34th Iranian Mathematical Olympiad

2016-2017

Young Scholars Club
Ministry of Education, I.R.Iran
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34^{th} Iranian Mathematical Olympiad

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Selected Problems with Solutions

This booklet is prepared by Benyamin Ghasemi Nia.
With special thanks to Hamed Abdi, Atrin Arya, Mostafa Eynollahzadeh, Taha Miranzadeh, Hesameddin Rajabzadeh, Morteza Saghafian and Fateme Sajadi.

Iranian Team Members at the 58$^{\text{th}}$ IMO
(Rio De Janeiro - Brazil)

From left to right:

- Aryo Lotfi
- Mohammad Sadegh Mahdavi
- Taha Miranzade
- Farhood Rostamkhani
- Amirmojtaba Sabour
- Soroush Taslimi

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Preface

The 34^{th} Iranian National Mathematical Olympiad consisted of four rounds. The First Round was held on 18^{th} of February 2016 nationwide. The exam consisted of 30 multiple-choice questions and a time of 3.5 hours. In total, more than 10000 students participated in the exam and more than 1500 of them were admitted for participation in the next round.

The Second Round was held on 28^{th} and 29^{th} of April 2016. In each day, participants were given 3 problems and 4.5 hours to solve them. After this round, the top 81 students were selected to participate in the Third Round.

The examination of the Third Round consisted of six separate exams, and a Final Exam with 6 problems. At the end of this round, 35 students were awarded a bronze medal, 24 students were awarded a silver medal, and the top 17 students were awarded a gold medal. The following list represents the names of the gold medalists:

1. Hamed Abdi
2. Atrin Arya
3. Kiarash Banihashem
4. Benyamin Ghasemi Nia
5. Shayan Kiani
6. Aryo Lotfi
7. Mohammad Sadegh Mahdavi
8. Taha Miranzade
9. Ashkan Mirzaei
10. Amir Mohammad Nazari
11. Farhood Rostamkhani
12. Amirmojtaba Sabour
13. Fateme Sajadi
14. Yousef Shakiba
15. Mohammad Amin Sharifi
16. Sina Taslimi
17. Soroush Taslimi

The Team Selection Test was held on 6 days, having the same structure as the International Mathematical Olympiad (IMO). In the end, the top 6 participants were selected to become members of the Iranian Team at the 58^{th} IMO.

In this booklet, we present the 6 problems of the Second Round, the 6 problems of the Final Exam of the Third Round, and 18 proposed problems of the Team Selection Test, together with their solutions.

It's a pleasure for the authors to offer their grateful appreciation to all the people who have contributed to the conduction of the 34^{th} Iranian Mathematical Olympiad, including the National Committee of Mathematics Olympiad, problem proposers, problem selection groups, exam preparation groups, coordinators, editors, instructors and all those who have shared their knowledge and effort to increase the Mathematics enthusiasm in our country, and assisted in various ways to the conduction of this scientific event.

Problems

Second Round

1. (Omid Naghshineh Arjmand) Let a, b and c be positive real numbers with $c \geq b \geq a$. Prove that

$$\frac{(c-a)^2}{6c} \leq \frac{a+b+c}{3} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

(→ p.18)

2. (Mahdi Ghasemi) ABC is a triangle with circumcircle ω_1 and $\widehat{C} = 2\widehat{B}$. A tangent line to ω_1 at A intersects BC at E. Let ω_2 be a circle passing through B and tangent to AC at C. This circle intersects AB for the second time at F. A line through E is tangent to ω_2 at K (where BC lies between A and K). Let M be the midpoint of arc BC of ω_1 (not containing A). Prove that $MFAK$ is a cyclic quadrilateral.

(→ p.19)

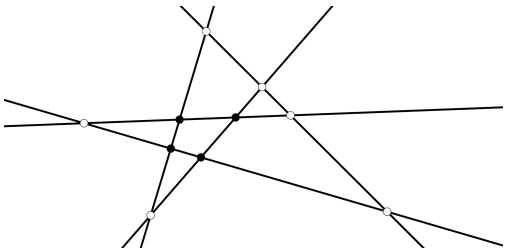
3. (Omid Naghshineh Arjmand) A council has 6 members and decisions are made based on agreeing and disagreeing votes. We call a decision making method an **Acceptable way to decide** if it satisfies the two following conditions.

- **Ascension**: If in some case, the final result is positive, it also stays positive if someone changes disagreeing vote to agreeing vote.
- **Symmetry**: If all members change their votes, the result will also change.

Weighted Voting for example, is an **Acceptable way to decide**. In this method, members are allotted with non-negative weights like $\omega_1, \omega_2, \cdots, \omega_6$ and the final decision is made with comparing the weight sum of agreeing votes, and disagreeing votes. For instance if $\omega_1 = 2$ and for all $i \geq 2, \omega_i = 1$, decision is based on the majority of the votes, and in case when votes are equal, the vote of the first member will be the decider. Give an example of some **Acceptable way to decide** that cannot be represented as a **Weighted Voting**.

(→ p.20)

4. (Mohammad Pourmohammadi) There are $n \geq 3$ lines on the plane, any two of them intersect each other and non three of them are concurrent. An intersection point is called **interior** whenever on each side of this point , on both two lines passing through it, there exists some other intersection point(s). (e.g. in the following figure with 5 lines, there are 4 **interior** points marked with filled circles.)



Prove that there are at least $\frac{(n-2)(n-3)}{2}$ **interior** points between the intersection points of these n lines.

(→ p.21)

5. (Ali Zamani) Quadrilateral $ABCD$ and point T in its interior is given such that AC is the angle bisector of $\widehat{BCD}$ and

$$\widehat{ABC} - \widehat{ATD} = \widehat{DAC}, \ \widehat{ADC} - \widehat{ATB} = \widehat{BAC}.$$

Prove that $\widehat{BAT} = \widehat{DAC}$.

(→ p.22)

6. (Mohsen Jamali) Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying the following conditions:

- For all $x, y \in \mathbb{Z}^+$, $f(x) + f(y)$ is divisible by $x + y$.
- For any integer $x \geq 1395$, the inequality $2f(x) \leq x^3$ holds.

(→ p.24)

Third Round

1. (Ali Behrooz) Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be two functions such that for all positive real numbers x and y

$$f\left(x + g(y)\right)^2 = f(x^2) + y^2.$$

Prove that the range of g is not bounded from above.

($\to$ p.26)

2. (Morteza Saghafian, Mahdi Etesami Fard) $ABCD$ is a square that is partitioned into rectangles such that no point is a corner of 4 rectangles. All corner points of rectangles are colored with 2 colors such that any two diagonal corners in a rectangle (of the partition) have different colors. If A and C have the same color, prove that B and D also have the same color.

($\to$ p.26)

3. (Mostafa Eynollahzadeh) Let p^m be a power of a prime number. Find the lowest value of d such that there exists a monic polynomial $Q(x)$ of degree d with integer coefficients such that for any positive integer n, p^m divides $Q(n)$.

($\to$ p.27)

4. (Iman Maghsoudi) An arbitrary point P lies on side BC of triangle ABC. Angle bisectors of $\widehat{APB}$ and $\widehat{APC}$ intersect the external angle bisector of $\widehat{A}$ at X and Y, respectively. Circumcircle of triangle PXY meets BC for the second time at Q. Prove that $\widehat{BAP} = \widehat{CAQ}$.

($\to$ p.28)

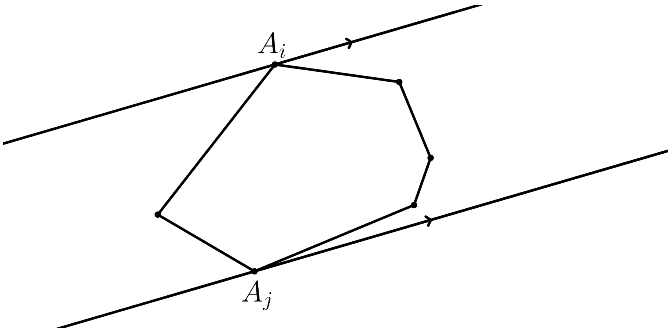
5. (Ali Sayyadi, Mahyar Sefidgaran) *a*) A number m is called **mirror-symmetry** if it is possible to divide the reverse decimal expansion of m into some blocks such that the multiply of these blocks is equal to m. For instance, numbers $6, 543$ and 21 are such blocks for number 123456, if the multiply of these 3 numbers was equal to 123456, we would call

it a **mirror-symmetry** number. Find all **mirror-symmetry** numbers with decimal digits of $\{1, 2, 3\}$.

b) A number m is called **good** if it is possible to divide m itself into some blocks with multiply of $\frac{m}{7}$. Prove that there are infinitely many **good** numbers.

($\rightarrow$ p.28)

6. (Morteza Saghafian) Let $A_1A_2\cdots A_n$ be a convex n-gon with no two sides parallel to each other. A graph G with vertices $V_1, V_2, \ldots, V_n$ is corresponded to this n-gon as following: An edge connects V_i to V_j whenever it is possible to draw two parallel lines passing through A_i, A_j such that the whole n-gon lies between these two lines (and except for A_i, A_j, not on them). Find the number of all labelled graphs with vertices $V_1, V_2, \ldots, V_n$ that are corresponded to some n-gon. (e.g. the answer for $n = 3, 4$ is $1, 4$ respectively.)



($\rightarrow$ p.29)

Team Selection Test

1. (Mohammad Jafari) Let a, b, c and d be positive real numbers with $a+b+c+d=2$. Prove that

$$\frac{(a+c)^2}{ad+bc}+\frac{(b+d)^2}{ac+bd}+4\geq 4\left(\frac{a+b+1}{c+d+1}+\frac{c+d+1}{a+b+1}\right).$$

(→ p. 35)

2. (Morteza Saghafian) In the country of Sugarland, there are 13 students in the IMO team selection camp. 6 team selection tests were taken and the results have came out. Assume that no students have the same score on the same test. To select the IMO team, the national committee of math Olympiad have decided to choose a permutation of these 6 tests and starting from the first test, the person with the highest score between the remaining students will become a member of the team. The committee is having a session to choose the permutation.
Is it possible that all 13 students have a chance of being a team member?

(→ p. 35)

3. (Hooman Fattahi) In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A and I_a and intersects the extensions of sides AB and AC (extended from B and C) at X and Y, respectively. Let S and T be points on segments I_aB and I_aC, respectively, such that $\widehat{AXI_a}=\widehat{BTI_a}$ and $\widehat{AYI_a}=\widehat{CSI_a}$. Suppose that lines BT and CS intersect at K, and lines KI_a and TS intersect at Z. Prove that X,Y,Z are collinear.

(→ p.36)

4. (Mahyar Sefidgaran, Mohyeddin Motevasel) We arranged all the prime numbers in the ascending order: $p_1=2<p_2<p_3<\cdots$.
Also assume that $n_1<n_2<\cdots$ is a sequence of positive integers that for all $i=1,2,3,\ldots$ the equation

$$x^{n_i}\overset{p_i}{\equiv}2,$$

Has a solution for x. Is there always a number x that satisfies all the equations?

($\rightarrow$ p.37)

5. (Iman Maghsoudi) In triangle ABC, arbitrary points P, Q lie on side BC such that $BP = CQ$ and P lies between B, Q. The circumcircle of triangle APQ intersects sides AB and AC at E and F, respectively. The point T is the intersection point of EP and FQ. Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X and Y, respectively. Prove that the circumcircles of triangles TXY and APQ are tangent to each other.

($\rightarrow$ p.38)

6. (Morteza Saghafian) In the unit squares of a transparent 1×100 tape, numbers $1, 2, \ldots, 100$ are written in the ascending order. We fold this tape on its lines with arbitrary order and arbitrary directions until we reach a 1×1 tape with 100 layers. A permutation of the numbers $1, 2, \ldots, 100$ can be seen on the tape, from the top to the bottom. Prove that the number of possible permutations is between 2^{100} and 4^{100}. (e.g. We can produce all permutations of numbers $1, 2, 3$ with a 1×3 tape)

($\rightarrow$ p.41)

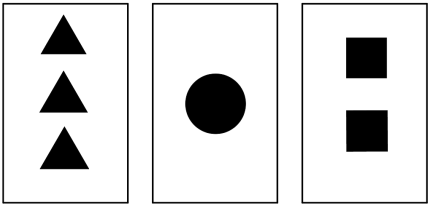
7. (Kasra Ahmadi) $ABCD$ is a trapezoid with $AB \parallel CD$. Supposet that the diagonals intersect at P. Let ω_1 be a circle passing through B and tangent to AC at A. Let ω_2 be a circle passing through C and tangent to BD at D. ω_3 is the circumcircle of triangle BPC. Prove that the common chord of circles ω_1, ω_3 and the common chord of circles ω_2, ω_3 intersect each other on AD.

($\rightarrow$ p.43)

8. (Morteza Saghafian) Find the largest natural number n for which there exist n positive integers such that non of them divides another one, but in any triplet of these numbers, one divides the sum of the other two.

($\rightarrow$ p.44)

9. (Amin Bahjati) There are 27 cards, each has some amount of (1 or 2 or 3) shapes (a circle, a square or a triangle) with some color (white, grey or black) on them. We call a triple of cards a **match** such that all of them have the same amount of shapes or mutually distinct amount of shapes, have the same shape or mutually distinct shapes and have the same color or mutually distinct colors. For instance, three cards shown in the figure are a **match** because they have distinct amount of shapes, distinct shapes but the same color of shapes.



What is the maximum number of cards that we can choose such that non of the triples make a **match**?

($\rightarrow$ p.46)

10. (Alireza Shavali) A $(n+1)$-tuple $(h_1, h_2, \dots, h_{n+1})$ where for every $1 \le i \le n+1$, $h_i\,(x_1, x_2, \dots, x_n)$ is a n variable polynomial with real coefficients is called **good** if the following condition holds.
For any n functions $f_1, f_2, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$ if for all $1 \le i \le n+1$, $P_i(x) = h_i\,(f_1(x), f_2(x), \dots, f_n(x))$ is a polynomial with variable x, then $f_1(x), f_2(x), \dots, f_n(x)$ are polynomials.
a) Prove that for all positive integers n, there exists a **good** $(n+1)$-tuple $(h_1, h_2, \dots, h_{n+1})$ such that the degree of each h_i is more than one.
b) Prove that there does not exist any integer $n > 1$ that for which there is a **good** $(n+1)$-tuple $(h_1, h_2, \dots, h_{n+1})$ such that all h_i are symmetric polynomials.

($\rightarrow$ p.49)

11. (Aryan Tajmir) k, n are two arbitrary positive integers. Prove that there exists at least $(k-1)(n-k+1)$ positive integers that can be produced by n number of k's and using only $+, -, \times, \div$ operations and adding parentheses between them, but cannot be produced using $n-1$ number of k's.

($\rightarrow$ p.51)

12. (Amirhossein Pooya) Let $k > 1$ be an integer. The sequence $\{a_i\}_{i=1}^{\infty}$ is defined as $a_1 = 1,\ a_2 = k$, and for all $n > 1$ we have

$$a_{n+1} - (k+1)a_n + a_{n-1} = 0.$$

Find all positive integers n such that a_n is a power of k.

($\rightarrow$ p.52)

13. (Navid Safaei) Let $n > 1$ be an integer. Prove that there exists an integer $n-1 \ge m \ge \left\lfloor \frac{n}{2} \right\rfloor$ such that the following equation has integer solutions with $a_m > 0$

$$\frac{a_m}{m+1} + \frac{a_{m+1}}{m+2} + \cdots + \frac{a_{n-1}}{n} = \frac{1}{\operatorname{lcm}(1, 2, \dots, n)}.$$

($\rightarrow$ p.54)

14. (Ali Zamani) Suppose that P is a point in the interior of quadrilateral $ABCD$ such that

$$\widehat{BPC} = 2\widehat{BAC}, \quad \widehat{PCA} = \widehat{PAD}, \quad \widehat{PDA} = \widehat{PAC}.$$

Prove that

$$\widehat{PBD} = \left|\widehat{PCA} - \widehat{BCA}\right|.$$

(→ p.56)

15. (Mojtaba Zare, Ali Daei Nabi) Find all functions $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions for all positive real numbers x, y, z

$$f\left(f(x,y), z\right) = x^2 y^2 f(x,z),$$

$$f\left(x, 1 + f(x,y)\right) \geq x^2 + xyf(x,x).$$

(→ p.57)

16. (Morteza Saghafian) There are 6 points on the plane such that no three of them are collinear. We know that among every 4 points of them, there exists a point that its power with respect to the circle passing through the other three points is a constant value k (power of a point in the interior of a circle has a negative value). Prove that $k = 0$ and all 6 points lie on a circle.

(→ p.59)

17. (Navid Safaei) Let $\{c_i\}_{i=0}^{\infty}$ be a sequence of non-negative real numbers with $c_{2017} > 0$. A sequence of polynomials if defined as

$$P_{-1}(x) = 0, \; P_0(x) = 1,$$
$$P_{n+1}(x) = xP_n(x) + c_n P_{n-1}(x). \quad n \geq 0$$

Prove that there does not exist any integer $n > 2017$ and some real number c such that

$$P_{2n}(x) = P_n(x^2 + c).$$

(→ p.60)

18. (Iman Maghsoudi) In triangle ABC denote by O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E and F lie on sides AB and AC, respectively, such that $BE = PC$ and $CF = PB$. Let K be the intersection point of AP and OH. Prove that $\widehat{EKF} = 90°$.

(→ p.61)

Solutions

Second Round

1. **First solution.** Just to simplify the calculations, define $S = a + c$ and $P = ac$. In this case, the RHS of the inequality becomes

$$RHS = \frac{a+b+c}{3} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{S+b}{3} - \frac{3Pb}{Sb+P}.$$

The goal is to minimize the above phrase as a function of b; Note that the phrase, as a function of b, has a linear term which is $\frac{S+b}{3}$, and a fractional term with the denominator $Sb + P$. Now we re-write the phrase using the new variable $t = Sb + P$ and with a little calculation we shall have

$$RHS = \frac{S+b}{3} - \frac{3Pb}{Sb+P} = \frac{S^2 - 10P}{3S} + \frac{t}{3S} + \frac{P^2}{St}.$$

Note that the multiply of the last two terms is a phrase without the variable t. Therefore with using the AM-GM inequality we obtain

$$RHS \geq \frac{S^2 - 10P}{3S} + 2\sqrt{\frac{P^2}{S^2}} = \frac{S^2 - 4P}{3S} = \frac{(c-a)^2}{3(c+a)} \geq \frac{(c-a)^2}{6c}. \qquad \blacksquare$$

Second solution. For the easement of calculations, for any arbitrary formula $F(a, b, c)$ with three variables a, b and c, the cyclic sum

$$F(a,b,c) + F(b,c,a) + F(c,a,b),$$

is shown with $\sum F(a, b, c)$. Using this notation, we expand the RHS of the inequality

$$\begin{aligned}\frac{\sum a}{3} - \frac{3}{\sum \frac{1}{a}} &= \frac{\sum a}{3} - \frac{3abc}{\sum ab} = \frac{(\sum a)(\sum ab) - 9abc}{3\sum ab} \\ &= \frac{\sum(a^2b + ab^2) - 6abc}{3\sum ab} = \frac{\sum a(b-c)^2}{3\sum ab}.\end{aligned}$$

Therefore, the claim of the problem is equivalent to the following inequality

$$\sum a(b-c)^2 \geq \left(\frac{\sum ab}{2c}\right)(c-a)^2. \tag{1}$$

But note that with condition $a \le b \le c$, $\frac{ab}{2c}$ is not greater than $\frac{b}{2}$ so for the RHS of the above inequality we have

$$\left(\frac{\sum ab}{2c}\right)(c-a)^2 = \left(\frac{ab}{2c} + \frac{a+b}{2}\right)(c-a)^2 \le \left(\frac{a}{2} + b\right)(c-a)^2. \quad (2)$$

For the LHS of (1), we have

$$\sum a(b-c)^2 \ge b(c-a)^2 + a\left((c-b)^2 + (b-a)^2\right). \quad (3)$$

Note that for any two numbers X, Y,

$$(X-Y)^2 \ge 0 \implies X^2 + Y^2 - 2XY \ge 0 \implies 2(x^2 + Y^2) \ge (X+Y)^2$$
$$\implies X^2 + Y^2 \ge \frac{(X+Y)^2}{2}.$$

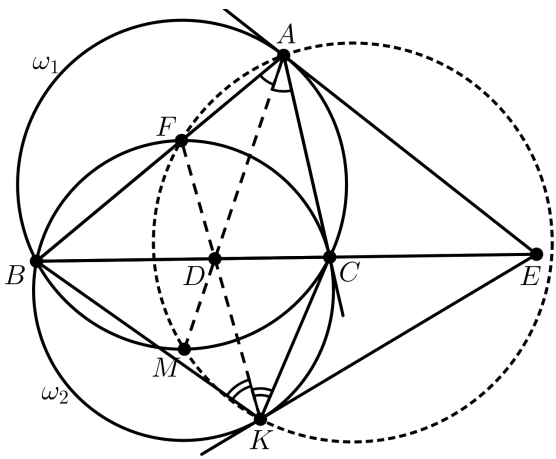
Therefore, if $X = c-b, Y = b-a$ using (3), we obtain

$$\sum a(b-c)^2 \ge b(c-a)^2 + a\left((c-b)^2 + (b-a)^2\right) \ge b(c-a)^2 + \frac{a}{2}(c-a)^2.$$

This inequality along with (2) implies (1) which is equivalent to the claim of the problem. ■

2. Considering the power of point E with respect to circles ω_1, ω_2 we have

$$\left.\begin{array}{rcl} EC \cdot EB &=& EA^2 \\ EC \cdot EB &=& EK^2 \end{array}\right\} \implies EA = EK.$$



Now we claim that the angle bisectors of angles $\widehat{BAC}$ and $\widehat{BKC}$ intersect each other on side BC. For this purpose, it suffices to show that

$$\frac{KB}{KC} = \frac{AB}{AC}.$$

Triangles ECK and EKB are similar therefore $\frac{KB}{KC} = \frac{EB}{EK}$. Triangles EAC and EBA are also similar therefore $\frac{AB}{AC} = \frac{EB}{EA}$.

And since $EK = EA$ we obtain $\frac{KB}{KC} = \frac{AB}{AC}$. Let D be the intersection point of these angle bisectors with BC, points F and M are the midpoint of arc BC in the two circles. Therefore K ,D and F are collinear and A, D and M are also collinear. Now, consider the power of D in both circles

$$\left.\begin{array}{rcl} BD \cdot DC & = & AD \cdot DM \\ BD \cdot DC & = & FD \cdot DK \end{array}\right\} \Longrightarrow AD \cdot DM = FD \cdot DK,$$

Hence $AFMK$ is cyclic. ∎

3. We number the members with $1, 2, \ldots, 6$. Consider the following decision method:

If person number 1 to the third person have the same vote, their vote is considered as the final result and in other case, the result is based on the majority of the votes between forth to the sixth persons.

This method clearly satisfies both **Ascension** and **Symmetry** conditions, so it is an **Acceptable way to decide**. We claim that this method is impossible to be represented as a **Weighted Voting**.

Assume the contrary, and let $(\omega_1, \ldots, \omega_6)$ be a weighting for this method (Means the weight of the first person is ω_1, the second one is ω_2 and so on the person i is ω_i). Note that the first to the third person have the same role on this method, also the next three persons have the same role. Therefore with this method,

$$(\omega_2, \omega_3, \omega_1, \omega_5, \omega_6, \omega_4)$$

and also

$$(\omega_3, \omega_1, \omega_2, \omega_6, \omega_4, \omega_5)$$

are other ways to represent the method as a **Weighted Voting**. It's easy to see that if some weightings lead to decide the same result, the sum of these weightings also lead to that result. So the sum of the three weightings we discussed is also a weighting for the presented method. Which means if we set $a = \omega_1 + \omega_2 + \omega_3$ and $b = \omega_4 + \omega_5 + \omega_6$, (a, a, a, b, b, b) is a weighting for the method.

Now first consider a case when first three persons have agreeing votes, and the last three persons have disagreeing votes. Since the result is set to be positive, by the given weighting we must have $3a > 3b$. Consider another case where two of the first three and one of the last three persons have agreeing votes and others have disagreeing votes. In this case the result is set to be negative, but the weight sum of the agreeing votes and disagreeing votes are $2a + b$ and $2b + a$, respectively, therefore we must have $2b + a > 2a + b$ which implies $b > a$. This inequality is in contradiction with the last one, therefore our contrary assumption leads to contradiction. So the given example is not a **Weighted Voting**. ■

Comment. There are other methods that can also be presented, here is two of them:

- If some decision (agree or disagree) has more than three votes, it will be the final result and in case of equality between the votes, the decision that an odd number of first to third persons have made is the result.

- Assume that the first five persons are sitting around a circular table. In this case if three adjacent persons have the same vote, that vote will be the result and in other case, the vote of the sixth person will be the result.

4. Consider a graph G with vertices of the intersection points in the statement of the problem and with edges of the segments made on these lines (means segments that have their terminal points chosen between the intersection points, and have no other intersection points on them). Any two of these n lines have exactly one intersection point, therefore G is a graph with $\frac{n(n-1)}{2}$ vertices. On the other hand, any line has $n - 1$ intersection points on itself, hence the segments between these points give $n - 2$ edges of G. So there are a total of $n(n - 2)$ edges in G.

Since there are at least three lines, any vertex of G has degree 2, 3 or 4, let the number of vertices with each of degree be a, b and c, respectively. Vertices with the highest degree are the **interior** points. Note that having the number of vertices of G and the fact that the sum of the degrees is twice as the number of the edges, we obtain

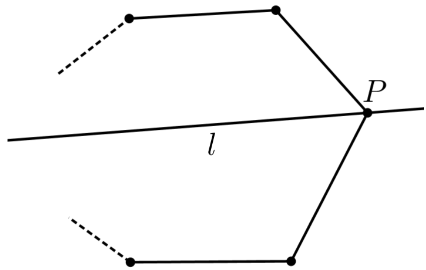
$$a + b + c = \frac{n(n-1)}{2}, \quad 2a + 3b + 4c = 2n(n-2).$$

If we subtract triple of the first equation from the second one we get

$$c - a = \frac{n^2 - 5n}{2} \implies c = (a - 3) + \frac{(n-2)(n-3)}{2}.$$

So the problem is equivalent to prove that there are at least 3 vertices with degree 2. To prove this, consider the convex hull of these points.

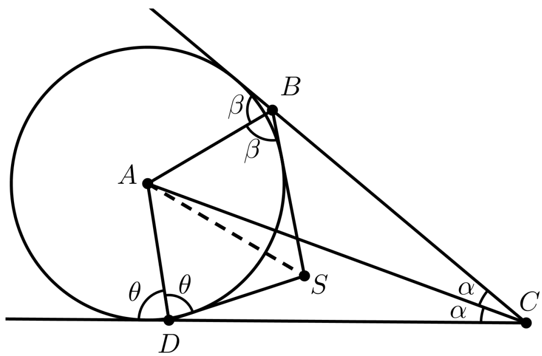
Let it be a k-gon with $k \geq 3$, it is easy to see that all vertices of the convex hull are of degree 2. Because if l is one of the two lines passing through P, a vertex of the convex hull, all the intersection points on it, lie inside of the k-gon and therefore, on the same side of P. So P has only one adjacent on this line and considering the other line passing through P, we conclude that P has the degree of 2 in the graph.



Therefore $a \geq k \geq 3$ and the proof is complete. ■

5. **First solution.** Vertex A lies on the angle bisector of $\widehat{C}$ and has equal distances to the lines CB and CD. So there exists a circle with center A that is tangent to CB and CD.

Let S be the intersection of second tangent lines through points B and D to this circle.



$$
\begin{aligned}
\widehat{BSD} &= \widehat{BCD} + \widehat{CBS} + \widehat{CDS} = 2\alpha + 180^\circ - 2\beta + 180^\circ - 2\theta \\
&= 2(180^\circ + \alpha - \beta - \theta),
\end{aligned}
$$

$$
\Longrightarrow \quad \widehat{BSA} = \widehat{DSA} = 180^\circ + \alpha - \beta - \theta,
$$

$$
\Longrightarrow \quad \begin{cases} \widehat{ABC} - \widehat{ASD} = (180^\circ - \beta) - (180^\circ + \alpha - \beta - \theta) = \theta - \alpha = \widehat{DAC}, \\ \widehat{ADC} - \widehat{ASB} = (180^\circ - \beta) - (180^\circ + \alpha - \beta - \theta) = \beta - \alpha = \widehat{BAC}, \end{cases}
$$

$$\Longrightarrow \begin{cases} \widehat{ASD} &= \widehat{ATD}, \\ \widehat{ASB} &= \widehat{ATB}. \end{cases} \tag{1}$$

Note that

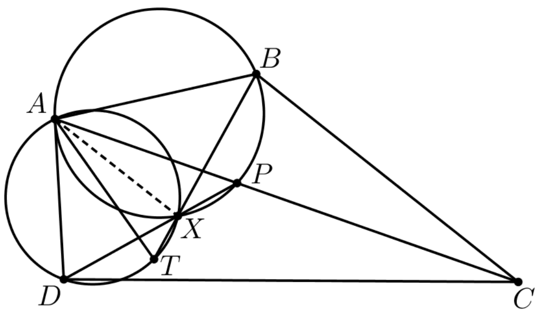
$$\widehat{BAS} = 180^\circ - \widehat{ABS} - \widehat{ASB} = 180^\circ - \beta - (180^\circ + \alpha - \beta - \theta) = \theta - \alpha = \widehat{DAC}.$$

So it suffices to show that $S = T$. If S and T are not coincident, then using (1), quadrilaterals $ABST$ and $ADST$ are cyclic, therefore $BADS$ is also cyclic and

$$\begin{aligned} \beta + \theta = 180^\circ \quad &\Longrightarrow \widehat{BSA} = \widehat{DSA} = 180^\circ + \alpha - \beta - \theta = \alpha, \\ &\Longrightarrow \widehat{BSD} = \widehat{BSA} + \widehat{DSA} = 2\alpha, \\ &\Longrightarrow \widehat{BTD} = \widehat{BSD} = 2\alpha = \widehat{BCD}. \end{aligned}$$

But according to the statement of the problem, T lies in the interior of $ABCD$ and $\widehat{BTD} > \widehat{BCD}$. Therefore S and T are coincident on each other, hence the claim of the problem. ■

Second solution.



$$\begin{aligned} \widehat{BCA} = \widehat{DCA} \quad &\Longrightarrow \widehat{ABC} + \widehat{BAC} = \widehat{ADC} + \widehat{DAC}, \\ &\Longrightarrow \widehat{ABC} - \widehat{DAC} = \widehat{ADC} - \widehat{BAC}, \\ &\Longrightarrow \widehat{ATD} = \widehat{ATB} = \alpha. \end{aligned}$$

Point P lies on diagonal AC such that $\widehat{PBC} = \widehat{DAC}$, so we have

$$\widehat{ABP} = \widehat{ABC} - \widehat{PBC} = \widehat{ABC} - \widehat{DAC} = \alpha,$$

$$\overset{\triangle}{BPC} \sim \overset{\triangle}{ADC} \Longrightarrow \frac{BC}{AC} = \frac{PC}{DC}.$$

Hence,

$$\left.\begin{array}{c}\widehat{BCA} = \widehat{DCA} \\ \frac{BC}{PC} = \frac{AD}{DC}\end{array}\right\} \implies \overset{\triangle}{DPC} \sim \overset{\triangle}{ABC},$$

$$\implies \widehat{PDC} = \widehat{BAC},$$

$$\implies \widehat{ADP} = \widehat{ADC} - \widehat{PDC} = \widehat{ADC} - \widehat{BAC} = \alpha.$$

Let X be the intersection point of DP and BT, we have

$$\widehat{ADX} = \widehat{ATX} = \alpha \implies ADTX \text{ is cyclic.}$$

$$\widehat{AXD} = \widehat{ATD} = \widehat{ABP} = \alpha \implies ABPX \text{ is cyclic.}$$

Therefore,

$$\widehat{BAP} = \widehat{BXP} = \widehat{DXT} = \widehat{DAT} \implies \widehat{BAT} = \widehat{DAC}.$$

■

6. First note that setting $x = y$ in the first assumption implies $f(x)$ is divisible by x and therefore we can write $f(x) = xg(x)$ where $g(x)$ is always an integer. Now set $g(1) = a$ and $g(2) = b$. We are going to show that for an odd and sufficiently large number x, $g(x)$ is equal to a. To prove this, we use the first assumption twice, for $y = 1$ and $y = 2$, and an odd number x

$$\left.\begin{array}{r}x+1 \mid xg(x)+a \\ x+1 \mid (x+1)g(x)\end{array}\right\} \implies x+1 \mid g(x)-a,$$

$$\left.\begin{array}{r}x+2 \mid xg(x)+2b \\ x+2 \mid (x+2)g(x)\end{array}\right\} \implies x+2 \mid 2(g(x)-b) \implies x+2 \mid g(x)-b.$$

(The last result is true because $x+2$ is odd.) Therefore, according to the fact that $x+1$ and $x+2$ are relatively prime, the remainder of $g(x)$ when divided by $(x+1)(x+2)$ is uniquely found knowing the remainder when divided by $x+1$ and $x+2$. The number $a(x+2)-b(x+1)$ as $g(x)$ satisfies both the above relations. So we have

$$g(x) = a(x+2)-b(x+1)+c(x+1)(x+2) = cx^2+(3c+a-b)x+(2c+2a-b).$$

Where c is an integer. According to the statement of the problem, for sufficiently large x we have $0 < g(x) \leq \frac{x^2}{2}$, therefore c is a non-negative number and is at most $\frac{1}{2}$, since it is an integer, $c = 0$ and we have

$$g(x) = (a-b)x + (2a-b).$$

(For sufficiently large odd values of x.)

Lemma. *If X, Y are relatively prime, then $X + Y \mid g(X) - g(Y)$.*

Proof. By the statement of the problem, $Xg(X) + Yg(Y)$ is divisible by $X+Y$. Adding $-(X+Y)g(Y)$ to that number, implies $X(g(X)-g(Y))$ is divisible by $X+Y$. But X and $X+Y$ are relatively prime, therefore $X + Y \mid g(X) - g(Y)$.

Now assume that x' is an odd and sufficiently large number that is relatively prime to x , in this case according to the lemma

$$x + x' \mid (a - b)(x - x'),$$

but since $\gcd(x+x', x-x') = 2$ we obtain that $\frac{x+x'}{2} | (a-b)$. Now choose x' such that it is greater than $2|a-b|$, it implies $a-b=0$ and therefore for large odd numbers x we have $g(x) = 2a - b = a$ and the claim is proved.

Now let y be an arbitrary positive integer, if x is a sufficiently large number, relatively prime with $2y$, according to the lemma

$$x + y \mid g(x) - g(y) = a - g(y).$$

Setting x such that it is greater than $|a - g(y)|$, implies $g(y) = a$ and hence $f(y) = ay$.

Finally, using the second assumption we get that the only solutions are $f(x) = ax$ where a is a positive integer not exceeding $\frac{1395^2}{2}$. ■

Third Round

1. Let $P(x, y)$ be the assertion

$$f(x+g(y))^2 = f(x^2) + y^2$$

If $g(f(t)) < t$ for some $t > 0$, then $P(t - g(f(t)), f(t))$ implies

$$f(t)^2 = f\big((t-g(t))^2\big) + f(t)^2 \implies f\big((t-g(t))^2\big) = 0$$

which is impossible.

So $g(f(x)) \geq x$, for all positive real numbers x. Hence the solution is complete. ■

2. Consider a graph G as follows, vertices of rectangles of the partition are vertices of G, two vertices are connected if they are opposite corners of a rectangle of the partition. Obviously, there are $2n$ edges in G where n is the number of rectangles used in the partition.

Note that if some point is a vertex of 3 rectangles, it must be a vertex of exactly 4 rectangles. By the statement of the problem, we conclude that each vertex of G has degree equal to 1 or 2 and also, the only points with degree 1 are A, B, C, D. Therefore, it is easy to see that G can be partitioned into some cycles and some paths. Each cycle has an even number of vertices, since any two adjacent vertices have the opposite colors. Also, since the first and the last vertices of each path are of degree 1, we conclude that there are exactly two paths in G, with ending points of A, B, C, D. There are two possibilities.

- A and C are in the same path.

  We know that the total number of edges is an even number ($2n$) and each cycle also has an even number of edges, and since A and C have the same color. the path containing A and C also has an even number of edges. Therefore, the remaining path of the graph, starts and ends with B and D and must have an even number of edges. Since any two adjacent vertices have different colors, we conclude that B and D are of the same color.

- A and C are not in the same path.

  Without loss of generality, assume that pairs A and D are in the same path and therefore B and C are also in the same path. With similar argument, we find out that either both paths have an odd number of edges, or both have an even number of edges.

  In the first case we conclude that the color of A is different with D, and the color of C is different with B. Since A and C have the same color, this implies that B and D have the same color.

  Similarly in the second case we conclude that A, B, C, D all have the same color.

So in each case, the claim of the problem is proved. ■

3. We prove the problem using the following lemma, that is actually a more general form of the problem.

Lemma. *Let l be a fixed non-zero integer and $P(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients such that for all integers k, $P(k)$ is divisible by l. Then we have $l \mid a_n \cdot n!$.*

Proof. We use induction on n. For $n = 0$ the claim of the lemma is obvious. Now we assume that the claim is true for n.

Let $P(x) = a_{n+1}x^{n+1} + \cdots + a_0$ be a polynomial of degree $n+1$ such that for all integers k, $l \mid P(k)$. We define two polynomials $R(x)$ and $S(x)$ as

$$P(x+1) - P(x) = R(x) = (n+1)a_{n+1}x^n + S(x),$$

then R is a polynomial of degree n and S has a degree less than n. Now according to the lemma for n, since $l \mid R(x)$ we obtain

$$l \mid ((n+1)a_{n+1}) \cdot n! = a_{n+1}(n+1)! \ ,$$

so the lemma is proved.

Now applying the Lemma for $l = p^m$ and $Q(x) = x^d + \cdots$ from the problem, we get $p^m \mid d!$. Let s be the smallest integer such that $p^m \mid s!$, we obtained $d \geq s$. Now consider the following polynomial

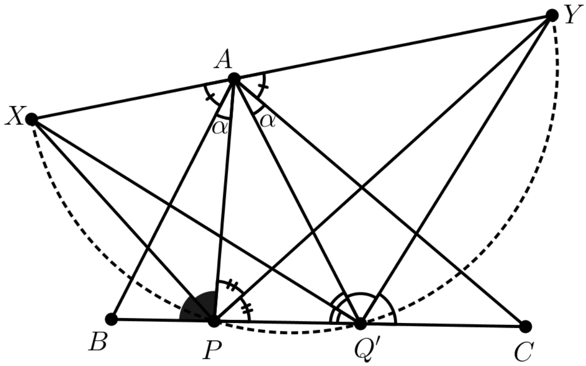
$$Q(x) = (x-s)(x-(s-1)) \cdots (x-1).$$

For any integer n, the number $t = Q(n)$ is a multiply of s consecutive integers and it is well-known that for such t, $s! \mid t$. Thus we have

$$p^m \mid s! \mid Q(n).$$

Therefore the answer of the problem is the smallest number d such that $p^m \mid d!$. ■

4. Let Q' be a point on side BC such that $\widehat{BAP} = \widehat{CAQ'} = \alpha$.



Note that X, Y also lie on the exterior angle bisector of $\widehat{PAQ'}$, that's because

$$\widehat{PAX} = \widehat{Q'AY} = \left(90^\circ - \tfrac{\widehat{A}}{2}\right) + \alpha \ .$$

Also PX, PY are exterior and interior angle bisector of vertex P in triangle APQ'. Therefore X is the Q'-excenter, and Y is the P-excenter of this triangle. So we get

$$\widehat{XPY} = \widehat{XQ'Y} = 90^\circ \implies XPQ'Y \text{ is cyclic.}$$

Since the intersection of $C_{\underset{XPQ}{\triangle}}$ with BC (other than P), is a unique point, we get $Q = Q'$. Therefore $\widehat{BAP} = \widehat{CAQ'} = \widehat{CAQ}$ ■

5. *a*) For any number A, let $\overleftarrow{A}$ be the reverse decimal expansion of A. Assume that $A = \overline{A_n A_{n-1} \cdots A_1}$ is a **mirror-symmetry** number with m digits, all from $\{1, 2, 3\}$, and $A_n, A_{n-1}, \ldots, A_1$ are blocks of A with number of digits $m_n, \ldots, m_1$ such that

$$A = \overleftarrow{A_n} \times \overleftarrow{A_{n-1}} \times \cdots \times \overleftarrow{A_1}.$$

Note that for all $1 \leq i \leq n$, we have

$$\overleftarrow{A_i} \leq \underbrace{333\cdots33}_{m_i} = \frac{10^{m_i} - 1}{3}.$$

On the other hand,

$$A \geq \underbrace{111\cdots11}_{m} = \frac{10^m - 1}{9}.$$

Therefore we obtain

$$\frac{10^m - 1}{9} \leq \frac{10^{m_n} - 1}{3} \times \cdots \times \frac{10^{m_1} - 1}{3}.$$

If $n \geq 2$ we have

$$\begin{aligned} 3^{n-2}(10^m - 1) &\leq (10^{m_n} - 1)(10^{m_{n-1}} - 1)\cdots(10^{m_1} - 1) \\ &< 10^{m_n} \times 10^{m_{n-1}} \times \cdots \times 10^{m_2} \times (10^{m_1} - 1) \\ &< 10^{m_n + m_{n-1} + \cdots + m_1} - 1 = 10^m - 1. \end{aligned}$$

Which is impossible. Therefore $n = 1$. So the only possible case is when $A = \overleftarrow{A}$, that means A is a **Palindromic number** (a number that remains the same when its digits are reversed). Clearly, all Palindromic numbers with digits of $\{1, 2, 3\}$ satisfy the conditions.

b) This part is a test of effort! Note that if we could find a **good** number $m = \overline{A_1 A_2 \cdots A_n}$ where A_i's are blocks of m such that

$$\frac{m}{7} = A_1 \times \cdots \times A_n,$$

then $10m$ is also a **good** number because

$$\frac{10m}{7} = A_1 \times \cdots \times \overline{A_n 0}.$$

Therefore $m, 10m, 100m, \ldots$ are all **good** numbers . So indeed, we just need to find a single **good** number. Now if we start to check the multiplies of 7 one by one, we shall finally reach $7 \times 45 = 315$ that for which

$$\frac{315}{7} = 3 \times 15.$$

Therefore by putting $m = 315$, we can find infinitely many **good** numbers, $\{315, 3150, 31500, \ldots\}$.

■

6. Let $A_1 A_2 \cdots A_n$ be an arbitrary n-gon and assume that G is the graph corresponded to this n-gon. First, a lemma is proved. Note that throughout the solution, indices are considered modulo n.
(i.e. $A_{n+1} = A_1$, $V_{n+1} = V_1$,... .) We start with some lemmas.

Lemma. *There is no isolated vertex in G, i.e. every vertex V_i of G is connected to at least one other vertex.*

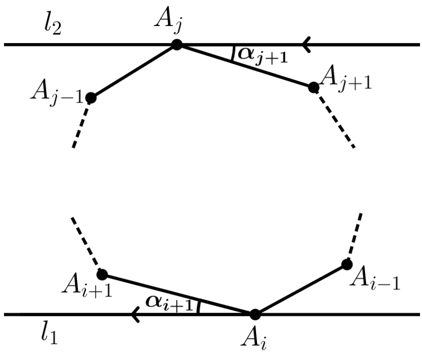
Proof. Since the polygon is convex, there exist a line l passing through A_i such that the whole polygon (except point A_i) is placed in one side of l. We can choose l such that it is not parallel to any side of the polygon. Now let A_j be the farthest vertex of the polygon from the line l. Let l' be the parallel line to l, passing through A_j. Since l was not parallel to any side of our polygon, there are no other vertex of the polygon on l'. So the pair (l, l') satisfies the desired property and so V_i is connected to V_j.

Lemma. *If V_iV_j is an edge of the graph, then exactly one of V_iV_{j+1} or $V_{i+1}V_j$ is an edge in G. Similarly, exactly one of $V_{i-1}V_j$ and V_iV_{j-1} is an edge in G.*

Proof. Since V_iV_j is an edge of the graph, there are two parallel lines l_1 and l_2, passing through A_i and A_j such that the whole polygon is placed between l_1 and l_2.
Define $\alpha_{i+1} = \measuredangle(A_{i+1}A_i, l_1)$ and $\alpha_{j+1} = \measuredangle(A_{j+1}A_j, l_2)$. Note that $\alpha_{i+1} \neq \alpha_{j+1}$, because otherwise we must have

$$A_iA_{i+1} \parallel A_jA_{j+1},$$

Which is impossible because of the statement of the problem.



We claim that if $\alpha_{i+1} > \alpha_{j+1}$, then V_iV_{j+1} is an edge of the graph (and vice versa). That is because there exist two parallel lines l and l', such that the strip bounded by l and l' contains the polygon; Consider a line l intersecting the n-gon only at A_{j+1} such that

$$\measuredangle(l, A_jA_{j+1}) < \alpha_{i+1} - \alpha_{j+1} \ .$$

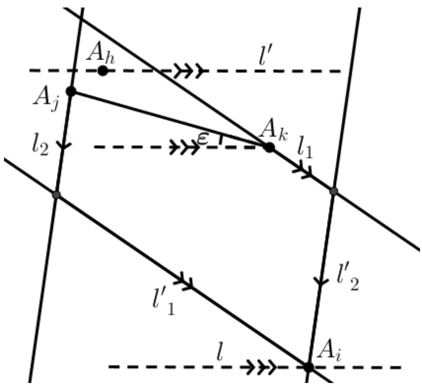
And l' is a parallel line to l, passing through A_i. Also, any line l such that the polygon is placed between l and a parallel line to l through A_i,

satisfies the given inequalities.
Similarly, in other case we have $\alpha_{j+1} > \alpha_{i+1}$ if and only if $V_{i+1}V_j$ is an edge of the graph.
So the lemma is proved.

Now it is deduced, from the lemma, that if V_iV_j is an edge of the graph, then at least one of the $\deg(V_i) \geq 2$ or $\deg(V_j) \geq 2$ inequalities hold.
Another lemma is needed to continue the solution.

Lemma. *Suppose that V_i is adjacent to two other vertices V_j and V_k in G such that A_j lies between A_i and A_k on the perimeter of the polygon, when we move from A_i to A_k clockwise. Then V_i is also connected to V_h, for any h such that A_h lies between A_j and A_k in clockwise order (if such h exists). Also* $\deg(V_h) = 1$.

Proof. To prove the lemma, we claim that if V_iV_j and V_iV_k are two edges in graph and A_jA_k is not a side of the polygon, then there exists some h (where A_h is between A_j and A_k) such that V_iV_h is also an edge in G. If there are a total of s points on the perimeter of polygon, between A_j and A_k, then applying this claim for s times will lead to the conclusion of the first part of the lemma.
Since V_iV_j and V_iV_k are edges of the graph, there are lines l_1 and l_2 passing through A_j and A_k, respectively, and lines l'_1 and l'_2 passing through A_i such that $l_1 \parallel l'_1$, $l_2 \parallel l'_2$ and two pairs (l_1, l'_1) and (l_2, l'_2) have the desired property. Hence, the polygon lies inside of the parallelogram bounded by these four lines.



Consider a line l passing through A_i such that the difference between the line scopes of l and A_iA_j is sufficiently small and moreover l is not parallel to any side of the polygon; Hence, the whole polygon is on one side of l. Between the points on the perimeter of the polygon, when we move from A_j to A_k clockwise, consider the point A_h with the farthest distance to l. Let l' be a line through A_h and parallel to l, hence similar to the first lemma, the pair (l, l') has the desired property, therefore V_iV_h is an edge of the graph. So the first part of the lemma is proved. Hence, V_i is connected to a consecutive set of vertices $\{V_j, V_{j+1}, \ldots, V_{k-1}, V_k\}$.

Since both V_iV_{h-1} and V_iV_h are edges of the graph, according to the first lemma, $V_{i-1}V_h$ cannot be an edge. Also since both V_iV_{h+1} and V_iV_h are edges of the graph, then $V_{i+1}V_h$ cannot be an edge of the graph. Now if V_h is connected to any other vertex, according to the previous part of this lemma, it should be connected to one of V_{i-1} or V_{i+1}, which is impossible. So $\deg(V_h) = 1$. Therefore the lemma is proved.

So G is a graph where (according to the third lemma,) any vertex V_i is connected to a consecutive set of vertices $\{V_j, V_{j+1}, \ldots, V_{j+k}\}$ with $k \geq 0$. Note that since V_iV_{j-1} is not an edge of the graph, according to the second lemma we conclude that $V_{i-1}V_j$ is an edge, therefore $\deg(V_j) > 1$, similarly $V_{i+1}V_{j+k}$ is an edge, hence $\deg(V_{j+k}) > 1$. So the lemma says that the neighbours of a fixed vertex, are consecutive vertices in the polygon, also, only the first and the last vertex of the list have a degree more than 1, and the others are of degree 1.

In summary we have

(1) Referring to the first lemma, for every vertex V_i of G, $\deg(V_i) \geq 1$.

(2) If $\deg(V_i) = 1$, for some i, then according to the second lemma, the only neighbour of V_i has degree more than 1.

(3) If $\deg(V_i) \geq 2$, for some i, then V_i is connected to exactly two other vertices of degree at least two, say V_j and V_k ($i \neq \{j, j+1, \ldots, k\}$), such that all vertices of the form V_{j+r} for $1 \leq r \leq k-j-1$, have degree one and they are connected only to V_i.

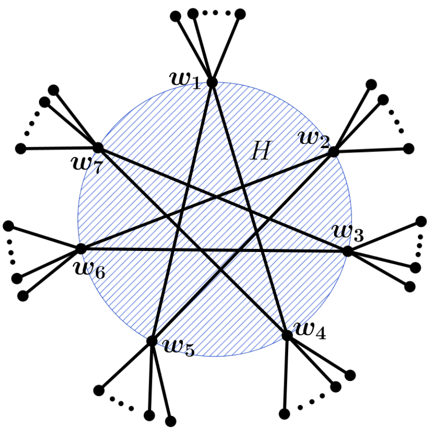
Now just focus on vertices of G that have a degree of at least 2, and set H to be the induced sub-graph of G on these vertices. Assume that $w_1 = V_{i_1}, w_2 = V_{i_2}, \ldots, w_m = V_{i_m}$ are all vertices of H, where $1 \leq i_1 \leq \cdots \leq i_m \leq n$. (Indices of w_i's are considered modulo m.) Note that if $w_{l-1}w_kw_lw_{k-1}$ is a path is H, then by the third lemma, $V_{i_l-1}V_{i_k}V_{i_l}V_{i_k-1}$ is a path in G, which is impossible by the second lemma. So the second lemma is also true for H, means

$$\text{If } w_iw_j \text{ is an edge of } H, \text{ then exactly one of the } w_{i+1}w_j \text{ or } w_iw_{j+1} \text{ is also an edge of } H. \text{ Similarly exactly one of } w_{i-1}w_j \text{ and } w_iw_{j-1} \text{ is an edge of } H. \qquad (*)$$

Now assume that w_1 is connected to w_a, w_{a+1} (in H). By $(*)$, w_{a+1} cannot be connected to w_m, so its second neighbour is w_2. Similarly

the second neighbour of w_2 cannot be w_a and so w_2 is connected to w_{a+2}. Repeating these arguments implies that for any $1 \le t \le m$, the neighbours of w_t are $\{w_{t+a-1}, w_{t+a}\}$.
Now, on one hand w_1 and w_m are both connected to w_a and on the other hand, neighbours of w_a are w_{2a-1} and w_{2a}. Hence $w_{2a} = w_1$ and therefore $m \mid 2a-1$, but since $1 < a < m$, $2a-1 < 2m-1$, we must have $2a-1 = m \ge 3$.
So we have found the figuration of H. Any subset of $\{1, 2, \dots, n\}$ with odd number of members (and at least 3), is corresponded with a graph G as following.

Assume that $S = \{i_1, i_2, \dots, i_m\}$ is a subset such that m is an odd number and $i_1 < i_2 < \cdots < i_m$. A graph H with m vertices is corresponded with S such that vertex V_{i_k} is connected to $V_{i_{k+\frac{m-1}{2}}}$ and $V_{i_{k+\frac{m+1}{2}}}$, whatever $1 \le k \le m$ is (indices are considered modulo m). Also, the graph G with n vertices is uniquely found by H, for all $1 \le k \le m$, vertex V_{i_k} is connected to all vertices between $V_{i_{k+\frac{m-1}{2}}}$ and $V_{i_{k+\frac{m+1}{2}}}$ (when we move clockwise). Any other vertex V_j among these is a vertex of degree 1, and its only neighbour is V_{i_k}).
The following figure for example, is what G looks like when $m = 7$.



Also, for such graph G, we can correspond a n-gon as following.

First, consider a regular m-gon $A_{i_1}A_{i_2}\dots A_{i_m}$ such that vertices are labelled clockwise (The graph corresponded to regular m-gon is H itself). Set $a = \frac{m+1}{2}$. Now, for all $1 \le k \le m$, put $\deg(V_{i_k})$ more points on the circle with center A_{i_k} and radius $A_{i_k}A_{i_{k+a-1}} = A_{i_k}A_{i_{k+a}}$ on the smaller arc $A_{i_{k+a-1}}A_{i_{k+a}}$ in this circle. Finally, label these points clockwise by $i_{k+a-1}+1, \dots, i_{k+a}-1$ (modulo n). Since $V_{i_k}V_{i_{k+a-1}}$ and $V_{i_k}V_{i_{k+a}}$ are edges of G, according to the third lemma, there exist two parallel lines passing trough A_{i_k} and every point in $A_{i_{k+a-1}}, \dots, A_{i_{k+a}}$ satisfying desired properties . So indeed, this figuration is corresponded with the desired graph G.

Therefore, the answer of the problem is the number of subsets of $\{1, 2, \ldots, n\}$ with odd amount of members, an at least three members; Which is equal to $2^{n-1} - n$. ■

Team Selection Test

1. According to Cauchy-Schwarz inequality

$$\left(\frac{(a+c)^2}{ad+bc}+\frac{(b+d)^2}{ac+bd}+4\right)\Big((ad+bc)+(ac+bd)+1\Big)$$
$$\geq (a+c+b+d+2)^2 = 16$$

$$\Longrightarrow \frac{(a+c)^2}{ad+bc}+\frac{(b+d)^2}{ac+bd}+4 \geq \frac{16}{ad+bc+ac+bd+1}.$$

Set $a+b=x$, $c+d=y$ (therefore $x+y=2$, $x^2+y^2=4-2xy$). It suffices to show that

$$\frac{16}{ad+bc+ac+bd+1}=\frac{16}{xy+1}\geq 4\left(\frac{x+1}{y+1}+\frac{y+1}{x+1}\right).$$

This is equivalent to

$$\begin{array}{rrl} & 4(x+1)(y+1)\geq & (xy+1)\left((x+1)^2+(y+1)^2\right) \\ \Longleftrightarrow & 4(xy+3)\geq & (xy+1)(10-2xy) \\ \Longleftrightarrow & 4xy+12\geq & 8xy-2(xy)^2+10 \\ \Longleftrightarrow & (xy)^2+1\geq & 2xy \\ \Longleftrightarrow & (xy-1)^2\geq & 0. \end{array}$$

Which is true, hence the claim of the problem. ■

2. The answer of the problem is yes.

Although the statement is discussed on 13 students, here is an example for 14 students, all having a chance of being a team member.
(students are labelled by $1, 2, \ldots, 14$.)

Rank \ Test	#1	#2	#3	#4	#5	#6
#1	1	1	1	1	1	1
#2	2	2	2	2	2	2
#3	3	3	4	4	5	5
#4	6	7	6	8	7	8
#5	9	10	11	12	13	14
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Note that the fifth person in each test can become a team member, only if all the other four persons above already have been chosen as a team member. Therefore, to prove that the example works, it suffices to show that the fifth person in each test can become a team member. For each of the students $9, 10, 11, 12, 13, 14$, consider these permutations of tests

$$\begin{array}{rcl}
\text{Student } 9: \overbrace{5 \to 6 \to 2 \to 4 \to 3 \to 1}^{\text{Permutation of Tests}} & \Longrightarrow & \overbrace{\{1,2,3,4,6,9\}}^{\text{Team Members}}, \\
\text{Student } 10: 3 \to 4 \to 1 \to 6 \to 5 \to 2 & \Longrightarrow & \{1,2,3,5,7,10\}, \\
\text{Student } 11: 5 \to 6 \to 2 \to 4 \to 1 \to 3 & \Longrightarrow & \{1,2,3,4,6,11\}, \\
\text{Student } 12: 1 \to 2 \to 3 \to 5 \to 6 \to 4 & \Longrightarrow & \{1,2,4,5,8,12\}, \\
\text{Student } 13: 3 \to 4 \to 1 \to 6 \to 2 \to 5 & \Longrightarrow & \{1,2,3,5,7,13\}, \\
\text{Student } 14: 1 \to 2 \to 3 \to 5 \to 4 \to 6 & \Longrightarrow & \{1,2,4,5,8,14\}.
\end{array}$$

■

3. Let M be the reflection of X with respect to I_aB.

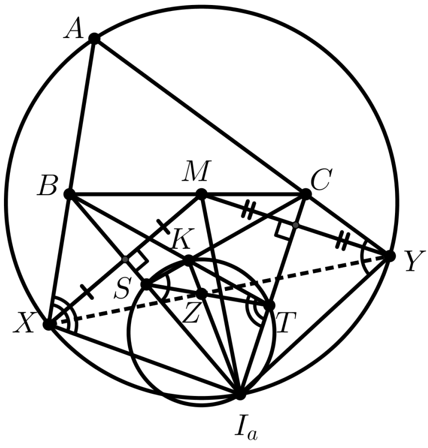
Clearly, M is a point on BC, since I_aB is angle bisector of both angles $\widehat{XBM}$ and $\widehat{XBC}$. Note that

$$\widehat{I_aMC} = 180^\circ - \widehat{I_aMB} = 180^\circ - \widehat{I_aXA} = \widehat{CYI_a}.$$

And since $\widehat{I_aCM} = \widehat{I_aCY}$, we get $C\overset{\triangle}{Y}I_a \equiv C\overset{\triangle}{M}I_a$. Therefore M is also the reflection of Y over I_aC. For quadrilateral KSI_aT we have

$$\widehat{KSI_a} + \widehat{KTI_a} = \widehat{AYI_a} + \widehat{AXI_a} = 180^\circ.$$

Therefore, KSI_aT is cyclic. Now since $\widehat{BMI_a} = \widehat{BXI_a} = \widehat{BTI_a}$, we get $BMTI_a$ and similarly, $CMSI_a$ are cyclic. This means M is the Miquel point of the complete quadrilateral KSI_aT.



Let ω_1 and ω_2 be the circumcircles of triangles CSI_a and BTI_a, respectively. The point M has the same Simson line with respect to each of these two triangles, and since X and Y are the reflections of M in I_aB and I_aC, respectively, line XY is homothetic with the Simson line of M, with center M and radius 2. So the problem is to prove that the Simson line of M with respect to ω_1 and ω_2 passes through the midpoint of MZ.

It is well-known that the Simson line of any point P lying on the circumcircle of some triangle QRU, passes through the midpoint of PH where H is the orthocenter of QRU. So according to this fact, the Simson line of M passes through the midpoint of MH_1 and MH_2 where H_1 and H_2 are orthocenters of triangles BTI_a and CSI_a, respectively. Now, it is sufficient to show that Z lies on H_1H_2. Consider two circles Γ and Ω with diameters ST and KI_a, respectively. Let T' be the foot of the perpendicular line from T to BI_a. We know that $T' \in \Gamma$. Also let I'_a be the foot of the perpendicular line from I_a to BT. Similarly $I'_a \in \Omega$. Note that Z is a point on the radical axis of Γ and Ω, and since $I_aI'_a \cap TT' = H_1$, we get $\mathcal{P}_\Gamma(H_1) = H_1T \cdot H_1T' = HI_a \cdot HI'_a = \mathcal{P}_\Omega(H_1)$, where $\mathcal{P}_\lambda(Q)$ is the power of point Q with respect to circle λ. therefore H_1 and similarly H_2 lie on the radical axis of Γ and Ω. So the line passing through Z, H_1 and H_2, is the radical axis of Γ and Ω and hence the claim of the problem. ■

4. Consider a sequence $n_1 < n_2 < \cdots$ of integers such that

$$\forall i \neq 3 : n_i = p_i + \varphi(p_i) = 2p_i - 1, \; n_3 = 7.$$

For all $i \neq 3$, $x_i = 2$ and for $i = 3$, $x_i = 3$ are solutions for

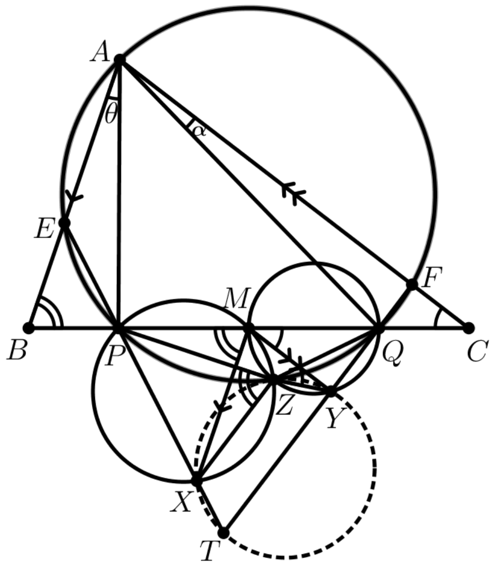
$$x_i^{n_i} \overset{p_i}{\equiv} 2.$$

Now assume that there exists a number x that satisfies all the equations. Note that $2^{n_3} \not\equiv 2 \pmod{p_3}$, therefore $x \neq 2$. Let p_k be a prime divisor of $x-1$, we have

$$x^{n_k} \overset{p_k}{\equiv} 1 \overset{p_k}{\equiv} 2,$$

which is clearly impossible, this contradiction shows us that the answer of the problem is no. ■

5. Let M be the midpoint of BC. Since $BP = CQ$, it is clear that $MP = MQ$ and $BQ = CP$. Let Z be the second intersection point of circumcircles of triangles MPX and MQY. We claim that the circumcircles of triangles APQ and XYT are tangent to each other at Z.



We have

$$\begin{cases} \widehat{PZQ} = \widehat{PZM} + \widehat{QZM}, \\ C_{\underset{MQY}{\triangle}} : \widehat{QZM} = \widehat{MYQ} = \widehat{QFC}, \\ C_{\underset{APQ}{\triangle}} : \widehat{QFC} = \widehat{APQ}, \\ C_{\underset{MPX}{\triangle}} : \widehat{PZM} = \widehat{MXP} = \widehat{PEB}, \\ C_{\underset{APQ}{\triangle}} : \widehat{PEB} = \widehat{AQP}, \end{cases}$$

And so $\widehat{PZQ} = \widehat{APQ} + \widehat{AQP} = 180^\circ - \widehat{PAQ}$. Which means Z lies on the circumcircle of triangle APQ. Also

$$\begin{cases} C_{\underset{MQY}{\triangle}} : \widehat{MZY} = 180^\circ - \widehat{MQT}, \\ C_{\underset{MPX}{\triangle}} : \widehat{MZX} = 180^\circ - \widehat{MPT}. \end{cases}$$

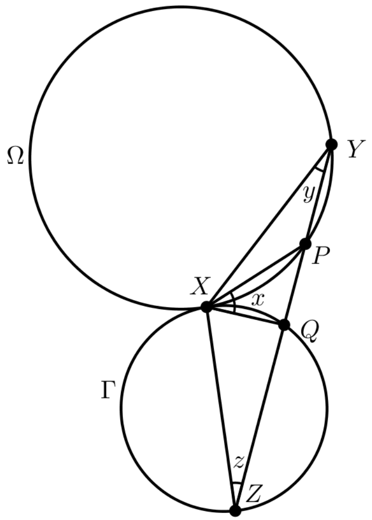
These imply

$$\widehat{XZY} = 360^\circ - (\widehat{MZY} + \widehat{MZX}) = \widehat{MQT} + \widehat{MPT} = 180^\circ - \widehat{PTQ}.$$

Which means Z also lies on the circumcircle of triangle XYT. We use a lemma to prove the problem.

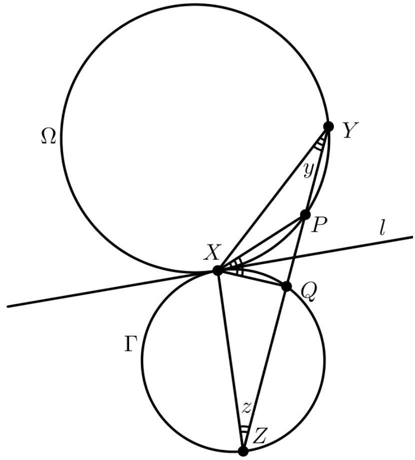
Lemma. *Given two circles Ω and Γ with a common point X and a line intersecting Ω at Y and P, and intersecting Γ at Q and Z (P and Q lie between Y and Z). In this case, these circles are tangent to each other at X if*

$$\widehat{PXQ} = \widehat{XYZ} + \widehat{XZY}.$$



Proof. Let l be the tangent line to Ω through X, therefore $\widehat{PXl} = \widehat{XYP}$ (with a fixed direction on l, and angles are considered with respect to this direction). l is also tangent to Γ, if and only if $\widehat{XZQ} = \widehat{QXl}$.
But $\widehat{PXl} = \widehat{PXQ} - \widehat{QXl}$. Therefore

$$\widehat{XZQ} = \widehat{PXQ} - \widehat{XYZ} = \widehat{PXQ} - \widehat{PXl} = \widehat{QXl}.$$



Now back to the problem, according to the lemma (for two circles $C_{\overset{\triangle}{MQY}}, C_{\overset{\triangle}{APQ}}$), it suffices to show that

$$\widehat{QZY} = \widehat{ZFT} + \widehat{ZTF}.$$

We have

$$\begin{cases} \widehat{QZY} = \widehat{QMY} = \widehat{C}, \\ C_{\overset{\triangle}{TXY}} : \widehat{ZTF} = \widehat{ZXY}, \\ C_{\overset{\triangle}{APQ}} : \widehat{TFZ} = \widehat{QPZ}, \\ C_{\overset{\triangle}{MPX}} : \widehat{QPZ} = \widehat{MXZ}. \end{cases}$$

Note that $\widehat{XMY} = \widehat{A}$ and the problem is equivalent to prove that

$$\begin{aligned} \widehat{C} = \widehat{MXZ} + \widehat{ZXY} = \widehat{MXY} &\iff \overset{\triangle}{MXY} \sim \overset{\triangle}{ACB}, \\ &\iff \frac{MY}{MX} = \frac{AB}{AC}. \end{aligned}$$

We also have:

$$\left.\begin{array}{c} MY \parallel AC \implies \frac{MY}{FC} = \frac{MQ}{QC} \\ MX \parallel AB \implies \frac{MX}{EB} = \frac{MP}{PB} \\ MP = MQ\ ,\ BP = CQ \end{array}\right\} \implies \frac{MY}{MX} = \frac{FC}{EB}.$$

So the problem is equivalent to show that $\frac{FC}{EB} = \frac{AB}{AC}$, or equivalently $FC \cdot AC = EB \cdot AB$. In order to prove this, we have

$$CF \cdot CA = \mathcal{P}_{C_{\overset{\triangle}{APQ}}}(C) = CQ \cdot CP = BP \cdot BQ = \mathcal{P}_{C_{\overset{\triangle}{APQ}}}(B) = BE \cdot BA,$$

(Where $\mathcal{P}_{\lambda}(S)$ is the power of point S with respect to circle λ.) Hence the claim. ■

6. To prove that at least 2^{100} permutations of numbers can be reached, we use induction. We can see that for $n=4$, there are exactly 2^4 possible permutations of a 1×4 tape, and for $n=5$, this number is something greater than 2^5. Therefore, assume that the claim of the problem is true for $n=k$, that there are more than 2^k possible permutations. For $n=k+1$, consider a possible permutation of numbers $1,2,\dots,n-1$. We can first fold number n above or below $n-1$, and then repeat the process of folding to get that permutation for $n-1$. Means, each permutation of $n-1$ layers can lead to at least two permutations of n layers. Therefore if P_n is the total number of possible permutations of a $1\times n$ tape, we obtain $P_n \geq 2P_{n-1}$. Therefore

$$P_{100} \geq 2P_{99} \geq 4P_{98} \geq \cdots \geq 2^{95}P_5 > 2^{100},$$

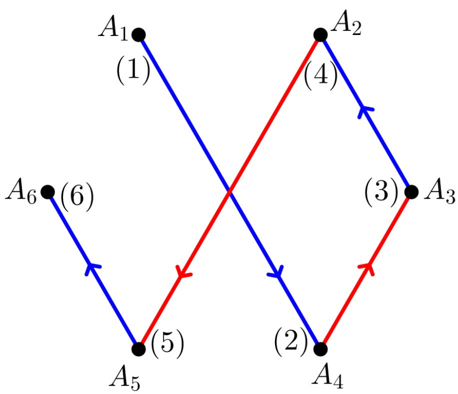
Hence, the first claim of the problem is proved.

Now for the second part, consider a graph on a convex 100-gon, with labelled vertices $A_1, A_2, \dots, A_{100}$. Assume that after folding the tape, there is a permutation of $\{1,2,\dots,100\}$ such that the place of number i is equal to a_i. Then on the graph, draw the directed path

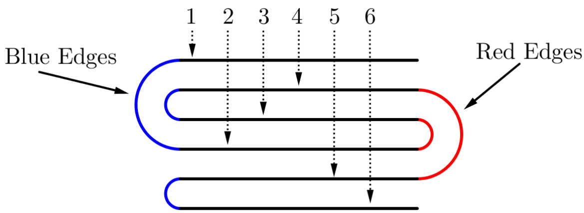
$$A_{a_1} \to A_{a_2} \to \cdots \to A_{a_{100}},$$

with edges alternatively colored blue, red, blue, red,... Then, the permutation $a_1a_2\dots a_{100}$ can be seen on the perimeter of the polygon, starting from A_1.

For example, the permutation $1,4,3,2,5,6$ of six numbers is corresponded with the following graph. (This permutation is a valid one, and is reachable by folding a 1×6 tape.)

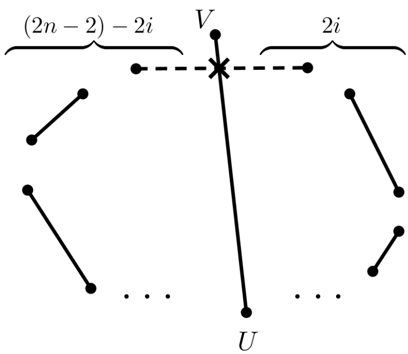


There are no two edges with the same color that intersect each other, because every blue edge corresponds to a piece of tape connecting an odd number like $2k-1$ in the tape to $2k$ and every red edge corresponds to a piece of tape connecting an even number in the tape like $2k$ to $2k+1$. The following figure shows the folded tape corresponded to the graph above.



Since all possible permutations are uniquely corresponded with a path on the polygon, we just need to show that the total number of valid paths is less than 4^{100}.

Note that there are a total of 50 blue edges, pairing all the 100 vertices, and there are a total of 49 red edges, other than the first and the last vertex in the path, pairing the other 98 vertices. Set C_n to be the possible ways to draw n edges in a convex polygon with $2n$ vertices, such that each vertex is connected to exactly one edge, and no two edges intersect each other. We can obtain a recurrence relation for C_n; Fix a vertex V in the graph, assume that U is the vertex that is connected to V. Since we are dealing with a convex polygon, the edge UV bisects the graph into two sets of vertices. There must be no edge between the two sets, otherwise UV will intersect that edge. Therefore, each vertex must be connected to another vertex on its own side. So the first conclusion is that each side has an even number of vertices, assume that there are $2i$ vertices on one side and $(2n-2)-2i$ other points on the other side.



There are C_i ways to connect the points on the first side, and $C_{(n-1)-i}$ ways to connect the points on the other side. Also in case when $i = 0$, we just need to connect the remaining $2n - 2$ points in C_{n-1} ways. Therefore, if we set $C_0 = 1$, we obtain the following recurrence relation.

$$C_0 = 1,\ C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}.$$

These numbers are known for Catalan Numbers, and their exact value in terms of binomial coefficients is

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

So there are a total of C_{50} ways to just draw the blue edges. Also, we need to set the initial and the final vertices of the path (in 100×99 ways) and then pair the remaining 98 vertices in C_{49} ways. So the following inequality is obtained.

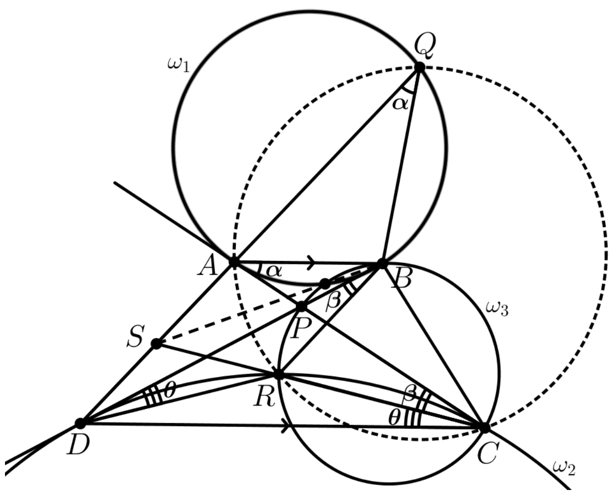
$$P_{100} \leq 100 \times 99 \cdot C_{49} \cdot C_{50} = 100 \times 99 \cdot \frac{\binom{98}{49}}{50} \cdot \frac{\binom{100}{50}}{51}$$

Note that $\binom{2n}{n} < \sum_{i=0}^{2n} \binom{2n}{i} = (1+1)^{2n} = 4^n$.
So we can finally get

$$P_{100} \leq \frac{100}{51} \times \frac{99}{50} \cdot \binom{98}{49} \cdot \binom{100}{50} < 2 \times 2 \times 4^{49} \times 4^{50} = 4^{100}.$$

Therefore $2^{100} < P_{100} < 4^{100}$. ■

7. Let Q be the second intersection point of AD with ω_1, and let R be the second intersection point of ω_2 and ω_3.



We have

$$\left.\begin{array}{rl} \widehat{DRB} = & 180^\circ - (\widehat{BDR} + \widehat{DBR}) \\ \omega_2: & \widehat{BDR} = \widehat{RCD} = \theta \\ \omega_3: & \widehat{DBR} = \widehat{PCR} = \beta \end{array}\right\} \Longrightarrow \widehat{DRB} = 180^\circ - \widehat{ACD}. \ (\star)$$

$$\left.\begin{array}{rl} AB \parallel CD \Longrightarrow & \widehat{ACD} = \widehat{BAC} = \alpha \\ \omega_1: & \widehat{BAC} = \widehat{BQD} = \alpha \end{array}\right\} \stackrel{\star}{\Longrightarrow} \widehat{DRB} = 180^\circ - \widehat{DQB}.$$

This implies that the quadrilateral $DQBR$ is cyclic.
So $\widehat{DQR} = \widehat{DBR} = \beta = \widehat{ACR}$, which means $AQCR$ is also cyclic. Let CR meet AD at S. We have

$$\mathcal{P}_{\omega_3}(S) = SR \cdot SC = SA \cdot SQ = \mathcal{P}_{\omega_1}(S).$$

(Where $\mathcal{P}_\lambda(U)$ is the power of point U with respect to circle λ.) That implies S lies on the common chord of ω_1, ω_3.

Therefore the common chord of ω_1 and ω_3, common chord of ω_2 and ω_3, and the line AD are concurrent at S. ■

Comment. By considering the power of points A, D with respect to circles ω_2, ω_1, we can also prove that S is the midpoint of AD.

8. The answer is $n = 6$. First, we will give an example for $n = 6$ and then we will prove that there are not 7 numbers with the desired property.

In our example, four numbers are $2, 3, 7, 17$ and the fifth number x is a number satisfying the following relations. Such x exists due to the **Chinese Remainder Theorem**.

$$3 \mid 7 + x \ , 7 \mid 17 + x \ , 17 \mid 3 + x \ , 2 \nmid x.$$

(For instance, $x = 473$.) Note that x is relatively prime with each of the previous four numbers. Now consider a number y satisfying the following relations. Again, such y exists due to the **Chinese Remainder Theorem**.

$$\begin{cases} 3 \mid 17 + y \Longrightarrow 3 \mid x + y, \ \text{(since } 3 \mid 7 + x.\text{)} \\ 7 \mid 3 + y \Longrightarrow 7 \mid 17 + y, \\ 17 \mid x + y \ , x \mid 7 + y \ , 2 \nmid y. \end{cases}$$

In this case y is also relatively prime with other numbers. So these 6 numbers satisfy the conditions.

Now assume that there is a set with (at least) 7 numbers satisfying the condition, we can divide them to their greatest common divisor, thus we can assume that at least one of the numbers is odd. Three following claims prove the problem:

- There are at most two even numbers in the set.

  Assume that a, b, c are three even numbers in the set. Consider an odd number k in the set, between k, a, b only k can divide the sum of the other two. Therefore $k \mid a+b$ and similarly $k \mid b+c$, $k \mid c+a$. Since k is odd, we obtain $k \mid a$, $k \mid b$ and $k \mid c$ which is impossible. So there are at most two even numbers in the set.

- There is at most five odd numbers in the set.

  Assume that there exists six odd numbers $a < b < c < d < e < f$. There are a total of 20 dividing relations between these numbers. On the other hand, it is easy to see that the largest number , f, cannot divide the sum of any two odd numbers smaller than itself, e can divide at most one sum and d can divide at most three sums. Therefore, at least 16 of the dividings were made by a, b, c. So one of them divides at least 6 sums. We denote this number by x. Consider a graph with vertices of the other five odd numbers, two vertices are connected if x divides the sum of these two numbers. So this graph has five vertices and at least six edges. It is easy to see that such graph has a vertex with degree of at least 3. Therefore there exists odd numbers y, u, v and w, different from x, such that

  $$x \mid y+w, \ x \mid y+u, \ x \mid y+v.$$

  Without loss of generality, assume that $w < v < u$. So we have

  $$x \mid v-w, \ x \mid u-v, \ x \mid u-w.$$

  Since x is odd and it must not divide any of the other numbers, so x cannot divide the sum of any pair among $\{u, v, w\}$. On the other hand, x is not the largest number in any triplet, and the largest odd number cannot divide the sum of the other two, therefore

  $$w \mid x+v, \ w \mid x+u, \ v \mid x+u.$$

  So $w \mid u-v$ that implies $w \nmid u+v$, therefore $v \mid w+u$. This implies that $v \mid w-x$. But because w and x are both less than u, this result is impossible.

- Two even numbers and five odd numbers do not satisfy the conditions.

  Assume that two even numbers are $a < b$ and five odd numbers are $c < d < e < f < g$. Similar to the previous arguments we conclude that all of these odd numbers must divide $a+b$ and since they are odd numbers, they must divide $\frac{a+b}{2}$. Which implies b is the largest number of all.

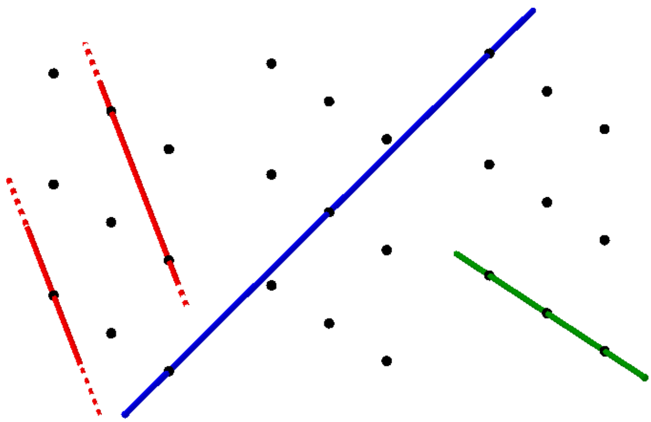
Now assume that b divides the sum of two of the odd numbers, hence b must be equal to the sum of those two numbers. But in this case, the larger number of two (and thus g itself) must be greater than or equal to $\frac{b}{2}$. So $2g \geq b > \frac{a+b}{2}$. But g was a divisor of $\frac{a+b}{2}$, and so we must have $g = \frac{a+b}{2}$ which means all other odd numbers are divisors of g, which is impossible.

So b cannot divide the sum of any two of the odd numbers. Now consider all pairs of odd numbers along with b. We obtain 10 dividing relations and the divisor is one of the 5 odd numbers, in other hand, g can only divide a single sum. So there is a number that divides at least 3 sums, which similar to the previous cases, leads that this number also divides the difference of any pair chosen of 3 odd numbers, and the same contradiction is made.

So indeed, there is no set with at least 7 elements, hence the answer of the problem is $n = 6$. ■

9. We will prove that the answer of the problem is 9.

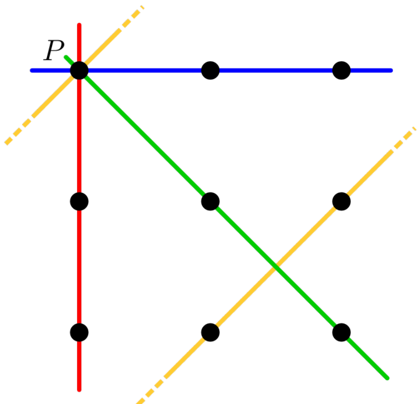
Each card can be corresponded with a point in $\mathbb{Z}_3^3$. A line in $\mathbb{Z}_3^3$ is defined to be a subset of the form $\{P, P+V, P+2V\}$ where P and V are two elements in $\mathbb{Z}_3^3$, such that $V \neq \vec{0}$ (in $\mathbb{Z}_3^3$). It is easy to see that three elements of $\mathbb{Z}_3^3$ form a line, if and only if their sum (in $\mathbb{Z}_3^3$) is equal to $\vec{0}$. For instance, in the following figure we can see three different lines. In this new language, the problem is to find the maximum number of points in $\mathbb{Z}_3^3$ such that no three of them are collinear.



The following lemma is the statement of the problem in $\mathbb{Z}_3^2$, and proof that the answer is not greater than 4.

Lemma. *There are no* 5 *points in* $\mathbb{Z}_3^2$ *such that no three of them are collinear (definition of line in* $\mathbb{Z}_3^2$ *is similar).*

Proof. Assume to the contrary that $\mathcal{S}$ is a set in $\mathbb{Z}_3^2$ with more than 4 elements such that no three points of $\mathcal{S}$ are collinear. let P be one point in $\mathbb{Z}_3^2$. There are exactly 4 lines (in $\mathbb{Z}_3^2$) passing through P. For example, if P is a point in the corner, these are the lines containing P.



Since no three points of $\mathcal{S}$ are collinear, each line passing through $P \in \mathcal{S}$ contains at most one other point in this set. So there are at most five points in $\mathcal{S}$ (Note that for any point in $\mathbb{Z}_3^2$ like $Q \neq P$, there is a unique line containing both P and Q).

Now assume that $\mathcal{S}$ has exactly 5 elements. Consider $P \in \mathcal{S}$. Again, since there are only four lines passing through P, and there are exactly four other points in $\mathcal{S}$, every line passing through P will contain another point of $\mathcal{S}$.
It means that every line in $\mathbb{Z}_3^2$, intersects $\mathcal{S}$ in exactly zero or two points. So there are $\binom{5}{2} = 10$ lines in $\mathbb{Z}_3^2$ containing two points of $\mathcal{S}$. On the other hand, the total number of lines in $\mathbb{Z}_3^2$ is 12 (for each of 9 points in $\mathbb{Z}_3^2$, there are 4 lines containing that point, and every line consists of 3 different points). Therefore there are two lines l_1 and l_2 in $\mathbb{Z}_3^2$ having empty intersection with $\mathcal{S}$. But the union of l_1 and l_2 contains at least 5 points. This contradicts $|\mathcal{S}| = 5$.

Back to the problem. First we review some definitions and properties for planes in $\mathbb{Z}_3^3$.

- We call a subset $\mathcal{P}$ of $\mathbb{Z}_3^3$ a plane, if there exist $a, b, c, d \in \mathbb{Z}_3$ with at least one of a, b and c is non-zero (in $\mathbb{Z}_3$) and such that

$$\mathcal{P} = \mathcal{P}_d(a,b,c) = \Big\{(x,y,z) : ax + by + cz = d \pmod 3\Big\}.$$

It is easy to see that every plane consists of exactly 9 points and for every three points A, B and C in $\mathbb{Z}_3^3$, either there exists a unique plane passing through them, or they are collinear. In the latter case there are exactly four different planes containing them.

- We call two planes parallel, if they have no common points. Again, it is easy to check that if two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel, there exist two different numbers $i, j \in \mathbb{Z}_3$ and $(a,b,c) \neq \vec{0}$ in $\mathbb{Z}_3^3$ such that

$$\mathcal{P}_1 = \mathcal{P}_i(a,b,c), \quad \mathcal{P}_2 = \mathcal{P}_j(a,b,c).$$

For any $(a,b,c) \neq \vec{0}$ (in $\mathbb{Z}_3^3$), all 27 points in $\mathbb{Z}_3^3$ can be partitioned into three parallel planes $\mathcal{P}_0(a,b,c)$, $\mathcal{P}_1(a,b,c)$ and $\mathcal{P}_2(a,b,c)$. But since $\mathcal{P}_i(a,b,c) = \mathcal{P}_{-i}(-a,-b,-c)$, two points (a,b,c) and $(-a,-b,-c)$ give the same partition. Therefore, there are exactly $\frac{26}{2} = 13$ ways to partition $\mathbb{Z}_3^3$ into three parallel planes. Note that if we consider all these 13 partitions, every plane in $\mathbb{Z}_3^3$ will appear exactly in one of the partitions.

Now assume that there exist a set $\mathcal{S}$ with 10 elements satisfying the desired properties. Consider one of above partitions. According to the lemma, each plane in this partition contains at most 4 points of $\mathcal{S}$. Since there are 10 points in total, there are two different type of partitions.

(1) Partitions such that two planes contain 4 points of $\mathcal{S}$, and the third one contains 2.

(2) Partitions such that two planes contain 3 points of $\mathcal{S}$, and the third one contains 4.

Let m and n be the total number of partitions of types (1) and (2), respectively. There are totally 13 ways to partition the points. So we obtain the following equation.

$$m + n = 13.$$

We are going to find another equation for m, n.
For this reason, we count the number of elements of the following set in two different ways.

$$W = \Big\{\big(\{A,B\}, \mathcal{P}\big) : \mathcal{P} \text{ is a plane, } A, B \in \mathcal{P} \cap S\Big\}$$

On one hand, if we count the number of pairs of points of $\mathcal{S}$ in each plane of partitions of type (1), we get a total of

$$\binom{4}{2}+\binom{4}{2}+\binom{2}{2}=13$$

pairs of points for each partition of this type. And for the type (2), this number is

$$\binom{4}{2}+\binom{3}{2}+\binom{3}{2}=12.$$

So the cardinality of W is

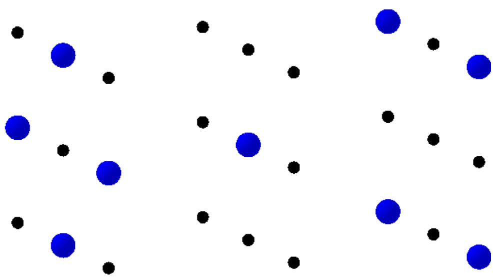
$$13m+12n.$$

On the other hand, for any two points $A, B \in \mathcal{S}$, the total amount of planes containing both A and B is 4. Since there are totally $\binom{10}{2}=45$ pairs, the cardinality of W is $4\times 45=180$, so we obtain $13m+12n=180$. But this equation is clearly impossible when $m+n=13$ because

$$169=13m+13n\geq 13m+12n=180,$$

Contradiction. So there are at most 9 points with the desired properties. Also the following set of points is a valid example for 9 points (look at the figure below).

$$\begin{cases}(0,0,1),(0,1,0),(0,1,2),(0,2,1),\\(1,1,1),\\(2,0,0),(2,0,2),(2,2,0),(2,2,2),\end{cases}$$



Translating the solution into the main problem, the total number of cards we can choose is 9. ■

10. *a*) We set an example using induction on n. For $n=1$, consider the following pair of polynomials of degree 2

$$h_1(x)=x^2+x,\ h_2(x)=-x^2.$$

Now for any function $f:\mathbb{R}\to\mathbb{R}$ if $P_1(x)=h_1(f(x))$ and $P_2(x)=h_2(f(x))$ are polynomials, then $P_1(x)+P_2(x)=f(x)$ is a polynomial.

Now assume that $(h_1,h_2,\dots,h_{n+1})$ is a **good** $(n+1)$-tuple of n variable polynomials. Define the following $(n+2)$-tuple $(g_1,g_2,\dots,g_{n+2})$ of polynomials with $n+1$ variables as

$$\begin{cases} g_i(x_1,x_2,\dots,x_{n+1})=h_i(x_1,x_2,\dots x_n), & 1\le i\le n+1,\\ g_{n+2}(x_1,x_2,\dots,x_{n+1})=x_1^2+x_{n+1}, & i=n+2. \end{cases}$$

Now let $f_1,f_2,\dots,f_{n+1}:\mathbb{R}\to\mathbb{R}$ be some functions such that for all $1\le i\le n+2$

$$P_i(x_1,\dots,x_{n+1})=g_i(f_1,f_2,\dots,f_{n+1}),$$

is a polynomial. By induction hypothesis we obtain that $f_1(x),f_2(x),\dots,f_n(x)$ are polynomials. Note that $f_1(x)^2$ is a polynomial.
Therefore, $f_{n+1}=P_{n+2}-f_1^2$ is also a polynomial.

The degree of all g_i's are more than 1. Therefore the given example is a **good** $(n+2)$-tuple.

b) According to the **Fundamental Theorem of Symmetric Polynomials**, for any symmetric polynomial $h_i(x_1,\dots,x_n)$ there exists another polynomial $g_i(x_1,\dots,x_n)$ such that

$$h_i(x_1,\dots,x_n)=g_i\Big(\sum_{sym}x_i,\sum_{sym}x_ix_j,\dots,x_1x_2\cdots x_n\Big).$$

Now assume that for some $n>1$ there exists a **good** $(n+1)$-tuple of symmetric polynomials. Consider the following functions

$$f_1(x)=|x|\,,\ f_2(x)=-|x|\,,\ f_3(x)=f_4(x)=\cdots=f_{n+1}(x)=0.$$

Note that

$$\begin{cases} \sum\limits_{sym} f_if_j=-x^2,\\ \sum\limits_{sym} f_{i_1}f_{i_2}\cdots f_{i_k}=0, \quad \forall k\neq 2\ . \end{cases}$$

Therefore,

$$P_i(x)=h_i\big(f_1(x),\dots,f_n(x)\big)=g_i(0,-x^2,0,\dots,0)=H_i(-x^2).$$

Where H_i is a polynomial with real coefficients. So for any i, $P_i(x)$ is a polynomial, and therefore all f_i's are polynomials, too. But f_1 and f_2 are not polynomials. Contradiction, hence the claim. ■

11. Consider numbers of the form $k^m(k^i + j)$ for $i \in \{1, \dots, n-k+1\}$, $j \in \{1, \dots, k-1\}$ and $m \geq 0$.
Note that for such i and j in these intervals, we have $i + j \leq n$. Based on $n - (i + j)$ is an odd number or an even number, we have two cases.

- If $n - (i + j)$ is odd, there is some integer $t \geq 0$ such that

  $$n - (i + j) = 2t + 1.$$

  In this case, the number $k^i + j$ can be represented using n number of k's as

  $$k^i + j = \underbrace{(k \times \cdots \times k)}_{i+t} \div \underbrace{(k \times \cdots \times k)}_{t} + \underbrace{(k + \cdots + k)}_{j} \div k.$$

- If $n - (i + j)$ is even, there is some integer $t \geq 0$ such that

  $$n - (i + j) = 2t.$$

  In this case, the number $k(k^i + j)$ can be shown using n number of k's as

  $$k(k^i + j) = k \times (\underbrace{(k \times \cdots \times k)}_{i+t-1} \div \underbrace{(k \times \cdots \times k)}_{t-1} + \underbrace{(k + \cdots + k)}_{j} \div k).$$

  (In case when $t = 0$, we have $t - 1 = -1$, so in the above equation, instead of multiplying the parentheses to -1 t's, we divide it by one t.)

So in both cases, there is $m \geq 0$ such that $k^m(k^i + j)$ can be be represented using n number of k's. Set m to be the largest number such that $k^m(k^i + j)$ can be represented using only n number of k's (for fixed i and j). There are a total of $(n - k + 1)(k - 1)$ numbers created with given properties. We prove that they are all distinct numbers and also cannot be represented using $n - 1$ number of k's.
If some number $u = k^m(k^i + j)$ can be shown using $n - 1$ number of k's, then clearly $k \times u = k^{m+1}(k^i + j)$ has a representation, using n number of k's, therefore m cannot be the largest number such that $k^m(k^i + j)$ has a representation using n number of k's, so we conclude that it is impossible to reach these numbers with only $n - 1$ number of k's.

Finally, if for two such triples $(i_1, j_1, m_1) \neq (i_2, j_2, m_2)$ with $m_1 \geq m_2$, we have

$$k^{m_1}\left(k^{i_1} + j_1\right) = k^{m_2}\left(k^{i_2} + j_2\right),$$

then

$$k^{m_1 - m_2}\left(k^{i_1} + j_1\right) = k^{i_2} + j_2.$$

But if $m_1 - m_2 > 0$, the above equation implies $k \mid j_2$, which is impossible because $j_2 < k$. Therefore $m_1 = m_2$ and so $k^{i_1} + j_1 = k^{i_2} + j_2$. Without loss of generality, we assume $i_1 \geq i_2$. So

$$k^{i_2}\left(k^{i_1 - i_2} - 1\right) = j_2 - j_1.$$

Again, if $i_1 > i_2$, we get $k \mid j_2 - j_1$. Note that $j_1, j_2 \in \{1, \ldots, k-1\}$, therefore $-k < |j_1 - j_2| < k$, so we must have $j_1 - j_2 = 0$ and thus $k^{i_1} = k^{i_2}$ which means $i_1 = i_2$, that is impossible in this case be cause we assumed that $i_1 > i_2$. The only possible case is when $i_1 = i_2$ which leads to $j_1 = j_2$. Therefore $(i_1, j_1, m_1) = (i_2, j_2, m_2)$, contradiction.
So indeed, all of the created numbers are distinct and hence the claim of the problem. ■

12. The answer is $n = 1, 2$ as $a_1 = k^0,\ a_2 = k^1$.

We are going to find the general term of the sequence. Note that since $k > 1$, the characteristic polynomial $P(x) = x^2 - (k+1)x + 1$ has two different roots

$$r_1 = \frac{k + 1 + \sqrt{(k-1)(k+3)}}{2},\ r_2 = \frac{k + 1 - \sqrt{(k-1)(k+3)}}{2}.$$

So a_n has a general form as

$$a_n = c_1 r_1^n + c_2 r_2^n,$$

where c_1, c_2 are fixed numbers. Considering the values of a_1, a_2, the values of c_1, c_2 are found

$$c_1 = \frac{\sqrt{k+3} - \sqrt{k-1}}{2\sqrt{k+3}},\ c_2 = \frac{\sqrt{k+3} + \sqrt{k-1}}{2\sqrt{k+3}}.$$

Lemma. *The sequence has a general form as*

$$a_n = \frac{\varphi^{2n-1} + \frac{1}{\varphi^{2n-1}}}{\varphi + \frac{1}{\varphi}},$$

where $\varphi = \frac{\sqrt{k+3} + \sqrt{k-1}}{2}$.

Proof. Note that $r_1 + r_2 = k + 1$ and $r_1 r_2 = 1$. If we set $\varphi = \frac{\sqrt{k+3}+\sqrt{k-1}}{2}$, we have

$$\begin{aligned}
\varphi^2 &= \frac{k + 1 + \sqrt{(k-1)(k+3)}}{2} = r_1 = \frac{1}{r_2}, \\
\frac{1}{\varphi} &= \frac{\sqrt{k+3} - \sqrt{k-1}}{2}, \\
\varphi + \frac{1}{\varphi} &= \sqrt{k+3}.
\end{aligned}$$

Therefore
$$c_1 = \frac{1}{\varphi} \cdot \frac{1}{\varphi + \frac{1}{\varphi}}, \ c_2 = \varphi \cdot \frac{1}{\varphi + \frac{1}{\varphi}}.$$
So we get
$$a_n = \frac{1}{\varphi(\varphi + \frac{1}{\varphi})} \cdot \left(\varphi^2\right)^n + \frac{\varphi}{\varphi + \frac{1}{\varphi}} \cdot \left(\frac{1}{\varphi^2}\right)^n = \frac{\varphi^{2n-1} + \frac{1}{\varphi^{2n-1}}}{\varphi + \frac{1}{\varphi}} .$$

It is easy to see, by induction, that if we consider the sequence modulo k we have
$$(a_{6n+1}, a_{6n+2}, a_{6n+3}, a_{6n+4}, a_{6n+5}, a_{6n+6}) \overset{k}{\equiv} (1, 0, -1, -1, 0, 1) .$$
Therefore another lemma is concluded.

Lemma. *For any positive integer n, we have*
$$k \mid a_n \iff n \overset{3}{\equiv} 2, \ \textit{and} \ \gcd(k, a_n) = 1 \iff n \overset{3}{\not\equiv} 2.$$

Now we prove the third and the final lemma.

Lemma. *For any two odd numbers m and n such that $n \mid m$, we have*
$$a_{\frac{n+1}{2}} \mid a_{\frac{m+1}{2}}$$

Proof. First, we claim that for any integer n, $\varphi^{2n} + \frac{1}{\varphi^{2n}} \in \mathbb{Z}^+$. For $n = 0, 1$ the claim is correct. Assume that the claim is true for $n-1$, $n-2$. We have
$$\underbrace{\left(\varphi^{2(n-2)} + \frac{1}{\varphi^{2(n-2)}}\right)}_{\in \mathbb{Z}^+} + \left(\varphi^{2n} + \frac{1}{\varphi^{2n}}\right) = \underbrace{\left(\varphi^{2(n-1)} + \frac{1}{\varphi^{2(n-1)}}\right)\left(\varphi^2 + \frac{1}{\varphi^2}\right)}_{\in \mathbb{Z}^+}.$$
Therefore $\varphi^{2n} + \frac{1}{\varphi^{2n}} \in \mathbb{Z}^+$, so the claim is proved, using induction on n.

Now set $m = nl$ (therefore l is also odd). We have
$$a_{\frac{n+1}{2}} = \frac{\varphi^n + \frac{1}{\varphi^n}}{\varphi + \frac{1}{\varphi}} \ , \ a_{\frac{m+1}{2}} = \frac{\varphi^{nl} + \frac{1}{\varphi^{nl}}}{\varphi + \frac{1}{\varphi}}.$$
Therefore
$$\begin{aligned} \frac{a_{\frac{m+1}{2}}}{a_{\frac{n+1}{2}}} &= \frac{\varphi^{nl} + \frac{1}{\varphi^{nl}}}{\varphi^n + \frac{1}{\varphi^n}} \\ &= \varphi^{(l-1)n} - \varphi^{(l-3)n} + \cdots - \frac{1}{\varphi^{(l-3)n}} + \frac{1}{\varphi^{(l-1)n}} \\ &= \sum_{i \overset{2}{\equiv} \frac{l-1}{2}} \left(\varphi^{2in} + \frac{1}{\varphi^{2in}}\right) - \sum_{j \overset{2}{\equiv} \frac{l-3}{2}} \left(\varphi^{2jn} + \frac{1}{\varphi^{2jn}}\right) = s \in \mathbb{Z}, \end{aligned}$$
So $a_{\frac{m+1}{2}} = s \cdot a_{\frac{n+1}{2}}$, hence the lemma.

Back to the problem, let n be an integer such that $a_n = k^m$ is a power of k. Assume that $n \geq 3$.

Consider a prime factor of $2n-1$ like p. Since both p and $2n-1$ are odd numbers, according to the final lemma we have

$$a_{\frac{p+1}{2}} \mid a_n.$$

Again, consider a prime factor of $a_{\frac{p+1}{2}}$ like q, we have

$$q \mid a_{\frac{p+1}{2}} \mid a_n = k^m \implies q \mid k.$$

Therefore $\gcd(k, a_{\frac{p+1}{2}}) \geq q$. So according to the second lemma, since $\gcd(k, a_{\frac{p+1}{2}}) \neq 1$ we have

$$\frac{p+1}{2} \overset{3}{\equiv} 2 \implies p \overset{3}{\equiv} 0 \implies p = 3.$$

So $2n-1 = 3^v$ for some integer v. Since $n \geq 3$ we have $v \geq 2$. Therefore

$$9 \mid 3^v \mid 2n-1.$$

Again, according to the final lemma we obtain

$$a_5 = a_{\frac{9+1}{2}} \mid a_n.$$

But $a_5 = k(k^3 + 3k^2 - 3)$. So we must have $t = k^3 + 3k^2 - 3 \mid k^{m-1}$. Note that $k \geq 2$, so $t \geq 17$. Thus $m - 1 > 0$. Let r be a prime factor of t. We have

$$\left.\begin{array}{r} r \mid t \mid k^{m-1} \implies r \mid k \\ r \mid k^3 + 3k^2 - 3 \end{array}\right\} \implies r \mid 3 \implies r = 3.$$

Therefore $k^3 + 3k^3 - 3 = 3^h$, since $t \geq 17$ we have $h \geq 3$. The final equation implies $k = 3u$ for some integer u. So we can rewrite it as

$$27u^3 + 27u - 3 = 3^h.$$

Since $h \geq 3$, we have $27 \mid 3^h$ but $27 \nmid 27u^3 + 27u - 3$. Contradiction.

This means our assumption that such $n \geq 3$ exists was incorrect. So $n = 1, 2$ are the only answers of the problem. ■

13. Two simple lemmas are needed to prove the problem.

Lemma. *For all integers $k > 1$, if $x_1, x_2, \ldots, x_k$ are integers with*

$$\gcd(x_1, \ldots, x_k) = 1,$$

then there are integers $a_1, \ldots, a_k$ such that

$$a_1x_1 + \cdots + a_kx_k = 1.$$

Proof. For $k = 2$ the statement of the lemma is **Bézout's Lemma** for relatively prime numbers, that is well-known. Now we use induction on k. Knowing that the lemma is true for $k = n$, for numbers $x_1, \dots, x_{n+1}$ that are relatively prime, we have

$$\gcd\big(x_1, x_2, \dots, x_{n-1}, \gcd(x_n, x_{n+1})\big) = 1.$$

Now applying the lemma for $k = n$, there are numbers $a_1, \dots, a_n$ such that

$$a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} + a_n\gcd(x_n, x_{n+1}) = 1$$

Using the **Bézout's Lemma**, there are b_n, b_{n+1} such that

$$\gcd(x_n, x_{n+1}) = b_nx_n + b_{n+1}x_{n+1}.$$

Set $y_i = a_i$ for $i < n$, $y_n = a_nb_n$ and $y_{n+1} = a_nb_{n+1}$ to get

$$y_1x_1 + \dots + y_{n+1}x_{n+1} = 1.$$

Which is the statement of the lemma for $k = n + 1$. So the lemma is proved using induction.

Now the second lemma.

Lemma. *Set* $L = \operatorname{lcm}(1, 2, \dots, n)$, *then* $\gcd\left(\frac{L}{m+1}, \frac{L}{m+2}, \dots, \frac{L}{n}\right) = 1$ *where* $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. First note that $\operatorname{lcm}(1, 2, \dots, n) = \operatorname{lcm}(m+1, m+2, \dots, n)$. That is because for all $1 \leq k \leq m$, the number $s = \lceil \log_2(\frac{m+1}{k}) \rceil$ is an integer such that

$$2^s \cdot k \in \{m + 1, m + 2, \dots, n\}.$$

Now assume that

$$d = \gcd\left(\frac{L}{m+1}, \frac{L}{m+2}, \dots, \frac{L}{n}\right).$$

For all $m + 1 \leq i \leq n$ we have

$$d \mid \frac{L}{i} \implies \exists k;\ d \cdot k = \frac{L}{i} \implies i \cdot k = \frac{L}{d} \implies i \mid \frac{L}{d}.$$

Which implies

$$L = \operatorname{lcm}(m + 1, m + 2, \dots, n) \mid \frac{L}{d} \implies d = 1.$$

Back to the problem, according to both lemmas, since

$$\gcd\left(\frac{L}{m+1}, \frac{L}{m+2}, \ldots, \frac{L}{n}\right) = 1,$$

we obtain there are some integers $x_m, x_{m+1}, \ldots, x_{n-1}$ such that

$$x_m \cdot \frac{L}{m+1} + x_{m+1} \cdot \frac{L}{m+2} + \cdots + x_{n-1} \cdot \frac{L}{n} = 1.$$

Now consider a natural number t such that $x_m + t \cdot \frac{L}{m+2} > 0$ and set

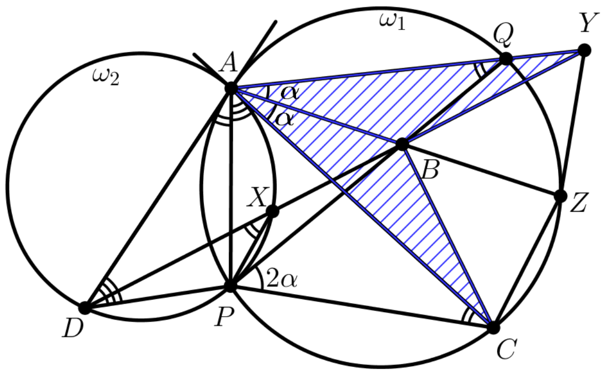
$$\begin{cases} a_m = x_m + t \cdot \frac{L}{m+2}, & \\ a_{m+1} = x_{m+1} - t \cdot \frac{L}{m+1}, & \\ a_i = x_i, & \forall i > m+1 \end{cases}$$

to get

$$\frac{a_m}{m+1} + \frac{a_{m+1}}{m+2} + \cdots + \frac{a_{n-1}}{n} = \frac{1}{\operatorname{lcm}(1, 2, \ldots, n)}.$$

Since $a_m = x_m + t \cdot \frac{L}{m+2} > 0$, this is the desired equation for $m = \lfloor \frac{n}{2} \rfloor$. ■

14. Label the circumcircles of triangles APC and APD with ω_1 and ω_2, respectively. Q is the second intersection point of PB with ω_1. DB cuts ω_2 and the line AQ at X and Y, respectively. Also let Z be the second intersection point of AB with ω_1. Set $\widehat{PAD} = \widehat{PCA} = \beta$ and $\widehat{PDA} = \widehat{PAC} = \theta$. (Note that Y exists, otherwise $AQ \parallel DB$ which implies $\widehat{DXP} = \widehat{AQB} = \widehat{DBP}$, that means $B = X$, therefore we can get $\widehat{ABD} = \widehat{AXD} = \widehat{APD} = 180° - (\beta + \theta)$. Thus, in triangle ABD, we obtain $\widehat{BAC} + \widehat{ADB} = 0°$, which is impossible, because it means A, B, C, D are collinear.)



We have

$$\left.\begin{array}{rr} & \widehat{PCA}=\widehat{PAD} \\ \omega_1: & \widehat{PCA}=\widehat{PQA} \\ \omega_2: & \widehat{PAD}=\widehat{PXD} \end{array}\right\} \implies \widehat{PXD}=\widehat{PQA},$$

Which means $PXQY$ is cyclic. Therefore

$$\left.\begin{array}{r} \implies BX\cdot BY=BP\cdot BQ \\ \mathcal{P}_{\omega_1}(B)=BP\cdot BQ=BZ\cdot BA \end{array}\right\} \implies BX\cdot BY=BZ\cdot BA.$$

Which means $AXZY$ is also cyclic. So we have

$$\left.\begin{array}{r} \implies \widehat{AZY}=\widehat{AXY}=180^\circ-\widehat{AXD} \\ \omega_2:\widehat{AXD}=\widehat{APD}=180^\circ-(\beta+\theta) \end{array}\right\} \implies \widehat{AZY}=\beta+\theta.$$

Also in circle ω_1

$$\widehat{AZC}=\widehat{PAC}+\widehat{PCA}=\beta+\theta.$$

Therefore,

$$\left.\begin{array}{c} \implies \widehat{AZY}=\widehat{AZC}=\beta+\theta \\ \widehat{ZAC}=\widehat{ZAY}=\frac{\widehat{QPC}}{2}=\alpha \\ ZA=ZA \end{array}\right\} \implies \overset{\triangle}{AZC}\equiv\overset{\triangle}{AZY}.$$

Which implies $AC=AY$. Now we have

$$\left.\begin{array}{c} AB=AB \\ \widehat{BAC}=\widehat{BAY}=\alpha. \\ AC=AY \end{array}\right\} \implies \overset{\triangle}{BAY}\equiv\overset{\triangle}{BAC}.$$

That implies $\widehat{BCA}=\widehat{BYA}$. Finally we get

$$\widehat{PBD}=\widehat{QBY}=\left|\widehat{BQA}-\widehat{BYA}\right|=\left|\widehat{PCA}-\widehat{BCA}\right|.$$

$\blacksquare$

15. Let $P(x,y,z)$ be the first assertion and $Q(x,y)$ be the second one. First we prove two claims.

- The function $g(x)=f(x,1)$ is bijective.

  Assume that a,b are two positive numbers with $f(a,1)=f(b,1)$. By comparing $P(a,1,1),P(b,1,1)$ we obtain

  $$a^2f(a,1)=f(f(a,1),1)=f(f(b,1),1)=b^2f(b,1) \implies a=b.$$

So $f(a,1)$ is injective. Also

$$P(1,y,1):\ f(f(1,y),1)=y^2 f(1,1).$$

The RHS of the above equation can be any positive real number, so $f(a,1)$ is surjective.

- The function $h(x)=f(1,x)$ is bijective.

  Note that for any positive real number t, we have

  $$P\Bigg(1,\underbrace{\sqrt{\tfrac{f(t,1)}{f(1,1)}}}_{y},1\Bigg):\ f(f(1,y),1)=f(t,1).$$

  According to the previous claim, we must have $h(y)=f(1,y)=t$ so $h(x)$ is surjective. Now

  $$P(1,1,1):\ f(f(1,1),1)=f(1,1)\implies f(1,1)=1.$$

  Now if for some positive numbers a,b we have $f(1,a)=f(1,b)$, by comparing $P(1,a,1),P(1,b,1)$ we obtain

  $$a^2=f(f(1,a),1)=f(f(1,b),1)=b^2\implies a=b.$$

  So $h(x)$ is injective.

We have

$$P(1,y,z):\ f(f(1,y),z)=y^2 f(1,z),$$

And also had

$$P(1,y,1):\ f(f(1,y),1)=y^2.$$

So we obtain

$$f(h(y),z)=f(f(1,y),z)=y^2 f(1,z)=g\left(h(y)\right)h(z).$$

Since h is surjective, we can write $a=h(y)$ and get

$$\forall a,z\in\mathbb{R}^+:f(a,z)=g(a)h(z).$$

Now using the above equation, we rewrite the first assertion $P(x,y,z)$ and get

$$g\left(g(x)h(y)\right)=x^2y^2g(x)\ ,\ g(1)=h(1)=1$$

Now set $y=1$ to get $g(g(x))=x^2g(x)$, also we had $g(h(y))=y^2$, so we can rewrite the above equation as

$$g(g(x)h(y))=g(g(x))g(h(y))\ \underset{\text{are surjective}}{\overset{g,h}{\Longrightarrow}}\ \forall x,y\in\mathbb{R}^+:g(xy)=g(x)g(y).$$

We can also get

$$\begin{aligned}
g(h(y)) = y^2 &\implies g(h(xy)) = x^2y^2 = g(h(x))g(h(y)) = g(h(x)h(y)), \\
&\underset{\text{is injective}}{\overset{g}{\implies}} h(xy) = h(x)h(y). \\
g(g(x)) = x^2 g(x) &\implies g(g(h(x))) = h(x)^2 g(h(x))), \\
&\implies g(x^2) = x^2 h(x^2) = x^2 h(x^2), \\
&\implies \forall x \in \mathbb{R}^+ : g(x) = xh(x), \\
&\implies h(y)h(h(y)) = y^2.
\end{aligned}$$

Now rewrite $Q(x, y)$

$$h(x + x^2 h(xy)) \geq x + xyh(x)^2.$$

Set $y \to \frac{y}{x}$ to get

$$h(x + x^2 h(y)) \geq x + yh(x)^2 \implies h(1 + xh(y)) \geq \frac{x}{h(x)} + yh(x).$$

Note that $h(1) = h(x)h\left(\frac{1}{x}\right)$, so $h\left(\frac{1}{x}\right) = \frac{1}{h(x)}$. Set $x = \frac{1}{h(y)}$ above to get

$$h(2) \geq \frac{h(h(y))}{h(y)} + \frac{y}{h(h(y))} \geq 2\sqrt{\frac{y}{h(y)}} \implies \exists c \in \mathbb{R}^+ : h(y) \geq cy.$$

Also since $h(xy) = h(x)h(y)$, we obtain $h(x^n) = h(x)^n$, for all positive integers n. Therefore

$$\begin{gathered}
h(y)^n = h(y^n) \geq cy^n \implies h(y) \geq \sqrt[n]{c}y \overset{n \to \infty}{\implies} h(y) \geq y \\
\implies y^2 = h(y)h(h(y)) \geq y^2 \implies h(y) = y \implies g(x) = x^2 \\
\implies f(x, y) = g(x)h(y) = x^2 y
\end{gathered}$$

So $\boxed{f(x, y) = x^2 y}$ is the only answer of the problem which is indeed a solution. ■

16. In any quadruple of the points, consider the point which has the power k with respect to the circle passing through the other three, name these points **good** points. We claim that there are two quadruples with the same **good** point, and two other common points.

There are $\binom{6}{4} = 15$ quadruples, each has at least one **good** point, since there are six points in total, there exists a point P which is a **good** point in at least $\left\lceil \frac{15}{6} \right\rceil = 3$ quadruples.

The other five points need to fill the remaining place in each of these quadruples, there are a total of $3 \times 3 = 9$ places, so there is a point Q that is is in at least $\left\lceil \frac{9}{5} \right\rceil = 2$ of the quadruples.

Consider these two quadruples containing P and Q. There are a total of 4 places in these quadruples need to be filled with the other 4 points. The claim is directly proved if some point R is a member of both of these quadruples. In other case, the remaining 4 points must appear exactly once in these 4 remaining places. So we have two quadruples

$$\mathcal{Q}_1 = (P, Q, R, S),\ \ \mathcal{Q}_2 = (P, Q, T, U).$$

Where P, Q, R, S, T, U are all six points of the problem.
We said P is a **good** point in at least 3 quadruples. Consider the third quadruple $\mathcal{Q}_3$ containing P as a **good** point. 3 remaining places must be filled with the other five points Q, R, S, T or U. If both R, S or T, U are members of $\mathcal{Q}_3$, then (respectively) $\mathcal{Q}_1$ or $\mathcal{Q}_2$ have three common points with $\mathcal{Q}_3$. Otherwise, only one member of $\{R, S\}$, one member of $\{T, U\}$ along with Q are members of $\mathcal{Q}_3$, in this case, both $\mathcal{Q}_1, \mathcal{Q}_2$ have three common points with $\mathcal{Q}_3$. Therefore the claim is proved in any case.

So we have found two quadruples with exactly three common points and also the same **good** point P. Without loss of generality, assume that these two quadruples are $(P, Q, R, S),\ (P, Q, R, T)$.
Let ω_1 and ω_2 be the circumcircles of triangles QRS and QRT, respectively. We have

$$\mathcal{P}_{\omega_1}(P) = \mathcal{P}_{\omega_2}(P) = k,$$

(Where $\mathcal{P}_\lambda(X)$ is the power of point X with respect to circle λ.) If $\omega_1 \neq \omega_2$, the above equation implies P is a point on the radical axis of these two circles, that means P lies on QR which is impossible since there are no three points on a same line. So the only possible case is when $\omega_1 = \omega_2$, means points Q, R, S and T lie on a circle. Therefore by considering the quadruple (Q, R, S, T), we obtain $k = 0$.
The rest of the problem is clear, $k = 0$ means between any quadruple, one point lies on the circle passing through the other three point, in other words, any four points are concyclic. Now fix three points P, Q, R, we get that any other point is concyclic with these three points, hence all six points lie on the circumcircle of triangle PQR. ■

17. It is immediately deduced that

$$P_d(x) = x^d + (c_1 + c_2 + \cdots + c_{d-1})x^{d-2} + \\ (c_3c_1 + c_4c_1 + c_4c_2 + \cdots + c_{d-1}c_{d-3})x^{d-4} + \cdots .$$

The coefficient of x^{d-4} could be written as $\sum_{k=3}^{d-1} \left(c_k \sum_{l=1}^{k-2} c_l\right)$ which is equal to

$$\frac{1}{2}\left((c_1 + c_2 + \cdots + c_{d-1})^2 - \sum_{k=1}^{d-1} c_k^2\right) - \sum_{k=1}^{d-2} c_k c_{k+1}.$$

Now assume the contrary, that there is such n and c, by comparing the coefficients of x^{2n-2} and x^{2n-4} we obtain

$$c = \frac{1}{n}\sum_{k=1}^{2n-1} c_k,$$
$$\sum_{k=3}^{2n-1}\left(c_k \sum_{l=1}^{k-2} c_l\right) = \frac{n(n-1)}{2}c^2 + \sum_{k=1}^{n-1} c_k.$$

Using the first equality in the second one, we find that

$$\frac{1}{n}\left(\sum_{k=1}^{2n-1} c_k\right)^2 = \sum_{k=1}^{2n-1} c_k^2 + 2\sum_{k=1}^{2n-2} c_k c_{k+1} + 2\sum_{k=1}^{n-1} c_k. \quad (\bigstar)$$

Now we prove that the above equality could not be established. Note that by Cauchy-Schwarz inequality, we have

$$\frac{1}{n}\left(\sum_{k=1}^{2n-1} c_k\right)^2 \leq c_1^2 + \sum_{k=1}^{n-1}(c_{2k+1} + c_{2k})^2,$$

and also

$$\frac{1}{n}\left(\sum_{k=1}^{2n-1} c_k\right)^2 \leq c_{2n-1}^2 + \sum_{k=1}^{n-1}(c_{2k-1} + c_{2k})^2.$$

Therefore we have

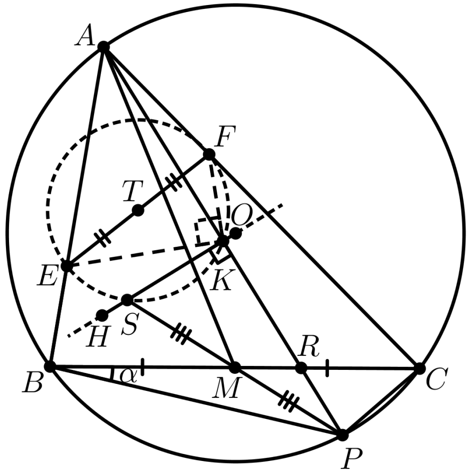
$$\frac{2}{n}\left(\sum_{k=1}^{2n-1} c_k\right)^2 \leq 2\sum_{k=1}^{2n-1} c_k^2 + 2\sum_{k=1}^{2n-2} c_k c_{k+1}.$$

Applying the above inequality to $\bigstar$ we obtain

$$\sum_{k=1}^{2n-2} c_k c_{k+1} + 2\sum_{k=1}^{n-1} c_k \leq 0.$$

But since $0 \leq c_k$ and $0 < c_{2017}$, the final result is impossible when $n > 2017$. Contradiction, hence the claim of the problem. ■

18. Let M be the midpoint of BC and S be the reflection of P over M. First we show that S lies on OH.



We have

$$\begin{rcases} PM = MS \\ PK = KA \end{rcases} \implies MK \parallel AS \ , \ MK = \frac{AS}{2}.$$

Also, it is well-known that $MO \parallel AH$, $MO = \frac{AH}{2}$. Combining it with the last result we obtain

$$\overset{\triangle}{MKO} \sim \overset{\triangle}{AHS}.$$

Because of the facts that $MK \parallel AS$ and $MO \parallel AH$, we conclude that $KO \parallel HS$ which means S is a point on OH.

Now we show that $\widehat{ESF} = 90°$. Note that since $MB = MC$ and $MS = MP$, $BSCP$ is a parallelogram, therefore

$$\begin{cases} \widehat{BSC} = \widehat{BPC} = 180° - \widehat{A}, \\ \widehat{SBC} = \widehat{PCB} \implies \widehat{SBE} = \widehat{B} - \widehat{PCB}, \\ \widehat{SCB} = \widehat{PBC} \implies \widehat{SCF} = \widehat{C} - \widehat{PBC}. \end{cases}$$

So we have

$$\begin{cases} BS = PC = BE \implies \widehat{ESB} = 90° - \frac{\widehat{SBE}}{2} = 90° - \frac{\widehat{B} - \widehat{PCB}}{2}, \\ CS = PB = CF \implies \widehat{FSC} = 90° - \frac{\widehat{SCF}}{2} = 90° - \frac{\widehat{C} - \widehat{PBC}}{2}. \end{cases}$$

Therefore

$$\begin{aligned} \widehat{ESF} &= 360° - \widehat{ESB} - \widehat{FSC} - \widehat{BSC} \\ &= 360° - \left(90° - \tfrac{\widehat{B} - \widehat{PCB}}{2}\right) - \left(90° - \tfrac{\widehat{C} - \widehat{PBC}}{2}\right) - \left(180° - \widehat{A}\right) \\ &= \tfrac{\widehat{B}}{2} + \tfrac{\widehat{C}}{2} + A - \tfrac{\widehat{PCB} + \widehat{PBC}}{2} \\ &= \frac{\widehat{B}}{2} + \frac{\widehat{C}}{2} + \frac{\widehat{A}}{2} = 90°. \end{aligned}$$

Let T be the circumcenter of triangle ESF, since BT and CT are the perpendicular bisectors of ES and FS respectively, and since $\widehat{ESF} = 90°$ we obtain $\widehat{BTS} = 90°$.

Now we show that $MT \parallel AP$. Let R be the intersection of AK with BC. It suffices to show that $\widehat{TMB} = \widehat{ARB}$.

If $\widehat{PBC} = \alpha$ we have

$$\widehat{ARB} = \alpha + \widehat{APB} = \alpha + \widehat{C},$$

$$\left.\begin{array}{l}\widehat{BTC} = 90° \\ MB = MC\end{array}\right\} \implies MB = MC = MT,$$

and so

$$\widehat{TMB} = 2\widehat{TCM} = 2\left(\widehat{SCM} + \widehat{TCS}\right) = 2\left(\alpha + \tfrac{\widehat{C}-\alpha}{2}\right) = \alpha + \widehat{C} = \widehat{ARB}.$$

Therefore, $MT \parallel AP$. We have $\widehat{SKP} = 90°$. Also $MP = MS$, so M is the circumcenter of triangle SKP and thus it lies on the perpendicular bisector of KS. Also since $MT \parallel AP$, and since $AP \perp SK$ we conclude that TM is the perpendicular bisector of KS and so $TS = TK$. We also had $TS = TF = TE$. Therefore, points E, S, K and F lie on a circle with center T. So finally we obtain $\widehat{ESF} = \widehat{EKF} = 90°$. ■