end{aligned}$$

d. Deduce that the sequence (x_n) converges to a number between 2 and 3. Have you seen this number before?
The student is being asked informally whether he/she recognizes that this limit is the number e. The number e will be seen in Chapter 10.

6. This exercise concerns the sequence (x_n) defined by the fact that $x_1 = 1$ and that, for each $n \geq 1$ we have

$$x_{n+1} = \sqrt[5]{4x_n - 2}.$$

a. Use *Scientific Notebook* to work out the first twenty terms in the sequence (x_n).

b. Prove that $1 \leq x_n < 2$ for every n.

We use mathematical induction. Since $x_1 = 1$, the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n is true. We see that

$$1 < \sqrt[5]{4(1) - 2} < \sqrt[5]{4x_n - 2} < \sqrt[5]{4(2) - 2} < 2$$

and so the assertion p_{n+1} is true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

c. Prove that the sequence (x_n) is strictly increasing.

We use mathematical induction. For each positive integer n we take p_n to be the assertion that $x_n < x_{n+1}$. Since

$$x_1 = 1 < \sqrt[5]{2} = x_2$$

we conclude that the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n happens to be true. Then

$$x_{n+2} = \sqrt[5]{4x_{n+1} - 2} > \sqrt[5]{4x_n - 2} = x_{n+1}$$

and so the assertion p_{n+1} must be true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

d. Prove that the sequence has a limit x that satisfies the equation $x^5 - 4x + 2 = 0$. Ask *Scientific Notebook* to make a 2D plot of the expression $x^5 - 4x + 2$ on the interval $[-2, 2]$ and to solve the equation

$$x^5 - 4x + 2 = 0$$
$$x \in [1, 2]$$

numerically. Compare the answer obtained here with the results that you obtained in part a.

7. a. Given that

$$f(x) = \frac{x}{2} + \frac{9}{2x}$$

for every number $x > 0$, prove that $f(x) \geq 3$ for each n and that the equation $f(x) = 3$ holds if and only if $x = 3$.

Solution: ***The desired result follows at once from the fact that whenever*** $x > 0$ ***we have***

$$f(x) = \frac{x}{2} + \frac{9}{2x} = \frac{(x-3)^2}{2x} + 3.$$

b. Given that $x_1 = 4$ and, for each $n \geq 1$, we have

$$x_{n+1} = \frac{x_n}{2} + \frac{9}{2x_n},$$

prove that the sequence (x_n) is decreasing and that the sequence converges to the number 3.

Solution: ***Since*** $x_{n+1} = f(x_n)$ ***for each*** n ***and since*** $f(x) > 3$ ***for every number*** $x \neq 3$ ***we see at once that*** $x_n > 3$ ***for every*** n***. To see that*** (x_n) ***is decreasing we observe that if*** n ***is any positive integer then***

$$x_n - x_{n+1} = x_n - \left(\frac{x_n}{2} + \frac{9}{2x_n}\right) = \frac{x_n^2 - 9}{2} > 0.$$

Since the sequence (x_n) ***is a decreasing sequence in the interval*** $(3, \infty)$ ***we know that*** (x_n) ***is convergent. If we write the limit of this sequence as*** x ***then it follows from the relationship***

$$x_{n+1} = \frac{x_n}{2} + \frac{9}{2x_n}$$

that

$$x = \frac{x}{2} + \frac{9}{2x}$$

from which we deduce that $x = 3$***.***

8. This exercise is a study of the sequence (x_n) for which $x_1 = 0$ and

$$x_{n+1} = \frac{1}{2 + x_n}$$

for every positive integer n. We note that this sequence is bounded below by 0 and above by 1/2.

a. Supply the definition

$$f(x) = \frac{1}{2 + x}$$

to ***Scientific Notebook***. Then open your Compute menu, click on Calculus, and choose to iterate the function f ten times, starting at the number 0. Evaluate the column of numbers that you have obtained accurately to ten decimal places and, in this way, show the first ten members of the sequence (x_n).

b. Show that

$$x_{n+2} = \frac{2 + x_n}{5 + 2x_n}$$

for every integer $n \geq 1$, and then show that the sequence (x_{2n-1}) is increasing and that the sequence (x_{2n}) is decreasing and that these two sequences have the same limit $\sqrt{2} - 1$.

Solution: ***For every positive number*** x ***we define***

$$g(x) = \frac{2 + x}{5 + 2x}.$$

Whenever $0 < t < x$ ***we see that***

$$g(x) - g(t) = \frac{2 + x}{5 + 2x} - \frac{2 + t}{5 + 2t} = \frac{x - t}{(5 + 2x)(5 + 2t)} > 0$$

and so the function g ***is strictly increasing. Since*** $x_1 < x_3$ ***we have*** $g(x_1) < g(x_3)$ ***from which we deduce that*** $x_3 < x_5$***. Continuing in this way we see that the sequence*** (x_{2n-1}) ***is increasing. Since*** $x_2 > x_4$ ***we have*** $g(x_2) > g(x_4)$ ***from which deduce that*** $x_4 > x_5$***. Continuing in this way we see that the sequence*** (x_{2n}) ***is decreasing. Therefore the sequences*** (x_{2n-1}) ***and*** (x_{2n}) ***are convergent. If we write***

$$x = \lim_{n\to\infty} x_{2n-1}$$

then it follows from the identity

$$x_{2n+1} = \frac{2 + x_{2n-1}}{5 + 2x_{2n-1}}$$

that

$$x = \frac{2 + x}{5 + 2x}$$

and we can see that the only positive solution of this equation is $\sqrt{2} - 1$***. Thus***

$$\lim_{n\to\infty} x_{2n-1} = \sqrt{2} - 1$$

and we can see similarly that

$$\lim_{n\to\infty} x_{2n} = \sqrt{2} - 1.$$

c. Deduce that $x_n \to \sqrt{2} - 1$ as $n \to \infty$.

Exercises on the Cantor Intersection Theorem

1. Suppose that (H_n) is a sequence (not necessarily contracting) of closed bounded sets and that for every positive integer n we have

$$\bigcap_{i=1}^{n} H_i \neq \emptyset.$$

Prove that

$$\bigcap_{i=1}^{\infty} H_i \neq \emptyset.$$

For each n we define

$$K_n = \bigcap_{i=1}^{n} H_i$$

Since the sequence (K_n) is a contracting sequence of nonempty closed bounded sets, we know from the Cantor intersection theorem that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

We observe finally that, since

$$\bigcap_{n=1}^{\infty} H_n = \bigcap_{n=1}^{\infty} K_n$$

the intersection of all of the sets H_n must also be nonempty.

2. Suppose that H is a closed bounded set of real numbers and that (U_n) is an expanding sequence of open sets.

   a. Explain why the sequence of sets $H \setminus U_n$ is a contracting sequence of closed bounded sets.
   For each n, it follows at once from the inclusion $U_n \subseteq U_{n+1}$ that

   $$H \setminus U_{n+1} \subseteq H \setminus U_n.$$

   b. Use the Cantor intersection theorem to deduce that if $H \setminus U_n \neq \emptyset$ for every n then

   $$\bigcap_{n=1}^{\infty} (H \setminus U_n) \neq \emptyset.$$

   Since H is closed and bounded and since

   $$H \setminus U_n = H \cap (\mathbf{R} \setminus U_n)$$

   for each n, the sets $H \setminus U_n$ must be closed and bounded. The desired result therefore follows at once from the Cantor intersection theorem.

   c. Prove that if

   $$H \subseteq \bigcup_{n=1}^{\infty} U_n$$

   then there exists an integer n such that $H \subseteq U_n$.
   We suppose that

   $$H \subseteq \bigcup_{n=1}^{\infty} U_n.$$

   Since the set

   $$\bigcap_{n=1}^{\infty} (H \setminus U_n) = H \setminus \bigcup_{n=1}^{\infty} U_n$$

   is empty we deduce from part b that there is a value of n for which $H \setminus U_n = \emptyset$ and for any such integer n we have $H \subseteq U_n$.

3. Suppose that (U_n) is a sequence of open sets (not necessarily expanding) and that H is a closed bounded set

and that

$$H \subseteq \bigcup_{n=1}^{\infty} U_n.$$

Prove that there exists a positive integer N such that

$$H \subseteq \bigcup_{n=1}^{N} U_n.$$

For each n we define

$$V_n = \bigcup_{i=1}^{n} U_i.$$

The sequence (V_n) is expanding and, since

$$H \subseteq \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} V_n$$

it follows from Exercise 2 that there exists a positive integer n for which

$$H \subseteq V_n = \bigcup_{i=1}^{n} U_i.$$

4. The Cantor intersection theorem depends upon the completeness of the real number system. Where in the proof of the theorem is the completeness used?
We use the completeness to guarantee that a nonempty closed bounded set has a least member and then we use the completeness again to guarantee that the sequence of all the least members of the given sets has a supremum.

Exercises on the Bolzano-Weierstrass Theorem

1. The Bolzano-Weierstrass theorem does not tell us that if a set S is bounded and infinite then at least one member of S must be a limit point of S. Give an example of a bounded infinite set S such that no member of S is a limit point of S.
The set

$$\left\{\frac{1}{n} \mid n \in \mathbf{Z}^{+}\right\}$$

has only one limit point; the number 0, that does not belong to the set.

2. Prove that if H is a closed bounded infinite set then

$$H \cap \mathbf{L}(H) \neq \emptyset.$$

Suppose that H is a closed bounded infinite set. We know from the Bolzano-Weierstrass theorem that H has a limit point. Since H is closed, every limit point of H must belong to H and so H must have at least one limit point that belongs to H.

3. This exercise suggests a different proof of the Bolzano-Weierstrass theorem:

   a. Prove that if E is a nonempty bounded set with no least member then $\inf E$ is a limit point of E.
   Suppose that E is a nonempty bounded set that has no least member. We know that the number $\inf E$ is close to the set E but, since E has no least member, $\inf E$ does not belong to E. As we know, a number that does not belong to a given set but is close to that S must be a limit point of the set.

   b. Prove that if a bounded set S has a nonempty subset that does not have a least member then S has a limit point.
   Suppose that S is a bounded set that has a nonnegative subset E that has no least member. From part a we know that E has a limit point. Since any limit point of E must also be a limit point of S we can conclude that S has a limit point.

c. Given that S is a bounded infinite set and that every nonempty subset of S has a least member, find an example of a strictly increasing sequence in the set S. By considering the limit of this sequence, prove that S must have a limit point.

Since S is a nonempty subset of itself, it has a least member. We define x_1 to be the least member of S. Since S is infinite, the set $S \setminus \{x_1\}$ is a nonempty subset of S. We define x_2 to be the least member of $S \setminus \{x_1\}$. We note that $x_1 < x_2$. Since S is infinite, the set $S \setminus \{x_1, x_2\}$ is a nonempty subset of S. We define x_3 to be the least member of $S \setminus \{x_1, x_2\}$ and note that $x_2 < x_3$. Continuing in this way we obtain a strictly increasing sequence (x_n) in the set S. As we know, an increasing bounded sequence must converge. We define

$$x = \lim_{n\to\infty} x_n.$$

Given any number $\varepsilon > 0$ we know that the condition

$$x_n \in (x - \varepsilon, x + \varepsilon)$$

must hold for all sufficiently large integers n and, since the sequence (x_n) is one-one we deduce that the interval $(x - \varepsilon, x + \varepsilon)$ must contain more than one member of S. Therefore x is a limit point of S.

Some Further Exercises on Limit Points

The exercies that appear here require the student to be familiar with the concept of

1. a. Given that S is an uncountable set of real numbers, explain why there must exist an integer n such that the set $S \cap [n, n+1]$ is uncountable.

   Solution: ***If the set*** $S \cap [n, n+1]$ ***were countable for every*** n ***then, since***

   $$S = \bigcup_{n=-\infty}^{\infty} S \cap [n, n+1],$$

   the set S***, being the union of a countable family of countable sets, would be countable by*** *an earlier theorem.*

   b. Prove that every uncountable set of real numbers must have a limit point.

   Hint: ***Apply part a and the Bolzano-Weierstrass theorem to an uncountable set of the form*** $S \cap [n, n+1]$.

2. a. Suppose that S is a set of real numbers, that $\varepsilon > 0$ and that for any two different members x and t of the set S we have $|x - t| \geq \varepsilon$. Prove that S is a countable set.

   b. Suppose that S is an uncountable set of real numbers and, for each positive integer n, suppose that

   $$S_n = \left\{ x \in S \mid |x - y| \geq \frac{1}{n} \text{ for every } y \in S \setminus \{x\} \right\}.$$

   Prove that each set S_n is countable and that

   $$S \setminus \mathbf{L}(S) = \bigcup_{n=1}^{\infty} S_n.$$

   c. Improve Exercise 1b by showing that if S is an uncountable set of real numbers then S has an uncountable set of limit points that belong to S.

   Solution: ***We deduce from part b that the set*** $S \setminus L(S)$ ***is countable. Therefore, since***

   $$S = \left(S \setminus \mathbf{L}(S)\right) \cup \left(S \cap \mathbf{L}(S)\right)$$

   and since the set S ***is countable, the set*** $S \cap L(S)$ ***must be uncountable.***

3. Suppose that S is a set of real numbers and that $\varepsilon > 0$. For each integer n, in the event that the set

$$\{x \in S \mid n\varepsilon \le x \le (n+1)\varepsilon\}$$

is nonempty, suppose that a number x_n has been chosen in this set. Prove that for every member x of the set S there is an integer n for which the number x_n is defined and for which

$$x \in [x_n - \varepsilon, x_n + \varepsilon].$$

4. Prove that if S is any set of real numbers and $\varepsilon > 0$ then there exists a countable subset E of S such that

$$S \subseteq \bigcup_{x \in E} [x_n - \varepsilon, x_n + \varepsilon].$$

5. Suppose that S is a set of real numbers.

   a. Prove that for every positive integer n it is possible to find a countable subset E_n of the set S such that

   $$S \subseteq \bigcup_{x \in E_n} \left[x - \frac{1}{n}, x + \frac{1}{n}\right]$$

   b. Prove that there exists a countable subset E of S such that $S \subseteq \overline{E}$.

6. Prove that every closed subset of **R** is the closure of some countable set.

Exercises on Upper and Lower Limits

1. Prove that if (x_n) is a sequence of real numbers then (x_n) has a limit if and only if

$$\operatorname*{lmsup}_{n\to\infty} x_n = \operatorname*{lminf}_{n\to\infty} x_n.$$

Solution: ***This exercises follows at once from*** *an earlier theorem.*

2. Prove that a sequence (x_n) of real numbers is bounded above if and only if

$$\operatorname*{lmsup}_{n\to\infty} x_n < \infty.$$

We know that a sequence is bounded above if and only if ∞ is not a partial limit of the sequence. There a sequence is bounded above if and only if its largest partial limit is not ∞.

3. Suppose that (x_n) is a sequence of real numbers.

   a. Prove that if x is a partial limit of the sequence (x_n) then the number $-x$ is a partial limit of the sequence $(-x_n)$.
   if ∞ is a partial limit of a sequence (x_n) then (x_n) is unbounded above and so $(-x_n)$ is unbounded below, making $-\infty$ a partial limit of $(-x_n)$.
   Now suppose that a real number x is a partial limit of a given sequence (x_n). To show that the number $-x$ is a partial limit of the sequence $(-x_n)$, suppose that $\varepsilon > 0$.
   We know that there are infinitely many integers n for which

   $$x - \varepsilon < x_n < x + \varepsilon$$

   and for each of these integers n we have

   $$-x - \varepsilon < -x_n < -x + \varepsilon.$$

   Therefore the sequence $(-x_n)$ is frequently in the interval $(-x-\varepsilon, -x+\varepsilon)$ and we conclude that $-x$ is a partial limit of $(-x_n)$.

   b. Prove that

   $$\operatorname*{lmsup}_{n\to\infty}(-x_n) = -\operatorname*{lminf}_{n\to\infty} x_n.$$

   Since $\operatorname{lminf}_{n\to\infty} x_n$ is a partial limit of (x_n) we know from part a that $-\operatorname{lminf}_{n\to\infty} x_n$ is a partial limit of the sequence $(-x_n)$. Now given any partial limit q of $(-x_n)$, we know that $-q$, being a partial limit of (x_n), cannot be less than $\operatorname{lminf}_{n\to\infty} x_n$. In other words, whenever q is a partial limit of $(-x_n)$ we have

$$\liminf_{n\to\infty} x_n \le -q$$

which gives us

$$q \le -\liminf_{n\to\infty} x_n.$$

Therefore $-\liminf_{n\to\infty} x_n$ is the largest partial limit of the sequence $(-x_n)$.

4. Suppose that (x_n) is a sequence of real numbers, that

$$x = \limsup_{n\to\infty} x_n$$

and that $u < x < v$.

a. Prove that the sequence (x_n) must be bounded above.
The given inequality $u < x < v$ tells us that $x \ne \infty$ and so (x_n) is bounded above.

b. Prove that the sequence (x_n) must be frequently in the interval $[u,\infty)$.
The interval $[u,\infty)$ is a neighborhood of the partial limit x of (x_n).

c. Prove that the sequence (x_n) cannot be frequently in the interval $[v,\infty)$.

Hint: ***Choose an upper bound α of (x_n). Now use*** *this theorem.*
If (x_n) were frequently in the interval $[v,\infty]$ then it would have to have a partial limit in this interval, contradicting the fact that x is the largest partial limit of (x_n).

5. Suppose that (x_n) is a sequence of real numbers, that x is a real number and that, whenever $u < x < v$, the sequence (x_n) is frequently in the interval $[u,\infty)$ but is not frequently in the interval $[v,\infty)$. Prove that

$$x = \limsup_{n\to\infty} x_n.$$

The fact that (x_n) must be frequently in the interval (u,v) whenever $u < x < v$ tells us that (x_n) is frequently in every neighborhood of x and so x must be a partial limit of (x_n). To show that x is the largest partial limit of (x_n) we shall show that no number larger than x can be a partial limit of (x_n). Suppose that $x < p$ and choose a number v between x and p. Since the interval $[v,\infty)$ is a neighborhood of p and (x_n) is not frequently in $[v,\infty)$ we conclude that p can't be a partial limit of (x_n).

6. Suppose that (x_n) is a bounded sequence of real numbers and that, for each integer n in the domain of this sequence we have defined

$$y_n = \sup\{x_m \mid m \ge n\}.$$

Prove that the sequence (y_n) is decreasing and that its limit is the lower limit of the sequence (x_n).
For each positive integer n, it follows from the fact that

$$\{x_m \mid m \ge n+1\} \subseteq \{x_m \mid m \ge n\}$$

that

$$\sup\{x_m \mid m \ge n+1\} \le \sup\{x_m \mid m \ge n\}$$

and so the sequence (y_n) must be decreasing. Since every lower bound of (x_n) is also a lower bound of (y_n), the sequence (y_n) must be bounded below. Therefore (y_n) is convergent. We define y to be the limit of (y_n). To show that y is $\limsup_{n\to\infty} x_n$ we shall use Exercise 5. Suppose that $u < y < v$. Using the fact that $y_n \to y$ as $n \to \infty$ we choose N such that the inequality $y_n < v$ holds whenever $n \ge N$.

u $\qquad$ y $\qquad$ y_n $\qquad$ v

Since

$$x_m \le y_N < v$$

whenever $m \ge N$ we know that the sequence (x_n) cannot be frequently in the interval $[v,\infty)$.
On the other hand, given any integer $n \ge N$ it follows from the fact that

$$u < y_n = \sup\{x_m \mid m \ge n\}$$

that there exists an integer $m \geq n$ such that $u < x_m$.

$$u \qquad x_m \qquad y \qquad\qquad\qquad y_n \qquad\qquad v$$

Thus the set of integers m for which $u < x_m$ is unbounded above and we have shown that (x_n) is frequently in the interval $[u, \infty)$.

7. State and prove an analogue of the preceding exercise for lower limits.
   Suppose that (x_n) is a bounded sequence that that, for each n we define
   $$y_n = \inf\{x_m \mid m \geq n\}.$$
   Then the sequence (y_n) is decreasing and the limit of the sequence (y_n) is $\operatorname{lminf}_{n\to\infty} x_n$.
   This assertion can be proved by a mirror image of the proof that was used in Exercise 6 and it can also be obtained from the statement of Exercise 6, in view of Exercise 3b.

8. Given that
   $$z_n = x_n + y_n$$
   for every positive integer n, prove that
   $$\operatorname*{lmsup}_{n\to\infty} z_n \leq \operatorname*{lmsup}_{n\to\infty} x_n + \operatorname*{lmsup}_{n\to\infty} y_n.$$

Solution: ***To obtain a contradiction we assume that***
$$\operatorname*{lmsup}_{n\to\infty} z_n > \operatorname*{lmsup}_{n\to\infty} x_n + \operatorname*{lmsup}_{n\to\infty} y_n.$$
Now we choose a number $\varepsilon > 0$ ***such that***
$$\operatorname*{lmsup}_{n\to\infty} z_n > \operatorname*{lmsup}_{n\to\infty} x_n + \operatorname*{lmsup}_{n\to\infty} y_n + \varepsilon.$$
Since
$$\operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2} > \operatorname*{lmsup}_{n\to\infty} x_n,$$
there are at most finitely many integers n ***for which***
$$x_n > \operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2}$$
and we see, in the same way, that there are at most finitely many integers n ***for which***
$$y_n > \operatorname*{lmsup}_{n\to\infty} y_n + \frac{\varepsilon}{2}.$$
Thus, for all but at most finitely many integers n ***we have***
$$z_n = x_n + y_n \leq \operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2} + \operatorname*{lmsup}_{n\to\infty} y_n + \frac{\varepsilon}{2}$$
which is impossible since
$$\operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2} + \operatorname*{lmsup}_{n\to\infty} y_n + \frac{\varepsilon}{2} < \operatorname*{lmsup}_{n\to\infty} z_n.$$

Alt 7: Limits of Sequences in Metric Spaces

Some Exercises on Subsequences

Decide whether each of the following assertions is true or false. If it is true, prove it. If it is false, illustrate this fact by giving an example.

1. If a sequence (x_n) is eventually in a given set S then every subsequence of (x_n) is eventually in S.
   The assertion is true. Suppose that (x_n) is a sequence that is eventually in a given set S and that (x_{n_i}) is a subsequence of (x_n). Choose an integer N such that the condition $x_n \in S$ holds whenever $n \geq N$. Now choose an integer j such that the condition $n_i \geq N$ holds whenever $i \geq j$. We see that

$x_{n_i} \in S$ whenever $i \geq j$.

2. If a sequence (x_n) is frequently in a given set S then every subsequence of (x_n) is frequently in S.
   The assertion is false. If we define $x_n = (-1)^n$ for every positive integer n then, although (x_n) is frequently in the set $\{-1\}$, the subsequence (x_{2n}) fails to be frequently in the set $\{-1\}$.

3. If every subsequence of a sequence is frequently in a given set S then (x_n) is frequently in S.
   The statement is obviously true because every sequence is a subsequence of itself.

4. If every subsequence of a sequence (x_n) is frequently in a given set S then (x_n) is eventually in S.
   The assertion is true. We shall show that if (x_n) fails to be eventually in S then (x_n) must have a subsequence that is not frequently in S. Suppose that (x_n) is a sequence that fails to be eventually in a given set S. We know that (x_n) is frequently in the set $\mathbf{R} \setminus S$ and from the preceding theorem we deduce that (x_n) has a subsequence in the set $\mathbf{R} \setminus S$. Such a subsequence cannot be frequently in S.

5. If (x_n) is a sequence of real numbers and $S \subseteq \mathbf{R}$, and if (x_n) is not eventually in S then (x_n) has a subsequence that is eventually in $\mathbf{R} \setminus S$.
   Since a sequence that is not eventually in S must be frequently in $\mathbf{R} \setminus S$, the desired result follows at once from a theorem on subsequences.

Some Exercises on Limits and Partial Limits

1. Given that
$$x_n = 3 + \frac{1}{n}$$
   for each positive integer n, prove that 3 is a limit of (x_n).
   We need to prove that for every number $\varepsilon > 0$ the sequence (x_n) will eventually be in the interval $(3 - \varepsilon, 3 + \varepsilon)$. So we start: Suppose that $\varepsilon > 0$.
   Before we go any further we need to ask ourselves what it means to say that $x_n \in (3 - \varepsilon, 3 + \varepsilon)$. We observe that the inequality $3 - \varepsilon < x_n < 3 + \varepsilon$ is equivalent to the assertion that
$$3 - \varepsilon < 3 + \frac{1}{n} < 3 + \varepsilon$$
   which holds when $1/n < \varepsilon$. This tells how how to continue:
   Using the fact that the number $1/\varepsilon$ is not an upper bound of the set $\mathbf{Z}$ of integers we choose an integer N such that $N > 1/\varepsilon$, in other words,
$$\frac{1}{N} < \varepsilon.$$
   Then, whenever $n \geq N$ we have
$$3 - \varepsilon < 3 < 3 + \frac{1}{n} \leq 3 + \frac{1}{N} < 3 + \varepsilon$$
   and so we have shown that (x_n) is eventually in $(3 - \varepsilon, 3 + \varepsilon)$.

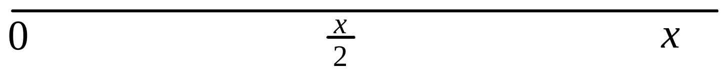
2. Given that
$$x_n = 3 + \frac{2}{n}$$
   for each positive integer n, prove that 3 is a limit of (x_n).
   We need to prove that for every number $\varepsilon > 0$ the sequence (x_n) will eventually be in the interval $(3 - \varepsilon, 3 + \varepsilon)$. So we start: Suppose that $\varepsilon > 0$. This time we make the observation that the inequality
$$3 - \varepsilon < 3 + \frac{2}{n} < 3 + \varepsilon$$
   holds when $2/n < \varepsilon$.
   Using the fact that the number $2/\varepsilon$ is not an upper bound of the set $\mathbf{Z}$ of integers we choose an integer N such that $N > 2/\varepsilon$, in other words,
$$\frac{2}{N} < \varepsilon.$$
   Then, whenever $n \geq N$ we have

$$3-\varepsilon<3<3+\frac{2}{n}\le 3+\frac{2}{N}<3+\varepsilon$$

and so we have shown that (x_n) is eventually in $(3-\varepsilon,3+\varepsilon)$.

3. Given that $x_n = 1/n$ for each positive integer n and that $x \ne 0$, prove that x is not a partial limit of (x_n).

   Solution: ***In the event that $x<0$, the interval $(-\infty,0)$ is a neighborhood of x and it is clear that (x_n) fails to be frequently in this neighborhood. Therefore no negative number can be a partial limit of (x_n). Suppose now that $x>0$. The interval $(x/2,\infty)$ is a neighborhood of x***

   and the condition $x_n \in (x/2,\infty)$ must fail to hold whenever $n>2/x$. Therefore the sequence (x_n) cannot be frequently in the interval $(x/2,\infty)$ and the number x cannot be a partial limit of (x_n).

4. Given that

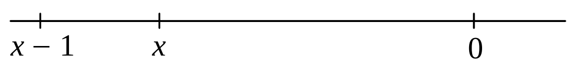
$$x_n=\begin{cases}(-1)^n n^3 & \text{if } \ n \text{ is a multiple of 3}\\ 0 & \text{if } \ n \text{ is one more than a multiple of 3}\\ 4 & \text{if } \ n \text{ is two more than a multiple of 3}\end{cases},$$

   Prove that the partial limits of (x_n) are $-\infty$, ∞, 0 and 4. Since (x_n) is unbounded both above and below, it follows from the discussion of infinite partial limits we saw earlier that both ∞ and $-\infty$ are partial limits of (x_n). Since the equation $x_n = 0$ holds for infinitely many values of n we know that (x_n) is frequently in every neighborhood of 0 and so 0 is a partial limit of (x_n). In the same way we can see that 4 is a partial limit of (x_n).

   Now we need to explain why any real number other than 0 and 4 must fail to be a partial limit of (x_n). Suppose that $x \in \mathbf{R}\setminus\{0,4\}$.

   In the event that $0<x<4$, the fact that x is not a partial limit of (x_n) follows from the fact that (x_n) is not frequently (or ever) in the interval $(0,4)$ which is a neighborhood of x.

   Now suppose that $x<0$.

   In order to show that x is not a partial limit of (x_n) we shall make the observation that (x_n) is not frequently in the interval $(x-1,0)$ which is a neighborhood of x. In fact, the inequality

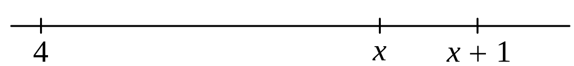
$$x-1<x_n<0$$

   can hold only when n is odd and

$$x-1<-n^3$$

   which is equivalent to saying that $n^3<1-x$. Since there are only finitely many such positive integers n we conclude that (x_n) is not frequently in the interval $(x-1,0)$

   Finally we must consider the case $x>4$.

   In this case we observe that there can be only finitely many positive integers n for which

$$4<n^3<x+1$$

   and so, once again, x can't be a partial limit of (x_n).

5. Give an example of a sequence of real numbers whose set of partial limits is the set $\{1\}\cup[4,5]$.

   Hint: ***For each positive integer n, if n can be written in the form***

$$n=2^m 3^k$$

   for some positive integers m and k and if

$$4\le\frac{m}{k}\le 5$$

then we define

$$x_n = \frac{m}{k}.$$

In all other cases we define $x_n = 1$***. Observe that the range of the sequence*** (x_n) ***is the set***

$$\{1\} \cup (\mathbf{Q} \cap [4,5])$$

and then show that the set of partial limits of (x_n) ***is*** $\{1\} \cup [4,5]$***.***
Since the equation $x_n = 1$ holds for infinitely many positive integers n the number 1 must be a partial limit of (x_n). To see that every number in the interval $[4,5]$ must be a partial limit of (x_n), suppose that $x \in [4,5]$ and suppose that $\varepsilon > 0$. Since the interval $(x - \delta, x + \delta)$ must contain infinitely many members of the set $\mathbf{Q} \cap [4,5]$ we know that the condition $x_n \in (x - \delta, x + \delta)$ must hold for infinitely many positive integers n and so x must be a partial limit of (x_n).
Finally we observe that if $x \in \mathbf{R} \setminus (\{1\} \cup [4,5])$ then the open set $\mathbf{R} \setminus (\{1\} \cup [4,5])$ is a neighborhood of x and that (x_n) fails to be frequently (or ever) in $\mathbf{R} \setminus (\{1\} \cup [4,5])$ and so x can't be a partial limit of (x_n).

6. Given that

$$x_n = \frac{3 + 2n}{5 + n}$$

for every positive integer n, prove that $x_n \to 2$ as $n \to \infty$.
We begin by observing that if n is a positive integer then

$$\left| \frac{3 + 2n}{5 + n} - 2 \right| = \frac{7}{5 + n}$$

Now suppose that $\varepsilon > 0$. The inequality

$$\left| \frac{3 + 2n}{5 + n} - 2 \right| < \varepsilon$$

says that

$$\frac{7}{5 + n} < \varepsilon$$

which holds when

$$\frac{5 + n}{7} > \frac{1}{\varepsilon}$$

in other words

$$n > \frac{7}{\varepsilon} - 5$$

With these inequalities in mind we choose an integer N such that

$$N > \frac{7}{\varepsilon} - 5$$

and we observe that whenever n is an integer and $n \geq N$ we have

$$\left| \frac{3 + 2n}{5 + n} - 2 \right| < \varepsilon.$$

7. Given that

$$x_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even} \\ \frac{1}{n^2+1} & \text{if } n \text{ is odd} \end{cases}$$

prove that $x_n \to 0$ as $n \to \infty$.
We observe that if n is a positive integer then

$$0 \leq x_n \leq \frac{1}{n}.$$

Now suppose that $\varepsilon > 0$. Choose an integer N such that $N > 1/\varepsilon$. We observe that whenever $n \geq N$,

$$|x_n - 0| = x_n < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

8. Suppose that (x_n) is a sequence of real numbers and that $x \in \mathbf{R}$. Prove that the following conditions are equivalent:

a. $x_n \to x$ as $n \to \infty$.

b. For every number $\varepsilon > 0$ the sequence (x_n) is eventually in the interval $(x - 5\varepsilon, x + 5\varepsilon)$.

To prove that condition a implies condition b we assume that $x_n \to x$ as $n \to \infty$. Suppose that $\varepsilon > 0$ and, using the fact that $x_n \to x$ as $n \to \infty$, choose N such that the condition

$$x_n \in (x - \varepsilon, x + \varepsilon)$$

will hold whenever $n \geq N$. Then, whenever $n \geq N$ we have

$$x_n \in (x - \varepsilon, x + \varepsilon) \subseteq (x - 5\varepsilon, x + 5\varepsilon).$$

Now to prove that condition b implies condition a we assume that condition b holds. To prove that condition a holds, suppose that $\varepsilon > 0$. Using the fact that $\varepsilon/5$ is a positive number, we choose an integer N such that the condition

$$x_n \subseteq \left(x - 5\left(\frac{\varepsilon}{5}\right), x + 5\left(\frac{\varepsilon}{5}\right)\right)$$

holds whenever $n \geq N$. Thus for every $n \geq N$ we have

$$x_n \subseteq \left(x - 5\left(\frac{\varepsilon}{5}\right), x + 5\left(\frac{\varepsilon}{5}\right)\right) = (x - \varepsilon, x + \varepsilon).$$

9. Prove that

$$\frac{n^2 + 3n + 1}{2n^2 + n + 4} \to \frac{1}{2}$$

as $n \to \infty$.

We begin by observing that if n is any positive integer then

$$\left| \frac{n^2 + 3n + 1}{2n^2 + n + 4} - \frac{1}{2} \right| = \left| \frac{5n - 2}{4n^2 + 2n + 8} \right| \leq \frac{5n}{4n^2} = \frac{5}{4n}.$$

Now suppose that $\varepsilon > 0$. The inequality

$$\left| \frac{n^2 + 3n + 1}{2n^2 + n + 4} - \frac{1}{2} \right| < \varepsilon$$

will hold when

$$\frac{5}{4n} < \varepsilon$$

which requires that

$$n > \frac{5}{4\varepsilon}$$

Choose an integer N such that $N > 5/(4\varepsilon)$ and observe that, whenever $n \geq N$ we have

$$\left| \frac{n^2 + 3n + 1}{2n^2 + n + 4} - \frac{1}{2} \right| \leq \frac{5}{4n} \leq \frac{5}{4N} < \varepsilon.$$

10. Given that (x_n) is a sequence in a metric space X and that $x \in X$, prove that $x_n \to x$ as $n \to \infty$ if and only if $d(x_n, x) \to 0$

The condition $x_n \to x$ as $n \to \infty$ says that for every $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ we have $d(x_n, x) < \varepsilon$.

The condition $d(x_n, x) \to 0$ as $n \to \infty$ says that for every $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ we have $|d(x_n, x) - 0| < \varepsilon$.

These two conditions clearly say the same thing. as $n \to \infty$.

11. Suppose that (x_n) is a sequence in a metric space X, that $x \in X$, and that (y_n) is a sequence of real numbers converging to 0. Suppose that the inequality

$$d(x, x_n) \leq y_n$$

holds for every n. Prove that $x_n \to x$ as $n \to \infty$.

Suppose that $\varepsilon > 0$. Using the fact that $y_n \to 0$ as $n \to \infty$, choose an integer N such that the inequality $y_n < \varepsilon$ will hold whenever $n \geq N$. We see at once that the inequality $d(x_n, x) < \varepsilon$ also holds whenever $n \geq N$.

12. Given that (x_n) and (y_n) are sequences of real numbers, that $x_n \geq y_n$ for each n and that $y_n \to \infty$ as $n \to \infty$, prove that $x_n \to \infty$ as $n \to \infty$..
Suppose that w is a real number. Using the fact that $y_n \to \infty$ as $n \to \infty$ we choose an integer N such that the inequality $y_n > w$ holds whenever $n \geq N$. Then whenever $n \geq N$ we have $x_n \geq y_n > w$.

13. Suppose that (x_n) and (y_n) are sequences in a metric space X and that x and y are points of X.

a. Prove that for every n we have
$$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y)$$
and deduce that
$$d(x,y) - d(x_n,y_n) \leq d(x,x_n) + d(y_n,y).$$
Then prove that for every n we have
$$d(x_n,y_n) - d(x,y) \leq d(x,x_n) + d(y_n,y)$$
and deduce finally that
$$|d(x_n,y_n) - d(x,y)| \leq d(x,x_n) + d(y_n,y).$$
Given any n we have
$$\begin{aligned} d(x,y) &\leq d(x,x_n) + d(x_n,y) \\ &\leq d(x,x_n) + d(x_n,y_n) + d(y_n,y) \end{aligned}$$
and the inequality
$$d(x,y) - d(x_n,y_n) \leq d(x,x_n) + d(y_n,y).$$
follows at once. In the same way we can show that
$$d(x_n,y_n) - d(x,y) \leq d(x_n,x) + d(y,y_n).$$
Therefore
$$|d(x_n,y_n) - d(x,y)| \leq d(x,x_n) + d(y_n,y).$$

b. Prove that if $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $d(x_n,y_n) \to d(x,y)$ as $n \to \infty$.
We assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Suppose that $\varepsilon > 0$. Choose an integer N_1 such that the inequality $d(x_n,x) < \frac{\varepsilon}{2}$ holds whenever $n \geq N_1$. Choose an integer N_2 such that the inequality $d(y_n,y) < \frac{\varepsilon}{2}$ holds whenever $n \geq N_2$. We define N to be the larger of the integers N_1 and N_2. Then whenever $n \geq N$ we have
$$|d(x_n,y_n) - d(x,y)| \leq d(x,x_n) + d(y_n,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

14. Prove that if $(\mathbf{x}_n)$ is a sequence in $\mathbf{R}^k$ and $\mathbf{x} \in \mathbf{R}^k$ and if $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$ then $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$ as $n \to \infty$.
Since $\|\mathbf{x}_n\| = \|\mathbf{x}_n - \mathbf{O}\|$ and $\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{O}\|$ the desired result follows at once from Exercise 13.

15. Prove that if a sequence (x_n) in a metric space X converges to a point $x \in X$ then every subsequence of (x_n) also converges to x.
Suppose that (x_n) is a sequence converging to a point x in a metric space X and that (x_{n_i}) is a subsequence of (x_n). Suppose that $\varepsilon > 0$ and, using the fact that $x_n \to x$ as $n \to \infty$, choose an integer N such that the inequality $d(x_n,x) < \varepsilon$ holds whenever $n \geq N$. Using the fact that (n_i) is a strictly increasing sequence of positive integers, choose an integer k such that the inequality $n_i \geq N$ will hold whenever $i \geq k$. Then, whenever $i \geq k$ we have $d(x_{n_i},x) < \varepsilon$.

16. a. Prove that if (x_n) is a sequence in a metric space X and if a point x is a limit point of the range of (x_n) then x must be a partial limit of the sequence (x_n).

Solution: ***Suppose that*** (x_n) ***is a sequence in a metric space*** X ***and that a point*** x ***is a limit point of the range of*** (x_n)***. Suppose that*** $\varepsilon > 0$***. Since there are infinitely many members of the range of*** (x_n) ***that lie in the ball*** $B(x,\varepsilon)$***, there must be infinitely many positive integers*** n ***for which*** $x_n \in B(x,\varepsilon)$***.***

b. Give an example of a sequence (x_n) in a metric space X and a partial limit x of (x_n) such that x fails to be a limit point of the range of the sequence (x_n).

If a sequence is constant then it converges but its range, being finite, has no limit point.

c. Prove that if a sequence (x_n) in a metric space X is one-one and if a point x is a partial limit of the sequence (x_n) then x must be a limit point of the range of (x_n).
Suppose that (x_n) is a sequence in a metric space X and that x is a partial limit of (x_n). Suppose that $\varepsilon > 0$. Since there are infinitely many integers n for which $x_n \in B(x,\varepsilon)$ and since the sequence (x_n) is one-one we see that there is more than one member of the range of (x_n) in the ball $B(x,\varepsilon)$. Therefore x is a limit point of the range of (x_n).

17. For each positive integer n, if n can be written in the form

$$n = 2^m 3^k$$

where m and k are postive integers and $m \leq k$ then we define

$$x_n = \frac{m}{k}.$$

Otherwise we define $x_n = 0$. Prove that the set of partial limits of the sequence (x_n) is $[0,1]$.
Since the range of (x_n) is the set of all rational numbers in the interval $[0,1]$, every neighborhood of a number x in the interval $[0,1]$ must contain infinitely many members of the range of (x_n) and must, therefore, contain the number x_n for infinitely many integers n. Thus every member of the interval $[0,1]$ is a partial limit of (x_n).
If x is any number in the set $\mathbf{R} \setminus [0,1]$ then, since $\mathbf{R} \setminus [0,1]$ is a neighborhood of x and (x_n) is not frequently (or, indeed, ever) in the set $\mathbf{R} \setminus [0,1]$, the number x must fail to be a partial limit of (x_n).

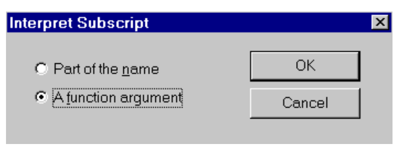
Exercises on the Elementary Properties of Limits

1. The purpose of this exercise is to use *Scientific Notebook* to gain an intuitive feel for the limit behaviour of a rather difficult sequence.

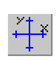
a. Point at the equation

$$x_n = \frac{n^n\sqrt{n}}{(n!)e^n}$$

and then click on the button  to supply the definition to *Scientific Notebook*. When you see the screen



make the selection "A function argument" so that *Scientific Notebook* knows that you are defining a sequence.

b. Point at the expression x_n and click on the button  to display the sequence graphically.
Revise your graph and set the domain interval as $[1,500]$. Double click into your graph to make the buttons

appear in the top right corner and click on the bottom button to select it. Trace your graph with the mouse and show graphically that

$$\lim_{n\to\infty}\frac{n^n\sqrt{n}}{(n!)e^n}\approx 0.3989.$$

c. Point at the expression

$$\lim_{n\to\infty}\frac{n^n\sqrt{n}}{(n!)e^n}$$

and ask *Scientific Notebook* to evaluate it numerically. Compare the result with the limit value you found graphically.

d. Point at the expression and ask *Scientific Notebook* to evaluate it exactly to show that the limit is $1/\sqrt{2\pi}$.

2. Prove that $5^n/n! \to 0$ as $n \to \infty$.

Solution: ***Whenever*** $n \geq 5$ ***we have***

$$0 \leq \frac{5^n}{n!} = \left(\frac{5}{1}\right)\left(\frac{5}{2}\right)\left(\frac{5}{3}\right)\left(\frac{5}{4}\right)\left(\frac{5}{5}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{5}{n}\right) \leq \left(\frac{5^4}{4!}\right)\left(\frac{5}{n}\right)$$

Now, to prove that $5^n/n! \to 0$ ***as*** $n \to \infty$***, suppose that*** $\varepsilon > 0$***.***
Choose an integer N ***such that***

$$N > \frac{5^5}{4!}\varepsilon.$$

Then whenever $n \geq N$ ***we have***

$$0 \leq \frac{5^n}{n!} \leq \left(\frac{5^4}{4!}\right)\left(\frac{5}{n}\right) \leq \left(\frac{5^4}{4!}\right)\left(\frac{5}{N}\right) < \varepsilon.$$

3. Prove that $n!/n^n \to 0$ as $n \to \infty$.

Hint: ***Make use of the fact that, for each*** n ***we have***

$$0 \leq \frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right) \leq \frac{1}{n}.$$

4. Given that (x_n) is a sequence of real numbers, that $x > 0$ and that $x_n \to x$ as $n \to \infty$, prove that there exists an integer N such that the inequality $x_n > 0$ holds for all integers $n \geq N$.

Hint: ***The interval*** $(0,\infty)$ ***is a neighborhood of the number*** x***.***

5. Given that $x_n \geq 0$ for every positive integer n and that x is a partial limit of the sequence (x_n), prove that $x \geq 0$.
We need to show that no negative number can be a partial limit of (x_n). Suppose that $y < 0$. Since the interval $(-\infty, 0)$ is a neighborhood of y and (x_n) is not frequently (or, indeed, ever) in the interval $(-\infty, 0)$, we deduce that y is not a partial limit of (x_n).

6. Suppose that (x_n) is a sequence of real numbers and that $x \in \mathbf{R}$. Prove that the following conditions are equivalent:

   a. $x_n \to x$ as $n \to \infty$.

   b. $|x_n - x| \to 0$ as $n \to \infty$.
   Condition a says that for every $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ we have $|x_n - x| < \varepsilon$.
   Condition b says that for every $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ we have $||x_n - x| - 0| < \varepsilon$.
   These two conditions clearly say the same thing.

7. Suppose that (x_n) is a sequence of real numbers, that $x \in \mathbf{R}$ and that $x_n \to x$ as $n \to \infty$. Prove that $|x_n| \to |x|$ as $n \to \infty$.

Hint: ***Make use of the fact that for each n we have***

$$0 \leq ||x_n| - |x|| \leq |x_n - x|.$$

8. Suppose that (x_n) is a sequence of real numbers, that $x \in \mathbf{R}$ and that $x_n \to x$ as $n \to \infty$. Suppose that p is an integer and that for every positive integer n we have

$$y_n = x_{n+p}.$$

Prove that $y_n \to x$ as $n \to \infty$.
Suppose that $\varepsilon > 0$. Using the fact that $x_n \to x$ as $n \to \infty$ we choose an integer N such that whenever $n \geq N$ we have $|x_n - x| < \varepsilon$. We observe that the inequality $|y_n - x| < \varepsilon$ will hold whenever $n + p \geq N$ which is the same as saying that $n \geq N - p$.

9. Given that $a_n \leq b_n$ for every positive integer n and given that $a_n \to \infty$, prove that $b_n \to \infty$.
Suppose that w is a real number. Using the fact that $a_n \to \infty$ as $n \to \infty$ we choose an integer N such that the inequality $a_n > w$ holds whenever $n \geq N$. Then whenever $n \geq N$ we have $b_n \geq a_n > w$.

10. Suppose that (a_n) and (b_n) are sequences of real numbers and that $|a_n - b_n| \leq 1$ for every positive integer n and that ∞ is a partial limit of the sequence (a_n). Prove that ∞ is a partial limit of (b_n).
We know that $b_n \geq a_n - 1$ for each n. Suppose that w is a real number. Since ∞ is a partial limit of (a_n) there are infinitely many integers n for which $a_n > w + 1$ and for all such integers we have $b_n > w$.

11. Two sequences (a_n) and (b_n) in a metric space X are said to be *eventually close* if for every number $\varepsilon > 0$ there exists an integer N such that the inequality $d(a_n, b_n) < \varepsilon$ holds for all integers $n \geq N$.

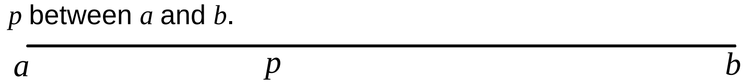
   a. Prove that if two sequences (a_n) and (b_n) in a metric space X are eventually close and if a point x is the limit of the sequence (a_n) then x is also the limit of the sequence (b_n).
   Suppose that (a_n) and (b_n) are eventually close and that $a_n \to x$ as $n \to \infty$. To show that $b_n \to x$ as $n \to \infty$, suppose that $\varepsilon > 0$.
   Choose N_1 such that the inequality $d(a_n, x) < \frac{\varepsilon}{2}$ holds whenever $n \geq N_1$. Choose N_2 such that the inequality $d(a_n, b_n) < \frac{\varepsilon}{2}$ holds whenever $n \geq N_2$. We define N to be the larger of N_1 and N_2 and we see that whenever $n \geq N$ we have

$$d(b_n, x) \leq d(b_n, a_n) + d(a_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   b. Prove that if two sequences (a_n) and (b_n) in a metric space X are eventually close and if a point x is a partial limit of the sequence (a_n) then x is also a partial limit of the sequence (b_n).
   Suppose that (a_n) and (b_n) are eventually close and that x is a partial limit of (a_n). Suppose that $\varepsilon > 0$. Choose an integer N such that the inequality $d(a_n, b_n) < \frac{\varepsilon}{2}$ holds whenever $n \geq N$. Since there are infinitely many integers n for which $d(a_n, x) < \frac{\varepsilon}{2}$ there must be infinitely many integers $n \geq N$ for which $d(a_n, x) < \frac{\varepsilon}{2}$. For every one of these integers n we have

$$d(b_n, x) \leq d(b_n, a_n) + d(a_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

12. Suppose that (a_n) and (b_n) are sequences of real numbers, that $a_n \to a$ and that $b_n \to b$ as $n \to \infty$, and that $a < b$. Prove that there exists an integer N such that the inequality $a_n < b_n$ holds for all integers $n \geq N$.
Choose a number p between a and b.



Using the fact that $a_n \to a$ as $n \to \infty$ and the fact that the interval $(-\infty, p)$ is a neighborhood of a, choose an integer N_1 such that $a_n < p$ whenever $n \geq N_1$. Similarly, choose an integer N_2 such that $b_n > p$ whenever $n \geq N_2$. We define N to be the larger of N_1 and N_2 and we observe that $a_n < p < b_n$ whenever $n \geq N$.

13. Give an example of two sequences (a_n) and (b_n) of real numbers satisfying $a_n > b_n$ for every positive integer n even though the sequence (a_n) has a partial limit a that is less than a partial limit b of the sequence (b_n).
We define $a_n = 2 + (-1)^n$ and $b_n = 3 + (-1)^n$ for every positive integer n. We see that $a_n < b_n$ for

each n and that 3 is a partial limit of (a_n) and that 2 is a partial limit of (b_n).

Some Exercises on the Algebraic Rules for Limits

1. Suppose that $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ are sequences in $\mathbf{R}^k$, that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$ and that $\mathbf{y}$ is a partial limit of $(\mathbf{y}_n)$. Prove that $\mathbf{x} + \mathbf{y}$ is a partial limit of the sequence $(\mathbf{x}_n + \mathbf{y}_n)$.
   Suppose that $\varepsilon > 0$. Using the fact that $(\mathbf{x}_n)$ converges to the point $\mathbf{x}$ we choose an integer N such that the inequality
   $$\|\mathbf{x}_n - \mathbf{x}\| < \frac{\varepsilon}{2}$$
   holds whenever $n \geq N$. Since $\mathbf{y}$ is a partial limit of the sequence $(\mathbf{y}_n)$ and since there are only finitely many positive integers $n < N$ there must be infinitely many integers $n \geq N$ for which the inequality
   $$\|\mathbf{y}_n - \mathbf{y}\| < \frac{\varepsilon}{2}$$
   holds. For each of these infinitely many integers we have
   $$\begin{aligned}\|\mathbf{x}_n + \mathbf{y}_n - (\mathbf{x} + \mathbf{y})\| &= \|\mathbf{x}_n - \mathbf{x} + \mathbf{y}_n - \mathbf{y}\| \\ &\leq \|\mathbf{x}_n - \mathbf{x}\| + \|\mathbf{y}_n - \mathbf{y}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

2. Is it true that if $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ are sequences in $\mathbf{R}^k$ and $\mathbf{x}$ is a partial limit of $(\mathbf{x}_n)$ and that $\mathbf{y}$ is a partial limit of $(\mathbf{y}_n)$ then $\mathbf{x} + \mathbf{y}$ is a partial limit of the sequence $(\mathbf{x}_n + \mathbf{y}_n)$?
   The answer is no; even in the metric space $\mathbf{R}$.
   Define $x_n = y_n = (-1)^n$ for each n. We observe that the numbers -1 and 1 are partial limits of (x_n) and (y_n) respectively but that $-1 + 1$ fails to be a partial limit of $(x_n + y_n)$.

3. State and prove some analogues of Exercise 1 for differences and inner products in $\mathbf{R}^k$ and for products and quotients in $\mathbf{R}$.
   Suppose that (x_n) and (y_n) are sequences of numbers, that (x_n) converges to a number x and that a real number y is a partial limit of (y_n). We shall prove that the number xy must be a partial limit of the sequence $(x_n y_n)$.
   Using the fact that (x_n) is convergent, and therefore bounded, we choose a number p such that $|x_n| < p$ for every n. For each n we observe that
   $$\begin{aligned}|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &\leq p|y_n - y| + |y||x_n - x|\end{aligned}$$
   Now, to show that xy is a partial limit of the sequence $(x_n y_n)$, suppose that $\varepsilon > 0$. Using the fact that $x_n \to x$ as $n \to \infty$ we choose an integer N such that the inequality
   $$|x_n - x| < \frac{\varepsilon}{2|y| + 1}$$
   holds whenever $n \geq N$. Since there are only finitely many positive integers $n < N$, and y is a partial limit of the sequence (y_n), there must be infinitely many integers $n \geq N$ for which the inequality
   $$|y_n - y| < \frac{\varepsilon}{2p}$$
   holds. For each of these infinitely many integers n we have
   $$\begin{aligned}|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &\leq p|y_n - y| + |y||x_n - x| \\ &< p\left(\frac{\varepsilon}{2p}\right) + |y|\left(\frac{\varepsilon}{2|y| + 1}\right) < \varepsilon.\end{aligned}$$

4. Suppose that $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ are sequences in $\mathbf{R}^3$ that converge respectively to points $\mathbf{x}$ and $\mathbf{y}$. Prove that

$$\lim_{n\to\infty}(\mathbf{x}_n \times \mathbf{y}_n) = \mathbf{x} \times \mathbf{y}.$$

We write each point $\mathbf{x}_n$ in the form

$$\mathbf{x}_n = (a_n, b_n, c_n)$$

and each point $\mathbf{y}_n$ in the form

$$\mathbf{y}_n = (u_n, v_n, w_n)$$

and

$$\mathbf{x} = (a, b, c)$$

and

$$\mathbf{y} = (u, v, w).$$

The given information tells us that as $n \to \infty$ we have $a_n \to a$, $b_n \to b$, $c_n \to c$, $u_n \to u$, $v_n \to v$ and $w_n \to w$. Therefore

$$\begin{aligned}\lim_{n\to\infty}\mathbf{x}_n \times \mathbf{y}_n &= \lim_{n\to\infty}(b_n w_n - c_n v_n, c_n u_n - a_n w_n, a_n v_n - b_n u_n)\\ &= (bw - cv, cu - aw, av - bu) = (a,b,c) \times (u,v,w) = \mathbf{x} \times \mathbf{y}.\end{aligned}$$

5. Give an example of two divergent sequences (x_n) and (y_n) in $\mathbf{R}$ such that the sequence $(x_n + y_n)$ is convergent.
We define $x_n = (-1)^n$ and $y_n = (-1)^{n-1}$ for each n. Note that the sequence $(x_n + y_n)$ is the sequence with constant value 0.

6. Give an example of two sequences (x_n) and (y_n) in $\mathbf{R}$ such that $x_n \to 0$ and $y_n \to \infty$ and

   a. $x_n y_n \to 0$
   We define $x_n = n$ and $y_n = 1/n^2$ for each n.

   b. $x_n y_n \to 6$
   We define $x_n = n$ and $y_n = 6/n$ for each n.

   c. $x_n y_n \to \infty$
   We define $x_n = n^2$ and $y_n = 1/n$ for each n.

   d. The sequence $(x_n y_n)$ is bounded but has no limit.
   We define $x_n = n$ and $y_n = (-1)^n/n$ for each n.

7. Given two sequences (x_n) and (y_n) of real numbers such that both of the sequences (x_n) and $(x_n + y_n)$ are convergent, is it true that the sequence (y_n) must be convergent?
Yes. Since

$$y_n = (x_n + y_n) - x_n$$

for each n it follows at once that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}(x_n + y_n) - \lim_{n\to\infty} x_n$$

Perhaps this exercise should have been given in $\mathbf{R}^k$.

8. Given that (x_n) is a sequence of real numbers and that $x_n \to 0$, prove that

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \to 0.$$

Solution: ***Suppose that*** $\varepsilon > 0$***. Using the fact that*** $x_n \to 0$ ***as*** $n \to \infty$***, choose an integer*** N_1 ***such that the inequality*** $|x_n| < \varepsilon/2$ ***holds whenever*** $n \ge N_1$***. Whenever*** $n \ge N_1$ ***we see that***

$$\begin{aligned}\left|\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right| &= \left|\frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{n} + \frac{x_{N_1} + x_{N_1+1} + \cdots + x_n}{n}\right|\\ &\le \left|\frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{n}\right| + \left|\frac{x_{N_1} + x_{N_1+1} + \cdots + x_n}{n}\right|.\end{aligned}$$

Now we choose an integer N_2 ***such that***

$$N_2 \geq \frac{2|x_1 + x_2 + x_3 + \cdots + x_{N_1}|}{\varepsilon}$$

and we define N ***to be the larger of the two numbers*** N_1 ***and*** N_2***. Then whenever*** $n \geq N$ ***we have***

$$\begin{aligned} \left| \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} - 0 \right| &\leq \left| \frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{n} \right| + \left| \frac{x_{N_1} + x_{N_1+1} + \cdots + x_n}{n} \right| \\ &\leq \left| \frac{x_1 + x_2 + x_3 + \cdots + x_{N_1}}{N_2} \right| + \frac{|x_{N_1+1}| + |x_{N_1+2}| + \cdots + |x_n|}{n} \\ &< \frac{\varepsilon}{2} + \left(\frac{n - N_1}{n} \right) \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

9. Given that (x_n) is a sequence of real numbers, that x is a real number and that $x_n \to x,$ prove that

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \to x.$$

Solution: ***Since*** $x_n - x \to 0$ ***as*** $n \to \infty$ ***we have***

$$\frac{(x_1 - x) + (x_2 - x) + (x_3 - x) + \cdots + (x_n - x)}{n} \to 0$$

as $n \to \infty$***. Therefore***

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} - x = \frac{(x_1 - x) + (x_2 - x) + (x_3 - x) + \cdots + (x_n - x)}{n} \to 0$$

as $n \to \infty$***.***

10. Given that (x_n) and (y_n) are sequences of real numbers and that $x_n - y_n \to 0,$ prove that (x_n) and (y_n) have the same set of partial limits.
Since $x_n = (x_n - y_n) + y_n$ and $y_n = x_n - (x_n - y_n)$ for each n the present result follows at once from Exercise 1.

11. Suppose that (x_n) and (y_n) are sequences of real numbers, that $x_n - y_n \to 0$ and that the number 0 fails to be a partial limit of at least one of the sequences (x_n) and (y_n). Prove that

$$\frac{x_n}{y_n} \to 1$$

as $n \to \infty$.

Solution: ***From Exercise 10 we know that*** (x_n) ***and*** (y_n) ***have the same sets of partial limits and we know, therefore, that*** 0 ***is not a partial limit of either of these two sequences.***
Using the fact that 0 ***is not a partial limit of*** (y_n)***, choose an integer*** N_1 ***and a number*** $\delta > 0$ ***such that the inequality***

$$|y_n| \geq \delta$$

holds whenever $n \geq N_1$***. For every*** $n \geq N_1$ ***we see that***

$$\left| \frac{x_n}{y_n} - 1 \right| = \left| \frac{x_n - y_n}{y_n} \right| \leq \frac{|x_n - y_n|}{\delta}$$

and so the fact that

$$\frac{x_n}{y_n} \to 1$$

as $n \to \infty$ ***follows from the sandwich theorem.***

12. Give an example to show that the requirement in Exercise 11 that 0 not be a partial limit of at least one of the two sequences is really needed.
We define $x_n = 2/n$ and $y_n = 1/n$ for each n and observe that, even though $x_n - y_n \to 0$ as $n \to \infty$, $x_n/y_n \to 2$ as $n \to \infty$.

13. Suppose that (x_n) and (y_n) are sequences of real numbers, that $x_n/y_n \to 1$ and that at least one of the sequences (x_n) and (y_n) is bounded. Prove that $x_n - y_n \to 0.$ Give an example to show that the conclusion $x_n - y_n \to 0$ can fail if both (x_n) and (y_n) are unbounded.

Hint: ***Make the observation that both of the sequences*** (x_n) ***and*** (y_n) ***must be bounded. Explain carefully how you arrive at this conclusion. Now make use of the fact that***

$$|x_n - y_n| = |y_n| \left| \frac{x_n}{y_n} - 1 \right|$$

for every n.

14. Suppose that (x_n) and (y_n) are sequences of real numbers, that $y_n \to 1$ and that for each n we have $z_n = x_n y_n$. Prove that the sequences (x_n) and (z_n) have the same set of partial limits.
It follows from the part of Exercise 3 for which a solution is provided above that if x is an partial limit of (x_n) then the number $x(1) = x$ is a partial limit of the sequence (z_n). At the same time, since $x_n = z_n / y_n$ for all sufficiently large n we know that if x is any partial limit of (z_n) then the number $x/1 = x$ is a partial limit of the sequence (x_n).

15. Given that $(\mathbf{x}_n)$ is a sequence in $\mathbf{R}^k$ and that for every point $\mathbf{x}$ in $\mathbf{R}^k$ we have $\mathbf{x}_n \cdot \mathbf{x} \to 0$ as $n \to \infty$, prove that the sequence $(\mathbf{x}_n)$ converges to the point $\mathbf{O}$.
Taking $\mathbf{x} = (1, 0, \cdots, 0)$ we see that the sequence of first coordinates of the points $\mathbf{x}_n$ converges to zero. We can repeat this argument for each of the other coordinates.

Exercises on Sequences and the Topology

1. Prove that a set S of real numbers is unbounded below if and only if there exists a sequence (x_n) in S such that $x_n \to -\infty$.

Solution: ***We want to show that, for a given set*** S ***of real numbers, the following conditions are equivalent:***

a. ***The set*** S ***is unbounded below.***

b. ***There exists a sequence*** (x_n) ***in the set*** S ***such that*** $x_n \to -\infty$ ***as*** $n \to \infty$.

To show that condition b implies condition a, assume that condition b holds and choose a sequence (x_n) ***in*** S ***such that*** $x_n \to -\infty$ ***as*** $n \to \infty$***. If*** w ***is any real number then it follows at once from the fact that the sequence*** (x_n) ***is eventually in the interval*** $(-\infty, w)$ ***that*** w ***is not a lower bound of*** S***. Therefore*** S ***is not bounded above.***

Now to show that condition a implies condition b, assume that condition a holds. For each positive integer n ***we use the fact that the number*** $-n$ ***is not a lower bound of*** S ***to choose a number that we shall call*** x_n ***such that*** $x_n < -n$***. The sequence*** (x_n) ***that we have made in this way is in the set*** S ***and it is clear that*** $x_n \to -\infty$.

2. Suppose that S is a nonempty set of real numbers and that α is an upper bound of S. Prove that the following conditions are equivalent:

a. We have $\alpha = \sup S$.

b. There exists a sequence (x_n) in S such that $x_n \to \alpha$ as $n \to \infty$.
In Exercise 7 of the exercises on closure we saw that an upper bound α of set S is close to S if and only if α is equal to $\sup S$. We also saw in a recent theorem that a number α is close to a set S if and only if there exists a sequence in S that converges to α. The present exercise follows at once from these two facts. Of course, we can write this exercise out directly if we wish.

4. Given a sequence (x_n) that is frequently in a subset S of a metric space X, and given a partial limit x of the sequence (x_n), is it necessarily true that $x \in \overline{S}$?
No, this statement need not be true. If we define $x_n = (-1)^n$ for each n then, although (x_n) is frequently in the set $\{-1\}$ it has the partial limit 1 that is not close to $\{-1\}$.

4. Prove that a subset U of a metric space X is open if and only if every sequence in X that converges to a

member of U must be eventually in U.
We know from a recent theorem that a set H is closed if and only if no sequence that is frequently in H can have a limit that doesn't belong to H. Therefore if U is a set of real numbers then the set $\mathbf{R} \setminus U$ is closed if and only if no sequence with a limit in U can fail to be eventually in U.
Of course the exercise can also be done directly.

5. Given that S is a subset of a metric space X and that $x \in X$, prove that the following conditions are equivalent:

   a. The point x is a limit point of the set S.

   b. There exists a sequence (x_n) in the set $S \setminus \{x\}$ such that $x_n \to x$ as $n \to \infty$.

Solution: ***The assertion in this exercise follows at once from the corresponding theorem about limits of sequences and closure of a set, and from the fact that*** x ***is a limit point of*** S ***if and only if*** $x \in \overline{S \setminus \{x\}}$.

6. Prove that if (x_n) is a sequence in a metric space X then the set of all partial limits of (x_n) is closed.
Suppose that (x_n) is a sequence in a metric space X and write the set of partial limits of (x_n) as H. In order to show that H is closed we shall show that $X \setminus H$ is open. Suppose that $x \in X \setminus H$. In other words, suppose that x is not a partial limit of (x_n). Choose a number $\varepsilon > 0$ such that the condition

$$x_n \in B(x, \varepsilon)$$

holds for at most finitely many integers n. Given any number $y \in B(x, \varepsilon)$, it follows from the fact that $B(x, \varepsilon)$ is a neighborhood of y and the fact that x_n belongs to this neighborhood of y for at most finitely many integers n that y is not a partial limit of (x_n). In other words,

$$B(x, \varepsilon) \subseteq \mathbf{R} \setminus H$$

and we have shown, as promised, that the set $X \setminus H$ is open.

7. *(This exercise is more advanced than the others.)* Prove that if X is a separable metric space then every closed subset of X is the set of partial limits of some sequence.

1. **Solution:** ***Suppose that*** S ***is a closed subset of a separable metric space*** X***. Using the fact that the subspace*** S ***of*** X ***must also be separable, choose a countable dense subset of*** S***. We express this countable set in the form*** $\{x_n \mid n = 1, 2, \cdots\}$***. Now we look at the sequence***

$$(y_n) = (x_1, x_1, x_2, x_1, x_2, x_3, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_5, \cdots).$$

A point of X ***lies in the closure of the set*** $\{x_n \mid n = 1, 2, \cdots\}$ ***if and only if it is a partial limit of the sequence*** (y_n).

8. Suppose that A and B are nonempty sets of real numbers and that for every number $x \in A$ and every number $y \in B$ we have $x < y$. Prove that the following conditions are equivalent:

   a. We have $\sup A = \inf B$.

   b. There exists a sequence (x_n) in the set A and a sequence (y_n) in the set B such that $y_n - x_n \to 0$ as $n \to \infty$.

Solution: ***From the information given about*** A ***and*** B ***we know that*** $\sup A \leq \inf B$***. In the event that*** $\sup A < \inf B$ ***we know that for every sequence*** (x_n) ***in*** A ***and every sequence*** (y_n) ***in*** B ***we have***

$$y_n - x_n \to \inf B - \sup A > 0$$

for every n ***and so we can't have*** $y_n - x_n \to 0$ ***as*** $n \to \infty$***. Therefore if condition a is false then so is condition b.***

Suppose now that condition a is true; in other words, that $\sup A = \inf B$***. For every positive integer*** n***, we use the fact that***

$$\sup A - \frac{1}{n}$$

is not an upper bound of A ***to choose a member*** x_n ***of*** A ***such that***

$$\sup A - \frac{1}{n} < x_n.$$

For every positive integer n we use the fact that

$$\inf B + \frac{1}{n}$$

is not a lower bound of B to choose a member y_n of B such that

$$y_n < \inf B + \frac{1}{n}.$$

Thus for each n we have

$$\sup A - \frac{1}{n} < x_n \le y_n < \inf B + \frac{1}{n}$$

and therefore

$$0 \le y_n - x_n < \frac{2}{n}$$

from which it follows that $y_n - x_n \to 0$ as $n \to \infty$.

9. Suppose that S is a nonempty bounded set of real numbers. Prove that there exist two sequences (x_n) and (y_n) in the set S such that

$$y_n - x_n \to \sup S - \inf S$$

as $n \to \infty$.

We have already seen that there exists a sequence (x_n) in S such that $x_n \to \sup S$ as $n \to \infty$. In the same way we can see that there exists a sequence (y_n) in S such that $y_n \to \inf S$ as $n \to \infty$. We now have

$$y_n - x_n \to \sup S - \inf S$$

as $n \to \infty$.

Exercises on Monotone Sequences

1. Given that $c > 1$, use the following method to prove that $c^n \to \infty$:

   a. Write $\delta = c - 1$, so that $c = 1 + \delta$, and then use mathematical induction to prove that, if n is any positive integer then $c^n \ge 1 + n\delta$.
   For each positive integer n we take p_n to be the assertion that $c^n \ge 1 + n\delta$. Since $c = 1 + \delta$ we know that the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n is true. We observe that

$$\begin{aligned} c^{n+1} &= cc^n \ge (1+\delta)(1+n\delta) \\ &= 1 + (n+1)\delta + n\delta^2 > 1 + (n+1)\delta \end{aligned}$$

   and so the assertion p_{n+1} is also true. We deduce from mathematical induction that the inequality $c^n \ge 1 + n\delta$ is true for every positive integer n.

   b. Explain why $1 + n\delta \to \infty$ and then use this exercise in this subsection to show that $c^n \to \infty$.
   To prove that $1 + n\delta \to \infty$ as $n \to \infty$, suppose that w is a real number. The inequality

$$1 + n\delta > w$$

   will hold when

$$n > \frac{w-1}{\delta}.$$

   We choose a positive integer N such that

$$N > \frac{w-1}{\delta}$$

   and we observe that $1 + n\delta > w$ whenever $n \ge N$.

2. Given that $c > 1$, use the following method to prove that $c^n \to \infty$:

   a. Explain why the sequence (c^n) is increasing and deduce that it has a limit.

The fact that the sequence (c^n) is increasing follows at once from the inequality

$$c^{n+1} = cc^n > 1c^n.$$

It now follows from the monotone sequences theorem that the sequence (c^n) has a limit.

b. Call the limit x and show that if x is finite then the equation

$$c^{n+1} = cc^n$$

leads to the equation $x = cx$, which implies that $x = 0$. But x cannot be equal to zero? Why not?
From the equation

$$c^{n+1} = cc^n$$

we obtain

$$\lim_{n\to\infty} c^{n+1} = \lim_{n\to\infty}(cc^n)$$

which gives us $x = cx$. Since $c^n > 1$ for every n we know that $x \geq 1$ and therefore $x \neq 0$.

3. Suppose that $|c| < 1$ and that, for every positive integer n,

$$x_n = \sum_{i=1}^{n} c^{i-1}.$$

Explain why

$$x_n \to \frac{1}{1-c}.$$

From the identity

$$(1-c)\sum_{i=1}^{n} c^{i-1} = 1 - c^n$$

we deduce that

$$\sum_{i=1}^{n} c^{i-1} = \frac{1-c^n}{1-c}$$

Now since $1/|c| > 1$ we know that $1/|c|^n \to \infty$ as $n \to \infty$ and we conclude that $|c|^n \to 0$ as $n \to \infty$.
Therefore $c^n \to 0$ as $n \to \infty$ and we conclude that

$$\lim_{n\to\infty} \frac{1-c^n}{1-c} = \frac{1}{1-c}.$$

4. Suppose that (x_n) is a given sequence of real numbers, that $x_1 = 0$ and that the equation

$$8x_{n+1}^3 = 6x_n + 1$$

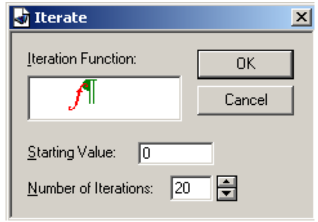
holds for every positive integer n.

a. Use *Scientific Notebook* to work out the first twenty terms in the sequence (x_n).

To use *Scientific Notebook* for this purpose, place your cursor in the equation

$$f(x) = \sqrt[3]{\frac{6x+1}{8}}$$

and click on the button in your computing toolbar to supply the definition of the function f to *Scientific Notebook*. Then open the *Compute* menu, scroll down to *Calculus* and move across to Iterate. Fill the dialogue box giving the name of the function as f, the number of iterations as 20 and the starting value as 0.



a. Prove that $x_n < 1$ for every positive integer n.
We use mathematical induction. For each positive integer n we take p_n to be the assertion that $x_n < 1$. Since $x_1 = 0$, the assertion p_1 is true. Now suppose that n is any positive integer for which the statement p_n is true. Then

$$8x_{n+1}^3 = 6x_n + 1 < 6(1) + 1$$

and so

$$x_{n+1} < \sqrt[3]{\frac{7}{8}} < 1$$

from which it follows that the assertion p_{n+1} is also true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

b. Prove that the sequence (x_n) is strictly increasing.
Once again we use mathematical induction. For each positive integer n we take p_n to be the assertion that $x_n < x_{n+1}$. Since $x_1 = 0 < \frac{1}{2} = x_2$, the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n is true. Since

$$x_{n+2} = \sqrt[3]{\frac{6x_{n+1} + 1}{8}} > \sqrt[3]{\frac{6x_n + 1}{8}} = x_{n+1}$$

we see that the assertion p_{n+1} must also be true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

c. Deduce that the sequence (x_n) is convergent and discuss its limit. Assuming an unofficial knowledge of the trigonometric functions, prove that the limit of the sequence (x_n) is $\cos\frac{\pi}{9}$.

Solution to part d: ***We write the limit of this sequence as*** x***. from the fact that***

$$8x_{n+1}^3 = 6x_n + 1$$

for each n ***we obtain***

$$8x^3 - 6x - 1 = 0.$$

This equation has one positive solution and two negative solutions and from the fact that

$$\begin{aligned} 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1 &= 2\left(4\cos^3\frac{\pi}{9} - 3\cos\frac{\pi}{9}\right) - 1 \\ &= 2\left(\cos\left(3\left(\frac{\pi}{9}\right)\right)\right) - 1 = 2\cos\frac{\pi}{3} - 1 = 0 \end{aligned}$$

we see that the positive root is $\cos\frac{\pi}{9}$.

5. In this exercise we study the sequence (x_n) defined by the equation

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

for every integer $n \geq 1$. You will probably want to make use of the binomial theorem when you do this exercise.

a. Ask *Scientific Notebook* to make a 2D plot of the graph of the function f defined by the equation

$$f(n) = \left(1 + \frac{1}{n}\right)^n$$

for $1 \leq n \leq 100$.

b. Prove that $x_n < 3$ for every n.

Solution: ***For each*** $n \geq 2$ ***we see that***

$$x_n = \left(1+\frac{1}{n}\right)^n = \sum_{j=0}^{n}\binom{n}{j}\left(\frac{1}{n}\right)^j = 1+1+\sum_{j=2}^{n}\frac{n(n-1)\cdots(n-j+1)}{(j!)n^j}$$
$$= 1+1+\sum_{j=2}^{n}(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)\frac{1}{j!}$$
$$< 2+\sum_{j=2}^{n}\frac{1}{j!} < 2+\sum_{j=2}^{n}\frac{1}{2^{j-1}} < 3$$

c. Prove that the sequence (x_n) is increasing.

Solution: ***For each*** $n \geq 2$ ***we see that***

$$x_n = 1+1+\sum_{j=2}^{n}(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)\frac{1}{j!}$$
$$< 1+1+\sum_{j=2}^{n}(1)\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{j-1}{n+1}\right)\frac{1}{j!} = x_{n+1}.$$

d. Deduce that the sequence (x_n) converges to a number between 2 and 3. Have you seen this number before?
The student is being asked informally whether he/she recognizes that this limit is the number e. The number e will be seen in Chapter 10.

6. This exercise concerns the sequence (x_n) defined by the fact that $x_1 = 1$ and that, for each $n \geq 1$ we have

$$x_{n+1} = \sqrt[5]{4x_n - 2}.$$

a. Use *Scientific Notebook* to work out the first twenty terms in the sequence (x_n).

b. Prove that $1 \leq x_n < 2$ for every n.
We use mathematical induction. Since $x_1 = 1$, the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n is true. We see that

$$1 < \sqrt[5]{4(1)-2} < \sqrt[5]{4x_n-2} < \sqrt[5]{4(2)-2} < 2$$

and so the assertion p_{n+1} is true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

c. Prove that the sequence (x_n) is strictly increasing.
We use mathematical induction. For each positive integer n we take p_n to be the assertion that $x_n < x_{n+1}$. Since

$$x_1 = 1 < \sqrt[5]{2} = x_2$$

we conclude that the assertion p_1 is true. Now suppose that n is any positive integer for which the assertion p_n happens to be true. Then

$$x_{n+2} = \sqrt[5]{4x_{n+1}-2} > \sqrt[5]{4x_n-2} = x_{n+1}$$

and so the assertion p_{n+1} must be true. We deduce from mathematical induction that the assertion p_n is true for every positive integer n.

d. Prove that the sequence has a limit x that satisfies the equation $x^5 - 4x + 2 = 0$. Ask *Scientific Notebook* to make a 2D plot of the expression $x^5 - 4x + 2$ on the interval $[-2, 2]$ and to solve the equation

$$x^5 - 4x + 2 = 0$$
$$x \in [1,2]$$

numerically. Compare the answer obtained here with the results that you obtained in part a.

7. a. Given that

$$f(x) = \frac{x}{2} + \frac{9}{2x}$$

for every number $x > 0$, prove that $f(x) \geq 3$ for each n and that the equation $f(x) = 3$ holds if and only if $x = 3$.

Solution: ***The desired result follows at once from the fact that whenever*** $x > 0$ ***we have***

$$f(x) = \frac{x}{2} + \frac{9}{2x} = \frac{(x-3)^2}{2x} + 3.$$

b. Given that $x_1 = 4$ and, for each $n \geq 1$, we have

$$x_{n+1} = \frac{x_n}{2} + \frac{9}{2x_n},$$

prove that the sequence (x_n) is decreasing and that the sequence converges to the number 3.

Solution: ***Since*** $x_{n+1} = f(x_n)$ ***for each*** n ***and since*** $f(x) > 3$ ***for every number*** $x \neq 3$ ***we see at once that*** $x_n > 3$ ***for every*** n***. To see that*** (x_n) ***is decreasing we observe that if*** n ***is any positive integer then***

$$x_n - x_{n+1} = x_n - \left(\frac{x_n}{2} + \frac{9}{2x_n}\right) = \frac{x_n^2 - 9}{2} > 0.$$

Since the sequence (x_n) ***is a decreasing sequence in the interval*** $(3,\infty)$ ***we know that*** (x_n) ***is convergent. If we write the limit of this sequence as*** x ***then it follows from the relationship***

$$x_{n+1} = \frac{x_n}{2} + \frac{9}{2x_n}$$

that

$$x = \frac{x}{2} + \frac{9}{2x}$$

from which we deduce that $x = 3$***.***

8. This exercise is a study of the sequence (x_n) for which $x_1 = 0$ and

$$x_{n+1} = \frac{1}{2 + x_n}$$

for every positive integer n. We note that this sequence is bounded below by 0 and above by 1/2.

a. Supply the definition

$$f(x) = \frac{1}{2 + x}$$

to ***Scientific Notebook***. Then open your Compute menu, click on Calculus, and choose to iterate the function f ten times, starting at the number 0. Evaluate the column of numbers that you have obtained accurately to ten decimal places and, in this way, show the first ten members of the sequence (x_n).

b. Show that

$$x_{n+2} = \frac{2 + x_n}{5 + 2x_n}$$

for every integer $n \geq 1$, and then show that the sequence (x_{2n-1}) is increasing and that the sequence (x_{2n}) is decreasing and that these two sequences have the same limit $\sqrt{2} - 1$.

Solution: ***For every positive number*** x ***we define***

$$g(x) = \frac{2 + x}{5 + 2x}.$$

Whenever $0 < t < x$ ***we see that***

$$g(x) - g(t) = \frac{2+x}{5+2x} - \frac{2+t}{5+2t} = \frac{x-t}{(5+2x)(5+2t)} > 0$$

and so the function g is strictly increasing. Since $x_1 < x_3$ we have $g(x_1) < g(x_3)$ from which we deduce that $x_3 < x_5$. Continuing in this way we see that the sequence (x_{2n-1}) is increasing. Since $x_2 > x_4$ we have $g(x_2) > g(x_4)$ from which deduce that $x_4 > x_5$. Continuing in this way we see that the sequence (x_{2n}) is decreasing. Therefore the sequences (x_{2n-1}) and (x_{2n}) are convergent. If we write

$$x = \lim_{n\to\infty} x_{2n-1}$$

then it follows from the identity

$$x_{2n+1} = \frac{2 + x_{2n-1}}{5 + 2x_{2n-1}}$$

that

$$x = \frac{2+x}{5+2x}$$

and we can see that the only positive solution of this equation is $\sqrt{2} - 1$. Thus

$$\lim_{n\to\infty} x_{2n-1} = \sqrt{2} - 1$$

and we can see similarly that

$$\lim_{n\to\infty} x_{2n} = \sqrt{2} - 1.$$

c. Deduce that $x_n \to \sqrt{2} - 1$ as $n \to \infty$.

Exercises on Upper and Lower Limits

1. Prove that if (x_n) is a sequence of real numbers then (x_n) has a limit if and only if

$$\operatorname*{lmsup}_{n\to\infty} x_n = \operatorname*{lminf}_{n\to\infty} x_n.$$

Solution: ***This exercises follows at once from*** *an earlier theorem.*

2. Prove that a sequence (x_n) of real numbers is bounded above if and only if

$$\operatorname*{lmsup}_{n\to\infty} x_n < \infty.$$

We know that a sequence is bounded above if and only if ∞ is not a partial limit of the sequence. There a sequence is bounded above if and only if its largest partial limit is not ∞.

3. Suppose that (x_n) is a sequence of real numbers.

a. Prove that if x is a partial limit of the sequence (x_n) then the number $-x$ is a partial limit of the sequence $(-x_n)$.
if ∞ is a partial limit of a sequence (x_n) then (x_n) is unbounded above and so $(-x_n)$ is unbounded below, making $-\infty$ a partial limit of $(-x_n)$.
Now suppose that a real number x is a partial limit of a given sequence (x_n). To show that the number $-x$ is a partial limit of the sequence $(-x_n)$, suppose that $\varepsilon > 0$.
We know that there are infinitely many integers n for which

$$x - \varepsilon < x_n < x + \varepsilon$$

and for each of these integers n we have

$$-x - \varepsilon < -x_n < -x + \varepsilon.$$

Therefore the sequence $(-x_n)$ is frequently in the interval $(-x-\varepsilon, -x+\varepsilon)$ and we conclude that $-x$ is a partial limit of $(-x_n)$.

b. Prove that

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n.$$

Since $\liminf_{n\to\infty} x_n$ is a partial limit of (x_n) we know from part a that $-\liminf_{n\to\infty} x_n$ is a partial limit of the sequence $(-x_n)$. Now given any partial limit q of $(-x_n)$, we know that $-q$, being a partial limit of (x_n), cannot be less than $\liminf_{n\to\infty} x_n$. In other words, whenever q is a partial limit of $(-x_n)$ we have

$$\liminf_{n\to\infty} x_n \le -q$$

which gives us

$$q \le -\liminf_{n\to\infty} x_n.$$

Therefore $-\liminf_{n\to\infty} x_n$ is the largest partial limit of the sequence $(-x_n)$.

4. Suppose that (x_n) is a sequence of real numbers, that

$$x = \limsup_{n\to\infty} x_n$$

and that $u < x < v$.

a. Prove that the sequence (x_n) must be bounded above.
The given inequality $u < x < v$ tells us that $x \neq \infty$ and so (x_n) is bounded above.

b. Prove that the sequence (x_n) must be frequently in the interval $[u, \infty)$.
The interval $[u, \infty)$ is a neighborhood of the partial limit x of (x_n).

c. Prove that the sequence (x_n) cannot be frequently in the interval $[v, \infty)$.

Hint: ***Choose an upper bound*** α ***of*** (x_n). ***Now use*** *this theorem.*
If (x_n) were frequently in the interval $[v, \infty]$ then it would have to have a partial limit in this interval, contradicting the fact that x is the largest partial limit of (x_n).

5. Suppose that (x_n) is a sequence of real numbers, that x is a real number and that, whenever $u < x < v$, the sequence (x_n) is frequently in the interval $[u, \infty)$ but is not frequently in the interval $[v, \infty)$. Prove that

$$x = \limsup_{n\to\infty} x_n.$$

The fact that (x_n) must be frequently in the interval (u, v) whenever $u < x < v$ tells us that (x_n) is frequently in every neighborhood of x and so x must be a partial limit of (x_n). To show that x is the largest partial limit of (x_n) we shall show that no number larger than x can be a partial limit of (x_n). Suppose that $x < p$ and choose a number v between x and p. Since the interval $[v, \infty)$ is a neighborhood of p and (x_n) is not frequently in $[v, \infty)$ we conclude that p can't be a partial limit of (x_n).

6. Suppose that (x_n) is a bounded sequence of real numbers and that, for each integer n in the domain of this sequence we have defined

$$y_n = \sup\{x_m \mid m \ge n\}.$$

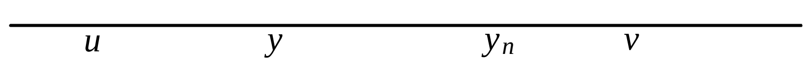
Prove that the sequence (y_n) is decreasing and that its limit is the lower limit of the sequence (x_n).
For each positive integer n, it follows from the fact that

$$\{x_m \mid m \ge n+1\} \subseteq \{x_m \mid m \ge n\}$$

that

$$\sup\{x_m \mid m \ge n+1\} \le \sup\{x_m \mid m \ge n\}$$

and so the sequence (y_n) must be decreasing. Since every lower bound of (x_n) is also a lower bound of (y_n), the sequence (y_n) must be bounded below. Therefore (y_n) is convergent. We define y to be the limit of (y_n). To show that y is $\limsup_{n\to\infty} x_n$ we shall use Exercise 5. Suppose that $u < y < v$. Using the fact that $y_n \to y$ as $n \to \infty$ we choose N such that the inequality $y_n < v$ holds whenever $n \ge N$.



Since

$$x_m \le y_N < v$$

whenever $m \ge N$ we know that the sequence (x_n) cannot be frequently in the interval $[v, \infty)$.
On the other hand, given any integer $n \ge N$ it follows from the fact that

$$u < y_n = \sup\{x_m \mid m \ge n\}$$

that there exists an integer $m \ge n$ such that $u < x_m$.

Thus the set of integers m for which $u < x_m$ is unbounded above and we have shown that (x_n) is frequently in the interval $[u, \infty)$.

7. State and prove an analogue of the preceding exercise for lower limits.
   Suppose that (x_n) is a bounded sequence that that, for each n we define

$$y_n = \inf\{x_m \mid m \ge n\}.$$

   Then the sequence (y_n) is decreasing and the limit of the sequence (y_n) is $\operatorname{lminf}_{n\to\infty} x_n$.
   This assertion can be proved by a mirror image of the proof that was used in Exercise 6 and it can also be obtained from the statement of Exercise 6, in view of Exercise 3b.

8. Given that

$$z_n = x_n + y_n$$

   for every positive integer n, prove that

$$\operatorname*{lmsup}_{n\to\infty} z_n \le \operatorname*{lmsup}_{n\to\infty} x_n + \operatorname*{lmsup}_{n\to\infty} y_n.$$

   Solution: ***To obtain a contradiction we assume that***

$$\operatorname*{lmsup}_{n\to\infty} z_n > \operatorname*{lmsup}_{n\to\infty} x_n + \operatorname*{lmsup}_{n\to\infty} y_n.$$

   Now we choose a number $\varepsilon > 0$ ***such that***

$$\operatorname*{lmsup}_{n\to\infty} z_n > \operatorname*{lmsup}_{n\to\infty} x_n + \operatorname*{lmsup}_{n\to\infty} y_n + \varepsilon.$$

   Since

$$\operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2} > \operatorname*{lmsup}_{n\to\infty} x_n,$$

   there are at most finitely many integers n ***for which***

$$x_n > \operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2}$$

   and we see, in the same way, that there are at most finitely many integers n ***for which***

$$y_n > \operatorname*{lmsup}_{n\to\infty} y_n + \frac{\varepsilon}{2}.$$

   Thus, for all but at most finitely many integers n ***we have***

$$z_n = x_n + y_n \le \operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2} + \operatorname*{lmsup}_{n\to\infty} y_n + \frac{\varepsilon}{2}$$

   which is impossible since

$$\operatorname*{lmsup}_{n\to\infty} x_n + \frac{\varepsilon}{2} + \operatorname*{lmsup}_{n\to\infty} y_n + \frac{\varepsilon}{2} < \operatorname*{lmsup}_{n\to\infty} z_n.$$

Exercises on the Cantor Intersection Theorem

1. Suppose that (H_n) is a sequence (not necessarily contracting) of closed bounded sets and that for every positive integer n we have

$$\bigcap_{i=1}^{n} H_i \ne \emptyset.$$

Prove that

$$\bigcap_{i=1}^{\infty} H_i \neq \emptyset.$$

For each n we define

$$K_n = \bigcap_{i=1}^{n} H_i$$

Since the sequence (K_n) is a contracting sequence of nonempty closed bounded sets, we know from the Cantor intersection theorem that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

We observe finally that, since

$$\bigcap_{n=1}^{\infty} H_n = \bigcap_{n=1}^{\infty} K_n$$

the intersection of all of the sets H_n must also be nonempty.

2. Suppose that H is a closed bounded set of real numbers and that (U_n) is an expanding sequence of open sets.

   a. Explain why the sequence of sets $H \setminus U_n$ is a contracting sequence of closed bounded sets.
   For each n, it follows at once from the inclusion $U_n \subseteq U_{n+1}$ that

   $$H \setminus U_{n+1} \subseteq H \setminus U_n.$$

   b. Use the Cantor intersection theorem to deduce that if $H \setminus U_n \neq \emptyset$ for every n then

   $$\bigcap_{n=1}^{\infty} (H \setminus U_n) \neq \emptyset.$$

   Since H is closed and bounded and since

   $$H \setminus U_n = H \cap (\mathbf{R} \setminus U_n)$$

   for each n, the sets $H \setminus U_n$ must be closed and bounded. The desired result therefore follows at once from the Cantor intersection theorem.

   c. Prove that if

   $$H \subseteq \bigcup_{n=1}^{\infty} U_n$$

   then there exists an integer n such that $H \subseteq U_n$.
   We suppose that

   $$H \subseteq \bigcup_{n=1}^{\infty} U_n.$$

   Since the set

   $$\bigcap_{n=1}^{\infty} (H \setminus U_n) = H \setminus \bigcup_{n=1}^{\infty} U_n$$

   is empty we deduce from part b that there is a value of n for which $H \setminus U_n = \emptyset$ and for any such integer n we have $H \subseteq U_n$.

3. Suppose that (U_n) is a sequence of open sets (not necessarily expanding) and that H is a closed bounded set and that

   $$H \subseteq \bigcup_{n=1}^{\infty} U_n.$$

   Prove that there exists a positive integer N such that

$$H \subseteq \bigcup_{n=1}^{N} U_n.$$

For each n we define

$$V_n = \bigcup_{i=1}^{n} U_i.$$

The sequence (V_n) is expanding and, since

$$H \subseteq \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} V_n$$

it follows from Exercise 2 that there exists a positive integer n for which

$$H \subseteq V_n = \bigcup_{i=1}^{n} U_i.$$

4. The Cantor intersection theorem depends upon the completeness of the real number system. Where in the proof of the theorem is the completeness used?
   We use the completeness to guarantee that a nonempty closed bounded set has a least member and then we use the completeness again to guarantee that the sequence of all the least members of the given sets has a supremum.

Exercises on Complete Metric Spaces

1. Prove that the metric space $\mathbf{R}^k$ with the ∞-metric is complete.
   Hint: ***From the inequality***

   $$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\| \le \sqrt{k}\,\|\mathbf{x}\|_\infty$$

   that holds for every point x in R^k you can show easily that a sequence $(\mathbf{x}_n)$ in R^k is a Cauchy sequence if and only if $(\mathbf{x}_n)$ is a Cauchy sequence in R^k with the ∞-metric. The assertion this this exercise also follows at once from the one that appears in Exercise 2.

2. Prove that if S is any set then the metric space $\ell^\infty(S)$ is complete. We saw the definition of this space earlier.
   Solution: ***Suppose that S is a set and that (f_n) is a Cauchy sequence in the metric space $\ell^\infty(S)$. For every member x of the set S we see from the fact that***

   $$|f_m(x) - f_n(x)| \le \|f - g\|_\infty$$

   for all m and n that the sequence $(f_n(x))$ is a Cauchy sequence in R. Since R is complete the sequence $(f_n(x))$ converges for every $x \in S$. For every $x \in S$ we define

   $$f(x) = \lim_{n\to\infty} f_n(x)$$

   and, in this way, we have defined a function $f : S \to R$. Now we need to show that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Suppose that $\varepsilon > 0$.

1. ***Using the fact that (f_n) is a Cauchy sequence in $\ell^\infty(S)$, choose an integer N such that the inequality***

   $$\|f_n - f_m\|_\infty < \varepsilon$$

   holds whenever m and n are integers and $m \ge N$ and $n \ge N$. Given any $n \ge N$ and any member x of the set S we have $|f_n(x) - f_m(x)| \le \varepsilon$ whenever $m \ge N$ and since

   $$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)|$$

   we have $|f_n(x) - f(x)| \le \varepsilon$. Therefore if $n \ge N$ we have

   $$\|f_n - f\| = \sup\{|f_n(x) - f(x)| \mid x \in S\} \le \varepsilon$$

   and we have shown that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

Finally we need to observe that the function f belongs to $\ell^\infty(S)$, in other words, that f is bounded. Using the fact that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, we choose an integer n such that $\|f_n - f\|_\infty < 1$. We see that

$$\|f\|_\infty = \|f - f_n + f_n\|_\infty \le \|f - f_n\|_\infty + \|f_n\|_\infty < 1 + \|f_n\|_\infty < \infty.$$

3. Prove that if (x_n) is a Cauchy sequence in a metric space X and if $x \in X$ then the sequence of numbers $d(x_n, x)$ converges in the metric space $\mathbf{R}$.
   Given any positive integers m and n we have
   $$|d(x_m, x) - d(x_n, x)| \le d(x_m, x_n)$$
   and it is therefore clear that the sequence $(d(x_n, x))$ is a Cauchy sequence in $\mathbf{R}$.

4. Give an example of a complete metric space X and a bounded sequence (x_n) in X such that the sequence (x_n) has no partial limit.
   Hint: ***Look for an example in a space of the form $\ell^\infty(S)$.***
   We take S to be the set $\mathbf{Z}^+$ of positive integers. For each $n \in S$ we define the function $f_n : S \to \mathbf{R}$ by the equation
   $$f_n(x) = \begin{cases} 1 & \text{if} \quad x = n \\ 0 & \text{if} \quad x \in S \setminus \{n\} \end{cases}$$
   From the fact that $\|f_m - f_n\|_\infty = 1$ whenever $m \neq n$ we see that (f_n) has no partial limit in the space $\ell^\infty(S)$.

5. Give an example of a complete metric space X and a contracting sequence (H_n) of nonempty closed bounded sets such that
   $$\bigcap_{n=1}^{\infty} H_n = \emptyset.$$
   We return to the functions f_n that were defined in the solution to Exercise 4. For each n we define
   $$H_n = \{f_m \mid m \ge n\}.$$

6. If X is a metric space then a function $f : X \to X$ is said to be a **contraction** on X if there exists a number $\alpha < 1$ such that whenever x and t belong to X we have
   $$d(f(t), f(x)) \le \alpha d(t, x).$$

   a. Suppose that f is a contraction on a metric space X and that $t \in X$. Suppose that we have $x_1 = t$ and that for every positive integer n we have
      $$x_{n+1} = f(x_n).$$
      Prove that the sequence (x_n) is a Cauchy sequence.
      We observe first that
      $$d(x_2, x_3) = d(f(x_1), f(x_2)) \le \alpha d(x_1, x_2)$$
      $$d(x_3, x_4) = d(f(x_2, f(x_3))) \le \alpha d(x_2, x_3) \le \alpha^2 d(x_1, x_2)$$
      and, in general,
      $$d(x_n, x_{n+1}) \le \alpha^{n-1} d(x_1, x_2).$$
      Now suppose that m and n are positive integers and that $n > m$. We see that
      $$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &< \alpha^{m-1} d(x_1, x_2) + \alpha^m d(x_1, x_2) + \alpha^{m+1} d(x_1, x_2) + \cdots + \alpha^{n-2} d(x_1, x_2) \\ &= \alpha^{m-1} d(x_1, x_2)(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m}) \\ &< \frac{\alpha^{m-1} d(x_1, x_2)}{1 - \alpha}. \end{aligned}$$

To show that (x_n) is a Cauchy sequence, suppose that $\varepsilon > 0$ and choose an integer N such that

$$\frac{\alpha^{N-1} d(x_1, x_2)}{1-\alpha} < \varepsilon.$$

Then whenever $m \geq N$ and $n \geq N$, regardless of whether $m < n$ or $m = n$ or $m > n$, we have $d(x_m, x_n) < \varepsilon$.

b. Suppose that f is a contraction on a complete metric space X. Prove that there exists a point $x \in X$ such that $f(x) = x$.

Choose a point $t \in X$. We define the sequence (x_n) as in part a. Since X is complete and (x_n) is a cauchy sequence in X we know that (x_n) is convergent. We define

$$x = \lim_{n \to \infty} x_n.$$

In order to show that $f(x) = x$

c. Suppose that f is a contraction on a complete metric space X. Prove that there is one and only one point $x \in X$ such that $f(x) = x$.

7. Prove that if X is a nonempty complete metric space and if every point of X is a limit point of X then X is uncountable.

Solution: ***Suppose that X is a complete metric space and that every point of X is a limit point of X. To obtain a contradiction, suppose that (x_n) is a sequence in X and that X is the range of the sequence (x_n).***

1. ***Since x_1 is a limit point of X the set $X \setminus \{x_1\}$ must be nonempty. Choose a point $y_1 \in X \setminus \{x_1\}$ and, using the fact that $X \setminus \{x_1\}$ is a neighborhood of y_1, choose a positive number $\delta_1 \leq 1$ such that the closed ball $B[y_1, \delta_1]$ is included in $X \setminus \{x_1\}$, in other words,***

$$B[y_1, \delta_1] \subseteq X \setminus \{x_1\}$$

Since y_1 is a limit point of X and $B(y_1, \delta_1)$ is a neighborhood of y_1, the ball $B(y_1, \delta_1)$ must contain infinitely many points and so the set $B(y_1, \delta_1) \setminus \{x_2\}$ must be nonempty. Choose a point

$$y_2 \in B(y_1, \delta_1) \setminus \{x_2\}$$

and, using the fact that $B(y_1, \delta_1) \setminus \{x_2\}$ is a neighborhood of y_2, choose a positive number $\delta_2 \leq \frac{1}{2}$ such that

$$B[y_2, \delta_2] \subseteq B(y_1, \delta_1) \setminus \{x_2\}.$$

Continuing this process we obtain a sequence (y_n) in the space X and a sequence (δ_n) of positive numbers such that, for each n,

$$\delta_n \leq \frac{1}{n}$$

and

$$B[y_{n+1}, \delta_{n+1}] \subseteq B(y_n, \delta_n) \setminus \{x_{n+1}\}.$$

We observe that

$$B[y_n, \delta_n] \subseteq X \setminus \{x_1, x_2, x_3, \cdots, x_n\}$$

for each n.

We shall obtain our contradiction by showing that the sequence (y_n) is a Cauchy sequence and then we shall observe that the limit of the sequence (y_n) cannot belong to the range of the sequence (x_n).

To show that (y_n) is a Cauchy sequence, suppose that $\varepsilon > 0$. Choose an integer $N > 2/\varepsilon$. Whenever m and n are integers and $m \geq N$ and $n \geq N$ we know that both y_m and y_n belong to the ball $B(y_N, \delta_N)$ and so

$$d(y_m, y_n) \leq d(y_m, y_N) + d(y_N, y_n) < \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

Thus (y_n) is a Cauchy sequence and, since the space X is complete, the sequence (y_n) is convergent. We define

$$y = \lim_{n \to \infty} y_n.$$

For each positive integer N, since the sequence (y_n) is eventually in the closed ball $B(y_N, \delta_N)$ we have

$$y \in B(y_N, \delta_N) \subseteq \{x_1, x_2, x_3, \cdots, x_N\}$$

and so** y **cannot belong to the range of the sequence** (x_n) **and we have reached the desired contradiction.

8. This exercise is an improvement on Exercise 7 and is known as Baire's theorem.

1. Suppose that (H_n) is a sequence of closed subsets of a complete metric space X and that none of the sets H_n have any interior points. prove that the set

$$\bigcup_{n=1}^{\infty} H_n$$

also fails to have any interior points.

Solution: ***We want to show that the set***

$$\bigcup_{n=1}^{\infty} H_n$$

cannot include any nonempty open set. Suppose that** U **is a nonemty open subset of** X **and, using the fact that** H_1 **has no interior points we choose a point** $y_1 \in U \setminus H_1$ **and using the fact that the set** $U \setminus H_1$ **is a neighborhood of** y_1 **we choose a positive number** $\delta_1 \leq 1$ **such that

$$B[y_1, \delta_1] \subseteq U \setminus H_1.$$

Using the fact that the open set** $B(y_1, \delta_1)$ **cannot be included in the set** H_2 **(because** H_2 **has no interior points) we choose a point

$$y_2 \in B(y_1, \delta_1) \setminus H_2 = B(y_1, \delta_1) \cap (X \setminus H_1)$$

and, using the fact that the latter set is a neighborhood of** y_2 **we choose a positive number** $\delta_2 \leq \frac{1}{2}$ **such that

$$B[y_2, \delta_2] \subseteq B(y_1, \delta_1) \setminus H_2.$$

Continuing this process we obtain a sequence** (y_n) **in the space** X **and a sequence** (δ_n) **of positive numbers such that, for each** n**,

$$\delta_n \leq \frac{1}{n}$$

and

$$B[y_{n+1}, \delta_{n+1}] \subseteq B(y_n, \delta_n) \setminus H_{n+1}.$$

We observe that

$$B[y_n, \delta_n] \subseteq U \setminus \bigcup_{j=1}^{n} H_j$$

In order to show that the set

$$U \setminus \bigcup_{n=1}^{\infty} H_n$$

is nonempty we shall show that the sequence** (y_n) **is a Cauchy sequence and then we shall observe that the limit of the sequence** (y_n) **must belong to the set

$$U \setminus \bigcup_{n=1}^{\infty} H_n.$$

To show that** (y_n) **is a Cauchy sequence, suppose that** $\varepsilon > 0$**. Choose an integer** $N > 2/\varepsilon$**. Whenever** m **and** n **are integers and** $m \geq N$ **and** $n \geq N$ **we know that both** y_m **and** y_n **belong to the ball** $B(y_N, \delta_N)$ **and so

$$d(y_m, y_n) \leq d(y_m, y_N) + d(y_N, y_n) < \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

Thus** (y_n) **is a Cauchy sequence and, since the space** X **is complete, the sequence** (y_n) **is convergent. We define

$$y = \lim_{n\to\infty} y_n.$$

For each positive integer N, since the sequence (y_n) is eventually in the closed ball $B(y_N, \delta_N)$ we have

$$y \in B(y_N, \delta_N) \subseteq U \setminus \bigcup_{j=1}^{N} H_j.$$

Thus

$$y \in U \setminus \bigcup_{n=1}^{\infty} H_n$$

and the proof is complete.

Exercises on Compact Metric Spaces

1. Prove the following converse of the Cantor intersection theorem: If X is a metric space and if every contracting sequence of nonempty closed subsets of X has a nonempty intersection then X is compact.

Solution: ***To show that every sequence in X has a partial limit, suppose that (x_n) is a sequence in X. For each positive integer n we define***

$$H_n = \overline{\{x_m \mid m \geq n\}}.$$

Using the fact that the sequence (H_n) is a contracting sequence of nonempty closed subsets of X, choose a point x such that

$$x \in \bigcap_{n=1}^{\infty} H_n.$$

It is easy to see that x is a partial limit of the sequence (x_n). Write out the details.

2. Given a metric space X, prove that the following conditions are equivalent:

   a. The space X is compact.

   b. For every family $\mathfrak{I}$ of open balls that covers the space X there exists a finite subfamily of $\mathfrak{I}$ that still covers X.

It is obvious that condition a implies condition b because every open ball is an open set. Now suppose that condition b holds. Suppose that $\mathfrak{I}$ is a family of open sets that covers X. We define $\mathfrak{I}^*$ to be the family of open balls B that are included in at least one member of the family $\mathfrak{I}$. Given any point $x \in X$, since $\mathfrak{I}$ covers X we know that x belongs to at least one member U of the family $\mathfrak{I}$ and, since such a set U is open, there exists an open ball B such that $x \in B \subseteq U$. Therefore the family $\mathfrak{I}^*$ covers the space X and, using condition b, we choose finitely many members, that we shall call $B_1, B_2, \cdots, B_n$, such that

$$X \subseteq \bigcup_{j=1}^{n} B_j.$$

For each j we choose a member U_j of the family $\mathfrak{I}$ such that $B_j \subseteq U_j$ and, since

$$X \subseteq \bigcup_{j=1}^{n} U_j$$

we have found finitely many members of $\mathfrak{I}$ that cover the space X.

3. Suppose that X is a compact metric space and that $\mathfrak{I}$ is a family of closed subsets of X. Prove that if

$$\bigcap \mathfrak{I} = \emptyset$$

then there must exist a finite subfamily $\mathfrak{I}^*$ of $\mathfrak{I}$ such that

$$\bigcap \mathfrak{I}^* = \emptyset.$$

We observe first that

$$\bigcup\{X \setminus H \mid H \in \Im\} = X \setminus \bigcap\{H \mid H \in \Im\} = X \setminus \emptyset = X.$$

Therefore, since X is compact, it is possible to find finitely many members $H_1, H_2, \cdots, H_n$ of the family $\Im$ such that

$$X = \bigcup_{j=1}^{n}(X \setminus H_j)$$

and we see at once that

$$\bigcap_{j=1}^{n} H_j = \emptyset.$$

4. In a theorem we have just studied we saw that if X is a compact metric space then every sequence in X that has no more than one partial limit must be convergent.
   Is it true that if every sequence in a given metric space having no more than one partial limit must be convergent, that X must be compact?
   Yes, it is true. If every sequence that has no more than one partial limit has to be convergent then every sequence must have at least one partial limit.
   Now let's change the question. Suppose that we know that, in a given metric space, that every sequence with precisely one partial limit must be convergent. Must the space be compact. Again the answer is yes. To see why, suppose that a metric space X fails to be compact and choose a sequence (x_n) in X such that (x_n) has no partial limit. We now look at the sequence

   $$x_1, x, x_2, x, x_3, x, \cdots.$$

   This sequence has precisely one partial limit but is not convergent.

5. Suppose that X is a compact metric space and that $S \subseteq X$. Prove that the metric space S is compact if and only if S is closed in X.
   The result follows at once from Theorem 7.12.14

6. Prove that if A and B are compact subspaces of $\mathbf{R}$ then $A + B$ is also compact.
   Hint: ***Show that every sequence in the set*** $A + B$ ***has a subsequence that converges to a point of*** $A + B$.
   Suppose that (x_n) is a sequence in the set $A + B$. For each n, choose numbers $a_n \in A$ and $b_n \in B$ such that $x_n = a_n + b_n$. Using the fact that A is compact, choose a subsequence $\left(a_{n_j}\right)$ of (a_n) that converges to a point a of the set A. For each j we have $x_{n_j} = a_{n_j} + b_{n_j}$. Using the fact that B is compact, choose a subsequence (b_{j_i}) of $\left(b_{n_j}\right)$ that converges to a point b of the set B. The subsequence $\left(x_{n_{j_i}}\right)$ of (x_n) converges to the point $a + b$ of the set $A + B$.

7. Given a metric space X, is it true that every bounded infinite subset of X has a limit point if and only if every closed bounded subset of X is compact as a subspace of X?
   Yes, this assertion is true.
   Suppose that every closed bounded subset of X is compact as a subspace of X and suppose that S is a bounded infinite subset of X. Since $\bar{S}$ is compact, every infinite subset of $\bar{S}$ must have a limit point. Therefore S must have a limit point.
   Suppose that every bounded infinite subset of X has a limit point and that Y is a closed bounded subset of X. Every infinite subset of Y, being bounded, must have a limit point and, since Y is closed, any such limit point must belong to Y. Therefore the metric space Y must be compact.

8 Limits and Continuity of Functions

Some Exercises on Limits of Functions

1. Write careful proofs of each of the following assertions:

   a. $x^3 - 3x \to 2$ as $x \to -1$.

   We observe that if x is any given number we have

$$\begin{aligned}|(x^3 - 3x) - 2| &= |x^3 - 3x - 2| \\ &= |(x+1)(x^2 - x - 2)|\end{aligned}$$

   The idea of the proof is to make the observation that the factor $x + 1$ will be small when x is close to -1 and that the factor $x^2 - x - 2$ is not too large. In fact, if $|x+1| < 1$ then, since $-2 < x < 0$ we have

$$|x^2 - x - 2| \le 2^2 + 2 + 2 = 8$$

   which gives us

$$|(x^3 - 3x) - 2| = |(x+1)(x^2 - x - 2)| \le 8|x+1|.$$

   Now suppose that $\varepsilon > 0$. We define ε to be the smaller of the two numbers 1 and $\varepsilon/8$. Then, whenever $|x+1| < \delta$ we have

$$|(x^3 - 3x) - 2| = |(x+1)(x^2 - x - 2)| \le 8|x+1| < 8\left(\frac{\varepsilon}{8}\right) = \varepsilon.$$

   We conclude that the inequality

$$|(x^3 - 3x) - 2| < \varepsilon$$

   holds whenever $x \ne -1$ and $|x+1| < \delta$ (and, as a matter of fact, the inequality is also true when $x = -1$).

   b. $\frac{1}{x} \to \frac{1}{3}$ as $x \to 3$.

   We begin with the observation that if x is any nonzero number we have

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3-x|}{3|x|}.$$

   To keep the denominator of this fraction from becoming too small we need to keep x away from 0. In fact, if $|x-3| < 1$ then, since $2 < x < 4$ we have

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3-x|}{3|x|} < \frac{|3-x|}{6}.$$

   Now suppose that $\varepsilon > 0$. We define δ to be the smaller of the two numbers 1 and ε and we observe that whenever $|x-3| < \delta$ we have

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3-x|}{3|x|} < \frac{|3-x|}{6} < \frac{\varepsilon}{6} < \varepsilon.$$

   We conclude that the inequality

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$$

   holds whenever $x \ne 3$ and $|x-3| < \delta$ (and, as a matter of fact, the inequality is also true when $x = 3$).

   c. $\frac{x^3 - 8}{x^2 + x - 6} \to \frac{12}{5}$ as $x \to 2$.

   We begin by observing that if x is any number for which $x^2 + x - 6 \ne 0$ then we have

$$\begin{aligned}\left|\frac{x^3 - 8}{x^2 + x - 6} - \frac{12}{5}\right| &= \left|\frac{(x-2)(x^2+2x+4)}{(x-2)(x+3)} - \frac{12}{5}\right| \\ &= \left|\frac{x^2+2x+4}{x+3} - \frac{12}{5}\right| \\ &= \left|\frac{(x-2)(5x+8)}{5(x+3)}\right|\end{aligned}$$

   In order to keep the denominator of this fraction from becoming too small we need to keep x

away from -3 and, in order to keep $|5x+8|$ from becoming too large we need to keep x from being too large. In fact, if $x \neq 2$ and $|x-2| < 1$ we have

$$\left|\frac{x^3-8}{x^2+x-6} - \frac{12}{5}\right| = \left|\frac{(x-2)(5x+8)}{5(x+3)}\right| < \left|\frac{(x-2)(5(3)+8)}{5(3+3)}\right| < |x-2|.$$

Now suppose that $\varepsilon > 0$ and define δ to be the smaller of the two numbers 1 and ε. We see that whenever $x \neq 2$ and $|x-2| < \delta$ we have

$$\left|\frac{x^3-8}{x^2+x-6} - \frac{12}{5}\right| < |x-2| < \varepsilon.$$

2. Given that

$$f(x) = \begin{cases} x & \text{if } 0 < x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

prove that $f(x) \to 1$ as $x \to 1$ and that this function f has no limit at the number 2.

Solution: ***Before we prove that*** $f(x) \to 1$ ***as*** $x \to 1$ ***we make the observation that whenever*** $|x-1| < 1$ ***we have***

$$|x^2-1| = |x-1||x+1| < 3|x-1|.$$

To prove that $f(x) \to 1$ ***as*** $x \to 1$***, suppose that*** $\varepsilon > 0$***. We observe that whenever*** $x \neq 2$ ***and*** $|x-1| < \varepsilon/3$ ***then, regardless of whether*** $|f(x)-1| = |x-1|$ ***or*** $|f(x)-1| = |x^2-1|$ ***we have***

$$|f(x)-1| \leq 3|x-1| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

With this fact in mind we define $\delta = \varepsilon/3$ ***and we observe that*** $|f(x)-1| < \varepsilon$ ***whenever*** x ***lies in the domain of*** f ***and*** $x \neq 1$ ***and*** $|x-1| < \delta$***.***

Note that we also have the inequality $|f(x)-1| < \varepsilon$ ***when*** $x = 1$ ***but we do not need this fact.***

Now we want to show that the function f ***has no limit at the number*** 2***. To obtain a contradiction, suppose that*** λ ***is a limit of the function*** f ***at*** 2***. The key to the desired contradiction is the fact that when*** $x > 2$ ***we have*** $f(x) = x^2$ ***which is close to*** 4 ***when*** x ***is close to*** 2***, and when*** $x < 2$ ***we have*** $f(x) = x$ ***which is close to*** 2 ***when*** x ***is close to*** 2***. We shall use this observation to argue that both of the numbers*** 2 ***and*** 4 ***must lie close to the limit value*** λ***, in spite of the fact that the distance from*** 2 ***to*** 4 ***is*** 2***.***

Using the fact that $1 > 0$ ***and that*** λ ***is a limit of*** f ***at*** 2***, we choose a number*** $\delta > 0$ ***such that the inequality***

$$|f(x)-\lambda| < 1$$

holds whenever $x \neq 2$ ***and*** $|x-2| < \delta$***. Choose a number*** $x_1 < 2$ ***and a number*** $x_2 > 2$ ***such that*** $|x_1-2| < \delta$ ***and*** $|x_2-2| < \delta$***. Then***

$$\begin{aligned} |f(x_1)-f(x_2)| &= |f(x_1)-\lambda+\lambda-f(x_2)| \\ &\leq |f(x_1)-\lambda| + |\lambda-f(x_2)| < 1+1 = 2. \end{aligned}$$

On the other hand, $f(x_1) = x_1 < 2$ ***and*** $f(x_2) = (x_2)^2 > 4$ ***which gives us***

$$|f(x_1)-f(x_2)| > 2.$$

We have therefore reached the desired contradiction.

3. Given that

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $f(x) \to 1$ as $x \to 1$ and prove that this function f has no limit at the number 2.

Solution: This solution is very similar to the solution of Exercise 2 but we have to exercise a little more care with the inequalities at the end.
Before we prove that $f(x) \to 1$ as $x \to 1$ we make the observation that whenever $|x-1| < 1$ we have

$$|x^2 - 1| = |x - 1||x + 1| < 3|x - 1|.$$

To prove that $f(x) \to 1$ as $x \to 1$, suppose that $\varepsilon > 0$. We observe that whenever $x \neq 2$ and $|x - 1| < \varepsilon/3$ then, regardless of whether $|f(x) - 1| = |x - 1|$ or $|f(x) - 1| = |x^2 - 1|$ we have

$$|f(x) - 1| \leq 3|x - 1| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

With this fact in mind we define $\delta = \varepsilon/3$ and we observe that $|f(x) - 1| < \varepsilon$ whenever x lies in the domain of f and $x \neq 1$ and $|x - 1| < \delta$**.**

Note that we also have the inequality $|f(x) - 1| < \varepsilon$ when $x = 1$ but we do not need this fact.

Now we want to show that the function f has no limit at the number 2. To obtain a contradiction, suppose that λ is a limit of the function f at 2. The key to the desired contradiction is the fact that when x is irrational we have $f(x) = x^2$ which is close to 4 when x is close to 2, and when x is rational we have $f(x) = x$ which is close to 2 when x is close to 2. We shall use this observation to argue that both of the numbers 2 and 4 must lie close to the limit value λ, in spite of the fact that the distance from 2 to 4 is 2.

Using the fact that $1 > 0$ and that λ is a limit of f at 2, we choose a number $\delta > 0$ such that the inequalities

$$|f(x) - \lambda| < \frac{1}{2}$$

$$|x^2 - 4| < \frac{1}{2}$$

$$|x - 2| < \frac{1}{2}$$

all hold whenever $x \neq 2$ and $|x - 2| < \delta$. Choose a rational number x_1 and an irrational number x_2 such that $|x_1 - 2| < \delta$ and $|x_2 - 2| < \delta$. Then

$$\begin{aligned}|f(x_1) - f(x_2)| &= |f(x_1) - \lambda + \lambda - f(x_2)| \\ &\leq |f(x_1) - \lambda| + |\lambda - f(x_2)| < \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

On the other hand, $f(x_1) = x_1$ and $f(x_2) = (x_2)^2$ and so

$$4 - 2 \leq |4 - f(x_2)| + |f(x_1) - f(x_2)| + |f(x_2) - 2| < \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

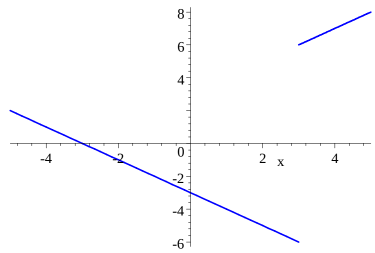
$$|f(x_1) - f(x_2)| > 2.$$

and we have therefore reached the desired contradiction.

4. Given that

$$f(x) = \frac{x^2 - 9}{|x - 3|}$$

for every number $x \neq 3$, prove that f has no limit at the number 3. Ask *Scientific Notebook* to draw the graph of this function.



The Graph $y = \frac{x^2-9}{|x-3|}$

We observe that

$$f(x) = \begin{cases} -x-3 & \text{if } x < 3 \\ x+3 & \text{if } x > 3 \end{cases}$$

To obtain a contradiction, suppose that the function f has a limit λ at 3. Choose $\delta > 0$ such that $\delta < 1$ and such that whenever $|x-3| < \delta$ and $x \neq 3$ we have

$$|f(x) - \lambda| < 1.$$

Choose numbers x_1 and x_2 such that

$$3 - \delta < x_1 < 3 < x_2 < 3 + \delta.$$

We observe that

$$|f(x_1) - f(x_2)| \leq |f(x_1) - \lambda| + |f(x_2) - \lambda| < 1 + 1 = 2.$$

On the other hand

$$f(x_1) = -x_1 - 3 < -2 - 3 = -5$$

and

$$f(x_2) = x_2 + 3 > 6$$

and so

$$|f(x_1) - f(x_2)| < 6 - (-5) > 2.$$

This is the desired contradiction.

5. Given that S is a set of real numbers, that $f : S \to \mathbf{R}$, that λ is a real number and that a is a limit point of S, prove that the following conditions are equivalent:

   a. $f(x) \to \lambda$ as $x \to a$.

   b. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality $|f(x) - \lambda| < 3\varepsilon$ holds for every number x in the set $S \setminus \{a\}$ that satisfies the inequality $|x - a| < \delta$.

   c. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality $|f(x) - \lambda| < 3\varepsilon$ holds for every number x in the set $S \setminus \{a\}$ that satisfies the inequality $|x - a| < 5\delta$.

Solution: ***We shall provide the proof that condition c implies condition a. Suppose that condition c holds. To prove that condition a holds, suppose that*** $\varepsilon > 0$***. Using the fact that the number*** $\varepsilon/3$ ***is positive we now apply condition c to choose a number*** $\delta > 0$ ***such that the inequality***

$$|f(x) - \lambda| < 3\left(\frac{\varepsilon}{3}\right)$$

holds whenever $x \in S \setminus \{a\}$ ***and*** $|x - a| < 5\delta$***. We see that*** $|f(x) - \lambda| < \varepsilon$ ***whenever*** $x \in S \setminus \{a\}$ ***and*** $|x - a| < \delta$***.***

6. Given that S is a set of real numbers, that $f : S \to \mathbf{R}$, that λ is a real number and that a is a limit point of S, prove that the following conditions are equivalent:

a. $f(x) \to \lambda$ as $x \to a$.

b. For every number $\varepsilon > 0$ there exists a neighborhood U of the number a such that the inequality $|f(x) - \lambda| < \varepsilon$ holds for every number x in the set $U \cap S \setminus \{a\}$.

c. For every neighborhood V of the number λ there exists a number $\delta > 0$ such that the condition $f(x) \in V$ holds for every number x in the set $S \setminus \{a\}$ that satisfies the inequality $|x - a| < \delta$.

To show that condition a implies condition b we assume that $f(x) \to \lambda$ as $x \to a$. Suppose that $\varepsilon > 0$. From condition a and the fact that the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ is a neighborhood of λ we deduce that there exists a neighborhood U of a such that the condition $f(x) \in (\lambda - \varepsilon, \lambda + \varepsilon)$ holds whenever $x \in U \cap S \setminus \{a\}$.

To show that condition b implies condition a we assume that condition b holds. Suppose that V is a neighborhood of λ. Choose a number $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \subseteq V$. Now, using condition b, choose a neighborhood U of a such that the condition $f(x) \in (\lambda - \varepsilon, \lambda + \varepsilon)$ holds whenever $x \in U \cap S \setminus \{a\}$. Then, whenever $x \in U \cap S \setminus \{a\}$ we have $f(x) \in V$.

To show that condition a implies condition c we assume that condition a holds. Suppose that V is a neighborhood of λ and, using condition a, choose a neighborhood U of a such that the condition $f(x) \in V$ will hold whenever $x \in U \cap S \setminus \{a\}$. Choose $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$. We observe that whenever $x \in S \setminus \{a\}$ and $|x - a| < \delta$, we must have $x \in U \cap S \setminus \{a\}$ and so $f(x) \in V$.

To show that condition c implies condition a we assume that condition c holds. Suppose that V is a neighborhood of λ. Using condition c we choose a number $\delta > 0$ such that the condition $f(x) \in V$ will hold whenever $x \in S \cap (a - \delta, a + \delta) \setminus \{a\}$. Since the interval $(a - \delta, a + \delta)$ is a neighborhood of a, condition a must hold.

7. Given that S is a set of real numbers, that $f : S \to \mathbf{R}$, that λ is a real number and that a is an interior point of S, prove that the following conditions are equivalent:

a. $f(x) \to \lambda$ as $x \to a$.

b. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality $|f(x) - \lambda| < \varepsilon$ will hold for every number x that satisfies the inequality $|x - a| < \delta$.

The only way in which condition b differs from the ε, δ form of the assertion that $f(x) \to \lambda$ as $x \to a$ is that it requires that $|f(x) - \lambda| < \varepsilon$ for **all** numbers within a distance δ of a and unequal to a. It does not merely assert that $|f(x) - \lambda| < \varepsilon$ when x is a member of the set S unequal to a and within a distance δ of a.

It is obvious that condition b implies condition a. To show that condition a implies condition b we assume that $f(x) \to \lambda$ as $x \to a$. Suppose that $\varepsilon > 0$. Choose a number $\delta_1 > 0$ such that the condition $|f(x) - \lambda| < \varepsilon$ will hold whenever $x \in S \cap (a - \delta_1, a + \delta_1) \setminus \{a\}$. Now, using the fact that a is an interior point of S, choose a number $\delta_2 > 0$ such that $(a - \delta_2, a + \delta_2) \subseteq S$. We define δ to be the smaller of the two numbers δ_1 and δ_2 and we observe that the inequality $|f(x) - \lambda| < \varepsilon$ will hold for every number x that satisfies the inequality $|x - a| < \delta$.

8. Suppose that S is a set of real numbers, that a is a limit point of S, that $f : S \to \mathbf{R}$ and that λ is a real number. Prove that if $f(x) \to \lambda$ as $x \to a$ then $|f(x)| \to |\lambda|$ as $x \to a$. Compare this exercise with an earlier exercise.

The key to this exercise is the fact that if x is any number in S then

$$||f(x)| - |\lambda|| \leq |f(x) - \lambda|.$$

To show that $|f(x)| \to |\lambda|$ as $x \to a$, suppose that $\varepsilon > 0$. Choose $\delta > 0$ such that the condition $|f(x) - \lambda| < \varepsilon$ will hold whenever $x \in S \cap (a - \delta, a + \delta) \setminus \{a\}$. Then for all such numbers x we have

$$||f(x)| - |\lambda|| \leq |f(x) - \lambda| < \varepsilon.$$

9. Suppose that S is a set of real numbers, that a is a limit point of S, that $f : S \to \mathbf{R}$ and that λ is a real number. Complete the following sentence: *The function f fails to have a limit of λ at the number a when there exists a number $\varepsilon > 0$ such that for every number $\delta > 0$ ......*

The function f fails to have a limit of λ at the number a when there exists a number $\varepsilon > 0$ such that for every number $\delta > 0$ there is at least one number x in the set $S \cap (a - \delta, a + \delta) \setminus \{a\}$ for which

$|f(x) - \lambda| \geq \varepsilon.$

Some Further Exercises on Limits

1. Given that

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 2 \\ 0 & \text{if} \quad x > 2 \end{cases}$$

prove that f has a limit from the left at 2 and also has a limit from the right at 2 but does not have a limit at 2.
The fact that f does not have a limit at 2 will be clear when we have seen that $f(x) \to 1$ as $x \to 2-$ and $f(x) \to 0$ as $x \to 2+$. Suppose that $\varepsilon > 0$. We define $\delta = 3$ (or just take δ to be any positive number you like). Whenever $x < 2$ and $|x - 2| < \delta$ we have

$$|f(x) - 1| = |1 - 1| = 0 < \varepsilon$$

and whenever $x > 2$ and $|x - 2| < \delta$ we have

$$|f(x) - 0| = |0 - 0| = 0 < \varepsilon.$$

2. Given that

$$f(x) = \frac{1}{|x-3|}$$

for all numbers $x \neq 3$, explain why f has a limit (an infinite limit) at 3.
We need to show that $f(x) \to \infty$ as $x \to 3$. Suppose that w is a real number. Given $x \neq 3$, the inequality $f(x) > w$ says that

$$\frac{1}{|x-3|} > w.$$

We can't simply turn these expressions over because w need not be positive but we can make the observation that the inequality

$$\frac{1}{|x-3|} > w$$

will certainly hold when

$$\frac{1}{|x-3|} > |w| + 1$$

which says that

$$|x - 3| < \frac{1}{|w|+1}.$$

We therefore define $\delta = \frac{1}{|w|+1}$ and observe that the condition $f(x) > w$ will hold whenever $x \neq 3$ and $|x - 3| < \delta$.

3. Given that

$$f(x) = \frac{1}{x-3}$$

for all numbers $x \neq 3$, explain why f has an infinite limit from the left at 3 and also has an infinite limit from the right at 3 but does not have a limit at 3.
The reason f has no two-sided limit at 3 is that the limits of f at 3 from the left and from the right are not equal to each other. In fact, the limit from the right is ∞ and the limit from the left is $-\infty$.
To see why the $f(x) \to \infty$ as $x \to 3+$, suppose that w is any real number. Given $x > 3$, the inequality $f(x) > w$ says that

$$\frac{1}{x-3} > w.$$

We can't simply turn these expressions over because w need not be positive but we can make the observation that the inequality

$$\frac{1}{x-3} > w$$

will certainly hold when

$$\frac{1}{x-3} > |w|+1$$

which says that

$$x-3 < \frac{1}{|w|+1}.$$

We therefore define $\delta = \frac{1}{|w|+1}$ and observe that the condition $f(x) > w$ will hold whenever $3 < x < 3+\delta$.

To see why the $f(x) \to -\infty$ as $x \to 3-$, suppose that w is any real number. Given $x < 3$, the inequality $f(x) < w$ says that

$$\frac{1}{x-3} < w$$

which we can express as

$$\frac{1}{3-x} > -w$$

We can't simply turn these expressions over because $-w$ need not be positive but we can make the observation that the inequality

$$\frac{1}{3-x} > -w$$

will certainly hold when

$$\frac{1}{x-3} > |w|+1$$

which says that

$$3-x < \frac{1}{|w|+1}.$$

We therefore define $\delta = \frac{1}{|w|+1}$ and observe that the condition $f(x) < w$ will hold whenever $3-\delta < x < 3$.

4. Prove that

$$\frac{x^3-8}{x^2+x-6} \to \infty$$

as $x \to \infty$.

We begin by observing that $x^2+x-6 > 0$ whenever $x > 2$. Given any number $x > 2$ we have

$$\frac{x^3-8}{x^2+x-6} = \frac{x^2+2x+4}{x+3} > \frac{x^2}{x} = x.$$

Now suppose that w is any real number and define v to be the larger of the two numbers 2 and w. The inequality

$$\frac{x^3-8}{x^2+x-6} > w$$

will hold whenver $x > v$.

5. Prove that

$$\frac{x^4-4x^3-x^2+x+7}{x^3-2x^2-2x-3} \to \infty$$

as $x \to \infty$.

Given any number x we have

$$x^3-2x^2-2x-3 = (x-3)(x^2+x+1)$$

and so x^3-2x^2-2x-3 will be positive whenever $x > 3$. Now given any number $x > 3$ we have

$$\frac{x^4-4x^3-x^2+x+7}{x^3-2x^2-2x-3} > \frac{x^4-4x^3-x^2}{x^3} = \frac{x^4-4x^3-x^3}{x^3} > x-\frac{5}{x} > x-\frac{5}{3}.$$

Now suppose that w is any real number. We define v to be the larger of the two numbers 3 and $w-\frac{5}{3}$ and observe that the inequality

$$\frac{x^4 - 4x^3 - x^2 + x + 7}{x^3 - 2x^2 - 2x - 3} > w$$

will hold whenever $x > v$.

6. Prove that

$$\frac{3x^2 + x - 1}{5x^2 + 4} \to \frac{3}{5}$$

as $x \to \infty$.

Given any number x we have

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| = \frac{|5x - 17|}{5(5x^2 + 4)}.$$

Whenever $x > \frac{17}{5}$ we observe that

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| = \frac{|5x - 17|}{5(5x^2 + 4)}$$

$$= \frac{5x - 17}{5(5x^2 + 4)} < \frac{5x}{25x^2} = \frac{1}{5x}.$$

Now suppose that $\varepsilon > 0$. As long as $x > \frac{17}{5}$, the inequality

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| < \varepsilon$$

will hold whenever $\frac{1}{5x} < \varepsilon$ which says that $x > \frac{1}{5\varepsilon}$. We define v to be the larger of the two numbers $\frac{17}{5}$ and $\frac{1}{5\varepsilon}$ and oberve that the inequality

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| < \varepsilon$$

holds whenever $x > v$.

7. Given that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

explain why f does not have a limit from the right at 2.

Solution: ***To obtain a contradiction, suppose that*** λ ***is a limit of the function*** f ***from the right at*** 2***. Using the fact that*** $1/2 > 0$ ***we choose a number*** $\delta > 0$ ***such that the inequality***

$$|f(x) - \lambda| < \frac{1}{2}$$

will hold for every number $x \in (2, 2 + \delta)$***. Choose a rational number*** x ***and an irrational number*** t ***in the interval*** $(x, x + \delta)$***. Then***

$$1 = |f(x) - f(t)| \le |f(x) - \lambda| + |\lambda - f(t)| < \frac{1}{2} + \frac{1}{2} = 1$$

and we have reached the desired contradiction.

8. Suppose that a is an interior point of a set S of real numbers and that $f : S \to \mathbf{R}$. Suppose that $f(x) \to 0$ as $x \to a-$ and that $f(x) \to 1$ as $x \to a+$. Prove that the function f does not have a limit at the number a.
Since x must be a limit point of each of the sets $(-\infty, a) \cap S$ and $S \cap (a, \infty)$, the desired result follows at once from Theorem 8.3.2.

Some Exercises on Continuity

1. Given that

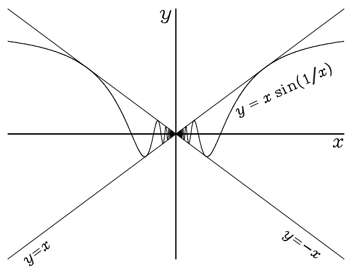
$$f(x) = \frac{x - 1}{x^2 + 3}$$

for every number x, prove that the function f is continuous at the number 2.
All we have to show is that $f(x) \to \frac{1}{7}$ as $x \to 2$. This fact can be deduced directly in the same way that we did the earlier exercises on limits or, if you prefer, it can be deduced at once from Theorem 8.5.4.

2. Given that

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

prove that the function f is continuous at the number 0. Hint: Use the fact that $|f(x)| \leq |x|$ for every number x and use the sandwich theorem. The graph of this function is illustrated in the following figure.



From the inequality

$$0 \leq |f(x) - f(0)| \leq |x|$$

that holds for every number x and from the sandwich theorem we see at once that $f(x) \to f(0)$ as $x \to 0$.

3. Given that f is the ruler function defined earlier, explain why f is continuous at every irrational number in the interval $[0,1]$ and discontinuous at every rational number in $[0,1]$.

Solution: ***Since the equation $f(x) = 0$ holds for $x \in [0,1]$ if and only if the number x is irrational, all we have to do is show that for every number $x \in [0,1]$ we have***

$$\lim_{t \to x} f(t) = 0.$$

Suppose that $x \in [0,1]$ and suppose that $\varepsilon > 0$. Choose an integer $N > 1/\varepsilon$. Since there are only finitely many rational numbers that have a reduced form m/n for which $n \leq N$ we know that there are at most finitely many numbers x in the interval $[0,1]$ for which $f(x) \geq \varepsilon$. We shall call this finite set S. Since the set $S \setminus \{x\}$, being finite, is closed, the set

$$\mathbf{R} \setminus (S \setminus \{x\})$$

must be open and is therefore a neighborhood of x. We also know that whenever

$$t \in [0,1] \cap \left(\mathbf{R} \setminus (S \setminus \{x\})\right)$$

we have

$$|f(t) - 0| = f(t) < \varepsilon$$

and so we have shown that $f(t) \to 0$ as $t \to x$.

4. Suppose that f and g are functions from a given set S of real numbers into $\mathbf{R}$ and that the inequality

$$|f(t) - f(x)| \leq |g(t) - g(x)|$$

holds for all numbers t and x in S. Prove that f must be continuous at every number at which the function g is continuous.
Suppose that the function g is continuous at a given number x. Suppose that $\varepsilon > 0$. Choose $\delta > 0$ such that the inequality $|g(t) - g(x)| < \varepsilon$ will hold whenever $t \in S \cap (x - \delta, x + \delta)$. Then for every such number t we have

$$|f(t) - f(x)| \leq |g(t) - g(x)| < \varepsilon.$$

5. Given that f is a continuous function from a set S into **R**, prove that the function $|f|$ is also continuous from S into **R**.

Hint: ***Use the preceding exercises and the fact that whenever t and x belong to S we have***

$$||f(t)| - |f(x)|| \leq |f(t) - f(x)|.$$

6. Suppose that a and b are real numbers, that $a < b$ and that

$$f : [a,b] \to \mathbf{R}.$$

Prove that the following conditions are equivalent:

a. The function f is continuous at the number a.

b. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number x in the interval $[a,b]$ that satisfies the inequality $x - a < \delta$ we have $|f(x) - f(a)| < \varepsilon$.

This exercise is obvious because of the fact that $|x - a| = x - a$ whenever $x \in [a,b]$. You can convert this exercise into one that is a shade more interesting by removing the words "in the interval" from condition b and replace them by the condition $x > a$. This change would require a slightly more careful choice of δ to ensure that it does not exceed $b - a$ so that any number $x > a$ that lies within a distance δ of a would automatically belong to the interval $[a,b]$.

7. Given that f is continuous on a closed set H and that (x_n) is a convergent sequence in the set H, prove that the sequence $(f(x_n))$ is also convergent. Prove that this assertion is false if we omit the assumption that S is closed.

Solution: ***Suppose that f is continuous on a closed set H and that (x_n) is a convergent sequence in H. We define***

$$x = \lim_{n\to\infty} x_n.$$

Since H is closed we must have $x \in H$ and therefore f is continuous at the number x. Therefore the fact that $x_n \to x$ as $n \to \infty$ guarantees that $f(x_n) \to f(x)$ as $n \to \infty$.

To see why the assertion does not remain true without the assumption that H be closed we look at the example in which

$$H = \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\}$$

and we define

$$f\left(\frac{1}{n}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Since no member of the set H is a limit point of H we know that f is continuous at every member of H. However, in spite of the fact that the sequence $(1/n)$ converges (to 0), the sequence $(f(1/n))$ fails to converge.

Actually, much more can be proved: If H is any set of real numbers and H fails to be closed then there exists a convergent sequence (x_n) in H and a continuous function f on H such that the sequence $(f(x_n))$ fails to converge. This stronger assertion is harder to prove and we omit the proof at this point. If you elect to read the optional section on the distance function of a set that follows this set of exercises then you will find a solution of this stronger assertion there.

8. Prove that if a set S has no limit points then every function $f : S \to \mathbf{R}$ is continuous on S.
This exercise follows at once from the fact stated in Theorem 8.7.5 that every function must be continuous at a number that is not a limit point of its domain.

9. Prove that if S is a set of real numbers and if no limit point of S belongs to the set S then every function

$f : S \to \mathbf{R}$ is continuous on S.
This exercise follows in exactly the same way as Exercise 8.

10. Suppose that f and g are functions from a set S to $\mathbf{R}$ and that f is continuous at a given number a at which the function g fails to be continuous.

   a. What can we say about the continuity of the function $f+g$ at the number a?
   Were the function $f+g$ to be continuous at a, it would follow from the fact that
   $$g = (f+g) - f$$
   that g is continuous at a. Therefore $f+g$ cannot be continuous at a.

   b. What can we say about the continuity of the function fg at the number a?
   If $f(a) \neq 0$ then, since the equation
   $$g(x) = \frac{f(x)g(x)}{f(x)}$$
   would hold for every number x sufficiently close to a, we could use an argument like the one we used in Part a to deduce that the function fg cannot be continuous at a.
   However, execise 2 shows us to functions, one continuous at 0 and the other discontinuous at 0 whose product is continuous at 0.

   c. What can we say about the continuity of the function fg at the number a if $f(a) = 0$ and g is a bounded function?
   Choose a number p such that $|g(x)| < p$ for each $x \in S$. Whenever $x \in S$ we have
   $$|f(x)g(x) - f(a)g(a)| = |f(x)g(x)| \leq p|f(x)|$$
   and we can now use this inequality to prove that fg must be continuous at a.
   Suppose that $\varepsilon > 0$. Using the fact that f is continuous at w and the fact that $\varepsilon/p > 0$ we choose a number $\delta > 0$ such that the condition $|f(x)| < \frac{\varepsilon}{p}$ whenever $x \in S \cap (a-\delta, a+\delta)$. Then, whenever $x \in S \cap (a-\delta, a+\delta)$ we have
   $$|f(x)g(x) - f(a)g(a)| \leq p|f(x)| < p\left(\frac{\varepsilon}{p}\right) = \varepsilon.$$

   d. What can be said about the continuity of the function fg if $f(a) \neq 0$?
   The purpose of this part of the question is to prod people who may not have done Part a.

11. Give an example of two functions f and g that are both discontinuous at a given number a such that their sum $f+g$ is continuous at a.
We define
$$f(x) = \begin{cases} -1 & \text{if } x < 2 \\ 0 & \text{if } x = 2 \\ 1 & \text{if } x > 2 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1 & \text{if } x < 2 \\ 0 & \text{if } x = 2 \\ -1 & \text{if } x > 2 \end{cases}$$

12. Given that f is a continuous function from a closed set H into $\mathbf{R}$ and that $a \in H$, prove that the set
$$E = \{x \in H \mid f(x) = f(a)\}$$
is closed. Hint: Consider the behavior of a convergent sequence in the set E.

Solution: ***Suppose that (x_n) is any convergent sequence in the set E. We shall show that the limit of this sequence, that we shall call x, must also belong to E. Since the set H is closed we know that $x \in H$ and therefore we know that f is continuous at the number x. Therefore***
$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(a) = f(a)$$
and so $x \in E$ as we promised.

13. Given that f and g are continuous functions from $\mathbf{R}$ to $\mathbf{R}$ and that
$$E = \{x \in \mathbf{R} \mid f(x) = g(x)\}$$

prove that the set E must be closed.
This exercise is very similar to Exercise 12. Suppose that (x_n) is any convergent sequence in the set E. We shall show that the limit of this sequence, that we shall call x, must also belong to E. We now observe that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

and so $x \in E$ as we promised.

14. Given that f and g are continuous functions from $\mathbf{R}$ to $\mathbf{R}$ and that $f(x) = g(x)$ for every rational number x, prove that $f = g$.
The set of numbers x for which $f(x) = g(x)$ is closed and includes the set $\mathbf{Q}$ of rational numbers. Therefore this set is all of $\mathbf{R}$.

15. Given that $f : \mathbf{Z}^+ \to \mathbf{R}$, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x) + f(t)$$

holds for all positive integers x and t, prove that $f(x) = x$ for every positive integer x.

Solution: ***If you are familiar with the method of proof by mathematical induction which is available in this book in an*** optional section ***then you should use mathematical induction to do this exercise. Otherwise we can use the method of proof by contradiction as follows: To obtain a contradiction, suppose that their are positive integers*** n ***for which the equation*** $f(n) = n$ ***fails to hold and define*** k ***to be the least of these positive integers. Since we are given that*** $f(1) = 1$***, we know that*** $k > 1$***. Therefore*** $k-1$ ***is a positive integer that is less than*** k ***and we conclude that*** $f(k-1) = k-1$***. Therefore***

$$f(k) = f((k-1)+1) = f(k-1) + f(1) = k-1+1 = k$$

which contradicts the way in which the integer k ***was chosen.***

16. Given that $f : \mathbf{Z} \to \mathbf{R}$, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x) + f(t)$$

holds for all integers x and t, prove that $f(x) = x$ for every integer x.
First we observe that, since

$$f(0) = f(0+0) = f(0) + f(0)$$

we have $f(0) = 0$. Now we know from Exercise 15 that $f(x) = x$ for every positive integer x. If x is a negative integer then, since $f(-x) = -x$ we have

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x) = f(x) - x,$$

and so $f(x) = x$.

17. Given that $f : \mathbf{Q} \to \mathbf{R}$, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x) + f(t)$$

holds for all rational numbers x and t, prove that $f(x) = x$ for every rational number x.

Solution: ***By the preceding exercise we know that the equation*** $f(x) = x$ ***holds whenever*** x ***is an integer. We shall now show that if*** x ***is any real number and*** n ***is a positive integer then*** $f(nx) = nf(x)$***. Once again, you can use the method of proof by mathematical induction if you are familiar with it but we shall use the method of proof by contradiction.***
Suppose that x ***is any real number and, to obtain a contradiction, suppose that there are positive integers*** n ***for which the equation*** $f(nx) = nf(x)$ ***fails to hold. We define*** k ***to the least of these integers. Since we know that the equation*** $f(nx) = nf(x)$ ***holds when*** $n = 1$***, we know that*** $k > 1$***. Therefore*** $k-1$ ***is a positive integer less than*** k ***and so***

$$f(kx) = f((k-1)x + x) = f((k-1)x) + f(x) = (k-1)x + x = kx$$

which contradicts the way in which k ***was chosen.***
To complete the exercise, suppose that x ***is any rational number and choose integers*** m ***and*** n ***such that*** $n > 0$ ***and*** $x = m/n$***. We see that***

$$f(x) = \frac{1}{n} nf(x) = \frac{1}{n} f(nx) = \frac{1}{n} f(m) = \frac{m}{n}.$$

18. Given that f is a continuous function from **R** to **R**, that $f(1) = 1$ and that the equation
$$f(x+t) = f(x) + f(t)$$
holds for all rational numbers x and t, prove that $f(x) = x$ for every real number x.
In view of the fact that $f(x) = x$ for every rational number x (by Exercise 17), the desired result follows at once from Exercise 13.

19. Given that f is an increasing function from **R** to **R**, that $f(1) = 1$ and that the equation
$$f(x+t) = f(x) + f(t)$$
holds for all rational numbers x and t, prove that $f(x) = x$ for every real number x.
Solution: ***Suppose that*** x ***is any real number and choose two sequences*** (a_n) ***and*** (b_n) ***of rational numbers such that***
$$a_n \le x \le b_n$$
for each n ***and***
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x.$$
Since the function f ***is increasing we know that for each*** n ***we have***
$$a_n = f(a_n) \le f(x) \le f(b_n) \le b_n$$
and so it follows from the sandwich theorem for limits that $f(x) = x$.

Some additional exercises on continuity can be found by clicking on the icon .

Further Exercises on Continuity

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$. Prove that the following conditions are equivalent:
   a. The function f is continuous on **R**.
   b. For every open set V the set $f^{-1}[V]$ is also open.
   c. For every closed set H the set $f^{-1}[H]$ is also closed.

   To prove that condition a implies condition b we assume that condition a holds. Suppose that V is an open set. Suppose that $x \in f^{-1}[V]$. We need to show that $f^{-1}[V]$ is a neighborhood of x. Using the fact that f is continuous at the number x and the fact that V is a neighborhood of $f(x)$ we choose a number $\delta > 0$ such that the condition $f(t) \in V$ holds whenever $|t - x| < \delta$. Since
$$(x - \delta, x + \delta) \subseteq f^{-1}[V]$$
we have shown, as promised, that x is an interior point of the set $f^{-1}[V]$.
To prove that condition b implies condition c we assume that condition b holds. Suppose that H is a closed set. Since $\mathbf{R} \setminus H$ is open, the set $f^{-1}[\mathbf{R} \setminus H]$ is open. Therefore, since
$$\mathbf{R} \setminus f^{-1}[H] = f^{-1}[\mathbf{R} \setminus H],$$
the set $\mathbf{R} \setminus f^{-1}[H]$ is open and we conclude that $f^{-1}[H]$ is closed.
The fact that condition c implies condition b follows in exactly the same way.
Finally we shall show that condition b implies condition a. We assume that condition b holds. Suppose that x is a real number. We want to show that f is continuous at x. Suppose that $\varepsilon > 0$. Since the set $(f(x) - \varepsilon, f(x) + \varepsilon)$ is open, so is the set $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$. Therefore the latter set is a neighborhood of thenumber x. Choose $\delta > 0$ such that
$$(x - \delta, x + \delta) \subseteq f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$$
and observe that if t is any number satisfying the inequality $|t - x| < \delta$ we have $|f(t) - f(x)| < \varepsilon$.

2. Give an example of a continuous function f on the set **R** of all real numbers and a closed set H such that the set $f[H]$ fails to be closed.

We define

$$f(x) = \frac{1}{x}$$

for every number $x \geq 1$. This function f is continuous from the closed set $[1,\infty)$ onto the set $(0,1]$ which fails to be closed.

3. Given that f is a continuous function from $\mathbf{R}$ to $\mathbf{R}$ and that S is a set of real numbers, prove that

$$f[\bar{S}] \subseteq \overline{f[S]}.$$

Solution: ***Suppose that f is continuous from $\mathbf{R}$ to $\mathbf{R}$ and that $S \subseteq \mathbf{R}$. Sinse the set $\overline{f[S]}$ is a closed set we know that the set $f^{-1}\left[\overline{f[S]}\right]$ is closed. Therefore since***

$$S \subseteq f^{-1}\left[\overline{f[S]}\right]$$

we have

$$\bar{S} \subseteq f^{-1}\left[\overline{f[S]}\right]$$

and we conclude that

$$f[\bar{S}] \subseteq \overline{f[S]}$$

Now we prove the "if" part of the exercise. Suppose that the inequality

$$f[\bar{S}] \subseteq \overline{f[S]}$$

holds for every subset S of $\mathbf{R}$. To prove that f is continuous, suppose that H is closed. We shall show that the set $f^{-1}[H]$ is closed. Now

$$f\left[\overline{f^{-1}[H]}\right] \subseteq \overline{f[f^{-1}[H]]} \subseteq \overline{H} = H$$

and therefore

$$\overline{f^{-1}[H]} \subseteq f^{-1}[H]$$

and we have shown that $f^{-1}[H]$ is closed.

4. Given that $f : \mathbf{R} \to \mathbf{R}$ and that for all numbers x and t we have

$$|f(x) - f(t)| \leq |x - t|^2,$$

prove that the function f must be constant. *Note that although this exercise is quite difficult right now, it will become considerably easier after we have studied the concept of a derivative.*
I am resisting the urge to write a direct solution of this exercise. Those students who wish to attempt it now will probably want to be left alone. All others can wait until after the mean value theorem when the fact that f is constant will follow at once from the obvious fact that $f'(x) = 0$ for every number x.

Exercises on the Distance Function

1. Two sets A and B of real numbers are said to be **separated** from each other if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

Prove that if two sets A and B are separated from each other then

$$\rho_A(x) - \rho_B(x) < 0$$

whenever $x \in A$ and

$$\rho_A(x) - \rho_B(x) > 0$$

whenever $x \in B$.

2. Prove that if two sets A and B are separated from each other then there exist two open sets U and V that are disjoint from each other such that $A \subseteq U$ and $B \subseteq V$.

Solution: ***We define***

$$U = \{x \in \mathbf{R} \mid \rho_A(x) - \rho_B(x) < 0\}$$

and

$$V = \{x \in \mathbf{R} \mid \rho_A(x) - \rho_B(x) > 0\}.$$

Since the function $\rho_A - \rho_B$ ***is continuous on*** R ***we deduce from the first of*** some earlier exercises ***that the sets*** U ***and*** V ***are open.***

3. Given two sets A and B of real numbers, prove that the following conditions are equivalent:

   a. We have

   $$\overline{A} \cap \overline{B} = \emptyset.$$

   b. There exists a continuous function $f : \mathbf{R} \to [0,1]$ such that

   $$f(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ 1 & \text{if} \quad x \in B \end{cases}.$$

   We deduce at once from Urysohn's lemma that condition a implies condition b. On the other hand, if f is a continuous function that takes the value 0 at every number $x \in A$ and takes the value 1 at every number $x \in B$ then

   $$\overline{A} \subseteq \left\{x \mid f(x) \leq \frac{1}{3}\right\}$$

   and

   $$\overline{B} \subseteq \left\{x \mid f(x) \geq \frac{2}{3}\right\}.$$

4. Suppose that A, B and C are closed sets of real numbers and no two of these three sets intersect. Prove that there exists a continuous function f on $\mathbf{R}$ such that

   $$f(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 2 & \text{if} \quad x \in B \\ 3 & \text{if} \quad x \in C \end{cases}.$$

   Using Urysohn's lemma we choose two continuous functions g_1 and g_2 from $\mathbf{R}$ into $[0,1]$ such that $g_1(x) = 0$ whenever $x \in A$ and $g_1(x) = 1$ whenever $x \in B \cup C$ and $g_2(x) = 0$ whenever $x \in A \cup B$ and $g_2(x) = 1$ whenever $x \in C$. We define $g_3(x) = 1$ for every number x. We now define

   $$f = g_1 + g_2 + g_3.$$

5. Suppose that S is a set of real numbers and that S fails to be closed. Prove that there exists a convergent sequence (x_n) in S and a continuous function f on the set S such that the sequence $(f(x_n))$ fails to converge.

   Solution: ***We begin by choosing a number*** $w \in \overline{S} \setminus S$ ***and we choose a sequence*** (x_n) ***in*** S ***that converges to the number*** w***. We define*** E ***to be the range of the sequence*** (x_n) ***and, Using the fact that that the set*** E ***must be infinite, we choose an infinite subset*** A ***of*** E ***such that the set*** $B = E \setminus A$ ***is also infinite. Since no member of*** S ***can lie in the closures of both*** A ***and*** B***, the function***

   $$f = \frac{\rho_A}{\rho_A + \rho_B}$$

   is continuous on the set S***. Furthermore, since there must be infinitely many integers*** n ***for which*** $x_n \in A$ ***and infinitely many integers*** n ***for which*** $x_n \in B$ ***the sequence*** $f((x_n))$ ***has no limit.***

Some Exercises on Continuous Functions on Closed Bounded Sets

1. Give an example of a function f that is continuous on a closed set H such that the range $f[H]$ of the function f

fails to be closed.
We can define $f(x) = \frac{1}{x}$ for $x \geq 1$. This function f is continuous on the closed set $[1, \infty)$ and its range is $(0, 1]$ which is not closed.

2. Give an example of a function f that is continuous on a closed set H such that the range $f[H]$ of the function f fails to be bounded.
We can define $f(x) = x$ for every $x \in \mathbf{R}$.

3. Give an example of a function f that is continuous on a bounded set H such that the range $f[H]$ of the function f fails to be closed.
We can define $f(x) = x$ for $0 < x < 1$.

4. Give an example of a function f that is continuous on a bounded set H such that the range $f[H]$ of the function f fails to be bounded.
We can define $f(x) = \frac{1}{x}$ for $0 < x < 1$.

5. Prove that if a set H of real numbers is unbounded above and $f(x) = x$ for every number x in H, then f is a continuous function on H and f fails to have a maximum.
Since the range of f is the set H which is assumed to be unbounded, the function f must be unbounded above.

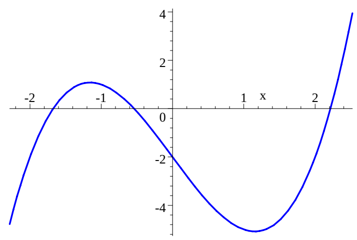
6. Prove that if H is a set of real numbers and a number a is close to H but does not belong to H, and if we define
$$f(x) = \frac{1}{|x - a|}$$
for every $x \in H$ then f is a continuous function on H but f has no maximum.
Solution: ***We can see that f has no maximum by showing that f is unbounded above. Given any positive number q the inequality***
$$f(x) > q$$
says that
$$\frac{1}{|x - a|} > q$$
which holds when
$$|x - a| < \frac{1}{q}$$
But since a is close to the set H we know that there do indeed exist members x of H for which the inequality
$$|x - a| < \frac{1}{q}$$
holds. Therefore there are members x in H for which $f(x) > q$ and we have shown that f fails to be bounded above.

Exercises on Continuity of Functions on Intervals

1. Given that S is a set of positive numbers and that $f(x) = \sqrt{x}$ for all $x \in S$, prove that f is a one-one continuous function on S. Prove that S is an interval if and only if the set $f[S]$ is an interval.

Hint: ***Once you have shown that f is strictly increasing and continuous on S, the fact that S is an interval if and only if $f[S]$ is an interval will follow from the Bolzano intermediate value*** *theorem* ***and*** *this theorem****.***

2. Prove that there are three real numbers x satisfying the equation
$$x^3 - 4x - 2 = 0.$$
Solution: ***First look at a sketch of the graph*** $y = x^3 - 4x - 2$



We now define $f(x) = x^3 - 4x - 2$ ***for every real number*** x ***and observe that*** $f(-2) < 0$ ***and*** $f(-1) > 0$***. Therefore Bolzano's intermediate theorem guarantees that the equation*** $x^3 - 4x - 2 = 0$ ***has a solution between*** -2 ***and*** -1***. Since*** $f(-1) > 0$ ***and*** $f(0) < 0$ ***we know that there is a solution of the equation between*** -1 ***and*** 0***. Finally, from the fact that*** $f(0) < 0$ ***and*** $f(3) > 0$ ***we know that there is a solution of the equation between*** 0 ***and*** 3***.***

3. Is it true that if a set S of real numbers is not an interval then there must exist a one-one continuous function on S whose inverse function fails to be continuous?
   Not at all. In fact, we know that if S is closed and bounded then every one-one continuous function S must have a continuous inverse function. Look also at the case in which S is the union of two open intervals that do not intersect with each other. If f is a one-one continuous function on S then the range of f is also the union of two open intervals that do not intersect with each other and the inverse function of f will be continuous.

4. Is it true that if a set S of real numbers is not an interval and is not closed then there must exist a one-one continuous function on S whose inverse function fails to be continuous?

   Hint: ***Look at the case in which*** S ***is the union of two mutually disjoint open intervals.***
   See the remarks about Exercise 3.

5. Is it true that if a set S of real numbers is not an interval and is not bounded then there must exist a one-one continuous function on S whose inverse function fails to be continuous?
   Again the answer is no. The remarks about Exercise 3 are not confined to bounded intervals.

6. Prove that if f is a continuous function from the interval $[0,1]$ into $[0,1]$ then there must be at least one number $x \in [0,1]$ such that $f(x) = x$. This assertion is the one-dimensional form of the **Brouwer fixed point theorem.**

   Solution: ***For every number*** $x \in [0,1]$ ***we define***
   $$g(x) = f(x) - x.$$
   The function g ***defined in this way is continuous on the interval*** $[0,1]$***. We see that***
   $$g(0) = f(0) - 0 \geq 0$$
   and
   $$g(1) = f(1) - 1 \leq 0$$
   and we conclude from the Bolzano intermediate value theorem that there is at least one number $x \in [0,1]$ ***for which*** $g(x) = 0$***.***

Exercises on Uniform Continuity

1. Is it true that if S is an unbounded set of real numbers and $f(x) = x^2$ for every number $x \in S$ then the function f fails to be uniformly continuous?
   The assertion given here is false. Every function defined on the set $\mathbf{Z}$ of integers must be uniformly continuous.

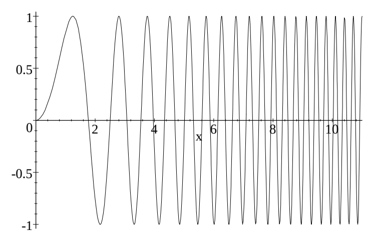
2. Given that

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 2 \\ 0 & \text{if} \quad 2 < x \le 3 \end{cases}$$

prove that f is continuous but not uniformly continuous on the set $[0,2) \cup (2,3]$.
Since the number 2 does not belong to the domain of f, the function f is constant in a neighborhood of every number in its domain. Therefore f is continuous on the set $[0,2) \cup (2,3]$. To see why f fails to be uniformly continuous we observe that given any positive number δ we can find a number $t \in [0,2)$ and a number $x \in (2,3]$ such that $|t - x| < \delta$, and for any such choice of numbers x and t we must have

$$|f(t) - f(x)| = |1 - 0| = 1.$$

3. Given that $f(x) = \sin(x^2)$ for all real numbers x, prove that f is not uniformly continuous on the set $\mathbf{R}$.



We define

$$x_n = \sqrt{2n\pi + \frac{\pi}{2}}$$

and

$$t_n = \sqrt{2n\pi}$$

for every positive integer n. Since $f(x_n) = 1$ and $f(t_n) = 0$ for every n we know that $f(x_n) - f(t_n)$ does not approach 0 as $n \to \infty$. Now

$$\begin{aligned} x_n - t_n &= \sqrt{2n\pi + \frac{\pi}{2}} - \sqrt{2n\pi} \\ &= \frac{\left(\sqrt{2n\pi + \frac{\pi}{2}} - \sqrt{2n\pi}\right)\left(\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}\right)}{\left(\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}\right)} \\ &= \frac{\frac{\pi}{2}}{\left(\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}\right)} \to 0 \end{aligned}$$

as $n \to \infty$ and so it follows from the relationship between limits of sequences and uniform continuity that the function f fails to be uniformly continuous.

4. Ask *Scientific Notebook* to make some 2D plots of the function f defined by the equation

$$f(x) = \sin(x \log x)$$

for $x > 0$. Plot the function on each of the intervals $[0,50]$, $[50,100]$, $[100,150]$ and $[150,200]$. Revise your plot and increase its sample size if it appears to contain errors. Why do these graphs suggest that f fails to be unformly continuous on the interval $(0,\infty)$? Prove that this function does, indeed, fail to be uniformly continuous.

Solution: ***To prove that f fails to be uniformly continuous we shall show that for every number $\delta > 0$ there exist two positive numbers a and b such that $|a - b| < \delta$ and $|f(a) - f(b)| \ge 1$. We begin by choosing a number p such that whenever $x \ge p$ we have***

$$\log x > \frac{\pi}{2\delta}.$$

Now choose a positive integer n such that

$$p\log p < n\pi.$$

Since $x\log x > n\pi$ for x sufficiently large x, we can use the Bolzano intermediate value theorem to choose a number $a > p$ such that $a\log a = n\pi$. Now since

$$\begin{aligned}(a+\delta)\log(a+\delta) &= a\log(a+\delta) + \delta\log(a+\delta)\\ &> a\log a + \delta\left(\frac{\pi}{2\delta}\right) = n\pi + \frac{\pi}{2}\end{aligned}$$

we can use the Bolzano intermediate value theorem again to choose a number $b \in (a, a+\delta)$ such that

$$b\log b = n\pi + \frac{\pi}{2}.$$

We now observe that

$$|f(a) - f(b)| = \left|\sin n\pi - \sin\left(n\pi + \frac{\pi}{2}\right)\right| = 1$$

and so the proof is complete.

5. a. A function f is said to be **Lipschitzian** on a set S if there exists a number k such that the inequality

$$|f(t) - f(x)| \leq k|t - x|$$

holds for all numbers t and x in S. Prove that every Lipschitzian function is uniformly continuous.
Suppose that f is a function defined on a set S, that k is a positive number and that the inequality

$$|f(t) - f(x)| \leq k|t - x|$$

holds for all numbers t and x in the set S. Suppose that $\varepsilon > 0$. We define $\delta = \varepsilon/k$ and observe that, whenever t and x belong to S and $|t - x| < \delta$ we have

$$|f(t) - f(x)| \leq k|t - x| < k\left(\frac{\varepsilon}{k}\right) = \varepsilon.$$

b. Given that $f(x) = \sqrt{x}$ for all $x \in [0,1]$ prove that f is uniformly continuous but not lipschitzian on $[0,1]$.

Solution: ***The fact that f is uniformly continuous on the closed bounded set $[0,1]$ follows at once from the fact that f is continuous there. Now, to prove that f fails to be Lipschitzian, suppose that k is any positive number. Given $x \in (0,1]$ we see that***

$$\frac{|f(x) - f(0)|}{|x - 0|} = \frac{1}{\sqrt{x}}$$

and this exceeds k whenever $x < 1/k^2$.

6. a. Suppose that f is uniformly continuous on a set S, that (x_n) is a sequence in the set S and that (x_n) has a partial limit $x \in \mathbf{R}$. Prove that it is impossible to have $f(x_n) \to \infty$ as $n \to \infty$.
Using the fact that f is uniformly continuous on S we choose a number $\delta > 0$ such that the inequality

$$|f(t) - f(s)| < 1$$

holds whenever t and s belong to S and $|t - s| < \delta$. Since x is a partial limit of the sequence (x_n) we know that the condition

$$x_n \in \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$$

holds for infinitely many positive integers n. Choose a positive integer N such that

$$x_N \in \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right).$$

For every one of the infinitely many positive integers n for which the condition

$$x_n \in \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$$

holds, since

$$|x_n - x_N| \le |x_n - x| + |x - x_N| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

we have

$$f(x_n) \le 1 + f(x_N).$$

Therefore the sequence of numbers $f(x_n)$ is frequently in the interval $(-\infty, 1 + f(x_N))$ and so it cannot approach ∞.
Of course, this sequence cannot approach $-\infty$ either.

b. Did you assume that $x \in S$ in Part a? If you did, go back and do the exercise again. You have no information that $x \in S$. If you didn't assume $x \in S$, you can sit this question out.

c. Suppose that f is uniformly continuous on a bounded set S and that (x_n) is a sequence in S. Prove that it is impossible to have $f(x_n) \to \infty$ as $n \to \infty$.
The assertion follows at once from Part a and the fact that every bounded sequence of numbers has a partial limit in $\mathbf{R}$.

d. Prove that if f is uniformly continuous on a bounded set S then the function f is bounded.

Solution: ***To obtain a contradiction, assume that f is unbounded above. Choose a sequence (x_n) in S such that $f(x_n) > n$ for each n. Now use part c of this exercise.***
Well, of course, such a choice of (x_n) would make $f(x_n) \to \infty$ which we know now to be impossible.

7. a. Given that S is a set of real numbers, that $a \in \bar{S} \setminus S$ and that

$$f(x) = \frac{1}{x - a}$$

for all $x \in S$, prove that f is continuous on S but not uniformly continuous.

Hint: ***Use the preceding exercise. Choose a sequence (x_n) in $S \setminus \{a\}$ that converges to x. Note that $f(x_n) \to \infty$ as $n \to \infty$.***

b. Given that S is a set of real numbers and that S fails to be closed, prove that there exists a continuous function on S that fails to be uniformly continuous on S.
Choose a number $a \in \bar{S} \setminus S$ and use Part a.

c. Is it true that if S is an unbounded set of real numbers then there exists a continuous function on S that fails to be uniformly continuous on S?
No, as we remarked after Exercise 1, every function on the set $\mathbf{Z}$ of integers must be uniformly continuous.

8. Is it true that the composition of a uniformly continuous function with a uniformly continuous function is uniformly continuous?
Yes, this assertion is true. Suppose that f is a uniformly continuous function on a set S and that g is a uniformly continuous function on a set T that includes the range $f[S]$ of f. To show that the function $g \circ f$ is uniformly continuous on S, suppose that $\varepsilon > 0$.
Using the uniform continuity of g on the set T we choose a number $\delta > 0$ such that the inequality $|g(u - g(v))| < \varepsilon$ holds whenever u and v belong to T and $|u - v| < \delta$. Now, using the uniform continuity of f on the set S we choose a positive number γ such that the inequality $|f(s) - f(t)| < \delta$ will hold whenever s and t belong to S and $|s - t| < \gamma$. Then whenever s and t belong to S and $|s - t| < \gamma$ we have $|g(f(s)) - g(f(t))| < \varepsilon$.

9. a. Suppose that f is uniformly continuous on a set S and that (x_n) is a convergent sequence in S. Prove that the sequence $(f(x_n))$ cannot have more than one partial limit.
We know from the result proved in Exercise 6 d that the sequence $(f(x_n))$ is bounded. We choose a partial limit y of $(f(x_n))$ and we want to prove that y is the only partial limit of $(f(x_n))$.
Suppose that z is any number unequal to y. We define

$$\varepsilon = \frac{1}{3}|y - z|.$$

Choose a number $\delta > 0$ such that the inequality

$$|f(s) - f(t)| < \varepsilon$$

holds whenever s and t belong to S and $|s - t| < \delta/2$.
We write the limit of (x_n) as x and choose an integer N such that the inequality $|x_n - x| < \frac{\delta}{2}$ holds whenever $n \geq N$. Using the fact that y is a partial limit of the sequence $(f(x_n))$ we choose an integer $m \geq N$ such that $|f(x_m) - y| < \varepsilon$. Now given any integer $n \geq N$, since

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

we must have

$$|f(x_n) - f(x_m)| < \varepsilon$$

and consequently

$$|y - f(x_n)| \leq |y - f(x_m)| + |f(x_n) - f(x_m)| < 2\varepsilon$$

$y \qquad f(x_m) \qquad f(x_n) \qquad\qquad\qquad\qquad z$

Thus if $n \geq N$, the number $f(x_n)$ cannot lie in the neighborhood $(z - \varepsilon, z + \varepsilon)$ of z and we conclude that z fails to be a partial limit of the sequence $(f(x_n))$.

b. In Part a, did you assume that the limit of the sequence (x_n) belongs to S? If so, go back and do the problem again.

c. Prove that if f is uniformly continuous on a set S and (x_n) is a convergent sequence in S then the sequence $(f(x_n))$ is also convergent. Do *not* assume that the limit of (x_n) belongs to S.
In view of Part a, the present result follows at once from an earlier theorem on limits of sequences.

d. Suppose that f is uniformly continuous on a set S, that x is a real number and that (x_n) and (t_n) are sequences in S that converge to the number x. Prove that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(t_n).$$

The existence of these limits was guaranteed in Part c. Now since $t_n - x_n \to 0$ as $n \to \infty$ we deduce from the relationship between uniform continuity and limits of sequences that $f(t_n) - f(x_n) \to 0$ as $n \to \infty$.

e. Suppose that f is uniformly continuous on a set S and that $x \in \bar{S} \setminus S$. Explain how we can use Part d to extend the definition of the function f to the number x in such a way that f is continuous on the set $S \cup \{x\}$.
We know that there exists a sequence (x_n) in S that converges to x and we know that there is a single limit for all of the sequences $(f(x_n))$ that can be made in this way. We define $f(x)$ to be this common limit value. This extension of the function f to the set $S \cup \{x\}$ is uniformly continuous. The proof will be given in the more extended case that we consider below in Part f.

f. Prove that if f is uniformly continuous on a set S then it is possible to extend f to the closure $\bar{S}$ of S in such a way that f is uniformly continuous on $\bar{S}$.
For every number $x \in \bar{S} \setminus S$ we define $f(x)$ by the method described in Part e. To show that the extension of f to $\bar{S}$ is uniformly continuous, suppose that $\varepsilon > 0$. Using the uniform continuity of f on S we choose $\delta > 0$ such that the inequality

$$|f(t) - f(x)| < \frac{\varepsilon}{2}$$

holds whenever t and x belong to S and $|t - x| < \delta$. We shall now observe that whenever t and x belong to $\bar{S}$ and $|t - x| < \delta$ we must have $|f(t) - f(x)| < \varepsilon$. To make this observation, suppose that t and x belong to $\bar{S}$ and that $|t - x| < \delta$.
choose a sequence (t_n) in S that converges to t and a sequence (x_n) in S that converges to x. Since

$$|t - x| = \lim_{n\to\infty} |t_n - x_n|$$

we know that the inequality $|t_n - x_n| < \delta$ must hold for all n sufficiently large and therefore, since

$$|f(t) - f(x)| = \lim_{n\to\infty} |f(t_n) - f(x_n)|$$

and since

$$|f(t_n) - f(x_n)| < \frac{\varepsilon}{2}$$

for all n sufficiently large we must have

$$|f(t) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

10. Suppose that f is a continuous function on a bounded set S. rove that the following two conditions are equivalent:

    a. The function f is uniformly continuous on S.

    b. It is possible to extend f to a continuous function on the set $\bar{S}$.

    The fact that condition a implies condition b follows from Exercise 9. On the other hand, if f has a continuous extension to the set $\bar{S}$ then, this extension, being a continuous function on a closed bounded set, must be uniformly continuous; and so f must be uniformly continuous on S.

11. Given that f is a function defined on a set S of real numbers, prove that the following conditions are equivalent:

    a. The function f fails to be uniformly continuous on the set S.

    b. There exists a number $\varepsilon > 0$ and there exist two sequences (t_n) and (x_n) in S such that $t_n - x_n \to 0$ as $n \to \infty$ and

    $$|f(x_n) - f(t_n)| \geq \varepsilon$$

    for every n.

    At the suggestion of my good friend Sean Ellermeyer this exercise was upgraded to a theorem. I have left in the exercise. Sometimes I find it interesting to see which of my students recognize that an item is the same as one they have already seen.

Alt. 8 Limits and Continuity in Metric Spaces

Some Exercises on Limits of Functions

1. Write careful proofs of each of the following assertions:

    a. $x^3 - 3x \to 2$ as $x \to -1$.

    We observe that if x is any given number we have

    $$\begin{aligned} |(x^3 - 3x) - 2| &= |x^3 - 3x - 2| \\ &= |(x+1)(x^2 - x - 2)| \end{aligned}$$

    The idea of the proof is to make the observation that the factor $x + 1$ will be small when x is close to -1 and that the factor $x^2 - x - 2$ is not too large. In fact, if $|x + 1| < 1$ then, since $-2 < x < 0$ we have

    $$|x^2 - x - 2| \leq 2^2 + 2 + 2 = 8$$

    which gives us

$$|(x^3-3x)-2| = |(x+1)(x^2-x-2)| \le 8|x+1|.$$

Now suppose that $\varepsilon > 0$. We define ε to be the smaller of the two numbers 1 and $\varepsilon/8$. Then, whenever $|x+1| < \delta$ we have

$$|(x^3-3x)-2| = |(x+1)(x^2-x-2)| \le 8|x+1| < 8\left(\frac{\varepsilon}{8}\right) = \varepsilon.$$

We conclude that the inequality

$$|(x^3-3x)-2| < \varepsilon$$

holds whenever $x \ne -1$ and $|x+1| < \delta$ (and, as a matter of fact, the inequality is also true when $x = -1$).

b. $\frac{1}{x} \to \frac{1}{3}$ as $x \to 3$.

We begin with the observation that if x is any nonzero number we have

$$\left|\frac{1}{x}-\frac{1}{3}\right| = \frac{|3-x|}{3|x|}.$$

To keep the denominator of this fraction from becoming too small we need to keep x away from 0. In fact, if $|x-3| < 1$ then, since $2 < x < 4$ we have

$$\left|\frac{1}{x}-\frac{1}{3}\right| = \frac{|3-x|}{3|x|} < \frac{|3-x|}{6}.$$

Now suppose that $\varepsilon > 0$. We define δ to be the smaller of the two numbers 1 and ε and we observe that whenever $|x-3| < \delta$ we have

$$\left|\frac{1}{x}-\frac{1}{3}\right| = \frac{|3-x|}{3|x|} < \frac{|3-x|}{6} < \frac{\varepsilon}{6} < \varepsilon.$$

We conclude that the inequality

$$\left|\frac{1}{x}-\frac{1}{3}\right| < \varepsilon$$

holds whenever $x \ne 3$ and $|x-3| < \delta$ (and, as a matter of fact, the inequality is also true when $x = 3$).

c. $\frac{x^3-8}{x^2+x-6} \to \frac{12}{5}$ as $x \to 2$.

We begin by observing that if x is any number for which $x^2+x-6 \ne 0$ then we have

$$\begin{aligned}\left|\frac{x^3-8}{x^2+x-6}-\frac{12}{5}\right| &= \left|\frac{(x-2)(x^2+2x+4)}{(x-2)(x+3)}-\frac{12}{5}\right| \\ &= \left|\frac{x^2+2x+4}{x+3}-\frac{12}{5}\right| \\ &= \left|\frac{(x-2)(5x+8)}{5(x+3)}\right|\end{aligned}$$

In order to keep the denominator of this fraction from becoming too small we need to keep x away from -3 and, in order to keep $|5x+8|$ from becoming too large we need to keep x from being too large. In fact, if $x \ne 2$ and $|x-2| < 1$ we have

$$\left|\frac{x^3-8}{x^2+x-6}-\frac{12}{5}\right| = \left|\frac{(x-2)(5x+8)}{5(x+3)}\right| < \left|\frac{(x-2)(5(3)+8)}{5(3+3)}\right| < |x-2|.$$

Now suppose that $\varepsilon > 0$ and define δ to be the smaller of the two numbers 1 and ε. We see that whenever $x \ne 2$ and $|x-2| < \delta$ we have

$$\left|\frac{x^3-8}{x^2+x-6}-\frac{12}{5}\right| < |x-2| < \varepsilon.$$

2. Given that

$$f(x) = \begin{cases} x & \text{if } 0 < x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

prove that $f(x) \to 1$ as $x \to 1$ and that this function f has no limit at the number 2.

Solution: ***Before we prove that $f(x) \to 1$ as $x \to 1$ we make the observation that whenever $|x-1| < 1$ we have***

$$|x^2 - 1| = |x-1||x+1| < 3|x-1|.$$

To prove that $f(x) \to 1$ as $x \to 1$, suppose that $\varepsilon > 0$. We observe that whenever $x \neq 2$ and $|x-1| < \varepsilon/3$ then, regardless of whether $|f(x)-1| = |x-1|$ or $|f(x)-1| = |x^2-1|$ we have

$$|f(x) - 1| \leq 3|x-1| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

With this fact in mind we define $\delta = \varepsilon/3$ and we observe that $|f(x)-1| < \varepsilon$ whenever x lies in the domain of f and $x \neq 1$ and $|x-1| < \delta$.

Note that we also have the inequality $|f(x)-1| < \varepsilon$ when $x = 1$ but we do not need this fact.

Now we want to show that the function f has no limit at the number 2. To obtain a contradiction, suppose that λ is a limit of the function f at 2. The key to the desired contradiction is the fact that when $x > 2$ we have $f(x) = x^2$ which is close to 4 when x is close to 2 and when $x < 2$ we have $f(x) = x$ which is close to 2 when x is close to 2. We shall use this observation to argue that both of the numbers 2 and 4 must lie close to the limit value λ, in spite of the fact that the distance from 2 to 4 is 2.

Using the fact that $1 > 0$ and that λ is a limit of f at 2, we choose a number $\delta > 0$ such that the inequality

$$|f(x) - \lambda| < 1$$

holds whenever $x \neq 2$ and $|x-2| < \delta$. Choose a number $x_1 < 2$ and a number $x_2 > 2$ such that $|x_1 - 2| < \delta$ and $|x_2 - 2| < \delta$. Then

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - \lambda + \lambda - f(x_2)| \\ &\leq |f(x_1) - \lambda| + |\lambda - f(x_2)| < 1 + 1 = 2. \end{aligned}$$

On the other hand, $f(x_1) = x_1 < 2$ and $f(x_2) = (x_2)^2 > 4$ which gives us

$$|f(x_1) - f(x_2)| > 2.$$

We have therefore reached the desired contradiction.

3. Given that

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $f(x) \to 1$ as $x \to 1$ and prove that this function f has no limit at the number 2.

Solution: This solution is very similar to the solution of Exercise 2 but we have to exercise a little more care with the inequalities at the end.
Before we prove that $f(x) \to 1$ as $x \to 1$ we make the observation that whenever $|x-1| < 1$ we have

$$|x^2 - 1| = |x-1||x+1| < 3|x-1|.$$

To prove that $f(x) \to 1$ as $x \to 1$, suppose that $\varepsilon > 0$. We observe that whenever $x \neq 2$ and $|x-1| < \varepsilon/3$ then, regardless of whether $|f(x)-1| = |x-1|$ or $|f(x)-1| = |x^2-1|$ we have

$$|f(x) - 1| \leq 3|x-1| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

With this fact in mind we define $\delta = \varepsilon/3$ and we observe that $|f(x)-1| < \varepsilon$ whenever x lies in the domain of f and $x \neq 1$ and $|x-1| < \delta$.

Note that we also have the inequality $|f(x)-1| < \varepsilon$ when $x = 1$ but we do not need this fact.

Now we want to show that the function f has no limit at the number 2. To obtain a contradiction, suppose that λ is a limit of the function f at 2. The key to the desired contradiction is the fact that when x is irrational we have $f(x) = x^2$ which is close to 4 when x is close to 2, and when x is rational we have $f(x) = x$ which is close to 2 when x is close to 2. We shall use this observation to argue that both of the numbers 2 and 4 must lie close to the limit value λ, in spite of the fact that the distance from 2 to 4 is 2.

Using the fact that $1 > 0$ and that λ is a limit of f at 2, we choose a number $\delta > 0$ such that the inequalities

$$|f(x) - \lambda| < \frac{1}{2}$$
$$|x^2 - 4| < \frac{1}{2}$$
$$|x - 2| < \frac{1}{2}$$

all hold whenever $x \neq 2$ and $|x - 2| < \delta$. Choose a rational number x_1 and an irrational number x_2 such that $|x_1 - 2| < \delta$ and $|x_2 - 2| < \delta$. Then

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - \lambda + \lambda - f(x_2)| \\ &\leq |f(x_1) - \lambda| + |\lambda - f(x_2)| < \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

On the other hand, $f(x_1) = x_1$ and $f(x_2) = (x_2)^2$ and so

$$4 - 2 \leq |4 - f(x_2)| + |f(x_1) - f(x_2)| + |f(x_2) - 2| < \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

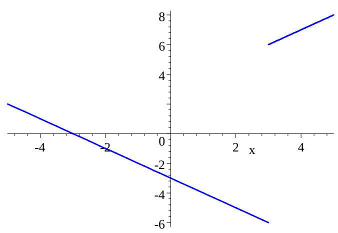
$$|f(x_1) - f(x_2)| > 2.$$

and we have therefore reached the desired contradiction.

4. Given that

$$f(x) = \frac{x^2 - 9}{|x - 3|}$$

for every number $x \neq 3$, prove that f has no limit at the number 3.



The Graph $y = \frac{x^2-9}{|x-3|}$

We observe that

$$f(x) = \begin{cases} -x - 3 & \text{if } x < 3 \\ x + 3 & \text{if } x > 3 \end{cases}$$

To obtain a contradiction, suppose that the function f has a limit λ at 3. Choose $\delta > 0$ such that $\delta < 1$ and such that whenever $|x - 3| < \delta$ and $x \neq 3$ we have

$$|f(x) - \lambda| < 1.$$

Choose numbers x_1 and x_2 such that

$$3 - \delta < x_1 < 3 < x_2 < 3 + \delta.$$

We observe that

$$|f(x_1) - f(x_2)| \leq |f(x_1) - \lambda| + |f(x_2) - \lambda| < 1 + 1 = 2.$$

On the other hand

$$f(x_1) = -x_1 - 3 < -2 - 3 = -5$$

and

$$f(x_2) = x_2 + 3 > 6$$

and so

$$|f(x_1) - f(x_2)| < 6 - (-5) > 2.$$

This is the desired contradiction.

5. Given that S is a set of real numbers, that $f : S \to \mathbf{R}$, that α is a real number and that a is a limit point of S, prove that the following conditions are equivalent:

   a. $f(x) \to \alpha$ as $x \to a$.

   b. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality $|f(x) - \alpha| < 3\varepsilon$ holds for every number x in the set $S \setminus \{a\}$ that satisfies the inequality $|x - a| < \delta$.

   c. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality $|f(x) - \alpha| < 3\varepsilon$ holds for every number x in the set $S \setminus \{a\}$ that satisfies the inequality $|x - a| < 5\delta$.

Solution: ***We shall provide the proof that condition c implies condition a. Suppose that condition c holds. To prove that condition a holds, suppose that*** $\varepsilon > 0$***. Using the fact that the number*** $\varepsilon/3$ ***is positive we now apply condition c to choose a number*** $\delta > 0$ ***such that the inequality***

$$|f(x) - \lambda| < 3\left(\frac{\varepsilon}{3}\right)$$

holds whenever $x \in S \setminus \{a\}$ ***and*** $|x - a| < 5\delta$***. We see that*** $|f(x) - \lambda| < \varepsilon$ ***whenever*** $x \in S \setminus \{a\}$ ***and*** $|x - a| < \delta$***.***

6. Given that S is a set of real numbers, that $f : S \to \mathbf{R}$, that α is a real number and that a is a limit point of S, prove that the following conditions are equivalent:

   a. $f(x) \to \alpha$ as $x \to a$.

   b. For every number $\varepsilon > 0$ there exists a neighborhood U of the number a such that the inequality $|f(x) - \alpha| < \varepsilon$ holds for every number x in the set $U \cap S \setminus \{a\}$.

   c. For every neighborhood V of the number α there exists a number $\delta > 0$ such that the condition $f(x) \in V$ holds for every number x in the set $S \setminus \{a\}$ that satisfies the inequality $|x - a| < \delta$.

To show that condition a implies condition b we assume that $f(x) \to \lambda$ as $x \to a$. Suppose that $\varepsilon > 0$. From condition a and the fact that the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ is a neighborhood of λ we deduce that there exists a neighborhood U of a such that the condition $f(x) \in (\lambda - \varepsilon, \lambda + \varepsilon)$ holds whenever $x \in U \cap S \setminus \{a\}$.

To show that condition b implies condition a we assume that condition b holds. Suppose that V is a neighborhood of λ. Choose a number $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \subseteq V$. Now, using condition b, choose a neighborhood U of a such that the condition $f(x) \in (\lambda - \varepsilon, \lambda + \varepsilon)$ holds whenever $x \in U \cap S \setminus \{a\}$. Then, whenever $x \in U \cap S \setminus \{a\}$ we have $f(x) \in V$.

To show that condition a implies condition c we assume that condition a holds. Suppose that V is a neighborhood of λ and, using condition a, choose a neighborhood U of a such that the condition $f(x) \in V$ will hold whenever $x \in U \cap S \setminus \{a\}$. Choose $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$. We observe that whenever $x \in S \setminus \{a\}$ and $|x - a| < \delta$, we must have $x \in U \cap S \setminus \{a\}$ and so $f(x) \in V$.

To show that condition c implies condition a we assume that condition c holds. Suppose that V is a neighborhood of λ. Using condition c we choose a number $\delta > 0$ such that the condition $f(x) \in V$ will hold whenever $x \in S \cap (a - \delta, a + \delta) \setminus \{a\}$. Since the interval $(a - \delta, a + \delta)$ is a neighborhood of a, condition a must hold.

7. Given that S is a subset of a metric space X, that f is a function from X into a metric space Y, that $\alpha \in Y$ and that a is an interior point of S, prove that the following conditions are equivalent:

   a. $f(x) \to \alpha$ as $x \to a$.

   b. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality $d(f(x), \alpha) < \varepsilon$ will hold for every point $x \in X \setminus \{a\}$ that satisfies the inequality $d(x, a) < \delta$.

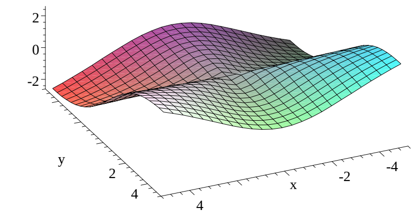
The only way in which condition b differs from the ε, δ form of the assertion that $f(x) \to \lambda$ as $x \to a$ is that it requires that $d(f(x), \alpha) < \varepsilon$ for **all** points x within a distance δ of a and unequal to a. It does not merely assert that $d(f(x), \alpha) < \varepsilon$ when x is a member of the set S unequal to a and within a distance δ of a.
It is obvious that condition b implies condition a. To show that condition a implies condition b we assume that $f(x) \to \lambda$ as $x \to a$. Suppose that $\varepsilon > 0$. Choose a number $\delta_1 > 0$ such that the condition $d(f(x), \alpha) < \varepsilon$ will hold whenever $x \in S \cap B(a, \delta_1) \setminus \{a\}$. Now, using the fact that a is an interior point of S, choose a number $\delta_2 > 0$ such that $B(a, \delta_2) \subseteq S$. We define δ to be the smaller of the two numbers δ_1 and δ_2 and we observe that the inequality $d(f(x), \alpha) < \varepsilon$ will hold for every point x that satisfies the inequality $d(a, x) < \delta$.

8. Suppose that S is a set of real numbers, that a is a limit point of S, that $f : S \to \mathbf{R}$ and that α is a real number. Complete the following sentence: *The function f fails to have a limit of α at the number a when there exists a neighborhood V of α such that ......*
The function f fails to have a limit of λ at the number a when there exists a number $\varepsilon > 0$ such that for every number $\delta > 0$ there is at least one number x in the set $S \cap (a - \delta, a + \delta) \setminus \{a\}$ for which $|f(x) - \lambda| \geq \varepsilon$.

9. Prove that
$$\frac{1 - x + xy}{1 + x^2 + xy + y^2} \to \frac{1}{4}$$
as $(x, y) \to (1, 2)$.
We begin with the observation that if $u = x - 1$ and $v = y - 2$ then
$$\left| \frac{1 - x + xy}{1 + x^2 + xy + y^2} - \frac{1}{4} \right| = \left| \frac{1 - (u+1) + (u+1)(v+2)}{1 + (u+1)^2 + (u+1)(v+2) + (v+2)^2} - \frac{1}{4} \right|$$
$$= \left| \frac{1}{4} \frac{u^2 + v^2 + v - 3uv}{8 + u^2 + 4u + uv + 5v + v^2} \right|.$$
In the event that $|x - 2| < \frac{1}{2}$ and $|y - 2| < \frac{1}{2}$ we see that
$$|8 + u^2 + 4u + uv + 5v + v^2| \geq 8 + u^2 + v^2 - 4\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - 5\left(\frac{1}{2}\right) \geq 3$$
and so
$$\left| \frac{1}{4} \frac{u^2 + v^2 + v - 3uv}{8 + u^2 + 4u + uv + 5v + v^2} \right| \leq \frac{|u^2 + v^2 + v - 3uv|}{12}.$$
Suppose that $\varepsilon > 0$ and define δ to be the smaller of the two numbers $\frac{1}{2}$ and ε. Then whenever $(x, y) \neq (1, 2)$ and $\|(x, y) - (1, 2)\| < \delta$, defining, $u = x - 1$ and $v = y - 2$, we have
$$\left| \frac{1 - x + xy}{1 + x^2 + xy + y^2} - \frac{1}{4} \right| \leq \frac{|u^2 + v^2 + v - 3uv|}{12} < \frac{6\varepsilon}{12} < \varepsilon.$$

10. In each of the following cases, determine whether or not the function f has a limit at $(0, 0)$. Use *Scientific Notebook* to view the graph of the function f in each case.

   a. For each point $(x, y) \neq (0, 0)$ we define
$$f(x, y) = \frac{x^2 y}{x^2 + y^2}.$$



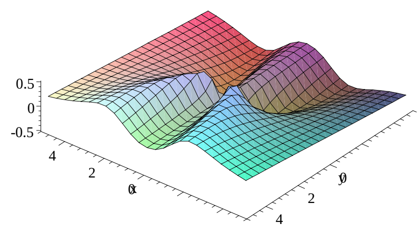
Since

$$\frac{x^2}{x^2+y^2} \le 1$$

whenever $(x,y) \neq (0,0)$ we have $|f(x,y)| \le |y|$ whenever $(x,y) \neq (0,0)$. Now we can show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Suppose that $\varepsilon > 0$. We define $\delta = \varepsilon$ and observe that, whenever $(x,y) \neq (0,0)$ and $\|(x,y)-(0,0)\| < \delta$ we have $|f(x,y)| < \varepsilon$.

b. For each point $(x,y) \neq (0,0)$ we define

$$f(x,y) = \frac{x^2 y}{x^4+y^2}.$$



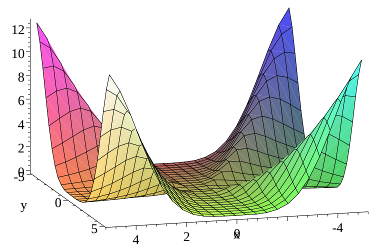
Whenever $y \neq 0$ we have $f(0,y) = 0$ and so, if f has a limit at $(0,0)$ then this limit is 0. Note also that if $x \neq 0$ then $f(x,0) = 0$ which again suggests that, if there is a limit, then the limit is 0. However, whenever $x \neq 0$ we have

$$f(x,x^2) = \frac{x^2 x^2}{x^4+(x^2)^2} = \frac{1}{2}$$

and so,if there is a limit, then the limit must be $\frac{1}{2}$. Therefore f has no limit at the point $(0,0)$. The figure illustrates how the function values remain at $\frac{1}{2}$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$.

c. For each point $(x,y) \neq (0,0)$ we define

$$f(x,y) = \frac{x^4 y^4}{x^6+y^6}.$$



If $|x| \le |y|$, then

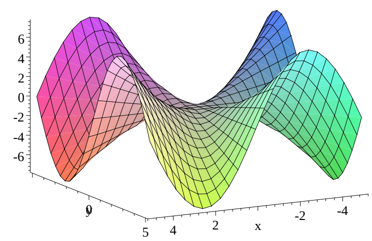
$$f(x,y) = \frac{x^4y^4}{x^6+y^6} \le \frac{y^8}{y^6} = y^2$$

and if $|y| \le |x|$, then $f(x,y) \le x^2$. Now we can show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Suppose that $\varepsilon > 0$ and define δ to be the smaller of the two numbers 1 and ε. Then, whenever $(x,y) \ne (0,0)$ and $\|(x,y)\| < \delta$ we have

$$|f(x,y)| < \delta^2 \le \varepsilon.$$

d. For each point $(x,y) \ne (0,0)$ we define

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}.$$



Since

$$\left|\frac{xy}{x^2+y^2}\right| \le \frac{1}{2}$$

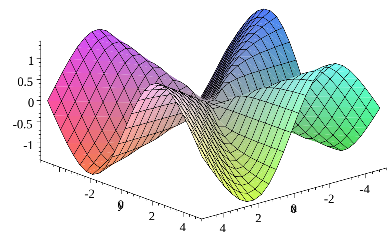
whenever $(x,y) \ne (0,0)$ we have

$$|f(x,y)| \le \frac{|x^2-y^2|}{2} \le \frac{x^2+y^2}{2}.$$

Now we can show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Suppose that $\varepsilon > 0$. We define δ to be the smaller of the numbers 1 and ε. Whenever $(x,y) \ne (0,0)$ and $\|(x,y)\| < \varepsilon$ we have $|f(x,y)| < \varepsilon$.

e. For each point $(x,y) \ne (0,0)$ we define

$$f(x,y) = \frac{xy(x^2-y^2)}{(x^2+y^2)^{3/2}}.$$



Since

$$\left|\frac{xy}{x^2+y^2}\right| \le \frac{1}{2}$$

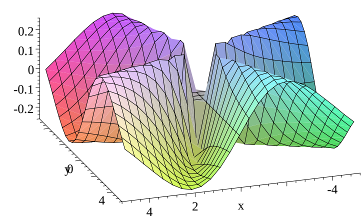
whenever $(x,y) \neq (0,0)$ we have

$$|f(x,y)| \le \frac{1}{2}\frac{|x^2-y^2|}{\sqrt{x^2+y^2}} \le \frac{\sqrt{x^2+y^2}}{2}$$

Now we can show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Suppose that $\varepsilon > 0$. We define $\delta = \varepsilon$. Whenever $(x,y) \neq (0,0)$ and $\|(x,y)\| < \varepsilon$ we have $|f(x,y)| < \varepsilon$.

f. For each point $(x,y) \neq (0,0)$ we define

$$f(x,y) = \frac{xy(x^2-y^2)}{(x^2+y^2)^2}.$$



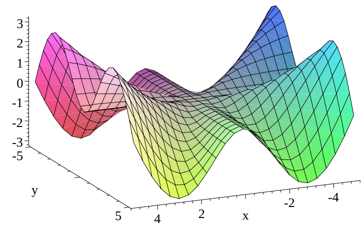
Whenever $x \neq 0$ we have $f(x,0) = 0$. Therefore, if f has a limit at $(0,0)$ then the limit is zero. However, if $x \neq 0$ then

$$\left|f\left(x, \frac{x}{2}\right)\right| = \left|\frac{x\left(\frac{x}{2}\right)\left(x^2 - \frac{x^2}{4}\right)}{\left(x^2 + \frac{x^2}{4}\right)^2}\right| = \frac{6}{25}.$$

Therefore if f has a limit at $(0,0)$ then the limit must be $\frac{6}{25}$ and we conclude that f has no limit at $(0,0)$.

g. For each point $(x,y) \neq (0,0)$ we define

$$f(x,y) = \frac{x^2y^2(x^2-y^2)}{(x^2+y^2)^2}.$$



Whenever $(x,y) \neq (0,0)$, since

$$\frac{x^2y^2}{(x^2+y^2)^2} \le \frac{1}{4}$$

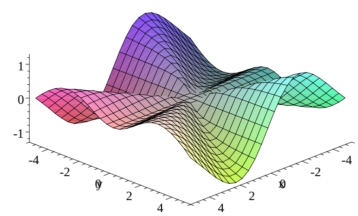
we have

$$|f(x,y)| \le \frac{1}{4}|x^2-y^2|$$

and, arguing as above, we see that $f(x,y) \to 0$ as $(x,y) \to (0,0)$.

h. For each point $(x,y) \neq (0,0)$ we define

$$f(x,y) = \frac{xy^2(x^2-y^2)}{(x^2+y^2)^2}.$$



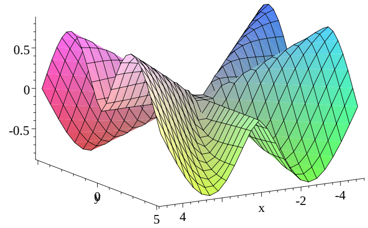
Whenever $(x,y) \neq (0,0)$ we have

$$\begin{aligned} |f(x,y)| &= \left|\frac{xy^2(x^2-y^2)}{(x^2+y^2)^2}\right| \\ &= \left|\frac{xy^2(x+y)(x-y)}{(x^2+y^2)^2}\right| \\ &\le \left|\frac{x^2y^2(x-y)}{(x^2+y^2)^2}\right| + \left|\frac{xy^3(x-y)}{(x^2+y^2)^2}\right| \end{aligned}$$

and we can argue as in the preceding exercises to show that each of these terms approaches 0 as $(x,y) \to (0,0)$.

i. For each point $(x,y) \neq (0,0)$ we define

$$f(x,y) = \frac{(x^2y^2)^{3/4}(x^2-y^2)}{(x^2+y^2)^2}.$$



Whenever $(x,y) \neq (0,0)$ we have

$$|f(x,y)| \leq \frac{|x|^{3/2}|y|^{3/2}(|x|+|y|)(|x|+|y|)}{(x^2+y^2)^2}$$
$$\leq \frac{|x|^{3/2}|y|^{3/2}|x|(|x|+|y|)}{(x^2+y^2)^2} + \frac{|x|^{3/2}|y|^{3/2}|y|(|x|+|y|)}{(x^2+y^2)^2}$$
$$\leq \frac{|x|^{3/2}|y|^{3/2}|x||y|}{(x^2+y^2)^2} + \frac{|x|^{3/2}|y|^{3/2}|y||x|}{(x^2+y^2)^2} = 2\frac{|x|^{5/2}|y|^{5/2}}{(x^2+y^2)^2}$$

and, by considering the cases $|x| \leq |y|$ and $|y| \leq |x|$, we see that

$$|f(x,y)| \leq |x| + |y|$$

and we conclude that $f(x,y) \to 0$ as $(x,y) \to (0,0)$.

11. Suppose that S is a set of real numbers, that a is a limit point of S, that $f : S \to \mathbf{R}^k$ and that $\alpha \in \mathbf{R}^k$. Prove that if $f(x) \to \alpha$ as $x \to a$ then $\|f(x)\| \to \|\alpha\|$ as $x \to a$. Compare this exercise with the corresponding exercise for sequences.
The key to this exercise is the fact that, whenever x is in the domain of f we have
$$|\|f(x)\| - \|\alpha\|| \leq \|f(x) - \alpha\|$$
and the proof runs along the same lines as the one used for sequences.

12. Suppose that X is a metric space, that S is a subset X and that both of the sets S and $X \setminus S$ are dense in X. Suppose that $c \in X$ and that
$$f(x) = \begin{cases} x & \text{if} \quad x \in S \\ c & \text{if} \quad x \in X \setminus S \end{cases}.$$
Prove that f has a limit at c but does not have a limit at any other point of the space X.
To show that $f(x) \to c$ as $x \to c$, suppose that $\varepsilon > 0$. We define $\delta = \varepsilon$ and observe that whenever $x \neq c$ and $d(c,x) < \delta$ we have
$$d(c, f(x)) = d(c,x) < \delta = \varepsilon$$
when $x \in S$ and we have
$$d(c, f(x)) = d(c,c) = 0 < \varepsilon$$
when $x \in X \setminus S$.
Now suppose that $x \in X \setminus \{c\}$. We want to show that f has no limit at x. To obtain a contradiction, suppose that f has a limit λ at x. We define $\varepsilon = d(c,x)$ and note that $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta < \varepsilon/2$, and such that for every point $t \in X \cap B(c,\delta) \setminus \{c\}$, we have $d(f(t),\lambda) < \varepsilon/4$. Choose points t_1 and t_2 in the set $X \cap B(x,\delta) \setminus \{x\}$ such that $t_1 \in S$ and $t_2 \in X \setminus S$. We see that
$$d(x,c) \leq d(x,t_1) + d(t_1,c)$$
$$\varepsilon < \frac{\varepsilon}{2} + d(t_1,c)$$
from which we conclude that $d(t_1,c) > \varepsilon/2$. Therefore
$$d(f(t_1), f(t_2)) = d(t_1,c) > \frac{\varepsilon}{2}.$$
On the other hand,

$$d(f(t_1), f(t_2)) \le d(f(t_1), \lambda) + d(\lambda, f(t_2)) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

This is the desired contradiction.

Some Further Exercises on Limits

1. Given that

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 2 \\ 0 & \text{if} \quad x > 2 \end{cases}$$

prove that f has a limit from the left at 2 and also has a limit from the right at 2 but does not have a limit at 2.
The fact that f does not have a limit at 2 will be clear when we have seen that $f(x) \to 1$ as $x \to 2-$ and $f(x) \to 0$ as $x \to 2+$. Suppose that $\varepsilon > 0$. We define $\delta = 3$ (or just take δ to be any positive number you like). Whenever $x < 2$ and $|x-2| < \delta$ we have

$$|f(x) - 1| = |1 - 1| = 0 < \varepsilon$$

and whenever $x > 2$ and $|x-2| < \delta$ we have

$$|f(x) - 0| = |0 - 0| = 0 < \varepsilon.$$

2. Given that

$$f(x) = \frac{1}{|x-3|}$$

for all numbers $x \neq 3$, explain why f has a limit (an infinite limit) at 3.
We need to show that $f(x) \to \infty$ as $x \to 3$. Suppose that w is a real number. Given $x \neq 3$, the inequality $f(x) > w$ says that

$$\frac{1}{|x-3|} > w.$$

We can't simply turn these expressions over because w need not be positive but we can make the observation that the inequality

$$\frac{1}{|x-3|} > w$$

will certainly hold when

$$\frac{1}{|x-3|} > |w| + 1$$

which says that

$$|x-3| < \frac{1}{|w|+1}.$$

We therefore define $\delta = \frac{1}{|w|+1}$ and observe that the condition $f(x) > w$ will hold whenever $x \neq 3$ and $|x-3| < \delta$.

3. Given that

$$f(x) = \frac{1}{x-3}$$

for all numbers $x \neq 3$, explain why f has an infinite limit from the left at 3 and also has an infinite limit from the right at 3 but does not have a limit at 3.
The reason f has no two-sided limit at 3 is that the limits of f at 3 from the left and from the right are not equal to each other. In fact, the limit from the right is ∞ and the limit from the left is $-\infty$.
To see why the $f(x) \to \infty$ as $x \to 3+$, suppose that w is any real number. Given $x > 3$, the inequality $f(x) > w$ says that

$$\frac{1}{x-3} > w.$$

We can't simply turn these expressions over because w need not be positive but we can make the observation that the inequality

$$\frac{1}{x-3} > w$$

will certainly hold when

$$\frac{1}{x-3} > |w| + 1$$

which says that

$$x - 3 < \frac{1}{|w|+1}.$$

We therefore define $\delta = \frac{1}{|w|+1}$ and observe that the condition $f(x) > w$ will hold whenever $3 < x < 3 + \delta$.

To see why the $f(x) \to -\infty$ as $x \to 3-$, suppose that w is any real number. Given $x < 3$, the inequality $f(x) < w$ says that

$$\frac{1}{x-3} < w$$

which we can express as

$$\frac{1}{3-x} > -w$$

We can't simply turn these expressions over because $-w$ need not be positive but we can make the observation that the inequality

$$\frac{1}{3-x} > -w$$

will certainly hold when

$$\frac{1}{x-3} > |w| + 1$$

which says that

$$3 - x < \frac{1}{|w|+1}.$$

We therefore define $\delta = \frac{1}{|w|+1}$ and observe that the condition $f(x) < w$ will hold whenever $3 - \delta < x < 3$.

4. Prove that

$$\frac{x^3-8}{x^2+x-6} \to \infty$$

as $x \to \infty$.

We begin by observing that $x^2 + x - 6 > 0$ whenever $x > 2$. Given any number $x > 2$ we have

$$\frac{x^3-8}{x^2+x-6} = \frac{x^2+2x+4}{x+3} > \frac{x^2}{x} = x.$$

Now suppose that w is any real number and define v to be the larger of the two numbers 2 and w. The inequality

$$\frac{x^3-8}{x^2+x-6} > w$$

will hold whenver $x > v$.

5. Prove that

$$\frac{x^4-4x^3-x^2+x+7}{x^3-2x^2-2x-3} \to \infty$$

as $x \to \infty$.

Given any number x we have

$$x^3 - 2x^2 - 2x - 3 = (x-3)(x^2+x+1)$$

and so $x^3 - 2x^2 - 2x - 3$ will be positive whenever $x > 3$. Now given any number $x > 3$ we have

$$\frac{x^4-4x^3-x^2+x+7}{x^3-2x^2-2x-3} > \frac{x^4-4x^3-x^2}{x^3} = \frac{x^4-4x^3-x^3}{x^3} > x - \frac{5}{x} > x - \frac{5}{3}.$$

Now suppose that w is any real number. We define v to be the larger of the two numbers 3 and

$w - \frac{5}{3}$ and observe that the inequality

$$\frac{x^4 - 4x^3 - x^2 + x + 7}{x^3 - 2x^2 - 2x - 3} > w$$

will hold whenever $x > v$.

6. Prove that

$$\frac{3x^2 + x - 1}{5x^2 + 4} \to \frac{3}{5}$$

as $x \to \infty$.
Given any number x we have

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| = \frac{|5x - 17|}{5(5x^2 + 4)}.$$

Whenever $x > \frac{17}{5}$ we observe that

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| = \frac{|5x - 17|}{5(5x^2 + 4)}$$
$$= \frac{5x - 17}{5(5x^2 + 4)} < \frac{5x}{25x^2} = \frac{1}{5x}.$$

Now suppose that $\varepsilon > 0$. As long as $x > \frac{17}{5}$, the inequality

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| < \varepsilon$$

will hold whenever $\frac{1}{5x} < \varepsilon$ which says that $x > \frac{1}{5\varepsilon}$. We define v to be the larger of the two numbers $\frac{17}{5}$ and $\frac{1}{5\varepsilon}$ and oberve that the inequality

$$\left| \frac{3x^2 + x - 1}{5x^2 + 4} - \frac{3}{5} \right| < \varepsilon$$

holds whenever $x > v$.

7. Given that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

explain why f does not have a limit from the right at 2.

Solution: ***To obtain a contradiction, suppose that λ is a limit of the function f from the right at 2. Using the fact that $1/2 > 0$ we choose a number $\delta > 0$ such that the inequality***

$$|f(x) - \lambda| < \frac{1}{2}$$

will hold for every number $x \in (2, 2 + \delta)$. Choose a rational number x and an irrational number t in the interval $(x, x + \delta)$. Then

$$1 = |f(x) - f(t)| \le |f(x) - \lambda| + |\lambda - f(t)| < \frac{1}{2} + \frac{1}{2} = 1$$

and we have reached the desired contradiction.

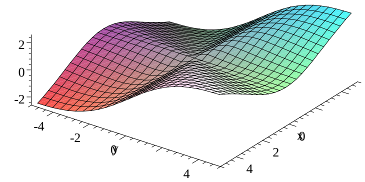
8. Suppose that a is an interior point of a set S of real numbers and that $f : S \to \mathbf{R}$. Suppose that $f(x) \to 0$ as $x \to a-$ and that $f(x) \to 1$ as $x \to a+$. Prove that the function f does not have a limit at the number a.
Since x must be a limit point of each of the sets $(-\infty, a) \cap S$ and $S \cap (a, \infty)$, the desired result follows at once from Theorem 8.3.2.

Some Exercises on Continuity

1. In each of the following cases, determine whether or not the function f is continuous at $(0, 0)$

   a. We define

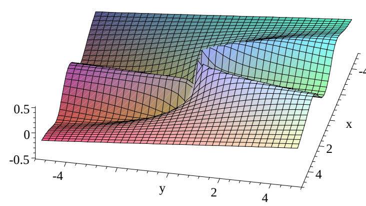
$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$



Since $|f(x,y)| \leq |y|$ for each point (x,y) we see that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ and so f is continuous at $(0,0)$.

b. We define

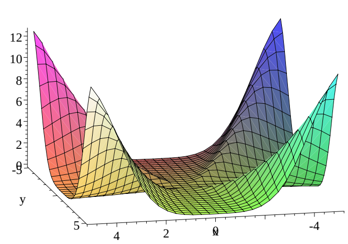
$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$



We saw in an earlier exercise that the function f has no limit at $(0,0)$ and therefore f can't be continuous at $(0,0)$.

c. We define

$$f(x,y) = \begin{cases} \frac{x^4y^4}{x^6+y^6} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$



Solution: ***From the inequality***

$$|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$$

that holds for all real numbers a ***and*** b ***we observe that whenever*** $(x,y) \neq (0,0)$ ***we have***

$$|f(x,y)| = |xy|\frac{|x^3y^3|}{x^6+y^6} \le \frac{|xy|}{2}.$$

The continuity of the function f at $(0,0)$ now follows simply from the sandwich theorem.

2. Given that X is a metric space, that $a \in X$ and that
$$f(x) = d(a,x)$$
for every point $x \in X$, prove that the function f is continuous on X.

Hint: ***First make the observation that whenever x and u belong to the space X we have***
$$|f(x) - f(u)| \le d(x,u).$$
This inequality has come up several times already. To show that f is continuous at a given point $x \in X$, suppose that $\varepsilon > 0$. We define $\delta = \varepsilon$ and observe that whenever $u \in B(x,\delta)$ we have
$$d(f(x),f(u)) \le d(x,u) < \varepsilon.$$

3. Suppose that f and g are functions from a metric space X into a metric space Y and that the inequality
$$d(f(t),f(x)) \le d(g(t),g(x))$$
holds for all numbers t and x in S. Prove that f must be continuous at every point at which the function g is continuous.
Suppose that $x \in X$ and that g is continuous at x. To show that f is continuous at x, suppose that $\varepsilon > 0$. Choose $\delta > 0$ such that, whenever $t \in B(x,\delta)$ we have
$$d(g(t),g(x)) < \varepsilon.$$
Then, for every $t \in B(x,\delta)$ we have
$$d(f(t),f(x)) < \varepsilon.$$

4. Prove that if f is a continuous function from a metric space X into $\mathbf{R}^k$ then the function $\|f\|$ is continuous from X into $\mathbf{R}$.
Since
$$|\|f(t)\| - \|f(x)\|| \le \|f(t) - f(x)\|$$
whenever t and x belong to the space X, the continuity of $\|f\|$ follows from Exercise 4.

5. Give an example of a continuous function f from a metric space X to a metric space Y and a closed subset H of X such that set $f[H]$ fails to be closed in Y.
We take $X = \mathbf{R}$ and $Y = [0,1]$ and we define
$$f(x) = \frac{1}{1+x^2}$$
for every $x \in \mathbf{R}$. We see that $f[\mathbf{R}] = (0,1)$ which is not closed in Y.

6. Give an example of a continuous function f from a metric space X to a metric space Y and an open subset U of X such that set $f[U]$ fails to be open in Y.
We take $X = [0,1] \cup [2,3]$ and we define
$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ x-1 & \text{if } 2 \le x \le 3. \end{cases}$$
This function f is continuous from X to the metric space $[0,2]$ and, even though $[0,1]$ is an open subset of X, the set $f[[0,1]]$ fails to be open in the space $[0,2]$.
Of course we could have taken a discrete space for X. A challenge question would be to ask whether the student can come up with an example in which the space X is connected. In fact, if $X = [0,1]$ then X is connected and, if we define
$$f(x) = (x,0)$$
for all $x \in X$, then f fails to send open sets to open subsets of $\mathbf{R}^2$.

7. Suppose that X is a metric space, that k is a positive integer and that $f_j : X \to \mathbf{R}$ for each $j = 1,2,\cdots,k$.

Suppose that

$$\mathbf{f}(x) = (f_1(x), f_2(x), \cdots, f_k(x))$$

for every point $x \in X$. Prove that the function $\mathbf{f}$ is continuous on X if and only if each of the functions f_j is continuous on X.

This exercise follows almost at once from the corresponding fact about limits.

8. Suppose that a and b are real numbers, that $a < b$ and that f is a function from the metric space $[a,b]$ to a metric space Y. Prove that the following conditions are equivalent:

   a. The function f is continuous at the number a.

   b. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every number x in the interval $[a,b]$ that satisfies the inequality $x - a < \delta$ we have $d(f(x), f(a)) < \varepsilon$.

   This exercise is obvious because of the fact that $|x-a| = x - a$ whenever $x \in [a,b]$. You can convert this exercise into one that is a shade more interesting by removing the words "in the interval" from condition b and replace them by the condition $x > a$. This change would require a slightly more careful choice of δ to ensure that it does not exceed $b-a$ so that any number $x > a$ that lies within a distance δ of a would automatically belong to the interval $[a,b]$.

9. Prove that if a metric space X has no limit point and f is a function from X to a metric space Y then f must be continuous on X.

   This result follows at once from the fact that a function is always continuous at a point in its domain if that point is not a limit point of the domain.

10. Suppose that f is a continuous function from a closed subset H of a metric space X into a metric space Y and suppose that (x_n) is a convergent sequence in the set H. Prove that the sequence $(f(x_n))$ is convergent in the space Y. Is this conclusion still valid if we don't assume that H is closed?

    Solution: ***Suppose that*** f ***is continuous on a closed set*** H ***and that*** (x_n) ***is a convergent sequence in*** H***. We define***

    $$x = \lim_{n\to\infty} x_n.$$

    Since H ***is closed we must have*** $x \in H$ ***and therefore*** f ***is continuous at the number*** x***. Therefore the fact that*** $x_n \to x$ ***as*** $n \to \infty$ ***guarantees that*** $f(x_n) \to f(x)$ ***as*** $n \to \infty$***.***

    To see why the assertion does not remain true without the assumption that H ***be closed we look at the example in which***

    $$H = \left\{ \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\}$$

    and we define

    $$f\left(\frac{1}{n}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

    Since no member of the set H ***is a limit point of*** H ***we know that*** f ***is continuous at every member of*** H***. However, in spite of the fact that the sequence*** $(1/n)$ ***converges (to*** 0***) in the metric space*** $\mathbf{R}$***, the sequence*** $(f(1/n))$ ***fails to converge.***

    Actually, much more can be proved: If H ***is any subset of a metric space*** X ***and*** H ***fails to be closed then there exists a convergent sequence*** (x_n) ***in*** H ***and a continuous function*** f ***on*** H ***such that the sequence*** $(f(x_n))$ ***fails to converge. This stronger assertion is harder to prove and we omit the proof at this point. This stronger assertion will be clear after we have studied the distance function of a set in Section 8.10.***

11. Is it true that if f is a continuous function from a metric space X to a metric space Y and if (x_n) is a Cauchy sequence in X then the sequence $(f(x_n))$ is a Cauchy sequence in the space Y?

    No. This statement is false. Define $f(x) = 1/x$ for every number $x \in (0,1)$ and look at the sequence $(1/n)$ in the space $(0,1)$.

12. Suppose that f is a continuous function from an open subset S of a metric space X into a metric space Y and that $x \in S$. Prove that for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $t \in B(x,\delta)$ we

have $d(f(t), f(x)) < \varepsilon$. Is this conclusion still valid if we don't assume that S is open?
Suppose that $\varepsilon > 0$. Using the fact that f is continuous at the point x of the metric space S, choose $\delta_1 > 0$ such that the inequality $d(f(t), f(x)) < \varepsilon$ will hold whenever $t \in S$ and $d(t,x) < \delta_1$. Using the fact that S is open in X, choose $\delta_2 > 0$ such that $B(x, \delta_2) \subseteq X$. Define δ to be the smaller of the two numbers δ_1 and δ_2.
No, this statement is not true if S fails to be open in X because it may happen that no ball $B(x,\delta)$ can be included in S; in which case, no assertion can be made about $f(t)$ for every $t \in B(x,\delta)$.

13. Given that f is a function from a metric space X into a metric space Y, prove that f is continuous on X if and only if for every subset S of X we have

$$f[\overline{S}] \subseteq \overline{f[S]}$$

Solution: ***Suppose that f is continuous from X to Y and that $S \subseteq X$. Since the set $\overline{f[S]}$ is a closed subset of the space Y we know that the set $f^{-1}\left[\overline{f[S]}\right]$ is closed in the space X. Therefore since***

$$S \subseteq f^{-1}\left[\overline{f[S]}\right]$$

we have

$$\overline{S} \subseteq f^{-1}\left[\overline{f[S]}\right]$$

and we conclude that

$$f[\overline{S}] \subseteq \overline{f[S]}$$

Now we prove the "if" part of the exercise. Suppose that the inequality

$$f[\overline{S}] \subseteq \overline{f[S]}$$

holds for every subset S of X. To prove that f is continuous, suppose that H is a closed subset of the space Y. We shall show that the set $f^{-1}[H]$ is closed in X. Now

$$f\left[\overline{f^{-1}[H]}\right] \subseteq \overline{f[f^{-1}[H]]} \subseteq \overline{H} = H$$

and therefore

$$\overline{f^{-1}[H]} \subseteq f^{-1}[H]$$

and we have shown that $f^{-1}[H]$ is closed in X.

14. Given that

$$f(t) = (\cos t, \sin t)$$

for each number t in the metric space $[0, 2\pi)$ explain why, although f is a continuous one-one function from the metric space $[0, 2\pi)$ onto the metric space

$$Y = \{(x,y) \in \mathbf{R} \mid x^2 + y^2 = 1\},$$

the inverse function f^{-1} fails to be continuous from Y to $[0, 2\pi)$.
The function f^{-1} fails to be continuous at the point $(1,0)$. To see why, suppose that $\delta > 0$. Using the fact that

$$\lim_{t\to 0}(\cos t, \sin t) = (1,0),$$

choose a number t between 0 and 1 such that

$$\|(\cos t, \sin t) - (1,0)\| < \delta.$$

Using the fact that

$$\lim_{t\to 2\pi}(\cos t, \sin t) = (1,0),$$

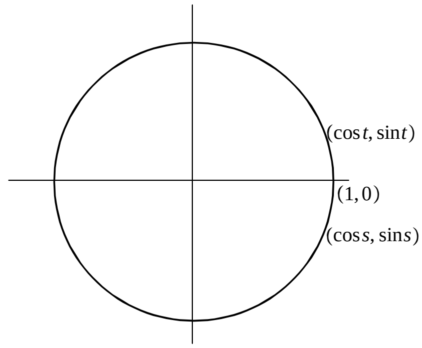
choose a number s between $2\pi - 1$ and 2π such that

$$\|(\cos s, \sin s) - (1,0)\| < \delta.$$

We observe that

$$|f^{-1}(t) - f^{-1}(s)| > 2\pi - 2.$$

Therefore the function f^{-1} can't have a limit at the point $(1,0)$.



15. Suppose that f and g are functions from a metric space X to $\mathbf{R}$ and that f is continuous at a given point a at which the function g fails to be continuous.

a. What can we say about the continuity of the function $f+g$ at the point a?
Were the function $f+g$ to be continuous at a, it would follow from the fact that
$$g = (f+g) - f$$
that g is continuous at a. Therefore $f+g$ cannot be continuous at a.

b. What can we say about the continuity of the function fg at the point a?
If $f(a) \neq 0$ then, since the equation
$$g(x) = \frac{f(x)g(x)}{f(x)}$$
would hold for every number x sufficiently close to a, we could use an argument like the one we used in Part a to deduce that the function fg cannot be continuous at a.
However, execise 2 shows us to functions, one continuous at 0 and the other discontinuous at 0 whose product is continuous at 0.

c. What can we say about the continuity of the function fg at the point a if $f(a) = 0$ and g is a bounded function?
Choose a number p such that $|g(x)| < p$ for each $x \in S$. Whenever $x \in S$ we have
$$|f(x)g(x) - f(a)g(a)| = |f(x)g(x)| \leq p|f(x)|$$
and we can now use this inequality to prove that fg must be continuous at a.
Suppose that $\varepsilon > 0$. Using the fact that f is continuous at w and the fact that $\varepsilon/p > 0$ we choose a number $\delta > 0$ such that the condition $|f(x)| < \frac{\varepsilon}{p}$ whenever $x \in S \cap (a-\delta, a+\delta)$. Then, whenever $x \in B(a,\delta)$ we have
$$|f(x)g(x) - f(a)g(a)| \leq p|f(x)| < p\left(\frac{\varepsilon}{p}\right) = \varepsilon.$$

d. What can be said about the continuity of the function fg if $f(a) \neq 0$?
The purpose of this part of the question is to prod people who may not have done Part a.

16. Give an example of two functions f and g that are both discontinuous at a given point a such that their sum $f+g$ is continuous at a.
We define
$$f(x) = \begin{cases} -1 & \text{if } x < 2 \\ 0 & \text{if } x = 2 \\ 1 & \text{if } x > 2 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1 & \text{if } x < 2 \\ 0 & \text{if } x = 2 \\ -1 & \text{if } x > 2 \end{cases}$$

17.

a. Given that f is a continuous function from a closed subset H of $\mathbf{R}$ into $\mathbf{R}$ and that $a \in H$, prove that the set

$$E = \{x \in H \mid f(x) = f(a)\}$$

is closed.

Solution: ***Suppose that (x_n) is any convergent sequence in the set E. We shall show that the limit of this sequence, that we shall call x, must also belong to E. Since the set H is closed we know that $x \in H$ and therefore we know that f is continuous at the number x. Therefore***

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(a) = f(a)$$

and so $x \in E$ as we promised.

b. Given that f and g are continuous functions from a metric space X to a metric space Y and that

$$E = \{x \in X \mid f(x) = g(x)\}$$

prove that the set E must be closed in X.

This exercise is very similar to part a. Suppose that (x_n) is any convergent sequence in the set E. We shall show that the limit of this sequence, that we shall call x, must also belong to E. We now observe that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

and so $x \in E$ as we promised.

18.

a. Given that f and g are continuous functions from $\mathbf{R}$ to $\mathbf{R}$ and that $f(x) = g(x)$ for every rational number x, prove that $f = g$.

The set of numbers x for which $f(x) = g(x)$ is closed and includes the set $\mathbf{Q}$ of rational numbers. Therefore this set is all of $\mathbf{R}$.

b. Given that $f : \mathbf{Z}^+ \to \mathbf{R}$, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x) + f(t)$$

holds for all positive integers x and t, prove that $f(x) = x$ for every positive integer x.

Solution: ***If you are familiar with the method of proof by mathematical induction which is available in this book in an*** optional section ***then you should use mathematical induction to do this exercise.***

Otherwise we can use the method of proof by contradiction as follows: To obtain a contradiction, suppose that their are positive integers n for which the equation $f(n) = n$ fails to hold and define k to be the least of these positive integers. Since we are given that $f(1) = 1$, we know that $k > 1$. Therefore $k - 1$ is a positive integer that is less than k and we conclude that $f(k-1) = k - 1$. Therefore

$$f(k) = f((k-1)+1) = f(k-1) + f(1) = k - 1 + 1 = k$$

which contradicts the way in which the integer k was chosen.

c. Given that $f : \mathbf{Z} \to \mathbf{R}$, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x) + f(t)$$

holds for all integers x and t, prove that $f(x) = x$ for every integer x.

First we observe that, since

$$f(0) = f(0+0) = f(0) + f(0)$$

we have $f(0) = 0$. Now we know from part b that $f(x) = x$ for every positive integer x. If x is a negative integer then, since $f(-x) = -x$ we have

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x) = f(x) - x,$$

and so $f(x) = x$.

d. Given that $f : \mathbf{Q} \to \mathbf{R}$, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x)+f(t)$$

holds for all rational numbers x and t, prove that $f(x) = x$ for every rational number x.

Solution: ***By the preceding exercise we know that the equation $f(x) = x$ holds whenever x is an integer. We shall now show that if x is any real number and n is a positive integer then $f(nx) = nf(x)$. Once again, you can use the method of proof by mathematical induction if you are familiar with it but we shall use the method of proof by contradiction.***

Suppose that x is any real number and, to obtain a contradiction, suppose that there are positive integers n for which the equation $f(nx) = nf(x)$ fails to hold. We define k to the least of these integers. Since we know that the equation $f(nx) = nf(x)$ holds when $n = 1$, we know that $k > 1$. Therefore $k-1$ is a positive integer less than k and so

$$f(kx) = f((k-1)x+x) = f((k-1)x)+f(x) = (k-1)x+x = kx$$

which contradicts the way in which k was chosen.

To complete the exercise, suppose that x is any rational number and choose integers m and n such that $n > 0$ and $x = m/n$. We see that

$$f(x) = \frac{1}{n}nf(x) = \frac{1}{n}f(nx) = \frac{1}{n}f(m) = \frac{m}{n}.$$

e. Given that f is a continuous function from **R** to **R**, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x)+f(t)$$

holds for all rational numbers x and t, prove that $f(x) = x$ for every real number x.

In view of the fact that $f(x) = x$ for every rational number x (by part d), the desired result follows at once from Exercise 17b.

f. Given that f is an increasing function from **R** to **R**, that $f(1) = 1$ and that the equation

$$f(x+t) = f(x)+f(t)$$

holds for all rational numbers x and t, prove that $f(x) = x$ for every real number x.

Solution: ***Suppose that x is any real number and choose two sequences (a_n) and (b_n) of rational numbers such that***

$$a_n \leq x \leq b_n$$

for each n and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x.$$

Since the function f is increasing we know that for each n we have

$$a_n = f(a_n) \leq f(x) \leq f(b_n) \leq b_n$$

and so it follows from the sandwich theorem for limits that $f(x) = x$.

19. Given that $f : \mathbf{R} \to \mathbf{R}$ and that for all numbers x and t we have

$$|f(x)-f(t)| \leq |x-t|^2,$$

prove that the function f must be constant. *Note that although this exercise is quite difficult right now, it will become considerably easier after we have studied the concept of a derivative.*

I am resisting the urge to write a direct solution of this exercise. Those students who wish to attempt it now will probably want to be left alone. All others can wait until after the mean value theorem when the fact that f is constant will follow at once from the obvious fact that $f'(x) = 0$ for every number x.

Exercises on the Distance Function

1. Two subsets A and B of a metric space X are said to be **separated** from each other if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

Prove that if two sets A and B are separated from each other then

$$\rho_A(x) - \rho_B(x) < 0$$

whenever $x \in A$ and

$$\rho_A(x) - \rho_B(x) > 0$$

whenever $x \in B$.

We know that if $x \in A$ then $\rho_A(x) = 0$, and that, since x does not belong to $\overline{B}$, we have $\rho_B(x) > 0$. Therefore the inequality

$$\rho_A(x) - \rho_B(x) < 0$$

holds whenever $x \in A$ and the same kind of argument shows that

$$\rho_A(x) - \rho_B(x) > 0$$

whenever $x \in B$.

2. Prove that if two sets A and B are separated from each other then there exist two open sets U and V that are disjoint from each other such that $A \subseteq U$ and $B \subseteq V$.

Solution: ***Suppose that** A **and** B **are subsets of a metric space** X **and that** A **and** B **are separated from each other. Define***

$$U = \{x \in X \mid \rho_A(x) - \rho_B(x) < 0\}$$

and

$$V = \{x \in X \mid \rho_A(x) - \rho_B(x) > 0\}$$

3. Given two subsets A and B of a metric space X, prove that the following conditions are equivalent:

   a. We have
   $$\overline{A} \cap \overline{B} = \emptyset.$$

   b. There exists a continuous function $f : X \to [0, 1]$ such that
   $$f(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ 1 & \text{if} \quad x \in B. \end{cases}$$

The fact that condition a implies condition b is just Urysohn's lemma. Now suppose that condition b holds and choose a continuous function $f : \mathbf{R} \to [0, 1]$ such that

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ 1 & \text{if} \quad x \in B. \end{cases}$$

Since

$$A \subseteq \{x \in X \mid f(x) = 0\}$$

and since the set $\{x \in X \mid f(x) = 0\}$ is closed, we have

$$\overline{A} \subseteq \{x \in X \mid f(x) = 0\}.$$

In the same way,

$$\overline{B} \subseteq \{x \in X \mid f(x) = 1\}$$

and therefore $\overline{A} \cap \overline{B} = \emptyset$.

4. Suppose that A, B and C are closed subsets of a metric space X, and that no two of these three sets intersect. Prove that there exists a continuous function $f : X \to \mathbf{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 2 & \text{if} \quad x \in B \\ 3 & \text{if} \quad x \in C \end{cases}.$$

Define

$$f = 1 + \frac{\rho_A \rho_C}{\rho_A \rho_C + \rho_B} + \frac{2\rho_A \rho_B}{\rho_A \rho_B + \rho_C}.$$

5. Suppose that S is a subset of a metric space and that S fails to be closed. Prove that there exists a convergent sequence (x_n) in S and a continuous function f from S to $\mathbf{R}$ such that the sequence $(f(x_n))$ fails to converge.

Solution: ***We begin by choosing a point*** $w \in \bar{S} \setminus S$ ***and we choose a sequence*** (x_n) ***in*** S ***that converges to the point*** w***. We define*** E ***to be the range of the sequence*** (x_n) ***and, Using the fact that that the set*** E ***must be infinite, we choose an infinite subset*** A ***of*** E ***such that the set*** $B = E \setminus A$ ***is also infinite. Since no member of*** S ***can lie in the closures of both*** A ***and*** B***, the function***

$$f = \frac{\rho_A}{\rho_A + \rho_B}$$

is continuous on the set S***. Furthermore, since there must be infinitely many integers*** n ***for which*** $x_n \in A$ ***and infinitely many integers*** n ***for which*** $x_n \in B$ ***the sequence*** $f((x_n))$ ***has no limit.***

Exercises on Continuous Functions on Compact Spaces

1. Give an example of a function f that is continuous from a closed set H of real numbers into $\mathbf{R}$ such that range $f[H]$ of f fails to be closed in $\mathbf{R}$.
We can define $f(x) = \frac{1}{x}$ for $x \geq 1$. This function f is continuous on the closed set $[1, \infty)$ and its range is $(0, 1]$ which is not closed.

2. Give an example of a function f that is continuous from a closed set H of real numbers into $\mathbf{R}$ such that range $f[H]$ of f fails to be bounded.
We can define $f(x) = x$ for every $x \in \mathbf{R}$.

3. Give an example of a function f that is continuous from a bounded set H of real numbers into $\mathbf{R}$ such that range $f[H]$ of f fails to be closed in $\mathbf{R}$.
We can define $f(x) = x$ for $0 < x < 1$.

4. Give an example of a function f that is continuous from a bounded set H of real numbers into $\mathbf{R}$ such that range $f[H]$ of f fails to be bounded.
We can define $f(x) = \frac{1}{x}$ for $0 < x < 1$.

5. Prove that if a set H of real numbers is unbounded above and $f(x) = x$ for every number x in H, then f is a continuous function on H and f fails to have a maximum.
Since the range of f is the set H which is assumed to be unbounded, the function f must be unbounded above.

6. Prove that if S is a subset of a metric space X and if a is a point of X that is close to S but not a member of S then the function f defined by the equation

$$f(x) = \frac{1}{d(a, x)}$$

for every $x \in X$ is unbounded.We can see that f has no maximum by showing that f is unbounded above. Given any positive number q the inequality

$$f(x) > q$$

says that

$$\frac{1}{d(a, x)} > q$$

which holds when

$$d(a, x) < \frac{1}{q}$$

But, since a is close to the set S, we know that there do indeed exist members x of S for which the inequality

$$|x-a| < \frac{1}{q}$$

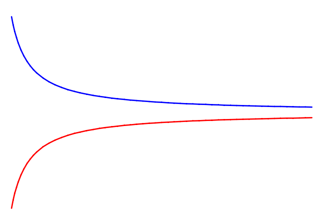
holds. Therefore there are members x in H for which $f(x) > q$ and we have shown that f fails to be bounded above.

7. Is it true that if every continuous function from a given metric space X to $(0,\infty)$ has a minimum then X must be compact?
   Yes, it's true. The theorems of this section tell us that if a metric space X is not compact then there exists a continuous unbounded real function on X. For any such function f, the function
   $$g = \frac{1}{1+f^2}$$
   is continuous (and bounded) on X and has no minimum.

8. Is it true that if X is a complete metric space and f is a continuous function from X onto a metric space Y then Y must also be complete?
   The answer is no. Define
   $$f(x) = \frac{1}{x}$$
   for every number x in the complete metric space $[1,\infty)$. The range of f is the metric space $(0,1]$, which is not complete.

9. Is it true that if X is a totally bounded metric space and f is a continuous function from X onto a metric space Y, then Y must also be totally bounded?
   Of course not. We have seen continuous functions on totally bounded spaces such as $(0,1]$ whose ranges are not bounded.

10. Give an example of a metric space X and two closed subsets H and K of X such that $H \cap K = \emptyset$ in spite of the fact that
    $$\inf\left\{d(x,y) \mid x \in H \text{ and } y \in K\right\} = 0.$$
    Take $X = \mathbf{R}$ and
    $$H = \{n \in \mathbf{Z}^+ \mid n \geq 2\}$$
    and
    $$K = \left\{n + \frac{1}{n} \in \mathbf{Z}^+ \mid n \geq 2\right\}$$
    For another example, take $X = \mathbf{R}^2$ and
    $$H = \left\{\left(x, \frac{1}{x}\right) \mid x \geq 1\right\}$$
    and
    $$K = \left\{\left(x, -\frac{1}{x}\right) \mid x \geq 1\right\}$$



11. Suppose that X is a metric space, that H and K are closed subsets of X and are disjoint from each other and that the subspace K of X is compact. Prove that
    $$\inf\left\{d(x,y) \mid x \in H \text{ and } y \in K\right\} > 0.$$
    The function ρ_H is continuous on the compact space K and therefore has a mininimum. Since $\rho_H(y) > 0$ for every $y \in K$ there must be a positive number δ such that $\rho_H(y) \geq \delta$ for every $y \in H$.

For such a number δ we must have $d(x,y) \geq \delta$ for all $x \in H$ and all $y \in K$.

12. a. Suppose that S is a compact subspace of a metric space X and that f is a continuous function from X to $\mathbf{R}$. Suppose that c is a point of the set S, that $f(c) = 1$ and that $f(x) > 1$ for all $x \in X \setminus S$. Prove that the function f has a minimum.
On the compact space S, the function f must have a minimum. Since $f(c) = 1$ we know that the minimum value of f on S cannot exceed 1. Since $f(x) > 1$ whenever $x \in X \setminus S$, we conclude that the minimum value of f on the subspace S is the minimum value of f on the whole space X.

b. Prove that if f is a quartic polynomial of the form
$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$
for every number x then the function f must have a minimum.
Since
$$f(x) = x^4\left(1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{x^4}\right)$$
for all $x \neq 0$, it is clear that $f(x) \to \infty$ as $n \to \infty$. Using this fact we choose a number $p > 0$ such that $f(x) > d$ whenever $|x| > p$.

$-p$ 0 p

The minimum value that f has on the interval $[-p,p]$ is the minimum of f on the entire line $\mathbf{R}$.

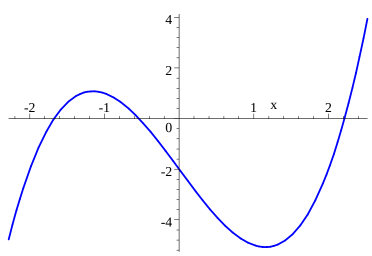
Exercises on Continuity of Functions on Intervals

1. Given that S is a set of positive numbers and that $f(x) = \sqrt{x}$ for all $x \in S$, prove that f is a one-one continuous function on S. Prove that S is an interval if and only if the set $f[S]$ is an interval.

Hint: ***Once you have shown that*** f ***is strictly increasing and continuous on*** S***, the fact that*** S ***is an interval if and only if*** $f[S]$ ***is an interval will follow from the Bolzano intermediate value*** *theorem* ***and*** *this theorem*.

2. Prove that there are three real numbers x satisfying the equation
$$x^3 - 4x - 2 = 0.$$

Solution: ***First look at a sketch of the graph*** $y = x^3 - 4x - 2$



We now define $f(x) = x^3 - 4x - 2$ ***for every real number*** x ***and observe that*** $f(-2) < 0$ ***and*** $f(-1) > 0$***. Therefore Bolzano's intermediate theorem guarantees that the equation*** $x^3 - 4x - 2 = 0$ ***has a solution between*** -2 ***and*** -1***. Since*** $f(-1) > 0$ ***and*** $f(0) < 0$ ***we know that there is a solution of the equation between*** -1 ***and*** 0***. Finally, from the fact that*** $f(0) < 0$ ***and*** $f(3) > 0$ ***we know that there is a solution of the equation between*** 0 ***and*** 3***.***

3. Is it true that if a set S of real numbers is not an interval then there must exist a one-one continuous function on S whose inverse function fails to be continuous?
Not at all. In fact, we know that if S is closed and bounded then every one-one continuous function S must have a continuous inverse function. Look also at the case in which S is the union of two open intervals that do not intersect with each other. If f is a one-one continuous function on S then

the range of f is also the union of two open intervals that do not intersect with each other and the inverse function of f will be continuous.

4. Is it true that if a set S of real numbers is not an interval and is not closed then there must exist a one-one continuous function on S whose inverse function fails to be continuous?

   Hint: ***Look at the case in which S is the union of two mutually disjoint open intervals.***
   See the remarks about Exercise 3.

5. Is it true that if a set S of real numbers is not an interval and is not bounded then there must exist a one-one continuous function on S whose inverse function fails to be continuous?
   Again the answer is no. The remarks about Exercise 3 are not confined to bounded intervals.

6. Prove that if f is a continuous function from the interval $[0,1]$ into $[0,1]$ then there must be at least one number $x \in [0,1]$ such that $f(x) = x$. This assertion is the one-dimensional form of the **Brouwer fixed point theorem.**

   Solution: ***For every number $x \in [0,1]$ we define***

$$g(x) = f(x) - x.$$

   The function g defined in this way is continuous on the interval $[0,1]$. We see that

$$g(0) = f(0) - 0 \geq 0$$

   and

$$g(1) = f(1) - 1 \leq 0$$

   and we conclude from the Bolzano intermediate value theorem that there is at least one number $x \in [0,1]$ for which $g(x) = 0$.

Exercises on Uniform Continuity

1. Is it true that if S is an unbounded set of real numbers and $f(x) = x^2$ for every number $x \in S$ then the function f fails to be uniformly continuous?
   The assertion given here is false. Every function defined on the set **Z** of integers must be uniformly continuous.

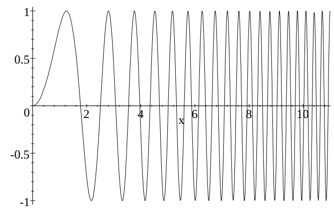
2. Given that

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x < 2 \\ 0 & \text{if} \quad 2 < x \leq 3 \end{cases}$$

   prove that f is continuous but not uniformly continuous on the set $[0,2) \cup (2,3]$.
   Since the number 2 does not belong to the domain of f, the function f is constant in a neighborhood of every number in its domain. Therefore f is continuous on the set $[0,2) \cup (2,3]$. To see why f fails to be uniformly continuous we observe that given any positive number δ we can find a number $t \in [0,2)$ and a number $x \in (2,3]$ such that $|t - x| < \delta$, and for any such choice of numbers x and t we must have

$$|f(t) - f(x)| = |1 - 0| = 1.$$

3. Given that $f(x) = \sin(x^2)$ for all real numbers x, prove that f is not uniformly continuous on the set **R**.



We define

$$x_n = \sqrt{2n\pi + \frac{\pi}{2}}$$

and

$$t_n = \sqrt{2n\pi}$$

for every positive integer n. Since $f(x_n) = 1$ and $f(t_n) = 0$ for every n we know that $f(x_n) - f(t_n)$ does not approach 0 as $n \to \infty$. Now

$$\begin{aligned} x_n - t_n &= \sqrt{2n\pi + \frac{\pi}{2}} - \sqrt{2n\pi} \\ &= \frac{\left(\sqrt{2n\pi + \frac{\pi}{2}} - \sqrt{2n\pi}\right)\left(\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}\right)}{\left(\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}\right)} \\ &= \frac{\frac{\pi}{2}}{\left(\sqrt{2n\pi + \frac{\pi}{2}} + \sqrt{2n\pi}\right)} \to 0 \end{aligned}$$

as $n \to \infty$ and so it follows from the relationship between limits of sequences and uniform continuity that the function f fails to be uniformly continuous.

4. Ask *Scientific Notebook* to make some 2D plots of the function f defined by the equation

$$f(x) = \sin(x \log x)$$

for $x > 0$. Plot the function on each of the intervals $[0,50]$, $[50,100]$, $[100,150]$ and $[150,200]$. Revise your plot and increase its sample size if it appears to contain errors. Why do these graphs suggest that f fails to be unformly continuous on the interval $(0,\infty)$? Prove that this function does, indeed, fail to be uniformly continuous.

Solution: ***To prove that f fails to be uniformly continuous we shall show that for every number $\delta > 0$ there exist two positive numbers a and b such that $|a-b| < \delta$ and $|f(a) - f(b)| \geq 1$. We begin by choosing a number p such that whenever $x \geq p$ we have***

$$\log x > \frac{\pi}{2\delta}.$$

Now choose a positive integer n such that

$$p \log p < n\pi.$$

Since $x \log x > n\pi$ for x sufficiently large x, we can use the Bolzano intermediate value theorem to choose a number $a > p$ such that $a \log a = n\pi$. Now since

$$\begin{aligned} (a+\delta)\log(a+\delta) &= a\log(a+\delta) + \delta\log(a+\delta) \\ &> a\log a + \delta\left(\frac{\pi}{2\delta}\right) = n\pi + \frac{\pi}{2} \end{aligned}$$

we can use the Bolzano intermediate value theorem again to choose a number $b \in (a, a+\delta)$ such that

$$b \log b = n\pi + \frac{\pi}{2}.$$

We now observe that

$$|f(a) - f(b)| = \left|\sin n\pi - \sin\left(n\pi + \frac{\pi}{2}\right)\right| = 1$$

and so the proof is complete.

5. a. A function f from a metric space X to a metric space Y is said to be **Lipschitzian** on a set S if there exists a number k such that the inequality

$$d(f(t), f(x)) \leq kd(t, x)$$

holds for all points t and x in S. Prove that every lipschitzian function is uniformly continuous.

Suppose that f is a function from a metric space X to a metric space Y, that k is a positive number and that the inequality

$$d(f(t), f(x)) \leq kd(t, x)$$

holds for all points t and x in the space X. Suppose that $\varepsilon > 0$. We define $\delta = \varepsilon/k$ and observe that, whenever t and x belong to X and $d(t, x) < \delta$ we have

$$d(f(t), f(x)) \leq kd(t, x) < k\left(\frac{\varepsilon}{k}\right) = \varepsilon.$$

b. Given that $f(x) = \sqrt{x}$ for all $x \in [0, 1]$ prove that f is uniformly continuous but not lipschitzian on $[0, 1]$.

Solution: ***The fact that*** f ***is uniformly continuous on the closed bounded set*** $[0, 1]$ ***follows at once from the fact that*** f ***is continuous there. Now, to prove that*** f ***fails to be Lipschitzian, suppose that*** k ***is any positive number. Given*** $x \in (0, 1]$ ***we see that***

$$\frac{|f(x) - f(0)|}{|x - 0|} = \frac{1}{\sqrt{x}}$$

and this exceeds k ***whenever*** $x < 1/k^2$**.**

6. a. Prove that if f is a uniformly continuous function from a totally bounded metric space X onto a metric space Y then Y is also totally bounded.

We suppose that f is a uniformly continuous function from a totally bounded metric space X onto a metric space Y. To show that Y is totally bounded, suppose that $\varepsilon > 0$. Choose $\delta > 0$ such that whenever t and x belong to X and $d(x, t) < \delta$ we have $d(f(x), f(t)) < \varepsilon$. Using the fact that X is totally bounded, choose finitely many points $x_1, x_2, \cdots, x_n$ in the space X such that

$$X = \bigcup_{j=1}^{n} B(x_j, \delta).$$

We shall now show that

$$Y = \bigcup_{j=1}^{n} B(f(x_j), \varepsilon).$$

Suppose that $y \in Y$. Using the fact that the function f is onto the space Y, we choose a member x of X such that $y = f(x)$. Choose j such that $x \in B(x_j, \delta)$. We see that $y \in B(f(x_j), \varepsilon)$.

b. Give an example of a uniformly continuous function from a bounded metric space onto an unbounded metric space.

If X is the discrete space in which

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

then every function from X to another metric space is continuous.

c. Prove that if a function f is uniformly continuous on a bounded subset S of $\mathbf{R}^k$ into a metric space Y then the range of f is a bounded subset of Y.

We know that whenever S is a bounded subset of $\mathbf{R}^k$, the subspace S is totally bounded.

d. Give a quick proof that if $f(x) = 1/x$ for all $x \in (0, 1)$ then f fails to be uniformly continuous on the interval $(0, 1)$.

Had this function been uniformly continuous then its range would have had to be bounded. But its range is $(0,\infty)$.

7. Suppose that f is a uniformly continuous function from a subset S of a metric space X into a metric space Y.

a. Prove that if (x_n) is a Cauchy sequence in the set S then the sequence $(f(x_n))$ is a Cauchy sequence in the space Y.
Suppose that (x_n) is a Cauchy sequence in the set S, and, to show that $(f(x_n))$ is a Cauchy sequence in the space Y, suppose that $\varepsilon > 0$. Using the fact that f is uniformly continuous on S, choose $\delta > 0$ such that whenever t and x belong to S and $d(t,x) < \delta$, we have $d(f(t),f(x)) < \varepsilon$. Using the fact that (x_n) is a Cauchy sequence, choose an integer N such that, whenever $m \geq N$ and $n \geq N$, we have $d(x_m,x_n) < \delta$. Then, whenever $m \geq N$ and $n \geq N$, we have $d(f(x_m),f(x_n)) < \varepsilon$.

b. Prove that if the space Y is complete then whenever a sequence (x_n) in the set S converges to a point $x \in X$ the sequence $(f(x_n))$ converges in the space Y.
We assume that Y is complete and that (x_n) is a sequence in S that converges to a point x of X. Note that we cannot simply claim that $f(x_n) \to f(x)$ because we have not said that the point x must belong to the set S. However, the sequence (x_n) must be a Cauchy sequence and therefore, by part a, the sequence $(f(x_n))$ must be a Cauchy sequence in Y, and must therefore converge in Y because Y is complete.

c. Prove that if $x \in \overline{S}$ and if (x_n) and (t_n) are two sequences in the set S both of which converge to x and if the space Y is complete then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(t_n).$$

We assume that (x_n) and (t_n) are sequences in the set S and that both of these sequences converge to x. We now consider the sequence $x_1, t_1, x_2, t_2, \cdots$. More precisely, we define

$$u_n = \begin{cases} t_{n/2} & \text{if } n \text{ is even} \\ x_{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Since the sequence (u_n) also converges to x, it is also a Cauchy sequence. Therefore the sequence $(f(u_n))$ is convergent in the space Y. Since both $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(t_n)$ are partial limits of the sequence $(f(u_n))$ we deduce that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} f(u_n).$$

d. Given that Y is complete and that $x \in \overline{S} \setminus S$, explain how we can use Part c to extend the definition of the function f to the point x in such a way that the extension is continuous on the set $S \cup \{x\}$.
The continuity of f at the point x is really very simple. If (y_n) is any sequence in the set $S \cup \{x\}$ converging to x then it is clear that $f(y_n) \to f(x)$. This part of the exercise also follows at once from part e that we shall prove below.

e. Prove that if Y is complete then there exists a uniformly continuous function g from the set $\overline{S}$ into Y such that $g(x) = f(x)$ for every point $x \in S$.
For each point $x \in \overline{S} \setminus S$ we define $f(x)$ by choosing a sequence (x_n) in S that converges to x and defining

$$f(x) = \lim_{n\to\infty} f(x_n).$$

We shall now show that this extension of the function f to $\overline{S}$ is uniformly continuous on $\overline{S}$. Suppose that $\varepsilon > 0$.
First Proof (a bit messy, perhaps). Using the fact that f is uniformly continuous on S, choose $\delta > 0$ such that whenever u and v belong to S and $d(u,v) < \delta$ we have $d(f(u),f(v)) < \varepsilon/3$. Now suppose that u and v are any points of $\overline{S}$ satisfying the inequality $d(u,v) < \delta/3$. Choose a sequence (u_n) in S converging to u and a sequence (v_n) in S converging to v. Using the facts that $u_n \to u$ and $v_n \to v$ and $f(u_n) \to f(u)$ and $f(v_n) \to f(v)$ as $n \to \infty$, choose N such that whenever $n \geq N$ we have

$$d(f(u_n), f(u)) < \frac{\varepsilon}{3}$$
$$d(f(v_n), f(v)) < \frac{\varepsilon}{3}$$
$$d(u_n, u) < \frac{\delta}{3}$$
and $d(v_n, v) < \frac{\delta}{3}$.

Since
$$d(u_N, v_N) \leq d(u_N, u) + d(u, v) + d(v, v_N) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta$$
and therefore
$$d(f(u_N), f(v_N)) < \frac{\varepsilon}{3}.$$
We conclude that
$$d(f(u), f(v)) \leq d(f(u), f(u_N)) + d(f(u_N), f(v_N)) + d(f(v_N), f(v)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Second Proof. (less messy but a little too slick, perhaps) Using the fact that f is uniformly continuous on S, choose $\delta > 0$ such that whenever u and v belong to S and $d(u, v) < \delta$ we have $d(f(u), f(v)) < \varepsilon$. Now suppose that u and v are any points of $\bar{S}$ satisfying the inequality $d(u, v) < \delta$. Choose a sequence (u_n) in S converging to u and a sequence (v_n) in S converging to v. Since $u_n \to u$ and $v_n \to v$ and $f(u_n) \to f(u)$ and $f(v_n) \to f(v)$ as $n \to \infty$, we see that $d(u_n, v_n) \to d(u, v)$ and $d(f(u_n), f(v_n)) \to d(f(u), f(v))$. Using the fact that $d(u, v) < \delta$, choose N such that the inequality $d(u_n, v_n) < \delta$ holds whenever $n \geq N$. Thus, for all $n \geq N$ we have $d(f(u_n), f(v_n)) < \varepsilon$ and therefore
$$d(f(u), f(v)) = \lim_{n \to \infty} d(f(u_n), f(v_n)) \leq \varepsilon.$$

8.

a. Given that S is a set of real numbers, that $a \in \bar{S} \setminus S$ and that
$$f(x) = \frac{1}{x - a}$$
for all $x \in S$, prove that f is continuous on S but not uniformly continuous. (Use this exercise.)
The result follows at once.

b. Given that S is a set of real numbers and that S fails to be closed, prove that there exists a continuous function on S that fails to be uniformly continuous on S.
The result follows from part a.

c. Is it true that if S is an unbounded set of real numbers then there exists a continuous function on S that fails to be uniformly continuous on S?
No, it isn't true. Every function defined on the set **Z** of integers is uniformly continuous there.

9. Given that f is a function defined on a metric space X, prove that the following conditions are equivalent:
This exercise was left in accidentally after being elevated to the status of a theorem. It appears as Theorem 8.15.4.

a. The function f fails to be uniformly continuous on the space X.

b. There exists a number $\varepsilon > 0$ and two sequences (t_n) and (x_n) in X such that $d(x_n, t_n) \to 0$ as $n \to \infty$ and
$$d(f(x_n), f(t_n)) \geq \varepsilon$$
for every n.

10. Is it true that the composition of a uniformly continuous function with a uniformly continuous function is uniformly continuous?
Yes, it's true. Suppose that f is a uniformly continuous function from a metric space X to a metric space Y and that g is a uniformly continuous function from Y to a metric space Z. To show that the composition $g \circ f$ is uniformly continuous, suppose that $\varepsilon > 0$. Using the fact that g is uniformly

continuous on the space Y, choose $\delta > 0$ such that, whenever y_1 and y_2 belong to Y, and $d(y_1, y_2) < \delta$, we have $d(g(y_1), g(y_2)) < \varepsilon$. Now, using the fact that f is uniformly continuous on the space X, choose $\gamma > 0$ such that, whenever x_1 and x_2 belong to X and $d(x_1, x_2) < \gamma$, we have $d(f(x_1), f(x_2)) < \delta$. Then, whenever x_1 and x_2 belong to X and $d(x_1, x_2) < \gamma$ we have $d(g(f(x_1)), g(f(x_2))) < \varepsilon$.

11. Suppose that f is a continuous function on a subset S of a compact metric space X. Prove that the following two conditions are equivalent:

    a. The function f is uniformly continuous on S.

    b. It is possible to extend f to a continuous function on the set $\bar{S}$.

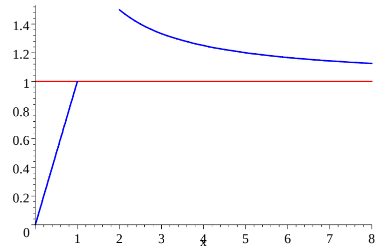
    The fact that condition a implies condition b follows at once from Exercise 7. Since $\bar{S}$ is compact, a continuous function on $\bar{S}$ must be uniformly continuous. Therefore, if f has a continuous extension to $\bar{S}$, then f must be uniformly continuous on S.

12. Is it true that if f is a uniformly continuous function from a bounded metric space X onto a metric space Y then Y is bounded?
    Not a chance! If X is the discrete space in which
$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$
    then every function from X to another metric space must be uniformly continuous.

13. Is it true that if f is a uniformly continuous function from a complete metric space X onto a metric space Y then Y is complete?
    No, this statement is false. For example, if we define
$$f(x) = \frac{1}{1+x^2}$$
    for $x \in [0, \infty)$ then f is a uniformly continuous function from $[0, \infty)$ onto $(0, 1]$.

14. Suppose that f is a one-one uniformly continuous function from a metric space X onto a metric space Y and that the inverse function f^{-1} of f is continuous on Y. Prove that if Y is complete then so is X.
    Suppose that (x_n) is a Cauchy sequence in the space X. Since f is uniformly continuous from X to Y, it is easy to see that the sequence $(f(x_n))$ is a Cauchy sequence in the space Y. Since Y is complete, the sequence $(f(x_n))$ must converge. We define
$$y = \lim_{n\to\infty} f(x_n).$$
    Since the function f^{-1} is continuous at the point y we have $f^{-1}(f(x_n)) \to f^{-1}(y)$ as $n \to \infty$. In other words, $x_n \to f^{-1}(y)$ as $n \to \infty$ and we have shown that the sequence (x_n) is convergent.

15. True or false: If f is a uniformly continuous one-one function from a complete metric space X onto a complete metric space Y then the inverse function f^{-1} of f must be continuous.
    No, this statement is false. We take $X = [0,1] \cup [2, \infty)$ and we define
$$f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 1 \\ 1 + \frac{1}{x} & \text{if} \quad x \ge 2 \end{cases}$$



The above figure also shows the line $y = 1$ in red. We see that f is a uniformly continuous one-one function from the complete space $[0,1] \cup [2,\infty)$ onto the complete space $\left[0, \frac{3}{2}\right]$.

16. We shall say that a subset S of a metric space X is **compressed** if for every number $\varepsilon > 0$ there exist two different points x and t in S such that $d(t,x) < \varepsilon$.

    a. Prove that every compressed subset of a metric space must be infinite.
    This statment is obvious because any finite set positive numbers has a least member that is positive.

    b. Prove that if a subset S of a metric space X has a limit point then S must be compressed.
    Suppose that S is a subset of a metric space X and that S has a limit point x. Suppose that $\varepsilon > 0$. Choose a point $u \in B(x, \varepsilon/2) \setminus \{x\}$. Now, using the fact that the number $d(x,u)$ is positive, choose a point $v \in B(x, d(x,u)) \setminus \{x\}$. Since both u and v belong to the ball $B(x, \varepsilon/2)$ we have $d(u,v) < \varepsilon$.

    c. Give an example of a compressed subset S of the metric space $\mathbf{R}$ such that S has no limit point.
    Look at the set
    $$\mathbf{Z}^+ \cup \left\{ n + \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\}.$$

    d. Is it true that if neither of two subsets A and B of a metric space X is compressed then the set $A \cup B$ cannot be compressed?
    No! Again, look at the set
    $$\mathbf{Z}^+ \cup \left\{ n + \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\}.$$

    e. Prove that if A is a finite subset of a metric space X and B is a subset of X and B is not compressed then the set $A \cup B$ is not compressed.
    Choose a number $\delta_1 > 0$ such that whenever x and y belong to B and $x \neq y$ we have $d(x,y) \geq \delta_1$. We know that, since neither of the sets A and B has a limit point, nor does the set $A \cup B$. Using the fact that A is finite and the fact that no member of A is a limit point of $A \cup B$, choose a number $\delta_2 > 0$ such that, whenever $x \in A$ we have
    $$B(x, \delta_2) \cap (A \cup B) \setminus \{x\} = \emptyset.$$
    We now define δ to be the smaller of the two numbers δ_1 and δ_2 and we observe that whenever x and y belong to $A \cup B$ and $x \neq y$ we have $d(x,y) \geq \delta$.

17. We shall say that a metric space X is **strongly complete** if every compressed subset of X has a limit point.

    a. Prove that every strongly complete metric space is complete.
    Suppose that X is a strongly complete metric space and that (x_n) is a Cauchy sequence in X. From the Cauchy condition we see that, unless the sequence (x_n) is constant from some point on, the range of (x_n) is compressed and has a limit point to which the sequence (x_n) must converge.

    b. Prove that every compact metric space is strongly complete.
    Every compressed set is infinite and every infinite subset of a compact space must have a limit point.

c. Prove that the metric space $\mathbf{Z}$ of integers is strongly complete but not compact.

The metric space $\mathbf{Z}$ has no compressed subsets and therefore cannot have a compressed subset that fails to have a limit point. Therefore $\mathbf{Z}$ is strongly complete. The fact that $\mathbf{Z}$ fails to be compact is obvious: No subset of $\mathbf{Z}$ can have a limit point, even though $\mathbf{Z}$ is infinite.

d. Prove that if d is the discrete metric on an infinite set X then the metric space (X, d) is strongly complete but not compact.

We can repeat, almost word for word, what we said about the space $\mathbf{Z}$.

e. Improve the principal theorem on uniform continuity by proving that every continuous function from a strongly complete metric space X to a metric space Y must be uniformly continuous on X.

a. **Solution:** ***Suppose that X is a strongly complete metric space and that f is a continuous function from X to a metric space Y. To obtain a contradiction we shall assume that the function f fails to be uniformly continuous on X. Choose a number $\varepsilon > 0$ such that for every number $\delta > 0$ there exist points x and t in the space X such that $d(t, x) < \delta$ and $d(f(t), f(x)) \geq \varepsilon$.***

For every positive integer n we now choose points that we shall call t_n and x_n in X such that $d(t_n, x_n) < \delta$ and

$$d\big(f(t_n), f(x_n)\big) \geq \varepsilon.$$

Now we define

$$S = \{t_n \mid n = 1, 2, \cdots\} \cup \{x_n \mid n = 1, 2, \cdots\}$$

and, using the fact that S is compressed and the fact that the space X is strongly complete, we choose a limit point u of the set S.

Using the fact that the function f is uniformly continuous at the point u we choose a number $\delta > 0$ such that whenever $x \in X$ and $d(u, x) < \delta$ we have

$$d\big(f(t), f(x)\big) < \frac{\varepsilon}{2}.$$

Now since u is a limit point of S we know that infinitely many members of S must lie in the ball $B(u, \delta/2)$ and, using this fact, we choose a positive integer $n > 2/\delta$ such that either t_n or x_n lies in the ball $B(u, \delta/2)$. Since one of the points t_n and x_n lies within a distance $\delta/2$ or u and since

$$d(t_n, x_n) < \frac{1}{n} < \frac{\delta}{2}$$

we know that both t_n and x_n lie within a distance δ of u. Thus

$$\begin{aligned} d(f(t_n), f(x_n)) &\leq d(f(t_n), f(u)) + d(f(u), f(x_n)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which contradicts the way in which the points t_n and x_n were chosen.

f. Suppose that S is a compressed subset of a metric space X. Prove that it is possible to find two sequences (a_n) and (b_n) in the set S such that

$$\{a_n \mid n \in \mathbf{Z}^+\} \cap \{b_n \mid n \in \mathbf{Z}^+\} = \emptyset$$

and such that the inequality $d(a_n, b_n) < 1/n$ holds for every $n \in \mathbf{Z}^+$.

Solution: ***Using the fact that S is compressed we choose two members of S, that we shall call a_1 and b_1 such that $a_1 \neq b_1$ and $d(a_1, b_1) < 1$. Since S is compressed and the set $\{a_1, b_1\}$ is finite we know that the set***

$$S \setminus \{a_1, b_1\}$$

is compressed. Using this fact we choose two members that we shall call a_2 and b_2 of the set $S \setminus \{a_1, b_1\}$ such that $a_2 \neq b_2$ and

$$d(a_2, b_2) < \frac{1}{2}.$$

By continuing this process we arrive at the desired sequences (a_n) and (b_n).

g. Suppose that A and B are subsets of a metric space X, that $A \cup B$ has no limit point, that $A \cap B = \emptyset$ and that for every positive number δ it is possible to find a member a of the set A and a member b of the set B such that $d(a,b) < \delta$. Prove that there exists a continuous function f from X to $\mathbf{R}$ such that f is not uniformly continuous on X. Hint: Use Urysohn's lemma.
Since $A \cup B$ has no limit point, the two sets A and B are closed. The continuous function
$$\frac{\rho_A}{\rho_A + \rho_B}$$
fails to be uniformly continuous on X.

h. Prove that if every continuous function from a given metric space X is uniformly continuous then the space X must be strongly complete.
This part follows at once from parts f and g.

9 Differentiation

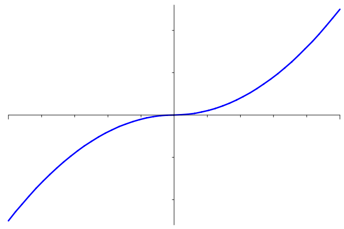
Exercises on Derivatives

1. Given that $f(x) = |x|$ for every number x, prove that $f'(0)$ does not exist.
Since
$$\lim_{t\to 0+} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0+} \frac{t}{t} = 1$$
and
$$\lim_{t\to 0-} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0+} \frac{-t}{t} = -1$$
the two sided limit
$$\lim_{t\to 0} \frac{f(t)-f(0)}{t-0}$$
fails to exist.

2. Given that $f(x) = |x|$ for all $x \in [-2,-1] \cup [0,1]$, prove that $f'(0)$ does exist.

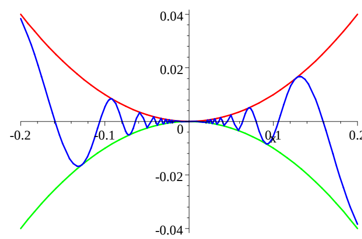
Hint: ***Observe that whenever*** $x \in [-2,-1] \cup [0,1]$ ***and*** $|x-0| < 1$ ***we have*** $x \geq 0$ ***and therefore, if*** $x \neq 0$ ***we have***
$$\frac{f(x)-f(0)}{x-0} = \frac{x-0}{x-0} = 1.$$

3. Given that $f(x) = x|x|$ for every number x, determine whether or not $f'(0)$ exists.
We observe that
$$\lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0} \frac{t|t|}{t} = \lim_{t\to 0} |t| = 0$$
and so $f'(0) = 0$. Students may want to sketch the graph of this function using ***Scientific Notebook***.
$x|x|$



4. This exercise concerns the function f defined by the equation
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$
You should assume all the standard formulas for the derivatives of the functions sin and cos.

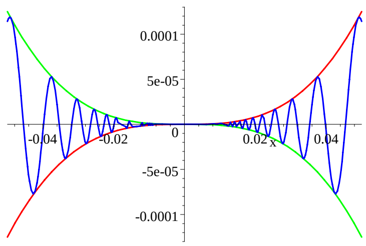
   a. Ask *Scientific Notebook* to make a 2D plot of the expression $x^2 \sin \frac{1}{x}$ on the interval $[-.2, .2]$ and then drag each of the expressions x^2 and $-x^2$ into your plot. Revise the plot and give the components different colors.



   b. Prove that the function f is differentiable on $\mathbf{R}$ but that the function f' is not continuous at the number 0.
   For each $t \neq 0$ we have
$$\frac{f(t) - f(0)}{t - 0} = \frac{t^2 \sin \frac{1}{t}}{t} = t \sin \frac{1}{t}$$
   and, since
$$\left| \frac{f(t) - f(0)}{t - 0} \right| = \left| t \sin \frac{1}{t} \right| \leq |t|$$
   for each $t \neq 0$ we can deduce from the sandwich theorem that
$$\lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = 0.$$
   Now given any $x \neq 0$ we have
$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$
   and the latter expression does not approach a limit as $x \to 0$. Therefore f' is not continuous at 0.

5. This exercise concerns the function f defined by the equation
$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}.$$

a. Ask *Scientific Notebook* to make a 2D plot of the expression $x^3 \sin \frac{1}{x}$ on the interval $[-.05, .05]$ and then drag each of the expressions x^3 and $-x^3$ into your plot. Revise the plot and give the components different colors.$x^3 \sin \frac{1}{x}$



b. Prove that the function f' is continuous at the number 0 but does not have a derivative there.

The fact that $f'(0) = 0$ follows in exactly the same way as Exercise 4. Now given any $x \neq 0$ we have

$$f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$$

and since

$$|f'(x)| \leq 3x^2 + |x|$$

for $x \neq 0$ we have

$\lim_{x \to 0} f'(x) = 0 = f'(0)$. To see why f'has no derivative at 0 we can argue as in Exercise 4.

6. Suppose that f is a function defined on an open interval (a, b) and that $x \in (a, b)$.

a. Prove that if $f'(x)$ exists then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Solution: ***We assume that*** $f'(x)$ ***exists. Thus***

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

To show that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

suppose that $\varepsilon > 0$***. Choose a number*** $\delta > 0$ ***such that the inequality***

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

holds whenever $t \in (a, b)$ ***and*** $t \neq x$ ***and*** $|t - x| < \delta$***. Then whenever*** $h \neq 0$ ***and*** $|h| < \delta$ ***and*** $h \in (a - x, b - x)$ ***we deduce from the fact that*** $x + h \in (a, b)$ ***and*** $|(x + h) - x| < \delta$ ***that***

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = \left| \frac{f(x+h) - f(x)}{(x+h) - x} - f'(x) \right| < \varepsilon.$$

b. Prove that if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

exists then $f'(x)$ exists and is equal to this limit.

c. Prove that if $f'(x)$ exists then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

Hint: ***Use the fact that whenever*** $h \neq 0$ ***and is sufficiently small we have***

$$\frac{f(x+h)-f(x-h)}{2h} = \frac{f(x+h)-f(x)-(f(x-h)-f(x))}{2h}$$
$$= \frac{1}{2}\frac{f(x+h)-f(x)}{h} + \frac{1}{2}\frac{f(x+(-h))-f(x)}{-h}$$

d. Prove that if $f'(x)$ exists then

$$f'(x) = \lim_{t\to x}\left(\lim_{u\to x}\frac{f(t)-f(u)}{t-u}\right)$$

Since $f'(x)$ exists, the function f must be continuous at x. Therefore the right side is

$$\lim_{t\to x}\left(\lim_{u\to x}\frac{f(t)-f(u)}{t-u}\right) = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right) = f'(x).$$

Exercises on the Mean Value Theorem

1. a. Given that f is a function defined on an interval S and that $f'(x) = 0$ for every $x \in S$, prove that f must be constant on S.
   Suppose that a and b are numbers in the interval S and $a < b$. We shall show that $f(a) = f(b)$. Applying the mean value theorem to the function f on the interval $[a,b]$ we choose a number c between a and b such that
   $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$
   Since $f'(c) = 0$ we deduce that $f(b)-f(a) = 0$ which gives us $f(a) = f(b)$.

   b. Given that f is a function defined on an interval S and that $f'(x) > 0$ for every $x \in S$, prove that f must be strictly increasing on S.

   Solution: ***Suppose that*** a ***and*** b ***are numbers in the interval*** S ***and*** $a < b$***. We shall show that*** $f(a) < f(b)$***. Applying the mean value theorem to the function*** f ***on the interval*** $[a,b]$ ***we choose a number*** c ***between*** a ***and*** b ***such that***
   $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$
   Since $f'(c) > 0$ ***and*** $b-a > 0$ ***we deduce that*** $f(b)-f(a) > 0$ ***which gives us*** $f(a) < f(b)$***.***

   c. Given that f is a function defined on an interval S and that $f'(x) < 0$ for every $x \in S$, prove that f must be strictly decreasing on S.
   Suppose that a and b are numbers in the interval S and $a < b$. We shall show that $f(a) > f(b)$. Applying the mean value theorem to the function f on the interval $[a,b]$ we choose a number c between a and b such that
   $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$
   Since $f'(c) < 0$ and $b-a > 0$ we deduce that $f(b)-f(a) < 0$ which gives us $f(a) > f(b)$.

2. Suppose that f and g are functions defined on an interval S and that $f'(x) = g'(x)$ for every number $x \in S$. Prove that there exists a real number c such that the equation
   $$f(x) = g(x) + c$$
   holds for every number $x \in S$.

   Hint: ***Apply the preceding exercise to the function*** $f-g$***.***

3. Suppose that f is continuous on an interval $[a,b]$ and differentiable on the interval (a,b) and that $f(a) = f(b)$. Suppose that $a < c < b$ and that $f'(x) > 0$ when $a < x < c$ and $f'(x) < 0$ when $c < x < b$. Prove that $f(c)$ is the maximum value of the function f.
   Since f is strictly increasing on the interval $[a,c]$ we have $f(x) < f(c)$ whenever $a \le x < c$ and since f is strictly decreasing on the interval $[c,b]$ we have $f(c) > f(x)$ whenever $c < x \le b$.

4. Given that f is a strictly increasing differentiable function on an interval S, is it true that $f'(x)$ must be positive for every $x \in S$?
   No. If we define $f(x) = x^3$ for every number x then, although f is strictly increasing, we have $f'(0) = 0$.

5. Prove that if f is a differentiable function on an interval S and $f'(x) \neq 0$ for every $x \in S$ then the function f must be one-one.

   Hint: ***You should be able to make your conclusion very quickly from Rolle's theorem.***
   We assume that $f'(x) \neq 0$ for every number x in an interval S. Given numbers a and b in the interval, if $a \neq b$ and $f(a) = f(b)$ then we could apply Rolle's theorem to find a number c between a and b such that $f'(c) = 0$, which is impossible. Therefore f is one-one.
   Note: Had this exercise told us that $f'(x) > 0$ for every $x \in S$ we could have used Exercise 1 to deduce that f is strictly increasing and if we had been told that $f'(x) < 0$ for each x then we would know that f is strictly decreasing. Some students have wanted to argue that the given information, that $f'(x) \neq 0$ for every $x \in S$ guarantees that $f'(x)$ must either be always positive or always negative. This is true but we don't know it yet. The intermediate value theorem for derivatives is deduced in Exercises 10 and 11 below.

6. Given that f is differentiable on an interval S and that the function f' is bounded on S, prove that f must be lipschitzian on S.
   Using the fact that f' is bounded we choose a positive number k such that $|f'(x)| \le k$ for every number $x \in S$. Now suppose that t and x are any numbers in the interval S. We shall show that
   $$|f(t) - f(x)| \le k|t - x|.$$
   If $t = x$ this inequality is obvious. Suppose that $t \neq x$. Using the mean value theorem we choose a number c between t and x such that
   $$f'(c) = \frac{f(t) - f(x)}{t - x}$$
   and observe that
   $$\left|\frac{f(t) - f(x)}{t - x}\right| \le |f'(c)| \le k$$

7. Given that f is a function defined on an interval S and that the inequality
   $$|f(t) - f(x)| \le |t - x|^2$$
   holds for all numbers t and x in S, prove that f must be constant.
   Given any number $x \in S$ we deduce from the inequality
   $$0 \le \left|\frac{f(t) - f(x)}{t - x}\right| \le |t - x|$$
   that holds for all $t \in S \setminus \{x\}$ and the sandwich theorem that
   $$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = 0.$$
   Since $f'(x) = 0$ for every x in the interval S, the function f must be constant.

8. Suppose that f and g are functions defined on $\mathbf{R}$ and that $f'(x) = g(x)$ and $g'(x) = -f(x)$ for every real number x.

   a. Prove that $f''(x) = -f(x)$ for every number x.
      Quite simply, for each x
      $$f''(x) = g'(x) = -f(x).$$

b. Prove that the function $f^2 + g^2$ is constant.
If we define $h = f^2 + g^2$ then $h' = 2ff' + 2gg' = 2fg - 2fg = 0$ and so h is constant.

9. Given that f is differentiable on the interval $(0, \infty)$ and that $f'(x) \to \lambda$ as $x \to \infty$, prove that

$$f(x+1) - f(x) \to \lambda$$

as $x \to \infty$.

Solution: ***Suppose that*** $\varepsilon > 0$ ***and, using the fact that*** $f'(x) \to \lambda$ ***as*** $x \to \infty$***, choose a number*** w ***such that the inequality***

$$|f'(x) - \lambda| < \varepsilon$$

holds whenever $x \geq w$***. We shall now show that the inequality***

$$\left|\left(f(x+1) - f(x)\right) - \lambda\right| < \varepsilon$$

holds whenever $x \geq w$***. Suppose that*** $x \geq w$***.***

Applying the mean value theorem to f on the interval $[x, x+1]$ ***we now choose a number*** c ***between*** x ***and*** $x+1$ ***such that***

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f'(c)$$

and we observe that

$$\left|\left(f(x+1) - f(x)\right) - \lambda\right| = \left|\frac{f(x+1) - f(x)}{(x+1) - x} - \lambda\right| = |f'(c) - \lambda| < \varepsilon.$$

10. Given that f is continuous on $[a, b]$ and differentiable on (a, b), and that $f'(x)$ approaches a limit $w \in \mathbf{R}$ as $x \to a$, prove that f must be differentiable at the number a and that $f'(a) = w$.
We need to show that

$$\frac{f(t) - f(a)}{t - a} \to w$$

as $t \to a+$. Suppose $\varepsilon > 0$ and choose $\delta > 0$ such that the condition $|f'(x) - w| < \varepsilon$ whenever $x \in [a, b]$ and $x < a + \delta$. Given any number $t \in [a, b]$ satisfying the inequality $a < t < a + \delta$ we choose x between a and t such that

$$\frac{f(t) - f(a)}{t - a} = f'(x)$$

and conclude that

$$\left|\frac{f(t) - f(a)}{t - a} - w\right| < \varepsilon.$$

a $\qquad$ x $\qquad$ t $\qquad$ $a + \delta$

11. Prove that if f is differentiable on an interval $[a, b]$ and $f'(a) < 0$ and $f'(b) > 0$ then there must be at least one number $c \in (a, b)$ for which $f'(c) = 0$.

Hint: ***Look at the number at which the function f takes its minimum value.***
We know from Fermat's theorem that f does not have its minimum at a or at b and so f must have its minimum at some number c between a and b. From Fermat's theorem again we deduce that $f'(c) = 0$.

12. Prove that if f is differentiable on an interval S then the range of the function f' must be an interval.

Hint: ***Use the preceding exercise.***
This argument should now be an exact parallel to the Bolzano intermediate value theorem for continuous functions.

13. Suppose that f is differentiable on the interval $[0, \infty)$, that $f(0) = 0$ and that f' is increasing on $[0, \infty)$. Prove

that if

$$g(x) = \frac{f(x)}{x}$$

for all $x > 0$ then the function g is increasing on $(0,\infty)$.

Solution: ***In order to prove that the differentiable function g is increasing on $[0,\infty)$ we shall show that $g'(x) \geq 0$ for every number $x > 0$. We therefore need to show that***

$$\frac{xf'(x) - f(x)}{x^2} \geq 0$$

for all $x > 0$ and we can express this desired inequality as

$$\frac{f(x)}{x} \leq f'(x).$$

Suppose that $x > 0$. We now apply the mean value theorem to the function f on the interval $[0,x]$ to choose a number c between 0 and x such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

Since $c < x$ and since the function f' is increasing we know that $f'(c) \leq f'(x)$ and we have therefore deduced that the inequality

$$\frac{f(x)}{x} \leq f'(x)$$

holds.

14. Suppose that f is defined on an open interval S and that $f''(x) < 0$ for every number $x \in S$. Suppose that $a \in S$. Prove that if $x \in S$ and $x > a$ then

$$f(x) < f(a) + (x-a)f'(a).$$

In other words, explain why, to the right of the point $(a, f(a))$, the graph of f lies below the tangent line to the graph at $(a, f(a))$.

Suppose that $x \in S$ and that $a < x$. Using the mean value theorem we choose a number c between a and x such that

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

Now since f'' is everywhere negative in S the function f' must be strictly decreasing and so

$$\frac{f(x) - f(a)}{x - a} = f'(c) < f'(a)$$

from which we deduce that

$$f(x) < f(a) + (x-a)f'(a).$$

15. Suppose that f is defined on an open interval S and that $f''(x) < 0$ for every number $x \in S$. Suppose that a and b belong to S and that $a < b$. Suppose that

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

for all $x \in [a,b]$.

a. Prove that there exists a number $c \in (a,b)$ such that $g'(c) = 0$.
We apply Rolle's theorem to g exactly as in the proof of the mean value theorem.

b. Prove that the function g' is strictly decreasing on the interval $[a,b]$.
The function g' is strictly decreasing because $g''(x) = f''(x) < 0$ for each x.

c. Prove that the function g is strictly increasing on $[a,c]$ and strictly decreasing on $[c,b]$.
Since $g'(c) = 0$ and g' is strictly decreasing we know that $g'(x) > 0$ whenever $a \leq x < c$ and $g'(x) < 0$ whenever $c < x \leq b$.

d. Prove that $g(x) > 0$ for every $x \in (a,b)$.
Since $g(a) = 0$ and g is strictly increasing on $[a,c]$ we have $g(x) > 0$ whenever $a < x \leq c$. Since

$g(b) = 0$ and g is strictly decreasing on $[c,b]$ we have $g(x) > 0$ whenever $c \le x < b$.

e. Prove that the straight line segment that joins the points $(a, f(a))$ and $(b, f(b))$ lies under the part of the graph of f that lies between the two points $(a, f(a))$ and $(b, f(b))$.
This exercise is asking us to prove that

$$f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) < f(x)$$

whenever $a < x < b$ and this fact follows at once from Part d.

16. A function f defined on an interval S is said to be **convex** on S if whenever a, x and b belong to S and $a < x < b$ we have

$$\frac{f(x) - f(a)}{x - a} \le \frac{f(b) - f(x)}{b - x}.$$

Prove that if f is differentiable on an interval S then f is convex on S if and only if the function f' is increasing.

This exercise is asking us for two proofs.

Part 1: We assume that f is a convex function on an interval S and we want to prove that f' is increasing on S. We begin by making the observation that if a and s and b and t belong to S and if $a < t < s < b$

$a \qquad t \qquad\qquad s \qquad b$

then

$$\frac{f(t) - f(a)}{t - a} \le \frac{f(s) - f(t)}{s - t} \le \frac{f(b) - f(s)}{b - s}$$

Suppose now that a and b are any two numbers in S and that $a < b$. If t and s are numbers unequal to a and b but sufficiently close to a and b, respectively, then $t < s$ and, regardless of the order in which a and t appear and the order in which s and b appear we have

$$\frac{f(t) - f(a)}{t - a} \le \frac{f(s) - f(b)}{s - b}.$$

Therefore

$$f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} \le \lim_{s \to b} \frac{f(s) - f(b)}{s - b} = f'(b)$$

Part 2: We assume that the function f' is increasing and we want to prove that f is a convex function. Suppose that a and x and b lie in the interval S and that $a < x < b$. Using the mean value theorem we choose a number c between a and x and a number d between x and b such that

$$\frac{f(x) - f(a)}{x - a} = f'(c) \qquad \text{and} \qquad \frac{f(b) - f(x)}{b - x} = f'(d).$$

Since f' is increasing we have

$$\frac{f(x) - f(a)}{x - a} = f'(c) \le f'(d) \le \frac{f(b) - f(x)}{b - x}.$$

17. Prove that if f is a convex function on an open interval S then f must be continuous on S.

a. **Solution:** ***Suppose that*** $x \in S$***. We shall show that*** f ***must be continuous at*** x***. Choose numbers*** a***,*** b***,*** c ***and*** d ***in*** S ***such that*** $a < b < x < c < d$***.***

$a \qquad b \qquad x \qquad\qquad c \qquad\qquad d$

Now given any number t ***between*** x ***and*** c

$a \qquad b \qquad x \qquad t \qquad c \qquad\qquad d$

we have

$$\frac{f(x) - f(b)}{x - b} \le \frac{f(t) - f(x)}{t - x} \le \frac{f(c) - f(t)}{c - t} \le \frac{f(d) - f(c)}{d - c}$$

from which we deduce that

$$\frac{f(x)-f(b)}{x-b}(t-x) \le f(t)-f(x) \le \frac{f(d)-f(c)}{d-c}(t-x)$$

and the fact that

$$\lim_{t\to 0+}\left(f(t)-f(x)\right) = 0$$

follows at once from the sandwich theorem. We can show similarly that

$$\lim_{t\to 0-}\left(f(t)-f(x)\right) = 0$$

and we therefore know that** f **is continuous at the number** x**.

18. By clicking on the icon you can reach some exercises that introduce Newton's method for approximating roots of an equation.

Some Exercises on Taylor Polynomials

1. Suppose that f is a polynomial whose degree does not exceed a given positive integer k and that $f^{(j)}(0) = 0$ for every $j = 0, 1, 2, \cdots, k$. Prove that f is the constant function zero.
   The fact that f is the constant function 0 follows at once from the equation

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}x^j$$

   that holds for every real number x.

2. Suppose that f and g are two polynomials whose degrees do not exceed a given positive integer k and that $f^{(j)}(0) = g^{(j)}(0)$ for every $j = 0, 1, 2, \cdots, k$. Prove that $f(x) = g(x)$ for every number x.

   Hint: ***Apply the preceding exercise to the polynomial** $f-g$.*

3. Prove that if f is a polynomial whose degree does not exceed a given positive integer k and n is an integer satisfying $n \ge k$ then the nth Taylor polynomial of f is f itself.
   Since $k \le n$, we can find numbers $a_0, a_1, \cdots, a_n$ such that the equation

$$f(x) = \sum_{j=0}^{n} a_j x^j$$

   holds for every number x.

4. Given a nonnegative integer n and that

$$f(x) = (1+x)^n$$

   for every number x, work out the nth Taylor polynomial of f and obtain a simple proof of the **binomial theorem**

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

   As you may know, the expression $\binom{n}{j}$, which is called the n, j **binomial coefficient,** is defined by the equation

$$\binom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!} = \frac{n!}{(n-j)!j!}$$

   whenever n and j are integers and $0 \le j \le n$. We shall define binomial coefficients more generally later.
   All we have to observe is that if x is any number and j is a nonnegative integer then

$$f^{(j)}(x) = n(n-1)\cdots(n-j+1)(1+x)^{n-j}$$

   and that, consequently,

$$\frac{f^{(j)}(0)}{n!} = \frac{n(n-1)\cdots(n-j+1)}{n!}$$

Some Exercises on Indeterminate Forms

1. Evaluate each of the following limits. In each case, use *Scientific Notebook* to verify that your limit value is correct.

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right) \qquad \lim_{x\to 0}\left(\frac{\tan x - x}{x-\sin x}\right)$$

$$\lim_{x\to 0}\left(\frac{\tan x-\sin x}{x^3}\right) \qquad \lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right)$$

$$\lim_{x\to 0}\left((1+x)^{1/x}\right) \qquad \lim_{x\to 0}\left(\frac{e-(1+x)^{1/x}}{x}\right)$$

To find the limit

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right)$$

we observe first that

$$\lim_{x\to 0}(x-\sin x) = \lim_{x\to 0}x^3 = 0$$

and so L'Hôpital's rule guarantees that

$$\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{3x^2}\right)$$

provided that the latter limit exists. Since

$$\lim_{x\to 0}(1-\cos x) = \lim_{x\to 0}(3x^2) = 0$$

L'Hôpital's rule guarantees that

$$\lim_{x\to 0}\left(\frac{1-\cos x}{3x^2}\right) = \lim_{x\to 0}\left(\frac{\sin x}{6x}\right)$$

provided that the latter limit exists. You may know already that the latter limit is $\frac{1}{6}$ or you may obtain it by using L'Hôpital's rule one more time.

To find the limit

$$\lim_{x\to 0}\left(\frac{\tan x - x}{x-\sin x}\right)$$

we observe that

$$\lim_{x\to 0}(\tan x - x) = \lim_{x\to 0}(x-\sin x) = 0$$

and so L'Hôpital's rule guarantees that

$$\lim_{x\to 0}\left(\frac{\tan x - x}{x-\sin x}\right) = \lim_{x\to 0}\left(\frac{\sec^2 x - 1}{1-\cos x}\right)$$

provided that the latter limit exists. Although it is possible to use L'Hôpital's rule on this latest expresssion, it is better to simplify instead.

$$\lim_{x\to 0}\left(\frac{\sec^2 x-1}{1-\cos x}\right)=\lim_{x\to 0}\left(\frac{1-\cos^2 x}{(1-\cos x)\cos^2 x}\right)$$
$$=\lim_{x\to 0}\left(\frac{(1-\cos x)(1+\cos x)}{(1-\cos x)\cos^2 x}\right)=\lim_{x\to 0}\left(\frac{1+\cos x}{\cos^2 x}\right)=2$$

To find the limit
$$\lim_{x\to 0}\left((1+x)^{1/x}\right)$$
we make the observation that if $x>-1$ and $x\neq 0$ we have
$$(1+x)^{1/x}=\exp\left(\frac{\log(1+x)}{x}\right)$$
Since
$$\lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right)=1$$
we have
$$\lim_{x\to 0}\left((1+x)^{1/x}\right)=\lim_{x\to 0}\left(\exp\left(\frac{\log(1+x)}{x}\right)\right)$$
and from the composition theorem for limits we deduce that
$$\lim_{x\to 0}\left((1+x)^{1/x}\right)=\lim_{x\to 0}\left(\exp\left(\frac{\log(1+x)}{x}\right)\right)=\exp(1)=e.$$

Solution: ***To find the limit***
$$\lim_{x\to 0}\left(\frac{e-(1+x)^{1/x}}{x}\right)$$
we make use of the fact that whenever $1+x>0$ ***we have***
$$(1+x)^{1/x}=\exp\left(\frac{\log(1+x)}{x}\right).$$
It is easy to see that
$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1$$
and so we see that
$$\lim_{x\to 0}\left(e-(1+x)^{1/x}\right)=\lim_{x\to 0}x=0$$
and L'Hôpital's rule rule guarantees us the equality
$$\lim_{x\to 0}\left(\frac{e-(1+x)^{1/x}}{x}\right)=\lim_{x\to 0}\left(\frac{0-\left(\frac{x}{1+x}-\log(1+x)\right)\exp\left(\frac{\log(1+x)}{x}\right)}{x^2}\right)$$
as long as the latter limit exists. Now
$$\lim_{x\to 0}\left(\frac{0-\left(\frac{x}{1+x}-\log(1+x)\right)\exp\left(\frac{\log(1+x)}{x}\right)}{x^2}\right)=\lim_{x\to 0}\left(\frac{(1+x)\log(1+x)-x}{(1+x)x^2}\right)\exp\left(\frac{\log(1+x)}{x}\right)$$
$$=\left[\lim_{x\to 0}\left(\frac{(1+x)\log(1+x)-x}{(1+x)x^2}\right)\right]\left[\lim_{x\to 0}\exp\frac{1}{1+x}\left(\frac{\log(1+x)}{x}\right)\right]$$
$$=e\lim_{x\to 0}\left(\frac{(1+x)\log(1+x)-x}{x^2}\right)$$

Since

$$\lim_{x\to 0}((1+x)\log(1+x)-x) = \lim_{x\to 0}(x^2) = 0,$$

L'Hôpital's rule guarantees us the equality

$$e\lim_{x\to 0}\left(\frac{(1+x)\log(1+x)-x}{x^2}\right) = e\lim_{x\to 0}\left(\frac{1+\log(1+x)-1}{2x}\right)$$

as long as the latter limit exists. One more application of L'Hôpital's rule (or a direct method) shows that

$$\lim_{x\to 0}\left(\frac{1+\log(1+x)-1}{2x}\right) = \frac{1}{2}$$

and we conclude that

$$\lim_{x\to 0}\left(\frac{e-(1+x)^{1/x}}{x}\right) = \frac{e}{2}$$

$$\lim_{x\to\infty}\left(\frac{x^{100}}{\exp\left[(\log x)^2\right]}\right) \qquad \lim_{x\to\pi/2}(\sin x)^{\tan x}$$

$$\lim_{x\to\infty}\left(x^{\frac{\log x}{x}}\right) \qquad \lim_{x\to\infty}\frac{x^{\log x}}{(x+1)^{\log(x+1)}}$$

$$\lim_{x\to 0}\left(\frac{e^x-\log(1+x)-1}{x^2(x+2)}\right) \qquad \lim_{x\to\infty}\left(\sqrt[4]{x^4-5x^3+8x^2-2x+1}-x\right)$$

We shall now discuss the limit

$$\lim_{x\to\infty}\left(\frac{x^{100}}{\exp\left[(\log x)^2\right]}\right).$$

It is possible to use L'Hôpital's rule to yield the equation

$$\lim_{x\to\infty}\left(\frac{x^{100}}{\exp\left[(\log x)^2\right]}\right) = \lim_{x\to\infty}\left(\frac{100x^{99}}{\left(\exp\left[(\log x)^2\right]\right)\left(\frac{2\log x}{x}\right)}\right)$$

but this fact doesn't seem to be particularly useful. On the other hand, if we make the substitution $u = \log x$ then the limit we are trying to find takes on the form

$$\lim_{u\to\infty}\left(\frac{\exp(100u)}{\exp(u^2)}\right) = \lim_{u\to\infty}\left(\exp(100u-u^2)\right)$$

and so the limit is obviously 0. The only question that remains is whether the idea of substitution $u = \log x$ really makes sense and whether the procedure is actually valid. What we are really saying, when making the substitution is that the desired limit can be expressed as

$$\lim_{x\to\infty}\left(\frac{(\exp(\log x))^{100}}{\exp\left[(\log x)^2\right]}\right)$$

and that the equation

$$\lim_{x\to\infty}\left(\frac{(\exp(\log x))^{100}}{\exp\left[(\log x)^2\right]}\right) = \lim_{u\to\infty}\left(\frac{(\exp u)^{100}}{\exp(u^2)}\right)$$

follows from the composition theorem for limits.

We now discuss the limit

$$\lim_{x\to\pi/2}(\sin x)^{\tan x} = \lim_{x\to\pi/2}\left(\exp\left[(\tan x)\log\sin x\right]\right)$$

We look first at the limit

$$\lim_{x\to\pi/2}(\tan x)\log\sin x = \lim_{x\to\pi/2}\left((\sin x)\left(\frac{\log\sin x}{\cos x}\right)\right)$$
$$= \left(\lim_{x\to\pi/2}\sin x\right)\left(\lim_{x\to\pi/2}\left(\frac{\log\sin x}{\cos x}\right)\right) = \lim_{x\to\pi/2}\left(\frac{\log\sin x}{\cos x}\right)$$

Applying L'Hôpital's rule to the latter limit we obtain

$$\lim_{x\to\pi/2}\left(\frac{\log\sin x}{\cos x}\right) = \lim_{x\to\pi/2}\left(\frac{\cot x}{-\sin x}\right) = 0$$

and so it follows from the composition theorem for limits that

$$\lim_{x\to\pi/2}(\sin x)^{\tan x} = \exp(0) = 1$$

Now we discuss the limit

$$\lim_{x\to\infty}\left(x^{\frac{\log x}{x}}\right).$$

Since this limit can be expressed as

$$\lim_{x\to\infty}\left(\exp\left(\frac{(\log x)^2}{x}\right)\right)$$

and since

$$\lim_{x\to\infty}\frac{(\log x)^2}{x} = 0$$

we have

$$\lim_{x\to\infty}\left(x^{\frac{\log x}{x}}\right)=\exp(0) = 1.$$

Now we discuss the limit

$$\lim_{x\to\infty}\frac{x^{\log x}}{(x+1)^{\log(x+1)}} = \lim_{x\to\infty}\frac{\exp\left((\log x)^2\right)}{\exp\left((\log(x+1))^2\right)}$$
$$= \lim_{x\to\infty}\exp\left((\log x)^2 - (\log(x+1))^2\right)$$

We look first at

$$\lim_{x\to\infty}\left((\log(x+1))^2 - (\log x)^2\right)$$

It is possible to find this limit using L'Hôpital's rule by expressing it as

$$\lim_{x\to\infty}(\log(x+1) - \log x)(\log(x+1) + \log x) = \lim_{x\to\infty}\left(\frac{\log(1+\frac{1}{x})}{\left(\frac{1}{\log\left(x^2+x\right)}\right)}\right)$$

which gives us

$$\lim_{x\to\infty}\left(\frac{\left(\frac{x}{x+1}\right)\left(-\frac{1}{x^2}\right)}{\left(\frac{-1}{\left(\log\left(x^2+x\right)\right)^2}\right)\left(\frac{2x+1}{x^2+x}\right)}\right)$$

which can be seen easily to be 0. Almost every student will want to do the problem this way. On the other hand, one can also use the mean value theorem to choose a number that we shall call $f(x)$ between x and $x+1$ such that

$$(\log(x+1))^2 - (\log x)^2 = \frac{(\log(x+1))^2 - (\log x)^2}{(x+1)-x} = \frac{2\log(f(x))}{f(x)}.$$

Since the function g defined by the equation

$$g(u) = \frac{\log u}{u}$$

for $u > e$, having a negative derivative, must be decreasing we have

$$\frac{1-\log x}{x} \geq \frac{2\log(f(x))}{f(x)} \geq \frac{1-\log(x+1)}{x+1}$$

and it follows that

$$\lim_{x\to\infty}\left(\frac{2\log(f(x))}{f(x)}\right) = 0.$$

From either method we deduce that

$$\lim_{x\to\infty}\frac{x^{\log x}}{(x+1)^{\log(x+1)}} = \exp(0) = 1.$$

2. Given that $\alpha > 0$, evaluate the limit

$$\lim_{x\to\infty} x\left(\frac{(2x+2)^{\alpha}-(2x+1)^{\alpha}}{(2x+2)^{\alpha}}\right).$$

Solution: ***Although L'Hôpital's rule can be used to solve this problem, it is not the best approach. Instead we should define***

$$f(x) = x^{\alpha}$$

for all $x > 0$ ***and, for a given positive number*** x***, apply the mean value theorem to the function*** f ***on the interval*** $[2x+1, 2x+2]$ ***to choose a number*** c ***between*** $2x+1$ ***and*** $2x+2$ ***such that***

$$(2x+2)^{\alpha}-(2x+1)^{\alpha} = \frac{f(2x+2)-f(2x+1)}{(2x+2)-(2x+1)} = f'(c) = \alpha c^{\alpha-1}$$

In the event that $\alpha \geq 1$ ***we know from the fact that*** $2x+1 < c < 2x+2$ ***that***

$$\alpha(2x+1)^{\alpha-1} \leq \alpha c^{\alpha-1} \leq \alpha(2x+2)^{\alpha-1}$$

and so

$$\frac{\alpha x(2x+1)^{\alpha-1}}{(2x+2)^{\alpha}} \leq x\left(\frac{(2x+2)^{\alpha}-(2x+1)^{\alpha}}{(2x+2)^{\alpha}}\right) \leq \frac{\alpha x(2x+2)^{\alpha-1}}{(2x+2)^{\alpha}}$$

which we can express as

$$\frac{\alpha x}{2x+2}\left(\frac{2x+1}{2x+2}\right)^{\alpha-1} \leq x\left(\frac{(2x+2)^{\alpha}-(2x+1)^{\alpha}}{(2x+2)^{\alpha}}\right) \leq \frac{\alpha x}{2x+2}$$

and we deduce from the sandwich theorem that

$$\lim_{x\to\infty} x\left(\frac{(2x+2)^{\alpha}-(2x+1)^{\alpha}}{(2x+2)^{\alpha}}\right) = \frac{\alpha}{2}$$

In the event that $0 < \alpha < 1$ ***the inequalities are reversed but the result is the same. Try to handle this case yourself.***

3. Evaluate the limits

$$\lim_{x\to\infty}((\log(x+1))^{\alpha} - (\log x)^{\alpha})$$

$$\lim_{x\to\infty}\left(\frac{\log(x+1)}{\log x}\right)^{x} \qquad \lim_{x\to\infty}\left(\frac{\log(x+1)}{\log x}\right)^{x\log x}$$

For the latter two limits, check the limit value with *Scientific Notebook*.

The limit

$$\lim_{x\to\infty}((\log(x+1))^{\alpha} - (\log x)^{\alpha})$$

is an extension of the limit

$$\lim_{x\to\infty}\left((\log(x+1))^{2} - (\log x)^{2}\right)$$

that is discussed above. We understand α to be a given constant in this exercise. In the case $\alpha = 2$ we approached the problem using L'Hôpital's rule and again using the mean value theorem. This time we shall use only the mean value theorem. For each $x > 0$ we choose a number $f(x)$ between x and $x+1$ such that

$$(\log(x+1))^\alpha - (\log x)^\alpha = \frac{\alpha(\log(f(x)))^{\alpha-1}}{f(x)}.$$

If we define

$$g(u) = \frac{(\log u)^{\alpha-1}}{u}$$

for $u > 0$ then since

$$g'(u) = \frac{(\alpha-1)(\log u)^{\alpha-2} - (\log u)^{\alpha-1}}{u^2} = (\log u)^{\alpha-2}\left(\frac{\alpha-1-\log u}{u^2}\right) < 0$$

whenever $u > \exp(\alpha-1)$ we have

$$\frac{\alpha(\log x)^{\alpha-1}}{x} \geq (\log(x+1))^\alpha - (\log x)^\alpha \geq \frac{\alpha(\log(x+1))^{\alpha-1}}{x+1}$$

whenever $x > \exp(\alpha-1)$ and so

$$\lim_{x\to\infty}\left((\log(x+1))^\alpha - (\log x)^\alpha\right) = 0.$$

Now we discuss the limit

$$\lim_{x\to\infty}\left(\frac{\log(x+1)}{\log x}\right)^x = \lim_{x\to\infty}\left(\exp\left(x\log\left(\frac{\log(x+1)}{\log x}\right)\right)\right)$$

$$= \lim_{x\to\infty}\left(\exp\left(x\left(\log\log(x+1) - \log\log x\right)\right)\right)$$

We look first at the limit

$$\lim_{x\to\infty}\left(x\left(\log\log(x+1) - \log\log x\right)\right) = \lim_{x\to\infty}\left(\frac{\log\log(x+1) - \log\log x}{\frac{1}{x}}\right)$$

Once again, L'Hôpital's rule can be used but the mean value theorem presents an attractive alternative. For each $x > 0$ we choose a number $f(x)$ between x and $x+1$ such that

$$\log\log(x+1) - \log\log x = \frac{1}{f(x)\log(f(x))}$$

from which we see (in the same way we argued in the preceding exercises) that

$$\frac{x}{x\log x} \leq \frac{\log\log(x+1) - \log\log x}{\frac{1}{x}} \leq \frac{x}{(x+1)\log(x+1)}$$

and so

$$\lim_{x\to\infty}\left(\frac{\log\log(x+1) - \log\log x}{\frac{1}{x}}\right) = 0$$

and we conclude that

$$\lim_{x\to\infty}\left(\frac{\log(x+1)}{\log x}\right)^x = \exp(0) = 1.$$

Now we consider the limit

$$\lim_{x\to\infty}\left(\frac{\log(x+1)}{\log x}\right)^{x\log x} = \lim_{x\to\infty}\left(\exp\left((x\log x)\log\left(\frac{\log(x+1)}{\log x}\right)\right)\right)$$

$$= \lim_{x\to\infty}\left(\exp\left((x\log x)\left(\log\log(x+1) - \log\log x\right)\right)\right)$$

We look first at the limit

$$\lim_{x\to\infty}\left((x\log x)\left(\log\log(x+1) - \log\log x\right)\right) = \lim_{x\to\infty}\left(\frac{\log\log(x+1) - \log\log x}{\frac{1}{x\log x}}\right)$$

Arguing as above we see that

$$\frac{x\log x}{x\log x} \le \frac{\log\log(x+1) - \log\log x}{\frac{1}{x\log x}} \le \frac{x\log x}{(x+1)\log(x+1)}$$

and so

$$\lim_{x\to\infty}\left(\frac{\log\log(x+1) - \log\log x}{\frac{1}{x\log x}}\right) = 1$$

and we conclude that

$$\lim_{x\to\infty}\left(\frac{\log(x+1)}{\log x}\right)^{x\log x} = \exp(1) = e.$$

4. a. Prove that if $0 < a < b$ and $x \ge 1$ then

$$xa^{x-1} \le \frac{b^x - a^x}{b-a} \le xb^{x-1}.$$

Suppose that $x \ge 1$. For every positive number t we define

$$f(t) = t^x$$

and we apply the mean value theorem to f on the interval $[a,b]$.

b. Prove that

$$\lim_{x\to\infty}\left((\log(x+1))^{\log x} - (\log x)^{\log x}\right) = \infty.$$

Solution: ***From part a we know that, whenever x is sufficiently large,***

$$(\log x)(\log x)^{\log x-1} \le \frac{(\log(x+1))^{\log x} - (\log x)^{\log x}}{\log(x+1) - \log x} \le (\log x)(\log(x+1))^{\log x-1}$$

Thus

$$(\log(x+1))^{\log x} - (\log x)^{\log x} \ge \left(\log\left(\frac{x+1}{x}\right)\right)(\log x)^{\log x}$$

$$(\log(x+1))^{\log x} - (\log x)^{\log x} \ge \left(\log(x+1) - \log x\right)(\log x)^{\log x}.$$

To see that the latter expression clearly approaches ∞ as $x \to \infty$, we write it as

$$\left(\frac{\log(x+1) - \log x}{\frac{1}{x}}\right)\left(\frac{\exp((\log x)(\log\log x))}{x}\right).$$

It is easy to see that

$$\lim_{x\to\infty}\left(\frac{\log(x+1) - \log x}{\frac{1}{x}}\right) = 1$$

and since $\log\log x > 2$ for all sufficiently large x we have

$$\left(\frac{\exp((\log x)(\log\log x))}{x}\right) \ge \frac{\exp(2\log x)}{x} = x$$

for all x sufficiently large and so

$$\lim_{x\to\infty}\left(\frac{\exp((\log x)(\log\log x))}{x}\right) = \infty.$$

c. Prove that

$$\lim_{x\to\infty}\left((\log(x+1))^{\log\log x} - (\log x)^{\log\log x}\right) = 0.$$

5. Prove that if q is any given number then

$$\lim_{x\to\infty} x^q\left(1 - \frac{q\log x}{x}\right)^x = 1$$

Solution: ***The assertion we want to prove is obviously true when $q = 0$. We now consider the case***

$q>0$.

We need to show that

$$q\log x + x\log\left(1-\frac{q\log x}{x}\right)\to 0$$

as $x\to\infty$***. Since***

$$\lim_{x\to\infty}\frac{\log x}{x}=0$$

we know that

$$1-\frac{q\log x}{x}>0$$

whenever x ***is sufficiently large. Thus, for*** x ***sufficiently large we have***

$$\begin{aligned} q\log x + x\log\left(1-\frac{q\log x}{x}\right) &= q\log x + x\log\left(\frac{x-q\log x}{x}\right)\\ &= q\log x + x\log(x-q\log x) - x\log x \end{aligned}$$

Now given any sufficiently large number x***, we can apply the mean value theorem to the function*** $\log$ ***on the interval*** $[x-q\log x, \log x]$ ***to obtain a number*** $f(x)$ ***between*** $x-q\log x$ ***and*** $\log x$ ***at which the derivative of the function*** $\log$ ***is equal to***

$$\frac{\log x-\log(x-q\log x)}{x-(x-q\log x)}$$

and we can express $f(x)$ ***in the form*** $x-g(x)$ ***where*** $0\le g(x)\le q\log x$***. Thus***

$$\frac{\log x-\log(x-q\log x)}{q\log x}=\frac{1}{x-g(x)}$$

and we conclude that

$$\begin{aligned} q\log x + x\log(x-q\log x) - x\log x &= q\log x - x\frac{q\log x}{x-g(x)}\\ &= q(\log x)\left(1-\frac{x}{x-g(x)}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \left|q\log x + x\log\left(1-\frac{q\log x}{x}\right)\right| &= q(\log x)\left(\frac{x}{x-g(x)}-1\right)\\ &\le q(\log x)\left(\frac{x}{x-q\log x}-1\right)\\ &= \frac{q^2(\log x)^2}{x-q\log x}=\frac{q^2\frac{(\log x)^2}{x}}{1-q\frac{\log x}{x}} \end{aligned}$$

and the latter expression approaches 0 ***as*** $x\to\infty$***. This completes our consideration of the case*** $q>0$***.***

Finally we consider the case $q<0$***. We write*** $p=-q$***. Then*** $p>0$ ***and we want to show that***

$$\lim_{x\to\infty}\frac{1}{x^p}\left(1+\frac{p\log x}{x}\right)^x=1.$$

We need to show that

$$x\log\left(1+\frac{p\log x}{x}\right)-p\log x\to 0$$

as $x\to\infty$***. Now whenever*** x ***is sufficiently large we have***

$$\begin{aligned} x\log\left(1+\frac{p\log x}{x}\right)-p\log x &= x\log\left(\frac{x+p\log x}{x}\right)-p\log x\\ &= x\log(x+p\log x)-x\log x-p\log x \end{aligned}$$

and for some number $g(x)$ ***satisfying*** $0\le g(x)\le p\log x$ ***we have***

$$x\log\left(1+\frac{p\log x}{x}\right)-p\log x = \frac{px\log x}{x+g(x)}-p\log x$$

and so

$$\begin{aligned}\left|x\log\left(1+\frac{p\log x}{x}\right)-p\log x\right| &= p(\log x)\left(1-\frac{x}{x+g(x)}\right)\\ &\le p(\log x)\left(1-\frac{x}{x+p\log x}\right)\\ &= \frac{p\frac{(\log x)^2}{x}}{1+p\frac{\log x}{x}}\end{aligned}$$

***and the latter expression approaches** 0 **as** $x\to\infty$**.*

6. Suppose that we want to evaluate the limit

$$\lim_{x\to\infty}\frac{3}{1+x+x^2}.$$

Would it be correct to use L'Hôpital's rule by taking $f(x) = 3$ and $g(x) = 1+x+x^2$ for every x and then arguing that

$$\lim_{x\to\infty}\frac{3}{1+x+x^2} = \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{0}{1+2x} = 0?$$

The answer is certainly correct, but is the reasoning correct? Have we made a valid use of L'Hôpital's rule?

Solution: ***Yes, we have made a valid application of L'Hôpital's rule. The fact that the denominator of the fraction***

$$\frac{3}{1+x+x^2}$$

***approaches** ∞ **as** $x\to\infty$ **is all we need; even though most elementary calculus books give the impression that L'Hôpital's rule requires both the numerator and the denominator to approach** ∞**.*

7. Evaluate the limits

$$\lim_{x\to\infty}\frac{e^{\sin x}}{x} \qquad \text{and} \qquad \lim_{x\to\infty}\frac{xe^{\sin x}}{\log x}.$$

Should L'Hôpital's rule be used to evaluate these limits?

L'Hôpital's rule would say that

$$\lim_{x\to\infty}\frac{e^{\sin x}}{x} = \lim_{x\to\infty}\frac{(e^{\sin x})\cos x}{1}$$

provided that the latter limit exists. But it doesn't. So L'Hôpital's rule can't be used here. Note, however that

$$\lim_{x\to\infty}\frac{e^{\sin x}}{x} = 0.$$

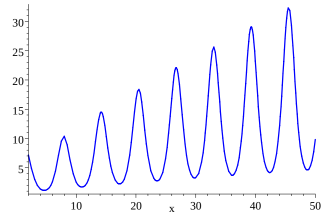
L'Hôpital's rule would say that

$$\lim_{x\to\infty}\frac{xe^{\sin x}}{\log x} = \lim_{x\to\infty}\frac{e^{\sin x}+x(e^{\sin x})\cos x}{\frac{1}{x}}$$

and the latter limit does not exist. Nor does the original limit. In fact, the graph

$$y = \frac{xe^{\sin x}}{\log x}$$

is



10 The Exponential and Logarithmic Functions

Some Exercises on the Exponential and Logarithmic Functions

1. Given that a is a positive number and that $f(x) = a^x$ for every real number x, prove that

$$f'(x) = a^x \log a$$

for every number x. In other words, prove that the number we called $\phi(a)$ in the preceding sections is just $\log a$.
For each x we have

$$f(x) = \exp(x \log a)$$

and so

$$f'(x) = \left(\exp(x \log a)\right) \log a = a^x \log a.$$

2. Given that a is a positive number and that $a \neq 1$, and given that $f(x) = \log_a x$ for every $x > 0$, prove that

$$f'(x) = \frac{1}{x(\log a)}$$

for every number $x > 0$.
For each $x > 0$ we have

$$a^{f(x)} = x$$

which gives us

$$\log(a^{f(x)}) = \log x.$$

Therefore

$$f(x) \log a = \log x$$

and so

$$f(x) = \frac{\log x}{\log a}$$

from which the desired result follows at once.

3. Given that f and g are differentiable functions and f is positive, use the fact that

$$f(x)^{g(x)} = \exp[g(x)(\log(f(x)))]$$

for each x to find a formula for the derivative of the function f^g.
The value of the derivative of f^g at a given number x is

$$\left(\exp[g(x)(\log(f(x)))]\right)\left(g'(x)\log(f(x)) + \frac{g(x)}{f(x)}f'(x)\right) = f(x)^{g(x)}g'(x)\log(f(x)) + g(x)f(x)^{g(x)-1}f'(x).$$

4. Given that f is differentiable on $\mathbf{R}$, that $f(0) = 1$ and that $f' = f$, prove that the function $\frac{f}{\exp}$ is constant and then conclude that $f = \exp$.
   Since
   $$\left(\frac{f}{\exp}\right)' = \frac{f'\exp - f\exp'}{\exp^2} = \frac{f\exp - f\exp}{\exp^2} = 0$$
   the function $\frac{f}{\exp}$ must be constant. To see that the constant is 1 we observe that
   $$\frac{f(0)}{\exp 0} = 1.$$

5. Suppose that $f : \mathbf{R} \to \mathbf{R}$ and that $f''(x) = f(x)$ for every real number x.
   a. Given that $g = f' + f$, prove that $g' = g$ and deduce that there exists a real number a such that $g'(x) = 2ae^x$ for every number x.
      We see that
      $$g' = f'' + f' = f + f' = g.$$
      From Exercise 4 we know that the function $g'/\exp$ is constant. If we call this constant $2a$ then we have $g'(x) = 2ae^x$ for every number x.
   b. Given that $h(x) = f(x)e^x$ for every real number x, prove that the equation $h'(x) = 2ae^{2x}$ holds for every number x and deduce that there is there is a number b such that the equation
      $$f(x) = ae^x + be^{-x}$$
      holds for every real number x.
      For every number x we have
      $$h'(x) = f'(x)e^x + f(x)e^x = g(x)e^x = 2ae^{2x}.$$
      It follows that the function whose value at every number x is $h(x) - ae^{2x}$ has a zero derivative at every number and must therefore be constant. We call this constant b. Thus we have
      $$h(x) = ae^{2x} + b$$
      for every x, in other words
      $$f(x) = ae^x + be^{-x}.$$

6. Suppose that $f : \mathbf{R} \to \mathbf{R}$ and that for all numbers t and x we have
   $$f(t + x) = f(t)f(x).$$
   a. Prove that either $f(x) = 0$ for every real number x or $f(x) \neq 0$ for every real number x.
      Suppose that there exists a number at which the function f is zero. Choose such a number and call it c. Given any number x we have
      $$f(x) = f(x - c + c) = f(x - c)f(c) = 0$$
      and so f is zero everywhere.
   b. Prove that if f is not the constant zero, then $f(0) = 1$ and that $f(x) > 0$ for every number x.
      We suppose that f is not the constant zero. From the equation
      $$f(0) = f(0 + 0) = f(0)f(0)$$
      and from the fact that $f(0) \neq 0$ we deduce that $f(0) = 1$. Finally, if x is any real number then the equation
      $$f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = \left(f\left(\frac{x}{2}\right)\right)^2$$
      that $f(x) > 0$.

c. Prove that if f is not the constant zero and if $a = f(1)$ then for every rational number x we have $f(x) = a^x$. Deduce that if f is continuous on the set $\mathbf{R}$ then the equation $f(x) = a^x$ holds for every real number x. Compare this exercise with the last few exercises on continuity.
This exercise follows in exactly the same way as that earlier sequence of exercises.

7. Suppose that α is a nonzero real number and that

$$S = \{x \in \mathbf{R} \mid 1 + \alpha x > 0\}.$$

a. Prove that if

$$g(x) = \alpha x - (1 + \alpha x)\log(1 + \alpha x)$$

for all $x \in S$ then $g(x) < 0$ for every nonzero number $x \in S$.

Solution: ***We begin by observing that if*** $x \in S$ ***then***

$$g'(x) = -\alpha \log(1 + \alpha x).$$

We look first at the case $\alpha > 0$***. In this case we have*** $S = \left(-\frac{1}{\alpha}, \infty\right)$***. Whenever*** $-\frac{1}{\alpha} < x < 0$ ***we have*** $1 + \alpha x < 1$ ***which gives us*** $g'(x) > 0$***. Furthermore, whenever*** $x > 0$ ***we have*** $1 + \alpha x > 1$ ***which gives us*** $g'(x) < 0$***. Therefore, since*** g ***is strictly increasing on the interval*** $\left(-\frac{1}{\alpha}, 0\right)$ ***and strictly decreasing on the interval*** $(0, \infty)$***, the function*** g ***must have a maximum value only at*** 0 ***. Since*** $g(0) = 0$ ***we deduce that*** $g(x) < 0$ ***for all*** $x \in S \setminus \{0\}$***.***

Now we look at the case $\alpha < 0$***. In this case we have*** $S = \left(-\infty, -\frac{1}{\alpha}\right)$***. Whenever*** $x < 0$ ***we have*** $1 + \alpha x > 1$ ***which gives us*** $g'(x) > 0$***. Furthermore, whenever*** $0 < x < -\frac{1}{\alpha}$ ***we have*** $1 + \alpha x < 1$ ***which gives us*** $g'(x) < 0$***. Thus*** g ***is strictly increasing on the interval*** $(-\infty, 0)$ ***and strictly descreasing on the interval*** $\left(0, -\frac{1}{\alpha}\right)$ ***and so, once again,*** g ***has its maximum only at*** 0***.***

b. Prove that if

$$f(x) = \frac{\log(1 + \alpha x)}{x}$$

for every nonzero number $x \in S$ and if $f(0) = \alpha$ then the function f is continuous and strictly decreasing on S. Deduce that the inequality $f(x) < \alpha$ holds for every positive number $x \in S$.

Solution: ***For every number*** $x \in S \setminus \{0\}$ ***we have***

$$f'(x) = \frac{\frac{\alpha x}{1+\alpha x} - \log(1 + \alpha x)}{x^2} = \frac{g(x)}{(1 + \alpha x)x^2}$$

and so it follows from part a that $f'(x) < 0$ ***for every nonzero number*** $x \in S$***. The fact that*** f ***is continuous at the number*** 0 ***follows from the observation that***

$$\lim_{x \to 0} \frac{\log(1 + \alpha x)}{x} = \alpha.$$

Now, to see why f ***must be strictly decreasing on*** S***, look first at the case in which*** $\alpha > 0$***. Since*** f ***has a negative derivative everywhere in the interval*** $\left(-\frac{1}{\alpha}, 0\right)$ ***and is continuous on the interval*** $\left(-\frac{1}{\alpha}, 0\right]$ ***we know that*** f ***is strictly decreasing on*** $\left(-\frac{1}{\alpha}, 0\right]$***. In a similar way we know that*** f ***is strictly decreasing on the interval*** $[0, \infty)$***. Therefore*** f ***is strictly decreasing on*** S***. Now fill in the details why*** f ***must be strictly decreasing on*** S ***when*** $\alpha < 0$***.***

Finally, since $f(0) = \alpha$***, the fact that*** $f(x) < \alpha$ ***whenever*** x ***is a positive member of*** S ***follows at once from the fact that*** f ***is strictly descreasing.***

c. Prove that the inequality

$$(1 + \alpha x)^{1/x} < e^\alpha$$

holds for every positive number $x \in S$.

Solution: ***This fact follows at once from the fact that***

$$(1+\alpha x)^{1/x} = \exp\left(\frac{\log(1+\alpha x)}{x}\right) = \exp(f(x))$$

whenever $x \in S \setminus \{0\}$

8. a. By applying the higher order mean value theorem to the function exp on the interval $[0,1]$, show that if n is a positive integer then there must exist a number $c \in (0,1)$ such that

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{e^c}{(n+1)!}.$$

The higher order mean value theorem tells us that if n is a positive integer then there is a number c between 0 and 1 such that

$$\exp(1) = \sum_{j=0}^{n}\frac{\exp^{(j)}(0)}{j!}+\frac{\exp^{(n+1)}(c)}{(n+1)!}$$

and this equation says that

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{e^c}{(n+1)!}.$$

b. Deduce that if n is a positive integer then

$$0 < e-\sum_{j=0}^{n}\frac{1}{j!} < \frac{e}{(n+1)!}.$$

The desired inequality follows at once from the fact that if $c < 1$ and n is a positive integer then

$$\frac{e^c}{(n+1)!} < \frac{e}{(n+1)!}$$

c. By putting $n = 2$ in the latter inequality, prove that $e < 3$.

From the fact that

$$0 < e-\sum_{j=0}^{2}\frac{1}{j!} < \frac{e}{(2+1)!}$$

we deduce that

$$e-\frac{e}{6} < 1+1+\frac{1}{2}$$

which gives us $e < 3$.

d. Prove that if n is a positive integer we have

$$0 < e-\sum_{j=0}^{n}\frac{1}{j!} < \frac{3}{(n+1)!}$$

and deduce that

$$\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!} = e.$$

The inequality

$$0 < e-\sum_{j=0}^{n}\frac{1}{j!} < \frac{3}{(n+1)!}$$

follows at once from the inequality

$$0 < e-\sum_{j=0}^{n}\frac{1}{j!} < \frac{e}{(n+1)!}$$

and the fact that $e < 3$. The fact that

$$\lim_{n\to\infty}\left(e-\sum_{j=0}^{n}\frac{1}{j!}\right)=0$$

now follows at once from the sandwich theorem.

e. Prove that the number e is irrational.

Solution: ***To obtain a contradiction, suppose that*** e ***is rational. Choose two positive integers*** p ***and*** q ***such that*** $e=\frac{p}{q}$***. Since***

$$\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}=e,$$

we have

$$\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{1}{j!}\right)=\lim_{n\to\infty}\left(\sum_{j=0}^{n}\frac{1}{j!}-\sum_{j=0}^{q}\frac{1}{j!}\right)=e-\sum_{j=0}^{q}\frac{1}{j!}$$

and therefore, multiplying by $q!$***, we obtain***

$$\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{q!}{j!}\right)=\left(\frac{p}{q}\right)q!-\sum_{j=0}^{q}\frac{q!}{j!}$$

Since both of the numbers $\sum_{j=0}^{q}\frac{q!}{j!}$ ***and*** $\left(\frac{p}{q}\right)q!$ ***are integers, we deduce that the number***

$$\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{q!}{j!}\right)$$

must also be an integer. Now, on the one hand, the number

$$\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{q!}{j!}\right)$$

must be positive because

$$\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{q!}{j!}\right)>\sum_{j=q+1}^{q+1}\frac{q!}{j!}=\frac{1}{q+1}>0.$$

However, whenever $j\geq q+1$ ***we have***

$$\frac{q!}{j!}=\frac{1}{(q+1)(q+2)\cdots(j)}<\frac{1}{(q+1)^{j-q}}$$

and therefore

$$\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{q!}{j!}\right)<\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{1}{(q+1)^{j-q}}\right)=\lim_{n\to\infty}\left(\sum_{j=1}^{n-q}\frac{1}{(q+1)^{j}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{q+1}\right)\left(\frac{1-\left(\frac{1}{q+1}\right)^{n-q}}{1-\frac{1}{q+1}}\right)=\frac{1}{q}\leq 1$$

We have therefore reached the impossible conclusion that the number

$$\lim_{n\to\infty}\left(\sum_{j=q+1}^{n}\frac{q!}{j!}\right)$$

is an integer that lies between 0 ***and*** 1 ***and so we have reached the desired contradiction.***

9. Prove that if (x_n) is a sequence of positive numbers and if

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha$$

then

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \alpha.$$

Deduce that

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

Solution: ***We look first at the case*** $\alpha > 0$***. In order to show that***

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \alpha$$

we shall show that no number other than α ***can be a partial limit of the sequence*** $(\sqrt[n]{x_n})$***. Suppose that*** u ***is any number that is unequal to*** α ***and choose a number*** δ ***between*** 0 ***and*** 1 ***such that the interval***

$$\left[\alpha\delta, \frac{\alpha}{\delta}\right]$$

does not contain the number u***. We now use the facts that***

$$\alpha\delta < \alpha < \frac{\alpha}{\delta}$$

and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha$$

to choose a positive integer N ***such that the inequality***

$$\alpha\delta < \frac{x_{n+1}}{x_n} < \frac{\alpha}{\delta}$$

holds whenever $n \geq N$***.***

Now whenever $n > N$ ***we have***

$$x_n = (x_N)\left(\frac{x_{N+1}}{x_N}\right)\left(\frac{x_{N+2}}{x_{N+1}}\right)\cdots\left(\frac{x_n}{x_{n-1}}\right)$$

and so

$$(x_N)(\alpha\delta)^{n-N} < x_n < (x_N)\left(\frac{\alpha}{\delta}\right)^{n-N}$$

which we can express as

$$\left(\sqrt[n]{\frac{x_N}{(\alpha^N)(\delta^N)}}\right)\alpha\delta < \sqrt[n]{x_n} < \left(\sqrt[n]{\frac{(\delta^N)x_N}{\alpha^N}}\right)\frac{\alpha}{\delta}.$$

Since

$$\lim_{n\to\infty} \sqrt[n]{\frac{x_N}{(\alpha^N)(\delta^N)}} = \lim_{n\to\infty} \sqrt[n]{\frac{(\delta^N)x_N}{\alpha^N}} = 1$$

we deduce that no partial limit of the sequence $(\sqrt[n]{x_n})$ ***can lie outside of the interval***

$$\left[\alpha\delta, \frac{\alpha}{\delta}\right]$$

and we conclude that u ***is not a partial limit of*** $(\sqrt[n]{x_n})$***.***

We now consider the case $\alpha = 0$***. In order to show that***

$$\lim_{n\to\infty} \sqrt[n]{x_n} = 0$$

and, once again, we shall demonstrate that no number unequal to α ***can be a partial limit of the sequence*** $(\sqrt[n]{x_n})$***. Since it is already clear that a negative number can't be a partial limit of*** $(\sqrt[n]{x_n})$***, we can confine our attention to positive numbers. Suppose that*** $u > 0$ ***and choose a number*** δ ***between*** 0 ***and*** 1 ***such that*** $\delta < u$***. We now use the fact that***

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$$

to choose an integer N ***such that the inequality***

$$\frac{x_{n+1}}{x_n} < \delta$$

holds whenever $n \geq N$***.***

Now whenever $n > N$ ***we have***

$$x_n = (x_N)\left(\frac{x_{N+1}}{x_N}\right)\left(\frac{x_{N+2}}{x_{N+1}}\right)\cdots\left(\frac{x_n}{x_{n-1}}\right) < (x_N)(\delta^{n-N})$$

which we can write as

$$\sqrt[n]{x_n} < \left(\sqrt[n]{\frac{x_N}{\delta^N}}\right)\delta.$$

Since

$$\lim_{n\to\infty} \sqrt[n]{\frac{x_N}{\delta^N}} = 1$$

we deduce that no partial limit of the sequence $(\sqrt[n]{x_n})$ ***can be more than*** δ ***and, therefore, that the number*** u ***cannot be a partial limit of the sequence*** $(\sqrt[n]{x_n})$***.***

11 The Riemann Integral

Some Exercises on Step Functions

1. True or false? If f is a step function on an interval $[a,b]$ and $[c,d]$ is a subinterval of $[a,b]$ then f is a step function on $[c,d]$.

Solution: ***The statement is true.***

a —— c ———— d —— b

Choose a partition $\mathbf{P}$ ***of*** $[a,b]$ ***within which the function*** f ***steps and then refine*** $\mathbf{P}$ ***by adding to it the two numbers*** c ***and*** d***. Then drop all of the points of this partition that lie outside of the interval*** $[c,d]$ ***and we obtain a partition*** $\mathbf{Q}$ ***of*** $[c,d]$ ***within which*** f ***steps.***

2. True or false? If f is a step function then given any interval $[a,b]$, the function f is a step function on $[a,b]$.

Hint: ***This statement is true. Write a short proof.***
Choose an interval $[c,d]$ such that f is a step function on $[c,d]$ and such that $f(x) = 0$ whenever $x \in \mathbf{R} \setminus [c,d]$. Choose numbers p and q such that p is less than both of the numbers a and c and q is greater than both of the numbers b and d

p a c b d q

Choose a partition $\mathbf{P}$ of the interval $[c,d]$ such that f steps within $\mathbf{P}$. If we add the two numbers p and q to the partition $\mathbf{P}$ then we obtain a partition of the larger interval $[p,q]$ within which f steps.
Since f is a step function on the interval $[p,q]$, it follows from Exercise 1 that f is a step function on $[a,b]$.

3. Give an example of a step function on the interval $[0,2]$ that does not step within any regular partition of $[0,2]$.

Solution: ***We define***

$$f(x) = \begin{cases} 0 & \textbf{\textit{if}} \quad 0 \leq x \leq \sqrt{2} \\ 1 & \textbf{\textit{if}} \quad \sqrt{2} < x \leq 2 \end{cases}$$

Now if* $\mathbf{P}$ *is any regular partition of the interval* $[0,2]$ *then, since the irrationality of* $\sqrt{2}$ *makes it impossible to find integers* n *and* j *such that

$$\sqrt{2} = 0 + \frac{2j}{n}$$

we know that* $\sqrt{2}$ *can't be a point of* $\mathbf{P}$. *In other words, the number* $\sqrt{2}$ *must be in one of the open intervals of* $\mathbf{P}$ *and* f *fails to be constant in that interval.

4. Explain why a step function must always be bounded.
   Suppose that f is a step function on an interval $[a,b]$. Choose a partition $\mathbf{P}$ of $[a,b]$ such that f steps with $\mathbf{P}$. We express $\mathbf{P}$ in the form $(x_0, x_1, \cdots, x_n)$. Since f is constant in each subinterval (x_{j-1}, x_j), the range of f must be a finite set and therefore f is bounded.

5. Prove that if f and g are step functions on an interval $[a,b]$ then so are their sum $f+g$ and their product fg.

   Hint: ***You can find a proof of this assertion in the section on linearity of integration of step functions.***

6. Prove that if f and g are step functions then so are their sum $f+g$ and their product fg.
   Choose an interval $[a,b]$ such that both of the functions f and g take the value 0 at every number in $\mathbf{R} \setminus [a,b]$. We deduce from Exercise 2 that both f and g are step functions on the interval $[a,b]$ and it follows from Exercise 5 that $f+g$ and fg are step functions on $[a,b]$ and we conclude that these functions are step functions.

7. Prove that a continuous step function on an interval must be constant on that interval.
   Suppose that f is a continuous step function on an interval $[a,b]$ and choose a partition

   $$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

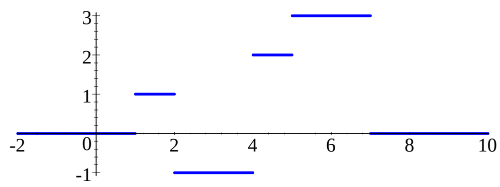
   of $[a,b]$ within which f steps. If c is the constant value of f on the interval (x_0, x_1) then, since f is continuous at x_0 and at x_1 we have $f(x_0) = f(x_1) = c$. Therefore, since f is continuous at x_1, the number c must also be the constant value of f on (x_1, x_2). Continuing in this way we see that f has the constant value c throughout the interval $[a,b]$.

Exercises on Integration of Step Functions

1. Given that f is the function defined by the equation

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } 1 \le x \le 2 \\ -1 & \text{if } 2 < x < 4 \\ 2 & \text{if } 4 \le x < 5 \\ 3 & \text{if } 5 \le x < 7 \\ 0 & \text{if } x \ge 7 \end{cases}$$

   whose graph appears iin the figure



evaluate $\int_{-\infty}^{\infty} f$.

Solution: ***We sum the function f over the partition***

$$\mathbf{P} = (-2, 1, 2, 4, 5, 7, 10)$$

of the interval $[-2, 10]$**.**

$$\int_0^\infty f = 0(1-(-2)) + 1(2-1) + (-1)(4-2) + 2(5-4) + 3(7-5) + 0(10-7) = 7.$$

2. Prove that if f is a step function then so is the function $|f|$ and we have

$$\left|\int_{-\infty}^{\infty} f\right| \leq \int_{-\infty}^{\infty} |f|.$$

Choose an interval $[a, b]$ outside of which the function f is zero. We deduce from Theorem 11.3.8 that

$$\left|\int_{-\infty}^{\infty} f\right| = \left|\int_a^b f\right| \leq \int_a^b |f| = \int_{-\infty}^{\infty} |f|.$$

3. Given that f is a function defined on $\mathbf{R}$ and that the set of numbers x for which $f(x) \neq 0$ is finite, explain why f must be a step function and why

$$\int_{-\infty}^{\infty} f = 0.$$

We define a and b to be the smallest and largest members, respectively, of the set $\{x \mid f(x) \neq 0\}$. If we arrange the members of the set $\{x \mid f(x) \neq 0\}$ in ascending order then we obtain a partition of $[a, b]$ within which f steps and we see at once that the sum of f over this partition is zero. Since f is zero outside the interval $[a, b]$ we conclude that f is a step function and that

$$\int_{-\infty}^{\infty} f = 0.$$

4. Given that f is a nonnegative step function and that

$$\int_{-\infty}^{\infty} f = 0,$$

prove that the set of numbers x for which $f(x) \neq 0$ must be finite.

Hint: ***We begin by choosing an interval $[a, b]$ and a partition $\mathbf{P}$ of $[a, b]$ such that $f(x) = 0$ whenever a number x lies outside the interval $[a, b]$ and such that f steps within the partition $\mathbf{P}$. Now explain briefly why f must have the constant value 0 on each of the open intervals of $\mathbf{P}$.***

If α_j is the constant value of f in the interval (x_{j-1}, x_j) for each $j = 1, 2, \cdots, n$, then, since each number α_j is nonnegative we see that for each j,

$$0 = \int_a^b f = \sum_{i=1}^{n} (x_i - x_{i-1})\alpha_i \geq (x_j - x_{j-1})\alpha_j \geq 0$$

from which it follows that each number α_j must be zero.

5. Given that f and g are step functions and that c is a real number, prove that

$$\int_{-\infty}^{\infty} cf = c\int_{-\infty}^{\infty} f$$

and

$$\int_{-\infty}^{\infty} (f+g) = \int_{-\infty}^{\infty} f + \int_{-\infty}^{\infty} g.$$

These results follow at once when we choose an interval $[a,b]$ outside of which both f and g are zero and then replace $\int_{-\infty}^{\infty}$ by $\int_a^b$.

Exercises on Elementary Sets

1. Given that A and B are elementary sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B).$$

Solution: ***We begin by choosing a lower bound** a **and an upper bound** b **of the set** $A \cup B$**. By looking at the different cases we can see easily that whenever** $x \in [a,b]$ **we have***

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}.$$

Therefore

$$\begin{aligned} m(A \cup B) &= \int_a^b \chi_{A\cup B} = \int_a^b (\chi_A + \chi_B - \chi_{A\cap B}) \\ &= \int_a^b \chi_A + \int_a^b \chi_B - \int_a^b \chi_{A\cap B} \\ &= m(A) + m(B) - m(A \cap B). \end{aligned}$$

2. Prove that if E is an elementary set and $m(E) = 0$ then E must be finite.
Choose an interval $[a,b]$ such that the function χ_E is zero outside $[a,b]$ and a partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ such that χ_E steps within $\mathbf{P}$. Since χ_E is nonnegative and its sum over $\mathbf{P}$ is zero, the constant value of χ_E in each interval $(x_{j-1}x_j)$ must be zero. Therefore χ_E can be nonzero only at points of $\mathbf{P}$, in other words, every member of E is a point of $\mathbf{P}$ and we conclude that the set E is finite.

3. Explain why the set of all rational numbers in the interval $[0,1]$ is not elementary.
The desired result will be clear when we have proved the stronger assertion that is made in Exercise 4.

4. Prove that if E is an elementary subset of $[0,1]$ and if every rational number in the interval belongs to E then the set $[0,1] \setminus E$ must be finite.
We assume that E is an elementary subset of $[0,1]$ and that every rational number in $[0,1]$ belongs to E. Choose a partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ such that χ_E steps within $\mathbf{P}$. For each j, since the interval (x_{j-1}, x_j) contains some rational numbers, the constant value of χ_E in (x_{j-1}, x_j) must be 1. Since the numbers in $[0,1]$ that do not belong to E have to be points of $\mathbf{P}$, the set $[0,1] \setminus E$ must be finite.

5. Give an example of a set A of numbers such that if E is any elementary subset of A we have $m(E) = 0$ and if E is any elementary set that includes A we have $m(E) \geq 1$.

Hint: ***Take another look at Exercise 4.***
The set $[0,1] \cap \mathbf{Q}$ has the desired properties.

6. Given that E is an elementary set that is not closed and that F is a closed elementary subset of E, prove that $m(E \setminus F) > 0$.

Solution: ***Choose a lower bound a and an upper bound b of the set E. Since the set $E \setminus F$ is elementary, if we want to show that $m(E \setminus F) > 0$ then, from Exercise 2, all we have to show is that the set $E \setminus F$ cannot be finite. The fact that $E \setminus F$ is not finite follows from the fact that finite sets are always closed, that F is closed and that the set E, which isn't closed is the union of the two sets F and $E \setminus F$.***

7. Given that f is a step function, that E is an elementary set and that $f(x) = 0$ whenever $x \in \mathbf{R} \setminus E$, prove that
$$\int_E f = \int_{-\infty}^{\infty} f.$$

Solution: ***The desired equality follows at once from the definitions and the fact that***
$$f = f\chi_E.$$

8. Given that f and g are step functions, that E is an elementary set and that $f(x) \le g(x)$ whenever $x \in E$, prove that
$$\int_E f \le \int_E g.$$
Choose an interval $[a,b]$ outside of which both of the functions f and g are zero. Since $f\chi_E \le g\chi_E$, it follows from the nonnegativity property of integrals of step functions that
$$\int_E f = \int_a^b f\chi_E \le \int_a^b g\chi_E = \int_E g$$

9. Given that f is a nonnegative step function, that A and B are elementary sets and that $A \subseteq B$, prove that
$$\int_A f \le \int_B f.$$
The desired inequality follows at once from the fact that $f\chi_A \le f\chi_B$. We choose an interval $[a,b]$ that includes the set B and use the nonnegativity property to obtain
$$\int_A f = \int_a^b f\chi_A \le \int_a^b f\chi_B = \int_B f$$

10. Given that f is a step function and that E is an elementary set, prove that
$$\left|\int_E f\right| \le \int_E |f|.$$

Hint: ***Use the fact that***
$$-|f|\chi_E \le f\chi_E \le |f|\chi_E.$$

11. Given that A and B are elementary sets, prove that
$$\int_A \chi_B = \int_B \chi_A = m(A \cap B).$$

Solution: ***Choose a lower bound a and an upper bound b of the set $A \cup B$. We see that***
$$\int_A \chi_B = \int_a^b \chi_B\chi_A = \int_a^b \chi_A\chi_B = \int_B \chi_A.$$
The fact that these expressions are equal to $m(A \cap B)$ follows at once from the fact that
$$\chi_A\chi_B = \chi_{A\cap B}.$$

The on-screen version of this book contains a special group of exercises that are designed to be done as a special project. These exercises require you to have read some of the chapter on infinite series. To reach this group of exercises, click on the icon .

Additional Exercises on Elementary Sets and Infinite

Series

1. Given that H is a closed elementary set and (U_n) is a sequence of open elementary sets and that
$$H \subseteq \bigcup_{n=1}^{\infty} U_n,$$
use this earlier exercise to deduce that, for some positive integer N we have
$$m(H) \leq \sum_{n=1}^{N} m(U_n)$$
and deduce that
$$m(H) \leq \sum_{n=1}^{\infty} m(U_n).$$
Choose a positive integer N such that
$$H \subseteq \bigcup_{n=1}^{N} U_n.$$
We have
$$m(H) \leq m\left(\bigcup_{n=1}^{N} U_n\right) \leq \sum_{n=1}^{N} m(U_n) \leq \sum_{n=1}^{\infty} m(U_n).$$

2. Given that E is an elementary set and that (U_n) is a sequence of open elementary sets and that
$$E \subseteq \bigcup_{n=1}^{\infty} U_n,$$
prove that
$$m(E) \leq \sum_{n=1}^{\infty} m(U_n).$$
Hint: ***Make use of the theorem on*** *approximation by open sets and closed sets.*
Given any closed subset H of E we know from Exercise 1 that
$$m(H) \leq \sum_{n=1}^{\infty} m(U_n).$$
Since $\sum_{n=1}^{\infty} m(U_n)$ is an upper bound of the set
$$\{m(H) \mid H \text{ is elementary and closed and } H \subseteq E\}$$
and since $m(E)$ is the *least* upper bound of this set we have
$$m(E) \leq \sum_{n=1}^{\infty} m(U_n).$$

3. Given that (A_n) is a sequence of elementary sets and that $\varepsilon > 0$ and that the series $\sum m(A_n)$ is convergent, and given that for each positive integer n the set U_n is an open elementary set that includes A_n and satisfies the inequality
$$m(U_n) < m(A_n) + \frac{\varepsilon}{2^n},$$
prove that
$$\sum_{n=1}^{\infty} m(U_n) < \sum_{n=1}^{\infty} m(A_n) + \varepsilon.$$
The desired result follows at once from the fact that

$$\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

4. Given that E is an elementary set and that (A_n) is a sequence of elementary sets and that
$$E \subseteq \bigcup_{n=1}^{\infty} A_n,$$
prove that
$$m(E) \le \sum_{n=1}^{\infty} m(A_n).$$
To obtain a contradiction, assume that
$$m(E) > \sum_{n=1}^{\infty} m(A_n).$$
Choose $\varepsilon > 0$ such that
$$m(E) > \sum_{n=1}^{\infty} m(A_n) + \varepsilon = \sum_{n=1}^{\infty} m(A_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}.$$
For each n, choose an open elementary set U_n that includes A_n such that
$$m(U_n) < m(A_n) + \frac{\varepsilon}{2^n}.$$
We see that
$$m(E) > \sum_{n=1}^{\infty} m(A_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} > \sum_{n=1}^{\infty} m(U_n)$$
which, in view of Exercise 2, is impossible because
$$E \subseteq \bigcup_{n=1}^{\infty} U_n.$$

5. Suppose that E is an elementary set and that (A_n) is a sequence of elementary sets with the property that whenever i and j are positive integers and $i \ne j$ we have
$$A_i \cap A_j = \emptyset.$$
Suppose that E is an elementary set and that
$$E = \bigcup_{n=1}^{\infty} A_n.$$
Prove that
$$m(E) = \sum_{n=1}^{\infty} m(A_n).$$
Given any postiive integer N we see that
$$m(E) \ge m\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} m(A_n).$$
Therefore
$$m(E) \ge \lim_{N\to\infty}\left(\sum_{n=1}^{N} m(A_n)\right) = \sum_{n=1}^{\infty} m(A_n),$$
and the desired result therefore follows from Exercise 4.

Some Exercises on the Riemann Integral

1. Prove that the integral

$$\int_1^4 3x^2 dx$$

exists and has the value 63.

Solution: ***This exercise will become obsolete when we reach the fundamental theorem of calculus later on in the chapter. The solution given here is a bare hands approach and repeats portions of the proof that monotone functions are integrable.***

We define $f(x) = 3x^2$ ***for each number*** $x \in [1,4]$***. Given any positive integer*** n***, if***

$$\mathbf{P}_n = (x_0, x_1, \cdots, x_n)$$

is the regular n***-partition of the interval*** $[1,4]$ ***then we define two step functions*** s_n ***and*** S_n ***by making***

$$s_n(x) = S_n(x) = f(x)$$

whenever x ***is a point of the partition*** P_n ***and, in each interval*** (x_{j-1}, x_j) ***of the partition*** P_n ***we make*** s_n ***and*** S_n ***take the constant values*** $f(x_{j-1})$ ***and*** $f(x_j)$ ***respectively. Since***

$$\begin{aligned}\int_1^4 (S_n - s_n) &= \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))\frac{4-1}{n} \\ &= \frac{3}{n}\sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) \\ &= \frac{3(f(4) - f(1))}{n} = \frac{135}{n} \to 0\end{aligned}$$

as $n \to \infty$ ***we know that*** f ***is integrable and that***

$$\lim_{n\to\infty} \int_1^4 S_n = \lim_{n\to\infty} \int_1^4 s_n = \int_1^4 f.$$

Now since

$$x_j = 1 + \frac{3j}{n}$$

for each $j = 0, \cdots, n$***, we have***

$$\begin{aligned}\int_1^4 S_n &= \sum_{j=1}^{n} \left(\frac{3}{n}\right)(3)\left(1 + \frac{3j}{n}\right)^2 \\ &= \frac{9(14n^2 + 15n + 3)}{2n^2} \to 63\end{aligned}$$

as $n \to \infty$***.***

2. In this exercise we take $f(x) = \sqrt{x}$ for $x \in [0,1]$. Given a positive integer n, we shall take $\mathbf{P}_n$ to be the partition of $[0,1]$ defined by the equation

$$\mathbf{P}_n = \left(\frac{0^2}{n^2}, \frac{1^2}{n^2}, \frac{2^2}{n^2}, \cdots, \frac{n^2}{n^2}\right).$$

Prove that if we define a step function S_n on $[0,1]$ by making

$$S_n(x) = \sqrt{x}$$

whenever x is a point of the partition $\mathbf{P}_n$ and giving S_n the constant value j/n in each interval

$$\left(\frac{(j-1)^2}{n^2}, \frac{j^2}{n^2}\right)$$

of the partition $\mathbf{P}_n$, then

$$\int_0^1 \sqrt{x}\,dx = \lim_{n\to\infty}\int_0^1 S_n = \frac{2}{3}.$$

Solution: ***For each positive integer*** n ***we define the function*** S_n ***as described in the exercise and we define a step function*** s_n ***on*** $[0,1]$ ***by making***

$$S_n(x) = \sqrt{x}$$

whenever x ***is a point of the partition*** $\mathbf{P}_n$ ***and giving*** s_n ***the constant value*** $(j-1)/n$ ***in each interval***

$$\left(\frac{(j-1)^2}{n^2}, \frac{j^2}{n^2}\right)$$

We see at once that $s_n \le f \le S_n$ ***for each*** n ***and that***

$$\lim_{n\to\infty}\int_0^1 (S_n - s_n) = \sum_{j=1}^{n}\left(\frac{j}{n} - \frac{j-1}{n}\right)\left(\frac{j^2}{n^2} - \frac{(j-1)^2}{n^2}\right) = \frac{1}{n} \to 0$$

as $n\to\infty$***. We conclude that the pair of sequences***(s_n) ***and*** (S_n) ***squeezes*** f ***on the interval*** $[0,1]$ ***and that***

$$\lim_{n\to\infty}\int_0^1 S_n = \int_0^1 f.$$

Therefore

$$\int_0^1 f = \lim_{n\to\infty}\int_0^1 S_n = \lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{j}{n}\right)\left(\frac{j^2}{n^2} - \frac{(j-1)^2}{n^2}\right)$$
$$= \lim_{n\to\infty}\frac{4n^2+3n-1}{6n^2} = \frac{2}{3}.$$

3. Prove that

$$\int_0^1 \sqrt[3]{x}\,dx = \frac{3}{4}.$$

This exercise is very similar to Exercise 2. This time we take

$$\mathbf{P}_n = \left(\frac{0^3}{n^3}, \frac{1^3}{n^3}, \frac{3^3}{n^3}, \cdots, \frac{n^3}{n^3}\right)$$

for each n and, for each n, we define S_n to be the step function that takes the value $\sqrt[3]{x}$ at each point of $\mathbf{P}_n$ and whose constant value in each interval

$$\left(\frac{(j-1)^3}{n^3}, \frac{j^3}{n^3}\right)$$

is j/n. We observe that

$$\lim_{n\to\infty}\int_0^1 S_n = \lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{j}{n}\right)\left(\frac{j^3}{n^3} - \frac{(j-1)^3}{n^3}\right)$$
$$= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{3j^3-3j^2+j}{n^4}\right)$$
$$= \lim_{n\to\infty}\left(\frac{3n^2+2n-1}{4n^2}\right) = \frac{3}{4}.$$

In the same way we can show that if s_n $\sqrt[3]{x}$ at each point of $\mathbf{P}_n$ and whose constant value in each interval

$$\left(\frac{(j-1)^3}{n^3}, \frac{j^3}{n^3}\right)$$

is $(j-1)/n$ then

$$\lim_{n\to\infty}\int_0^1 s_n = \frac{3}{4}$$

and so the pair of sequences (s_n) and (S_n) squeezes the cube root function on $[0,1]$.

Some Exercises on Riemann Integrability

1. Suppose that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ has the form } \frac{1}{n} \text{ for some positive integer } n \\ 0 & \text{otherwise} \end{cases}.$$

Prove that f is integrable on the interval $[0,1]$ and that $\int_0^1 f = 0$.

Solution: ***For each positive integer*** n ***we define*** $\mathbf{P}_n$ ***to be the following partition of the interval*** $[0,1]$***:***

$$\mathbf{P}_n = \left(0, \frac{1}{n}, \frac{1}{n-1}, \cdots, \frac{1}{3}, \frac{1}{2}, 1\right).$$

For each n ***we see easily that***

$$\int_0^1 l(\mathbf{P}_n, f) = 0 \qquad \textbf{\textit{and}} \qquad \int_0^1 u(\mathbf{P}_n, f) = \frac{1}{n}$$

and therefore

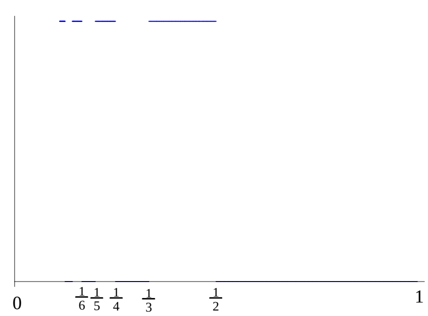
$$\lim_{n\to\infty} \int_0^1 l(\mathbf{P}_n, f) = \lim_{n\to\infty} \int_0^1 u(\mathbf{P}_n, f) = 0.$$

2. Suppose that f is defined on the interval in such a way that whenever $x \in [0,1]$ and x has the form $\frac{1}{n}$ for some positive integer n we have $f(x) = 0$ and whenever x belongs to an interval of the form $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ for some positive integer n we have

$$f(x) = 1 + (-1)^n.$$

Draw a rough sketch of the graph of this function and explain why it is integrable on the interval $[0,1]$.

Solution:



For each positive integer n ***we define*** $\mathbf{P}_n$ ***to be the following partition of the interval*** $[0,1]$***:***

$$\mathbf{P}_n = \left(0, \frac{1}{n}, \frac{1}{n-1}, \cdots, \frac{1}{3}, \frac{1}{2}, 1\right).$$

For each n ***we see easily that***

$$\int_0^1 l(\mathbf{P}_n, f) = \sum_{j=1}^{n-1} \left(\frac{1}{j} - \frac{1}{j+1}\right)\left(1 + (-1)^j\right)$$

and

$$\int_0^1 u(\mathbf{P}_n, f) = \frac{2}{n} + \sum_{j=1}^{n-1} \left(\frac{1}{j} - \frac{1}{j+1}\right)\left(1 + (-1)^j\right)$$

and so

$$\lim_{n\to\infty}\left(\int_0^1 u(\mathbf{P}_n,f)-\int_0^1 l(\mathbf{P}_n,f)\right)=\lim_{n\to\infty}\frac{2}{n}=0$$

and we have shown that f is integrable on $[0,1]$***.***

Incidentally, we have also shown that

$$\int_0^1 f=\lim_{n\to\infty}\sum_{j=1}^{n-1}\left(\frac{1}{j}-\frac{1}{j+1}\right)\left(1+(-1)^j\right)$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{2}{2j(2j+1)}\right)$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{1}{j(2j+1)}\right)$$

In the chapter on infinite series you will learn how to show that the latter limit is $2-2\log 2$***.***

3. Given that f is a bounded function on an interval $[a,b]$, prove that the following conditions are equivalent:

   a. The function f is integrable on the interval $[a,b]$.

   b. For every number $\varepsilon>0$ there exist step functions s and S on the interval $[a,b]$ such that $s\le f\le S$ and
   $$\int_a^b (S-s)<\varepsilon.$$

   c. For every number $\varepsilon>0$ there exist step functions s and S on the interval $[a,b]$ such that $s\le f\le S$ and such that if
   $$E=\{x\in[a,b]\mid S(x)-s(x)\ge\varepsilon\},$$
   we have $m(E)<\varepsilon$.

   To show that condition a implies condition b we assume that condition a holds. In other words, we assume that f is integrable on $[a,b]$. Suppose that $\varepsilon>0$. Using the first criterion for integrability we choose a partition $\mathbf{P}$ of $[a,b]$ such that
   $$\int_a^b w(\mathbf{P},f)<\varepsilon.$$
   We define $S=u(\mathbf{P},f)$ and $s=l(\mathbf{P},f)$ and observe that the functions s and S have the desired properties.
   The proof that condition a implies condition c is very similar. This time, the partition $\mathbf{P}$ is chosen using the second criterion for integrability.
   To prove that condition b implies condition a we assume that condition b holds. What we shall show is that the first criterion for integrability holds. Suppose that $\varepsilon>0$. Using condition b we choose step functions s and S on the interval $[a,b]$ such that $s\le f\le S$ and
   $$\int_a^b (S-s)<\varepsilon.$$
   Choose a partition $\mathbf{P}$ of $[a,b]$ such that both s and S step within $\mathbf{P}$ and observe that since
   $$s\le l(\mathbf{P},f)\le u(\mathbf{P},f)\le S$$
   we have
   $$\int_a^b w(\mathbf{P},f)=\int_a^b (u(\mathbf{P},f)-l(\mathbf{P},f))\le\int_a^b (S-s)<\varepsilon.$$
   The proof that condition c implies condition a is very similar. This time we show that f is integrable by showing that the second criterion for integrability holds.

4. Suppose that f is a bounded function on an interval $[a,b]$ and that for every number $\varepsilon>0$ there exists an elementary subset E of $[a,b]$ such that $m(E)<\varepsilon$ and such that the function $f(1-\chi_E)$ is Riemann integrable on $[a,b]$. Prove that f must be Riemann integrable on the interval $[a,b]$.

Solution: ***In order to show that** f **is integrable on** $[a,b]$ **we shall show that** f **satisfies the** second criterion **for integrability. Suppose that** $\varepsilon > 0$.*

Using the given property of** f **we choose an elementary subset** A **of** $[a,b]$ **such that** $m(A) < \varepsilon/2$ **and such that the function** $f(1-\chi_A)$ **is integrable on** $[a,b]$**. Choose a partition** $\mathbf{P}_1$ **of** $[a,b]$ **such that the function** χ_A **steps within** $\mathbf{P}$**. Now, using the fact that** $f(1-\chi_A)$ **satisfies the second criterion for integrability we choose a partition** $\mathbf{P}_2$ **of** $[a,b]$ **such that if we define

$$B = \{x \in [a,b] \mid w(\mathbf{P}, f(1-\chi_A))(x) \geq \varepsilon\}.$$

then** $m(B) < \varepsilon/2$**. We now define** P **to be the common refinement of** $\mathbf{P}_1$ **and** $\mathbf{P}_2$ **and we express** $\mathbf{P}$ **as

$$\mathbf{P} = (x_0, x_1, \cdots, x_n).$$

For each** $j = 1, 2, \cdots, n$**, if the open interval** (x_{j-1}, x_j) **is not included in** $A \cup B$ **then, since the functions** f **and** $f(1-\chi_A)$ **agree in the interval** (x_{j-1}, x_j) **the condition

$$w(\mathbf{P}, f)(x) = w(\mathbf{P}, f(1-\chi_A))(x) < \varepsilon$$

must hold whenever** $x \in (x_{j-1}, x_j)$**. Since

$$m(A \cup B) \leq m(A) + m(B) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

we have succeeded in showing that** f **satisfies the second criterion for integrability.

5.

a. Suppose that f is a nonnegative function defined on an interval $[a,b]$ and that for every number $\varepsilon > 0$, the set

$$\{x \in [a,b] \mid f(x) \geq \varepsilon\}$$

is finite. Prove that f must be integrable on $[a,b]$ and that $\int_a^b f = 0$.

We shall show that f satisfies the first criterion for integrability. Suppose that $\varepsilon > 0$. Choose a finite set S such that

$$f(x) < \frac{\varepsilon}{b-a}$$

whenever $x \in [a,b] \setminus S$ and define $\mathbf{P}$ to be the partition of $[a,b]$ whose points are the numbers a and b and the members of S arranged in increasing order. Since $l(\mathbf{P}, f)$ is nonnegative and $u(\mathbf{P}, f)$ never exceeds the value $\varepsilon/(b-a)$, we have

$$\int_a^b w(\mathbf{P}, f) = \int_a^b (u(\mathbf{P}, f) - l(\mathbf{P}, f)) \leq \int_a^b u(\mathbf{P}, f) \leq \varepsilon.$$

Since f satisfies the first criterion for integrability, f is integrable on $[a,b]$ and the same argument shows that whenever $\varepsilon > 0$ we have $\int_a^b f \leq \varepsilon$ from which we deduce that $\int_a^b f = 0$.

b. Prove that if f is the ruler function that was introduced in an earlier example then f is an integrable function on the interval $[0,1]$, even though f is discontinuous at every rational number in the interval.
The ruler function obviously has the property described in part a.

6. Given that f is a bounded nonnegative function defined on an interval $[a,b]$, prove that the following conditions are equivalent:

a. The function f is integrable on the interval $[a,b]$ and $\int_a^b f = 0$.

b. For every number $\varepsilon > 0$ there exists an elementary set E such that $m(E) < \varepsilon$ and such that

$$\{x \in [a,b] \mid f(x) \geq \varepsilon\} \subseteq E.$$

To show that condition a implies condition b we assume that f is integrable on $[a,b]$. Suppose that $\varepsilon > 0$. To obtain the desired set E we shall use the same sort of technique as was used in the proof of Theorem 11.8.4. Choose a step function $S \geq f$ such that

$$\int_a^b S < \varepsilon^2$$

and define

$$E = \{x \in [a,b] \mid S(x) \geq \varepsilon\}.$$

We observe that

$$\{x \in [a,b] \mid f(x) \geq \varepsilon\} \subseteq E.$$

Now since S is a step function, the set E is elementary and we have

$$\varepsilon^2 > \int_a^b S \geq \int_E S \geq \int_E \varepsilon = \varepsilon m(E)$$

from which we deduce that $m(E) < \varepsilon$.

To show that condition b implies condition a we assume that condition b holds. Once again we borrow from the proof of proof of Theorem 11.8.4. Using the fact that f is bounded we choose a number k such that $f(x) < k$ for every $x \in [a,b]$. For each positive integer n we choose an elementary set E_n such that $m(E_n) < \frac{1}{n}$ and such that

$$\left\{x \in [a,b] \mid f(x) \geq \frac{1}{n}\right\} \subseteq E_n.$$

For each n we have

$$\begin{aligned}\int_a^b u(\mathbf{P}_n, f) &= \int_{E_n} u(\mathbf{P}_n, f) + \int_{[a,b]\setminus E_n} u(\mathbf{P}_n, f) \\ &\leq \int_{E_n} k + \int_{[a,b]\setminus E_n} \frac{1}{n} \leq km(E_n) + \int_a^b \frac{1}{n} \\ &< \frac{k}{n} + \frac{b-a}{n}\end{aligned}$$

Since the latter expression approaches 0 as $n \to \infty$ we deduce that

$$\inf\left\{\int_a^b S \mid S \text{ is a step function and } f \leq S\right\} = 0$$

and this shows that f is integrable and that $\int_a^b f = 0$.

If you are reading the on-screen version of this book you can find a special group of exercises that are designed to be done as a special project. These exercises require you to have read some of the chapter on infinite series and they depend upon the special group of exercises on elementary sets that appeared earlier. The main purpose of these exercises is to invite you to prove the following interesting fact about integrals: *If f is a nonnegative integrable function on an interval $[a,b]$, where $a < b$, and if $\int_a^b f = 0$ then there must be at least one number $x \in [a,b]$ for which $f(x) = 0$.*

To reach this special group of exercises, click on the link .

Positive Integrable Functions Have Positive Integrals

Suppose that f is a nonnegative integrable function on an interval $[a,b]$ where $a < b$ and that

$$\int_a^b f = 0.$$

1. Prove that for every number $\varepsilon > 0$ there exists an elementary set E such that $m(E) < \varepsilon$ and such that

$$\{x \in [a,b] \mid f(x) \geq \varepsilon\} \subseteq E.$$

This exercise is a duplicate of the last exercise in the exercises on exercises on integrability.

2. Prove that if, for every positive integer n, we choose an elementary set E_n such that

$$m(E_n) < \frac{b-a}{2^n}$$

and such that

$$\left\{x \in [a,b] \mid f(x) \geq \frac{b-a}{2^n}\right\} \subseteq E_n$$

then for every elementary E satisfying

$$E \subseteq \bigcup_{n=1}^{\infty} E_n$$

we have $m(E) < b-a$. To obtain this proof you will need to make use of the special group of exercises on elementary sets that can be reached by clicking on the icon .

The existence of the sets E_n follows from Exercise 1. Now if

$$E \subseteq \bigcup_{n=1}^{\infty} E_n$$

then the special exercises on elementary sets guarantee that

$$m(E) \leq \sum_{n=1}^{\infty} m(E_n) < \sum_{n=1}^{\infty} \frac{b-a}{2^n} = b-a.$$

3. Prove that if the sets E_n are defined as in Exercise 2, the set

$$[a,b] \setminus \bigcup_{n=1}^{\infty} E_n$$

must be nonempty and deduce that there must exist a number $x \in [a,b]$ such that $f(x) = 0$.

The elementary set $[a,b]$ can't be a subset of $\bigcup_{n=1}^{\infty} E_n$ because we do not have

$$m([a,b]) < b-a.$$

Thus there must be numbers x in $[a,b]$ that do not belong to any of the sets E_n and since any such number x must satisfy the inequality

$$f(x) < \frac{b-a}{2^n}$$

for every positive integer n we have $f(x) = 0$ for such numbers x.

4. Improve on the preceding exercises by proving that for every number $\varepsilon > 0$ there exists a sequence (E_n) of elementary sets such that the following two conditions hold:

   a. For every number satisfying

   $$x \in [a,b] \setminus \bigcup_{n=1}^{\infty} E_n$$

   we have $f(x) = 0$.

   b. For every elementary set E satisfying

   $$E \subseteq \bigcup_{n=1}^{\infty} E_n$$

   we have $m(E) < \varepsilon$.

Suppose that $\varepsilon > 0$. We choose a sequence (E_n) of elementary sets such that the conditions

$$m(E_n) < \frac{\varepsilon}{2^n}$$

and

$$\left\{x \in [a,b] \mid f(x) \geq \frac{\varepsilon}{2^n}\right\} \subseteq E_n$$

hold for each n. Continue as above. Of course, we don't have a guarantee that $[a,b] \setminus \bigcup_{n=1}^{\infty} E_n$ is nonempty unless $\varepsilon < b-a$.

Some Exercises on the Junior Lebesgue Criterion

1. True or false? Every step function satisfies the junior Lebesgue criterion.
   Of course this statement is true. The set of discontinuities of a step function, being finite, is an elementary set with zero measure.

2. Suppose that (x_n) is a convergent sequence in an interval $[a,b]$ and that f is a bounded function on $[a,b]$ that is continuous at every member of $[a,b]$ that does not lie in the range of the sequence (x_n). Prove that f is integrable on $[a,b]$.

   Solution: ***See the solution to Exercise 3.***

3. Suppose that (x_n) is a sequence in an interval $[a,b]$, that (x_n) has only finitely many partial limits, and that f is a bounded function on $[a,b]$ that is continuous at every member of $[a,b]$ that does not belong to the range of the sequence (x_n). Prove that f is integrable on $[a,b]$.

   Solution: ***We shall write the set of partial limits of*** (x_n) ***as*** $\{y_1, y_2, \cdots, y_k\}$. ***To prove that*** f ***satisfies the junior Lebesgue criterion, suppose that*** $\varepsilon > 0$.

   We define

$$U = \bigcup_{j=1}^{k} \left(y_j - \frac{\varepsilon}{2k}, y_j - \frac{\varepsilon}{2k} \right)$$

   and we observe that

$$m(U) \le \sum_{j=1}^{k} m\left(\left(y_j - \frac{\varepsilon}{2k}, y_j - \frac{\varepsilon}{2k} \right) \right) = \sum_{j=1}^{k} \frac{\varepsilon}{k} = \varepsilon.$$

   Since the set $[a,b] \setminus U$ ***is closed and bounded and since the sequence*** (x_n) ***has no partial limits in*** $[a,b] \setminus U$ ***we know that*** (x_n) ***cannot be frequently in the set*** $[a,b] \setminus U$. ***Therefore, if***

$$F = \{x_n \mid n = 1, 2, \cdots\} \setminus U$$

   then the set F ***is finite and so*** $m(F) = 0$. ***We have thus found an elementary subset*** $U \cup F$ ***of*** $[a,b]$ ***such that*** $m(U \cup F) \le \varepsilon$ ***and such that*** f ***is continuous at every number*** $x \in [a,b] \setminus (U \cup F)$.

4. This exercise does not ask you for a proof. Suppose that (x_n) is a sequence in an interval $[a,b]$ and that f is a bounded function on $[a,b]$ that is continuous at every member of $[a,b]$ that does not belong to the range of the sequence (x_n). Do you think that the function f has to be integrable on $[a,b]$? What does your intuition tell you?

   Solution: ***The function f must be integrable. This fact will follow from the full version of the Lebesgue criterion for integrability that will appear in the chapter*** on sets of measure zero.

Some Exercises on the Composition Theorem

1. Given two functions f and g defined on a set S, we define the functions $f \vee g$ and $f \wedge g$ as follows:

$$f \vee g(x) = \begin{cases} f(x) & \text{if } f(x) \ge g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

   and

$$f \wedge g(x) = \begin{cases} f(x) & \text{if } f(x) \le g(x) \\ g(x) & \text{if } f(x) > g(x) \end{cases}.$$

Given Riemann integrable functions f and g on an interval $[a,b]$, make the observations

$$f \vee g = \frac{f+g+|f-g|}{2}$$

and

$$f \wedge g = \frac{f+g-|f-g|}{2}$$

and deduce that the functions $f \vee g$ and $f \wedge g$ are also integrable on $[a,b]$.

There really isn't much to do in this exercise. The equation

$$(f \vee g)(x) = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

and

$$(f \wedge g)(x) = \frac{f(x)+g(x)-|f(x)-g(x)|}{2}$$

for each x follow at once when we consider the cases $f(x) \le g(x)$ and $f(x) > g(x)$.

2. Given that f is a nonnegative integrable function on an interval $[a,b]$, explain why the function $\sqrt{f}$ is integrable.
   This exercise follows at once from the fact that the square root function is uniformly continuous on the range of f.

3. Given that f is integrable on an interval $[a,b]$, that $f(x) \ge 1$ for every $x \in [a,b]$ and that
   $$g(x) = \log(f(x))$$
   for every $x \in [a,b]$, explain why the function g must be integrable on $[a,b]$.
   Since the function $\log$ is uniformly continuous on the interval $[1,\infty)$, the integrability of g follows at once from the composition theorem.

4. Suppose that f is integrable on an interval $[a,b]$ and that $\alpha \le f(x) \le \beta$ for every $x \in [a,b]$. Show how the junior version of the composition theorem for integrability can be used to show that if h is any continuous function on the interval $[\alpha,\beta]$ the function $h \circ f$ is integrable on $[a,b]$.
   The result follows at once from the fact that any continuous function on the interval $[\alpha,\beta]$ must be uniformly continuous.

Exercises on the Change of Variable Theorem

1. a. Given that f is a continuous function on the interval $[-1,1]$, prove that
   $$\int_0^{2\pi} f(\sin x)\cos x\,dx = 0.$$

   Solution: ***We define*** $u(x) = \sin x$ ***for every number*** $x \in [0,2\pi]$ ***and observe that***

   $$\int_0^{2\pi} f(\sin x)\cos x\,dx = \int_0^{2\pi} f(u(x))u'(x)\,dx$$
   $$= \int_{u(0)}^{u(2\pi)} f = \int_0^0 f = 0$$

   b. Given that f is a continuous function on the interval $[0,1]$, prove that
   $$\int_0^{\pi/2} f(\sin x)\,dx = \int_{\pi/2}^{\pi} f(\sin x)\,dx.$$

   Solution: ***We define*** $u(x) = \pi - x$ ***for every number*** $x \in \left[0, \frac{\pi}{2}\right]$ ***and observe that***

$$\int_0^{\pi/2} f(\sin x)dx = -\int_0^{\pi/2} f(\sin(\pi - x))u'(x)dx$$
$$= -\int_0^{\pi/2} f(\sin(u(x)))u'(x)dx$$
$$= -\int_{u(0)}^{u(\pi/2)} f(\sin t)dt = -\int_{\pi}^{\pi/2} f(\sin t)dt$$
$$= \int_{\pi/2}^{\pi} f(\sin t)dt$$

Of course, it makes no difference whether we write t or x in the integral $\int_{\pi/2}^{\pi} f(\sin t)dt$.

c. Given that $\alpha > 0$, prove that

$$\int_0^{\pi/2} \sin^{\alpha} x dx = 2^{\alpha} \int_0^{\pi/2} \sin^{\alpha} x \cos^{\alpha} x dx.$$

Solution: ***We define $u(x) = 2x$ for all $x \in [0, \pi/2]$ and observe that***

$$2^{\alpha} \int_0^{\pi/2} \sin^{\alpha} x \cos^{\alpha} x dx = \int_0^{\pi/2} 2^{\alpha} \sin^{\alpha} x \cos^{\alpha} x dx = \int_0^{\pi/2} (2 \sin x \cos x)^{\alpha} dx$$
$$= \frac{1}{2} \int_0^{\pi/2} (\sin^{\alpha} 2x) 2dx = \frac{1}{2} \int_0^{\pi/2} (\sin^{\alpha}(u(x)))u'(x)dx$$
$$= \frac{1}{2} \int_{u(0)}^{u(\pi/2)} \sin^{\alpha} t dt = \frac{1}{2} \int_0^{\pi} \sin^{\alpha} t dt$$
$$= \frac{1}{2} \left(\int_0^{\pi/2} \sin^{\alpha} t dt + \int_{\pi/2}^{\pi} \sin^{\alpha} t dt \right)$$

and from part b we deduce that the latter expression is equal to

$$\frac{1}{2} \left(\int_0^{\pi/2} \sin^{\alpha} t dt + \int_0^{\pi/2} \sin^{\alpha} t dt \right) = \int_0^{\pi/2} \sin^{\alpha} t dt.$$

2. Given that u is a differentiable function on an interval $[a, b]$ and that its derivative u' is integrable on $[a, b]$ and given that $u(a) = u(b)$ and that f is integrable on the range of u, prove that

$$\int_a^b f(u(t))u'(t)dt = 0.$$

From the change of variable theorem we see that

$$\int_a^b f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f(x)dx = \int_{u(a)}^{u(a)} f(x)dx = 0.$$

3. Given that f is integrable on an interval $[a, b]$ and that c is any number, prove that

$$\int_a^b f(t)dt = \int_{a+c}^{b+c} f(t - c)dt$$

For every number t we define $u(t) = t - c$. We observe that $u'(t) = 1$ for every t. Now

$$\int_{a+c}^{b+c} f(t - c)dt = \int_{a+c}^{b+c} f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f(x)dx = \int_a^b f(t)dt.$$

I have changed the name of the dummy variable back to t to match the expression in the exercise.

4. Given that a, b and c are real numbers, that $ac < bc$ and that f is a continuous function on the interval $[ac, bc]$, prove that

$$\int_{ac}^{bc} f(t)dt = c \int_a^b f(ct)dt.$$

Hint: ***Look at the definition $u(t) = ct$ for each t.***

We assume that $c > 0$. We define $u(t) = ct$ for every number t. We see that $u'(t) = c$ for each t and we deduce from the change of variable theorem that

$$c\int_a^b f(ct)dt = \int_a^b f(u(t))u'(t)dt = \int_{ac}^{bc} f(x)dx = \int_{ac}^{bc} f(t)dt$$

5. a. Suppose that f is a continuous function on an interval $[a,b]$, that g is nonnegative and integrable on $[a,b]$. Prove that, if m and M are, respectively, the minimum and maximum values of f, then

$$m\int_a^b g \le \int_a^b fg \le M\int_a^b g$$

and deduce that there exists a number $c \in [a,b]$ such that

$$\int_a^b fg = f(c)\int_a^b g.$$

This fact is sometimes called the **mean value theorem for integrals**.
Since

$$m \le f(x) \le M$$

for every $x \in [a,b]$ we have

$$\int_a^b mg \le \int_a^b fg \le \int_a^b Mg$$

which gives us

$$m\int_a^b g \le \int_a^b fg \le M\int_a^b g$$

and so

$$m \le \frac{\int_a^b fg}{\int_a^b g} \le M.$$

The existence of the number c follows at once from the Bolzano intermediate value theorem.

b. Given that f is continuous on an interval $[a,b]$, prove that there exits a number $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

6. a. Given that f is a nonnegative continuous function on an interval $[a,b]$ where $a < b$ and that $\int_a^b f = 0$, prove that f is the constant zero function.
To obtain a contradiction, suppose that $c \in [a,b]$ and that $f(c) > 0$. Using the fact that f is continuous at c, choose $\delta > 0$ such that the inequality

$$f(x) > \frac{f(c)}{2}$$

holds for every number $x \in [a,b] \cap (c-\delta, c+\delta)$. The set $[a,b] \cap (c-\delta, c+\delta)$ is an interval with positive length that we shall write as $[c,d]$. In fact, c is the larger of the two numbers a and $c-\delta$ and d is the smaller of the two numbers b and $c+\delta$. Since

$$\int_a^b f \ge \int_c^d f \ge \int_c^d \frac{f(c)}{2} = \frac{f(c)}{2}(d-c) > 0$$

which gives us our desired contradiction.

b. Given that f is a continuous function on an interval $[a,b]$ where $a < b$ and that $\int_a^x f = 0$, for every $x \in [a,b]$, prove that f is the constant zero function.
We know that if

$$F(x) = \int_a^x f$$

for each x then $F'(x) = f(x)$ for each x. Since F is the constant function zero we conclude that $f(x) = 0$ for each x.

7. In this exercise we consider another proof of the "u decreasing" form of the change of variable theorem.

a. Given that f is an integrable function on an interval $[a,b]$ and that $g(t) = f(-t)$ whenever $-b \le t \le -a$, give a direct proof that g is integrable on the interval $[-b,-a]$ and that

$$\int_{-b}^{-a} g(t)dt = \int_a^b f(x)dx.$$

Suppose first that f is a step function on $[a,b]$ that steps within the partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

taking the constant value α_j on each interval (x_{j-1}, x_j). We define

$$\mathbf{Q} = (-x_n, -x_{n-1}, \cdots, -x_1, -x_0)$$

and observe that $\mathbf{Q}$ is a partition of the interval $[-b,-a]$. Now we define $g(t) = f(-t)$ for each $t \in [a,b]$ and we observe that g is a step function that takes the value α_j on each interval $(-x_j, x_{j-1})$. We see at once that

$$\int_a^b f = \sum_{j=1}^{n} \alpha_j (x_j - x_{j-1}) = \sum_{j=1}^{n} (-x_{j-1} - (-x_j))\alpha_j = \int_{-b}^{-a} g$$

We can now handle the general case. Using the fact that f is integrable on the interval $[a,b]$, choose a pair of sequences of step functions that squeezes f on the interval $[b,a]$. In other words,

$$s_n \le f \le S_n$$

for each n and

$$\lim_{n\to\infty} \int_a^b (S_n - s_n) = 0.$$

For each n we define $s_n^*(t) = s_n(-t)$ and $S_n^*(t) = S_n(-t)$ and we observe that

$$s_n^* \le g \le S_n^*$$

and, by the case we have already considered we deduce that

$$\int_{-b}^{-a} (S_n^* - s_n^*) = \int_a^b (S_n - s_n) \to 0$$

as $n \to \infty$.

It follows that the function g is integrable on the interval $[-b,-a]$ and that

$$\int_{-b}^{-a} g = \lim_{n\to\infty} \int_{-b}^{-a} s_n^* = \lim_{n\to\infty} \int_a^b s_n = \int_a^b f.$$

1. a. Suppose that u is a decreasing differentiable function on an interval $[a,b]$ and that the derivative u' of u is integrable on $[a,b]$. Apply the form of the monotone version of the change of variable theorem proved above to the function v defined by the equation $v(t) = -u(t)$ for $-b \le t \le -a$ to show that the equation

$$\int_a^b f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f(x)dx$$

holds for every function f integrable on the interval $[u(b), u(a)]$.
We define $g(t) = f(-t)$ for all $t \in [a,b]$. Then g is integrable on $[-u(a), -u(b)]$ and we have

$$\int_{-u(a)}^{-u(b)} g = \int_{u(b)}^{u(a)} f = -\int_{u(a)}^{u(b)} f$$

From the monotone version of the theorem proved earlier we also see that

$$\int_{-u(a)}^{-u(b)} g = \int_{v(a)}^{v(b)} g = \int_a^b g(v(t))v'(t)dt$$

$$= \int_a^b g(-u(t))(-u'(t))dt = -\int_a^b f(u(t))u'(t)dt.$$

We conclude that

$$\int_a^b f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f$$

To reach some additional exercises that invite you to develop some important inequalities, click on the icon .

Additional Exercises on the Change of Variable Theorem

The exercises in this document invite you to derive some inequalities that have important applications in mathematics and other disciplines.

1. Given that f is a strictly increasing continuous function on an interval $[a,b]$, prove that

$$\int_a^b f + \int_{f(a)}^{f(b)} f^{-1} = bf(b) - af(a).$$

Solution:

a. ***First we shall motivate this problem by looking at the special case in which f has a continuous derivative. In this case we can change variable in the integral***

$$\int_{f(a)}^{f(b)} f^{-1}$$

to obtain

$$\int_{f(a)}^{f(b)} f^{-1} = \int_a^b f^{-1}(f(t))f'(t)dt = \int_a^b tf'(t)dt$$

which we can integrate by parts to obtain

$$\int_a^b tf'(t)dt = bf(b) - af(a) - \int_a^b 1f(t)dt.$$

So the desired identity is true in this special case. Incidentally, this special case is all that we need for the application of the identity to Exercise 3.

b. ***Now we consider the general case. The proof in this case is not difficult but it depends upon Darboux's theorem. (As a matter of fact, all the proof needs is the application of Darboux's theorem to continuous functions and this form of Darboux's theorem is easier to prove.) Suppose that***

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

is a the regular n-partition of the interval $[a,b]$ for each n and, for each j, suppose that $y_j = f(x_j)$. Then if Q_n is the partition

$$\mathbf{Q}_n = (y_0, y_1, \cdots, y_n)$$

of the interval $[f(a), f(b)]$ then, since f is uniformly continuous on $[a,b]$, the mesh of the partition Q_n must approach 0 as $n \to \infty$. Now for each n we see that

$$\begin{aligned}
\sum_{j=1}^{n}(x_j - x_{j-1})f(x_j) + \sum_{j=1}^{n}(y_j - y_{j-1})f^{-1}(y_{j-1}) &= \sum_{j=1}^{n}(x_j - x_{j-1})y_j + \sum_{j=1}^{n}(y_j - y_{j-1})x_{j-1} \\
&= \sum_{j=1}^{n}\Big((x_j - x_{j-1})y_j + (y_j - y_{j-1})x_{j-1}\Big) \\
&= \sum_{j=1}^{n}\Big(x_j y_j - y_{j-1}x_{j-1}\Big) = x_n y_n - x_0 y_0 \\
&= bf(b) - af(a)
\end{aligned}$$

and so, letting $n \to \infty$, we see that

$$\int_a^b f + \int_{f(a)}^{f(b)} f^{-1} = bf(b) - af(a).$$

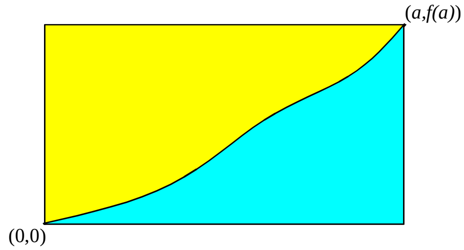
2. Given that f is a strictly increasing continuous unbounded function on the interval $[0,\infty)$ and that $f(0) = 0$, prove that for all positive numbers a and b we have

$$ab \le \int_0^a f + \int_0^b f^{-1}.$$

Solution:

The Case $b = f(a)$:

The case $b = f(a)$ is illustrated in the following figure.



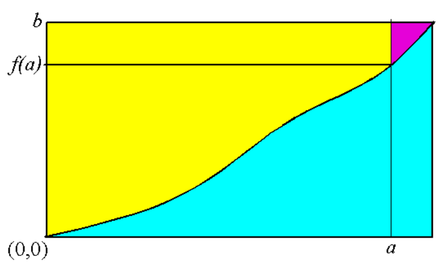
The case $b = f(a)$

The area of the yellow region is $\int_0^{f(a)} f^{-1}$ and the area of the turquoise region is $\int_0^a f$. The sum of these two areas is ab. Now we say it precisely: It follows from the preceding exercise that

$$ab = af(a) - 0f(0) = \int_0^a f + \int_0^{f(a)} f^{-1} = \int_0^a f + \int_0^b f^{-1}.$$

The Case $b > f(a)$:

The case $b > f(a)$ is illustrated in the following figure.



The case $b > f(a)$

In this case, the number ab is the area of the yellow region plus the area of the turquoise region to the left of the vertical line at a. If we add on the area of the purple region then we obtain the sum of the two integrals $\int_0^a f$ and $\int_0^b f^{-1}$ and so

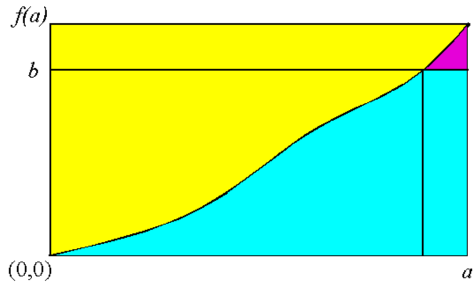
$$ab \le \int_0^a f + \int_0^b f^{-1}.$$

Now we say it precisely:
It follows from the preceding exercise that

$$ab = af(a) + a(b - f(a)) = \int_0^a f + \int_0^{f(a)} f^{-1} + \int_{f(a)}^b a$$
$$\le \int_0^a f + \int_0^{f(a)} f^{-1} + \int_{f(a)}^b f^{-1}(y)dy = \int_0^a f + \int_0^b f^{-1}.$$

The case $b < f(a)$:

The case $b < f(a)$ is illustrated in the following figure.



The case $b < f(a)$

In this case the number ab ***is the area of the yellow region below the horizontal line at*** b ***plus the area of the turquoise region. If we add on the area of the purple region then we obtain the sum of the two integrals*** $\int_0^a f$ ***and*** $\int_0^b f^{-1}$ ***and so***

$$ab \leq \int_0^a f + \int_0^b f^{-1}.$$

Now we say it precisely: We write $g = f^{-1}$***. Since*** $a > g(b)$ ***we can apply the case just considered to the function*** g ***to obtain***

$$ab \leq \int_0^a g + \int_0^b g^{-1}$$

and this is exactly the desired result.

3. Given that a and b are positive numbers, that p and q are positive numbers satisfying the equation

$$\frac{1}{p} + \frac{1}{q} = 1,$$

prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

This inequality is known as **W.H. Young's inequality.**

Hint:

Apply the preceding exercise to the function f ***defined by the equation*** $f(x) = x^{p-1}$ ***for all*** $x > 0$***.***

The inequality

$$ab \leq \int_0^a f + \int_0^b f^{-1}$$

gives us

$$ab \leq \int_0^a x^{p-1}dx + \int_0^b t^{1/(p-1)}dt$$

which yields

$$ab \leq \frac{a^p}{p} + \frac{b^{1+1/(p-1)}}{1 + 1/(p-1)} = \frac{a^p}{p} + \frac{b^q}{q}$$

4. If f is an integrable function on an interval $[a, b]$ then the p**-norm** $\|f\|_p$ of f is defined by the equation

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}.$$

Prove that if f and g are integrable on an interval $[a, b]$ and if p and q are positive numbers satisfying the equation

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then the following assertions are true:

a. For every number $x \in [a,b]$ we have

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \le \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q.$$

This inequality follows at once from Exercise 3.

b. Integrating both sides of the preceding inequality and applying additivity yields the inequality

$$\int_a^b fg \le \|f\|_p\|g\|_q.$$

This inequality is known as **Hölder's inequality.**

We have

$$\int_a^b \frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}dx \le \frac{1}{p}\int_a^b\left(\frac{|f(x)|}{\|f\|_p}\right)^p dx + \frac{1}{q}\int_a^b\left(\frac{|g(x)|}{\|g\|_q}\right)^q dx$$

$$= \frac{1}{p\|f\|_p^p}\int_a^b |f(x)|^p dx + \frac{1}{q\|g\|_q^q}\int_a^b |g(x)^q| dx$$

$$= \frac{1}{p\|f\|_p^p}\|f\|_p^p + \frac{1}{q\|g\|_q^q}\|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1$$

and so

$$\int_a^b fg \le \int_a^b |fg| \le \|f\|_p\|g\|_q$$

5. This exercise makes use of the concept of a convex function that was introduced in earlier exercises. Suppose that f is integrable on the interval $[0,1]$ and that

$$\alpha \le f(x) \le \beta$$

for every number $x \in [0,1]$ and that

$$\int_0^1 f = I.$$

Prove that if h is a convex function on an open interval that includes the interval $[\alpha,\beta]$ then the following assertions hold:

a. There is a number k such that the inequality

$$\frac{h(y)-h(I)}{y-I} \le k$$

holds whenever $\alpha \le y < I$ and the inequality

$$\frac{h(y)-h(I)}{y-I} \ge k$$

holds whenever $I \le y \le \beta$.

We suppose that h is a convex function on an interval (p,q) where $p < \alpha$ and $\beta < q$.

Define

$$A = \left\{\frac{h(y)-h(I)}{y-I} \;\middle|\; \alpha \le y < q\right\}$$

and

$$B = \left\{\frac{h(y)-h(I)}{y-I} \;\middle|\; p < y \le \beta\right\}.$$

Since h is a convex function we know that whenever $p < y < I < z < q$

$p \qquad y \qquad I \qquad z \qquad q$

we have

$$\frac{h(y)-h(I)}{y-I} \le \frac{h(z)-h(I)}{z-I}$$

and therefore $\sup A \leq \inf B$ and any number k in the interval $[\sup A, \inf B]$ will have the desired properties.

b. If k is chosen with the property specified in part a then for every number $x \in [0,1]$ we have
$$h(f(x)) - h(I) \geq k(f(x) - I).$$
The desired inequality becomes clear when we consider the cases $f(x) < I$, $f(x) = I$ and $f(x) > I$.

c. Integrating both sides of the preceding inequality yields the inequality
$$\int_0^1 h(f(x))dx \geq h\left(\int_0^1 f(x)dx\right).$$
This inequality is known as **Jensen's inequality.**
When we integrate both sides we obtain
$$\int_0^1 \left(h(f(x)) - h(I)\right)dx \geq k\int_0^1 \left(f(x) - I\right)dx$$
which yields
$$\int_0^1 h(f(x))dx - h(I) \geq 0$$
and this is the desired result.

6. a. Prove that if f is integrable on the interval $[0,1]$ then
$$\int_0^1 \exp(f(x))dx \geq \exp\left(\int_0^1 f(x)dx\right).$$
This inequality follows at once from Jensen's inequality because the function $\exp$, having an increasing derivative, must be convex.

b. Prove that if f is integrable on the interval $[0,1]$ and if for some number $\delta > 0$ we have $f(x) > \delta$ for every $x \in [0,1]$ then
$$\int_0^1 f(x)dx \geq \exp\left(\int_0^1 \log(f(x))dx\right).$$
Since the function $\log$ is uniformly continuous on the interval $[\delta, \infty)$ we know that the function
$$g = \log \circ f$$
is integrable on $[0,1]$. It follows from Part a that
$$\int_0^1 \exp(g(x))dx \geq \exp\left(\int_0^1 g(x)dx\right)$$
which gives us
$$\int_0^1 f(x)dx \geq \exp\left(\int_0^1 \log(f(x))dx\right).$$

c. Given positive numbers $c_1, c_2, \cdots, c_n$, apply part b to an appropriate step function f on $[0,1]$ to obtain the inequality
$$\frac{c_1 + c_2 + \cdots + c_n}{n} \geq (c_1 c_2 \cdots c_n)^{1/n}.$$
In other words, the arithmetic mean of the numbers $c_1, c_2, \cdots, c_n$ is not less than the geometric mean.
We define
$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$
to be the regular n-partition of $[0,1]$ and we define f to be the step function on $[0,1]$ that takes the constant value c_j on each interval (x_{j-1}, x_j) and we apply Part b.

Alternative 11 The Riemann-Stieltjes Integral

Some Exercises on Step Functions

1. True or false? If f is a step function on an interval $[a,b]$ and $[c,d]$ is a subinterval of $[a,b]$ then f is a step function on $[c,d]$.

   Solution: ***The statement is true.***

   a ——— c ——————— d ——— b

   Choose a partition $\mathbf{P}$ of $[a,b]$ within which the function f steps and then refine $\mathbf{P}$ by adding to it the two numbers c and d. Then drop all of the points of this partition that lie outside of the interval $[c,d]$ and we obtain a partition $\mathbf{Q}$ of $[c,d]$ within which f steps.

2. True or false? If f is a step function then given any interval $[a,b]$, the function f is a step function on $[a,b]$.

   Hint: ***This statement is true. Write a short proof.***
   Choose an interval $[c,d]$ such that f is a step function on $[c,d]$ and such that $f(x) = 0$ whenever $x \in \mathbf{R} \setminus [c,d]$. Choose numbers p and q such that p is less than both of the numbers a and c and q is greater than both of the numbers b and d

   p a c b d q

   Choose a partition $\mathbf{P}$ of the interval $[c,d]$ such that f steps within $\mathbf{P}$. If we add the two numbers p and q to the partition $\mathbf{P}$ then we obtain a partition of the larger interval $[p,q]$ within which f steps.
   Since f is a step function on the interval $[p,q]$, it follows from Exercise 1 that f is a step function on $[a,b]$.

3. Give an example of a step function on the interval $[0,2]$ that does not step within any regular partition of $[0,2]$.

   Solution: ***We define***

   $$f(x) = \begin{cases} 0 & \textbf{\textit{if}} \quad 0 \le x \le \sqrt{2} \\ 1 & \textbf{\textit{if}} \quad \sqrt{2} < x \le 2 \end{cases}$$

   Now if $\mathbf{P}$ is any regular partition of the interval $[0,2]$ then, since the irrationality of $\sqrt{2}$ makes it impossible to find integers n and j such that

   $$\sqrt{2} = 0 + \frac{2j}{n}$$

   we know that $\sqrt{2}$ can't be a point of $\mathbf{P}$. In other words, the number $\sqrt{2}$ must be in one of the open intervals of $\mathbf{P}$ and f fails to be constant in that interval.

4. Explain why a step function must always be bounded.
   Suppose that f is a step function on an interval $[a,b]$. Choose a partition $\mathbf{P}$ of $[a,b]$ such that f steps with $\mathbf{P}$. We express $\mathbf{P}$ in the form $(x_0, x_1, \cdots, x_n)$. Since f is constant in each subinterval (x_{j-1}, x_j), the range of f must be a finite set and therefore f is bounded.

5. Prove that if f and g are step functions on an interval $[a,b]$ then so are their sum $f+g$ and their product fg.

   Hint: ***You can find a proof of this assertion in the section on linearity of integration of step functions.***

6. Prove that if f and g are step functions then so are their sum $f+g$ and their product fg.
   Choose an interval $[a,b]$ such that both of the functions f and g take the value 0 at every number in $\mathbf{R} \setminus [a,b]$. We deduce from Exercise 2 that both f and g are step functions on the interval $[a,b]$ and it follows from Exercise 5 that $f+g$ and fg are step functions on $[a,b]$ and we conclude that these

functions are step functions.

7. Prove that a continuous step function on an interval must be constant on that interval.
Suppose that f is a continuous step function on an interval $[a,b]$ and choose a partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ within which f steps. If c is the constant value of f on the interval (x_0, x_1) then, since f is continuous at x_0 and at x_1 we have $f(x_0) = f(x_1) = c$. Therefore, since f is continuous at x_1, the number c must also be the constant value of f on (x_1, x_2). Continuing in this way we see that f has the constant value c throughout the interval $[a,b]$.

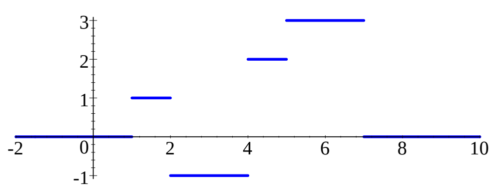
Exercises on Integration of Step Functions

1. Given that

$$\phi(x) = \begin{cases} x & \text{if} \quad x \le 3 \\ x^2 & \text{if} \quad x > 3 \end{cases},$$

that $\mathbf{P}$ is the partition $(-2,1,2,4,5,7,10)$ of the interval $[-2,10]$, that

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 & \text{if} \quad 1 \le x \le 2 \\ -1 & \text{if} \quad 2 < x < 4 \\ 2 & \text{if} \quad 4 \le x < 5 \\ 3 & \text{if} \quad 5 \le x < 7 \\ 0 & \text{if} \quad x \ge 7 \end{cases}$$



and that $\mathbf{Q}$ is the refinement of $\mathbf{P}$ given by

$$\mathbf{Q} = (-2,1,2,3,4,5,6,7,10)$$

work out the sums $\Sigma(\mathbf{P}, f, \phi)$ and $\Sigma(\mathbf{Q}, f, \phi)$ and verify that they are equal to each other.
We have

$$\Sigma(\mathbf{P}, f, \phi) = 0\big(1-(-1)\big) + 1(2-1) + (-1)(16-2) + 2(25-16) + 3(49-25) + 0(100-49) = 77$$

and

$$\begin{aligned} \Sigma(\mathbf{Q}, f, \phi) &= (-1)J(\phi,3) + 0\big(1-(-1)\big) + 1(2-1) + (-1)(3-2) + (-1)(16-9) + 2(25-16) + 3(49-25) + 0(100-49) \\ &= (-1)(9-3) + 0\big(1-(-1)\big) + 1(2-1) + (-1)(3-2) + (-1)(16-9) + 2(25-16) + 3(49-25) + 0(100-49) \end{aligned}$$

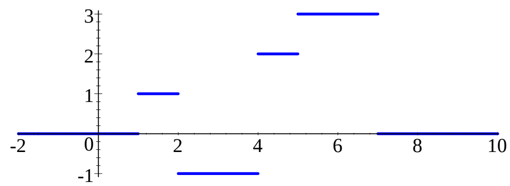
and since

$$(-1)(9-3)(-1)(3-2) + (-1)(16-9) = (-1)(16-2)$$

we have

$$\Sigma(\mathbf{Q}, f, \phi) = \Sigma(\mathbf{P}, f, \phi) = 77.$$

2. Given that f is the function whose graph appears in the figure, evaluate $\int_{-\infty}^{\infty} f$.



We sum the function f over the partition

$$\mathbf{P} = (-2, 1, 2, 4, 5, 7, 10)$$

of the interval $[-2, 10]$.

$$\int_{0}^{\infty} f = 0(1-(-2)) + 1(2-1) + (-1)(4-2) + 2(5-4) + 3(7-5) + 0(10-7) = 7.$$

3. In each of the following cases evaluate $\int_{-\infty}^{\infty} f d\phi$.

a. We define

$$f(x) = \begin{cases} 2 & \text{if } -1 \le x < 1 \\ 0 & \text{if } x \in \mathbf{R} \setminus [-1, 1) \end{cases}$$

and

$$\phi(x) = \begin{cases} 0 & \text{if } x < -1 \\ 3 & \text{if } x \ge -1 \end{cases}.$$

$$\int_{-\infty}^{\infty} f d\phi = \int_{-1}^{1} f d\phi = 2(3-0) + 2(3-3) = 6$$

b. We define

$$f(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{if } x \in \mathbf{R} \setminus [-1, 1] \end{cases}$$

and

$$\phi(x) = \begin{cases} 0 & \text{if } x < -1 \\ 3 & \text{if } x \ge -1 \end{cases}.$$

$$\int_{-\infty}^{\infty} f d\phi = \int_{-1}^{1} f d\phi = 0(3-0) + 2(3-3) = 0$$

c. We define

$$f(x) = \begin{cases} 2 & \text{if } -1 \le x < 1 \\ 0 & \text{if } x \in \mathbf{R} \setminus [-1, 1) \end{cases}$$

and

$$\phi(x) = \begin{cases} 0 & \text{if } x \le -1 \\ 3 & \text{if } x > -1 \end{cases}.$$

$$\int_{-\infty}^{\infty} f d\phi = \int_{-1}^{1} f d\phi = 2(3-0) + 2(3-3) = 6$$

d. We define

$$f(x) = \begin{cases} 2 & \text{if } -1 \le x < 1 \\ 0 & \text{if } x \in \mathbf{R} \setminus [-1, 1) \end{cases}$$

and

$$\phi(x) = \begin{cases} 0 & \text{if } x < -1 \\ 2 & \text{if } x = -1 \\ 3 & \text{if } x \ge -1 \end{cases}.$$

$$\int_{-\infty}^{\infty} f d\phi = \int_{-1}^{1} f d\phi = 2(3-0) + 2(3-3) = 6$$

e. We define

$$f(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ -1 & \text{if } 2 \le x \le 6 \\ 0 & \text{if } x \in \mathbf{R} \setminus [[0,1) \cup [2,6]] \end{cases}$$

and

$$\phi(x) = \begin{cases} -5 & \text{if } x \le 0 \\ 0 & \text{if } 0 < x < 3 \\ x & \text{if } x \ge 3 \end{cases}$$

——0——1——2——3————6——

$$\begin{aligned} \int_{-\infty}^{\infty} f d\phi &= \int_{0}^{6} f d\phi \\ &= 2(0-(-5)) + (-1)(3-0) + 2(0-0) + (-1)(0-0) + (-1)(6-3) = 4. \end{aligned}$$

4. Prove that if ϕ is an increasing function and f is a step function then the function $|f|$ is a step function and

$$\left| \int_{-\infty}^{\infty} f d\phi \right| \le \int_{-\infty}^{\infty} |f| d\phi.$$

The fact that $|f|$ is a step function whenever f is a step function follows at once from the fact that $|f|$ is constant on any interval on which the function f is constant. Since $-|f| \le f \le |f|$, it follows from nonnegativity that

$$-\int_{-\infty}^{\infty} |f| d\phi \le \int_{-\infty}^{\infty} f d\phi \le \int_{-\infty}^{\infty} |f| d\phi$$

and we conclude that

$$\left| \int_{-\infty}^{\infty} f d\phi \right| \le \int_{-\infty}^{\infty} |f| d\phi.$$

5. Given that f is a function defined on $\mathbf{R}$ and that the set of numbers x for which $f(x) \neq 0$ is finite, explain why f must be a step function and why if ϕ is a continuous increasing function we must have

$$\int_{-\infty}^{\infty} f d\phi = 0.$$

We define a and b to be the smallest and largest members, respectively, of the set $\{x \mid f(x) \neq 0\}$. If we arrange the members of the set $\{x \mid f(x) \neq 0\}$ in ascending order then we obtain a partition of $[a, b]$ within which f steps. Since the jump of ϕ at each of the numbers in the set $\{x \mid f(x) \neq 0\}$ is zero, the sum of f over this partition is zero. Since f is zero outside the interval $[a, b]$ we conclude that f is a step function and that

$$\int_{-\infty}^{\infty} f = 0.$$

6. Given that f is a nonnegative step function and that ϕ is a strictly increasing function and that

$$\int_{-\infty}^{\infty} f d\phi = 0,$$

prove that the set of numbers x for which $f(x) \neq 0$ must be finite.
We begin by choosing an interval $[a,b]$ and a partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ such that $f(x) = 0$ whenever a number x lies outside the interval $[a,b]$ and such that f steps within the partition $\mathbf{P}$. If the constant value of f in each interval (x_{j-1}, x_j) is α_j then

$$\sum_{j=0}^{n} f(x_j) J(\phi, x_j) + \sum_{j=1}^{n} \alpha_j \Big(\phi(x_j -) - \phi(x_{j-1} +) \Big) = \int_{-\infty}^{\infty} f d\phi = 0$$

and, since every term in this summation is nonnegative we know that every term must be zero. For each j, the fact that ϕ is strictly increasing and the fact that

$$\alpha_j \Big(\phi(x_j -) - \phi(x_{j-1} +) \Big) = 0$$

guarantees that $\alpha_j = 0$.

7. Given that f and g are step functions, that ϕ is an increasing function and that c is a real number, prove that

$$\int_{-\infty}^{\infty} cf d\phi = c \int_{-\infty}^{\infty} f d\phi$$

and

$$\int_{-\infty}^{\infty} (f+g) d\phi = \int_{-\infty}^{\infty} f d\phi + \int_{-\infty}^{\infty} g d\phi.$$

These results follow at once when we choose an interval $[a,b]$ outside of which both f and g are zero and then replace $\int_{-\infty}^{\infty}$ by $\int_a^b$.

8. Given that f is a step function and that ϕ and ψ are increasing functions, prove that

$$\int_{-\infty}^{\infty} f d(\phi + \psi) = \int_{-\infty}^{\infty} f d\phi + \int_{-\infty}^{\infty} f d\psi.$$

We begin by choosing an interval $[a,b]$ and a partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ such that $f(x) = 0$ whenever a number x lies outside the interval $[a,b]$ and such that f steps within the partition $\mathbf{P}$. If the constant value of f in each interval (x_{j-1}, x_j) is α_j then

$$\begin{aligned}
\int_{-\infty}^{\infty} f d(\phi + \psi) &= \sum_{j=0}^{n} f(x_j) J(\phi + \psi, x_j) + \sum_{j=1}^{n} \alpha_j \Big((\phi + \psi)(x_j -) - (\phi + \psi)(x_{j-1} +) \Big) \\
&= \sum_{j=0}^{n} f(x_j) \Big(J(\psi, x_j) + J(\phi, x_j) \Big) + \sum_{j=1}^{n} \alpha_j \Big(\phi(x_j -) + \psi(x_j -) - \phi(x_{j-1} +) - \psi(x_{j-1} +) \Big) \\
&= \int_{-\infty}^{\infty} f d\phi + \int_{-\infty}^{\infty} f d\psi.
\end{aligned}$$

9. Given that f is a step function and that ϕ is an increasing function and that c is a nonnegative number, prove that

$$\int_{-\infty}^{\infty} f d(c\phi) = c \int_{-\infty}^{\infty} f d\phi.$$

We begin by choosing an interval $[a,b]$ and a partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ such that $f(x) = 0$ whenever a number x lies outside the interval $[a,b]$ and such that f steps within the partition $\mathbf{P}$. If the constant value of f in each interval (x_{j-1}, x_j) is α_j then

$$\int_{-\infty}^{\infty} fd(c\phi) = \sum_{j=0}^{n} f(x_j)J(c\phi, x_j) + \sum_{j=1}^{n} \alpha_j \Big((c\phi)(x_j -) - (c\phi)(x_{j-1} +) \Big)$$
$$= c\sum_{j=0}^{n} f(x_j)J(\phi, x_j) + c\sum_{j=1}^{n} \alpha_j \Big(\phi(x_j -) - \phi(x_{j-1} +) \Big) = c\int_{-\infty}^{\infty} fd\phi.$$

Exercises on Elementary Sets

1. Given that A and B are elementary sets and ϕ is an increasing function, prove that
$$\operatorname{var}(\phi, A \cup B) = \operatorname{var}(\phi, A) + \operatorname{var}(\phi, B) - \operatorname{var}(\phi, A \cap B).$$

   Solution: ***We begin by choosing a lower bound** a **and an upper bound** b **of the set** $A \cup B$**. By looking at the different cases we can see easily that whenever** $x \in [a,b]$ **we have***
$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}.$$
   Therefore
$$\operatorname{var}(\phi, A \cup B) = \int_a^b \chi_{A \cup B} d\phi = \int_a^b (\chi_A + \chi_B - \chi_{A \cap B}) d\phi$$
$$= \int_a^b \chi_A d\phi + \int_a^b \chi_B d\phi - \int_a^b \chi_{A \cap B} d\phi$$
$$= \operatorname{var}(\phi, A) + \operatorname{var}(\phi, B) - \operatorname{var}(\phi, A \cap B).$$

2. Prove that if E is an elementary set and $m(E) = 0$ then E must be finite.
   Choose an interval $[a,b]$ such that the function χ_E is zero outside $[a,b]$ and a partition
$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$
   of $[a,b]$ such that χ_E steps within $\mathbf{P}$. Since χ_E is nonnegative and its sum over $\mathbf{P}$ is zero, the constant value of χ_E in each interval $(x_{j-1} x_j)$ must be zero. Therefore χ_E can be nonzero only at points of $\mathbf{P}$, in other words, every member of E is a point of $\mathbf{P}$ and we conclude that the set E is finite.

3. Explain why the set of all rational numbers in the interval $[0,1]$ is not elementary.
   The desired result will be clear when we have proved the stronger assertion that is made in Exercise 4.

4. Prove that if E is an elementary subset of $[0,1]$ and if every rational number in the interval belongs to E then the set $[0,1] \setminus E$ must be finite.
   We assume that E is an elementary subset of $[0,1]$ and that every rational number in $[0,1]$ belongs to E. Choose a partition
$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$
   of $[a,b]$ such that χ_E steps within $\mathbf{P}$. For each j, since the interval (x_{j-1}, x_j) contains some rational numbers, the constant value of χ_E in (x_{j-1}, x_j) must be 1. Since the numbers in $[0,1]$ that do not belong to E have to be points of $\mathbf{P}$, the set $[0,1] \setminus E$ must be finite.

5. Give an example of a set A of numbers such that if E is any elementary subset of A we have $m(E) = 0$ and if E is any elementary set that includes A we have $m(E) \geq 1$.
   The set $[0,1] \cap \mathbf{Q}$ has the desired properties.

6. Given that E is an elementary set that is not closed and that F is a closed elementary subset of E, prove that $m(E \setminus F) > 0$.

   Solution: ***Choose a lower bound** a **and an upper bound** b **of the set** E**. Since the set** $E \setminus F$ **is elementary, if we want to show that** $m(E \setminus F) > 0$ **then, from Exercise 2, all we have to show is that the set** $E \setminus F$ **cannot be finite. The fact that** $E \setminus F$ **is not finite follows from the fact that finite sets are always***

closed, that F is closed and that the set E, which isn't closed is the union of the two sets F and $E \setminus F$.

7. Given that ϕ is an increasing function, that f is a step function, that E is an elementary set and that $f(x) = 0$ whenever $x \in \mathbf{R} \setminus E$, prove that
$$\int_E f d\phi = \int_{-\infty}^{\infty} f d\phi.$$

Solution: ***The desired equality follows at once from the definitions and the fact that***
$$f = f\chi_E.$$

8. Given that ϕ is an increasing function, that f and g are step functions, that E is an elementary set and that $f(x) \leq g(x)$ whenever $x \in \mathbf{R}$, prove that
$$\int_E f d\phi \leq \int_E g d\phi.$$
Choose an interval $[a,b]$ outside of which both of the functions f and g are zero. Since $f\chi_E \leq g\chi_E$, it follows from the nonnegativity property of integrals of step functions that
$$\int_E f d\phi = \int_a^b f\chi_E d\phi \leq \int_a^b g\chi_E d\phi = \int_E g d\phi$$

9. Given that ϕ is an increasing function, that f is a nonnegative step function, that A and B are elementary sets and that $A \subseteq B$, prove that
$$\int_A f d\phi \leq \int_B f d\phi.$$
The desired inequality follows at once from the fact that $f\chi_A \leq f\chi_B$. We choose an interval $[a,b]$ that includes the set B and use the nonnegativity property to obtain
$$\int_A f = \int_a^b f\chi_A \leq \int_a^b f\chi_B = \int_B f$$

10. Given that ϕ is an increasing function, that f is a step function and that E is an elementary set, prove that
$$\left|\int_E f d\phi\right| \leq \int_E |f| d\phi.$$

Hint: ***Use the fact that***
$$-|f|\chi_E \leq f\chi_E \leq |f|\chi_E.$$

11. Given that A and B are elementary sets, prove that
$$\int_A \chi_B = \int_B \chi_A = m(A \cap B).$$

Solution: ***Choose a lower bound a and an upper bound b of the set $A \cup B$. We see that***
$$\int_A \chi_B d\phi = \int_a^b \chi_B \chi_A d\phi = \int_a^b \chi_A \chi_B d\phi = \int_B \chi_A d\phi.$$
The fact that these expressions are equal to $m(A \cap B)$ follows at once from the fact that
$$\chi_A \chi_B = \chi_{A \cap B}.$$

For some additional exercises on the variation of a function ϕ on elementary sets click on the following icon.

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Additional Exercises on Elementary Sets and Infinite Series

Suppose that ϕ is an increasing function.

1. Given that H is a closed elementary set and (U_n) is a sequence of open elementary sets and that

$$H \subseteq \bigcup_{n=1}^{\infty} U_n,$$

use this earlier exercise to deduce that, for some positive integer N we have

$$\operatorname{var}(\phi, H) \leq \sum_{n=1}^{N} \operatorname{var}(\phi, U_n)$$

and deduce that

$$\operatorname{var}(\phi, H) \leq \sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n).$$

Choose a positive integer N such that

$$H \subseteq \bigcup_{n=1}^{N} U_n.$$

We have

$$\operatorname{var}(\phi, H) \leq \operatorname{var}\left(\phi, \bigcup_{n=1}^{N} U_n\right) \leq \sum_{n=1}^{N} \operatorname{var}(\phi, U_n) \leq \sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n).$$

2. Given that E is an elementary set and that (U_n) is a sequence of open elementary sets and that

$$E \subseteq \bigcup_{n=1}^{\infty} U_n,$$

prove that

$$\operatorname{var}(\phi, E) \leq \sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n).$$

Hint: ***Make use of the theorem on*** approximating by closed sets and open sets
Given any closed subset H of E we know from Exercise 1 that

$$\operatorname{var}(\phi, H) \leq \sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n).$$

Since $\sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n)$ is an upper bound of the set

$$\left\{\operatorname{var}(\phi, H) \mid H \text{ is elementary and closed and } H \subseteq E\right\}$$

and since $\operatorname{var}(\phi, E)$ is the *least* upper bound of this set we have

$$\operatorname{var}(\phi, E) \leq \sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n).$$

3. Given that (A_n) is a sequence of elementary sets and that $\varepsilon > 0$ and that the series $\sum \operatorname{var}(\phi, A_n)$ is convergent, and given that for each positive integer n the set U_n is an open elementary set that includes A_n and satisfies the inequality

$$\operatorname{var}(\phi, U_n) < \operatorname{var}(\phi, A_n) + \frac{\varepsilon}{2^n},$$

prove that

$$\sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n) < \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n) + \varepsilon.$$

The desired result follows at once from the fact that

$$\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

4. Given that E is an elementary set and that (A_n) is a sequence of elementary sets and that

$$E \subseteq \bigcup_{n=1}^{\infty} A_n,$$

prove that

$$\operatorname{var}(\phi, E) \leq \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n).$$

To obtain a contradiction, assume that

$$\operatorname{var}(\phi, E) > \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n).$$

Choose $\varepsilon > 0$ such that

$$\operatorname{var}(\phi, E) > \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n) + \varepsilon = \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}.$$

For each n, choose an open elementary set U_n that includes A_n such that

$$\operatorname{var}(\phi, U_n) < \operatorname{var}(\phi, A_n) + \frac{\varepsilon}{2^n}.$$

We see that

$$\operatorname{var}(\phi, E) > \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} > \sum_{n=1}^{\infty} \operatorname{var}(\phi, U_n)$$

which, in view of Exercise 2, is impossible because

$$E \subseteq \bigcup_{n=1}^{\infty} U_n.$$

5. Suppose that E is an elementary set and that (A_n) is a sequence of elementary sets with the property that whenever i and j are positive integers and $i \neq j$ we have

$$A_i \cap A_j = \emptyset.$$

Suppose that E is an elementary set and that

$$E = \bigcup_{n=1}^{\infty} A_n.$$

Prove that

$$\operatorname{var}(\phi, E) = \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n).$$

Given any postiive integer N we see that

$$\operatorname{var}(\phi, E) \geq \operatorname{var}\left(\phi, \bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \operatorname{var}(\phi, A_n).$$

Therefore

$$\operatorname{var}(\phi, E) \geq \lim_{N \to \infty} \left(\sum_{n=1}^{N} \operatorname{var}(\phi, A_n)\right) = \sum_{n=1}^{\infty} \operatorname{var}(\phi, A_n),$$

and the desired result therefore follows from Exercise 4.

Some Exercises on Integrals with Respect to the Cantor

Function

1. Prove the claim that was made earlier that if ϕ is the Cantor function and I is one of the component intervals of the elementary set E_n then

$$\operatorname{var}(\phi, I) = \frac{1}{2^n}.$$

Solution: ***We can express the left endpoint of I in the form***

$$\sum_{j=1}^{n} \frac{a_j}{3^j}$$

where each number a_j is either 0 or 2. The right endpoint of I is

$$\sum_{j=1}^{n} \frac{a_j}{3^j} + \frac{1}{3^n} = \sum_{j=1}^{n} \frac{a_j}{3^j} + \sum_{j=n+1}^{\infty} \frac{2}{3^j}$$

and we observe that

$$\phi\left(\sum_{j=1}^{n} \frac{a_j}{3^j} + \sum_{j=n+1}^{\infty} \frac{2}{3^j}\right) - \phi\left(\sum_{j=1}^{n} \frac{a_j}{3^j}\right) = \frac{1}{2}\sum_{j=1}^{n} \frac{a_j}{2^j} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} - \frac{1}{2}\sum_{j=1}^{n} \frac{a_j}{2^j}$$

$$= \sum_{j=n+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^n}.$$

2. The preceding examples show that a function f can be Riemann integrable on the interval $[0,1]$ even though it fails to be Riemann-Stieltjes integrable with respect to the Cantor function ϕ. Can you give an example of a function f that is Riemann-Stieltjes integrable with respect to ϕ on $[0,1]$ but fails to be Riemann integrable on $[0,1]$?
We can make use of the fact that the Cantor function is constant in the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$. We define

$$f(x) = \begin{cases} 0 & \text{if } \; x \text{ is irrational} \\ 0 & \text{if } \; x \in [0,1] \setminus \left[\frac{1}{3}, \frac{2}{3}\right] \\ 1 & \text{if } \; x \text{ is rational and } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \end{cases}$$

3. Prove that if ϕ is the Cantor function then

$$\int_0^1 x^2 d\phi(x) = \frac{3}{8}.$$

Solution:

4. Prove that if ϕ is the Cantor function then

$$\int_0^1 x^3 d\phi(x) = \frac{5}{16}.$$

Solution:

5. Prove that if ϕ is the Cantor function then

$$\int_0^1 x^3 d\phi(x) = \frac{39}{160}.$$

Solution:
The solutions to Exercises 3, 4, and 5 are given after the solution to Exercise 6 below.

6. Prove that if ϕ is the Cantor function then

$$\int_0^1 \phi(x) d\phi(x) = \frac{1}{2}.$$

Once again we recall that C is the intersection of the family of sets E_n that were defined in our discussion of the Cantor set and that each set E_n is the union of 2^n closed intervals. Look, once again, at the set E_2:

The function ϕ takes the value $\frac{1}{4}$ at each of the points $\frac{1}{9}$ and $\frac{2}{9}$ and takes the value $\frac{1}{2}$ at each of the points $\frac{1}{3}$ and $\frac{2}{3}$ and takes the value $\frac{3}{4}$ at each of the points $\frac{7}{9}$ and $\frac{8}{9}$. Therefore, if I is any one of the four component intervals of E_2 then we have

$$\operatorname{var}(\phi, I) = \frac{1}{4}.$$

More generally we may see that for each positive integer n, if I is any one of the 2^n component intervals of E_n then

$$\operatorname{var}(\phi, I) = \frac{1}{2^n}.$$

For each natural number n we define $\mathbf{P}_n$ to be the partition of $[0,1]$ whose points are the endpoints of the component intervals of the set E_n. For example,

$$\mathbf{P}_2 = \left(0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1\right).$$

For each n, if the partition $\mathbf{P}_n$ is expressed as

$$\mathbf{P} = (x_0, x_1, \cdots, x_{2^{n+1}})$$

then we define two step functions s_n and S_n on $[0,1]$ by defining

$$s_n(x_j) = S_n(x_j) = \phi(x_j)$$

for every $j = 0, 1, 2, \cdots, 2^n$ and by defining

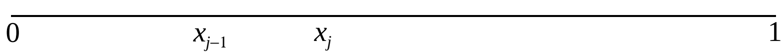
$$s_n(x) = \phi(x_{j-1})$$

and

$$S_n(x) = \phi(x_j)$$

whenever

$$x_{j-1} < x < x_j$$



We observe that $s_n \leq \phi \leq S_n$. Now given any two consecutive points x_{j-1} and x_j of the partition $\mathbf{P}_n$ there are two possibilities: Either the interval $[x_{j-1}, x_j]$ is a component interval of the set E_n; in which case

$$\operatorname{var}(\phi, [x_{j-1}, x_j]) = \frac{1}{2^n}$$

or the interval $(x_{j-1}x_j)$ is a gap between two component intervals of E_n; in which case

$$\operatorname{var}(\phi, [x_{j-1}, x_j]) = 0.$$

We deduce that

$$\int_0^1 (S_n - s_n)\,d\phi = \left(\frac{1}{2^n}\right)\frac{1}{2^n} = \frac{1}{4^n}.$$

Since the latter expression approaches 0 as $n \to \infty$ we know that the pair of sequences (s_n) and (S_n) squeezes ϕ with respect to ϕ and so ϕ is Riemann-Stieltjes integrable with respect to itself on the interval $[0,1]$. To find the value of the integral $\int_0^1 \phi\,d\phi$ we observe that for each n we have

$$\int_0^1 S_n\,d\phi = \sum_{j=1}^{2^n} \frac{j}{2^n}\left(\frac{1}{2^n}\right) = \frac{1}{4^n}\left(\frac{2^n(2^n+1)}{2}\right)$$

and so

$$\lim_{n\to\infty} \int_0^1 S_n d\phi = \frac{1}{2}.$$

Therefore

$$\int_0^1 \phi d\phi = \frac{1}{2}.$$

The fact that this integral is $\frac{1}{2}$ becomes obvious after one has studied the integration by parts identity. As a matter of fact, if ϕ is any increasing continuous function on an interval $[a,b]$ then

$$\int_a^b \phi d\phi = \frac{(\phi(b))^2 - (\phi(a))^2}{2}.$$

Integrating with Respect to the Cantor Function Solutions to Exercises 3, 4 and 5

Quick Review of the Cantor set and the Cantor function

The Cantor set C is the set of all those real numbers that can be expressed in the form

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

where (a_n) is a sequence in the set $\{0,2\}$. The least member of the Cantor set is

$$\sum_{n=1}^{\infty} \frac{0}{3^n} = 0$$

and the greatest member is

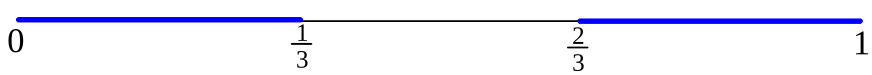
$$\sum_{n=1}^{\infty} \frac{2}{3^n} = 1.$$

The least member $\sum_{n=0}^{\infty} \frac{a_n}{3^n}$ of C for which $a_1 = 2$ is $\frac{2}{3}$ and the greatest member for which $a_1 = 0$ is

$$\sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3}.$$

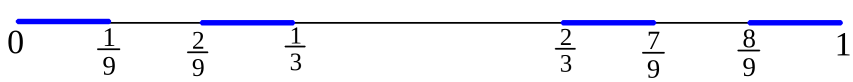
Thus the set C is included in the set

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$



By looking at the two cases $a_2 = 0$ and $a_2 = 2$ we can go a step further and see that C is included in the set

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$



and, in general, if n is any positive integer then C is included in the set E_n that is the union of 2^n closed intervals of length $1/3^n$ and whose left endpoints are the numbers

$$\sum_{j=1}^{n} \frac{a_j}{3^j}$$

where each number a_j is either 0 or 2. It is not hard to show that

$$C = \bigcap_{n=1}^{\infty} E_n.$$

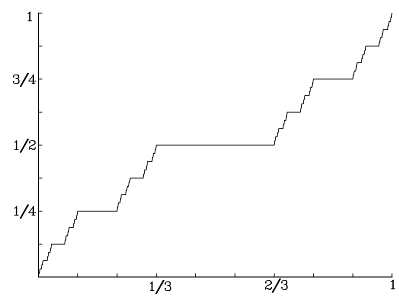
The value of the **Cantor function** ϕ at each member $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ of the set C is defined by the equation

$$\phi\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}.$$

Thus

$$\begin{aligned} \phi(0) &= 0 \\ \phi(1) &= \phi\left(\sum_{n=1}^{\infty} \frac{2}{3^n}\right) = \sum_{n=1}^{\infty} \frac{2}{2^{n+1}} = 1 \\ \phi\left(\frac{1}{3}\right) &= \phi\left(\frac{2}{3}\right) = \frac{1}{2} \\ \phi\left(\frac{1}{9}\right) &= \phi\left(\frac{2}{9}\right) = \frac{1}{4} \\ \phi\left(\frac{7}{9}\right) &= \phi\left(\frac{8}{9}\right) = \frac{3}{4}. \end{aligned}$$

The function ϕ is a continuous strictly increasing function from C onto the interval $[0,1]$. We now extend ϕ to an increasing function from $[0,0]$ onto $[0,1]$ by making ϕ constant on every component interval of the open set $[0,1] \setminus C$. The graph of ϕ is shown in the following figure:

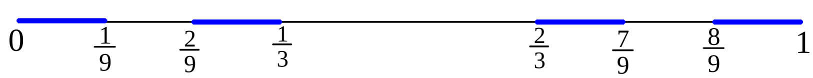


Graph of the Cantor Function

A General Discussion of the integral $\int_0^1 x^p d\phi(x)$

The purpose of this document is to explore some integrals of the form $\int_0^1 x^p d\phi(x)$ where p is a positive integer. Note that such an integral always exists because the integrand is an increasing function. However, it is worth showing the existence of the integrals directly. We define $f(x) = x^p$ for each $x \in [0,1]$. For each positive integer n we define $\mathbf{P}_n$ to be the partition of $[0,1]$ whose points are the endpoints of the component intervals of the set E_n. For example,

$$\mathbf{P}_2 = \left(0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1\right).$$



For each n, if the partition $\mathbf{P}_n$ is expressed as

$$\mathbf{P} = (x_0, x_1, \cdots, x_{2^{n+1}})$$

then we define two step functions s_n and S_n on $[0,1]$ by defining

$$s_n(x_j) = S_n(x_j) = x_j^p$$

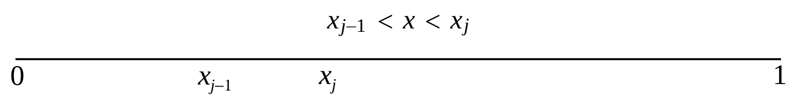
for every $j = 0, 1, 2, \cdots, 2^n$ and by defining

$$s_n(x) = x_{j-1}^p$$

and

$$S_n(x) = x_j^p$$

whenever



We observe that $s_n \le f \le S_n$. Now given any two consecutive points x_{j-1} and x_j of the partition $\mathbf{P}_n$ there are two possibilities: Either the interval $[x_{j-1}, x_j]$ is a component interval of the set E_n; in which case

$$\operatorname{var}(\phi, [x_{j-1}, x_j]) = \frac{1}{2^n}$$

or the interval $(x_{j-1} x_j)$ is a gap between two component intervals of E_n; in which case

$$\operatorname{var}(\phi, [x_{j-1}, x_j]) = 0.$$

When $[x_{j-1}, x_j]$ is a component interval of E_n we have

$$x_j^p - x_{j-1}^p = \frac{1}{3^n}\left(x_j^{p-1} + x_j^{p-2} x_{j-1}^1 + x_j^{p-3} x_{j-1}^2 + \cdots + x_{j-1}^{p-1}\right) < \frac{p}{3^n}$$

and so

$$\begin{aligned}\int_0^1 (S_n - s_n)\,d\phi &= \left(\sum_{j=1}^{2^n}\left(x_j^p - x_{j-1}^p\right)\right)\frac{1}{2^n}\\ &= \left(\sum_{j=1}^{2^n}\frac{1}{3^n}\left(x_j^{p-1} + x_j^{p-2} x_{j-1}^1 + x_j^{p-3} x_{j-1}^2 + \cdots + x_{j-1}^{p-1}\right)\right)\frac{1}{2^n}\\ &< \frac{p}{3^n}.\end{aligned}$$

Since the latter expression approaches 0 as $n \to \infty$ we know that the pair of sequences (s_n) and (S_n) squeezes f with respect to ϕ and so f is Riemann-Stieltjes integrable with respect to ϕ on the interval $[0,1]$. We deduce that

$$\lim_{n\to\infty}\int_0^1 s_n\,d\phi = \int_0^1 x^p\,d\phi(x)$$

We now look at the step functions s_n more closely. We write the 2^n left endpoints of the component intervals of E_n as

$$c_1, c_2, c_3, \cdots, c_{2^n}$$

and for each $k = 1, 2, 3, \cdots, 2^n$ we express c_k in the form

$$c_k = \sum_{j=1}^{n}\frac{a_{kj}}{3^j}.$$

Where each of the numbers a_{kj} is either 2 or 0. Thus

$$\int_0^1 s_n\,d\phi = \sum_{k=1}^{2^n}\left(\sum_{j=1}^{n}\frac{a_{kj}}{3^j}\right)^p\frac{1}{2^n}$$

The latter expression can be expressed in the form

$$\begin{aligned}&\frac{1}{2^n}\sum_{k=1}^{2^n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_p=1}^{n}\frac{a_{kj_1}}{3^{j_1}}\frac{a_{kj_2}}{3^{j_2}}\cdots\frac{a_{kj_p}}{3^{j_p}}\\ &= \frac{1}{2^n}\sum_{j_1=1}^{n}\frac{1}{3^{j_1}}\sum_{j_2=1}^{n}\frac{1}{3^{j_2}}\cdots\sum_{j_p=1}^{n}\frac{1}{3^{j_p}}\sum_{k=1}^{2^n}a_{kj_1}a_{kj_2}\cdots a_{kj_p}\end{aligned}$$

and, in the special cases that follow, we shall write the number

$$\frac{1}{2^n}\sum_{k=1}^{2^n}a_{kj_1}a_{kj_2}\cdots a_{kj_p}$$

as $g(j_1, j_2, j_3, \cdots, j_p)$. As will be explained below in the special cases, the value of the function g at any member $(j_1, j_2, j_3, \cdots, j_p)$ of its domain depends upon the number of (distinct) members in the set $\{j_1, j_2, j_3, \cdots, j_p\}$.

$$g(j_1,j_2,j_3,\cdots,j_p) = \begin{cases} 1 & \text{if } \{j_1,j_2,j_3,\cdots,j_p\} \text{ has } p \text{ members} \\ 2 & \text{if } \{j_1,j_2,j_3,\cdots,j_p\} \text{ has } p-1 \text{ members} \\ 4 & \text{if } \{j_1,j_2,j_3,\cdots,j_p\} \text{ has } p-2 \text{ members} \\ \vdots & \vdots \quad \vdots \\ 2^p & \text{if } \{j_1,j_2,j_3,\cdots,j_p\} \text{ has only 1 member} \end{cases}$$

The Integral $\int_0^1 x d\phi(x)$

When $p = 1$ we have

$$\int_0^1 s_n d\phi = \sum_{k=1}^{2^n} \sum_{j=1}^{n} \frac{a_{kj}}{3^j} \frac{1}{2^n}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{2^n} \frac{a_{kj}}{3^j} \frac{1}{2^n} = \frac{1}{2^n} \sum_{j=1}^{n} \frac{1}{3^j} \sum_{k=1}^{2^n} a_{kj}$$

Now for each j, exactly 2^{n-1} of the integers $k \in \{1,2,\cdots,2^n\}$ give us $a_{kj} = 2$ and the other 2^{n-1} values of k give us $a_{kj} = 0$. Therefore

$$\int_0^1 s_n d\phi = \sum_{j=1}^{n} \frac{1}{3^j} = -\frac{1}{2} 3^{-n} + \frac{1}{2} \to \frac{1}{2}$$

as $n \to \infty$, and we conclude that

$$\int_0^1 x d\phi(x) = \frac{1}{2}.$$

Some Special Cases

The Integral $\int_0^1 x^2 d\phi(x)$

When $p = 2$ we have

$$\int_0^1 s_n d\phi = \sum_{k=1}^{2^n} \left(\sum_{j=1}^{n} \frac{a_{kj}}{3^j} \right)^2 \frac{1}{2^n} = \sum_{k=1}^{2^n} \frac{1}{2^n} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{a_{kj}}{3^j} \frac{a_{ki}}{3^i} \right)$$

$$= \frac{1}{2^n} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{3^i} \frac{1}{3^j} \sum_{k=1}^{2^n} a_{kj} a_{ki}$$

Now for each i, there are exactly 2^{n-1} values of k for which $a_{ki} = 2$ and, whenever $j \neq i$, exactly 2^{n-2} of these 2^{n-1} values of k give us $a_{jk} = 2$. Therefore

$$\frac{1}{2^n} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{3^i} \frac{1}{3^j} \sum_{k=1}^{2^n} a_{kj} a_{ki} = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{3^i} \frac{1}{3^j} g(i,j)$$

where

$$g(i,j) = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

and we conclude that

$$
\begin{aligned}
\int_0^1 s_n d\phi &= \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{3^i}\frac{1}{3^j}g(i,j) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{3^i}\frac{1}{3^j}+\sum_{j=1}^{n}\left(\frac{1}{3^j}\right)^2 \\
&= \left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)-\frac{1}{8}9^{-n}+\frac{1}{8}
\end{aligned}
$$

and we conclude that

$$\int_0^1 x^2 d\phi(x) = \lim_{n\to\infty}\int_0^1 s_n d\phi = \frac{3}{8}.$$

The Integral $\int_0^1 x^3 d\phi(x)$

When $p = 3$ we have

$$
\begin{aligned}
\int_0^1 s_n d\phi &= \sum_{k=1}^{2^n}\left(\sum_{j=1}^{n}\frac{a_{kj}}{3^j}\right)^3\frac{1}{2^n} \\
&= \frac{1}{2^n}\sum_{k=1}^{2^n}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\frac{a_{kr}}{3^r}\frac{a_{ks}}{3^s}\frac{a_{kt}}{3^t} \\
&= \frac{1}{2^n}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\frac{1}{3^r3^s3^t}\sum_{k=1}^{2^n}a_{kr}a_{ks}a_{kt} \\
&= \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\frac{1}{3^r3^s3^t}g(r,s,t)
\end{aligned}
$$

where

$$
g(r,s,t) = \begin{cases} 4 & \text{if} \quad r = s = t \\ 2 & \text{if} \quad \text{the set } \{r,s,t\} \text{ has two members} \\ 1 & \text{if} \quad \text{the set } \{r,s,t\} \text{ has three members} \end{cases}
$$

We observe that

$$
\begin{aligned}
&\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\frac{1}{3^r3^s3^t}g(r,s,t) = \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\frac{1}{3^r3^s3^t}+3\sum_{r=1}^{n}\sum_{t=1}^{n}\frac{1}{(3^r)^2 3^t} \\
&= \left(\sum_{r=1}^{n}\frac{1}{3^r}\right)\left(\sum_{s=1}^{n}\frac{1}{3^s}\right)\left(\sum_{t=1}^{n}\frac{1}{3^t}\right)+3\left(\sum_{r=1}^{n}\left(\frac{1}{3^r}\right)^2\right)\left(\sum_{t=1}^{n}\frac{1}{3^t}\right) \\
&= \left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right) \\
&\quad +3\left(-\frac{1}{8}9^{-n}+\frac{1}{8}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)
\end{aligned}
$$

and so

$$\int_0^1 x^3 d\phi(x) = \lim_{n\to\infty}\int_0^1 s_n d\phi = \frac{5}{16}.$$

The Integral $\int_0^1 x^4 d\phi(x)$

When $p = 4$ we have

$$\int_0^1 s_n d\phi = \sum_{k=1}^{2^n}\left(\sum_{j=1}^{n}\frac{a_{kj}}{3^j}\right)^4\frac{1}{2^n}$$
$$= \frac{1}{2^n}\sum_{k=1}^{2^n}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\frac{a_{kr}}{3^r}\frac{a_{ks}}{3^s}\frac{a_{kt}}{3^t}\frac{a_{ku}}{3^u}$$
$$= \frac{1}{2^n}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\frac{1}{3^r3^s3^t3^u}\sum_{k=1}^{2^n}a_{kr}a_{ks}a_{kt}a_{ku}$$
$$= \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\frac{1}{3^r3^s3^t3^u}g(r,s,t,u)$$

where

$$g(r,s,t,u) = \begin{cases} 8 & \text{if } \{r,s,t,u\} \text{ has one member} \\ 4 & \text{if } \{r,s,t,u\} \text{ has two members} \\ 2 & \text{if } \{r,s,t,u\} \text{ has three members} \\ 1 & \text{if } \{r,s,t,u\} \text{ has four members} \end{cases}$$

We observe that

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\frac{1}{3^r3^s3^t3^u}g(r,s,t,u)$$
$$= \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\frac{1}{3^r3^s3^t3^u} + 4\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\frac{1}{3^r}\right)^2\left(\frac{1}{3^s}\right)\left(\frac{1}{3^t}\right)$$
$$+ 6\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\frac{1}{3^r}\right)^2\left(\frac{1}{3^s}\right)^2 - 3\sum_{r=1}^{n}\left(\frac{1}{3^r}\right)^4$$
$$= \left(\sum_{r=1}^{n}\frac{1}{3^r}\right)\left(\sum_{s=1}^{n}\frac{1}{3^s}\right)\left(\sum_{t=1}^{n}\frac{1}{3^t}\right)\left(\sum_{u=1}^{n}\frac{1}{3^u}\right)$$
$$+ 4\left(\sum_{r=1}^{n}\frac{1}{3^{2r}}\right)\left(\sum_{s=1}^{n}\frac{1}{3^s}\right)\left(\sum_{t=1}^{n}\frac{1}{3^t}\right)$$
$$+ 6\left(\sum_{r=1}^{n}\frac{1}{3^{2r}}\right)\left(\sum_{s=1}^{n}\frac{1}{3^{2s}}\right) - 3\sum_{r=1}^{n}\left(\frac{1}{3^{4r}}\right)$$
$$= \left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)$$
$$+ 4\left(-\frac{1}{8}9^{-n}+\frac{1}{8}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)\left(-\frac{1}{2}3^{-n}+\frac{1}{2}\right)$$
$$+ 6\left(-\frac{1}{8}9^{-n}+\frac{1}{8}\right)\left(-\frac{1}{8}9^{-n}+\frac{1}{8}\right) - 3\left(-\frac{1}{80}81^{-n}+\frac{1}{80}\right)$$
$$= \frac{11}{160}81^{-n} + \frac{3}{16}9^{-n} - \frac{1}{2}3^{-n} + \frac{39}{160}$$

and we conclude that

$$\int_0^1 x^4 d\phi(x) = \lim_{n\to\infty}\int_0^1 s_n d\phi = \frac{39}{160}.$$

Some Exercises on the Riemann-Stieltjes Integral

1. Prove that the integral

$$\int_1^4 3x^2 dx$$

exists and has the value 63.

Solution: ***This exercise will become obsolete when we reach the fundamental theorem of calculus later on in the chapter. The solution given here is a bare hands approach and repeats portions of the proof that monotone functions are integrable.***

We define $f(x) = 3x^2$ ***for each number*** $x \in [1,4]$***. Given any positive integer*** n***, if***

$$\mathbf{P}_n = (x_0, x_1, \cdots, x_n)$$

is the regular n***-partition of the interval*** $[1,4]$ ***then we define two step functions*** s_n ***and*** S_n ***by making***

$$s_n(x) = S_n(x) = f(x)$$

whenever x ***is a point of the partition*** P_n ***and, in each interval*** (x_{j-1}, x_j) ***of the partition*** P_n ***we make*** s_n ***and*** S_n ***take the constant values*** $f(x_{j-1})$ ***and*** $f(x_j)$ ***respectively. Since***

$$\begin{aligned}\int_1^4 (S_n - s_n) &= \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))\frac{4-1}{n} \\ &= \frac{3}{n}\sum_{j=1}^{n}(f(x_j) - f(x_{j-1})) \\ &= \frac{3(f(4) - f(1))}{n} = \frac{135}{n} \to 0\end{aligned}$$

as $n \to \infty$ ***we know that*** f ***is integrable and that***

$$\lim_{n\to\infty} \int_1^4 S_n = \lim_{n\to\infty}\int_1^4 s_n = \int_1^4 f.$$

Now since

$$x_j = 1 + \frac{3j}{n}$$

for each $j = 0, \cdots, n$***, we have***

$$\begin{aligned}\int_1^4 S_n &= \sum_{j=1}^{n}\left(\frac{3}{n}\right)(3)\left(1 + \frac{3j}{n}\right)^2 \\ &= \frac{9(14n^2 + 15n + 3)}{2n^2} \to 63\end{aligned}$$

as $n \to \infty$***.***

2. In this exercise we take $f(x) = \sqrt{x}$ for $x \in [0,1]$. Given a positive integer n, we shall take $\mathbf{P}_n$ to be the partition of $[0,1]$ defined by the equation

$$\mathbf{P}_n = \left(\frac{0^2}{n^2}, \frac{1^2}{n^2}, \frac{2^2}{n^2}, \cdots, \frac{n^2}{n^2}\right).$$

Prove that if we define a step function S_n on $[0,1]$ by making

$$S_n(x) = \sqrt{x}$$

whenever x is a point of the partition $\mathbf{P}_n$ and giving S_n the constant value j/n in each interval

$$\left(\frac{(j-1)^2}{n^2}, \frac{j^2}{n^2}\right)$$

of the partition $\mathbf{P}_n$, then

$$\int_0^1 \sqrt{x}\, dx = \lim_{n\to\infty}\int_0^1 S_n = \frac{2}{3}.$$

Solution: ***For each positive integer*** n ***we define the function*** S_n ***as described in the exercise and we***

define a step function s_n ***on*** $[0,1]$ ***by making***

$$S_n(x) = \sqrt{x}$$

whenever x ***is a point of the partition*** $\mathbf{P}_n$ ***and giving*** s_n ***the constant value*** $(j-1)/n$ ***in each interval***

$$\left(\frac{(j-1)^2}{n^2}, \frac{j^2}{n^2} \right)$$

We see at once that $s_n \le f \le S_n$ ***for each*** n ***and that***

$$\lim_{n\to\infty} \int_0^1 (S_n - s_n) = \sum_{j=1}^{n} \left(\frac{j}{n} - \frac{j-1}{n} \right)\left(\frac{j^2}{n^2} - \frac{(j-1)^2}{n^2} \right) = \frac{1}{n} \to 0$$

as $n \to \infty$***. We conclude that the pair of sequences***(s_n) ***and*** (S_n) ***squeezes*** f ***on the interval*** $[0,1]$ ***and that***

$$\lim_{n\to\infty} \int_0^1 S_n = \int_0^1 f.$$

Therefore

$$\int_0^1 f = \lim_{n\to\infty} \int_0^1 S_n = \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{j}{n} \right)\left(\frac{j^2}{n^2} - \frac{(j-1)^2}{n^2} \right)$$

$$= \lim_{n\to\infty} \frac{4n^2 + 3n - 1}{6n^2} = \frac{2}{3}.$$

3. Prove that

$$\int_0^1 \sqrt[3]{x}\, dx = \frac{3}{4}.$$

This exercise is very similar to Exercise 2. This time we take

$$\mathbf{P}_n = \left(\frac{0^3}{n^3}, \frac{1^3}{n^3}, \frac{3^3}{n^3}, \cdots, \frac{n^3}{n^3} \right)$$

for each n and, for each n, we define S_n to be the step function that takes the value $\sqrt[3]{x}$ at each point of $\mathbf{P}_n$ and whose constant value in each interval

$$\left(\frac{(j-1)^3}{n^3}, \frac{j^3}{n^3} \right)$$

is j/n. We observe that

$$\lim_{n\to\infty} \int_0^1 S_n = \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{j}{n} \right)\left(\frac{j^3}{n^3} - \frac{(j-1)^3}{n^3} \right)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{3j^3 - 3j^2 + j}{n^4} \right)$$

$$= \lim_{n\to\infty} \left(\frac{3n^2 + 2n - 1}{4n^2} \right) = \frac{3}{4}.$$

In the same way we can show that if s_n $\sqrt[3]{x}$ at each point of $\mathbf{P}_n$ and whose constant value in each interval

$$\left(\frac{(j-1)^3}{n^3}, \frac{j^3}{n^3} \right)$$

is $(j-1)/n$ then

$$\lim_{n\to\infty} \int_0^1 s_n = \frac{3}{4}$$

and so the pair of sequences (s_n) and (S_n) squeezes the cube root function on $[0,1]$.

4. Prove that the integral

$$\int_0^1 x d(x^3)$$

exists and evaluate it.

Solution: ***This exercise will become obsolete when we reach the theorem on*** reduction ***of Riemann-Stieltjes integrals to Riemann integrals. The solution given here is a bare hands approach. We define*** $\phi(x) = x^3$ ***for each*** x***. For each positive integer*** n ***we define***

$$\mathbf{P}_n = \left(\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\right),$$

we define s_n ***and*** S_n ***to be the step functions that take the value*** $\frac{j}{n}$ ***at each point*** $\frac{j}{n}$ ***of the partition*** $\mathbf{P}_n$ ***and that takes the constant values*** $\frac{j-1}{n}$ ***and*** $\frac{j}{n}$ ***respectively on each interval*** $\left(\frac{j-1}{n}, \frac{j}{n}\right)$ ***of the partition.***
For each n ***we see that***

$$\int_0^1 (S_n - s_n)d\phi = \sum_{j=1}^{n}\left(\frac{j}{n} - \frac{j-1}{n}\right)\left(\left(\frac{j}{n}\right)^3 - \left(\frac{j-1}{n}\right)^3\right) = \frac{1}{n}$$

and the latter expression approaches 0 ***as*** $n \to \infty$***. Therefore the required integral exists. Now for each*** n ***we see that***

$$\int_0^1 S_n d\phi = \sum_{j=1}^{n}\left(\frac{j}{n}\right)\left(\left(\frac{j}{n}\right)^3 - \left(\frac{j-1}{n}\right)^3\right) = \frac{3n^2+2n-1}{4n^2}$$

and the latter expression approaches $\frac{3}{4}$ ***as*** $n \to \infty$***. Therefore***

$$\int_0^1 x d(x^3) = \frac{3}{4}.$$

5. Suppose that

$$\phi(x) = \begin{cases} x & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases}.$$

Prove that the integral

$$\int_{-1}^1 (1+x)d\phi(x)$$

exists and evaluate it.

Solution: ***For each positive integer*** n ***we define*** $\mathbf{P}_n$ ***to be the regular*** $2n$ ***partition of the interval*** $[-1,1]$***. Thus for each*** j***, the*** j***th point of*** P_n ***is***

$$-1 + \frac{2j}{2n}$$

and

$$\mathbf{P}_n = \left(-1, -1 + \frac{2}{2n}, -1 + \frac{4}{2n}, \cdots, -1 + \frac{4n}{2n}\right)$$
$$= \left(\frac{-2n}{2n}, \frac{-2n+1}{2n}, \cdots, \frac{-1}{2n}, 0, \frac{1}{2n}, \cdots, \frac{2n-1}{2n}, \frac{2n}{2n}\right).$$

Our reason for taking the $2n$***-partition instead of the*** n***-partition is that we want to guarantee that the number*** 0 ***is a point of the partition.***
For each n ***we define*** s_n ***and*** S_n ***to be the step functions that take the value*** $1 - 1 + \frac{2j}{2n}$ ***at each point*** $-1 + \frac{2j}{2n}$ ***of the partition*** $\mathbf{P}_n$ ***and that take the constant values*** $1 - 1 + \frac{2(j-1)}{2n}$ ***and*** $1 - 1 + \frac{2j}{2n}$ ***respectively in each interval*** $\left(-1 + \frac{2(j-1)}{2n}, -1 + \frac{2j}{2n}\right)$ ***of the partition.***
We see that

$$\int_{-1}^1 (S_n - s_n)d\phi = \sum_{j=1}^{n}\left(\frac{2}{2n}\right)\left(\frac{1}{n}\right) + \sum_{j=n+1}^{2n}\left(\frac{2}{2n}\right)\left(\frac{2}{n}\right) \to 0$$

as $n \to \infty$***. Now for each*** n ***we have***

$$\int_{-1}^1 S_n d\phi = \sum_{j=1}^{n}\left(\frac{2j}{2n}\right)\left(\frac{1}{n}\right) + \sum_{j=n+1}^{2n}\left(\frac{2j}{2n}\right)\left(\frac{2}{n}\right) = \frac{7n+3}{2n}$$

and the latter expression approaches $\frac{7}{2}$ ***as*** $n \to \infty$***.***

6. Suppose that

$$\phi(x) = \begin{cases} x & \text{if } x \le 0 \\ x+1 & \text{if } x > 0 \end{cases}.$$

Prove that the integral

$$\int_{-1}^{1} (1+x)\,d\phi(x)$$

exists and evaluate it.

Solution: ***For each positive integer*** n ***we define*** P_n ***to be the regular*** $2n$ ***partition of the interval*** $[-1,1]$***. Thus for each*** j***, the*** j***th point of*** P_n ***is***

$$-1 + \frac{2j}{2n}$$

and

$$\mathbf{P}_n = \left(-1, -1+\frac{2}{2n}, -1+\frac{4}{2n}, \cdots, -1+\frac{4n}{2n}\right)$$
$$= \left(\frac{-2n}{2n}, \frac{-2n+1}{2n}, \cdots, \frac{-1}{2n}, 0, \frac{1}{2n}, \cdots, \frac{2n-1}{2n}, \frac{2n}{2n}\right).$$

Our reason for taking the $2n$***-partition instead of the*** n***-partition is that we want to guarantee that the number*** 0 ***is a point of the partition.***

For each n ***we define*** s_n ***and*** S_n ***to be the step functions that take the value*** $1 - 1 + \frac{2j}{2n}$ ***at each point*** $-1 + \frac{2j}{2n}$ ***of the partition*** $\mathbf{P}_n$ ***and that take the constant values*** $1 - 1 + \frac{2(j-1)}{2n}$ ***and*** $1 - 1 + \frac{2j}{2n}$ ***respectively in each interval*** $\left(-1+\frac{2(j-1)}{2n}, -1+\frac{2j}{2n}\right)$ ***of the partition.***

We see that

$$\int_{-1}^{1} (S_n - s_n)\,d\phi = 0J(\phi,0) + \sum_{j=1}^{2n}\left(\frac{2}{2n}\right)\left(\frac{1}{n}\right) = \frac{2}{n} \to 0$$

as $n \to \infty$***. Now for each*** n ***we have***

$$\int_{-1}^{1} S_n\,d\phi = 1J(\phi,0) + \sum_{j=1}^{2n}\left(\frac{2j}{2n}\right)\left(\frac{1}{n}\right) = 1 + \frac{2n+1}{n}$$

and the latter expression approaches 3 ***as*** $n \to \infty$***.***

7. Prove that the integral

$$\int_0^1 x\,d\sqrt{x}$$

exists and evaluate it.

We define $\phi(x) = \sqrt{x}$ whenever $x \in [0,1]$. For each positive integer n we define

$$\mathbf{P}_n = \left(\left(\frac{0}{n}\right)^2, \left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \cdots, \left(\frac{n}{n}\right)^2\right),$$

we define s_n and S_n to be the step functions that take the value $\left(\frac{j}{n}\right)^2$ at each point $\left(\frac{j}{n}\right)^2$ of the partition $\mathbf{P}_n$ and that takes the constant values $\left(\frac{j-1}{n}\right)^2$ and $\left(\frac{j}{n}\right)^2$ respectively on each interval $\left(\left(\frac{j-1}{n}\right)^2, \left(\frac{j}{n}\right)^2\right)$ of the partition.

For each n we see that

$$\int_0^1 (S_n - s_n)\,d\phi = \sum_{j=1}^{n}\left(\left(\frac{j}{n}\right)^2 - \left(\frac{j-1}{n}\right)^2\right)\left(\frac{j}{n} - \frac{j-1}{n}\right) = \frac{1}{n}$$

and the latter expression approaches 0 as $n \to \infty$. Therefore the required integral exists. Now for each n we see that

$$\int_0^1 S_n d\phi = \sum_{j=1}^{n} \left(\frac{j}{n}\right)^2 \left(\frac{j}{n} - \frac{j-1}{n}\right) = \frac{2n^2+3n+1}{6n^2}$$

and the latter expression approaches $\frac{1}{3}$ as $n \to \infty$. Therefore

$$\int_0^1 x d\sqrt{x} = \frac{1}{3}.$$

Some Exercises on Riemann-Stieltjes Integrability

1. Suppose that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ has the form } \frac{1}{n} \text{ for some positive integer } n \\ 0 & \text{otherwise} \end{cases}.$$

Prove that f is Riemann integrable on the interval $[0,1]$ and that $\int_0^1 f = 0$.

Solution: ***For each positive integer** n **we define** $\mathbf{P}_n$ **to be the following partition of the interval** $[0,1]$**:***

$$\mathbf{P}_n = \left(0, \frac{1}{n}, \frac{1}{n-1}, \cdots, \frac{1}{3}, \frac{1}{2}, 1\right).$$

For each** n **we see easily that

$$\int_0^1 l(\mathbf{P}_n, f) = 0 \qquad \textbf{and} \qquad \int_0^1 u(\mathbf{P}_n, f) = \frac{1}{n}$$

and therefore

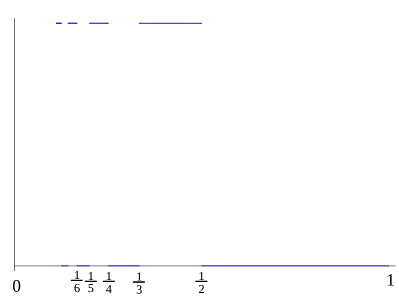
$$\lim_{n\to\infty} \int_0^1 l(\mathbf{P}_n, f) = \lim_{n\to\infty} \int_0^1 u(\mathbf{P}_n, f) = 0.$$

2. Suppose that f is defined on the interval in such a way that whenever $x \in [0,1]$ and x has the form $\frac{1}{n}$ for some positive integer n we have $f(x) = 0$ and whenever x belongs to an interval of the form $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ for some positive integer n we have

$$f(x) = 1 + (-1)^n.$$

Draw a rough sketch of the graph of this function and explain why it is Riemann integrable on the interval $[0,1]$.

Solution:



For each positive integer** n **we define** $\mathbf{P}_n$ **to be the following partition of the interval** $[0,1]$**:

$$\mathbf{P}_n = \left(0, \frac{1}{n}, \frac{1}{n-1}, \cdots, \frac{1}{3}, \frac{1}{2}, 1\right).$$

For each** n **we see easily that

$$\int_0^1 l(\mathbf{P}_n, f) = \sum_{j=1}^{n-1} \left(\frac{1}{j} - \frac{1}{j+1}\right)\left(1 + (-1)^j\right)$$

and

$$\int_0^1 u(\mathbf{P}_n,f) = \frac{2}{n} + \sum_{j=1}^{n-1}\left(\frac{1}{j} - \frac{1}{j+1}\right)\left(1+(-1)^j\right)$$

and so

$$\lim_{n\to\infty}\left(\int_0^1 u(\mathbf{P}_n,f) - \int_0^1 l(\mathbf{P}_n,f)\right) = \lim_{n\to\infty}\frac{2}{n} = 0$$

and we have shown that f ***is integrable on*** $[0,1]$**.**

Incidentally, we have also shown that

$$\begin{aligned}\int_0^1 f &= \lim_{n\to\infty}\sum_{j=1}^{n-1}\left(\frac{1}{j} - \frac{1}{j+1}\right)\left(1+(-1)^j\right)\\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{2}{2j(2j+1)}\right)\\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{1}{j(2j+1)}\right)\end{aligned}$$

In the chapter on infinite series you will learn how to show that the latter limit is $2-2\log 2$**.**

3. Given that ϕ is an increasing function f is a bounded function on an interval $[a,b]$, prove that the following conditions are equivalent:

   a. The function f is integrable with respect to ϕ on the interval $[a,b]$.

   b. For every number $\varepsilon > 0$ there exist step functions s and S on the interval $[a,b]$ such that $s \le f \le S$ and
   $$\int_a^b (S-s)d\phi < \varepsilon.$$

   c. For every number $\varepsilon > 0$ there exist step functions s and S on the interval $[a,b]$ such that $s \le f \le S$ and such that if
   $$E = \{x \in [a,b] \mid S(x) - s(x) \ge \varepsilon\},$$
   we have $\operatorname{var}(\phi,E) < \varepsilon$.

   d. For every number $\varepsilon > 0$ there exist functions g and h that are Riemann-Stieltjes integrable with respect to ϕ on the interval $[a,b]$ such that $g \le f \le h$ and
   $$\int_a^b (h-g)d\phi < \varepsilon.$$

To show that condition a implies condition b we assume that condition a holds. In other words, we assume that f is integrable with respect to ϕ on $[a,b]$. Suppose that $\varepsilon > 0$. Using the first criterion for integrability we choose a partition $\mathbf{P}$ of $[a,b]$ such that

$$\int_a^b w(\mathbf{P},f)d\phi < \varepsilon.$$

We define $S = u(\mathbf{P},f)$ and $s = l(\mathbf{P},f)$ and observe that the functions s and S have the desired properties.

The proof that condition a implies condition c is very similar. This time, the partition $\mathbf{P}$ is chosen using the second criterion for integrability.

To prove that condition b implies condition a we assume that condition b holds. What we shall show is that the first criterion for integrability holds. Suppose that $\varepsilon > 0$. Using condition b we choose step functions s and S on the interval $[a,b]$ such that $s \le f \le S$ and

$$\int_a^b (S-s)d\phi < \varepsilon.$$

Choose a partition $\mathbf{P}$ of $[a,b]$ such that both s and S step within $\mathbf{P}$ and observe that since

$$s \leq l(\mathbf{P},f) \leq u(\mathbf{P},f) \leq S$$

we have

$$\int_a^b w(\mathbf{P},f)d\phi = \int_a^b (u(\mathbf{P},f) - l(\mathbf{P},f))d\phi \leq \int_a^b (S-s)d\phi < \varepsilon.$$

The proof that condition c implies condition a is very similar. This time we show that f is integrable by showing that the second criterion for integrability holds.

4. Suppose that ϕ is an increasing function, that f is a bounded function on an interval $[a,b]$ and that for every number $\varepsilon > 0$ there exists an elementary subset E of $[a,b]$ such that $\operatorname{var}(\phi,E) < \varepsilon$ and such that the function $f(1-\chi_E)$ is integrable with respect to ϕ on $[a,b]$. Prove that f must be integrable with respect to ϕ on the interval $[a,b]$.

Solution: ***In order to show that*** f ***is integrable on*** $[a,b]$ ***we shall show that*** f ***satisfies the*** second criterion ***for integrability. Suppose that*** $\varepsilon > 0$***.***
Using the given property of f ***we choose an elementary subset*** A ***of*** $[a,b]$ ***such that*** $\operatorname{var}(\phi,A) < \varepsilon/2$ ***and such that the function*** $f(1-\chi_A)$ ***is integrable with respect to*** ϕ ***on*** $[a,b]$***. Choose a partition*** $\mathbf{P}_1$ ***of*** $[a,b]$ ***such that the function*** χ_A ***steps within*** $\mathbf{P}$. ***Now, using the fact that*** $f(1-\chi_A)$ ***satisfies the second criterion for integrability we choose a partition*** $\mathbf{P}_2$ ***of*** $[a,b]$ ***such that if we definè***

$$B = \{x \in [a,b] \mid w(\mathbf{P},f(1-\chi_A))(x) \geq \varepsilon\}.$$

then $\operatorname{var}(\phi,B) < \varepsilon/2$***. We now define*** P ***to be the common refinement of*** $\mathbf{P}_1$ ***and*** $\mathbf{P}_2$ ***and we express*** $\mathbf{P}$ ***as***

$$\mathbf{P} = (x_0,x_1,\cdots,x_n).$$

For each $j = 1,2,\cdots,n$***, if the open interval*** (x_{j-1},x_j) ***is not included in*** $A \cup B$ ***then, since the functions*** f ***and*** $f(1-\chi_A)$ ***agree in the interval*** (x_{j-1},x_j) ***the condition***

$$w(\mathbf{P},f)(x) = w(\mathbf{P},f(1-\chi_A))(x) < \varepsilon$$

must hold whenever $x \in (x_{j-1},x_j)$***. Since***

$$\operatorname{var}(\phi,A\cup B) \leq \operatorname{var}(\phi,A) + \operatorname{var}(\phi,B) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

we have succeeded in showing that f ***satisfies the second criterion for integrability.***

5.

   a. Suppose that ϕ is an increasing function, that f is a nonnegative function defined on an interval $[a,b]$ and that for every number $\varepsilon > 0$, the set

$$\{x \in [a,b] \mid f(x) \geq \varepsilon\}$$

   is finite. Prove that f must be integrable with respect to ϕ on $[a,b]$ and that $\int_a^b f d\phi = 0$.
   We shall show that f satisfies the first criterion for integrability. Suppose that $\varepsilon > 0$. Choose a finite set S such that

$$f(x) < \frac{\varepsilon}{\operatorname{var}(\phi,[a,b])}$$

   whenever $x \in [a,b] \setminus S$ and define $\mathbf{P}$ to be the partition of $[a,b]$ whose points are the numbers a and b and the members of S arranged in increasing order. Since $l(\mathbf{P},f)$ is nonnegative and $u(\mathbf{P},f)$ never exceeds the value $\varepsilon/(b-a)$, we have

$$\int_a^b w(\mathbf{P},f)d\phi = \int_a^b (u(\mathbf{P},f) - l(\mathbf{P},f))d\phi \leq \int_a^b u(\mathbf{P},f)d\phi \leq \varepsilon.$$

   Since f satisfies the first criterion for integrability, f is integrable on $[a,b]$ and the same argument shows that whenever $\varepsilon > 0$ we have $\int_a^b f d\phi \leq \varepsilon$ from which we deduce that $\int_a^b f d\phi = 0$.

   b. Prove that if f is the ruler function that was introduced in an earlier example then f is a Riemann integrable function on the interval $[0,1]$, even though f is discontinuous at every rational number in the interval.
   The ruler function obviously has the property described in part a.

6. Given that ϕ is an increasing function and that f is a bounded nonnegative function defined on an interval

$[a,b]$, prove that the following conditions are equivalent:

a. The function f is integrable with respect to ϕ on the interval $[a,b]$ and $\int_a^b f d\phi = 0$.

b. For every number $\varepsilon > 0$ there exists an elementary set E such that $\operatorname{var}(\phi, E) < \varepsilon$ and such that
$$\{x \in [a,b] \mid f(x) \geq \varepsilon\} \subseteq E.$$

To show that condition a implies condition b we assume that f is integrable with respect to ϕ on $[a,b]$. Suppose that $\varepsilon > 0$. To obtain the desired set E we shall use the same sort of technique as was used in the proof of Theorem 11.9.4. Choose a step function $S \geq f$ such that
$$\int_a^b S d\phi < \varepsilon^2$$
and define
$$E = \{x \in [a,b] \mid S(x) \geq \varepsilon\}.$$
We observe that
$$\{x \in [a,b] \mid f(x) \geq \varepsilon\} \subseteq E.$$
Now since S is a step function, the set E is elementary and we have
$$\varepsilon^2 > \int_a^b S d\phi \geq \int_E S d\phi \geq \int_E \varepsilon d\phi = \varepsilon \operatorname{var}(\phi, E)$$
from which we deduce that $\operatorname{var}(\phi, E) < \varepsilon$.

To show that condition b implies condition a we assume that condition b holds. Once again we borrow from the proof of proof of Theorem 11.9.4. Using the fact that f is bounded we choose a number k such that $f(x) < k$ for every $x \in [a,b]$. For each positive integer n we choose an elementary set E_n such that $\operatorname{var}(\phi, E_n) < \frac{1}{n}$ and such that
$$\left\{x \in [a,b] \mid f(x) \geq \frac{1}{n}\right\} \subseteq E_n.$$
For each n we have
$$\begin{aligned}\int_a^b u(\mathbf{P}_n, f) d\phi &= \int_{E_n} u(\mathbf{P}_n, f) d\phi + \int_{[a,b]\setminus E_n} u(\mathbf{P}_n, f) d\phi \\ &\leq \int_{E_n} k d\phi + \int_{[a,b]\setminus E_n} \frac{1}{n} d\phi \leq k \operatorname{var}(\phi, E_n) + \int_a^b \frac{1}{n} d\phi \\ &< \frac{k}{n} + \frac{\operatorname{var}(\phi, [a,b])}{n}\end{aligned}$$
Since the latter expression approaches 0 as $n \to \infty$ we deduce that
$$\inf\left\{\int_a^b S \mid S \text{ is a step function and } f \leq S\right\} = 0$$
and this shows that f is integrable with respect to ϕ and that $\int_a^b f d\phi = 0$.

This chapter provides a special group of exercises that are designed to be done as a special project and which depend upon the special group of exercises on elementary sets that appeared earlier. The main purpose of these exercises is to invite you to prove the following interesting fact about integrals:

If f is a nonnegative function on an interval $[a,b]$ and f is Riemann-Stieltjes integrable with respect to an increasing function ϕ and if $\operatorname{var}(\phi, [a,b]) > 0$ and if $\int_a^b f d\phi = 0$ then there must be at least one number $x \in [a,b]$ for which $f(x) = 0$.

To reach this special group of exercises, click on the icon .

Positive Integrable Functions Have Positive Integrals

Suppose that ϕ is an increasing function, that f is a nonnegative function that is integrable with respect to ϕ on an interval $[a,b]$. Suppose that $\operatorname{var}(\phi,[a,b]) > 0$ and that

$$\int_a^b f d\phi = 0.$$

1. Prove that for every number $\varepsilon > 0$ there exists an elementary set E such that $\operatorname{var}(\phi, E) < \varepsilon$ and such that

$$\{x \in [a,b] \mid f(x) \geq \varepsilon\} \subseteq E.$$

This exercise is a duplicate of the last exercise in the exercises on exercises on integrability

2. Prove that if, for every positive integer n, we choose an elementary set E_n such that

$$\operatorname{var}(\phi, E_n) < \frac{\operatorname{var}(\phi,[a,b])}{2^n}$$

and such that

$$\left\{x \in [a,b] \mid f(x) \geq \frac{\operatorname{var}(\phi,[a,b])}{2^n}\right\} \subseteq E_n$$

then for every elementary E satisfying

$$E \subseteq \bigcup_{n=1}^{\infty} E_n$$

we have

$$\operatorname{var}(\phi, E) < \operatorname{var}(\phi,[a,b]).$$

To obtain this proof you will need to make use of the special group of exercises on elementary sets that can be reached by clicking on the icon .

The existence of the sets E_n follows from Exercise 1. Now if

$$E \subseteq \bigcup_{n=1}^{\infty} E_n$$

then the special exercises on elementary sets guarantee that

$$\operatorname{var}(\phi, E) \leq \sum_{n=1}^{\infty} \operatorname{var}(\phi, E_n) < \sum_{n=1}^{\infty} \frac{\operatorname{var}(\phi,[a,b])}{2^n} = \operatorname{var}(\phi,[a,b]).$$

3. Prove that if the sets E_n are defined as in Exercise 2, the set

$$[a,b] \setminus \bigcup_{n=1}^{\infty} E_n$$

must be nonempty and deduce that there must exist a number $x \in [a,b]$ such that $f(x) = 0$.
The elementary set $[a,b]$ can't be a subset of $\bigcup_{n=1}^{\infty} E_n$ because we do not have

$$\operatorname{var}(\phi,[a,b]) < \operatorname{var}(\phi,[a,b]).$$

Thus there must be numbers x in $[a,b]$ that do not belong to any of the sets E_n and since any such number x must satisfy the inequality

$$f(x) < \frac{\operatorname{var}(\phi,[a,b])}{2^n}$$

for every positive integer n we have $f(x) = 0$ for such numbers x.

4. Improve on the preceding exercises by proving that for every number $\varepsilon > 0$ there exists a sequence (E_n) of elementary sets such that the following two conditions hold:

   a. For every number satisfying

$$x \in [a,b] \setminus \bigcup_{n=1}^{\infty} E_n$$

we have $f(x) = 0$.

b. For every elementary set E satisfying

$$E \subseteq \bigcup_{n=1}^{\infty} E_n$$

we have $\mathrm{var}(\phi, E) < \varepsilon$.

Suppose that $\varepsilon > 0$. We choose a sequence (E_n) of elementary sets such that the conditions

$$\mathrm{var}(\phi, E_n) < \frac{\varepsilon}{2^n}$$

and

$$\left\{ x \in [a,b] \mid f(x) \geq \frac{\varepsilon}{2^n} \right\} \subseteq E_n$$

hold for each n. Continue as above. Of course, we don't have a guarantee that $[a,b] \setminus \bigcup_{n=1}^{\infty} E_n$ is nonempty unless $\varepsilon < b - a$.

Some Exercises on the Junior Lebesgue Criterion

1. True or false: Every step function satisfies the junior Lebesgue criterion.
   Of course this statement is true. The set of discontinuities of a step function, being finite, is an elementary set. If we express this set as

   $$E = \{t_j \mid j = 1, 2, \cdots, n\}$$

   then

   $$\mathrm{var}(\phi, E) = \sum_{j=1}^{n} J(\phi, t_j).$$

2. Suppose that (x_n) is a convergent sequence in an interval $[a,b]$ and that f is a bounded function on $[a,b]$ that is continuous at every member of $[a,b]$ that does not lie in the range of the sequence (x_n). Prove that f is Riemann integrable on $[a,b]$.

   Solution: ***See the solution to Exercise 4.***

3. Suppose that (x_n) is a convergent sequence in an interval $[a,b]$ that ϕ is an increasing function and that f is a bounded function on $[a,b]$ that is continuous at every member of $[a,b]$ that does not lie in the range of the sequence (x_n). Prove that f is Riemann-Stieltjes integrable with respect to ϕ on $[a,b]$.

4. Suppose that (x_n) is a sequence in an interval $[a,b]$ and that (x_n) has only finitely many partial limits. Suppose that ϕ is an increasing function and that f is a bounded function on $[a,b]$ that is continuous at every member of $[a,b]$ that does not belong to the range of the sequence (x_n). Prove that f is Riemann-Stieltjes integrable with respect to ϕ on $[a,b]$.

   Solution: ***We shall write the set of partial limits of (x_n) as $\{y_1, y_2, \cdots, y_k\}$. To prove that f satisfies the junior Lebesgue criterion, suppose that $\varepsilon > 0$.***

   For each $j = 1, 2, \cdots, k$, we choose a number $u_j < y_j$ and a number $v_j > y_j$ such that

   $$\phi(v_j) - \phi(u_j) < J(\phi, y_j) + \frac{\varepsilon}{k}.$$

   We define

$$U = \bigcup_{j=1}^{k} (u_j, v_j)$$

and we observe that

$$\mathrm{var}(\phi, U) \leq \sum_{j=1}^{k} \mathrm{var}(\phi, (u_j, v_j)) < \varepsilon + \sum_{j=1}^{k} J(\phi, y_j).$$

Since the set $[a,b] \setminus U$ ***is closed and bounded and since the sequence*** (x_n) ***has no partial limits in*** $[a,b] \setminus U$ ***we know that*** (x_n) ***cannot be frequently in the set*** $[a,b] \setminus U$***. Therefore, if***

$$F = \{x_n \mid n = 1, 2, \cdots\} \setminus U$$

then the set F ***is finite. We have thus found an elementary subset*** $U \cup F$ ***of*** $[a,b]$ ***such that***

$$\begin{aligned} \mathrm{var}(\phi, U \cup F) &\leq \mathrm{var}(\phi, U) + \mathrm{var}(\phi, F) \\ &< \varepsilon + \sum_{j=1}^{k} J(\phi, y_j) + \sum_{x \in F} J(\phi, x) \end{aligned}$$

and such that f ***is continuous at every number*** $x \in [a,b] \setminus (U \cup F)$***.***

5. This exercise does not ask you for a proof. Suppose that ϕ is an increasing function, that (x_n) is a sequence in an interval $[a,b]$ and that f is a bounded function on $[a,b]$ that is continuous at every member of $[a,b]$ that does not belong to the range of the sequence (x_n). Do you think that the function f has to be integrable with respect to ϕ on $[a,b]$. What does your intuition tell you?

Solution: ***The function*** f ***must be integrable. This fact will follow from the full version of the Lebesgue criterion for integrability that will appear in the chapter*** on sets of measure zero***.***

Some Exercises on the Composition Theorem

1. Given two functions f and g defined on a set S, we define the functions $f \vee g$ and $f \wedge g$ as follows:

$$f \vee g(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

and

$$f \wedge g(x) = \begin{cases} f(x) & \text{if } f(x) \leq g(x) \\ g(x) & \text{if } f(x) > g(x) \end{cases}.$$

Given that f and g are Riemann-Stieltjes integrable with respect to an increasing function ϕ on an interval $[a,b]$, make the observations

$$f \vee g = \frac{f + g + |f - g|}{2}$$

and

$$f \wedge g = \frac{f + g - |f - g|}{2}$$

and deduce that the functions $f \vee g$ and $f \wedge g$ are also integrable with respect to ϕ on $[a,b]$.
There really isn't much to do in this exercise. The equation

$$(f \vee g)(x) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

and

$$(f \wedge g)(x) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

for each x follow at once when we consider the cases $f(x) \leq g(x)$ and $f(x) > g(x)$.

2. Given that f is a nonnegative function that is integrable on an interval $[a,b]$ with respect to an increasing function ϕ, explain why the function $\sqrt{f}$ is integrable with respect to ϕ on $[a,b]$.
   This exercise follows at once from the fact that the square root function is uniformly continuous on the range of f.

3. Given that f is integrable with respect to an increasing function ϕ on an interval $[a,b]$, that $f(x) \geq 1$ for every $x \in [a,b]$ and that
$$g(x) = \log(f(x))$$
for every $x \in [a,b]$, explain why the function g must be integrable with respect to ϕ on $[a,b]$.
   Since the function $\log$ is uniformly continuous on the interval $[1,\infty)$, the integrability of g follows at once from the composition theorem.

4. Suppose that f is integrable with respect to an increasing function ϕ on an interval $[a,b]$ and that $\alpha \leq f(x) \leq \beta$ for every $x \in [a,b]$. Show how the junior version of the composition theorem for integrability can be used to show that if h is any continuous function on the interval $[\alpha,\beta]$ then the function $h \circ f$ is integrable with respect to ϕ on $[a,b]$.
   The result follows at once from the fact that any continuous function on the interval $[\alpha,\beta]$ must be uniformly continuous.

Exercises on the Change of Variable Theorem

1. a. Given that f is a continuous function on the interval $[-1,1]$, prove that
$$\int_0^{2\pi} f(\sin x)\cos x dx = 0.$$

Solution: ***We define*** $u(x) = \sin x$ ***for every number*** $x \in [0,2\pi]$ ***and observe that***
$$\begin{aligned}\int_0^{2\pi} f(\sin x)\cos x dx &= \int_0^{2\pi} f(u(x))u'(x)dx \\ &= \int_{u(0)}^{u(2\pi)} f = \int_0^0 f = 0\end{aligned}$$

b. Given that f is a continuous function on the interval $[0,1]$, prove that
$$\int_0^{\pi/2} f(\sin x)dx = \int_{\pi/2}^{\pi} f(\sin x)dx.$$

Solution: ***We define*** $u(x) = \pi - x$ ***for every number*** $x \in \left[0, \frac{\pi}{2}\right]$ ***and observe that***
$$\begin{aligned}\int_0^{\pi/2} f(\sin x)dx &= -\int_0^{\pi/2} f(\sin(\pi - x))u'(x)dx \\ &= -\int_0^{\pi/2} f(\sin(u(x)))u'(x)dx \\ &= -\int_{u(0)}^{u(\pi/2)} f(\sin t)dt = -\int_{\pi}^{\pi/2} f(\sin t)dt \\ &= \int_{\pi/2}^{\pi} f(\sin t)dt\end{aligned}$$

Of course, it makes no difference whether we write t ***or*** x ***in the integral*** $\int_{\pi/2}^{\pi} f(\sin t)dt$***.***

c. Given that $\alpha > 0$, prove that
$$\int_0^{\pi/2} \sin^\alpha x dx = 2^\alpha \int_0^{\pi/2} \sin^\alpha x \cos^\alpha x dx.$$

Solution: ***We define*** $u(x) = 2x$ ***for all*** $x \in [0,\pi/2]$ ***and observe that***

$$2^\alpha \int_0^{\pi/2} \sin^\alpha x \cos^\alpha x dx = \int_0^{\pi/2} 2^\alpha \sin^\alpha x \cos^\alpha x dx = \int_0^{\pi/2} (2\sin x \cos x)^\alpha dx$$
$$= \frac{1}{2}\int_0^{\pi/2} (\sin^\alpha 2x) 2 dx = \frac{1}{2}\int_0^{\pi/2} (\sin^\alpha (u(x))) u'(x) dx$$
$$= \frac{1}{2}\int_{u(0)}^{u(\pi/2)} \sin^\alpha t dt = \frac{1}{2}\int_0^{\pi} \sin^\alpha t dt$$
$$= \frac{1}{2}\left(\int_0^{\pi/2} \sin^\alpha t dt + \int_{\pi/2}^{\pi} \sin^\alpha t dt\right)$$

and from part b we deduce that the latter expression is equal to

$$\frac{1}{2}\left(\int_0^{\pi/2} \sin^\alpha t dt + \int_0^{\pi/2} \sin^\alpha t dt\right) = \int_0^{\pi/2} \sin^\alpha t dt.$$

2. Given that u is a differentiable function on an interval $[a,b]$ and that its derivative u' is integrable on $[a,b]$ and given that $u(a) = u(b)$ and that f is integrable on the range of u, prove that

$$\int_a^b f(u(t))u'(t)dt = 0.$$

From the change of variable theorem we see that

$$\int_a^b f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f(x)dx = \int_{u(a)}^{u(a)} f(x)dx = 0.$$

3. Given that f is integrable on an interval $[a,b]$ and that c is any number, prove that

$$\int_a^b f(t)dt = \int_{a+c}^{b+c} f(t-c)dt$$

For every number t we define $u(t) = t - c$. We observe that $u'(t) = 1$ for every t. Now

$$\int_{a+c}^{b+c} f(t-c)dt = \int_{a+c}^{b+c} f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f(x)dx = \int_a^b f(t)dt.$$

I have changed the name of the dummy variable back to t to match the expression in the exercise.

4. Given that $a,$ b and c are real numbers, that $ac < bc$ and that f is a continuous function on the interval $[ac, bc]$, prove that

$$\int_{ac}^{bc} f(t)dt = c\int_a^b f(ct)dt.$$

Hint: ***Look at the definition*** $u(t) = ct$ ***for each*** t.

We assume that $c > 0$. We define $u(t) = ct$ for every number t. We see that $u'(t) = c$ for each t and we deduce from the change of variable theorem that

$$c\int_a^b f(ct)dt = \int_a^b f(u(t))u'(t)dt = \int_{ac}^{bc} f(x)dx = \int_{ac}^{bc} f(t)dt$$

5. a. Suppose that f is a continuous function on an interval $[a,b]$, that g is nonnegative and integrable on $[a,b]$. Prove that, if m and M are, respectively, the minimum and maximum values of f, then

$$m\int_a^b g \le \int_a^b fg \le M\int_a^b g$$

and deduce that there exists a number $c \in [a,b]$ such that

$$\int_a^b fg = f(c)\int_a^b g.$$

This fact is sometimes called the **mean value theorem for integrals**.
Since

$$m \le f(x) \le M$$

for every $x \in [a,b]$ we have

$$\int_a^b mg \le \int_a^b fg \le \int_a^b Mg$$

which gives us

$$m\int_a^b g \le \int_a^b fg \le M\int_a^b g$$

and so

$$m \le \frac{\int_a^b fg}{\int_a^b g} \le M.$$

The existence of the number c follows at once from the Bolzano intermediate value theorem.

b. Given that f is continuous on an interval $[a,b]$, prove that there exits a number $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

6. a. Given that f is a nonnegative continuous function on an interval $[a,b]$ where $a < b$ and that $\int_a^b f = 0$, prove that f is the constant zero function.
To obtain a contradiction, suppose that $c \in [a,b]$ and that $f(c) > 0$. Using the fact that f is continuous at c, choose $\delta > 0$ such that the inequality

$$f(x) > \frac{f(c)}{2}$$

holds for every number $x \in [a,b] \cap (c-\delta, c+\delta)$. The set $[a,b] \cap (c-\delta, c+\delta)$ is an interval with positive length that we shall write as $[c,d]$. In fact, c is the larger of the two numbers a and $c-\delta$ and d is the smaller of the two numbers b and $c+\delta$. Since

$$\int_a^b f \ge \int_c^d f \ge \int_c^d \frac{f(c)}{2} = \frac{f(c)}{2}(d-c) > 0$$

which gives us our desired contradiction.

b. Given that f is a continuous function on an interval $[a,b]$ where $a < b$ and that $\int_a^x f = 0$, for every $x \in [a,b]$, prove that f is the constant zero function.
We know that if

$$F(x) = \int_a^x f$$

for each x then $F'(x) = f(x)$ for each x. Since F is the constant function zero we conclude that $f(x) = 0$ for each x.

7. In this exercise we consider another proof of the "u decreasing" form of the change of variable theorem.

a. Given that f is an integrable function on an interval $[a,b]$ and that $g(t) = f(-t)$ whenever $-b \le t \le -a$, give a direct proof that g is integrable on the interval $[-b,-a]$ and that

$$\int_{-b}^{-a} g(t)dt = \int_a^b f(x)dx.$$

Suppose first that f is a step function on $[a,b]$ that steps within the partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

taking the constant value α_j on each interval (x_{j-1}, x_j). We define

$$\mathbf{Q} = (-x_n, -x_{n-1}, \cdots, -x_1, -x_0)$$

and observe that $\mathbf{Q}$ is a partition of the interval $[-b,-a]$. Now we define $g(t) = f(-t)$ for each $t \in [a,b]$ and we observe that g is a step function that takes the value α_j on each interval $(-x_j, x_{j-1})$. We see at once that

$$\int_a^b f = \sum_{j=1}^n \alpha_j(x_j - x_{j-1}) = \sum_{j=1}^n (-x_{j-1} - (-x_j))\alpha_j = \int_{-b}^{-a} g$$

We can now handle the general case. Using the fact that f is integrable on the interval $[a,b]$, choose a pair of sequences of step functions that squeezes f on the interval $[b,a]$. In other words,

$$s_n \le f \le S_n$$

for each n and

$$\lim_{n\to\infty} \int_a^b (S_n - s_n) = 0.$$

For each n we define $s_n^*(t) = s_n(-t)$ and $S_n^*(t) = S_n(-t)$ and we observe that

$$s_n^* \le g \le S_n^*$$

and, by the case we have already considered we deduce that

$$\int_{-b}^{-a} (S_n^* - s_n^*) = \int_a^b (S_n - s_n) \to 0$$

as $n \to \infty$.

It follows that the function g is integrable on the interval $[-b, -a]$ and that

$$\int_{-b}^{-a} g = \lim_{n\to\infty} \int_{-b}^{-a} s_n^* = \lim_{n\to\infty} \int_a^b s_n = \int_a^b f.$$

b. Suppose that u is a decreasing differentiable function on an interval $[a,b]$ and that the derivative u' of u is integrable on $[a,b]$. Apply the form of the monotone version of the change of variable theorem proved above to the function v defined by the equation $v(t) = -u(t)$ for $-b \le t \le -a$ to show that the equation

$$\int_a^b f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f(x)dx$$

holds for every function f integrable on the interval $[u(b), u(a)]$.

We define $g(t) = f(-t)$ for all $t \in [a,b]$. Then g is integrable on $[-u(a), -u(b)]$ and we have

$$\int_{-u(a)}^{-u(b)} g = \int_{u(b)}^{u(a)} f = -\int_{u(a)}^{u(b)} f$$

From the monotone version of the theorem proved earlier we also see that

$$\int_{-u(a)}^{-u(b)} g = \int_{v(a)}^{v(b)} g = \int_a^b g(v(t))v'(t)dt$$

$$= \int_a^b g(-u(t))(-u'(t))dt = -\int_a^b f(u(t))u'(t)dt.$$

We conclude that

$$\int_a^b f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f$$

To reach some additional exercises that invite you to develop some important inequalities, click on the icon .

Exercises that Yield Another Version of Darboux's Theorem

The exercises in this section depend upon the material on Darboux's theorem. Their purpose is to show that if f is Riemann-Stieltjes integrable with respect to an increasing function ϕ and if ϕ is continuous at every number at which ϕ jumps then Darboux's theorem can be stated using the ordinary mesh $\|\mathbf{P}\|$ of a partition $\mathbf{P}$ instead of the ϕ-mesh.

1. Prove the following special case of Darboux's theorem that applies when ϕ is continuous: Suppose that f is Riemann-Stieltjes integrable with respect to an increasing continuous function ϕ on an interval $[a,b]$. Then for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ satisfying the inequality $\|\mathbf{P}\| < \delta$ and every choice of numbers $t_j \in [x_{j-1}, x_j]$ for each j we have

$$\left|\sum_{j=1}^{n} f(t_j)(\phi(x_j) - \phi(x_{j-1})) - \int_a^b f d\phi\right| < \varepsilon.$$

Since the function ϕ is continuous on the interval $[a,b]$, it is uniformly continuous there. Suppose that $\varepsilon > 0$. Using Darboux's theorem, choose a number $\delta_1 > 0$ such that the inequality

$$\left|\sum_{j=1}^{n} f(t_j)(\phi(x_j) - \phi(x_{j-1})) - \int_a^b f d\phi\right| < \varepsilon$$

will hold whenever the partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

satisfies the condition $\|\phi, \mathbf{P}\| < \delta_1$. Using the fact that ϕ is uniformly continuous on the interval $[a,b]$, choose $\delta > 0$ such that the inequality $\|\phi, \mathbf{P}\| < \delta_1$ will hold whenever $\|\mathbf{P}\| < \delta$.

2. Suppose that ϕ is an increasing function that varies discretely on an interval $[a,b]$. Suppose that f is a bounded function on $[a,b]$ and that f is continuous at every number at which the function ϕ is discontinuous. Prove that for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of $[a,b]$ satisfying the inequality $\|\mathbf{P}\| < \delta$ and for every choice of numbers $t_j \in [x_{j-1}, x_j]$ for each j, if we define g to be the step function that takes the value $f(x_j)$ at each number x_j and that takes the constant value $f(t_j)$ in each interval (x_{j-1}, x_j) then we have

$$\left|\int_a^b g d\phi - \int_a^b f d\phi\right| < \varepsilon.$$

Suppose that $\varepsilon > 0$. We write the set of discontinuities of ϕ in the interval $[a,b]$ as $\{y_1, y_2, y_3, \cdots\}$. Choose a number k such that the inequality $|f(x)| \le k$ holds for every number x in the interval $[a,b]$ and choose an integer N such that

$$\sum_{n=N}^{\infty} kJ(\phi, y_n) < \frac{\varepsilon}{4}.$$

Using the fact that f is continuous at each of the numbers y_n for $n \le N$, choose $\delta > 0$ such that whenever $n \le N$ and $|t - y_n| < \delta$ we have

$$|f(t) - f(yn)| < \frac{\varepsilon}{2\operatorname{var}(\phi, [a,b])}.$$

Now suppose that

$$\mathbf{P} = (x_0, x_1, \cdots, x_p)$$

is any partition of $[a,b]$ for which $\|\mathbf{P}\| < \delta$ and that $t_j \in [x_{j-1}, x_j]$ for each j, and define g to be the step function that takes the value $f(x_j)$ at each number x_j and that takes the constant value $f(t_j)$ in each interval (x_{j-1}, x_j). We see that

$$\begin{aligned}
\left|\int_a^b g d\phi - \int_a^b f d\phi\right| &= \left|\sum_{n=1}^{\infty} g(y_n) J(\phi, y_n) - \sum_{n=1}^{\infty} f(y_n) J(\phi, y_n)\right| \\
&\le \left|\sum_{n=1}^{N} g(y_n) J(\phi, y_n) - \sum_{n=1}^{N} f(y_n) J(\phi, y_n)\right| + \sum_{n=N+1}^{\infty} |g(y_n)| J(\phi, y_n) + \sum_{n=N+1}^{\infty} |f(y_n)| J(\phi, y_n) \\
&\le \sum_{n=1}^{N} |g(y_n) - f(y_n)| J(\phi, y_n) + 2\sum_{n=N+1}^{\infty} kJ(\phi, y_n) \\
&< \sum_{n=1}^{N} \frac{\varepsilon}{2\operatorname{var}(\phi, [a,b])} J(\phi, y_n) + \frac{\varepsilon}{2} \le \varepsilon
\end{aligned}$$

3. By combining the preceding two exercises, obtain an analog of Exercise 2 that does not require the assumption that ϕ varies discretely on $[a,b]$.

The desired extension to the case in which ϕ is an arbitrary increasing function follows at once when we split ϕ into its continuous and discrete parts.

12 Infinite Series

Some Elementary Exercises on Series

1. Find the nth partial sum of the series
$$\sum \frac{n-3}{n(n+1)(n+3)}.$$
Deduce that this series is convergent and find its sum.

Solution: ***Given any positive integer j we have***
$$\frac{j-3}{j(j+1)(j+3)} = \frac{2}{j+1} - \frac{1}{j} - \frac{1}{j+3}$$
and therefore
$$\begin{aligned}\sum_{j=1}^{n} \frac{j-3}{j(j+1)(j+3)} &= \sum_{j=1}^{n}\left(\frac{2}{j+1} - \frac{1}{j} - \frac{1}{j+3}\right)\\ &= \sum_{j=2}^{n+1}\frac{2}{j} - \sum_{j=1}^{n}\frac{1}{j} - \sum_{j=4}^{n+3}\frac{1}{j}\\ &= \frac{2}{2} + \frac{2}{3} + \frac{2}{n+1} - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\\ &\to \frac{2}{2} + \frac{2}{3} - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} = -\frac{1}{6}\end{aligned}$$
as $n \to \infty$.

2. a. Find the derivative of the nth partial sum of the series $\sum x^n$.
If n is any positive integer and $x \neq 1$ then
$$\sum_{j=1}^{n} x^j = \frac{x(1-x^n)}{1-x}.$$
Differentiating we obtain
$$\sum_{j=1}^{n} jx^{j-1} = \frac{1-x^n-nx^n+nx^{n+1}}{(1-x)^2} = \frac{1-x^n-nx^n(1-x)}{(1-x)^2}$$

b. Find the nth partial sum of the series $\sum nx^{n-1}$. Deduce that if $|x| < 1$ we have
$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$
In order to deduce the identity
$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$
we need to take the limit as $n \to \infty$ of each side of of the identity
$$\sum_{j=1}^{n} jx^{j-1} = \frac{1-x^n-nx^n+nx^{n+1}}{(1-x)^2}$$

and for this purpose we need to know that

$$\lim_{n\to\infty} nx^n = 0$$

whenever $|x| < 1$. Later in this chapter, we shall see some simple ways of finding the latter limit. Perhaps the simplest way to find it right now is to use L'Hôpital's rule. Suppose that $|x| < 1$. To show that $n|x|^n \to 0$ as $n \to \infty$ we use the fact that

$$n|x|^n = \frac{n}{(1/|x|)^n}$$

We define $c = \log(1/|x|)$. Note that $c > 0$ and

$$n|x|^n = \frac{n}{e^{cn}}$$

and the fact that $n|x|^n \to 0$ as $n \to \infty$ follows from the fact that

$$\lim_{t\to\infty} \frac{t}{e^{ct}} = 0$$

which follows easily from L'Hôpital's rule.

An alternative to L'Hôpital's rule is to give the students the assignment of proving the inequality

$$e^{ct} \geq 1 + ct + \frac{c^2t^2}{2}$$

for all $t \geq 0$. This inequality follows easily from the mean value theorem and from it we obtain the above limit easily.

3. Given that $|x| < 1$ and that

$$s_n = \sum_{j=1}^{n} (3j-1)x^{2j}$$

for every positive integer n, obtain the identity

$$s_n(1-x^2) = 2x^2 + \frac{3x^4 - 3x^{2n+4} - (3n-1)x^{2n+2} - x^{2n+4}}{1-x^2}$$

and deduce that

$$\sum_{j=1}^{\infty} (3j-1)x^{2j} = 2x^2 + \frac{3x^4}{1-x^2}$$

We observe that

$$\begin{aligned}
s_n(1-x^2) &= \sum_{j=1}^{n} (3j-1)x^{2j} - x^2 \sum_{j=1}^{n} (3j-1)x^{2j} \\
&= (2x^2 + 5x^4 + 8x^6 + \cdots + (3n-1)x^{2n}) - (2x^4 + 5x^6 + 8x^8 + \cdots + (3n-1)x^{2n+2}) \\
&= 2x^2 + 3(x^4 + x^6 + \cdots + x^{2n}) - (3n-1)x^{2n+2} \\
&= 2x^2 + \frac{3x^4(1-x^{2n})}{1-x^2} - (3n-1)x^{2n+2} \\
&= 2x^2 + \frac{3x^4(1-x^{2n}) - (3n-1)x^{2n+2}(1-x^2)}{1-x^2} \\
&= 2x^2 + \frac{3x^4 - 3x^{2n+4} - (3n-1)x^{2n+2} - x^{2n+4}}{1-x^2}
\end{aligned}$$

and the final result follows by letting $n \to \infty$.

Exercises on the Integral Test

1. For every positive integer n we have

$$\sum_{j=n+1}^{2n} \frac{1}{j} = \left(\sum_{j=1}^{2n} \frac{1}{j} - \log 2n \right) - \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) + \log 2.$$

Using this fact, deduce that

$$\lim_{n\to\infty} \sum_{j=n+1}^{2n} \frac{1}{j} = \log 2.$$

The identity

$$\sum_{j=n+1}^{2n} \frac{1}{j} = \left(\sum_{j=1}^{2n} \frac{1}{j} - \log 2n \right) - \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) + \log 2.$$

is clear and the final result follows at once from the fact that

$$\lim_{n\to\infty} \left(\sum_{j=1}^{2n} \frac{1}{j} - \log 2n \right) = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right) = \gamma.$$

2. Explain why

$$\sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j} = \sum_{j=n+1}^{2n} \frac{1}{j}$$

for very positive integer n and deduce that

$$\lim_{n\to\infty} \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j} = \log 2.$$

Deduce that

$$\lim_{n\to\infty} \sum_{j=1}^{2n+1} \frac{(-1)^{j-1}}{j}$$

is also equal to $\log 2$ and conclude that

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} = \log 2.$$

$$\begin{aligned}
\sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j} &= \sum_{j=1}^{n} \frac{1}{2j-1} - \sum_{j=1}^{n} \frac{1}{2j} \\
&= \sum_{j=1}^{n} \frac{1}{2j-1} + \sum_{j=1}^{n} \frac{1}{2j} - 2\sum_{j=1}^{n} \frac{1}{2j} \\
&= \sum_{j=1}^{2n} \frac{1}{j} - \sum_{j=1}^{n} \frac{1}{j} = \sum_{j=n+1}^{2n} \frac{1}{j}
\end{aligned}$$

The fact that

$$\lim_{n\to\infty} \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j} = \log 2.$$

follows at once from Exercise 1. Since

$$\lim_{n\to\infty} \sum_{j=1}^{2n+1} \frac{(-1)^{j-1}}{j} = \lim_{n\to\infty} \left(\sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j} + \frac{1}{2n+1} \right)$$

we conclude that

$$\lim_{n\to\infty} \sum_{j=1}^{2n+1} \frac{(-1)^{j-1}}{j} = \log 2.$$

Finally, to prove that

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} = \log 2,$$

suppose that $\varepsilon > 0$. Choose N such that the inequalities

$$\left| \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j} - \log 2 \right| < \varepsilon$$

and

$$\left| \sum_{j=1}^{2n+1} \frac{(-1)^{j-1}}{j} - \log 2 \right| < \varepsilon$$

hold whenever $n \geq N$. We see that

$$\left| \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} - \log 2 \right| < \varepsilon$$

whenever $n \geq 2N + 1$.

3. Express the rational expression

$$\frac{1}{j(2j-1)}$$

in partial fractions and show that if n is any positive integer we have

$$\sum_{j=1}^{n} \frac{1}{j(2j-1)} = 2 \sum_{j=n+1}^{2n} \frac{1}{j}.$$

Deduce that

$$\sum_{j=1}^{\infty} \frac{1}{j(2j-1)} = \log 4.$$

Solution: ***From the identity***

$$\frac{1}{j(2j-1)} = \frac{2}{2j-1} - \frac{1}{j}$$

we see that if n ***is any positive integer then***

$$\begin{aligned} \sum_{j=1}^{n} \frac{1}{j(2j-1)} &= \sum_{j=1}^{n} \frac{2}{2j-1} - \sum_{j=1}^{n} \frac{1}{j} \\ &= \sum_{j=1}^{n} \frac{2}{2j-1} + \sum_{j=1}^{n} \frac{2}{2j} - \sum_{j=1}^{n} \frac{2}{2j} - \sum_{j=1}^{n} \frac{1}{j} \\ &= \sum_{j=1}^{2n} \frac{2}{j} - \sum_{j=1}^{n} \frac{2}{j} = 2 \sum_{j=n+1}^{2n} \frac{1}{j} \end{aligned}$$

From Exercise 1 we deduce that

$$\sum_{j=1}^{\infty} \frac{1}{j(2j-1)} = \lim_{n \to \infty} 2 \sum_{j=n+1}^{2n} \frac{1}{j} = 2 \log 2 = \log 4$$

4. Evaluate the sums

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)}.$$

Solution: ***From the identity***

$$\frac{1}{j(2j+1)} = \frac{1}{j} - \frac{2}{2j+1}$$

we see that if** n **is any positive integer then

$$\begin{aligned}\sum_{j=1}^{n} \frac{1}{j(2j+1)} &= \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} \frac{2}{2j+1}\\ &= \sum_{j=1}^{n} \frac{1}{j} + \sum_{j=1}^{n} \frac{2}{2j} - \sum_{j=1}^{n} \frac{2}{2j} - \sum_{j=1}^{n} \frac{2}{2j+1}\\ &= \sum_{j=1}^{n} \frac{2}{j} - \sum_{j=1}^{2n+1} \frac{2}{j} + 2 = 2 - 2\sum_{j=n+1}^{2n} \frac{1}{j} - \frac{2}{2n+1}\end{aligned}$$

Therefore

$$\sum_{j=1}^{\infty} \frac{1}{j(2j+1)} = \lim_{n\to\infty}\left(2 - 2\sum_{j=n+1}^{2n} \frac{1}{j} - \frac{2}{2n+1}\right) = 2 - \log 4.$$

We conclude that

$$\sum_{j=1}^{\infty} \frac{1}{j(2j-1)} - \sum_{j=1}^{\infty} \frac{1}{j(2j+1)} = 4\log 2 - 2$$

and combining the terms on the left side we obtain

$$\sum_{j=1}^{\infty} \frac{1}{j(4j^2-1)} = 2\log 2 - 1.$$

5. Prove that if $1 \le q \le p$ then

$$\lim_{n\to\infty} \sum_{j=qn+1}^{pn} \frac{1}{j} = \log\left(\frac{p}{q}\right).$$

We see that

$$\begin{aligned}\lim_{n\to\infty} \sum_{j=qn+1}^{pn} \frac{1}{j} &= \lim_{n\to\infty}\left(\sum_{j=1}^{pn} \frac{1}{j} - \sum_{j=1}^{qn} \frac{1}{j}\right)\\ &= \lim_{n\to\infty}\left(\left(\sum_{j=1}^{pn} \frac{1}{j} - \log(pn)\right) - \left(\sum_{j=1}^{qn} \frac{1}{j} - \log(qn)\right) + \log\left(\frac{p}{q}\right)\right)\\ &= \gamma - \gamma + \log\left(\frac{p}{q}\right) = \log\left(\frac{p}{q}\right).\end{aligned}$$

6. Given that $p > 1$, evaluate the limit

$$\lim_{n\to\infty} \sum_{j=n+1}^{n^p} \frac{1}{j\log j}.$$

Solution: ***We know that the limit***

$$\lim_{n\to\infty}\left(\sum_{j=2}^{n} \frac{1}{j\log j} - \log\log n\right)$$

exists and is finite. Now whenever** $n \ge 2$ **we have

$$\sum_{j=n+1}^{n^p} \frac{1}{j\log j} = \sum_{j=2}^{n^p} \frac{1}{j\log j} - \log\log(n^p) - \left(\sum_{j=2}^{n} \frac{1}{j\log j} - \log\log n\right) + \log p$$

and we conclude that

$$\lim_{n\to\infty} \sum_{j=n+1}^{n^p} \frac{1}{j\log j} = \log p.$$

Some Exercises on The Comparison Test

Test each of the following series for convergence.

1. $\sum \frac{1}{n^{3/2}+n}$

Solution: ***We define***

$$a_n = \frac{1}{n^{3/2}+n}$$

and

$$b_n = \frac{1}{n^{3/2}}$$

for each n. Since the series $\sum b_n$ is a convergent p-series and since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{3/2}+n}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}+n}$$
$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1$$

we see that $\sum a_n$ is convergent.

Note that we could have achieved this solution more rapidly by observing that $a_n \le b_n$ for each n. However, taking the limit was worth while because it applies just as well to Exercise 2.

2. $\sum \frac{1}{n^{3/2}-n}$

We define

$$a_n = \frac{1}{n^{3/2}-n}$$

and

$$b_n = \frac{1}{n^{3/2}}$$

for each n. Since $\sum b_n$ is a convergent p-series and since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

we deduce that $\sum a_n$ is convergent.

3. $\sum \frac{n}{\sqrt{n^4-n^2+2}}$

Solution: ***In order to see how to proceed observe that, for large n, the expression***

$$\frac{n}{\sqrt{n^4-n^2+2}}$$

behaves like

$$\frac{n}{\sqrt{n^4}} = \frac{1}{n}$$

because $-n^2+2$ is much smaller than n^4. Now we begin.

We define

$$a_n = \frac{n}{\sqrt{n^4 - n^2 + 2}}$$

and

$$b_n = \frac{1}{n}$$

for each n***. Since*** $\sum b_n$ ***is a divergent*** p***-series and since***

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^4 - n^2 + 2}}}{\frac{1}{n}} = \lim_{n\to\infty} \sqrt{\frac{n^4}{n^4 - n^2 + 2}} = 1$$

we deduce that $\sum a_n$ ***is divergent.***

4. $\sum \frac{n \log n}{\sqrt{n^5 - n^2 + 2}}$

Solution: ***The key to this exercise is the fact that, although*** $n \log n$ ***is larger than*** n***, it is not much larger because of the fact that***

$$\lim_{n\to\infty} \frac{\log n}{n^\delta} = 0$$

whenever $\delta > 0$***. The denominator of the fraction***

$$\frac{n \log n}{\sqrt{n^5 - n^2 + 2}}$$

behaves like $n^{5/2}$***. We can therefore consider that***

$$\frac{n \log n}{\sqrt{n^5 - n^2 + 2}}$$

behaves like

$$\frac{n \log n}{n^{5/2}} = \frac{\log n}{n^{3/2}} = \left(\frac{\log n}{n^{1/4}}\right)\left(\frac{1}{n^{5/4}}\right).$$

With this in mind, we define

$$a_n = \frac{n \log n}{\sqrt{n^5 - n^2 + 2}}$$

and

$$b_n = \frac{1}{n^{5/4}}$$

for each n***. We observe that*** $\sum b_n$ ***is a convergent*** p***-series and that***

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\log n}{n^{1/4}} = 0$$

and so $\sum a_n$ ***is convergent.***

5. $\sum \frac{1}{n^{(1+(\log n)/n)}}$

Solution: ***We recall that if*** a ***and*** b ***are any positive numbers then***

$$a^b = \exp(b \log a).$$

We begin by observing that if n ***is any positive integer then***

$$n^{(1+(\log n)/n)} = n(n^{(\log n)/n}) = n \exp\left(\left(\frac{\log n}{n}\right)(\log n)\right) = n \exp\left(\frac{(\log n)^2}{n}\right)$$

Since

$$\lim_{n\to\infty}\exp\left(\frac{(\log n)^2}{n}\right) = \exp\left(\lim_{n\to\infty}\frac{(\log n)^2}{n}\right) = \exp 0 = 1$$

we conclude that

$$\frac{1}{n^{(1+(\log n)/n)}}$$

behaves like $1/n$ ***for large*** n***. We define***

$$a_n = \frac{1}{n^{(1+(\log n)/n)}}$$

and

$$b_n = \frac{1}{n}$$

for each n***. Since*** $\sum b_n$ ***is a divergent*** p***-series and since***

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\exp\left(\frac{(\log n)^2}{n}\right)} = 1$$

we conclude that $\sum a_n$ ***is divergent.***

6. $\sum \frac{1}{n^{\left(1+(\log n)^2/n\right)}}$

Hint: ***This exercise is similar to Exericse 5 because of the identity***

$$n^{\left(1+(\log n)^2/n\right)} = n\exp\left(\frac{(\log n)^3}{n}\right)$$

and because

$$\lim_{n\to\infty}\exp\left(\frac{(\log n)^3}{n}\right) = \exp\left(\lim_{n\to\infty}\frac{(\log n)^3}{n}\right) = \exp 0 = 1.$$

Thus $\sum \frac{1}{n^{\left(1+(\log n)^2/n\right)}}$ ***is divergent.***

7. $\sum \frac{1}{n^{\left(1+\left((\log n)^{(\log\log n)}\right)/n\right)}}$

Solution: ***For each*** n ***we see that***

$$n^{\left(1+\left((\log n)^{(\log\log n)}\right)/n\right)} = n\exp\left(\frac{(\log n)\left((\log n)^{(\log\log n)}\right)}{n}\right)$$

Now

$$(\log n)^{\log\log n} = \exp\left((\log\log n)^2\right) = \exp\left(\left(\frac{(\log\log n)^2}{\log n}\right)(\log n)\right)$$

and since

$$\lim_{n\to\infty}\frac{(\log\log n)^2}{\log n} = 0$$

we conclude that if n ***is sufficiently large then***

$$\begin{aligned}(\log n)^{\log\log n} &= \exp\left(\left(\frac{(\log\log n)^2}{\log n}\right)(\log n)\right)\\ &< \exp\left(\frac{1}{2}(\log n)\right) = n^{1/2}\end{aligned}$$

Therefore, for n ***sufficiently large we have***

$$0 < \frac{(\log n)\left((\log n)^{(\log\log n)}\right)}{n} < \frac{(\log n)n^{1/2}}{n} = \frac{\log n}{n^{1/2}}$$

and we conclude that

$$\lim_{n\to\infty}\frac{(\log n)\left((\log n)^{(\log\log n)}\right)}{n} = 0$$

Now we define

$$a_n = \frac{1}{n^{\left(1+\left((\log n)^{(\log\log n)}\right)/n\right)}}$$

and

$$b_n = \frac{1}{n}$$

for each n***. Since*** $\sum b_n$ ***is a divergent*** p***-series and since***

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\exp\left(\frac{(\log n)\left((\log n)^{(\log\log n)}\right)}{n}\right)} = 1$$

we deduce that $\sum a_n$ ***is divergent.***

8. $\sum\left(\frac{n}{n+1}\right)^{n\log n}$

Solution: ***For each*** n ***we have***

$$\left(\frac{n}{n+1}\right)^{n\log n} = \exp\left((n\log n)\log\left(\frac{n}{n+1}\right)\right).$$

We now observe that

$$\lim_{n\to\infty} n\log\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\frac{\log n - \log(n+1)}{\frac{1}{n}}$$

and an easy application of L'Hôpital's rule shows that

$$\lim_{n\to\infty} n\log\left(\frac{n}{n+1}\right) = -1.$$

We can therefore regard the expression

$$\left(\frac{n}{n+1}\right)^{n\log n}$$

as being of the order of

$$\exp\left((-1)\log n\right) = \frac{1}{n}$$

for large values of n ***and this observation suggests how we can solve the problem. For each*** n ***we define***

$$a_n = \left(\frac{n}{n+1}\right)^{n\log n}$$

and

$$b_n = \frac{1}{n}$$

Now

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\exp\left((n\log n)\log\left(\frac{n}{n+1}\right)\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\exp\left((n\log n)\log\left(\frac{n}{n+1}\right)\right)}{\exp(-\log n)}$$

$$= \lim_{n\to\infty}\exp\left(\log n + (n\log n)\log\left(\frac{n}{n+1}\right)\right)$$

Now

$$\lim_{n\to\infty}\left(\log n + (n\log n)\log\left(\frac{n}{n+1}\right)\right) = \lim_{n\to\infty}\left((\log n)\left(1 + n\log\left(\frac{n}{n+1}\right)\right)\right)$$

$$= \lim_{n\to\infty}\frac{1 + n\log\left(\frac{n}{n+1}\right)}{\frac{1}{\log n}}$$

and, using L'Hôpital's rule twice, we can see that

$$\lim_{n\to\infty}\left(\log n + (n\log n)\log\left(\frac{n}{n+1}\right)\right) = 0$$

Thus

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\exp\left(\log n + (n\log n)\log\left(\frac{n}{n+1}\right)\right) = \exp 0 = 1$$

Since $\sum b_n$ ***is a divergent*** p***-series, it follows that*** $\sum a_n$ ***is divergent.***

9. $\sum\left(\frac{n}{n+1}\right)^{n(\log n)^2}$

Solution: ***We have already seen that***

$$\lim_{n\to\infty}\frac{\left(\frac{n}{n+1}\right)^{n(\log n)}}{\frac{1}{n}} = 1$$

and so for all sufficiently large n ***we have***

$$\left(\frac{n}{n+1}\right)^{n(\log n)} < \frac{e}{n}.$$

For all such n ***we have***

$$\left(\frac{n}{n+1}\right)^{n(\log n)^2} = \left(\left(\frac{n}{n+1}\right)^{n(\log n)}\right)^{\log n} < \left(\frac{e}{n}\right)^{\log n} < \frac{n}{n^{\log n}}$$

As long as n ***is also large enough to make*** $\log n > 3$ ***we have***

$$\left(\frac{n}{n+1}\right)^{n(\log n)^2} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since $\sum\frac{1}{n^2}$ ***is a convergent*** p***-series we deduce that*** $\sum\left(\frac{n}{n+1}\right)^{n(\log n)^2}$ ***is convergent.***

10. $\sum\left(\frac{1}{\log n}\right)^3$

Solution: ***As*** $n\to\infty$***, the expression*** $\log n$ ***increases much more slowly than*** n***. So we proceed as follows. For each*** n ***we define***

$$a_n = \left(\frac{1}{\log n}\right)^3$$

and

$$b_n = \frac{1}{n}.$$

Since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{(\log n)^3} = \infty$$

and since $\sum b_n$ ***is a divergent*** p***-series we deduce that*** $\sum a_n$ ***is divergent.***

11. $\sum\left(\frac{1}{\log n}\right)^n$

Solution: ***Since*** $\log n > 2$ ***whenever*** $n > e^2$***, the inequality***

$$\left(\frac{1}{\log n}\right)^n < \frac{1}{2^n}$$

holds whenever n ***is sufficiently large. Since*** $\sum 1/2^n$ ***is a convergent (geometric) series, we deduce that*** $\sum\left(\frac{1}{\log n}\right)^n$ ***is convergent.***

12. $\sum\left(\frac{1}{\log n}\right)^{\log n}$

Solution: ***Since*** $\log\log n > 2$ ***whenever*** $n > \exp(\exp 2)$ ***we know that the inequality***

$$(\log n)^{\log n} = \exp((\log n)(\log \log n)) > \exp(2 \log n) = n^2$$

holds whenever** n **is sufficiently large. Therefore

$$\left(\frac{1}{\log n}\right)^{\log n} < \frac{1}{n^2}$$

for all sufficiently large** n **and it follows from the fact that** $\sum 1/n^2$ **is a convergent** p**-series that the series** $\sum\left(\frac{1}{\log n}\right)^{\log n}$ **is convergent.

13. $\sum\left(\frac{1}{\log \log n}\right)^{\log n}$

Hint: Use the fact that** $\log \log \log n > 2$ **whenever** $n > \exp(\exp(\exp 2))$**.

Thus, for n sufficiently large we have

$$\left(\frac{1}{\log \log n}\right)^{\log n} = \frac{1}{\exp((\log n)(\log \log \log n))} < \frac{1}{\exp(2(\log n))} = \frac{1}{n^2}.$$

Since $\sum \frac{1}{n^2}$ is a convergent p-series it follows that the series $\sum\left(\frac{1}{\log \log n}\right)^{\log n}$ is convergent.

14. $\sum\left(\frac{1}{\log n}\right)^{\log \log n}$

Solution: ***Since***

$$\lim_{n\to\infty} \frac{(\log \log n)^2}{\log n} = 0$$

we know that the inequality** $(\log \log n)^2 < \log n$ **holds whenever** n **is sufficiently large. Therefore, if** n **is sufficiently large we have

$$(\log n)^{\log \log n} = \exp\left((\log \log n)^2\right) < \exp(\log n) = n$$

which gives us

$$\left(\frac{1}{\log n}\right)^{\log \log n} > \frac{1}{n}.$$

Since** $\sum 1/n$ **is a divergent** p**-series we conclude that** $\sum\left(\frac{1}{\log n}\right)^{\log \log n}$ **is divergent.

15. $\sum\left(\frac{1}{\log \log \log n}\right)^{\log n}$

For n sufficiently large we have $\log \log \log \log n > 2$ and for all such n we have

$$\left(\frac{1}{\log \log \log n}\right)^{\log n} = \frac{1}{\exp((\log n)(\log \log \log \log n))} < \frac{1}{\exp(2(\log n))} = \frac{1}{n^2}.$$

Since $\sum \frac{1}{n^2}$ is a convergent p-series it follows that the series $\sum\left(\frac{1}{\log \log n}\right)^{\log n}$ is convergent.

16. $\sum\left(\frac{1}{\log \log n}\right)^{\log \log n}$

Since $\log \log n < \log n$ for all n sufficiently large we know from Exe5rcise 14 that if n is sufficiently large then

$$\left(\frac{1}{\log \log n}\right)^{\log \log n} > \left(\frac{1}{\log n}\right)^{\log \log n} > \frac{1}{n}$$

and so $\sum\left(\frac{1}{\log \log n}\right)^{\log \log n}$ is divergent.

17. $\sum\left(\left(\frac{1}{\log\log n}\right)^{\log\log n}\right)^{\log\log n}$

Solution: ***We observe first that if*** n ***is sufficiently large then***

$$\left((\log\log n)^{\log\log n}\right)^{\log\log n} = (\log\log n)^{(\log\log n)^2} = \exp\left((\log\log n)^2(\log\log\log n)\right).$$

Since

$$\lim_{n\to\infty}\frac{\log\log\log n}{\log\log n} \to 0$$

as $n \to \infty$ ***we know that***

$$\log\log\log n < \log\log n$$

whenever n ***is sufficiently large and, therefore, that the inequality***

$$\left((\log\log n)^{\log\log n}\right)^{\log\log n} < \exp\left((\log\log n)^3\right)$$

must hold whenever n ***is sufficiently large. Now since***

$$\lim_{n\to\infty}\frac{(\log\log n)^3}{\log n} = 0$$

we know that

$$(\log\log n)^3 < \log n$$

whenever n ***is sufficiently large and, therefore, that the inequality***

$$\left((\log\log n)^{\log\log n}\right)^{\log\log n} < \exp\left((\log\log n)^3\right) < \exp(\log n) = n$$

must hold whenever n ***is sufficiently large.***

Therefore

$$\left(\left(\frac{1}{\log\log n}\right)^{\log\log n}\right)^{\log\log n} > \frac{1}{n}$$

whenever n ***is sufficiently large and, since*** $\sum 1/n$ ***is a divergent*** p***-series, we conclude that the series*** $\sum\left(\left(\frac{1}{\log\log n}\right)^{\log\log n}\right)^{\log\log n}$ ***is divergent.***

18. $\sum\left(\frac{1}{\log\log n}\right)^{\left((\log\log n)^{(\log\log n)}\right)}$

Hint: ***Whenever*** $n > \exp(\exp e)$ ***we have***

$$\begin{aligned}(\log\log n)^{(\log\log n)} &= \exp((\log\log n)(\log\log\log n))\\ &> \exp(\log\log n) = \log n.\end{aligned}$$

Now use the conclusion you obtained in Exercise 13.

19. $\sum(\sin\frac{x}{n})^{\alpha}$ where x and α are given positive numbers.

Solution: ***Since***

$$\lim_{t\to 0}\frac{\sin t}{t} = 1$$

we have

$$\lim_{n\to\infty}\frac{(\sin\frac{x}{n})^{\alpha}}{\frac{1}{n^{\alpha}}} = \lim_{n\to\infty}\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^{\alpha}x^{\alpha} = x^{\alpha}.$$

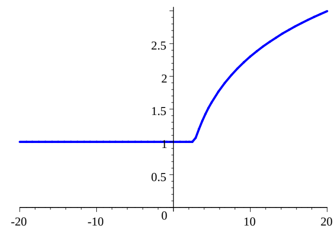
Comparing with the p***-series*** $\sum 1/x^{\alpha}$ ***we conclude that*** $\sum(\sin\frac{x}{n})^{\alpha}$ ***converges if*** $\alpha > 1$ ***and diverges if*** $\alpha \le 1$***.***

20. Prove that if (a_n) is a sequence of positive numbers and $\sum a_n$ converges then so does the series $\sum a_n^2$.
We assume that (a_n) is a sequence of positive numbers and that $\sum a_n$ is convergent. Since $a_n \to 0$ as $n \to \infty$ we have $a_n < 1$ for n sufficiently large and for all such n we have $a_n^2 < a_n$. Since $\sum a_n$ is convergent we deduce from the comparison test that $\sum a_n^2$ is convergent.

21. In this exercise we encounter a series that diverges very slowly. We begin by defining

$$L(x) = \begin{cases} \log x & \text{if} \quad x \geq e \\ 1 & \text{if} \quad x < e \end{cases}.$$

The graph of this function is illustrated in the following figure:



If k is any positive number we shall write L^k for the composition of the function L with itself k times. Thus if x is a given number then $L^3(x) = L(L(L(x)))$. We also define $L^0(x) = x$ for every x.

a. Given any number x, explain why we must have $L^k(x) = 1$ for all sufficiently large values of k. For a given positive integer n, give a simple meaning to the "infinite product"

$$\prod_{k=0}^{\infty} L^k(n) = L^0(n)L^1(n)L^2(n)\cdots.$$

If we define $f(x) = e^x - 2x$ for every number x then, since $f(1) = e - 2 > 0$ and $f'(x) > 0$ whenever $x \geq 1$ it follows that $f(x) > 0$ whenever $x \geq 1$.
(As a matter of fact it is easy to show that $f(x) > 0$ for every real number x.) For each positive integer k we define

$$a_k = \exp^k(1)$$

where $\exp^k$ stands for the composition of $\exp$ with itself k times. Since $a_k > 2^k$ for each k it follows that $a_k \to \infty$ as $k \to \infty$. Now if x is any positive number and k is a positive integer and $L^k(x) \neq 1$ then

$$L^k(x) = \log^k(x) > 1$$

and so $x > a_k$. Since the inequality $x < a_k$ must hold for sufficiently large k we deduce that $L^k(x) = 1$ for sufficiently large k. the meaning of the expression

$$\prod_{k=0}^{\infty} L^k(n)$$

is now obvious.

b. For each positive integer n, we define

$$a_n = \frac{1}{\prod_{k=0}^{\infty} L^k(n)}.$$

Using the integral test (or otherwise), show that the series $\sum a_n$ is divergent.
For every number $x \geq 1$ we define

$$g(x) = \frac{1}{\prod_{k=0}^{\infty} L^k(x)}.$$

The function g is a positive decreasing function on $[1, \infty)$. Now we observe that

$$\int_1^e g(x)dx = \int_1^e \frac{1}{x}dx = 1$$
$$\int_e^{e^e} g(x)dx = \int_e^{e^e} \frac{1}{x\log x}dx = 1$$
$$\int_{\exp^2(1)}^{\exp^3(1)} g(x)dx = \int_{\exp^2(1)}^{\exp^3(1)} \frac{1}{x\log x\log\log x}dx = 1$$

and, in general,

$$\int_{\exp^n(1)}^{\exp^{n+1}(1)} g(x)dx = 1$$

for each integer $n \geq 0$. Since

$$\int_1^{\exp^n(1)} g(x)dx \geq n$$

for each n we deduce that

$$\lim_{n\to\infty} \int_1^n g(x)dx = \infty$$

and so it follows from the integral test that the series $\sum a_n$ ia divergent.

Some Exercises on the Ratio Tests

Test the following series for convergence. In the event that the series diverges, determine whether or not its nth term approaches 0 as $n \to \infty$.

1. $\sum \frac{((2n)!)^3}{((3n)!)^2}$

Solution: ***We define***

$$a_n = \frac{((2n)!)^3}{((3n)!)^2}$$

for each n. For each n we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((2(n+1))!)^3}{((3(n+1))!)^2}}{\frac{((2n)!)^3}{((3n)!)^2}} = \frac{(2n+2)^3(2n+1)^3}{(3n+3)^2(3n+2)^2(3n+1)^2} \to \frac{64}{729}$$

as $n \to \infty$. We deduce from d'Alembert's test that $\sum a_n$ is convergent.

2. $\sum \frac{3^{(n^2)}}{n!}$

For each n we have

$$\frac{\frac{3^{((n+1)^2)}}{(n+1)!}}{\frac{3^{(n^2)}}{n!}} = \frac{3^{2n+1}}{n+1}$$

and since the latter expression approaches ∞ as $n \to \infty$, it follows from d'Alembert's test that $\sum \frac{3^{(n^2)}}{n!}$ is divergent.

3. $\sum \frac{3^{(n\log n)}}{n!}$

Solution: ***For each*** n ***we define***

$$a_n = \frac{3^{(n\log n)}}{n!}.$$

Since $e < 3$ ***we have the inequality***

$$a_n = \frac{3^{(n\log n)}}{n!} > \frac{e^{(n\log n)}}{n!} = \frac{n^n}{n!}$$

To show that $\sum a_n$ ***is divergent, it is sufficient to show that*** $\sum n^n/n!$ ***is divergent. But this fact follows at once from d'Alembert's test and the fact that***

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e > 1.$$

4. $\displaystyle\sum \frac{(\log n)^n}{e^n(\log 2)(\log 3)\cdots(\log n)}$

Solution: ***For each*** n ***we define***

$$a_n = \frac{(\log n)^n}{e^n(\log 2)(\log 3)\cdots(\log n)}$$

Then for each n ***we have***

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(\log(n+1))^n}{e(\log n)^n} \\ &= \frac{1}{e}\frac{\exp(n\log\log(n+1))}{\exp(n\log\log n)} = \frac{1}{e}\exp(n\log\log(n+1) - n\log\log n)\end{aligned}$$

Now for each n ***we observe that***

$$n\log\log(n+1) - n\log\log n = \frac{\log\log(n+1) - \log\log n}{\frac{1}{n}}$$

and by using L'Hôpital's rule we can deduce that

$$\lim_{n\to\infty}(n\log\log(n+1) - n\log\log n) = 0.$$

We deduce that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$

and it follows from d'Alembert's test that $\sum a_n$ ***diverges.***

5. $\displaystyle\sum\left(\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\right)^2$ where α is a given number

6. $\displaystyle\sum\left|\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\right|^p$ where α and p are given numbers.

Solution: ***For each*** n ***we define***

$$a_n = \left|\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\right|^p.$$

Then for each n ***we have***

$$\frac{a_{n+1}}{a_n} = \left|\frac{\alpha-n}{n+1}\right|^p$$

and for sufficiently large n ***we have***

$$n\left(1-\frac{a_{n+1}}{a_n}\right) = n\left(1-\frac{(n-\alpha)^p}{(n+1)^p}\right)$$
$$= n\left(\frac{(n+1)^p-(n-\alpha)^p}{(n+1)^p}\right)$$
$$= n\left(\frac{(n+1)^p-(n-\alpha)^p}{(n+1)-(n-\alpha)}\right)\left(\frac{(n+1)-(n-\alpha)}{(n+1)^p}\right)$$

Using the mean value theorem we choose a number u_n ***between*** $n-\alpha$ ***and*** $n+1$ ***such that***

$$\frac{(n+1)^p-(n-\alpha)^p}{(n+1)-(n-\alpha)} = p(u_n)^{p-1}.$$

Thus for each n ***we have***

$$n\left(1-\frac{a_{n+1}}{a_n}\right) = p(\alpha+1)\left(\frac{n}{n+1}\right)\left(\frac{u_n}{n+1}\right)^{p-1}$$

Since

$$\lim_{n\to\infty}\left(\frac{n-\alpha}{n+1}\right)^{p-1} = \lim_{n\to\infty}\left(\frac{n+1}{n+1}\right)^{p-1} = 1$$

we deduce from the sandwich theorem for limits of sequences that

$$\lim_{n\to\infty}\left(\frac{u_n}{n+1}\right)^{p-1} = 1$$

and we conclude that

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = p(1+\alpha).$$

We deduce from Raabe's test that

$$\sum\left|\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\right|^p$$

converges when $p(1+\alpha) > 1$ ***and diverges when*** $p(1+\alpha) < 1$***.***

7. a. $\sum \frac{n^\alpha}{n!}$ where α is a given number

This problem is an easy application of d'Alembert's test and is there to lead into parts b and c. Since

$$\lim_{n\to\infty}\frac{\frac{(n+1)^\alpha}{(n+1)!}}{\frac{n^\alpha}{n!}} = \lim_{n\to\infty}\frac{1}{n}\left(\frac{n+1}{n}\right)^{\alpha-1} = 0$$

the series $\sum \frac{n^\alpha}{n!}$ is convergent.

b. $\sum \frac{n^{\alpha n}}{n!}$ where α is a given number

For each n we have

$$\frac{\frac{(n+1)^{\alpha(n+1)}}{(n+1)!}}{\frac{n^{\alpha n}}{n!}} = (n+1)^{\alpha-1}\left(\left(\frac{n+1}{n}\right)^n\right)^\alpha$$

Since

$$\lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^n\right)^\alpha = e^\alpha$$

we see that if $\alpha > 1$ then

$$\lim_{n\to\infty}\left(\frac{\frac{(n+1)^{\alpha(n+1)}}{(n+1)!}}{\frac{n^{\alpha n}}{n!}}\right) = \infty$$

and $\sum \frac{n^{\alpha n}}{n!}$ diverges in this case. If $\alpha < 1$ then

$$\lim_{n\to\infty}\left(\frac{\frac{(n+1)^{\alpha(n+1)}}{(n+1)!}}{\frac{n^{\alpha n}}{n!}}\right)=0$$

and $\sum \frac{n^{\alpha n}}{n!}$ converges. In the event that $\alpha = 1$ we have

$$\lim_{n\to\infty}\left(\frac{\frac{(n+1)^{\alpha(n+1)}}{(n+1)!}}{\frac{n^{\alpha n}}{n!}}\right)=e>1$$

and the series $\sum \frac{n^{\alpha n}}{n!}$ diverges in this case.

c. $\sum \frac{n^{n-\log n}}{n!}$

For each n we have

$$\frac{\frac{(n+1)^{(n+1)-\log(n+1)}}{(n+1)!}}{\frac{n^{n-\log n}}{n!}}=\left(\frac{n+1}{n}\right)^n\left(\frac{n^{\log n}}{(n+1)^{\log(n+1)}}\right)$$

and we deduce from one of the parts of Exercise 1 of the exercises in indeterminate forms that

$$\lim_{n\to\infty}\left(\frac{\frac{(n+1)^{(n+1)-\log(n+1)}}{(n+1)!}}{\frac{n^{n-\log n}}{n!}}\right)=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\left(\frac{n^{\log n}}{(n+1)^{\log(n+1)}}\right)=e>1$$

from which we deduce that $\sum \frac{n^{n-\log n}}{n!}$ diverges.

One can actually go a step further and consider the series $\sum \frac{n^{n-\log n}}{e^n n!}$ which requires Raabe's test (Theorem 12.6.7). This series converges. In fact, we can do even better by replacing $\log n$ by a constant p giving us the series $\sum \frac{n^{n-p}}{e^n n!}$ which Raabe's test shows to be convergent when $p > \frac{1}{2}$ and divergent when $p < \frac{1}{2}$. If $p = \frac{1}{2}$ then the stronger form of Raabe's test (Theorem 12.6.9) shows the series to be divergent.

8. $\sum \frac{(2n)!}{4^n (n!)^2} x^n$ where x is a given positive number

A simple application of d'Alembert's test shows that this series converges if $x < 1$ and diverges if $x > 1$. When $x = 1$ the series reduces to the one we considered in Example 1 of Subsection 12.6.8.

9. $\sum \frac{n!}{x(x+1)(x+2)\cdots(x+n-1)}$ where x is a given positive number

If we define

$$a_n=\frac{n!}{x(x+1)(x+2)\cdots(x+n-1)}$$

for each n then for each n we have

$$\frac{a_{n+1}}{a_n}=\frac{\frac{(n+1)!}{x(x+1)(x+2)\cdots(x+n-1)}}{\frac{n!}{x(x+1)(x+2)\cdots(x+(n+1)-1)}}=\frac{n+1}{n+x}$$

Therefore

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty} n\left(1-\frac{n+1}{n+x}\right)=x-1$$

and we deduce from Raabe's test that $\sum a_n$ is convergent when $x > 1$ and divergent when $x < 1$.

When $x = 1$ we have $a_n = 1$ for all n and, of course, $\sum a_n$ diverges.

10. a. $\sum \frac{e^n n!}{n^n}$

This series is a special case of the series mentioned in the discussion that follows Exercise 7c above. If we define

$$a_n = \frac{e^n n!}{n^n}$$

for each n then

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1 - \frac{\frac{e^{n+1}(n+1)!}{(n+1)^{(n+1)}}}{\frac{e^n n!}{n^n}}\right)$$

which can be shown to be $-\frac{1}{2}$. We can deduce the divergence of $\sum a_n$ from Raabe's test but there is really no need for such big guns. Since the limit above is negative we know that

$$\frac{a_{n+1}}{a_n} > 1$$

for sufficiently large n and so a_n cannot approach 0 as $n \to \infty$.

b. $\sum \frac{n^n}{e^n n!}$

If we define

$$a_n = \frac{n^n}{e^n n!}$$

for each n then

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1 - \frac{\frac{(n+1)^{(n+1)}}{e^{n+1}(n+1)!}}{\frac{n^n}{e^n n!}}\right)$$

$$= \frac{1}{e}\lim_{n\to\infty} n\left(e - \left(1 + \frac{1}{n}\right)^n\right) = \frac{1}{2}$$

and so it follows from Raabe's test that $\sum a_n$ diverges.

11. $\sum \left(\frac{(2n)!}{4^n (n!)^2}\right)^p$ where p is a given number

We already know that this series diverges when $p = 1$ and from the comparison test it follows that the series also diverges when $p < 1$. From now on we suppose that $p > 1$. We define

$$a_n = \left(\frac{(2n)!}{4^n (n!)^2}\right)^p$$

for each n and observe that

$$\frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n+2}\right)^p$$

and so

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1 - \left(\frac{2n+1}{2n+2}\right)^p\right)$$

$$= \lim_{n\to\infty} n\left(\frac{(2n+2)^p - (2n+1)^p}{(2n+2)^p}\right)$$

To see that the latter limit is $\frac{p}{2}$, we use mean value theorem to chooce a number c_n between $2n+1$ and $2n+2$ such that

$$n\left(\frac{(2n+2)^p - (2n+1)^p}{(2n+2)^p}\right) = n\left(\frac{pc_n^{p-1}}{(2n+2)^p}\right)$$

and observe that

$$n\left(\frac{p(2n+1)^{p-1}}{(2n+2)^p}\right) \le n\left(\frac{pc_n^{p-1}}{(2n+2)^p}\right) \le n\left(\frac{p(2n+2)^{p-1}}{(2n+2)^p}\right).$$

We deduce from Raabe's test that the given series converges when $p > 2$ and diverges when $p < 2$.

We must now consider the case $p = 2$. In this case we shall use the more powerful form of Raabe test:

$$\lim_{n\to\infty} n(\log n)\left(1 - \frac{1}{n} - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n(\log n)\left(1 - \frac{1}{n} - \left(\frac{2n+1}{2n+2}\right)^2\right)$$

$$= \lim_{n\to\infty} n(\log n)\left(-\frac{5n+4}{4n(n+1)^2}\right) = 0$$

and we conclude that the series diverges when $p = 2$.

12. $\sum\left(\frac{\pi^{\frac{n}{2}}}{2^{n(n+1)}e^{\frac{n(n+3)}{2}}n^{11/12}}\right)\prod_{j=1}^{n}\frac{(2j+1)^{2j+\frac{1}{2}}}{(j^j)(j!)}$

Solution: ***Start off by supplying the definition***

$$a_n = \left(\frac{\pi^{\frac{n}{2}}}{2^{n(n+1)}e^{\frac{n(n+3)}{2}}n^{11/12}}\right)\prod_{j=1}^{n}\frac{(2j+1)^{2j+\frac{1}{2}}}{(j^j)(j!)}$$

to Scientific Notebook and make the selection that the subscript n ***is a function argument. You then have***

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{\pi^{\frac{(n+1)}{2}}}{2^{(n+1)((n+1)+1)}e^{\frac{(n+1)((n+1)+3)}{2}}(n+1)^{11/12}}\right)\prod_{j=1}^{(n+1)}\frac{(2j+1)^{2j+\frac{1}{2}}}{(j^j)(j!)}}{\left(\frac{\pi^{\frac{n}{2}}}{2^{n(n+1)}e^{\frac{n(n+3)}{2}}n^{11/12}}\right)\prod_{j=1}^{n}\frac{(2j+1)^{2j+\frac{1}{2}}}{(j^j)(j!)}}.$$

Do not ask Scientific Notebook to simplify this expression as it stands. First observe that

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{\pi^{\frac{(n+1)}{2}}}{2^{(n+1)((n+1)+1)}e^{\frac{(n+1)((n+1)+3)}{2}}(n+1)^{11/12}}\right)\frac{(2n+3)^{2n+2+\frac{1}{2}}}{\left((n+1)^{(n+1)}\right)((n+1)!)}}{\left(\frac{\pi^{\frac{n}{2}}}{2^{n(n+1)}e^{\frac{n(n+3)}{2}}n^{11/12}}\right)}$$

and then evaluate to obtain

$$\frac{a_{n+1}}{a_n} = 4^{-n-1}\sqrt{\pi}\,(n+1)^{-\frac{23}{12}-n}(4n^2+12n+9)^n\frac{\left(\sqrt{(2n+3)}\right)^5}{\Gamma(n+2)}\left(\sqrt[12]{n}\right)^{11}e^{-n-2}$$

and then supply the definition

$$g(n) = 4^{-n-1}\sqrt{\pi}\,(n+1)^{-\frac{23}{12}-n}(4n^2+12n+9)^n\frac{\left(\sqrt{(2n+3)}\right)^5}{\Gamma(n+2)}\left(\sqrt[12]{n}\right)^{11}e^{-n-2}$$

to Scientific Notebook. Thus

$$g(n) = \frac{a_{n+1}}{a_n}.$$

Finally you can use the Evaluate button to obtain

$$\lim_{n\to\infty} g(n) = 1$$

$$\lim_{n\to\infty} n(1 - g(n)) = 1$$

$$\lim_{n\to\infty} n(\log n)\left(1 - \frac{1}{n} - g(n)\right) = 0 < 1$$

and so the series diverges.

13. $\sum\left|\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}\right|$ where α and β are given numbers

Our understanding of this problem is that β is automatically prohibited from being 0 or any negative integer.
For each n we define

$$a_n = \left|\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}\right|$$

and we observe that

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1-\frac{|\alpha+n|}{|\beta+n|}\right)$$
$$= \lim_{n\to\infty} n\left(1-\frac{\alpha+n}{\beta+n}\right) = \beta-\alpha.$$

We deduce from Raabe's test that $\sum a_n$ converges if $\beta > \alpha+1$ and diverges if $\beta < \alpha+1$. In the event that $\beta = \alpha+1$ we have

$$a_n = \left|\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}\right| = \left|\frac{\alpha}{\alpha+n}\right|$$

Except for the trivial case $\alpha = 0$, the series $\sum\left|\frac{\alpha}{\alpha+n}\right|$ is divergent.

14. $\sum\left(\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}\right)^2$ where α and β are given numbers

Solution: ***For each*** n ***we define***

$$a_n = \left(\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}\right)^2$$

and we observe that

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1-\frac{(\alpha+n)^2}{(\beta+n)^2}\right) = 2(\beta-\alpha).$$

We deduce from Raabe's test that $\sum a_n$ ***is convergent when*** $\beta > \alpha+\frac{1}{2}$ ***and is divergent when*** $\beta < \alpha+\frac{1}{2}$***.***

Assume now that $\beta = \frac{1}{2}+\alpha$***. In this case we apply the*** more powerful form of ***Raabe test. Since***

$$\lim_{n\to\infty} n(\log n)\left(1-\frac{1}{n}-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n(\log n)\left(1-\frac{1}{n}-\frac{\left(\frac{1}{2}+\alpha+n\right)^2-(\alpha+n)^2}{\left(\frac{1}{2}+\alpha+n\right)^2}\right)$$
$$= \infty$$

we deduce that $\sum a_n$ ***is convergent in this case. Therefore*** $\sum\left(\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+n-1)}\right)^2$ ***is convergent when*** $\alpha \geq \beta+\frac{1}{2}$ ***and divergent when*** $\alpha < \beta+\frac{1}{2}$***.***

15. **Cauchy's root test** says that if (a_n) is a sequence of nonnegative numbers and if $\sqrt[n]{a_n} \to \alpha$ as $n\to\infty$ then the series $\sum a_n$ converges if $\alpha < 1$ and diverges if $\alpha > 1$.

   a. Prove Cauchy's root test.
   Suppose that $\sqrt[n]{a_n} \to \alpha$ as $n\to\infty$. In the event that $\alpha > 1$ we have $\sqrt[n]{a_n} > 1$ for all sufficiently large n which tells us that $a_n > 1$ for sufficiently large n. Therefore, in the case $\alpha > 1$ the series $\sum a_n$ is divergent.
   Now suppose that $\alpha < 1$. Choose a number p such that $\alpha < p < 1$. For all n sufficiently large we have $\sqrt[n]{a_n} < p$ and for all such n we have $a_n < p^n$. Since the series $\sum p^n$ is a convergent geometric series the comparison test guarantees that $\sum a_n$ is convergent.

   b. Review an earlier exercise and then prove that if Cauchy's root test can be used to test a given series for

convergence then so can d'Alembert's ratio test.
The desired assertion follows at once from that previous exercise which is where all the hard work lies.

16. Prove the following more powerful root test:
If $a_n \geq 0$ for all n and if

$$\frac{n}{\log n}\left(1-(a_n)^{1/n}\right) \to p$$

as $n \to \infty$, then the series $\sum a_n$ converges if $p > 1$ and diverges if $p < 1$.
This form of the root test is one of the results that are developed in the special document on ratio and root tests and that can be reached by clicking on the icon .

Solution: ***We suppose that $p > 1$ and that***

$$\frac{n}{\log n}\left(1-(a_n)^{1/n}\right) \to p$$

as $n \to \infty$. Choose a number q such that $1 < q < p$. We know that the inequality

$$\frac{n}{\log n}\left(1-(a_n)^{1/n}\right) > q$$

must hold for all sufficiently large n. For all such n we have

$$a_n < \left(1-\frac{q\log n}{n}\right)^n$$

From** the fact **that

$$\lim_{n\to\infty} n^q\left(1-\frac{q\log n}{n}\right)^n = 1$$

and from the fact that $\sum 1/n^q$ is a convergent p-series we deduce that the series $\sum\left(1-\frac{q\log n}{n}\right)^n$ is convergent; and it follows from the comparison test that $\sum a_n$ is convergent.

Now try the case $p < 1$.

Exercises on Conditionally Convergent Series

1. A common test for convergence that one encounters in an elementary calculus course is the **alternating series test**, sometimes known as the **Leibniz test** which says that if (a_n) is a decreasing sequence of positive numbers and if $a_n \to 0$ as $n \to \infty$ then the series $\sum(-1)^n a_n$ is convergent. Prove that the alternating series test follows at once from Dirichlet's test.
Since

$$\left|\sum_{j=1}^{n}(-1)^j\right| \leq 1$$

for every n, the Leibniz test follows at once from Dirichlet's test.

2. Given that (a_n) is a decreasing sequence of positive numbers and that for each n we have

$$b_n = 1-\frac{a_{n+1}}{a_n},$$

prove that the series $\sum(-1)^n a_n$ is convergent if and only if the series $\sum b_n$ is divergent.
If the series $\sum(-1)^n a_n$ is convergent then $(-1)^n a_n \to 0$ as $n \to \infty$ which implies that $a_n \to 0$ as $n \to \infty$. Since (a_n) is a decreasing sequence of positive numbers, it follows from the special case of Dirichlet's test discussed in Exercise 1 that $\sum(-1)^n a_n$ will converge if $a_n \to 0$ as $n \to \infty$. Now we know from Theorem 12.6.11 that the criterion for a_n to approach 0 as $n \to \infty$ is that $\sum b_n$ should diverge. Therefore $\sum(-1)^n a_n$ is convergent if and only if $\sum b_n$ is divergent.

3. Test the following series for convergence and for absolute convergence:

a. $\sum \frac{(-1)^n \log n}{n}$

If
$$f(x) = \frac{\log x}{x}$$
for $x > 0$ then for each $x > 0$ we have
$$f'(x) = \frac{1 - \log x}{x^2}$$
and we see easily that f is decreasing on the interval $[e, \infty)$. Since
$$\lim_{n\to\infty} \frac{\log n}{n} = 0,$$
the convergence of $\sum \frac{(-1)^n \log n}{n}$ follows at once from Dirichlet's test.

b. $\sum \frac{\sin(n\pi/4)}{n}$

Of course this series can be handled directly; but it is also a special case of the series $\sum \frac{\sin nx}{n}$ that was discussed in Example 2 of Subsection 12.7.9.

c. $\sum\left(\frac{1}{2} - 1\right)\left(\frac{1}{3} - 1\right)\cdots\left(\frac{1}{n} - 1\right)$

Solution: ***For each n we define***
$$a_n = \left(\frac{1}{2} - 1\right)\left(\frac{1}{3} - 1\right)\cdots\left(\frac{1}{n} - 1\right)$$
and observe that
$$a_n = (-1)^{n-1}\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right)$$
and that
$$|a_n| = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right).$$
Since the sequence $(|a_n|)$ ***is decreasing and since it follows from the divergence of*** $\sum \frac{1}{n}$ ***and*** *an earlier theorem* ***that*** $|a_n| \to 0$ ***as*** $n \to \infty$ ***we deduce that*** $\sum a_n$ ***is convergent. Now to show that*** $\sum a_n$ ***fails to converge abolutely we shall use the*** *more powerful* ***form of Raabe's test:***
$$\begin{aligned}\lim_{n\to\infty} n(\log n)\left(1 - \frac{1}{n} - \frac{|a_{n+1}|}{|a_n|}\right) &= \lim_{n\to\infty} n(\log n)\left(1 - \frac{1}{n} - \left(1 - \frac{1}{n+1}\right)\right)\\ &= \lim_{n\to\infty} n(\log n)\left(-\frac{1}{n(n+1)}\right) = 0 < 1\end{aligned}$$
and so the series $\sum |a_n|$ ***must be divergent.***

d. $\sum\left(\frac{1}{2^\delta} - 1\right)\left(\frac{1}{3^\delta} - 1\right)\cdots\left(\frac{1}{n^\delta} - 1\right)$ where $\delta > 0$.

Hint: ***The series converges absolutely when*** $0 \le \delta < 1$***, converges conditionally when*** $\delta = 1$ ***and diverges when*** $\delta > 1$***. Use Raabe's test when*** $\delta < 1$ ***and show that the*** n***th term fails to approach*** 0 ***as*** $n \to \infty$ ***when*** $\delta > 1$***.***

The case $0 \le \delta < 1$:

For each n we define
$$a_n = \left(\frac{1}{2^\delta} - 1\right)\left(\frac{1}{3^\delta} - 1\right)\cdots\left(\frac{1}{n^\delta} - 1\right)$$
and observe that
$$|a_n| = \left(1 - \frac{1}{2^\delta}\right)\left(1 - \frac{1}{3^\delta}\right)\cdots\left(1 - \frac{1}{n^\delta}\right)$$
If $\delta = 0$ then the series $\sum |a_n|$ is obviously convergent. Assume now that $0 < \delta < 1$. Since

$$\lim_{n\to\infty} n\left(1-\frac{|a_{n+1}|}{|a_n|}\right) = \lim_{n\to\infty} n\left(1-\frac{\left(1-\frac{1}{2^\delta}\right)\left(1-\frac{1}{3^\delta}\right)\cdots\left(1-\frac{1}{n^\delta}\right)\left(1-\frac{1}{(n+1)^\delta}\right)}{\left(1-\frac{1}{2^\delta}\right)\left(1-\frac{1}{3^\delta}\right)\cdots\left(1-\frac{1}{n^\delta}\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{(n+1)^\delta}\right) = \infty$$

the convergence of $\sum|a_n|$ follows from Raabe's test.

The case $\delta > 1$:

For each n we define

$$a_n = \left(\frac{1}{2^\delta}-1\right)\left(\frac{1}{3^\delta}-1\right)\cdots\left(\frac{1}{n^\delta}-1\right)$$

and observe that

$$|a_n| = \left(1-\frac{1}{2^\delta}\right)\left(1-\frac{1}{3^\delta}\right)\cdots\left(1-\frac{1}{n^\delta}\right)$$

it follows from an earlier theorem and the convergence of $\sum\frac{1}{n^\delta}$ that $|a_n|$ fails to approach 0 as $n\to\infty$.

e. $\sum\frac{(2\log 2-1)(3\log 3-1)\cdots(n\log n-1)}{(n!)(\log 2)(\log 3)\cdots(\log n)}$

For each n we define

$$a_n = \frac{(2\log 2-1)(3\log 3-1)\cdots(n\log n-1)}{(n!)(\log 2)(\log 3)\cdots(\log n)}$$

and we observe that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\log(n+1)-1}{(n+1)\log(n+1)}$$

and since

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1-\frac{(n+1)\log(n+1)-1}{(n+1)\log(n+1)}\right)$$

$$= \lim_{n\to\infty}\frac{n}{(n+1)\ln(n+1)} = 0 < 1$$

we conclude that $\sum a_n$ is divergent.

4. Determine for what values of x the following series converge and for what values of x the series converge absolutely.

a. $\sum\frac{(3x-2)^n}{n}$

The given series must be divergent if $|3x-2| > 1$ because, in this case, the expression $\frac{(3x-2)^n}{n}$ fails to approach 0 as $n\to\infty$. In the event that $|3x-2| < 1$, the series $\sum\frac{|3x-2|^n}{n}$ converges by a simple application of d'Alembert's test. We are left with the case $|3x-2| = 1$ which occurs when $x = \frac{1}{3}$ or $x = 1$. When $x = \frac{1}{3}$ the given series becomes $\sum\frac{(-1)^n}{n}$ which is conditionally convergent and when $x = 1$ it becomes $\sum\frac{1}{n}$ which is divergent.

Thus the series converges absolutely when $\frac{1}{3} < x < 1$, converges conditionally when $x = \frac{1}{3}$ and diverges otherwise.

b. $\sum\frac{(\log x)^n}{n}$

This series converges absolutely when $|\log x| < 1$, converges conditionally when $\log x = -1$ and diverges otherwise. In other words, the series converges absolutely when $\frac{1}{e} < x < e$, converges conditionally when $x = \frac{1}{e}$ and diverges otherwise.

c. $\sum \frac{(-1)^n x^n}{(\log n)^x}$

For each $n \geq 2$ we define

$$a_n = \frac{(-1)^n x^n}{(\log n)^x}$$

and observe that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x|\left(\frac{\log n}{\log(n+1)}\right)^x = |x|.$$

Therefore $\sum a_n$ converges absolutely when $|x| < 1$. If $|x| > 1$ then, since a_n fails to approach 0 as $n \to \infty$, the series $\sum a_n$ diverges. When $x = -1$ the series is $\sum \log n$ which diverges. When $x = 1$ the series is $\sum \frac{(-1)^n}{\log n}$ which is conditionally convergent.

d. $\sum \frac{((3n)!)x^n}{((2n)!)(n!)}$

For each n we define

$$a_n = \frac{((3n)!)x^n}{((2n)!)(n!)}$$

and observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = |x|\frac{(3n+3)(3n+2)(3n+1)}{(2n+2)(2n+1)(n+1)}$$

and so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{27|x|}{4}.$$

Therefore the series $\sum \frac{((3n)!)x^n}{((2n)!)(n!)}$ converges absolutely when $|x| < \frac{4}{27}$. If $|x| > \frac{4}{27}$ then, since a_n fails to approach 0 as $n \to \infty$, the series $\sum a_n$ diverges.

Now suppose $x = \frac{4}{27}$. The series $\sum a_n$ becomes

$$\sum \frac{((3n)!)4^n}{((2n)!)(n!)27^n}$$

and in this case

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1 - \frac{(3n+3)(3n+2)(3n+1)4}{(2n+2)(2n+1)(n+1)27}\right) = \frac{1}{2} < 1$$

Thus it follows from the criterion for the nth term to approach 0 that a_n decreases to 0 as $n \to \infty$. In spite of this fact, the series $\sum a_n$ is divergent.

Finally, suppose that $x = -\frac{4}{27}$. The series $\sum a_n$ becomes

$$\sum (-1)^n \frac{((3n)!)4^n}{((2n)!)(n!)27^n}$$

which converges by Dirichlet's test. Therefore the series $\sum \frac{((3n)!)x^n}{((2n)!)(n!)}$ converges absolutely when $|x| < \frac{4}{27}$, converges conditionally when $x = -\frac{4}{27}$ and diverges for other values of x.

e. $\sum \frac{n^n x^n}{n!}$

For each n we define

$$a_n = \frac{n^n x^n}{n!}$$

and observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = |x|\frac{\left(1+\frac{1}{n}\right)^n}{e}$$

and so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = e|x|.$$

Therefore the series $\sum \frac{n^n x^n}{n!}$ converges absolutely when $|x| < \frac{1}{e}$. If $|x| > \frac{1}{e}$ then, since a_n fails to approach 0 as $n \to \infty$, the series $\sum a_n$ diverges.

Now suppose that $x = \frac{1}{e}$. The series $\sum a_n$ becomes $\sum \frac{n^n}{e^n n!}$. Since

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1 - \frac{\frac{(n+1)^{n+1}}{e^{n+1}(n+1)!}}{\frac{n^n}{e^n n!}}\right) = \frac{1}{2}$$

we know from Raabe's test that $\sum a_n$ is divergent but we also know from the criterion for the nth term to approach 0 that a_n decreases to 0 as $n \to \infty$.

Finally suppose that $x = -\frac{1}{e}$. The series $\sum a_n$ becomes $\sum (-1)^n \frac{n^n}{e^n n!}$ which converges by Dirichlet's test.

Therefore the series $\sum \frac{n^n x^n}{n!}$ converges if $|x| < \frac{1}{e}$, converges conditionally if $x = -\frac{1}{e}$ and diverges for other values of x.

5. Find the values of x for which the following series converge and when they converge absolutely:

   a. $\sum \frac{\sin nx \cos nx}{n}$

   This series is just $\sum \frac{1}{2}\frac{\sin 2nx}{n}$ and so it follows from this earlier example that $\sum \frac{\sin nx \cos nx}{n}$ converges for every number x and converges conditionally unless x is an integer multiple of $\frac{\pi}{2}$.

   b. $\sum \frac{(-1)^n \cos nx}{n}$

   The series we are considering here can be expressed as

$$\sum \frac{\cos n(\pi + x)}{n}$$

   and from the case considered above we see that the series converges if and only if $\pi + x$ is not an integer multiple of 2π. In other words, the series converges if and only if x is not an odd multiple of π.

   c. $\sum \frac{\cos^2 nx}{n}$

   Hint: ***Use the identity***

$$\cos^2 nx = \frac{1}{2} + \frac{1}{2}\cos 2nx$$

   and show that the series is divergent for every number x.

   We see from the identity

$$\frac{1}{n} = \frac{2\cos^2 nx}{n} - \frac{\cos 2nx}{n}$$

   that, unless x is an integer multiple of π, the convergence of $\sum \frac{\cos 2nx}{n}$ and the divergence of $\sum \frac{1}{n}$ guarantee that the series

$$\sum \frac{\cos^2 nx}{n}$$

   is divergent. On the other hand, if x is an integer multiple of π then the given series reduces to $\sum \frac{1}{n}$ which diverges. Therefore the series

$$\sum \frac{\cos^2 nx}{n}$$

   diverges for every number x.

   d. $\sum \frac{|\cos nx|}{n}$

Since

$$0 \le \frac{\cos^2 nx}{n} \le \frac{|\cos nx|}{n}$$

for all n and x we deduce from Part c that the series

$$\sum \frac{|\cos nx|}{n}$$

diverges for every number x.

e. $\sum \frac{\cos^3 nx}{n}$

Hint: ***Use the identity***

$$\cos^3 nx = \frac{3}{4}\cos nx + \frac{1}{4}\cos 3nx.$$

If x is an integer multiple of 2π then the series

$$\sum \frac{\cos^3 nx}{n}$$

is

$$\sum \frac{1}{n}$$

which diverges. If $3x$ is an integer multiple of 2π but x is not then, since

$$\sum \frac{\cos nx}{n}$$

is convergent and

$$\sum \frac{\cos 3nx}{n}$$

is divergent, the series

$$\sum \frac{\cos^3 nx}{n} = \sum\left(\frac{3}{4}\frac{\cos nx}{n} + \frac{1}{4}\frac{\cos 3nx}{n}\right)$$

is divergent. If $3x$ is not an integer multiple of 2π then nor is x and, since both $\sum \frac{\cos nx}{n}$ and $\sum \frac{\cos 3nx}{n}$ are convergent, so is

$$\sum \frac{\cos^3 nx}{n}.$$

f. $\sum \frac{\cos^4 nx}{n}$

Given any number x positive integer n we have

$$\frac{\cos^4 nx}{n} = \frac{3}{8n} + \frac{1}{2}\frac{\cos 2nx}{n} + \frac{1}{8}\frac{\cos 4nx}{n}.$$

In the event that x is an integer multiple of $\pi/2$ then the series

$$\sum \frac{\cos^4 nx}{n}$$

is either

$$\sum \frac{1}{n} \quad \text{or} \quad \sum \frac{1}{2n}$$

and is therefore divergent. In the event that x is not an integer multiple of $\pi/2$, both of the series

$$\sum \frac{\cos 2nx}{n} \quad \text{and} \quad \sum \frac{\cos 4nx}{n}$$

are convergent and, since

$$\sum \frac{3}{8n}$$

is divergent, the series

$$\sum \frac{\cos^4 nx}{n}$$

is divergent. We conclude that the series

$$\sum \frac{\cos^4 nx}{n}$$

diverges for every number x.

6. With an eye on the preceding exercise give an example of a convergent series $\sum a_n$ such that the series $\sum a_n^3$ is divergent.
   The series
   $$\sum \frac{\cos \frac{2n\pi}{3}}{\sqrt[3]{n}}$$
   is convergent but the series
   $$\sum \left(\frac{\cos \frac{2n\pi}{3}}{\sqrt[3]{n}} \right)^3$$
   is divergent.

7. Find the values of x and α for which the **binomial series**
   $$\sum \binom{\alpha}{n} x^n$$
   is convergent.
   If α is a nonnegative integer then, since $\binom{\alpha}{n} = 0$ whenever $n > \alpha$ the series $\sum \binom{\alpha}{n} x^n$ converges for every number x. From now on we assume that α fails to be a nonnegative integer. Since
   $$\lim_{n\to\infty} \frac{\left|\binom{\alpha}{n+1} x^{n+1}\right|}{\left|\binom{\alpha}{n} x^n\right|} = |x|$$
   we know that the series
   $$\sum \binom{\alpha}{n} x^n$$
   is divergent whenever $|x| > 1$ and is absolutely convergent when $|x| < 1$. Now we need to consider the cases $x = \pm 1$.
   Now we saw earlier that the series
   $$\sum \binom{\alpha}{n}$$
   is absolutely convergent when $\alpha > 0$. We also saw that the nth term of this series fails to approach 0 as $n \to \infty$ when $\alpha \le -1$. Finally we saw that the series
   $$\sum \binom{\alpha}{n}$$
   is conditionally convergent when $-1 < \alpha < 0$ and that
   $$\binom{\alpha}{n} = (-1)^n \left|\binom{\alpha}{n}\right|$$
   for each α and n. We can summarize these facts as follows:

   - If α is a nonnegative integer then the given series converges for all x.
   - If α is not an integer and $\alpha > 0$ then the series converges absolutely when $|x| \le 1$ and diverges if $|x| > 1$.
   - If $\alpha \le -1$ then the series converges absolutely when $|x| < 1$ and diverges if $|x| \ge 1$.
   - If $-1 < \alpha < 0$ then the series converges absolutely when $|x| < 1$, converges conditionally when $x = 1$ and diverges when either $x \le -1$ or $x > 1$.

8. Prove that if x is not an integer multiple of 2π then
   $$\left| \sum_{j=1}^{\infty} \frac{\sin jx}{j} \right| \le \frac{1}{\left|\sin \frac{x}{2}\right|}$$

Hint: ***Use the inequality*** obtained after Dirichlet's test.

9. Prove **Abel's test** for convergence of a series which states that if (a_n) is a decreasing sequence of positive numbers and if $\sum b_n$ is a convergent series then the series $\sum a_n b_n$ is convergent. This theorem may be proved by the method of proof of Dirichlet's test but it also follows very simply from the statement of Dirichlet's test. Which proof do you prefer?
   We show the proof that uses the statement of Dirichlet's test. Assume that
$$\lim_{n\to\infty} a_n = a.$$
   From Dirichlet's test we know that
$$\sum (a_n - a) b_n$$
   is convergent. Therefore, since the series
$$\sum a b_n$$
   is convergent and
$$a_n b_n = (a_n - a) b_n + a b_n$$
   for each n, it follows that
$$\sum a_n b_n$$
   converges.

10. Give an example of a sequence (a_n) of positive numbers and a sequence (b_n) of real numbers such that each of the following conditions holds:
    a. We have $a_n \to 0$ as $n \to \infty$.
    b. The sequence of number $\sum_{j=1}^{n} b_j$ is bounded.
    c. The series $\sum a_n b_n$ is divergent.

    We define $b_n = (-1)^n$ for each n and we define
$$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ \frac{1}{2^n} & \text{if } n \text{ is odd} \end{cases}$$

11. Give an example of sequences (a_n) and (b_n) such that the following conditions hold:
    a. The sequence (a_n) is a decreasing sequence of positive numbers.
    b. The sequence of number $\sum_{j=1}^{n} b_j$ is bounded.
    c. The series $\sum a_n b_n$ is divergent.

    We define $b_n = (-1)^n$ and we define
$$a_n = 1 + \frac{1}{n}$$
    for each n.

Some Exercises on Products of Series

1. Calculate the Cauchy product of the series $\sum (-1)^n x^n$ and $\sum x^n$. By looking at the sums of these three series, verify that Cauchy's theorem is true for these series when $|x| < 1$.
   We know that

$$\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}$$

and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

The nth term of the Cauchy product of $\sum(-1)^n x^n$ and $\sum x^n$ is

$$\sum_{j=0}^{n}(-1)^{n-j}x^{n-j}x^j = \begin{cases} x^n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

and so the Cauch product is $\sum x^{2n}$. We observe that

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} = \left(\frac{1}{1+x}\right)\left(\frac{1}{1-x}\right) = \left(\sum_{n=0}^{\infty}(-1)^n x^n\right)\left(\sum_{n=0}^{\infty} x^n\right).$$

2. This exercises requires a knowledge of the binomial theorem. Show that the Cauchy product of the two series $\sum x^n/n!$ and $\sum y^n/n!$ is $\sum(x+y)^n/n!$. As you may know, the sums of these series are e^x and e^y and e^{x+y} respectively, and you will see this fact officially in a later subsection. What does Cauchy's theorem say for these three series?

   The nth term of the Cauchy product of $\sum x^n/n!$ and $\sum y^n/n!$ is

$$\sum_{j=0}^{n}\left(\frac{x^{n-j}}{(n-j)!}\right)\left(\frac{y^j}{j!}\right) = \sum_{j=0}^{n}(x^{n-j}y^j)\left(\frac{n!}{(n-j)!j!}\right)\left(\frac{1}{n!}\right)$$
$$= \left(\frac{1}{n!}\right)\sum_{j=0}^{n}(x^{n-j}y^j)\binom{n}{j} = \frac{(x+y)^n}{n!}$$

   Assuming that the three series $\sum x^n/n!$ and $\sum y^n/n!$ and $\sum(x+y)^n/n!$ converge, respectively, to e^x and e^y and e^{x+y} we can interpret Cauchy's theorem as saying that

$$e^{x+y} = (e^x)(e^y).$$

Some Exercises on Cauchy Products

1. Suppose that (f_n) is a sequence of functions defined on the set P of nonnegative integers and that f is a function defined on P. Suppose that the condition

$$\lim_{n\to\infty} f_n(j) = f(j)$$

   holds for every nonnegative integer j. Suppose that it is possible to find a sequence (α_n) of nonnegative numbers such that the series $\sum \alpha_n$ converges and such that

$$|f_n(j)| \le \alpha_j$$

   for all j and n. Prove that

$$\lim_{n\to\infty}\sum_{j=0}^{\infty} f_n(j) = \sum_{j=0}^{\infty} f(j).$$

   This exercise is a very slight variation of the proof of the technical lemma that precedes Mertens' theorem. The solution below is almost a carbon copy of the proof of that lemma.

   We remark first that for every j we have $|f(j)| \le \alpha_j$ and so the absolute convergence of the series $\sum f(j)$ and the series $\sum f_n(j)$ for any given positive integer n is guaranteed by the comparison test. In order to show that

$$\lim_{n\to\infty}\sum_{j=0}^{\infty} f_n(j) = \sum_{j=0}^{\infty} f(j),$$

suppose that $\varepsilon > 0$. Choose a positive integer N_1 such that

$$\sum_{j=N_1}^{\infty}\alpha_j < \frac{\varepsilon}{3}.$$

Now that N_1 has been chosen, choose a positive integer N_2 such that whenever $n \geq N_2$ we have

$$\left|\sum_{j=0}^{N_1} f_n(j) - \sum_{j=0}^{N_1} f(j)\right| < \frac{\varepsilon}{3}.$$

We define N to be the larger of the two numbers N_1 and N_2. Now we observe that whenever $n \geq N$ we have

$$\left|\sum_{j=0}^{\infty} f_n(j) - \sum_{j=0}^{\infty} f(j)\right| = \left|\sum_{j=0}^{N_1} f_n(j) - \sum_{j=0}^{N_1} f(j) + \sum_{j=N_1+1}^{\infty} f_n(j) - \sum_{j=N_1+1}^{\infty} f(j)\right|$$

$$\leq \left|\sum_{j=0}^{N_1} f_n(j) - \sum_{j=0}^{N_1} f(j)\right| + \left|\sum_{j=N_1+1}^{\infty} f_n(j)\right| + \left|\sum_{j=N_1+1}^{\infty} f(j)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

2. Prove that if $\alpha > 0$ and $\beta > 1$ then the Cauchy product of the series

$$\sum \frac{(-1)^n}{(n+1)^{\alpha}} \qquad \text{and} \qquad \sum \frac{(-1)^n}{(n+1)^{\beta}}$$

is convergent.
Since both series converge and the second series converges absolutely, the desired result follows at once from Mertens' theorem

3. Given that

$$S_n = \sum_{j=0}^{n} \frac{1}{j+1}$$

for each n, prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} S_n = \frac{1}{2}(\log 2)^2$$

Hint: ***Make use of an exercise proved in the document that provides a sharper form of the integral test. To reach the exercise,*** *click here*. ***If you have not read that integral test document, you can look for the same exercise when we study power series in*** *this chapter* ***of the text.***

4. Prove that if α and β are positive numbers and $\alpha + \beta \leq 1$ then the Cauchy product of the series

$$\sum \frac{(-1)^n}{(n+1)^{\alpha}} \qquad \text{and} \qquad \sum \frac{(-1)^n}{(n+1)^{\beta}}$$

diverges.
The nth term of this Cauchy product is

$$\sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j+1)^{\alpha}} \frac{(-1)^j}{(j+1)^{\beta}} = (-1)^n \sum_{j=0}^{n} \frac{1}{(n-j+1)^{\alpha}(j+1)^{\beta}}$$

and since

$$(n-j+1)^{\alpha}(j+1)^{\beta} \leq (n+1)^{\alpha}(n+1)^{\beta} = (n+1)^{\alpha+\beta} \leq n+1$$

whenever $0 \leq j \leq n$ we see that

$$\left|(-1)^n \sum_{j=0}^{n} \frac{1}{(n-j+1)^{\alpha}(j+1)^{\beta}}\right| \geq \sum_{j=0}^{n} \frac{1}{n+1} = 1.$$

Since the nth term of the Cauchy product does not approach 0 as $n \to \infty$, the Cauchy product diverges.

5. Suppose that $\sum c_n$ is the Cauchy product of two convergent series $\sum a_n$ and $\sum b_n$ and suppose that, for some number $\delta > 0$, the sequences $(na_n \log n)$ and $(n^{\delta} b_n)$ are bounded. Prove that the series $\sum c_n$ converges.

   Hint: ***Apply Neder's theorem with $\phi(n) = n^{\delta}$ for each n.***

Some Exercises on the Cantor Set

1. Prove that the Cantor set does not include any interval of positive length.
   Given any positive integer n, an interval that is included in the set E_n must have a length not exceeding $1/2^n$. Therefore any interval that is included in the Cantor set must have length less than $1/2^n$ for every positive integer n. In other words, such an interval must be a singleton.

2. Prove that every member of the Cantor set is a limit point of the Cantor set.
   Suppose that x belongs to the Cantor set C. We express x in the ternary decimal form
   $$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$
   where each number a_n is either 0 or 2. To show that x is a limit point of C, suppose that $\delta > 0$. Choose a positive integer k such that
   $$\frac{2}{3^k} < \delta.$$
   Given any positive integer n we define
   $$b_n = \begin{cases} a_n & \text{if } n \neq k \\ 2 - a_n & \text{if } n = k \end{cases}.$$
   The number
   $$y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$$
   belongs to C and is unequal to x and we have
   $$|x-y| \leq \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{3^n} = \frac{2}{3^k} < \delta.$$

3. True or false? If a number x in the interval $[0,1]$ can be expressed as a ternary decimal expansion of the form
   $$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$
   and if there exists a value of n for which $a_n = 1$ then the number x can't belong to the Cantor set.
   This statement is false. For example, we have seen that
   $$\frac{1}{3} = \sum_{n=2}^{\infty} \frac{2}{3^n} \in C.$$

4. At which numbers x is the Cantor function differentiable?
   Given any number $x \in [0,1] \setminus C$, since the Cantor function ϕ is constant in a neighborhood of x it must be differentiable at x and, in fact, $\phi'(x) = 0$. Now suppose that $x \in C$. We express x in the

ternary decimal form

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

where each number a_n is either 0 or 2. Now if for a given positive integer k we define

$$b_n = \begin{cases} a_n & \text{if } n \neq k \\ 2 - a_n & \text{if } n = k \end{cases}$$

and if we define the number y by the equation

$$y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$$

then y belongs to C and is unequal to x and we have

$$\frac{\phi(y) - \phi(x)}{y - x} = \frac{\frac{1}{2}\left(\frac{a_k}{2^k} - \frac{b_k}{2^k}\right)}{\frac{a_k}{3^k} - \frac{b_k}{3^k}} = \frac{1}{2}\left(\frac{3}{2}\right)^k.$$

Therefore, since the difference quotients of ϕ at x form an unbounded set, the limit

$$\lim_{y \to x} \frac{\phi(y) - \phi(x)}{y - x}$$

cannot exist (and be finite) and so ϕ fails to be differentiable at x whenever $x \in C$.

13 Improper Integrals

Some Exercises on Improper Integrals

1. Evaluate each of the following improper integrals, when possible, and specify those that diverge. If you can't see how to evaluate the integral exactly yourself, ask *Scientific Notebook* to evaluate it for you. (Before asking *Scientific Notebook* to evaluate one of these intgrals, remove the arrow sign from the limits of integration.)

a. $\int_0^{\to\infty} \frac{1}{(1+x^2)^{3/2}} dx$

The substitution $u = \arctan x$ gives us

$$\int_0^{\to\infty} \frac{1}{(1+x^2)^{3/2}} dx = \lim_{w\to\infty} \int_0^w \frac{1}{(1+x^2)^{3/2}} dx$$

$$= \lim_{w\to\infty} \int_0^{\arctan w} \frac{1}{(1+\tan^2 u)^{3/2}} \sec^2 u du = \lim_{w\to\infty} (\sin(\arctan w) - 0) = 1.$$

b. $\int_2^{\to\infty} \frac{1}{x\sqrt{x^2-1}} dx$

The substitution $u = \operatorname{arcsec} x$ gives us

$$\int_2^{\to\infty} \frac{1}{x\sqrt{x^2-1}} dx = \lim_{w\to\infty} \int_2^w \frac{1}{x\sqrt{x^2-1}} dx$$

$$= \lim_{w\to\infty} \int_{\operatorname{arcsec} 2}^{\operatorname{arcsec} w} \frac{1}{\sec u \tan u} \sec u \tan u du = \frac{\pi}{2} - \operatorname{arcsec} 2.$$

c. $\int_{1\leftarrow}^{2} \frac{1}{x\sqrt{x^2-1}} dx$

The substitution $u = \operatorname{arcsec} x$ gives us

$$\int_{1\leftarrow}^{2} \frac{1}{x\sqrt{x^2-1}} dx = \lim_{w\to 1+} \int_{w}^{2} \frac{1}{x\sqrt{x^2-1}} dx$$
$$= \lim_{w\to 1+} \int_{\operatorname{arcsec} w}^{\operatorname{arcsec} 2} 1 du = \operatorname{arcsec} 2 - 1$$

d. $\int_{1\leftarrow}^{\to\infty} \frac{1}{x\sqrt{x^2-1}} dx$

This integral is defined to be

$$\int_{1\leftarrow}^{2} \frac{1}{x\sqrt{x^2-1}} dx + \int_{2}^{\to\infty} \frac{1}{x\sqrt{x^2-1}} dx = \frac{\pi}{2} - 1.$$

e. $\int_{0}^{\to\pi/2} \tan x dx$

This integral is

$$\lim_{w\to\pi/2-} \int_{0}^{w} \tan x dx = \lim_{w\to\pi/2-} (\log \sec w) = \infty$$

which means that the given integral diverges.

f. $\int_{0}^{\to\pi/2} \sqrt{\tan x \sin x}\, dx$

This integral is

$$\lim_{w\to\pi/2-} \int_{0}^{w} \frac{\sin x}{\sqrt{\cos x}} dx = \lim_{w\to\pi/2-} (2 - 2\sqrt{\cos w}\,) = 2.$$

g. $\int_{0\leftarrow}^{\pi/2} \frac{x\cos x - \sin x}{x^2} dx$

This integral is

$$\lim_{w\to 0+} \int_{w}^{\pi/2} \frac{x\cos x - \sin x}{x^2} dx = \lim_{w\to 0+} \left(\frac{\sin \pi/2}{\pi/2} - \frac{\sin w}{w} \right) = \frac{2}{\pi} - 1$$

h. $\int_{1}^{\to\infty} e^{-x} \sin x dx$

This integral is

$$\lim_{w\to\infty} \int_{1}^{w} e^{-x} \sin x dx = \lim_{w\to\infty} \left(\left(-\frac{1}{2} e^{-w} \cos w - \frac{1}{2} e^{-w} \sin w \right) - \left(-\frac{1}{2} e^{-1} \cos 1 - \frac{1}{2} e^{-1} \sin 1 \right) \right)$$
$$= \frac{\cos 1}{2e} + \frac{\sin 1}{2e}$$

2. a. Prove that the integral

$$\int_{0\leftarrow}^{1} \frac{1}{x^p} dx$$

converges when $p < 1$ and diverges when $p \geq 1$.

As long as $p \neq 1$ we have

$$\int_{0\leftarrow}^{1} \frac{1}{x^p} dx = \lim_{w\to 0-} \int_{w}^{1} \frac{1}{x^p} dx = \lim_{w\to 0-} \left(\frac{1}{1-p} - \frac{1}{1-p} w^{1-p} \right).$$

This limit is $\frac{1}{1-p}$ if $p < 1$ and is ∞ if $p > 1$. We see also that

$$\int_{0\leftarrow}^{1} \frac{1}{x} dx = \lim_{w\to 0-} (\log 1 - \log w) = \infty.$$

b. Prove that the integral

$$\int_1^{\to\infty} \frac{1}{x^p}\,dx$$

converges when $p > 1$ and diverges when $p \le 1$.

As long as $p \neq 1$ we have

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{w\to\infty} \int_1^{w} \frac{1}{x^p}\,dx = \lim_{w\to\infty}\left(\frac{1}{1-p} w^{1-p} - \frac{1}{1-p} \right).$$

This limit is $\frac{1}{p-1}$ if $p > 1$ and is ∞ if $p < 1$. We see also that

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{w\to\infty}(\log w - \log 1) = \infty.$$

3. Prove that the integral

$$\int_2^{\to\infty} \frac{1}{x(\log x)^p}\,dx$$

is convergent when $p > 1$ and divergent when $p \le 1$.

As long as $p \neq 1$ we have

$$\int_2^{\to\infty} \frac{1}{x(\log x)^p}\,dx = \lim_{w\to\infty} \int_2^{w} \frac{1}{x(\log x)^p}\,dx$$
$$= \lim_{w\to\infty}\left(\frac{1}{1-p}(\log w)^{1-p} - \frac{1}{1-p}(\log 2)^{1-p} \right)$$

This limit is $\frac{1}{p-1}(\log 2)^{1-p}$ when $p > 1$ and is ∞ when $p < 1$.

We see also that

$$\int_2^{\infty} \frac{1}{x\log x}\,dx = \lim_{w\to\infty}(\log\log w - \log\log 2) = \infty.$$

4. Interpret the integral

$$\int_0^{2} \frac{1}{(x-1)^{1/3}}\,dx$$

as the sum of two improper integrals and evaluate it.

The two integrals are $\int_0^{\to 1} \frac{1}{(x-1)^{1/3}}\,dx$ and $\int_{1\leftarrow}^{2} \frac{1}{(x-1)^{1/3}}\,dx$ and can easily be seen to be 0.

5. Prove that if f is bounded on an interval $[a,b]$ and is improper Riemann integrable on $[a,b)$ then f is Riemann integrable on $[a,b]$ and

$$\int_a^{\to b} f = \int_a^{b} f.$$

To show that f satisfies the second criterion for integrability, suppose that $\varepsilon > 0$. Using the fact that f is integrable on the interval $\left[a, b - \frac{\varepsilon}{2}\right]$ choose a partition

$$\mathbf{P} = (x_0, x_1, \cdots, x_n)$$

of the interval $\left[a, b - \frac{\varepsilon}{2}\right]$ such that if

$$E = \left\{ x \in \left[a, b - \frac{\varepsilon}{2}\right] \;\middle|\; w(\mathbf{P}, f)(x) \ge \frac{\varepsilon}{2} \right\}$$

then $m(E) < \frac{\varepsilon}{2}$. Now we extend the partition $\mathbf{P}$ to make a partition $\mathbf{Q}$ of the interval $[a,b]$ by adding the point b. Thus

$$\mathbf{Q} = (x_0, x_1, \cdots, x_n, b)$$

and since

$$\left\{ x \in \left[a, b - \frac{\varepsilon}{2}\right] \;\middle|\; w(\mathbf{Q}, f)(x) \ge \frac{\varepsilon}{2} \right\} \subseteq E \cup \left[b - \frac{\varepsilon}{2}, b\right]$$

the measure of the latter set is less than ε.

Now to show that

$$\lim_{w\to b-}\int_a^w f = \int_a^b f$$

we need only note that if $a \leq w < b$ then

$$\left|\int_a^b f - \int_a^w f\right| = \left|\int_w^b f\right| \leq (\sup|f|)(b-w)$$

and that the latter expression approaches 0 as $w \to b$.

6. In the discussion of improper integrals that appears above we used the words

... Somewhat less precisely, we also say that the integral $\int_a^{\to b} f(x)dx$ is convergent. ...

Why is the statement that the integral $\int_a^{\to b} f(x)dx$ is convergent less precise than the statement that f is improper Riemann integrable on $[a,b)$? Hint: In our study of infinite series we made a careful distinction between the symbols $\sum a_n$ and $\sum_{n=1}^{\infty} a_n$.
The symbol $\int_a^{\to b} f(x)dx$ stand for the limit

$$\lim_{w\to b-}\int_a^w f(x)dx$$

which is a number. When we say that $\int_a^{\to b} f(x)dx$ is convergent we do not really mean what we are saying. Literally, the statement that $\int_a^{\to b} f(x)dx$ is convergent is the statement that the value of the limit

$$\lim_{w\to b-}\int_a^w f(x)dx$$

is convergent. Thus, if the limit is 2 then we should be saying that the number 2 is convergent. Of course such an interpretation is not intended. Instead, we mean that the limit exists and is finite. In that sense, saying that $\int_a^{\to b} f(x)dx$ is convergent is not as precise as saying that f is improper Riemann integrable on $[a,b)$.

Some Exercises on the Comparison Test for Integrals

1. Determine the convergence or divergence of the following integrals:

   a. $\displaystyle\int_1^{\to\infty} \frac{\sqrt{x}}{x^2-x+1}dx$

   Since

   $$\lim_{x\to\infty}\frac{\frac{\sqrt{x}}{x^2-x+1}}{\frac{1}{x^{3/2}}} = 1$$

   and since $\int_1^{\to\infty} \frac{1}{x^{3/2}}dx$ is convergent, the integral $\displaystyle\int_1^{\to\infty} \frac{\sqrt{x}}{x^2-x+1}dx$ is convergent.

   b. $\displaystyle\int_{0\leftarrow}^{1} \frac{1}{x+x^2}dx$

   Since

   $$\lim_{x\to 0+}\frac{\frac{1}{x+x^2}}{\frac{1}{x}} = 1$$

   and since $\int_{0\leftarrow}^{1} \frac{1}{x}dx$ is divergent, the integral $\displaystyle\int_{0\leftarrow}^{1} \frac{1}{x+x^2}dx$ is divergent.

   c. $\displaystyle\int_1^{\to\infty} \frac{\sin^2 x}{x^2}dx$

Since

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$$

for all $x \ge 1$ and since the integral $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent, the integral $\int_1^{\to\infty} \frac{\sin^2 x}{x^2} dx$ is convergent.

d. $\int_{0\leftarrow}^{\to\infty} \frac{\sin^2 x}{x^2\sqrt{x}} dx$

We break thisintegral into the two parts $\int_{0\leftarrow}^{1} \frac{\sin^2 x}{x^2\sqrt{x}} dx$and $\int_{1}^{\to\infty} \frac{\sin^2 x}{x^2\sqrt{x}} dx$.

Since

$$\lim_{x\to 0+} \frac{\frac{\sin^2 x}{x^2\sqrt{x}}}{\frac{1}{\sqrt{x}}} = 1$$

and since the integral $\int_{0\leftarrow}^{1} \frac{1}{\sqrt{x}} dx$ is convergent, the integral $\int_{0\leftarrow}^{1} \frac{\sin^2 x}{x^2\sqrt{x}} dx$ is convergent.

Since

$$0 \le \frac{\sin^2 x}{x^2\sqrt{x}} \le \frac{1}{x^{5/2}}$$

whenever $x \ge 1$ and since the integral $\int_1^{\to\infty} \frac{1}{x^{5/2}} dx$ is convergent, the integral $\int_1^{\to\infty} \frac{\sin^2 x}{x^2\sqrt{x}} dx$ is convergent. Therefore the given integral $\int_{0\leftarrow}^{\to\infty} \frac{\sin^2 x}{x^2\sqrt{x}} dx$ is convergent.

e. $\int_0^{\to\pi/2} \sqrt{\tan x}\, dx$

From the fact that

$$\lim_{x\to\pi/2-} \frac{\cos x}{\frac{\pi}{2} - x} = 1$$

we see that

$$\lim_{x\to\pi/2-} \frac{\sqrt{\tan x}}{\sqrt{\frac{\pi}{2} - x}} = 1$$

and since the integral $\int_0^{\to\pi/2} \frac{1}{\sqrt{\frac{\pi}{2} - x}} dx$ is convergent, the integral $\int_0^{\to\pi/2} \sqrt{\tan x}\, dx$ is convergent.

Alternatively, we could have evaluated this integral directly. It is an elementary integral that can be evaluated by making the substitution $u = \sqrt{\tan x}$ and then breaking the integral down with partial fractions.

f. $\int_{1\leftarrow}^{2} \frac{1}{\log x} dx$

Since

$$\lim_{x\to 1+} \frac{\frac{1}{\log x}}{\frac{1}{x-1}} = 1$$

and since theintegral $\int_{1\leftarrow}^{2} \frac{1}{x-1} dx$ is divergent, the integral $\int_{1\leftarrow}^{2} \frac{1}{\log x} dx$ is divergent.

g. $\int_{0\leftarrow}^{\pi/2} \log(\sin x) dx$

We observe that $\log \sin x \le 0$for each $x \in \left(0, \frac{\pi}{2}\right]$ and so the comparision test can't be applied to this integral as it stands. On the other hand, we can apply the comparison test to the integral $\int_{0\leftarrow}^{\pi/2} (-\log \sin x) dx$. Since

$$\lim_{x\to 0} \frac{-\log \sin x}{\frac{1}{\sqrt{x}}} = 0$$

and since the integral $\int_{0\leftarrow}^{1} \frac{1}{\sqrt{x}} dx$ is convergent, the integral $\int_{0\leftarrow}^{\pi/2} (-\log \sin x) dx$ is convergent and therefore the integral $\int_{0\leftarrow}^{\pi/2} \log \sin x dx$ is convergent.

h. $\int_{2}^{\to\infty} \frac{1}{(\log x)^{\log x}} dx$

Hint: ***This problem is the integral analogue of the problems that occurred*** *earlier* ***on the comparison test for series.***

For x sufficiently large we have

$$\frac{1}{(\log x)^{\log x}} = \frac{1}{\exp((\log x)(\log \log x))} \le \frac{1}{\exp(2\log x)} = \frac{1}{x^2}.$$

Since the integral $\int_{2}^{\to\infty} \frac{1}{x^2} dx$ is convergent, the integral $\int_{2}^{\to\infty} \frac{1}{(\log x)^{\log x}} dx$ is convergent.

i. $\int_{3}^{\to\infty} \frac{1}{(\log \log x)^{\log x}} dx$

For x sufficiently large we have

$$\frac{1}{(\log \log x)^{\log x}} = \frac{1}{\exp((\log x)(\log \log \log x))} \le \frac{1}{\exp(2\log x)} = \frac{1}{x^2}.$$

Since the integral $\int_{2}^{\to\infty} \frac{1}{x^2} dx$ is convergent, the integral $\int_{2}^{\to\infty} \frac{1}{(\log \log x)^{\log x}} dx$ is convergent.

j. $\int_{30}^{\to\infty} \frac{1}{(\log \log \log x)^{\log x}} dx$

For x sufficiently large we have

$$\frac{1}{(\log \log \log x)^{\log x}} = \frac{1}{\exp((\log x)(\log \log \log \log x))} \le \frac{1}{\exp(2\log x)} = \frac{1}{x^2}.$$

Since the integral $\int_{2}^{\to\infty} \frac{1}{x^2} dx$ is convergent, the integral $\int_{2}^{\to\infty} \frac{1}{(\log \log \log x)^{\log x}} dx$ is convergent.

k. $\int_{3}^{\to\infty} \frac{1}{(\log x)^{\log \log x}} dx$

Since

$$\lim_{x\to\infty} \frac{(\log \log x)^2}{\log x} = 0$$

we know that $(\log \log x)^2 \le \frac{1}{2} \log x$ whenever x is sufficiently large. For such x we have

$$\frac{1}{(\log x)^{\log \log x}} = \frac{1}{\exp\left((\log \log x)^2\right)} \ge \frac{1}{\exp\left(\frac{1}{2} \log x\right)} = \frac{1}{\sqrt{x}}.$$

Since the integral $\int_{3}^{\to\infty} \frac{1}{\sqrt{x}} dx$ is divergent, the integral $\int_{3}^{\to\infty} \frac{1}{(\log x)^{\log \log x}} dx$ is divergent.

l. $\int_{1}^{\to\infty} \frac{1}{\exp\left(\sqrt{\log x}\right)} dx$

Since

$$\sqrt{\log x} \le \frac{1}{2} \log x$$

whenever x is sufficiently large we have

$$\frac{1}{\exp\left(\sqrt{\log x}\right)} \ge \frac{1}{\exp\left(\frac{1}{2} \log x\right)} = \frac{1}{\sqrt{x}}$$

for x sufficiently large. Therefore, since the integral $\int_1^{\to\infty} \frac{1}{\sqrt{x}}dx$ is divergent, the integral $\int_1^{\to\infty} \frac{1}{\exp\left(\sqrt{\log x}\right)}dx$ is divergent.

2. a. Prove that the integral

$$\int_1^{\to\infty} x^{\alpha-1}e^{-x}dx$$

converges for every number α.

The convergence of this integral follows at once from the convergence of the integral $\int_1^{\to\infty} \frac{1}{x^2}dx$ and the fact that

$$\lim_{x\to\infty} \frac{x^{\alpha-1}e^{-x}}{\frac{1}{x^2}} = 0.$$

b. Prove that the integral

$$\int_{0\leftarrow}^{1} x^{\alpha-1}e^{-x}dx$$

converges if and only if $\alpha > 0$ and deduce that the integral

$$\int_{0\leftarrow}^{\to\infty} x^{\alpha-1}e^{-x}dx$$

converges if and only if $\alpha > 0$. The latter integral defines the value at α of the **gamma function** and is denoted as $\Gamma(\alpha)$. You will see more about this important function later.

The convergence of the integral $\int_{0\leftarrow}^{1} x^{\alpha-1}e^{-x}dx$ if and only if $\alpha > 0$ follows at once from the corresponding fact about the integral $\int_{0\leftarrow}^{1} x^{\alpha-1}dx$ and the fact that

$$\lim_{x\to 0} \frac{x^{\alpha-1}e^{-x}}{x^{\alpha-1}} = 1.$$

3. Prove that the integral

$$\int_{0\leftarrow}^{\to 1}(1-t)^{\alpha-1}t^{\beta-1}dt$$

converges if and only if both α and β are positive. This integral defines the value at the point (α, β) of the **beta function** and is denoted as $B(\alpha, \beta)$.

Some Further Exercises on Improper Integrals

1. Determine the convergence or divergence of the following integrals:

a. $\int_0^{\to\infty} \frac{\sin x}{\sqrt{x}}dx$

This integral converges by Diriclet's test. We can use exactly the same argument that was used in Subsection 13.4.7.

b. $\int_2^{\to\infty} \frac{\sin^3 x}{x}dx$

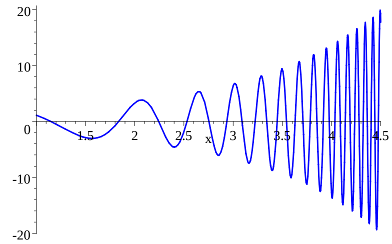
In order to use Dirichlet's test for this integral we observe that if $w \geq 2$ then

$$\left|\int_2^w \sin^3 x dx\right| = \left|-\cos w + \frac{1}{3}\cos^3 w + \cos 2 - \frac{1}{3}\cos^3 2\right| \leq \frac{8}{3}.$$

In this way we see that the integral $\int_2^{\to\infty} \frac{\sin^3 x}{x}dx$ is convergent.

c. $\int_1^{\to\infty} \frac{e^x \sin(e^x)}{x}dx$

It is instructive to ask ***Scientific Notebook*** to sketch the graph $y = \frac{e^x \sin(e^x)}{x}$.



As we move from left to right, the function oscillates with a rapidly expanding amplitude but the peaks, as they become higher, also become very narrow and, as we shall see in a moment, the integral $\int_1^{\to\infty} \frac{e^x \sin(e^x)}{x} dx$ is convergent.

Given any number $w \geq 1$ we have

$$\left| \int_1^w e^x \sin e^x dx \right| = \left| \int_e^{e^w} \sin u du \right| \leq 2$$

and therefore Dirichlet's test guarantees that the integral $\int_1^{\to\infty} \frac{e^x \sin(e^x)}{x} dx$ is convergent.

2. Integrate by parts to obtain the identity

$$\int_0^w \frac{\sin^2 x}{x^2} dx = -\frac{\sin^2 w}{w} + \int_0^w \frac{2 \sin x \cos x}{x} dx$$

and deduce that each of the following four improper integrals equals

$$\int_0^{\to\infty} \frac{\sin x}{x} dx.$$

a. $\int_0^{\to\infty} \frac{2 \sin x \cos x}{x} dx$

$$\begin{aligned} \int_0^{\to\infty} \frac{2 \sin x \cos x}{x} dx &= 2 \int_0^{\to\infty} \frac{\sin 2x}{2x} dx \\ &= 2 \lim_{w \to \infty} \int_0^w \frac{\sin 2x}{2x} dx \end{aligned}$$

and the substitution $u = 2x$ yields

$$\lim_{w \to \infty} \int_0^{2w} \frac{\sin u}{u} du = \int_0^{\to\infty} \frac{\sin x}{x} dx.$$

b. $\int_0^{\to\infty} \frac{\sin^2 x}{x^2} dx$

$$\lim_{w \to \infty} \int_0^w \frac{2 \sin x \cos x}{x} dx = \lim_{w \to \infty} \int_0^w \frac{\sin^2 x}{x^2} dx + \lim_{w \to \infty} \frac{\sin^2 w}{w} = \int_0^{\to\infty} \frac{\sin^2 x}{x^2} dx.$$

c. $\int_0^{\to\infty} \frac{2 \sin^2 x \cos^2 x}{x^2} dx$

$$\begin{aligned} \int_0^{\to\infty} \frac{2 \sin^2 x \cos^2 x}{x^2} dx &= \lim_{w \to \infty} \int_0^w \frac{2 \sin^2 x \cos^2 x}{x^2} dx \\ &= \lim_{w \to \infty} 2 \int_0^w \frac{\sin^2 2x}{(2x)^2} dx \\ &= \lim_{w \to \infty} \int_0^{2w} \frac{\sin^2 u}{u^2} du = \int_0^{\to\infty} \frac{\sin^2 x}{x^2} dx. \end{aligned}$$

d. $\int_0^{\to\infty} \frac{2 \sin^4 x}{x^2} dx$

$$\int_0^{\to\infty} \frac{2\sin^4 x}{x^2}dx = \int_0^{\to\infty} \frac{2\sin^2 x(1-\cos^2 x)}{x^2}dx$$
$$= \int_0^{\to\infty} \frac{2\sin^2 x}{x^2}dx - \int_0^{\to\infty} \frac{2\sin^2 x\cos^2 x}{x^2}dx$$
$$= \int_0^{\to\infty} \frac{2\sin^2 x}{x^2}dx - \int_0^{\to\infty} \frac{\sin^2 x}{x^2}dx = \int_0^{\to\infty} \frac{\sin^2 x}{x^2}dx.$$

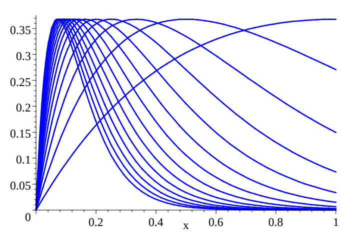
14 Sequences and Series of Functions

Exercises on Convergence of Sequences of Functions

1. For each of the following definitions of the function f_n on the interval $[0,1]$ prove that the sequence (f_n) converges pointwise to the function 0 on $[0,1]$ and determine whether the sequence converges boundedly and whether it converges uniformly. In each case, determine whether or not we have
$$\lim_{n\to\infty}\int_0^1 f_n = 0.$$
In each case, use *Scientific Notebook* to sketch some graphs of the given function and ask yourself whether your conclusion is compatible with what you see in the graphs.

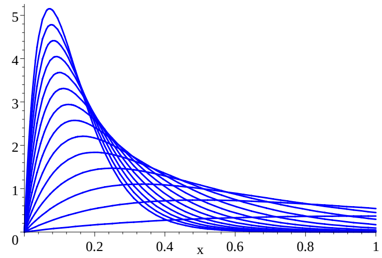
a. $f_n(x) = nx\exp(-nx)$ for each $x \in [0,1]$ and each positive integer n.

Hint: ***To see that the sequence converges boundedly but not uniformly to the constant 0, observe that each function f_n has its maximum when***
$$n\exp(-nx) - n^2x\exp(-nx) = 0$$
which occurs when $x = 1/n$. Note that $f_n(1/n) = 1/e$ for each n.



When $x = 0$ the condition
$$\lim_{n\to\infty} nx\exp(-nx) = 0$$
is obvious and when $x \neq 0$ the condition
$$\lim_{n\to\infty} nx\exp(-nx) = 0$$
follows from the fact that
$$\lim_{u\to\infty}\frac{u}{e^u} = 0.$$
Therefore the sequence (f_n) converges pointwise and boundedly but not uniformly to the constant function 0.
Finally we observe that
$$\int_0^1 f_n(x)dx = \int_0^1 nxe^{-nx}dx = \frac{1}{n} - \frac{1}{e^n} - \frac{1}{ne^n} \to 0$$
as $n \to \infty$.

b. $f_n(x) = n^2 x \exp(-nx)$ for each $x \in [0,1]$ and each positive integer n.



Arguing as we did in Part a, we see that each function f_n has its maximum value $\frac{n}{e}$ at the number $\frac{1}{n}$. Since $\sup f_n \to \infty$ as $n \to \infty$ we see that (f_n) fails to converge boundedly.
When $x = 0$ the condition

$$\lim_{n\to\infty} n^2 x \exp(-nx) = 0$$

is obvious and when $x \neq 0$ the condition

$$\lim_{n\to\infty} n^2 x \exp(-nx) = \frac{1}{x} \lim_{n\to\infty} \frac{n^2 x^2}{e^{nx}} = 0$$

follows from the fact that

$$\lim_{u\to\infty} \frac{u^2}{e^u} = 0.$$

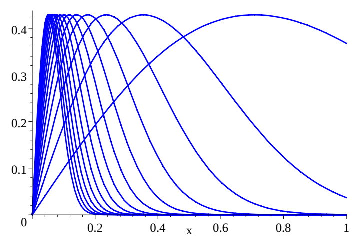
Therefore the sequence (f_n) converges pointwise to the constant function 0. Finally we observe that

$$\int_0^1 f_n(x)dx = \int_0^1 n^2 x e^{-nx} dx = 1 - \frac{n}{e^n} - \frac{1}{e^n} \to 1$$

as $n \to \infty$. So even though the sequence (f_n) converges pointwise to the constant function 0 we do **not** have

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 0$$

c. $f_n(x) = nx \exp(-n^2 x^2)$ for each $x \in [0,1]$ and each positive integer n.



For each x and n we have

$$f_n'(x) = n \exp(-n^2 x^2) - 2n^3 x^2 \exp(-n^2 x^2)$$

and so the equation $f_n'(x) = 0$ holds when $x = \frac{1}{\sqrt{2}\, n}$. Since

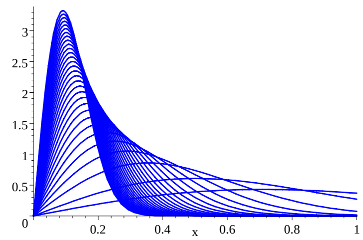
$$f_n\left(\frac{1}{\sqrt{2}\, n}\right) = \frac{1}{\sqrt{2e}}.$$

From this observation we see that the sequence (f_n) converges boundedly to the constant function 0.
Finally we observe that

$$\int_0^1 f_n(x)dx = \int_0^1 nxe^{-n^2x^2}dx = -\frac{1}{2}\frac{e^{-n^2}-1}{n} \to 0$$

as $n \to \infty$.

d. $f_n(x) = nx\exp(-nx^2)$ for each $x \in [0,1]$ and each positive integer n.



For each n and x the equation $f_n'(x) = 0$ says that

$$n\exp(-nx^2) - 2n^2x^2\exp(-nx^2) = 0$$

which gives us $x = \frac{1}{\sqrt{2n}}$ and we observe that

$$f_n\left(\frac{1}{\sqrt{2n}}\right) = \sqrt{\frac{n}{2}}\exp\left(-\frac{1}{2}\right)$$

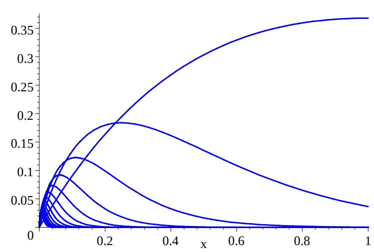
from which we deduce that the sequence (f_n) fails to be boundedly convergent on the interval $[0,1]$. As in the earlier examples the sequence (f_n) converges pointwise on $[0,1]$ to the constant function 0.

Finally we observe that

$$\int_0^1 f_n(x)dx = \int_0^1 nx\exp(-nx^2)dx = -\frac{1}{2}e^{-n} + \frac{1}{2} \to \frac{1}{2}$$

as $n \to \infty$.

e. $f_n(x) = nx\exp(-n^2x)$ for each $x \in [0,1]$ and each positive integer $n.x\exp(-x)$



For each n and x, the equation $f_n'(x) = 0$ says that

$$n\exp(-n^2x) - n^3x\exp(-n^2x) = 0$$

which gives us $x = \frac{1}{n^2}$ and we observe that

$$f_n\left(\frac{1}{n^2}\right) = \frac{1}{ne}.$$

We deduce that $\sup f_n \to 0$ as $n \to \infty$ and therefore the sequence (f_n) converges uniformly to the constant function 0 on $[0,1]$.

Finally we observe that

$$\begin{aligned}\int_0^1 f_n(x)dx &= \int_0^1 nx\exp(-n^2x)dx \\ &= \frac{1}{n^3\exp(n^2)} - \frac{1}{n\exp(n^2)} - \frac{1}{n^3} \to 0\end{aligned}$$

as $n \to \infty$.

2. Given that $f_n(x) = x^n$ for all x and n, prove that the series $\sum f_n$ converges pointwise, but not uniformly, on the interval $[0,1)$ and that $\sum f_n$ converges uniformly on the interval $[0,\delta]$ whenever $0 \le \delta < 1$.
The fact that $\sum f_n$ converges pointwise on the interval $[0,1)$ follows from the fact that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

whenever $0 \le x < 1$. Now given any n we have

$$\sup\left\{\left|\frac{1}{1-x} - \sum_{j=0}^{n} x^n\right| \mid 0 \le x < 1\right\} = \sup\left\{\left|\frac{1}{1-x} - \frac{1-x^{n+1}}{1-x}\right| \mid 0 \le x < 1\right\}$$

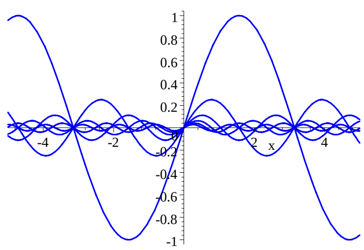
$$= \sup\left\{\left|\frac{x^{n+1}}{1-x}\right| \mid 0 \le x < 1\right\} = \infty$$

and so, certainly, the sequence of numbers

$$\sup\left\{\left|\frac{1}{1-x} - \sum_{j=0}^{n} x^n\right| \mid 0 \le x < 1\right\}$$

does not approach 0 as $n \to \infty$.

3. Given that $f_n(x) = (\sin nx)/n^2$ for all n and x, prove that the series $\sum f_n$ converges uniformly on $\mathbf{R}$. Use *Scientific Notebook* to sketch some of the graphs of these functions to motivate your conclusions.



Since

$$|f_n(x)| \le \frac{1}{n^2}$$

for all n and x the uniform convergence of $\sum f_n$ follows at once from the comparison test.

4. Prove that the series $\sum x^n/n!$ converges uniformly in x on every bounded interval but does not converge uniformly in x on the entire line $\mathbf{R}$.

Hint: ***You do not need to know that the sum of this series is e^x for each x in order to answer this question. Note that whenever $x > 0$ and n is a positive integer we have***

$$\left|\sum_{j=0}^{\infty} \frac{x^j}{j!} - \sum_{j=0}^{n} \frac{x^j}{j!}\right| \ge \frac{x^{n+1}}{(n+1)!}$$

and therefore if n is a positive integer we have

$$\sup\left\{\left|\sum_{j=0}^{\infty} \frac{x^j}{j!} - \sum_{j=0}^{n} \frac{x^j}{j!}\right| \mid x > 0\right\} = \infty.$$

5. Prove that the series

$$\sum \frac{(2n)!}{4^n (n!)^2} x^n$$

does not converge uniformly in x on the interval $(-1,1)$ but that it does converge uniformly on the interval $[-\delta,\delta]$ whenever $0 \le \delta < 1$.

Hint: ***Whenever*** $0 \le x < 1$ ***and*** n ***is a positive integer we have***

$$\left|\sum_{j=1}^{\infty}\frac{(2j)!}{4^j(j!)^2}x^j - \sum_{j=1}^{n}\frac{(2j)!}{4^j(j!)^2}x^j\right| = \sum_{j=n+1}^{\infty}\frac{(2j)!}{4^j(j!)^2}x^j.$$

Now suppose that n ***is any given positive integer. Using the fact that the series*** $\sum \frac{(2j)!}{4^j(j!)^2}$ ***diverges, we choose an integer*** $k > n+1$ ***such that***

$$\sum_{j=n+1}^{k}\frac{(2j)!}{4^j(j!)^2} > 1.$$

Using the fact that

$$\lim_{x\to 1}\sum_{j=n+1}^{k}\frac{(2j)!}{4^j(j!)^2}x^j = \sum_{j=n+1}^{k}\frac{(2j)!}{4^j(j!)^2}$$

we choose a number $t < 1$ ***such that***

$$\sum_{j=n+1}^{k}\frac{(2j)!}{4^j(j!)^2}t^j > 1.$$

We therefore know that

$$\sup\left\{\left|\sum_{j=1}^{\infty}\frac{(2j)!}{4^j(j!)^2}x^j - \sum_{j=1}^{n}\frac{(2j)!}{4^j(j!)^2}x^j\right| \mid 0 \le x < 1\right\} \ge \sum_{j=n+1}^{\infty}\frac{(2j)!}{4^j(j!)^2}t^j > 1.$$

6. Prove that the series $\sum (x\log x)^n$ converges uniformly in x on the interval $(0,1]$.
   We begin by observing that $x\log x \to 0$ as $x \to 0$ (from the right). Now the expression $x\log x$ takes its minimum value when $\log x + 1 = 0$, in other words, when $x = \frac{1}{e}$. Therefore the maximum value of $|x\log x|$ is $\frac{1}{e}$ and the fact that

$$(x\log x)^n \le \frac{1}{e^n}$$

   for every positive integer n and every number $x \in (0,1]$ allows us to use the comparison test to deduce that $\sum (x\log x)^n$ converges uniformly in x on the interval $(0,1]$.

7. Given that (f_n) and (g_n) are sequences of real valued functions defined on a set S, that f and g are functions defined on S and that $f_n \to f$ and $g_n \to g$ pointwise as $n \to \infty$, prove that

   a. $f_n + g_n \to f + g$ pointwise as $n \to \infty$.
      Given any $x \in S$, the fact that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ as $n \to \infty$ guarantees that $f_n(x) + g_n(x) \to f(x) + g(x)$.

   b. $f_n - g_n \to f - g$ pointwise as $n \to \infty$.

   c. $f_n g_n \to fg$ pointwise as $n \to \infty$.

   d. In the event that $g(x) \neq 0$ for every number x in the set S we have $f_n/g_n \to f/g$ pointwise as $n \to \infty$.

8. Given that (f_n) and (g_n) are sequences of real valued functions defined on a set S, that f and g are functions defined on S and that $f_n \to f$ and $g_n \to g$ boundedly as $n \to \infty$, prove that

   a. $f_n + g_n \to f + g$ boundedly as $n \to \infty$.

   b. $f_n - g_n \to f - g$ boundedly as $n \to \infty$.

   c. $f_n g_n \to fg$ boundedly as $n \to \infty$.

   d. In the event that there exists a number $\delta > 0$ such that $|g_n(x)| \ge \delta$ for each n and every number x in the

set S we have $f_n/g_n \to f/g$ boundedly as $n \to \infty$.
These assertions follow at once from Exercise 7 after we have observed that if (f_n) and (g_n) are bounded sequences of functions then so are $(f_n + g_n)$ etc.

9. Suppose that (f_n) is a sequence of real valued functions defined on a set S and that f is a given function defined on S. Prove that the following conditions are equivalent:

   a. The sequence (f_n) converges uniformly to the function f on the set S.

   b. For every number $\varepsilon > 0$ there exists an integer N such that the inequality
   $$\sup|f_n - f| < \varepsilon$$
   holds for all $n \geq N$.

   c. For every number $\varepsilon > 0$ there exists an integer N such that the inequality
   $$|f_n(x) - f(x)| < \varepsilon$$
   holds for all $n \geq N$ and all $x \in S$.
   Conditions a and b are obviously the same and it is clear that these imply condition c. To show that condition c implies condition b, suppose that $\varepsilon > 0$. Using the fact that $\frac{\varepsilon}{2}$ is a positive number, choose an integer N such that the inequality
   $$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$$
   holds whenever $n \geq N$ and $x \in S$. Then for all $n \geq N$ we have
   $$\sup|f_n - f| \leq \frac{\varepsilon}{2} < \varepsilon.$$

10. Suppose that (f_n) is a sequence of real valued functions defined on a set S and that f is a given function defined on S. Examine the following two conditions:

   - For every number $\varepsilon > 0$ there exists an integer N such that the inequality
   $$|f_n(x) - f(x)| < \varepsilon$$
   holds for all $n \geq N$ and all $x \in S$.

   - For every number $\varepsilon > 0$ and every number $x \in S$ there exists an integer N such that the inequality
   $$|f_n(x) - f(x)| < \varepsilon$$
   holds for all $n \geq N$.

   The first of these conditions asserts that the sequence (f_n) converges uniformly to the function f while the second one asserts that (f_n) converges pointwise to f. Make sure that you can distinguish between the two conditions and see that they are not saying the same thing.
   This exercise departs from the conventional mould. It doesn't have a solution in the conventional sense. Instead, it asks the student to think about the statements and make sure that the distinction between them has been appreciated.

11. Given that a sequence (f_n) converges uniformly to a function f on a set S and that the function f is bounded, prove that (if we start the sequence at a sufficiently large value of n) the sequence (f_n) converges boundedly to f.
   Suppose that the sequence (f_n) converges uniformly on a set S to a bounded function f. Choose a number p such that the inequality $|f(x)| \leq p$ holds for all $x \in S$. Using the fact that (f_n) converges uniformly to f, choose N such that the inequality
   $$\sup|f_n - f| < 1$$
   holds whenever $n \geq N$. Then for all $n \geq N$ we have
   $$\begin{aligned}\sup|f_n| &= \sup|f_n - f + f| \\ &\leq \sup|f_n - f| + \sup|f| \leq p + 1.\end{aligned}$$
   Thus, if we start the sequence (f_n) at the integer N then (f_n) converges boundedly to its limit f.

12. Prove that if sequences (f_n) and (g_n) converge uniformly on a set S to functions f and g respectively then

$f_n + g_n \to f + g$ uniformly on S.
Suppose that (f_n) and (g_n) converge uniformly on S to functions f and g respectively. To show that $f_n + g_n \to f + g$ uniformly on S, suppose that $\varepsilon > 0$.
Choose integers N_1 and N_2 such that $\sup|f_n - f| < \frac{\varepsilon}{2}$ whenever $n \geq N_1$ and $\sup|g_n - g| < \frac{\varepsilon}{2}$ whenever $n \geq N_2$. We define N to be the larger of N_1 and N_2 and we observe that whenever $n \geq N$ we have

$$\sup|(f_n + g_n) - (f + g)| \leq \sup|f_n - f| + \sup|g_n - g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

13. Give an example of sequences (f_n) and (g_n) that converge uniformly on a set S to functions f and g respectively such that the sequence $(f_n g_n)$ fails to converge uniformly to the function fg.

Solution: ***We define*** $f(x) = x$ ***whenever*** $x > 0$ ***and for each positive integer*** n ***we define***

$$f_n(x) = x - \frac{1}{n}$$

whenever $x > 0$***. Since***

$$\sup|f - f_n| = \frac{1}{n} \to 0$$

and

$$\sup|f^2 - f_n^2| = \sup\left\{\left|\frac{2nx - 1}{n^2}\right| \mid x > 0\right\} = \infty$$

we see that (f_n) ***converges uniformly to*** f ***on*** $(0,\infty)$ ***but that*** (f_n^2) ***fails to converge uniformly to*** f^2 ***on*** $(0,\infty)$***.***

14. Prove that if sequences (f_n) and (g_n) converge uniformly and boundedly on a set S to functions f and g respectively then $f_n g_n \to fg$ uniformly on S.
Suppose that $f_n \to f$ and $g_n \to g$ uniformly and boundedly on a set S. Choose a number p such that $\sup|f_n| \leq p$ and $\sup|g_n| \leq p$ for every n. We observe that

$$\begin{aligned}\sup|f_n g_n - fg| &= \sup|f_n g_n - f g_n + f g_n - fg| \\ &\leq \sup|f_n g_n - f g_n| + \sup|f g_n - fg| \\ &\leq \sup|f_n - f||g_n| + \sup|f||g_n - g| \\ &\leq \sup p|f_n - f| + \sup p|g_n - g|\end{aligned}$$

and the latter expression approaches 0 as $n \to \infty$.

15. Given that (f_n) is a decreasing sequence of nonnegative continuous functions on a closed bounded set S and that (f_n) converges pointwise to the function 0, prove that (f_n) converges uniformly to the function 0.

Solution: ***We need to show that*** $\sup f_n \to 0$ ***as*** $n \to \infty$***. Note that the sequence*** $(\sup f_n)$ ***is decreasing. To obtain a contradiction, suppose that the sequence*** $(\sup f_n)$ ***fails to converge to*** 0 ***and, using this assumption, choose a number*** $\varepsilon > 0$ ***such that the inequality***

$$\sup f_n > \varepsilon$$

holds for every n***. For each*** n***, choose a number*** $x_n \in S$ ***such that*** $f_n(x_n) > \varepsilon$ ***and, using the fact that*** (x_n) ***is a sequence in the closed bounded set*** S***, choose a partial limit*** x ***of the sequence*** (x_n)***.***

Using the fact that $f_n(x) \to 0$ ***as*** $n \to \infty$***, choose an integer*** N ***such that*** $f_N(x) < \varepsilon$***. Now, using the fact that the function*** f_N ***is continuous at the number*** x***, choose a number*** $\delta > 0$ ***such that the inequality*** $f_N(t) < \varepsilon$ ***holds for every number*** $t \in S \cap (x - \delta, x + \delta)$***.***

Using the fact that x ***is a partial limit of the sequence*** (x_n) ***we now choose an integer*** $n > N$ ***such that*** $x_n \in (x - \delta, x + \delta)$***. Then we have***

$$f_n(x_n) \leq f_N(x_n) < \varepsilon$$

contradicting the way in which the number x_n ***was chosen.***

Some Exercises on the Tests for Uniform Convergence

1. Prove that the series

$$\sum \frac{(-1)^{n-1}\sin(1+\frac{x}{n})}{\sqrt{n}}$$

converges uniformly in x on the interval $(-1,1)$ and converges pointwise on the entire line **R**.

If $x \geq 0$ then, starting at a sufficiently large value of n, the sequence of numbers $\sin(1+\frac{x}{n})$ is decreasing (with limit $\sin 1$) and so the sequence

$$\left(\frac{\sin(1+\frac{x}{n})}{\sqrt{n}}\right)$$

is a decreasing sequence of positive numbers with limit 0 and it follows from Dirichlet's test that the series

$$\sum \frac{(-1)^{n-1}\sin(1+\frac{x}{n})}{\sqrt{n}}$$

converges. If p is any given positive number and if we define

$$f_n(x) = \frac{\sin(1+\frac{x}{n})}{\sqrt{n}}$$

then starting at a sufficiently large integer N the sequence of functions decreases uniformly to 0 on the interval $[0,p]$ as $n \to \infty$. By Dirichlet's test for uniform convergence the series

$$\sum \frac{(-1)^{n-1}\sin(1+\frac{x}{n})}{\sqrt{n}}$$

converges uniformly in x on the interval $[0,p]$.
Now suppose that $x < 0$. It is clear that

$$\lim_{n\to\infty} \frac{\sin(1+\frac{x}{n})}{\sqrt{n}} = 0$$

but, before we can use Dirichlet's test, we need to show that this sequence is decreasing. We write

$$f(t) = \frac{\sin(1+\frac{x}{t})}{\sqrt{t}}$$

for $t > 0$ and we observe that for each t we have

$$f'(t) = \frac{-x\cos(1+\frac{x}{t}) - \frac{t}{2}\sin(1+\frac{x}{t})}{t^{5/2}}$$

and it is clear that $f'(t) < 0$ if t is sufficiently large. A similar argument shows that if $p > 0$ then it is possible to find an integer N such that starting at $n = N$, the sequence

$$\left(\frac{\sin(1+\frac{x}{n})}{\sqrt{n}}\right)$$

decreases to zero for every number $x \in [-p,0]$ and it is clear that this convergence is uniform. It therefore follows from Dirichlet's test for uniform convergence that

$$\sum \frac{(-1)^{n-1}\sin(1+\frac{x}{n})}{\sqrt{n}}$$

converges uniformly on the interval $[-p,0]$.
We conclude that the series converges uniformly on every bounded set of real numbers, which is more than the exercise requested.

2. Prove that if we define

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(1+\frac{x}{n})}{\sqrt{n}}$$

for every real number x then the function f is differentiable at every real number and for every number x we have

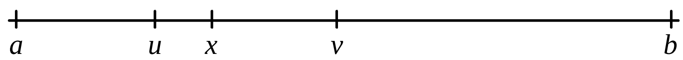
$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(1 + \frac{x}{n})}{n^{3/2}}$$

The solution of this exercise follows at once from the theorem on term by term differentiation of series, from Exercise 1 and from the uniform convergence in x of the series

$$\sum \frac{(-1)^{n-1} \cos(1 + \frac{x}{n})}{n^{3/2}}.$$

Exercises on Continuity of a Limit Function

1. Suppose that a and b are real numbers and that $a < b$, that H is a closed subset of $\mathbf{R}$ and that $x \in [a,b] \setminus H$. Prove that there exist numbers u and v such that $a \le u \le x \le v \le b$ and $u < v$ and $[u,v] \cap H = \emptyset$.



Using the fact that x belongs to the open set $\mathbf{R} \setminus H$, choose $\delta > 0$ such that

$$(x - \delta, x + \delta) \cap H = \emptyset.$$

We now define u to be the larger of the numbers a and $x - \delta/2$ and v to be the smaller of the numbers b and $x + \delta/2$.

2. Suppose that (H_n) is a sequence of closed subsets of an interval $[a,b]$ where $a < b$ and that none of the sets H_n has any interior points. Find a contracting sequence of subintervals $[a_n, b_n]$ of the interval $[a,b]$ such that

$$H_n \cap [a_n, b_n] = \emptyset$$

for each n. By looking at a number that lies in the intersection of all the intervals $[a_n, b_n]$, prove that there exists a number in the interval $[a,b]$ that does not belong to the set

$$\bigcup_{n=1}^{\infty} H_n.$$

Using the fact that the set $H_1 \neq [a,b]$, choose a number $x_1 \in [a,b] \setminus H_1$. Using Exercise 1, choose two numbers a_1 and b_1 such that $a_1 < b_1$ and $a \le a_1 \le x_1 \le b_1 \le b$ and

$$H_1 \cap [a_1, b_1] = \emptyset.$$

Since $a_1 < b_1$ and since the set H_2 has no interior point, the open interval (a_1, b_1) must contain numbers that do not belong to H_2. Choose $x_2 \in [a_1, b_1] \setminus H_2$. We now use Exercise 1 again to choose two numbers a_2 and b_2 such that $a_2 < b_2$ and $a_1 \le a_2 \le x_2 \le b_2 \le b_1$. Continuing in this way we obtain the desired sequence of intervals $[a_n, b_n]$.

3. Suppose that (H_n) is a sequence of closed subsets of an interval $[a,b]$ where $a < b$ and that none of the sets H_n has any interior points. Prove that the set

$$[a,b] \setminus \bigcup_{n=1}^{\infty} H_n$$

is dense in the interval $[a,b]$.

Hint: ***If $[c,d]$ is a subinterval of the interval $[a,b]$, apply Exercise 2 to the sequence of sets***

$$H_n \cap [c,d].$$

We can observe that whenever (c,d) is a nonempty open interval of $[a,b]$, the set

$$[c,d] \setminus \bigcup_{n=1}^{\infty} H_n \neq \emptyset.$$

4. Suppose that (H_n) is a sequence of closed sets, that $a < b$ and that

$$[a,b] = \bigcup_{n=1}^{\infty} H_n.$$

For each n, suppose that U_n is the set of interior points of the set H_n. Prove that the set

$$\bigcup_{n=1}^{\infty} U_n$$

is a dense open subset of the interval $[a,b]$.

Suppose that (c,d) is a nonempty open subinterval of $[a,b]$. Since

$$[c,d] = \bigcup_{n=1}^{n} H_n \cap [c,d],$$

it follows from Exercise 2 that at least one of the set $H_n \cap [c,d]$ must have an interior point and an interior point of $H_n \cap [c,d]$ must belong to U_n.

5. Suppose that (f_n) is a sequence of continuous functions on an interval $[a,b]$ where $a < b$ and that (f_n) converges pointwise on $[a,b]$ to a function f. Suppose that $\varepsilon > 0$ and that for each n, suppose that H_n is defined to be the set of all those numbers $x \in [a,b]$ for which the inequality

$$|f_i(x) - f_j(x)| \le \frac{\varepsilon}{3}$$

holds whenever $i \ge n$ and $j \ge n$. Suppose finally that U_n is the set of interior points of H_n for each n.

a. Prove that the set

$$V = \bigcup_{n=1}^{\infty} U_n$$

is an open dense subset of the interval $[a,b]$.

This assertion follows at once from Exercise 4 and the fact that

$$[a,b] = \bigcup_{n=1}^{\infty} H_n.$$

b. Prove that for every number $x \in V$ there exists a number $\delta > 0$ such that for every number t satisfying the inequality $|t - x| < \delta$ we have $|f(t) - f(x)| \le \varepsilon$.

Choose n such that $x \in U_n$. Choose $\delta_1 > 0$ such that $(x - \delta, x + \delta) \subseteq U_n$. Using the fact that the function f_n is continuous at the number x, choose $\delta_2 > 0$ such that the inequality

$$|f_n(t) - f_n(x)| < \frac{\varepsilon}{3}$$

holds whenever $|t - x| < \delta_2$. We now define δ to be the smaller of the the numbers δ_1 and δ_2. Whenever $|t - x| < \delta$ we have

$$|f(t) - f(x)| \le |f(t) - f_n(t)| + |f_n(t) - f_n(x)| + |f_n(x) - f(x)|.$$

Now, whenever $|t - x| < \delta$, the fact that both t and x belong to the set U_n gives us

$$|f(t) - f_n(t)| = \lim_{j \to \infty} |f_j(t) - f_n(t)| \le \frac{\varepsilon}{3}$$

and

$$|f(x) - f_n(x)| = \lim_{j \to \infty} |f_j(x) - f_n(x)| \le \frac{\varepsilon}{3}$$

and we conclude that $|f(t) - f(x)| \le \varepsilon$.

6. Suppose that (f_n) is a sequence of continuous functions on an interval $[a,b]$ where $a < b$ and that (f_n) converges pointwise on $[a,b]$ to a function f. Suppose that for every positive number ε, the set V introduced in Exercise 4 is now called $V(\varepsilon)$. Apply Exercise 2 to the sequence of sets

$$[a,b] \setminus V\left(\frac{1}{n}\right)$$

and deduce that if

$$D = \bigcap_{n=1}^{\infty} V\left(\frac{1}{n}\right)$$

then D is a dense subset of the interval $[a,b]$. Prove that the function f is continuous at every number in that belongs to this dense set D.

From Exercise 5a we know that each of the set $V(\frac{1}{n})$ is an open dense subset of $[a,b]$ and it follows from Exercise 3 that the set

$$D = \bigcap_{n=1}^{\infty} V\left(\frac{1}{n}\right) = [a,b] \setminus \bigcup_{n=1}^{\infty}\left(\mathbf{R} \setminus V\left(\frac{1}{n}\right)\right)$$

is dense in $[a,b]$. To see that f is continuous on D, suppose that $x \in D$ and that $\varepsilon > 0$. Choose a positive integer n such that $1/n < \varepsilon$, and, using Exercise 5b, and the fact that $x \in V(\frac{1}{n})$, choose $\delta > 0$ such that, whenever $|t - x| < \delta$ we have

$$|f(t) - f(x)| \le \frac{1}{n}.$$

Some Exercises on the Properties of Uniform Convergence

1. Determine whether the following statement is true or false:
If f_n is uniformly continuous on a set S for every positive integer n and if the sequence (f_n) converges uniformly on S to a function f then f must be uniformly continuous on S.
The statement is true. Suppose that $\varepsilon > 0$. Using the fact that $f_n \to f$ uniformly on S, choose an integer N such that the inequality
$$\sup|f_n - f| < \frac{\varepsilon}{3}$$
holds whenever $n \ge N$. Using the fact that the function f_N is uniformly continuous on S, choose $\delta > 0$ such that the inequality
$$|f_N(t) - f_N(x)| < \frac{\varepsilon}{3}$$
holds whenever t and x belong to S and $|t - x| < \delta$. Then, whenever t and x belong to S and $|t - x| < \delta$ we have
$$\begin{aligned} |f(t) - f(x)| &= |f(t) - f_N(t) + f_N(t) - f_N(x) + f_N(x) - f(x)| \\ &\le |f(t) - f_N(t)| + |f_N(t) - f_N(x)| + |f_N(x) - f(x)| < \varepsilon. \end{aligned}$$

2. Determine whether the following statement is true or false:
If f_n is uniformly continuous on a set S for infinitely many positive integers n and if the sequence (f_n) converges uniformly on S to a function f then f must be uniformly continuous on S.
The statement is true and the proof is nearly identical to the one used in Exercise 1. The only difference is that, instead of using the uniform continuity of the function f_N we have to choose an integer $n \ge N$ such that the function f_n is uniformly continuous.

3. A family $\mathfrak{I}$ of functions is said to be **equicontinuous** on a set S if for every $\varepsilon > 0$ and every number $x \in S$ there exists a number $\delta > 0$ such that whenever $f \in \mathfrak{I}$ and whenever t lies in the set $S \cap (x - \delta, x + \delta)$ we have
$$|f(t) - f(x)| < \varepsilon.$$
Prove that if a sequence (f_n) converges uniformly on S and if each function f_n is continuous on S then the family $\{f_n \mid n = 1, 2, 3, \cdots\}$ is equicontinuous on S.
Suppose that $x \in S$ and that $\varepsilon > 0$. Choose an integer N such that the inequality
$$\sup|f_n - f| < \frac{\varepsilon}{5}$$
holds whenever $n \ge N$. Using the fact that the function f_N is continuous at the number x, choose a number $\delta_1 > 0$ such that the inequality

$$|f_N(t) - f_N(x)| < \frac{\varepsilon}{5}$$

holds whenever $t \in S$ and $|t - x| < \delta_1$. Now we use the fact that there are only finitely many positive integers $n < N$ to choose a number $\delta_2 > 0$ such that the inequality

$$|f_n(t) - f_n(x)| < \frac{\varepsilon}{5}$$

holds whenever $t \in S$ and $|t - x| < \delta_2$ and $1 \leq n < N$. Finally, we define δ to be the smaller of the two numbers δ_1 and δ_2. Now given any member t of S satisfying the condition $|t - x| < \delta$, the condition

$$|f_n(t) - f_n(x)| < \frac{\varepsilon}{5} < \varepsilon$$

is already known for $1 \leq n < N$ and if $n \geq N$ then we have

$$\begin{aligned} |f_n(t) - f_n(x)| &= |f_n(t) - f(t) + f(t) - f_N(t) + f_N(t) - f_N(x) + f_N(x) - f(x) + f(x) - f_n(x)| \\ &\leq |f_n(t) - f(t)| + |f(t) - f_N(t)| + |f_N(t) - f_N(x)| + |f_N(x) - f(x)| + |f(x) - f_n(x)| < \varepsilon. \end{aligned}$$

4. Invent a meaning for *equi-uniform continuity* of a family $\mathfrak{I}$ on a set S and decide whether or not your definition provides an analogue of the preceding exercise.

   A family $\mathfrak{I}$ of functions on a set S is said to be equi-uniformly continuous on S if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that whenever t and x belong to S and $|t - x| < \delta$ and $f \in \mathfrak{I}$ we have

   $$|f(t) - f(x)| < \varepsilon.$$

   The point of this exercise is to invite the student to show that if (f_n) is a uniformly convergent sequence of uniformly continuous functions on a set S then the family $\{f_n \mid n = 1, 2, 3, \cdots\}$ is equi-uniformly continuous on S. The proof is similar to the one used in Exercise 3.

Some Exercises on Bounded Convergence

1. Prove that the sequence (f_n) in the bounded convergence theorem for derivatives actually converges boundedly to the function f.

   Using the fact that the sequence (f_n') converges boundedly, choose a number p such that the inequality

   $$|f_n'(x)| < p$$

   holds for every positive integer n and every $x \in [a, b]$. Using the fact that the sequence $(f_n(c))$, being convergent, is bounded, choose a number q such that

   $$|f_n(c)| < q$$

   for every positive integer n.

   Now given any positive integer n and any member x of $[a, b]$ we have

   $$\begin{aligned} |f_n(x)| &= \left| f_n(c) + \int_c^x f_n'(t)dt \right| \\ &\leq |f_n(c)| + \left| \int_c^x f_n'(t)dt \right| \leq q + p(b - a). \end{aligned}$$

2. Suppose that (a_n) is a strictly increasing sequence of positive integers and that

   $$f_n(x) = (\sin a_n x - \sin a_{n+1} x)^2$$

   for every positive integer n and every number x.

   a. Work out the integral

      $$\int_0^{2\pi} f_n(x)dx$$

      and deduce that there must be at least one number $x \in [0, 2\pi]$ such that the sequence $(f_n(x))$ does not converge to zero.

      For each n and x we have

$$|f_n(x)| = \left|(\sin a_n x - \sin a_{n+1} x)^2\right| \leq 4.$$

If the sequence (f_n) converged pointwise (and therefore boundedly) on the interval $[0, 2\pi]$ to the constant function 0 then the bounded convergence theorem would give us

$$\lim_{n\to\infty} \int_0^{2\pi} f_n(x)dx = \int_0^{2\pi} 0 = 0.$$

However,

$$\int_0^{2\pi} f_n(x)dx = \int_0^{2\pi} (\sin a_n x - \sin a_{n+1} x)^2 dx = 2\pi$$

for each n and we conclude that the sequence (f_n) cannot converge pointwise to 0.

b. Prove that there must be at least one number $x \in [0, 2\pi]$ for which the sequence $(\sin a_n x)$ diverges.
Given any number x, if the sequence $(\sin a_n x)$ converges then the sequence $(f_n(x))$ must converge to the number 0 and we know from part a that there are numbers x for which $(f_n(x))$ does not converge to 0.

3. Suppose that for each positive integer n we have

$$f_n(x) = \begin{cases} \sum_{j=0}^{n} (-x)^j & \text{if } -1 < x < 1 \\ \frac{1}{2} & \text{if } x = 1 \end{cases}.$$

a. Prove that if

$$f(x) = \frac{1}{1+x}$$

for $-1 < x \leq 1$ then the sequence (f_n) converges boundedly to the function f on the interval $[0, 1]$.
This statement is obvious.

b. Explain why each function f_n is integrable on the interval $[0, 1]$ and why

$$\int_0^1 f_n = \sum_{j=0}^{n} \frac{(-1)^j}{j+1}$$

and deduce that

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} = \log 2.$$

Each function f_n is bounded and is continuous at all but one number in the interval $[0, 1]$. Therefore each function f_n is integrable on $[0, 1]$. We see at once that, for each n,

$$\int_0^1 f_n = \int_0^1 \sum_{j=0}^{n} (-x)^j dx = \sum_{j=0}^{n} \frac{(-1)^j}{j+1}.$$

From the bounded convergence theorem we deduce that

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(-1)^j}{j+1} = \lim_{n\to\infty} \int_0^1 f_n$$
$$= \int_0^1 \frac{1}{1+x} dx = \log 2.$$

c. Given any number x satisfying $-1 < x < 1$, prove that the series $\sum f_n$ converges boundedly to f on the closed interval running between 0 and x and deduce that the equation

$$\sum_{j=0}^{\infty} \frac{(-1)^j x^{j+1}}{j+1} = \log(1+x)$$

holds whenever $-1 < x \leq 1$.

This part of the exercise is simpler than part b because the comparison test guarantees that the series

$$\sum \frac{(-1)^j t^{j+1}}{j+1}$$

converges absolutely and uniformly in t on the closed interval running between 0 and x. We repeat the argument used in part b but with integration from 0 to x.

4. Suppose that for each positive integer n we have

$$f_n(x) = \begin{cases} \sum_{j=0}^{n} (-x^2)^j & \text{if } \ 0 \le x < 1 \\ \frac{1}{2} & \text{if } \ x = 1 \end{cases}.$$

Repeat the steps of the preceding for this sequence of functions and deduce that

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = \frac{\pi}{4}.$$

The sequence (f_n) converges boundedly to the function f defined by the equation

$$f(x) = \frac{1}{1+x^2}$$

for $0 \le x \le 1$. Applying the bounded convergence theorem to this sequence of functions yields the desired result.

5. Prove that if

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } \ 0 \le x \le n \\ 0 & \text{if } \ x > n \end{cases}$$

whenever n is a positive integer then the sequence (f_n) converges uniformly on the interval $[0, \infty)$ to the constant function 0 even though

$$\lim_{n\to\infty} \int_0^{\to\infty} f_n \ne \int_0^{\to\infty} 0.$$

There really isn't much to do in this exercise. All the assertions should be obvious, or nearly so. The point of the exercise is to point out that uniform convergence isn't enough to guarantee interchangability of limits and integrals on an unbounded domain interval.

6. Suppose that $\alpha > 0$ and that

$$f_n(x) = \begin{cases} \left(1 - \frac{x}{n}\right)^n x^{\alpha-1} & \text{if } \ \frac{1}{n} \le x \le n \\ \\ 0 & \text{if } \ x < \frac{1}{n} \text{ or } x > n \end{cases}$$

whenever n is a positive integer.

   a. Prove that if x is any positive number then

   $$\lim_{n\to\infty} f_n(x) = e^{-x} x^{\alpha-1}$$

   and that for each n we have

   $$|f_n(x)| \le e^{-x} x^{\alpha-1}.$$

   Solution: ***A simple application of L'Hôpital's rule can be used to show that if*** $x > 0$ ***then***

   $$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}.$$

   We now want to show that whenever $0 \le x \le n$ ***we have***

   $$\left(1 - \frac{x}{n}\right)^n \le e^{-x}.$$

   Suppose that x ***is a positive number. We define***

   $$f(t) = \left(1 - \frac{x}{t}\right)^t$$

whenever $t \geq x$***. The fact that*** $f(t) \leq e^{-x}$ ***whenever*** $t \geq x$ ***will follow if we can show that the function*** f ***is increasing. Now whenever*** $t \geq x$ ***we have***

$$f(t) = \exp\left(t\log\left(1 - \frac{x}{t}\right)\right)$$

and so all we have to show is that if we define

$$g(t) = t\log\left(1 - \frac{x}{t}\right)$$

for $t \geq x$ ***then the function*** g ***is increasing. Since*** $g(x) = 0$ ***and since*** g ***is continuous on the interval*** $[x, \infty)$ ***it will follow that*** g ***is increasing when we have shown that*** $g'(t) \geq 0$ ***whenever*** $t > x$***. Now whenever*** $t > x$ ***we have***

$$g'(t) = \log\left(1 - \frac{x}{t}\right) + \frac{\frac{x}{t}}{1 - \frac{x}{t}}$$

and therefore what we want to show is that the inequality

$$\log(1-u) + \frac{u}{1-u} \geq 0$$

holds whenever $0 \leq u < 1$***. We define***

$$h(u) = \log(1-u) + \frac{u}{1-u}$$

and observe that $h(0) = 0$ ***and that whenever*** $0 < u < 1$ ***we have***

$$h'(u) = \frac{-1}{1-u} + \frac{(1-u)+u}{(1-u)^2} = \frac{u}{(u-1)^2} > 0$$

Therefore $h(u) > 0$ ***whenever*** $0 < u < 1$ ***and our proof is complete.***

b. Use the dominated convergence theorem to show that

$$\lim_{n\to\infty} \int_{1/n}^{n} \left(1 - \frac{x}{n}\right)^n x^{\alpha-1} = \int_0^{\to\infty} e^{-x}x^{\alpha-1}dx = \Gamma(\alpha).$$

For the definition of the gamma function Γ, see an earlier exercise.

Some Exercises on Power Series

1. Find the radius of convergence and the interval of convergence of each of the following series:

a. $\sum \frac{(n!)^2 x^n}{(2n)!}$

An easy application of d'Alembert's test shows that the series converges when $|x| < 4$. To see what happens when $x = 4$ we define

$$a_n = \frac{(n!)^2 4^n}{(2n)!}$$

for each n. Since

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((n+1)!)^2 4^{n+1}}{(2n+2)!}}{\frac{(n!)^2 4^n}{(2n)!}} = \frac{2n+2}{2n+1} > 1$$

we see that a_n does not approach 0 as $n \to \infty$ and so the given series must diverge when $x = \pm 4$. Thus the interval of convergence is $(-4, 4)$ and the radius of convergence is 4.

b. $\sum \frac{((2n)!)x^n}{(n!)^2}$An easy application of d'Alembert's test shows that the series converges when $|x| < \frac{1}{4}$. To see what happens when $x = \frac{1}{4}$ we define

$$a_n = \frac{(2n)!}{4^n (n!)^2}$$

for each n. Since

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{4^{n+1}((n+1)!)^2}}{\frac{(2n)!}{4^n(n!)^2}} = \frac{1}{2}\frac{2n+1}{n+1} = \frac{2n+1}{2n+2} < 1$$

and so the sequence (a_n) is decreasing. Since

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \frac{1}{2} < 1$$

Raabe's test guarantees that $\sum a_n$ is divergent but the criterion for terms to approach zero guarantees that $a_n \to 0$ as $n \to \infty$. It therefore follows from Dirichlet's testDirichlet's test that, although $\sum a_n$ is divergent, the series $\sum(-1)^n a_n$ is convergent. The interval of convergence of the series $\sum \frac{((2n)!)x^n}{(n!)^2}$ is therefore $\left[-\frac{1}{4}, \frac{1}{4}\right)$.

c. $\sum \frac{(n!)x^n}{n^n}$

An easy application of d'Alembert's test shows that the series converges when $|x| < e$. To see what happens when $x = e$ we define

$$a_n = \frac{(n!)e^n}{n^n}$$

for each n. Since

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1 - \frac{\frac{((n+1)!)e^{n+1}}{(n+1)^{n+1}}}{\frac{(n!)e^n}{n^n}}\right)$$

$$= \lim_{n\to\infty} n\left(1 - \frac{e}{\left(1+\frac{1}{n}\right)^n}\right) = -\frac{1}{2}$$

we know that the sequence (a_n) does not approach zero and $\sum a_n$ diverges. When $x = -e$ we have

$$\frac{(n!)x^n}{n^n} = (-1)^n \frac{(n!)e^n}{n^n}$$

and, once again, because this nth term fails to approach zero the series diverges. The interval of convergence of the series $\sum \frac{(n!)x^n}{n^n}$ is therefore $(-e, e)$.

d. $\sum \frac{n^n x^n}{n!}$

We saw in an earlier group of exercises that the interval of convergence of this series is $\left[-\frac{1}{e}, \frac{1}{e}\right)$.

2. Given that $c \neq 1$ and that

$$f(x) = \frac{1}{1-x}$$

whenever $x \neq 1$, expand the function f in a power series with center c and find the interval of convergence of this series.

Hint: ***Use the identity***

$$\frac{1}{1-x} = \left(\frac{1}{1-c}\right)\left(\frac{1}{1-\frac{x-c}{1-c}}\right).$$

Thus

$$\frac{1}{1-x} = \left(\frac{1}{1-c}\right)\sum_{n=0}^{\infty}\left(\frac{x-c}{1-c}\right)^n$$

The series $\sum\left(\frac{x-c}{1-c}\right)^n$ converges if and only if

$$\left|\frac{x-c}{1-c}\right| < 1$$

which occurs when

$$c - |1-c| < x < c + |1-c|.$$

If $c < 1$ then the latter inequality says that $2c - 1 < x < 1$ and if $c > 1$ the inequality says that $1 < x < 2c - 1$.

3. Does a power series have to have the same interval of convergence as its derived series?
   No. Look at the intervals of convergence of $\sum \frac{x^{n+1}}{n+1}$ and of $\sum x^n$. The first of these intervals contains the number -1 but the second does not.

4. Suppose that f and g are given functions, that c is a given number and that $r > 0$. Suppose that both of the functions f and g are the sums of their Taylor series at every number x in the interval $(c-r, c+r)$. Suppose finally that there exists a number $\delta > 0$ such that $f(x) = g(x)$ whenever x belongs to the interval $(c-\delta, c+\delta)$.

   $c - r$ $\qquad$ $c - \delta$ $\qquad$ c $\qquad$ $c + \delta$ $\qquad$ $c + r$

   Prove that $f(x) = g(x)$ for every number x in the interval $(c-r, c+r)$.
   The given information tells us that

   $$f^{(n)}(c) = g^{(n)}(c)$$

   for every nonnegative integer n.

5. True or false: If f and g have derivatives or all orders in a neighborhood of a number c and if $f^{(n)}(c) = g^{(n)}(c)$ for every nonnegative integer n then we have $f(x) = g(x)$ for every number x sufficiently close to c.
   The statement if false. Define f to be the function introduced in this example and g to be the constant function 0.

6. Use *Scientific Notebook* to calculate a variety of nth partial sums of the Maclaurin series of the function f defined by the equation

   $$f(x) = \frac{e^x \sin x}{1 + x^6}$$

   for every real number x. Then use *Scientific Notebook* to plot some of these nth partial sums with the graph of f on the interval $[-2, 2]$ and explore the accuracy of these partial sums as approximation so f on the interval.

Some Exercises on the Series Expansion of exp

1. Prove that if c and x are any real numbers then

   $$\exp(x) = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n.$$

Solution:

$$\begin{aligned}\exp(x) &= \exp(c)\exp((x-c)) \\ &= e^c \sum_{n=0}^{\infty} \frac{1}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n.\end{aligned}$$

2. Prove that

   $$\int_0^1 e^{-x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)}$$

   From the fact that

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

and from the fact that the series $\sum \frac{(-1)^n x^{2n}}{n!}$ converges bounded (in fact, uniformly) in x on the inteval $[0, 1]$, we see that

$$\int_0^1 e^{-x^2} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} dx$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$$

3. Show that even before we have showed that the function f defined in the proof of the above theorem is the exponential function exp we could have seen from the binomial theorem and Cauchy's theorem on products of series (click here to see it) that for all numbers x and y we have

$$f(x)f(y) = f(x+y).$$

Solution: ***Suppose that x and y are real numbers.***

$$f(x)f(y) = \left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right)$$

From Cauchy's theorem on products of series we know that the latter product is

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{x^{n-j}}{(n-j)!} \frac{y^j}{j!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} x^{n-j} y^j\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\right)$$

and from the binomial theorem we know that

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\right) = \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n = f(x+y).$$

4. a. Prove that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

b. Prove that if m is any positive integer then the number

$$(m!)\sum_{j=0}^{m} \frac{1}{j!}$$

is an integer and prove that

$$0 < (m!)\sum_{j=m+1}^{\infty} \frac{1}{j!} < 1.$$

c. Prove that if m is any positive integer then the number $(m!)e$ is not an integer and deduce that the number e is irrational.

Solution: ***This problem also appeared in Chapter 10 where it was provided with*** a solution.

Some Exercises on Binomial Series

1. Given that $\alpha > -1$, prove that

$$2^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n}.$$

Solution: ***We know from*** earlier work ***that the series*** $\sum \binom{\alpha}{n}$ ***converges and it therefore follows from*** *Abel's theorem* ***that***

$$2^{\alpha} = \lim_{x\to 1-}(1+x)^{\alpha} = \lim_{x\to 1-} \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = \sum_{n=0}^{\infty} \binom{\alpha}{n}.$$

2. Given that $\alpha > 0$, prove that

$$\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} = 0.$$

3. Given that α and β are any real numbers and $|x| < 1$, apply Cauchy's theorem on products of series to the Maclaurin expansions of $(1+x)^{\alpha}$ and $(1+x)^{\beta}$ to deduce that the equation

$$\sum_{j=0}^{n} \binom{\alpha}{n-j}\binom{\beta}{j} = \binom{\alpha+\beta}{n}$$

holds for every positive integer n.
Since the nth term of the Cauchy product of the series $\sum \binom{\alpha}{n} x^n$ and $\sum \binom{\beta}{n} x^n$ is

$$\sum_{j=0}^{n} \binom{\alpha}{n-j} x^{n-j} \binom{\beta}{j} x^j = \sum_{j=0}^{n} \binom{\alpha}{n-j}\binom{\beta}{j} x^n$$

and since this Cauchy product must converge on the interval $(-1,1)$ to

$$\left(\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n\right)\left(\sum_{n=0}^{\infty} \binom{\beta}{n} x^n\right) = (1+x)^{\alpha}(1+x)^{\beta} = (1+x)^{\alpha+\beta}$$

the equation

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \binom{\alpha}{n-j}\binom{\beta}{j}\right) x^n = \sum_{n=0}^{\infty} \binom{\alpha+\beta}{n} x^n$$

holds for all $x \in (-1,1)$. From the theorem on uniqueness of coefficients of a power series we deduce that

$$\sum_{j=0}^{n} \binom{\alpha}{n-j}\binom{\beta}{j} = \binom{\alpha+\beta}{n}.$$

4. The motivation and notation used in this exercise is the theory of Cesàro summability of series that can be found in the book *Divergent Series* by G. H. Hardy.
In this exercise we define

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}$$

whenever $\alpha > -1$ and n is a positive integer. We also define $A_0^{\alpha} = 1$.

a. Prove that if $\alpha > -1$ and n is a nonnegative integer we have

$$\binom{-\alpha-1}{n} = (-1)^n A_n^{\alpha}.$$

This identity is obvious.

b. Use the preceding exercise to prove that if α and β are greater than -1 and n is a nonnegative integer then

$$\sum_{j=0}^{n} A_{n-j}^{\alpha} A_j^{\beta} = A_n^{\alpha+\beta+1}.$$

We observe that

$$\begin{aligned}\sum_{j=0}^{n} A_{n-j}^{\alpha} A_j^{\beta} &= \sum_{j=0}^{n} (-1)^{n-j}(-1)^{j} \binom{-\alpha-1}{n-j}\binom{-\beta-1}{j} \\ &= (-1)^{n} \sum_{j=0}^{n} \binom{-\alpha-1}{n-j}\binom{-\beta-1}{j} \\ &= (-1)^{n} \binom{-\alpha-\beta-2}{n} = A_n^{\alpha+\beta+1}\end{aligned}$$

c. Prove that if α and n are nonnegative integers then

$$A_n^{\alpha} = \frac{(n+1)(n+2)\cdots(n+\alpha)}{\alpha!}$$

where the numerator of the right side is understood to be 1 when $\alpha = 0$.
This identity is obvious.

d. Prove that if α, β and n are nonnegative integers we have

$$\sum_{j=0}^{n} \left(\frac{(j+1)\cdots(j+\alpha)}{\alpha!} \right)\left(\frac{(j+1)\cdots(j+\beta)}{\beta!} \right) = \frac{(n+1)\cdots(n+\alpha+\beta+1)}{\alpha!}$$

This identity follows at once from parts b and c.

5. a. Prove that if

$$f(x) = \sqrt{1-x}$$

whenever $0 \le x \le 1$ then there exists a sequence of polynomials that converges uniformly to f on the interval $[0,1]$.

Solution: ***Since the series***

$$\sum \binom{1/2}{n}$$

is absolutely convergent, the uniform convergence of the Maclaurin series of f on the interval $[0,1]$ ***follows from the*** *comparison test* ***for uniform convergence.***

b. Prove that if f is the function defined in part a and if

$$g(x) = f(1-x^2)$$

whenever $-1 \le x \le 1$ then there exists a sequence of polynomials that converges uniformly to g on the interval $[-1,1]$.
Choose a sequence (p_n) of polynomials that converges uniformly on the interval $[0,1]$ to the function f. For each positive integer n and for each $x \in [-1,1]$ we now define

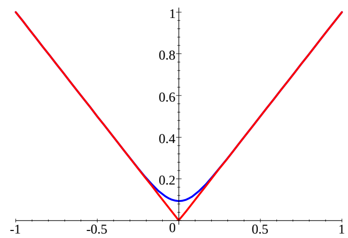
$$q_n(x) = p_n(1-x^2)$$

and we observe that (q_n) converges uniformly on $[-1,1]$ to the function g. Of course, each function q_n is also a polynomial.

c. Prove that if $g(x) = |x|$ for all $x \in [-1,1]$ then there exists a sequence of polynomials that converges uniformly to g on the interval $[-1,1]$.
This part follows at once from part b, in the light of the fact that if g is the function defined in part b then for each $x \in [-1,1]$ we have

$$g(x) = f(1 - x^2) = \sqrt{1 - (1 - x^2)} = |x|.$$

d. Use *Scientific Notebook* to calculate some nth Maclaurin polynomials of the function f defined in part a. For each chosen value of n, if f_n is the nth Maclaurin polynomial, and $h_n(x) = f_n(1 - x^2)$ for each x, ask *Scientific Notebook* to sketch the graph of the function h together with the graph of the absolute value function and observe graphically that the sequence (h_n) is converging uniformly to the absolute value function on the interval $[-1, 1]$. The case $n = 35$ is illustrated in the following figure.



This exercise is of considerable importance because it may be used as the starting point for a major theorem known as the **Stone-Weierstrass** . You can find an elementary presentation of the Stone-Weierstrass theorem in Rudin reference starting with Corollary 7.27.

Exercises on the Trigonometric Functions

1. Given any real number x, prove that $\sin x = 0$ if and only if x is an integer multiple of π. Prove that $\cos x = 0$ if and only if x is an odd multiple of $\pi/2$. Prove that if n is any integer then $\cos n\pi = (-1)^n$.

Solution: ***We know that*** $\sin \pi = \sin 0 = 0$ ***and that*** $\sin x > 0$ ***whenever*** $0 < x < \pi$***. We also know that whenever*** x ***and*** y ***are numbers and*** $\sin x = \sin y = 0$ ***we have***

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x = 0$$

and from this fact we see that $\sin n\pi = 0$ ***for every integer*** n***. Now suppose that*** x ***is any number for which the equation*** $\sin x = 0$ ***holds. If we define*** n ***to the the greatest integer that does not exceed the number*** x/π ***then we have***

$$0 \le x - n\pi < \pi$$

and since $\sin(x - n\pi) = 0$ ***we conclude that*** $x = n\pi$***.***

A similar argument may be used to show that

$$\{x \in \mathbf{R} \mid \cos x = 0\} = \left\{\frac{n\pi}{2} \mid n \text{ is an integer}\right\}.$$

Finally, if G ***is the set of integers*** n ***for which the equation*** $\cos n\pi = (-1)^n$ ***is true then we know that*** $1 \in G$ ***and that the sum and difference of any two members of*** G ***must belong to*** G***. Therefore the equation*** $\cos n\pi = (-1)^n$ ***holds for every integer*** n***.***

2. Prove that if α is any real number then the equation

$$\sin(x + \alpha) = \sin x$$

holds for every real number x if and only if α is an even multiple of π.

Solution: ***We already know that the equation***

$$\sin(x + 2n\pi) = \sin x$$

holds for every number x ***and every integer*** n***. Now suppose that*** α ***is a number for which the equation***

$$\sin(x + \alpha) = \sin x$$

holds for every real number** x**. Since

$$\sin\alpha = \sin(0+\alpha) = \sin 0 = 0$$

we know that** α **is an integer multiple of** π**. We can therefore express** α **as** $n\pi$ **for some integer** n**. Since

$$1 = \sin\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + n\pi\right) = \cos n\pi = (-1)^n$$

we know that** n **is even.

3. Prove that the restriction of the function sin to the interval $[-\pi/2, \pi/2]$ is a strictly increasing function from $[-\pi/2, \pi/2]$ onto the interval $[-1,1]$. Prove that if the function arcsin is now defined to be the inverse function of this restriction of sin then for every number $u \in (-1,1)$ we have

$$\arcsin u = \int_0^u \frac{1}{\sqrt{1-t^2}}\,dt.$$

Solution: ***We have already seen that the restriction of the function** sin **to the interval** $[-\pi/2, \pi/2]$ **is strictly increasing and that the range of this function is the interval** $[-1,1]$**. The function** arcsin **is therefore a strictly increasing continuous function from** $[-1,1]$ **onto** $[-\pi/2, \pi/2]$ **and, from the facts about differentiation of inverse functions** studied earlier **we know that for every number** u **between** -1 **and** 1 **we have***

$$\arcsin'(u) = \frac{1}{\cos(\arcsin u)}$$

Since the function** cos **is nonnegative on the interval** $[-\pi/2, \pi/2]$ **we deduce that

$$\arcsin'(u) = \frac{1}{\cos(\arcsin u)} = \frac{1}{\sqrt{1-\sin^2(\arcsin u)}} = \frac{1}{\sqrt{1-u^2}}$$

Therefore whenever** $-1 < u < 1$ **we have

$$\arcsin u = \int_0^u \frac{1}{\sqrt{1-t^2}}\,dt.$$

4. By analogy with the preceding exercise, give a definition of the function arctan and deduce that if u is any real number then

$$\arctan u = \int_0^u \frac{1}{1+t^2}\,dt$$

The function tan is, of course, defined by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

whenever $\cos x \neq 0$. At any such number x we have

$$\tan' x = \frac{1}{\cos^2 x} > 0$$

and we conclude that the restriction of the function tan to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a strictly increasing function whose derivative is everywhere positive. Furthermore, since

$$\lim_{x\to\pi/2-} \tan x = \lim_{x\to\pi/2-} \frac{\sin x}{\cos x} = \infty$$

and

$$\lim_{x\to-\pi/2+} \tan x = \lim_{x\to-\pi/2+} \frac{\sin x}{\cos x} = -\infty,$$

the range of the restriction of tan to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the entire set $\mathbf{R}$.

We define arctan to be the inverse function of the restriction of tan to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We deduce from the theorem on differentiation of inverse functions that if x is any real number we have

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)} = \frac{1}{\frac{1}{\cos^2(\arctan x)}} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}.$$

It follows from the fundamental theorem that if u is any real number we have

$$\arctan u = \int_0^u \frac{1}{1+t^2}\,dt.$$

5. Prove that the restriction of the function cos to the interval $[0,\pi]$ is a strictly decreasing function from the interval $[0,\pi]$ onto the interval $[-1,1]$. Prove that if the function arccos is now defined to be the inverse function of this restriction of cos then for every number $u \in (-1,1)$ we have

$$\arccos u = \frac{\pi}{2} - \int_0^u \frac{1}{\sqrt{1-t^2}}\,dt.$$

We know that $\cos 0 = 1$ and $\cos \pi = -1$ and that

$$\cos' x = -\sin x < 0$$

whenever $0 < x < \pi$. Therefore $\cos$ is a strictly decreasing differentiable function from the interval $[0,\pi]$ onto $[-1,1]$ whose derivative is positive at every number between -1 and 1. We define $\arccos$ to be the inverse function of the restriction of the function $\cos$ to $[0,\pi]$. Note that because $\sin u \geq 0$ whenever $0 \leq u \leq \pi$ we have the identity

$$\sin(\arccos x) = \sqrt{1-\cos^2(\arccos x)} = \sqrt{1-x^2}$$

whenever $-1 \leq x \leq 1$. We deduce from the theorem on differentiation of inverse functions that if $-1 < x < 1$ then

$$\arccos'(x) = \frac{1}{\cos'(\arccos x)} = -\frac{1}{\sin(\arccos x)} = -\frac{1}{\sqrt{1-x^2}}$$

and it follows from the fundamental theorem that if $-1 < u < 1$ then

$$\begin{aligned}\arccos u &= \arccos 0 + \int_0^u \left(-\frac{1}{1-t^2}\right) dt \\ &= \frac{\pi}{2} - \int_0^u \frac{1}{\sqrt{1-t^2}}\,dt.\end{aligned}$$

6. Prove that if x and y are real numbers that are not both zero and if

$$\alpha = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

then

$$\sin\alpha = \pm\frac{y}{\sqrt{x^2+y^2}}.$$

Deduce that if x and y are real numbers that are not both zero then there exists a positive number r and a real number $\theta \in [0,2\pi)$ such that $x = r\cos\theta$ and $y = r\sin\theta$.

Suppose that x and y are real numbers that are not both zero and that

$$\alpha = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

We see that

$$\begin{aligned}\sin\alpha &= \pm\sqrt{1-\cos^2\alpha} \\ &= \pm\sqrt{1-\cos^2\left(\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)} \\ &= \pm\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2} = \pm\frac{y}{\sqrt{x^2+y^2}}.\end{aligned}$$

We define $r = \sqrt{x^2+y^2}$ and we observe that

$$x = r\cos\alpha$$

and

$$y = \pm r\sin\alpha.$$

In the event that the latter equation says that $y = r\sin\alpha$ we define $\theta = \alpha$ and if the equation $y = r\sin\alpha$ is false (in which case $y = -r\sin\alpha$) we define $\theta = \pi + \alpha$. In either event we have the two equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Finally we observe that since $0 \leq \alpha \leq \pi$ we must have $0 \leq \theta \leq 2\pi$. However, if $\alpha = \pi$ then the equation $y = r\sin\alpha$ is true and $\theta = \alpha = \pi$. Thus the case $\theta = 2\pi$ cannot occur and we conclude that $0 \leq \theta < 2\pi$.

Exercises on Analytic Functions

Prove that the function arctan is analytic on $\mathbf{R}$.

Suppose that c is any number. Using the fact that the rational function whose value at every number x is $\frac{1}{1+x^2}$ is analytic, choose $\delta > 0$ and a sequence (a_n) such that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} a_n (x-c)^n$$

whenever $|x-c| < \delta$. If we now define

$$f(x) = \arctan x - \arctan c - \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

for $|x-c| < 1$ then we see at once that f is the constant 0 and so

$$\arctan x = \arctan c + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

whenever $|x-c| < \delta$.

1. Given that

$$f(x) = \sqrt[3]{x}$$

for every number x, prove that f is analytic on $\mathbf{R} \setminus \{0\}$ but is not analytic on $\mathbf{R}$.

Suppose that $c \neq 0$. The equation

$$f(x) = (c + x - c)^{1/3} = c^{1/3}\left(1 + \frac{x-c}{c}\right)^{1/3} = c^{1/3}\sum_{n=0}^{\infty} \frac{(-1)^n}{c^n}(x-c)^n$$

holds whenever $|x-c| < |c|$. Therefore f is analytic on the set $\mathbf{R} \setminus \{0\}$. Since f is not differentiable at 0, the function f can't be analytic on any open interval that contains 0.

2. Prove that the function tan is analytic in some neighborhood of the number 0.

The function $\tan$, being the quotient of the two analytic functions $\sin$ and $\cos$, is analytic on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

3. Given that

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases},$$

prove that f is analytic on $\mathbf{R} \setminus \{0\}$ but is not analytic on $\mathbf{R}$.

In spite of the fact that this function f has derivatives of all orders at every number, we saw in an earlier example that f can't be expressed as the sum of it's Taylor series center 0 in any neighborhood of 0. On the other hand, the composition theorem for analytic functions guarantees that f is analytic on the set $\mathbf{R} \setminus \{0\}$.

4. Prove that a rational function is analytic on any open set in which its denominator does not vanish.
   This fact follows at once from the fact that the quotient of two analytic functions is analytic as long as the denominator is not zero.

5. Given that

$$f(x) = \begin{cases} (1+x)^{1/x} & \text{if } x \neq 0 \\ e & \text{if } x = 0 \end{cases}$$

   prove that f is analytic on the interval $(-1,1)$. Use *Scientific Notebook* to work out the first few terms of some of the series expansions of this function. To do so, point at the expression $(1+x)^{1/x}$, open the Maple menu and click on the Power Series option.
   We begin by observing that

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

   whenever $|x| < 1$. We now define

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n$$

   and we observe that, since $g(0) = 1$ and

$$g(x) = \frac{\log(1+x)}{x}$$

   for every $x \in (-1,1) \setminus \{0\}$ we have

$$f(x) = \exp(g(x))$$

   whenever $|x| < 1$. Since g is analytic on the interval $(-1,1)$ the composition theorem for analytic functions guarantees that f is analytic on $(-1,1)$.

6. Suppose that $\sum a_n x^n$ and $\sum b_n x^n$ are two power series that converge in an interval $(-r,r)$ where $r > 0$, and suppose that the set of numbers $x \in (-r,r)$ for which the equation

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

   has a limit point in $(-r,r)$. Prove that $a_n = b_n$ for every n.
   The desired result follows at once from the fact that the sum of a power series is analytic and that two analytic functions on an open interval must be identical if they agree on a set that has a limit point in that interval.

7. Given that f is analytic and nonconstant on an open interval U and that $c \in U$, prove that there exists a positive integer n such that $f^{(n)}(c) \neq 0$.
   Since the equation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

   must hold for all x sufficiently close to c the desired result follows at once.

8. Given that f is analytic and nonconstant on an open set U and that K is a closed bounded subset of U, prove that the set

$$\{x \in K \mid f(x) = 0\}$$

   is finite.
   If the set

$$\{x \in K \mid f(x) = 0\}$$

were infinite then it would have to have a limit point in K and so f would have to be the constant function 0.

9. Given that f is analytic and nonconstant on an open interval U prove that the set

$$\{x \in U \mid f(x) = 0\}$$

is countable.

This solution makes use of the distance function of a set. For each positive integer n we define

$$K_n = [-n, n] \cap \left\{x \in U \mid \rho_{\mathbf{R} \setminus U}(x) \le \frac{1}{n}\right\}.$$

Each set K_n is closed and bounded and

$$U = \bigcup_{n=1}^{\infty} K_n.$$

Since

$$\{x \in U \mid f(x) = 0\} = \bigcup_{n=1}^{\infty} \{x \in K_n \mid f(x) = 0\}$$

we deduce that the set

$$\{x \in U \mid f(x) = 0\}$$

is countable.

16 Integration of Functions of Two Variables

Some Exercises on Iterated Riemann Integrals

1. Given that

$$f(x) = \int_0^x \exp(-t^2)dt \qquad g(x) = \int_0^1 \frac{\exp(-x^2(t^2+1))}{t^2+1}dt$$

and given

$$h(x) = (f(x))^2 + g(x)$$

for every real number x, prove that the function h must be constant. What is the value of this constant? Deduce that

$$\int_0^{\to\infty} \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}.$$

For every number x we see from the theorem on differentiating a partial integral that

$$\begin{aligned}
h'(x) &= 2f(x)f'(x) + g'(x) \\
&= 2(\exp(-x^2)) \int_0^x \exp(-t^2)dt + \int_0^1 \frac{-2x(t^2+1)\exp(-x^2(t^2+1))}{t^2+1}dt \\
&= 2(\exp(-x^2)) \int_0^x \exp(-t^2)dt - 2(\exp(-x^2)) \int_0^1 x\exp(-x^2t^2)dt
\end{aligned}$$

In the latter integral we make the substitution $u = tx$ and we deduce that

$$h'(x) = 2(\exp(-x^2)) \int_0^x \exp(-t^2)dt - 2(\exp(-x^2)) \int_0^x \exp(-u^2)du = 0$$

for every number x. Therefore the function h is constant. Now since

$$h(0) = \left(\int_0^0 \exp(-t^2)dt\right)^2 + \int_0^1 \frac{\exp(-0^2(t^2+1))}{t^2+1}dt = \frac{\pi}{4}$$

We deduce that $h(x) = \frac{\pi}{4}$ for every number x.
Now we use the fact that $h(n) = \frac{\pi}{4}$ for every positive integer n. From the bounded convergence theorem we see that

$$\lim_{n\to\infty}\int_0^1 \frac{\exp(-n^2(t^2+1))}{t^2+1}dt = \int_0^1 \frac{0}{t^2+1}dt = 0$$

and so

$$\frac{\pi}{4} = \lim_{n\to\infty} h(n) = \left(\int_0^\infty \exp(-t^2)dt\right)^2 + 0$$

from which we deduce that

$$\int_0^\infty \exp(-t^2)dt = \frac{\sqrt{\pi}}{2}.$$

2. Given that

$$g(y) = \int_0^{\to\infty} \exp(-x^2)\cos 2xydx \qquad \text{and} \qquad f(y) = \exp(y^2)g(y)$$

for every real number y, prove that the function f must be constant. What is the value of this constant?
Since

$$|\exp(-x^2)\cos 2xy| \le \exp(-x^2)$$

and

$$|D_2(\exp(-x^2)\cos 2xy)| = |-2x\exp(-x^2)\sin 2xy| \le 2x\exp(-x^2)$$

for all x and y we see that the hypotheses of the theorem on differentiation of a Partial Improper Riemann Integral are satisfied and so if y is any real number we have

$$g'(y) = -\int_0^{\to\infty} 2x\exp(-x^2)\sin 2xydx.$$

Therefore, for each y we have

$$\begin{aligned}
f'(y) &= 2y\exp(y^2)g(y) + \exp(y^2)g'(y) \\
&= 2y\exp(y^2)g(y) - \exp(y^2)\int_0^{\to\infty} 2x\exp(-x^2)\sin 2xydx \\
&= 2y\exp(y^2)g(y) - \exp(y^2)\lim_{w\to\infty}\int_0^w 2x\exp(-x^2)\sin 2xydx
\end{aligned}$$

We now apply the method of integration by parts to the last integral and obtain

$$\begin{aligned}
f'(y) &= 2y\exp(y^2)g(y) + \exp(y^2)\lim_{w\to\infty}\left(\exp(-w^2)\sin 2wy - 0 - \int_0^w 2y\exp(-x^2)\cos 2xydx\right) \\
&= 2y\exp(y^2)g(y) - \exp(y^2)\left(\int_0^{\to\infty} 2y\exp(-x^2)\cos 2xydx\right) = 0
\end{aligned}$$

and we conclude that the function f is constant. To find the value of this constant we observe that

$$f(0) = \exp(0^2)g(0) = \int_0^{\to\infty} \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}.$$

3. Find an explicit formula for the integral

$$\int_0^{\to\infty} \exp(-x^2)\cos 2xydx.$$

From Exercise 2 we see at once that if y is any real number then

$$\int_0^{\to\infty} \exp(-x^2)\cos 2xydx = \frac{\sqrt{\pi}}{2}\exp(-y^2).$$

4. Evaluate the integral

$$\int_0^{\to\infty}\int_0^{\to\infty}\exp(-x^2)\cos 2xydxdy$$

What happens in this integral if we invert the order of integration?

$$\int_0^{\to\infty}\int_0^{\to\infty}\exp(-x^2)\cos 2xydxdy = \int_0^{\to\infty}\frac{\sqrt{\pi}}{2}\exp(-y^2)dy = \frac{\pi}{4}.$$

If we invert the order of integration, the inside integral diverges. Since the order of integration can't be inverted we know that the given integral converges conditionally.

5. Apply Fichtenholz's theorem to the integral $\int_0^{\to\infty}\int_0^{\to\infty} f(x,y)dxdy$ where

$$f(x,y) = \begin{cases} \exp(-y^3) & \text{if } x < y^2 \\ 0 & \text{if } x \geq y^2 \end{cases}.$$

Now evaluate the integral

$$\int_0^{\to\infty}\int_{\sqrt{x}}^{\to\infty}\exp(-y^3)dydx.$$

We apply the version of the Fichtenholz theorem for improper integrals to obtain

$$\int_0^{\to\infty}\int_0^{\to\infty} f(x,y)dxdy = \int_0^{\to\infty}\int_0^{\to\infty} f(x,y)dydx$$

which gives us

$$\int_0^{\to\infty}\int_0^{y^2}\exp(-y^3)dxdy = \int_0^{\to\infty}\int_{\sqrt{x}}^{\to\infty}\exp(-y^3)dydx$$

and we conclude that

$$\int_0^{\to\infty}\int_{\sqrt{x}}^{\to\infty}\exp(-y^3)dydx = \int_0^{\to\infty} e^{-y^3}y^2dy = \frac{1}{3}$$

6. a. Express the integrand of the following integral in partial fractions and show that if x and y are positive numbers then

$$\int_0^{\to\infty}\frac{1}{(1+t^2x^2)(1+t^2y^2)}dt = \frac{\pi}{2(x+y)}.$$

$$\frac{1}{(1+t^2x^2)(1+t^2y^2)} = \frac{x^2}{(x^2-y^2)(1+t^2x^2)} - \frac{y^2}{(x^2-y^2)(1+t^2y^2)}$$

and so

$$\int_0^{\to\infty}\frac{1}{(1+t^2x^2)(1+t^2y^2)}dt = \lim_{w\to\infty}\int_0^w\frac{x^2}{(x^2-y^2)(1+t^2x^2)}dt - \lim_{w\to\infty}\int_0^w\frac{y^2}{(x^2-y^2)(1+t^2y^2)}dt$$

$$= \lim_{w\to\infty}\frac{x}{x^2-y^2}\arctan(tx) - \lim_{w\to\infty}\frac{y}{x^2-y^2}\arctan(ty)$$

$$= \frac{\pi}{2}\left(\frac{x}{x^2-y^2} - \frac{y}{x^2-y^2}\right) = \frac{\pi}{2(x+y)}.$$

b. Apply Fichtenholz's theorem (more than once) to the integral

$$\int_{\leftarrow 0}^1\int_0^1\frac{\pi}{2(x+y)}dxdy$$

and deduce that

$$\int_0^{\to\infty}\frac{(\arctan x)^2}{x^2}dx = \pi\log 2.$$

We begin by observing that

$$\int_{\leftarrow 0}^{1}\int_{0}^{1}\frac{\pi}{2(x+y)}dxdy = \int_{\leftarrow 0}^{1}\int_{0}^{1}\int_{0}^{\to\infty}\frac{1}{(1+t^2x^2)(1+t^2y^2)}dtdxdy$$
$$= \int_{\leftarrow 0}^{1}\int_{0}^{\to\infty}\int_{0}^{1}\frac{1}{(1+t^2x^2)(1+t^2y^2)}dxdtdy$$
$$= \int_{0}^{\to\infty}\int_{\leftarrow 0}^{1}\int_{0}^{1}\frac{1}{(1+t^2x^2)(1+t^2y^2)}dxdydt$$
$$= \int_{0}^{\to\infty}\int_{\leftarrow 0}^{1}(1+t^2y^2)\frac{1}{t}\arctan t dydt$$
$$= \int_{0}^{\to\infty}\frac{(\arctan t)^2}{t^2}dt$$

On the other hand,

$$\int_{\leftarrow 0}^{1}\int_{0}^{1}\frac{\pi}{2(x+y)}dxdy = \frac{\pi}{2}\int_{\leftarrow 0}^{1}(\log(1+y)-\log y)dy = \pi\log 2.$$

c. Evaluate the integrals

$$\int_{0}^{\pi/2}\frac{x^2}{\sin^2 x}dx \qquad \text{and} \qquad \int_{0}^{\pi/2}x\cot x dx \qquad \text{and} \qquad \int_{\leftarrow 0}^{\pi/2}\log\sin x dx.$$

Solution: ***From the*** change of variable ***theorem we observe that whenever***

$$0 < u < v < \frac{\pi}{2}$$

we have

$$\int_{u}^{v}\frac{x^2}{\sin^2 x}dx = \int_{\tan u}^{\tan v}\frac{(\arctan t)^2}{\sin^2(\arctan t)}\arctan'(t)dt.$$

From the fact that

$$\sin^2(\arctan t) = \cos^2(\arctan t)\tan^2(\arctan t)$$
$$= \frac{t^2}{\sec^2(\arctan t)} = \frac{t^2}{1+t^2}$$

we deduce that

$$\int_{u}^{v}\frac{x^2}{\sin^2 x}dx = \int_{\tan u}^{\tan v}\frac{(\arctan t)^2}{t^2}dt$$

Therefore

$$\int_{0}^{v}\frac{x^2}{\sin^2 x}dx = \lim_{u\to 0+}\int_{u}^{v}\frac{x^2}{\sin^2 x}dx$$
$$= \lim_{u\to 0+}\int_{\tan u}^{\tan v}\frac{(\arctan t)^2}{t^2}dt$$
$$= \int_{0}^{\tan v}\frac{(\arctan t)^2}{t^2}dt.$$

Therefore

$$\int_{0}^{\pi/2}\frac{x^2}{\sin^2 x}dx = \lim_{v\to\pi/2-}\int_{0}^{v}\frac{x^2}{\sin^2 x}dx$$
$$= \lim_{v\to\pi/2-}\int_{0}^{\tan v}\frac{(\arctan t)^2}{t^2}dt$$
$$= \int_{0}^{\to\infty}\frac{(\arctan t)^2}{t^2}dt = \pi\log 2.$$

If $f(x) = -\cot x$ ***for*** $0 < x \le \frac{\pi}{2}$ ***then, for each such number*** x ***we have*** $f'(x) = 1/\sin^2 x$***. Therefore, integrating by parts, we obtain***

$$\int_0^{\pi/2} \frac{x^2}{\sin^2 x} dx = \lim_{u\to 0+} \int_u^{\pi/2} x^2 f'(x) dx$$
$$= \lim_{u\to 0+} \Big[x^2 f(x) \Big]_u^{\pi/2} - \lim_{u\to 0+} \int_u^{\pi/2} 2x f(x) dx$$
$$= \lim_{u\to 0+} \Big[-\frac{x^2}{\sin x} \cos x \Big]_u^{\pi/2} + \lim_{u\to 0+} \int_u^{\pi/2} 2x \cot x dx$$
$$= 2\int_0^{\pi/2} x \cot x dx$$
$$\int_0^{\pi/2} \frac{x^2}{\sin^2 x} dx = 2\int_0^{\pi/2} x \cot x dx$$

and therefore

$$\pi \log 2 = 2\int_0^{\pi/2} x \cot x dx$$

from which we see that

$$\int_0^{\pi/2} x \cot x dx = \frac{\pi \log 2}{2}.$$

Finally, applying integration by parts to the integral

$$\int_u^{\pi/2} \log \sin x dx$$

for $0 < u < \frac{\pi}{2}$***, we obtain***

$$\int_u^{\pi/2} \log \sin x dx = \Big[x \log \sin x \Big]_u^{\pi/2} - \int_u^{\pi/2} x \cot x dx$$

and so

$$\int_{\leftarrow 0}^{\pi/2} \log \sin x dx = \lim_{u\to 0+} \int_u^{\pi/2} \log \sin x dx$$
$$= \lim_{u\to 0+} - u \log \sin u - \frac{\pi \log 2 - 1}{2}$$
$$= -\frac{\pi \log 2}{2}.$$

7. Prove that if f is a function defined on a rectangle $[a,b) \times [c,d)$ then the identity
$$\int_a^{\to b} \int_c^{\to d} f(x,y) dy dx = \int_c^{\to d} \int_a^{\to b} f(x,y) dx dy$$
will hold as long as both sides exist as iterated Improper Riemann integrals and the left side converges absolutely. Hint: Use the fact that $f = (|f| + f) - |f|$.
The desired result follows at once from the fact that the repeated integration can be reversed for each of the functions $|f| + f$ and $|f|$.

8. Given that f is improper Riemann integrable on $[0,\infty)$, that $a \geq 0$ and that $g(u) = f(u-a)$ whenever $u > a$, prove that g is improper Riemann integrable on the interval $[a,\infty)$ and that
$$\int_{\leftarrow 0}^{\to\infty} f(x) dx = \int_{\leftarrow a}^{\to\infty} g(u) du.$$

9. In this exercise we suppose that f and g are nonnegative improper Riemann integrable functions on the interval $(0,\infty)$ and that the function h is defined on the set $(0,\infty) \times (0,\infty)$ by the equation
$$h(x,y) = \begin{cases} f(x-y)g(y) & \text{if } \; y \leq x \\ 0 & \text{if } \; y > x \end{cases}.$$

   a. Prove that

$$\int_{\leftarrow 0}^{\to\infty}\int_{\leftarrow 0}^{\to\infty} h(x,y)dxdy = \int_{\leftarrow 0}^{\to\infty}\int_{\leftarrow y}^{\to\infty} f(x-y)g(y)dxdy = \left(\int_{\leftarrow 0}^{\to\infty} f\right)\left(\int_{\leftarrow 0}^{\to\infty} g\right).$$

All we have to notice is that if $w > 0$ then

$$\int_{\leftarrow y}^{\to\infty} f(x-y)dx = \lim_{w\to 0+}\int_{y+w}^{\to\infty} f(x-y)dx = \lim_{w\to 0+}\int_{w}^{\to\infty} f(t)dt = \int_{\leftarrow 0}^{\to\infty} f.$$

b. Apply Fichtenholz's theorem for improper integrals to the first integral in part a and deduce that the integral is equal to

$$\int_{\leftarrow 0}^{\to\infty}\int_{\leftarrow 0}^{\to x} f(x-y)g(y)dydx$$

To invert the integral we apply the Fichtenholz theorem for improper integrals to each of the integrals

$$\int_{\leftarrow 0}^{1}\int_{\leftarrow 0}^{1} h(x,y)dxdy$$

and

$$\int_{1}^{\to\infty}\int_{1}^{\to\infty} h(x,y)dxdy$$

to obtain

$$\int_{\leftarrow 0}^{\to\infty}\int_{\leftarrow 0}^{\to\infty} h(x,y)dydx = \int_{\leftarrow 0}^{\to\infty}\int_{\leftarrow 0}^{\to\infty} h(x,y)dxdy$$

This equation becomes

$$\int_{\leftarrow 0}^{\to\infty}\int_{\leftarrow 0}^{\to x} f(x-y)g(y)dydx = \left(\int_{\leftarrow 0}^{\to\infty} f\right)\left(\int_{\leftarrow 0}^{\to\infty} g\right).$$

Some Exercises that Explore the Gamma Function

The exercises in this subsection can be used to develop most of the basic facts about the **gamma function** and the related **beta function** that were defined earlier. In these exercises we assume that α and β are given positive numbers. The expressions $\Gamma(\alpha)$ and $B(\alpha,\beta)$ are defined as follows:

$$\Gamma(\alpha) = \int_{0\leftarrow}^{\to\infty} x^{\alpha-1}e^{-x}dx$$

$$B(\alpha,\beta) = \int_{0\leftarrow}^{\to 1} (1-t)^{\alpha-1}t^{\beta-1}dt.$$

1. Apply the above exercise to the functions f and g defined by the equations

$$f(x) = x^{\alpha-1}e^{-x} \quad\text{and}\quad g(x) = x^{\beta-1}e^{-x}$$

for all $x > 0$. Deduce that

$$\Gamma(\alpha)\Gamma(\beta) = \int_{0\leftarrow}^{\to\infty}\int_{0\leftarrow}^{\to x} e^{-x}(x-y)^{\alpha-1}y^{\beta-1}dydx.$$

There really isn't anything to do here. The desired equation

$$\Gamma(\alpha)\Gamma(\beta) = \int_{0\leftarrow}^{\to\infty}\int_{0\leftarrow}^{\to x} e^{-x}(x-y)^{\alpha-1}y^{\beta-1}dydx$$

follows at once from the earlier exercise.

2. By making the substitution $y = ux$ in the inside integral in Exercise 1, deduce that

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta).$$

The substitution $y = ux$ gives us

$$\int_{0\leftarrow}^{\to\infty}\int_{0\leftarrow}^{\to x} e^{-x}(x-y)^{\alpha-1}y^{\beta-1}dydx = \int_{0\leftarrow}^{\to\infty}\int_{0\leftarrow}^{\to 1} e^{-x}(x-ux)^{\alpha-1}(ux)^{\beta-1}xdudx$$
$$= \int_{0\leftarrow}^{\to\infty} e^{-x}x^{\alpha-1+\beta-1+1}\left(\int_{0\leftarrow}^{\to 1}(1-u)^{\alpha-1}u^{\beta-1}du\right)dx$$
$$= \Gamma(\alpha+\beta)B(\alpha,\beta).$$

3. Apply the method of integration by parts to the integral that defines $\Gamma(\alpha)$ and deduce that
$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$
$$\Gamma(\alpha+1) = \int_{0\leftarrow}^{\to\infty} t^\alpha e^{-t}dt = \lim_{n\to\infty}\int_{1/n}^{n} t^\alpha e^{-t}dt$$
$$= \lim_{n\to\infty}\left(-n^\alpha e^{-n} + \left(\frac{1}{n}\right)^\alpha e^{-1/n} - \int_{1/n}^{n}\alpha t^{\alpha-1}(-e^{-t})dt\right) = \alpha\Gamma(\alpha).$$

4. Make the substitution $t = \sin^2 y$ in the definition of the beta function, deduce that
$$B(\alpha,\beta) = 2\int_{0\leftarrow}^{\to\pi/2}\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta d\theta.$$
This exercise is a routine manipulation.
$$\int_0^1 (1-t)^{\alpha-1}t^{\beta-1}dt$$

5. Use Exercise 4 to evaluate $B\left(\frac{1}{2},\frac{1}{2}\right)$ and deduce that
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$
This exercise is a routine manipulation.

6. Make the substitution $t = u^2$ in the definition of $\Gamma(\alpha)$ and deduce that
$$\Gamma(\alpha) = 2\int_{0\leftarrow}^{\to\infty} u^{2\alpha-1}\exp(-u^2)du$$
and then use Exercise 5 to find another way of showing that
$$\int_{0\leftarrow}^{\to\infty}\exp(-x^2)dx = \frac{\sqrt{\pi}}{2}.$$
Recall that we obtained this identity in an earlier exercise.
This exercise is a routine manipulation.

7. With an eye on the exercises in an earlier subsection, prove that if $p > -1$ then
$$\int_{0\leftarrow}^{\to\pi}\sin^p\theta d\theta = 2\int_{0\leftarrow}^{\pi/2}\sin^p\theta d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}.$$
Suppose that $p > -1$. Using Exercise 2 we obtain
$$\int_{0\leftarrow}^{\to\pi}\sin^p\theta d\theta = \int_{0\leftarrow}^{\to\pi}\sin^{2\left(\frac{p+1}{2}\right)-1}\theta\cos^{2(1/2)-1}\theta d\theta$$
$$= \frac{1}{2}B\left(\frac{p+1}{2},\frac{1}{2}\right)$$
$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}.$$

8. Prove that if $\alpha > 0$ then
$$B(\alpha,\alpha) = 2^{1-2\alpha}B\left(\alpha,\frac{1}{2}\right)$$
and deduce that
$$\sqrt{\pi}\,\Gamma(2\alpha) = 2^{2\alpha-1}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right).$$

Hint: ***Suppose that*** $\alpha > 0$***. First observe that***

$$\begin{aligned} B(\alpha,\alpha) &= 2\int_0^{\pi/2} \cos^{2\alpha-1}\theta \sin^{2\alpha-1}\theta d\theta \\ &= \frac{2}{2^{2\alpha-1}} \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2\alpha-1} d\theta \\ &= \frac{2}{2^{2\alpha-1}} \int_0^{\pi/2} (\sin 2\theta)^{2\alpha-1} d\theta \end{aligned}$$

Then make the substitution $u = 2\theta$ ***and use Exercise 2.***

$$\begin{aligned} B(\alpha,\alpha) &= \frac{2}{2^{2\alpha-1}} \int_0^{\pi/2} (\sin 2\theta)^{2\alpha-1} \cos^{2(1/2)-1}\theta d\theta \\ &= 2^{1-2\alpha} B\left(\alpha, \frac{1}{2}\right) \end{aligned}$$

and so

$$\frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(\alpha+\alpha)} = 2^{1-2\alpha} \frac{\Gamma(\alpha)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)}$$

which gives us the desired result.

9. Prove that

$$\int_0^{\to\pi/2} \sqrt{\tan x}\, dx = \frac{\pi}{\sqrt{2}}.$$

The intention of this exercise is to express the left side as

$$\begin{aligned} \int_0^{\to\pi/2} \sin^{2(3/4)-1}\theta \cos^{2(1/4)-1}\theta d\theta &= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) \\ &= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \end{aligned}$$

Now if we put $\alpha = \frac{1}{4}$ in the identity

$$\sqrt{\pi}\,\Gamma(2\alpha) = 2^{2\alpha-1}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)$$

we see that

$$\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right) = 2^{2\left(\frac{1}{4}\right)-1}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

and so

$$\int_0^{\to\pi/2} \sqrt{\tan x}\, dx = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sqrt{2}}$$

It is worth mentioning that this integral could have been evaluated with the substitution $u = \sqrt{\tan x}$. As a matter of fact, even the indefinite integral

$$\int \sqrt{\tan x}\, dx$$

can be evaluated in this way to give

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \arccos(\cos x - \sin x) - \frac{1}{\sqrt{2}} \log\left(\cos x + \sqrt{2}\sqrt{\cos x \sin x} + \sin x\right)$$

and you might want to challenge your students to come up with

$$\int \sqrt[3]{\tan x}\, dx = -\frac{1}{2}\log\left(\sqrt[3]{\tan^2 x}+1\right) + \frac{1}{4}\log\left(\sqrt[3]{\tan^4 x} - \sqrt[3]{\tan^2 x}+1\right) + \frac{\sqrt{3}}{2}\arctan\frac{1}{\sqrt{3}}\left(2\sqrt[3]{\tan^2 x}-1\right)$$

and that

$$\int_0^{\pi/2} \sqrt[3]{\tan x}\, dx = \frac{\pi}{\sqrt{3}}.$$

10. In this exercise we define

$$\phi(p) = \int_0^\pi \sin^p\theta d\theta$$

whenever $p > 0$.

a. Prove that ϕ is a decreasing function on the interval $(0, \infty)$.
The fact that ϕ is decreasing follows from the fact that whenever $0 \le \theta \le \pi$ we have $0 \le \sin\theta \le 1$ and so $\sin^p\theta$ decreases as p increases.

b. Prove that

$$\lim_{p\to\infty} \frac{\phi(2p+2)}{\phi(2p)} = 1.$$

Given any positive number p we integrate by parts and obtain

$$\begin{aligned}\phi(2p+2) &= \int_0^\pi \sin^{2p+2}\theta d\theta = \int_0^\pi (\sin^{2p+1}\theta)(\sin\theta)d\theta \\ &= (\sin^{2p+1}\pi)(-\cos\pi) - (\sin^{2p+1}0)(-\cos 0) - \int_0^\pi ((2p+1)\sin^{2p}\theta\cos\theta)(-\cos\theta)d\theta \\ &= (2p+1)\int_0^\pi \sin^{2p}\theta(1-\sin^2\theta)d\theta = (2p+1)\phi(2p) - (2p+1)\phi(2p+2)\end{aligned}$$

and so

$$\lim_{p\to\infty} \frac{\phi(2p+2)}{\phi(2p)} = \lim_{p\to\infty} \frac{2p+1}{2p+2} = 1.$$

c. Combine the first two parts of this question and deduce that

$$\lim_{p\to\infty} \frac{\phi(2p+1)}{\phi(2p)} = 1.$$

For each $p > 0$ we have

$$\phi(2p+2) \le \phi(2p+1) \le \phi(2p)$$

and so

$$\frac{\phi(2p+2)}{\phi(2p)} \le \frac{\phi(2p+1)}{\phi(2p)} \le 1$$

and the fact that

$$\lim_{p\to\infty} \frac{\phi(2p+1)}{\phi(2p)} = 1$$

follows from the sandwich rule.

d. Prove that

$$\lim_{p\to\infty} \frac{4^p\Gamma^2(p+1)}{\sqrt{p}\,\Gamma(2p+1)} = \sqrt{\pi}.$$

This assertion is known as **Wallis's formula**
From Exercise 7 we know that whenever $p > 0$,

$$\phi(2p+1) = \int_{0\leftarrow}^{\to\pi} \sin^{2p+1}\theta d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{2p+1+1}{2}\right)}{\Gamma\left(\frac{2p+1}{2}+1\right)}$$

and

$$\phi(2p) = \int_{0\leftarrow}^{\to\pi} \sin^{2p}\theta d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{2p+1}{2}\right)}{\Gamma\left(\frac{2p}{2}+1\right)}$$

which gives us

$$\frac{\phi(2p+1)}{\phi(2p)} = \frac{\frac{\sqrt{\pi}\,\Gamma\left(\frac{2p+1+1}{2}\right)}{\Gamma\left(\frac{2p+1}{2}+1\right)}}{\frac{\sqrt{\pi}\,\Gamma\left(\frac{2p+1}{2}\right)}{\Gamma\left(\frac{2p}{2}+1\right)}} = \frac{\Gamma^2(p+1)}{\left(p+\frac{1}{2}\right)\Gamma^2\left(p+\frac{1}{2}\right)}$$

Now from Exercise 8 we have

$$\sqrt{\pi}\,\Gamma(2p) = 2^{2p-1}\Gamma(p)\Gamma\left(p+\frac{1}{2}\right)$$

and so

$$\begin{aligned}\frac{\phi(2p+1)}{\phi(2p)} &= \frac{\Gamma^2(p+1)}{\left(p+\frac{1}{2}\right)\Gamma^2\left(p+\frac{1}{2}\right)} = \frac{\Gamma^2(p+1)}{\left(p+\frac{1}{2}\right)\left(\sqrt{\pi}\,\frac{\Gamma(2p)}{2^{2p-1}\Gamma(p)}\right)^2} \\ &= \frac{2^{4p-2}\Gamma^2(p+1)\Gamma^2(p)}{\pi\left(p+\frac{1}{2}\right)\Gamma^2(2p)} = \frac{2^{4p}\Gamma^2(p+1)(p\Gamma(p))^2}{\pi\left(p+\frac{1}{2}\right)(2p\Gamma(2p))^2} \\ &= \frac{2^{4p}\Gamma^4(p+1)}{\left(p+\frac{1}{2}\right)\pi\Gamma^2(2p+1)}\end{aligned}$$

and so

$$\lim_{p\to\infty}\frac{2^{4p}\Gamma^4(p+1)}{\left(p+\frac{1}{2}\right)\pi\Gamma^2(2p+1)} = 1$$

from which we deduce that

$$\lim_{p\to\infty}\frac{4^p\Gamma^2(p+1)}{\sqrt{p}\,\Gamma(2p+1)} = \lim_{p\to\infty}\frac{4^p\Gamma^2(p+1)}{\sqrt{p+\frac{1}{2}}\,\Gamma(2p+1)}\sqrt{\frac{p+\frac{1}{2}}{p}} = \sqrt{\pi}$$

e. Assuming that p is restricted to be a positive integer, rewrite Wallis's formula in terms of factorials.

If p is restricted to be a positive integer then Wallis's formula becomes

$$\lim_{p\to\infty}\frac{4^p(p!)^2}{\sqrt{p}\,(2p)!} = \sqrt{\pi}$$

11. The purpose of this exercise is to encourage you to read a proof of an interesting theorem known as **Stirling's formula** that states that

$$\lim_{x\to\infty}\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{x}} = \sqrt{2\pi}\,.$$

Note that if n is a large natural number then Stirling's formula suggests that an approximate value for $n!$ is

$$\frac{\sqrt{2\pi n}\,n^n}{e^n}.$$

The proof of Stirling's formula that is provided by the preceding link is based on a proof that is provided on page 195 of Walter Rudin's classic text Principles of Mathematical Analysis. The proof provided here is actually a little simpler because it makes use of the improper integral form of the bounded convergence theorem.

17 Sets of Measure Zero

Some Exercises on the Measure Zero Concept

1. Prove that an elementary set E has measure zero if and only if $m(E) = 0$.

Suppose that E is an elementary set. In the event that $m(E) = 0$, it is clear that E has (nineteenth century) measure zero. On the other hand, if $m(E) > 0$ then E includes a closed bounded interval of positive length and it follows from the discussion on closed bounded sets of measure zero that E can't have measure zero.

2. Prove that every countable set has measure zero.
   This fact follows at once from the theorem on unions of sets of measure zero the fact that a singleton has measure zero.

3. Give an example of an uncountable closed bounded set that has measure zero.
   We have already observed that the Cantor set has measure zero.

4. Prove that if U is a nonempty open set then U has a closed bounded subset H that does not have measure zero.
   A nonempty open set must include a closed bounded interval with positive length.

5. Prove that the set of all irrational numbers in the interval $[0,1]$ does not have measure zero.
   The set $[0,1]\cap\mathbf{Q}$, being countable, must have measure zero. Since
$$[0,1] = \left([0,1] \cap \mathbf{Q}\right) \cup \left([0,1] \setminus \mathbf{Q}\right)$$
   and since $[0,1]$ does not have measure zero, we deduce from the theorem on unions that the set $[0,1] \setminus \mathbf{Q}$ does not have measure zero.

6. For the purposes of this exercise we agree to call two sets A and B **almost equal to each other** if both of the sets $A \setminus B$ and $B \setminus A$ have measure zero. Prove that if (A_n) and (B_n) are sequences of sets and if A_n and B_n are almost equal to each other then the sets
$$\bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} B_n$$
   are almost equal to each other. Can the same assertion be made for intersections?
   We observe that
$$\bigcup_{n=1}^{\infty} A_n \setminus \bigcup_{j=1}^{\infty} B_j = \bigcup_{n=1}^{\infty}\bigcap_{j=1}^{\infty} A_n \setminus B_j.$$
   Now, given any n, since the set $A_n \setminus B_j$ has measure zero for all j, we see at once that the set
$$\bigcap_{j=1}^{\infty} A_n \setminus B_j$$
   has measure zero. It follows from the theorem on unions that the set
$$\bigcup_{n=1}^{\infty} A_n \setminus \bigcup_{j=1}^{\infty} B_j$$
   has measure zero. We see similarly that the set
$$\bigcup_{j=1}^{\infty} B_j \setminus \bigcup_{n=1}^{\infty} A_n$$
   has measure zero.
   Now we consider intersections. Since
$$\bigcap_{n=1}^{\infty} A_n \setminus \bigcap_{j=1}^{\infty} B_j = \bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty} A_n \setminus B_j$$
   we see easily that this set too has measure zero.

7. Prove that if A has measure zero and if $\varepsilon > 0$ then there exists a sequence (U_n) of open elementary sets whose union includes A and for which $m(U_n) < \varepsilon$ for every n.
   Suppose that A has measure zero and that $\varepsilon > 0$. Choose an expanding sequence (E_n) of elementary sets such that $m(E_n) < \varepsilon/2$ for every n and such that

$$A \subseteq \bigcup_{n=1}^{\infty} E_n.$$

For each n, choose an open elementary set U_n that includes E_n such that $m(U_n) < \varepsilon$.

8. Prove that if A has measure zero and if $\varepsilon > 0$ then there exists an expanding sequence (U_n) of open elementary sets whose union includes A and for which $m(U_n) < \varepsilon$ for every n.

   Hint: ***Suppose that*** A ***has measure zero and that*** $\varepsilon > 0$***. Follow the step by step procedure outlined below:***

   a. ***Choose an expanding sequence*** (E_n) ***of elementary sets such that*** $m(E_n) < \varepsilon/2$ ***for every*** n ***and such that***
   $$A \subseteq \bigcup_{n=1}^{\infty} E_n.$$
   For each n***, choose an open elementary set*** U_n ***such that*** $E_n \subseteq U_n$ ***and such that***
   $$m(U_n) < m(E_n) + \frac{\varepsilon}{2^{n+1}}.$$

   b. ***For each*** n ***define***
   $$V_n = \bigcup_{j=1}^{n} U_j,$$
   observe that
   $$V_n = U_n \cup (U_1 \setminus E_1) \cup (U_2 \setminus E_2) \cup \cdots \cup (U_{n-1} \setminus E_{n-1})$$
   and then deduce that $m(V_n) < \varepsilon$ ***for each*** n***.***

Some Exercises on Integrability

1. If S is a set of real numbers then the *boundary* of S is defined to be the set
   $$\bar{S} \cap \overline{\mathbf{R} \setminus S}.$$
   Prove that if S is a subset of an interval $[a,b]$ then the function χ_S is Riemann integrable on $[a,b]$ if and only if the boundary of S has measure zero.
   The function χ_S fails to be continuous at a number x if and only if x belongs to the boundary of S.

2. If S is a set of real numbers then the *boundary* of S is defined to be the set
   $$\bar{S} \cap \overline{\mathbf{R} \setminus S}.$$
   Prove that if S is a subset of an interval $[a,b]$ then the function χ_S is Riemann integrable on $[a,b]$ if and only if the boundary of S has nineteenth century measure zero.
   The boundary of a set is closed and bounded. Therefore Exercises 1 and 2 are saying exactly the same thing.

3. Suppose that (E_n) is a contracting sequence of closed elementary subsets of $[0,1]$ whose intersection C does not contain any interval of positive length. Prove that the function χ_C is Riemann integrable on $[0,1]$ if and only if the set C has measure zero.
   The fact that C does not contain any interval of positive length tells us that
   $$[0,1] \subseteq \overline{[0,1] \setminus C}.$$
   Therefore, since C is closed, the boundary of C must be C itself. The desired result therefore follows at once from Exercise 1.

4. Suppose that $0 < \delta \leq 1/3$. The δ-*Cantor set* $C(\delta)$ is defined as follows: We begin by defining
   $$E_0(\delta) = [0,1].$$

For every nonnegative integer n, if the elementary set E_n has already been defined then we obtain $E_{n+1}(\delta)$ from $E_n(\delta)$ by removing from each of its component intervals the centrally located open interval of length $\delta(1/3)^n$. Then we define

$$C(\delta) = \bigcap_{n=1}^{\infty} E_n(\delta).$$

a. Show that $\delta = 1/3$ then the δ-Cantor set is just the "usual" Cantor set C that was defined earlier.

Solution: ***This assertion was proved in our earlier discussion of the Cantor set.***

b. Prove that the δ-Cantor set contains no interval of positive length.

Solution: ***Since each set $E_n(\delta)$ is the union of 2^n closed intervals each of length less than $1/2^n$, that C contains no interval of positive length follows from the discussion of the Cantor set that was given earlier.***

c. Evaluate the integral

$$\int_0^1 \chi_{E_n(\delta)}$$

and deduce that the δ-Cantor set has measure zero if and only if $\delta = 1/3$. Because of this fact, it is traditional to call the Cantor set $C(1/3)$ the *thin Cantor set* and to call the sets $C(\delta)$ *fat Cantor sets* when $0 < \delta < 1/3$.

Solution: ***For each n, the set $E_n(\delta)$ is obtained by removing 2^n non-overlapping open intervals of length $\delta(1/3)^{n-1}$ from the set $E_{n-1}(\delta)$. Thus $E_n(\delta)$ is obtained from $E_{n-1}(\delta)$ by removing an elementary set with measure $\delta(2/3)^{n-1}$. Therefore, if n is any positive integer, the set $E_n(\delta)$ is obtained from the interval $[0,1]$ by removing an elementary set with measure***

$$\sum_{j=1}^{n} \delta\left(\frac{2}{3}\right)^{j-1} = 3\delta\left(1 - \left(\frac{2}{3}\right)^n\right)$$

and we conclude that

$$m(E_n(\delta)) = 1 - 3\delta + 3\delta\left(\frac{2}{3}\right)^n$$

In the event that $\delta = 1/3$, it follows from the fact that $m(E_n(\delta)) \to 0$ as $n \to \infty$ that the set $C(\delta)$ has measure zero.

Suppose now that $\delta < 1/3$. Since the set $C(\delta)$ is closed and bounded, in order to show that $C(\delta)$ fails to have measure zero, all we have to do is show that whenever an elementary set E includes $C(\delta)$, we have $m(E) \geq 1 - 3\delta$. To obtain a contradiction, suppose that E is an elementary set, that $C(\delta) \subseteq E$, and that $m(E) < 1 - 3\delta$. Choose an open elementary set U such that $E \subseteq U$ and $m(U) < 1 - 3\delta$. Since

$$\bigcap_{n=1}^{\infty}(E_n(\delta) \setminus U) = \bigcap_{n=1}^{\infty}(E_n(\delta)) \setminus U = C(\delta) \setminus U = \emptyset$$

it follows from the Cantor intersection theorem that, for some n, we have $E_n \subseteq U$ and we deduce that

$$m(U) \geq m(E_n(\delta)) = 1 - 3\delta + 3\delta\left(\frac{2}{3}\right)^n > 1 - 3\delta.$$

This is the desired contradiction.

d. Prove that the function $\chi_{C(\delta)}$ is Riemann integrable on $[0,1]$ if and only if $\delta = 1/3$.

Solution: ***This assertion follows at once from part c and the fact that the set $C(\delta)$ is the set of discontinuities of the function $\chi_{C(\delta)}$.***

e. Given $0 < \delta \leq 1/3$, prove that there is a strictly increasing continuous function from $[0,1]$ onto $[0,1]$ that sends the set $C(\delta)$ onto the usual Cantor set C.

Solution: ***For each number $x \in C(\delta)$ we define the function $\hat{x}$ from Z^+ into $\{0,2\}$ as follows: Given any positive integer n, if the number x belongs to one of the 2^n component intervals I of $E_{n-1}(\delta)$ and if, after I is split into two subintervals by the removal of its centrally located open subinterval of I***

with length $\delta(1/3)^{n-1}$, the number x lies in the left subinterval we define $\hat{x}(n) = 0$ and if x lies in the right subinterval we define $\hat{x}(n) = 2$. We call the function $\hat{x}$ the address of the number x. We now define

$$u(x) = \sum_{n=1}^{\infty} \frac{\hat{x}(n)}{3^n}$$

for every number $x \in C(\delta)$ and we observe that u is a strictly increasing continuous function from $C(\delta)$ onto $C(1/3)$.
We now extend u to be a function defined on the entire interval $[0,1]$ as follows: Given $x \in [0,1] \setminus C(\delta)$, if p is the greatest member of $C(\delta)$ that is less than x and q is the least member of $C(\delta)$ that is greater than x then we define

$$u(x) = u(p) + \left(\frac{u(q) - u(p)}{q - p}\right)(x - p).$$

f. Prove that if $f = \chi_C$ where C is the usual Cantor set and if u is the function found in part e then, although u is continuous on $[0,1]$ and f is Riemann integrable on the range of u, the function $f \circ u$ is not Riemann integrable on $[0,1]$.
Solution: ***This assertion follows at once from the fact that***

$$f \circ u = \chi_{C(\delta)}.$$

5. In this exercise we introduce a further extension of the notion of extended Riemann integrability that was introduced in some earlier optional reading. We shall say that a function f defined on an interval $[a,b]$ is *almost extended Riemann integrable* on $[a,b]$ if there exists a sequence (f_n) of Riemann integrable functions and a number K such that the following two conditions hold

- For each n we have $|f_n(x)| \leq K$ for almost every $x \in [a,b]$.
- For almost every $x \in [a,b]$ we have $f_n(x) \to f(x)$ as $n \to \infty$.

a. Given an almost extended Riemann integrable function f on an interval $[a,b]$, give a reasonable definition of the integral $\int_a^b f$. Take care to say why your definition makes sense. Which theorem are you using for this purpose?

b. Prove that the integration of almost extended Riemann integrable functions is linear, nonnegative and additive.

c. Give an example of a function that is almost extended Riemann integrable on $[0,1]$ but is not Riemann integrable on $[0,1]$.

6. An example of Sierpinski shows that it is possible to find a subset S of the square

$$[0,1] \times [0,1]$$

such that for every $x \in [0,1]$ the vertical x-section S_x of S has measure zero and for every $y \in [0,1]$ the horizontal y-section S^y contains almost every number in $[0,1]$. This example is quite easy to produce if you have studied enough set theory and it can be found here. Unless you have read the set theory, must assume the existence of the Sierpinski example for the purposes of this exercise.
Using Sierpinski's example, prove that the statement of Fichtenholz's theorem becomes false if we widen it to include functions that are almost Riemann integrable.

Exercises 5 and 6 are offered as a resource for a special project.

Alternative 17 Sets of Measure Zero

Some Exercises on the Measure Zero Concept

1. Prove that an elementary set E has measure zero if and only if $m(E) = 0$.
   Suppose that E is an elementary set. In the event that $m(E) = 0$, it is clear that E has (nineteenth century) measure zero. On the other hand, if $m(E) > 0$ then E includes a closed bounded interval of positive length and it follows from the discussion on closed bounded sets of measure zero that E can't have measure zero.

2. Prove that every countable set has measure zero.
   This fact follows at once from the theorem on unions of sets of measure zero the fact that a singleton has measure zero.

3. Give an example of an uncountable closed bounded set that has measure zero.
   We have already observed that the Cantor set has measure zero.

4. Prove that if U is a nonempty open set then U has a closed bounded subset H that does not have measure zero.
   A nonempty open set must include a closed bounded interval with positive length.

5. Prove that the set of all irrational numbers in the interval $[0,1]$ does not have measure zero.
   The set $[0,1]\cap\mathbf{Q}$, being countable, must have measure zero. Since
$$[0,1] = \left([0,1]\cap\mathbf{Q}\right)\cup\left([0,1]\setminus\mathbf{Q}\right)$$

6. For the purposes of this exercise we agree to call two sets A and B **almost equal to each other** if both of the sets $A\setminus B$ and $B\setminus A$ have measure zero. Prove that if (A_n) and (B_n) are sequences of sets and if A_n and B_n are almost equal to each other then the sets
$$\bigcup_{n=1}^{\infty}A_n \qquad \text{and} \qquad \bigcup_{n=1}^{\infty}B_n$$
   are almost equal to each other. Can the same assertion be made for intersections?
   We observe that
$$\bigcup_{n=1}^{\infty}A_n\setminus\bigcup_{j=1}^{\infty}B_j = \bigcup_{n=1}^{\infty}\bigcap_{j=1}^{\infty}A_n\setminus B_j.$$
   Now, given any n, since the set $A_n\setminus B_j$ has measure zero for all j, we see at once that the set
$$\bigcap_{j=1}^{\infty}A_n\setminus B_j$$
   has measure zero. It follows from the theorem on unions that the set
$$\bigcup_{n=1}^{\infty}A_n\setminus\bigcup_{j=1}^{\infty}B_j$$
   has measure zero. We see similarly that the set
$$\bigcup_{j=1}^{\infty}B_j\setminus\bigcup_{n=1}^{\infty}A_n$$
   has measure zero.
   Now we consider intersections. Since
$$\bigcap_{n=1}^{\infty}A_n\setminus\bigcap_{j=1}^{\infty}B_j = \bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty}A_n\setminus B_j$$
   we see easily that this set too has measure zero.

7. Prove that if A has measure zero and if $\varepsilon > 0$ then there exists a sequence (U_n) of open elementary sets whose union includes A and for which $m(U_n) < \varepsilon$ for every n.

8. Prove that if A has measure zero and if $\varepsilon > 0$ then there exists an expanding sequence (U_n) of open elementary sets whose union includes A and for which $m(U_n) < \varepsilon$ for every n.

Hint: ***Suppose that A has measure zero and that $\varepsilon > 0$. Follow the step by step procedure outlined below:***

1. a. ***Choose an expanding sequence (E_n) of elementary sets such that $m(E_n) < \varepsilon/2$ for every n and such that***

$$A \subseteq \bigcup_{n=1}^{\infty} E_n.$$

For each n, choose an open elementary set U_n such that $E_n \subseteq U_n$ and such that

$$m(U_n) < m(E_n) + \frac{\varepsilon}{2^{n+1}}.$$

b. ***For each n define***

$$V_n = \bigcup_{j=1}^{n} U_j,$$

observe that

$$V_n = U_n \cup (U_1 \setminus E_1) \cup (U_2 \setminus E_2) \cup \cdots \cup (U_{n-1} \setminus E_{n-1})$$

and then deduce that $m(V_n) < \varepsilon$ for each n.

9. Prove that if ϕ is an increasing function and A is ϕ-nul and if $\varepsilon > 0$ then there exists an expanding sequence (U_n) of open elementary sets whose union includes A and for which $\operatorname{var}(\phi, U_n) < \varepsilon$ for every n.
The solution to this exercise is almost identical to that of Exercise 8.

10. Prove that if ϕ is an increasing function and if H is a closed bounded ϕ-null set then for every $\varepsilon > 0$ there exists an elementary set E such that $H \subseteq E$ and $\operatorname{var}(\phi, E) < \varepsilon$.
The solution to this exercise is almost a carbon copy of the proof of the theorem on closed bounded sets of measure zero.

11. Prove that if ϕ is an increasing function and x is a real number then the set $\{x\}$ is ϕ-null if and only if ϕ is continuous at x.
A direct proof of this assertion is very simple but it also follows at once from Exercise 10.

12. Prove that if ϕ is an increasing function and U is an interval then U is ϕ-null if and only if ϕ is constant on U.
In the event that an increasing function ϕ is constant on an interval U, the set U is clearly ϕ-null. In the event that an increasing function ϕ is not constant on an interval U, then ϕ fails to be constant on a closed bounded interval H that is included in U, and it follows from Exercise 10 that the subset H fails to be ϕ-null.

13. Prove that if ϕ is an increasing function and A is a countable set then A is ϕ-null if and only if ϕ is continuous on A.
This assertion follows at once from Exercise 11 and the theorem on unions.

14. Prove that if ϕ is the Cantor function and C is the Cantor set then, although C has measure zero, it is not ϕ-null.
As we know, the Cantor set C has measure zero. Now suppose that ϕ is the Cantor function. To obtain a contradiction, suppose that the Cantor set C is ϕ null, and, using Exercise 10, choose an elementary set E that includes C such that $\operatorname{var}(\phi, E) < 1$. Choose an open elementary set U such that $E \subseteq U$ and $\operatorname{var}(\phi, U) < 1$. We now define the elementary sets E_n as in the discussion of the Cantor set. Since

$$\bigcap_{n=1}^{\infty} (E_n \setminus U) = \left(\bigcap_{n=1}^{\infty} E_n \right) \setminus U = \emptyset,$$

it follows from the the Cantor intersection theorem that for some positive integer we have $E_n \subseteq U$.

Since $\operatorname{var}(\phi, E_n) = 1$ we have contradicted the fact that $\operatorname{var}(\phi, U) < 1$.

15. Prove that if ϕ is an increasing function and U is an open set that is not ϕ-null then U has a closed bounded subset that is not ϕ-null.
We shall show that if every closed bounded subset of a given open set U is ϕ-null then U is ϕ-null. Suppose that U is open and that every closed bounded subset of U is ϕ-null. Suppose that $\varepsilon > 0$. For each positive integer n we define

$$H_n = \left\{x \in \mathbf{R} \mid \rho_{\mathbf{R}\setminus U}(x) \geq \frac{1}{n}\right\} \cap [-n, n]$$

where $\rho_{\mathbf{R}\setminus U}$ stands for the distance function of the set $\mathbf{R} \setminus U$. Since each set H_n is closed and bounded, it is ϕ-null. Therefore, since

$$U = \bigcup_{n=1}^{\infty} H_n$$

it follows from the theorem on unions that U is ϕ-null.

Some Exercises on Integrability

1. If S is a set of real numbers then the *boundary* of S is defined to be the set

$$\overline{S} \cap \overline{\mathbf{R} \setminus S}.$$

Prove that if S is a subset of an interval $[a, b]$ then the function χ_S is Riemann integrable on $[a, b]$ if and only if the boundary of S has measure zero.
The function χ_S fails to be continuous at a number x if and only if x belongs to the boundary of S.

2. If S is a set of real numbers then the *boundary* of S is defined to be the set

$$\overline{S} \cap \overline{\mathbf{R} \setminus S}.$$

Prove that if S is a subset of an interval $[a, b]$ then the function χ_S is Riemann integrable on $[a, b]$ if and only if the boundary of S has nineteenth century measure zero.
The boundary of a set is closed and bounded. Therefore Exercises 1 and 2 are saying exactly the same thing.

3. Suppose that (E_n) is a contracting sequence of closed elementary subsets of $[0, 1]$ whose intersection C does not contain any interval of positive length. Prove that the function χ_C is Riemann integrable on $[0, 1]$ if and only if the set C has measure zero.
The fact that C does not contain any interval of positive length tells us that

$$[0, 1] \subseteq \overline{[0, 1] \setminus C}.$$

Therefore, since C is closed, the boundary of C must be C itself. The desired result therefore follows at once from Exercise 1.

4. Suppose that $0 < \delta \leq 1/3$. The *δ-Cantor set* $C(\delta)$ is defined as follows: We begin by defining

$$E_0(\delta) = [0, 1].$$

For every nonnegative integer n, if the elementary set E_n has already been defined then we obtain $E_{n+1}(\delta)$ from $E_n(\delta)$ by removing from each of its component intervals the centrally located open interval of length $\delta(1/3)^n$. Then we define

$$C(\delta) = \bigcap_{n=1}^{\infty} E_n(\delta).$$

a. Show that $\delta = 1/3$ then the δ-Cantor set is just the "usual" Cantor set C that was defined earlier.
Solution: ***This assertion was proved in our earlier discussion of the Cantor set.***

b. Prove that the δ-Cantor set contains no interval of positive length.

Solution: ***Since each set $E_n(\delta)$ is the union of 2^n closed intervals each of length less than $1/2^n$, that C contains no interval of positive length follows from the discussion of the Cantor set that was given earlier.***

c. Evaluate the integral

$$\int_0^1 \chi_{E_n(\delta)}$$

and deduce that the δ-Cantor set has measure zero if and only if $\delta = 1/3$. Because of this fact, it is traditional to call the Cantor set $C(1/3)$ the *thin Cantor set* and to call the sets $C(\delta)$ *fat Cantor sets* when $0 < \delta < 1/3$.

Solution: ***For each n, the set $E_n(\delta)$ is obtained by removing 2^n non-overlapping open intervals of length $\delta(1/3)^{n-1}$ from the set $E_{n-1}(\delta)$. Thus $E_n(\delta)$ is obtained from $E_{n-1}(\delta)$ by removing an elementary set with measure $\delta(2/3)^{n-1}$. Therefore, if n is any positive integer, the set $E_n(\delta)$ is obtained from the interval $[0,1]$ by removing an elementary set with measure***

$$\sum_{j=1}^{n} \delta\left(\frac{2}{3}\right)^{j-1} = 3\delta\left(1-\left(\frac{2}{3}\right)^n\right)$$

and we conclude that

$$m(E_n(\delta)) = 1 - 3\delta + 3\delta\left(\frac{2}{3}\right)^n$$

In the event that $\delta = 1/3$, it follows from the fact that $m(E_n(\delta)) \to 0$ as $n \to \infty$ that the set $C(\delta)$ has measure zero.
Suppose now that $\delta < 1/3$. Since the set $C(\delta)$ is closed and bounded, in order to show that $C(\delta)$ fails to have measure zero, all we have to do is show that whenever an elementary set E includes $C(\delta)$, we have $m(E) \geq 1 - 3\delta$. To obtain a contradiction, suppose that E is an elementary set, that $C(\delta) \subseteq E$, and that $m(E) < 1 - 3\delta$. Choose an open elementary set U such that $E \subseteq U$ and $m(U) < 1 - 3\delta$. Since

$$\bigcap_{n=1}^{\infty}(E_n(\delta) \setminus U) = \bigcap_{n=1}^{\infty}(E_n(\delta)) \setminus U = C(\delta) \setminus U = \emptyset$$

it follows from the Cantor intersection theorem that, for some n, we have $E_n \subseteq U$ and we deduce that

$$m(U) \geq m(E_n(\delta)) = 1 - 3\delta + 3\delta\left(\frac{2}{3}\right)^n > 1 - 3\delta.$$

This is the desired contradiction.

d. Prove that the function $\chi_{C(\delta)}$ is Riemann integrable on $[0,1]$ if and only if $\delta = 1/3$.

Solution: ***This assertion follows at once from part c and the fact that the set $C(\delta)$ is the set of discontinuities of the function $\chi_{C(\delta)}$.***

e. Given $0 < \delta \leq 1/3$, prove that there is a strictly increasing continuous function from $[0,1]$ onto $[0,1]$ that sends the set $C(\delta)$ onto the usual Cantor set C.

Solution: ***For each number $x \in C(\delta)$ we define the function $\hat{x}$ from Z^+ into $\{0,2\}$ as follows: Given any positive integer n, if the number x belongs to one of the 2^n component intervals I of $E_{n-1}(\delta)$ and if, after I is split into two subintervals by the removal of its centrally located open subinterval of I with length $\delta(1/3)^{n-1}$, the number x lies in the left subinterval we define $\hat{x}(n) = 0$ and if x lies in the right subinterval we define $\hat{x}(n) = 2$. We call the function $\hat{x}$ the address of the number x. We now define***

$$u(x) = \sum_{n=1}^{\infty} \frac{\hat{x}(n)}{3^n}$$

for every number $x \in C(\delta)$ and we observe that u is a strictly increasing continuous function from $C(\delta)$ onto $C(1/3)$.
We now extend u to be a function defined on the entire interval $[0,1]$ as follows: Given $x \in [0,1] \setminus C(\delta)$, if p is the greatest member of $C(\delta)$ that is less than x and q is the least member of

$C(\delta)$ ***that is greater than*** x ***then we define***

$$u(x) = u(p) + \left(\frac{u(q) - u(p)}{q - p}\right)(x - p).$$

f. Prove that if $f = \chi_C$ where C is the usual Cantor set and if u is the function found in part e then, although u is continuous on $[0,1]$ and f is Riemann integrable on the range of u, the function $f \circ u$ is not Riemann integrable on $[0,1]$.

Solution: ***This assertion follows at once from the fact that***

$$f \circ u = \chi_{C(\delta)}.$$

5. In this exercise we introduce a further extension of the notion of extended Riemann integrability that was introduced in some earlier optional reading. We shall say that a function f defined on an interval $[a,b]$ is *almost extended Riemann integrable* on $[a,b]$ if there exists a sequence (f_n) of Riemann integrable functions and a number K such that the following two conditions hold

- For each n we have $|f_n(x)| \leq K$ for almost every $x \in [a,b]$.
- For almost every $x \in [a,b]$ we have $f_n(x) \to f(x)$ as $n \to \infty$.

a. Given an almost extended Riemann integrable function f on an interval $[a,b]$, give a reasonable definition of the integral $\int_a^b f$. Take care to say why your definition makes sense. Which theorem are you using for this purpose?

b. Prove that the integration of almost extended Riemann integrable functions is linear, nonnegative and additive.

c. Give an example of a function that is almost extended Riemann integrable on $[0,1]$ but is not Riemann integrable on $[0,1]$.

6. An example of Sierpinski shows that it is possible to find a subset S of the square

$$[0,1] \times [0,1]$$

such that for every $x \in [0,1]$ the vertical x-section S_x of S has measure zero and for every $y \in [0,1]$ the horizontal y-section S^y contains almost every number in $[0,1]$. This example is quite easy to produce if you have studied enough set theory and it can be found here. Unless you have read the set theory, must assume the existence of the Sierpinski example for the purposes of this exercise.
Using Sierpinski's example, prove that the statement of Fichtenholz's theorem becomes false if we widen it to include functions that are almost Riemann integrable.

Exercises 5 and 6 are offered as a resource for a special project.

18 Calculus of Several Variables

Exercises on Velocity, Speed and Curve Length

1. Point at the equation

$$\gamma(t) = (t\cos t, t\sin t, t^2)$$

and assign it as a definition in *Scientific Notebook*. Then point at the integral $\int_0^{\pi} \|\gamma'(t)\| dt$ and click on Evaluate and then point at the result and then click on Simplify to show that the length of the curve is

$$\frac{1}{2}\pi\sqrt{(5\pi^2+1)} + \frac{1}{10}\sqrt{5}\,\ln\left(\sqrt{5}\,\pi + \sqrt{(5\pi^2+1)}\right)$$

2. Prove that every piecewise smooth curve in $\mathbf{R}^k$ is rectifiable.
   This assertion follows at once from the relationship between speed and curve length and the fact that a function that is piecewise continuous on an interval must be itegrable.

3. Given that

$$\gamma(t) = \begin{cases} (t, t\sin\frac{1}{t}) & \text{if } \ 0 < t \le \pi \\ (0,0) & \text{if } \ t = 0 \end{cases},$$

   prove that γ is not rectifiable.

1. **Solution:** ***We observe first that for each*** $t \in (0,\pi]$ ***we have***

$$\dot{\gamma}(t) = \|\gamma'(t)\| = \sqrt{1 + \frac{1}{t^2}\left(t\sin\frac{1}{t} - \cos\frac{1}{t}\right)^2} \ge \frac{1}{t}\left|t\sin\frac{1}{t} - \cos\frac{1}{t}\right|$$

and therefore, whenever $0 < t < \frac{1}{4}$ ***we have***

$$\dot{\gamma}(t) \ge \frac{1}{t}\left|t\sin\frac{1}{t} - \cos\frac{1}{t}\right| \ge \frac{1}{t}\left(\left|\cos\frac{1}{t}\right| - \frac{1}{4}\right)$$

Now given any positive integer n***, if***

$$2n\pi - \frac{\pi}{4} \le \frac{1}{t} \le 2n\pi + \frac{\pi}{4}$$

then

$$\left|\cos\frac{1}{t}\right| = \cos\frac{1}{t} \ge \frac{1}{\sqrt{2}}.$$

In other words, if n ***is a positive integer and if***

$$\frac{4}{(8n+1)\pi} \le t \le \frac{4}{(8n-1)\pi}$$

then

$$\dot{\gamma}(t) \ge \frac{1}{t}\left(\left|\cos\frac{1}{t}\right| - \frac{1}{4}\right) \ge \frac{1}{t}\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)$$

and so

$$\int_{\frac{4}{(8n+1)\pi}}^{\frac{4}{(8n-1)\pi}} \dot{\gamma}(t)dt \ge \int_{\frac{4}{(8n+1)\pi}}^{\frac{4}{(8n-1)\pi}} \frac{1}{t}\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)dt$$

$$= \left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)\left(\log\left(\frac{4}{(8n-1)\pi}\right) - \log\left(\frac{4}{(8n+1)\pi}\right)\right)$$

$$= \left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)\left(\log\left(\frac{8n+1}{8n-1}\right)\right) = \left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)\left(\log\left(1 + \frac{2}{8n-1}\right)\right)$$

Now since

$$\lim_{n\to\infty}\frac{\log\left(1+\frac{2}{8n-1}\right)}{\frac{1}{n}} = \frac{1}{4}$$

it follows from the *comparison test* ***that the series***

$$\sum \log\left(1 + \frac{2}{8n-1}\right)$$

is divergent and we deduce that the series

$$\sum \int_{\frac{4}{(8n+1)\pi}}^{\frac{4}{(8n-1)\pi}} \dot{\gamma}(t)dt$$

is divergent. Therefore

$$\lim_{\delta\to 0+}\int_{\delta}^{\pi}\dot{\gamma}(t)dt = \infty.$$

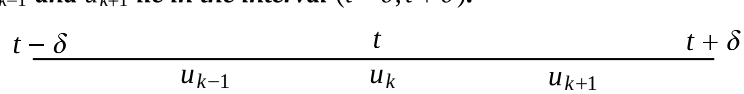
4. Prove that if f is an increasing continuous function from an interval $[a,b]$ into $\mathbf{R}$ and if
$$\gamma(t) = (t, f(t))$$
for all $t \in [a,b]$ then γ is rectifiable.

1. **Solution:** ***Suppose that*** $(t_0, t_1, t_2, \cdots, t_n)$ ***is a partition of the interval*** $[a,b]$***. We have***
$$\begin{aligned}\sum_{j=1}^{n}\|\gamma(t_j)-\gamma(t_{j-1})\| &= \sum_{j=1}^{n}\|(t_j-t_{j-1}, f(t_j)-f(t_{j-1}))\| \\ &\leq \sum_{j=1}^{n}\Big(|t_j-t_{j-1}|+|f(t_j)-f(t_{j-1})|\Big) \\ &= \sum_{j=1}^{n}\Big((t_j-t_{j-1})+(f(t_j)-f(t_{j-1}))\Big) \\ &\leq b-a+f(b)-f(a).\end{aligned}$$

5. Suppose that γ is a rectifiable curve with domain $[a,b]$ in a metric space X and that, for each $t \in [a,b]$ we have defined $\phi(t)$ to be the length of the restriction of the curve γ to the interval $[a,t]$. Prove that the function ϕ must be continuous on the interval $[a,b]$.

1. **Solution:** ***We shall explain the continuity of*** ϕ ***at each number*** t ***in the open interval*** (a,b) ***and leave as an exercise the task of proving the one sided continuity of*** ϕ ***at the numbers*** a ***and*** b***. Suppose that*** $a < t < b$***. To show that*** ϕ ***is continuous at the number*** t ***we shall show that for every number*** $\varepsilon > 0$ ***we have***
$$\lim_{u\to t+}\phi(u) - \lim_{u\to t-}\phi(u) < \varepsilon.$$
Suppose that $\varepsilon > 0$***.***

Using the fact that the function γ ***is continuous at*** t***, we choose a number*** $\delta > 0$ ***such that***
$$a < t-\delta < t < t+\delta < b$$
and such that the inequality
$$d(\gamma(t), \gamma(u)) < \frac{\varepsilon}{4}$$
holds whenever $t-\delta < u < t+\delta$***. Now we choose a partition***
$$(u_0, u_1, u_2, \cdots, u_n)$$
of the interval $[a,b]$ ***such that***
$$\phi(b) - \sum_{j=1}^{n} d(\gamma(u_{u-1}), \gamma(u_j)) < \frac{\varepsilon}{4}.$$
Since the inclusion of extra points in this partition makes the sum
$$\sum_{j=1}^{n} d(\gamma(u_{j-1}), \gamma(u_j))$$
larger, we may assume, without loss of generality that t ***is a point in the partition and that if*** $t = u_k$ ***then both of the points*** u_{k-1} ***and*** u_{k+1} ***lie in the interval*** $(t-\delta, t+\delta)$***.***



Now given any $m = 1, 2, \cdots, n$ ***we clearly have***

$$\sum_{j=1}^{m} d(\gamma(u_{j-1}),\gamma(u_j)) \le \phi(u_m)$$

and, since $\phi(b)-\phi(u_m)$ ***is the length of the restriction of the curve*** γ ***to the interval*** $[u_m,b]$***, we also have***

$$\sum_{j=m+1}^{n} d(\gamma(u_{j-1}),\gamma(u_j)) \le \phi(b)-\phi(u_m).$$

Therefore if $m=1,2,\cdots,n$ ***then since***

$$\phi(b)-\sum_{j=1}^{n} d(\gamma(u_{u-1}),\gamma(u_j)) = \left[\phi(u_m)-\sum_{j=1}^{m} d(\gamma(u_{j-1}),\gamma(u_j))\right]+\left[\phi(b)-\phi(u_m)-\sum_{j=m+1}^{n} d(\gamma(u_{j-1}),\gamma(u_j))\right]$$

we conclude that

$$\phi(u_m)-\sum_{j=1}^{m} d(\gamma(u_{j-1}),\gamma(u_j)) < \frac{\varepsilon}{4}.$$

We conclude that

$$\begin{aligned}\phi(u_{k+1}) &< \sum_{j=1}^{k+1} d(\gamma(u_{j-1}),\gamma(u_j))+\frac{\varepsilon}{4}\\ &= \sum_{j=1}^{k-1} d(\gamma(u_{j-1}),\gamma(u_j))+d(\gamma(u_{k-1}),\gamma(u_k))+d(\gamma(u_k),\gamma(u_{k+1}))+\frac{\varepsilon}{4}\\ &< \phi(u_{k-1})+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\end{aligned}$$

Thus

$$\lim_{u\to t+}\phi(u) \le \phi(u_{k+1}) < \phi(u_{k-1})+\frac{3\varepsilon}{4} < \lim_{u\to t-}\phi(u)+\varepsilon.$$

6. Suppose that γ is a rectifiable curve with domain $[a,b]$ in a metric space X. Prove that for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that whenever $\mathbf{P}$ is a partition of $[a,b]$ and $\|\mathbf{P}\| < \delta$ we have

$$|L(\mathbf{P},\gamma)-L(\gamma)| < \varepsilon.$$

Hint: Examine the method of proof of Darboux's theorem*.*

7. Suppose that γ_1 and γ_2 are curves in $\mathbf{R}^k$ with domain $[a,b]$. The *sum* $\gamma_1+\gamma_2$ and dot product $\gamma_1\cdot\gamma_2$ of γ_1 and γ_2 are defined by the equations

$$(\gamma_1+\gamma_2)(t) = \gamma_1(t)+\gamma_2(t)$$

and

$$(\gamma_1\cdot\gamma_2)(t) = \gamma_1(t)\cdot\gamma_2(t)$$

for each $t\in[a,b]$. If f is a real valued function defined on the interval $[a,b]$ then we define $f\gamma_1$ by the equation

$$(f\gamma_1)(t) = f(t)\gamma_1(t)$$

for every $t\in[a,b]$. Prove that if, for a given number $t\in[a,b]$ the derivatives $\gamma_1'(t)$ and $\gamma_2(t)$ and $f(t)$ exist then

$$(\gamma_1+\gamma_2)'(t) = \gamma_1'(t)+\gamma_2'(t)$$

$$(\gamma_1\cdot\gamma_2)'(t) = \gamma_1'(t)\cdot\gamma_2(t)+\gamma_1(t)\cdot\gamma_2'(t)$$

$$(f\gamma_1)'(t) = f'(t)\gamma_1(t)+f(t)\gamma_1'(t)$$

All of these facts follow directly from the definitions.

8. Prove that if γ is a differentiable curve in $\mathbf{R}^k$ and if the function $\|\gamma\|$ is constant then $\gamma\cdot\gamma'$ is the constant function zero. Interpret this fact geometrically.

If we define $f(t) = \|\gamma(t)\|^2$ then, for each t,

$$0 = f'(t) = \gamma'(t) \cdot \gamma(t) + \gamma(t) \cdot \gamma'(t) = 2\gamma'(t) \cdot \gamma(t).$$

A geometric interpretation of this fact is that if the path γ runs around in a sphere with center $\mathbf{O}$ then the velocity γ', being perpendicular to the line that runs from $\mathbf{O}$ to γ, is tangential to the sphere.

9. If a curve γ in $\mathbf{R}^k$ is differentiable at a number t and if $\gamma'(t) \neq O$ then the *unit tangent* $T(t)$ of θ at the number t is defined by the equation

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}.$$

This is standard stuff and can be found in any elementary calculus text.

a. Prove that if $\gamma''(t)$ exists then the derivative $T'(t)$ of T exists at the number t and we have

$$T(t) \cdot T'(t) = 0.$$

In this event, the *curvature* $K(t)$ of the curve γ at the number t is defined by the equation

$$K(t) = \frac{\|T'(t)\|}{\|\gamma'(t)\|}.$$

In the event that $K(t) \neq 0$ then the *principal normal* $N(t)$ of γ at the number t is defined by the equation

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{T'(t)}{K(t)\|\gamma'(t)\|}.$$

b. Prove that if $\gamma''(t)$ exists and $K(t) \neq 0$ then

$$\gamma''(t) = \dot{\gamma}'(t)T(t) + (\dot{\gamma}(t))^2 K(t)N(t)$$

Exercises on Integrals on Curves

1. Point at each of the equations

$$\gamma_1(t) = (t\cos \pi t, t\sin \pi t, t)$$

and

$$\gamma_2(t) = (t^2 \cos \pi t \sin \pi t, t^2 \sin \pi t, t^2 \cos \pi t)$$

and

$$\mathbf{f}(x,y,z) = (xy, yz, zx)$$

and supply them as definitions to *Scientific Notebook*. Taking the domain of each of the curves γ_1 and γ_2 to be the interval $[0,1]$, use *Scientific Notebook* to evaluate each of the intervals $\int_{\gamma_1} \mathbf{f}(x,y,z) \cdot d(x,y,z)$ and $\int_{\gamma_2} \mathbf{f}(x,y,z) \cdot d(x,y,z)$. For example, the first of these integrals is

$$\int_0^1 \mathbf{f}(t\cos \pi t, t\sin \pi t, t) \cdot \gamma'(t)dt.$$

Are the two integrals you have just evaluated equal to one another?

2. For each of the following curves γ and point (a,b) in $\mathbf{R}^2$, evaluate the integral

$$\frac{1}{2\pi} \int_\gamma \frac{1}{\|(x,y) - (a,b)\|^2} (-(y-b), x-a) \cdot d(x,y).$$

Ask *Scientific Notebook* to show you a sketch of each curve.

a. We define

$$\gamma(t) = (3\cos t, 3\sin t)$$

for $0 \le t \le 2\pi$ and $(a,b) = (1,1)$.

b. We define

$$\gamma(t) = ((1+2\cos t)\cos t, (1+2\cos t)\sin t)$$

for $0 \le t \le 2\pi$ and $(a,b) = \left(\frac{1}{2}, 0\right)$.

c. We define

$$\gamma(t) = ((1+2\cos t)\cos t, (1+2\cos t)\sin t)$$

for $0 \le t \le 2\pi$ and $(a,b) = (2,0)$.

d. We define

$$\gamma(t) = ((1+2\cos t)\cos t, (1+2\cos t)\sin t)$$

for $0 \le t \le 2\pi$ and $(a,b) = (4,0)$.

3. Given that

$$\gamma(t) = (\gamma_1(t, \gamma_2(t), , \gamma_n(t)))$$

for each t in a given domain $[a,b]$ and given that each of the functions γ_j is Riemann integrable on $[a,b]$ we define

$$\int_a^b \gamma(t)dt = \left(\int_a^b \gamma_1(t)dt, \int_a^b \gamma_2(t)dt, \cdots, \int_a^b \gamma_n(t)dt\right).$$

a. Prove that the integral just defined has the properties of linearity and additivity that we obtained for Riemann integrals of real functions.
These properties follow at once from the definitions.

b. State and prove an analog of the fundamental theorem of calculus for this kind of integral.
Suppose that

$$\Gamma(t) = (\Gamma_1(t), \Gamma_2(t), \cdots, \Gamma_n(t))$$

for $a \le t \le b$ and that each function Γ_j has a Riemann integrable derivative on $[a,b]$. Then

$$\begin{aligned}
\int_a^b \Gamma'(t)dt &= \int_a^b (\Gamma_1'(t), \Gamma_2'(t), \cdots, \Gamma_n'(t))dt \\
&= \left(\int_a^b \Gamma_1'(t)dt, \int_a^b \Gamma_2'(t)dt, \cdots, \int_a^b \Gamma_n'(t)dt\right) \\
&= \left(\Gamma_1(b) - \Gamma_1(a), \Gamma_2(b) - \Gamma_2(a), \cdots, \Gamma_n(b) - \Gamma_n(a)\right) \\
&= \Gamma(b) - \Gamma(a)
\end{aligned}$$

c. Assuming that

$$\int_a^b \gamma_j(t)dt = q_j$$

for each j, obtain the identity

$$\left\|\int_a^b \gamma(t)dt\right\|^2 = \int_a^b \left(\sum_{j=1}^n q_j\gamma_j(t)\right)dt.$$

We observe that

$$\begin{aligned}
\left\|\int_a^b \gamma(t)dt\right\|^2 &= \left(\int_a^b \gamma_1(t)dt\right)^2 + \left(\int_a^b \gamma_2(t)dt\right)^2 + \cdots + \left(\int_a^b \gamma_n(t)dt\right)^2 \\
&= \sum_{j=1}^n q_j^2 = \sum_{j=1}^n q_j \int_a^b \gamma_j(t)dt = \int_a^b \left(\sum_{j=1}^n q_j\gamma_j(t)\right)dt.
\end{aligned}$$

d. By applying the Cauchy-Schwarz inequality to the latter expression, deduce that

$$\left\| \int_a^b \gamma(t)dt \right\| \leq \int_a^b \|\gamma(t)\| dt.$$

We observe that

$$\begin{aligned} \left\| \int_a^b \gamma(t)dt \right\|^2 &= \int_a^b \left(\sum_{j=1}^n q_j \gamma_j(t) \right) dt \leq \int_a^b \sqrt{\sum_{j=1}^n q_j^2} \sqrt{\sum_{j=1}^n \gamma_j^2(t)}\, dt \\ &= \left\| \int_a^b \gamma(t)dt \right\| \int_a^b \|\gamma(t)\| dt. \end{aligned}$$

and it follows that

$$\left\| \int_a^b \gamma(t)dt \right\| \leq \int_a^b \|\gamma(t)\| dt.$$

Some Exercises on Exact Functions

1. a. Find a potential function for the function $\mathbf{f}$ defined to be

$$\left(\frac{2xz}{x^2+y^2}, \frac{2yz}{x^2+y^2} + \sin yz + yz\cos yz, \log(x^2+y^2) + y^2\cos yz + 2z \right)$$

at every point (x,y,z) of $\mathbf{R}^3$ for which $x^2+y^2 \neq 0$.

We are looking for a function F for which the equations

$$\begin{aligned} D_1F(x,y,z) &= \frac{2xz}{x^2+y^2} \\ D_2F(x,y,z) &= \frac{2yz}{x^2+y^2} + \sin yz + yz\cos yz \\ D_3F(x,y,z) &= \log(x^2+y^2) + y^2\cos yz + 2z \end{aligned}$$

hold whenever $x^2+y^2 \neq 0$. From the first of these equations we see that

$$D_1(F(x,y,z) - z\log(x^2+y^2)) = 0$$

for each point (x,y,z) and therefore, for each point (y,z), the expression $F(x,y,z) - z\log(x^2+y^2)$ is independent of x. We write $F(x,y,z) - z\log(x^2+y^2)$ as $\phi(y,z)$. In other words,

$$F(x,y,z) = z\log(x^2+y^2) + \phi(y,z)$$

for each point (x,y,z). Thus

$$D_2F(x,y,z) = \frac{2yz}{x^2+y^2} + D_2\phi(y,z)$$

and we conclude that

$$\frac{2yz}{x^2+y^2} + D_2\phi(y,z) = \frac{2yz}{x^2+y^2} + \sin yz + yz\cos yz$$

which gives us

$$D_2\phi(y,z) = \sin yz + yz\cos yz$$

and so we can express $\phi(y,z)$ in the form $y\sin yz + \psi(z)$ which gives us

$$F(x,y,z) = z\log(x^2+y^2) + y\sin yz + \psi(z).$$

Therefore

$$D_3F(x,y,z) = \log(x^2+y^2) + y^2\cos yz + \psi'(z)$$

which gives us

$$\log(x^2+y^2) + y^2\cos yz + \psi'(z) = \log(x^2+y^2) + y^2\cos yz + 2z$$

and we conclude that the function F defined by the equation

$$F(x,y,z) = z\log(x^2+y^2) + y\sin yz + z^2$$

whenever $x^2+y^2 \neq 0$ is a potential function for **f**.

b. Given that **f** is the function defined in part a and that γ is any piecewise smooth curve in the set

$$\{(x,y,z) \in \mathbf{R}^3 \mid x^2+y^2 > 0\}$$

running from the point $(1,0,1)$ to the point $\left(3,\frac{\pi}{6},2\right)$, evaluate the integral

$$\int_\gamma \mathbf{f}(\mathbf{x}) \cdot d\mathbf{x}.$$

Using the function v that we found in part a we see that

$$\int_\gamma \mathbf{f}(\mathbf{x}) \cdot d\mathbf{x} = v\left(3,\frac{\pi}{6},3\right) - v(1,0,1) = -3\log 36 + 3\log(324+\pi^2) + \frac{1}{6}\pi + 8$$

2. Find a potential function for the function **f** defined to be

$$\left(x^2\ln(x^2+y^2z+uv) + \frac{vux^2}{x^2+y^2z+uv}, \frac{x^2u^2}{x^2+y^2z+uv}, 2ux\ln(x^2+y^2z+uv) + \frac{2ux^3}{x^2+y^2z+uv}, \frac{2ux^2yz}{x^2+y^2z+uv}, \frac{ux^2y^2}{x^2+y^2z+uv}\right)$$

at every point (u,v,x,y,z) or $\mathbf{R}^5$ for which the latter expression is defined.
If F is a potential function for **f** then, for each point (u,v,x,y,z) we want

$$D_1F(u,v,x,y,z) = x^2\ln(x^2+y^2z+uv) + \frac{vux^2}{x^2+y^2z+uv}$$

$$D_2F(u,v,x,y,z) = \frac{x^2u^2}{x^2+y^2z+uv}$$

$$D_3F(u,v,x,y,z) = 2ux\ln(x^2+y^2z+uv) + \frac{2ux^3}{x^2+y^2z+uv}$$

$$D_4F(u,v,x,y,z) = \frac{2ux^2yz}{x^2+y^2z+uv}$$

$$D_5F(u,v,x,y,z) = \frac{ux^2y^2}{x^2+y^2z+uv}$$

The last of these five equations gives us

$$F(u,v,x,y,z) = ux^2\log(x^2+y^2z+uv) + \phi_1(u,v,x,y).$$

To cut a long story short we observe that if

$$F(u,v,x,y,z) = ux^2\log(x^2+y^2z+uv)$$

for each point (u,v,x,y,z) then the function F is a potential for **f**.

3. In this exercise you will show that the necessary condition for exactness is sufficient if the domain of the given function is a rectangle.
Suppose that $\mathbf{f} = (f_1,f_2)$ has continuous partial derivatives on the rectangle

$$E = \left\{(x,y) \mid a_1 \leq x \leq b_1 \text{ and } a_2 \leq y \leq b_2\right\}$$

and that the condition

$$D_1f_2(x,y) = D_2f_1(x,y)$$

holds for every point $(x,y) \in E$. Prove that the function F defined at each point $(x,y) \in E$ by the equation

$$F(x,y) = \int_{a_1}^{x} f_1(t,y)dt + \int_{a_2}^{y} f_2(a_1,t)dt$$

is a potential of **f** on E.
The result follows at once from the theorem on differentiation of a partial Riemann integral.

4. Repeat the preceding exercise for a function $\mathbf{f} = (f_1,f_2,f_3)$ defined on the rectangular box

$$E = \left\{(x,y,z) \mid a_1 \leq x \leq b_1 \text{ and } a_2 \leq y \leq b_2 \text{ and } a_3 \leq z \leq b_3\right\}.$$

This time we take

$$F(x,y,z) = \int_{a_1}^{x} f_1(t,y,z)dt + \int_{a_2}^{y} f_2(a_1,t,z)dt + \int_{a_3}^{z} f_3(a_1,a_2,t)dt.$$

Can you extend this idea to n dimensions?

Some Exercises on Multiple Integrals

1. Evaluate the integral

$$\iint_S xy^2 dxdy$$

where

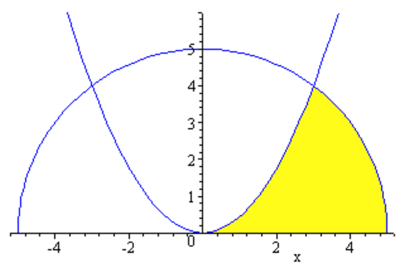
$$S = \left\{(x,y) \mid x \ge 0 \text{ and } y \ge 0 \text{ and } x+y \le 1\right\}.$$

$$\iint_S xy^2 dxdy = \int_0^1 \int_0^{1-x} xy^2 dydx = \frac{1}{60}$$

2. a. Convert the expression

$$\int_0^3 \int_0^{4x^2/9} f(x,y)dydx + \int_3^5 \int_0^{\sqrt{25-x^2}} f(x,y)dydx$$

into an integral of the form $\iint_S f$ where S is an appropriate subset of $\mathbf{R}^2$. You may assume that f is a continuous function.



The set S is

$$\left\{(x,y) \in \mathbf{R}^2 \mid 0 \le x \le 3 \text{ and } 0 \le y \le \frac{4x^2}{9}\right\} \cup \left\{(x,y) \in \mathbf{R}^2 \mid 3 \le x \le 5 \text{ and } 0 \le y \le \sqrt{25-x^2}\right\}$$

b. Convert the integral

$$\int_0^3 \int_0^{4x^2/9} f(x,y)dydx + \int_3^5 \int_0^{\sqrt{25-x^2}} f(x,y)dydx$$

into a repeated integral with order of integration reversed.

Reversing the integral we obtain

$$\int_0^3 \int_0^{4x^2/9} f(x,y)dydx + \int_3^5 \int_0^{\sqrt{25-x^2}} f(x,y)dydx = \int_0^4 \int_{3\sqrt{y}/2}^{\sqrt{25-y^2}} f(x,y)dxdy.$$

3. Given that Q^3 is the standard simplex in $\mathbf{R}^3$ defined by the equation

$$Q^3 = \left\{(x,y,z) \in \mathbf{R}^3 \mid x \ge 0 \text{ and } y \ge 0 \text{ and } z \ge 0 \text{ and } x+y+z \le 1\right\},$$

evaluate $\operatorname{vol}(Q^3)$.

The solution to this exercise was given in the preceding examples.

4. Given that S is the disc in $\mathbf{R}^2$ with center at $(0,0)$ and radius $r > 0$, show that the area $\operatorname{vol}(S)$ of S is given by the equation

$$\mathrm{vol}(S) = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 1\,dy\,dx = \pi r^2.$$

The evaluation of this integral is a routine problem in elementary calculus. Note that the integral needs to be evaluated as a repeated integral. Students should be strongly discouraged from making a two-variable change to polar coordinates, even if they learned such a procedure in their elementary calculus courses. That procedure will not be available till much later in this chapter.

5. Given that A and B are subsets of $\mathbf{R}^n$ and $\mathbf{R}^k$ respectively where n and k are positive integers and given that the volumes $\mathrm{vol}(A)$ and $\mathrm{vol}(B)$ exist, prove that the measure $\mathrm{vol}(A \times B)$ exists in the space $\mathbf{R}^{n+k}$ and that
$$\mathrm{vol}(A \times B) = \mathrm{vol}(A)\,\mathrm{vol}(B).$$
Suppose that E is a nk cell that includes the set $A \times B$. We can express E in the form $E_1 \times E_2$ where E_1 is an n-cell that includes A and E_2 is a k-cell that includes B. Then
$$\mathrm{vol}(A \times B) = \int_E \chi_{A\times B} = \int_{E_1\times E_2} \chi_A \chi_B = \left(\int_{E_1} \chi_A\right)\left(\int_{E_2} \chi_B\right) = \left(\mathrm{vol}(A)\right)\left(\mathrm{vol}(B)\right)$$

6. Suppose that for each (A_n) is an expanding sequence of subsets of $\mathbf{R}^k$, that
$$A = \bigcup_{n=1}^{\infty} A_n$$
and that the set A is bounded. Suppose that each of the sets A_n is a finite union of k-cells. Prove that
$$\mathrm{vol}(A) = \lim_{n\to\infty} \mathrm{vol}(A_n).$$
We choose a k-cell E that includes A. Since the repeated integral of χ_{A_n} over E exists for each n and since χ_{A_n} converges boundedly to χ_A as $n \to \infty$, we have
$$\mathrm{vol}(A) = \int_E \chi_A = \lim_{n\to\infty} \int_{E_n} \chi_{A_n} = \lim_{n\to\infty} \mathrm{vol}(A_n).$$

7. Suppose that for each (A_n) is an contracting sequence of subsets of $\mathbf{R}^k$, that
$$A = \bigcup_{n=1}^{\infty} A_n$$
and that the set A_1 is bounded. Suppose that each of the sets A_n is a finite union of k-cells. Prove that
$$\mathrm{vol}(A) = \lim_{n\to\infty} \mathrm{vol}(A_n).$$
Choose a k-cell E that includes A_1. Since the sequence $(E \setminus A_n)$ is expanding and since the union of this sequence is
$$E \setminus \bigcap_{n=1}^{\infty} A_n,$$
the desired result follows at once from Exercise 6.

Some Further Exercises on Multiple Integrals

The exercises in this subsection are similar to those that appear above but make use of the **gamma function** and **beta function** that were introduced earlier. They also refer to the **standard simplex** Q^k in $\mathbf{R}^k$ that is defined to be the set of all points
$$\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbf{R}^k$$
for which $x_j \geq 0$ for each j and
$$x_1 + x_2 + \cdots + x_k \leq 1.$$

1. Prove that if a and b are nonnegative numbers then

$$\iint_{Q^2} x^a y^b d(x,y) = \frac{1}{a+1} B(a+2,b+1) = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+3)}.$$

We have

$$\iint_{Q^2} x^a y^b d(x,y) = \int_0^1 \int_0^{1-x} x^a y^b dydx = \frac{1}{b+1} \int_0^1 x^a (1-x)^{b+1} dydx$$

$$= \frac{1}{b+1} B(a+1,b+2) = \frac{1}{b+1} \frac{\Gamma(a+1)\Gamma(b+2)}{\Gamma(a+b+3)}$$

$$= \frac{1}{b+1} \frac{\Gamma(a+1)(b+1)\Gamma(b+1)}{\Gamma(a+b+3)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+3)}$$

2. Prove that if a, b and c are nonnegative numbers then

$$\iiint_{Q^3} x^a y^b z^c d(x,y,z) = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)}{\Gamma(a+b+c+4)}.$$

We have

$$\iiint_{Q^3} x^a y^b z^c d(x,y,z) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^a y^b z^c dzdydx$$

$$= \frac{1}{c+1} \int_0^1 \int_0^{1-x} x^a y^b (1-x-y)^{c+1} dzdydx.$$

In the inside integral we make the substitution $u = y/(1-x)$ and we obtain

$$\iiint_{Q^3} x^a y^b z^c d(x,y,z) = \frac{1}{c+1} \int_0^1 \int_0^1 x^a (1-x)^b u^b (1-x)^{c+1} (1-u)^{c+1} (1-x) dudx$$

$$= \frac{1}{c+1} B(a+1,b+c+3) B(b+1,c+2)$$

$$= \frac{1}{c+1} \frac{\Gamma(a+1)\Gamma(b+c+3)\Gamma(b+1)\Gamma(c+2)}{\Gamma(a+b+c+4)\Gamma(b+c+3)}$$

$$= \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)}{\Gamma(a+b+c+4)}$$

3. Given that k is a positive integer and given nonnegative numbers $a_1, a_2, \cdots, a_k$, guess the value of the integral

$$\int_{Q^k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} d(x_1, x_2, \cdots, x_k)$$

and then prove (by induction, perhaps) that your guess is correct.

We expect the equation

$$\int_{Q^k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} d(x_1, x_2, \cdots, x_k) = \frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_k+1)}{\Gamma(a_1+a_2+\cdots+a_k+k+1)}$$

for every positive integer k. When $k \geq 2$, the integral

$$\int_{Q^k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} d(x_1, x_2, \cdots, x_k)$$

can be expressed as

$$\int_{Q^{k-1}} \int_0^{1-x_1-x_2-\cdots-x_{k-1}} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} dx_k d(x_1, x_2, \cdots, x_{k-1})$$

$$= \frac{1}{a_k+1} \int_{Q^{k-1}} x_1^{a_1} x_2^{a_2} \cdots x_{k-1}^{a_{k-1}} (1-x_1-x_2-\cdots-x_{k-1})^{a_k+1} d(x_1, x_2, \cdots, x_{k-1})$$

and so the equation

$$\int_{Q^k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} d(x_1, x_2, \cdots, x_k) = \frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_k+1)}{\Gamma(a_1+a_2+\cdots+a_k+k+1)}$$

can be expressed as

$$\frac{1}{a_k+1}\int_{Q^{k-1}} x_1^{a_1}x_2^{a_2}\cdots x_{k-1}^{a_{k-1}}(1-x_1-x_2-\cdots-x_{k-1})^{a_k+1}d(x_1,x_2,\cdots,x_{k-1}) = \frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_k+1)}{\Gamma(a_1+a_2+\cdots+a_k+k+1)}.$$

We write the assertion that this equation holds for any choice of nonnegative numbers $a_1,\cdots,a_k$ as $p(k)$. We have seen that the assertions $p(1)$ and $p(2)$ are true. Now suppose that k is any positive integer for which the assertion $p(k)$ is true and suppose that $a_1,\cdots,a_{k+1}$ are nonnegative numbers. Then

$$\frac{1}{a_{k+1}+1}\int_{Q^k} x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}(1-x_1-x_2-\cdots-x_k)^{a_{k+1}+1}d(x_1,x_2,\cdots,x_k)$$
$$= \frac{1}{a_{k+1}+1}\int_{Q^{k-1}}\int_0^{1-x_1-x_2-\cdots-x_{k-1}} x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}(1-x_1-x_2-\cdots-x_k)^{a_{k+1}+1}dx_k d(x_1,x_2,\cdots,x_{k-1})$$

In the inside integral we make the substitution

$$u = \frac{x_k}{1-x_1-x_2-\cdots-x_{k-1}}$$

and we obtain

$$\frac{1}{a_{k+1}+1}\int_{Q^{k-1}}\int_0^1 x_1^{a_1}x_2^{a_2}\cdots x_{k-1}^{a_{k-1}}u^{a_k}(1-x_1-x_2-\cdots-x_{k-1})^{2+a_k+a_{k+1}}(1-u)^{a_{k+1}+1}du\,d(x_1,x_2,\cdots,x_{k-1})$$
$$= \frac{1}{a_{k+1}+1}\left(\int_0^1 u^{a_k}(1-u)^{a_{k+1}+1}du\right)\left(\int_{Q^{k-1}} x_1^{a_1}x_2^{a_2}\cdots x_{k-1}^{a_{k-1}}(1-x_1-x_2-\cdots-x_{k-1})^{2+a_k+a_{k+1}}d(x_1,x_2,\cdots,x_{k-1})\right)$$
$$= \left(\frac{1}{a_{k+1}+1}\right)(2+a_k+a_{k+1})B(a_k+1,a_{k+1}+2)\frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_{k-1}+1)\Gamma(2+a_k+a_{k+1})}{\Gamma(a_1+a_2+\cdots+a_{k-1}+a_k+a_{k+1}+1+k+1)}$$
$$= \left(\frac{1}{a_{k+1}+1}\right)(2+a_k+a_{k+1})\frac{\Gamma(a_k+1)\Gamma(a_{k+1}+2)}{\Gamma(a_k+a_{k+1}+3)}\frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_{k-1}+1)\Gamma(2+a_k+a_{k+1})}{\Gamma(a_1+a_2+\cdots+a_{k-1}+a_k+a_{k+1}+1+k+1)}$$
$$= \frac{1}{a_{k+1}+1}\frac{\Gamma(a_k+1)(a_{k+1}+1)\Gamma(a_{k+1}+1)}{\Gamma(a_k+a_{k+1}+2)}\frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_{k-1}+1)\Gamma(2+a_k+a_{k+1})}{\Gamma(a_1+a_2+\cdots+a_{k-1}+a_k+a_{k+1}+1+k+1)}$$
$$= \frac{\Gamma(a_1+1)\Gamma(a_2+1)\cdots\Gamma(a_{k-1}+1)\Gamma(a_k+1)\Gamma(a_{k+1}+1)}{\Gamma(a_1+a_2+\cdots+a_{k-1}+a_k+a_{k+1}+1+k+1)}$$

Thus the assertion $p(k+1)$ is true and the truth of $p(k)$ for each k follows by mathematical induction.

4. Prove that if k is any positive integer then

$$\mathrm{vol}(Q^k) = \frac{1}{k!}.$$

This assertion follows at once from Exercise 3 when we take $a_j = 0$ for every j.
The hard work has now been done. The rest of the exercises in this section follow quite simply.

5. Prove that if a and b are nonnegative numbers and

$$S = \left\{(x,y)\in\mathbf{R}^2 \mid x\ge 0 \text{ and } y\ge 0 \text{ and } x^2+y^2\le 1\right\}$$

then

$$4\iint_S x^a y^b d(x,y) = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b+4}{2}\right)}.$$

Hint: Make the substitutions $x = \sqrt{u}$ and $y = v = \sqrt{v}$ one at a time to reduce the integral to one that is taken over the standard simplex Q^2.

6. Given that a, b and c are nonnegative numbers and that

$$S = \left\{(x,y,z)\in\mathbf{R}^3 \mid x\ge 0 \text{ and } y\ge 0 \text{ and } z\ge 0 \text{ and } x^2+y^2+z^2\le 1\right\}$$

then

$$8\iiint_S x^a y^b z^c d(x,y,z) = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{a+b+c+5}{2}\right)}.$$

Deduce that the volume of a ball with radius 1 in $\mathbf{R}^3$ is $4\pi/3$.

7. Given that

$$S = \left\{\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbf{R}^k \mid x_j \geq 0 \text{ for each } j \text{ and } \|\mathbf{x}\| \leq 1\right\}.$$

and given nonnegative numbers $a_1, a_2, \cdots, a_k$, express the integral

$$\int_S x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} d\mathbf{x}$$

in terms of gamma functions. Deduce that the volume of the ball in $\mathbf{R}^k$ that has center $\mathbf{O}$ and radius 1 is

$$\frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^k}{\Gamma\left(1 + \frac{k}{2}\right)}.$$

Work out this expression for a few values of k.

8. a. Suppose that B is the ball in $\mathbf{R}^k$ with center $\mathbf{O}$ and radius $r > 0$ then

$$\operatorname{vol}(B) = \frac{r^k \left(\Gamma\left(\frac{1}{2}\right)\right)^k}{\Gamma\left(1 + \frac{k}{2}\right)}.$$

b. Prove that if B is the ball introduced in part a then, in the event that k is even and $k = 2n$ then we have

$$\operatorname{vol}(B) = \frac{\pi^n r^{2n}}{n!}.$$

c. Prove that if B is the ball introduced in part a then, in the event that k is odd and $k = 2n + 1$ then we have

$$\operatorname{vol}(B) = \frac{\pi^n r^{2n+1} 2^{2n+1}}{(2n + 1)!}.$$

9. Suppose that $r > 0$ and for each positive integer n, suppose that B_n is the ball with center $\mathbf{O}$ and radius r in the space $\mathbf{R}^n$. Show that if we agree to define $m(B_0) = 1$ then

$$\sum_{n=0}^{\infty} \operatorname{vol}(B_{2n}) = \exp(\pi r^2)$$

and

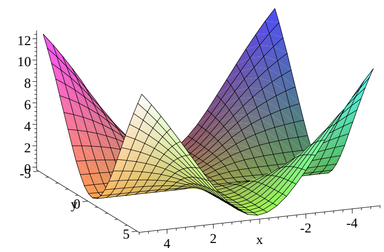
$$\sum_{n=0}^{\infty} \operatorname{vol}(B_{2n+1}) = \frac{\sinh(2r\sqrt{\pi})}{\sqrt{\pi}}.$$

Exercises on Total Derivatives

1. For each of the following functions f, determine whether or not f is continuous at the point $(0,0)$, whether or not the partial derivatives of f exist at $(0,0)$ and whether or not f is differentiable at $(0,0)$. Use *Scientific Notebook* to sketch the graph of each of these functions.

a. We define

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$



This function is continuous at $(0,0)$. Now we have

$$D_1 f(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = 0$$

and, similarly,

$$D_2 f(0,0) = 0.$$

Thus, if

$$r(x,y) = f(x,y) - f(0,0) - Df(0,0)\Big((x,y) - (0,0)\Big)$$

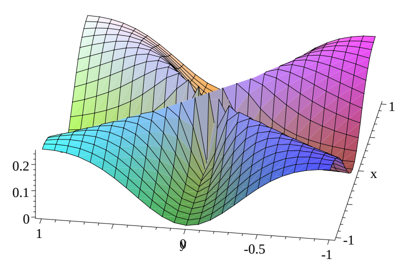
for each point (x,y) then we have

$$\frac{r(x,y)}{\|(x,y)\|} = \frac{x^2y^2}{(x^2+y^2)^{3/2}}$$

and, since the latter expression approaches 0 as $(x,y) \to (0,0)$, the function f is differentiable at $(0,0)$.

b. We define

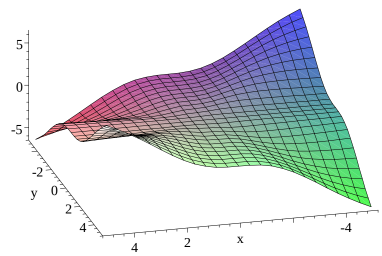
$$f(x,y) = \begin{cases} \frac{x^2y^2}{(x^2+y^2)^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$



This function is not continuous at $(0,0)$.

c. We define

$$f(x,y) = \begin{cases} \frac{x^3y^3}{(x^2+y^2)^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$



This function is continuous at $(0,0)$. Now we have

$$D_1 f(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = 0$$

and, similarly,

$$D_2 f(0,0) = 0.$$

Thus, if

$$r(x,y) = f(x,y) - f(0,0) - Df(0,0)\Big((x,y) - (0,0)\Big)$$

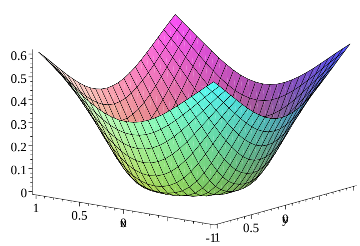
for each point (x,y) then we have

$$\frac{r(x,y)}{\|(x,y)\|} = \frac{x^3 y^3}{(x^2+y^2)^{5/2}}$$

and, since the latter expression approaches 0 as $(x,y) \to (0,0)$, the function f is differentiable at $(0,0)$.

d. We define

$$f(x,y) = \begin{cases} \exp\left(\frac{-1}{x^2+y^2}\right) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$



This function is continuous at $(0,0)$. Now we have

$$D_1 f(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = 0$$

and, similarly,

$$D_2 f(0,0) = 0.$$

Thus, if

$$r(x,y) = f(x,y) - f(0,0) - Df(0,0)\Big((x,y) - (0,0)\Big)$$

for each point (x,y) then we have

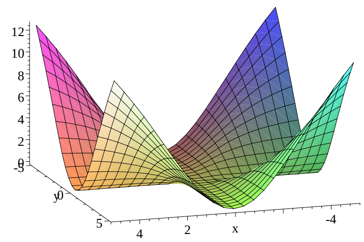
$$\frac{r(x,y)}{\|(x,y)\|} = \frac{\exp\left(\frac{-1}{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$$

and, since the latter expression approaches 0 as $(x,y) \to (0,0)$, the function f is differentiable at

$(0,0)$.

e. We define

$$f(x,y) = \begin{cases} x^2y^2\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$



This function is continuous at $(0,0)$. Now we have

$$D_1f(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = 0$$

and, similarly,

$$D_2f(0,0) = 0.$$

Thus, if

$$r(x,y) = f(x,y) - f(0,0) - Df(0,0)\big((x,y)-(0,0)\big)$$

for each point (x,y) then we have

$$\frac{r(x,y)}{\|(x,y)\|} = \frac{x^2y^2\sin\left(\frac{1}{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$$

and, since the latter expression approaches 0 as $(x,y)\to(0,0)$, the function f is differentiable at $(0,0)$.

2. Prove that if A is an $m\times n$ matrix and if $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}\in\mathbf{R}^n$ then for every such point $\mathbf{x}$ we have $\mathbf{f}'(\mathbf{x}) = A$.
   We see easily that
   $$D_jf_i(\mathbf{x}) = a_{ij}$$
   at each point $\mathbf{x}$ and for all i and j.

3. Suppose that U is a neighborhood of a point $\mathbf{a}$ in $\mathbf{R}^n$ and that $\mathbf{f}: U\to\mathbf{R}^m$. Suppose that A is a $k\times m$ matrix and that $g(\mathbf{x}) = A\mathbf{f}(\mathbf{x})$ for every $\mathbf{x}\in U$. Prove that if $\mathbf{f}$ is differentiable at the point $\mathbf{a}$ then so is $\mathbf{g}$ and we have $\mathbf{g}'(\mathbf{a}) = A\mathbf{f}'(\mathbf{a})$. (Your proof should be very short.)
   This assertion follows at once from the chain rule.

4. State true or false: If the partial derivatives of a function $\mathbf{f}$ are all zero at every point in an open connected set U then $\mathbf{f}$ must be constant in U.
   This statement is true. If the partial derivatives are all zero then they are all continuous. Thus the function $\mathbf{f}$ is differentiable with a zero derivative at every point of U.

5. Given an example of a nonconstant function $\mathbf{f}$ on an open set $U\subseteq\mathbf{R}^n$ such that $\mathbf{f}'(\mathbf{x}) = \mathbf{O}$ for every $\mathbf{x}\in U$.
   We take $n = 1$ and define
   $$f(x) = \begin{cases} 1 & \text{if } x<0 \\ 0 & \text{if } x>0. \end{cases}$$

6. State true of false: if U is a neighborhood of a point $\mathbf{a}$ in $\mathbf{R}^n$ and if the directional derivative of a function $\mathbf{f}$

exists at **a** in every direction then **f** is differentiable at **a**.
This statement is false. The function does not even have to be continuous at **a**.

7. Prove that if **f** and **g** are both differentiable at a point **a** in $\mathbf{R}^n$ then so is the function $\mathbf{f}+\mathbf{g}$ and we have
$$(\mathbf{f}+\mathbf{g})'(\mathbf{a}) = \mathbf{f}'(\mathbf{a}) + \mathbf{g}'(\mathbf{a}).$$
This assertion is very easy.