Kd-trees and range trees

Computational Geometry

Lecture 6: Kd-trees and range trees

Databases

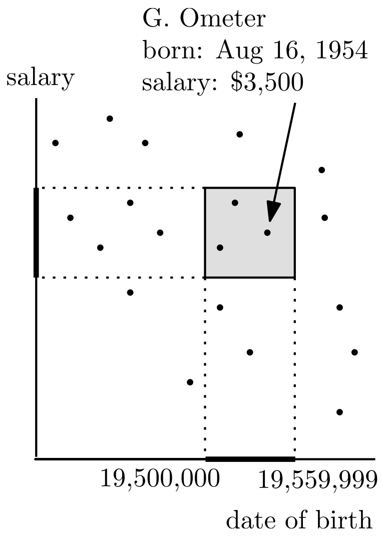
Databases store records or objects

Personnel database: Each employee has a name, id code, date of birth, function, salary, start date of employment, ...

Fields are textual or numerical

Database queries

A database query may ask for all employees with age between a_1 and a_2, and salary between s_1 and s_2



Database queries

When we see numerical fields of objects as coordinates, a database stores a point set in higher dimensions

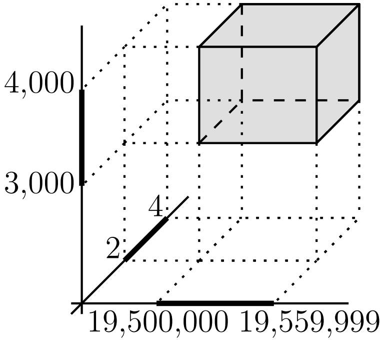
Exact match query: Asks for the objects whose coordinates match query coordinates exactly

Partial match query: Same but not all coordinates are specified

Range query: Asks for the objects whose coordinates lie in a specified query range (interval)

Database queries

Example of a 3-dimensional (orthogonal) range query: children in $[2, 4]$, salary in $[3000, 4000]$, date of birth in $[19{,}500{,}000\ ,\ 19{,}559{,}999]$



Data structures

Idea of data structures

- Representation of structure, for convenience (like DCEL)
- Preprocessing of data, to be able to solve future questions really fast (sub-linear time)

A (search) data structure has a storage requirement, a query time, and a construction time (and an update time)

1D range query problem

1D range query problem: Preprocess a set of n points on the real line such that the ones inside a 1D query range (interval) can be answered fast

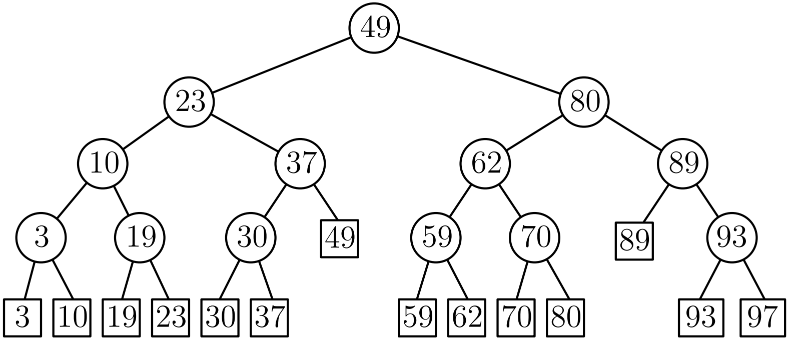
The points $p_1, \ldots, p_n$ are known beforehand, the query $[x, x']$ only later

A solution to a query problem is a data structure, a query algorithm, and a construction algorithm

Question: What are the most important factors for the *efficiency* of a solution?

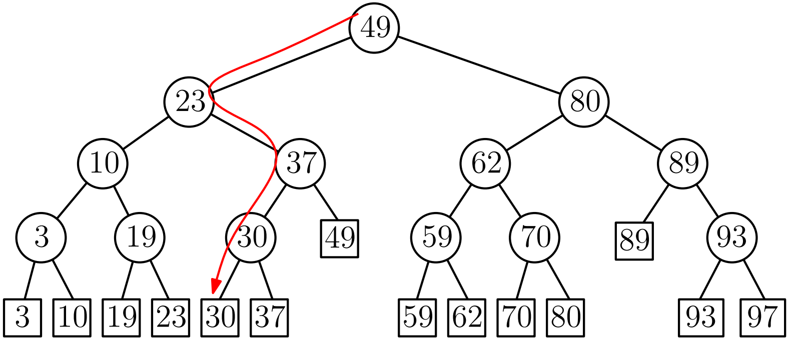
Balanced binary search trees

A balanced binary search tree with the points in the leaves



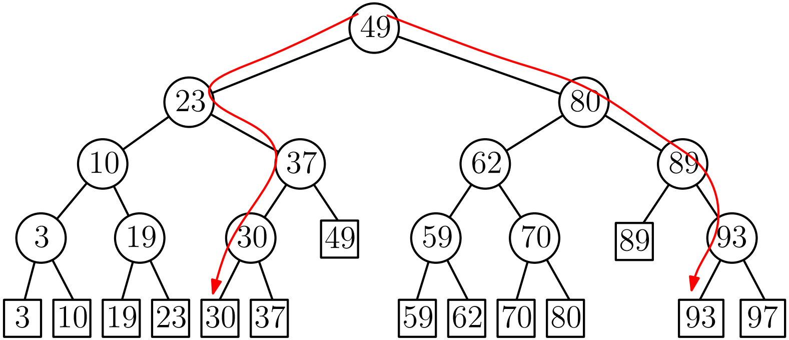
Balanced binary search trees

The search path for 25



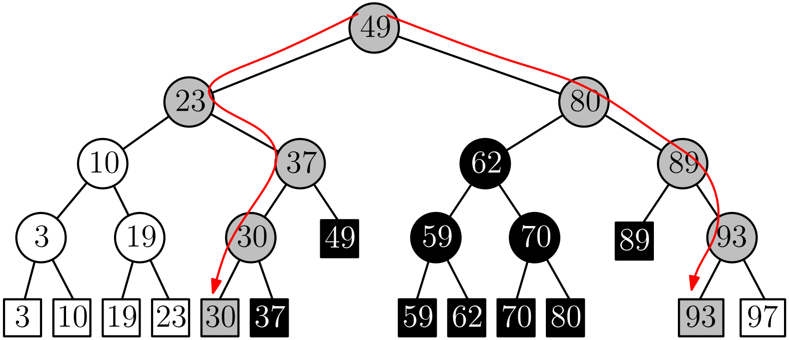
Balanced binary search trees

The search paths for 25 and for 90



Example 1D range query

A 1-dimensional range query with $[25, 90]$



Node types for a query

Three types of nodes *for a given query*:

- **White nodes:** never visited by the query
- **Grey nodes:** visited by the query, unclear if they lead to output
- **Black nodes:** visited by the query, whole subtree is output

Question: What query time do we hope for?

Node types for a query

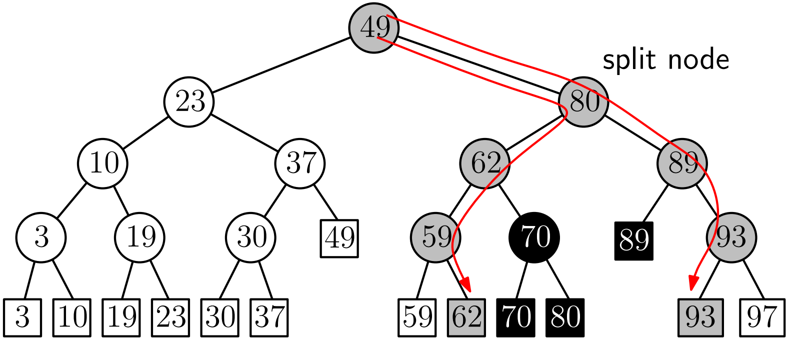
The query algorithm comes down to what we do at each type of node

Grey nodes: use query range to decide how to proceed: to not visit a subtree (pruning), to report a complete subtree, or just continue

Black nodes: traverse and enumerate all points in the leaves

Example 1D range query

A 1-dimensional range query with $[61, 90]$



1D range query algorithm

```
Algorithm 1DRANGEQUERY($\mathcal{T}, [x : x']$)
1.  $v_{\text{split}} \leftarrow$ FINDSPLITNODE($\mathcal{T}, x, x'$)
2.  if $v_{\text{split}}$ is a leaf
3.     then Check if the point in $v_{\text{split}}$ must be reported.
4.     else  $v \leftarrow lc(v_{\text{split}})$
5.           while $v$ is not a leaf
6.              do if $x \leq x_v$
7.                    then REPORTSUBTREE($rc(v)$)
8.                         $v \leftarrow lc(v)$
9.                    else  $v \leftarrow rc(v)$
10.          Check if the point stored in $v$ must be reported.
11.          Similarly, follow the path to $x'$, and ...
```

Query time analysis

The efficiency analysis is based on counting the numbers of nodes visited for each type

- **White nodes:** never visited by the query; no time spent
- **Grey nodes:** visited by the query, unclear if they lead to output; time determines dependency on n
- **Black nodes:** visited by the query, whole subtree is output; time determines dependency on k, the output size

Query time analysis

Grey nodes: they occur on only two paths in the tree, and since the tree is balanced, its depth is $O(\log n)$

Black nodes: a (sub)tree with m leaves has $m-1$ internal nodes; traversal visits $O(m)$ nodes and finds m points for the output

The time spent at each node is $O(1) \Rightarrow O(\log n + k)$ query time

Storage requirement and preprocessing

A (balanced) binary search tree storing n points uses $O(n)$ storage

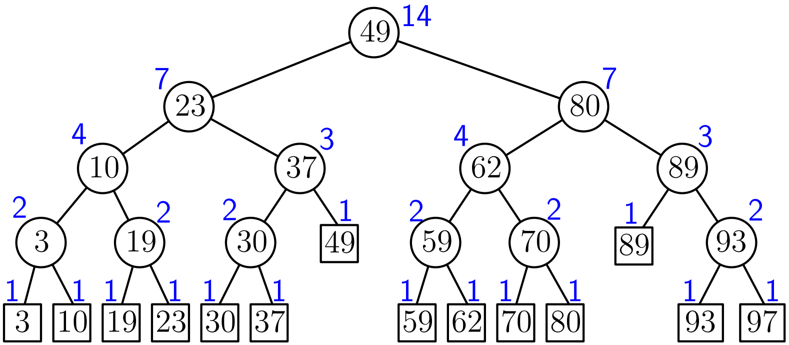
A balanced binary search tree storing n points can be built in $O(n)$ time after sorting

Result

Theorem: A set of n points on the real line can be preprocessed in $O(n \log n)$ time into a data structure of $O(n)$ size so that any 1D range query can be answered in $O(\log n + k)$ time, where k is the number of answers reported

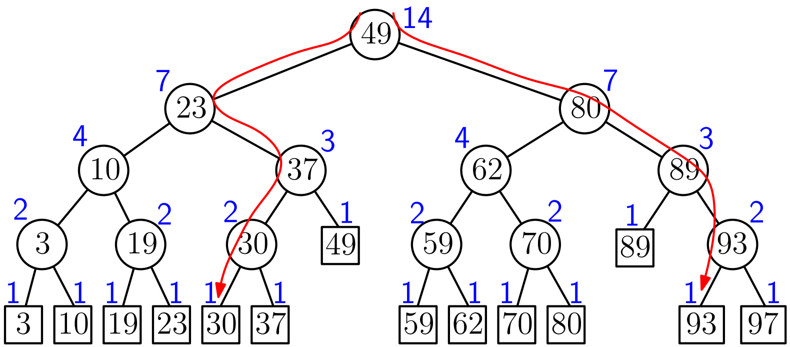
Example 1D range counting query

A 1-dimensional range tree for **range counting queries**



Example 1D range counting query

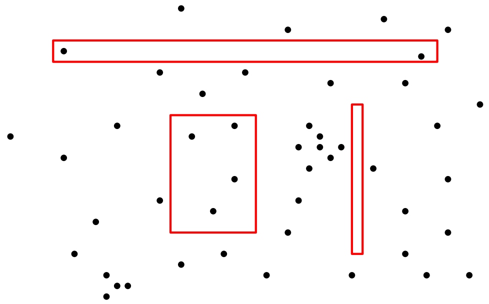
A 1-dimensional range counting query with $[25, 90]$



Result

Theorem: A set of n points on the real line can be preprocessed in $O(n\log n)$ time into a data structure of $O(n)$ size so that any 1D range counting query can be answered in $O(\log n)$ time

Range queries in 2D



Range queries in 2D

Question: Why can't we simply use a balanced binary tree in x-coordinate?

Or, use one tree on x-coordinate and one on y-coordinate, and query the one where we think querying is more efficient?

Kd-trees

Kd-trees, the idea: Split the point set alternatingly by x-coordinate and by y-coordinate

split by x-coordinate: split by a vertical line that has half the points left and half right

split by y-coordinate: split by a horizontal line that has half the points below and half above

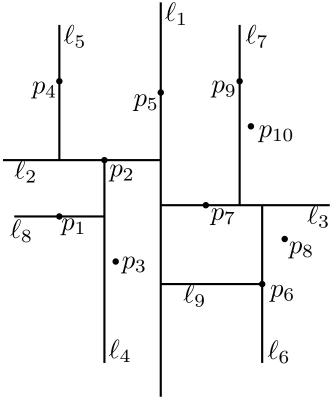
Kd-trees

Kd-trees, the idea: Split the point set alternatingly by x-coordinate and by y-coordinate

split by x-coordinate: split by a vertical line that has half the points left or on, and half right

split by y-coordinate: split by a horizontal line that has half the points below or on, and half above

Kd-trees


ℓ_1
ℓ_5
ℓ_7
p_4
p_5
p_9
p_{10}
ℓ_2
p_2
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p_1
p_8
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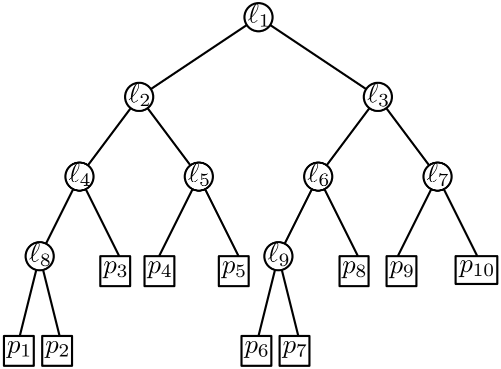
p_6
ℓ_4
ℓ_6


ℓ_1
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ℓ_9
p_8
p_9
p_{10}
p_1
p_2
p_6
p_7

Kd-tree construction

Algorithm $\textsc{BuildKdTree}(P, \mathit{depth})$

```
1.  if $P$ contains only one point
2.     then return a leaf storing this point
3.     else if $\mathit{depth}$ is even
4.             then Split $P$ with a vertical line $\ell$ through the
                    median $x$-coordinate into $P_1$ (left of or
                    on $\ell$) and $P_2$ (right of $\ell$)
5.             else Split $P$ with a horizontal line $\ell$ through
                    the median $y$-coordinate into $P_1$ (below
                    or on $\ell$) and $P_2$ (above $\ell$)
6.          $v_{\text{left}} \leftarrow \textsc{BuildKdTree}(P_1, \mathit{depth}+1)$
7.          $v_{\text{right}} \leftarrow \textsc{BuildKdTree}(P_2, \mathit{depth}+1)$
8.          Create a node $v$ storing $\ell$, make $v_{\text{left}}$ the left
            child of $v$, and make $v_{\text{right}}$ the right child of $v$.
9.          return $v$
```

Kd-tree construction

The median of a set of n values can be computed in $O(n)$ time (randomized: easy; worst case: much harder)

Let $T(n)$ be the time needed to build a kd-tree on n points

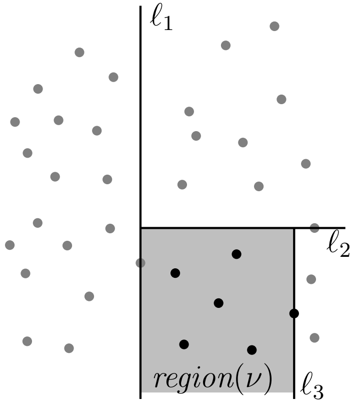
$$T(1) = O(1)$$

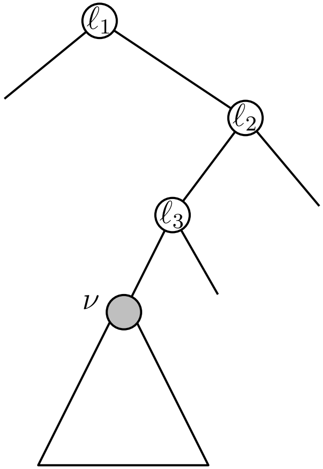
$$T(n) = 2 \cdot T(n/2) + O(n)$$

A kd-tree can be built in $O(n \log n)$ time

Question: What is the storage requirement?

Kd-tree regions of nodes





Kd-tree regions of nodes

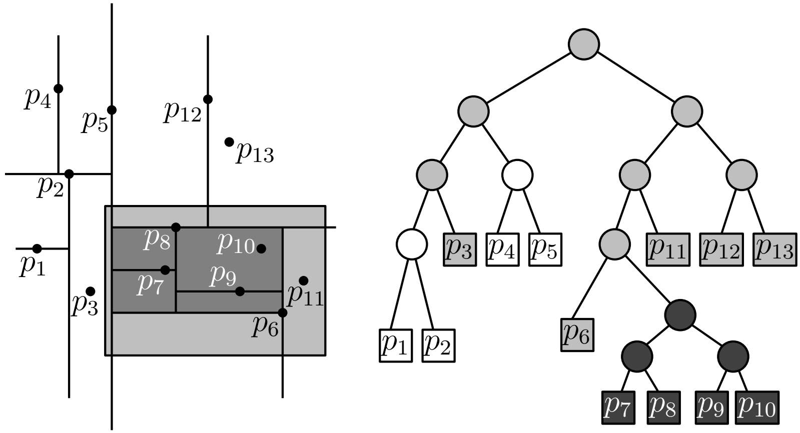
How do we know $region(v)$ when we are at a node v?

Option 1: store it explicitly with every node

Option 2: compute it on-the-fly, when going from the root to v

Question: What are reasons to choose one or the other option?

Kd-tree querying



Kd-tree querying

```
Algorithm SEARCHKDTREE($\nu$,$R$)
Input. The root of (a subtree of) a kd-tree, and a range $R$
Output. All points at leaves below $\nu$ that lie in the range.
1.  if $\nu$ is a leaf
2.     then Report the point stored at $\nu$ if it lies in $R$
3.     else if $region(lc(\nu))$ is fully contained in $R$
4.             then REPORTSUBTREE($lc(\nu)$)
5.             else if $region(lc(\nu))$ intersects $R$
6.                     then SEARCHKDTREE($lc(\nu)$,$R$)
7.          if $region(rc(\nu))$ is fully contained in $R$
8.             then REPORTSUBTREE($rc(\nu)$)
9.             else if $region(rc(\nu))$ intersects $R$
10.                    then SEARCHKDTREE($rc(\nu)$,$R$)
```

Kd-tree querying

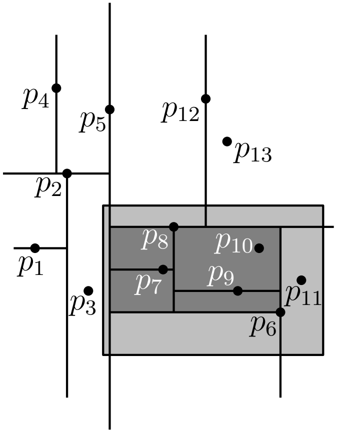
Question: How about a range *counting* query?
How should the code be adapted?

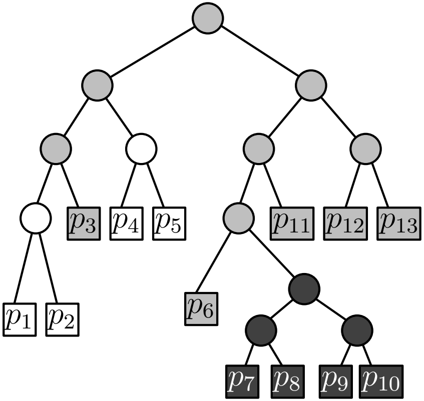
Kd-tree query time analysis

To analyze the query time of kd-trees, we use the concept of white, grey, and black nodes

- **White nodes:** never visited by the query; no time spent
- **Grey nodes:** visited by the query, unclear if they lead to output; time determines dependency on n
- **Black nodes:** visited by the query, whole subtree is output; time determines dependency on k, the output size

Kd-tree query time analysis



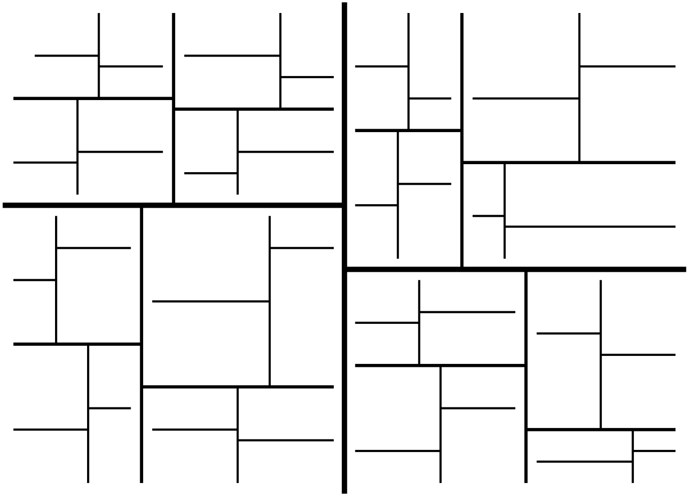


Kd-tree query time analysis

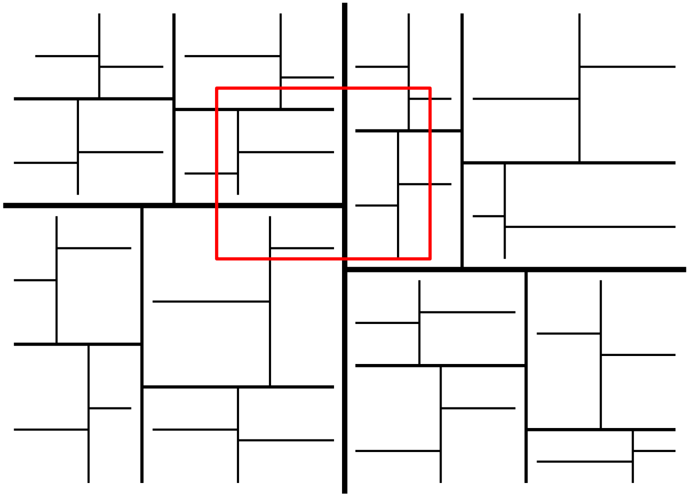
White, grey, and black nodes with respect to $region(\nu)$:

- **White node ν:** R does not intersect $region(\nu)$
- **Grey node ν:** R intersects $region(\nu)$, but $region(\nu) \not\subseteq R$
- **Black node ν:** $region(\nu) \subseteq R$

Kd-tree query time analysis

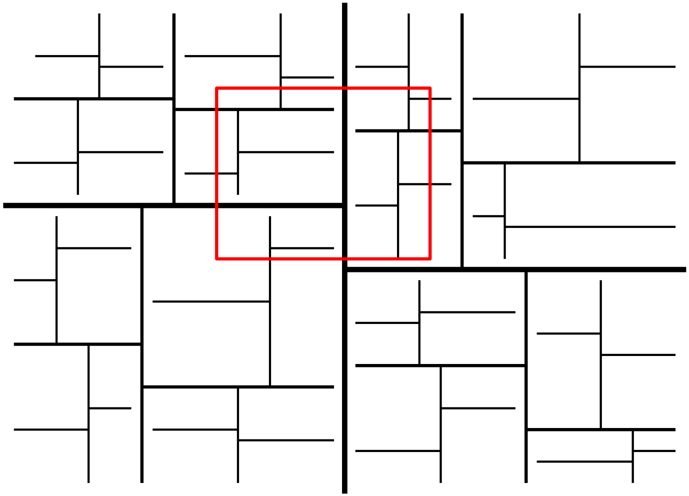


Kd-tree query time analysis



Question: How many grey and how many black *leaves*?

Kd-tree query time analysis



Question: How many grey and how many black *nodes*?

Kd-tree query time analysis

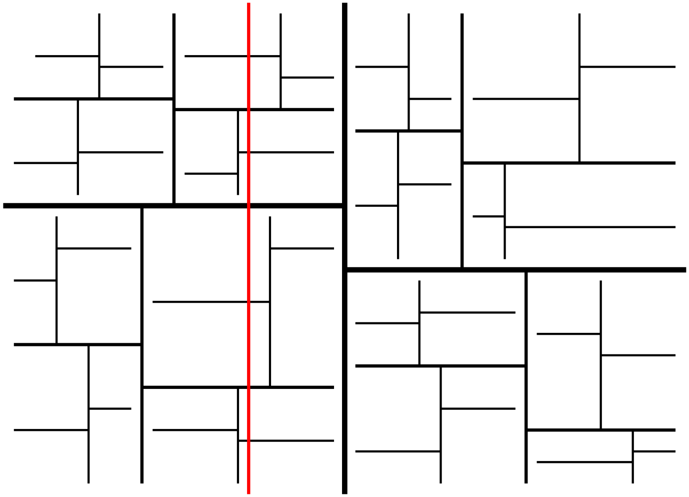
Grey node v: R intersects $region(v)$, but $region(v) \not\subseteq R$

It implies that the boundaries of R and $region(v)$ intersect

Advice: If you don't know what to do, simplify until you do

Instead of taking the boundary of R, let's analyze the number of grey nodes if the query is with a vertical line ℓ

Kd-tree query time analysis



Question: How many grey and how many black *leaves*?

Kd-tree query time analysis

We observe: At every vertical split, ℓ is only to one side, while at every horizontal split ℓ is to both sides

Let $G(n)$ be the number of grey nodes in a kd-tree with n points (leaves)

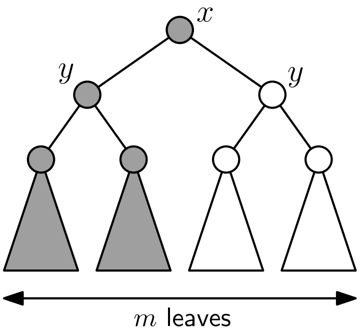
If a subtree has m leaves: $G(m) = 1 + G(n/2)$ at even depth

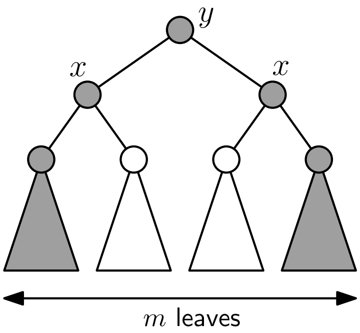
If a subtree has m leaves: $G(m) = 1 + 2 \cdot G(n/2)$ at odd depth

If we use *two levels at once,* we get:

$$G(m) = 2 + 2 \cdot G(m/4) \quad \text{or} \quad G(m) = 3 + 2 \cdot G(m/4)$$

Kd-tree query time analysis





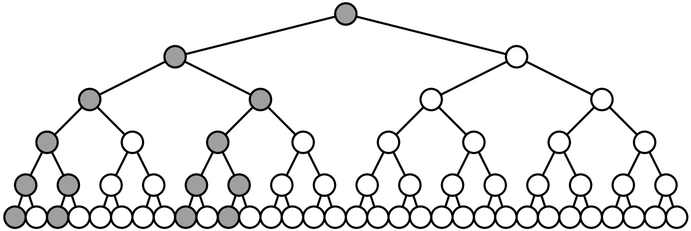
Kd-tree query time analysis

$G(1) = 1$

$G(n) = 2 \cdot G(n/4) + O(1)$

Question: What does this recurrence solve to?

Kd-tree query time analysis



The grey subtree has unary and binary nodes

Kd-tree query time analysis

The depth is $\log n$, so the binary depth is $\frac{1}{2} \cdot \log n$

Counting only binary nodes, there are
$2^{\frac{1}{2} \cdot \log n} = 2^{\log n^{1/2}} = n^{1/2} = \sqrt{n}$

Every unary grey node has a unique binary parent (except the root) $\cdots$

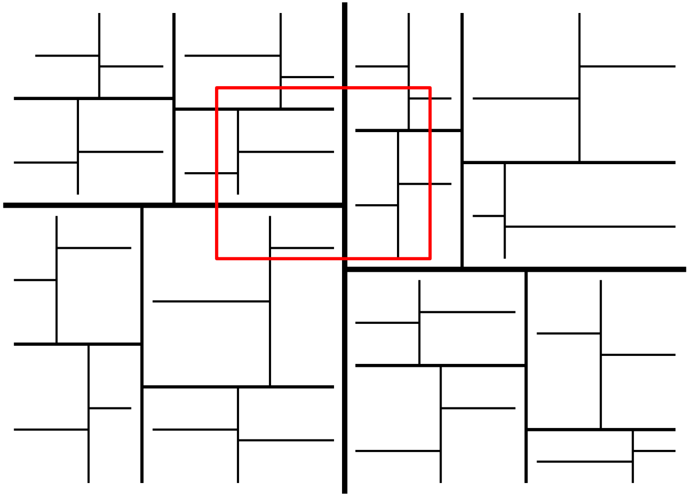
Kd-tree query time analysis

The number of grey nodes if the query were a vertical line is $O(\sqrt{n})$

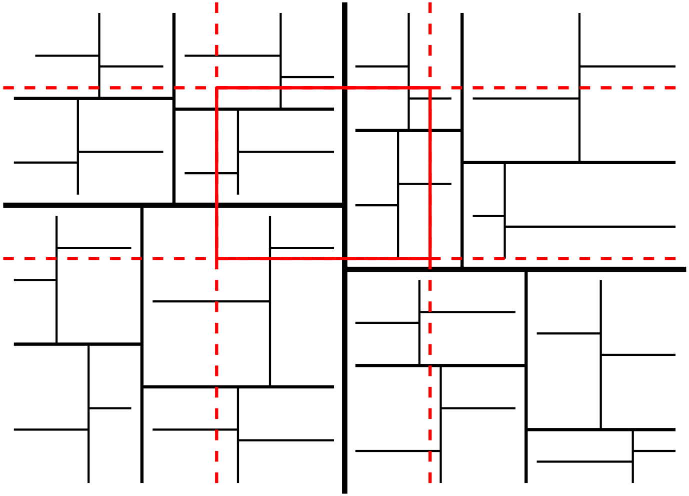
The same is true if the query were a horizontal line

How about a query rectangle?

Kd-tree query time analysis



Kd-tree query time analysis



Kd-tree query time analysis

The number of grey nodes for a query rectangle is at most the number of grey nodes for two vertical and two horizontal lines, so it is also $O(\sqrt{n})$!

For black nodes, reporting a whole subtree with k leaves takes $O(k)$ time (there are $k-1$ internal black nodes)

Result

Theorem: A set of n points in the plane can be preprocessed in $O(n \log n)$ time into a data structure of $O(n)$ size so that any 2D range query can be answered in $O(\sqrt{n} + k)$ time, where k is the number of answers reported

For range counting queries, we need $O(\sqrt{n})$ time

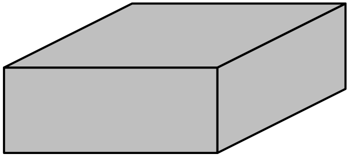
Efficiency

n	$\log n$	$\sqrt{n}$
4	2	2
16	4	4
64	6	8
256	8	16
1024	10	32
4096	12	64
1.000.000	20	1000

Higher dimensions

A 3-dimensional kd-tree alternates splits on x-, y-, and z-coordinate

A 3D range query is performed with a box



Higher dimensions

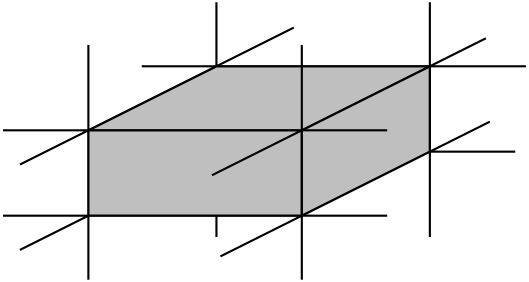
The construction of a 3D kd-tree is a trivial adaptation of the 2D version

The 3D range query algorithm is exactly the same as the 2D version

The 3D kd-tree still requires $O(n)$ storage if it stores n points

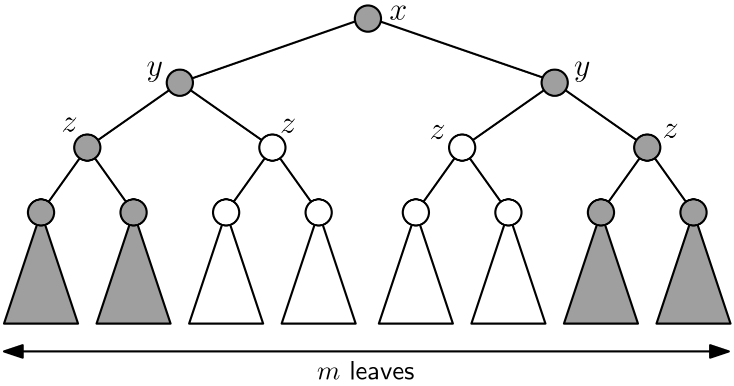
Higher dimensions

How does the query time analysis change?



Intersection of B and $region(\nu)$ depends on intersection of facets of B $\Rightarrow$ analyze by axes-parallel planes (B has no more grey nodes than six planes)

Higher dimensions



Kd-tree query time analysis

Let $G_3(n)$ be the number of grey nodes for a query with an axes-parallel plane in a 3D kd-tree

$G_3(1) = 1$

$G_3(n) = 4 \cdot G_3(n/8) + O(1)$

Question: What does this recurrence solve to?

Question: How many leaves does a perfectly balanced binary search tree with depth $\frac{2}{3}\log n$ have?

Result

Theorem: A set of n points in d-space can be preprocessed in $O(n \log n)$ time into a data structure of $O(n)$ size so that any d-dimensional range query can be answered in $O(n^{1/d} + k)$ time, where k is the number of answers reported