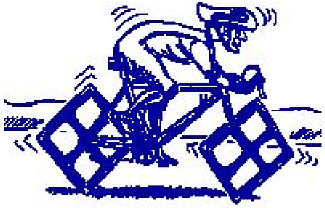
# Smooth Rides on Square Wheels

Stacy Hoehn Fonstad

Vanderbilt University

October 4, 2011



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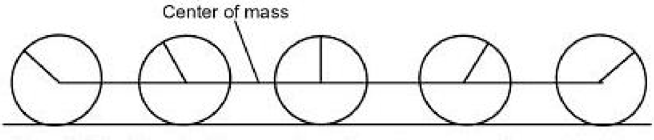

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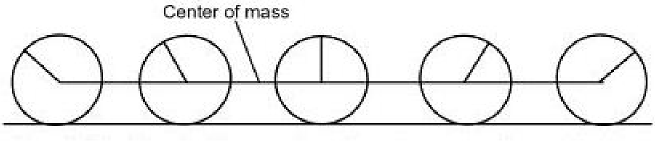


As the wheel rolls, the center of the wheel stays at a constant height, allowing the bicycle to ride smoothly.

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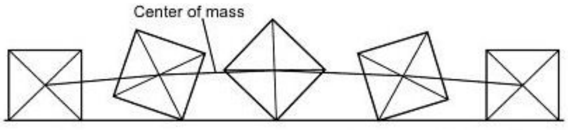

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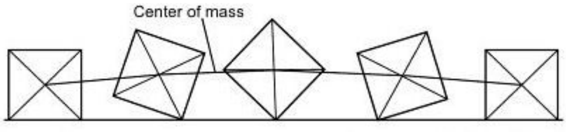


As the square wheel rolls across a flat surface, the center of the square changes elevation. To compensate for these elevation changes and to smooth the ride, the road's surface needs to be uneven.

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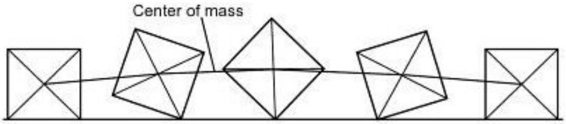


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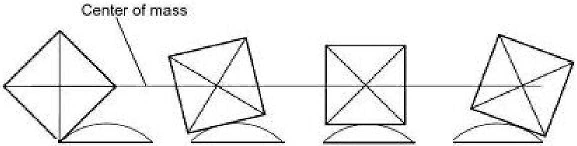
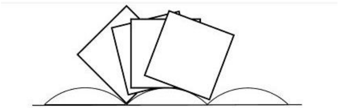


# Reinventing the Road

A series of these "bumps" forms a road that a square can roll smoothly on.



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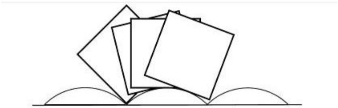
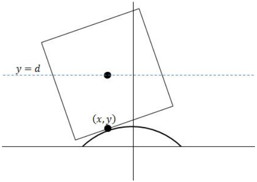


How do we determine the exact shape of these “bumps”?

# Determining the Shape of the "Bumps"



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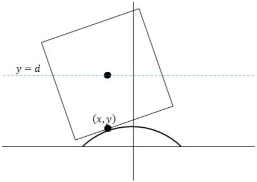


Design Criteria:

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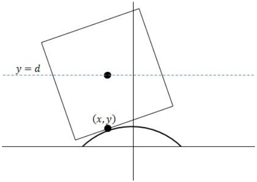


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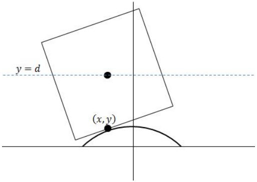


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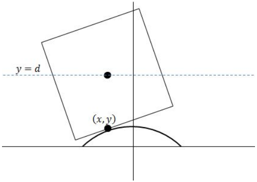


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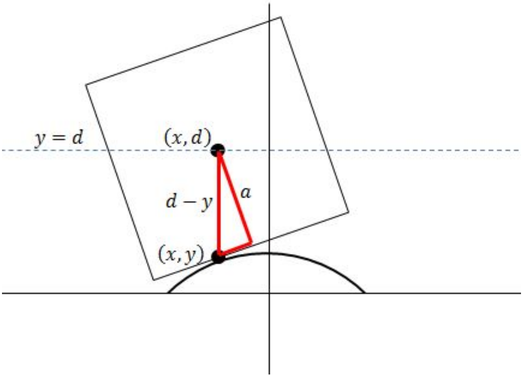
- The distance along the surface of the "bump" must equal the length of one side of the square.

# The Geometry Behind the Shape



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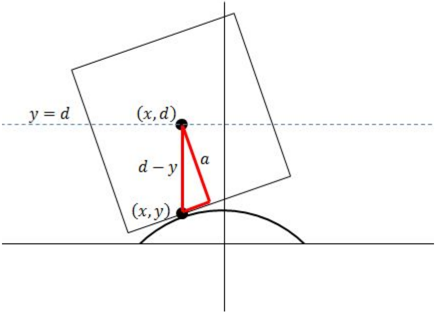


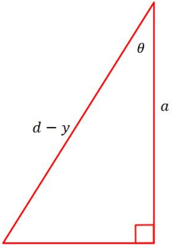

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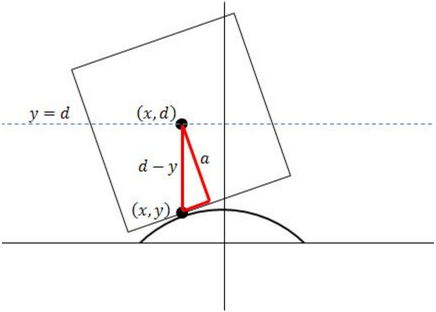


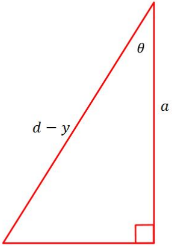


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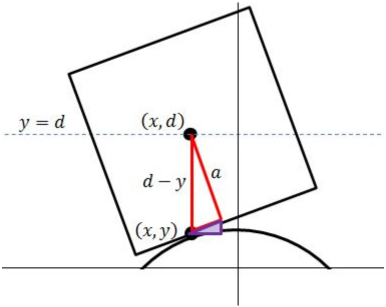


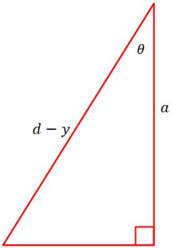


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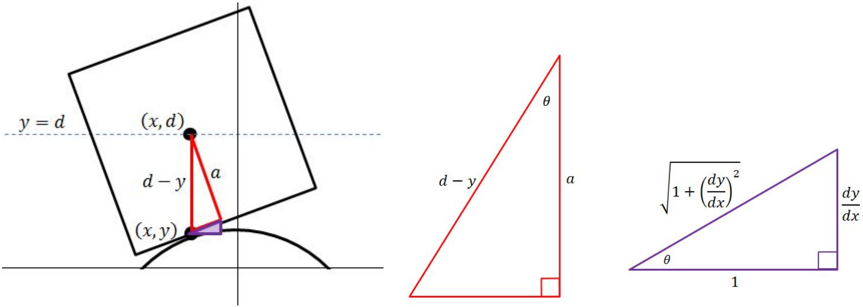


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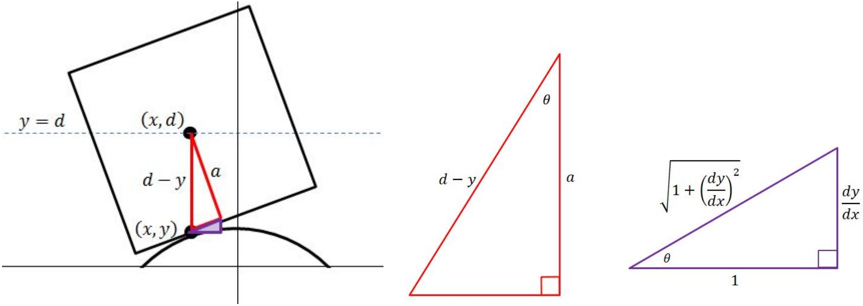


$$a = (d - y)\cos(\theta) = (d - y)\frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

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Integrate both sides:

$$\int \frac{1}{\sqrt{\left(\frac{d-y}{a}\right)^2 - 1}}\, dy = \int dx$$

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What is an antiderivative of $g(u) = \frac{1}{\sqrt{u^2-1}}$?

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What is an antiderivative of $g(u) = \frac{1}{\sqrt{u^2-1}}$?
$G(u) = cosh^{-1}(u)$ (inverse hyperbolic cosine)!!

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Solving for $y$, we get that $y = d - a\ cosh(\frac{x}{a} + c)$, where $c$ is some constant.

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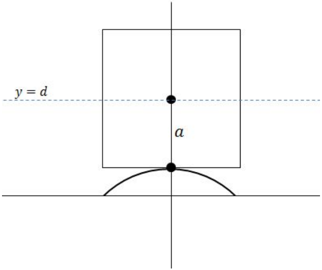

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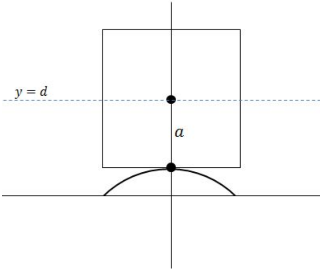
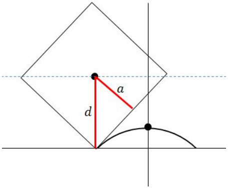



Initial Value: $y(0) = d - a$, so $c = 0$. Thus, $y = d - a\ cosh(\frac{x}{a})$.
(This type of curve is called an inverted catenary.)

# Example:

For example, if the wheel is a square with sides of length 2, $a = 1$ and $d = \sqrt{2}$.



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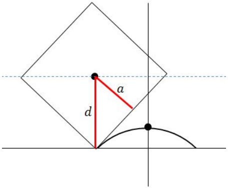
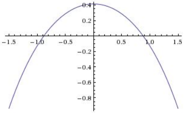
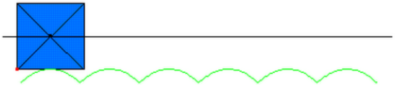



Thus, $y = \sqrt{2} - cosh(x)$.



# How Much of the Graph Do We Need to Use?

Because of how we set up this problem, we only need to use the portion of the graph lying above the $x$-axis (i.e. the portion of the graph with $-b \leq x \leq b$, where $b = cosh^{-1}(\sqrt{2}) \approx 0.8814$) and repeat it.



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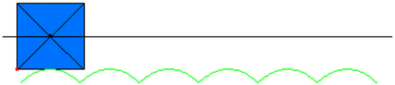

Remarks:

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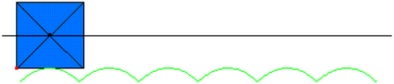

Remarks:

- The arclength of $y = \sqrt{2} - cosh(x)$ from $x = -b$ to $x = b$ equals 2, which is the length of a single side of the square.
- The slope of the graph of $y = \sqrt{2} - cosh(x)$ at $x = b$ is 1, and the slope at $x = -b$ is -1, so the angle between two consecutive "bumps" of the road is 90°.

# Squares on a Roll

Mathematica Demonstration

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Mathematica Demonstration

Figure 1: Macalester College

Figure 2: St. Norbert College

Texas A & M Video

# Hyperbolic Functions and Catenaries

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A catenary is the shape you get when you let a chain or a string hang freely between two endpoints.

# Catenaries (continued)

- Pyramids: There is evidence that ancient Egyptians might have used tracks of wood cut into quarter-circles (whose shapes are very close to those of inverted catenary curves) to move large blocks of stone for the pyramids.

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- Arches: Inverted catenaries $y = d - a\ cosh(\frac{x}{a})$ form the strongest arches.

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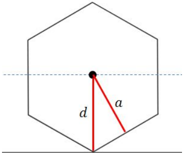
The Gateway Arch is actually a weighted catenary of the form $y = d - b\cosh(\frac{x}{a})$ because it is narrower at the top than at its base. Its shape corresponds to the shape that a weighted chain, having lighter links in the middle, would form.

# What about Other Regular Polygons?



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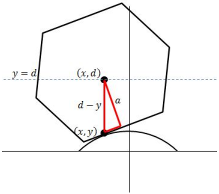

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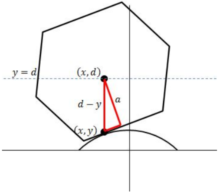


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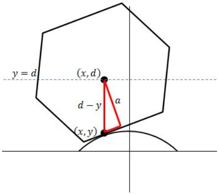


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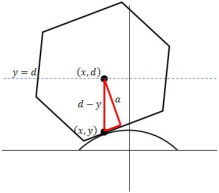
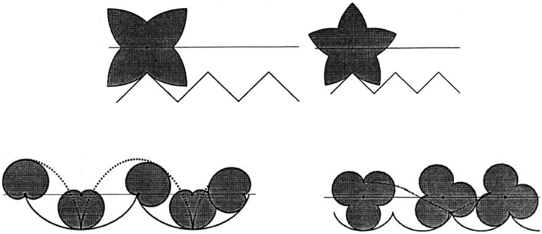



- The same procedure that we used for a square will work for any regular polygon (except for a triangle!)
- Instead of $d = \sqrt{2}a$, now $d = \frac{a}{\cos(\pi/n)}$, where $n$ is the number of sides of the polygon.
- As the number of sides of the polygon increases, the bumps on the catenary road will become flatter and flatter.

# Other Things to Try

- Given some other wheel shape, can we figure out the shape of the corresponding road?



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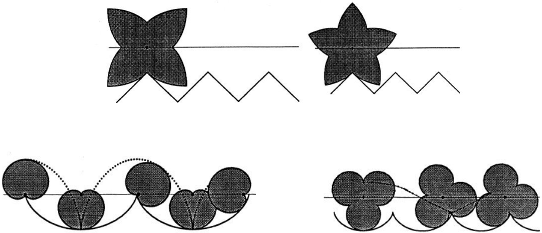

- Given a road shape, can we find a wheel that will ride smoothly over that road?

# A Few References


- Leon Hall and Stan Wagon. "Roads and Wheels." *Mathematics Magazine*, Vol. 65, No. 5 (Dec., 1992), pp. 283-301.
- G.B. Robison. "Rockers and Rollers." *Mathematics Magazine*, Vol. 33 (1960), pp. 139-144.