The 46th Annual Iranian Mathematics Conference

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Proceedings of the Conference

Talk

Preface

The Annual Iranian Mathematics Conference (AIMC) has been held since 1970. It is the oldest Iranian scientific gathering which takes place regularly each year at one of Iranian universities. The 36^{th} annual Iranian mathematics conference was held at Yazd University and now we are pleased to organize the 46^{th} conference. The 46^{th} AIMC will be held at Yazd University in Yazd (the most beautiful and historical city of Iran) from August 25 until August 28, 2015. The Iranian Mathematical Society and Yazd University have jointly sponsored the 46^{th} AIMC. This conference is an international conference and includes Keynote speakers, Invited speakers, Presentations of contributed research papers, and Poster presentations.

It is our pleasure to publish the Proceedings of the 46^{th} AIMC. More than 700 mathematicians from our country and abroad have taken part in the conference. By kind cooperation of contributors, more than 1100 papers were received. The scientific committee put a considerable effort on referral process in order to arrange a conference of excellent scientific quality. We have 15 plenary speakers from universities of Iran, as well as from Australia, South Korea, Canada, China, Czech Republic, India, Serbia and Spain. Moreover, our invited speakers are about 12.

The Scientific Committee of
46^{th} Annual Iranian Mathematics Conference

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A medley of group actions*

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Abstract

Most of my interaction and collaborative research with Iranian mathematicians has been linked with symmetric structures, and has involved group actions. The lecture will be a tribute to my Iranian colleagues.

Keywords: Group actions, symmetric structures, Iranian mathematicians
Mathematics Subject Classification [2010]: 20B25, 05C25

1 My first visit to Iran

My first mathematical colleague from Iran was Dr Akbar Hassani, who had been a graduate student with me in Oxford. His sabbatical leave spent at the University of Western Australia in 1986 led to my first visit to Tehran in 1994. Dr Hassani worked in Perth with me and Dr Luz Nochefranca on 2-arc transitive graphs.

Definition 1.1. A graph Γ is $(G, 2)$-*arc-transitive*, for some subgroup G of automorphisms, if G is transitive on all vertex triples (α, β, γ) such that $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ are both edges and $\alpha \neq \gamma$.

Previous work of mine had shown that every non-bipartite $(G, 2)$-arc transitive graph is a normal cover of a basic one where the group G has a special from. Hassani, Luz and I classified all possible basic examples for an infinite family of almost simple groups G.

Theorem 1.2. *[1] All $(G, 2)$-arc-transitive graphs such that $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$ are known.*

My lecture course in Tehran in 1994 was on the movement and separation of subsets under group actions, and some open problems on this theme became the topic of the PhD thesis for Mehdi Khayaty, now Professor Mehdi Alaeiyan.

Definition 1.3. Let G be a permutation group on a finite set Ω such that G has no fixed points in Ω, and let $\Gamma \subseteq \Omega$. The *movement of* Γ is $\mathrm{move}(\Gamma) = \max_{g \in G} |\Gamma^g \setminus \Gamma|$, and the *movement of* G is the maximum value of $\mathrm{move}(\Gamma)$ over all subsets Γ.

*Will be presented in English
†Speaker

In earlier work I had shown that both the number of G-orbits in Ω and the length of each G-orbit are bounded above by linear functions of the movement of G. In particular, if G is transitive on Ω with movement m, and if G not a 2-group and p is the smallest odd prime dividing its order $|G|$, then I had shown that $|\Omega| \leq \lfloor \frac{2mp}{p-1} \rfloor$. The main result of Mehdi's thesis was a classification of all groups which attain this upper bound.

Theorem 1.4. *[2] Let p be a prime, $p \geq 5$, let m be a positive integer, and let G be a transitive permutation group on a set Ω of size $\lfloor \frac{2mp}{p-1} \rfloor$ such that G has movement m, G is not a 2-group and p is the least odd prime dividing $|G|$. Then either G is known explicitly, or G is a p-group of exponent bounded in terms of p only.*

The second of Akbar Hassani's students who worked with me in the 1990s was Associate Professor Mohammadali Iranmanesh. Mohammadali's thesis topic was vertex-transitive non-Cayley graphs, namely deciding whether such graphs exist of certain orders [3].

Definition 1.5. Let G be a group and S an inverse-closed subset of G such that $1_G \notin S$. The *Cayley graph* $\mathrm{Cay}(G, S)$ is the graph with vertex set G such that $\{x, y\}$ is an edge if and only if $xy^{-1} \in S$. The group G acts by right multiplication as a *regular* subgroup of automorphisms (that is, G is transitive and only the identity fixes a vertex).

A graph Γ is a Cayley graph (for some group) if and only if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup. As a result of Mohammadali's work (extending work of Brendan McKay, Alice Miller, Greg Gamble, Ákos Seress, Akbar Hassani and myself) we know precisely when such graphs exist for a large class of orders. Mohammadali has worked on several other research projects with me since this time [5, 6, 7, 16].

Theorem 1.6. *[4] All integers n are known such that n has at most three distinct prime divisors, and there exists a vertex-transitive graph on n vertices which is not a Cayley graph.*

2 Professor Mehdi Behzad

In 2005 I participated in the Annual Iranian Mathematical Society Conference in Yazd. At that conference I met four Iranian mathematicians who have since visited me in Perth. The first is Professor Mehdi Behzad, with whom I wrote two papers [8, 9] jointly also with Professor Behzad's son Arash. The most interesting one, for me, was the paper [9] in which we discussed nine different fundamental domination parameters for a graph Γ. (A vertex/edge subset A *dominates* a graph Γ if each vertex/edge is either in A or adjacent to an element of A.) We interpreted these parameters in terms of the *total graph* $T(\Gamma)$ of Γ introduced by Professor Behzad, namely, the vertices of $T(\Gamma)$ are the vertices and edges of Γ, with two (vertices or edges) being adjacent in $T(\Gamma)$ if they are either adjacent or incident in Γ. We concluded that, arguably, the most fundamental of these parameters is the vertex-vertex domination parameter.

In addition, I spent hundreds of hours editing an English version of Professor Behzad's play "The Legend of the King and the Mathematician" [10]. Based on the puzzle of the Wolf, Sheep and Cabbage, the play is a wonderful initiative of Professor Behzad aimed at inspiring young people to enjoy and engage with the mathematical strategies behind the main story.

3 My work with younger Iranian colleagues

Dr Seyed Hassan Alavi worked with me and Dr John Bamberg on triple factorisations of groups of the form $G = ABA$ (for proper subgroups A, B). A surprising equivalence is that a triple factorisation is directly associated with a G-flag-transitive point-line incidence structure in which each point-pair is incident with at least one line. If the latter property holds we say that the geometry is *collinearly complete*. Part of Hassan's development, of a theory of these geometries, is his fundamental paper [11] which connects these geometries with primitive permutation groups, with restricted movement of point-subsets, and with flag-transitive symmetric designs. One very interesting class of examples arises for general linear groups: note that, for given collections of points and lines there are often several possible notions of incidence. In [12], Hassan identifies all possibilities for subspace actions, producing new collinearly complete geometries. He also find new examples when the points or lines are subspace bisections.

Theorem 3.1. *[12] Let $G = \mathrm{GL}(n, q)$, and $V = \mathrm{GF}(q)^n$, and consider the geometry with m-dimensional subspaces as 'points', k-dimensional subspaces as 'lines', and incidence between a 'point' and a 'line' when the intersection has dimension j. This geometry is collinearly complete if and only if* $\max\{0, m + k - n\} \leq j \leq \frac{k}{2} + \max\{0, m - \frac{n}{2}$.

Associate Professor Ashraf Daneshkhah worked with me and Associate Professor Alice Devillers in Perth on subdivision graphs $S(\Sigma)$ of a given graph Σ, that is, the graph obtained by 'adding a vertex' in the middle of each edge of Σ. Formally, the vertices of $S(\Sigma)$ are the vertices and edges of Σ, and edges of $S(\Sigma)$ are those vertex-edge pairs (α, e) such that the vertex α lies on the edge e. The paper [13] elucidates connections between various symmetry properties of Σ and of its subdivision graph $S(\Sigma)$, in particular local s-arc-transitivity, and local s-distance transitivity.

Theorem 3.2. *[13] Let Σ be a connected graph, s a positive integer, and $G \leq \mathrm{Aut}(\Sigma)$. Then $S(\Sigma)$ is locally (G, s)-arc transitive if and only if Σ is $(G, \lceil \frac{s+1}{2} \rceil)$-arc transitive. Moreover, provided Σ has diameter at least $\frac{s+1}{2}$, either of these conditions holds if and only $S(\Sigma)$ is locally (G, s)-distance transitive.*

Ashraf and Alice then extended this study further and obtained a complete classification of locally distance transitive subdivision graphs, which highlighted their connection with projective planes, generalised quadrangles and generalised hexagons.

Dr Moharram Iradmusa and I worked on a very interesting generalisation of Cayley graphs, called 2-*sided group digraphs*. Start with a group G and two subsets L, R of G. The corresponding 2-sided group digraphs $\overrightarrow{2S}(G; L, R)$ has vertex set G and an arc from a vertex x to a vertex y if and only if $y = \ell^{-1}xr$ for some $\ell \in L, r \in R$. Despite the similarities to Definition 1.3, these digraphs need not be vertex-transitive, and we give in [14, Example 2.1] a surprising example with 12 vertices, and with connected components of sizes 4 and 8 (see Figure 11). We also determine conditions under which $\overrightarrow{2S}(G; L, R)$ is a graph (that is, the joining relation is symmetric), and conditions for it to be connected, and to be a Cayley graph or digraph. We pose several open problems about these digraphs.

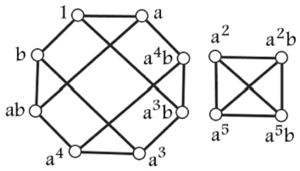

Figure 1: Disconnected two-sided group graph with non-isomorphic components

I have worked also with Dr Azizollah Azad on non-commuting graphs for general linear groups [15, 16], and with Dr Marzieh Akbari on codes in Hamming graphs. I thank all my Iranian colleagues for their great collaborations and their friendship.

Acknowledgement

I thank the University of Yazd for their generous support for my presence at this conference. I also thank the University of Western Australia for granting my sabbatical leave which has allowed me to visit Iran this year.

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On Laplacian eigenvalues of graphs

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Abstract

Let $G = (V, E)$ be a simple graph. Denote by $D(G)$ the diagonal matrix of its vertex degrees and by $A(G)$ its adjacency matrix. Then the Laplacian matrix of G is $L(G) = D(G) - A(G)$. Denote the spectrum of $L(G)$ by $S(L(G)) = (\mu_1, \mu_2, \ldots, \mu_n)$, where we assume the eigenvalues to be arranged in nonincreasing order: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. Let a be the algebraic connectivity of graph G. Then $a = \mu_{n-1}$. Among all eigenvalues of the Laplacian matrix of a graph, the most studied is the second smallest, called the algebraic connectivity (a(G)) of a graph [5]. In this talk we show some results on $\mu_1(G)$ and $a(G)$ of graph G. We obtain some integer and real Laplacian eigenvalues of graphs. Moreover, we discuss several relations between Laplacian eigenvalues and graph parameters. Finally, we give some conjectures on the Laplacian eigenvalues of graphs.

Keywords: Graph, Largest Laplacian eigenvalue, Algebraic connectivity, Diameter, Minimum degree

Mathematics Subject Classification [2010]: 05C50

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Partition of Unity Parametrics: A framework for meta-modeling in computer graphics

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In the past three decades, the field of Computer Graphics (CG) has experienced a revolution, benefiting from significant research and technical achievements. Creating detailed digital content is a major task in CG related industries such as Game, Film, GIS and CAD, and requires well-constructed, high quality geometric models. However, even with sophisticated software packages, geometric modeling is still a challenging and time consuming task. This challenge is due to the mathematical foundation of geometric models, our way of interacting with them, and more specifically, the augmenting of these geometric models with respect to their macro- and microscopic character. Therefore, geometric modeling - as a main pillar of CG - still requires evaluation to rectify foundation issues.

We present Partition of Unity Parametrics (PUPs), a natural and more flexible extension of NURBS (which are widely used in industry) that maintains affine invariance. NURBS inherit many useful properties from B-spline basis functions, and extend B-splines by allowing a scalar weight to be associated with each control point, indicating its relative importance to the curve. For these reasons NURBS have emerged as the predominant choice for modeling in computer graphics. Despite their widespread use, it is difficult to modify the characteristics of NURBS models. In practice, it is complex to toggle between sharp and smooth features, as well as to interpolate and approximate control points. Likewise, it is difficult to control the local character of curves and surfaces, and not possible to increase NURBS smoothness without increasing its support.

PUPs replace the weighted basis functions of NURBS with arbitrary weight-functions (WFs). By choosing appropriate WFs, PUPs yield a comprehensive geometric modeling framework, accounting for a variety of beneficial properties, such as local-support, specified smoothness, arbitrary sharp features and approximating or interpolating curves. This serves as a basis for metamodeling systems where users model the tools used for modeling (ie. weight functions) in tandem with the model itself. PUPs allow common geometric requirements and operations to be phrased succinctly, including: the addition of control points, arbitrary sharp features, increasing smoothness without increasing support, approximation and interpolation. For surfaces, PUPs permit non-tensor weight functions and allow control points to be added anywhere (without introducing other control points). This facilitates simple methods for sketching features and converting a planar mesh into a parametric surface of arbitrary smoothness.

As an important class of PUPs, we introduce CINAPCT-spline, based on bump-functions, which is C-infinity but with compact-support. The underlying weight functions

*Speaker

are similar in form to B-spline basis functions, and are parameterized by a degree-like shape parameter. We examine approximating and interpolating curves created using CINAPCT-spline. Furthermore, we propose and demonstrate a method to specify the tangents and higher order derivatives of the curve at control points for CINPACT and PUPs curves.

An eigenvalue problem

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Abstract

This talk is motivated by the following nonlinear Lorentz invariant wave equation:

$$\Box_2 u + \epsilon \Box_6 u - V'(u) = 0, \tag{1}$$

where

$$\Box_p u = \frac{\partial}{\partial t}\left[(c^2|\nabla u|^2 - |u_t|^2)^{p-2}u_t\right] - c^2\nabla\cdot\left[(c^2|\nabla u|^2 - |u_t|^2)^{p-2}\nabla u\right],$$

and V is an appropriate function. In the last equation, $u : \mathbb{R}^{3+1} \to \mathbb{R}^4$, $u = u(x,t)$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, ∇u denotes the Jacobian with respect to x, and u_t is the derivative with respect to t.

A static solution of (1) is a function $Z : \mathbb{R}^3 \to \mathbb{R}^4$ that satisfies

$$-c^2\Delta Z - \epsilon c^{10}\,\Delta_{10} Z - V'(Z) = 0, \tag{2}$$

where $\Delta_p = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$ is the well-known p-Laplace operator. The differential operator in (2) is a linear combination of Δ and Δ_{10}.

Here we are interested in a class of scalar equations similar to (2), in which the differential operator is a *convex* combination of $-\Delta_p$ and $-\Delta$. More precisely, we consider the eigenvalue problem

$$\begin{cases} -t\Delta_p u - (1-t)\Delta u = \lambda u & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \qquad (p \neq 2) \tag{3}$$

where $D \subseteq \mathbb{R}^n$ is a smooth bounded domain. We will show that the set of eigenvalues of (3) is continuous for $t \in (0,1]$. In fact, if λ_1 is the first eigenvalue of $-\Delta$, then we will prove the striking result that the spectrum of (3) is $((1-t)\lambda_1, \infty)$, even when t is very close to zero. This result is surprising because when t approaches zero the differential operator

$$\mathfrak{C}_t := -t\Delta_p - (1-t)\Delta$$

approaches $-\Delta$ and the expectation would be that when t is very near zero the spectrum $\sigma(\mathfrak{C}_t)$ of $\mathfrak{C}_t$ would be the union $\bigcup I_i$ of some intervals I_i each containing the i^{th}-eigenvalue of $-\Delta$. Recall that the spectrum of the Laplacian is a discrete set:

$$\sigma(-\Delta) = \{\lambda_j \mid j \in \mathbb{N}\} \text{ where } \lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \le \cdots \to \infty.$$

*Speaker

In other words, when the convex parameter t moves from 1 to 0 in the interval $[0,1]$, the spectrum $\sigma(\mathfrak{C}_t)$ will keep containing the interval $[\lambda_1,\infty)$ until t takes the exact value 0, in which case $\sigma(\mathfrak{C}_t)$ suddenly snaps into the discrete set $\sigma(-\Delta)$.

The eigenvalue problems of type (3) are new in the mathematics literature. Recently, the following eigenvalue problem was investigated:

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } D \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D, \end{cases}$$

where ν denotes the unit outward normal to the boundary ∂D. It was proved that the spectrum is $\{0\} \cup (\lambda_1^N, \infty)$, where λ_1^N denotes the first non-zero eigenvalue of $-\Delta$ with respect to the Neumann boundary condition. Our approach toward solving the eigenvalue problem (3) is different; our approach is based on the fibering method that was introduced in the early 1990's by the late S. Pohozaev. The fibering method is far more powerful than the Nehari-manifold method as it is applicable to a much broader range of boundary value problems than we discuss here. To help with a geometric intuition of the material, we introduce the δ-plane, which we denote by δ_π. This plane has two axes, the $-\Delta_p$-axis and the $-\Delta$-axis. The δ-plane is naturally equivalent to $\mathbb{R}^2$ in the sense that there exists a canonical map $\eta : \mathbb{R}^2 \to \delta_\pi$ as follows:

$$\eta(a,b) = -a\Delta_p - b\Delta.$$

In particular, we have

$$\mathfrak{C}_t = \eta(t, 1-t),$$

which is a *convex* combination of $-\Delta_p$ and $-\Delta$.[1]

The unit square S is the square with vertices at points $O = \eta(0,0)$, $A = \eta(1,0)$, $B = \eta(1,1)$, and $C = \eta(0,1)$. The main diagonal of S, joining $\eta(0,1)$ to $\eta(1,0)$, is what we are interested in.

The following is a summary of what is known about the spectrum of some of the operators in the δ-plane:

(i) $\sigma(\eta(0,1)) = \{\lambda_j \mid j \in \mathbb{N}\}$ in which $\lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \le \cdots \to \infty$, with respect to both Dirichlet and Neumann boundary conditions. In the latter case, $\lambda_1 = 0$ and $\lambda_2 < \lambda_3$.

(ii) $\sigma(\eta(1,1)) = \{0\} \cup (\lambda_1^N, \infty)$, with respect to the Neumann boundary conditions.

(iii) $\sigma(\eta(1,0)) = [0,\infty)$, provided that $p \in (\frac{2n}{n+2}, \infty) \setminus \{2\}$.

Note that every operator in the first quadrant of the δ-plane $\eta(\mathbb{R}_+ \times \mathbb{R}_+)$ is a translate of one in S. The same goes with those in the third quadrant, since $\eta(-a,-b) = -\eta(a,b)$. Hence it makes sense to focus on S in this talk. On the other hand, the operators in the second and the fourth quadrants need to be treated separately.

The main result of this presentation is the following:

Theorem 0.1. *Let $p \in (1,\infty) \setminus \{2\}$ and $t \in (0,1)$. Then the following hold:*

(i) If $\lambda \in [0, (1-t)\lambda_1]$, then $\lambda \notin \sigma(\mathfrak{C}_t)$.

(ii) If $\lambda \in ((1-t)\lambda_1, \infty)$, then $\lambda \in \sigma(\mathfrak{C}_t)$.

Here λ_1 denotes the first eigenvalue of $-\Delta$ with respect to the Dirichlet boundary conditions on ∂D.

[1] hence the use of the calligraphic 'C' with a 't' subscript in $\mathfrak{C}_t$.

We prove the theorem using variational methods. For this purpose we will consider an energy functional associated with (3), and prove that the critical points of this functional will give rise to non-trivial solutions of (3). The challenge is the parameter p. More precisely, for $p > 2$, the energy functional is coercive, hence the direct method applies. However, for the case $p < 2$, the lack of coercivity will render the direct method ineffective. Hence, we will apply the fibering method of Pohozaev.

We will derive *a priori* bounds and regularity results on the eigenfunctions. We will show that the behavior of the eigenfunctions are totally different between the case of $p \in (1, 2)$ and that of $p > 2$. More precisely, it turns out that when λ approaches the threshold $(1 - t)\lambda_1$, then

$$\begin{cases} \sup_D |u| \to 0, & (p > 2) \\ \sup_D |u| \to \infty, & (1 < p < 2). \end{cases}$$

Key Words: Lorentz invariant wave equation, continuous eigenvalues, Laplacian, p-Laplacian, fibering method, coercivity, existence, bounds, regularity.

MSC 2010: 81Q05, 35J60, 35P30

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Set-theoretic methods of homological algebra and their applications to module theory

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Abstract

We present some of the recent tools of set-theoretic homological algebra together with their applications, notably to the approximation theory of modules, and to (infinite dimensional) tilting.

Keywords: approximations of modules, set-theoretic homological algebra, infinite dimensional tilting theory
Mathematics Subject Classification [2010]: 16DXX, 18G25, 13D07, 03E75

1 Introduction

A major topic of module theory concerns existence and uniqueness of direct sum decompositions. Positive results provided by the Krull-Remark-Schmidt-Azumaya theorems, the Faith-Walker Theorem, and Kaplansky theorems, form the cornerstones of the classic theory. However, there are a number of important classes of (not necessarily finitely generated) modules to which the theory does not apply, because their modules do not decompose into (possibly infinite) direct sums of indecomposable, or small, submodules.

While such direct sum decompositions are rare, there do exist more general structural decompositions that are almost ubiquitous. The point is to replace direct sums by transfinite extensions. For example, taking direct sums of copies of the group $\mathbb{Z}_p$, one obtains all $\mathbb{Z}_p$-modules whose sole isomorphism invariant is the vector space dimension. In contrast, transfinite extensions of copies of $\mathbb{Z}_p$ yield the much richer class of all abelian p-groups whose isomorphism invariants are known basically only in the totally-projective case (the Ulm-Kaplansky invariants).

Starting with the solution of the Flat Cover Conjecture [5], numerous classes $\mathcal{C}$ of modules have been shown to be deconstructible, that is, expressible as transfinite extensions of small modules from $\mathcal{C}$. Basic tools for deconstruction come from set-theoretic homological algebra and originate in abelian group theory [6], but have since been expanded and generalized to module categories, and even beyond that setting.

Each deconstructible class is precovering, so it provides for approximations of modules. By choosing appropriately the class $\mathcal{C}$, one can tailor these approximations to the needs of various particular structural problems, cf. [12].

*Speaker

Approximations can also be employed in developing relative homological algebra in module categories. In the case when minimal approximations exist, ones obtains new invariants of modules, generalizing classic invariants such as the Betti numbers, or the (dual) Bass invariants, cf. [8]. Further applications in this direction involve model category structures associated to deconstructible classes in the setting of Grothendieck categories, such as the category of all unbounded chain complexes of modules, or the category of all quasi-coherent sheaves on a scheme. They yield new ways of computing cohomology of quasi-coherent sheaves via the approach of Quillen and Hovey, cf. [9], [11], [15].

But deconstructibility has its limits. This has first been observed by Eklof and Shelah [7] who proved that it is consistent with ZFC that the class of all Whitehead groups is not precovering. The latter fact, however, is not provable in ZFC, because it is also consistent that all Whitehead groups are free. More recent results show that non-deconstructibility is a phenomenon occuring in ZFC, and it is much more widespread than expected earlier. There is also a surprising connection to another important part of module theory: the tilting theory, [2], [14].

Our goal here is to explain these developments in more detail, and present some of the techniques of set-theoretic homological algebra and approximation theory of modules that have been developed over the past two decades. We will also consider several applications, notably to (infinite dimensional) tilting theory [1] and to representation theory [13].

2 Filtrations and approximations

2.1 Filtrations and the Hill Lemma

For an (associative, but not necessarily commutative) ring R with 1, we denote by Mod-R the category of all (unitary right R-) modules. Moreover, given an infinite cardinal κ and a class of modules $\mathcal{C}$, we will use the notation $\mathcal{C}^{<\kappa}$ to denote the subclass of $\mathcal{C}$ consisting of all modules possessing a projective resolution consisting of less than κ-generated projective modules. In particular, mod-R := (Mod-R)$^{<\omega}$ will denote the category of all *strongly finitely presented* modules, i.e, the modules possessing a projective resolution consisting of finitely generated projective modules.

Note that if R is right noetherian, then mod-R is just the category of all finitely generated modules, while if R is right coherent, then mod-R is the category of all finitely presented modules.

Definition 2.1. Let $\mathcal{C}$ be a class of modules. A module M is said to be $\mathcal{C}$*-filtered* (or a *transfinite extension* of the modules in $\mathcal{C}$), provided there exists an increasing chain $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ of submodules of M with the following properties: $M_0 = 0$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, $M_{\alpha+1}/M_\alpha \cong C_\alpha$ for some $C_\alpha \in \mathcal{C}$ for each $\alpha < \sigma$, and $M_\sigma = M$.

The chain $\mathcal{M}$ is called a $\mathcal{C}$*-filtration* of the module M of length σ. If σ is finite, then M is said to be finitely $\mathcal{C}$-filtered. The class of all $\mathcal{C}$-filtered modules will be denoted by Filt($\mathcal{C}$). We will say that $\mathcal{C}$ is *closed under transfinite extensions* provided that $\mathcal{C}$ =Filt($\mathcal{C}$).

For example, if $\mathcal{C}$ is the class of all simple modules, then Filt($\mathcal{C}$) is the class of all semiartinian modules, and finitely $\mathcal{C}$-filtered modules coincide with the modules of finite length.

As mentioned in the Introduction, given a class of modules $\mathcal{C}$ and $M \in \mathcal{C}$, it is rarely possible to decompose M into a direct sum of small, or indecomposable, modules from $\mathcal{C}$. Deconstructibility is much more feasible:

Definition 2.2. Let $\mathcal{C}$ be a class of modules and κ an infinite cardinal. Then $\mathcal{C}$ is κ-*deconstructible* provided that $\mathcal{C}$ =Filt($\mathcal{C}^{<\kappa}$). The class $\mathcal{C}$ is called *deconstructible*, if $\mathcal{C}$ is κ-deconstructible for some infinite cardinal κ.

For example, the class of all projective modules $\mathcal{P}_0$ is $\aleph_1$-deconstructible, because each projective module is a direct sum of countably generated projective modules by a classic theorem of Kaplansky. Let $n \geq 0$ and κ be an uncountable cardinal. If each right ideal of R is $< \kappa$-generated, then the class $\mathcal{P}_n$ of all modules of projective dimension at most n is κ-deconstructible. Similarly, if R has cardinality $< \kappa$, then the class $\mathcal{F}_n$ of all modules of flat dimension at most n is κ-deconstructible, [12].

A module equipped with a $\mathcal{C}$-filtration often possess many other $\mathcal{C}$-filtrations, and their lengths may vary in general. There is however a way to organize some of these $\mathcal{C}$-filtrations in a family that makes it possible to develop a sort of infinite dimensional Jordan-Hölder theory in this generality:

Lemma 2.3. (Hill Lemma) *Let R be a ring, M a module, κ a regular infinite cardinal, and $\mathcal{C}$ a class of $< \kappa$–presented modules. Let $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ be a $\mathcal{C}$-filtration of M.*

Then there exists a family $\mathcal{H}$ consisting of submodules of M such that (i) $\mathcal{M} \subseteq \mathcal{H}$, (ii) $\mathcal{H}$ forms a complete distributive sublattice of the complete modular lattice of all submodules of M, (iii) P/N is $\mathcal{C}$-filtered for all $N \subseteq P$ in $\mathcal{H}$, and (iv) if $N \in \mathcal{H}$ and S is a subset of M of cardinality $< \kappa$, then there is $P \in \mathcal{H}$ such that $N \cup S \subseteq P$ and P/N is $< \kappa$–presented.

Proof. (sketch) For each $\alpha < \sigma$ take an arbitrary $< \kappa$-generated submodule A_α of $M_{\alpha+1}$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$. (So $M_\alpha = \sum_{\beta<\alpha} A_\beta$ in particular.)

A subset $S \subseteq \sigma$ is called *closed* in case each $\alpha \in S$ satisfies $M_\alpha \cap A_\alpha \subseteq \sum_{\beta<\alpha,\beta\in S} M_\beta$. Define $\mathcal{H} = \{\sum_{\alpha\in S} A_\alpha \mid S \text{ closed }\}$. □

Hill Lemma makes it possible to replace a given $\mathcal{C}$-filtration of M by a different one fitting better the particular problem in case. We refer to [12, Chap.7] for various applications of the Hill Lemma. Here, we present only one (due to Enochs and Šťovíček) that makes it possible to replace any $\mathcal{C}$-filtration of M by a new filtration of (shorter) length $\leq \kappa$ on the account of making the consecutive factors of the new filtration thicker. (In the particular case when $\mathcal{C}$ = the class of all simple modules, an instance of the new filtration is provided by the socle sequence of a semiartinian module.)

Corollary 2.4. *In the setting of Lemma 2.3, let Sum($\mathcal{C}$) denote the class of all direct sums of copies of the modules from $\mathcal{C}$. Then M possesses a Sum($\mathcal{C}$)–filtration of length $\leq \kappa$.*

2.2 Approximations and complete cotorsion pairs

Definition 2.5. (i) A class of modules $\mathcal{A}$ is *precovering* if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through f:

$$\begin{array}{ccc} A & \xrightarrow{f} & M \\ {\scriptstyle g}\uparrow & \nearrow {\scriptstyle f'} & \\ A' & & \end{array}$$

The map f is called an $\mathcal{A}$*-precover* of M (or a *right* $\mathcal{A}$*-approximation* of M).

(ii) An $\mathcal{A}$-precover is *special* in case it is surjective, and its kernel K satisfies $\mathrm{Ext}^1_R(A, K) = 0$ for each $A \in \mathcal{A}$.

(iii) Let $\mathcal{A}$ be precovering. Assume that in the setting of (i), if $f' = f$ then each factorization g is an automorphism. Then f is an $\mathcal{A}$*-cover* of M. $\mathcal{A}$ is called a *covering* class in case each module has an $\mathcal{A}$-cover. We note that each covering class containing $\mathcal{P}_0$ and closed under extensions is necessarily special precovering.

For example, the class $\mathcal{P}_0$ is easily seen to be precovering, while $\mathcal{F}_0$ is covering by [5]. By a classic result of Bass, $\mathcal{P}_0$ is covering, iff $\mathcal{P}_0 = \mathcal{F}_0$, i.e., iff R is a right perfect ring.

Dually, we define *(special) preenveloping* and *enveloping* classes of modules. For example, $\mathcal{I}_0$, the class of all injective modules, is an enveloping class.

Precovering classes are ubiquitous because of the following

Theorem 2.6. *Let $\mathcal{S}$ be a set of modules and $\mathcal{C} = Filt(\mathcal{S})$. Then $\mathcal{C}$ is precovering. Moreover, if $\mathcal{C}$ is closed under direct limits, then $\mathcal{C}$ is covering.*

Example 2.7. The classes $\mathcal{P}_n$ $(n < \omega)$ for any ring R, as well as $\mathcal{GP}$, the class of all Gorenstein projective modules for R Iwanaga–Gorenstein, are special precovering. The classes $\mathcal{F}_n$ $(n < \omega)$ over any ring, and $\mathcal{GF}$ of all Gorenstein flat modules for R Iwanaga–Gorenstein, are covering. The classes $\mathcal{I}_n$ $(n < \omega)$ for any ring R (resp. $\mathcal{GI}$ for R Iwanaga–Gorenstein) are special preenveloping (resp. enveloping).

Precovering classes $\mathcal{C}$, and preenveloping classes $\mathcal{E}$, can be employed in developing relative homological algebra similarly as the classes of all projective and injective modules are used in the classic (absolute) case, cf. [8].

Besides the formal duality between the definitions of precovering and preenveloping classes, there is also an explicit duality discovered by Salce, mediated by complete cotorsion pairs:

Definition 2.8. Let R be a ring. A pair of classes of modules $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a (hereditary) *cotorsion pair* provided that

1. $\mathcal{A} = {}^{\perp}\mathcal{B} := \{A \in \text{Mod-}R \mid \mathrm{Ext}^i_R(A, B) = 0 \text{ for all } i \geq 1 \text{ and } B \in \mathcal{B}\}$, and
2. $\mathcal{B} = \mathcal{A}^{\perp} := \{B \in \text{Mod-}R \mid \mathrm{Ext}^i_R(A, B) = 0 \text{ for all } i \geq 1 \text{ and } A \in \mathcal{A}\}$.

If moreover 3. *For each module M, there exists an exact sequences $0 \to B \to A \to M \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$*, then $\mathfrak{C}$ is called *complete*.

Condition 3. implies that $\mathcal{A}$ is a special precovering class. In fact, 3. is equivalent to its dual: $3'$. *For each module M there is an exact sequence $0 \to M \to B \to A \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$*, which in turn implies that $\mathcal{B}$ is a special preenveloping class.

Complete cotorsion pairs, and hence special precovering and special preenveloping classes, are abundant:

Theorem 2.9. *For each set of modules $\mathcal{S}$, there is a complete cotorsion pair of the form $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ in Mod-R.*

3 Applications

3.1 Infinite dimensional tilting

For a module T, denote by Add(T) (resp. add(T)) the class of all direct summands of arbitrary (resp. finite) direct sums of copies of T.

Definition 3.1. A module T is *tilting* provided that

(T1) T has finite projective dimension.

(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i$ and all cardinals κ.

(T3) There exist $r < \omega$ and an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ where $T_i \in \mathrm{Add}(T)$ for each $i \leq r$.

The class $\mathcal{T}_T := T^{\perp}$ is the *tilting class*, and the cotorsion pair $\mathfrak{C}_T := ({}^{\perp}\mathcal{T}_t, \mathcal{T}_T)$ the *tilting cotorsion pair*, induced by T. If T has projective dimension $\leq n$, then the tilting module T is called *n-tilting*, and similarly for $\mathcal{T}_T$ and $\mathfrak{C}_T$. If T and T' are tilting modules, then T is *equivalent* to T' in case T and T' induce the same tilting class.

Strongly finitely presented tilting modules are called *classical*. A tilting module T is *good* provided that all the modules T_i in condition (T3) can be taken in add(T). We note that each classical tilting module is good, and each tilting module is equivalent to a good one.

Tilting theory originated in the realm finitely generated modules/representations of finite dimensional algebras, but many of its aspects extend to the general setting of possibly infinitely generated modules over arbitrary rings. Such extension is especially desired for commutative rings, because each finitely generated tilting module over a commutative ring is projective, that is, 0-tilting.

A classic result of Miyashita says that each classical n-tilting module induces (via the functors $\mathrm{Ext}^i_R(T, -)$ and $\mathrm{Tor}^S_i(-, T)$ for $i = 0, \ldots, n$) an $n+1$-tuple of category equivalences between certain subcategories of Mod-R and Mod-S where $S = \mathrm{End}(T_R)$. For $n = 0$, this is just the well known Morita equivalence between Mod-R and Mod-S. Miyashita's result has recently been extended to good n-tilting modules in [4].

Rather than studying equivalences induced by large tilting modules, we will consider here approximation properties of the corresponding tilting classes. The first result concerns 1-tilting and torsion classes of modules:

Proposition 3.2. *Let R be a ring and $\mathcal{T}$ be a torsion class of modules. Then $\mathcal{T}$ is 1-tilting, iff $\mathcal{T}$ is special preenveloping.*

A much more complex argument is needed to prove the following characterization of general tilting classes and tilting cotorsion pairs:

Theorem 3.3. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is tilting, iff $\mathcal{A} \subseteq \mathcal{P}_n$ for some $n < \omega$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

Even though tilting modules are allowed to be infinitely generated, there is always a grain of finiteness preserved. Indeed, the following result, proved by set-theoretic methods in a series of papers in 2005-7, says that each n-tilting class $\mathcal{T}$ is *of finite type*, that is, there exists a set $\mathcal{S}$ consisting of strongly finitely presented modules of projective dimension $\leq n$ such that $\mathcal{T} = \mathcal{S}^{\perp}$. In particular, $\mathcal{T}$ is axiomatizable, by a (possibly infinite) set of formulas of the language of the first order theory of modules:

Theorem 3.4. *Let R be a ring, T be an n-tilting module, and $\mathcal{T} = T^{\perp}$ the induced n-tilting class. Then $\mathcal{T}$ is of finite type.*

Theorem 3.4 makes it possible to classify tilting modules and classes over Dedekind domains, because finitely presented modules are classified in this case. Further tools are needed to handle the general commutative noetherian case. The main recent result from [1] offers the following classification. (A sequence $\mathcal{P} = (P_0, \dots, P_{n-1})$ consisting of subsets of the spectrum Spec(R) is called *characteristic* provided that $P_0 \subseteq P_1 \subseteq \dots \subseteq P_{n-1}$, and for each $i < n$, P_i is a lower subset of the poset $(\mathrm{Spec}(R), \subseteq)$ such that P_i contains all associated primes of the ith cosyzygy in the minimal injective coresolution of R.

Theorem 3.5. *Let R be a commutative noetherian ring and $n < \omega$. Then n-tilting classes are parametrized by characteristic sequences: the tilting class $\mathcal{T}$ corresponding to a characteristic sequence $\mathcal{P} = (P_0, \dots, P_{n-1})$ is defined by the formula*

$$\mathcal{T} = \{M \in \mathit{Mod}\text{-}R \mid \mathit{Tor}_i^R(M, R/p) = 0 \text{ for all } i < n \text{ and } p \in \mathit{Spec}(R) \setminus P_i\}.$$

3.2 Flat Mittag-Leffler modules and local freeness

Having defined tilting modules, we can now proceed to locally T-free modules:

Definition 3.6. Let R be a ring. A system $\mathcal{S}$ consisting of countably presented submodules of a module M is a *dense system* provided that $\mathcal{S}$ is closed under unions of well-ordered countable ascending chains, and each countable subset of M is contained in some $N \in \mathcal{S}$.

Let $\mathcal{F}$ be a set of countably presented modules. Denote by $\mathcal{C}$ the class of all modules possessing a countable $\mathcal{F}$-filtration. A module M is *locally $\mathcal{F}$-free* provided that M contains a dense system of submodules from $\mathcal{C}$. (Notice that if M is countably presented, then M is locally $\mathcal{F}$-free, iff $M \in \mathcal{C}$.)

If $\mathcal{F} = \mathcal{A}^{<\aleph_1}$ for a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, then $\mathcal{C} = \mathcal{A}^{<\aleph_1}$, and a module is locally $\mathcal{A}^{<\aleph_1}$-free, iff it admits a dense system of countably presented submodules from $\mathcal{A}$. In particular, if T is a tilting module with the induced tilting cotorsion pair $\mathfrak{C}_T = (\mathcal{A}, \mathcal{B})$, then the locally $\mathcal{A}^{<\aleph_1}$-free modules are called *locally T-free* modules.

For example, if $T = R$, then the locally T-free modules coincide with flat Mittag-Leffler modules, [10]. So in this particular case, the following theorem says that the class of all flat Mittag-Leffler modules is precovering, iff R is a right perfect ring:

Theorem 3.7. *[2] Let R be a ring and T be a tilting module. Then the class of all locally T-free modules is precovering, iff T is locally split (i.e., each pure embedding in $Add(T)$ splits).*

The proof of Theorem 3.7 uses the notion of a *tree module* M from [14], that is, of a module constructed by a particular decoration of the tree T_κ of all finite sequences of ordinals less that a given infinite cardinal κ. While the initial combinatorial object is T_κ, the initial algebraic object used for its decoration is a *Bass module*, i.e., a fixed countable direct limit B of the modules from $\mathcal{A}^{<\aleph_1}$. The key property of the tree module M is the fact that M contains a direct sum D of κ (= the number of nodes of T_κ) elements of $\mathcal{A}^{<\aleph_1}$, while M/D contains κ^ω (= the number of branches of T_κ) copies of the Bass module B.

3.3 Almost split morphisms

We finish with a rather surprising application of the tree module construction to solving a long-standing open problem from representation theory going back to Auslander.

Definition 3.8. Given a non-projective module N, an epimorphism of modules $f : M \to N$ is said to be *right almost split* provided that f is not split, and if $g : P \to N$ is not a split epimorphism, then g factorizes through f. Dually, we define a *left almost split monomorphism* $f' : N' \to M'$ for N' non-injective.

A short exact sequence of modules $0 \to N' \overset{f'}{\to} M \overset{f}{\to} N \to 0$ is *almost split* provided that it does not split, f is a right almost split epimorphism, and f' is a left almost split monomorphism.

Auslander proved that if N is an (indecomposable) finitely presented non-projective module with local endomorphism ring, then there always exists a right almost split epimorphism $f : M \to N$. This result is the basis of the celebrated Auslander-Reiten theory of almost split maps and sequences [3], with a number of far reaching consequences in the representation theory of algebras.

Already in 1977, Auslander asked, whether there are other cases where a right almost split epimorphism ending in a non-projective module N exists. Only recently, Šaroch was able to give a negative answer. The key ingredient in his proof employs generalized tree modules. (The term generalized refers to the fact that unlike the trees T_κ above, the generalized trees may have branches of length bigger than ω in order to capture also uncountable well-ordered direct limits of modules rather than just the Bass modules.)

Theorem 3.9. *[13] Let R be a ring and N be a non-projective module. Then there exists a right almost split epimorphism $f : M \to N$, iff N is finitely presented and its endomorphism ring is local.*

Theorem 3.9 has a corollary concerning the structure of almost split sequences in Mod-R:

Corollary 3.10. *[13] Let R be a ring and $0 \to N' \to M \to N \to 0$ an almost split sequence in Mod-R. Then N is finitely presented with local endomorphism ring, and N' is pure-injective.*

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Biological Networks

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Extended Abstract

The theory of complex networks has a wide range of applications in a variety of disciplines such as communications and power system engineering, the internet and worldwide web (www), food webs, human social networks, molecular biology, population biology and biological networks. The focus of this talk is on biological applications of the theory of graphs and networks. Network analysis leads to a better understanding of the critical role of these networks in many key questions.

we present some of the popular biological networks which have been investigated by several authors.

Protein-Protein Interaction network (PPI-Network) is a graph $G = (V, E)$ where V is a set of proteins and two proteins are joined by an edge if they interact physically. The interaction between viral proteins and human proteins can be represented as a bipartite graph G. The vertex set of G is $V_1 \cup V_2$, where V_1 is the set of viral proteins and V_2 is the set of all human proteins. A viral protein $v \in V_1$ is joined to a human protein $w \in V_2$ if v interacts with w. This bipartite graph is called viral-human protein interaction network and this network has been investigated by Mukhopadhyay and Maulik [2].

Human protein and disease association network is a bipartite graph G whose vertex is $V_1 \cup V_2$, where V_1 is the set of human proteins and V_2 is the set of diseases and $v_1 \in V_1$ is joined by an edge to $v_2 \in V_2$, if the human protein v_1 is associated with the disease v_2. This network has been investigated by Mukhopadhyay and Maulik [2].

Metabolome based reaction network is a directed graph $D = (V, A)$ where V is a set of metabolites and a vertex v is joined to a vertex w by an arc (v, w) if there is a reaction or interaction which transforms the metabolite v to the metabolite w. This network has been investigated by Veeky Baths et al. [4].

Gene regulation is a general term for cellular control of the synthesis of protein at the transcription step. Often one gene is regulated by another gene via the corresponding protein. Thus gene regulation leads to the concept of gene regulatory network, which has been investigated by Yue and Chunmei [5]. Gene regulatory network is a directed graph $D = (V, A)$ where V is the set of genes and two genes $g_1, g_2 \in V$ are joined by an arc if there is a regulatory relationship between g_1 and g_2, or more precisely g_1 regulates g_2.

*Speaker

The regulatory relationship between two genes may be either positive direct regulatory influence or inverse causality or no correlation. Hence gene regulatory network can also be represented as a directed weighted graph, where the weight of an arc is an estimate of the probability of relationship between the genes in the network. This network has been investigated by Raza and Jaiswal [3]. Positive regulatory relationship represents activation and negative regulatory relationship represents inhibition. This leads to the representation of a gene regulatory network as a signed directed graph where an arc (g_1, g_2) is assigned a positive sign if the corresponding regulatory relationship is activation and is assigned a negative sign if the corresponding relationship is inhibition. A study of gene regulatory network leads to a better understanding of the regularity mechanism of the genes and prediction of the behavior of some unknown genes.this network has been studied in Christensen et al. [1].

There are several centrality measures such as Stress, Betweenness, Edge betweenness, Diameter, Average distance, Closeness, Eigenvector Centrality and Eccentricity which are used for analyzing biological networks.

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Covering properties defined by stars

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Abstract

We discuss covering properties in topological spaces defined by stars. Special attention is paid to two star covering properties related to the Gerlits-Nagy property GN. Some examples in this connection are given.

Keywords: Star selection principles, star-GN, strongly star-GN
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1 Introduction

If A is a subset of a topological space X, and $\mathcal{P}$ is a family of subsets of X, then $\mathrm{St}(A,\mathcal{P}) := \cup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$; when $A = \{x\}$, $x \in X$, one writes $\mathrm{St}(x,\mathcal{P})$ instead of $\mathrm{St}(\{x\},\mathcal{P})$. In the literature one can find a big number of topological properties which are defined or characterized in terms of stars. In particular, it is the case with many covering properties of topological spaces. We consider here an application of this method in the theory of star selection principles introduced in [4]. For more details on star selection principles and for undefined notions see the survey paper [5].

Selection Principles Theory has roots in the papers by Menger [7], Hurewicz [3], Rothberger [9], but in the last two-three decades a big number of mathematicians work systematically in this field of mathematics.

Following [4] and [5] we have the following definitions.

Let $\mathcal{O}$ be the collection of all open covers of a space X, $\mathcal{B}$ a subfamily of $\mathcal{O}$, and $\mathcal{K}$ a family of subsets of X. Then:

1. The symbol $\mathsf{S}^*_{fin}(\mathcal{O},\mathcal{B})$ denotes the selection hypothesis: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{O}$ there is a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\{\mathrm{St}(\cup\mathcal{V}_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;
2. $\mathsf{S}^*_1(\mathcal{O},\mathcal{B})$ denotes the selection hypothesis: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{O}$ there is a sequence $\langle U_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{\mathrm{St}(U_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;
3. $\mathsf{SS}^*_{\mathcal{K}}(\mathcal{O},\mathcal{B})$ denotes the following selection hypothesis: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{O}$ there exists a sequence $\langle K_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{K}$ such that $\{\mathrm{St}(K_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.
When $\mathcal{K}$ is the collection of all finite (resp. one-point, compact) subspaces of X we write $\mathsf{SS}^*_{fin}(\mathcal{O},\mathcal{B})$ (resp., $\mathsf{SS}^*_1(\mathcal{O},\mathcal{B})$, $\mathsf{SS}^*_{\mathrm{K}}(\mathcal{O},\mathcal{B})$) instead of $\mathsf{SS}^*_{\mathcal{K}}(\mathcal{O},\mathcal{B})$.

Let Γ denotes the collection of γ-covers of a space X. (An open cover $\mathcal{U}$ of X is a *γ-cover* if for each $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.) Let X be a space. The following terminology and notation (for X) we borrow from the above mentioned papers.

SR: the *star-Rothberger property* $= \mathsf{S}_1^*(\mathcal{O}, \mathcal{O})$;
SSR: the *strongly star-Rothberger property* $= \mathsf{SS}_1^*(\mathcal{O}, \mathcal{O})$;
SH: the *star-Hurewicz property* $= \mathsf{S}_{fin}^*(\mathcal{O}, \Gamma)$;
SSH: the *strongly star-Hurewicz property* $= \mathsf{SS}_{fin}^*(\mathcal{O}, \Gamma)$.

In [2], Gerlits and Nagy introduced several covering properties of a topological spaces. One of these properties, denoted $(*)$ and nowadays called the *Gerlits-Nagy property* (or GN-*property* for short), has been characterized in [8] in a form more convenient for use: a space X is Gerlits-Nagy if and only if it is Hurewicz and Rothberger. Other characterizations of GN properrty were obtained in [6]. One of these characterizations is: a space X is GN if and only if it satisfies the selection property $\mathsf{S}_1(\mathcal{O}, \mathcal{O}^{gp})$. Here, $\mathcal{O}^{\mathrm{gp}}$ denotes the family of groupable open covers of X: an open cover $\mathcal{U}$ of X is *groupable* if it can be represented in the form $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, so that $\mathcal{U}_n$'s are finite, pairwise disjoint, and each x belongs to all but finitely many $\cup\mathcal{U}_n$.

Following the first of these two results we introduce the following definition.

Definition 1.1. A space X is said to be:

1. *star-Gerlits-Nagy*, denoted $X \in \mathcal{C}_{\mathsf{SGN}}$, if X is SH and SR;
2. *strongly star-Gerlits-Nagy*, denoted $X \in \mathcal{C}_{\mathsf{SSGN}}$, if X is SSH and SSR.

2 Main results

We need also the following known uncountable small cardinal

$$\mathsf{add}(\mathcal{M}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{M} \ \& \ \cup\mathcal{F} \notin \mathcal{M}\},$$

where $\mathcal{M}$ is the ideal of meager subsets of $\mathbb{R}$.

Recall a known topological construction. A family of infinite subsets of $\mathbb{N}$ is *almost disjoint* if the intersection of any two distinct elements is finite. For an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$, set $\Psi(\mathcal{A}) = \mathbb{N}\cup\mathcal{A}$. Topologize $\Psi(\mathcal{A})$ so that the points of $\mathbb{N}$ are isolated and a basic neighbourhood of a point $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$, where F is a finite set in $\mathbb{N}$.

Theorem 2.1. *If* $|\mathcal{A}| < \mathsf{add}(\mathcal{M})$*, then* $\Psi(\mathcal{A}) \in \mathcal{C}_{\mathsf{SSGN}}$.

Proof. Matveev proved: (a) $\Psi(\mathcal{A})$ is SSH if and only if $|\mathcal{A}| < \mathfrak{b}$; (b) if $|\mathcal{A}| < \mathsf{cov}(\mathcal{M})$, then $\Psi(\mathcal{A})$ is SSR (see [5]). Combining these results with the Miller-Truss theorem (see [1]) saying that $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})$ we have the proof of the theorem. □

We do not know if the converse of this theorem true.

Theorem 2.2. *There is a Tychonoff space which is in* $\mathcal{C}_{\mathrm{SGN}}$ *but is not in* $\mathcal{C}_{\mathrm{SSGN}}$ *(in fact, it is not neither* SSR *nor* SSH*).*

Proof. Let $\alpha D(\mathfrak{c}) = D(\mathfrak{c}) \cup \{\infty\}$ be the one-point compactification of the discrete space $D(\mathfrak{c})$ of cardinality $\mathfrak{c}$. Set $Y = \alpha D(\mathfrak{c}) \times [0, \mathfrak{c}^+)$, $Z = D(\mathfrak{c}) \times \{\mathfrak{c}^+\}$. Endow $X = Y \cup Z$ with the relative topology of the product $\alpha D(\mathfrak{c}) \times [0, \mathfrak{c}^+]$.

Claim 1. X is SH

Let $\mathcal{U}$ be an open cover of X. Since $\alpha D(\mathfrak{c}) \times [0, \mathfrak{c}^+)$ is countably compact (the product of a compact and a countably compact space), there is a finite set $F \subset X$ such that $\mathrm{St}(F;\mathcal{U}) \supset \alpha D(\mathfrak{c}) \times [0, \mathfrak{c}^+)$. For each $\alpha \in D(\mathfrak{c})$ one can choose $U_\alpha \in \mathcal{U}$ such that $(\alpha, \mathfrak{c}^+) \in U$. Take $x_\alpha = (\alpha, \beta_\alpha) \in U_\alpha \setminus \{(\alpha, \mathfrak{c}^+)\}$. Let $\beta = \sup\{\beta_\alpha : \alpha \in D(\mathfrak{c})\}$. Then $\beta < \mathfrak{c}^+$, because $\mathfrak{c}^+$ is regular. The set $K = \mathrm{Cl}_{D(\alpha)\times[0,\beta]}\{x_\alpha : \alpha \in D(\mathfrak{c})\}$ is compact, and thus there exists a finite set $E \subset X$ such that $\mathrm{St}(E,\mathcal{U}) \supset K$. The set $A = F \cup E$ is finite and $\mathsf{St}(A,\mathcal{U}) = X$.

Claim 2. X is SR.

It is known that every ordinal space $[0, \alpha)$ is SSR, hence SR. Therefore, Z is SR.

Let $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ be a sequence of open covers of X. For every $\alpha < \mathfrak{c}$ take β_α having property $\{\alpha\} \times [\beta_\alpha, \mathfrak{c}^+] \subset V$ for some $V \in \mathcal{U}_1$. Let $\beta = \sup\{\beta_\alpha : \alpha < \mathfrak{c}\}$, and let $(\infty, \mathfrak{c}^+) \in U_1 \in \mathcal{U}_1$. The set $\mathrm{St}(U_1, \mathcal{U}_1)$ contains all but finitely many elements $x_\alpha = (\alpha, \mathfrak{c}^+)$, $\alpha < \mathfrak{c}$, say $x_{\alpha_2}, \cdots, x_{\alpha_m}$. For each $i = 2, \cdots m$ pick an element $U_i \in \mathcal{U}_i$ such that $x_{\alpha_i} \in U_i$, and any $U_j \in \mathcal{U}_j$ for $j > m$. Then the sequence $\langle U_n : n \in \mathbb{N}\rangle$ witnesses for $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ that $Y \subset \mathrm{St}(U_n, \mathcal{U}_n)$. This implies that $X = Y \cup Z$ is SR.

It follows from Claims 1 and 2 that $X \in \mathcal{C}_{\mathrm{SGN}}$.

Claim 3. X is not in $\mathcal{C}_{\mathrm{SSGN}}$.

It is enough to prove that X is not SSR. For each n let $\mathcal{U}_n = \mathcal{U} = \{\alpha D(\mathfrak{c}) \times [0, \mathfrak{c}^+)\} \cup \{\{\alpha\} \times [0, \mathfrak{c}^+] : \alpha \in D(\mathfrak{c})\}$. Then we have a sequence of open covers $\mathcal{U}_n$, $n \in \mathbb{N}$, of X. Suppose that we have chosen an element $x_n \in X$ for each $n \in \mathbb{N}$. Set $A = \{x_n : n \in \mathbb{N}\}$. We prove that $\mathrm{St}(A, \mathcal{U}) \neq X$. Let π be the projection of X onto $\alpha D(\mathfrak{c})$. As $\pi(A)$ is countable, there is a point $u \in X \setminus \pi(A)$. Then, as it is easily checked, $(u, \mathfrak{c}^+) \notin \mathrm{St}(A, \mathcal{U})$, hence X is not SSR.

This completes the proof of the theorem. □

Remark 2.3. The product of a compact SSGN spaces X and a compact space Y need not be SSGN. Take X to be a compact Rothberger space. It is well known that a compact space is Rothberger if and only if it is scattered (i.e. each nonempty subspace has an isolated point). Further, in the class of (para)compact spaces the Rothberger property coincides with the SSR property [4], so that X is SSR. On the other hand, since X is compact, it is Hurewicz, hence strongly star-Hurewicz. Therefore, X is an SSGN space.

Let Y be a a non-scattered compact space. Then $X \times Y$ is not SSGN space. Suppose to the contrary, that $X \times Y \in \mathsf{SSGN}$. By the results mentioned above $X \times Y$ must be scattered, being compact and Rothberger. By the fact that a compact space which is a continuous image of a compact scattered space is also (compact) scattered, it would follow that Y is scattered. A contradiction.

Theorem 2.4. *There is a space $X \in \mathcal{C}_{\mathsf{SSGN}}$ and a Lindelöf space Y such that $X \times Y$ is not in $\mathcal{C}_{\mathsf{SGN}}$.*

Proof. Let $X = [0; \omega_1)$ with the usual order topology and Y the one-point Lindelöfication of X (i.e. $Y = [0; \omega_1]$ with the following topology: each point α with $\alpha < \omega_1$ is isolated, and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable). Then X is countably compact, Y is Lindelöf, and $X \times Y$ is not in this class, even not in the class $\mathcal{C}_{\mathsf{SGN}}$.

The space X is SSR because every ordinal space is SSR. On the other hand, X is SSH being (Hausdorff) countably compact and so strongly starcompact. Therefore, $X \in \mathcal{C}_{\mathsf{SSGN}}$. According to [5], the product $X \times Y$ is not SH, hence $X \times Y$ is not in the class $\mathcal{C}_{\mathsf{SGN}}$. □

The following result regarding SSH spaces (see [5])

Theorem 2.5. *A space X is* SSH *if and only if* $X \in \mathsf{SS}^*_{\rm fin}(\mathcal{O}, \mathcal{O}^{gp})$

suggests the following

Problem. Is it true that $\mathsf{S}^*_{\rm fin}(\mathcal{O}, \Gamma) = \mathsf{S}^*_{\rm fin}(\mathcal{O}, \mathcal{O}^{gp})$? Is it true that $X \in \mathcal{C}_{\rm SSGN}$ if and only if $\mathsf{SS}^*_1(\mathcal{O}, \mathcal{O}^{gp})$ if and only if $\mathsf{SS}^*_1(\mathcal{O}, \Gamma)$?

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Some question on the reduction of elliptic curves

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Abstract

An elliptic curve E over the rationals gives, in a natural way, a family of elliptic curves over finite fields simply considering the reduction Ep of the curve modulo prime numbers. And many interesting question arises regarding this family. For example, one could ask for the number of primes up to X so that Ep has a prime number of points, and try to solve an open problem stated long back by Koblitz. Recall that this question has a direct interest in building elliptic curves interesting for cryptographic purposes. Another problems related with this family are the famous Sato-Tate conjecture, or the Lang-Trotter conjectures on the trace of the Frobenius element and the Frobenius ring. In the talk, after a review of the ingredients, i will talk about some contributions that i could do, on these problems.

Nonlinear Separation for Constrained Optimization

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Abstract

We give a brief survey of image space analysis and its applications to constrained optimization problems. By introducing some class of nonlinear separation functions in the image space associated with an infinite analysis, we investigate con constrained optimization problems. Furthermore, the equivalence between the existence of nonlinear separation function and a saddle point condition for a generalized Lagrangian function associated with the given problem is obtained. Some open problems for the vector variational inequalities with constraints are mentioned.

Keywords: Nonlinear separation for Image space analysis, Scalarization of vector optimization, Generalized Lagrangian function, Exact penalty
Mathematics Subject Classification [2010]: 90C26, 90C29, 26B25, 49J40

1 Introduction

The image space analysis(ISA) approach has been proved to be a fruitful method in many topics of optimization theory (e.g., optimality condition, existence of solution, duality, vector variational inequalities and vector equilibrium problems); see [1-13] and [18-20]. Moreover, it has been shown that several theoretical aspects of a constrained extremum problem as duality , penalty methods , regularity and Lagrangian- type optimality can be developed by Image space Analysis .
Furthermore, (ISA) has received considerable attention in the optimization community and has become a powerful tool and a unifying scheme for studying constrained optimization problems . In the (ISA) method, the optimality condition for constrained optimization problems is expressed under the form of the impossibility of a parametric system. The impossibility of such a system is reduced to the disjunction of two suitable subsets of the image space (IS) associated with the given problem; such a disjunction can be proved by showing that they lie in two disjoint level sets of a nonlinear separation function (see[11]). Here, we focus our attentions on some nonlinear separation functions for the constrained extremum problem. We extend a nonlinear regular weak separation function that has been discussed in [12], to use in set-valued optimization in normed linear spaces. Then, we define two new nonlinear (regular) weak separation functions based on the oriented distance function $\triangle$ and derive some optimality conditions, in particular, some saddle point sufficient optimality conditions for the constrained extremum problem.
Let X be a topological vector space and let Y and Z be two normed linear spaces with

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normed dual spaces Y^* and Z^*, respectively. Let $C \subset Y$ and $D \subset Z$ be pointed, closed and convex cones with nonempty interiors. The space of continuous linear operators from Z to Y is denoted by $L(Z, Y)$ and

$$L_+(Z, Y) := \{T \in L(Z, Y) : \quad T(D) \subseteq C\}.$$

The positive dual cone of C is defined by

$$C^+ := \{p \in Y^* : p(y) \geq 0, \ \forall y \in C\},$$

and the set of all positive linear functionals in C^+ is

$$C^{+i} := \{p \in Y^* : p(y) > 0, \ \forall y \in C \setminus \{0\}\}.$$

Note that, if C is a convex cone in Y, then int $C^+ \subseteq C^{+i}$ and the equality holds if int $C^+ \neq \emptyset$. A partial order $\leq_C$ in Y is defined by

$$y_1 \leq_C y_2 \quad \Leftrightarrow \quad y_2 - y_1 \in C, \quad \forall y_1, y_2 \in Y.$$

For simplicity, throughout this talk, we denote $\mathring{C} :=$ int C and $C_0 := C \setminus \{0\}$.
In the sequel, we suppose that $F : U \rightrightarrows Y$ is a multifunction defined on a nonempty convex subset U of X with values in Y.

Definition 1.1. Let $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ be two multifunctions with nonempty values. We consider the following vector optimization problem:

$$\min_C F(x) \quad s.t. \qquad x \in R := \{x \in U : G(x) \cap (-D) \neq \emptyset\}, \tag{1}$$

where R is called the feasible region of Problem (1).

Definition 1.2. A point $\bar{x} \in R$ is called a minimum point of Problem (1) iff

$$\exists \bar{y} \in F(\bar{x}) \quad s.t. \quad (F(R)) \cap (\bar{y} - C_0) = \emptyset.$$

In this case we say that $(\bar{x}, \bar{y})$ is a minimizer for Problem (1) and a point $\bar{x} \in R$ is called a weak minimum point of Problem (1) iff

$$\exists \bar{y} \in F(\bar{x}) \quad \text{s.t.} \quad (F(R)) \cap (\bar{y} - \mathring{C}) = \emptyset.$$

In this case we say that $(\bar{x}, \bar{y})$ is a weak minimizer for Problem (1).

The following result presents a necessary and sufficient condition for a vector to be a minimum point or a weak minimum point of Problem (1).

Lemma 1.3. *[17] Let $\bar{x} \in R$ and $(\bar{x}, \bar{y}) \in$ gr F. Then*

(i) $\quad (\bar{x}, \bar{y})$ *is a minimizer of Problem (1) iff*

$$(\bar{y} - C_0, -D) \cap (F(x), G(x)) = \emptyset \quad \forall x \in U.$$

(ii) $(\bar{x}, \bar{y})$ *is a weak minimizer of problem (1) iff*

$$(\bar{y} - \mathring{C}, -D) \cap (F(x), G(x)) = \emptyset \quad \forall x \in U.$$

Here, we develop the image space analysis for vector optimization with multifunction constraints and multifunction objective. Let $\bar{x} \in R$ and $\bar{p} := (\bar{x}, \bar{y}) \in \text{gr } F$. We introduce the multifunction $A_{\bar{p}} : U \rightrightarrows Y \times Z$, defined by

$$A_{\bar{p}}(x) := \{(\bar{y} - y, -z) : \ y \in F(x) \ , \ z \in G(x), \ x \in U\},$$

and we associate the following sets to $\bar{p} \in \text{gr } F$

$$\mathcal{H} = C_0 \times D \quad , \quad \mathcal{K}_{\bar{p}} = A_{\bar{p}}(U).$$

The set $\mathcal{K}_{\bar{p}}$ is called the image space associated with Problem (1). By Lemma 1.3, $\bar{p} = (\bar{x}, \bar{y})$ is a minimizer of Problem (1) iff

$$\mathcal{K}_{\bar{p}} \cap \mathcal{H} = \emptyset. \tag{2}$$

and $\bar{p} = (\bar{x}, \bar{y})$ is a weak minimizer of Problem (1) iff

$$\mathcal{K}_{\bar{p}} \cap \mathcal{H}_{ic} = \emptyset,$$

where, $\mathcal{H}_{ic} = \mathring{C} \times D$.

Definition 1.4. Let Γ be a set of parameters and $\mathcal{H} = C_0 \times D$.The class of all the functions $\omega : Y \times Z \times \times Y^* \times \Gamma \longrightarrow \mathbb{R}$ such that

$$\mathcal{H} \subseteq \text{lev}_{\geq 0} \ \ \omega(.,.,.,\gamma), \qquad \forall \gamma \in \Gamma, \tag{3}$$

and

$$\bigcap\nolimits_{\gamma \in \Gamma} \text{lev}_{>0} \ \ \omega(.,.,.,\gamma) \subseteq \mathcal{H}, \tag{4}$$

is called the class of weak separation functions and is denoted by $\mathcal{W}(\Gamma)$, in which $\text{lev}_{>0} \ \omega(.,.,\bar{\theta},\bar{\gamma}) := \{(u, v) \in Y \times Z : \omega(u, v, \bar{\theta}, \bar{\gamma}) > 0\}$ denotes the level set of $\omega(.,.,\bar{\theta},\bar{\gamma})$.

Definition 1.5. The class of all the functions $\omega : Y \times Z \times \times Y^* \times \Gamma \longrightarrow \mathbb{R}$, such that

$$\bigcap\nolimits_{\gamma \in \Gamma} \text{lev}_{>0} \ \ \omega(.,.,.,\gamma) = \mathcal{H}, \tag{5}$$

is called the class of regular weak separation functions and is denoted by $\mathbb{W}_r(\Gamma)$.

Suppose that Γ is the given set of parameters and the class of functions $\omega_1 : Y \times Z \times Y^* \times \Gamma \mapsto \mathbb{R}$ is given by:

$$\omega_1(u, v, \theta, \gamma) := \langle \theta, u \rangle + \omega_0(v, \gamma).$$

where ω_0 fulfils the following conditions

$$\forall \gamma \in \Gamma, \ \ \forall \alpha \in \mathbb{R}_+, \ \ \exists \gamma_\alpha \in \Gamma \ \ s.t. \ \ \alpha\omega_0(v, \gamma) = \omega_0(v, \gamma_\alpha) \ \ \forall v \in Z. \tag{6}$$

$$\bigcap\nolimits_{\gamma \in \Gamma} \text{lev}_{\geq 0} \ \ \omega_0(., \gamma) = D. \tag{7}$$

In sequel, we consider the following assumptions

$$\inf_{\gamma\in\Gamma} \omega_0(v,\gamma) = -\infty \quad \forall v \notin D. \tag{8}$$

$$\inf_{\gamma\in\Gamma} \omega_0(v,\gamma) = 0 \quad \forall v \in D. \tag{9}$$

Definition 1.6. Suppose that $A \subseteq Y$ and $d_A(y) = \inf\{\|a-y\| : a \in A\}$ is the distance function from A. The function $\triangle_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\triangle_A(y) = d_A(y) - d_{Y\setminus A}(y),$$

is called the oriented distance function.

Now by the oriented distance function $\triangle$, we consider the nonlinear class of functions $\omega_2 : Y \times Z \times \Gamma \mapsto \mathbb{R}$ given by:

$$\omega_2(u,v,\gamma) := -\triangle_C(u) + \omega_0(v,\gamma).$$

The class of separation ω_1 and ω_2 are unified the following known linear or nonlinear separation functions; see [1, 15, 16]:

(i) $\omega_3(u,v,\theta,\gamma) := \langle\theta,u\rangle + \langle\gamma,v\rangle$,

(ii) $\omega_4(u,v,\theta,\gamma) := \langle\theta,u\rangle - \triangle_{\mathbb{R}_+}(\langle\gamma,v\rangle)$,

(iii) $\omega_5(u,v,\theta,\gamma) := \langle\theta,u\rangle - \gamma d_D(v)$,

(iv) $\omega_6(u,v,\theta) := \langle\theta,u\rangle - \delta_D(v)$, where, δ_D is indicator function of D.

(v) $\omega_7(u,v,\gamma) := -\triangle_C(u) + \langle\gamma,v\rangle$,

(vi) $\omega_8(u,v) := -\triangle_C(u) - \delta_D(v)$,

(vii) $\omega_9(u,v,\theta,\gamma) := \langle\theta,u\rangle - \triangle_C(Tv)$, where, $T \in L_+(Z,Y)$

2 Main results

Here, we obtain first some results for minimizing of Problem (1).

Proposition 2.1. *(a)- Let $\bar{x} \in R$, $\bar{p} = (\bar{x},\bar{y}) \in \operatorname{gr} F$. Let $\omega_1(u,v,\bar{\theta},\gamma) := \langle\bar{\theta},u\rangle + \omega_0(v,\gamma)$, be a class of regular nonlinear separation functions satisfying both conditions (8) and (9). If,*

$$\inf_{\gamma\in D^+} \sup_{(u,v)\in\mathcal{K}_{\bar{p}}} \omega_1(u,v,\bar{\theta},\gamma) \ \le 0,$$

then, $\bar{p}$ is a minimizer of Problem (1).
(b)- If $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ admit the following regular nonlinear separation functions

$$\omega_2(u,v,\bar{\gamma}) := -\triangle_{\mathcal{C}}(u) + \omega_0(v,\bar{\gamma}),$$

then $\bar{p}$ is a minimizer of Problem (1).
(c)- Let $\omega_2(u,v,\gamma) := -\triangle_{\mathcal{C}}(u) + \omega_0(v,\gamma)$, be a class of nonlinear separation functions satisfying both conditions (8) and (9). If for each $z \in G(x) \cap (-D)$,

$$\inf_{\gamma \in D^+} \sup_{\{y \in F(x): x \in R\}} \omega_2(\bar{y} - y, -z, \gamma) < 0,$$

then, $\bar{p}$ is a minimizer of Problem (1).

The next results shows that the existence of a nonlinear separation between $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ is equivalent to the existence of a saddle point for the generalized Lagrangian.

Theorem 2.2. *Let $\bar{p} = (\bar{x}, \bar{y}) \in \mathrm{gr}\ F$, and $\omega_1(u,v,\theta,\gamma) := \langle \theta, u \rangle + \omega_0(v,\gamma)$ be the class of nonlinear functions satisfying conditions (8) and (9).*

(i) If $(\bar{x}, \bar{\gamma})$ is a saddle point for the generalized Lagrangian function $\mathcal{L}_1 : U \times C^+ \times \Gamma \mapsto \mathbb{R}$ defined by

$$\mathcal{L}_1(x,\theta,\gamma) = \inf_{y \in F(x)} \langle \theta, y \rangle - \sup_{z \in G(x)} \omega_0(-z,\gamma),$$

where F is compact valued, i.e.

$$\mathcal{L}_1(\bar{x},\bar{\theta},\gamma) \leq \mathcal{L}_1(\bar{x},\bar{\theta},\bar{\gamma}) \leq \mathcal{L}_1(x,\bar{\theta},\bar{\gamma}), \quad \forall x \in U,\ \forall \gamma \in \Gamma,$$

for a fixed $\bar{\theta} \in C^*$ then, $\bar{x} \in R$ and $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, admit a nonlinear separation;

(ii) Suppose that $F(\bar{x}) \subseteq \{\bar{y}\} + C$, and there exists $(\bar{\theta}, \bar{\gamma}) \in C^* \times \Gamma$ which admits a nonlinear separation for $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, then $(\bar{x}, \bar{\gamma})$ is a saddle point for the generalized Lagrangian function ,i.e.

$$\mathcal{L}_1(\bar{x},\bar{\theta},\gamma) \leq \mathcal{L}_1(\bar{x},\bar{\theta},\bar{\gamma}) \leq \mathcal{L}_1(x,\bar{\theta},\bar{\gamma}), \quad \forall x \in U,\ \forall \gamma \in \Gamma,$$

Theorem 2.3. *Let $\bar{p} = (\bar{x}, \bar{y}) \in \mathrm{gr}\ F$, and $\omega_2(u,v,\gamma) := -\triangle_{\mathcal{C}}(u) + \omega_0(v,\gamma)$ be the class of functions satisfying two conditions (8) and (9).*

(i) If $\bar{x} \in R$, $F(\bar{x}) \subseteq \{\bar{y}\} + C$ and $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, admit a (regular) nonlinear separation then, $(\bar{x}, \bar{\gamma})$ is a saddle point for the generalized Lagrangian function $\mathcal{L}_2 : U \times \Gamma \mapsto \mathbb{R}$ defined by

$$\mathcal{L}_2(x,\gamma) = \inf_{y \in F(x)} \triangle_{\mathcal{C}}(\bar{y} - y) - \sup_{z \in G(x)} \omega_0(-z,\gamma),$$

where F is compact valued, i.e.

$$\mathcal{L}_2(\bar{x},\gamma) \leq \mathcal{L}_2(\bar{x},\bar{\gamma}) \leq \mathcal{L}_2(x,\bar{\gamma}), \quad \forall x \in U,\ \forall \gamma \in \Gamma.$$

(ii) Suppose that $F(\bar{x}) \subseteq \{\bar{y}\} + \overset{\circ}{C}$, and $(\bar{x}, \bar{\gamma})$ is a saddle point for the generalized Lagrangian function $\mathcal{L}_2$ then, $\bar{x} \in R$ and $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$, admit a regular nonlinear separation

In the following result, we suppose X and Z are reflexive and derive an exterior penalty method for the Problem (1).

Theorem 2.4. *Let* $\bar{x} \in R$, $\bar{p} = (\bar{x}, \bar{y}) \in \mathrm{gr}\ F, F(\bar{x}) \subseteq \{\bar{y}\} + C$, $\bar{\theta} \in C^{+i}$ *and the function* $\mathcal{L}^{\omega} : U \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *defined by*

$$\mathcal{L}^{\omega}(x, \gamma) := \inf_{y \in F(x)} \langle \bar{\theta}, y \rangle + \gamma \inf_{z \in G(x)} d_D(-z).$$

Then the following statements are equivalent:

(i) *cl cone* $\mathcal{E}_{\bar{p}} \cap \mathcal{H}_u = \emptyset$.

(ii) *there exists* $\bar{\gamma} \in \mathbb{R}_+ \setminus \{0\}$ *such that*

$$\sup_{y \in F(x)} \langle \bar{\theta}, \bar{y} - y \rangle \leq \bar{\gamma} \inf_{z \in G(x)} d_D(-z) \quad \forall x \in U.$$

(iii) *there exists* $\bar{\gamma} \in \Gamma := \mathbb{R}_+ \setminus \{0\}$ *such that*

$$\omega(u, v, \bar{\theta}, \bar{\gamma}) \leq 0, \qquad \forall (u, v) \in \mathcal{K}_{\bar{p}};$$

where

$$\omega(u, v, \theta, \gamma) = \langle \theta, u \rangle + \omega_0(v, \lambda) = \langle \theta, u \rangle - \gamma d_D(v)$$

(iv) $\mathcal{L}^{\omega}(x, \gamma)$ *is an exact penalty function of Problem (1) at* $\bar{x}$.

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Talk Derived Algebraic Structures from Algebraic Hyperstrutctures pp.: 1–8

Derived Algebraic Structures from Algebraic Hyperstrutctures

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Abstract

Given an algebraic hyperstructure (AHS) H. Let P be an algebaic property. In our talk we want to answer to Is there a smallest strongly regular relation ρ on H, such that the quotient H/ρ, the derived algebraic structure (AS) from H, satisfies in the property P? In this regards we try to answer to this question in general. In this regards first we review briefly some attempts to this diirection and we answer the questions for two specila manners for derived Engle groups and (pseduo) regular rings.

AMS:20N20, 16Y99.
Keywords: fundamental relation, multiplicative hyperring, Engel, pseudo regular.

1 Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [21]. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups and then many researchers have been worked on this new field of modern algebra and developed it. It was later observed that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi-hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed [21, 10, 11, 14, 33].
In [11] Corsini and Leoreanu-Fotea have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. A special equivalence relations which is called fundamental relations play important roles in the the theory of algebraic hyperstructures. The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures are derived from algebraic hyperstructures by them. The fundamental relation β^* on hypergroups was defined by Koskas [19], mainly studied by Corsini [21], Freni [18], Vougiouklis [34](for more details about hyperrings and fundamental relations

*Speaker

on hyperrings see [3, 12, 14, 32, 34]). Also, recently in [6] nilpotent groups derived from a polygroup studied; and R. Ameri and E. Mohamadzadeh in [1] introduced and studied Engel groups derived from hypergroups was .
In this note we start by the following important quoestion in the theory of algebraic hyperstructures: Consider an algebraic hyperstructure H and an algebraic property P. Is there an strongly regular relation ρ on H such that the quotient algebraic structure H/ρ satisfies in the property P?
In this talk we try to answer to this question in general. Also, we examine this question for two important case for derived Engel groups and pseudo regular rings .

Recall that a *hyperoperation* "." on nonempty set H is a mapping of $H \times H$ into the family of all nonempty subsets of H. Let "." be a hyperoperation on H. Then, $(H, .)$ is called a *hypergroupoid.* we can extend the hyperoperation on H to subsets of H as follows. For $A, B \subseteq H$ and $h \in H$, then $AB = \cup_{a \in A, b \in B} ab$,
$Ah = A\{h\}, hB = \{h\}B$. A *semihypergroup* is a hypergroupoid $(H, .)$, which is associative, that is $(a.b).c = a.(b.c)$ for all $a, b, c \in H$. A *hypergroup* is a semihypergroup $(H, .)$, that satisfies the *reproduction axioms*, that is $a.H = H = H.a$ for all $a \in H$.

A non-empty set R with two hyperoperations $+$ and . is said to be a *hyperring* if $(R, +)$ is a *canonical hypergroup,* $(R, .)$ is a semihypergroup with $r.0 = 0.r = 0$ for all $r \in R$ (0 as a bilaterally absorbing element) and the hyperoperation . is distributive with respect to $+$, i.e., for every $a, b, c \in R; a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.
A *multiplicative hyperring* is an additive commutative group $(R, +)$ endowed with a hyperoperation . which satisfies the following conditions:
(1.) $\forall a, b, c \in R : a(bc) = (ab)c$;
(2.) $\forall a, b, c \in R : (a + b)c \subseteq ac + bc, a(b + c) \subseteq ab + ac$;
(3.) $\forall a, b \in R : (-a)b = a(-b) = -(ab)$.
If in (2) we have equalities instead of inclusions, then we say that the multiplicative hyperring is *strongly distributive.*

2 Derived Engel Groups

Definition 2.1. let H be a hypergroup . We define for a fix element $s \in H$,
1) $L_{0,s}(H) = H$
1) $L_{k+1,s}(H) = \{h; h \in [x, s]; x \in L_{k,s}(H)\}$.
for all $k \geq 0$ suppose that $n \in N$, and $\omega_n = \bigcup_{m>1} \omega_{mn}$ where ω_{1n} is the diagonal relation and for every integer $m \geq 1$, ω_{mn} is the relation defined as follows:
$x\omega_{mn}y \iff \exists(z_1, ..., z_m) \in H^m; \exists\delta \in S_m : \delta(i) = i$ if z_i is not in $L_{n,s}(H)$ such that $x \in \prod_{i=1}^m z_i$, $y \in \prod_{i=1}^m z_{\delta(i)}$.

Obviously, for every $n \geq 1$, the relation ω_n is reflexive and symmetric. Now let ω_n^* be the transitive closure of ω_n.

Theorem 2.2. *For every $n \in N$, the relation ω_n^* is a strongly regular relation.*

Corollary 2.3. *If H is a commutative hypergroup, then $\beta^* = \omega_n^* = \nu_n^* = \gamma^*$.*

Definition 2.4. For any group G we define the subgroups $Z_i(G_y)$ for a fix element y, $i \in \{0, 1, , ...\}$ as follows. Define $Z_{0,y}(G) = \{e\}$,$Z_{1,y}(G) =< \{x \in G; [x, y] = e\} >$,...,$Z_k(G_y) =< \{x \in G; [x,_k y] = e\} >$.
Also we define $L_0(G_s) = G$, and for a fix $s \in G$, $L_{k+1}(G_s) = \{[x, s]; x \in L_k(G_s)\}$.

Theorem 2.5. *If H is a hypergroup and φ is a strongly regular relation on H, then for a fix $s \in H$,*
$L_{k+1,s}(\frac{H}{\varphi})) = \{[\bar{t}, \bar{s}]; t \in L_{k,s}((H)\}$.

Theorem 2.6. $\frac{H}{\omega_n^*}$ *is an n-Engel group.*

In this section we introduce the smallest strongly relation ω^* on a finite hypergroup H such that $\frac{H}{\omega^*}$ is an Engel group.

Definition 2.7. Let H be a finite hypergroup. Then we define the relation ω^* on H as follows:
$\omega^* = \bigcap_{n \geq 1} \omega_n^*$.

Theorem 2.8. *The relation ω^* is a strongly regular relation on a finite hypergroup H such that $\frac{H}{\omega^*}$ is an Engel group.*

Theorem 2.9. *The relation ω^* is the smallest strongly regular relation on a finite hypergroup H such that $\frac{H}{\omega^*}$ is an Engel group.*

3 Part II: Pseudo Regular Rings

Let R be a ring. An element $a \in R$ is *regular* if there exists $x \in R$, such that $a = axa$. R is a *regular ring* if every elements of R is regular. The set of all regular elements in R is denoted by $V(R)$. In this section we introduced the notation of pseudo regular rings. In 1950, Brown and McCoy [9], defined the set of elements of a ring such that generated ideal of that elements is regular and they denoted this set by $\mathcal{M}(R)$. They proved that $\mathcal{M}(R)$ is an ideal and clearly $\mathcal{M}(R) \subseteq V(R)$.

Definition 3.1. Let $(R, +, \cdot)$ be a ring. We define
(1) $T_0(R) = R$
(2) $T_{k+1}(R) = \{x - xrx | x \in T_k(R), r \in R\}$,
for $k \geq 0$.

Definition 3.2. Let R be a ring. An ideal series of R is a finite chain of ideals of R such that

$$\{0\} = R_0 \triangleleft R_1 \triangleleft \cdots \triangleleft R_k = R$$

such that $1 \leq i \leq k$, $R_{i-1} \triangleleft R_i$. Then k is said to be the *length* of series and denoted by $\ell(R)$.

Definition 3.3. Let R be a ring. An ideal series

$$\{0\} = R_0 \lhd R_1 \lhd \cdots \lhd R_k = R$$

is called *regular series,* if for all $1 \leq i \leq k$, $\frac{R_i}{R_{i-1}} \lhd \mathcal{M}(\frac{R}{R_{i-1}})$, where $\mathcal{M}(R) = \{x \in R| < x > \textit{ is a regular ideal}\}$.

Proposition 3.4. *([9]) Let R be a ring. Then $\mathcal{M}(\frac{R}{\mathcal{M}(R)}) = \{0\}$.*

Remark 3.5. Let R be a ring. We denote $\mathcal{M}_0(R) = \{0\}$.

Definition 3.6. Let R be a ring. A lower ideal series is an ideal series

$$R = R^0 \rhd R^1 \rhd R^2 \rhd \cdots ,$$

where $R^i =< T_i(R) >$, for all $1 \leq i \leq k$.

Definition 3.7. A ring R is said to be *pseudo regular* if it has a regular series. The smallest length of a regular series of R is called regularity class of R.

Example 3.8. Let R be a nontrivial pseudo regular ring. Then $\mathcal{M}(R) \neq \{0\}$, because on the otherwise R will be trivial.

Example 3.9. Let $R = \mathbb{Z}_p[i] = \{a + ib|a, b \in \mathbb{Z}_p\}$, $i = \sqrt{-1}$ be the Gaussian integer modulo p, for some odd prime p. Then by Corollary 3.11 of [23], $\mathcal{M}(R) \neq \{0\}$, and hence R is pseudo regular with length ≥ 1, where $\mathbb{Z}_p[i]$. If $R = \mathbb{Z}_{p^k}[i]$ for some odd prime p and $k \neq 1$, then $\mathcal{M}(R) = \{0\}$ and in this case R is not pseudo regular.

Example 3.10. Let $R = \mathbb{Z}_{2^k}[i]$ for all k. Then $\mathcal{M}(R) = \{0\}$. Therefore, R is not a pseudo regular ring.

Theorem 3.11. *Let R be a ring and $n \geq 1$. Then the following statements are equivalent:*
(i) $R^n = \{0\}$;
(ii) R is pseudo regular.

Corollary 3.12. *Let R be a ring and $n \geq 1$. If $\mathcal{M}(R) = R$, then R is pseudo regular.*

Proposition 3.13. *([9]) If r is an element of R such that $a - ara$ is regular, then a is regular.*

Theorem 3.14. *Let R be a ring. If $\mathcal{M}(R) = V(R)$ and R is pseudo regular, then $\mathcal{M}(R) = R$.*

Theorem 3.15. *Let R be a non trivial pseudo regular ring with an unitary and I an non zero ideal of R. Then $\mathcal{M}(I) \neq \{0\}$.*

Theorem 3.16. *Let I be an ideal of a pseudo regular ring R. Then I and $\frac{R}{I}$ are pseudo regular.*

Theorem 3.17. *If $R_1, R_2, \cdots, R_r$ are pseudo regular rings, then $R = \prod_{i=1}^{r} R_i$ is pseudo regular.*

Corollary 3.18. *Let R be a ring and I and J be two ideals of R. If $\frac{R}{I}$ and $\frac{R}{J}$ are pseudo regular, then $\frac{R}{I\bigcap J}$ is also pseudo regular.*

Proposition 3.19. *([10]) If $(H,\cdot)$ is a semihypergroup(resp. hypergroup) and ρ is a strongly regular relation on H, then the quotient H/ρ is a semigroup(resp. group) under the operation:*

$$\rho(x)\otimes\rho(y)=\rho(z),\quad \forall z\in x\cdot y.$$

We denote $\rho(x)$ by $\bar{x}$ and instead of $\bar{x}\otimes\bar{y}$ we write $\bar{x}\bar{y}$.

For all $n>1$, we define the relation β_n on a semihypergroup H, as follows:

$$a\beta_n b\Leftrightarrow \exists(x_1,\cdots,x_n)\in H^n:\{a,b\}\subseteq\prod_{i=1}^{n}x_i,\quad and\quad \beta=\bigcup_{n\geq1}\beta_n$$

where $\beta_1=\{(x,x);x\in H\}$, is the diagonal relation on H. This relation was introduced by Koskas [19] and studied by many researchers in the theory of algebraic hyperstructures(for more see [1, 3, 10, 11, 12, 13, 15, 33, 34]). Consider β^* as the transitive closure of β. It is proved that the relation β^* is a strongly regular relation, and it is called the fundamental relation of H [10].
Let $(R,+,.)$ be a hyperring. Define the relation γ as follows:

$$x\gamma_n y\Leftrightarrow \exists(x_1,\cdots,x_n)\in H^n,\exists\tau\in\mathbb{S}_n: x\in\prod_{i=1}^{n}x_i, y\in\prod_{i=1}^{n}x_{\tau(i)},$$

and $\gamma=\bigcup_{n\geq1}\gamma_n$. We denote the transitive closure of γ by γ^*. The relation γ^* is the smallest equivalence relation on a multiplicative hyperring $(R,+,.)$ such that the quotient R/γ^*, the set of all equivalence classes, is a ring. The relation γ^* is called fundamental relation on R, and R/γ^* is called the fundamental ring. Suppose that $\gamma^*(a)$ is the equivalence class containing $a\in R$. Then both the sum $\oplus$ and the product $\odot$ in R/γ^* are defined as follows:

$\gamma^*(a)\oplus\gamma^*(b)=\gamma^*(c)$ for all $c\in\gamma^*(a)+\gamma^*(b)$ and $\gamma^*(a)\odot\gamma^*(b)=\gamma^*(d)$ for all $d\in\gamma^*(a).\gamma^*(b)$. Then R/γ^* is a ring, which is called fundamental ring of R (for more details see also [33]).

Definition 3.20. Let R be a multiplicative hyperring. We say that $a\in R$ is *regular* if there exists $x\in R$ such that $a\in axa$, and R is a regular multiplicative hyperring, if all the elements of R are regular. The set of all regular elements in R is denoted by $V(R)$.

Definition 3.21. Let $(R,+,\cdot)$ be a multiplicative hyperring. Define
(1) $L_0(R)=R$; (2) $L_{k+1}(R)=\{h|h\in(x-xrx)\bigcap(xr-rx), x\in L_k(R), r\in R\}$;
for $k\geq0$. Suppose that $n\in\mathbb{N}$ and $\eta_n=\bigcup_{m>1}\eta_{m,n}$, where $\eta_{1,n}$ is the diagonal relation and for every integer $m\geq1$, the relation $\eta_{m,n}$ is defined as follows:
$x\eta_{m,n}y\Leftrightarrow\exists(z_1,\cdots,z_m)\in R^m,\exists\sigma\in\mathbb{S}_m:\sigma(i)=i\ if\ z_i\notin L_n(R)\ such\ that$
$x\in\prod_{i=1}^{m}z_i\ and\ y\in\prod_{i=1}^{m}z_{\sigma(i)}.$
Obviously, for every $n\geq1$, the relation η_n is reflexive and symmetric. Now suppose that η_n^* is the transitive closure of η_n.

Corollary 3.22. *For every $n\in\mathbb{N}$, we have $\beta^*\subseteq\eta_n^*\subseteq\gamma^*$.*

Theorem 3.23. *For every $n \in \mathbb{N}$, the relation η_n^* is a strongly regular relation.*

Theorem 3.24. *Let R be a multiplicative hyperring. Let ρ be a strongly regular relation on R. Then $L_{k+1}(R/\rho) = \{\bar{h} | \bar{h} = \bar{x} - \bar{x}\bar{r}\bar{x} = \bar{x}\bar{r} - \bar{r}\bar{x}, x \in L_k(R), r \in R\}$, where $\bar{r}$ is the class of r with respect to ρ.*

Theorem 3.25. *R/η_n^* is a pseudo regular ring of the class at most $n + 1$.*

Corollary 3.26. *The relation η^* is a strongly regular relation on a multiplicative hyperring R, such that R/η^* is a pseudo regular ring.*

Theorem 3.27. *The relation η^* is the smallest strongly regular relation on a multiplicative hyperring such that R/η^* is a pseudo regular ring.*

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A survey of simplicial cohomology for semigroup algebras

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Abstract

In this survey, we investigate the higher simplicial cohomology groups of the convolution algebra $\ell^1(S)$ for various semigroups S. The classes of semigroups considered are semilattices, Clifford semigroups, regular Rees semigroups and the additive semigroups of integers greater than a for some integer a. Our results are of two types: in some cases, we show that some cohomology groups are 0, while in some other cases, we show that some cohomology groups are Banach spaces.

1 Introduction

In this talk, we investigate the higher simplicial cohomology groups of the convolution algebra $\ell^1(S)$ for various semigroups S. Our results are of two types: in some cases, we show that some cohomology groups are 0, while in some other cases, we show that some cohomology groups are Banach spaces.

First we explain the general idea for showing that a cohomology group is a Banach space. Let $\delta : C^n(\mathcal{A}, \mathcal{X}) \longrightarrow C^{n+1}(\mathcal{A}, \mathcal{X})$ be the boundary map. Then $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space if and only if the range of δ is closed, which is the case if and only if δ is open onto its range, that is there exists a constant K such that if $\psi = \delta(\phi)$ is such that $\|\psi\| < 1$ then there exists $\phi_1 \in C^n(\mathcal{A}, \mathcal{X})$ such that $\|\phi_1\| < K$ and $\psi = \delta(\phi_1)$.

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}'$ be a Banach $\mathcal{A}$-bimodule in the usual way. An n-cochain is a bounded n-linear map T from $\mathcal{A}$ to $\mathcal{A}'$, which we denote by $T \in C^n(\mathcal{A}, \mathcal{A}')$. The map $\delta^n : C^n(\mathcal{A}, \mathcal{A}') \longrightarrow C^{n+1}(\mathcal{A}, \mathcal{A}')$ is defined by

$$\begin{aligned}(\delta^n T)(a_1, \ldots, a_{n+1})(a_0) \;=\;& T(a_2, a_3, \ldots, a_{n+1})(a_0 a_1) \\ & - T(a_1 a_2, a_3, \ldots, a_{n+1})(a_0) \\ & + T(a_1, a_2 a_3, a_4, \ldots, a_{n+1})(a_0) + \ldots \\ & + (-1)^n T(a_1, \ldots, a_{n-1}, a_n a_{n+1})(a_0) \\ & + (-1)^{n+1} T(a_1, \ldots, a_n)(a_{n+1} a_0)\,.\end{aligned}$$

The n-cochain T is an *n-cocycle* if $\delta^n T = 0$ and it is an *n-coboundary* if $T = \delta^{n-1} S$ for some $S \in C^{n-1}(\mathcal{A}, \mathcal{A}')$. The linear space of all n-cocycles is denoted by $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')$, and the linear space of all n-coboundaries is denoted by $\mathcal{B}^n(\mathcal{A}, \mathcal{A}')$. We also recall that $\mathcal{B}^n(\mathcal{A}, \mathcal{A}')$ is included in $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')$ and that the n^{th} simplicial cohomology group $\mathcal{H}^n(\mathcal{A}, \mathcal{A}')$ is defined by the quotient

$$\mathcal{H}^n(\mathcal{A}, \mathcal{A}') = \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')}{\mathcal{B}^n(\mathcal{A}, \mathcal{A}')}.$$

Definition 1.1. Let S be a semigroup and

$$\ell^1(S) = \{f : S \longrightarrow \mathbb{C} : \|f\|_1 = \sum_{s\in S} |f(s)| < \infty\}.$$

We define the convolution of two elements $f = \sum_{s\in S} f(s)\delta_s$ and $g = \sum_{t\in S} g(t)\delta_t$ in $\ell^1(S)$ by

$$\sum_{s\in S} f(s)\delta_s * \sum_{t\in S} g(t)\delta_t = \sum_{r\in S}\sum_{st=r} f(s)g(t)\delta_r,$$

where δ_s is the point mass function at s. Then $(\ell^1(S), *, \|\cdot\|_1)$ becomes a Banach algebra that is called the semigroup algebra of S.

2 Semilattice Algebra

Let S be a semigroup and let $E(S) = \{p \in S : p^2 = p\}$. We say that S is a semilattice if S is commutative and $E(S) = S$, that is, $e^2 = e$ for every $e \in S$.

Theorem 2.1. [Gourdeau, Pourabbas and White] Let $\mathcal{A} = \ell^1(S)$, where S is a semilattice, and let $\mathcal{X}$ be a commutative $\mathcal{A}-$module. Then $\mathcal{H}^3(\mathcal{A}, \mathcal{X})$ is a Banach space.

The idea of the proof, if one knows that the algebraic cohomology vanishes, this often implies that the coboundaries are dense in the space of cocycles. If only we can show that the coboundary map is open onto its range, then we will be able to show that the coboundary map has closed range. A method of showing that the map is open is to try the following strategy. Take a proof that $\mathcal{H}^n(A, A')$ is trivial, so that all cocycles are coboundaries. This will show that a coboundary map is surjective, so certainly open onto its range. Now try to rewrite this proof to show that if ϕ is an approximate n-cocycle, that is $\|\delta\phi\| < 1$, then it is approximately equal to a coboundary, i.e. there exists a ψ so that $\|\phi - \delta\psi\| < K$ (for some K). Then we will have a small $\phi' = \phi - \delta\psi$, which has $\delta\phi' = \delta\phi$.

Now let us see how this works in the particular case of Theorem 2.1. We take the standard proof that derivations vanish on symmetrically acting idempotents.

$$D(e) = D(e^2) = eD(e) + D(e)e = 2eD(e)$$

Hence $eD(e) = 2eD(e)$ and so $eD(e) = 0$ and so $D(e) = 0$.

Then if we are given a small 2-coboundary, $\delta\psi$, say $\|\delta\psi\| < 1$, we can think of this as saying that ψ is an approximate derivation. Then we have $\psi(e) = \psi(e^2) \approx 2e\psi(e)$, hence $e\psi(e) \approx 2e\psi(e)$, and so $e\psi(e) \approx \psi(e)$ and $\psi(e) \approx 0$. This shows that ψ is small on symmetrically acting idempotents.

Theorem 2.2 (Gourdeau, Pourabbas and White)**.** Let S be a semilattice. Then $\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = 0$.

Proof. Let $\mathcal{A} = \ell^1(S)$, where S is a semilattice, and let $T \in C^3(\mathcal{A}, \mathcal{A}')$. We define

$$\begin{aligned} t^2(T)(u, v) =& 2uvT(u, u, uv) + uvT(v, v, uv) - uvT(uv, v, v) \\ &+ uT(v, uv, uv) + uT(u, v, v) - uT(uv, uv, v) \\ &+ 2T(u, uv, uv) - T(u, v, uv) - T(u, u, v). \end{aligned}$$

We claim that $\delta^1 t^1 + t^2\delta^2 = id$, where $t^1 : C^2(\mathcal{A}, \mathcal{A}') \longrightarrow C^1(\mathcal{A}, \mathcal{A}')$ is defined by $t^1(\phi)(e) = (2e-1)\phi(e,e)$. To prove our claim for $\phi \in C^2(\mathcal{A}, \mathcal{A}')$ we have

$$\begin{aligned} t^2(\delta^2)(\phi)(u,v) =& 2uv\delta^2\phi(u,u,uv) + uv\delta^2\phi(v,v,uv) - uv\delta^2\phi(uv,v,v) \\ &+ u\delta^2\phi(v,uv,uv) + u\delta^2\phi(u,v,v) - u\delta^2\phi(uv,uv,v) \\ &+ 2\delta^2\phi(u,uv,uv) - \delta^2\phi(u,v,uv) - \delta^2\phi(u,u,v). \end{aligned}$$

Using the definition of boundary map δ^2 we obtain the value of all terms on the right-hand side of the above as follows

$$\begin{aligned} t^2(\delta^2\phi)(u,v) &= \phi(u,v) - [u(2v-1)\phi(v,v) - (2uv-1)\phi(uv,uv) + v(2u-1)\phi(u,u)] \\ &= (id - \delta^1 t^1)(\phi)(u,v), \end{aligned}$$

which proves our claim, and the proof is complete. □

Theorem 2.3 (Choi)**.** Let S be a semilattice. Then
(i) $\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) = 0$, for all $n \geq 1$.
(ii) $\mathcal{H}^n(\ell^1(S), X) = 0$, for all symmetric $\ell^1(S)$-bimodule X and all $n \geq 1$.

If S is a semilattice, Duncan and Namioka showed that $\ell^1(S)$ is amenable if and only if S is finite. Dales and Duncan observed that

$$\mathcal{H}^1(\ell^1(S), X) = \mathcal{H}^2(\ell^1(S), X) = 0,$$

for all symmetric $\ell^1(S)$-bimodule X and this has been extended to the third cohomology by [Gourdeau, Pourabbas and White].

3 Approximately additive functions and the semigroup $\mathbf{N}_a$

Definition 3.1. A real-valued function f defined on a subset X of a semigroup S is called *1-additive* if

$$|f(x) + f(y) - f(x+y)| < 1 \text{ when } x, y, x+y \in X,$$

and *additive* if

$$|f(x) + f(y) - f(x+y)| = 0 \text{ when } x, y, x+y \in X.$$

The following proposition will enable us to deduce that the boundary map

$$\delta : C^1(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a)) \longrightarrow C^2(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$$

is open onto its range, and hence that $\mathcal{H}^2(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ is a Banach space.

Proposition 3.2 (Gourdeau, Pourabbas and White)**.** Let f be a real-valued 1-additive function on $[s,t] = \{n \in \mathbf{N} : s \leq n \leq t\}$. Then there exists a universal constant K and an additive function g on $[s,t]$ such that $\|f - g\|_\infty < K$ where $\|f\|_\infty = \max_{x\in[s,t]} |f(x)|$.

Theorem 3.3 (Gourdeau, Pourabbas and White)**.** $\mathcal{H}^2(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ is a Banach space.

Proof. Let $\phi \in C^1(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ be such that $\|\delta\phi\| < 1$. Using the one-to-one correspondence between $C^n(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ and bounded functions from the n-fold product $\mathbf{N}_a \times \cdots \times \mathbf{N}_a$ into $\ell^\infty(\mathbf{N}_a)$, we write

$$|\delta\phi(x,y)(z)| < 1 \qquad \forall x,y,z \in \mathbf{N}_a,$$

which is

$$|\phi(y)(x+z) - \phi(x+y)(z) + \phi(x)(y+z)| < 1\,.$$

For each $N \geq 3a$, let $f_N : [a, N-a] \longrightarrow \mathbf{R}$ be given by

$$f_N(x) = \phi(x)(N-x).$$

Then f_N is 1-additive as, for $x, y, x+y \in [a, N-a]$, we have

$$\begin{aligned} |f_N(x) + f_N(y) - f_N(x+y)| &= |\delta\phi(x,y)(N-(x+y))| \\ &< 1 \end{aligned}$$

Therefore it follows from the previous Proposition that, for each $N \geq 3a$, there exists $g_N : [a, N-a] \longrightarrow \mathbf{R}$ additive such that $\|f_N - g_N\|_\infty < K$ for a fixed constant K.

Let $\psi \in C^1(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ be induced by

$$\psi(x)(y) = \begin{cases} \phi(x)(y) & \text{if } x+y < 3a; \\ g_N(x) & \text{else, where } N = x+y. \end{cases}$$

Then $\delta(\phi - \psi) = \delta(\phi)$ and $\|\phi - \psi\| < K$. The map δ is therefore open onto its range, which proves the theorem. □

4 Rees Semigroup Algebra

Let G be a group, I and Λ be index sets, and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. Let $P = (p_{\lambda i})$ be a regular sandwich matrix over G^0, so each row and column of P contains at least one nonzero entry. The associated Rees semigroup is defined by $S_\emptyset = I \times G \times \Lambda \cup \{\emptyset\}$, where $\emptyset$ acts as the zero element of S and

$$(i, g, \lambda)(j, h, \mu) = (i, g p_{\lambda j} h, \mu),$$

if $p_{\lambda,j} \neq 0$ and $\emptyset$ otherwise.

Theorem 4.1 (Gourdeau, Gronbaek and White)**.** Let $S_\emptyset$ be a regular Rees semigroup. Then the cohomology groups $\mathcal{H}^2(\ell^1(S_\emptyset), \ell^\infty(S_\emptyset))$ and $\mathcal{H}C^2(\ell^1(S_\emptyset))$ are Banach spaces.

To show that $\mathcal{H}^2(\ell^1(S_\emptyset), \ell^\infty(S_\emptyset))$ is a Banach space, we must show that the space $\mathcal{B}^2(\ell^1(S_\emptyset), \ell^\infty(S_\emptyset))$ is closed. We do this by showing that the map

$$\delta : C^1(\ell^1(S_\emptyset), \ell^\infty(S_\emptyset)) \longrightarrow C^2(\ell^1(S_\emptyset), \ell^\infty(S_\emptyset))$$

is an open map onto its range and hence has closed range.

Theorem 4.2 (Gourdeau, Gronbaek and White)**.** Let S be a Rees semigroup with underlying group G. Then we have

$$\mathcal{H}_n(\ell^1(S), \ell^1(S)) \simeq \mathcal{H}_n(\ell^1(G), \ell^1(G))$$

and

$$\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) \simeq \mathcal{H}^n(\ell^1(G), \ell^\infty(G)).$$

That is, the simplicial cohomology and homology of $\ell^1(S)$ is isomorphic of those underlying discrete group algebra.

As a consequence $\mathcal{H}^1(\ell^1(S), \ell^\infty(S)) = 0$ and $\mathcal{H}^2(\ell^1(S), \ell^\infty(S))$ is a Banach space.

5 Brandt semigroup

Let G be a group and let I be a non-empty set. Set

$$\mathcal{M}^0(G, I) = \{(g)_{ij} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{ij}$ denotes the $I \times I$-matrix with entry $g \in G$ in the (i, j) position and zero elsewhere. Then $\mathcal{M}^0(G, I)$ with the multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad (g, h \in G,\ i, j, k, l \in I),$$

is an inverse semigroup with $(g)^*_{ij} = (g^{-1})_{ji}$, that is called the Brandt semigroup over G with index set I.

If S is Brandt semigroup over G with a finite index set I, Duncan and Namioka showed that $\ell^1(S)$ is amenable if and only if G is finite.

The notion of approximate amenable Banach algebras was introduced by F. Ghahramani and R. J. Loy. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. A continuous derivation $D : \mathcal{A} \longrightarrow E$ is approximately inner if there is a net (D_ν) of inner derivations in $\mathcal{B}(\mathcal{A}, E)$ such that

$$D(a) = \lim_\nu D_\nu(a) \quad (a \in A),$$

where the limit is taken in the strong-operator topology of $\mathcal{B}(\mathcal{A}, E)$.

A Banach algebra $\mathcal{A}$ is called approximately amenable if for each Banach $\mathcal{A}$-bimodule E, every continuous derivation $D : \mathcal{A} \longrightarrow E'$ is approximately inner. That is,

$$\mathcal{H}^1_{app}(\mathcal{A}, \mathcal{A}') = \frac{\mathcal{Z}^1(\mathcal{A}, \mathcal{A}')}{\overline{\mathcal{B}^1(\mathcal{A}, \mathcal{A}')}^{strong}} = 0.$$

A Banach algebra $\mathcal{A}$ is pseudo-amenable if there is a net $(m_\alpha) \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$, called an approximate diagonal for $\mathcal{A}$, such that for each $a \in \mathcal{A}$

$$a \cdot m_\alpha - m_\alpha \cdot a \longrightarrow 0 \quad \text{and} \quad \pi(m_\alpha)a \longrightarrow a.$$

M. M. Sadr has shown that if G is an amenable group, then the Brandt semigroup algebra is pseudo-amenable. It remained open whether pseudo-amenability of the Brandt semigroup algebra implies the amenability of group G. Essmaili, Rostami and Pourabbas characterized pseudo-amenability of Brandt semigroup algebras and this characterization answers the question raised by Sadr.

Theorem 5.1 (Essmaili, Rostami and Pourabbas)**.** Let G be a group, I be a non-empty set and let $S = \mathcal{M}^0(G, I)$ be the Brandt semigroup over G with index set I. Then $\ell^1(S)$ is pseudo-amenable if and only if G is an amenable group.

Recently, Sadr and Pourabbas characterized the approximate amenability of Brandt semigroup algebras. Precisely, they have shown that for a Brandt semigroup $S = \mathcal{M}^0(G, I)$, the semigroup algebra $\ell^1(S)$ is approximate amenable if and only if G is amenable and I is finite. This fact and previous result gives an example of pseudo-amenable Banach algebra that is not approximate amenable.

Theorem 5.2 (Sadr and Pourabbas)**.** Let $S = \mathcal{M}^0(G, I)$ be a Brandt semigroup. Then the following are equivalent.

(1) $\ell^1(S)$ is amenable.

(2) $\ell^1(S)$ is approximately amenable.

(3) I is finite and G is amenable.

6 Clifford semigroup algebra

We recall that S is a Clifford semigroup if it is an inverse semigroup with each idempotent central, or equivalently, if it is a strong semilattice of groups. So we can write our Clifford semigroup as $S = \cup\{G_e : e \in E\}$ where E is the semilattice of idempotents and each G_e is a group with identity element e, and for every $e, e' \in E$, we have $G_e G_{e'} \subseteq G_{ee'}$.

Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup over a finite semilattice $E(S)$, Duncan and Namioka showed that $\ell^1(S)$ is amenable if and only if each G_e is amenable.

Theorem 6.1 (Gourdeau, Pourabbas and White)**.** Let S be a Clifford semigroup. Then $\mathcal{H}^2(\ell^1(S), \ell^\infty(S))$ is a Banach space.

To prove the theorem for every $\psi \in C^1(\ell^1(S), \ell^\infty(S))$ with $\|\delta\psi\| < 1$, we show that there exists a constant M and $\hat{\psi} \in C^1(\ell^1(S), \ell^\infty(S))$ such that $||\hat{\psi}|| < M$ and $\delta\hat{\psi} = \delta\psi$, which proves the result.

Theorem 6.2 (Choi, 2010)**.** (i) Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup. Suppose that each G_e is amenable. Then

$$\mathcal{H}_n(\ell^1(S), \ell^1(S)) = 0$$

and

$$\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) = 0$$

for all $n \geq 1$.

(ii) Let S be a commutative Clifford semigroup and let X be any symmetric Banach $\ell^1(S)$-bimodule. Then $\mathcal{H}^n(\ell^1(S), X) = 0$ for all $n \geq 1$.

(iii) Let S be a normal band (that is, a semigroup in which every element is idempotent and $abca = acba$ for all $a, b, c \in S$). Then $\ell^1(S)$ is simplicially trivial, that is $\mathcal{H}^n(\ell^1(S), \ell^\infty(S)) = 0$ for all $n \geq 1$.

Proposition 6.3 (Essmaili, Rostami and Pourabbas)**.** Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is pseudo-amenable if and only if G_e is amenable for every $e \in E(S)$.

Remark 6.1. A Theorem of Essmaili, Rostami and Pourabbas implies that $\ell^1(S)$ is pseudo-amenable whenever S is a uniformly locally finite semilattice and they claim that the converse does not hold in general.

Let $S = (\mathbb{N}, \min)$. Then $\ell^1(S)$ is approximate amenable, as has been by [Dales, *et. al.*]. But $(\delta_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\ell^1(S)$. On the other hand, any Banach algebra with a bounded approximate identity is approximate amenable if and only if it is pseudo-amenable. This shows that $\ell^1(S)$ is pseudo-amenable but $(S, \leq)$ is not uniformly locally finite

7 Inverse semigroup

The semigroup S is an inverse semigroup if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. For any inverse semigroup S, there is a partial order on S defined by

$$s \leq t \iff s = ss^*t \qquad (s, t \in S). \tag{7.1}$$

The canonical partial order on $E(S)$ is given by

$$s \leq t \iff s = st = ts \qquad (s, t \in E(S)). \tag{7.2}$$

It is easily verified that the partial order given on S coincides with that given on $E(S)$.

If $(S, \leq)$ is a partially ordered set, we set $(x] = \{y \in S : y \leq x\}$. The partially ordered set $(S, \leq)$ is called locally finite if $(x]$ is finite for every $x \in S$ and is called uniformly locally finite if $\sup\{|(x]| : x \in S\} < \infty$.

Theorem 7.1 (Essmaili, Rostami and Pourabbas)**.** Let S be an inverse semigroup such that $(E(S), \leq)$ is uniformly locally finite. Then the following are equivalent:

(i) $\ell^1(S)$ is pseudo-amenable.

(ii) Each maximal subgroup of S is amenable.

(iii) $\ell^1(S)$ is biflat.

The Banach algebra $(\ell^1(S), \bullet, \|\cdot\|_1)$ is called the restricted semigroup algebra and will be denoted by $B(S)$, where the multiplication $\bullet$ on $\ell^1(S)$ is defined by

$$\sum_{s\in S} f(s)\delta_s \bullet \sum_{t\in S} g(t)\delta_t = \sum_{r\in S} \sum_{\substack{st=r,\\ s^*s=tt^*}} f(s)g(t)\delta_r,$$

if there are no elements $t, s \in S$ with $st = r$ and $s^*s = tt^*$, the multiplication is taken as zero.

Theorem 7.2 (Rostami, Pourabbas and Essmaili)**.** Let S be an inverse semigroup. Then the following are equivalent:

(i) B(S) is approximately amenable.

(ii) $\ell^1(S)$ is amenable.

(iii) B(S) is amenable.

Theorem 7.3 (Rostami, Pourabbas and Essmaili)**.** Let S be a uniformly locally finite inverse semigroup. Then the following are equivalent:

(i) $\ell^1(S)$ is approximately amenable.

(ii) E(S) is finite and each maximal subgroup of S is amenable.

(iii) $\ell^1(S)$ is amenable.

(iv) $\ell^1(S)$ is boundedly approximate contractible.

(v) $\ell^1(S)$ is boundedly approximate amenable.

For proof we have

$$\ell^1(S) \cong \ell^1 - \bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\},$$

as Banach algebras. Thus for each $\lambda \in \Lambda$, $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is a homomorphic image of $\ell^1(S)$. This shows that $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is approximately amenable for each $\lambda \in \Lambda$. On the other hand, we have $E(S)$ is finite. This implies that $\ell^1(G_{p_\lambda})$ is approximately amenable for each $\lambda \in \Lambda$. Hence G_{p_λ} is amenable for each $\lambda \in \Lambda$ and thus (ii) holds.

Remark 7.1. The above Theorem is not valid if S is a locally finite but not uniformly locally finite inverse semigroup. For example the semigroup $S = (\mathbb{N}, min)$ is locally finite but is not uniformly locally finite. Also, $E(S) = S$ and it is shown in [4, Example 10.10] that $\ell^1(S)$ is approximately amenable.

Corollary 7.4. (i) Let $S = \mathcal{M}^0(G, I)$ be the Brandt semigroup over group G with index set I. Then $\ell^1(S)$ is approximately amenable if and only if I is finite and G is amenable.

(ii) Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is approximately amenable if and only if $E(S)$ is finite and G_e is amenable for every $e \in E(S)$.

(iii) Let S be a uniformly locally finite semilattice. Then $\ell^1(S)$ is approximately amenable if and only if S is finite.

Theorem 7.5. Let S be a band semigroup and let $\ell^1(S)$ be approximately amenable. Then S is a semilattice.

Corollary 7.6. Let S be a uniformly locally finite band semigroup. Then the following are equivalent:

(i) $\ell^1(S)$ is approximately amenable.

(ii) S is a finite semilattice.

(iii) $\ell^1(S)$ is amenable.

Duncan and Namioka showed that the amenability of $\ell^1(S)$ implies that S is an amenable semigroup, this extended to some classes of semigroups in the following theorem.

Theorem 7.7 (Essmaili, Rostami and Medghalchi)**.** Let S be an inverse semigroup. If $\ell^1(S)$ is pseudo amenable, then S is an amenable semigroup.

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Zero Divisors of Group Rings of Torsion-Free Groups

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Abstract

Irving Kaplansky proposed this conjecture that the group ring $\mathbb{F}[G]$ has no zero divisor for any field $\mathbb{F}$ and any torsion-free group G. We will talk on a recent approach to bound the size of the support of a possible zero divisor.

Keywords: Kaplansky's Zero Divisor Conjecture; Torsion-Free Groups; Group Rings
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1 Introduction

Let R be a ring and H be a group. Recall that the group ring $R[H]$ is the set of all functions α from H to R with finite supports, where the support of α is $\{x \in H \mid \alpha(x) \neq 0_R\}$ and denoted by $\mathrm{supp}(\alpha)$. The group ring $R[H]$ is a ring with pointwise addition and 'polynomial-like' multiplication. That is, if α and β in $R[H]$ then $\alpha + \beta$ is the function from H to R such that $(\alpha + \beta)(x) = \alpha(x) + \beta(x)$ for all $x \in H$; and $\alpha\beta$ is the function from H to R such that

$$(\alpha\beta)(x) = \sum_{(y,z)\in \mathrm{supp}(\alpha)\times \mathrm{supp}(\beta)} \alpha(y)\beta(z).$$

We call a non-zero element a of a ring, a zero divisor whenever $ab = 0$ or $ba = 0$ for some b in the ring.

Irving Kaplansky proposed the following conjecture [1].

Kaplansky's Zero Divisor Conjecture (KZDC) [1]. Let G be a torsion free group and $\mathbb{F}$ be any field. Then the group ring $\mathbb{F}[G]$ has no zero divisor.

Let α be a possible zero divisor of $\mathbb{F}[G]$. Then it is known that $|\mathrm{supp}(\alpha)| \geq 3$ (see e.g. [2, Theorem 2.1]), where $\mathrm{supp}(\alpha) = \{x \in G \mid \alpha(x) \neq 0_{\mathbb{F}}\}$.

Let α and β be non-zero elements of $\mathbb{F}_2[G]$ such that $\alpha\beta = 0$. It is proved in [2, Theorem 1.3] that

1. if $|\mathrm{supp}(\alpha)| = 3$, then $|\mathrm{supp}(\beta)| \geq 18$;
2. if $|\mathrm{supp}(\alpha)| = 4$, then $|\mathrm{supp}(\beta)| \geq 8$.

*Speaker

Assume that G is a torsion free group and $R = \mathbb{Q}$ or $\mathbb{F}_p$ the field with prime p elements, such that there exist non-zero elements $\alpha, \beta \in R[G]$ such that $\alpha\beta = 0$ and $|\mathrm{supp}(\alpha)| = 3$.

We would like to study the improvement of the above lower bound of $|\mathrm{supp}(\beta)|$ for such βs obtained in [2] and mentioned above.

2 Main results

Theorem 2.1. *Let G be a torsion-free group such that the group ring $\mathbb{Q}[G]$ contains a zero divisor with the support of size 3. Then there exist a zero divisor of the form $1+x+y$ or $1+x-y$ for some $x, y \in G$.*

Theorem 2.2. *Let G be a torsion-free group and $\mathbb{F}$ be a field such that the group ring $\mathbb{F}[G]$ contains a zero divisor α with support of size 3. Then $S := \{s^{-1}t \mid s, t \in supp(\alpha), s \neq t\}$ is of size 6. Suppose that $\beta \in \mathbb{F}[G] \setminus \{0\}$ is such that $\alpha\beta = 0$ and $\alpha\beta' \neq 0$ whenever $|supp(\beta')| < |supp(\beta)|$ for $\beta' \in \mathbb{F}[G]$. Let Γ be the induced subgraph Γ of the Cayley graph $Cay(G, S)$ on the support $supp(\beta)$ of β. If $\mathbb{F} = \mathbb{F}_2$ is the field of size 2 and the graph Γ has a cycle of length 4, then G contains two distinct elements x, y such that $x^2 = y^3$ and either $1 + x + y$ or $1 + y + y^{-1}x$ is a zero divisor in $\mathbb{F}_2[G]$.*

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Talk Geometry and Architecture pp.: 1–8

Geometry and Architecture

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Abstract

Minimal surfaces have an essential rule in the Industrial designs, architecture and biology. First we discuss about eight equivalent definitions of minimality and their connections to other branches of mathematics. Then we will refer to meshing of the minimal surface in the discrete geometry. Also we will point to the branche point and conjugate of minimal surfacs. Finally we will see the extension concept of minimal surface in Finsler geometry with their applications.

Keywords: minimal surface, Weierstrass Representations, Discrete minimal surface, adjoint of minimal surface, Finsler minimal surface.

1 Introduction

Minimal surfaces have many applications in architecture, industrial design and biology. Minimal surface is used in architecture for light roof constructions and tents for air exchange. The number of architects that know this way have been increasing. Among the famous buildings are designed according to this way include: RTV Headquarters in Zurich, Michael Schumacher tower, Japan Pavilion,... .
In this article we first discuss the equivalent definitions of minimal surfaces, then we explain Weierstrass Representations of minimal surface and their construction method. After that we mention definition of adjoint of minimal surface and branch points.
Since minimal surfaces are surfaces with minimum area relative to its boundary, we consider them from the perspective of calculus of variations and PDE.
But in architecture and applied issues we need to discrete continuous geometry. Therefore we give a quick review of minimal surface in discrete geometry.
Our main problem in this article is to generalize minimal surface in Finsler geometry. In Finsler geometry there are a few number of articles such as [13, 10, 1]. We want to study these surfaces from applied vision. Finally we present application of minimal surfaces in architecture and industrial designs.

2 Minimal surfaces

We can define a minimal surface from different point of view. Here we consider the eight equivalent definitions of minimality and their connections to other branches of mathemat-

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ics.
Let $S \subset \mathbb{R}^3$ be a surface and

$$X : \Omega \subset \mathbb{R}^2 \overset{C^\infty}{\longrightarrow} S \subset \mathbb{R}^3,$$
$$X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

be a parameterization of S, where Ω is an open domain in $\mathbb{R}^2$. S is called regular if $X_u \times X_v \neq 0$ for each $(u,v) \in U$. Put $w = (u,v)$, $DX(w) = [X_u(w), X_v(w)]$ then

$$T_pS := DX(w)(\mathbb{R}^2), \quad p = X(u,v),$$

is called tangent space of S at poit $p = X(u,v)$.

Definition 2.1. The surface S is called minimal sursace iff x_i is a harmonic map for each i, i.e. $\triangle x_i = 0$, where $\triangle$ is the Riemannian Laplacian operator.

Suppose $N(w) = \frac{X_u(w)\times X_v(w)}{|X_u(w)\times X_v(w)|}$ is a unit normal vector at point w (S is orientable). Then $N : \Omega \overset{C^\infty}{\longrightarrow} S \subset \mathbb{R}^3$ is a Gauss map and $dN_p : T_pS \longrightarrow T_pS$ is a self-adjoint linear map. $H(p) = trace\ dN_p$ is a mean curvature and we have $\triangle X = 2HN$.

Definition 2.2. A surface $S \subset \mathbb{R}^3$ is minimal iff its mean curvature vanishes identically. Any regular surface can be locally expressed as the graph of a function $u = u(x,y)$. In [9] the mean curvature to vanishe identically, the quasilinear, second order, elliptic partial differential equation

$$(1 + u_x^2)u_{yy} - 2u_xu_yu_{xy} + (1 + u_y^2)u_{xx} = 0 \tag{1}$$

which admits a divergence form version

$$div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0 \tag{2}$$

Definition 2.3. A surface $S \subset \mathbb{R}^3$ is minimal iff it can be locally expressed as the graph of $u = u(x,y)$ of a solution of the equation (1) or (2)

Let Ω be a domain with $\bar{\Omega}$ is compact, if $h \in C^\infty(\Omega)$ with compact support and $Y(t) = X + tuN$ is again an immersion whenever $H < \epsilon_0$ then

$$A(t) = \int\int_\Omega |Y_u(t) \times Y_v(t)|\ dA$$

is an area functional. We have

$$A'(0) = -2\int\int_\Omega hHdA$$

.

Definition 2.4. A surface $S \subset \mathbb{R}^3$ is minimal surface iff it is a critical point of the area functional.

With second variational of area functional we obtain:

Definition 2.5. A surface $S \subset \mathbb{R}^3$ is minimal surface iff for each point $p \in S$ there exist a neighborhood with least-area relative to its boundary.

Now we consider an other well-known functional in the calculus of variations is the Dirichlet energy

$$E = \int\int_{\Omega} |\nabla X|^2 dA$$

where Ω is compact closure. We have $E \geq 2A$ with equality iff $X : \Omega \longrightarrow S \subset \mathbb{R}^3$ is conformal.
The coordinate (u, v) on Ω are said to be isothermal if there exists a function $\lambda(u, v) > 0$ such that $< X_u, X_u >= \lambda^2 =< X_v, X_v >$ and $< X_u, X_v >= 0$.
Spivak [12] shows that for each differentiable surface S in $\mathbb{R}^3$ at each point $p \in S$, locally an isothermal coordinate exists.
Then if $X : \Omega \longrightarrow S \subset \mathbb{R}^3$ and $(u, v) \in \Omega$ is an isothermal coordinate then

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = \lambda^2(u, v)(du^2 + dv^2)$$

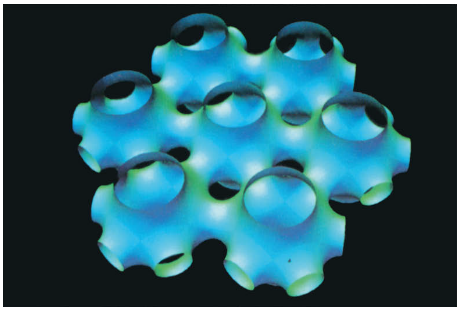
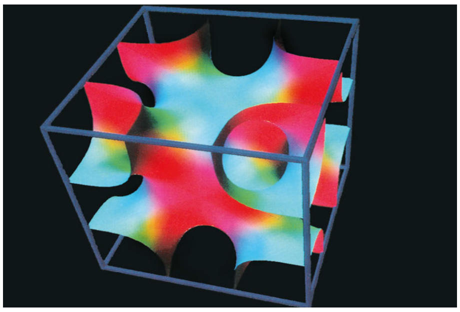
so X is conformal.
Then the existence of isothermal coordinate and conformality of X allow us to give

Definition 2.6. A conformal immersion $X : \Omega \longrightarrow S \subset \mathbb{R}^3$ is minimal surface iff it is a critical point of the Dirichlet energy (least energy).
From a physical point of view, the pressure at the two sides of the surface when are equal then membrane has zero mean curvature. Therefore, soap film (i.e. not bubbles) in space are physical realization of the ideal concept of a minimal surface.

Definition 2.7. A surface $S \subset \mathbb{R}^3$ is minimal surface iff every point $p \in S$ has a neighborhood D_p which is equal to the unique idealized soap film with boundary ∂D.

Suppose $A_p = -dN_p : T_pS \longrightarrow T_pS$ the shape operator. After identification of N with its stereographic projection $g : S \longrightarrow \mathbb{C} \cup \{+\infty\}$ the next result is given.

Definition 2.8. A Riemannian surface (complex manifold with complex dimension 1) is minimal surface iff its stereographically projected Gauss map $g : S \longrightarrow \mathbb{C} \cup \{+\infty\}$ is meromorphic.

3 Weierstrass Representations

The minimal surface $X : \Omega \longrightarrow S \subset \mathbb{R}^3$ in isothermal coordinate satisfies the equations

$$\triangle X = 0, \quad |X_u|^2 = |X_v|^2, \quad < X_u, X_v >= 0.$$

Let $z = u + iv$, $X(z, \bar{z}) = (x_1(z, \bar{z}), x_2(z, \bar{z}), x_2(z, \bar{z}))$ and

$$\phi(z) := \frac{\partial X}{\partial z}, \quad \phi_k(z) := \frac{\partial x_k}{\partial z}, \quad k = 1, 2, 3$$

where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$.
Then the conditions $|X_u|^2 = |X_v|^2$ and $< X_u, X_v >= 0$ are equivalent to $\phi_1^2(z) = \phi_2^2(z) = \phi_3^2(z) = 0$.
Put $f = \phi_1 - i\phi_2$ and $g = \phi_3/\phi_1 - i\phi_2$, then $fg = \phi_3$. f and $f.g^2$ are holomorphic and g is meromorphic.

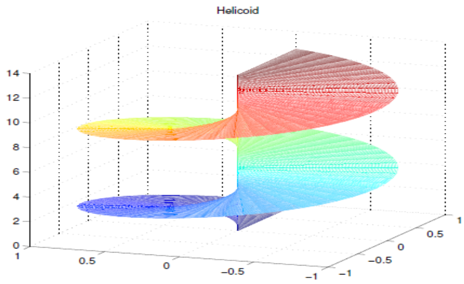
Theorem 3.1. ***(Weierstrass)****: If f be holomorphic and g be a meromorphic such that fg^2 be a holomorphic on the simply connected domain Ω then there exists a minimal surface $X(u, v) = (x_1, x_2, x_3)$ such that*

$$x_1 = Re \int_\Omega f(1 - g^2)dz$$
$$x_2 = \int_\Omega if(1 + g^2)dz$$
$$x_3 = \int_\Omega fgdz$$

Theorem 3.2. ***(Weierstrass)****: If g is holomorphic function and dh holomorphic $1-$form on the simply connected domain Ω then*

$$X(z) = Re \int_{z_0}^{z} (\frac{1}{2}(\frac{1}{g} - g), \frac{i}{2}(\frac{1}{g} + g), 1)dh$$

expresses the minimal surface [8].



4 The Adjoint surface of minimal surface

Let $X(u,v)$ for $(u,v)\in\Omega$ be a minimal surface in isothermal coordinate then the surface $X^*(u,v)$ is called the adjoint surface of $X(u,v)$ if $X_u = X^*_v$ and $X_v = X^*_u$.

$$(\triangle X = 0, \quad |X_u|^2 = |X_v|^2, \quad < X_u, X_v >= 0) \Longleftrightarrow$$
$$(\triangle X^* = 0, \quad |X^*_u|^2 = |X^*_v|^2, \quad < X^*_u, X^*_v >= 0) \Longleftrightarrow$$
$$(X_{z\bar{z}} = 0, \quad < X_u, X_v >= 0)$$

i.e. $X^*(u,v)$ is minimal surface.
Note: $X^{**} = -X$.

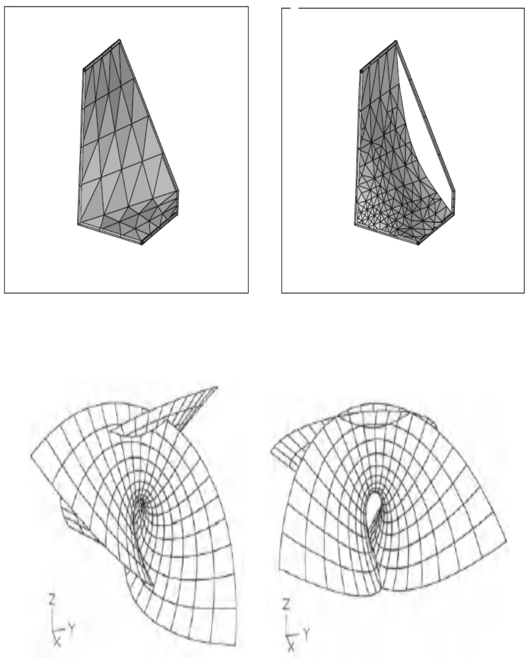


Figure 1: `Minimal surfaces and their adjoint`

Theorem 4.1. *The singular points z of a non constant minimal surface X on a domain Ω are isolated. They are exactly the zeros of the function $|X_u|$ in Ω [4].*

Definition 4.2. (**Branch points:**) The singular points of minimal surfaces are called branch points.
As we shall see the behavior of a minimal surface in the neighborhood of one of its singular points resembles the behavior of a holomorphic function $\varphi(z) = x_1(z) + ix_1^*(z)$ in the neighborhood of a zero of $\varphi'(z)$.

5 The Plateau Problem and the Partially Boundary Problem

Given in $\mathbb{R}^3$ a configuration $\Gamma =< \Gamma_1, \Gamma_2, ..., \Gamma_k >$ consisting of k closed and mutually disjoint Jordan curves Γ_j, find a minimal surface of prescribed Euler characteristic, orientable or not, that span Γ. If Γ is a closed Jordan curve that lies on a convex surface, then Γ bounds a disk-type minimal surface without self-intersections.
Another positive result, due to White:
If Γ is a closed Jordan curve in $\mathbb{R}^3$ with total curvature less or equal to 4π, then any minimal surface (independently of its topological type) is embedded up to and including the boundary, with no interior branch points.

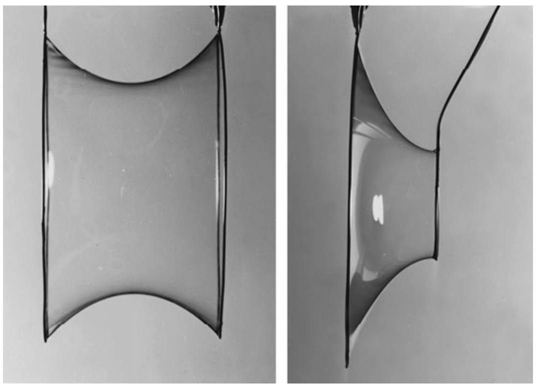


Figure 2: `A soap film experiment`

We will explain in the next sections discrete minimal surface and minimal surface in Finsler geometry. Also we will show the applications of minimal surface in architecture.

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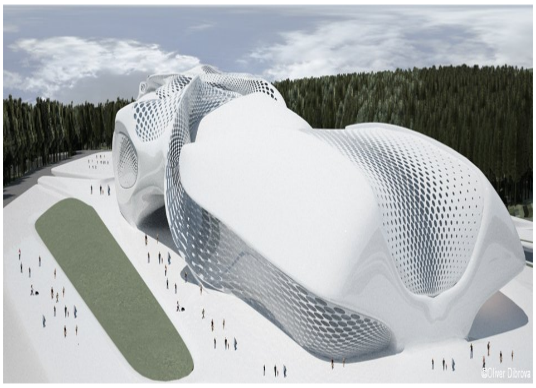

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Crámer's Probabilistic Model of Primes and The Zeta Function

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Abstract

In 1936 Harald Crámer proposed a probabilistic model to mimic the behavior of prime numbers. According to the prime number theorem we know that density of primes around a big number x is approximately $1/\ln x$. Crámer's model simply chooses every natural number n with probability $1/\ln n$, independently, and considers these numbers as "primes"! It is believed that this model captures some characteristics of distribution of primes e.g. asymptotics on size of large gaps in primes. In this paper we study the behavior of Zeta function for the Crámer's model. We prove that if $q_1, q_2, \cdots$ is a realization of primes from Crámer's model then the associated zeta function, $\zeta_C(s) = \prod_{i=1}^{\infty}(1 - q_i^{-s})^{-1}$, which is defined for $Re(s) > 1$, is almost surely continuable to a holomorphic function on $Re(s) > \frac{1}{2}$ but not to any larger domain.

Keywords: Zeta Function, Cramer's Model, Random Prime Numbers
Mathematics Subject Classification [2010]: 11M45, 30B20

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Talk Semigroups with apartness: constructive versions of some classical theorems pp.: 1–4

Semigroups with apartness: constructive versions of some classical theorems

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Abstract

The starting point of our work is the structure $(S, =, \neq, \cdot)$ called a semigroup with apartness. We examine and prove constructive analogues of some classical theorems, like, for example, isomorphism theorems and Cayley's theorem.

Keywords: Set with apartness, Semigroup with apartness, Coequivalence, Cocongruence.
Mathematics Subject Classification [2010]: 03F65, 20M99

1 Introduction

Following [10, Vol II], "The study of algebraic structures in an intuitionistic setting was undertaken by Heyting [7]." Within **BISH**, which forms the framework for our work, the history of constructive semigroups with an inequality began recently, [1]. In [3], [9] it is shown/announced that constructive algebraic structures with apartness can be applied in computer science (especially in computer programming) as well.

Definition 1.1. By an **apartness** on S (see [8]), we mean a binary relation $\not\simeq$ on S which satisfies the axioms of irreflexivity, symmetry and cotransitivity: $\neg(x \not\simeq x)$, $x \not\simeq y \Rightarrow y \not\simeq x$, $x \not\simeq z \Rightarrow \forall_y (x \not\simeq y \vee y \not\simeq z)$. We then say that $(S, \simeq, \not\simeq)$ is a **set with apartness**. An apartness is **tight** if $\neg(x \not\simeq y) \Rightarrow x \simeq y$.

Definition 1.2. Let $(A, \simeq, \not\simeq)$ be a set with apartness. A function $f : A \longrightarrow A$ is *strongly extensional*, or, for short, a *se-function* if whenever we have $f(a) \not\simeq f(b)$, then $a \not\simeq b$ follows, $a, b \in A$.

Following [6], [10], where the notion of commutative constructive semigroups with tight apartness has appeared, we define and put the notion of noncommutative constructive semigroups with "ordinary" apartness in the centre of our study.

Definition 1.3. A tuple $(S, \simeq, \not\simeq, \cdot)$ is a **semigroup with apartness** with $(S, \simeq, \not\simeq)$ as a set with apartness, $\cdot$ a binary operation on S which is associative, i.e. $\forall_{a,b,c \in S}$ $[(a \cdot b) \cdot c \simeq a \cdot (b \cdot c)]$, and strongly extensional, i.e. $\forall_{a,b,x,y \in S}$ $(a \cdot x \not\simeq b \cdot y \Rightarrow (a \not\simeq b \vee x \not\simeq y))$.

*Speaker

Theorem 1.4. *Let $(A, \backsimeq, \not\backsimeq)$ be a set with apartness, and let $f : A \longrightarrow A$ be an se-mapping. If S is a set of all se-functions from A to A, and $\circ$ composition of functions, then $(S, \backsimeq, \not\backsimeq, \circ)$ with*

$$f \backsimeq g \;\Leftrightarrow\; \forall_{x\in A}\,(f(x) \backsimeq g(x)) \quad \text{and} \quad f \not\backsimeq g \;\Leftrightarrow\; \exists_{x\in A}\,(f(x) \not\backsimeq g(x)),$$

is a semigroup with apartness.

Corollary 1.5. *Every semigroup with apartness se-embeds into the semigroup of all strongly extensional self-maps on a set.*

Remark 1.6. For undefined notions and notations as well as omitted proofs see [4], [5].

2 Main results

In order to give the constructive versions of the isomorphism theorems for sets and semigroups with apartness we need the following notions.

Definition 2.1. A binary relation α defined on semigroup with apartness S is
– consistent if $\alpha \subseteq \not\backsimeq$;
– cotransitive if $(x, z) \in \alpha \;\Rightarrow\; \forall_y\,((x, y) \in \alpha \,\vee\, (y, z) \in \alpha)$;
– coequivalence if it is consistent, symmetric and cotransitive;
– cocongruence if it is coequivalence that is cocompatible with multiplication, i.e. that is $\forall_{a,b,x,y\in S}\,((ax, by) \in \alpha \;\Rightarrow\; (a, b) \in \alpha \vee (x, y) \in \alpha)$.

Quotient sets (structures) are not part of **BISH**. In order to make them a part of **BISH** we need the following notions: equivalence, taken from **CLASS**, which behaves on constructive mathematics rules; coequivalence, a constructive notion, as well as link(s) between them - Theorem 2.3 (and Theorem 2.4) from [4]. Now we can formulate one of the main results - **Apartness Isomorphism Theorem** for sets with apartness.

Theorem 2.2. *Let $f : S \longrightarrow T$ be an se-mapping between sets with apartness. Then:*
(i) *the relation* $\operatorname{coker} f = \{(x, y) \in S \times S : f(x) \not\backsimeq f(y)\}$ *is a coequivalence on S (which we call the **cokernel** of f);*
(ii) $\operatorname{coker} f$ *is associated with the kernel of f, denoted, as usual, by* $\ker f$, *and* $\ker f \subseteq\sim \operatorname{coker} f$;
(iii) $(S/\ker f, \backsimeq, \not\backsimeq)$ *is a set with apartness, where*

$$a(\ker f) \backsimeq b(\ker f) \;\Leftrightarrow\; (a, b) \in \ker f$$
$$a(\ker f) \not\backsimeq b(\ker f) \;\Leftrightarrow\; (a, b) \in \operatorname{coker} f;$$

(iv) *the mapping $\theta : S/\ker f \longrightarrow T$, defined by $\theta(x(\ker f)) \backsimeq f(x)$, is a one-one, injective se-mapping such that $f \backsimeq \theta \circ \pi$; and*
(v) *if f maps S onto T, then θ is an apartness bijection.*

Proof: (i) The consistency of $\operatorname{coker} f$ is easy to prove: if $(x, y) \in \operatorname{coker} f$, then $f(x) \not\backsimeq f(y)$ and therefore $x \not\backsimeq y$. If $(x, y) \in \operatorname{coker} f$, then, by the symmetry of apartness in T, $f(y) \not\backsimeq f(x)$; so $(y, x) \in \operatorname{coker} f$. If $(x, y) \in \operatorname{coker} f$ and $z \in S$, i.e. $f(x) \not\backsimeq f(y)$

and $f(z) \in T$, then either $f(x) \not\simeq f(z)$ or $f(z) \not\simeq f(y)$; that is, either $(x,z) \in \operatorname{coker} f$ or $(z,y) \in \operatorname{coker} f$. Hence $\operatorname{coker} f$ is a coequivalence on S.

(ii) Let $(x,y) \in \operatorname{coker} f$ and $(y,z) \in \ker f$; then $f(x) \not\simeq f(y)$ and $f(y) \simeq f(z)$. Hence $f(x) \not\simeq f(z)$, i.e. $(x,z) \in \operatorname{coker} f$, and $\operatorname{coker} f$ is associated with $\ker f$. Now let $(x,y) \in \ker f$, so $f(x) \simeq f(y)$. If $(u,v) \in \operatorname{coker} f$, then, by the cotransitivity of $\operatorname{coker} f$, it follows that $(u,x) \in \operatorname{coker} f$ or $(x,y) \in \operatorname{coker} f$ or $(y,v) \in \operatorname{coker} f$. Thus either $(u,x) \in \operatorname{coker} f$ or $(y,v) \in \operatorname{coker} f$, and, by the consistency of $\operatorname{coker} f$, either $u \not\simeq x$ or $y \not\simeq v$; whence we have $(x,y) \not\simeq (u,v)$. Thus $(x,y) \triangleright \operatorname{coker} f$, or, equivalently $(x,y) \in \sim \operatorname{coker} f$.

(iii) This follows from the definition of $\not\simeq$ in $S/\ker f$ and (i).

(iv) Let us first prove that θ is well defined. Let $x(\ker f), y(\ker f) \in S/\ker f$ be such that $x(\ker f) \simeq y(\ker f)$; that is, $(x,y) \in \ker f$. Then we have $f(x) \simeq f(y)$, which, by the definition of θ, means that $\theta(x(\ker f)) \simeq \theta(y(\ker f))$. Now let $\theta(x(\ker f)) \simeq \theta(y(\ker f))$; then $f(x) \simeq f(y)$. Hence $(x,y) \in \ker f$, which implies that $x(\ker f) \simeq y(\ker f)$. Thus θ is one-one. Next let $\theta(x(\ker f)) \not\simeq \theta(y(\ker f))$; then $f(x) \not\simeq f(y)$. Hence $(x,y) \in \operatorname{coker} f$, which, by (iii), implies that $x(\ker f) \not\simeq y(\ker f)$. Thus θ is an se-mapping. Let $x(\ker f) \not\simeq y(\ker f)$; that is, by (iii), $(x,y) \in \operatorname{coker} f$. So we have $f(x) \not\simeq f(y)$, which, by the definition of θ means $\theta(x(\ker f)) \not\simeq \theta(y(\ker f))$. Thus θ is injective. On the other hand, by the definition of composition of functions, Theorem 2.4 ([4]), and the definition of θ, for each $x \in S$ we have $(\theta \circ \pi)(x) \simeq \theta(\pi(x)) \simeq \theta(x(\ker f)) \simeq f(x)$.

(v) Taking into account (iv), we have to prove only that θ is onto. Let $y \in T$. Then, as f is onto, there exists $x \in S$ such that $y \simeq f(x)$. On the other hand $\pi(x) \simeq x(\ker f)$. By (iv), we now have

$$y \simeq f(x) \simeq (\theta \circ \pi)(x) \simeq \theta(\pi(x)) \simeq \theta(x(\ker f)).$$

Thus θ is onto. □

Using this result we can prove another main result of this paper, **Apartness Isomorphism Theorem** for semigroups with apartness.

Theorem 2.3. *Let $f : S \longrightarrow T$ be an se-homomorphism between semigroups with apartness. Then:*

(i) *the relation* $\operatorname{coker} f$ *is a cocongruence on S associated with* $\ker f$*;*

(ii) $(S/\ker f, \simeq, \not\simeq, \cdot)$ *is a semigroup with apartness, where*

$$\begin{aligned} a(ker\, f) \simeq b(ker\, f) \ &\Leftrightarrow\ (a,b) \in ker\, f, \\ a(ker\, f) \not\simeq b(ker\, f) \ &\Leftrightarrow\ (a,b) \in \operatorname{coker} f, \\ a(ker\, f)\, b(ker\, f) \ &\simeq\ (ab)(ker\, f); \end{aligned}$$

(iii) *the mapping $\theta : S/ker\, f \longrightarrow T$, defined by $\theta(x(ker\, f)) \simeq f(x)$, is an apartness embedding such that $f \simeq \theta \circ \pi$; and*

(iv) *if f is onto, then θ is an apartness isomorphism.*

Results of several years long investigation, presented in [4], [5], present a semigroup facet of some relatively well established direction of constructive mathematics. Imprtant sourse of ideas and notions of our work is [2]. At the very end we want to emphasize that semigroups with apartness are a **new approach**, and **not** a new class of semigroups.

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On a conjecture of Richard Stanley*

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Abstract

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{K}$. In 1982, Stanley defined what is now called the Stanley depth of a multigraded S-module. He conjectured that Stanley depth is an upper for the depth of the module. This conjecture has been recently disproved by Duval et al., [2]. In this talk, we describe their counterexample. We also present the recent developments in this topic.

Keywords: Stanley depth, Monomial ideal, Cohen-Macaulay simplicial complex, Partitionable simplicial complex
Mathematics Subject Classification [2010]: 13C15, 13C13, 05E40

1 Introduction

Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \dots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[Z]$-module. Here, as usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by $\mathrm{sdepth}(\mathcal{D})$. The quantity

$$\mathrm{sdepth}(M) := \max\big\{\mathrm{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\big\}$$

is called the *Stanley depth* of M. For a reader friendly introduction to Stanley depth, we refer to [7] and for a nice survey on this topic, we refer to [3].

A $\mathbb{Z}^n$-graded S-module M is said to satisfies *Stanley's inequality* if

$$\mathrm{depth}(M) \leq \mathrm{sdepth}(M).$$

In fact, Stanley [11] conjectured that

Stanley depth conjecture. Every $\mathbb{Z}^n$-graded S-module satisfies Stanley's inequality.

This conjecture has been recently disproved in [2]. In this talk, we describe their counterexample. Time permitting, We will also present the recent developments in this topic.

*Will be presented in English
†Speaker

2 A counterexample for the Stanley's conjecture

In this section, we describe the counterexample invented in [2] to disprove the Stanley's conjecture. We first need to introduce some basic notions from the theory of simplicial complexes.

A *simplicial complex* Δ on the set of vertices $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ, the *size* of a face F is defined to be $|F|$ and its *dimension* is defined to be $|F| - 1$. (As usual, for a given finite set X, the number of elements of X is denoted by $|X|$.) The *dimension* of Δ which is denoted by $\dim \Delta$, is defined to be $d-1$, where $d = \max\{|F| \mid F \in \Delta\}$. A *facet* of Δ is a maximal face of Δ with respect to inclusion. We say that Δ is *pure* if all facets of Δ have the same cardinality.

One of the connections between the combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let Δ be a simplicial complex on $[n]$. For every subset $F \subseteq [n]$, we set $x_F = \prod_{i \in F} x_i$. The *Stanley–Reisner ideal of* Δ *over* $\mathbb{K}$ is the ideal I_Δ of S which is generated by those squarefree monomials x_F with $F \notin \Delta$. In other words, $I_\Delta = \langle x_F \mid F \in \mathcal{N}(\Delta) \rangle$, where $\mathcal{N}(\Delta)$ denotes the set of minimal nonfaces of Δ with respect to inclusion. The *Stanley–Reisner ring of* Δ *over* $\mathbb{K}$, denoted by $\mathbb{K}[\Delta]$, is defined to be $\mathbb{K}[\Delta] = S/I_\Delta$. We say that a simplicial complex Δ is *Cohen–Macaulay over* $\mathbb{K}$, if the Stanley–Reisner ring $\mathbb{K}[\Delta]$ of Δ is Cohen–Macaulay.

Definition 2.1. Let Δ be a pure simplicial complex with facets $F_1, \ldots, F_m$. A partitioning $\mathcal{P}$ of Δ is a decomposition into pairwise-disjoint Boolean intervals

$$\Delta = \bigsqcup_{i=1}^{m} [G_i, F_i],$$

where $G_1, \ldots, G_m$ are faces of Δ and

$$[G_i, F_i] = \{F \in \Delta \mid G_i \subseteq F \subseteq F_i\}.$$

Another well-known conjecture of Stanley [10] states that

Partitionability conjecture. Every Cohen-Macaulay simplicial complex is partitionable.

Herzog, Soleyman Jahan and Yassemi [4] proved that

Theorem 2.2. *The Stanley depth conjecture implies the Partitionability conjecture.*

Thus, in order to disprove the Stanley depth conjecture, it is enough to find a counterexample for the the Partitionability conjecture.

Let Δ be a simplicial complex. A *subcomplex* of Δ is a simplicial complex Γ with $\Gamma \subseteq \Delta$. A subcomplex is an *induced subcomplex* if it is of the form

$$\Delta\,|_W := \{\sigma \in \Delta \mid \sigma \subseteq W\},$$

for some $W \subseteq V$.

In the construction of the counterexample, one needs to work with the more general class of *relative simplicial complexes*. A relative complex Φ on V is a subset of 2^V that is convex: if $\rho, \tau \in \Phi$ and $\rho \subseteq \sigma \subseteq \tau$, then $\sigma \in \Phi$. Every relative complex can be expressed as a pair $\Phi = (\Delta, \Gamma) := \Delta \setminus \Gamma$, where Δ is a simplicial complex and $\Gamma \subseteq \Delta$ is a subcomplex. Note that there are infinitely many possibilities for the pair Δ, Γ.

The following technical lemma will be central to the construction.

Lemma 2.3. *[2, Proposition 2.3] Let $\Delta_1, \dots, \Delta_t$ be d-dimensional Cohen-Macaulay simplicial complexes on disjoint vertex sets. Let Γ be a Cohen-Macaulay simplicial complex of dimension $d-1$ or d, and suppose that each Δ_i contains a copy of Γ as an induced subcomplex. Then the complex obtained from $\Delta_1, \dots, \Delta_t$ by identifying the t copies of Γ is Cohen-Macaulay.*

The following theorem gives a general construction that reduces the problem of finding a counterexample to the problem of constructing a certain kind of non-partitionable Cohen-Macaulay relative complex.

Theorem 2.4. *[2, Theorem 3.1] Let $Q = (X, A)$ be a relative complex such that*

(i) X and A are Cohen-Macaulay;

(ii) A is an induced subcomplex of X of codimension at most 1; and

(iii) Q is not partitionable.

Let t be the total number of faces of A, let $N > t$, and let $C = C_N$ be the simplicial complex constructed from N disjoint copies of X identified along the subcomplex A. Then C is Cohen-Macaulay and not partitionable.

Thus, in order to construct the counterexample, it is enough to construct a relative complex which satisfies the conditions (i), (ii) and (iii) of Theorem 2.4.

Construction. The construction begins with Ziegler's nonshellable 3-ball Z, which is a nonshellable triangulation of the 3-ball with 10 vertices labeled $0, 1, \dots, 9$ and the following 21 facets:

$$\begin{array}{c} 0123,\ 0125,\ 0237,\ 0256,\ 0267,\ 1234,\ 1249 \\ 1256,\ 1269,\ 1347,\ 1457,\ 1458,\ 1489,\ 1569 \\ 1589,\ 2348,\ 2367,\ 2368,\ 3478,\ 3678,\ 4578 \end{array}$$

Then Z is Cohen-Macaulay. Let B be the induced subcomplex $Z\mid_{\{0,2,3,4,6,7,8\}}$. That is, B is the pure 3-dimensional complex with facets

$$0237,\ 0267,\ 2367,\ 2368,\ 2348,\ 3678,\ 3478$$

Then B is Cohen-Macaulay (in fact, the above order is a shelling of B) and one can check that the relative complex (Z, B) is not partitionable. Therefore, thanks to Lemma 2.3 and theorem 2.4, this construction provides a counterexample for the partitionability conjecture.

3 More results about the Stanley depth of monomial ideals

In this section, we list some recent results about the Stanley depth of monomial ideals and their quotients. The first result provides a method for comparing the Stanley depth of factors of monomial ideals.

Theorem 3.1. *[8, Theorem 2.1] Let $I_2 \subsetneq I_1$ and $J_2 \subsetneq J_1$ be monomial ideals in S. Assume that there exists a function $\phi : \mathrm{Mon}(S) \to \mathrm{Mon}(S)$, such that the following conditions are satisfied.*

(i) For every monomial $u \in \mathrm{Mon}(S)$, $u \in I_1$ if and only if $\phi(u) \in J_1$.

(ii) For every monomial $u \in \mathrm{Mon}(S)$, $u \in I_2$ if and only if $\phi(u) \in J_2$.

(iii) For every Stanley space $u\mathbb{K}[Z] \subseteq S$ and every monomial $v \in \mathrm{Mon}(S)$, $v \in u\mathbb{K}[Z]$ if and only if $\phi(v) \in \phi(u)\mathbb{K}[Z]$.

Then

$$\mathrm{sdepth}(I_1/I_2) \geq \mathrm{sdepth}(J_1/J_2).$$

Theorem 3.1 has interesting corollaries.

Corollary 3.2. *[1, 5] Let $J \subsetneq I$ be monomial ideals in S such that $\sqrt{I} \neq \sqrt{J}$. Then*

$$\mathrm{sdepth}(I/J) \leq \mathrm{sdepth}(\sqrt{I}/\sqrt{J}).$$

In the following corollary, $\overline{I}$ denotes the integral closure of the ideal I

Corollary 3.3. *[9] Let $J \subsetneq I$ be two monomial ideals in S such that $\overline{I} \neq \overline{J}$. Then for every integer $k \geq 1$*

$$\mathrm{sdepth}(\overline{I^k}/\overline{J^k}) \leq \mathrm{sdepth}(\overline{I}/\overline{J}).$$

Corollary 3.4. *Let $J \subsetneq I$ be monomial ideals in S and $v \in S$ be a monomial such that $(I : v) \neq (J : v)$. Then*

$$\mathrm{sdepth}(I/J) \leq \mathrm{sdepth}((I : v)/(J : v)).$$

Let I be a squarefree monomial ideal in S and suppose that I has the irredundant primary decomposition

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r,$$

where every $\mathfrak{p}_i$ is an ideal of S generated by a subset of the variables of S. Let k be a positive integer. The kth *symbolic power* of I, denoted by $I^{(k)}$, is defined to be

$$I^{(k)} = \mathfrak{p}_1^k \cap \ldots \cap \mathfrak{p}_r^k.$$

Corollary 3.5. *[8] Let $J \subseteq I$ be squarefree monomial ideals in S. Then for every pair of integers $k, s \geq 1$*

$$\mathrm{sdepth}(I^{(ks)}/J^{(ks)}) \leq \mathrm{sdepth}(I^{(s)}/J^{(s)}).$$

Definition 3.6. Let $J \subsetneq I$ be two monomial ideals. Assume that $G(I)$ and $G(J)$ are the sets of minimal monomial generators of I and J, respectively. The *lcm number* of I/J, denoted by $l(I/J)$, is the maximum integer t for which there exist monomials $u_1, \dots, u_t \in G(I) \cup G(J)$ such that

$$u_1 \neq \mathrm{lcm}(u_1, u_2) \neq \ldots \neq \mathrm{lcm}(u_1, u_2, \dots, u_t).$$

The following theorem gives a lower bound for the Stanley depth of factors of monomial ideals in terms of the lcm number.

Theorem 3.7. *[6, Theorem 2.4] Let $J \subsetneq I$ be two monomial ideals of S. Then* $\mathrm{depth}(I/J) \geq n - l(I/J) + 1$ *and* $\mathrm{sdepth}(I/J) \geq n - l(I/J) + 1$.

Using the above theorem, we are able to prove the Stanley's inequality for some classes of monomial ideals.

Theorem 3.8. *[6, Theorem 4.4] Let I be a monomial ideal of S. If $l(I) \leq 3$, then I and S/I satisfy Stanley's inequality.*

Theorem 3.9. *[6, Corollary 4.5] Let I be a monomial ideal of S such that S/I is Gorenstein. If $l(I) \leq 4$, then I and S/I satisfy Stanley's inequality.*

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Steiner triple systems with forbidden configurations

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Abstract

We discuss several characterizations of special classes of Steiner triple systems in terms of forbidden configurations. Among other things, we present such a characterization for strongly anti-Pasch Steiner triple systems.

Keywords: Steiner triple systems, Pasch configuration
Mathematics Subject Classification [2010]: 05B07, 05B05

1 Introduction

Steiner triple systems are classical objects in combinatorial design theory. A *Steiner triple system* (STS for short) is a pair $S = (X, \mathcal{B})$ where X is a set of v points and $\mathcal{B}$ is a set of 3-subsets of X, called the triples of S, such that every two distinct points are contained in exactly one triple of S. One of the most classical results in combinatorics asserts that a Steiner triple system with v points exists if and only if $v \equiv 1, 3 \pmod 6$, $v \geq 3$. See [2] for a through treatment of enormous results on Steiner triple systems.

There are several prominent classes of Steiner triple systems of which we recall projective, affine and Hall STS in what follows. A *projective Steiner triple system* $\mathrm{PG}(d, 2)$ is the Steiner triple system with $2^{d+1} - 1$ points corresponding to non-zero $(d + 1)$-dimensional vectors over $\mathbb{Z}_2$ for $d \geq 1$. Three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ form a triple of $\mathrm{PG}(d, 2)$ if $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. The smallest non-trivial projective Steiner triple system is $\mathrm{PG}(2, 2)$ which is indeed the Fano plane. An *affine Steiner triple system* $\mathrm{AG}(d, 3)$ is the Steiner triple system with 3^d points corresponding to d-dimensional vectors over $\mathbb{Z}_3$ for $d \geq 1$. Three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ form a triple of $\mathrm{AG}(d, 3)$ if $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. The smallest non-trivial affine Steiner triple system, $\mathrm{AG}(2, 3)$, is the unique Steiner triple system with nine points which we denote it by S_9. Another interesting family of Steiner triple systems is the class of *Hall triple systems*. A Steiner triple system S is a Hall triple system if for every point x of S, there exists an involutory automorphism of S that fixes only the point x. Hall [5] showed that Hall triple systems are "locally" affine Steiner triple systems. To be more precise, a STS is a Hall STS if and only if every Steiner triple system induced by the points of two non-disjoint triples of S is isomorphic to S_9.

There are several characterizations for certain classes of combinatorial objects in terms of well-described forbidden substructures. For instance, the celebrated Kuratowski's theorem asserts that a graph is planar if and only if it does not contain a subdivision of one of the graphs $K_{3,3}$ or K_5. In analogy, it is natural to ask whether special classes of Steiner triple systems can be characterized in terms of *forbidden configurations*. By a configuration C we mean a set of points and triples such that each pair of points is in at most one of the triples, and we say that a Steiner triple system S contains C if there is an injective mapping of the points of C to the points of S such that the image of any triple of C is a triple of S. An important configuration of triple systems is the *Pasch configuration* (or *quadrilateral*) which is a set of four triples

$$\{a, b, c\},\ \{a, d, e\},\ \{f, b, d\},\ \{f, c, e\}$$

such that all elements a, b, c, d, e, f are distinct. See Figure 1 for an illustration of this and two other configurations namely C_{14} and anti-mitre.

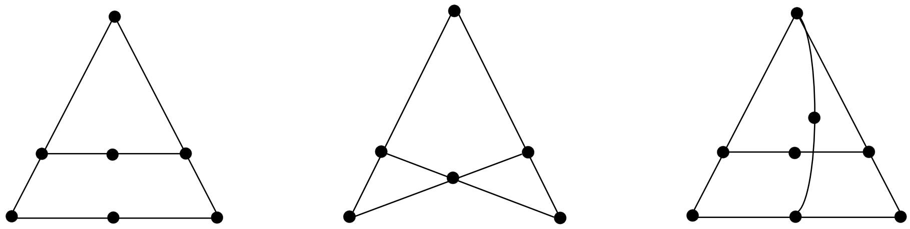

Figure 1: The configurations C_{14}, Pasch and anti-mitre, respectively

For the aforementioned classes of Steiner triple systems, characterization in terms of forbidden configurations is possible.

Theorem 1.1. ([3, 8]) *A Steiner triple system is projective if and only if it contains no configuration* C_{14}.

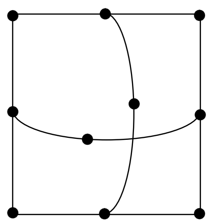

Figure 2: The configurations C_A and C_B, respectively, of Theorem 1.2

Theorem 1.2. ([6], see also [7]) *Let S be a Steiner triple system.*

(i) *S is an Hall STS if and only if it does not contain any Pasch or anti-mitre configuration.*

(ii) *S is an affine STS if and only if it does not contain any Pasch, C_A, or C_B configuration.*

2 Strongly anti-Pasch Steiner triple systems

Some STS contains Pasch configuration and some does not. In fact any projective STS on v points contains exactly $v(v-1)(v-3)/24$ distinct Pasch configurations and on the other hand any affine STS does not contain any. STS with no Pasch configurations are called *anti-Pasch* (or *quadrilateral-free*). For a long time it had been conjectured that for any admissible $v \equiv 1, 3 \pmod 6$, except $v = 7, 13$, an anti-Pasch STS exists. This conjecture was finally proved in [4].

Let $S = (X, \mathcal{B})$ be a STS on $v \geq 7$ points. Let $\binom{X}{3}$ denote the set of all 3-subsets of X. If $\{a, b, c\} \in \binom{X}{3} \setminus \mathcal{B}$, then there are distinct points $x, y, z \in X$ such that

$$\{\{a, b, x\}, \{a, c, y\}, \{b, c, z\}\} \subset \mathcal{B}. \tag{1}$$

Note that the three triples of (1) together with $\{x, y, z\}$ make a Pasch configuration.

We see that for a given STS $S = (X, \mathcal{B})$ and $\{x, y, z\} \in \binom{X}{3} \setminus \mathcal{B}$ it may happen that there exist three triples in $\mathcal{B}$ which make a Pasch configuration together with $\{x, y, z\}$. We are interested in the extremal case that this property holds for all $\{x, y, z\} \in \binom{X}{3} \setminus \mathcal{B}$. It is seen that in this extremal case, S must be anti-Pasch and the Pasch configuration C with $\{x, y, z\} \in C$ and $|C \cap \mathcal{B}| = 3$ is unique (see [1]). This motivates the following definition.

Definition 2.1. Let $S = (X, \mathcal{B})$ be a STS. We say that S is *strongly anti-Pasch* if for any $\{x, y, z\} \in \binom{X}{3} \setminus \mathcal{B}$ there exits $a, b, c \in X$ such that $\{\{a, b, x\}, \{a, c, y\}, \{b, c, z\}\} \subset \mathcal{B}$.

In analogy to the characterizations of projective, affine, and Hall STS in terms of forbidden configuration, we present the following for strongly anti-Pasch STS.

Theorem 2.2. *A STS is Strongly anti-Pasch if and only if it contains neither Pasch configuration nor the configuration Q of Figure 3.*

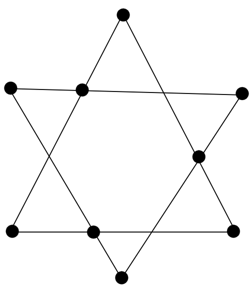

Figure 3: The configuration Q

3 Questions

Any strongly anti-Pasch STS which we are known of is a Hall STS. This motivates us to ask the following:

Question 1. Is it true that any strongly anti-Pasch STS is a Hall STS?

By Theorem 1.2 any STS S is Hall if and only if S contains neither Pasch nor the anti-mitre configuration. The question raises that:

Question 2. Is it true that in an anti-Pasch STS the existences of anti-mitre configuration and the configuration Q are equivalent?

Thus answering Question 2 would be a possible direction in studying Question 1.

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A new view of supremum, infimum, maximum and minimum

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Abstract

Let X be a set and let R be a relation on X (not necessary partially order relation). Let E be a subset of X. We define the left bound, right bound, supremum, infimum, maximum and minimum of E with respect to relation R. Also, we generalize the concept of lattices and by some examples we show that our definitions are real extensions of the old ones. We prove some new fixed point theorems. Among many other things, we investigate several results and theorems of set theory by replacing "relation R" instead of "partially order relation". The results of the present paper can be useful in economic, game theory, computer sciences and information sciences.

Keywords: Poset; supremum; infimum; maximum; minimum
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In mathematics, the supremum, infimum, maximum and minimum are define for subsets of partially ordered sets. These concepts are important in analysis (especially in Lebesgue integration), algebra, geometry, applied mathematics, mathematical physics and other sciences. In this paper, we define the left bound, right bound, supremum, infimum, maximum and minimum for subset E of the set X with respect to the relation $R \subseteq X \times X$. Hence, we would like to study the set theory by replacing "relation R" instead of "partially order relation".

From now on, we suppose that X is a nonempty set and $R \subseteq X \times X$ is a relation on X. The following definition is the main definition of this paper.

Definition 1.1. Let $E \subseteq X$ be a subset of X. Then

• $r \in X$ is called a right bound for E (with respect to relation R) if eRr for all $e \in E$. We denote by $\mathcal{R}(E)$ the set of all right bounds of E.

• $l \in X$ is called a left bound for E (with respect to relation R) if lRe for all $e \in E$. We denote by $\mathcal{L}(E)$ the set of all left bounds of E.

• $b \in X$ is called a bound for E (with respect to relation R) if $e \in \mathcal{R}(E) \cap \mathcal{L}(E)$; in the other words, eRb and bRe for all $e \in E$. We denote by $\mathcal{B}(E)$ the set of all bounds of E.

• We define supremum and infimum of E (with respect to relation R) as follows:

$$sup(E) := \{r \in \mathcal{R}(E) : r \in \mathcal{L}(\mathcal{R}(E))\} = \mathcal{R}(E) \cap \mathcal{L}(\mathcal{R}(E)),$$

*Speaker

$$inf(E) := \{l \in \mathcal{L}(E) : l \in \mathcal{R}(\mathcal{L}(E))\} = \mathcal{L}(E) \cap \mathcal{R}(\mathcal{L}(E)).$$

• Moreover, we define maximum and minimum of E as follows:

$$max(E) := E \cap sup(E),$$

$$min(E) := E \cap inf(E).$$

• Let $n \in \mathbb{N}$. Then X is said to be a $n - R-$lattice if for every subset E of X with card(E)=n, we have $sup(E) \neq \phi$ and $inf(E) \neq \phi$.
• X is said to be a permanently $R-$lattice if for every nonempty finite subset E of X, we have $sup(E) \neq \phi$ and $inf(E) \neq \phi$.
• X is said to be a complete $R-$lattice if for every nonempty subset E of X, we have $sup(E) \neq \phi$ and $inf(E) \neq \phi$.
• X is said to be a strongly complete $R-$lattice if for every nonempty subset E of X, we have $sup(E) \cap inf(E) \neq \phi$.
• X is said to be a high strongly complete $R-$lattice if we have $\cap_{E \subseteq X, E \neq \phi}[sup(E) \cap inf(E)] \neq \phi$.

Example 1.2. Let X be a set with $card(X) > 1$. Let $R := \{(x, x) : x \in X\}$. Let $E := \{x\}$ and $x \in X$ be fixed. Then we have $\mathcal{L}(E) = \mathcal{R}(E) = \{x\}$. It follows that

$$sup(E) = inf(E) = E \neq \phi.$$

It follows that X is 1-R-lattice. On the other hand for every subset F of X with $card(F) > 1$, we have $\mathcal{L}(F) = \mathcal{R}(F) = \phi$. It follows that

$$sup(F) = inf(F) = \phi.$$

Then X is not n-R-Lattice for $n > 1$.

Example 1.3. Let X be a set with $card(X) > 1$. Let $U \neq \phi$ be a subset of X. Put $R := X \times U \cup U \times X$. Then we have $\mathcal{L}(E) = \mathcal{R}(E) = U$ for all nonempty subset E of X. It follows that

$$sup(E) = inf(E) = U \neq \phi.$$

It follows that X is high strongly complete $R-$lattice.

2 some important results

In this section, we prove some basic results.

It is easy to see that $\mathcal{R}(E) = \cap_{e \in E}\mathcal{R}(\{e\})$ and $\mathcal{L}(E) = \cap_{e \in E}\mathcal{L}(\{e\})$ for all subset E of X.

Note that in posets a right bound is an upper bound and the left bound is the lower bound. Also, $sup(E) = a$ as an element of poset, if and only if $sup(E) = \{a\}$ by our definition. We have the same situation for $inf(E), max(E)$ and $min(E)$.

A mapping $f : X \to X$ is called $R-$preserving if

$$\forall x, y \in X; xRy \Rightarrow f(x)Rf(y).$$

We denote by $Fix(R)$ the set of all $x \in X$ such that xRx. Also, we denote by R^C the set $X \times X - R$ (the compliment of R). Moreover, we denote by R^* for dual of relation R given by $\{(b,a) : (a,b) \in R\}$. It is easy to see that for every nonempty set X and relation R on X, and for every subset E of X, we have

$$\mathcal{R}_R(E) = \mathcal{L}_{R^*}(E), \mathcal{L}_R(E) = \mathcal{R}_{R^*}(E),$$

it follows that

$$\mathcal{B}_R(E) = \mathcal{B}_{R^*}(E), sup_R(E) = inf_{R^*}(E), inf_R(E) = sup_{R^*}(E),$$

for all $E \subseteq X$.

Lemma 2.1. *Let $E \subseteq X$. Then we have*
(1) $max(E) = E \cap \mathcal{R}(E)$,
(2) $min(E) = E \cap \mathcal{L}(E)$,
(3) $sup(E) = min(\mathcal{R}(E))$,
(4) $inf(E) = max(\mathcal{L}(E))$.
(5) $max(E) \cup min(E) \cup sup(E) \cup inf(E) \subseteq Fix(R)$.

It is easy to see that the relation R is reflexive on X if and only if $Fix(R) = X$. It follows from (5) that X is $R-$reflexive if and only if

$$(\cup_{E \subseteq X} max(E)) \cup (\cup_{E \subseteq X} min(E)) \cup (\cup_{E \subseteq X} sup(E)) \cup (\cup_{E \subseteq X} inf(E)) = X.$$

Lemma 2.2. *R is antisymmetric if and only if for each subset E of X with $B(E) \neq \phi$, E is singleton and $B(E) = E$.*

Theorem 2.3. *X is strongly complete $R-$lattice if if and only if $B(X) \neq \phi$.*

Lemma 2.4. *$\mathcal{L}(E) \times E \subseteq R, inf(E) \times E \subseteq R, E \times \mathcal{R}(E) \subseteq R, E \times sup(E) \subseteq R, \mathcal{B}(E) \times E \subseteq R$ and $E \times \mathcal{B}(E) \subseteq R$ for all E contained in X.*

Lemma 2.5. *The following assertions are equivalent for every subset E of X.*
(1) E is contained in $\mathcal{L}(E)$,
(2) E is contained in $\mathcal{R}(E)$,
(3) E is contained in $\mathcal{B}(E)$,
(4) $E \times E$ is contained in R.

We have the following lemma for antisymmetric relations.

Lemma 2.6. *Let R be an anti-symmetric relation on a set X and let $E \subseteq X$. Then we have $card(max(E)) \leq 1$, $card(min(E)) \leq 1$, $card(sup(E)) \leq 1$ and $card(inf(E)) \leq 1$.*

Lemma 2.7. *Let X be a set and R relation on X. Then the following assertions hold. i) R is a function if and only if $card(\mathcal{L}(E)) \leq 1$ for all non-empty subset E of X.*
ii) R is reflexive if and only if $Fix(R) = X$.

Lemma 2.8. *Let R be a relation on a set X and let $U \subseteq V \subseteq X$. Then we have*

$$\mathcal{R}(V) \subseteq \mathcal{R}(U)$$

and

$$\mathcal{L}(V) \subseteq \mathcal{L}(U).$$

Proof. It is straightforward. □

Definition 2.9. Let X be a set and $R \subseteq X \times X$ be a relation on X. Let $f : X \to X$ be a mapping. Then
R is said to be $f-$antisymmetric if

$$\forall a \in X; ((aRf(a), f(a)Ra) \Rightarrow a = f(a)).$$

R is said to be $f-$transitive if

$$\forall a, b \in X; \big(((aRf(a), f(a)Rb) \Rightarrow aRb) \wedge ((bRf(a), f(a)Ra) \Rightarrow bRa)\big).$$

It is easy to see that
(i) every transitive relation is $f-$transitive;
(ii) every antisymmetric relation is $f-$antisymmetric.
By the following examples, we show that the converse of above statements are not correct.

Example 2.10. Let $X = \mathbb{R}$ the set of real numbers and let $f : X \to X$ defined by $f(x) = 1$ for all $x \in X$. Then one can easily to check that the relation

$$R_1 := \{(r, r+1) : r \in \mathbb{R}\} \cup \{(r+1, r) : r \in \mathbb{R}\} \cup \{(1, 1)\}$$

is $f-$antisymmetric, but R_1 is not antisymmetric. Moreover, we can see that R_1 is not $f-$transitive. To this end, put $a = 2, b = 0$. Then we have $2R_1f(2)$ and $f(2)R_10$. But we have not $2R_10$. Now, we put

$$R_2 := R_1 - \{(0, 1), (1, 0), (1, 2)\}.$$

Then it is easy to see that R_2 is $f-$transitive and it is not transitive (we have $4R_23$ and $3R_22$ but we have not $4R_22$).

Now, we generalize the Tarski fixed point theorem as follows.

Theorem 2.11. *Let (X, R) be a non-empty complete R-lattice.*
If $f : X \to X$ is a monotone mapping, such that R is $f-$transitive and $f-$antisymmetric. Then $Fix(f \neq \phi$.

Proof. i) Let $Fix(f)$ denote the set of fixed points of f. We show that $Fix(f)$ is non-empty and $max(Fix(f)) \neq \phi$ and $min(Fix(f)) \neq \phi$. Since X is a complete R-lattice, we have $sup(X) \neq \phi$ and $inf(X) \neq \phi$. On the other hand, we have $sup(X) = sup(X) \cap X = max(X)$, $inf(X) = inf(X) \cap X = min(X)$. Let $a_0 \in min(X)$ and $b_0 \in max(X)$. Let $A := \{x \in X : xRf(x) \quad or \quad x = f(x)\}$. Then we have $a_0Rf(a_0)$, and $a_0 \in A$. Hence A is non-empty. Since X is complete, then $sup(A) \neq \phi$. Let $\beta \in sup(A)$. We show that $\beta \in Fix(f)$. We claim first that $\beta \in A$. To this end, note that for any $x \in A$, since $xR\beta$ and f is monotone, then $f(x)Rf(\beta)$. Moreover, $x \in A$ then we have $xRf(x)$ or $x = f(x)$. On the other hand R is $f-$transitive, then $xRf(\beta)$. Since this holds for every $x \in A$, this establishes that $f(\beta)$ is an upper bound of A. On the other hand, β is a supremum of A. Then $\beta Rf(\beta)$; which means that β satisfies the condition for inclusion in A. We next claim that for any $x \in A$, $f(x) \in A$. To this end, note that by definition, if $x \in A$ then $xRf(x)$ or $x = f(x)$. Since f is monotone, then $f(x)Rf(f(x))$ or $f(x) = f(f(x))(= x)$, which is the condition for $f(x) \in A$, which establishes the claim. This implies, in particular, that since $\beta \in A$, then $f(\beta) \in A$. Since β is an upper bound for A, this means that $f(\beta)R\beta$. Since R is $f-$antisymmetric, then we have $\beta = f(\beta)$. Hence, $\beta \in Fix(f)$. On the other hand, by definition of A, we have $Fix(f) \subseteq A$ and hence, since β is an upper bound of A, it is an upper bound of $Fix(f)$. So, $\beta \in max(Fix(f))$. So, we have

$$(\phi \neq)sup(A) \subseteq max(Fix(f)) \subseteq Fix(f).$$

A similar argument establishes that $inf(B) \neq \phi$ if $B := \{x \in X : f(x)Rx \quad or \quad f(x) = x\}$ and $inf(B) \subseteq Fix(f)$. Let $\alpha \in inf(B)$. We have $Fix(f) \subseteq B$ and since α is a lower bound of B, it is a lower bound of $Fix(f)$. Also, we can show that $\alpha \in Fix(f)$. It follows that $\alpha \in minFix(f)$. So, we have

$$inf(B) \subseteq min(Fix(f)) \subseteq Fix(f).$$

□

Question: Is the set of fixed points of f a complete R-lattice?

From now on, we suppose that (X, d) is a metric space and R is a relation on X. We denote by (X, d, R) this metric space with this relation.

Definition 2.12. Let (X, d, R) be a metric space with relation R. a) A sequence $\{x_n\}$ is called $R - increasing$ if

$$\forall n \in \mathbb{N}, x_n R x_{n+1}.$$

b) A sequence $\{x_n\}$ is called $R - decreasing$ if

$$\forall n \in \mathbb{N}, x_{n+1} R x_n.$$

b) A sequence $\{x_n\}$ is called $R - monotone$ if it is $R - increasing$ or $R - decreasing$.
c) (X, d) is called weakly $R - complete$ if every Cauchy $R - monotone$ sequence in X is convergent.
d) Let $k \in [0, 1)$. A mapping $f : X \to X$ is called $R - k - contraction$ if

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ with xRy.
e) A mapping $f : X \to X$ is called weakly $R - continuous$ in $x \in X$ if for every $R - monotone$ sequence $\{x_n\}$ in X, if $x_n \to x$, then $f(x_n) \to f(x)$. Also, $f : X \to X$ is called weakly $R - continuous$ on X if it is weakly $R - continuous$ in x for all $x \in X$.

It is well known that that every sequence in $\mathbb{R}$ has a monotone subsequence and it has a key role in BolzanoWeierstrass theorem. We will show that this assertion is true in every chain (X, R).

Theorem 2.13. *Let R be a relation on X (not necessary metric space). Then we have the following assertions:*
i) If X is a chain, then every sequence in X has an $R - monotone$ subsequence.

ii) If in part i), the relation R is transitive, then every sequence in X has a $WR-$ subsequence.
iii) If every sequence in X has $WR-$ subsequence, then R is reflexive.
v) If every sequence in X has $R-$ monotone subsequence, then R is reflexive.

Proof. i) Let $\{x_n\}$ be a sequence in X. Let

$$S := \{n \in \mathbb{N} : \forall m \in \mathbb{N}, m > n \iff \quad x_m R x_n\}.$$

If S is infinite, then $\{x_n\}$ has $R-$ *increasing* subsequence. If S is finite, put $n_1 := max(S) + 1$. Then $n_1 \in \mathbb{N} - S$. So by definition of S, there exists $n_2 > n_1$ such that x_{n_2} has not relation R with x_{n_1}. On the other hand X is a chain. Then we have $x_{n_1} R x_{n_2}$. Also, we have $n_2 \in \mathbb{N} - S$. Similarly, there exists $n_3 > n_2$ such that $x_{n_2} R x_{n_3}$. Continuing this process, we can find an $R-$ *decreasing* subsequence of $\{x_n\}$.
ii) Let R be transitive. Then every $R-$ *increasing* sequence is $R-$ *sequence*; every $R-$ *decreasing* sequence is R^*- *sequence*. Hence, every $R-$ *monotone* sequence is $WR-$ *sequence*. Then easily ii) follows from i).
iii) Let $x \in X$. Put $x_n := x$ for all $n \in \mathbb{N}$. $\{x_n\}$ has $WR-$ *subsequence*. Then there exist $n_1, n_2 \in \mathbb{N}$ such that $x = x_{n_1} R x_{n_2} = x$. It follows that xRx.
The proof of v) is similar to iii). □

The following assertions are true in $\mathbb{R}$ with Euclidean metric.
1) Every bounded monotone sequence is convergent.
2) Every monotone sequence with a convergent subsequence is convergent.
3) If $sup(A)(or \quad inf(A))$ exists in $\mathbb{R}$, then $sup(A)(or \quad inf(A))$ belongs closer of A for all subset A of $\mathbb{R}$.

The same assertions are not true for general (X, d, R). For example, if $X := \mathbb{R}$ with Euclidean metric, $R := \{(-1, 1), (1, -1)\}$ and $x_n := (-1)^n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a bounded $R-$ *sequence* (with convergent subsequence) which is not convergent. Also, if $X := \mathbb{R}$ with Euclidean metric, $R := [0, 1] \times \mathbb{R} \cup \mathbb{R} \times [0, 1]$, then $sup([2, 3]) = inf([2, 3]) = [0, 1]$ is not contained in closer of $[2, 3]$.
The main result of papers [3] and [2], is the following theorem.

Theorem 2.14. *Let X be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. If F is continuous, monotone mapping from X into X such that*

- *there exists $k \in (0, 1)$ with $d(F(x), F(y)) \leq kd(x, y), \forall x \geq y$. If there exists $x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$, then F has a fixed point. Furthermore, if every pair $x, y \in X$ has a lower bound or an upper bound, then the fixed point of F is unique and F is a Picard operator (briefly PO), that is, F has a unique fixed point x^* and $\lim_{n\to\infty} F^n(x) = x^*$ for all $x \in X$.*

Now, we would like to generalize above theorem.
In the main theorem of [3], the authors consider a complete metric space (X, d). We can replace "weakly $R-$ *complete* metric space" instead of "complete metric space", also, the authors consider a continuous mapping where, we can replace $R-$ *continuous*

mapping instead of continuous mapping. Moreover, they consider only relation "$\geq$' ' and $\geq -k - contraction$, so, we can replace relation R instead of $\geq$ and $R - k - contraction$ instead of $\geq -k - contraction$ as follows.

Theorem 2.15. *Let (X, d, R) be a metric space with relation R such that for every pair $x, y \in X$, $\mathcal{L}(\{x, y\}) \neq \phi$, or $\mathcal{R}(\{x, y\}) \neq \phi$. Let $k \in (0, 1)$ be fixed. Let (X, d) be waekly $R-$complete and F be a weakly $R-$continuous, $R-$preserving and $R-k-$contraction from X into X. If there exists $x_0 \in X$ such that $x_0 R F(x_0)$ or $F(x_0) R x_0$, then F is a Picard operator (briefly PO), that is, F has a unique fixed point x^* and $\lim_{n\to\infty} F^n(x) = x^*$ for all $x \in X$.*

In [2], authors replaced the condition
$(*)$ $''$ if a non-increasing sequence $x_n \to x \in X$; then $x \leq x_n$ for all $n \in \mathbb{N}''$,
instead of continuity of F in the main results of paper. This condition study in metric space (X, d) with partially order relation $'' \leq ''$ (or $\geq$). We can write the general form of condition $(*)$ by replacing arbitrary relation R instead of partially order relation $\leq$ (or $\geq$) as follows:
$(**)$ if $\{x_n\}$ is a sequence such that $\forall n \in \mathbb{N}; x_n R x_{n+1}$ and $x_n \to x \in X$; then $x_n R x$, for all $n \in \mathbb{N}$,
and $(***)$ if $\{x_n\}$ is a sequence such that $\forall n \in \mathbb{N}; x_{n+1} R x_n$ and $x_n \to x \in X$; then $x R x_n$, for all $n \in \mathbb{N}$.

Using conditions $(**)$ and $(***)$ to prove the following theorem in metric space (X, d) with arbitrary relation R.

Theorem 2.16. *Let $k \in (0, 1)$ be fixed. Let (X, d, R) be a metric space with relation R such that conditions $(**)$ and $(***)$ hold. Then every $R - k - contraction$ from X into X is weakly $R - continuous$ on X.*

Then we have the following result.

Theorem 2.17. *Let (X, d, R) be a metric space with relation R with conditions $(**)$ and $(***)$ such that for every pair $x, y \in X$, $\mathcal{L}(\{x, y\}) \neq \phi$, or $\mathcal{R}(\{x, y\}) \neq \phi$. Let $k \in (0, 1)$ be fixed. Let (X, d) be waekly $R-$complete and F be an $R-$preserving and $R-k-$contraction from X into X. If there exists $x_0 \in X$ such that $x_0 R F(x_0)$ or $F(x_0) R x_0$, then F is a Picard operator (briefly PO), that is, F has a unique fixed point x^* and $\lim_{n\to\infty} F^n(x) = x^*$ for all $x \in X$.*

More recently, M. Eshaghi et al.[1] introduced the notion of orthogonally sets and then they gave an extension of Banach's fixed point theorem. A binary relation $\perp$ on X is called an orthogonality relation if

$$(\exists x_0 : \forall y, y \perp x_0) \ \ or \ \ (\exists x_0 : \forall y, x_0 \perp y),$$

then X is called an *orthogonal set* (briefly *O-set*). We denote this O-set by $(X, \perp)$.

Theorem 2.18. *(Theorem 3.11 of [1]) Let $\perp$ be an orthogonally relation on X, (X, d) be $\perp$-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be weakly $\perp$-continuous, $\perp - \lambda - contraction$ and $\perp$-preserving. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim f^n(x) = x^*$ for all $x \in X$.*

Now, we generalize this theorem as follows.

Theorem 2.19. *Let (X, d, R) be a metric space with relation R. Let (X, d) be $R-$complete metric space and $0 < \lambda < 1$ be fixed. Let $f : X \to X$ be a weakly $R-$ continuous, $R-\lambda-$ contraction and $R-$ preserving. If there exists $x_0 \in X$ such that $x_0 R y$ for all y in range(f), then f has a unique fixed point $x^* \in X$. Also, f is a PO.*

It is well known that Theorem 2.18 is a real generalization of Banach principle (see [1]). Also, it is easy to see that Theorem 2.19 is a generalization of Theorem 2.18. Now, we show that it is a real generalization. To this end, let $X = [0, 1)$ with Euclidean metric and $R := \{(0, x) : x \in \mathbb{Q} \cap X\}$. Define $f : X \to X$ by

$$f(x) = \begin{cases} \frac{x}{2} & , if \quad x \in \mathbb{Q} \cap X, \\ 0 & , if \ x \in \mathbb{Q}^c \cap X. \end{cases}$$

The mapping f is $R - \frac{1}{2} -$ *contraction*, $R -$ *preserving* and weakly $R -$ *continuous* on X. But R is not an orthogonal relation on X. Then Theorem 2.18 does not work to find fixed points of f. Indeed, by using Theorem 2.19, we can show that f has a unique fixed point, and f is a PO.

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On enumeration of complete semihypergroups and M-P-Hs.

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Abstract

In this paper, we compute the number of complete semihypergroups generated by semigroups of order 2 or 3. Also, we enumerate M-polysymmetrical hypergroups of order less than 6. We show that there are 7 isomorphism classes of M-polysymmetrical hypergroups of order 5 and calculate Cayley tables of them.

Keywords: complete semihypergroup, polysymmetrical hypergroup, semigroup.
Mathematics Subject Classification [2010]: 20N20

1 Introduction

The concept of a hyperstructures first was introduced by Marty at the 8^{th} international Congress of Scandinavian Mathematicians. The hyperstructure theory had applications to several domains of theoretical and applied mathematics[4, 5].

In [7] and [6] introduced K_H-hypergroups; particularly studied the relations of similitude in K_H-hypergroups. De Salvo [8] computed the number of K_H-hypergroups of given size $n \leq 4$.

J. Mittas in his paper[9], which has been announced in the French Academy of Sciences, has introduced a special type of hypergroup that he has named polysymmetrical. Polysymmetrical hypergroups are special class of K_H-hypergroups. Also, in the same paper J. Mittas has given certain fundamental properties of this hyperstructure.

Staring from the above paper and having called Mittas structure M-polysymmetrical hypergroup (in order to distinguish this polysymmetrical hypergroup from other types of polysymmetrical hypergroups) we have proceeded to a profound analysis of this hypergroup[10] and its subhypergroups[11].

We recall the construction of K_H-(semi)hypergroups[6]: let $(H, \circ)$ be a (semi)hypergroup and $\{A_a | a \in H\}$ a family of non-empty and pairwise disjoint sets, having as indexes the elements of H; the the set $K = \cup_{a \in H} A_a$ becomes a (semi)hypergroup under the following hyperoperation:

$$x * y = \cup_{c \in a \circ b} A_c, \ \forall x \in A_a, \ y \in A_b.$$

We say the $(K, *)$ is a K_H-(semi)hypergroup, generated by the (semi)hypergroup H.

If $(K, *)$ is a K_H-semihypergroup and H be a semigroup then we say that $(K, *)$ is a complete semihypergroup and $K/\beta^* \cong H$.

We recall definition of M-polysymmetrical hypergroup of [11] as follows:

*Speaker

A non-empty set H is called M-polysymmetrical hypergroup (M-P-H.) if it is endowed with a hyperoperation $+ : H \times H \to \mathcal{P}^*(H)$, when $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H, that satisfies the following axioms:

(1) $+$ is associative, i. e, for every $x, y, z \in H$ we have $x + (y + z) = (x + y) + z$;

(2) $+$ is commutative, i. e, for every $x, y \in H$, $x + y = y + x$;

(3) there exists $0 \in H$ such that for every $x \in H$ we have $x \in x + 0$;

(4) for every $x \in H$ there exists $x' \in H$ such that $0 = x + x'$, (x' is an opposite or symmetrical of x , with regard to considered 0, and the set of all the opposites $S(x) = \{x'|0 = x + x'\}$ is the symmetrical set of x),

(5) for every $x, y, z \in H, x' \in S(x), y' \in S(y)$ and $z' \in S(z)$, $x \in y + z$ implies that $x' \in y' + z'$.

Theorem 1.1. *[11] Let $(H, +)$ be a M-P-H, then for every $x, y, z, w \in H$ we have:*

(1) $S(0) = 0$, that means $0 + 0 = 0$;

(2) $0 \in 0 + x \Rightarrow x = 0$ and hence $y \in y + x \Rightarrow x = 0$;

(3) 0 is unique;

(4) $(x + y) \cap (z + w) \Rightarrow x + y = z + w$;

(5) for all $z' \in S(z)$, $x \in y + z$ implies that $y \in x + z'$;

(6) $0 \in x + y \Rightarrow x + y = 0$.

Let H be an M-polysymmetrical hypergroup and $H/(0) = \{c(0), c(x_2), \ldots, c(c_n)\}$. We call an M-polysymmetrical hypergroup H of order n of type $(k_1 = 1, k_2, \ldots, k_n)$ when $|H/(0)| = n$ and

Mittas [9] proved that, in general, M-polysymmetrical hypergroups are associated with abelian groups:

Theorem 1.2. *[9] Let $(H, +)$ be an M-polysymmetrical hypergroup. The set $C(x) = 0+x$ when x traverse H, from a partition of H and we have:*

$$x + y = 0 + x + y = (0 + x) + (0 + y),$$

moreover, $x + y$ is a class of partition and the set $G = \{C(x)|x \in H\}$ of these classes is an abelian group according to operation $C(x) + C(y)$.

We recall the following results from [9, 11]:

We symbolize with mod 0 or simply (0) the equivalence relation that the above mentioned partition defines, for which we have:

$$x \equiv y \;\Leftrightarrow\; 0 + x = 0 + y \;\Leftrightarrow\; C(x) = C(y).$$

Thus $G = H/(0)$ calling this group, group of reduction of H. So, mod 0 is a strongly regular equivalence relation to H. In group G for every $x, y \in H$, $x_1 \in C(x)$ and $y_1 \in C(y)$,we have

$$C(x) + C(y) = \bigcup_{z_1 \in x_1 + y_1} C(z_1) = C(z),$$

when $z \in x + y$.

We choose, foe every class C, mod 0, of H one element x_C as distinguished element of the class, let it be $\overline{G}$ the set of this elements. Then we consider the mapping

$$f : G \to \overline{G} \textit{ with } f(C) = x_C.$$

Obviously it is one-to-one and using this map we consider the following operation into $\overline{G}$:

$$x \oplus y = f[C(x) + C(y)], \forall x, y \in \overline{G}.$$

Consequently we have:

Theorem 1.3. *[11] To every M-polysymmetrical hypergroup $(H, +)$ there is subset $\overline{G}$ of H with abelian group's structure isomorphic to the group of reduction $H/(0)$. We call the group $(\overline{G}, \oplus)$ group of choice of $(H, +)$.*

Inversely, from an abelian group it is possible, under certain conditions to become an M-polysymmetrical hypergroup. The detailed study of this subject leads to the following theorem:

Theorem 1.4. *[11] Let E be a set and G its subset with the structure of an abelian group. Also, let 0 be its neutral element and for every $x \in G$, $-x$ be its opposite. If:*

> *there exist a partition R of E and mapping one-to-one of quotient set E/R on G such as for every $x \in G$, $f^{-1}(x) = C_R(x)$, [where $C_R(x)$ is the class of E mod R that contains the element x] and:*
>
> $C_R(0) = \{0\}$,

then the hyperoperation $x \oplus y = f^{-1}[f(C_R(x) + C_R(y))]$ defined on E, through the group G gives in E the structure of an M-polysymmetrical hypergroup of which the group of reduction $E/(0)$ coincides to E/R.

2 On enumeration of complete semihypergroups

In this section, we will find, up to equivalent (with equivalence being isomorphism or anti-isomorphism), the number of complete semihypergroups of given size n, which are generated by semigroups H, such that $|H| \leq 3$. We recall the following notations from [8].

If K is a complete semihypergroup generated by semigroup H, such that $K = \cup_{a \in H} A_a$, $|K| = n$, $H = \{x_1, \ldots, x_m\}$, $|A_{x_i}| = n_i$ then K turns out to be type $(n_1, \ldots, n_m)$, where obviously $n_1 + \ldots + n_m = n$. There are as many types, as m-ples of positive integers whose sum equal to n; this number is $\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}$.

Let $[n, m]$ be the number of similitude classes of complete semihypergroups of size n, generated by semigroup of size m. Beside, let $[[n, m]]$ denote the number of equivalent classes of the complete semihypergroups satisfying the same conditions. We have $[n, 1] = [[n, 1]] = 1$. Also $[n, n] = 1$ and $[[n, n]] = s_n$, where s_n denoted, up to equivalent, the number of the semigroups of size n.

Now, suppose $m = 2$ or $m = 3$, In order to compute $[n, m]$, we observe this number depends on the solutions of linear equation

$$x_1 + \ldots + x_m = x$$

such that for every i, $1 \leq i \leq m, x_i \in \mathbb{N}^*$. Denote $S(n, m)$, the number of such m-ples of positive integers . If $\alpha = (\alpha_1, \ldots, \alpha_m) \in S(m, n)$, then for every $i, 1 \leq i \leq m$, $r_\alpha(\alpha_i)$ indicates the number of times the α_i appairs in the m-ple α. Define in $S(n, m)$ the following relation ρ:

$$\forall(\alpha, \beta) \in S(n, m)^2, \alpha = (\alpha_1, \ldots, \alpha_m), \beta = (\beta_1, \ldots, \beta_m),$$

$$\alpha\rho\beta \Leftrightarrow \{[\{\alpha_1, \ldots, \alpha_m\} = \{\beta_1, \ldots, \beta_m\} = T] \; and \; [\forall w \in T, r_\alpha(w) = r_\beta(w)]\}.$$

ρ is an equivalence, and let $S^*(n, m)$ the quotient set of $S(n, m)$ relative to it. For the definition of similitude, $S^*(n, m) = [n, m]$

We have: $S^*(n, m) = \sum_{t=1}^{m} s_t(n, m)$, where for every t, $s_t(n, m)$ is the number of the equivalence classes relative to ρ, determined by the m-ples, whose underlying set is of size t. De salvo [8] enumerated $s_t(n, m)$ for $m = 2, 3$ and $t = 1, \ldots, m$ as follows:

$$s_1(n, 2) = \begin{cases} 0 & \text{if n is odd} \\ 1 & \text{if n is even.} \end{cases} \qquad s_2(n, 2) = \begin{cases} \frac{n}{2} - 1 & \text{if n is even} \\ \frac{n-1}{2} & \text{if n is odd.} \end{cases}$$

$$s_1(n, 3) = \begin{cases} 0 & \text{if 3 dos'nt divide n} \\ 1 & \text{if 3 divides n.} \end{cases}$$

We obtain $s_2(3, 3) = 0$; $s_2(4, 3) = 1$; $s_2(5, 3) = 2$ and for $n \geq 4$

$$s_2(n, 3) = s_2(n - 3, 3) + \begin{cases} 1 & \text{if n is even} \\ 2 & \text{if n is odd.} \end{cases}$$

Also, $s_3(n, 3) = 0$; and for all $n \geq 6$

$$s_3(n, 3) = s_3(n - 3, 3) + \begin{cases} \frac{n-4}{2} & \text{if n is even} \\ \frac{n-5}{2} & \text{if n is odd.} \end{cases}$$

Therefore by using the preceding formulas, we can compute the value of the numbers $[n, m]$. In the following, we will value the number $[[n, 2]]$ and $[[n, 3]]$.

De salvo enumerated K_H-hypergroups and complete hypergroups[8]. We compute the number of complete semihypergroups:

We begin the determine the number $[[n, 2]]$:

We can consider only the complete semihypergroups by the following four semigroup, which determine the equivalence classes relative to the relation of equivalent on the set of the semigroup of two elements:

$$H_1 : \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \qquad H_2 : \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \qquad H_3 : \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} \qquad H_4 : \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

Also, $|AUT(H_3)| = 2$ and $|AUT(H_1)| = |AUT(H_2)| = |AUT(H_4)| = 1$. So, the number of complete semihypergroups of order n, which is generated by semigroup H_3:

$$s_1(n, 2) + s_2(n, 2)$$

and for H_1, H_2 and H_4, we obtain

$$s_1(n, 2) + 2s_2(n, 2)$$

Therefore:

Theorem 2.1. *For every $n \geq 2$*

$$[[n, 2]] = 3(s_1(n, 2) + 2s_2(n, 2)) + s_1(n, 2) + s_2(n, 2) = 4s_1(n, 2) + 7s_2(n, 2)$$

Since there exist 18 non-equivalent semigroups (with equivalence being isomorphism or anti-isomorphism) and compute the automorphism groups are given:

$Tabel A.G.$	$Order 3$
$group$	$number$
$trivial$	12
$\mathbb{Z}_2$	5
$\mathbb{S}_3$	1

Therefore we by using the above table obtain:

Theorem 2.2. *For every $n \geq 3$*

$$\begin{aligned} [[n, 3]] &= 12(s_1(n, 3) + 3s_2(n, 3) + 6s_3(n, 3)) + 5(s_1(n, 3) + 2s_2(n, 3) + 3s_3(n, 3))+ \\ &\quad (s_1(n, 3) + s_2(n, 3) + s_3(n, 3)) \\ &= 18s_1(n, 3) + 47s_2(n, 3) + 88s_3(n, 3). \end{aligned}$$

3 On enumeration of M-P-Hs.

In this section we use the results of the papers [11] and [12] and we characterize the M-P-Hs. of order less than 6 up to isomorphism.

Theorem 3.1. *Every M-P-H. $(H, +)$ of order 2 is a group and so $H \cong \mathbb{Z}_2$.*

Proof. Let $(H = \{0, 1\}, +)$ be an M-P-H. of order 2. Since $0 = 0 + 0$ then by part (4) of Theorem 3.1 we have $0 + 1 = 1 + 0 = 1$ and $1 + 1 = 0$. Therefore $(H, +)$ is an group and it is isomorphism with $(\mathbb{Z}_2, +)$. □

Notice that there are 20 isomorphism classes of H_v-groups of order 2 and 8 isomorphism classes of hypergroups of order 2.

Theorem 3.2. *For every M-polysymmetrical hypergroup $(H, +)$ with $|H| \geq 2$, we have $|H/(0)| \geq 2$.*

Theorem 3.3. *Let $(H, +)$ be an M-polysymmetrical hypergroup. If $x_1 \in C(x)$ and $y_1 \in C(y)$ then $x_1 + y_1 = x + y = C(z)$, for every $z \in C(x) + C(y)$.*

Theorem 3.4. *There are 2 isomorphism classes of M-P-Hs. of order 3 with the following tables:*

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

+	0	1	2
0	0	12	12
1	12	0	0
2	12	0	0

Proof. Let $H = \{0, 1, 2\}$ be an M-polysymetrical hypergroup of order 3. Then we have $H/(0) \cong \mathbb{Z}_2$ or $H/(0) \cong \mathbb{Z}_3$.

If $H/(0) \cong \mathbb{Z}_2$ then $C(1) = C(2) = \{1, 2\}$ (because $C(0) = \{0\}$ and by Theorem 1.2). Thus $0 + 1 = 0 + 2 = \{1, 2\}$. By Theorem 3.3, we obtain $1 + 1 = 1 + 2 = 2 + 1 = 2 + 2 = C(0) = \{0\}$. If $H/(0) \cong \mathbb{Z}_3$ then H is an group of order 3 and so $H \cong \mathbb{Z}_3$. □

Bayon and Lygeros [1] show that there are 1.026.462 isomorphism classes of H_v-groups of order 3 and Tsitouras and Massouros [12] enumerated 23.192 isomorphism classes of hypergroups of order 3.

Theorem 3.5. *There are 4 isomorphism classes of M-P-Hs. of order 4 with the following tables:*

T_1 :

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

T_3 :

+	0	1	2	3
0	0	123	123	123
1	123	0	0	0
2	123	0	0	0
3	123	0	0	0

T_2 :

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

T_4 :

+	0	1	2	3
0	0	1	23	23
1	1	23	0	0
2	23	0	1	1
3	23	0	1	1

Proof. By the theory of abelian groups we have one of 4 cases for group of reduction $H/(0)$ of H :

Case 1. $H/(0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and so $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.(Table T_1)

Case 2. $H/(0) \cong \mathbb{Z}_4$ and so $H \cong \mathbb{Z}_4$.(Table T_2)

Case 3. $H/(0) \cong \mathbb{Z}_2$. Thus we have $C(1) = C(2) = C(3) = \{1, 2, 3\}$ and so by Theorem 3.3 we obtain $i + j = C(0) = \{0\}$ and $0 + i = C(i)$ for all $i, j \in \{1, 2, 3\}$. Therefore we obtain $(H, +)$ have the cayley table T_3.

Case 4. $H/(0) \cong \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$. Since $C(0) = \{0\}$ so there exists two equivalence classes $C(x)$ and $C(y)$ where $x, y \in \{1, 2, 3\}$, $|C(x)| = 1$ and $|C(y)| = 2$. This means $H/(0) = \{C(0), C(x), C(y)\}$. Thus by the cayley table of $\mathbb{Z}_3$ we obtain the following table for $(H/(0), +)$:

+	$C(0)$	$C(x)$	$C(y)$
$C(0)$	$C(0)$	$C(x)$	$C(y)$
$C(x)$	$C(x)$	$C(y)$	$C(0)$
$C(y)$	$C(y)$	$C(0)$	$C(x)$.

Now, we construction cayley table $(H_{(x,y,z)}, +)$ by the cayley table of $H/(0)$. We have $C(0) = \{0\}$, $C(x) = \{x\}$ and $C(y) = H - \{0, x\} = \{y, z\}$. By Theorem 3.3 and straightforward computing we obtain the cayley table of $(H_{(x,y,z)}, +)$:

+	0	x	y	z
0	0	x	yz	yz
x	x	yz	0	0
y	yz	0	x	x
z	yz	0	x	x

For any choice of x ,y and z we obtain an M-polysymmetrical $(H_{(x,y,z)}, +)$ isomorphic to the M-polysymmetrical hypergroup with table T_4. In fact $f : H \to H_{(x,y,z)}$ with $f(1) = x,\ f(2) = y$ and $f(3) = z$ is an isomorphism.

□

Bayon and Lygeros [2] show that there are 10.614.362 isomorphism classes of abelian hypergroups of order 4 and Bayon and Lygeros [3] enumerated 8.028.299.905 isomorphism classes of abelian H_v-groups of order 4.

Theorem 3.6. *There are 7 isomorphism classes of M-P-Hs. of order 5 with the following tables:*

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	0	1	2	4
4	0	1	2	3	4

+	0	1	2	3	4
0	0	1234	1234	1234	1234
1	1234	0	0	0	0
2	1234	0	0	0	0
3	1234	0	0	0	0
4	1234	0	0	0	0

+	0	1	2	3	4
0	0	12	12	34	34
1	12	34	34	0	0
2	12	34	34	0	0
3	34	0	0	12	12
4	34	0	0	12	12

+	0	1	2	3	4
0	0	1	2	34	34
1	1	2	34	0	0
2	2	34	0	1	1
3	34	0	1	2	2
4	34	0	1	2	2

+	0	1	2	3	4
0	0	1	234	234	234
1	1	234	0	0	0
2	234	0	1	1	1
3	234	0	1	1	1
4	234	0	1	1	1

+	0	1	2	3	4
0	0	1	23	23	4
1	1	23	4	4	0
2	23	4	0	0	1
3	23	4	0	0	1
4	4	0	1	1	23

+	0	1	2	3	4
0	0	1	2	34	34
1	1	0	34	2	2
2	2	34	0	1	1
3	34	2	1	0	0
4	34	2	1	0	0

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Closed non-vanishing ideals in $C_B(X)$

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Abstract

Let X be a completely regular space. For a closed non-vanishing ideal H in $C_B(X)$ we construct the spectrum $\mathfrak{sp}(H)$ of H as a subspace of the Stone–Čech compactification of X. The known construction of $\mathfrak{sp}(H)$ will then enable us to derive certain properties of $\mathfrak{sp}(H)$ which are not generally expected to be easily deducible from the standard Gelfand theory.

This paper is a rather self-contained extract from the research monograph [M. R. Koushesh, *Ideals in $C_B(X)$ arising from ideals in X*, 53 pp.] available as the arXiv preprint `arXiv:1508.07734 [math.FA]`, to which the reader may also be referred to.

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1 Introduction

Throughout this paper by a *space* we will mean a *topological space.*

Let X be a completely regular space. Let $C_B(X)$ be the algebra of all complex valued continuous bounded mappings on X equipped with the supremum norm. Also, let $C_0(X)$ be the subset of $C_B(X)$ consisting of all f which vanish at infinity (i.e., $|f|^{-1}([\epsilon,\infty))$ is compact for each $\epsilon > 0$). A subset H of $C_B(X)$ is said to be *non-vanishing* if for each x in X there is some h in H such that $h(x) \neq 0$.

The commutative Gelfand–Naimark theorem states that every commutative C^*-algebra A is isometrically $*$-isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y. Such a space Y is necessarily unique (up to homemorphism) by the Banach–Stone theorem and is identical to the spectrum of A. Here, using purely topological arguments, we prove that a closed non-vanishing ideal H of $C_B(X)$ is isometrically isomorphic to $C_0(Y)$ for a locally compact space Y. This in particular re-proves the commutative Gelfand–Naimark theorem in its special case. We construct Y as a subspace of the Stone–Čech compactification of X. The known construction of Y will then enable us to study it deeper and derived results which are not generally expected to be easily deducible from the standard Gelfand theory.

This paper is an extract from the research monograph [10]. However, it is rather self-contained, as it contains a complete proof for its main result (Theorem 2.7). For proofs of the remaining results (Theorems 2.9 and 2.10) we refer the interested reader to the original preprint [10]. (See [6]–[8] for further related results.)

In what follows we use the Stone–Čech compactification as the main tool. For its importance we define it in the following and refer to the texts [3], [4] and [11] for further information.

The Stone–Čech compactification

Let X be a completely regular space, i.e., a Hausdorff space such that for every closed subset C of X and every x in $X \setminus C$ there is a continuous mapping $f : X \to [0, 1]$ such that $f(x) = 0$ and $f|_C = \mathbf{1}$. A *compactification* of X is a compact Hausdorff space which contains X as a dense subspace. The *Stone–Čech compactification* of X, denoted by βX, is the compactification of X which is characterized among all compactifications of X by the fact that every continuous bounded mapping $f : X \to \mathbb{C}$ is extendable to a continuous mapping $F : \beta X \to \mathbb{C}$. This extension is necessarily unique, as any two such extensions agree on the dense subspace X of βX. The Stone–Čech compactification of a completely regular space always exists.

2 The representation theorem

The following is motivated by the definition of $\lambda_{\mathscr{P}} X$ as given in [5] and [9].

Definition 2.1. Let X be a completely regular space. For an ideal H of $C_B(X)$ define

$$\lambda_H X = \bigcup \big\{\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}|h|^{-1}\big((1,\infty)\big) : h \in H\big\},$$

which is considered as a subspace of βX.

Recall that a subset H of $C_B(X)$ is said to be *non-vanishing* if for each x in X there is an h in H such that $h(x) \neq 0$.

Note that if X is a space and D is a dense subspace of X, then

$$\mathrm{cl}_X U = \mathrm{cl}_X(U \cap D)$$

for any open subspace U of X.

Lemma 2.2. *Let X be a completely regular space and let H be an ideal of $C_B(X)$. Let h be in H and let $h_\beta : \beta X \to \mathbb{C}$ be the continuous extension of h. Then*

$$|h_\beta|^{-1}\big((1,\infty)\big) \subseteq \lambda_H X.$$

Proof. Observe that

$$\begin{aligned} |h_\beta|^{-1}\big((1,\infty)\big) &\subseteq \mathrm{cl}_{\beta X}|h_\beta|^{-1}\big((1,\infty)\big) \\ &= \mathrm{cl}_{\beta X}\big(X \cap |h_\beta|^{-1}\big((1,\infty)\big)\big) = \mathrm{cl}_{\beta X}|h|^{-1}\big((1,\infty)\big), \end{aligned}$$

as $|h_\beta|^{-1}((1,\infty))$ is open in βX and X is dense in βX. Therefore

$$|h_\beta|^{-1}\big((1,\infty)\big) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}|h|^{-1}\big((1,\infty)\big).$$

But $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}|h|^{-1}((1,\infty))$ is contained in $\lambda_H X$ by the way we have defined $\lambda_H X$. □

Lemma 2.3. *Let X be a completely regular space and let H be a non-vanishing ideal of $C_B(X)$. Then*

$$X \subseteq \lambda_H X.$$

Proof. Let x be in X. There is some h' in H such that $h'(x) \neq 0$. Choose some $n = 1, 2, \ldots$ such that $|h'(x)| > 1/n$. Denote $h = nh'$. Let $h_\beta : \beta X \to \mathbb{C}$ be the continuous extension of h. Then

$$|h_\beta|^{-1}\big((1,\infty)\big) \subseteq \lambda_H X$$

by Lemma 2.2. Therefore x is in $\lambda_H X$, as x is in $|h_\beta|^{-1}((1,\infty))$. □

Lemma 2.4. *Let X be a completely regular space and let H be an ideal in $C_B(X)$. Let K be a compact subspace of $\lambda_H X$. Then*

$$K \subseteq \mathrm{cl}_{\beta X} h^{-1}\big((1,\infty)\big)$$

for some h in H.

Proof. By compactness of K we have

$$K \subseteq \bigcup_{i=1}^{j} \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}|h_i|^{-1}\big((1,\infty)\big) \tag{1}$$

where h_i is in H for each $i = 1, \ldots, j$. Let

$$h = \sum_{i=1}^{j} h_i\overline{h_i} = \sum_{i=1}^{j} |h_i|^2.$$

Then h is in H, as H is an ideal in $C_B(X)$. We have

$$\bigcup_{i=1}^{j} |h_i|^{-1}\big((1,\infty)\big) \subseteq h^{-1}\big((1,\infty)\big).$$

In particular

$$\begin{aligned}\bigcup_{i=1}^{j} \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}|h_i|^{-1}\big((1,\infty)\big) \quad &\subseteq \quad \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}\Big[\bigcup_{i=1}^{j} |h_i|^{-1}\big((1,\infty)\big)\Big] \\ &\subseteq \quad \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} h^{-1}\big((1,\infty)\big).\end{aligned}$$

This together with (1) implies that

$$K \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} h^{-1}\big((1,\infty)\big).$$

The lemma now follows. □

Lemma 2.5. *Let X be a completely regular space and let f be in $C_B(X)$. Let $f_1, f_2, \ldots$ be a sequence in $C_B(X)$ such that*

$$|f|^{-1}\big([1/n,\infty)\big) \subseteq |f_n|^{-1}\big([1,\infty)\big)$$

for each $n = 1, 2, \ldots$. Then, there is a sequence $g_1, g_2, \ldots$ in $C_B(X)$ such that $g_n f_n \to f$.

Proof. Fix some $n = 1, 2, \ldots$. We define a mapping $u_n : X \to \mathbb{C}$ by

$$u_n(x) = \begin{cases} 1/f_n(x), & \text{if } x \in |f_n|^{-1}([1,\infty)); \\ \overline{f_n(x)}, & \text{if } x \in |f_n|^{-1}([0,1]). \end{cases}$$

The mapping u_n is well defined, as $1/f_n(x) = \overline{f_n(x)}$ for any x in the intersection

$$|f_n|^{-1}\big([1,\infty)\big) \cap |f_n|^{-1}\big([0,1]\big) = |f_n|^{-1}(1),$$

and u_n is continuous, as it is continuous on each of the two closed subspaces $|f_n|^{-1}([1,\infty))$ and $|f_n|^{-1}([0,1])$ of X whose union is the entire X. Note that $|u_n(x)| \leq 1$ for each x in X. In particular, u_n is in $C_B(X)$. We now verify that

$$\big|u_n f_n f(x) - f(x)\big| < 1/n \tag{2}$$

for each x in X. Let x be in X. We consider the following two cases:

Case 1 Suppose that x is in $|f_n|^{-1}([1,\infty))$. Then $u_n(x)f_n(x) = 1$ by the definition of u_n. Therefore $u_n f_n f(x) - f(x) = 0$, and thus (2) holds in this case.

Case 2 Suppose that x is in $|f_n|^{-1}([0,1))$. Then $f_n(x) = \overline{u_n(x)}$ by the definition of u_n. Therefore

$$u_n(x)f_n(x)f(x) - f(x) = \big|u_n(x)\big|^2 f(x) - f(x) = \big[\big|u_n(x)\big|^2 - 1\big] f(x).$$

But $|u_n(x)| \leq 1$ and $|f(x)| < 1/n$, as using our assumption

$$|f_n|^{-1}\big([0,1)\big) \subseteq |f|^{-1}\big([0,1/n)\big).$$

Therefore (2) holds in this case as well.

By (2) it follows that $\|u_n f_n f - f\| \leq 1/n$ for each $n = 1, 2, \ldots$ and consequently

$$\|u_n f_n f - f\| \to 0.$$

Let $g_n = u_n f$ for each $n = 1, 2, \ldots$. Then $g_1, g_2, \ldots$ is a sequence in $C_B(X)$ such that $g_n f_n \to f$. □

Lemma 2.6. *Let X be a completely regular space. Let $X \subseteq Y \subseteq \beta X$ and for any f in $C_B(X)$ let $f_Y = f_\beta|_Y$ where $f_\beta : \beta X \to \mathbb{C}$ is the continuous extension of f. Then, for any f and g in $C_B(X)$ we have*

(a) $(f + g)_Y = f_Y + g_Y$.

(b) $(fg)_Y = f_Y g_Y$.

(c) $\|f_Y\| = \|f\|$.

Proof. To show (a) observe that $(f+g)_Y$ and $f_Y + g_Y$ are identical, as they are continuous mappings which both coincide with $f + g$ on the dense subspace X of Y. That (b) holds follows analogously.

To show (c), note that

$$|f_Y|(Y) = |f_Y|(\mathrm{cl}_Y X) \subseteq \overline{|f_Y|(X)} = \overline{|f|(X)} \subseteq \big[0, \|f\|\big],$$

where the bar denotes the closure in $\mathbb{R}$. This yields $\|f_Y\| \leq \|f\|$. That $\|f\| \leq \|f_Y\|$ is clear, as f_Y is an extension of f. □

By a version of the Banach–Stone theorem, for locally compact Hausdorff spaces X and Y, the Banach algebras $C_0(X)$ and $C_0(Y)$ are isometrically isomorphic if and only if the spaces X and Y are homeomorphic; see Theorem 7.1 of [2]. (It turns out that for a locally compact Hausdorff space X even the ring theoretic structure of $C_0(X)$ suffices to determine the topology of the space X; see [1].) This will be used in the proof of the following theorem.

Theorem 2.7. *Let X be a completely regular space. Let H be a closed non-vanishing ideal in $C_B(X)$. Then H is isometrically isomorphic to $C_0(Y)$ for some unique locally compact Hausdorff space Y, namely for $Y = \lambda_H X$. In particular, Y is the spectrum of H. Furthermore, the following are equivalent:*

(a) *H is unital.*

(b) *H contains $\mathbf{1}$.*

(c) *Y is compact.*

(d) *$Y = \beta X$.*

Proof. For an f in $C_B(X)$ denote

$$f_H = f_\beta|_{\lambda_H X}$$

where $f_\beta : \beta X \to \mathbb{C}$ is the continuous extension of f. Observe that X is contained in $\lambda_H X$ by Lemma 2.3, thus, in particular, f_H extends f.

Claim. *For an f in $C_B(X)$ the following are equivalent:*

(i) *f is in H.*

(ii) *f_H is in $C_0(\lambda_H X)$.*

Proof of the claim. (i) *implies* (ii). Let $n = 1, 2, \ldots$. Note that

$$|f_\beta|^{-1}\big([1/n,\infty)\big) \subseteq |f_\beta|^{-1}\Big(\Big(\frac{1}{n+1},\infty\Big)\Big) = \big|(n+1)f_\beta\big|^{-1}\big((1,\infty)\big) \subseteq \lambda_H X$$

by Lemma 2.2. Thus

$$|f_H|^{-1}\big([1/n,\infty)\big) = \lambda_H X \cap |f_\beta|^{-1}\big([1/n,\infty)\big) = |f_\beta|^{-1}\big([1/n,\infty)\big)$$

is closed in βX and is therefore compact.

(ii) *implies* (i). Let $n = 1, 2, \ldots$. Since $|f_H|^{-1}([1/n,\infty))$ is a compact subspace of $\lambda_H X$, by Lemma 2.4 we have

$$|f_H|^{-1}\big([1/n,\infty)\big) \subseteq \mathrm{cl}_{\beta X} g_n^{-1}\big((1,\infty)\big)$$

for some g_n in H. Therefore, if we intersect the two sides of the above relation with X, it yields

$$\begin{aligned} |f|^{-1}\big([1/n,\infty)\big) &= X \cap |f_H|^{-1}\big([1/n,\infty)\big) \\ &\subseteq X \cap \mathrm{cl}_{\beta X} g_n^{-1}\big((1,\infty)\big) = \mathrm{cl}_X g_n^{-1}\big((1,\infty)\big). \end{aligned}$$

In particular,

$$|f|^{-1}\big([1/n,\infty)\big) \subseteq g_n^{-1}\big([1,\infty)\big).$$

By Lemma 2.5 there is a sequence $l_1, l_2, \ldots$ in $C_B(X)$ such that $l_n g_n \to f$. Note that $l_n g_n$ is in H for each $n = 1, 2, \ldots$, as g_n is in H. But then f is the limit of a sequence in H and is therefore in H, as H is closed in $C_B(X)$ by our assumption.

Claim. *Let*

$$\psi : H \to C_0(\lambda_H X)$$

be defined by $\psi(h) = h_H$ *for any* h *in* H. *Then* ψ *is an isometric isomorphism.*

Proof of the claim. The mapping ψ is clearly well defined by the first claim. The mapping ψ is a homomorphism by Lemma 2.6. Also, it is clear that ψ is injective. (Observe that $X \subseteq \lambda_H X$ by Lemma 2.3, and use the fact that any two real valued continuous mappings on $\lambda_H X$ coincide if they agree on its dense subspace X.) We show that ψ is surjective. Let g be in $C_0(\lambda_H X)$. Then $(g|_X)_H = g$ (as $(g|_X)_H$ and g are identical when restricted to X) and thus $g|_X$ is in H by the first claim. Observe that $\psi(g|_X) = g$. Finally, observe that $\|h_H\| = \|h\|$ for any h in H by Lemma 2.6. That is ψ is an isometry. This proves the claim.

The uniqueness of Y follows from the Banach–Stone theorem. (Note that $\lambda_H X$ is open in βX by its definition, and is therefore a locally compact Hausdorff space.)

To show the concluding assertion of the theorem, let H be unital with the unit element u. For each x in X choose some h_x in H such that $h_x(x) \neq 0$. Then $u(x)h_x(x) = h_x(x)$ which yields $u(x) = 1$. Thus $u = \mathbf{1}$. But then $\lambda_H X = \beta X$ by the way $\lambda_H X$ is defined. Observe that if Y is compact then $C_0(Y) = C_B(Y)$, and therefore H is unital, as it is isometrically isomorphic to $C_0(Y)$ and the latter is so. □

Remark 2.8. The existence of the space space Y in Theorem 2.7 may also be deduced from the commutative Gelfand–Naimark theorem in which Y is the spectrum (or the character space or the maximal ideal space) of H. The advantage of our method is that it constructs the space Y explicitly as a subspace of the Stone–Čech compactification of X. This known construction of Y enables us to derive certain properties of Y which are generally not expected to be deducible from the standard theories. (See [6]–[8] for examples.)

The following two results are to illustrate the advantage of our topological approach. The proofs, however, are relatively long and are therefore omitted. The interested reader is referred to [10] for the complete proofs.

Theorem 2.9. *Let* X *be a completely regular space. Let* H *be a closed non-vanishing ideal in* $C_B(X)$. *The following are equivalent:*

(a) *The spectrum of* H *is* σ*-compact.*

(b) *The ideal* H *is* σ*-generated, i.e.,*

$$H = \overline{\langle f_1, f_2, \ldots \rangle}$$

for some $f_1, f_2, \ldots$ *in* $C_B(X)$.

Let $\{Y_i : i \in I\}$ be a collection of topological spaces. We may assume that the spaces Y_i's are pairwise disjoint. The *topological direct sum* of $\{Y_i : i \in I\}$, denoted by $\bigoplus_{i\in I} Y_i$, is the set $Y = \bigcup_{i\in I} Y_i$ together with the family $\mathscr{O}$ of all $U \subseteq Y$ such that $U \cap Y_i$ is open in Y_i for every $i \in I$.

Let X be a completely regular space. The collection $\mathscr{H}$ of all ideals of $C_B(X)$ (partially ordered with set-theoretic inclusion $\subseteq$) is a complete upper semi-lattice, that is, together with any subcollection $\mathscr{G}$ of $\mathscr{H}$, $\mathscr{H}$ contains their least upper bound $\bigvee \mathscr{G}$. Indeed, let $\{H_i : i \in I\}$ be a collection of ideals in $C_B(X)$. Then

$$\bigvee_{i\in I} H_i = \Big\langle \bigcup_{i\in I} H_i \Big\rangle.$$

Also, we denote

$$\bigoplus_{i\in I} H_i = \Big\langle \bigcup_{i\in I} H_i \Big\rangle$$

if we further have $H_i \cap \langle \bigcup_{i\neq j\in I} H_j \rangle = \mathbf{0}$ for each $i \in I$.

Theorem 2.10. *Let X be a completely regular space. Let $\{H_i : i \in I\}$ be a collection of ideals in $C_B(X)$.*

(1) *Suppose that H_i is non-vanishing for each $i \in I$. Then*

$$\mathfrak{sp}\Big(\overline{\bigvee_{i\in I} H_i}\Big) = \bigcup_{i\in I} \mathfrak{sp}\big(\overline{H_i}\big).$$

(2) *Suppose that $\bigoplus_{i\in I} H_i$ is non-vanishing. Then*

$$\mathfrak{sp}\Big(\overline{\bigoplus_{i\in I} H_i}\Big) = \bigoplus_{i\in I} \mathfrak{sp}\big(\overline{H_i}\big).$$

(3) *Suppose that $\bigcap_{i\in I} H_i$ is non-vanishing. Then*

$$\mathfrak{sp}\Big(\overline{\bigcap_{i\in I} H_i}\Big) = \operatorname{int}_{\mathfrak{sp}(C_B(X))}\Big(\bigcap_{i\in I} \mathfrak{sp}\big(\overline{H_i}\big)\Big).$$

Here the bar denotes the closure in $C_B(X)$ and $\mathfrak{sp}(H)$ denotes the spectrum of H.

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Second derivative general linear methods for the numerical solution of IVPs

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Abstract

General linear methods (GLMs) were introduced as the natural generalizations of the classical RungeKutta and linear multistep methods. An extension of GLMs, so-called SGLMs (GLM with second derivative), was introduced to the case in which second derivatives, as well as first derivatives, can be calculated. In this paper, we introduce the basic concepts, construction and implementation of SGLMs.

Keywords: Stiff IVPs, General linear methods, Second derivative methods, stability aspects, Variable stepsize implementation.
Mathematics Subject Classification [2010]: 65L05

1 Introduction

Traditional numerical methods for solving an initial value problem

$$\begin{aligned} &y'(x) = f(y(x)), \qquad x \in [x_0, \overline{x}], \\ &y(x_0) = y_0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ and m is the dimensionality of the system, generally fall into two main classes: linear multistep (multivalue) and Runge–Kutta (multistage) methods. In 1966, Butcher [5] introduced general linear methods as a unifying framework for the traditional methods to study the properties of consistency, stability, and convergence and to formulate new methods with clear advantages over the these classes.

On the other hand, one of the main directions to construct methods with higher order and extensive stability region, is the using higher derivatives of the solutions, and some methods have been introduced that have good properties, especially for stiff problems. See [7, 8, 10]. Although the mentioned GLM includes linear multistep methods, Runge–Kutta and many other standard methods, but for the above reasons, it has be seemed that it be extended to the case in which second derivatives of solution, as well as first derivatives, can be calculated. These methods introduced by Butcher and Hojjati [6].

In this paper, we will review the basic concepts, types, construction and implementation issues of SGLMs.

*Speaker

2 Basic Concepts

An SGLM is characterized by six matrices denoted by $A, \overline{A} \in \mathbb{R}^{s\times s}$, $U \in \mathbb{R}^{s\times r}$, $B, \overline{B} \in \mathbb{R}^{r\times s}$ and $V \in \mathbb{R}^{r\times r}$. By denoting the second derivative stage value of step number n by $g(Y^{[n]})) = [g(Y_i^{[n]})]_{i=1}^{s}$, where $g(\cdot) = f'(\cdot)f(\cdot)$ and using of previous notations, the representation of SGLMs takes the form

$$\begin{aligned} Y^{[n]} &= h(A\otimes I_m)f(Y^{[n]}) + h^2(\overline{A}\otimes I_m)g(Y^{[n]}) + (U\otimes I_m)y^{[n-1]},\\ y^{[n]} &= h(B\otimes I_m)f(Y^{[n]}) + h^2(\overline{B}\otimes I_m)g(Y^{[n]}) + (V\otimes I_m)y^{[n-1]}. \end{aligned} \tag{2}$$

It is convenient to write coefficients of the method, that is elements of A, $\overline{A}$, U, B, $\overline{B}$ and V as a partitioned $(s+r)\times(2s+r)$ matrix

$$\left[\begin{array}{c|c|c} A & \overline{A} & U \\ \hline B & \overline{B} & V \end{array}\right].$$

In an SGLM we assumed that the ith subvector in $y^{[n-1]}$ represents $u_i y(x_{n-1}) + v_i h y'(x_{n-1}) + O(h^2)$. The vectors u and v are characteristic of any particular method.

Definition 2.1. An SGLM $(A, \overline{A}, U, B, \overline{B}, V)$ is 'pre-consistent' if V has an eigenvalue equal to 1 and u be a corresponding eigenvector and also $Uu = \mathrm{e}$, where $\mathrm{e} = [1, 1, \cdots, 1]^T \in \mathbb{R}^s$.

Definition 2.2. An SGLM $(A, \overline{A}, U, B, \overline{B}, V)$ is 'consistent' if it is pre-consistent with pre-consistency vector u and there exists a vector v (consistency vector) such that $B\mathrm{e} + Vv = u + v$.

Definition 2.3. An SGLM $(A, \overline{A}, U, B, \overline{B}, V)$ is 'stable' if there exists a constant k such that

$$\|V^n\| \le k, \qquad \text{for all } n = 1, 2, \ldots$$

Theorem 2.4. *[2] If the SGLM $(A, \overline{A}, U, B, \overline{B}, V)$ is convergent, then it is stable.*

Theorem 2.5. *[2] Let $(A, \overline{A}, U, B, \overline{B}, V)$ denote a convergent SGLM which is, moreover, covariant with pre-consistency vector u. Then it is consistent.*

Theorem 2.6. *[2] A consistent and stable SGLM is convergent.*

An SGLM has order p and stage order q if

$$y^{[n-1]} = \sum_{k=0}^{p} h^k\big(\alpha_k \otimes y^{(k)}(x_{n-1})\big) + O(h^{p+1}) \tag{3}$$

implies that

$$Y^{[n]} = \sum_{k=0}^{p} h^k\big(\frac{c^k}{k!} \otimes y^{(k)}(x_{n-1})\big) + O(h^{q+1}) \tag{4}$$

and

$$y^{[n]} = \sum_{k=0}^{p} h^k\big(\alpha_k \otimes y^{(k)}(x_n)\big) + O(h^{p+1}), \tag{5}$$

for some vectors $\alpha_0, \alpha_1, \ldots, \alpha_p \in \mathbb{R}^r$ associated with the method.

Theorem 2.7. *[3] An SGLM has order p equal to stage order q if and only if*

$$\begin{cases} C = ACK + \overline{A}CK^2 + UW, \\ WE = BCK + \overline{B}CK^2 + VW, \end{cases} \tag{6}$$

where

$$C := \begin{bmatrix} 1 & \frac{c}{1!} & \frac{c^2}{2!} & \cdots & \frac{c^p}{p!} \end{bmatrix}, \qquad \begin{array}{l} K := [0 \quad e_1 \quad e_2 \quad \cdots \quad e_p], \\ W := [\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_p], \end{array}$$

$$E := \exp(K) = \begin{bmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{p!} \\ 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(p-1)!} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{1!} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The stability matrix of SGLMs is obtained by applying these methods to the standard test problem of Dahlquist $y' = qy$, where q is a (possibly complex) number, which it is

$$M(z) = V + \left(zB + z^2\overline{B}\right)\left(I - zA - z^2\overline{A}\right)^{-1}U,$$

where $z = qh$.

Definition 2.8. If the characteristic polynomial of $M(z)$, known as the stability function, has the special form

$$p(w, z) = \det\left(wI - M(z)\right) = w^{r-1}\left(w - R(z)\right),$$

then the method is said to possess 'Runge–Kutta stability' (RKS).

We divide SGLMs into four types, depending on the nature of the differential system to be solved and the computer architecture that is used to implement these methods. For type 1 or 2 methods, matrices A and $\overline{A}$ have the form

$$A = \begin{bmatrix} \lambda & & & \\ a_{21} & \lambda & & \\ \vdots & \vdots & \ddots & \\ a_{s1} & a_{s2} & \cdots & \lambda \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} \mu & & & \\ \overline{a}_{21} & \mu & & \\ \vdots & \vdots & \ddots & \\ \overline{a}_{s1} & \overline{a}_{s2} & \cdots & \mu \end{bmatrix},$$

where $\lambda = \mu = 0$ or $\lambda > 0$, $\mu < 0$, respectively. For type 3 or 4 methods, $A = \lambda I$ and $\overline{A} = \mu I$, where $\lambda = \mu = 0$ or $\lambda > 0$, $\mu < 0$, respectively.

Some order barriers have been peroved for SGLMs.

- Let p be the order of an SGLM of type 2 with RKS property. Then

$$p \leq \begin{cases} 2s + 2, & \text{if } \mu < -\frac{\lambda^2}{4}, \\ 2s + 1, & \text{if } \mu \geq -\frac{\lambda^2}{4}, \end{cases}$$

  where s is the number of internal stages.

- The orders of types 3 and 4 SGLMs with RKS property cannot exceed two and four respectively.

3 Nordsieck SGLMs

If $p = q = s + 1 = r - 1$ and the matrix W equal to the identity matrix, the methods can be represented in the Nordsieck form with the output values as

$$y^{[n]} = \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ h^2 y''(x_n) \\ \vdots \\ h^p y^{(p)}(x_n) \end{bmatrix} + O(h^{p+1}).$$

For such methods order conditions (6) can be written as

$$\begin{cases} U = C - ACK - \overline{A}CK^2, \\ V = E - BCK - \overline{B}CK^2. \end{cases} \qquad (7)$$

The coefficients matrices of a single example of L–stable Nordsieck SGLMs with RKS with $s = 2$, $p = 3$, $c = [\frac{1}{2} \;\; 1]^T$ and the error constant $\mathcal{C} = -0.4 \times 10^{-5}$ take the following forms

$$A = \begin{bmatrix} 0.9320000000 & 0 \\ -0.4808609798 & 0.9320000000 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} -0.2860000000 & 0 \\ 0.1314734508 & -0.2860000000 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.4808609798 & 0.9320000000 \\ 0 & 1 \\ 0 & 0 \\ 2.1002864620 & 5.2426575276 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0.1314734508 & -0.2860000000 \\ 0 & 0 \\ 0 & 1 \\ 1.4715274391 & -2.6196282911 \end{bmatrix},$$

$$U = \begin{bmatrix} 1.0000000000 & -0.4320000000 & -0.0550000000 & 0.0473333333 \\ 1.0000000000 & 0.5488609798 & -0.0370429609 & -0.0189624363 \end{bmatrix},$$

$$V = \begin{bmatrix} 1.0000000000 & 0.5488609798 & -0.0370429609 & -0.0189624363 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -7.3429439896 & -5.1446999066 & 0 \end{bmatrix}.$$

The coefficients matrices of a single example of L–stable Nordsieck SGLMs with RKS with $s = 3$, $p = 4$, $c = [\frac{1}{4} \;\; \frac{1}{2} \;\; 1]^T$ and the error constant $\mathcal{C} = -10^{-5}$ take the following forms

$$A = \begin{bmatrix} 0.5000000000 & 0 & 0 \\ -0.1084646955 & 0.5000000000 & 0 \\ -21.4854212762 & 11.3754879183 & 0.5000000000 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} -0.0500000000 & 0 & 0 \\ -0.0514889392 & -0.0500000000 & 0 \\ -0.1268254314 & -1.5150262971 & -0.0500000000 \end{bmatrix},$$

$$B = \begin{bmatrix} -21.4854212762 & 11.3754879183 & 0.5000000000 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -175.3567237676 & 187.8806581465 & -12.5239343789 \\ -456.0798770012 & 442.7876468092 & -22.7655989812 \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} -0.1268254314 & -1.5150262971 & -0.0500000000 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -19.1546420631 & -25.0495110943 & 6.6269394049 \\ -45.4064679121 & -58.1223790607 & 10.7694811395 \end{bmatrix},$$

$$U = \begin{bmatrix} 1.0000000000 & -0.2500000000 & -0.0437500000 & -0.0005208333 & 0.0004231771 \\ 1.0000000000 & 0.1084646955 & 0.0036051130 & -0.0004049101 & 0.0003289895 \\ 1.0000000000 & 10.6099333579 & 1.3754630884 & 0.0053695982 & -0.0043627985 \end{bmatrix},$$

$$V = \begin{bmatrix} 1.0000000000 & 10.6099333579 & 1.3754630884 & 0.0053695982 & -0.0043627985 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0567408031 & 0.0461019020 \\ 0 & 36.0578291732 & 8.1511106602 & -0.0698348354 & 0.0567408031 \end{bmatrix}.$$

4 Implementation aspects

A Nordsieck SGLM in the variable stepsize mode takes the form

$$\begin{aligned} Y^{[n]} &= h_n(A \otimes I_m)f(Y^{[n]}) + h_n^2(\overline{A} \otimes I_m)g(Y^{[n]}) + (UD(\delta_n) \otimes I_m)y^{[n-1]}, \\ y^{[n]} &= h_n(B \otimes I_m)f(Y^{[n]}) + h_n^2(\overline{B} \otimes I_m)g(Y^{[n]}) + (VD(\delta_n) \otimes I_m)y^{[n-1]}, \end{aligned} \tag{8}$$

where $h_n = x_n - x_{n-1}$. Here $Y^{[n]}$ is an approximation of stage order $q = p$ to the vector $y(x_{n-1} + ch_n) = [y(x_{n-1} + c_i h_n)]_{i=1}^s$, $y^{[n]}$ is an approximation of order p to the Nordsieck vector $[h_n^{i-1} y^{(i-1)}(x_n)]_{i=1}^r$, and $D(\delta_n)$ is the rescaling matrix defined by

$$D(\delta_n) := \operatorname{diag}\left(1, \delta_n, \delta_n^2, \dots, \delta_n^p\right),$$

where δ_n is the ratio of consecutive stepsizes $\delta_n = h_n/h_{n-1}$.

To obtain a reliable approximation to the vector $y^{[0]}$, we carry out one step of SDIRK method of order $p^\star = 3$ which gives sufficient output information, $\widetilde{y}_1 \approx y(x_0 + h_0)$ and $\widetilde{Y}_i \approx y(x_0 + \widetilde{c}_i h_0)$, $i = 1, 2, \dots, p^\star$.

For the stage predictors, without any additional computational cost, we use the Taylor expansion to predict the stage values

$$\begin{aligned} Y_i^{[n],0} &= y(x_{n-1} + c_i h_n) + O(h_n^{p+1}) \\ &= C^{(i)} D(\delta_n) y^{[n-1]} + O(h_n^{p+1}) \\ &\approx C^{(i)} D(\delta_n) y^{[n-1]}, \end{aligned}$$

where $C^{(i)}$ is the ith row of the matrix C.

In order to control the stepsize, we need to estimate the local truncation error. To do this, we approximate the $h^{p+1}y^{(p+1)}(x_n)$, using linear combination of the known stage first and second derivatives, $hf(Y_i^{[n]})$ and $h^2 g(Y_i^{[n]})$, $i = 1, 2, \ldots, s$.

The used strategy to control the stepsize in the advancing from the step n to the step $n+1$ is according to the following control

$$\text{est}(x_n) \leq Rtol \cdot \max\{\|y_n\|, \|y_{n+1}\|\} + Atol, \tag{9}$$

where $Atol$ and $Rtol$ are given absolute and relative tolerances. If the control (9) is not satisfied, the current step is repeated with the halved stepsize. Otherwise, the current step is accepted and we carry our the next step with the new stepsize as the following

$$h_{n+1} = \delta_{n+1} h_n,$$

where

$$\delta_{n+1} = \min\Big\{\Delta, \Big(\frac{\rho \cdot tol}{\|\text{est}(x_n)\|}\Big)^{\frac{1}{p+1}}\Big\}.$$

In our numerical experiments we have used $Atol = Rtol = tol$, $\rho = 0.9$ and $\Delta = 2$. This value for Δ is a safe choice, since it guarantees the zero-stability of the constructed methods of orders 3and 4.

5 Numerical experiments

In this section we present the results of numerical experiments to show efficiency of the constructed methods of order 3 and 4 in the variable stepsize mode. To compare, we also present the results of numerical experiments of the L–stable Nordsieck GLM of order $p = q = 3$ given in [11] on the page 88. Computational experiments are done by applying methods on the stiff chemical reaction problem, called E5 [9],

$$\begin{cases} y_1' = -Ay_1 - By_1y_3, \\ y_2' = Ay_1 - MCy_2y_3, \\ y_3' = Ay_1 - By_1y_3 - MCy_2y_3 + Cy_4, \\ y_4' = By_1y_3 - Cy_4, \end{cases}$$

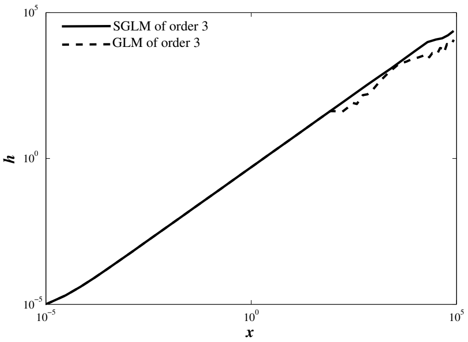
where $A = 7.89 \times 10^{-10}$, $B = 1.1 \times 10^7$, $C = 1.13 \times 10^3$, and $M = 10^6$. The initial values are $y(0) = [1.76 \times 10^{-3}, 0, 0, 0]^T$ and $x \in [0, 10^5]$. The variables of this problem are badly scaled ($y_1 \approx 10^{-3}$ at the beginning, all other components do not exceed the value 1.46×10^{-10}). The differential equations possess the invariant $y_2 - y_3 - y_4 = 0$, and because of eventual cancellation of digits, we use the relation $y_3' = y_2' - y_4'$ in solving.

Table 1: Numerical results for problem E5 solved by the methods of order 3 with $h_0 = 10^{-5}$.

tol	Method	*ns*	*nrs*	*nfe*	*nJe*	*ge*
10^{-6}	SGLM	38	5	516	432	1.96×10^{-7}
	GLM	42	3	649	473	1.00×10^{-6}
10^{-8}	SGLM	37	0	337	265	1.56×10^{-8}
	GLM	52	5	763	539	1.95×10^{-8}
10^{-10}	SGLM	48	1	482	386	7.38×10^{-10}
	GLM	103	4	1354	930	3.36×10^{-9}
10^{-12}	SGLM	84	2	1201	1031	2.28×10^{-11}
	GLM	276	10	3437	2297	1.16×10^{-10}

Table 2: Numerical results for problem E5 solved by the methods of order 4 with $h_0 = 10^{-5}$.

tol	Method	*ns*	*nrs*	*nfe*	*nJe*	*ge*
10^{-6}	SGLM	56	7	1746	1560	4.80×10^{-8}
	GLM	187	17	5138	4123	4.36×10^{-5}
10^{-8}	SGLM	57	8	1403	1211	2.71×10^{-9}
	GLM	228	29	5317	4037	3.11×10^{-7}
10^{-10}	SGLM	62	0	1432	1249	3.68×10^{-10}
	GLM	554	107	13461	10156	3.07×10^{-9}
10^{-12}	SGLM	83	1	1416	1167	1.21×10^{-11}
	GLM	429	60	8717	6277	8.94×10^{-11}



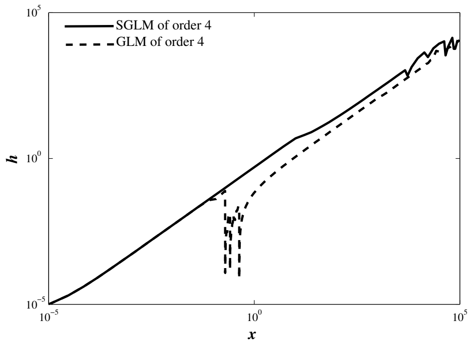


Figure 1: Accepted stepsizes versus x of the SGLM and GLM of order 3 (left) and order 4 (right) for problem P4 with $h_0 = 10^{-5}$ and $tol = 10^{-8}$.

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On a sub-projective Randers geometry

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Abstract

Projective manifolds form an important class of spaces in geometry and topology. Metric projective manifolds are are typical examples of spaces on which straight line segments are the shortest connection between two points, at least at a local scheme. Randers manifolds $(M, F = \alpha + \beta)$ are the ubiquitous in Finslerian geometry with applications. A notable sub-group of the projective group $\mathsf{Proj}(M, F)$ which is denoted by $\widehat{\mathsf{Proj}}(M, F)$ turns the projective Finsler geometry to be a finer geometry called special projective geometry. Some difficult results in projective Finsler geometry which are not proved yet, are established in this finer projective geometry; A Lichnérowicz-Obata type result is proved for Randers manifolds.

Keywords: Projective geometry, projective manifolds, projective group, Randers metric
Mathematics Subject Classification [2010]: 53B40, 53C60, 58J60

1 Introduction

Felix Klein's Erlanger program in 1872 upturns geometry to the study of those issues of a space which are invariant under a group of transformations. In a vastly structure free sense, a geometry due to Klein's manifest, is a pair (X, G), where X is a set and G is a group acting (usually transitively) on X. The set X and the group G may have geometric, topological, algebraic, analytic, combinatorial, etc., or even composite additional structures in any actual instances. The geometries $(\mathbb{R}^n, \mathsf{Isom}(\mathbb{R}^n, d_{Euclidean}))$, $(\mathbb{R}^n, \mathsf{Aff}(\mathbb{R}^n))$,$(\mathbb{R}\mathrm{P}^n, \mathsf{PGl}(n+1, \mathbb{R}))$, are called the *Euclidean geometry*, the *Affine geometry* and the *Projective geometry*, respectively. If $\mathbb{R}^n$ is equipped with a Minkowski norm and d denotes the associated metric, the $(\mathbb{R}^n, \mathsf{Isom}(\mathbb{R}^n, d))$-geometry is called a *Minkowski geometry.* One may also think of geometries which are infinitesimally modeled on the Euclidean (resp. Minkowskian) geometry $(\mathbb{R}^n, \mathsf{Isom}(\mathbb{R}^n, d_{Euclidean}))$ (or $(\mathbb{R}^n, \mathsf{Isom}(\mathbb{R}^n, d))$) when we deal with differentiable manifolds. These class of geometries are known by *Riemannian geometry* (resp. *Finslerian geometry*). It is even natural to consider spaces which have local (X, G) geometry; this is indeed, modeling the space locally on a (X, G)-geometry.

Hilbert's fourth problem, posted in International Congress of Mathematics 1n 1900, asks, in a modern version, to construct and study the geometries in which the straight line segment is the shortest connection between two points, cf. [5]. These geometries may be found in the wider geometries locally modeled on $(\mathbb{R}\mathrm{P}^n, \mathsf{PGl}(n+1, \mathbb{R}))$; although, the different modern approaches evokes the problem at the basis of integral geometry, inverse problems in the calculus of variations, and Finslerian geometry. There has been a longstanding history of research activities in solving Hilbert's fourth problem, cf. [13, 19]. This problem may have solutions in smooth and non-smooth synthetic settings. Smooth solutions of Hilbert's fourth problems are in fact Riemannian-Finslerian geometries in which, the space is covered by coordinates systems within the geodesics are rectilinear. These types of Riemannian-Finslerian geometries are said to be *projective*. *Beltrami*'s theorem asserts that projective Riemannian geometries are exactly those with constant sectional curvature and vice-versa, cf. [4, 9]. Therefore, within an immense framework, constant curvature geometries are sub-geometries of projective geometry in the Riemannian setting. However, this result fails in for Finslerian geometries, since there are non-projective Finslerian geometries with constant curvature, cf. [3]. Another important sub-geometries of the projective geometry are locally modeled on $(\mathbb{R}^n, \mathsf{Aff}(\mathbb{R}^n))$. These geometries not only possess the notions of projective geometry, but also enjoy the notion of parallelism. An (X, G)-structure on a manifold is an atlas of coordinates neighborhoods $\mathcal{A} = \{\phi_\alpha : U_\alpha \longrightarrow X\}_{\alpha\in I}$ such given any intersecting neighborhood U_α and U_β and a connected component C of $U_\alpha \cap U_\beta$, there is an element $g_{\alpha,\beta,C} \in G$, such that $\phi_\alpha \circ \phi_\beta^{-1} = g_{\alpha,\beta,C}$. Every $(\mathbb{R}^n, \mathsf{Aff}(\mathbb{R}^n))$ (called an *affine structure*) corresponds to a projective affine connection and every $(\mathbb{RP}^n, \mathsf{PGl}(\mathbb{R}^n))$ (called a *projective structure*) corresponds to a projective connection.

2 Preliminaries

Let M be a connected and smooth manifold of dimension $n \geq 2$. We denote the elements of the tangent manifold TM by (x, v) where $v \in T_xM$ with the natural projection $\pi : TM \to M$ is given by $\pi(x, v) := x$ and we set $TM_0 = TM \setminus \{0\}$. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ with the following properties: (1) F is C^∞ on TM_0, (2) F is positively 1-homogeneous on the fibers of tangent bundle TM and (3) the y-Hessian of F^2 with elements $g_{ij}(x, v) := \frac{\partial^2 F^2}{\partial v^i \partial v^j}$ is positive definite. The pair (M, F) is then called a *Finsler space*. We denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)v^i v^j}$ and a 1-form by $\beta = b_i(x)v^i$.
Two Finsler metrics F and $\tilde{F}$ on a smooth n-manifold M are said to be *projectively equivalent* (resp. *affine equivalent*) if they have the same forward geodesics (resp. they have the same forward geodesics with the same parametrization). A Finsler manifold (M, F) is said to be *projective* if M is covered by an atlas $\mathcal{A}$ of coordinates neighborhood U on which F and the Euclidean metric are projectively equivalent; A Finsler manifold (M, F) is said to be *flat* if M is covered by an atlas $\mathcal{A}$ of coordinates neighborhood U on which F and the Euclidean metric are affine equivalent; This terminology sometimes is called locally flat or locally Minkowski. Euclidean geodesics on U are straight lines, hence coordinates change in $\mathcal{A}$ may be viewed in the above cases, naturally as elements of $\mathsf{PGl}(n, \mathbb{R})$ (resp. as element of $\mathsf{Aff}(\mathbb{R}^n)$. Thus, a projective (resp. Affine) Finsler manifold on M is indeed a projective (rep. affine) structure on M and it can be modeled locally on $(\mathbb{RP}^n, \mathsf{PGl}(n + 1, \mathbb{R}))$ (rep. $(\mathbb{R}^n, \mathsf{Aff}(\mathbb{R}^n)))$. Given a Finsler space (M, F), a diffeomorphism $\phi : M \longrightarrow M$ is called a *projective transformation* (resp. *affine transformation*) if F and $\phi^* F$ are projectively (resp. affine equivalent). The collection of all projective (resp. affine) transformations is denoted by $\mathsf{Proj}(M, F)$ (resp. $(\mathsf{Aff}(M, F)$ and forms a finite dimensional Lie group with

respect to the compact-open topology, e.g. cf. [18]. Its connected component containing the identity map is denoted by $\mathsf{Proj}_0(M,F)$ (resp. $\mathsf{Aff}_0(M,F)$. A very natural and old fashion problem in differential geometry is to characterize (pseudo-)Riemannian manifolds (M,g) for which $\mathsf{Aff}(M,g) \subsetneq \mathsf{Proj}(M,g)$, namely,$M$ admits an essential projective transformation, cf. [11]. Upon a long public research history, (e.g. see [6, 7, 8]), the following rigidity result is announced:

Theorem 2.1. *(Projective Lichnérowicz conjecture) Let (M,g) be a compact (pseudo-) Riemannian manifold. Then, unless $(M;g)$ is a finite quotient of the Euclidean sphere, $\mathsf{Proj}(M,g)/\mathsf{Aff}(M,g)$ is finite. Same does hold when compactness is replaced by completeness.*

The another form of projective Lichnérowicz conjecture may be formulated in other forms: *if M is compact and $\mathsf{Aff}(M,g) \subsetneq \mathsf{Proj}(M,g)$, then (M,g) is covered by the Euclidean sphere by local isometry.*

Let us suppose that F is Finsler metric on the manifold M. Given any vector field $W \in \mathcal{X}(M)$ satisfying $F(x,W_x) < 1,\ x \in M$, there is a Finsler metric F_W on M such that we have $F(x, \frac{v}{F_W(x,v)} + W_x) = 1,\ x \in M,\ v \in T_xM \setminus \{0\}$. Hence, at every point $x \in M$, the indicatrix S_x of F equals the translation of the indicatrix S_x^W of F_W along the vector $W_x \in T_xM$. The Finsler metric F_W is called *the Zermelo transform of F with respect to W* and we write $Z_WF := F_W$, cf. [10]. The Zermelo transform of every Riemannian metric with respect to any appropriate vector field W is a Randers metric and vice-versa; this is called the so called *Zermelo correspondence* in the contexts, cf. [2]. Two Finsler metrics F and $\tilde{F}$ on M are said to be *weakly conformal* if there is a function $\sigma \in C^\infty(M)$ and a vector field W on M such that the pair (F,W) solves the Zermelo navigation problem $e^\sigma \tilde{F}(x, \frac{y}{F} + W_x) = 1,\ x \in M,\ y \in T_xM$, namely, $Z_W(e^\sigma \tilde{F}) = F$. Two Finsler manifold (M,F) and $(\tilde{M},\tilde{F})$ are said to be weakly conformal if there is diffeomorphism $\phi : M \longrightarrow N$ such that F and $\phi^*\tilde{F}$ are weakly conformal. The expression *weakly conformal* can be replaced by *weakly isometric* if we have $\sigma \equiv 0$. This terminology was first proposed by Zhongmin Shen and used in [17].

3 Main results

Given a Randers manifold $(M, F = \alpha + \beta)$, the group of projective transformations $\mathsf{Proj}(M,F)$ is a subgroup of $\mathsf{Proj}(M,\alpha)$, cf. [15, 16]; In fact, $\mathsf{Proj}(M,F)$ is the group of stabilizers of the tensor $\alpha s^i{}_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$. The equality holds if the 1-form β is closed. Some local issues of the projective group, such as local dimension, can be obtained using its Lie algebra of projective vector fields. The reference [1] is good introductory source for arriving projective vector fields in Finsler geometry. In an infinitesimal form it follows:

Theorem 3.1. *[14, 15] A vector field V is projective on a Randers space $(M, F = \alpha + \beta)$ if and only if V is projective in (M,α) and $\pounds_{\hat{V}}(\alpha s^i{}_j) = 0$.*

A classical result states that if $n = \mathrm{dim}M$, then $\mathsf{Proj}(M,\alpha)$ has at most $n(n+2)$ dimensions and the equality holds if and only if α is a projective metric. The latter is equivalent to constancy of the sectional curvature of α by Beltrami's theorem. We may generalize this to the following result:

Theorem 3.2. *An Randers metric $F = \alpha + \beta$ on a manifold M of dimension $(n \geq 3)$ is projective if and only if $\mathsf{Proj}(M, F)$ has (locally) dimension $n(n + 2)$.*

A generic Finsler manifold has in general several non-Riemannian quantities and each of which has a stabilizer group of transformations. The Berwald curvature for a Finsler metric an important non-Riemannian quantity which measures the deflection of the geodesic spray form being induced by a Riemannian metric, namely, the failure of being quadratic on tangent spaces. The collection of projective transformations which stabilize the Berwald curvature forms a subgroup of $\mathsf{Proj}(M, F)$, denoted by $\widehat{\mathsf{Proj}}(M, F)$. On the other hand, notice that the important non-Riemannian quantity which is called **S**-curvature play a fundamental role in Finsler spaces. It has fine link with Lott, Sturm and Villani's curvature-dimension condition $\mathsf{CD}(K, N)$ in its synthetic form, cf, [12]. Every Berwald spaces has vanishing **S**-curvature and thus, the projective group refers to be induced by a Riemannian metric. Einstein-Randers metrics have constant **S**-curvature. Hence, we are interested to consider Randers spaces whose S-curvature are nonzero constant and result the following characterization:

Theorem 3.3. *Let $(M, F = \alpha+\beta)$ be an n-dimensional $(n \geq 3)$ Randers space of non-zero constant* **S***-curvature. The special projective algebra of (M, F) has maximum dimension $\frac{n(n+1)}{2}$ if and only if F is (up to a rescaling) locally isometric the following locall projectively flat Randers metric:*

$$F(x, y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2} \pm \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \tag{1}$$

where, $y \in T_x\mathbb{R}^n,\ a \in \mathbb{R}^n, |a| < 1$.

Notice that, the Randers metric given by (1) is a asymmetric generalization of the Klein's metric on the unit disk.
Let $\mathsf{Fins}(M)$ (resp. $\mathsf{Riem}(M)$) denotes the collection of all Finsler metrics (resp. Riemannian metrics) on the manifold M. Any subgroup of diffeomorphism group $\mathsf{Diff}(M)$ can act naturally on $\mathsf{Fins}(M)$ by pull-back; So does the projective group $\mathsf{Proj}(M, F)$ and in particular the special projective group $\widehat{\mathsf{Proj}}(M, F)$. A dynamical issue of such an action can be considered by having fixed point or acting with no fixed points. In the former case, the stabilizer of fixed points are in fact isometries and the projective group is called essential in the later case. A classical result related to this dynamics is has been formulated by Lichnérowicz and later by Obata under the name *Projective Lichnérowicz-Obata conjecture*. In the sub-class Riemannian metrics, this conjecture was proved by Matveev in [7]. However, the proof for the large class Finsler metrics seems to be far away to be done, since the $\mathsf{Proj}(M, F)$-orbits in $\mathsf{Riem}(M)$ are finite dimensional manifold while $\mathsf{Fins}(M)$ may be infinite dimensional and this causes the analysis highly different. This may be related to more complex dynamical issues of the mentioned acting in comparison to Riemannian setting. Here, we prove this conjecture for Randers manifolds and in a version reduced to the special projective group:

Theorem 3.4. *Let us suppose that $(M, F = \alpha+\beta)$ be a Randers space of dimension $n \geq 2$ and is obtained by Zermelo transform $Z_W h = F$, where, h is a Riemannian metric and W is a vector field satisfying $h(W, W) < 1$. Then, at least one of the following statements*

holds:
(i) Every special projective vector field on (M, F) *is a conformal vector field of the Riemannian metric* h*,*
(ii) F *is of isotropic S-curvature.*

Notice that, the above result does not require any further topological assumptions such as completeness and has no counterparts in the Riemannian setting; Moreover, Theorem 3.4 is in fact an assertion about the acting connected component containing the identity of $\widehat{\mathsf{Proj}}(M, F)$. It is surprising that the special projective geometry is a sub-geometry of the conformal geometry whence the Randers metric is not of isotropic **S**-curvature.

Theorem 3.5. *Let us suppose that* $(M, F = \alpha + \beta)$ *be a closed and connected Randers space of dimension* $n \geq 2$ *and is obtained by Zermelo transform* $Z_W h = F$*, where,* h *is a Riemannian metric and* W *is a vector field satisfying* $h(W, W) < 1$*. Suppose that,* V *is a special projective vector field of* F*. Then, at least one of the following statements holds:*
(i) V *is a conformal vector field for the Riemannian metric* h*,*
(ii) There is a Randers metric $\hat{F}$ *such that* V *is a Killing vector field for* $\hat{F}$*,*
(iii) After an appropriate rescaling, F *is of the following local form:*

$$F(x, y) = \frac{\sqrt{|y|^2 + |x|^2|y|^2 - \langle x, y\rangle^2}}{1 + |x|^2} - \frac{f_{x^k} y^k}{\sqrt{1 - f^2(x)}}, \ y \in T_x M \cong \mathbb{R}^n, \tag{2}$$

where, f *is an eigenfunction of the standard Laplacian satisfying* $\Delta f = nf$ *and* $\|f\|_{L^\infty(M)} < 1$*. In particular,* (M, F) *is a projective manifold of positive flag curvature* $\mathbf{K}(x, y) = \frac{1}{4} + \frac{3F(x,-y)}{4(1-f(x)^2)F(x,y)}$*.*

Notice that, the case (iii) in Theorem 3.5 entails that (M, F) is weakly isometric tho the Euclidean sphere $(\mathbb{S}^n, h_1)$, where, h_1 is the standard Riemannian metric on $\mathbb{S}^n$ induced as a hypersurface in $\mathbb{R}^{n+1}$.

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Derivations of direct limits of Lie superalgebras

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Abstract

In this work, we study derivations of a direct limit of Lie superalgebras. As an application, we determine the derivation algebra of a direct union of finite dimensional basic classical simple Lie superalgebras.

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1 Derivations

Following the interest of physicists in the context of supersymmetries, in 1977, V. Kac [1] introduced Lie superalgebras (known as $\mathbb{Z}_2$-graded Lie algebras in Physics). He classified classical Lie superalgebras, i.e., finite dimensional simple Lie superalgebras whose even parts are reductive Lie algebras. These Lie superalgebras are a generalization of finite dimensional simple Lie algebras over an algebraically closed field of characteristic zero but classical Lie superalgebras are not necessarily equipped with nondegenerate invariant bilinear forms while Killing form on a finite dimensional simple Lie algebra over a field of characteristic zero is invariant and nondegenerate. To get a better super version of finite dimensional simple Lie algebras, one can work with those classical Lie superalgebras equipped with even nondegenerate invariant bilinear forms, called finite dimensional basic classical simple Lie superalgebras. It is known that all derivations of a finite dimensional Lie superalgebra with nondegenerate Killing form are inner. In [2], the author studies derivations of locally finite split simple Lie algebras [3]; a locally finite split simple Lie algebra is a direct union of finite dimensional split simple Lie algebras. In this work, we first study derivations of a direct limit of Lie superalgebras and then as an application, we determine the derivations of locally finite basic classical simple Lie superalgebras [4]. This work has been derived from the author's recent preprint on the topic.

Throughout this work, $\mathbb{F}$ is an algebraically closed field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over $\mathbb{F}$. We denote the dual space of a vector space V by V^*. We denote the degree of a homogenous element v of a superspace by $|v|$ and make a convention that if in an expression, we use $|u|$ for an element u of a superspace, by default we have assumed u is homogeneous. For two symbols i, j, by $\delta_{i,j}$, we mean the Kronecker delta.

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Suppose that $\mathfrak{L}$ is a Lie superalgebra and M is a superspace, we say M together with a bilinear map $\cdot : M \times \mathfrak{L} \longrightarrow M$ is a *right $\mathfrak{L}$-module* if

$$M_i \cdot \mathfrak{L}_j \subseteq M_{i+j}$$
$$a \cdot [x, y] = (a \cdot x) \cdot y - (-1)^{|x||y|}(a \cdot y) \cdot x$$

for $x, y \in \mathfrak{L}$, $a \in M$ and $i, j \in \{0, 1\}$. We also say M together with a bilinear map $* : \mathfrak{L} \times M \longrightarrow M$ is a *left $\mathfrak{L}$-module* if

$$\mathfrak{L}_i * M_j \subseteq M_{i+j}$$
$$[x, y] * a = x * (y * a) - (-1)^{|x||y|} y * (x * a)$$

for $x, y \in \mathfrak{L}$, $a \in M$ and $i, j \in \{0, 1\}$. We note that if $(M, \cdot)$ is a right $\mathfrak{L}$-module, then M together with the action $x \cdot a := -(-1)^{|x||a|} a * x$ $(x \in \mathfrak{L}, a \in M)$ is a left $\mathfrak{L}$-module. In the sequel, when we say M is an $\mathfrak{L}$-module, we mean that it is a right $\mathfrak{L}$-module; in this case, we use the left action of $\mathfrak{L}$ on M as we have just defined. For an $\mathfrak{L}$-module M, we set $M^{\mathfrak{L}} := \{a \in M \mid ax = 0; \quad \forall x \in \mathfrak{L}\}$ and note that

$$M^{\mathfrak{L}} := \{a \in M \mid xa = 0; \quad \forall x \in \mathfrak{L}\}. \tag{1.1}$$

Also for an $\mathfrak{L}$-module M, we say a bilinear form $(\cdot, \cdot) : M \times M \longrightarrow \mathbb{F}$ is $\mathfrak{L}$-invariant if

$$(ax, b) = (a, xb); \quad x \in \mathfrak{L}, \;\; a, b \in M.$$

Definition 1.1. Suppose that $\mathcal{L}$ is a Lie superalgebra and M is an $\mathcal{L}$-module. A *derivation* of $\mathcal{L}$ in M is a linear map $d : \mathcal{L} \longrightarrow M$ satisfying

$$d[x, y] = d(x)y - (-1)^{|x||y|} d(y)x$$

for all $x, y \in \mathcal{L}$. We denote the set of all derivations of $\mathcal{L}$ in M by $\mathrm{der}(\mathcal{L}, M)$. A derivation $d \in \mathrm{der}(\mathcal{L}, M)$ is called *inner* if there is $m \in M$ with $d(x) = mx$ for all $x \in \mathcal{L}$. If we consider $\mathcal{L}$ as an $\mathcal{L}$-module, we denote $\mathrm{der}(\mathcal{L}, \mathcal{L})$ by $\mathrm{der}(\mathcal{L})$.

We recall that *the first cohomology group* of a Lie superalgebra $\mathcal{L}$ with coefficients in an $\mathcal{L}$-module M is the quotient space $H^1(\mathcal{L}, M) := \mathrm{der}(\mathcal{L}, M)/\mathrm{Ider}(\mathcal{L}, M)$ in which $\mathrm{Ider}(\mathcal{L}, M)$ is the set of inner derivations of $\mathcal{L}$ in M.

The aim of this work is the study of the derivations of a direct limit $\mathcal{L}$ of Lie superalgebras in $\mathcal{L}$-modules. We first briefly explain the concepts of the direct limit and the inverse limit in a category $\mathcal{C}$. Suppose that $(I, \preccurlyeq)$ is a directed set and $\{A^i \mid i \in I\}$ is a class of objects of $\mathcal{C}$. For $i, j \in I$ with $i \preccurlyeq j$, suppose $f_{ji} : A^i \longrightarrow A^j$ is a morphism such that $f_{ii} = \mathrm{id}$ and $f_{kj} f_{ji} = f_{ki}$ for $i, j, k \in I$ with $i \preccurlyeq j \preccurlyeq k$. The pair $(\{A^i\}_{i \in I}, \{f_{ji}\}_{i \preccurlyeq j})$ is called a *directed system.* A *direct limit* of this directed system is an abject A together with a class $\{f_i : A^i \longrightarrow A \mid i \in I\}$ of morphisms such that

- for each $i, j \in I$ with $i \preccurlyeq j$, $f_j \circ f_{ji} = f_i$,
- if B is an object of $\mathcal{C}$ and $\{\varphi_i : A^i \longrightarrow B \mid i \in I\}$ is a class of morphisms such that for each $i, j \in I$ with $i \preccurlyeq j$, $\varphi_j \circ f_{ji} = \varphi_i$, then there is a unique morphism $\varphi : A \longrightarrow B$ such that $\varphi \circ f_i = \varphi_i$ for all $i \in I$.

We refer to f_i's as *canonical morphisms.* Direct limits of a directed system $(\{A^i\},\{f_{ji}\}_{i\preccurlyeq j})$ are equivalent, and so if there exists one, we refer to as "*the*" direct limit and denote it by $\varinjlim_{i\in I}A^i$. Direct limits exist in the category of Lie superalgebras. One knows that if $(\{A^i\},\{f_{ji}\}_{i\preccurlyeq j})$ is a directed system in a concrete category $\mathcal{C}$ such that the direct limit exists for this directed system, then $\varinjlim_{i\in I}A^i=\cup_{i\in I}f_i(A^i)$. Also if $f_i(a)=f_j(b)$, for some $i,j\in I$, $a\in A^i$ and $b\in A^j$, then there is $k\in I$ with $i\preccurlyeq k$, $j\preccurlyeq k$ and $f_{ki}(a)=f_{kj}(b)$.

Next for $i,j\in I$ with $i\preccurlyeq j$, suppose $p_{ij}:A^j\longrightarrow A^i$ is a morphism such that $p_{ii}=\mathrm{id}$ and $p_{ij}p_{jk}=p_{ik}$ for $i,j,k\in I$ with $i\preccurlyeq j\preccurlyeq k$. The pair $(\{A^i\},\{p_{ij}\}_{i\preccurlyeq j})$ is called an *inverse system.* An *inverse limit* of this inverse system is an abject A together with a class $\{p_i:A\longrightarrow A^i\mid i\in I\}$ of morphisms such that

- for each $i,j\in I$ with $i\preccurlyeq j$, $p_{ij}\circ p_j=p_i$,
- if B is an object of $\mathcal{C}$ and $\{\psi_i:B\longrightarrow A^i\mid i\in I\}$ is a class of morphisms such that for each $i,j\in I$ with $i\preccurlyeq j$, $p_{ij}\circ\psi_j=\psi_i$, then there is a unique morphism $\psi:B\longrightarrow A$ such that $p_i\circ\psi=\psi_i$ for all $i\in I$.

Two inverse limits of an inverse system $(\{A^i\}_{i\in I},\{p_{ij}\}_{i\preccurlyeq j})$ are equivalent, and so if an inverse limit exists, we refer to as "*the*" inverse limit and denote it by $\varprojlim_{i\in I}A^i$. One knows that if $(\{A^i\}_{i\in I},\{p_{ij}\}_{i\preccurlyeq j})$ is an inverse system in a concrete category $\mathcal{C}$ such that $\prod_{i\in I}A^i$ is a product of $\{A^i\mid i\in I\}$ in $\mathcal{C}$ (e.g. if $\mathcal{C}$ is the category of super vector spaces), then

$$\{(a_i)_i\in\prod_{i\in I}A^i\mid p_{ij}(a_j)=a_i;\ i\preccurlyeq j\}$$

together with the canonical projection maps corresponding to the direct product $\prod_{i\in I}A^i$ is the inverse limit of $(\{A^i\}_{i\in I},\{p_{ij}\}_{i\preccurlyeq j})$. In the sequel, by the inverse limit for such an inverse system, we mean the one we have just defined. From now on till the end of this section, we suppose I is a directed set and $(\{\mathfrak{L}^i\}_i,\{f_{ji}\}_{i\preccurlyeq j})$ is a directed system in the category of Lie superalgebras. Set $\mathfrak{L}:=\varinjlim_{i\in I}\mathfrak{L}^i$ with the canonical morphisms f_i $(i\in I)$. Suppose $\mathfrak{u}$ is an $\mathfrak{L}$-module whose module action is written as juxtaposition. For $i\in I$, consider $\mathfrak{u}$ as an $\mathfrak{L}^i$-module via the action $u\cdot_i x:=uf_i(x)$ for $x\in\mathfrak{L}^i$ and $u\in\mathfrak{u}$.

Proposition 1.2. We have the following:

(i) Suppose $i,j\in I$ with $i\preccurlyeq j$, then for $d\in\mathrm{der}(\mathfrak{L}^j,\mathfrak{u})$, $d\circ f_{ji}\in\mathrm{der}(\mathfrak{L}^i,\mathfrak{u})$.

(ii) Suppose that $i,j\in I$ with $i\preccurlyeq j$. Define $\mathfrak{d}_{ij}:\mathrm{der}(\mathfrak{L}^j,\mathfrak{u})\longrightarrow\mathrm{der}(\mathfrak{L}^i,\mathfrak{u})$ mapping $d\in\mathrm{der}(\mathfrak{L}^j,\mathfrak{u})$ to $d\circ f_{ji}$, then $(\{\mathrm{der}(\mathfrak{L}^i,\mathfrak{u})\}_i,\{\mathfrak{d}_{ij}\}_{i\preccurlyeq j})$ is an inverse system.

(iii) $\mathrm{der}(\varinjlim_{i\in I}\mathfrak{L}^i,\mathfrak{u})\simeq\varprojlim_{i\in I}\mathrm{der}(\mathfrak{L}^i,\mathfrak{u})$.

Proof. *(i)*, *(ii)* One can easily check it.

(iii) We recall that $\mathfrak{L}=\varinjlim_{i\in I}\mathfrak{L}^i$ and suppose that $d\in\mathrm{der}(\mathfrak{L},\mathfrak{u})$. Then for $i\in I$ and $x,y\in\mathfrak{L}^i$, we have

$$\begin{aligned}(d\circ f_i)[x,y] &= d[f_i(x),f_i(y)]\\ &= d(f_i(x))f_i(y)-(-1)^{|f_i(y)||f_i(x)|}d(f_i(y))f_i(x)\\ &= d(f_i(x))\cdot_i y-(-1)^{|y||x|}d(f_i(y))\cdot_i x\\ &= (d\circ f_i)(x)\cdot_i y-(-1)^{|y||x|}(d\circ f_i)(y)\cdot_i x.\end{aligned}$$

This means that $d \circ f_i \in \mathrm{der}(\mathfrak{L}^i, \mathfrak{u})$. Define $\mathfrak{d}_i : \mathrm{der}(\mathfrak{L}, \mathfrak{u}) \longrightarrow \mathrm{der}(\mathfrak{L}^i, \mathfrak{u})$ mapping $d \in \mathrm{der}(\mathfrak{L}^i, \mathfrak{u})$ to $d \circ f_i$. We claim that $\mathrm{der}(\varinjlim_{i \in I} \mathfrak{L}^i, \mathfrak{u})$ together with the maps $\mathfrak{d}_i$ $(i \in I)$ is the inverse limit of the inverse system $(\{\mathrm{der}(\mathfrak{L}^i, \mathfrak{u})\}_i, \{\mathfrak{d}_{ij}\}_{i \preccurlyeq j})$. It follows from the following:

• For $i, j \in I$ with $i \preccurlyeq j$ and $d \in \mathrm{der}(\mathfrak{L}^j, \mathfrak{u})$, we have

$$\mathfrak{d}_{ij} \circ \mathfrak{d}_j(d) = \mathfrak{d}_{ij}(d \circ f_j) = d \circ f_j \circ f_{ji} = d \circ f_i = \mathfrak{d}_i(d).$$

• Suppose that $\mathcal{V}$ is a vector superspace and that for each $i \in I$, $\psi_i : \mathcal{V} \longrightarrow \mathrm{der}(\mathfrak{L}^i, \mathfrak{u})$ is a Linear transformation such that for $i, j \in I$ with $i \preccurlyeq j$, $\mathfrak{d}_{ij} \circ \psi_j = \psi_i$. We know that $\varinjlim_{i \in I} \mathfrak{L}^i = \cup_{i \in I} f_i(\mathfrak{L}^i)$. Define

$$\begin{aligned} &\psi : \mathcal{V} \longrightarrow \mathrm{der}(\varinjlim_{i \in I} \mathfrak{L}^i, \mathfrak{u}) \\ &x \mapsto \psi(x) \end{aligned}$$

in which for $x \in \mathcal{V}$, $\psi(x)$ maps $f_i(y)$, for $y \in \mathfrak{L}^i$, to $\psi_i(x)(y)$. We show that ψ is well-defined. Suppose that $x \in \mathcal{V}$, $i, j \in I$, $z \in \mathfrak{L}^i$ and $y \in \mathfrak{L}^j$ such that $f_j(y) = f_i(z)$ and show that $\psi_j(x)(y) = \psi_i(x)(z)$. We know that there is $k \in I$ with $i \preccurlyeq k$ and $j \preccurlyeq k$ such that $f_{kj}(y) = f_{ki}(z)$. Now we have

$$\psi_j(x)(y) = \psi_i(x)(z).$$

• For $i \in I$, we have $\mathfrak{d}_i \circ \psi = \psi_i$. In fact, for $x \in \mathcal{V}$ and $y \in \mathfrak{L}^i$,

$$(\mathfrak{d}_i \circ \psi)(x)(y) = \mathfrak{d}_i(\psi(x))(y) = (\psi(x) \circ f_i)(y) = \psi(x)(f_i(y)) = \psi_i(x)(y).$$

• The linear transformation ψ with the mentioned property in the previous part is unique. Indeed, suppose that $\varphi : \mathcal{V} \longrightarrow \mathrm{der}(\varinjlim_{i \in I} \mathfrak{L}^i)$ is a linear transformation such that for each $i \in I$, $\mathfrak{d}_i \circ \varphi = \psi_i$. Then for $i \in I$, $x \in \mathcal{V}$ and $y \in \mathfrak{L}^i$, we have

$$\begin{aligned} \psi(x)(f_i(y)) = \psi_i(x)(y) = (\mathfrak{d}_i \circ \varphi)(x)(y) &= (\mathfrak{d}_i(\varphi(x))(y) \\ &= (\varphi(x) \circ f_i)(y) \\ &= \varphi(x)(f_i(y)). \end{aligned}$$

Therefore, we have $\psi = \varphi$. This completes the proof. □

Proposition 1.2. *Suppose that for each $i \in I$, $H^1(\mathfrak{L}^i, \mathfrak{u}) = \{0\}$. For $i \in I$, denote the equivalence class $u \in \mathfrak{u}$ in $\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$ by $[u]_i$. For $i, j \in I$ with $i \preccurlyeq j$, define $p_{ij} : \mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^j} \longrightarrow \mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$ mapping $[u]_j$ to $[u]_i$. Then $\{\{\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}\}_i, \{p_{ij}\}_{i \preccurlyeq j}\}$ is an inverse system and*

$$\mathrm{der}(\mathfrak{L}, \mathfrak{u}) \simeq \varprojlim_{i \in I} (\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}).$$

Proof. Suppose that $i, j \in I$ with $i \preccurlyeq j$. If $[u]_j = 0$ for some $u \in \mathfrak{u}$, then for each $x \in \mathfrak{L}^j$, $u \cdot_j x = u f_j(x) = 0$, so for each $x \in \mathfrak{L}^i$,

$$u \cdot_i x = u f_i(x) = u f_j(f_{ji}(x)) = 0.$$

This means that p_{ij} is well-defined. It is immediate that $p_{ii} = \mathrm{id}$ and that $p_{ij} p_{jk} = p_{ik}$ for $i, j \in I$ with $i \preccurlyeq j$. This is what we need for the first assertion. Next suppose that $d \in \operatorname{der}(\mathfrak{L}, \mathfrak{u})$. As we have already seen, for each $i \in I$, $d \circ f_i \in \operatorname{der}(\mathfrak{L}^i, \mathfrak{u})$. Since $H^1(\mathfrak{L}^i, \mathfrak{u}) = \{0\}$, there is $u_i \in \mathfrak{u}$ such that

$$(d \circ f_i)(x) = u_i \cdot_i x = u_i f_i(x); \quad x \in \mathfrak{L}^i.$$

We claim that

$$\eta_d := ([u_i]_i)_i \in \varprojlim_{i \in I}(\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}).$$

To show this, we assume $i, j \in I$ with $i \preccurlyeq j$ and show that $[u_i]_i = [u_j]_i$. We must show $u_i - u_j \in \mathfrak{u}^{\mathfrak{L}^i}$ or equivalently, $u_i f_i(x) = u_j f_i(x)$ for all $x \in \mathfrak{L}^i$. For each $x \in \mathfrak{L}^i$, we have

$$\begin{aligned} u_j f_i(x) = u_j f_j(f_{ji}(x)) = u_j \cdot_j f_{ji}(x) = (d \circ f_j)(f_{ji}(x)) &= (d \circ f_j \circ f_{ji})(x) \\ &= (d \circ f_i)(x) \\ &= u_i \cdot_i x \\ &= u_i f_i(x). \end{aligned}$$

This is what we need. Now $\eta : \operatorname{der}(\mathfrak{L}, \mathfrak{u}) \longrightarrow \varprojlim_{i \in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$ mapping d to η_d is a well-defined linear transformation. Next suppose $\alpha := ([u_i]_i)_{i \in I} \in \varprojlim_{i \in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$ and recall that $\mathfrak{L} = \varinjlim_{i \in I}\mathfrak{L}^i = \cup_{i \in I} f_i(\mathfrak{L}^i)$. Define $d_\alpha \in \operatorname{der}(\mathfrak{L}, \mathfrak{u})$ mapping $f_i(x)$ to $u_i f_i(x)$ if $i \in I$ and $x \in \mathfrak{L}^i$. We first show that d_α is well-defined. Suppose that $i, j \in I$, $x \in \mathfrak{L}^i$ and $y \in \mathfrak{L}^j$ are such that $f_i(x) = f_j(y)$. Then there is $k \in I$ with $i \preccurlyeq k$ and $j \preccurlyeq k$ such that $f_{ki}(x) = f_{kj}(y)$. Since $\alpha \in \varprojlim_{i \in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$, we have $[u_k]_i = [u_i]_i$ and $[u_k]_j = [u_j]_j$. So it follows that $u_i f_i(x) = u_k f_i(x)$ and $u_j f_j(y) = u_k f_j(y)$. Therefore, we have

$$\begin{aligned} u_i f_i(x) = u_k f_i(x) = u_k(f_k \circ f_{ki})(x) = u_k f_k(f_{ki}(x)) = u_k f_k(f_{kj}(y)) &= u_k f_j(y) \\ &= u_j f_j(y). \end{aligned}$$

This shows that d_α is well-defined. Next we show that d_α is a derivation. Suppose that $a, b \in \mathfrak{L}$, then there are $i, j \in I$, $x \in \mathfrak{L}^i$ and $y \in \mathfrak{L}^j$ with $a = f_i(x)$ and $b = f_j(y)$. Take $k \in I$ to be such that $i \preccurlyeq k$ and $j \preccurlyeq k$. Therefore, we have

$$\begin{aligned} d_\alpha[a, b] &= d_\alpha[f_i(x), f_j(y)] \\ &= d_\alpha[f_k(f_{ki}(x)), f_k(f_{kj}(y))] \\ &= d_\alpha(f_k[(f_{ki}(x)), (f_{kj}(y))]) \\ &= u_k f_k[f_{ki}(x), f_{kj}(y)] \\ &= u_k[f_k(f_{ki}(x)), f_k(f_{kj}(y))] \\ &= (u_k f_k(f_{ki}(x))) f_k(f_{kj}(y)) - (-1)^{|x||y|}(u_k f_k(f_{kj}(y))) f_k(f_{ki}(x) \\ &= (u_k f_k(f_{ki}(x))) f_j(y) - (-1)^{|x||y|}(u_k f_k(f_{kj}(y))) f_i(x) \\ &= d_\alpha(a)(b) - (-1)^{|a||b|} d_\alpha(b)a. \end{aligned}$$

Now we are done as $\eta_{d_\alpha} = \alpha$ and $d_{\eta_d} = d$ for $\alpha \in \varprojlim_{i \in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$ and $d \in \operatorname{der}(\mathfrak{L}, \mathfrak{u})$. □

Proposition 1.3. *Suppose that* $\mathfrak{u} = \cup_{i \in I} f_i(\mathfrak{L}^i)\mathfrak{u}$ *and that* $(\cdot, \cdot) : \mathfrak{u} \times \mathfrak{u} \longrightarrow \mathbb{F}$ *is an* $\mathfrak{L}$-*invariant nondegenerate bilinear form. For* $\alpha := ([u_i])_i \in \varprojlim_{i \in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$, *define* θ_α *to be an element of* $\mathfrak{u}^*$ *mapping* $a \in f_i(\mathfrak{L}^i)\mathfrak{u}$ $(i \in I)$ *to* (u_i, a). *Then*

$$\operatorname{der}(\mathfrak{L}) \simeq \{\theta_\alpha \mid \alpha \in \varprojlim_{i \in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}\} \subseteq \mathfrak{u}^*.$$

Proof. Using Proposition 1.2, we just need to show that each θ_α is a well-defined functional and that if $\theta_\alpha = \theta_\beta$, then $\alpha = \beta$.

Step 1. For $i \in I$, we have $(\mathfrak{u}^{\mathfrak{L}^i}, f_i(\mathfrak{L}^i)\mathfrak{u}) = \{0\}$: Suppose that $z \in \mathfrak{u}^{\mathfrak{L}^i}$ and $a = \sum_{t=1}^m f_i(a^t)v_t \in f_i(\mathfrak{L}^i)\mathfrak{u}$, where m is a positive integer and $a^t \in \mathfrak{L}^i$ for $t \in \{1, \dots, m\}$. Then using (1.1), we have

$$(z, a) = (z, \sum_{t=1}^m f_i(a^t)v_t) = \sum_{t=1}^m (zf_i(a^t), v_t) = \sum_{t=1}^m (z \cdot_i a^t, v_t) = 0$$

Step 2. Suppose that $([u_i]_i)_{i\in I} \in \varprojlim_{i\in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}_i}$. If $i_0, j_0 \in I$ with $i_0 \preccurlyeq j_0$ and $a \in f_{i_0}(\mathfrak{L}^{i_0})\mathfrak{u}$, then $(u_{i_0}, a) = (u_{j_0}, a)$: Suppose that $a = \sum_{t=1}^m f_{i_0}(a^t)v_t$, where m is a positive integer and $a^t \in \mathfrak{L}^{i_0}$ for $t \in \{1, \dots, m\}$. Since $[u_{i_0}]_{i_0} = [u_{j_0}]_{i_0}$, one finds $z \in \mathfrak{u}^{\mathfrak{L}^{i_0}}$ with $u_{i_0} = u_{j_0} + z$. Now using Step 1, we have

$$(u_{i_0}, a) = (u_{j_0}, a) + (z, a) = (u_{j_0}, a) + 0 = (u_{j_0}, a).$$

Step 3. Suppose that $([u_i]_i)_{i\in I} \in \varprojlim_{i\in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$. If $i, j \in I$, $a \in f_i(\mathfrak{L}^i)\mathfrak{u}$ and $b \in f_j(\mathfrak{L}^j)\mathfrak{u}$ such that $a = b$, then $(u_i, a) = (u_j, b)$: Take $k \in I$ to be such that $i \preccurlyeq k$ and $j \preccurlyeq k$, then by Step 2, we have $(u_i, a) = (u_k, a) = (u_k, b) = (u_j, b)$.

Step 4. If $\alpha = \beta \in \varprojlim_{i\in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$, then $\theta_\alpha = \theta_\beta$: Suppose that $\alpha = ([u_i]_i)_{i\in I}$, $\beta = ([u'_i]_i)_{i\in I}$, then for each $i \in I$, $[u_i]_i = [u'_i]_i$. Then for each $i \in I$, there is $z_i \in \mathfrak{u}^{\mathfrak{L}^i}$ such that $u'_i = z + u_i$. But by Step 1, $(z_i, a) = 0$ for all $a \in f_i(\mathfrak{L}^i)\mathfrak{u}$. So for all $a \in f_i(\mathfrak{L}^i)\mathfrak{u}$, we have

$$(u'_i, a) = (u_i + z_i, a) = (u_i, a) + (z_i, a) = (u_i, a).$$

This shows that $\theta_\alpha = \theta_\beta$.

Step 5. For $\alpha \in \varprojlim_{i\in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$, θ_α is linear: Suppose that $a, b \in \mathfrak{u} = \cup_{i\in I} f_i(\mathfrak{L}^i)\mathfrak{u}$ and $r \in \mathbb{F}$, then there are $i, j, k \in I$ with $a \in f_i(\mathfrak{L}^i)\mathfrak{u}$, $b \in f_j(\mathfrak{L}^j)\mathfrak{u}$ and $ra + b \in f_k(\mathfrak{L}^k)\mathfrak{u}$. Take $t \in I$ with $i \preccurlyeq t$, $j \preccurlyeq t$ and $k \preccurlyeq t$, then by Step 1, we have

$$\begin{aligned}\theta_\alpha(ra + b) = (u_k, ra + b) = (u_t, ra + b) = r(u_t, a) + (u_t, b) &= r(u_i, a) + (u_j, b)\\ &= r\theta_\alpha(a) + \theta(b).\end{aligned}$$

Step 6. Suppose that $\alpha = ([u_i]_i)_{i\in I}, \beta = ([u'_i]_i)_{i\in I} \in \varprojlim_{i\in I}\mathfrak{u}/\mathfrak{u}^{\mathfrak{L}^i}$ such that $\theta_\alpha = \theta_\beta$, then $\alpha = \beta$: For $i \in I$, $a \in \mathfrak{L}^i$ and $u \in \mathfrak{u}$, using Step 1, we have

$$\begin{aligned}(u_i f_i(a), u) = (u_i, f_i(a)u) = \theta_\alpha(f_i(a)u) = \theta_\beta(f_i(a)u) &= (u'_i, f_i(a)u)\\ &= (u'_i f_i(a), u).\end{aligned}$$

But the form is nondegenerate, so $u_i - u'_i \in \mathfrak{u}^{\mathfrak{L}^i}$. This completes the proof. □

2 An application

For index supersets I, J, by an $I \times J$-matrix with entries in $\mathbb{F}$, we mean a map $A : I \times J \longrightarrow \mathbb{F}$. For $i \in I, j \in J$, we set $a_{ij} := A(i, j)$ and call it the (i, j)-*th entry* of A. By a convention, we denote the matrix A by (a_{ij}). We also denote the set of all $I \times J$-matrices with entries in $\mathbb{F}$ by $\mathbb{F}^{I\times J}$ and by $\mathbb{F}^{I\times J}_{rc-fin}$, the set of all matrices (a_{ij}) such that for all $i \in I$ and $j \in J$,

$$\{t \in J \mid a_{i,t} \neq 0\} \quad \text{and} \quad \{r \in I \mid a_{r,j} \neq 0\}$$

are finite sets. For $A = (a_{ij}) \in \mathbb{F}^{I\times J}$, the matrix $B = (b_{ij}) \in \mathbb{F}^{J\times I}$ with

$$b_{ij} := \begin{cases} a_{ji} & |i| = |j| \\ a_{ji} & |i| = 1, |j| = 0 \\ -a_{ji} & |i| = 0, |j| = 1 \end{cases}$$

is called the *supertransposition* of A and denoted by A^{st}. If $A = (a_{ij}) \in \mathbb{F}^{I\times J}$ and $B = (b_{ij}) \in \mathbb{F}^{J\times K}$ are such that for all $i \in I$ and $k \in K$, at most for finitely many $j \in J$, $a_{ij}b_{jk}$'s are nonzero, we define the product AB of A and B to be the $I \times K$-matrix $C = (c_{ik})$ with $c_{ik} := \sum_{j\in J} a_{ij}b_{jk}$ for all $i \in I, k \in K$. We note that if A, B, C are three matrices such that AB, $(AB)C$, BC and $A(BC)$ are defined, then $A(BC) = (AB)C$. We make a convention that if I is a disjoint union of subsets $I_1, \ldots, I_t$ of I, then for an $I \times I$-matrix A, we write

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,t} \\ A_{2,1} & \cdots & A_{2,t} \\ \vdots & \vdots & \vdots \\ A_{t,1} & \cdots & A_{t,t} \end{bmatrix}$$

in which for $1 \leq r, s \leq t$, $A_{r,s}$ is an $I_r \times I_s$-matrix whose (i, j)-th entry coincides with (i, j)-th entry of A for all $i \in I_r, j \in I_s$. We note that the defined matrix product obeys the product of block matrices. For $i \in I, j \in J$, we define $e_{i,j}$ to be a matrix in $\mathbb{F}^{I\times J}$ whose (i, j)-th entry is 1 and other entries are zero. If $\{a_i \mid i \in I\} \subseteq \mathbb{F}$, by $\mathrm{diag}(a_i)$, we mean an $I \times I$-matrix whose (i, i)-th entry is a_i for all $i \in I$ and other entries are zero. We also set $1_I := \mathrm{diag}(1)$. Take $M_{I\times J}(\mathbb{F})$ to be the subspace of $\mathbb{F}^{I\times J}$ spanned by $\{e_{ij} \mid i \in I, j \in J\}$. Then $M_{I\times J}(\mathbb{F})$ is a superspace with $M_{I\times J}(\mathbb{F})_{\bar{i}} := \mathrm{span}_{\mathbb{F}}\{e_{rs} \mid |r| + |s| = \bar{i}\}$, for $i = 0, 1$. Also with respect to the multiplication of matrices, the vector superspace $M_{I\times I}(\mathbb{F})$ is an associative $\mathbb{F}$-superalgebra and so it is a Lie superalgebra under the Lie bracket $[A, B] := AB - (-1)^{|A||B|}BA$ for all $A, B \in M_{I\times I}(\mathbb{F})$. We denote this Lie superalgebra by $\mathfrak{pl}_{\mathbb{F}}(I)$ or $\mathfrak{pl}_{\mathbb{F}}(I_0, I_1)$. For an element $X \in \mathfrak{pl}_{\mathbb{F}}(I)$, we set $str(X) := \sum_{i\in I}(-1)^{|i|}x_{i,i}$ and call it the *supertrace* of X. In the case that I_0, I_1 or both are finite, we denote $\mathfrak{pl}_{\mathbb{F}}(I_0, I_1)$ by $\mathfrak{pl}_{\mathbb{F}}(|I_0|, I_1)$, $\mathfrak{pl}_{\mathbb{F}}(I_0, |I_1|)$ or $\mathfrak{pl}_{\mathbb{F}}(|I_0|, |I_1|)$ respectively.

$\mathfrak{sl}(J_0, J_1)$: Suppose that J is a superset with $J_0 \neq \emptyset$. Set

$$\mathcal{G} := \mathfrak{sl}(J_0, J_1) = \{X \in \mathfrak{pl}_{\mathbb{F}}(J_0, J_1) \mid str(X) = 0\}.$$

If $|J| < \infty$ and $|J_0| = |J_1| \neq 0$, take $K := \mathbb{F}\sum_{j\in J} e_{jj}$. Set

$$\mathfrak{sl}_s(J_{\bar{0}}, J_{\bar{1}}) := \begin{cases} \mathcal{G}/K & \text{if } |J| < \infty \text{ and } |J_0| = |J_1| \neq 0 \\ \mathcal{G} & J_0 \neq \emptyset \text{ and } J_1 \neq \emptyset. \end{cases}$$

$\mathfrak{sl}_s(J_{\bar{0}}, J_{\bar{1}})$ is a subsuperalgebra of $\mathfrak{pl}_{\mathbb{F}}(J_0, J_1)$ which is a direct union of finite dimensional basic classical simple Lie superalgebras.

$\underline{\mathfrak{osp}(2I,2J),\ \mathfrak{osp}(2I+1,2J):}$ For two disjoint index sets I,J with $J\neq\emptyset$, suppose that $\{0,i,\bar{i},\mid i\in I\cup J\}$ is a superset with $|0|=|i|=|\bar{i}|=0$ for $i\in I$ and $|j|=|\bar{j}|=1$ for $j\in J$. We set $\dot{I}:=I\cup\bar{I}$, $\dot{I}_0:=\{0\}\cup I\cup\bar{I}$ and $\dot{J}:=J\cup\bar{J}$ in which

$$\bar{I}:=\{\bar{i}\mid i\in I\}\quad\text{and}\quad\bar{J}:=\{\bar{j}\mid j\in J\}.$$

For $\mathcal{I}:=\dot{I}\cup\dot{J}$ or $\mathcal{I}:=\dot{I}_0\cup\dot{J}$, we set

$$Q_{\mathcal{I}}:=\begin{pmatrix} M_1 & 0\\ 0 & M_2\end{pmatrix}$$

in which

$$M_2:=\sum_{j\in J}(e_{j,\bar{j}}-e_{\bar{j},j})\quad\text{and}\quad M_1:=\begin{cases}-2e_{0,0}+\sum_{i\in I}(e_{i,\bar{i}}+e_{\bar{i},i}) & \text{if } \mathcal{I}=\dot{I}_0\cup\dot{J}\\ \sum_{i\in I}(e_{i,\bar{i}}+e_{\bar{i},i}) & \text{if } I\neq\emptyset, \mathcal{I}=\dot{I}\cup\dot{J}.\end{cases}$$

Now

$$\mathcal{G}_{\mathcal{I}}:=\{X\in\mathfrak{pl}_{\mathbb{F}}(\mathcal{I})\mid X^{st}Q_{\mathcal{I}}=-Q_{\mathcal{I}}X\}$$

is a subsuperalgebra of $\mathfrak{pl}_{\mathbb{F}}(\mathcal{I})$ which we refer to as $\mathfrak{osp}(2I,2J)$ or $\mathfrak{osp}(2I+1,2J)$ if $\mathcal{I}=\dot{I}\cup\dot{J}$ or $\mathcal{I}=\dot{I}_0\cup\dot{J}$ respectively.

Theorem 2.1. (i) suppose that J is an infinite superset with $J_0\neq\emptyset$, then

$$\operatorname{der}(\mathfrak{sl}_{\mathbb{C}}(J))\simeq\mathbb{C}^{I\times I}_{rc-fin}/\mathbb{C}1_I.$$

(ii) Suppose that I,J are two index sets with $J\neq\emptyset$ and $|I\cup J|=\infty$. Consider $\mathcal{I}$ and $Q_{\mathcal{I}}$ as above. Then

$$\operatorname{der}(\mathcal{G}_{\mathcal{I}})\simeq\{X\in\mathbb{C}^{\mathcal{I}\times\mathcal{I}}_{rc-fin}\mid X^{st}Q_{\mathcal{I}}=-Q_{\mathcal{I}}X\}.$$

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Local bifurcation control of nonlinear singularities*

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Abstract

In this talk we discuss the *local bifurcation control* of *singular smooth germs* and *singular germs of vector fields*. We describe how we may help to design an efficient nonlinear *bifurcation controller* for a *nonlinear singular system*.

Keywords: Normal form theory; Singularities; Bifurcation theory; controller design.
Mathematics Subject Classification [2010]: 34C20; 13P10; 14H20.

Designing an efficient nonlinear controller for linearly uncontrollable singular systems is an important challenging problem and has wide applications in different engineering disciplines. This is closely related to *universal unfolding* and *codimension* of singularities. Since many singular differential systems in engineering problems are not *finitely determined*, we have recently defined the notion of *asymptotic universal unfolding* and have used it to suggest designs of efficient controllers. In this talk we discuss how (asymptotic) universal unfolding of such systems can help to suggest efficient nonlinear controllers. The main tools for our study falls within the scope of *normal form theory* of singular systems; see [7–11, 21]. Our claims are theoretically proven and all results are computable. Thus, they can be used in practice.

Let

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \tag{1}$$

be a smooth germ with a rest point at the origin, *i.e.,* $f(0, 0) = 0$. We here address two categories of problems. One is related to steady-state solutions of systems governed by zeros of f, *i.e.,*

$$f(x, \alpha) = 0, \text{ for } x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m, \tag{2}$$

while the other is related to the differential system

$$\dot{x} := f(x, \alpha), \text{ for } x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m. \tag{3}$$

The main aim is to suggest a control system designed by the nonlinear (polynomial) map

$$P : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n, \text{ where } P(x, 0) = 0 \text{ for any } x \in \mathbb{R}^n, 0 \in \mathbb{R}^k, \tag{4}$$

so that appropriate choices for $u \in \mathbb{R}^k$ in $F(x, \alpha, u) := f(x, \alpha) + P(x, u)$ would lead to a desired dynamics for either the system $F(x, \alpha, u) = 0$ or $\dot{x} = F(x, \alpha, u)$, respectively;

*Will be presented in English

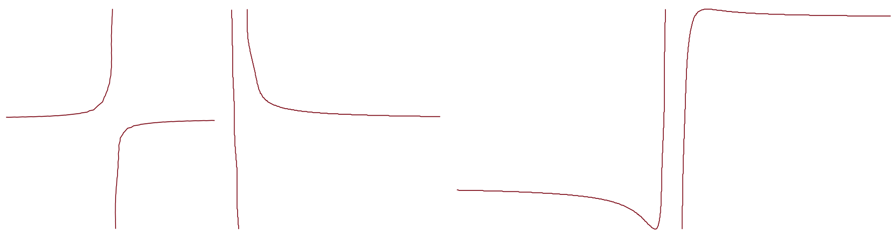
see [10]. Our proposal for bifurcation control of nonlinear singular systems (2) and (3) is closely related to, but different from, bifurcation control of *control systems* described in [1, 2, 14–18].

The rest of this conference paper is organized as follows. A brief introduction to a few concepts such as universal unfolding and bifurcation analysis of singular germs is given in Section 1. Section 2 treats problem types of Equations (2) and (3) for $n := 1$ and the cases of $n > 1$ are addressed in Section 3.

1 Introduction

There exist many engineering problems modeled by either (2) or (3). Further, equilibrium solutions of ODEs, steady-state solutions of PDEs, and periodic solutions of dynamical systems may be reduced to either of these equations by reduction techniques like *Liapunov-Schmidt,* traveling wave solution or similarity methods; see [13, Chapter VII] and [20]. Further, singular systems frequently occur in engineering problems and thus, a method to suggest designs of efficient nonlinear controllers is an important contribution. This paper addresses an approach for this goal.

Our suggested approach is closely related to the concept of universal unfoldings of singular germs and the asymptotic universal unfolding of germs of vector fields. However, our proposal is well beyond these and can be applied in other contexts with different applications. Therefore, we first digress to introduce a few related concepts and then, we describe how these are used to suggest bifurcation controllers' designs.



(a) (b) (c) (d)

Figure 1: Persistent bifurcation diagrams.

Generally speaking, the qualitative properties of solutions of a parametric system may change when its parameters are smoothly varied. When the qualitative properties of solutions change at certain points (we call the points by *bifurcation points*), we say that a *bifurcation* is occurred. Next, the system at the bifurcation point is called *singular* and we refer to the system a *singularity.* The qualitative properties are usually defined via an equivalence relation, that is, a property is called *qualitative property* when either all or non of elements of an equivalence class has the property. Many equivalence relations have been used in the literature due to their applications. We can mention a few of these equivalences that have been used to study of bifurcation analysis of singular germs like contact-, right-, right-left-, strategy-, topological-, orbital, formal normal form-, formal orbital-, formal parametric- and their associated N-asymptotic-equivalences. From now on, we assume that an appropriate equivalence relation has been chosen and fixed and thus, the associated qualitative properties are defined.

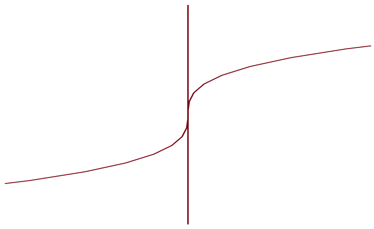


Figure 2: Transition set.

A bifurcation on a system modeling a real life problem demonstrates a surprising change on its solutions. For example, consider $f(x,\lambda,u_1,u_2) := x^3 + \lambda x + u_1 + u_2\lambda$ representing the dynamics of a real life problem subjected to a quasi-static changes of parameters. Therefore, the bifurcation diagrams merely shows the equilibria but not the transit solutions that the system experiences through stabilization. An end-user friendly Maple library, named "*Singularity*", is developed for local bifurcation analysis of nonlinear singular scalar germs. Using `Singularity`, a complete list of persistent bifurcation diagrams associated with f are given in Figure 1.

Each of these bifurcation diagrams are associated with arbitrary choices of parameters u_i from connected components in the complement of the transition set (regions) depicted in Figure 2.

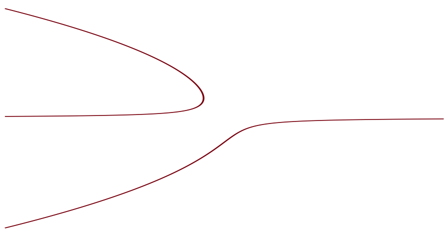


Figure 3: A bifurcation diagrams.

We explain how these bifurcations influence the dynamics of a real life system through a hypothetical scenario based on the bifurcation diagram given in Figure 3. When λ decreases from positive values, we have an stable equilibrium and at the bifurcation point (when another stable equilibrium is born), it looses its stability. At the bifurcation point two new branches of solutions are born, one is usually stable and the other is unstable. This means that the solution jumps up and follows the stable solution branch. This demonstrates how new and surprising changes in the dynamics of real life problems occur.

A parametric germ

$$G : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n, \text{ where } P(x,0) = 0 \text{ for any } x \in \mathbb{R}^n, 0 \in \mathbb{R}^k, \tag{5}$$

is called a *versal unfolding* for $f(x,0)$, $0 \in \mathbb{R}^m, x \in \mathbb{R}^n$, when for any small perturbation of $f(x,0)$, say $f(x,\alpha) + p(x,\beta), \alpha \in \mathbb{R}^m, \beta \in \mathbb{R}^l$, there exists smooth germs $\gamma : \mathbb{R}^{l+m} \to \mathbb{R}^k$, so that $G(x, \gamma(\alpha, \beta))$ is equivalent to $f(x, \alpha) + p(x, \beta)$. The smallest natural number k results in *universal unfolding* of $f(x, 0)$ and is called *codimension.* Sometimes an additional condition on *universal unfolding* is assumed so that it would be the *simplest* among its equivalent class, that is, the universal unfolding is a *parametric normal form* according to [6–9, 11, 12]; also see [10].

2 Scalar smooth germs

2.1 Systems given by Equation (2)

We have recently developed an end-user Maple library, named `Singularity`, for local bifurcation analysis of zeros of scalar smooth germs. `Singularity` will be made available to everyone, once the reference [5] is accepted for publication in a refereed journal. Assume that we have a system of the form (2) when $m := 1$ and we intend to design an efficient nonlinear controller for it.

Using the command `UniversalUnfolding(f(x, \alpha)+P(x, u))`, where $P(x, u)$ is a multi-variate polynomial germ, we may determine whether a parametric germ is a (uni)versal unfolding for $f(x, 0)$ or not. Therefore by increasing the density of the polynomial germ $P(x, u)$ and its degree, we may find a versal unfolding for $f(x, 0)$ provided that the equivalence relation (that we have chosen) results in a finite codimension problem.

We may work with $F := f(x, \alpha) + P(x, u)$ and choose P so that no extra unnecessary parameters are added into F. Further, we can determine the parametric terms on which they can play the role of universal unfolding terms. Then, we may replace P (and update F) with a polynomial germ of least density and degree. Using a reparametrization (u_i may depend on certain important parameters of α_j) we may describe $F(x, u, \alpha) := f(x, 0) + P(x, u) + g(x, \alpha)$, where $g(x, 0) = 0$. Next, $P(x, u)$ suggest an efficient nonlinear controller for the system.

Note that the choice of P is not usually unique and mostly, there are alternative choices for P. This is important for their application in real life problems and the possible user should be able to try any possible alternatives. The command `UniversalUnfolding` is designed with various built-in options to give us other possible choices suitable for applications. Then, using the command `TransitionSet(F(x, u, 0))` we obtain a partition for the parameters u_i-s so that each connected component of the partition represents a qualitative type of dynamics. `PersistentDiagram` generates a list of persistent bifurcation diagrams and provides a good insight about the system's dynamics. Furthermore, the parameters u depend to certain important parameters α_j of the original system (2) and their relations are computed. Therefore, by choosing the control parameters from a connected component of the partition (off course, it should be far from the partition's boundary), we expect to arrive at a desired and predicted dynamics for Equation (2). This is due to the fact that the parameters contributing into P are the ones playing the roles of universal unfolding terms and they are expected to dominate its dynamics.

2.2 Systems given by Equation (3)

This subsection is related to parametric single zero singularity. These systems are well-studied in [11, Section 5]. We proved that the universal unfolding of such systems is governed by

$$G(x,u) := \pm x^{k+1} + \sum_{i=1}^{k} u_i x^i \tag{6}$$

for some k. Thereby, for any parametric system (3), there exists germs of $u_i(\alpha)$ so that the parametric system (3) can be transformed into (6). The bifurcation analysis of Equation (6) may be performed by our Maple library, `Singularity`, via the commands `TransitionSet` and `PersistentDiagram`.

There are two suggested practical approaches here described as follows. First, a parametric system is transformed into its parametric normal form. Then, polynomial controllers of degree less than or equal to k are added to the parametric normal form when necessary. Then, the obtained universal unfolding may be transformed back to the original system so that we can accommodate the possible contribution of the added unfolding terms. This suggests a design for a potential nonlinear controller. The second approach is that we add arbitrary polynomial controllers to the original system and use their parametric normal forms to check if they can form a versal unfolding. By reducing the polynomial density of the controllers, we can obtain the universal unfolding. The relations between the original parameters and the universal unfolding parameters (in either of the two approaches) provide the transformations transforming the bifurcation diagrams of (6) into that of the original system.

3 Multi-dimensional state variables

Our proposal for systems described by (2) for multi-dimensional state variables is similar to one dimensional case, but it is an in-progress project. Therefore, we merely describe differential systems of type (3).

3.1 Degenerate nonlinear center

Section 3 in [12] considers the parametric normal form of an arbitrary parametric degenerate nonlinear center. By [12, Theorem 3.9], any such system can be transformed into the reduced system (ignoring phase equation)

$$F(\rho,u) := \pm \rho^{2k+1} + \sum_{i=1}^{k} u_i \rho^{2i-1} \tag{7}$$

for some k, and functions of u_i in terms of the parameters of the degenerate nonlinear center system. The zeros of Equation (7) constitute the equilibria and limit cycles of the nonlinear centers. These can be analyzed via `Singularity` while the assocaited transformations between the parametric nonlinear center and their parametric normal form can be computed via the Maple program developed by the method of formal decomposition method described in [12]. The bifurcation controller designs follows the suggested approach in Subsection 2.2.

3.2 Bogdanov-Takens singularity

One of the most challenging singularities in the long history of normal form theory is Bogdanov-Takens singularity. Many contributions have been made in the literature and there are still some problems that they remain to be addressed. We considered a general case of this singularity in [4, Chapter 2] and by [4, Corollary 2.3.9.], any such parametric systems can be transformed into

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^2 + xu_1 + yu_2 + axy + \sum_{i=0}^{\infty} b_i x^{3i+3} y u_{i+2}, \end{aligned} \tag{8}$$

where certain conditions relating the first few coefficients holds. Furthermore, another general case of this singularity has been considered in [6] in a different normal form style. We skip the presentation of the other parametric normal form and refer the reader to [6, Theorem 5.3]. The normal form style used in Equation (8) is useful for locating the equilibria of the normal form system while other normal form styles have other benefits and we may use them for bifurcation analysis of this singularity. For instance, the results in [6] uses $\mathfrak{sl}_2$-style normal form. This style is useful for detecting certain symmetries of systems, introduction of new and important families of systems, computation or estimating of first integrals for systems with a first integral, integrable or Hamiltonian systems; also see [7–9]. They can also be used for homoclinic bifurcation analysis and use of Melnikov functions.

Given different parametric normal forms and by using a N-equivalence relation defined by [22], we may call an N-degree truncated parametric normal form of (8) an *asymptotic universal unfolding* for Bogdanov-Takens singularity. Note that there are still some general cases of this singularity that their parametric normal forms and asymptotic universal unfolding normal forms remain to be derived. Certain dynamics of those systems can be detected by an N-degree truncated parametric normal form of (8) and our proposed approach is helpful to control those dynamics while certain dynamics may not be controllable by polynomial controllers. Parametric normal forms along with a thorough discussion about finite determinacy of its normal forms using different equivalence relations is required for certain bifurcation control of this singularity. This is an in-progress project and will be addressed in future.

3.3 Hopf-zero singularity

We have recently derived the infinite level normal form of a general family of this singularity. The normal forms use a dynamically meaningful decomposition of Hopf-zero vector fields and use a $\mathfrak{sl}_2$-type of normal form style. The orbital and parametric normal form of this family is divided into three cases. The first family are the ones with leading solenoidal terms and their orbital and parametric normal forms are obtained; see [10]. The orbital normal forms and parametric normal forms for the other two cases are also obtained and their dynamics and bifurcation control are in progress.

Hopf-zero singularity normal forms with leading solenoidal terms are a large family of vector fields with applications in different disciplines. In order to avoid technical details, we consider the most generic cases of this family, say v. The results have been obtained for more general cases. By Theorem 4.1 in [10], v can be transformed into a parametric normal form that its planner reduced 2-jet truncated part is given by

$$\begin{aligned} \dot{\rho} &= \frac{1}{2}u_2\rho - a_1 x\rho, \\ \dot{x} &= u_1 + 2\rho^2 + u_2 x + a_1 x^2. \end{aligned} \tag{9}$$

We proved that the systems governed by Equation (9) are 2-contact equivalent determined. Following the procedure in [10, Page 20], we successfully showed that we may suggest efficient controllers for controlling the limit cycles and equilibria bifurcating from a Hopf-zero equilibrium. Numerical and symbolic implementations verifies our claims. The basic idea is similar to Subsection 2.2. We first derive the 2-asymptotic universal unfolding normal form for such systems and assume an arbitrary polynomial controller for the system. The controller can be changed and chosen based on its potential applications. Next, the procedure in [10, Page 20] detects the parametric terms depending on α_i that they can play the role of asymptotic universal unfolding and recognizes the need for adding extra parametric unfolding terms (u_i) into the system. By deriving the transition sets of the proposed asymptotic universal unfolding normal form system, we may find our possible desired dynamics. Since the relations between the unfolding terms with original parameters of the system and the controllers' parameters are available, we may simply project our desired conditions to the original system's and controller's parameters. This controls our designed control system to behave as desired.

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Algebra

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Talk 2-absorbing ideal in lattice pp.: 1–4

2-absorbing ideal in lattice

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Abstract

In this paper, we define 2-absorbing, weakly 2-absorbing and n-absorbing ideals in a lattice. We also show that 2-absorbing and weakly 2-absorbing ideals are equvalent in a lattice. It is shown that a non-zero proper ideal I of L is a 2-absorbing ideal if and only if whenever $I_1 \wedge I_2 \wedge I_3 \subseteq I$ then $I_1 \wedge I_2 \subseteq I$ or $I_1 \wedge I_3 \subseteq I$ or $I_2 \wedge I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L.

Keywords: lattice , 2-absorbing ideal , n-absorbing ideal.
Mathematics Subject Classification [2010]: 03G10; 16D25.

1 Introduction

The concept of 2-absorbing ideals, in a commutative ring, was introduced by A. Badawi, in [1], as a generalization of prime ideals, and some properties of 2-absorbing ideals were studied. The definitions and related threads are taken from [1, 2, 3]. In this paper we introduced the 2-absorbing ideal of a lattice L. A proper ideal I of L is said to be 2-absorbing if $a \wedge b \wedge c \in I$ for $a, b, c \in L$ implies that $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

In this paper we introduce radical of ideal I in a lattice L and we show that $RadI = I$. In Section 2, a 2-absorbing ideal of a lattice L and also a weakly 2-absorbing ideal are defined. Particular, we show that if I be a 2-absorbing ideal, then $|MinI| \leq 2$, where $Min(I)$ denotes the set of minimal prime ideals of I in L.

Then , we introduce the concept n-absorbing ideal in a lattice L. It shown that an n-absorbing ideal is also an m-absorbing ideal for all integers $m \geq n$.

Definition 1.1. Let I be an ideal of a lattice L. The radical of I, denoted $RadI$, is the ideal $\bigcap P$, where the intersection is taken over all prime ideals P which contain I. If the set of prime ideals containing I is empty, then $RadI$ is defined to be L.

Proposition 1.2. *If I is an ideal of a lattice L, then $RadI = I$.*

Definition 1.3. Let I be an ideal of L. A prime ideal P in L is called a minimal prime ideal of I if $I \subseteq P$ and there is no prime ideal P' such that $I \subseteq P' \subset P$.

Proposition 1.4. *If an ideal I of a lattice L is contained in a prime ideal P of a lattice L, then P contains a minimal prime ideal of I.*

Proposition 1.5. [4] *Let I be an ideal of L. Let P be a prime ideal containing I. Then P is a minimal prime ideal belonging to I if and only if for each $x \in P$ there is a $y \notin P$ such that $x \wedge y \in I$.*

*Speaker

2 2-absorbing ideals

Definition 2.1. A proper ideal I of lattice L is said to be a 2-absorbing ideal if for any $a, b, c \in L$, $a \wedge b \wedge c \in L$ implies either $a \wedge b \in L$ or $b \wedge c \in L$ or $a \wedge c \in L$.

Example 2.2. Consider the lattice $L = \{\bot, a, b, c, d, e, f, \top\}$ whose Hasse diagram is given in the figure (1):

Consider the ideal $I = \{\bot, a, b, c, f\}$. It is clear that I is 2-absorbing ideal of L, but I is not prime ideal of L.

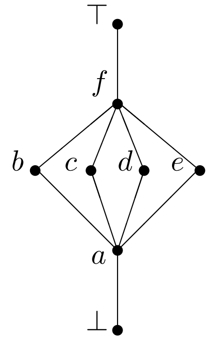

Figure (1)

Proposition 2.3. *Let I_1, I_2 be two prime ideals of lattice L, then $I_1 \cap I_2$ is a 2-absorbing ideal of lattice L.*

Proposition 2.4. *Let L and K be two lattices and $\varphi : L \to K$ be a lattice homomorphism. If J is a 2-absorbing ideal of K, then $\varphi^{-1}(J)$ is a 2-absorbing ideal of L. Furthermore, if φ is an onto lattice homomorphism and J is a 2-absorbing ideal of L such that $ker\varphi \subseteq J^2$, then $\varphi(J)$ is a 2-absorbing ideal of K.*

Proposition 2.5. *Let L and L' be two lattices. If I_1 is a 2-absorbing ideal of L, then $I_1 \times L'$ is a 2-absorbing ideal of $L \times L'$. Also if I_2 is a 2-absorbing ideal of L', then $L \times I_2$ is a 2-absorbing ideal of $L \times L'$.*

Proposition 2.6. *If I is a 2-absorbing ideal of lattice L, then $|Min(I)| \leq 2$.*

Corollary 2.7. *Suppose that I is a proper ideal of a lattice L. The following statements are equivalent*

1. *I is 2-absorbing ideal of lattice L.*
2. *If $I_1 \wedge I_2 \wedge I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L, then $I_1 \wedge I_2 \subseteq I$ or $I_1 \wedge I_3 \subseteq I$ or $I_2 \wedge I_3 \subseteq I$.*

Definition 2.8. A proper ideal I in lattice L is said to be a weakly 2-absorbing ideal if for any $a, b, c \in L$, $\bot \neq a \wedge b \wedge c \in I$ implies either $a \wedge b \in I$ or $b \wedge c \in I$ or $a \wedge c \in I$.

Let I be a weakly 2-absorbing ideal of a lattice L and $a, b, c \in L$. We say (a, b, c) is a triple-zero of I if $a \wedge b \wedge c = \bot$, $a \wedge b \notin I$, $b \wedge c \notin I$, and $a \wedge c \notin I$.

Proposition 2.9. *Let I be a weakly 2-absorbing ideal of a lattice L and suppose that that (a, b, c) is a triple-zero of I for some $a, b, c \in L$. Then*

1. $a \wedge b \wedge I = b \wedge c \wedge I = a \wedge c \wedge I = \{\perp\}$.

2. $a \wedge I = b \wedge I = c \wedge I = \{\perp\}$.

Proposition 2.10. *For every proper ideal $I \neq \{\perp\}$ in lattice L, I is a 2-absorbing ideal of lattice L if and only if I is a weakly 2-absorbing ideal of lattice L.*

Now, we give some basic properties of n-absorbing ideals.

Definition 2.11. Let n be a positive integer. Proper ideal I of a lattice L is an n-absorbing ideal of L if whenever $a_1 \wedge a_2 \wedge ... \wedge a_{n+1} \in I$ for $a_1, a_2, ..., a_{n+1} \in L$, then there are n of the a_i's whose meet is in I.

Proposition 2.12. *Let L be a lattice, and let m and n be positive integers.*

1. *A proper ideal I of L is n-absorbing if and only if whenever $a_1 \wedge a_2 \wedge ... \wedge a_m \in I$ for $a_1, ..., a_m \in I$ with $m > n$, then there are n of the a_i's whose meet is in I.*

2. *If I is an n-absorbing ideal, then I is an m-absorbing ideal, for all $m \geq n$.*

3. *If I_j is an n_j-absorbing ideal of L for each $1 \leq j \leq m$, then $I_1 \wedge I_2 \wedge ... \wedge I_m$ is an n-absorbing ideal of L for $n = n_1 + n_2 + ... + n_m$.*

Let I be a proper ideal of a lattice L. In Proposition 2.12, we observed that an n-absorbing ideal is also m-absorbing ideal for all integers $m \geqslant n$. If I is an n-absorbing ideal of L for some positive integer n, then define $\omega_L(I) = \min\{n|$ I is an n-absorbing ideal of $L\}$; otherwise, set $\omega_L(I) = \infty$.

Thus for any ideal I of L, we have $\omega_L(I) \in N \cup \{0, \infty\}$ with $\omega_L(I) = 1$ if and only if I is a prime ideal of L and $\omega_L(I) = 0$ if and only if $I = L$.

Proposition 2.13. *Let I be an n-absorbing ideal of L. Then there are at most n prime ideals of L minimal over I. Morover $|Min(I)| \leq \omega_L(I)$.*

Proposition 2.14. *Let $f : L \to R$ be a homomorphism of lattice.*

1. *Let J be an n-absorbing ideal of R. Then $f^{-1}(J)$ is an n-absorbing ideal of L. Moreover, $W_L(f^{-1}(J)) < W_R(J)$.*

2. *Let f be surjective and I be an n-absorbing ideal of L such that $ker f \subseteq I^2$. Then $f(I)$ is an n-absorbing ideal of R if and only if I is an n-absorbing ideal of L. Morover $W_R(f(I)) = W_L(I)$.*

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2-absorbing Submodules and Flat modules *

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Abstract

2-absorbing submodule is generalization of the notion of 2-absorbing ideal. We will study 2-absorbing submodules and we prove that 2-absorbing submodules are not too far from prime submodules, which are well-known and studied concepts. Also we find some properties of 2-absorbing submodules in flat modules.

Keywords: 2-absorbing submodule, Flat modules, Faithfully flat modules
Mathematics Subject Classification [2010]: 13E05, 13C99, 13C13, 13F05, 13F15.

1 Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider $n > 1$ a positive integer. Let N be a submodule of an $R-$module M. The set $\{r \in R | rM \subseteq N\}$ is denoted by $(N : M)$. Also we consider $T(M) = \{m \in M | \exists 0 \neq r \in R,\ rm = 0\}$. A module M is called torsion-free, if $T(M) = 0$.

According to [1] an ideal I of a ring R is called *2-absorbing*, if $abc \in I$ for $a, b, c \in I$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

A module version of 2-absorbing ideals is introduced as follows:

Definition 1.1. A proper submodule N of M will be called 2-absorbing if for $r, s \in R$ and $x \in M$, $rsx \in N$ implies that $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

In order to obtain our main results, we use some definitions and lemma such as the following:

Let F be an R-module. Writing φ to stand for a sequence $... \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow ...$ of R-modules and linear maps, we let $F \otimes \varphi$ stand for induced sequence $... \longrightarrow F \otimes N' \longrightarrow F \otimes N \longrightarrow F \otimes N'' \longrightarrow ...$

The R-module F is called flat, if for every sequence φ,

$$\varphi \text{ is exact} \Longrightarrow F \otimes \varphi \text{ is exact.}$$

According to [2, p. 45], F is called faithfully flat, if for every sequence φ,

$$\varphi \text{ is exact} \Longleftrightarrow F \otimes \varphi \text{ is exact.}$$

*Will be presented in English

Lemma 1.2. *[3, Lemma 2.14] Let N, K be two submodules of M and $r \in R$. Then for every flat R-module F, we have:*

(i) $(F \otimes N : r) = F \otimes (N : r)$.

If F is faithfully flat, then we have the following:

(ii) *If $F \otimes N \subseteq F \otimes K$, then $N \subseteq K$.*

(iii) $(F \otimes N : F \otimes M) = (N : M)$.

2 Main results

Here we study 2-absorbing submodules and we introduce the main results of our article.

Lemma 2.1. *Let N be a proper submodule of M. If for $r, s \in R$ and $x \in M$, $rsx \in N$ implies that $rs \in \sqrt{(N : M)}$ or $rx \in N$ or $sx \in N$, then:*

(i) *If $rst \in (N : M)$ for some $r, s, t \in R$, then $rs \in \sqrt{(N : M)}$ or $rt \in (N : M)$ or $st \in (N : M)$.*

(ii) *$\sqrt{(N : M)}$ is a 2-absorbing ideal of R and one of the following holds:*

  (a) *$\sqrt{(N : M)} = P$, where P is a prime ideal of R.*

  (b) *$\sqrt{(N : M)} = P_1 \cap P_2$, where P_1, P_2 are the only distinct minimal prime ideals over $(N : M)$.*

(iii) *If $\sqrt{(N : M)} = P$ and $P^2 \subseteq (N : M)$, then $(N : M)$ is 2-absorbing.*

Proof. (i) Let $s, t, r \in R$ and $str \in (N : M)$. If $sr, tr \notin (N : M)$, then there exist $x, y \in M \setminus N$ such that $srx, try \notin N$.

Since $st(r(x+y)) \in N$, by assumption $st \in \sqrt{(N : M)}$ or $sr(x+y) \in N$ or $tr(x+y) \in N$. If $sr(x + y) \in N$, then since $srx \notin N$, we have $sry \notin N$. So as $st(ry) \in N$ and $try \notin N$, $st \in \sqrt{(N : M)}$.

Similarly in case $tr(x + y) \in N$, we get $st \in \sqrt{(N : M)}$.

(ii) Let $s, t, r \in R$ and $str \in \sqrt{(N : M)}$. Then for some positive integer n we have $(str)^n \in (N : M)$ and by part(i), $(st)^n \in \sqrt{(N : M)}$ or $(sr)^n \in (N : M)$ or $(tr)^n \in (N : M)$ and therefore either $st \in \sqrt{(N : M)}$ or $sr \in \sqrt{(N : M)}$ or $tr \in \sqrt{(N : M)}$. Then $\sqrt{(N : M)}$ is a 2-absorbing ideal, hence as $\sqrt{\sqrt{(N : M)}} = \sqrt{(N : M)}$, the rest of result follows from ([1, Theorem 2.1]).

(iii) suppose that $rst \in (N : M)$ for some $r, s, t \in R$. Then $rst \in P$ and so we can assume that $r \in P$. If $s \in P$ or $t \in P$, then $rs \in P^2 \subseteq (N : M)$ or $rt \in P^2 \subseteq (N : M)$ and we have the result. Therefore we suppose that $s, t \notin P$. Hence $st \notin P$ and so by part(i), $sr \in (N : M)$ or $tr \in (N : M)$. Consequently $(N : M)$ is 2-absorbing.

□

Theorem 2.2. *Let R be an integral domain of dimension one and M a nonzero torsion free and notherian R-module. Then the following are equivalent.*

(i) *For every maximal ideal m of R and $s \in m \setminus m^2$, $mM_m = sM_m$.*

(ii) *If N is a 2-absorbing submodule of M, then $(N : M) = 0$ or $(N : M) = m$ or $(N : M) = m_1m_2$ or $(N : M) = m^2$ where m, m_1, m_2 are some maximal ideals.*

Proof. $(i) \Rightarrow (ii)$ Let N be a 2-absorbing submodule of M such that $(N : M) \neq 0$. Since $dimR = 1$ and by [4, Proposition 1], we have two cases.

Case 1: For some maximal ideal m of R, $\sqrt{(N : M)} = m$. Then by [4, Proposition 1], $m^2 \subseteq (N : M)$. If $(N : M) \neq m^2$, then by assumption for some $s \in (N : M) \setminus m^2$, we have $m_mM_m = mM_m = sM_m \subseteq (N : M)M_m = (N : M)_mM_m$ and so $(N : M)_m = m_m$. Hence as $(N : M)$ is primary, $(N : M) = m$.

Case 2: There exist maximal ideals m_1, m_2 of R such that $\sqrt{(N : M)} = m_1 \cap m_2 = m_1m_2$. Then [4, Proposition 1] implies that $m_1^2m_2^2 \subseteq (N : M)$ and since $(N : M)$ is 2-absorbing, so either $m_1m_2 \subseteq (N : M)$ or $m_1^2m_2 \subseteq (N : M)$ or $m_1m_2^2 \subseteq (N : M)$. If $m_1m_2 \subseteq (N : M) \subseteq m_1m_2$, then $m_1m_2 = (N : M)$ and we have the result.

Now suppose that $m_1^2m_2 \subseteq (N : M)$. Since $(N : M)$ is 2-absorbing, either $m_1^2 \subseteq (N : M)$ or $m_1m_2 \subseteq (N : M)$. But $m_1^2 \nsubseteq (N : M)$, since otherwise $m_1 = m_1m_2$ and hence as m_1 is maximal, $m_1 = m_2$. Thus $m_1 = m_1m_2 = m_1^2$ and so $m_1M = m_1^2M$. Since M is a notherian R-module, m_1M is finitely generated and since M is nonzero torsion free, hence by Nakayama lemma $m_1 = 0$ or $m_1 = R$, which is a contradiction. Then $m_1m_2 = (N : M)$.

Consequently we have the result.

$(ii) \Rightarrow (i)$ Suppose that m is a maximal submodule of R and $s \in m \setminus m^2$. We have $m^2M + sM \neq M$. Since otherwise $mM = M$ and so by Nakayama lemma $m = 0$ or $m = R$, which is impossible. We claim that $(m^2M + sM : M) = m$.

If $(m^2M + sM : M) = m^2$, then $s \in (m^2M + sM : M) = m^2$, which is a contradiction. Hence as $m^2 \subseteq (m^2M + sM : M)$, $\sqrt{(m^2M + sM : M)} = m$ and so $m^2M + sM$ is primary and then by [4, Lemm 4], $m^2M + sM$ is 2-absorbing. Therefore the hypothesis in (ii) implies that $(m^2M + sM : M) = 0$ or $(m^2M + sM : M) = m_1$ or $(m^2M + sMM : M) = m_1 \cap m_2$ or $(m^2M + sM : M) = m_3^2$, where m_1, m_2, m_3 are some maximal ideals. Clearly $(m^2M + sM : M) \neq o$, since otherwise $m^2 = 0$ and so $m = 0$, which is impossible. Therefore as $\sqrt{(m^2M + sM : M)} = m$, $m_1 = m$ or $m_1 = m_2 = m$ or $m_3 = m$ and since $(m^2M + sM : M) \neq m^2$, $(m^2M + sM : M) = m$. Thus $mM = m^2M + sM$ and so $m_mM_m = m_m^2M_m + Rs_mM_m$. Then by Nakayama lemma $mM_m = m_mM_m = Rs_mM_m = sM_m$.

□

Lemma 2.3. *Let N be a proper submodule of M. Then the following are equivalent.*

(i) *N is 2-absorbing.*

(ii) $(N : ab) = (N : a) \cup (N : b)$, *for every $a, b \in R$ and $ab \in R \setminus (N : M)$.*

(iii) $(N : ab) = (N : a)$ *or* $(N : ab) = (N : b)$, *for every $a, b \in R$ and $ab \in R \setminus (N : M)$.*

Proof. $(i) \Rightarrow (ii)$ Let $a, b \in R$ and $ab \in R \setminus (N : M)$ and $x \in (N : ab)$. Then $abx \in N$ and since $ab \notin (N : M)$ and N is 2-absorbing, $ax \in N$ or $bx \in N$. Therefore $(N : ab) \subseteq (N : a) \cup (N : b)$ and clearly we have the result.

(ii) $\Rightarrow$ *(iii)* The proof is clear, since a submodule cannot be written as the union of two distinct submodules.

(iii) $\Rightarrow$ *(i)* Let $a, b \in R$ and $ab \in R \setminus (N : M)$ with $aby \in N$. Then $y \in (N : ab)$ and so $y \in (N : a)$ or $y \in (N : b)$. Consequently $ay \in N$ or $by \in N$. □

Theorem 2.4. *Let N be a submodule of M.*

(i) *If F is a flat R-module and N an 2-absorbing submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a 2-absorbing submodule of $F \otimes M$.*

(ii) *Let F be a faithfully flat R-module. Then N is a 2-absorbing submodule of M if and only if $F \otimes N$ is a 2-absorbing submodule of $F \otimes M$.*

Proof. (i) Let Let $a, b \in R$ and $ab \in R \setminus (F \otimes N : F \otimes M)$. Hence as $(N : M) \subseteq (F \otimes N : F \otimes M)$, $ab \notin (N : M)$. By Lemma 2.3, $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$.

If $(N : ab) = (N : a)$, then Lemma 1.2(i), implies that $(F \otimes N : ab) = (F \otimes N : a)$.

Similarly in case $(N : ab) = (N : b)$, we have $(F \otimes N : ab) = (F \otimes N : b)$.

Consequently by Lemma 2.3, $F \otimes N$ is 2-absorbing.

(ii) ($\Longrightarrow$) Let N is a 2-absorbing submodule of M. By Lemma 1.2(ii), $F \otimes N \neq F \otimes M$. Now by part (i), $F \otimes N$ is 2-absorbing.

($\Longleftarrow$) Suppose that $F \otimes N$ is 2-absorbing. Since $F \otimes N \neq F \otimes M$, clearly $N \neq M$. Let $a, b \in R$ and $ab \in R \setminus (N : M)$. By Lemma 1.2(iii), $(F \otimes N : F \otimes M) = (N : M)$, then $ab \in R \setminus (F \otimes N : F \otimes M)$.

According to Lemma 2.3, $(F \otimes N : ab) = (F \otimes N : a)$ or $(F \otimes N : ab) = (F \otimes N : b)$.

So by Lemma(1.2)(i), $F \otimes (N : ab) = (F \otimes N : ab) = (F \otimes N : a) = F \otimes (N : a)$ or $F \otimes (N : ab) = (F \otimes N : ab) = (F \otimes N : b) = F \otimes (N : b)$.

Hence by Lemma 1.2(ii), $(N : ab) = (N : a)$ or $(N : ab) = (N : a)$. Now Lemma 2.3 implies that N is 2-absorbing. □

Corollary 2.5. *Let F be a flat R-module and I an ideal of R.*

(i) *If I is a 2-absorbing ideal of R and $IF \neq F$, then IF is a 2-absorbing submodule of F.*

(ii) *If F is faithfully flat, then I is a 2-absorbing ideal of R if and only if IF is a 2-absorbing submodule of F.*

Proof. Having that $IF \cong F \otimes I$, we put $M = R$. Now the proof follows from Theorem 2.4. □

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Talk 2-capability and 2-exterior center of a group pp.: 1–3

2-capability and 2-exterior center of a group

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Abstract

The aim of this talk is to obtain a characteristic subgroup of G to give a criteria for detecting 2-capability of G. We show that a relation between this subgroup and 2-epicenter of any group.

Keywords: 2-capability, 2-exterior center, 2-nilpotent multiplier.
Mathematics Subject Classification [2010]: 17B30.

1 Introduction and Motivation

The concept of epicenter $Z^*(G)$ is defined by Beyl and others in [1]. It gives a criteria for detecting capable groups. In fact G is capable if and only if $Z^*(G) = 1$. Ellis defined the exterior center $Z^\wedge(G)$ of G the set of all elements g of G for which $g \wedge h = 1$ for all $h \in G$ and he showed $Z^*(G) = Z^\wedge(G)$.

Similar to the concept of capability of group, a group G is called 2-capable if here exists a group H such that $G \cong H/Z_2(H)$. The concepts of $2-$capability and $2-$epicenter, $Z_2^*(G)$, were introduced by Ellis in [2]. Later Moghaddam and Kayvanfar in [4] showed that the $2-$epicenter $Z_2^*(G)$ of G is minimal subject to being the image of G of some $\mathcal{N}_2$ extensions of G, that is,

$$Z_2^*(G) = \bigcap\nolimits_{(E,\phi)\text{is } \mathcal{N}_2 \text{ extension of } G} \phi(Z_2(E)).$$

Let G be a finite group presented as the quotient of a free group F by a normal subgroup R, following the notation in [2], we may define

$$\gamma_3^*(G) = \gamma_3(F)/\gamma_3(R, F) \text{ and } Z_2^*(G) = \pi(Z_2(F/\gamma_3(R, F)))$$

where $\pi : F/\gamma_3(R, F) \twoheadrightarrow G \cong F/R$ is an epimorphism given by $\gamma_3(R, F)x \mapsto Rx$.

Recall that the 2-nilpotent multiplier of G is the abelian group $\mathcal{M}^{(2)}(G) = \frac{R\cap\gamma_3(F)}{[R,F,F]}$, and the following sequence is exact

$$\mathcal{M}^2(G) \hookrightarrow \gamma_3^*(G) \twoheadrightarrow \gamma_3(G).$$

The main result of [2] shows G is $2-$capable if and only if $Z_2^*(G) = 1$.

It the current note, we define 2-exterior center $Z_2^\wedge(G)$ of G, and then we get that $Z_2^*(G) = Z_2^\wedge(G)$.

*Speaker

2 Main results

The following result will be used in our notes and we give here for the convenience of the reader.

Proposition 2.1. *(See [2])*
If N is a normal subgroup of G contained in $Z_2^*(G)$, then the canonical

$$\mathcal{M}^{(2)}(G) \hookrightarrow \mathcal{M}^{(2)}(G/N)$$

is injection.

For any group G with normal subgroup N, $\gamma_3^\sharp(N,G)$ defined as the quotient of $(N \wedge G) \wedge G$ by imposing the relations

$$((x \wedge y) \wedge^y z)((y \wedge z) \wedge^z x)((z \wedge x) \wedge^x y) = 1, x, y, z \in N.$$

Since these relations correspond to the well-known Hall-Witt commutator relation, the homomorphism $\delta : (N \wedge G) \wedge G \to G$ induces a homomorphism $\sigma : \gamma_3^\sharp(N,G) \to G$. Here we denote $\gamma_3^\sharp(G)$ instead of $\gamma_3^\sharp(G,G)$.

Lemma 2.2. *(See [2]) Let G be a group and $N \trianglelefteq G$. Then*

$$\gamma_3^\sharp(N,G) \to \gamma_3^\sharp(G) \to \gamma_3^\sharp(G/N) \to 1.$$

It is well-known that $\mathcal{M}^{(1)}(G) \cong \ker(G \wedge G \to G)$. A corresponding isomorphism for $\mathcal{M}^{(2)}(G)$ is given in [2] as the following.

Lemma 2.3. *There exist cononical isomorphisms*

$$\gamma_3^\sharp(G) \cong \gamma_3(G) \text{ and } \mathcal{M}^2(G) \cong \ker(\sigma : \gamma_3^\sharp(G) \to G).$$

Definition 2.4. Let G be a group. Then

$$Z_2^\wedge(G) = \{x \in G \mid (x \wedge g_1) \wedge g_2 = 1_{\gamma_3^\sharp(G)} \text{ for all } g_1, g_2 \in G\}$$

and it is called is the 2-exterior center of G.

Using the above definition, it is easy to see that

Proposition 2.5. *(i) $Z_2^\wedge(G)$ is a characteristic subgroup of G contained in $Z_2(G)$.*
Let N be a normal subgroup of G.

(ii) $\dfrac{Z_2^\wedge(G)N}{N} \subseteq Z_2^\wedge(G/N)$ *and* $Z_2^\wedge(G/Z_2^\wedge(G)) = 1$.

(iii) The sequence

$$1 \to Z_2^\wedge(G) \cap N \to Z_2^\wedge(G) \to Z_2^\wedge(G/N)$$

is exact.

Lemma 2.6. *$N \subseteq Z_2^\wedge(G)$ if and only if the natural map $\gamma_3^\sharp(G) \to \gamma_3^\sharp(G/N)$ is a monomorphism.*

Corollary 2.7. *$N \subseteq Z_2^\wedge(G)$ if and only if the natural map*

$$\mathcal{M}^{(2)}(G) \hookrightarrow \mathcal{M}^{(2)}(G/N)$$

is a monomorphism.

Theorem 2.8. *For any group G, we have $Z_2^\wedge(G) = Z_2^*(G)$.*

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Talk A classification of cubic one-regular graphs pp.: 1–4

A classification of cubic one-regular graphs

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Abstract

A graph is *one-regular* if its automorphism group acts regularly on the set of its arcs. In this talk, we classify cubic one-regular graphs of order $2p^2q$.

Keywords: One-regular graphs, Symmetric graphs, Cayley graphs.
Mathematics Subject Classification [2010]: 05C25, 20B25

1 Introduction

Throughout this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph X we use $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set and its full automorphism group, respectively. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. By an n-cycle we shall always mean a cycle with n vertices. Also girth is the length of shortest cycle. For a subgroup $G \leq Aut(X)$, a graph X is said to be (G, s)*-arc-transitive* or (G, s)*-regular* if G acts transitively or regularly on the set of s-arcs of X, respectively. In the special case graph is *one-regular* if its automorphism group acts regularly on the set of its arcs.

Proposition 1.1. *Let $p \geq 7$ be a prime and X a cubic symmetric graph of order $2p$. Then X is a one-regular normal Cayley graph on the dihedral group D_{2p}.*

Proposition 1.2. *Let X be a connected cubic symmetric graph and let G be a s-regular subgroup of $Aut(X)$. Then the stabilizer G_v of $v \in V(X)$ in G is isomorphic to $\mathbb{Z}_3, S_3, S_3 \times \mathbb{Z}_2, S_4$ or $S_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5, respectively.*

Proposition 1.3. $N_{Aut(X)}(R(G)) = R(G) \rtimes Aut(G, S)$.

Proposition 1.4. *Let G be a finite group and let Q be an abelian Sylow subgroup contained in the center of its normalizer. Then Q has a normal complement K (indeed, K is even a characteristic subgroup of G).*

Proposition 1.5. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $Aut(H)$ of H.*

*Speaker

2 Main results

Lemma 2.1. *Let p and q be distinct odd primes and $p > q \geq 7$. Also let G be a group of order $2p^2q$ and G is not isomorphic to $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes D_{2q}$ or $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{2q}$. Then G has a normal subgroup of order p or q.*

Proof. Let G be a group of order $2p^2q$. Thus G is solvable and has a characteristic subgroup of order p^2q, say H. Clearly $G^{'} \leq H$ and $|G^{'}| \in \{1, p, p^2, p^2q, q, pq\}$. If $|G^{'}| = 1$, then G is an abelian group and so G has a normal subgroups of orders p and q, as desired. If $|G^{'}| \in \{p, q, pq, p^2q\}$, then $G^{'}$ has a characteristic subgroup of order p or q. Thus G has a normal subgroup of order p or q, as desired. Now assume that $|G^{'}| = p^2$. By our assumption $G^{'}$ cannot be isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Thus $G^{'} \cong \mathbb{Z}_{p^2}$, and so $G^{'}$ has characteristic subgroup of order p. Therefore G has normal subgroup of order p, as desired.

□

The following theorem is the main result of this paper. Also for construction of the graphs see [2, 3, 4]

Theorem 2.2. *Let X be a cubic one-regular graph of order $2p^2q$. Then X is isomorphic to $\mathcal{C}(\mathbb{Z}_p^3)$, $\mathcal{C}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$, CB_{p^2}, CQ_p, for $3|p-1$, $CI(p,k,q)$, where $q \equiv 1 \pmod 3$ and $k^2+k+1 = 0$ ($k \in \mathbb{Z}_q^*$), or $Cay(D_{2p^2q}, \{\tau, \tau\rho, \tau\rho^{k+1}\})$, where $D_{2p^2q} = \langle \tau, \rho \mid \tau^2 = \rho^{p^2q} = 1, \tau^{-1}\rho\tau = \rho^{-1}\rangle$ and $k^2+k+1 = 0$ ($k \in \mathbb{Z}_{p^2q}^*$).*

Proof. If $p = q$, then X has order $2p^3$ and by [4, Theorem 3.2], cubic one-regular graphs of this order is isomorphic to $\mathcal{C}(\mathbb{Z}_p^3)$, or $\mathcal{C}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$, where $3|p-1$. Thus we may assume that $p \neq q$. If $q = 2$, then X has order $4p^2$ and by [2, Theorem 6.2], there is no cubic one regular graph of this order. If $q = 3$, then X has order $6p^2$ and by [2, Theorem 5.3], cubic one-regular graph of this order is isomorphic to CB_{p^2}, where $3|p-1$. Also if $q = 5$, then X has order $10p^2$ and by [1, Theorem 5.1], there is no one-regular graph of this order. Finally if $p = 2$, then X has order $8q$ and by [3, Theorem 5.1], cubic one-regular graph of this order is isomorphic to CQ_p, where $3|p-1$. In what following we may assume that either $p > q \geq 7$ or $q > p > 2$. First assume that $p > q \geq 7$. Let $A = Aut(X)$. By the one-regularity of X, one has $|A| = 6p^2q$. Let P be Sylow p-subgroup of A.

Claim I: A Sylow p-subgroup P is normal in A.

Since $|A| = 2.3p^2q$, it follows that A has normal subgroup of order $3p^2q$, say H. Let n_p and n_q be the number of Sylow p-subgroups and Sylow q-subgroups of H, respectively. Now $n_p = 1 + rp$, and $n_q = 1 + sq$ for some integers r and s. Since $n_p \mid 3q$ and $p > q \geq 7$, we have $n_p = 1$ or $n_p = 3q$. Suppose that $n_p = 3q$ and so $1 + rp = 3q$. Thus $r = 2$ and so $1 + 2p = 3q$. On the other hand $n_q = p$, p^2, $3p$ or $3p^2$. If $n_q = p$, then $q \mid p - 1$, a contradiction. If $n_q = p^2$, then $q \mid p^2 - 1$. Since $q \mid 1 + 2p$, we get a contradiction. If $n_q = 3p$, then $q \mid 3p - 1$, a contradiction. Finally if $n_q = 3p^2$, then $q \mid 3p^2 - 1$. Also since $q \mid 1 + 2p$, we have $q \mid 2 + 3p$. Now by $q \mid 1 + 2p$, we have $q \mid p$, a contradiction. Thus $n_p = 1$, and $P \trianglelefteq A$, as claimed.

Let X_P be the quotient graph of X relative to the set of orbits of P. Thus $|V(X_P)| = 2q$ or $2pq$.

First assume that $|V(X_N)| = 2q$. X_N is A/N-arc-transitive, so X_N is a one-regular normal Cayley graph on the dihedral group D_{2q}. Now by Proposition 1.2, the stabilizer $Aut(X_N)_v$ of $v \in V(X_N)$ is isomorphic to $\mathbb{Z}_3$. Thus $Aut(X_N)$ has order $6q$, and so $A/N = Aut(X_N)$. Also A has normal subgroup G such that G/N acts regularly on $V(X_N)$, and so $|G/N| = 2q$. Therefore G acts regularly on $V(X)$ and one may assume that X is normal Cayley graph on the group G, say $X = Cay(G, S)$. Clearly $|G| = 2p^2q$. Since X has valency 3, S contains an involution. Since $Aut(G, S)$ is transitive on S and so S contains of three involutions. By the connectivity of X, G can be generated by three involutions.

Suppose that G is not isomorphic to $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes D_{2q}$ or $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{2q}$. By Lemma 2.1, G has a normal subgroup of order p or q. First suppose that G has a normal subgroup of order q, say Q. Then $G = \langle a \rangle \rtimes K$, where $|K| = 2p^2$ and $o(a) = q$.

If K is an abelian group, then $K \cong \mathbb{Z}_{2p^2}$, or $\mathbb{Z}_{2p} \times \mathbb{Z}_p$. If $K \cong \mathbb{Z}_{2p^2} = \langle b \rangle$, then $b^{-1}ab = a^i$, where $0 \le i \le q-1$. Thus $i^{2p^2} = 1 \pmod q$. If $i = 1 \pmod q$ or $i = -1 \pmod q$, then either $b^{-1}ab = a$ or $b^{-1}ab = a^{-1}$. For the former case G is an abelian group and all involutions of G are contained in the subgroup $\langle b \rangle$, a contradiction. So $b^{-1}ab = a^{-1}$. The elements of order 2 are $a^m b^{p^2}$, where $0 \le m \le q-1$. Clearly G cannot be generated by these elements. Thus we may assume that $i \ne \pm 1 \pmod q$. So $2p^2 \mid q-1$, a contradiction.

Now assume that $K \cong \mathbb{Z}_{2p} \times \mathbb{Z}_p = \langle b \rangle \times \langle c \rangle$, where $o(b) = 2p$ and $o(c) = p$. Since $Q \lhd G$, we have $b^{-1}ab = a^i$, $c^{-1}ac = a^j$, where $0 \le i \le q-1$ and $0 \le j \le q-1$. Thus $i^{2p} = 1 \pmod q$ and $j^p = 1 \pmod q$. If $i = \pm 1 \pmod q$, and $j = 1 \pmod q$, then we have the following cases:
(1) $b^{-1}ab = a$, $c^{-1}ac = a$;
(2) $b^{-1}ab = a^{-1}$, $c^{-1}ac = a^{-1}$;
(3) $b^{-1}ab = a$, $c^{-1}ac = a^{-1}$;
(4) $b^{-1}ab = a^{-1}$, $c^{-1}ac = a$.

For the first case G is an abelian group and all involution of G are contained in the subgroup $\langle b \rangle \times \langle c \rangle$, a contradiction. For case (2), the elements of order 2 are $a^i b^p$, where i is odd. Clearly G cannot be generated by these elements. For case (3), b^p is the only element of order 2. Clearly G cannot be generated by b^p. Finally for case (4), the elements of order 2 are $a^i b^p$, where i is odd, a contradiction. Thus we may suppose that $i \ne \pm 1$ and $j \ne 1$. So $p \mid q-1$, a contradiction.

If K is not abelian, then from elementary group theory we know that there are three non-abelian groups of order $2p^2$ up to isomorphism:
$G_1(p) = \langle b, c \mid b^2 = c^{p^2} = 1, bcb = c^{-1} \rangle$;
$G_2(p) = \langle b, c, d \mid b^p = c^p = d^2 = [b, c] = 1, d^{-1}bd = b^{-1}, d^{-1}cd = c^{-1} \rangle$;
$G_3(p) = \langle b, c, d \mid b^p = c^p = d^2 = 1, [b, c] = [b, d] = 1, d^{-1}cd = c^{-1} \rangle$.

Now by considering all cases we complete the proof. □

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A generalization of commutativity notion

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Abstract

Mason introduced the reflexive property for ideals. We in this article consider the reflexive ring property on nil ideals, introducing the concept of a *nil-reflexive* ring as a generalization of the reflexive ring property. It is proved that the polynomial and power series rings over right Noetherian (or NI) rings R are both shown to be nil-reflexive if $(aRb)^2 = 0$ implies $aRb = 0$ for all $a, b \in N(R)$. The structure of nil-reflexive rings is studied in relation to various sorts of ring extensions which have roles in ring theory.

Keywords: Nil-reflexive ring, Nil ideal, Polynomial ring, Power series ring, Right quotient ring
Mathematics Subject Classification [2010]: 16N40, 16S70

1 Introduction

Throughout this article all rings are associative with identity unless otherwise specified. Given a ring R, the polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). For any ring R and $n \geq 2$, denote the n by n full matrix ring over R by $Mat_n(R)$ and the n by n upper triangular matrix ring over R by $U_n(R)$. Let $D_n(R)$ denote the subring $\{A \in U_n(R) \mid \text{the diagonal entries of } A \text{ are all equal}\}$ of $U_n(R)$. We use $N^*(R)$ and $N(R)$ to denote the upper nilradical (i.e., the sum of all nil ideals) and the set of all nilpotent elements of R, respectively. It is well-known that $N^*(R) \subseteq N(R)$. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n).

The reflexive property for right ideals was first studied by Mason [8]. a right ideal I of a ring R is called *reflexive* if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$, and R is called *reflexive* if 0 is a reflexive ideal. Every semiprime ring is reflexive by an easy computation. Kwak and Lee [6] characterized the aspects of the reflexive and one-sided idempotent reflexive properties, and provided a method by which a reflexive ring, which is not semiprime, can always be constructed from any semiprime ring, and showed that the reflexive property is Morita invariant.

In [6], it is proved that a ring R is reflexive if and only if $IJ = 0$ implies $JI = 0$ for ideals I, J of R. We will consider the reflexive ring property on nil ideals of a ring.

Definition 1.1. A ring R is called *nil-reflexive* if $IJ = 0$ implies $JI = 0$ for nil ideals I, J of R.

*Speaker

Any reflexive ring is clearly nil-reflexive. But the converse need not hold by the following.

Example 1.2. Let F be a field and $F\langle a,b\rangle$ be the free algebra with noncommuting indeterminates a,b over F. Let I be the ideal of $F\langle a,b\rangle$ generated by ab. Set $R = F\langle a,b\rangle/I$ and let a,b coincide with their images in R for simplicity. We can show that R is nil-reflexive but not reflexive.

For a nonempty subset X of a ring R, we write $r_R(X) = \{a \in R | Xa = 0\}$, which is called the *right annihilator* of X in R. The left annihilator is defined similarly and denoted $\ell_R(X)$.

Proposition 1.3. *For a ring R the following are equivalent:*
(1) *R is a nil-reflexive ring.*
(2) *$aRb = 0$ for $a,b \in N^*(R)$ implies $bRa = 0$.*
(3) *For each $a \in N^*(R)$, $r_{N^*(R)}(aR) = \ell_{N^*(R)}(Ra)$.*
(4) *$ARB = 0$ implies $BRA = 0$ for any nonempty subsets A,B of $N^*(R)$.*

Recall that a ring is *reduced* if it has no nonzero nilpotent elements. Cohn [2] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for $a,b \in R$. Reduced rings are clearly reversible, and reversible rings are obviously reflexive. In [7], a ring R is called *NI* if $N^*(R) = N(R)$. The class of NI rings contains reversible rings, but we can see that the concepts of (nil-)reflexive rings and NI rings are independent of each other.

Proposition 1.4. *Let R be an NI ring. Then the following conditions are equivalent:*
(1) *R is nil-reflexive.*
(2) *$aRb = 0$ for $a,b \in N(R)$ implies $bRa = 0$.*
(3) *$IJ = 0$ implies $JI = 0$ for all nil right (or, left) ideals I,J of R.*

2 Main results

Theorem 2.1. (1) *If R is a nil-reflexive ring then so is eRe for each central $e^2 = e \in R$.*

(2) *If R is a nil-reflexive ring then so is $Mat_n(R)$ for any $n \geq 2$.*

(3) *Let $R = \bigoplus_{i\in I} R_i$ be a direct sum of rings R_i and I be a finite index set. Then R is a nil-reflexive ring if and only if R_i is a nil-reflexive ring for each $i \in I$.*

(4) *If R is a ring with an Abelian unit group, then $N(R)$ is commutative (and hence R is nil-reflexive).*

Corollary 2.2. *For a central idempotent e of a ring R, eR and $(1-e)R$ are nil-reflexive if and only if R is nil-reflexive.*

We can prove that both $U_n(R)$ and $D_n(R)$ for any ring R and $n \geq 3$ are not nil-reflexive, but we can construct reversible (hence (nil-)reflexive) subrings of $D_n(R)$ for $n \geq 3$ over reduced ring R. If R is a reduced ring, then $D_2(R)$ is reversible and so it is nil-reflexive. But the following example shows that there exists a reversible (and so nil-reflexive) R such that $D_2(R)$ is not nil-reflexive.

Example 2.3. Let

$$S = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra generated by noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Next let I be the ideal of S generated by

$$\begin{aligned}&a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\\&b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,\\&(a_0+a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \text{ and } r_1r_2r_3r_4,\end{aligned}$$

where the constant terms of $r, r_1, r_2, r_3, r_4 \in S$ are zero. Now set $R = S/I$. Then R is a reversible ring, but $D_2(R)$ is not nil-reflexive.

However, we have the following result.

Proposition 2.4. (1) *Let R be a ring. If $N^*(R)^2 = 0$ then R is nil-reflexive.*
(2) *If R is a reduced ring then $U_2(R)$ is nil-reflexive.*
(3) *If $U_2(R)$ is nil-reflexive, then R is nil-reflexive.*
(4) *If $D_2(R)$ is nil-reflexive, then R is nil-reflexive.*

Notice that $R = Mat_3(S)$ over a reduced ring S is a nil-reflexive ring by Theorem 2.1(2), but the subring $U_3(S)$ of R is not nil-reflexive. Therefore the nil-reflexivity is not closed under subrings. One may conjecture that the nil-reflexivity is closed under factor rings, but the following erases the possibility.

Example 2.5. Let F be a field and $R = F\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over F. Let I be the ideal of R generated by

$$ab, a^2 \text{ and } b^2.$$

Let a, b coincide with their elements in R/I for simplicity. Obviously R is nil-reflexive, but we can show that R/I is not nil-reflexive.

Proposition 2.6. *For a ring R and a proper ideal I of R, if R/I is a nil-reflexive ring and I is reduced as a ring without identity, then R is nil-reflexive.*

A ring is called *Abelian* if every idempotent is central. Reversible rings are Abelian through a simple computation, but not conversely in general. The concepts of an Abelian ring and a nil-reflexive ring do not imply each other. For, the ring $R = D_3(A)$, over a reduced ring A, is Abelian by help of [5, Proposition 1.2], but R is not nil-reflexive. On the other hand, the nil-reflexive ring $Mat_3(S)$ over a reduced ring S is not Abelian clearly.

Let R be an algebra over a commutative ring A. Due to Dorroh [3], the *Dorroh extension* of R by A is the Abelian group $R \oplus A$ with multiplication given by $(r_1, a_1)(r_2, a_2) = (r_1r_2 + a_1r_2 + a_2r_1, a_1a_2)$ for $r_i \in R$ and $a_i \in A$.

Theorem 2.7. *Let R be an algebra with identity over a commutative reduced ring A. Then R is nil-reflexive if and only if the Dorroh extension D of R by A is.*

We can show that the nil-reflexive property does not pass to polynomials by Example2.3.

A ring R is called *Armendariz* if whenever any polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, $a_ib_j = 0$ for all i, j.

Theorem 2.8. *Let R be an Armendariz ring. Then R is nil-reflexive if and only if so is $R[x]$.*

Note that the classes of Armendariz rings and nil-reflexive rings are independent of each other. The nil-reflexive property does not go up to power series rings by the same argument as polynomials, either. However, we have the following result.

Theorem 2.9. *Let R be a ring such that $(aRb)^2 = 0$ implies $aRb = 0$ for all $a, b \in N(R)$.*
(1) *If R is a right Noetherian ring then $IJ = JI = 0$ for all nil ideals I, J in $R[[x]]$ ($R[x]$).*
(2) *If R is a right Noetherian ring then $R[[x]]$ ($R[x]$) is nil-reflexive.*
(3) *If R is an NI ring then $R[[x]]$ ($R[x]$) is nil-reflexive.*

A multiplicatively closed (m.c. for short) subset X of a ring R is said to satisfy the *right Ore condition* if for each $r \in R$ and $x \in X$, there exist $r_1 \in R$ and $x_1 \in X$ such that $rx_1 = xr_1$. It is shown by [9, Theorem 2.1.12] that X satisfies the right Ore condition and X consists of regular elements if and only if the right quotient ring of R with respect to X exists.

Theorem 2.10. *Let X be an m.c. subset of a ring R, and suppose that X satisfies the right Ore condition and X consists of regular elements.*
(1) *If R is a reflexive ring then so is the right quotient ring Q of R with respect to X.*
(2) *If R is a nil-reflexive ring then so is the right quotient ring Q of R with respect to X.*

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Talk A new algorithm to compute secondary invariants pp.: 1–4

A New Algorithm to Compute Secondary Invariants

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Abstract

In this paper we present a new method to compute secondary invariants of invariant rings. The main advantage of our approach relies on using SAGBI-Gröbner basis in computation which against the Gröbner basis, keeps the invariant structure of polynomials. For this purpose, we use Molien's formula to compute Hilbert series and find the degree of secondary invariants. When the degrees are known, it is sufficient to compute partial SAGBI-Gröbner bases up to certain degrees to find a set of secondary invariants.

Keywords: Invariant ring, Secondary invariants, SAGBI-Gröbner basis
Mathematics Subject Classification [2010]: 13A50, 13P10

1 Introduction

Let G be a finite $n \times n$ matrix group, linearly acting on a polynomial ring R with n variables over the field K. The ring of all polynomials in R which are invariant under the action of G is called the invariant ring denoted by R^G, which has also an algebra structure. Thanks to the known Hilbert theorem, R^G is finitely generated as a $K-$algebra and furthermore, there are n algebraically independent homogeneous invariants $P = \{f_1, \ldots, f_n\}$ for which R^G is finitely generated module over sub-algebra $K[f_1, \ldots, f_n]$. The elements of P are called primary invariant, and any minimal system of homogeneous invariants $g_1, \ldots, g_t$ generating R^G as a $K[f_1, \ldots, f_n]-$module is called a system of secondary invariants.

There are some algorithms to compute secondary invariants each of which uses an special kind of Gröbner basis. Most of these algorithms like those stated in [5], use some extra auxiliary variables which increase the volume of computations. Furthermore, Gröbner basis breaks the invariant structure of polynomials. There is a generalization of Gröbner basis for ideals of sub-algebras of polynomial rings, which contains important information about the ideal, and also there are efficient algorithms to compute it [2, 3]. The main idea of this paper is to use SAGBI-Gröbner basis to compute secondary invariants. So, in the sequel we recall necessary concepts and then we state our new algorithm. The following definition states the main computational tool in invariant ring.

Definition 1.1. The Reynolds operator of G is the map $\mathcal{R} : R \to R^G$ mapping each $f \in R$ to $\mathcal{R}(f) = 1/|G|(\sum_{\sigma \in G} f(\sigma.X))$ where X is the column vector of variables.

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It is easy to see that the Reynolds operator is a $K-$linear map onto R^G which does not change invariants. We are going now to recall the definition of SAGBI-Gröbner basis. Fix an admissible monomial ordering $\prec$ that is a well-ordering and stable under monomial multiplication. For a polynomial $f \in R$, the greatest monomial w.r.t. $\prec$ contained in f is called the leading monomial of f, denoted by $\mathrm{LM}(f)$. Further, if F is a set of polynomials, $\mathrm{LM}(F)$ is defined to be $\{\mathrm{LM}(f) | f \in F\}$. Also, the monomials appearing in $\mathrm{LM}(R^G)$ are called initial monomials.

Definition 1.2. Let $I^G \subset R^G$ be an ideal and $F \subset I^G$ be a finite set. We call F a SAGBI-Gröbner basis for I^G whenever $\mathrm{LM}(F)$ generates the initial ideal $\langle \mathrm{LM}(I^G) \rangle$ as an ideal in $\langle \mathrm{LM}(R^G) \rangle$. Further, we call it a partial SAGBI-Gröbner basis up to degree D if $\mathrm{LM}(F)$ generates ideal $\langle \mathrm{LM}(I^G) \rangle$ up to degree D.

One of the most efficient algorithms for computing SAGBI-Gröbner bases is $\mathrm{G}^2\mathrm{V}$-Invariant algorithm mentioned in [3] which we use in this paper for computations. The following lemma states a nice property of SAGBI-Gröbner basis which is one of the base tools in this paper.

Lemma 1.3. *If F is a SAGBI-Gröbner basis for I^G then the set of initial monomials which are not divisible by* $\mathrm{LM}(F)$ *construct a basis for the $K-$vector space R^G/I^G.*

2 Description of the main idea

In this section we state our main result on computing secondary invariants. The cornerstone of our idea is the Nakayama's lemma [4, Lemma 2.1] as follows:

Lemma 2.1. *Suppose that a set of primary invariants, P is given. Then $\{g_1, \ldots, g_t\}$ is a set of secondary invariants if it generates R^G/I^G as a $K-$vector space.*

It is worth noting that to apply SAGBI-Gröbner basis, we must restrict ourselves to the cases for which the matrix group G is a monomial matrix group. By a monomial matrix group we mean a group which converts monomials to monomials. So, in the sequel we assume that the group G is a monomial matrix group. Using the above lemma together with Lemma 1.3, it is enough to know the degrees of each g_i appearing in the set of secondary invariants to compute them. In doing so, we can use the well-known Hilbert series and Molien's formula as mentioned in [5, Chapter 2]:

Proposition 2.2. *Let $d_1, \ldots, d_n$ be the degree of primary invariants of R^G, then*

- *in the non-modular case by Mollien's formula, the Hilbert series of R^G equals*

$$H(R^G, z) = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{\det(id - z\sigma)}$$

- *if $e_1, \ldots, e_t$ be the degrees of secondary invariants, then we have*

$$H(R^G, z) \prod_{i=1}^{n} (1 - z^{d_i}) = z^{e_1} + \cdots + z^{e_t}$$

We are ready now to state our new algorithm which we have implemented in Maple[1].

Theorem 2.3. *The following algorithm computes a set of secondary invariants for R^G:*

Algorithm 1 SECONDARY

Require: P, a set of primary invariants.
Ensure: $\{g_1, \ldots, g_t\}$, a set of secondary invariants.

```
S := {};
Compute {e_1, ..., e_t} using Proposition 2.2, sorted increasingly;
for i = 1, ..., t do
    Compute F, a SAGBI-Gröbner basis for ⟨P⟩ up to degree e_i;
    S := S union the set of generators of R^G/⟨F⟩ of degree e_i using Lemma 1.3;
end for
RETURN(S);
```

The following example shows the behaviour of our algorithm to compute secondary invariants.

Example 2.4. Let G be the cyclic group generated by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Suppose that the set of primary invariants $P = \{x^2 + y^2, z^2, x^4 + y^4\}$ is given. Using Proposition 2.2 we have

$$H(R^G, z)(1 - z^2)^2(1 - z^4) = 1 + 2z^3 + z^4$$

which implies that we must compute secondary invariants of degrees $0, 3$ and 4. It is obvious that the secondary invariant of degree 0 is $g_1 = 1$. To continue, we compute a SAGBI-Gröbner basis for $\langle P \rangle$ up to degree 3 which is:

$$\{x^2 + y^2, z^2, x^2y^2\}.$$

Therefore, the set of initial monomials of degree 3 generating $R^G/\langle P \rangle$ is $\{xyz, y^2z\}$. Thus we have $g_2 = \mathcal{R}(xyz) = xyz$ and $g_3 = \mathcal{R}(x^2z) = x^2z - y^2z$. For degree 4, we receive to the same SAGBI-Gröbner basis and so $g_4 = \mathcal{R}(x^3y) = x^3y - xy^3$.

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A new result of the intersection graph of subgroups of a finite group

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Abstract

For a non-trivial finite group G different from a cyclic group of prime order, the intersection graph $\Gamma(G)$ of G is the simple undirected graph whose vertices are the non-trivial proper subgroups of G and two vertices are joined by an edge if and only if they have a non-trivial intersection. In this paper we will survey many of the known results of this graph and we will provide references to the literature for their proofs. Also as a new result, we characterize all finite groups with planar intersection graphs. It turns out that few solvable groups have planar intersection graphs.

Keywords: Subgroups graph, Subgroups lattice, Intersection of subgroups
Mathematics Subject Classification [2010]: 20D99; 05C25, 05C83

1 Introduction

Csákány and Pollák [4], introduced the intersection graph of non-trivial proper subgroups of groups. For a group G, which is not cyclic of prime order the intersection graph of G, which is denoted by $\Gamma(G)$ is the graph whose vertex set is the set of all proper non-trivial subgroups of G, with two vertices H_1 and H_2 being adjacent if and only if $H_1 \cap H_2 \neq \{1\}$.

This study was inspired by the definition of the intersection of non-trivial proper subsemigroups due to Bosák [3]. Zelinka [7], continued the investigation of the intersection graph of subgroups of a finite abelian group.

The main result of [7], states that if $\Gamma(A)$ is known for a finite abelian group A, one can determine the number of factors in the expression of A as a direct product of Sylow groups and the intersection graph of any of these Sylow groups. The author concludes with the conjecture that two finite abelian groups with isomorphic intersection graphs, are isomorphic. This conjecture was invesigated by Bertholf and Walls in [2].

The authors gave a counterexample to this conjecture, namely non-isomorphic cyclic primary groups of the same height. Then they present a theorem: If G is a finite abelian group with no cyclic Sylow subgroups, then G is determined by its intersection graph.

In response to a question posed by Csákány and Pollák [4], Shen [6], classified finite groups with disconnected intersection graph of subgroups. These groups are classified as $\mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are primes, or a Frobenius group whose complement is a group

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of prime order and the kernel is a minimal normal subgroup. The prime graph of a non-abelian simple group plays a major role in the proof of the theorem.

Herzog et al. [5], defined a graph $\Gamma M(G)$, where G is a finitely generated group, whose vertex set is the set of all the maximal subgroups of G and two distinct vertices M_1 and M_2 are joined by an edge if and only if $M_1 \cap M_2 \neq \{1\}$. The paper has two main results. First of all the authors proved that if G is a finite simple group then $\Gamma M(G)$ is a connected graph with diameter at most 62. Secondly they proved that G is a finite group with $\Gamma M(G)$ disconnected if and only if: (i) G is elementary abelian of order p^2 (p a prime number), (ii) G is cyclic of order pq (p, q different prime numbers), (iii) G is the semi-direct product of an elementary abelian p-group P by a cyclic group Q of prime order q, where $q \neq p$, and Q acts irreducibly and fixed point freely on P. We emphasize that in this work G is a non-trivial finite group different from a cyclic group of prime order.

Definition 1.1. For a group G, the intersection graph of G, which is denoted by $\Gamma(G)$ is the graph whose vertex set is the set of all proper non-trivial subgroups of G, with two vertices H_1 and H_2 being adjacent if and only if $H_1 \cap H_2 \neq \{1\}$.

We classify groups with planar graphs in theorem 2.1, using a well-known theorem, due to Kuratowski:

Theorem 1.2. *(Theorem 8.6.5 [1]) A graph is planar if and only if it contains no subdivisions of K_5 or $K_{3,3}$.*

We start with the following useful lemmas:

Lemma 1.3. *$\Gamma(G)$ is non-planar if one of the following holds:*
(1) G has at least 5 distinct subgroups with mutually non-trivial intersection.
(2) G has distinct subgroups H_i and S_j, such that $H_i \cap S_j \neq \{1\}$; $1 \leq i \leq j \leq 3$.
(3) $\Gamma(H)$ is non-planar, for some subgroup H of G.
(4) G has a normal subgroup N such that $\Gamma(G/N)$ is non-planar.

Lemma 1.4. *If G is a finite solvable group and $\Gamma(G)$ is planar, then $|G| = p^\alpha q^\beta r^\gamma$, where p, q, r are distinct primes, α, β, γ are non-negative integers such that $2 \leq \alpha + \beta + \gamma \leq 5$. Also if $|\pi(G)| = 3$, then $3 \notin \{\alpha, \beta, \gamma\}$.*

Remark 1.5. For two groups G and H if $G \cong H$, then obviously $\Gamma(G) \cong \Gamma(H)$.

2 Main results

Our main result is the following:

Theorem 2.1. *The graph $\Gamma(G)$ is planar if and only if G is one of the following types:*

(1) $\mathbb{Z}_{p^\alpha}, \mathbb{Z}_{p^\beta q}, \mathbb{Z}_{pqr}$, where p, q, r are distinct primes, $2 \leq \alpha \leq 5$ and $1 \leq \beta \leq 2$,

(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$, where p is an odd prime number.

(3) Q_8 or D_8, where Q_8 and D_8 are the quaternion group and the dihedral group of order 8; respectively,

(4) $\langle a,\ b \mid a^p = b^q = 1,\ bab^{-1} = a^i,\ \mathrm{Ord}_p(i) = q\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$, *where* p, q *are distinct primes with* $q < p$,

(5) $\langle a,\ b \mid a^q = b^{p^2} = 1,\ bab^{-1} = a^i,\ \mathrm{Ord}_q(i) = p^2\rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$ *with* $p < q$ *and* $p^2 \mid (q-1)$,

(6) $\langle a,\ b,\ c \mid a^p = b^p = c^q = 1,\ ab = ba,\ cac^{-1} = a^ib^k,\ cbc^{-1} = a^jb^l\rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$, *with* $q \nmid (p-1)$ *and* $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ *has order* q *in* $GL(2,p)$,

(7) $\langle a,\ b,\ c \mid a^p = b^p = c^{q^2} = 1,\ ab = ba,\ cac^{-1} = a^ib^k,\ cbc^{-1} = a^jb^l\rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$, *with* $q < p$, $q \nmid (p-1)$ *and* $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ *has order* q^2 *in* $GL(2,p)$,

(8) $\langle a,\ b \mid a^r = b^{pq} = 1,\ bab^{-1} = a^i,\ \mathrm{Ord}_r(i) = pq\rangle \cong \mathbb{Z}_r \rtimes \mathbb{Z}_{pq}$, *where* p, q, r *are distinct primes and* $p < q < r$.

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Talk A note on the graph of equivalence classes of zero divisors of a ring pp.: 1–4

A note on the graph of equivalence classes of zero divisors of a ring

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Abstract

In this paper we study the graph of equivalence classes of zero divisors of a ring R, denoted by $\Gamma_E(R)$. We give some necessary conditions for finiteness of $\Gamma_E(R)$.

Keywords: Zero divisor, Annihilator, Associated prime
Mathematics Subject Classification [2010]: 13A15, 13A99, 05C12

1 Introduction

The graph of equivalence classes of zero divisors of a ring R, denoted by $\Gamma_E(R)$, is defined in [7] and studied in [4]. Let $Z(R)$ denotes the set of zero divisors of a ring R and $Z(R)^* = Z(R)\backslash\{0\}$. Define an equivalence relation $\sim$ on $Z(R)$ as follows[5]: $x \sim y$ if and only if $Ann(x) = Ann(y)$. $\Gamma_E(R)$ is a graph associated to R whose vertices are the classes of elements in $Z(R)^*$, and two distinct classes $[x] \neq [y]$ are joined by an edge if and only if $xy = 0$. Another interpretation of $\Gamma_E(R)$ is as follows: The vertices are the elements of $\{ann(a) : a \in Z(R)^*\}$ and two distinct elements $Ann(x)$ and $Ann(y)$ are adjacent if and only if $xy = 0$.

First we recall some facts and notations related to this paper. Throughout this paper R denotes a commutative ring with unit element. For any ideal $I, Ann(I) = \{r \in R : ri = 0 \forall i \in I\}$ is called an annihilator ideal. We say R satisfies $ACC(Ann)$ if every chain in the set of annihilator ideals has a maximal element. If R is a subring of a Noetherian ring then R satisfies $ACC(Ann)$. A prime ideal P is called an associated prime ideal if $P = Ann(x)$ for some $x \in Z(R)^*$. The set of associated prime ideals of R is denoted by $Ass(R)$. Also a vertex $[x]$ of $\Gamma_E(R)$ is called associated prime if $Ann(x) \in Ass(R)$.

Let Γ be a simple graph. The *degree* of $v \in V(\Gamma)$ denoted by $d(v)$. The set of vertices which are adjacent to v is denoted by $N_\Gamma(v)$. A complete subgraph of Γ is called a clique. The *clique number* of Γ, denoted by $\omega(\Gamma)$, is suprimum of size of cliques. A subset S of V is called a dominating set if every vertex in $V\backslash S$ has a neighbor in S. The minimum size of the dominating sets is called domination number and is denoted by $\gamma(\Gamma)$.

In [4] and [7] the ring R is Noetherian. In this paper we show that many results are true without Noetherian condition or true with a weaker condition.

*Speaker

2 Main results

In this section we state and prove our main results.

Theorem 2.1. *Let R be a ring. Let $S \subseteq R$ be a subset such that*

1. $0 \notin S$
2. *If $a \in R, s \in S$ and $as \neq 0$ then $as \in S$*

If $Ann(x)$ is a maximal element in $\{Ann(s) : s \in S\}$ then it is an associated prime.

Proof. Let $ab \in Ann(x)$. If $bx = 0$ then $b \in Ann(x)$. If $bx \neq 0$ then $bx \in S$ and $Ann(x) \subseteq Ann(bx)$. By maximality of $Ann(x)$ we conclude that $Ann(x) = Ann(bx)$. So $a \in Ann(bx) = Ann(x)$. □

Theorem 2.2. *[7] $Ass(R)$ is a clique in $\Gamma_E(R)$.*

Proof. Let $Ann(x) \neq Ann(y)$ be two elements of $Ass(R)$. Let $t \in Ann(x)\backslash Ann(y)$. Since $(t)Ann(t) = 0 \subseteq Ann(y)$, so $Ann(t) \subseteq Ann(y)$. Thus $xy = 0$. □

Theorem 2.3. *Let R be a ring. The following are equivalent:*

1. *$\Gamma_E(R)$ is finite.*
2. *$\{Ann(a) : a \in R\}$ is finite.*
3. *$\{Ann(I) : I \trianglelefteq R\}$ is finite.*

Proof.

1. $1 \Leftrightarrow 2$: Let $f : \Gamma_E(R) \to \{Ann(a) : a \in Z(R)^*\}$ be such that $f([a]) = Ann(a)$. This map is a one to one corresponding. So the result follows because $\{Ann(a) : a \in R\} = \{Ann(a) : a \in Z(R)^*\} \cup \{0, R\}$.
2. $2 \Rightarrow 3$: It is clear that $Ann(I) = \cap_{a\in I} Ann(a)$. Since $\{Ann(a) : a \in R\}$ is finite so $\{Ann(I) : I \trianglelefteq R\}$ is finite.
3. $3 \Rightarrow 2$:This is clear.

□

Theorem 2.4. *[7] Let R be a ring. If every $Ann(a)$ sits in an associated prime ideal i.e R satisfies $ACC(Ann)$ then $Ass(R)$ is a dominating set.*

Proof. Assume $Ann(a)$ is not an associated prime. Let $t \in Ann(a)$. If $Ann(t) \subseteq Ann(x) \in Ass(R)$ then $ax = 0$. Hence $Ann(a)$ is adjacent to $ann(x) \in Ass(R)$. □

Theorem 2.5. *Let R be a ring. If the degree of $[x]$ is finite then every chain in $\{ann(a) : a \in Ann(x)\}$ is finite and $[x]$ is adjacent to an associated prime. Also, If the degree of each vertex is finite then R satisfies $ACC(Ann)$.*

Proof. Let $S = Ann(x)\backslash\{0\}$. Since $d(x) < \infty$, so $\{Ann(a) : a \in S\}$ is finite. The maximal elements of this set are associated primes by theorem 2.1 which are adjacent to $[x]$. □

Corollary 2.6. *[7] Let R be a ring.*

1. *$\Gamma_E(R)$ is a finite graph if and only if each vertex of $\Gamma_E(R)$ has finite degree.*

2. *If $d([x]) = 1$ then only neighbor of $[x]$ is an associated prime.*

Proof. 1. One implication is trivial. Assume every vertex has finite degree. Hence R satisfies $ACC(Ann)$. Since $Ass(R)$ is a clique in $\Gamma_E(R)$, so $Ass(R)$ must be finite. Also $Ass(R)$ is a dominating set of $\Gamma_E(R)$ by Theorem 2.4. This implies that $\Gamma_E(R)$ is a finite graph.

2. This is clear.

□

Theorem 2.7. *Let R be a ring. If $\Gamma_E(R))$ contains a cycle of length three then there is a vertex such that is adjacent to only one of vertices of this cycle.*

Proof. Let $[x],[y],[z]$ be the vertices of the cycle and $ann(z)$ be a maximal element in $\{ann(x), Ann(y), Ann(z)\}$. Then $Ann(z) \nsubseteq Ann(x) \bigcup Ann(y)$. Let $w \in Ann(z) \backslash Ann(x) \bigcup Ann(y)$. So $[w] \neq [x],[y],[z]$. Hence $[w]$ is adjacent only to vertex $[z]$. □

Corollary 2.8. *[7] If $|\Gamma_E(R))| \geq 3$ then $\Gamma_E(R))$ is not a complete graph*

The following theorem is a theorem in [6][Theorem 3.2.24,p 364] which we give the commutative version of it here.

Theorem 2.9. *Let R be a reduced ring. If R satisfies $ACC(Ann)$ then $Ass(R) = \{P_1, \cdots, P_n\}$ is finite and every $Ann(I)$ is an intersection of some of the P_i.*

It is clear that if $\Gamma_E(R)$ is finite then R satisfies $ACC(Ann)(DCC(Ann))$. In the following theorem we prove a partial converse to this fact.

Theorem 2.10. *Let R be a reduced ring. Then $\Gamma_E(R)$ is finite if and only if R satisfies $ACC(Ann)$.*

Proof. If $\Gamma_E(R)$ is finite then $\{Ann(I) : I \trianglelefteq R\}$ is finite by Theorem 2.3. So R satisfies ACC on annihilator ideals. Conversely, If R satisfies ACC on annihilator ideals then $\{Ann(I) : I \trianglelefteq R\}$ is finite by Theorem 2.9. Thus $\Gamma_E(R)$ is finite by Theorem 2.3. □

Corollary 2.11. *Let R be a Noetherian reduced ring. Then $\Gamma_E(R)$ is a finite graph.*

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Annihilator Conditions in Noncommutative Ring Extensions

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Abstract

Let R be a ring, S a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. In [4], Marks, Mazurek and Ziembowski study the class of (S, ω)-Armendariz rings, as a generalization of the standard Armendariz condition from ordinary polynomial ring to skew generalized power series ring. We observe from results in [4], that the upper nilradical coincides with the prime radical in (S, ω)-Armendariz rings and also every one-sided nil ideal of these rings is contained in a two-sided nil ideal of the ring, namely satisfies in the Köthe's conjecture. Also it can be shown that the factor rings of an (S, ω)-Armendariz rings over its prime radical is also (S, ω)-Armendariz. We continue in this paper the study of rings with such property in skew generalized power series rings and bring some properties of these rings.

Keywords: Lower nilradical, Nilpotent elements, Skew generalized power series ring.
Mathematics Subject Classification [2010]: Primary 16N40, 20M25; Secondary 06F05.

1 Introduction

Throughout the present paper all rings considered, unless otherwise noted, shall be assumed to be associative and possess an identity; subrings of a ring need not have the same unit, *subrng* will denote a subring without unit, and "an order" on a set will always mean "a partial order". Our notation and terminology are standard and shall follow [3]. For instance, for such a ring R, the monoid of endomorphisms of R (with composition of endomorphisms as the operation) is denoted by $\mathrm{End}(R)$. We adopt the notations $Ni\ell(R)$, $Ni\ell_*(R)$ and $Ni\ell^*(R)$ to represent the set of all nilpotent elements, the lower nilradical (i.e., the prime radical) and the upper nilradical (i.e., the sum of all nil ideals) of a ring R, respectively. By $R[S]$, we mean the monoid ring of a monoid S over a ring R, while $R[x]$ denotes the ring of all polynomials over a ring R.

Let $(S, \leq)$ be an ordered set. Then $(S, \leq)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite and $(S, \leq)$ is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. An *ordered monoid* is a pair $(S, \leq)$ consisting of a monoid S (written multiplicatively) and an order $\leq$ on S such that for all $s_1, s_2, t \in S$, $s_1 \leq s_2$ implies $s_1 t \leq s_2 t$ and $t s_1 \leq t s_2$. An ordered monoid $(S, \leq)$ is said to be *strictly*

*Speaker

ordered if for all $s_1, s_2, t \in S$, $s_1 < s_2$ implies $s_1 t < s_2 t$ and $ts_1 < ts_2$. It is known that torsion-free nilpotent groups and free groups are ordered groups. Hence, any submonoid of a torsion-free nilpotent group or a free group is an ordered monoid. Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω, that is, $\omega_s = \omega(s)$. Let A be the set of all functions $f : S \to R$ such that the support $supp(f) = \{s \in S : f(s) \neq 0\}$ is an artinian and narrow set. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(x, y) \in supp(f) \times supp(g) : s = xy\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(x,y)\in X_s(f,g)} f(x).\omega_x(g(y))$$

(by convention, a sum over the empty set is 0). With multiplication as defined above and pointwise addition, A becomes a ring, called the *ring of skew generalized power series with coefficients in R and exponents in S*, denoted by $R[[S, \omega, \leq]]$ (see also [4]). The construction of the skew generalized power series rings generalizes some classical ring constructions such as polynomial rings ($S = \mathbb{N} \cup \{0\}$ under usual addition, with the trivial order, i.e., the order with respect to which any two distinct elements are incomparable, and ω is trivial, i.e., the monoid homomorphism that sends every element of S to the identity endomorphism), monoid rings (trivial order, and ω is trivial), skew polynomial ring $R[x; \sigma]$ for some $\sigma \in \mathrm{End}(R)$ ($S = \mathbb{N} \cup \{0\}$ under usual addition, with the trivial order, and $\omega_1 = \sigma$), skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$ for some $\sigma \in \mathrm{End}(R)$ ($S = \mathbb{Z}$ under usual addition, with the trivial order, and $\omega_1 = \sigma$), skew monoid rings (with trivial order), skew power series ring $R[[x; \sigma]]$ for some $\sigma \in \mathrm{End}(R)$ ($S = \mathbb{N} \cup \{0\}$ under usual addition, with the usual order, and $\omega_1 = \sigma$), skew Laurent power series ring $R[[x, x^{-1}; \sigma]]$ for some $\sigma \in \mathrm{End}(R)$ ($S = \mathbb{Z}$ with usual addition, with the usual order, and $\omega_1 = \sigma$), the Mal'cev-Neumann construction ($(S, . , \leq)$ a totally ordered group and trivial ω) the Mal'cev-Neumann construction of twisted Laurent series rings ($(S, . , \leq)$ a totally ordered group, and generalized power series rings. For each $r \in R$ and $s \in S$, let $c_r, e_s \in R[[S, \omega, \leq]]$ defined by

$$c_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{if } x \in S\setminus\{1\}, \end{cases} \quad e_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \in S\setminus\{s\}. \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $R[[S, \omega, \leq]]$ and $s \mapsto e_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S, \omega, \leq]]$, and also we have $e_s c_r = c_{\omega_s(r)} e_s$. Moreover, for any nonempty subset X of R we have

$$X[[S, \omega, \leq]] = \{f \in R[[S, \omega, \leq]] : \quad f(s) \in X \cup \{0\} \quad \text{for every} \quad s \in S\},$$

and for each nonempty subset Y of $R[[S, \omega, \leq]]$, we put $C_Y = \{g(t) : \quad g \in Y \quad , \quad t \in S\}$.

In their pioneering work [5] in the 1997's, Rege and Chhawchharia introduced Armendariz property of rings which have since become the most widely used tool for studying the annihilators of a ring extensions. Recall that a ring R is said to be *Armendariz* if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. This nomenclature was used by them since it was Armendariz [2, Lemma 1] who initially showed that a *reduced* ring (i.e., ring without non-zero nilpotent element) always satisfies this condition. Since its introduction, the concept of an Armendariz ring has been generalized and extended in many different ways. All these were unified by Marks et al. [4] calling this unified generalization an (S, ω)-Armendariz ring R, where $(S, \leq)$ is a strictly ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ is a monoid homomorphism. A ring R is called (S, ω)-*Armendariz* if whenever $fg = 0$ for $f, g \in R[[S, \omega, \leq]]$, then $f(s).\omega_s(g(t)) = 0$ for all $s, t \in S$ [4, Definition 2.1].

Antoine [1] continued to work in this area, introducing the concept of nil-Armendariz ring. A ring R is called *nil-Armendariz* if the product of two polynomials has coefficients in the set of nilpotent elements, then the product of the coefficients of the polynomials is also nilpotent. This condition was introduced by Antoine to develop an annihilator theory for polynomial rings, which is related to a question of Amitsur of whether polynomial rings over nil rings are nil. It was extensively studied in conjunction with another zero-divisor conditions. Our results continues this ongoing effort in the case of skew generalized power series ring with respect to lower nilradical.

2 Main results

We start our main results with the following definition.

Definition 2.1. Let R be any ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. We say that R is *lower nil* (S, ω)-*Armendariz* if whenever $fg \in Ni\ell_*(R)[[S, \omega, \leq]]$ for $f, g \in R[[S, \omega, \leq]]$, then $f(s).\omega_s(g(t)) \in Ni\ell_*(R)$ for all $s, t \in S$.

Lemma 2.2. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and also $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then we have the following statements:*

(a) [4, Proposition 4.5] *If R is S-compatible and (S, ω)-Armendariz, then $N_0(R) = Ni\ell_*(R) = Ni\ell^*(R)$.*

(b) *The class of (S, ω)-Armendariz rings is closed under subrings (possibly without unity) and direct products.*

Proposition 2.3. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and also $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible and lower nil (S, ω)-Armendariz, then for each elements $f_1, \ldots, f_n$ in $R[[S, \omega, \leq]]$ such that $f_1 f_2 \cdots f_n \in Ni\ell_*(R)[[S, \omega, \leq]]$, we have $f_1(s_1)f_2(s_2)\cdots f_n(s_n) \in Ni\ell_*(R)$, where $s_i \in S$ for each i.*

Theorem 2.4. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and also $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible, then the ring R is lower nil (S, ω)-Armendariz if and only if the factor ring $R/Ni\ell_*(R)$ is $(S, \bar{\omega})$-Armendariz, where $\bar{\omega} : S \to \mathrm{End}(R/Ni\ell_*(R))$ is the induced monoid homomorphism.*

Theorem 2.5. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and also $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatibe and lower nil (S, ω)-Armendariz ring, then $Ni\ell_*(R) = Ni\ell^*(R)$.*

Proposition 2.6. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and also $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. If a nil ring R is S-compatible and lower nil (S, ω)-Armendariz, then it is a prime radical ring.*

Let R be any ring, $(S, \leq)$ a strictly ordered monoid and also $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Recall that a subset $P \subseteq R$ is *S-stable* if for every $s \in S$ we have $\omega_s(P) = P$. Moreover, an ideal I of a ring R is S-compatible (or (S, ω)-compatible) if for all $a, b \in R$ and each $s \in S$, $ab \in I$ if and only if $a\omega_s(b) \in I$.

Proposition 2.7. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and also $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. If I is S-stable ideal of R such that $I \subseteq Ni\ell_*(R)$, then the ring R is lower nil (S, ω)-Armendariz if and only if the factor ring R/I is lower nil $(S, \bar{\omega})$-Armendariz, where $\bar{\omega} : S \to \mathrm{End}(R/I)$ is the induced monoid homomorphism.*

Proposition 2.8. *Let R be any ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that R is S-compatible. If $Ni\ell_*(R)$ is S-compatible and R is lower nil (S, ω)-Armendariz ring, then $Ni\ell(R)$ forms a subrng of R.*

The study of nil rings is one of the central topics in noncommutative ring theory because of the famous Köthe's conjecture which posits that a ring with no non-zero nil (two-sided) ideals has no non-zero nil one-sided ideals either. This problem has been open since 1930. We have the following related result.

Corollary 2.9. *Let R be any ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible and lower nil (S, ω)-Armendariz ring and also $Ni\ell_*(R)$ is S-compatible, then R satisfies the Köthe's conjecture.*

Hence by considering the monoid ring $R[S]$, we conclude that for a strictly ordered monoid $(S, \leq)$, each lower nil S-Armendariz ring satisfies the Köthe's conjecture.

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Baer invariants of certain class of groups

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Abstract

In this talk, we intend to investigate the Baer invariants of certain class of groups with respect to the variety of polynilpotent groups of class row (c_1, c_2), when $(c_2+1)n-(c_2+1) < c_1$. Moreover, an explicit formula for the Baer invariant of direct product of two finite cyclic groups with respect to the variety of metabelian groups is also given.

Keywords: Baer invariant, Nilpotent product, Basic commutator
Mathematics Subject Classification [2010]: 20E34,20E10,20F18

1 Introduction

Let $\mathcal{N}_{c_1,c_2}$ be the variety of polynilpotent groups of class row (c_1, c_2), and G be an arbitrary group with a free presentation

$$1 \to R \to F \to G \to 1.$$

The Baer invariant of G with respect to the variety of polynilpotent groups of class row (c_1, c_2), is defined to be

$$\mathcal{N}_{c_1,c_2}M(G) \cong \frac{R \cap \gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[R,\ _{c_1}\ F,\ _{c_2}\ \gamma_{c_1+1}(F)]}.$$

The Baer invariant of G with respect to this variety, is called a (c_1, c_2) *polynilpotent multiplier.*

Now let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of cyclic groups and A be the free product of this family. $n-$ nilpotent product of $\{A_\lambda\}_{\lambda\in\Lambda}$ is defined as follows,

$$\prod_{\lambda\in\Lambda}^{\overset{n}{*}} A_\lambda = \frac{A}{\gamma_{n+1}(A)}.$$

Assume that

$$\mathbf{Z}_r =< x \mid x^r = 1 > \qquad , \qquad \mathbf{Z}_s =< y \mid y^s = 1 >$$

*Speaker

be two cyclic groups of orders r and s, respectively. Also consider the following free presentations for $\mathbf{Z}_r$ and $\mathbf{Z}_s$ where $\mathbf{Z}_r \cong \frac{F_1}{R_1}$ and $\mathbf{Z}_s \cong \frac{F_2}{R_2}$ such that $F_1 =< x >$, $F_2 =< y >$, $F = F_1 * F_2$, $R_1 =< x^r >^{F_1}$ and $R_2 =< y^s >^{F_2}$. It is easy to check that

$$\mathbf{Z}_r \overset{n}{*} \mathbf{Z}_s =< x, y \mid x^r, y^s, \gamma_{n+1}(F) >,$$

is a free presentation for $\mathbf{Z}_r \overset{n}{*} \mathbf{Z}_s$ which is denoted by $G_{(r,s,n)}$.
Now put $S =< R_1, R_2 >^F$, and $R = S\gamma_{n+1}(F)$. With this notations $\mathbf{Z}_r \overset{n}{*} \mathbf{Z}_s \cong \frac{F}{R}$.

The following theorems are vital in our main results.

Theorem 1.1. (P.Hall [4]). Let $F =< x_1, x_2, \ldots, x_t >$ be a free group, then

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)} \quad , \qquad 1 \leq i \leq n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, \ldots, x_t\}$.

Theorem 1.2. (Witt Formula [4]). The number of basic commutators of weight n on t generators is given by the following formula:

$$\chi_n(t) = \frac{1}{n} \sum_{m|n} \mu(m) t^{n/m}$$

where $\mu(m)$ is the *Mobious function*, and defined to be

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & \text{if } m = p_1 \ldots p_s. \end{cases}$$

In this talk we find the structure of $(c_1, c_2)-$polynilpotent multiplier of the group $G_{(r,s,n)}$ under some conditions.

2 The Main Results

In this section, we intend to investigate the structure of

$$\mathcal{N}_{c_1,c_2} M(G_{(r,s,n)}),$$

where $c_2 < 5$ and $(c_2 + 1)n - (c_2 + 1) < c_1$.

Clearly the Baer invariant of $G_{(r,s,n)}$ with respect to the variety of polynilpotent groups of class row (c_1, c_2), is as follows.

$$\mathcal{N}_{c_1,c_2} M(G_{(r,s,n)}) \cong \frac{S\gamma_{n+1}(F) \cap \gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[S\gamma_{n+1}(F), \ _{c_1} F, \ _{c_2} \gamma_{c_1+1}(F)]}.$$

Now, let $\rho_{c_1+1}(S) = [S, {}_{c_1} F]$ for $c_1 \geq 0$, then we have

$$\begin{aligned}\mathcal{N}_{c_1,c_2}M(G_{(r,s,n)}) &\cong \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[\rho_{c_1+1}(S)\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)]} \\ &\cong \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))/[\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)]}{[\rho_{c_1+1}(S)\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)]/[\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)]}.\end{aligned}$$

In [2] We have determined the structure of the factor group $\frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[\gamma_{c_1+n+1},\ {}_{c_2}\ \gamma_{c_1+1}(F)]}$. One notes that the main problem is to find the structure of the factor group $\frac{[\rho_{c_1+1}(S)\gamma_{c_1+n+1}(F),\ {}_{c_2}\gamma_{c_1+1}(F)]}{[\gamma_{c_1+n+1}(F),\ {}_{c_2}\gamma_{c_1+1}(F)]}$. In order to find the structure of $\mathcal{N}_{c_1,c_2}M(\mathbf{Z}_r \overset{n}{*} \mathbf{Z}_s)$, we need the following notations and theorems.

Let $d = (r, s)$, Y be the set of all basic commutators on X of weights $c_1 + 1, \ldots, c_1 + n$ and L_j be the set of all dth powers of the basic commutators on Y of weight j.

Theorem 2.1. If $(c_2 + 1)n - (c_2 + 1) < c_1$ then we have

$$[\rho_{c_1+1}(S)\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)] \ \equiv\ < L_{c_2+1} > \quad (mod\ [\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)]).$$

The following theorem is proved in [2].

Theorem 2.2. There exists a set of basic commutators on X, Z_{c_2+1} say; with

$$[\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)] \subseteq < Z_{c_2+1} > \text{modulo } \gamma_{(c_2+1)c_1+(c_2+1)n+2}(F),$$

and $Z_{c_2+1} \cap M_{c_2+1} = \emptyset$.

Now, we are in a position to prove the following important theorem.

Theorem 2.3. With the above notation and assumption, if $c_2 < 5$ and $(c_2 + 1)n - (c_2 + 1) < c_1$, then

$$\frac{[\rho_{c_1+1}(S)\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)]}{[\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)]},$$

is a free abelian group with the following basis

$$\overline{L}_{c_2+1} = \{l[\gamma_{c_1+n+1}(F),\ {}_{c_2}\ \gamma_{c_1+1}(F)] \mid l \in L_{c_2+1}\}.$$

The immediate consequence of the Theorems 2.1 and 2.2 is as follows.

Theorem 2.4. With the above notations, if $c_2 < 5$ then
(i) For each odd integers r and s,

$$\mathcal{N}_{c_1,c_2}M(G_{(r,s,2)}) \cong \mathbf{Z}_d \oplus \ldots \oplus \mathbf{Z}_d \qquad (\chi_{c_2+1}(\sum_{i=1}^{2} \chi_{c_1+i}(2)) - copies),$$

in which $c_2 + 1 < c_1$.
(ii) For all non negative integers r and s, which are not divisible by 2 and 3, then

$$\mathcal{N}_{c_1,c_2}M(G_{(r,s,3)}) \cong \mathbf{Z}_d \oplus \ldots \oplus \mathbf{Z}_d \qquad (\chi_{c_2+1}(\sum_{i=1}^{3} \chi_{c_1+i}(2)) - copies),$$

where $2c_2 + 2 < c_1$.
(iii) For all non negative integers r and s, which are not divisible by 2 and 3, then

$$\mathcal{N}_{c_1,c_2}M(G_{(r,s,4)}) \cong \mathbf{Z}_d \oplus \ldots \oplus \mathbf{Z}_d \qquad (\chi_{c_2+1}(\sum_{i=1}^{4} \chi_{c_1+i}(2)) - copies),$$

where $3c_2 + 3 < c_1$.

In the end of this talk we state the following interesting results. Note that $\mathcal{S}_2$ is the variety of metabelian groups is in fact the variety of polynilpotent groups of class row $(1, 1)$.

Corollary 2.5. Let r and s be two arbitrary positive integers. Then for each $(c_2 + 1)n - (c_2 + 1) < c_1$ and $c_2 < 5$ we have

$$\mathcal{N}_{c_1,c_2}M(\mathbf{Z}_r \times \mathbf{Z}_s) \cong \mathbf{Z}_d \oplus \ldots \oplus \mathbf{Z}_d \qquad (\chi_{c_2+1}(\chi_{c_1+1}(2)) - copies),$$

in which $d = (r, s)$. In particular

$$\mathcal{S}_2 M(\mathbf{Z}_r \times \mathbf{Z}_s) \cong < 1 > .$$

Corollary 2.6. If $(r, s) = 1$ then for any n

$$\mathcal{N}_{c_1,c_2}M(G_{(r,s,n)}) \cong < 1 >,$$

where $c_2 < 5$ and $(c_2 + 1)n - (c_2 + 1) < c_1$.

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Behavior of Prime (Ideals)Filters of Hyperlattices under the Fundamental Relation

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Abstract

The purpose of this note is the study of some lattice properties such as distributivity and dual distributivity under the fundamental relation. Also, we investigate the behavior of prime (resp. ideals) filters under fundamental relation in hyperlattices. In particular, we construct a one to one correspondence between the prime (resp. ideals) of a hyperlattice L containing ω_ϕ, the heart of L, and the prime (resp. ideals) filters of the fundamental lattice $L\backslash\varepsilon^*$.

Keywords: Hyperlattice, Prime filter, Fundamental relation
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Hyperstructures theory was first introduced by F. Marty in the eighth congress of Scandinavians in 1934 [8]. This theory has been developed in various fields. R. Ameri and Zahedi in [1] introduced and studied hyperalgebraic systems as a general form of algebraic hyperstructures; R. Ameri and Nozari studied relationship between the categories of multialgebra and algebra [2]. Also, Ameri and Rosenberg studied congruences and strongly congruences of multialgebras [3]. The theory of hyperlattices, as a class of multialgebras, was introduced by Konstantinidou in [6]. Rahnemaei Barghi considered the prime ideal theorem for distributive hyperlattices in [9]. In [5], B. B. N. Koguep, C. Nkuimi, and C. Lele studied fuzzy ideals(filters) in hyperlattices. Rasouli and Davvaz in [10] introduced and studied fundamental relation on hyperlattices. In this note, we studied prime (resp. ideals) and filters. Also, we use the fundamental relation ϵ^* on a given hyperlattice L, as the smallest equivalence relation on L, such that the quotient $L\backslash\epsilon^*$ is a lattice, and study the behavior of (rep. dual)distributivity under this quotient. Also, we study the relationship prime filters and ideals of L and fundamental lattice $L\backslash\epsilon^*$.

Recall that for a nonempty set H, a *hyperoperation* on H is a mapping from $H \times H$ into $P^\star(H)$, where $P^\star(H)$ is the set of all nonempty subsets of H.

Definition 1.1. [6] Let L be a nonempty set, " $\wedge$ " be a binary operation, and " $\vee$ " be a hyperoperation on L. Then L is called a hyperlattice, if for all $a, b, c \in L$ the following conditions hold:

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$L1)$ $a \in a \vee a$, and $a \wedge a = a$;
$L2)$ $a \vee b = b \vee a$, and $a \wedge b = b \wedge a$;
$L3)$ $a \in [a \wedge (a \vee b)] \cap [a \vee (a \wedge b)]$;
$L4)$ $a \vee (b \vee c) = (a \vee b) \vee c$, and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
$L5)$ $a \in a \vee b \Longrightarrow a \wedge b = b$.

In the natural way, we can extend " $\wedge$ " and " $\vee$ " to subsets of L as follows:

$$A \vee B = \cup\{a \vee b \mid a \in A, b \in B\},$$
$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\},$$

where $A, B \in P^\star(L)$.

A nonempty subset I of L is an ideal, if the following conditions hold:
(i) If $a, b \in I$, then $a \vee b \subseteq I$;
(ii) If $a \in I, b \leq a$, and $b \in L$, then $b \in I$.

An ideal I is a prime ideal, if $a \wedge b \in I$, then $a \in I$ or $b \in I$, for all $a, b \in L$. Also, a nonempty subset F of L is a filter, if the following conditions hold:
(i) If $a, b \in F$, then $a \wedge b \in F$;
(ii) If $a \in F$, $a \leq b$, and $b \in L$, then $b \in F$.

A filter F is a prime filter if $(a \vee b) \cap F \neq \emptyset$, then $a \in F$ or $b \in F$ for all $a, b \in L$.

A hyperlattice L is distributive, if for all $a, b, c \in L$:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

for more see [9])

Example 1.2. [5] Let $L = \{0, a, b, 1\}$. " $\wedge$ " and " $\vee$ " are given with Table 1. Then $(L, \vee, \wedge, 0, 1)$ is a distributive hyperlattice.

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

(a)

$\vee$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{0, a\}$	$\{1\}$	$\{b, 1\}$
b	$\{b\}$	$\{1\}$	$\{0, b\}$	$\{a, 1\}$
1	$\{1\}$	$\{b, 1\}$	$\{a, 1\}$	L

(b)

Table 1

2 Fundamental relation and primeness

Let R be a reflexive and symmetric relation on a nonempty set L. As it is well known the transitive closure of R is the smallest equivalence relation which containing R and it is denoted by $R^\star$. Therefore,

$$xR^\star y \Longleftrightarrow \exists n \in \mathbf{N}, \exists (x_1, x_2, ..., x_n) \in L^n$$

such that $xRx_1Rx_2R...x_nRy$.

Let L be a hyperlattice. Then $\varepsilon^\star$, the smallest equivalence relation on L, such that the quotient $L/\varepsilon^\star$ is a lattice is called the *fundamental relation* on L and the quotient $L/\varepsilon^\star$ is said to be *fundamental lattice* of L. Let X be a nonempty subset of L and $\Sigma(X)$ denote the set of all finite combinations respect to "$\vee$" and "$\wedge$". For example, if $X = \{x, y\}$, then $\Sigma(X) = \{x \vee y, x \wedge y, (x \wedge y) \vee x, (x \wedge (y \vee x)) \vee x, ...\}$ (for more details see [10]).

Letting $\epsilon_1 = \{(x, x)|x \in L\}$, and for every integer $n > 1$, define the relation ϵ_n as follows:

$$x\epsilon_n y \Longleftrightarrow \exists(z_1, z_2, ..., z_n) \in L^n, \exists z \in \Sigma(\{z_1, z_2, ..., z_n\}) : \{x, y\} \subseteq z.$$

Obviously, for $n \geqslant 1$, the relations ϵ_n are symmetric, and the relation $\varepsilon = \bigcup_{n\geqslant 1} \epsilon_n$ is reflexive and symmetric. Let $\varepsilon^\star$ be the transitive closure of ε. [10].

Definition 2.1. [10] Let $(L, \vee, \wedge)$ be a hyperlattice and R be an equivalence relation on L. Define hyperoperations $\oplus, \otimes : L/R \times L/R \longrightarrow P^*(L/R)$ as follows:

$$R(x) \otimes R(y) = R(x \wedge y),$$

and

$$R(x) \oplus R(y) = R(x \vee y).$$

Clearly, if X and Y are nonempty subsets of L, then $R(X) \otimes R(Y) = R(X \wedge Y)$ and $R(X) \oplus R(Y) = R(X \vee Y)$.

Theorem 2.2. *If L is a distributive (resp. dual distributive) hyperlattice, then $L/\varepsilon^\star$ is so.*

Remark 2.3. The converse of Theorem 2.2, is not true. Because consider $(L, \vee, \wedge)$ as a non-distributive lattice. Then define $a \oplus b = L$, for all $a, b \in L$. clearly, $L/\varepsilon^\star = (0)$, is distributive, since $\forall a, b \in L, a\varepsilon^\star b, (a, b \in a \oplus b = L)$.

Theorem 2.4. *If P is a prime filter (resp. ideal), then $P/\varepsilon^\star$ is so.*

Lemma 2.5. *Let L and K be hyperlattices and $f : L \longrightarrow K$ be a good homomorphism.*
(i) If P is a prime ideal of L and f is onto, then
$f(P)$ is a prime ideal in K.
(ii)

Lemma 2.6. *Let L be a hyperlattice and $\phi_L : L \longrightarrow L/\varepsilon^\star$ define by $\phi_L(x) = \varepsilon^\star(x)$, for all $x \in L$. Then ϕ_L is an onto good homomorphism and it is called canonical map.*

Theorem 2.7. *If P is a prime filter (resp. ideal) of L, then $\phi_L(P)$ is a prime filter (resp. ideal) of $L/\varepsilon^\star$ and $\phi_L(P) = P/\varepsilon^\star$.*

Is the converse of Theorem 2 true Precisely, is every prime filter (resp. ideal) Q in $L/\varepsilon^\star$ is to the form $P/\varepsilon^\star$, where P is a prime filter (resp. ideal) of L.
By Lemma 2.5, $(\phi_L)^{-1}(Q)$ is a prime filter of L. Let $(\phi_L)^{-1}(Q) = P$. Then $\phi_L(\phi_L)^{-1}(Q) = \phi(P)$. So, $Q = P/\varepsilon^\star$.

We know that $(\phi_L)^{-1}(Q) = \{x \in L \mid \varepsilon^\star(x) \in Q\}$. We define $\omega_{\phi_L} = (\phi_L)^{-1}(0) = \{x \in L \mid \varepsilon^\star(x) = 0 = \varepsilon^\star(0)\}$. It is clear that $\omega_{\phi_L} \subseteq (\phi_L)^{-1}(Q)$ where, Q is a prime ideal of $L/\varepsilon^\star$.

Theorem 2.8. [Correspondence Theorem] *There is a correspondence between the set all of prime ideals of L and the prime ideals of $L/\varepsilon^\star$ that contains ω_{ϕ_L}*

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Capability of groups satisfying a certain bound for the index of the center

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Abstract

It is shown that for a capable group G, the index of the center is bounded above by the function $|G'|^{2log_2|G'|}$. In this talk, we intend to determine the sufficient conditions for capability of a group G which satisfies this inequality.

Keywords: Capable group, Schur's theorem
Mathematics Subject Classification [2010]: 20B05, 20D25 , 20E34

1 Introduction

In 1938, Bare[1] initiated a systematic investigation of the question which conditions a group G must fulfill in order to be the group of inner automorphisms of some group E ($G \cong E/Z(E)$). Following M. Hall and Senior [5] such a group G is called *capable*. Baer classified capable groups that are direct sums of cyclic groups. His characterisation of finitely generated abelian groups that are capable is given in the following theorem.

Theorem 1.1. *[1]. Let G be a finitely generated abelian group written as $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, such that each integer $n_i + 1$ is divisible by n_i, where $\mathbb{Z}_0 = \mathbb{Z}$, the infinite cyclic group. Then G is capable if and only if $k \geqslant 2$ and $n_{k-1} = n_k$.*

In 1940, P. Hall [4] introduced the concept of isoclinism of groups, which is one of the most significant methods for classification of groups. He showed that capable groups play an important role in characterizing p-groups. Also, capability has interesting connections to other branch of group theory. So some authors studied different aspects of capable groups. One of the interesting aspects is finding a relation between the index of $Z(G)$ and the order of G' in a capable group G.

Understanding the relationship between $G/Z(G)$ and G' goes back at least to 1904 when I. Schur[11] proved that the finiteness of $G/Z(G)$ implies the finiteness of G'. Infinite extra-special p-groups show that the converse of Schur's theorem does not hold in general. Isaacs [6] proved that if G is a finite capable group, then $|G/Z(G)|$ is bounded above by a function of $|G'|$. Podoski and Szegedy [8] extended Isaac's result and gave the following explicit bound for the index of the center in a capable group.

*Speaker

Theorem 1.2. *If G is a capable group and $|G'| = n$, then $|G/Z(G)| \leq n^{2log_2 n}$.*

Now, one should notice that the extra-special p-groups of order p^3 and exponent p^2 satisfy the inequality, but these groups are not capable. Therefore, the inequality $|G/Z(G)| \leq |G^{'}|^{2log_2|G^{'}|}$ is a necessary condition for capability of groups with finite derived subgroup, whereas it is not a sufficient condition.

Definition 1.3. Let χ denote a class of groups satisfying the inequality $|G/Z(G)| \leq |G^{'}|^2$.

It is clear that, each group in the class χ has the necessary condition for capability. Now, we intend to determine the sufficient conditions for capability of some groups belong to the class χ.

2 Main results

In this section, we introduce three subclasses of groups which belong to the class χ. Then, we intend to determine capable groups among them.

Theorem 2.1. *[7, Theorem A] Let G be a finite non-abelian group with all Sylow subgroups abelian. Then $|G/Z(G)| < |G^{'}|^2$.*

The first subclass of desirable capable groups is as follows.

Theorem 2.2. *Let G be a finite group with all Sylow subgroups abelian. If $G/G^{'}$ is a capable group, then so is G.*

Example 2.3. Let $G = (\oplus_1^t \mathbb{Z}_p) \ltimes \mathbb{Z}_q$, where p and q are two distinct prime and $t \geqslant 2$. Using Lemma 2.2, one can see that G is a capable group.

Theorem 2.4. *Let G be a soluble group all of whose Sylow subgroups are abelian and the smallest term of the lower central series of G is abelian. If the system normalizer of G is capable, then G too is capable.*

The second subclass of the class χ is obtained by the following theorem.

Theorem 2.5. *[3] Let G be a group such that $G^{'}$ is finite and $\phi(G) = 1$. Then $|G/Z(G)| \leq |G^{'}|^2$.*

Theorem 2.6. *Let G be a group such that $G^{'}$ is finite and $\phi(G) = 1$. If $G/G^{'}$ is a capable group, then so is G.*

Theorem 2.7. *Let G be a finite group with the abelian derived subgroup and $\phi(G) = 1$. If the complement of $G^{'}$ is capable, then G is capable.*

Beyl *et al.* [2] proved that every finite group G having a cyclic normal subgroup of order m with cyclic factor group of order n has a presentation

$$G(m, n, r, s) = \langle x, y; x^m = 1, y^{-1}xy = x^r, y^n = x^s \rangle,$$

where r and s are positive integer satisfying $r^n \equiv 1 \ (mod\ m)$ and $(m, 1+r+\ldots+r^{n-1}) \equiv 0 \ (mod\ s)$. They also described a finite capable metacyclic group $G(m, n, r, s)$ as follows.

Theorem 2.8. *[2, Corollary 9.3] The group $G(m, n, r, s)$ is capable if and only if $s = m$ and n is the smallest positive integer satisfying $1 + r + \ldots + r^{n-1} \equiv 0$ (mod m).*

The third subclass of the class χ is obtained by the following theorem.

Theorem 2.9. *[7, Theorem B] Let G be a finite non-abelian group such that G/G' is cyclic. Then $|G/Z(G)| < |G^{'}|^2$.*

The exact structure of some groups with cyclic abelianization is given in the following lemmas.

Theorem 2.10. *[10, 10.26] Let G be a finite group such that all Sylow subgroups of G are cyclic. Then $G^{'}$ and $G/G^{'}$ are both cyclic. So that G is metacyclic, G splits over $G^{'}$, and $G^{'}$ is a Hall subgroup of G.*

Theorem 2.11. *[9, 10.1.10] If G is a finite group such that all of whose Sylow subgroups are cyclic, then G has a presentation*

$$\langle a, b | a^m = 1 = b^n, b^{-1}ab = a^r \rangle,$$

where $r^n \equiv 1$ (mod m), m is odd, $0 \leq r < m$, and m and $n(r-1)$ are coprime. Conversely in a group with such a presentation all Sylow subgroups are cyclic.

Thus, the set of finite groups with all Sylow subgroups cyclic is a subclass of the class χ. Moreover, using Lemmas 2.8 and 2.11, one can describe the exact structure of capable groups in this set.

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Characterizations of interior hyperideals of semihypergroups towards fuzzy points

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Abstract

Using a generalized version of the notion of quasi-coincidence of a fuzzy point, we discuss on a generalization of $(\in, \in \vee q)$-fuzzy interior hyperideal, called $(\in, \in \vee q_k)$-fuzzy interior hyperideal in a semihypergroup. Also, we characterize this notion in different ways. Specially, by using a fuzzy subset of a semihypergroup, we discuss on the generated $(\in, \in \vee q_k)$-fuzzy interior hyperideal.

Keywords: Semihypergroup, Interior hyperideal, Quasi-coincidence, $(\in, \in \vee q_k)$-fuzzy interior hyperideal
Mathematics Subject Classification [2010]: 20N20, 08A72

1 Preliminaries and Notations

In this section, for the purpose of reference, we present some definitions and results about semihypergroups and fuzzy sets on which our research in this paper is based.

A *hypergroupoid* [1] is a non-empty set S together with a map $\cdot : S \times S \longrightarrow \mathcal{P}^*(S)$ where $\mathcal{P}^*(S)$ denotes the set of all the non-empty subsets of S. The image of the pair (x, y) is denoted by $x \cdot y$. If $x \in S$ and A, B are non-empty subsets of S, then $A \cdot B$ is defined by $A \cdot B = \cup_{a \in A, b \in B} a \cdot b$. Also $A \cdot x$ is used for $A \cdot \{x\}$ and $x \cdot A$ for $\{x\} \cdot A$. A hypergroupoid $(S, \cdot)$ is called a *semihypergroup* if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in S$. A non-empty subset $\mathcal{I}$ of a semihypergroup S is called a *subsemihypergroup* if $\mathcal{I} \cdot \mathcal{I} \subseteq \mathcal{I}$. A subsemihypergroup $\mathcal{I}$ of a semihypergroup S is called *interior hyperideal* if, for all $x, y \in S$ and $a \in \mathcal{I}$, we have $x \cdot a \cdot y \subseteq \mathcal{I}$. Let S and S' be semihypergroups. A function $f : S \longrightarrow S'$ is called a *homomorphism* if it satisfies the condition $f(x \cdot y) = f(x) \cdot f(y)$, for all $x, y \in S$.

According to [6], a function $\mu : X \longrightarrow [0, 1]$ is called a *fuzzy subset* of X. Let f be a mapping from a set X to a set Y and μ, λ be fuzzy subsets of X and Y, respectively. Then the *homomorphic preimage* $f^{-1}(\lambda)$ and *homomorphic image* $f(\mu)$ are fuzzy sets in X and Y, respectively, defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$ and

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$ and $y \in Y$.

*Speaker

Definition 1.1. [3] Let S be a semihypergroup and μ a fuzzy subset of S. Then μ is said to be a *fuzzy interior hyperideal* of S if, for all $a, x, y \in S$, the following axioms hold:
(1) $\wedge_{z\in x\cdot y}\mu(z) \geq \mu(x) \wedge \mu(y)$,
(2) $\wedge_{z\in x\cdot a\cdot y}\mu(z) \geq \mu(a)$.

Theorem 1.2. *[3] Let μ be a fuzzy subset of a semihypergroup S. Then μ is a fuzzy interior hyperideal of S if and only if, for every $t \in (0, 1]$, each non-empty level subset $\mu_t = \{x \in S \mid \mu(x) \geq t\}$ of μ is an interior hyperideal of S.*

Let $x \in S$ and $t \in (0, 1]$. A fuzzy set μ of a semihypergroup S of the form

$$\mu(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$$

is said to be a *fuzzy point* [2] with support x and value t and is denoted by $[x; t]$. A fuzzy point $[x; t]$ is said *to belong to* (resp. *to be quasicoincident with*) a fuzzy subset μ, written as $[x; t] \in \mu$ (resp. $[x; t]q\mu$), if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $[x; t] \in \mu$ or $[x; t]q\mu$, then we write $[x; t] \in \vee q\mu$. We write $[x; t]\overline{\alpha}\mu$, if $[x; t]\alpha\mu$ does not hold, for $\alpha \in \{\in, q, \in \vee q\}$.

Let $t \in (0, 1]$ and $k \in [0, 1)$. For a fuzzy point $[x; t]$ and a fuzzy subset μ of a semihypergroup S, we write $[x; t]q_k\mu$, if $\mu(x) + t + k > 1$ and $[x; t] \in \vee q_k\mu$, if $[x; t] \in \mu$ or $[x; t]q_k\mu$. We write $[x; t]\underline{q}\mu$ if $\mu(x) + t \geq 1$, $[x; t]\underline{q}_k\mu$ if $\mu(x) + t + k \geq 1$ and $[x; t]\overline{\alpha}\mu$ if $[x; t]\alpha\mu$ does not hold, for $\alpha \in \{q_k, \underline{q}_k, \in \vee q_k\}$.

Definition 1.3. [4] Let μ be a fuzzy subset of a semihypergroup S and $t \in (0, 1]$ and $k \in [0, 1)$. Then the set $\underline{Q}(\mu; t) := \{x \in S \mid [x; t]\underline{q}\mu\}$ is called *closed q- level subset* of S, the set $Q_k(\mu; t) := \{x \in S \mid [x; t]q_k\mu\}$ is called the *q_k-level subset* of S, the set $\underline{Q}_k(\mu; t) := \{x \in S \mid [x; t]\underline{q}_k\mu\}$ is called *closed q_k-level subset* of S, the set $U_k(\mu; t) := \{x \in S \mid [x; t] \in \vee q_k\mu\}$ is called $(\in \vee q_k)$*- level subset* of S, and the set $\underline{U}_k(\mu; t) := \{x \in S \mid [x; t] \in \vee \underline{q}_k\mu\}$ is called *closed* $(\in \vee q_k)$*-level subset* of S.

2 Main Results

In what follows let S denote a semihypergroup and k an arbitrary element of $[0, 1)$ unless otherwise specified. In this section, we concentrate on the notion of $(\in, \in \vee q_k)$-fuzzy interior hyperideal and give various characterizations of it.

Definition 2.1. [5]A fuzzy subset μ of S is called an $(\in, \in \vee q_k)$*-fuzzy interior hyperideal* of S if, for any $x, a, y \in S$ and $t, r \in (0, 1]$

(I1) $[x; t] \in \mu$, $[y; r] \in \mu \Longrightarrow [z; t \wedge r] \in \vee q_k\mu$, for all $z \in x \cdot y$

(I2) $[a; t] \in \mu \Longrightarrow [w; t] \in \vee q_k\mu$, for all $w \in x \cdot a \cdot y$

where $t \wedge r = \min\{t, r\}$.

Example 2.2. Let $S = \{a, b, c, d, e\}$. Then $(S, \cdot)$ is a semigroup, where $\cdot$ is defined by the Table 1.

It is a routine to check that $(S, \odot)$ is a semihypergroup where the hyperoperation $\odot$ is defined by $x \odot y = \{a, x \cdot c \cdot y, x \cdot d \cdot y\}$, for all $x, y \in S$. Now, if $\mu(a) = \mu(b) = \mu(d) = 0.9$, $\mu(c) = 0.8$ and $\mu(e) = 0.6$, then it is easy to verify that μ is an $(\in, \in \vee q_{0.6})$-fuzzy interior hyperideal of $(S, \odot)$.

Table 1: Tabl of Example 2.2

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	d	d	e
e	a	a	c	c	e

Theorem 2.3. *[5]A fuzzy subset μ of S is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S if and only if the following conditions hold:*

1. $\wedge_{z\in x\cdot y}\mu(z) \geq \mu(x) \wedge \mu(y) \wedge \frac{1-k}{2}$, *for all $x, y \in S$;*
2. $\wedge_{z\in x\cdot a\cdot y}\mu(z) \geq \mu(a) \wedge \frac{1-k}{2}$, *for all $x, a, y \in S$.*

In the next theorem, we characterize $(\in, \in \vee q_k)$-fuzzy interior hyperideals based on $\in$-level sets.

Theorem 2.4. *A fuzzy subset μ of S is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S if and only if the set $\mu_t(\neq \emptyset)$ is an interior hyperideal of S, for all $t \in (0, \frac{1-k}{2}]$.*

We say that μ_t is an $\in$-*level interior hyperideal* of μ in S.

In the following theorem, we investigate some equivalent conditions for μ_t as an interior hyperideal.

Theorem 2.5. *For a fuzzy subset μ of S, the following assertions are equivalent:*

1. $\mu_t(\neq \emptyset)$ *is an interior hyperideal of S, for all $t \in (\frac{1-k}{2}, 1]$.*
2. μ *satisfies the following conditions:*

   (2.1) $\wedge_{z\in x\cdot y}(\mu(z) \vee \frac{1-k}{2}) \geq \mu(x) \wedge \mu(y)$, *for all $x, y \in S$.*

   (2.2) $\wedge_{z\in x\cdot a\cdot y}(\mu(z) \vee \frac{1-k}{2}) \geq \mu(a)$, *for all $x, a, y \in S$.*

In the next theorem, we characterize $(\in, \in \vee q_k)$-fuzzy interior hyperideals based on closed q_k-level sets.

Theorem 2.6. *Let μ be an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S. Then $\underline{Q}_k(\mu; t)(\neq \emptyset)$ is an interior hyperideal of S, for all $t \in (\frac{1-k}{2}, 1]$.*

Now, we characterize $(\in, \in \vee q_k)$-fuzzy interior hyperideals based on closed $\in \vee q_k$-level sets.

Theorem 2.7. *A fuzzy subset μ of S is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S if and only if $\underline{U}_k(\mu; t)(\neq \emptyset)$ is an interior hyperideal of S, for all $t \in (0, 1]$.*

Corollary 2.8. *A fuzzy subset μ of S is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S if and only if $U_k(\mu; t)(\neq \emptyset)$ is an interior hyperideal of S, for all $t \in (0, 1]$.*

We say that $U_k(\mu;t)$ is an $\in \vee q_k$*-level interior hyperideal* of μ in S.

In the next theorem, we investigate the behavior of $(\in, \in \vee q_k)$-fuzzy interior hyperideals under the homomorphisms of semihypergroups.

Theorem 2.9. *Let $f : S_1 \to S_2$ be a semihypergroup homomorphism and μ and λ be $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S_1 and S_2, respectively. Then:*

(i) $f^{-1}(\lambda)$ is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S_1.

(ii) If f is onto and μ is f-invariant ($f(x) = f(y)$ implies that $\mu(x) = \mu(y)$), then $f(\mu)$ is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S_2.

Theorem 2.10. *For any chain $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = S$ of interior hyperideals of S there exists an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S whose $\in$-level interior hyperideals are precisely the members of the chain.*

Definition 2.11. Let S be a semihypergroup and X a subset of S. Let $\{H_i\}_{i\in I}$ be the family of all subsemihypergroups of S which contain X. Then $\cap_{i\in I} H_i$ is called the subsemihypergroup of S *generated* by the set X and denoted by $\langle X\rangle$. If $X = \{a_1, a_2, \cdots, a_n\}$, we write $\langle a_1, a_2, \cdots, a_n\rangle$ in place of $\langle X\rangle$. If $a_1, a_2, \cdots, a_n \in S$ and $S = \langle a_1, a_2, \cdots, a_n\rangle$, S is said to be *finitely generated.* If $a \in S$ and $S = \langle a\rangle$, then S is said to be *cyclic*. It is not difficult to see that, for every $a \in S$, $\langle a\rangle = \{a\} \cup a^2 \cup a^3 \cup \cdots$.

Theorem 2.12. *Let S be a semihypergroup and assume that there exists an element $a \in S$ such that $S = \langle a\rangle$. If μ is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of S such that $\mu(a) \geq \frac{1-k}{2}$, then $\mu(x) \geq \frac{1-k}{2}$, for all $x \in S$.*

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Class preserving automorphisms of finite p-groups

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Abstract

Let G be a finite non-abelian p-group and $\mathrm{Aut}_c(G)$ denote the group of all class preserving automorphisms of G. In this paper, using the notion of Frattinian groups, we give necessary condition for finite p-groups G for the groups $\mathrm{Aut}_c(G)$ and $\mathrm{Inn}(G)$ coincide when $(G, Z(G))$ is a Camina pair.

Keywords: automorphism, p-group, Class preserving
Mathematics Subject Classification [2010]: 20D45, 20D15, 20D25

1 Introduction

Let G be a finite p-group. For $x \in G$, x^G denotes the conjugacy class of x in G. By $\mathrm{Aut}(G)$ we denote the group of all automorphisms of G. An automorphism α of G is called class preserving if $\alpha(x) \in x^G$ for all $x \in G$. We let $\mathrm{Aut}_c(G)$ denote the set of all class preserving automorphisms of G. The group $\mathrm{Aut}_c(G)$ have been studied by several authors, see for example [3, 4, 10], [12, 13] . It is well known that if G is a finite p-group, then so is the group $\mathrm{Aut}_c(G)$, In this paper we study closely the groups $\mathrm{Aut}_c(G)$ for a finite non-abelian p-group G. We give necessary condition for finite p-groups G for the groups $\mathrm{Aut}_c(G)$ and $\mathrm{Inn}(G)$ coincide when $(G, Z(G))$ is a Camina pair. Throughout the paper all groups are assumed to be finite groups.

2 Main results

In this section we give some known results which will be used in the rest of the paper.

Let G be a finite p-group. Following Schmid, we call G Frattinian provided $Z(G) \neq Z(M)$ for all maximal subgroups M of G. In [11], P. Schmid proved the following structural theorem for the Frattinian groups.

Theorem 2.1 ([11])**.** *Suppose G is a non-abelian Frattinian p-group. Then one of the following holds:*

(i) G is the central product of non-abelian p-groups of order $p^2|Z(G)|$, amalgamating their centres.

*Speaker

(ii) $G = E{*}F$ *is the central product of Frattinian subgroups* E *and* F *with* $C_F(Z(\Phi(F))) = \Phi(F)$*,* $E = C_G(F)$ *and* $\Phi(E) \le Z(G)$*.*

It is worth noting that in case (i) of the above theorem the factors of the central product are minimal non-abelian p-groups. Accordingly, in this case we have $Z(G) = \Phi(G)$. Also in (ii) either $E = Z(G)$ (and therefore $G = F$) or E is a central product as in (i).

Camina groups were introduced by A.R. Camina in [2] and were studied in past (see for example [5, 6, 7, 8, 9]). Let G be a finite p-group and N be non-trivial proper normal subgroup of G. Then (G, N) is called a *Camina pair* if $xN \subseteq x^G$ for all $x \in G - N$, where x^G denotes the conjugacy class of x in G. It follows that (G, N) is a Camina pair if and only if $N \subseteq [x, G]$ for all $x \in G - N$, where $[x, G] = \{[x, g] | g \in G\}$.
We start with a result of I. D. Macdonald [6].

Theorem 2.2 ([6], Theorem 2.2)**.** *Let* (G, H) *be a Camina pair, let* $H = Z(G)$*, and let* G *have class* c*. Then* $Z_r(G)/Z_{r-1}(G)$ *has exponent* p *whenever* $1 \le r \le c$*.*

Theorem 2.3. *Let* G *be a finite* p*-group such that* $(G, Z(G))$ *is a Camina pair and* $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$*. Then one of the following holds.*

(i) G *is extraspecial.*

(ii) $Z_2(G)$ *is abelian and* $C_G(Z_2(G)) = \Phi(G)$*.*

(iii) $G = EF$*, where* $E = C_G(F)$*,* $Z_2(F) \le \Phi(F)$*,* $Z_2(F)$ *is abelian,* $C_G(Z_2(F)) = \Phi(F)$ *and both* E, F *are Frattinian* p*-groups. Moreover* $E = E_1...E_s, |E_i| = p^3$ *for all* $1 \le i \le s$*.*

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Congruence on a ternary monoid generated by a relation

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Abstract

In this paper we define the notion of congruence on a ternary monoid generated by a relation and we determine the method of obtaining a congruence on a ternary monoid T from a relation R on T. Making of congruences is important because we can gain new ternary monoid from them.

Keywords: Ternary monoid, Relation, Congruence
Mathematics Subject Classification [2010]: 20M99

1 Introduction

The theory of ternary algebraic systems was introduced by D. H. Lehmer [3] in 1932, but before that (1904) such structures were studied by E. Kanser [2] who gave the idea of n-ary algebras. Lehmer studied certain ternary algebraic systems called triplexes, commutative ternary groups, in fact. Ternary structures and their generalization, the so called n-ary structures, are outstanding for their application in physics. The notion of congruence was first introduced by Karl Fredrich Gauss in the beginning of the nineteenth century. Congruences are a special type of equivalence relations which play a vital role in the study of quotiont structures of different algebraic structures. In this paper we define the notion of congruence on a ternary monoid generated by a relation and we determine the method of obtaining a congruence on a ternary monoid T from a relation R on T. Making of congruences is important because we can gain new ternary monoid (in fact quotiont monoids) from them. The first we express some primary notions.

Definition 1.1. A non-empty set T is called a ternary semigroup if there exists a ternary operation $T \times T \times T \to T$, written as $(a, b, c) \to abc$ satisfying the following statement: $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in T$.

Definition 1.2. An element e of a ternary semigroup T is called,
(i) a left identity (left unital element) if $eex = x$ for all $x \in T$;
(ii) a right identity (right unital element) if $xee = x$ for all $x \in T$;
(iii) a lateral identity (lateral unital element) if $exe = x$ for all $x \in T$;
(iv) a two-sided identity (bi-unital element) if $eex = xee = x$ for all $x \in T$;
(v) an identity (unital element) if $eex = exe = xee = x$ for all $x \in T$.

*Speaker

Remark 1.3. There is no need any ternary semigroup to have unique identity. For example $\mathbb{Z}$, the set of all integers, with usual ternary multiplication of integers is a ternary semigroup and both of 1 and -1 are identity elements of $\mathbb{Z}$.

Definition 1.4. A ternary semigroup T is called a ternary monoid if it has an identity.

Example 1.5. $\{\bar{0}, \bar{1}, \bar{5}\} \subseteq \mathbb{Z}_{30}$ with ternary multiplication of $\mathbb{Z}_{30}$ is a ternary monoid

Definition 1.6. Let R be a relation on a set X. Then the smallest equivalence on X containing R (the intersection of all equivalence relations on X containing R) is called the equivalence relation on X generated by R and it is denoted by R^e.

Definition 1.7. Let S be a reflexive relation on a set X. Then we denote $\cup_{n\geq 1} S^n$ by S^∞ and we call it the transitive closure of the relation S.

Proposition 1.8. *For every relation R on a set X, $R^e = (R \cup R^{-1} \cup 1_X)^\infty$.*

Corollary 1.9. *Let R be a relation on a set X. Then $(x, y) \in R^e$ if and only if either $x = y$ or for some $n \in \mathbb{N}$, there is a sequence $x = z_1, z_2, \cdots, z_{n-1}, z_n = y$ of elements of T such that, for each $i \in \{1, 2, \cdots, n-1\}$, either $(z_i, z_{i+1}) \in R$ or $(z_{i+1}, z_i) \in R$.*

2 Main results

In this section we try to obtain a congruence on a ternary monoid T from a relation R on T.

Definition 2.1. A relation ρ on a ternary monoid T is said to be,
(i) a left compatible relation if for every $a, b \in T$, $a\rho b$ implies $at_1t_2 \,\rho\, bt_1t_2$ for all $t_1, t_2 \in T$;
(ii) a right compatible relation if for every $a, b \in T$, $a\rho b$ implies $t_1t_2a\rho t_1t_2b$ for all $t_1, t_2 \in T$;
(iii) a lateral compatible relation if for every $a, b \in T$, $a\rho b$ implies $t_1at_2 \,\rho\, t_1bt_2$ for all $t_1, t_2 \in T$;
(iv) a compatible relation if for all $a, b, c, a', b', c' \in T$, $a\rho a', b\rho b', c\rho c'$ imply $abc \,\rho\, a'b'c'$.

Proposition 2.2. *Let T be a ternary monoid. Then every left and right compatible relation on T is a lateral compatible relation on T.*

Proposition 2.3. *Let R be a left (right, lateral) compatible relation on a ternary monoid T. Then R^n is a left (right, lateral) compatible relation on T for every $n \geq 1$.*

Definition 2.4. An equivalence relation ρ on a ternary monoid T is said to be a right (left, lateral) congruence if it is a right (left, lateral) compatible relation. Furthermore a compatible equivalence relation ρ on a ternary monoid T is called a congruence on T.

Proposition 2.5. *An equivalence relation ρ on a ternary monoid T is a congruence if and only if it is left, right and lateral congruence.*

Definition 2.6. Let R be a relation on a ternary monoid T. Then the smallest congruence on T containing R (the intersection of all congruences on T containing R) is called the congruence generated by R and it is denoted by $R^\#$.

Lemma 2.7. *Let T be a ternary monoid and let R be a relation on T. Then $R^c = \{(xay, xby) \mid x, y \in T, (a, b) \in R\}$ is the smallest left, right and lateral compatible relation on T containing R.*

Lemma 2.8. *Let R and S be two relations on a ternary monoid T. Then*
(1) $R \subseteq S \Rightarrow R^c \subseteq S^c$.
(2) $(R^{-1})^c = (R^c)^{-1}$.
(3) $(R \cup S)^c = R^c \cup S^c$.

Theorem 2.9. *For every relation R on a ternary monoid T, $R^{\#} = (R^c)^e$.*

Corollary 2.10. *Let R be a relation on a ternary monoid T and $a, b \in T$. Then $(a, b) \in R^{\#}$ if and only if either $a = b$ or for some $n \in \mathbb{N}$, there is a sequence $a = c_1, c_2, \cdots, c_{n-1}, c_n = b$ of elements of T such that, for each $i \in \{1, 2, \cdots n-1\}$, either $(c_i, c_{i+1}) \in R^c$ or $(c_{i+1}, c_i) \in R^c$.*

Proposition 2.11. *Let T be a ternary monoid and let E be an equivalence on T. Then*

$$E^{\flat} = \{(a, b) \in T \times T \mid (xay, xby) \in E \text{ for all } x, y \in T\}$$

is the largest congruence on T contained in E.

Example 2.12. Let T be a ternary monoid and A be a subset of T. Also let π_A be an equivalence on T whose classes are A and $T \setminus A$. Then $\pi_A^{\flat} = \{(a, b) \in T \times T \mid xay \in A \Leftrightarrow xby \in A \text{ for all x, } \ y \in T\}$.

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Decomposing modules into modules with local endomorphism rings

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Abstract

Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular lifting module that does not have relatively projection component. Then $M = \oplus_{i=1}^{n} M_i$ has the exchange property and the decomposition complements direct summands, where each endomorphism ring $End(M_i)$ is local.

Keywords: noncosingular module; lifting module; local endomorphism ring.
Mathematics Subject Classification [2010]: 16D10, 16D80.

1 Introduction

Throughout this paper R will denote an associative ring with identity. Modules over R will be right R-modules. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L+N \neq M$). $Rad(M)$ will denote the Jacobson radical of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M. M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. It is clear that every local module is hollow. A module M is called *lifting* if for every submodule $A \leq M$, there exists a direct summand B of M such that $B \leq A$ and $A/B \ll M/B$. Lifting modules are dual notions of extending modules and [3] deals with different aspects of lifting modules. A module M is amply supplemented and every coclosed submodule of M is a direct summand of M if and only if M is lifting by [3, 22.3(d)]. In [5] Talebi and Vanaja defined $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = \mathrm{Re}\,(\mathrm{M}, \mathcal{S}) = \bigcap\{\mathrm{Ker(g)} \mid \mathrm{g} \in \mathrm{Hom}\,(\mathrm{M}, \mathrm{L}), \mathrm{L} \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the class of all small modules. They called M a *cosingular* (*noncosingular*) module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$).

A family $\{X_\lambda : \lambda \in \Lambda\}$ of submodules of a module M is called a *local summand* of M, if $\sum_{\lambda\in\Lambda} X_\lambda$ is direct and $\sum_{\lambda\in F} X_\lambda$ is a summand of M for every finite subset $F \subseteq \Lambda$. If even $\sum_{\lambda\in\Lambda} X_\lambda$ is a summand of M, we say that *the local summand is a summand.* A module M is said to have the *(finite) exchange property* if for any (finite) index set I, whenever $M \oplus N = \oplus_{i\in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i\in I} B_i)$ for submodules $B_i \leq A_i$. Let $M = \oplus_I M_i$ be a decomposition of the module M into nonzero summands M_i. This decomposition is said to *complement direct summands* if, whenever A is a direct summand of M, there is a subset J of I for which $M = (\oplus_J M_j) \oplus A$.

In Section 2, we prove the following theorem:

Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular lifting module which has no relatively projection component. Then $M = \oplus_{i=1}^{n} M_i$, where each endomorphism ring $End(M_i)$ is local and the following statements satisfy:

(1) The decomposition complements direct summands.

(2) Every local summand of M is a summand.

(3) M has the exchange property.

(4) The radical factor ring $S/J(S)$ of the endomorphism ring S of M is von Neumann regular, and idempotents lift modulo $J(S)$.

2 Main results

Lemma 2.1. [1, Lemma 2.2] *Let $M = \oplus_{i=1}^{\infty} M_i$, where each M_i is local noncosingular. If, for each i, there is an epimorphism $f_i : M_i \longrightarrow M_{i+1}$ which is non-isomorphism, then M is not lifting.*

Proposition 2.2. *Let R be an arbitrary ring and M a noncosingular local module. If M is not noetherian, then there exists a countable family $\{N_i \mid i \in \mathbb{N}\}$ of non-noetherian images of M such that $\oplus_{i\in\mathbb{N}} N_i$ is not lifting.*

Recall that a family of modules $\{M_i \mid i \in I\}$ is called *(locally) semi-T-nilpotent* if, for any countable set of non-isomorphisms $\{f_n : M_{i_n} \to M_{i_{n+1}}\}_{\mathbb{N}}$ with all i_n distinct in I, (and for any $x \in M_{i_1}$), there exists $k \in \mathbb{N}$ (depending on x) such that $f_k \dots f_1 = 0$ ($f_k \dots f_1(x) = 0$). It is obvious that if each M_i is a local module, then the family $\{M_i \mid i \in I\}$ of modules is locally semi-T-nilpotent if and only if it is semi-T-nilpotent.

Theorem 2.3. *Let $M = \oplus_{i=1}^{\infty} M_i$ with M_i local noncosingular and M_j-projective whenever $j \neq i$. If M is a lifting module, then:*

(1) $\{M_i\}$ is locally semi-T-nilpotent.

(2) M is quasi-discrete.

(3) $Rad(M) \ll M$.

(4) The decomposition $M = \oplus_{i=1}^{\infty} M_i$ complements summands.

Recall that a module M is said to be *Hopfian* if any epimorphism is an isomorphism.

Lemma 2.4. *Let R be a right artinian ring or a perfect commutative ring. Then every noncosingular hollow R-module M has a local endomorphism ring.*

A module M is said to have *finite hollow dimension* if there exists an epimorphism from M to a finite direct sum of n hollow factor modules with small kernel.

Theorem 2.5. [1, Theorem 2.1] *Let R be a right perfect ring. Let M be a noncosingular lifting module that does not have relatively projective component. Then M has finite hollow dimension.*

Theorem 2.6. *Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular lifting module that does not have relatively projection component. Then $M = \oplus_{i=1}^{n} M_i$, where each endomorphism ring $End(M_i)$ is local and the following statements satisfy:*

(1) *The decomposition complements direct summands.*
(2) *Every local summand of M is a summand.*
(3) *M has the exchange property.*
(4) *The radical factor ring $S/J(S)$ of the endomorphism ring S of M is von Neumann regular, and idempotents lift modulo $J(S)$.*

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Talk Divisibility Graph for some finite simple groups pp.: 1–3

Divisibility Graph for some finite simple groups*

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Abstract

The *divisibility graph* of a finite group G has vertex set the conjugacy class sizes of non-central elements in G and two vertices are adjacent if one divides the other. We determine the connected components of the divisibility graph of the finite simple groups of Lie type over a finite field of odd characteristic.

Keywords: Conjugacy class, Divisibility graph, Finite simple group, Prime graph.
Mathematics Subject Classification [2010]: 05C25, 20D05

1 Introduction

In [3] the *divisibility graph* which is related to a set of positive integers have been introduced. The divisibility graph, $\overrightarrow{\mathrm{D}}(X)$ is a graph with vertex set $X^* = X \setminus \{1\}$ and there is an arc between two vertices a and b if and only if a divides b. It is also asked for the structure and especially the number of connected components of this graph (see [3, Question 7]).

Let G be a finite group and $\mathrm{cs}(G)$ denotes the set of conjugacy class sizes of non-central elements in G. We show the underlying graph of $\overrightarrow{\mathrm{D}}(\mathrm{cs}(G))$ by $\mathrm{D}(G)$ without changing the name for convenience. Actually by the *divisibility graph* $\mathrm{D}(G)$ we mean a graph with vertex set $\mathrm{cs}(G)$ and two conjugacy class sizes are adjacent if one divides the other.

In [1], The structure of divisibility graph $\mathrm{D}(G)$, where G is a symmetric group or an alternating group is studied.

Theorem 1.1. *[1, Corollary 11] $D(S_n)$ has at most two connected components. If it is disconnected then one of its connected components is K_1.*

Theorem 1.2. *[1, Corollary 17] $D(A_n)$ has at most three connected components. If it is disconnected, then two of its connected components are K_1.*

Also in [2], the structure of divisibility graphs for $\mathrm{PSL}(2,q)$, $\mathrm{Sz}(q)$ and 26 sporadic simple groups have been described.

Theorem 1.3. *[2, Theorem 2.1] Let $G = \mathrm{PSL}(2,q)$. Then $\mathrm{D}(G)$ is either $3\mathrm{K}_1$ or $\mathrm{K}_2{+}2\mathrm{K}_1$.*

Theorem 1.4. *[2, Theorem 2.2] Let $G = \mathrm{Sz}(q)$. Then $\mathrm{D}(G) = \mathrm{K}_2 + 3\mathrm{K}_1$.*

*Will be presented in English
†Speaker

Here we are interested in determining the divisibility graph for other finite simple groups. The classification theorem of finite simple groups is well known.

Theorem 1.5. *[5, p. 6] Every finite simple group is one of the following:*

1) *a cyclic group $\mathbb{Z}_p$ of prime order p,*

2) *an alternating group A_n for $n \geq 5$,*

3) *a finite simple group of Lie type,*

4) *one of 26 sporadic simple groups.*

In the next section we will study the structure of divisibility graph for finite simple groups of Lie type.

2 Main results

For a finite group G let $\mathrm{cs}(G) = \{|x^G|;\ x \in G\} \setminus \{1\}$ denotes the set of conjugacy class sizes of non-central elements in G. Let $\mathrm{D}(G)$ denote the divisibility graph of G, which is a graph with vertex set $\mathrm{cs}(G)$ and edge set $E(G) = \{\{|x^G|, |y^G|\}$: either $|x^G|$ divides $|y^G|$ or $|y^G|$ divides $|x^G|\}$.

For two arbitrary elements $x, y \in G$, we say x is *equivalent* to y whenever $|x^G|$ and $|y^G|$ are in the same connected component of $\mathrm{D}(G)$.

We now reintroduce a well known graph, namely the *prime graph.* The vertex set of the prime graph of a finite group G, $\rho(G)$, is the set of primes dividing the order of the group and two vertices r and s are adjacent if and only if G contains an element of order rs. Williams [6, Lemma 6] investigated prime graphs of finite simple groups.

From now on let G be a finite simple group of Lie type over a finite field $\mathbb{F}_q$ in characteristic p where p is an odd and good prime, that is (see [4, p. 28])

1. $p \neq 2$ when G has type $A_\ell, {}^2A_\ell,\ B_\ell,\ C_\ell,\ D_\ell, {}^2D_\ell$,

2. $p \notin \{2, 3\}$ when G has type $G_2, F_4, E_6, {}^2E_6, E_7$,

3. $p \notin \{2, 3, 5\}$ when G has type E_8.

Lemma 2.1. $\mathrm{D}(\mathrm{PSL}(3, q))$ *and* $\mathrm{D}(\mathrm{PSU}(3, q^2))$ *are as Figure 1. In this figure* $\delta = 1$ *for* $G = \mathrm{PSL}(3, q)$ *and* $\delta = -1$ *for* $G = \mathrm{PSU}(3, q)$, $r = q - \delta$, $s = q + \delta$, $t = q^2 + \delta q + 1$, $r' = r/\gcd(3, r)$, *and* $t' = t/\gcd(3, r)$.

In the rest of the paper we assume G is not the groups $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$ and $\mathrm{PSU}(3, q^2)$.

Lemma 2.2. *All unipotent elements of G are equivalent.*

Lemma 2.3. *Every involution is equivalent to a unipotent element.*

Lemma 2.4. *for a semisimple element $s \in G$ two possibilities may arise:*

- *s is equivalent to a an involution.*

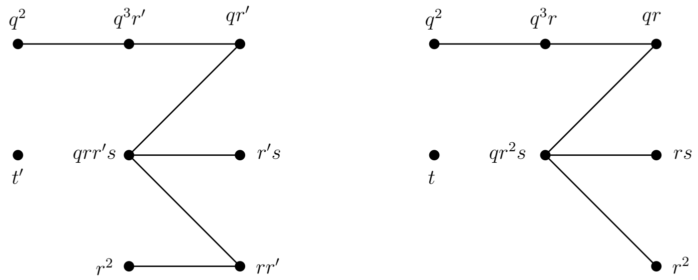

Figure 1: The Divisibility Graph for $\mathrm{PSL}(3,q)$ and $\mathrm{PSU}(3,q^2)$ (left: $\gcd(3,r) \neq 1$ right: $\gcd(3,r) = 1$).

- *s is not equivalent to an involution. In this case, the maximal torus containing s, namely T, is an isolated Hall subgroup of G and the conjugacy classes of all elements of T have the same length. So there is only one isolated vertex in $\mathrm{D}(G)$ related to all elements T. In this case the prime divisors of $|T|$ make a connected component of $\rho(G)$ not containing 2.*

Now we give our main result in the following theorem which shows the relation between the divisibility graph and the prime graph.

Theorem 2.5. *Let G be a finite simple group of Lie type over a finite field $\mathbb{F}_q$ in characteristic p where p is an odd and good prime. Then the divisibility graph $\mathrm{D}(G)$ is either connected or at most one of its connected components is not an isolated vertex. Moreover the number of connected components of $\mathrm{D}(G)$ is equal to the number of $\rho(G)$.*

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Talk Domination number of the order graph of a group pp.: 1–3

Domination number of the order graph of a group

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Abstract

The order graph of a group G, denoted by $\Gamma^*(G)$, is a graph whose vertices are non-trivial subgroups of G and two distinct vertices H and K are adjacent if and only if $|H|\big||K|$ or $|K|\big||H|$. In this paper, we study the domination number of this graph.

Keywords: Order graph, Domination number, Perfect group
Mathematics Subject Classification [2010]: 20A05, 05C25

1 Introduction

Let G be a finite group. The order graph of G is the (undirected) graph $\Gamma^*(G)$, whose vertices are non-trivial proper subgroups of G and two distinct vertices H and K are adjacent if and only if either $|H|\big||K|$ or $|K|\big||H|$. So $\Gamma^*(G)$ is the empty graph if and only if $|G|$ is a prime number. This graph has studied in [8] and [4]. In this paper, we study the domination number of this graph.

First we recall some facts and notations related to this paper. Throughout this paper G denotes a nontrivial finite group. Let $\pi(n)$ be the set of prime divisors of n. We denote $\pi(|G|)$ by $\pi(G)$. The cyclic group of order n is denoted by C_n. The symmetric group on n letters is denoted by S_n. D_n is the dihedral group of order $2n$. The alternative group is denoted by A_n. The finite field with q elements is denoted by $\mathbb{F}_q$.

Let Γ be a simple graph with vertex set V. A subset S of V is called a dominating set if every vertex in $V\backslash S$ has a neighbor in S. The minimum size of the dominating sets is called domination number and is denoted by $\gamma(\Gamma)$. We denote $\gamma(\Gamma^*(G))$ by $\gamma(G)$.

2 Main results

In this section we state and prove our main results.

Theorem 2.1. *Let S be a set of subgroups of G such that for each prime $p \in \pi(G)$ there is only one subgroup P of order p in S. Then S is a dominating set.*

Proof. Let H be a subgroup of G. Let p be a prime factor of $|H|$. If P is a subgroup of order p in S then $H = P$ or H is adjacent to P. □

Corollary 2.2. *The domination number of the order graph of G is at most $|\pi(G)|$.*

*Speaker

Theorem 2.3. *Let S be a set of maximal subgroups of G such that for each maximal subgroup M of G there is only one subgroup $M_1 \in S$ such that $|M| = |M_1|$. Then S is a dominating set.*

Proof. Let H be a proper subgroup of G. So $H \leq M$ for a maximal subgroup M of G. Since $|M| = |M_1|$ for some $M_1 \in S$, so $H = M_1$ or H is adjacent to M_1. □

Theorem 2.4. *G is a $p-$group if and only if $\gamma(G) = 1$.*

Proof. First assume that $\gamma(G) = 1$ and $|G| = n = p_1^{n_1} \cdots p_k^{n_k}, k \geq 2$. Let H be a subgroup which is adjacent to other vertices. If $p \in \pi(G)$ then $p \big| |H|$. Since every Sylow subgroup is adjacent to H, so $p_i^{n_i} \big| |H|$. Hence $|G| = |H|$ which is a contradiction. Thus $|\pi(G)| = 1$ i.e G is a $p-$group. Conversely, If G is a $p-$group then $\Gamma^*(G)$ is a complete graph. So $\gamma(G) = 1$. □

Corollary 2.5. *If $|\pi(G)| = 2$ then $\gamma(G) = 2$.*

Theorem 2.6. *If G is not a $p-$group and G has a subgroup H of a prime power index then $\gamma(G) = 2$.*

Proof. Assume $[G : H] = p^k$ and P is a subgroup of order p. Let K be a subgroup of G. If $p \big| |K|$ then K is adjacent to P. If $p \nmid |K|$ then $(p, |K|) = 1$. Hence $|K| \big| |H|$. So K is adjacent to H and the proof is complete. □

Remark 2.7. The groups with a prime power index subgroup are studied in many papers, see [1],[2],[6]. One of the main theorems is the Burnside's theorem which states that if $[G : C_g(a)] = p^k > 1$ then G is not a simple group. By using classification theorem of simple groups, Guralnick has classified all simple groups with a prime power index subgroup in [6].

Corollary 2.8. *If G is not a $p-$group and $G \neq G'$ i.e G is not a perfect group then $\gamma(G) = 2$.*

Proof. Since G/G' is a nontrivial abelian group, so it has a subgroup of prime index. Thus $\gamma(G) = 2$ by Theorem 2.6. □

Corollary 2.9. *If G is not a $p-$group and G is a solvable group then $\gamma(G) = 2$.*

Corollary 2.10. *If $n \geq 3$ then $\gamma(S_n) = 2$.*

Corollary 2.11. *If F is a finite field then $\gamma(GL_n(F)) = 2$.*

Remark 2.12. Let $G = A_5$. Let $H \cong A_4$ and $K \cong D_5$ be subgroups of order 12 and 10. Then $S = \{H, K\}$ is a dominating set by Theorem 2.3. So $\gamma(\Gamma^*(A_5)) = 2 < |\pi(A_5)| = 3$.

Question. Determine all the groups G such that $\gamma(G) = |\pi(G)|$.

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Talk   Extended annihilating-ideal graph of a ring   pp.: 1–3

Extended annihilating-ideal graph of a ring

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Abstract

In this paper we extend the concept of annihilating-ideal graph of a commutative ring and then we characterize commutative Artinian local ring whose Extended annihilating-ideal graph is star graph.

Keywords: Annihilating-ideal graph, Extended annihilating-ideal graph
Mathematics Subject Classification [2010]: 13A15, 13E15, 05C75

1 Introduction

A graph (simple graph)G is an ordered pair of disjoint sets (V, E) such that $V = V(G)$ is the vertex set of G and $E = E(G)$ is its edge set. If the graph G contains a vertex, say v, to which all other vertices are joined and has no other edges, it is called a star graph with center v.
Throughout this paper, all rings R are assumed to be commutative with identity 1_R. For a ring R, let $I(R)$ be the set of ideals of R, $A(R)$ the set of annihilating-ideals of R, where a nonzero ideal I of R is called an annihilating-ideal if there exists a non-zero ideal J of R such that $IJ = 0$. The annihilating-ideal graph $AG(R)$ of R is a simple graph with vertex set $A(R)$, such that distinct vertices I and J are adjacent if and only if $IJ = 0$. Annihilating-ideal graphs of rings, first introduced and studied in [3], provide an excellent setting for studying some aspects of algebraic property of a commutative ring, especially, the ideal structure of a ring. Some fundamental results on the concept have been established in [1, 3]. For example, $AG(R)$ is always a simple, connected and undirected graph with diameter less than four; if $AG(R)$ contains a cycle, then its girth is less than five; if R is a non-domain ring, then $AG(R)$ is a finite graph if and only if R has finitely many ideals, if and only if every vertex of $AG(R)$ has finite degree. In this paper we extend the concept of annihilating-ideal graph of a ring and then we characterize commutative Artinian local rings whose Extended annihilating-ideal graph is star graph.

2 Artinian local ring and Extended annihilating-ideal graph

In this section we first extend the concept of annihilating-ideal graphs of a ring and then we state some properties of this graph.

*Speaker

Definition 2.1. Extended annihilating-ideal graph $AG^*(R)$ of R is a (not necessarily simple) graph with vertex set $A(R)$, such that vertices I and J (not necessarily distinct) are adjacent if and only if $IJ = 0$.

Definition 2.2. A local ring (R, m) is called a special product of almost prime ideals ring (abbreviated, SPAP-ring), if for each $x \in m - m^2$, $(x^2) = m^2$ and $m^3 = 0$.

SPAP-rings were introduced in [2]. D. D. Anderson and Malik Bataineh in [2] characterize Noetherian rings whose proper ideals are a product of almost prime ideals. Thus almost prime ideals play an important role in commutative algebra.

Lemma 2.3. *Let (R, m) be an SPAP-ring with $m^2 \neq 0$. Then m^2 is a minimal ideal of R.*

Proof. If $m = m^2$, then $m^2 = m^3 = 0$, a contradiction. Therefore $m \neq m^2$, thus there exists $y \in m - m^2$. So $m^2 = (y^2)$. Thus m^2 is a cyclic R-mod and therefore it is a multiplication R-module. Now if J is a submodule (ideal of R) of m^2, there exists an ideal K of R, such that $J = Km^2$. If $K = R$ then $J = m^2$ and if $K \neq R$ then $J = Km^2 \subseteq m^3 = 0$, hence $J = 0$. Therefore m^2 is a minimal ideal of R. □

Lemma 2.4. [4, *Lemma* 2.1] *Let (R, m) is an Artinian local ring such that $AG(R)$ is a star graph. If $m^s \neq 0$ and $m^{s+1} = 0$, where either $s = 2$ or $s = 3$, then m^s is the unique minimum nonzero ideal of R.*

Lemma 2.5. *Let (R, m) be an SPAP-ring such that $m^2 \neq 0$. If I is an ideal of R then, $I = 0$ or $I = m^2$ or $I^2 = m^2$.*

Proof. Let (R, m) be an SPAP-ring such that $m^2 \neq 0$. By lemma 2.3, m^2 is a minimal ideal. Now let I be a proper ideal of (R, m). If $I \subseteq m^2$ then, $I = 0$ or $I = m^2$. If $I \nsubseteq m^2$, then there exists $y \in I - m^2$. Thus $m^2 = (y^2)$, hence $m^2 = (y^2) \subseteq I^2$. Thus $I^2 = m^2$. Therefore for any proper ideal I of R, we have $I = 0$ or $I = m^2$ or $I^2 = m^2$. □

Lemma 2.6. *Let (R, m) be an Artinian local ring with unique minimal ideal such that $m^2 = 0$. Then we have the following statements:*
i) The Extended annihilating-ideal graph $AG^*(R)$ of R has a unique loop;
ii) If we eliminate the loop of the Extended annihilating-ideal graph $AG^*(R)$ of R, then the remainder graph is a simple star graph;
iii)If v is the center of the remainder graph describe in (ii), then v has a loop in $AG^*(R)$.

Lemma 2.7. *Let (R, m) be an Artinian SPAP-ring with $m^2 \neq 0$, such that for all ideals I and J with $m^2 \subsetneq I, J$, $IJ \neq 0$. Then we have the following statements:*
i) The Extended annihilating-ideal graph $AG^*(R)$ of R has a unique loop;
ii) If we eliminate the loop of the Extended annihilating-ideal graph $AG^*(R)$ of R, then the remainder graph is a simple star graph;
iii) If v is the center of the remainder graph describe in (ii), then v has a loop in $AG^*(R)$.

Theorem 2.8. [*3, Theorem 2.6*] *Let R be an Artinian ring. Then $AG(R)$ is a star graph if and only if either $R \cong F_1 \oplus F_2$, where F_1, F_2 are fields, or (R, m) is a local ring and one of the following conditions holds.*
(i) $m^2 = (0)$ and m is the only nonzero proper ideal of R.

(ii) $m^3 = (0)$*,* m^2 *is the only minimal ideal of* R *and for every distinct proper ideals* I_1, I_2 *of* R *such that* $m^2 \neq I_i$ *(i = 1, 2),* $I_1 I_2 = m^2$*.*
(iii) $m^4 = (0)$*,* $m^3 \neq (0)$ *and* $AG(R) = \{m, m^2, m^3\}$*.*

Theorem 2.9. *Let* (R, m) *be an Artinian ring such that The Extended annihilating-ideal graph* $AG^*(R)$ *of* R *has a unique loop and if we eliminate the loop of the Extended annihilating-ideal graph* $AG^*(R)$ *of* R*, then the remainder graph is a simple star graph. Then* $m^3 = 0$ *and one of the following conditions holds:*
i) If $m = 0$*, then* R *is a field.*
ii) If $m^2 = 0$ *and* $m \neq 0$*, then* (R, m) *is a local ring with unique minimal ideal.*
iii) If $m^3 = 0$ *and* $m^2 \neq 0$*, then* (R, m) *is a SPAP-ring.*

Theorem 2.10. *Let* (R, m) *be an Artinian ring such that for all ideals* I *and* J *with* $m^2 \subsetneq I, J$*, we have* $IJ \neq 0$*. The following statements are equivalent.*

1) R is a field or SPAP-ring or a local ring with $m^2 = 0$ *and has a unique minimal ideal.*

2) The Extended annihilating-ideal graph $AG^*(R)$ *of* R *has a unique loop and if we eliminate the loop of the Extended annihilating-ideal graph* $AG^*(R)$ *of* R*, then the remainder graph is a simple star graph.*

3) One of the following conditions holds:

(i) (R, m) *is a PIR, where* $m \neq 0$ *and* m *has nilpotency index less than or equal to 4. (This is equivalent to saying that there exists an element* $\beta \in m$ *such that* $m = (\beta)$*,* $\beta^{s+1} = 0$ *and* $\beta^s \neq 0$ *for some* $1 \leq s \leq 3$*)*

(ii) $char(R) = 2$ *or* $char(R) = 4$*, and* m *has a minimal generating set* $\{\beta_1, \beta_2\}$ *with* $\beta_1\beta_2 \neq 0$*,* $\beta_1^2 = \beta_2^2 = 0$*. In this case,* $m^2 \neq 0, m^3 = 0$*.*

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First hochchild cohomology of square algebra

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Abstract

In this paper, we define the square algebra and describe the first hochschild cohomology of this algebra.

Keywords: First hochschild cohomology, Hochschild cohomology, Square algebra.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

If $\mathbb{A}$ and $\mathbb{B}$ are algebras, M is an A , B-module. and N is a B , A-module, then we will call $S = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ a square algebra. We study the structure of Hochschild cohomology groups of square algebra. This groups is important in many areas of mathematics, such as ring theory, commutative algebra, geometry, group theory and etc.
Although Hochschild cohomology for algebras has been studied extensively for many years, there are still few techniques available for explicitly calculating the various cohomology groups. The study of first cohomology group $H^1(A, X)$, where A is algebra and X is A-bimodule, is essentially the study of inner dervations.

2 Main results

Definition 2.1. Let A and B be algebra. Let M be an A, B-module and N be a B, A-module such that $M \otimes_B N = 0 = N \otimes_A M$. We put
$S = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} : a \in A, m \in M, n \in N, b \in B \right\}$.

If S is given the usual operations associated with 2×2 matrices, then S becomes an algebra. We shall call such an algebra a square algebra.
If A is algebra, a continuous derivation on A is a bounded linear operator $S : A \longrightarrow A$ such that $\delta(ab) = a\delta(b) + \delta(a)b$. Given $x \in A$, we define the map $\delta_x : A \longrightarrow A$ by
$\delta_x(a) = xa - ax$.
The map δ_x is easily seen to be a continuous derivations are said to be inner. Let Der (A) denote all continuous derivation of A and Let Inn (A) denote all inner derivations. We define $H^1(A, A)$, the first cohomology group of A by
$H^1(A, A) = Der(\text{A})/Inn(\text{A})$.

*Speaker

Proposition 2.2. *Let $\delta : S \longrightarrow S$ be a derivation.*
Then the map $\tau : M \longrightarrow M$ and $\sigma : N \longrightarrow N$ obtained above satisfies

$$\begin{aligned} &(i) \quad \tau(a.m) = \delta_A(a).m + a.\tau(m) \\ &(ii) \quad \tau(m.b) = \tau(m).b + m.\delta_B(b) \\ &(iii) \quad \sigma(n.a) = n\delta_A(a) + \sigma(n).a \\ &(iv) \quad \sigma(b.n) = b.\sigma(n) + \delta_B(b).n. \end{aligned}$$

Conversely, if δ_A and δ_B are continuous derivations of A and B respectively and if $\tau : M \longrightarrow M$ and $\sigma : N \longrightarrow N$ are any continuous linear maps satisfying (i),(ii),(iii) and (iv) then the map $\delta\Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\Big) = \begin{bmatrix} \delta_A(a) & \tau(m) \\ \sigma(n) & \delta_B(b) \end{bmatrix}$ define a continuous derivation on δ.

Proof. The proof of first statement follows immediately from [2, proposition 2.2] and for (3) and (4) we have

$$\delta\Big(\begin{bmatrix} 0 & 0 \\ n.a & 0 \end{bmatrix}\Big) = \delta\Big(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big) = n.\delta_A(a) + \sigma(n).a$$

and

$$\delta\Big(\begin{bmatrix} 0 & 0 \\ b.n & 0 \end{bmatrix}\Big) = \delta\Big(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\Big) = b.\sigma(n) + \delta_B(b).n$$

To prove the converse consider:

$$\begin{aligned} &\delta\Big(\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix}\begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix}\Big) = \delta\Big(\begin{bmatrix} a_1a_2 & a_1m_2 + m_1b-2 \\ n_1a_2 + b_1n_2 & b_1b_2 \end{bmatrix}\Big) \\ &= \begin{bmatrix} \delta_A(a_1a_2) & (a_1m_2 + m_1b-2) \\ \sigma(n_1a_2 + b_1n_2) & \delta_B(b_1b_2) \end{bmatrix}. \end{aligned}$$

Moreover,

$$\begin{aligned} &\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix}\delta\begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix} + \delta\Big(\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix}\Big)\begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1\delta_A(a_2) & a_1z(m_2) + m_1\delta_B(b_2) \\ n_1S_A(a_2 + b_1\sigma(n_2) & b_1\delta_B(b_2) \end{bmatrix} \\ &= \begin{bmatrix} a_1\delta_A(a_2) + \delta_A(a_1)a_2 & a_1z(m_2) + m_1\delta_B(b_2) + \delta_A(a_1)m_2 + z(M_1b_2 \\ n_1\delta_A(a_2) + b_1\sigma(n_2) + \sigma(n_1)a_2\delta_B(b_1)n_2 & b_1\delta_B(b_2) + \delta_B(b_1)b_2 \end{bmatrix} \\ &= \begin{bmatrix} \delta_A(a_1a_2) & z(a_1m_2) + z(m_1b-2) \\ \sigma(n_1a_2) + \sigma(b_1n_2) & \delta_B(b_1b_2) \end{bmatrix} \end{aligned}$$

by (1) and (2) thus δ is a derivation on δ. Continuity is clear. ☐

Lemma 2.3. *Let $\varphi \in HomA, B(M)$ and $\sigma \in HomB, A(N)$. Then the map $\delta_{\varphi,\sigma} : S \longrightarrow S$ given by*

$$\delta_{\varphi,\sigma}\Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & \varphi(m) \\ \sigma(n) & 0 \end{bmatrix}$$

is a continuous derivation. Moreover, $\delta_{\varphi,\sigma}$ is an inner derivation if and only if $\varphi = \tau_{x,z}$ and $\sigma = \tau_{z,x}$ where $\tau_{x,z} \in \tau R_{A,B}(M)$ and $\tau_{z,x} \in \tau\tau R_{B,A}(N)$.

Proof. The first statement follows immediately from Assume that $\varphi = \tau_{x,\tau}$ and $\sigma = \tau_{\tau,x}$ where $x \in \tau(A)$ and $\tau \in \tau(B)$. Then

$$\delta_{\begin{bmatrix} x & 0 \\ 0 & \tau \end{bmatrix}}(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} x & 0 \\ 0 & \tau \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} - \begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & \tau \end{bmatrix} = \begin{bmatrix} xa & xm \\ \tau n & \tau b \end{bmatrix} - \begin{bmatrix} ax & m\tau \\ nx & b\tau \end{bmatrix}$$
$$\begin{bmatrix} xa - ax & xm - m\tau \\ \tau n - nx & \tau b - b\tau \end{bmatrix} = \begin{bmatrix} 0 & xm - m\tau \\ \tau n - nx & 0 \end{bmatrix} = \begin{bmatrix} 0 & \varphi(m) \\ \varphi(n) & 0 \end{bmatrix}.$$

Hence $\delta_{\varphi,\sigma}$ is inner. Conversely, assume that $\delta_{\varphi,\sigma}$ is inner. Then there exists $\begin{bmatrix} x & y \\ w & \tau \end{bmatrix} \in \delta$ such that $\delta_{\varphi,\sigma} = \delta_{\begin{bmatrix} x & y \\ w & \tau \end{bmatrix}}$.

However

$$\delta_{\begin{bmatrix} x & y \\ w & \tau \end{bmatrix}}(\begin{bmatrix} a & m \\ n & b \end{bmatrix}) = \begin{bmatrix} x & y \\ w & \tau \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} - \begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} x & y \\ w & \tau \end{bmatrix}$$
$$= \begin{bmatrix} xa - ax & xm + yb - ay - m\tau \\ wa + \tau n - nx - bw & \tau b - b\tau \end{bmatrix}.$$

If $\delta_{\begin{bmatrix} x & y \\ w & \tau \end{bmatrix}} = \delta_{\varphi,\sigma}$, then $xa - ax = 0$ for each $a \in A$ and $\tau b - b\tau = 0$ for each $b \in B$. In particular, $x \in \tau(A)$ and $\tau \in \tau(B)$. Moreover, we have

$$\varphi(m) = xm + yb - ay - m\tau$$

and

$$\sigma(n) = wa + \tau n - nx - bw.$$

Since $\varphi \in H_{A,B}(M)$ and $\sigma \in H_{B,A}(N)$, it follows that $yb - ay = 0$ and $wa - bw = 0$.
Hence $\varphi(m) = xm - m\tau = \tau_{x,z}(m)$ and $\sigma(n) = \tau n - nx = \tau_{z,x}(n)$.
In particular, $\varphi \in \tau R_{A,B}(M)$ and $\sigma \in \tau R_{B,A}(N)$. □

We can now state the main result of this section for describe $H^1(S, S)$.

Theorem 2.4. *Let A be a with unit algebra and B be an algebra with a bounded approximate id. Let M be an essential A, B-module, N be an essential B,A-module and Let $S = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$. If $H^1(A, A) = 0 = H^1(B, B)$, then*

$$H^1(S, S) \cong \frac{Hom_{A,B}(M) \times Hom_{B,A}(N)}{ZR_{A,B}(M) \times ZR_{B,A}(N)}$$

Proof. Let $\phi : Hom_{A,B}(M) \times Hom_{B,A}(N) \longrightarrow H^1(S, S)$ be defined by

$$\phi(\varphi, \sigma) = \bar{\delta}_{\varphi,\sigma},$$

where $\bar{\delta}_\varphi$ represents the equivalence class of $\delta_{\varphi,\sigma}$ in $H^1(S,S)$. Clearly ϕ is linear. We first show that ϕ is surjective. Let S be a continuous derivation of S. Let δ_A , δ_B , $\sigma \overset{\tau:M\to M}{:} N \to N$, $m_\delta a$, n_δ be as in the statement of Proposition 2.2 Since $H^1(A,A) = H^1(B,B) = 0$, we can find $x \in A$ and $\tau \in B$ such that $\delta_A = \delta_x$ and $\delta_B = \delta_z$. Define $\delta_0 : S \longrightarrow S$ by

$$\delta_0\Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\Big) = \begin{bmatrix} \delta_x(a) & \tau_{x,z} + (am_\delta - m_\delta b) \\ \sigma_{z,x} + (n_\delta a - bn_\delta) & \delta_z(b) \end{bmatrix}.$$

Then δ_0 is the inner derivation of S induced by $\begin{bmatrix} x & -m_\delta \\ -n_\delta & \tau \end{bmatrix}$ and as such $_0$ is clearly continuous. Furthermore, if $\delta_1 = S - \delta_0$ then δ_1 is a derivation and by Proposition 2.2,

$$\delta_1\Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\Big) = \begin{bmatrix} \delta_x(a) & \tau(m) + (am_\delta - m_\delta b) \\ \sigma - n + (n_\delta a - bn_\delta) & \delta_z(b) \end{bmatrix}$$
$$- \begin{bmatrix} \delta_x(a) & \tau_{x,z}(m) + (am_\delta - m_\delta b) \\ \sigma_{z,x} n + (n_\delta a - bn_\delta) & \delta_z(b) \end{bmatrix} = \begin{bmatrix} 0 & \tau_{x,z}(m) - \tau_{x,z}(m) \\ \sigma n - \sigma_{z,x} n & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \tau_1(m) \\ \sigma_1 n & 0 \end{bmatrix}$$

$\tau_1 = \tau - \tau_{x,z}$ and $\sigma_1 = \sigma - \sigma_{z,x}$.
It follow from Proposition 2.2 that $\tau_1 \in Hom_{A,B}(M)$ and $\sigma_1 Hom_{B,A}(N)$.
Finally $\bar{\delta} = \bar{\delta}_1 = \phi(\varpi\sigma_1)$, and so ϕ is surjective.
We have shown that

$$H^1(S,S) \cong \frac{Hom_{A,B}(M) \times Hom_{B,A}(N)}{Ker\phi}.$$

However $(\varphi,\sigma) \in Ker\phi$ if and only if $\delta_{\varphi,\sigma}$ is inner. By Lemma 2.3, $Ker\phi = \tau R_{A,B}(M) \times \tau R_{B,A}(N)$. □

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Frobenius semirational groups

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Abstract

In this talk, we give a survey of some recent advances on the problem of studying semi-rational finite groups.

Keywords: Semi-rational groups, Frobenius groups, Simple groups.
Mathematics Subject Classification [2010]: 20E45, 20E34

1 Introduction

For a finite group G, an element x of G is called *rational* if all generators of the group $\langle x\rangle$ are conjugate in G. If all elements of G are rational, then G itself is called *rational*. The notion of rational elements and rational groups has been generalised by Chillag and Dolfi [3]. An element $x \in G$ is called *k-semi-rational* if the generators of $\langle x\rangle$ belongs to at most k conjugacy classes of G. The group G is said to be *k-semi-rational* if all its elements are k-semi-rational in G. In particular, a 2-semi-rational group is called *semi-rational* and its elements are called *semi-rational*.

It was proved by Gow [6] that if G is a rational solvable group then $\pi(|G|) \subseteq \{2, 3, 5\}$. Chillag and Dolfi extended Gow's result to semi-rational groups and proved that $\pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}$ when G is a semi-rational solvable group. They also posed the following problem:

Problem 1. [3, Problem 2] Let G be a solvable group, and let k be a positive integer. If G is a k-semi-rational, then is $\pi(|G|)$ bounded in terms of k?

This talk is based on the results in [1]. Indeed, we generalise the results of [4] to semi-rational Frobenius groups:

Theorem 1.1. *Let $G = HK$ be a Frobenius group with complement H and kernel K. Then G is semi-rational if and only if the following two properties hold:*

(a) H is itself semi-rational;

(b) Each element of K is semi-rational in G, that is, for every $x \in K$, the generators of $\langle x\rangle$ belong to at most two conjugacy classes of G.

We moreover give more details on the structure of semi-rational Frobenius groups:

*Speaker

Theorem 1.2. *Suppose that $G = HK$ is a semi-rational Frobenius group with complement H and kernel K. Then*

(a) if H is of even order, then H and K are known;

(b) if H is of odd order, then $H \cong \mathrm{C}_3$ and $|K| = 2^a \cdot 7^b > 1$ with $a \geqslant 0$ and $b \geqslant 0$. In particular, if $b \geqslant 1$, then K is not semi-rational;

Consequently, we answered Problem 1 for Frobenius groups G and showed that $|\pi(G)| \leqslant 5$.

In general, composition factors of rational group studied by Feit and Seitz [5], in particular, they determined all simple rational groups. In this direction, for semi-rational groups, Alavi, Burness and Daneshkhah [2] studied semi-rational simple groups.

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Group factorisations and associated geometries

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Abstract

Triple factorisations of groups G of the form $G = ABA$, for proper subgroups A and B, are fundamental in the study of Lie type groups, as well as in geometry. In this talk, we present recent studies of such factorisations in the context of both permutation group theory and geometry.

Keywords: Triple factorisation, Rank 2 geometry, Large subgroup
Mathematics Subject Classification [2010]: 20B15, 51E30

1 Introduction

For a group G with subgroups A and B, if $G = ABA$ then we say that (G, A, B) is a *triple factorisation*. For example, groups with BN-pairs give rise to triple factorisations $G = BNB$ where B is a Borel subgroup and $N/(N \cap B)$ is the Weyl subgroup. Geometrically, the study of flag-transitive rank 2 incidence geometries are closely related to triple factorisations of their automorphism groups.

Higman and McLaughlin [6] introduced the notion of *Geometric ABA-groups* and showed that a Geometric ABA-group acts primitively (as an automorphism group) on the point set of the associated linear space, see [6, Propositions 1-3]. As a generalisation, for a given triple factorisation $G = ABA$, Alavi and Praeger [5] introduced a reduction pathway to the case where A is maximal and core-free. This motivates us to investigate *large subgroups* H of finite simple groups G, that is $|G| \leq |H|^3$, see [4]. This talk is based on results in [2, 3, 4] in which we studied *parabolic triple factorisations* $G = ABA$ of general linear groups G and its classical subgroups with A and B maximal parabolic subgroups.

In connection with geometry, each triple factorisation $G = ABA$ gives rise to a *collinearly complete* coset geometry $\mathsf{Cos}(G; A, B)$ (with A the stabiliser of a point p and B the stabiliser of a line ℓ incident with p) in which 'each pair of points lies on at least one line', and vice versa [6, Lemma 3]. Interchanging the roles of points and lines, leads us to a dual completeness concept: a geometry is *concurrently complete* if 'each pair of lines is incident with at least one point'.

In this talk, we also establish above natural connection between triple factorisations and geometry, and apply such geometric method to obtain new triple factorisations. Consequently, in addition to the well-known examples (linear spaces, symmetric designs and projective spaces), our results leads us to new collinearly and/or concurrently complete spaces.

*Speaker

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Independence graph of a vector space

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Abstract

Let V be a vector space over field F. The independence graph of V, denoted by Γ_V is a graph with all elements of V minus zero as vertices, and two distinct vertices v_1 and v_2 are adjacent if and only if $\{ v_1, v_2\}$ is independent. In this paper we obtain some properties of the independence graph. For example it is shown that when the independent graph is complete.

Keywords: Independence graph, Vector space, Vertices
Mathematics Subject Classification [2010]: 97H60, 97K30

1 Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, see([1]-[3]). Throughout the paper V is a vector space over a field F. We define the independence graph of V to be graph Γ_V with all elements of V minus zero as vertices, and two distinct vertices v_1 and v_2 are adjacent if and only if $\{v_1, v_2\}$ is independent.

Let Γ be a graph with vertices x and y. We define $d(x, y)$ to be the length of the shortest path from x to y. The diameter of Γ is $diam(\Gamma) = \sup\{d(x, y)|x \text{ and } y \text{ are vertices of } \Gamma\}$. The girth of Γ, denoted by $gr(\Gamma)$, is the length of a shortest cycle in Γ.

In Section 2, we obtain some properties of the independence graph of a vector space. Basic references for graph theory is [5]; for linear algebra see [4].

2 Main results

It is clear that the independent graph of a vector space of dimension zero is empty graph.

Theorem 2.1. *Let V be a vector space of dimension greater or equal than 1 over field F. Then Γ_V is only a set of some vertices if and only if* $\dim(V) = 1$.

Proof. First, suppose $\dim(V) = 1$. Then there is $x \in V$ such that every element of V is cx which c is an scaler. Therefore every subset of V with at least 2 elements is not independence and so there is not any edges in Γ_V.

Now, Γ_V is only a set of some vertices. Therefore for every pair of vertices x and y, there exists c such that $x = cy$. Thus $\dim(V) = 1$. □

*Speaker

Theorem 2.2. *Let V be a vector space of dimension greater or equal than 2. For every v ad w of Γ_V we have $d(v, w) \leq 2$ and so Γ_V is connected.*

Proof. Suppose v and w are two vertices. If $\{v, w\}$ is independent, then there is an edge between them. In other case, $v = cw$ such that $c \neq 0$. In view of the dimension of V, there is a vector x such that $\{v, x\}$ and $\{x, w\}$ are independent. Therefore x and v are adjacent and also x and w. This ends the proof. □

Theorem 2.3. *Let V be a vector space of dimension greater or equal than 1. Then Γ_V is complete if and only if $|F| = 2$.*

Proof. First, suppose Γ_V is complete. Let x be a nonzero vector. If $|F| \geq 3$, there is c in F such that $c \neq 1$ and $c \neq 0$. Thus $\{x, cx\}$ is not independent, and so there is not any edge between x and cx which is a contradiction. Consequently $|F| = 2$.

Conversely, suppose $|F| = 2$ and x and y two nonzero distinct elements of V. If there is any edge between x and y, then there exists $c \in F$ such that $x = cy$. Since $x \neq 0$ and $c \neq 0$, $c = 1$ and so $x = y$ which is contradiction. Thus there is an edge between x and y and so Γ_V is complete. □

Proposition 2.4. *Let V be a vector space of dimension $n \geq 2$ and $|F| = r$. Then the cardinal of the set of all edges of Γ_V is*

$$(r^n - 1)(r^n - 1 - (r - 1))/2.$$

Proof. We know that the number of edges in a complete graph with the number of vertices s is $s(s - 1)/2$. On the other hand $s = r^n - 1$. Let v be a vector and $C(v)$ be the set of all vertices connected to v. Then the cardinal of $C(v)$ is $r^n - 1 - (r - 1)$ and therefore $(r^n - 1)(r^n - 1 - (r - 1))/2$ is the cardinal of the set of all edges of Γ_V.

□

Definition 2.5. Let v be a nonzero vector. The subspace generated by v minus zero is said to be the line which passes through from v.

Theorem 2.6. *Let V be a vector space of dimension $n \geq 2$ over field F and $|F| = r$. Then the number of triangle of Γ_V is $(r^n - 1)(r^n - r)(r^n - 2r + 1)$ and so $gr(\Gamma_V) = 3$.*

Proof. The number of vertices which do not connect to a vector v is the cardinal of F minus one. The number of lines in Γ_V is $s = r^n - 1/r - 1$. First, we choose three line. The number of this choice is $s(s - 1)(s - 2)$. Then we will choose one vertex of every line. Therefore we have $s(s - 1)(s - 2)(r - 1)^3 = (r^n - 1)(r^n - r)(r^n - 2r + 1)$ triangles. □

Let V be a vector space over field F and the cardinal of the set of vertices of Γ_V be finite. Let the cardinal of F be p^t and the cardinal of the set of vertices of Γ_V be s. Therefore $s + 1 = p^{tn}$ such that p^t is the cardinal of F and n is the dimension of V. In view of $p^{tn} = s + 1$, we obtain the characteristic of F. Let $\overline{\Gamma_V}$ be the complement graph of Γ_V. The number of vertices in every components of $\overline{\Gamma_V}$ is $p^t - 1$ and so we obtain t. Consequently, we obtain the dimension of V.

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Large non-nilpotent subsets of finite general linear groups

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Abstract

Let G be a group. A subset X of G is said to be non-nilpotent if, for any two distinct elements x and y in X, $\langle x, y\rangle$ is a non-nilpotent subgroup of G. Define ωG to be the order of the largest non-nilpotent set in G.

Using regular semisimple and regular unipotent elements we find a lower bound for $\omega(\mathcal{N}_G)$ for $G = GL_n(q)$

Keywords: non-nilpotent set, regular semisimple element, regular unipotent element
Mathematics Subject Classification [2010]: 20D60

1 Introduction

Let G be a group. A subset X of G is said to be a *non-nilpotent subset* if, for any two distinct elements x and y in X, $\langle x, y\rangle$ is a subgroup of G which is not nilpotent. Define ωG to be the order of the largest non-nilpotent set in G. If G is a nilpotent group we define $\omega(G) = 1$.

The value of ωG has been studied for various groups. Endimioni proved that if a finite group G satisfies $\omega G \leq 3$ then G is nilpotent, while if $\omega G \leq 20$ then G is soluble; furthermore these bounds cannot be improved [4]. Tomkinson proved that if G is a finitely generated soluble group such that $\omega G = n$, then $|G/Z^*(G)| \leq n^{n^4}$, where $Z^*(G)$ is the hypercentre of G [9]. Also, for a finite insoluble group G, it has been proved that G satisfies the condition $\omega G = 21$ if and only if $G/Z^*(G) \cong A_5$ [1, Theorem 1.2].

Definition 1.1. Let G is a linear algebraic group we can write $G \leq GL_n(K)$ for some integer n. An element $g \in G$ is then said to be *semisimple* if g is diagonalizable in $GL_n(K)$, and is said to be *unipotent* if all of its eigenvalues are equal to 1.

Theorem 1.2. *Suppose that H contains a set of subgroups $A_1, A_2, \ldots, A_n$ that form a partition of H. If $\mathrm{nil}_H(g) \leq A_i$, for all $g \in A_i \setminus \mathrm{nil}(H)$, then*

1. $\omega(\mathcal{N}_H) = \sum_{i=1}^{n} \omega(\mathcal{N}_{A_i})$.
2. *If A_i is nilpotent for all $i \in \{1, \ldots, n\}$, then every non-nilpotent subset of H can be extended to a maximal non-nilpotent subset of H.*

*Speaker

Lemma 1.3. *Suppose that p is a prime number dividing $|H|$. Let $P = P_1, P_2, \dots, P_{\nu_p(H)}$ be the Sylow p-subgroups of H. Suppose that*

$$P \setminus \bigcup_{i=2}^{\nu_p(H)} P_i \neq \emptyset. \tag{1}$$

Then there exists a non-nilpotent set $\Omega \subseteq H$ such that all elements of Ω are p-elements and $|\Omega| = \nu_p(H)$.

Proposition 1.4. *T Let $H \cong J \times K$ for two finite groups J and K. Then*

$$\omega(H) = \omega(J) \cdot \omega(K).$$

Definition 1.5. Let $g \in GL(n, q)$ where $q = p^k$, p a prime, and $|g| = q^n - 1$. Then $\langle g \rangle$ is called a Singer cycle subgroup of G.

Definition 1.6. Let V be a vector space over a finite field F with dimension n. We call $V = Vn_1 \oplus Vn_2 \oplus \dots \oplus Vn_k$ an $(n_1, n_2, \dots, n_k)$-decomposition if $(n_1, n_2, \dots, n_k)$ is a partition of n and for $i = 1, 2, \dots, k$, V_{n_i} is a subspace of V of dimension n_i.

Definition 1.7. Let V be an $n-$dimensional vector space over a finite field F with size q. An element g of $GL(n, q)$ is called an $(n_1, n_2, \dots, n_k)$-Singer generator if there is an $(n_1, n_2, \dots, n_k)$-decomposition $V = V_{n_1} \oplus V_{n_2} \oplus \dots \oplus V_{n_k}$ of V such that $g = g_{n_1} g_{n_2} \dots g_{n_k}$, where for each i, $\langle g_{n_i} \rangle$ is a Singer cycle subgroup of $GL(V_{n_i})$, or if $n_i = 1$ then g_{n_i} has eigenvalue 1, and if $n_i = n_j$ with $i \neq j$, then $c_{g_{n_i}}(t) \neq c_{g_{n_j}}(t)$, where $c_{g_{n_i}}(t)$ is the characteristic polynomial for g_{n_i} on V_{n_i}. We call $\Pi_{i=1}^{k} \langle g_{n_i} \rangle$ the $(n_1, n_2, \dots, n_k)-$maximal torus corresponding to g.

Note that $GL(n, q)$ has no $(1, 1, \dots, 1)$-Singer generator unless $q \geq n + 1$.

Definition 1.8. Let n be a natural number. We define a partition of n by $1_k = (1, 1, 1, \dots, 1, n - k)$ so that the first k elements are 1 and the last is $n - k$, with $k = 0, 1, 2, \dots, n - 1$.

Lemma 1.9. *Let $G = GL(n, q)$, with $q = p^k \geq n + 1$ and suppose that $g \in G$ is an 1_k-Singer generator, where $k = 0, 1, 2, \dots, n - 1$, with $g = g_{1_1} g_{1_2} \cdots g_{1_k} g_{n-k}$. Then $C_G(g) = \Pi_{i=1}^{k} \langle g_{1_i} \rangle \times \langle g_{n-k} \rangle$ is a subgroup of order $\Pi_{i=1}^{k} (q - 1)^k \times (q^{n-k} - 1)$ and p does not divide $|C_G(g)|$.*

Theorem 1.10. *Let $G = GL(n, q)$, where $q = p^k \geq n + 1$. Let N_{1_k} consist of one 1_k-Singer generator element of G corresponding to each 1_k-maximal torus of G. Then N_{1_k} is a non-nilpotent subset of size $\frac{|G|}{k!(q-1)^k (n-k)(q^{n-k}-1)}$.*

2 Main results

Theorem 2.1. *Let $G = GL(n, q)$, where $q \geq n+1$. Then $N = \bigcup_{k=0}^{n} N_{1_k}$ is a non-nilpotent subset of the regular semisimple elements of size*

$$|N| = \sum_{k=1}^{n} |N_{1_k}| = \frac{|G|}{n(q^n - 1)} + \sum_{k=1}^{n-2} \frac{|G|}{k!(q - 1)^k (n - k)(q^{n-k} - 1)} + \frac{|G|}{n!(q - 1)^n}.$$

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Talk Lie structure of smash products pp.: 1–4

Lie structure of smash products *

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Abstract

We investigate the conditions under which the smash product of an (ordinary or restricted) enveloping algebra and a group algebra is Lie solvable or Lie nilpotent.

Keywords: smash products, enveloping algebras, group algebras, Lie solvable
Mathematics Subject Classification [2010]: 17B60, 16S40, 16R40, 17B35

1 Introduction

Let A be an associative algebra over a field and regard A as a Lie algebra via the Lie product defined by $[x, y] = xy - yx$, for every $x, y \in A$. Then, A is said to be Lie solvable (respectively, Lie nilpotent) if it is solvable (nilpotent) as a Lie algebra. The Lie structure of associative algebras have been extensively studied over the years and considerable attention has been especially devoted to group algebras (see e.g. [2, 3, 5]) and restricted enveloping algebras (see e.g. [6, 7, 8, 9, 10]).

Let G be a group and $\mathbb{F}$ a field. We denote by $\mathbb{F}G$ the group algebra of G over $\mathbb{F}$. We also denote by G' the derived subgroup of G. Passi, Passman and Sehgal established in [5] when $\mathbb{F}G$ is Lie solvable and Lie nilpotent.

Theorem 1.1 ([5])**.** Let $\mathbb{F}G$ be the group algebra of a group G over a field $\mathbb{F}$ of characteristic $p \geq 0$. Then $\mathbb{F}G$ is Lie nilpotent if and only if one of the following conditions hold:

1. $p = 0$ and G is abelian;
2. $p > 0$, G is nilpotent and G' is a finite p-group;

Theorem 1.2 ([5])**.** Let $\mathbb{F}G$ be the group algebra of a group G over a field $\mathbb{F}$ of characteristic $p \geq 0$. Then $\mathbb{F}G$ is Lie solvable if and only if one of the following conditions hold:

1. $p = 0$ and G is abelian;
2. $p > 2$ and G' is a finite p-group;
3. $p = 2$ and G has a subgroup N of index at most 2 such that N' is a finite 2-group.

*Will be presented in English
†Speaker

Let L a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. The restricted enveloping algebras of L is denoted by $u(L)$. We recall that a subset S of L is said to be p-nilpotent if $S^{[p]^m} = \{x^{[p]^m} | x \in S\} = 0$, for some $m \geq 1$. We will also denote by L' the derived subalgebra $[L, L]$ of L. The Lie structure of restricted enveloping algebras has been investigated by Riley and Shalev in [6].

Theorem 1.3 ([6])**.** Let L be a restricted Lie algebra over a field of characteristic $p > 0$. Then $u(L)$ is Lie nilpotent if and only if L is nilpotent and L' is finite-dimensional and p-nilpotent.

Theorem 1.4 ([6])**.** Let L be a restricted Lie algebra over a field of characteristic $p > 2$. Then $u(L)$ is Lie solvable if and only if L' is finite-dimensional and p-nilpotent.

Let H be a Hopf algebra and suppose that A is a left H-module algebra via $\varphi : H \to \mathrm{End}_{\mathbb{F}}(A)$. For every $h \in H$ and $x \in A$, we set $h * x = \varphi(h)(x)$ and use the so-called Sweedler's notation $\Delta(h) = \sum h_1 \otimes h_2$ for the comultiplication of H . We recall that the smash product $A\#H$ is the vector space $A \otimes_{\mathbb{F}} H$ endowed with the following multiplication (we will write $a\#h$ for the element $a \otimes h$):

$$(a\#h)(b\#k) = \sum a(h_1 * b)\#h_2 k.$$

In particular, we consider $H = \mathbb{F}G$, where the action of $\mathbb{F}G$ on A is induced by a group homomorphism $\varphi : G \to \mathrm{Aut}(A)$. Conversely, every $\mathbb{F}G$-module algebra arises in this way. Since $\Delta(g) = g \otimes g$, the multiplication in $A\#\mathbb{F}G$ is just given by $(a\#g)(b\#h) = a(g*b)\#gh$, for all $a, b \in A$ and $g, h \in G$. See [4] for more details.

Now, suppose that a group G acts by automorphisms on a restricted Lie algebra L over a field $\mathbb{F}$ of positive characteristic. This action is naturally extended to the action of $\mathbb{F}G$ on $u(L)$ and one can form the smash product $u(L)\#\mathbb{F}G$. Necessary and sufficient conditions under which these smash products satisfy a nontrivial polynomial identity were provided by Bahturin and Petrogradsky in [1]. In this paper we determine necessary and sufficient conditions under which $u(L)\#\mathbb{F}G$ is Lie solvable or Lie nilpotent.

It is worth mentioning that smash products, sometimes referred as semidirect products, arise very frequently in the theory of Hopf algebras. A classical example is a celebrated structure theorem of Cartier-Kostant-Milnor-Moore, asserting that every cocommutative Hopf algebra over an algebraically closed field of characteristic zero can be presented as a smash product of a group algebra and an enveloping algebra (see e.g. [4, §5.6]). As an application of our results, we show that a cocommutative Hopf algebra over a field of characteristic zero is Lie solvable if and only if it is commutative.

2 Main results

In the main results of this paper we determine the conditions under which $u(L)\#\mathbb{F}G$ is Lie solvable in odd characteristic (Theorem 2.1) or Lie nilpotent (Theorem 2.2). We also deal with smash products $U(L)\#\mathbb{F}G$, where $U(L)$ is the ordinary enveloping algebra of a Lie algebra over any field. In particular, we establish when $U(L)\#\mathbb{F}G$ is Lie solvable (in characteristic different than 2) or Lie nilpotent.

Theorem 2.1. Let G be a group acting by automorphisms on a restricted Lie algebra L over a field $\mathbb{F}$ of characteristic $p > 2$. Then $u(L)\#\mathbb{F}G$ is Lie solvable if and only if the following conditions hold:

1. G' is a finite p-group;
2. L contains a finite-dimensional p-nilpotent G-stable restricted ideal P such that L/P is abelian and G acts trivially on L/P.

Let a group G act by automorphisms on a $\mathbb{F}$-vector space V. One says that G acts nilpotently on V if there exists a chain $0 = V_0 \subseteq V_1 \subseteq \cdots V_n = V$ of G-stable subspaces of V such that the induced action of G on each factor V_i/V_{i-1} is trivial. Note that this is tantamount to saying that $\omega(G)^m * V = 0$ for some m, where V is regarded as an $\mathbb{F}G$-module in the natural way.

Theorem 2.2. Let G be a group acting by automorphisms on a restricted Lie algebra L over a field $\mathbb{F}$ of characteristic $p > 0$. Then $u(L)\#\mathbb{F}G$ is Lie nilpotent if and only if the following conditions are satisfied:

1. G is nilpotent and G' is a finite p-group;
2. L is nilpotent;
3. L has a finite-dimensional p-nilpotent G-stable restricted ideal P such that L/P is abelian, and G acts nilpotently on L and trivially on L/P.

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Local dimension and direct sum of cyclic modules

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Abstract

In this paper we study rings with local dimension which certain of ideals are direct sum of cyclic modules. It is shown that for a commutative ring R with local dimension ω, every ideal is direct sum of cyclic modules if and only if R is a principal ideal domain. We show that for a non-local ring R with finite local dimension if every ideal of R is a direct sum cyclic right R-modules, then R is right Artinian.

Keywords: Local dimension, Cyclic modules, Principal ideal ring, Artinian Ring.
Mathematics Subject Classification [2010]: 13F10, 16P20, 16D25.

1. Introduction

Throughout this article, let R denote an arbitrary ring with identity. All modules are assumed to be unitary. For a module M_R, we write $\mathrm{Soc}(M)$ and $\mathrm{Rad}(M)$ for the socle and the Jacobson radical of M, respectively. Also, $\mathrm{J}(R)$ will be used for the Jacobson radical of a ring R. A local ring is a ring with only one maximal right (or left) ideal. The study of commutative rings which ideals are direct sum of cyclic modeles was initiated by Behboodi, Ghorbani and Moradzade in [1]. An interesting natural question of this sort is: "What is the class of non-local commutative rings R for which every ideal is a direct sum of cyclic modules?" We answer this question in the case where rings have local dimension less than or equal to ω. Recall that local dimension is a measure of how far a coatomic module deviates from being local. Let M be an R-module. If M has a largest proper submodule, i.e., a proper submodule which contains all proper submodules, then M is called a local module (see [5]). A module M is called coatomic if every proper submodule of M is contained in a maximal submodule (see [3]).

Definition 1.1. In order to define local dimension for coatomic modules over a ring R, we first define, by transfinite induction, classes ζ_α of coatomic R-modules for all ordinals α. To start with, let ζ_1 be the class of non-zero local modules. Next, consider an ordinal $\alpha > 1$; if ζ_β has been defined for all ordinals $\beta < \alpha$, let ζ_α be the class of those coatomic R-modules M such that, for every submodule $N < M$, $M/N \ncong M$ implies $M/N \in \bigcup_{\beta<\alpha} \zeta_\beta$. If a coatomic R-module M belongs to some ζ_α, then the least such α is the *local dimension of* M, denoted $\mathrm{l.dim}(M)$. For $M = 0$, we define $\mathrm{l.dim}(M) = 0$. If a coatomic module M does not belong to any ζ_α, then we say that "$\mathrm{l.dim}(M)$ is not defined," or that " M has no local dimension" (See [2]).

*Speaker

In this paper we study commutative rings with local dimension less than or equal to ω such that every ideal is direct sum of cyclic modules. In Theorem 2.4, we show that when local dimension of a commutative ring R is ω, every ideal of R is direct sum of cyclic modules if and only if R is a principal ideal ring. In continue, we study non-commutative rings with finite local dimension. We have obtained the conditions under which, noncommutative rings with finite local dimension are Artinian. By Proposition 3.2, if R is a non-local ring such that $\mathrm{l.dim}(R_R) < \infty$ and $\mathrm{J}(R)$ is finitely generated as righ R-module, then R is right Artinian and $\mathrm{length}(R_R) = \mathrm{l.dim}(R_R)$. This yields that in the same condition if $\mathrm{J}(R)^n$ is a direct sum of finitely generated right R-modules, for every $n \in \mathbb{N}$, then R is right Artinian and $\mathrm{lenght}(R_R) = \mathrm{l.dim}(R_R)$ (Theorem 3.4).

2. Commutative rings with countable local dimension

In this section we study commutative rings with local dimension less than or equal to ω whose ideals are direct sum of cyclic modules. Behboodi et al. in [1] showed that for a commutative Noetherian local ring $(R, \mathcal{M})$ every ideal is direct sum of cyclic modules if and only if $\mathcal{M} = Rw_1 \oplus \cdots \oplus Rw_n$ with at most two of Rw_i's not simple. On the other hand, we showed that a commutative ring R has finite local dimension if and only if R is Artinian or local (See [2, Theorem 4.12]). Therefore we have the following.

Proposition 2.1. *Suppose R is a commutative non-local ring with finite local dimension. Then every ideal of R is direct sum of cyclic modules if and only if $R = R_1 \times \cdots \times R_n$ such that for every $1 \leqslant i \leqslant n$, R_i is an Artinian local ring with maximal ideal $\mathcal{M}_i$ that $\mathcal{M}_i = Rw_{i1} \oplus \cdots \oplus Rw_{in_i}$ with at most two of Rw_{ij}'s not simple.*

Now we want to study commutative rings with local dimension equal to ω whose ideals are direct sum of cyclic modules. First we need some preliminary definition and results.

Lemma 2.2. (See [2, Theorem 2.8]) *If R is a ring such that R_R has finite local dimension, then R is a semilocal ring.*

In the above lemma replace R with $R/\mathrm{J}(R)$, and suppose $\mathrm{l.dim}(R/\mathrm{J}(R))$ is finite, then $R/\mathrm{J}(R)$ is semilocal and since $\mathrm{J}(R/\mathrm{J}(R)) = 0$ we conclude that $R/\mathrm{J}(R)$ is semisimple.

Corollary 2.3. *If R be a ring (not necessarily commutative) such that $\mathrm{l.dim}(R/\mathrm{J}(R))$ as right R-module is finite, then R is semilocal.*

Now we are in a position to state our main theorem.

Theorem 2.4. *Let R be a commutative ring such that $\mathrm{l.dim}(R) = \omega$. Then, every ideal of R is a direct sum of cyclic modules if and only if R is PID.*

Proof. We give here a sketch of proof. Consider that by [2, Corollary 4.11], R is right Noetherian. First we show that for every ideal I of R, if R/I is not local then I is cyclic. This implies that every ideal of R is a direct sum of two cyclic modules. In continue, we show that $\mathrm{Soc}(R) = \mathrm{J}(R) = 0$. From this we concluded that R is a principal ideal domain. □

3. Noncommutative rings with finite local dimension

As you saw in section 2, for a commutative ring R if $1 < \text{l.dim}(R) < \infty$, then R is Artinian and $\text{l.dim}(R) = \text{length}(R)$. In this section, we obtain the conditions under which noncommutative rings with finite local dimension are Artinian. First consider the following.

Lemma 3.1. *Let R be a ring such that $1 < \text{l.dim}(R_R) < \infty$. If $I_1 \supseteq I_2 \supseteq \cdots$ is a chain of two sided ideals of R such that R/I_1 is not local, then there exist $m \in \mathbb{N}$ such that $I_m = I_j$, for every $j \geqslant m$.*

Proposition 3.2. *Let R be a ring such that $1 < \text{l.dim}(R_R) < \infty$. If $\mathrm{J}(R)$ is a finitely generated right R-module, then R is right Artinian and $\text{l.dim}(R_R) = \text{length}(R_R)$.*

Lemma 3.3. [1, Lemma 2.3] *Let R be a ring and M be an R-module such that M is a direct sum of a family of finitely generated R-modules. Then Nakayamas lemma holds for M (i.e., for each $I \subseteq \mathrm{J}(R)$, if $MI = M$, then $M = (0)$).*

Now we can state the main result of this section.

Theorem 3.4. *Let R be a ring such that $1 < \text{l.dim}(R_R) < \infty$. If $\mathrm{J}(R)^n$ is a direct sum of finitely generated right R-moduls, for every $n \in \mathbb{N}$, then R is right Artinian and $\text{l.dim}(R_R) = \text{length}(R_R)$.*

Proof. Proof by induction hypothesis on local dimension of R_R. The base of induction is obvious by [2, lemma 4.6]. Suppose $\text{l.dim}(R_R) = n$ and the assertion is true for all rings with local dimension less than n. If $\mathrm{J}(R) = 0$, then R is semisimple and $\text{l.dim}(R_R) = \text{length}(R_R)$. Suppose $\mathrm{J}(R) \neq 0$, we show that $\mathrm{J}(R)$ is finitely generated right R-module and so by Proposition 3.2, R is right Artinian and $\text{l.dim}(R_R) = \text{length}(R_R)$. By Lemma 3.1, there exist $m \in \mathbb{N}$ such that $\mathrm{J}(R)^m = \mathrm{J}(R)^{m+1}$. Then $\mathrm{J}(R)^m\mathrm{J}(R) = \mathrm{J}(R)^m$ and by Lemma 3.3, $\mathrm{J}(R)^m = 0$, because $\mathrm{J}(R)^m$ is a direct sum of finitely generated modules. Assume that $\mathrm{J}(R) = \oplus_{i\in I} L_i$ such that L_i's are finitely generated modules, let $k = \text{l.dim}(R/\mathrm{J}(R))$ and $t = n - k$. Note that $\mathrm{J}(R)\mathrm{J}(R)^{m-1} = 0$ so $\mathrm{J}(R)$ is an $R/\mathrm{J}(R)^{m-1}$-module. Since $\text{l.dim}(R/\mathrm{J}(R)^{m-1}) < \text{l.dim}(R)$ by induction hypothesis $R/\mathrm{J}(R)^{m-1}$ has finite length, hence L_i has finite length as an $R/\mathrm{J}(R)^{m-1}$-module and so as an R-module. Without loss of generality, we can assume $\mathrm{J}(R) = H_1 \oplus H_2 \oplus \cdots H_{t+1} \oplus F$ such that $F = \oplus_{j\in J} H_j$ and $J = I \setminus \{1, \cdots, t + 1\}$. For $i = 1, \cdots, t + 1$, let $L_i = L_{i0} > L_{i1} > \cdots > L_{iq_i} = 0$ be a composition series for L_i. Then $\mathrm{J}(R) = L_{10} \oplus L_{20} \oplus \cdots \oplus L_{(t+1)0} \oplus F$. Let $h = q_1 + q_2 + \cdots + q_{t+1}$. We can show that there is a series $\mathrm{J}(R) = J_0 > J_1 > J_2 > \cdots > J_h = F$ such that J_i/J_{i+1} is simple, moreover $h \geqslant t + 1$. By induction hypothesis, $\text{length}(R/\mathrm{J}(R)) = \text{l.dim}(R/\mathrm{J}(R)) = k$. Consider that J_0/J_1 is simple, hence $\text{length}(R/J_1) = \text{length}(R/J_0) + 1 = k + 1$ and so $R/J_1 \ncong R/J_0$, then $\text{l.dim}(R/J_1) > \text{l.dim}(R/J_0)$. On the other hand, by [2, corollary 3.3], $\text{l.dim}(R/J_1) \leqslant \text{length}(R/J_1) = \text{length}(R/J_0) + 1 = \text{l.dim}(R/J_0) + 1$ which show that $\text{l.dim}(R/J_1) = \text{l.dim}(R/J_0)+1$. Similarly $\text{length}(R/J_{i+1}) = \text{length}(R/J_i)+1$, for every $0 \leqslant i < h$, because J_i/J_{i+1} is simple. Which show that $\text{length}(R/J_i) = k+i$, for every $0 \leqslant i \leqslant h$ and so $R/J_{i+1} \ncong R/J_i$. Therefore $= \text{l.dim}(R/J_i) < \text{l.dim}(R/J_{i+1}) \leqslant \text{length}(R/J_{i+1})$, for every $0 \leqslant i < h$, and so $\text{l.dim}(R/J_i) = \text{length}(R/J_i) = k + i$, for every $0 \leqslant i \leqslant h$. Now for $i = t + 1$ we have $\text{l.dim}(R/J_i) = k + i = k + t + 1 = k + n - k + 1 = n + 1 > n$ which is contradiction and hence $\mathrm{J}(R)$ is a finitely generated right R-module. □

Corollary 3.5. *Let R be a ring such that $1 < \mathrm{l.dim}(R_R) < \infty$. If every ideal of R is a direct sum of cyclic right R-modules, then R is right Artinian and $\mathrm{l.dim}(R_R) = \mathrm{length}(R_R)$.*

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Talk Minimum size of intersetion for covering groups by subgroups pp.: 1–4

Minimum size of intersetion for covering groups by subgroups

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Abstract

Let G denotes a semisimple $\mathfrak{C}_8$-group and $\{M_i \mid 1 \leq i \leq 8\}$ be a maximal irredundant 8-cover for G, with core-free intersection $D = \bigcap_{i=1}^{8} M_i$. Also for each i, $1 \leq i \leq 8$ we assume that $|G : M_i| = \alpha_i$ such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \leq \alpha_7 \leq \alpha_8$.

Let l is minimum positive integer such that $\bigcap_{i=1}^{l}(M_i)_G \neq 1$. We say that l is minimum size of intersetion and in this case we show that MSI(G)=l. In this paper we show that if G be a semisimple $\mathfrak{C}_8$-group and $\alpha_l \leq 4$ then MSI(G)$\leq$ 3

Keywords: covering groups by subgroups, Subdirect product, maximal iredundant cover, core-free intersection
Mathematics Subject Classification [2010]: 20F99

1 Introduction and history

Let G be a group. A set $\mathcal{C}$ of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G. If the size of $\mathcal{C}$ is n, we call $\mathcal{C}$ an n-cover for the group G. A cover $\mathcal{C}$ for a group G is called irredundant if no proper subset of $\mathcal{C}$ is a cover for G. A cover $\mathcal{C}$ for a group G is called core-free if the intersection $D = \bigcap_{M\in\mathcal{C}} M$ of $\mathcal{C}$ is core-free in G, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of G. A cover $\mathcal{C}$ for a group G is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of G. A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group. A finite group is called semisimple if it has no non-trivial normal abelian subgroups (see p. 86 of [9] for further information on such groups).

Also we use the usual notations ([9]); for example, C_n denotes the cyclic group of order n, $(C_n)^j$ is the direct product of j copies of C_n, the core of a subgroup H of G is denoted by H_G.

In [10], Scorza determined the structure of all groups having an irredundant 3-cover with core-free intersection.

Theorem 1.1. (Scorza [10]) *Let $\{A_i : 1 \leq i \leq 3\}$ be an irredundant cover with core-free intersection D for a group G. Then $D = 1$ and $G \cong C_2 \times C_2$.*

In [7], Greco characterized all groups having an irredundant 4-cover with core-free intersection. Bryce et al.[6], characterized groups with maximal irredundant 5-cover with core-free intersection.

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We characterized groups with maximal irredundant 6-cover with core-free intersection in [1]. Abdollahi et al.[3], characterized groups with maximal irredundant 7-cover with core-free intersection.

Also we characterized p-groups with maximal irredundant 8-cover with core-free intersection in [2].

Theorem 1.2. (*See [2]*). *Let G be a $\mathfrak{C}_8$-group. Then G is a p-group for a prime number p if and only if $G \cong (C_3)^4$ or $(C_7)^2$.*

Also we investigated covering groups by subgroups and semisimplity condition in [4] and subdirect product and covering groups by subgroups in [5].

Let l is minimum positive integer such that $\bigcap_{i=1}^{l}(M_i)_G \neq 1$. We say that l is minimum size of intersetion and in this case we show that MSI(G)=l. In this paper we show that if G be a semisimple $\mathfrak{C}_8$-group and $\alpha_l \leq 4$ then MSI(G)≤ 3

2 Main results

In the proofs of the main results we need the following lemmas:

Lemma 2.1. (*Lemma 2.2 of* [*6*]). *Let $\Gamma = \{A_i \ : \ 1 \leq i \leq m\}$ be an irredundant covering of a group G whose intersection of the members is D.*

(a) If p is a prime, x a p-element of G and $|\{i : x \in A_i\}| = n$, then either $x \in D$ or $p \leq m - n$.

(b) $\bigcap\limits_{j \neq i} A_j = D$ for all $i \in \{1, 2, ..., m\}$.

(c) If $\bigcap\limits_{i \in S} A_i = D$ whenever $|S| = n$, then $\left|\bigcap\limits_{i \in T} A_i : D\right| \leq m - n + 1$ whenever $|T| = n - 1$.

(d) If Γ is maximal and U is an abelian minimal normal subgroup of G, then if $|\{i : U \subseteq A_i\}| = n$, either $U \subseteq D$ or $|U| \leq m - n$.

Lemma 2.2. (*Lemma 3.1 of* [*11*]). *Let M be a proper subgroup of the finite group G and let H_1, H_2, ..., H_k be subgroups with $|G : H_i| = \beta_i$ and $\beta_1 \leq \beta_2 \leq ... \leq \beta_k$. If $G = M \cup H_1 \cup \cdots \cup H_k$ then $\beta_1 \leq k$. Furthermore if $\beta_1 = k$ then $\beta_1 = \beta_2 = \cdots = \beta_k = k$ and $H_i \cap H_j \leq M$ for all $i \neq j$.*

Lemma 2.3. (*Lemma 3.2 of* [*11*]). *Let N be a normal subgroup of the finite group G. Let U_1,...,U_h be proper subgroups of G containing N and V_1,..., V_k be subgroups such that $V_iN = G$ with $|G : V_i| = \beta_i$ and $\beta_1 \leq \beta_2 \leq ... \leq \beta_k$. If $G = U_1 \bigcup \cdots \bigcup U_h \bigcup V_1 \bigcup \cdots \bigcup V_k$ then $\beta_1 \leq k$. Furthermore if $\beta_1 = k$ then $\beta_1 = \beta_2 = \cdots = \beta_k = k$ and $V_i \bigcap V_j \subseteq U_1 \bigcup \cdots \bigcup U_h$ for all $i \neq j$.*

Remark 2.4. (1) The only primitive subgroups of degree 5 are C_5, $C_5 \rtimes C_2$, $C_5 \rtimes C_4$, Alt_5 and Sym_5.

(2) The only primitive subgroups of degree 6 are Alt_5, Alt_6, Sym_5 and Sym_6.

(3) The only primitive subgroups of of degree 7 are C_7, $C_7 \rtimes C_2$, $C_7 \rtimes C_3$, $AGL(1, 7)$, $PSL(3, 2)$, Alt_7 and Sym_7.

Lemma 2.5. *Let G be a semisimple $\mathfrak{C}_8$-group. Then for every subset S of $\{1, \ldots, 8\}$ such that $|S| = 4$, we have $|\bigcap_{i \in S}(M_i)_G| = 1$.*

Proof. Suppose, on the contrary, that $K := \bigcap_{i\in S}(M_i)_G \neq 1$ and $|S| = 4$. Therefore by Lemma 2.1 (a), K contains no 5-element and no 7-element. Thus K is a normal soluble subgroup of G, which contradicts the semisimplity of G. □

Lemma 2.6. *Let G be a semisimple $\mathfrak{C}_8$-group. If $\alpha_l \leq 4$ for $l \leq 8$, then MSI(G)≤ 3.*

Proof. In the first we show that $\bigcap_{i=1}^{l}(M_i)_G \neq 1$.

Let $\bigcap_{i=1}^{l}(M_i)_G = 1$, then

$$G = \frac{G}{\bigcap_{i=1}^{l}(M_i)_G} \hookrightarrow \underbrace{Sym_4 \times \cdots \times Sym_4}_{l}.$$

Thus G is soluble, which it is not possible since G is semisimple. It follows from Lemma 2.5 that $l \leq 3$.

□

Now we introduce one question for researchers, because answer to bellow question is very important for classification of $\mathfrak{C}_n$-group.

Question 2.7. *Let m and n are positive integer numbers and G be a primitive subgroups of degree m. Now for which of number m, G is a $\mathfrak{C}_n$-group?*

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Talk Monoids over which products of indecomposable acts are indecomposable pp.: 1–4

Monoids over which products of indecomposable acts are indecomposable

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Abstract

In this paper we prove that for a monoid S, products of indecomposable right S-acts are indecomposable if and only if S contain a right zero. Besides, we prove that subacts of indecomposable right S-acts are indecomposable if and only if S is left reversible. Ultimately, we prove that the one element right S-act Θ_S is product flat if and only if S contains a left zero.

Keywords: Indecomposable act, left reversible monoid, Baer criterion, product flat, super flat.
Mathematics Subject Classification [2010]: Primary: 20M30; Secondary: 20M50

1 Introduction

Throughout this paper, S stands for a monoid and 1 denotes its identity element. A nonempty set A together with a mapping $A \times S \to A,\ (a, s) \rightsquigarrow as$, is called a right S-act or simply an act (and is denoted by A_S) if $a(st) = (as)t$ and $a1 = a$ for all $a \in A,\ s, t \in S$. We refer the reader to [1, 6] for more details on the concepts mentioned in this paper.

Since for a given monoid S any right S-act A_S is uniquely the disjoint union of indecomposable acts called indecomposable components of A_S, analogous to the bricks forming a wall, indecomposable acts deserve to be taken into consideration. A pioneering work in this account goes back to [3], where collection of all indecomposable right S-acts are partitioned into equivalence classes correspond to components of the right S-act $\mathscr{R}$ formed by letting S act on its right congruences by translation.

As mentioned, every right S-act A_S has a unique decomposition into indecomposable subacts, indeed, indecomposable components of A_S are the equivalence classes of the relation $\sim$ on A_S defined in [8] by $a \sim b$ if there exist $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n \in S,\ a_1, a_2, \ldots, a_n \in A_S$ such that

$$a = a_1s_1,\ a_1t_1 = a_2s_2,\ a_2t_2 = a_3s_3, \ldots,\ a_nt_n = b$$

which we shall call this sequence of equalities a scheme of length n.

The paper comprises three sections as follows. In the first section we presented a short account of the needed notions. The second one concerns indecomposable acts over left

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reversible monoids which we prove that in Baer criterion for acts, the condition of possessing a zero element can be abandoned in case that S is not left reversible. In third section we engage in the main results of this paper that is conditions under which indecomposable, product flat and super flat properties are preserved under products. Furthermore we prove that for the one element act Θ_S, the tensor functor $\Theta_S \otimes -$ preserves limits if and only if it preserves products, equivalently; products of indecomposable left S-acts are indecomposable.

2 Main results

In what follows we investigate indecomposable acts over left reversible monoids and give some characterizations for left reversible monoids regarding indecomposable property. In the next proposition we show that for left reversible monoids the length of the preceding scheme can be considered 2.

Proposition 2.1. *For a monoid S the following are equivalent.*

i) S is a left reversible monoid,

ii) a right S-act A_S is indecomposable if and only if for any $a, a' \in A_S$ there exist $s, s' \in S$ such that $as = a's'$,

iii) any indecomposable right S-act contains at most one zero element.

Recall that Baer criterion for right S-acts asserts that a right S-act is injective if and only if it possesses a zero element and is injective relative to all inclusions into cyclic right S-acts. In what follows we prove that if S is not left reversible then the condition of possessing a zero element in Baer criterion could be omitted.

Proposition 2.2. *Let S be a monoid that is not left reversible. A right S-act Q_S is injective if and only if it is injective relative to all inclusions into cyclic right S-acts.*

Here a question can be posed that
whether a monoid S over which injective acts are precisely ones that are injective relative to all inclusions into cyclic acts, is not left reversible.

In the next proposition we characterize monoids over which subacts of indecomposable acts are indecomposable.

Proposition 2.3. *For a monoid S all subacts of indecomposable right S-acts are indecomposable if and only if S is left reversible.*

Corollary 2.4. *For a monoid S the category of indecomposable right S-acts is a full subcategory of* **Act**-*S if and only if S is left reversible.*

The next proposition characterizes monoids over which non-zero cofree acts are decomposable.

Proposition 2.5. *For a monoid S the following are equivalent.*

i) all Non-zero cofree S-acts are decomposable,

ii) there exists a non-zero decomposable cofree right S-act,

iii) S is left reversible.

Note that products of indecomposable acts are not indecomposable in general, for instance if S is a left zero semigroup with an identity element externally adjoint, then there is no scheme in $S \times S$ connecting $(1, a)$ to $(a, 1)$ for $1 \neq a \in S$.

Corollary 2.6. *For a monoid S, $S^{S\times S}$ is indecomposable if and only if S^I is indecomposable for each nonempty set I. In the case that S is finite, $S^{S\times S}$ is indecomposable if and only if $S \times S$ is indecomposable.*

A subject of interest in the study of tensor products is preservation of limits by tensor functor $A_S \otimes -$ for a right S-act A_S which is investigated in [3]. Following terms used in this reference a right S-act A_S is called (finitely) super flat if the functor $A_S \otimes -$ preserves all (finite) limits, and (finitely) product flat if it preserves all (finite) products. Now if finite products of indecomposable acts are indecomposable then $S \times S$ is indecomposable. In the next theorem we show that this is a sufficient condition for finite products of indecomposable acts to be indecomposable which is equivalent to the one element left S-act ${}_S\Theta$ is finitely product flat. Besides in the sequel we show that products of indecomposable acts are indecomposable if and only if the one element left S-act ${}_S\Theta$ is product flat.

Theorem 2.7. *For a monoid S the following are equivalent.*

i) finite products of indecomposable acts are indecomposable,

ii) finite products of cyclic acts are indecomposable,

iii) S^n is indecomposable for each $n \in \mathbb{N}$,

iv) S^n is indecomposable for some $1 \neq n \in \mathbb{N}$,

v) $S \times S$ is indecomposable,

vi) the one element left S-act ${}_S\Theta$ is finitely product flat.

If products of indecomposable acts are indecomposable, then S^I is indecomposable for each nonempty set I, though, in comparison with Theorem 2.7, this is a strict implication. Hereby, we need an additional condition on S to fill the gap namely *Condition right-FI* under which there exists a fixed natural number n such that any pair of elements in any indecomposable right S-act can be connected via a scheme of length n (see [3, Corollary 2.11]).

In the next proposition we characterize monoids satisfying Condition right-FI.

Proposition 2.8. *Monoids satisfying condition right-FI are precisely left reversible monoids which the associated natural number can be taken* 2.

In the next proposition we characterize monoids for which products of indecomposable acts are indecomposable.

Proposition 2.9. *For a monoid S the following are equivalent:*

i) products of indecomposable right S-acts are indecomposable,

ii) S is left reversible and $S^{S\times S}$ is indecomposable,

iii) S satisfies condition right-FI and $S^{S\times S}$ is indecomposable,

iv) non-zero cofree acts are decomposable and $S^{S\times S}$ is indecomposable,

v) All subacts of indecomposable right S-acts are indecomposable and $S^{S\times S}$ is indecomposable.

For commutative monoids, the left reversibility condition in Proposition 2.9 is fulfilled and the following corollary is obtained.

Corollary 2.10. *For a commutative monoid S products of indecomposable acts are indecomposable if and only if $S^{S\times S}$ is indecomposable.*

Lemma 2.11. *For a left reversible monoid S, finite products of indecomposable right S-acts are indecomposable if and only if S is right collapsible.*

Theorem 2.12. *For a monoid S products of indecomposable right S-acts are indecomposable if and only if S has a right zero.*

Note that in [3, Proposition 3.8] states that for a proper right ideal K of a monoid S if the Rees factor act S/K is finitely product flat then S/K is super flat. So a naturally come question to the mind is the case that $K = S$. In the next proposition we show that in this case product flatness is equivalent to super flatness. Indeed in [3] it is proved that the one element left S-act ${}_S\Theta$ is product flat if and only if S satisfies condition right-FI and S^I is indecomposable for each set I. Hereby we give the next proposition which is an improvement of this result.

Proposition 2.13. *For a monoid S the following are equivalent:*

i) the one element right S-act Θ_S is super flat,
ii) the one element right S-act Θ_S is product flat,
iii) S contains a left zero.
iv) products of indecomposable left S-acts are indecomposable.

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On direct products of S-posets

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Abstract

In this paper we investigate on direct products of (po-)torsion free, principally weakly and weakly (po-)flat and strongly flat S-posets. Moreover, a characterization of pomonoids over which direct products of S-posets satisfying conditions (P), (E), and (P_w) again satisfy that conditions is given.

Keywords: Pomonoid, S-poset, Direct product
Mathematics Subject Classification [2010]: 06F05, 20M30

1 Introduction

A monoid S that is also a partially ordered set, in which the binary operation and the order relation are compatible, is called a pomonoid. A right S-poset A_S is a right S-act A equipped with a partial order $\leq$ and, in addition, for all $s,t \in S$ and $a,b \in A$, if $s \leq t$ then $as \leq at$, and if $a \leq b$ then $as \leq bs$. An S-subposet of a right S-poset A is a subset of A that is closed under the S-action. The definition of ideal is the same for the act case. Moreover, $X \subseteq S$ and take $(X] = \{p \in S \mid \ \exists x \in X, p \leq x\}$. Finally, an S-morphism from S-poset A to S-poset C is a monotonic map that preserves S-action.

A right S-poset A_S is weakly po-flat if $a \otimes s \leq a' \otimes t$ in $A_S \otimes S$ implies that the same inequality holds also in $A_S \otimes {}_S(Ss \cup St)$ for $a, a' \in A_S, s, t \in S$. A right S-poset A_S is principally weakly po-flat if $as \leq a's$ implies that $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_S Ss$ for $a, a' \in A_S, s \in S$. Weakly flat and principally weakly flat can be defined as same as the previous by replacing $\leq$ by $=$.

An S-poset A_S satisfies condition (P_w) if, for all $a,b \in A$ and $s,t \in S$, $as \leq bt$ implies $a \leq a'u,\ a'v \leq b$ for some $a' \in A,\ u,v \in S$ with $us \leq vt$. A right S-poset A_S satisfies condition (P) if, for all $a,b \in A$ and $s,t \in S$, $as \leq bt$ implies $a = a'u, b = a'v$ for some $a' \in A,\ u,v \in S$ with $us \leq vt$, and it satisfies condition (E) if, for all $a \in A$ and $s,t \in S$, $as \leq at$ implies $a = a'u$ for some $a' \in A,\ u \in S$ with $us \leq ut$. A right S-poset is called strongly flat if it satisfies both conditions (P) and (E).

If S is a pomonoid, the cartesian product S^Γ is a right and left S-poset equipped with the order and the action componentwise where Γ is a non-empty set. Moreover, $(s_\gamma)_{\gamma\in\Gamma} \in S^\Gamma$ is dented simply by (s_γ), and the right S-poset $S \times S$ will be denoted by $D(S)$.

Recall that an S-poset morphism $f : A_S \to B_S$ is called an *order-embedding* if $f(a) \leq f(a')$ implies $a \leq a'$, for all $a, a' \in A$. The proof of the following lemma is routine.

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Lemma 1.1. *Let S be a pomonoid, Γ any non-empty set, and I a left ideal of S. Then the following are equivalent:*
(*i*) $S^\Gamma \otimes I \to S^\Gamma \otimes S$ *is order-embedding;*
(*ii*) $S^\Gamma \otimes I \to I^\Gamma$ *is order-embedding.*

Proposition 1.2. *Let S be a pomonoid and $s \in S$. Then the following are equivalent:*
(*i*) $f_s : S^\Gamma \otimes Ss \to (Ss)^\Gamma$ *is order-embedding for all* $\Gamma \neq \emptyset$;
(*ii*) *there exist* $(s_1, t_1), \ ..., (s_n, t_n) \in D(S)$ *such that*
(1) $s_i s \leq t_i s$ *for all* $1 \leq i \leq n$, *and*
(2) *if* $us \leq vs$ *for some* $u, v \in S$, *then there exist* $u_1, ..., u_n \in S$ *such that*

$$\begin{aligned} u &\leq u_1 s_1 \\ u_1 t_1 &\leq u_2 s_2 \\ &\vdots \\ u_n t_n &\leq v. \end{aligned}$$

2 Main results

First, we begin our investigation with the weakest of the flatness properties. An element c of a pomonoid S will be called *right po-cancellable* if, for all $s, t \in S$, $sc \leq tc$ implies $s \leq t$. A right S-poset A_S is called po-torsion (torsion) free if, for $a, a' \in A$ and a right po-cancellable (cancellable) element c of S, from $ac \leq a'c$ ($ac = a'c$) it follows that $a \leq a'$ ($a = a'$). The proof of the following result is immediately evident.

Proposition 2.1. *For any pomonoid S direct products of po-torsion (torsion) free S-posets are again po-torsion (torsion) free.*

Recall that a pomonoid S is call a left PSF pomonoid if all principal left ideals of a pomonoid S are strongly flat. Let S be a pomonoid. An element $u \in S$ is called *right semi-po-cancellable* if for $s, t \in S, su \leq tu$ implies that there exists $r \in S$ such that $ru = u, \ sr \leq tr$. In [7], it is shown that a pomonoid S is left PSF pomonoid if and only if every element of S is right semi-po-cancellable.

Lemma 2.2. *([7]) Over a left PSF pomonoid S a right S-poset A_S is principally weakly po-flat if and only if for any $a, a' \in A_S, s \in S$, if $as \leq a's$, then there exists $r \in S$ such that $rs = s$ and $ar \leq a'r$.*

Proposition 2.3. *If S is a left PSF pomonoid, then the S-poset S^n is principally weakly po-flat for each $n \in \mathbb{N}$.*

Since principally weakly po-flat implies principally weakly flat, over a left PSF pomonoid S, S^n is also principally weakly flat.

Proposition 2.4. *The following are equivalent for a pomonoid S:*
(*i*) S^Γ_S *is principally weakly po-flat for each non-empty set* Γ;
(*ii*) *For any $s \in S$, the mapping* $f_s : S^\Gamma \otimes Ss \longrightarrow (Ss)^\Gamma$ *is order-embedding for each non-empty set* Γ;
(*iii*) *For any $s \in S$ there exist* $(s_1, t_1), \ldots, (s_n, t_n) \in D(S)$ *such that*

(1) $s_i s \leq t_i t$ *for all* $1 \leq i \leq n$, *and*

(2) *if* $us \leq vs$ $(u, v \in S)$, *then there exist* $u_1, ..., u_n \in S$ *such that*

$$\begin{aligned} u &\leq u_1 s_1 \\ u_1 t_1 &\leq u_2 s_2 \\ &\vdots \\ u_n t_n &\leq v. \end{aligned}$$

In [7], it is shown that a right S-poset A_S is weakly po-flat if and only if it is principally weakly po-flat and satisfies condition (W):

If $as \leq a't$ for $a, a' \in A_S,\ s, t \in S$, then there exist $a'' \in A_S,\ p \in Ss$ and $q \in St$ such that $p \leq q,\ as \leq a''p,\ a''q \leq a't$.

For each $(p, q) \in D(S)$, $\{(u, v) \in D(S) |\ \exists w \in S, u \leq wp, wq \leq v\}$ is a left S-poset and will be denoted by $\widehat{S(p, q)}$ from now on. Clearly $\widehat{S(p, q)}$ contains the cyclic S-poset $S(p, q)$. Moreover, if $Ss \cap (St] \neq \emptyset$, $\{(as, a't)|\ as \leq a't\}$ is denoted by $H(s, t)$.

Proposition 2.5. *The diagonal S-poset $D(S)$ is weakly po-flat if and only if it is principally weakly po-flat and $Ss \cap (St] = \emptyset$ or for each $(as, a't)$ and $(bs, b't)$ in $H(s, t)$ there exist $(p, q) \in H(s, t)$ such that $(as, a't), (bs, b't) \in \widehat{S(p, q)}$.*

Definition 2.6. Let S be a pomonoid. A finitely generated left S-poset ${}_SB$ is called *finitely definable (FD)* if the S-morphism $S^\Gamma \otimes B \to B^\Gamma$ is order-embedding for all non-empty set Γ.

Theorem 2.7. *The following are equivalent for a pomonoid S:*

(*i*) *S^Γ is weakly po-flat right S-poset for each $\Gamma \neq \emptyset$;*

(*ii*) *every finitely generated left ideal of S is FD;*

(*iii*) *Ss is FD for each $s \in S$, and*

for every $s, t \in S$, if $Ss \cap (St] \neq \emptyset$, then $H(s, t) \subseteq \widehat{S(p, q)}$ for some $(p, q) \in H(s, t)$.

The ordered version of locally cyclic acts is called *weakly locally cyclic* S-poset as an S-poset A that every finitely generated S-subposet of A is contained in a cyclic S-poset. Moreover, a left ideal of S that is also weakly locally cyclic is called *weakly locally principal left ideal.* The set $L(a, b) := \{(u, v) \in D(S) |\ ua \leq vb\}$ is a left S-subposet of $D(S)$, and the set $(l(a, b) := \{u \in S |\ ua \leq ub\})$ is a left ideal of S.

Proposition 2.8. *For any pomonoid S the following are equivalent:*

(*i*) *any finite product of right S-posets satisfying condition (P) (condition (E)) satisfies condition (P) (condition (E));*

(*ii*) *the diagonal S-poset $D(S)$ satisfies condition (P) (condition (E));*

(*iii*) *for every $a, b \in S$ the set $L(a, b)$ $(l(a, b))$ is either empty or a weakly locally cyclic left S-poset (weakly locally principal left ideal of S).*

Proposition 2.9. *For any pomonoid S the following are equivalent:*

(*i*) *any finite product of right S-posets satisfying condition (P_w) satisfies condition (P_w);*

(*ii*) *the diagonal S-poset $D(S)$ satisfies condition (P_w);*

(*iii*) *for every* $a, b \in S$ *the set* $L(a, b)$ *is either empty or for each two elements* $(u, v), (u', v') \in L(a, b)$ *there exists* $(p, q) \in L(a, b)$ *such that* $(u, v),\ (u', v') \in \widehat{S(p, q)}$.

Theorem 2.10. *The following are equivalent for a pomonoid S:*

(*i*) *the direct product of every non-empty family of right S-posets satisfying condition (P) (condition (E)) satisfies condition (P) (condition (E));*

(*ii*) $(S^{\Gamma})_S$ *satisfies condition (P) (condition (E)) for every non- empty set* Γ*;*

(*iii*) *for every* $a, b \in S$ *the set* $L(a; b)$ $(l(a, b))$ *is either empty or a cyclic left S-poset (principal left ideal of S).*

Theorem 2.11. *The following are equivalent for a pomonoid S:*

(*i*) *the direct product of every non-empty family of right S-posets satisfying condition* (P_w) *satisfies condition* (P_w)*;*

(*ii*) $(S^{\Gamma})_S$ *satisfies condition* (P_w) *for every non- empty set* Γ*;*

(*iii*) *for every* $a, b \in S$ *the set* $L(a, b)$ *is either empty or there exists* $(p, q) \in L(a, b)$ *such that* $L(a, b) = \widehat{S(p, q)}$.

Corollary 2.12. *The following are equivalent for a pomonoid S:*

(*i*) *every product* S^{Γ} *is strongly flat right S-poset for a non-empty set* Γ*;*

(*ii*) *every product* $\prod_{i\in I} A_i$ *of strongly flat right S-posets* A_i, $i \in I$, *is strongly flat;*

(*iii*) *for all* $(a, b) \in D(S)$, $L(a, b)$ *is either empty or a cyclic left S-poset and* $l(a, b)$ *is either empty or a principal left ideal of S.*

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On graded generalized local cohomology modules

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Abstract

Let M and N be two finitely generated graded modules over a standard graded Noetherian ring $R = \bigoplus_{n\geq 0} R_n$. In this paper we show that if R_0 is semi-local of dimension ≤ 2 then, the set $\mathrm{Ass}_{R_0}\Big(H^i_{R_+}(M,N)_n\Big)$ is asymptotically stable for $n \to -\infty$ in some special cases. Also, we study the torsion-freeness of graded generalized local cohomology modules $H^i_{R_+}(M,N)$. Finally, the tame loci $T^i(M,N)$ of (M,N) are introduced and some sufficient conditions are proposed for the openness of these sets in Zariski topology.

Keywords: generalized local cohomology modules, associated prime ideals, tame loci
Mathematics Subject Classification [2010]: 13D45, 13A02

1 Introduction

Assume that R is a commutative Noetherian ring with identity and all modules are unitary. Let $\mathfrak{a}$ be an ideal of R and $R - Mod$ the category of R-modules and R-homomorphisms. We denote by $\mathbb{N}_0$ and $\mathbb{N}$ the sets of non-negative and positive integers, respectively.

For $i \in \mathbb{N}_0$, the i-th generalized local cohomology functor with respect to $\mathfrak{a}$ is a generalization of the i-th local cohomology functor with respect to $\mathfrak{a}$, i.e. $H^i_{\mathfrak{a}}(-) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, -)$([1], [5]). It is defined, by Herzog ([6]), as follows:

$$H^i_{\mathfrak{a}}(-,-) : R - Mod \times R - Mod \to R - Mod$$

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

For all R-modules M and N, $H^i_{\mathfrak{a}}(M,N)$ is called the i-th generalized local cohomology module of M and N with respect to $\mathfrak{a}$. These functors coincide when $M = R$ and have been studied by many authors (see for instance [2], [3].

Now, let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a standard graded Noetherian ring and let M and N be two finitely generated graded R-modules. Also, assume that $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ denotes the irrelevant ideal of R. It is well known that for each $i \in \mathbb{N}_0$, $H^i_{R_+}(M,N)$ carries a natural

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grading. Then, according to [8], $H^i_{R_+}(M,N)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and it vanishes for all sufficiently large values of n. Therefore, the R_0-modules $H^i_{R_+}(M,N)_n$ are asymptotically trivial if $n \to +\infty$.

One basic question in this respect is to ask for the asymptotic behavior of the graded components $H^i_{R_+}(M,N)_n$ for $n \to -\infty$. The concept of tameness is the most fundamental concept related to the asymptotic behavior of cohomology. A graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is said to be tame, or asymptotically gap free, if either $T_n \neq 0$ for all $n \ll 0$ or else $T_n = 0$ for all $n \ll 0$. In this paper we are interested to the study of the tame loci $T^i(M,N)$ with respect to a pair of modules (M,N), that is, the sets of all primes $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$ for which the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M,N)_{\mathfrak{p}_0}$ is tame. Tame loci $T^i(R,N)$ have been studied in [4].

The paper is organized as follows: in the second section, we study the asymptotic behavior of $\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ as $n \to -\infty$. More precisely, we show that if R_0 is semi-local and $\dim R_0 \leq 2$ then the set $\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ is asymptotically stable in each of the following cases:

(1) $\operatorname{depth}(R_0) > 0$ and $\Gamma_{\mathfrak{m}_0}(M) = 0 = \Gamma_{\mathfrak{m}_0}(N)$,

(2) $\dim_{R_0}\left(H^{i-1}_{R_+}(M,N)_n\right) \leq 1$ for all $n \ll 0$ (Theorem 2.9).

Section 3 deals with the torsion-freeness of $H^i_{R_+}(M,N)$. In this section we show that if R_0 is a domain and $\dim H^i_{R_+}(N) \leq 2$, then there is some $t \in R_0 - \{0\}$ such that the $(R_0)_t$-module $H^i_{R_+}(M,N)_t$ is torsion-free (or vanishes) for each $i \in \mathbb{N}_0$ (Theorem 3.2).

Section 4 is devoted to the study of Tame loci $T^i(M,N)$. In this section we use the results in previous sections to show that these sets are open in Zariski topology in some special cases.

Throughout the paper, $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a standard graded Noetherian ring, $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ is the irrelevant ideal of R and M and N denote two finitely generated graded R-modules.

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Talk On hypergroups with trivial fundamental group pp.: 1–4

On hypergroups with trivial fundamental group

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Abstract

Let $(H, \circ)$ be a hypergroup. Consider the fundamental relation β^*, as the smallest equivalence relation on H, such that the quotient algebraic structures $(H/\beta^*, \otimes)$, the fundamental group of H, is a group. In this paper we investigate some conditions such that for a given finite hypergroup H, its fundamental group $(H/\beta^*, \otimes)$ is a trivial group.

Keywords: OC-Hypergroup, Adapted Hypergroup, TS-Hypergroup, Identical Hypergroup, m^n-Hypergroup
Mathematics Subject Classification [2010]: 20N20

1 Introduction

The concept of hypersrtucture was defined by Marty in 1934 [6]. A non-empty set H together with a mapping $\circ$ (namely hyperproduct) from $H \times H$ into $P^*(H)$, the set of all non-empty subsets of H, is called a *hypergroupoid* and denoted by $(H, \circ)$. If there is no ambiguity, we simply write H instead of $(H, \circ)$. For two non-empty subsets $A, B \subseteq H$, define $A \circ B = \bigcup_{(a,b)\in A\times B} a \circ b$. By abuse of notation, $a \circ b = \{x\}$, $A \circ \{a\}$ and $\{a\} \circ A$ are denoted by $a \circ b = x$, $A \circ a$ and $a \circ A$, respectively. A hypergroupoid $(H, \circ)$ is called a *hypergroup* if $\circ$ is associative and $H \circ x = x \circ H = H$, for every $x \in H$ (*reproduction axiom*). From now on, if there is no ambiguity, by xy (for $x, y \in H$) and H, we mean $x \circ y$ and hypergroup $(H, \circ)$, respectively. A hypergroup H is *commutative* if $xy = yx$ for every $x, y \in H$. Many books and papers has been written about the applications of hyperstructures theory in mathemathics and even other sciences ([1, 2, 3]). The purpose of this paper is to study some finite hepergroups that have trivial fundamental group. In this regards, we introduce the notion of overlapped covering of a hypergrpup, which leads us to class of *OC-hypergroups*, and then some special subclasses, namely class of *adapted hypergroups* and class of *TS-hypergroups*. First, we need some general and basic concepts of hyperstructures theory.
A non-empty subset A of the hypergroup H is called a *complete part* of H if for all positive integer n and for all $(x_1, x_2, ..., x_n) \in H^n$, $\prod_{i=1}^n x_i \bigcap A \neq \emptyset$ implies $\prod_{i=1}^n x_i \subseteq A$. The complete closure of A in H is the intersection of all complete parts containing A and is denoted by $C(A)$ and is equal to $K(A)$ that is obtained as the following way:

$$K_1(A) = A,$$

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$$K_n(A) = \{x \in H| \quad \exists n \in \mathbb{N}, \exists z_1, z_2, \dots, z_n \in H : \quad x \in \prod_{i=1}^{n} z_i, \quad \prod_{i=1}^{n} z_i \bigcap K_{n-1}(A) \neq \emptyset\}$$

and $K(A) = \bigcup_{n \geq 1} K_n(A)$.
A hypergroup H is called *complete* if for all $x, y \in H$, $C(xy) = xy$. For a hypergroup H, let $\mathcal{U}(H)$ be the set of all finite hyperproducts of the elements of H. Define the relation $\beta = \bigcup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation and for every integer $n > 1$, β_n is the relation defined as the following:

$$x \beta_n y \Longleftrightarrow \exists (z_1, z_2, \dots, z_n) \in H^n : x, y \in \prod_{i=1}^{n} z_i.$$

The relation β was introduced by Koskas [5] and was studied mainly by Corsini [1] and Freni [4]. Consider β^* as the *transitive closure* of β. Indeed,

$$x \beta^* y \Longleftrightarrow \exists x_1, x_2, \dots, x_n \in H : \quad x = x_1 \beta x_2 \beta \cdots x_{n-1} \beta x_n = y$$

in which $x_i, x_{i+1} \in u_i \in \mathcal{U}(H)$ for $1 \leq i \leq n-1$.
Let R be an equivalence relation on H and $\emptyset \neq A, B \subseteq H$. Then $A\overline{\overline{R}}B$ if and only if xRy for all $(x, y) \in A \times B$. An equivalence relation R on H is said to be *strongly regular* if for all $(x, a, b) \in H^3$, aRb implies $ax\overline{\overline{R}}bx$ and $xa\overline{\overline{R}}xb$. Refering to [7], it is well-known that the relation β^* is called the *fundamental relation* of hypergroup H, as the smallest strongly regular equivalence relation such that the quotient $(H/\beta^*, \otimes)$ is a group, where

$$\beta^*(x) \otimes \beta^*(y) = \beta^*(z) \quad \forall x, y \in H, \quad \forall z \in xy.$$

The group $(H/\beta^*, \otimes)$ is called the *fundamental group* of H. Freni in [4] proved that β is transitive on hypergroups, i.e., $\beta^* = \beta$.
Consider ϕ_H as the canonical map $\phi_H : H \longrightarrow H/\beta^*$, where $\phi_H(a) = \beta^*(a)$. The set $\omega_H = \{a \in H| \quad \phi_H(a) = 1_{H/\beta^*}\}$ is called the *heart* or *core* of H. Let H be a hypergroup. An element $e \in H$ is called an *identity* such that $x \in ex \cap xe$ for each $x \in H$. For $x \in H$, if there exists $y \in H$ such that $e \in xy \cap yx$, then x is said to be *invertible* and y is an inverse of x. The set of all identities of H is denoted by E. A hypergroup H is *regular* if $E \neq \emptyset$ and every element of H has an inverse. A hypergroup H is called *identical* if $E = H$. Let H be a hypergroup. We say $x \in H$ is *adapted* if there exists $e \in E$ and $k \in \mathbb{N}$ such that $e, x \in x^k$. In this case, we say that x is e-adapted or adapted with respect to e and $\delta_e(x)$ denotes the smallest element of k's satisfying in $e, x \in x^k$. A hypergroup H is adapted if each $x \in H$ is adapted. In other words, H is adapted if each $x \in H$ is e-adapted for some $e \in E$. A hypergroup H with $e \in E$ is *e-adapted* if each $x \in H$ is e-adapted, i.e., for all $x \in H$ there exists $n \in \mathbb{N}$ such that $e, x \in x^n$. A hypergroup H with $e \in E$ is called *strongly e-adapted* if there exists a fixed $n \in \mathbb{N}$ such that $e, x \in x^n$ for all $x \in H$. Obviously if H is strongly e-adapted, then it is e-adapted. Let H be strongly e-adapted and set $M = \{i \in \mathbb{N}| \quad \forall x \in H \quad e, x \in x^i\}$. Clearly, M is non-empty. For the strongly e-adapted hypergroup H, we set $\delta_e(H) = min(M)$ and say H is e-adapted of power $\delta_e(H)$.

Definition 1.1. Let $x_1, x_2, \dots, x_m \in H$ be a sequence of not necessarily distinct elements. Set $P = \prod_{i=1}^{m} x_i$. With notation $x^r := \prod_{i=1}^{r} x$ for each $r \in \mathbb{N}$, we rewrite and denote P by $S_P = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k}$ in which $n_1 + n_2 + \cdots + n_k = m$, $x_{i_1} \neq x_{i_2} \neq \cdots \neq x_{i_k}$

and $\{i_1, i_2, \dots i_k\} \subseteq \{1, 2, \dots, m\}$ with the same order $i_1 \leq i_2 \leq \cdots \leq i_k$ (for some $k \in \{1, 2, \dots, m\}$). We say $S_P = \prod_{j=1}^{k} x_{i_j}^{n_j}$ is the *simplified form* of hyperproduct $P = \prod_{i=1}^{m} x_i$.

The following example illustrates what we defined.

Example 1.2. Let $x_1, x_2, \dots, x_n \in H$ be distinct. The simplified forms of hyperproducts $x_1x_2x_5x_5x_1$, $x_2x_3x_2x_3x_2x_3$ and $x_1x_1x_3x_3x_3x_1$ are $x_1x_2x_5^2x_1$, $x_2x_3x_2x_3x_2x_3$ and $x_1^2x_3^3x_1$, respectively.

Definition 1.3. A sequence $x_1, x_2, \dots, x_m$ of not necessarily distinct elements of hypergroup H is called a *total sequence* or briefly *T-sequence* if $\prod_{i=1}^{m} x_i = H$ and $\prod_{i=1}^{m} x_i$ is called a *T-hyperproduct*. We call the sequence $n_1, n_2, \dots, n_k \in \mathbb{N}$ appeared in the simplified form $S_P = \prod_{j=1}^{k} x_{i_j}^{n_j}$ of the hyperproduct $P = \prod_{i=1}^{m} x_i$, the *T-power sequence* of the T-sequence $x_1, x_2, \dots, x_m \in H$.

Definition 1.4. We say a hypergroup H is a *TS-hypergroup* of *T-power* $(n_1, n_2, \dots, n_k)$ if there is a T-sequence $x_1, x_2, \dots, x_m \in H$ with simplified form $\prod_{j=1}^{k} x_{i_j}^{n_j}$.

Definition 1.5. Let H be a hypergroup and $(m, n) \in \mathbb{N}^2$. We say H is an *m^n-hypergroup* if there exist not necessarily distinct elements $x_1, x_2, \dots, x_m \in H$ such that $\prod_{j=1}^{n} \prod_{i=1}^{m} x_i$ is a T-hyperproduct. In this case we write $(\prod_{i=1}^{m} x_i)^n = H$.

Definition 1.6. Let A be a set and $A_1, A_2, \dots, A_n \subseteq A$ with $1 < n \in \mathbb{N}$ such that $A = \bigcup_{i=1}^{n} A_i$, and $A_i \cap A_{i+1} \neq \emptyset$ for $i = 1, 2, ..., n-1$. Then, we say $(A_1, A_2, \dots, A_n)$ is an *overlapped covering of length n* and A has an overlapped covering.

Remark 1.7. Note that in 1.6 A_i's can be repeated. Also, $(A_1, \dots, A_n)$ is said non-trivial if $A_i \neq A$ for some i.

Definition 1.8. Let H be a hypergroup. We say H is an *OC-hypergroup* if $\mathcal{U}(H)$ contains an overlapped covering of H.

2 Main results

As the first results, we have the following statements:

Proposition 2.1. *Every e-adapted hypergroup has trivial fundamental group.*

Proposition 2.2. *Every TS-hypergroup has trivial fundamental group.*

Proposition 2.3. *A hypergroup H is an OC-hypergroup if and only if it has trivial fundamental group.*

Proposition 2.4. *Every commutative e-adapted hypergroup is a regular hypergroup.*

Let H be a hypergroup and set

$$\widehat{E} = \{e \in E| \quad \forall x \in H \quad \exists n \in \mathbb{N} : \quad e \in x^n\}.$$

Even we can restrict $\widehat{E}$ to

$$\widehat{\widehat{E}} = \{e \in E| \quad \forall x \in H \quad \exists n \in \mathbb{N} : \quad e, x \in x^n\}.$$

Theorem 2.5. *Let H be a hypergroup with $E \neq \emptyset$.*

1. *If H is a TS-hypergroup with complete part E, then H is an identical hypergroup with trivial fundamental group.*
2. *If H is an e-adapted hypergroup with complete parts $\widehat{E}$ or $\widehat{\widehat{E}}$, then H is an identical hypergroup with trivial fundamental group.*

Proposition 2.6. *Let H be a complete commutative hypergroup with at least two elements. If H is e-adapted of power $n \in \mathbb{N}$, then*

1. *H is a regular 2^1-hypergroup as well as a regular 2^n-hypergroup, or*
2. *H is a regular 1^2-hypergroup as well as a regular 1^{2n}-hypergroup.*

Note that in 2.6 in the first case, H is of T-powers $(1,1)$, (n,n) and $(1,1,\ldots,1)$ with $2n$ components 1, and in the second case, H is of T-powers (2) and $(2n)$.

Proposition 2.7. *Let H be an OC-hypergroup. Then H does not have any proper complete part.*

Theorem 2.8. *Let H be a complete OC-hypergroup. Then H is a regular identical hypergroup that has trivial fundamental group and ω_H is the only complete part of H.*

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Talk On prime submodules and hypergraphs pp.: 1–4

on prime submodules and hypergraphs

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Abstract

In this paper, for every free R-module F of finite rank, we associate a hypergraph $PH_Q(F)$ called the prime submodules hypergraph of F with respect to Q, where Q is a prime ideal of comutative ring R. We then investigate the interplay between the module-theoretic properties of F and the graph-theoretic properties of $PH_Q(F)$. We also show that $PH_Q(F)$ is the union of Steiner systems and use their properties for counting the number of Q-prime submodules of F when Q is a maximal ideal of R and $[R:Q]$ (number of cosets R in Q) is finite.

Keywords: Hypergraphs, Prime submodules, Turán graphs, Steiner systems .
Mathematics Subject Classification [2010]: 05C65, 05C15, 13C99, 51E10.

1 Introduction

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Throughout this article, all rings are assumed to be commutative with identity and F denotes a free R-module of finite rank. Let M be an R-module and Q be a prime ideal of R. A proper submodule N of M is called Q-prime if, for $r \in R$, $m \in M$ and $rm \in N$ we have $m \in N$ or $r \in Q = (N : M)$, where $(N : M) = \{r \in R \mid rM \subseteq N\}$. We use the notation $R^{(n)}$ for $\underbrace{R \oplus \cdots \oplus R}_{n-times}$.

A hypergraph is a pair $H = (V, E)$ of disjoint sets where the elements of E are nonempty subsets (of any cardinality) of V. The elements of V are the vertices and the elements of E are the edges of hypergraph. Note that, if the cardinality of each edge is two, then we have a simple graph. For $x \in V$ the degree of x denoted by $d_H(x)$, is the number of edges in E containing x. A hypergraph in which all vertices have the same degree r is said to be regular of degree r or r-regular. A hypergraph is called an intersecting if every pair of edges intersects nontrivially. The hypergraph $H = (V, E)$ is called k-uniform whenever every edge e of H is a $k-$subset of V. A k-uniform hypergraph H is called complete if every k-subset of the vertices is an edge of H. The hypergraph $H' = (V', E')$ is a subhypergraph of the hypergraph $H = (V, E)$, whenever $V' \subset V$ and $E' \subset E$. The union of two hypergraphs H and H' is the hypergraph $H \cup H'$ with $V(H \cup H') = V(H) \cup V(H')$ and $E(H \cup H') = E(H) \cup E(H')$.

Let H be a k-uniform hypergraph. A subset A of $V(H)$ is called a clique of H if every k-subset of A is an edge of H. A path of a hypergraph H is an alternating sequence of

*Speaker

distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k$ such that for all $1 \leq i \leq k-1$, v_i and v_{i+1} are in e_i. The number of edges of a path is called its length. The distance between two vertices x and y of H, denoted by $d_H(x, y)$, is the length of the shortest path from x to y. If no such path between x and y exists, we set $d_H(x, y) = \infty$. The greatest distance between any two vertices in H is called the diameter of H and is denoted by $diam(H)$. The hypergraph H is said to be connected whenever $diam(H) < \infty$.

A Steiner system $S(t, k, n)$ $(1 < t < k < n)$ is a k-uniform hypergraph on n vertices with the property that every t-element subsets of vertices is contained in exactly one edge. If $t = 2$ then we have a projective plane and $S(2, 3, 7)$ is called Fano plane. In combinatorial mathematics, a set S of $k-$subsets of an $n-$set X is a block design with parameters (t, k, n, λ) if every $t-$subset of X belongs to exactly λ elements of S. A Steiner system is a type of block design, specifically a $t-$design, with $\lambda = 1$ and $t \geq 2$[see 2].

We recall that a complete multipartite graph $K_{a_1,\ldots,a_s}$ has a vertex-set which may be partitioned into s parts $B_1, B_2, \ldots, B_s$, where $|B_i| = a_i$ $(1 \leq i \leq s)$. Two vertices are adjacent if they belong to different parts. This graph is not regular in general but its complement consists of regular connected components [see 3].

The Turán graph $T(n, r)$ is a complete multipartite graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible.If n is divisible by r, then it is a regular graph.

2 Main results

Definition 2.1 Let $F = R^{(n)}$, Q be a prime ideal of R and $H_Q(F)$ denote the hypergraph with vertices $F^* = F \backslash Q^{(n)}$. A subset $\{X_1, \ldots, X_k\}$ $(2 \leq k \leq n)$ of F^* is an edge of $H_Q(F)$, if the determinant of every submatrix $k \times k$ of matrix $B = [X_1 \ldots X_k]$ is in Q.

Remark 2.2 Let $H_Q(F)$ be the hypergraph in Definition 2.1 and $H_Q^k(F)$ $(2 \leq k \leq n)$ denote a subhypergraph of $H_Q(F)$ with $V(H_Q^k(F)) = F^*$ and $E(H_Q^k(F)) = \{e \in E(H_Q(F)) \mid |e| = k\}$. Then $H_Q^k(F)$ is a k-uniform hypergraph, for $2 \leq k \leq n$. It follows that $H_Q(F)$ is the union of k-uniform hypergraphs $H_Q^k(F)$, $2 \leq k \leq n$. Furthermore, if $k = 2$ then $H_Q^2(F)$ is a simple graph.We call it $PG_Q(F)$.

Theorem 2.3 Let $F = R^{(n)}$ and N be a submodule of F. Then N is a prime submodule of F if and only if $(N : F) = Q$ is a prime ideal of R and $N = Q^{(n)}$ or there exists a positive integer $1 \leq k \leq n-1$ such that $N^* = N \backslash Q^{(n)}$ is a clique of a $(k+1)$-uniform hypergraph $H_Q^{k+1}(F)$, that is not strictly contained in any clique of $H_Q^{k+1}(F)$.

Theorem 2.4 Let $F = R^{(n)}$ $(n \geq 2)$ and Q be a prime ideal of R. Then $PG_Q(F)$ is a disconnected graph with complete connected components. Furthermore, $\overline{PG_Q(F)}$ is a complete multipartite graph, if R is finite.

Corollary 2.5 Let $F = R^{(n)}$ and Q be a prime ideal of a finite ring R. Let $[R : Q] = m$. Then $\overline{PG_Q(F)}$ is a regular Turán graph with parameter $(|Q|^n(m^n - 1), \sum_{i=1}^{n} m^{n-i})$.

Proposition 2.6 Let $F = R^{(n)}$ and Q be a prime ideal of R. Let $N \neq Q^{(n)}$ be a Q-prime submodule of F. Then $N^* = N \backslash Q^{(n)}$ is the union of components of $PG_Q(F)$ which have a vertex in N^*.

Definition 2.7 Let $F = R^{(n)}$ and Q be a prime ideal of R. The prime submodule hypergraph of F with respect to Q (denoted by $PH_Q(F)$) is the hypergraph with vertex set

$V(PH_Q(F)) = \{[X] \mid [X]$ is a connected component of $PG_Q(F)\}$. A subset $e = \{[X_i] \mid i \in I\}$ of $V(PH_Q(F))$ is an edge of $PH_Q(F)$, if $\bigcup_{i\in I}[X_i] \cup Q^{(n)}$ is a prime submodule of F (equivalently, $\bigcup_{i\in I}[X_i]$ is a clique of $H^k_Q(F)$ that is not strictly contained in other cliques, for some $2 \le k \le n$).

Remark 2.8 Let $PH_Q(F)$ be as above. We use $PH^k_Q(F)(1 \le k \le n-1)$ as a subhypergraph of $PH_Q(F)$ with $V(PH^k_Q(F)) = V(PH_Q(F))$ and $E(PH^k_Q(F)) = \{e \in E(PH_Q(F)) \mid \bigcup_{[x]\in e}[x] \cup Q^{(n)}$ is a Q-prime submodule of $Q-height$ equal to $k\}$. Indeed, $e \in E(PH_Q(F))$ is an edge of $PH^k_Q(F)$ if and only if $\bigcup_{[x]\in e}[x]$ is a clique of $H^{k+1}_Q(F)$ that is not strictly contained in other cliques, $2 \le k \le n-1$. If $k = 1$ then $PH^1_Q(F)$ is a 1-uniform hypergraph which has only loops as edges.

Theorem 2.9 Let $F = R^{(n)}$ $(n \ge 2)$ and Q be a maximal ideal of R such that $[R : Q] = m$. Then $PH^k_Q(F)$ is a $\sum_{i=0}^{k-1} m^i$-uniform hypergraph, $1 \le k \le n-1$.

Corollary 2.10 Let $F = R^{(n)}$ $(n \ge 3)$ and Q be a maximal ideal of R such that $[R : Q] = m$. Then $PH^k_Q(F)$ is a Steiner system with parameters $(k, \sum_{i=0}^{k-1} m^i, \sum_{i=0}^{n-1} m^i)$, $2 \le k \le n-1$.

Corollary 2.11 Let $F = R^{(n)}$ and Q be a maximal ideal of R such that $[R : Q] = m$. Then $PH^k_Q(F)$ is a $\sum_{i=0}^{k-1} m^i$-uniform $r_k = \dfrac{\binom{(\sum_{i=0}^{n-1} m^i) - 1}{k-1}}{\binom{(\sum_{i=0}^{k-1} m^i) - 1}{k-1}}$-regular hypergraph with

$b_k = \dfrac{\binom{\sum_{i=0}^{n-1} m^i}{k}}{\binom{\sum_{i=0}^{k-1} m^i}{k}}$ edges, $(2 \le k \le n-1)$.

Corollary 2.12 Let $F = R^{(n)}$ and Q, m, b_k $(2 \le k \le n-1)$ be as in Corollary 2.11. Then F has b_k, Q-prime submodules of Q-height equal to k , $2 \le k \le n-1$.

Corollary 2.13 Let $F = R^{(n)}$ and Q, m, b_k $(2 \le k \le n-1)$ be as in Corollary 2.11. Then F has $\sum_{k=1}^{n-1} b_k$, Q-prime submodules, where $b_1 = \sum_{i=0}^{n-1} m^i$.

Proposition 2.14 Let $F = R^{(n)}$ $(n \ge 3)$ and Q be a prime ideal of R. Then, for $2 \le k \le n-1$, $PH^k_Q(F)$ is a connected hypergraph with diameter one that is not complete.

Example 2.15 Let $F = Z^{(3)}$ and $Q = 2Z$ be a maximal ideal of Z. Then $[Z : 2Z] = 2$. By Corollary 2.12 , $PH^2_{2Z}(F)$ is a Fano Plane . Also by Corollary 2.15 , F has fourteen $2Z$ - prime submodues .

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Talk On split Clifford algebras with involution in characteristic two pp.: 1–4

On split Clifford algebras with involution in characteristic two

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Abstract

In characteristic two, the involutions on split Clifford algebras induced by the involutions of orthogonal group are investigated. Orthogonal and symplectic involutions on these algebras are classified up to isomorphism by invariants of involutions in orthogonal group.

Keywords: Clifford algebra, involution, quadratic form, matrix algebra.
Mathematics Subject Classification [2010]: 16W10, 11E88.

1 Introduction

Let A be a central simple algebra over a field F. An anti-automorphism $\sigma : A \to A$ is called an *involution* if $\sigma^2 = \mathrm{id}$. Every nondegenerate bilinear form $B : V \times V \to F$ on a finite-dimensional F-vector space V induces a unique involution σ_B on $\mathrm{End}_F(V)$ which satisfies $B(x, f(y)) = B(\sigma_B(f)(x), y)$ for every $x, y \in V$ and $f \in \mathrm{End}_F(V)$. This involution is called the *adjoint involution* of $\mathrm{End}_F(V)$ with respect to B. The map $B \mapsto \sigma_B$ defines a one-to-one correspondence between the similarity classes of nondegenerate bilinear forms over F and the isomorphism classes of split F-algebras with involution (see [2, p. 1]).

Let (V, q) be a quadratic space over a field F. The group of all isometries of (V, q) is called the *orthogonal group* of (V, q) and is denoted by $O(V, q)$. An isometry $\tau \in O(V, q)$ is called an *involution* if $\tau^2 = \mathrm{id}$. Every involution $\tau \in O(V, q)$ induces a *natural* involution J_τ on the Clifford algebra $C(V, q)$ which satisfies $J_\tau(v) = v$ for every $v \in V$. The natural involutions were studied in [6] and [7] in connection with the Pfister Factor Conjecture, which was finally settled in [1]. Some properties of these involutions were also investigated in [3] and [5]. It is shown that for every multiquaternion algebra with involution $(A, \sigma) := (Q_1, \sigma_1) \otimes \cdots \otimes (Q_n, \sigma_n)$, there exists a quadratic space (V, q) and an involution $\tau \in O(V, q)$ such that $(A, \sigma) \simeq (C(V, q), J_\tau)$ (see [3, (6.3)] and [5, (6.3)]). This shows that properties of multiquaternion algebras with involution are reflected in properties of Clifford algebras with natural involution.

The main object of this work is to study the natural involutions of split Clifford algebras in characteristic 2. The transpose involution is the most elementary involution on the matrix algebra $M_n(F)$ over a field F. For a quadratic space (V, q) over a field F of characteristic 2, we obtain a necessary and sufficient condition to have $(C(V, q), J_\tau) \simeq (M_{2^n}(F), t)$. More generally, we characterize orthogonal and symplectic natural involutions on split Clifford algebras.

Following an approach based on the ideas of [3] and [5], we start with some observations on involutions of orthogonal group in characteristic 2. In [8, Theorem 1] it is shown that

for every involution τ in $O(V,q)$, there exists a decomposition $V = W \perp V_1 \perp V_2 \perp \cdots$ to τ-invariant subspaces of V, where $\tau|_W = \mathrm{id}$ and exactly one of the following is true:
(1) each V_i is a two-dimensional subspace of V and the restriction of τ to V_i is nontrivial;
(2) each V_i is a four-dimensional subspace of V and the fixed subspace of the restriction of τ to V_i is a totally isotropic space of dimension 2.

This decomposition is called a *Wiitala decomposition* of (V,τ) and the subspace W is called a *Wiitala subspace* of V.

For an involution σ on a central simple F-algebra A, the set of *alternating* elements of A is defined as follows:

$$\mathrm{Alt}(A,\sigma) = \{a - \sigma(a) | a \in A\}.$$

If $\mathrm{Char}\, F = 2$, an involution σ on A is symplectic if $1 \in \mathrm{Alt}(A,\sigma)$. Otherwise σ is orthogonal (see [2, (2.6)]). If σ is orthogonal and A is of even degree $n = 2m$ over F, then the *discriminant* of σ is defined as follows:

$$\mathrm{disc}\, \sigma = (-1)^m \mathrm{Nrd}_A(a) F^{\times 2} \in F^\times / F^{\times 2} \quad \text{for } a \in \mathrm{Alt}(A,\sigma) \cap A^*,$$

where $\mathrm{Nrd}_A(a)$ is the reduced norm of a and A^* is the unit group of A.

2 Main results

Definition 2.1. Let F be a field. The *canonical involution* γ on $M_2(F)$ is defined by

$$\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

for $a, b, c, d \in F$.

It is known that the canonical involution is the unique symplectic involution on $M_2(F)$ and it is characterized by the property $\gamma(x)x \in F$ for every $x \in M_2(F)$ (see [2, Ch. I]).

Definition 2.2. Let F be a field of characteristic 2 and let $\alpha \in F^\times$. Define the involution $T_\alpha : M_2(F) \to M_2(F)$ via

$$T_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c\alpha^{-1} \\ b\alpha & d \end{pmatrix}.$$

In particular $T_1 = t$ is the transpose involution.

Note that T_α is, up to isomorphism, the unique orthogonal involution on $M_2(F)$ such that $\mathrm{disc}\, T_\alpha = \alpha F^{\times 2} \in F^\times / F^{\times 2}$ (see [2, (7.4)]).

Definition 2.3. Let (V,q) be a quadratic space over a field F of characteristic 2 and let $u \in V$ be an anisotropic vector. The involution $\tau_u \in O(V,q)$ defined by $\tau_u(v) = v + \frac{b(v,u)}{q(u)} u$ for every $v \in V$, is called the *reflection* along u. Also the class of $q(u)$ in the quotient group $F^\times / F^{\times 2}$ is called the *spinor norm* of τ_u and is denoted by $\theta(\tau_u)$.

Remark 2.4. Let (V,q) be a 2-dimensional quadratic space over a field F of characteristic 2. Then $C(V,q)$ splits if and only if q represents 1. It follows that $(C(V,q), J_\tau) \simeq (M_2(F), t)$ if and only if τ is a reflection and $\theta(\tau) = 1 \in F^\times / F^{\times 2}$. More generally, if $C(V,q)$ splits then $(C(V,q), J_\tau) \simeq (M_2(F), T_\alpha)$ if and only if τ is a reflection and $\theta(\tau) = \alpha F^{\times 2}$, also $(C(V,q), J_{\mathrm{id}}) \simeq (M_2(F), \gamma)$.

Lemma 2.5. *Let F be a field of characteristic 2 and let $A \in M_n(F)$ such that $A^t = A$ and $A^2 \in F$. Then $A^2 \in F^2$.*

Notation. Let (V, q) be a quadratic space over a field F. For an isometry $\tau \in O(V, q)$ we use the notation $\mathrm{Fix}(V, \tau) = \{v \in V | \tau(v) = v\}$.

Proposition 2.6. ([5, (4.7)]) *Let (V, q) be a quadratic space over a field F of characteristic 2 and let τ be an involution in $O(V, q)$. Then the involution J_τ on $C(V, q)$ is orthogonal if and only if (V, τ) has trivial Wiitala subspace if and only if $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$.*

Definition 2.7. Let (V, q) be a 4-dimensional quadratic space over a field F of characteristic 2. An involution $\tau \in O(V, q)$ is called an *interchange isometry* if $\mathrm{Fix}(V, \tau)$ is a totally isotropic space of dimension 2.

The next result follows from [5, (6.10)] and (2.4).

Proposition 2.8. *Let (V, q) be a 4-dimensional quadratic space over a field F of characteristic 2 and let τ be an interchange isometry of (V, q). Then $(C(V, q), J_\tau) \simeq (M_4(F), t)$.*

Theorem 2.9. *Let (V, q) be a quadratic space over a field F of characteristic 2 and let $\tau \in O(V, q)$ be an involution. Then $(C(V, q), J_\tau) \simeq (M_{2^n}(F), t)$ if and only if $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$ and $q(x) \in F^2$ for every $x \in \mathrm{Fix}(V, \tau)$.*

Proof. Since the involution t is of orthogonal type, if $f : (C(V, q), J_\tau) \simeq (M_{2^n}(F), t)$ is an isomorphism, then by (2.6), we have $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$. Let $x \in \mathrm{Fix}(V, \tau)$, i.e., $\tau(x) = x$ and set $A = f(x) \in M_{2^n}(F)$. Then $A^2 = f(x)^2 = q(x) \in F$ and $A^t = A$, so by (2.5), $A^2 \in F^2$, i.e., $q(x) = x^2 \in F^2$.

Conversely suppose that $\dim \mathrm{Fix}(V, \tau) = \frac{1}{2} \dim V$ and $q(x) \in F^2$ for every $x \in \mathrm{Fix}(V, \tau)$. By (2.6), (V, τ) has trivial Wiitala subspace. So $\tau = \tau_1 \perp \tau_2 \perp \cdots$, where either every τ_i is a reflection on a two-dimensional subspace V_i of V, or every τ_i is an interchange isometry on a four-dimensional subspace $\mathbb{A}_i$ of V. If τ_i is an interchange isometry, by (2.8) we have $(C(\mathbb{A}_i, q|_{\mathbb{A}_i}), J_{\tau_i}) \simeq (M_4(F), t)$, $i = 1, \cdots, s$. Also if τ_i is a reflection, as $q(x) \in F^2$ for every $x \in \mathrm{Fix}(V, \tau)$, we obtain $\theta(\tau_i) = 1 \in F^\times / F^{\times 2}$. So by (2.4) we have $(C(V_i, q|_{V_i}), J_{\tau_i}) \simeq (M_2(F), t)$, $i = 1, \cdots, r$. This completes the proof. □

The following result characterizes the symplectic involutions on split Clifford algebras. The idea of proof is that if $C(V, q)$ is split and J_τ is symplectic, then $(C(V, q), J_\tau)$ is hyperbolic and isomorphic to $\bigotimes_{i=1}^n (M_2(F), \gamma)$, see [2, (12.35)].

Theorem 2.10. *Let (V, q) be a quadratic space of dimension n over a field F of characteristic 2 and let $\tau \in O(V, q)$ be an involution. Suppose that $C(V, q)$ splits. Then the following statements are equivalent:*

(*i*) *J_τ is of symplectic type.*

(*ii*) *$\dim \mathrm{Fix}(V, \tau) > \frac{1}{2} \dim V$.*

(*iii*) *$(C(V, q), J_\tau) \simeq \bigotimes_{i=1}^n (M_2(F), \gamma)$.*

Definition 2.11. Let F be a field and let $q : V \to F$ be a quadratic form. We say that q is *totally singular* if $q(u+v) = q(u) + q(v)$ for every $u, v \in V$. For $\alpha_1, \cdots, \alpha_n \in F$, the isometry class of the n-dimensional totally singular quadratic form q over F defined by $q(v_1, \ldots, v_n) = \alpha_1 v_1^2 + \cdots + \alpha_n v_n^2$ is denoted by $[\alpha_1] \perp \cdots \perp [\alpha_n]$.

The following result characterizes the orthogonal involutions on split Clifford algebras. Note that by [4, (3.6)], up to isomorphism, every involution of orthogonal type on $M_{2^n}(F)$ is of the form $\bigotimes_{i=1}^{n}(M_2(F), T_{\alpha_i})$ for some $\alpha_1, \cdots, \alpha_n \in F^\times$.

Theorem 2.12. *Let (V, q) be a quadratic space of dimension $n = 2m$ over a field F of characteristic 2 and let $\tau \in O(V, q)$ be an involution. Let $L = \mathrm{Fix}(V, \tau)$ and let (V', q') be the quadratic form $[\alpha_1] \perp \cdots \perp [\alpha_m]$, where $\alpha_1, \cdots, \alpha_m \in F^\times$. Suppose that $C(V, q)$ splits and $\dim L = m$. Then the following statements are equivalent:*

(i) $(C(V, q), J_\tau) \simeq \bigotimes_{i=1}^{n}(M_2(F), T_{\alpha_i})$.

(ii) $C(L, q|_L) \simeq C(V', q')$.

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ON STRONGLY CLEAN TRIANGULAR MATRIX RINGS

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Abstract

Let R be a associative ring with identity. We prove that for $(a_0, a_1, \cdots, a_{n-1}) \in \frac{R[x]}{(x^n)} \cong T(R,n)$, if a_0 or $1 - a_0$ is strongly π-regular in R, then $(a_0, a_1, \cdots, a_{n-1})$ is a strongly clean element in the triangular matrix ring $\frac{R[x]}{(x^n)} \cong T(R,n)$. As a corollary, we deduce that if R is a strongly π-regular ring, then $\frac{R[x]}{(x^n)} \cong T(R,n)$ is a strongly clean ring. We also show that the $(k, g(x))$-clean property of a ring R and $\frac{R[x]}{(x^n)} \cong T(R,n)$ is equivalent.

Keywords: Triangular matrix ring, Strongly clean ring, $(k, g(x))$-clean
Mathematics Subject Classification [2010]: Primary: 16S36, 16N60; Secondary: 16U80

1 Introduction

According to Nicholson [11], a ring R is called clean if every element of R can be written as a sum of a unit and an idempotent. Nicholson [13] also defined the notion of strong cleanness. An element of a ring R is strongly clean if it is the sum of an idempotent and a unit that commute. A ring R is strongly clean if every element of R is strongly clean. Local rings are obviously strongly clean. An element $a \in R$ is called right π-regular if the chain $aR \supseteq a^2R \supseteq \cdots$ terminates. The left π-regular elements are defined analogously. An element $a \in R$ is called strongly π-regular if it is both left and right π-regular, and R is called a strongly π-regular ring if every element is strongly π- regular. According to Burgess and Menal (Proposition 2.6 [4]) and (Theorem 1, [13]), strongly π-regular rings are strongly clean. It was a question in $\frac{R[x]}{(x^n)} \cong T(R,n)$[13] whether the matrix ring over a strongly clean ring is again strongly clean. The answer is 'No' by [14] where it was shown that for the localization $\mathbb{Z}_{(2)}$ of $\mathbb{Z}$ at (2), $M_2(\mathbb{Z}_{(2)})$ is not strongly clean. In [10] A. R. Nasr-Isfahani and A. Moussavi introduced $T(R,n)$ as below,

$$T(R,n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1 \end{pmatrix} | a_i \in R \right\}.$$

with $n \geq 2$. It is easy to see that $T(R,n)$ is a subring of the triangular matrix ring, with matrix addition and multiplication. We can denote elements of $T(R,n)$by

$(a_0, a_1, \cdots, a_{n-1})$. Then $T(R, n)$ is a ring with addition pointwaise and multiplication given by $(a_0, a_1, \cdots, a_{n-1})(b_0, b_1, \cdots, b_{n-1}) = (a_0b_0, a_0b_1+a_1b_0, \cdots, a_0b_{n-1}+\cdots+a_{n-1}b_0)$, for each $a_i, b_j \in R$.

On the other hand, there is a ring isomorphism $\varphi : \frac{R[x]}{(x^n)} \to T(R, n)$, given by, $\varphi(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = (a_0, a_1, \cdots, a_{n-1})$, with $a_i \in R$, $0 \le i \le n-1$. So $\frac{R[x]}{(x^n)} \cong T(R, n)$, where $R[x]$ is the rings of polynomials in an indeterminant x, and (x^n) is the ideal generated by x^n.

We prove that for $(a_0, a_1, \cdots, a_{n-1}) \in \frac{R[x]}{(x^n)} \cong T(R, n)$, if a_0 or $1 - a_0$ is strongly π-regular in R, then $(a_0, a_1, \cdots, a_{n-1})$ is a strongly clean element in the triangular matrix ring $\frac{R[x]}{(x^n)} \cong T(R, n)$. As a corollary, we deduce that if R is a strongly π-regular ring, then $\frac{R[x]}{(x^n)} \cong T(R, n)$ is a strongly clean ring.

2 STRONGLY CLEAN TRIANGULAR MATRIX RING

A ring R is strongly π-regular if for each $a \in R$ there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$. By results of Azumaya [2] and Dischinger [6], the element x can be chosen to commute with a. In particular, this definition is left-right symmetric. Strongly π-regular rings were introduced by Kaplansky [8] as a common generalization of algebraic algebras and Artinian rings. Following [15], a ring R is an exchange ring if ${}_RR$ satisfies the (finite) exchange property. By [[15], Corollary 2], this definition is left-right symmetric. Every strongly π-regular ring is an exchange ring [$\frac{R[x]}{(x^n)} \cong T(R, n)$[13], Example 2.3]. The strong π-regularity has roles in module theory and ring theory as we see in Ara [1], Azumaya [2], Birkenmeier et al. [3], Burgess and Menal [4], Hirano [7], $\frac{R[x]}{(x^n)} \cong T(R, n)$[13], and so on.

Lemma 2.1. *An element $r \in R$ is strongly π-regular if and only if there exists $m \ge 1$ such that $r^m = fw = wf$, where $f^2 = f \in R, w \in U(R)$ and r, f and w all commute.*

Proof. By [2] or (proposition 1, [11]) hold. □

Theorem 2.2. *Let R be a ring and $(a_0, a_1, \cdots, a_{n-1}) \in \frac{R[x]}{(x^n)} \cong T(R, n)$. If either a_0 or $1 - a_0$ is a strongly π-regular element of R, then $(a_0, a_1, \cdots, a_{n-1})$ is a strongly clean element of $\frac{R[x]}{(x^n)} \cong T(R, n)$.*

Corollary 2.3. *If R is a strongly π-regular ring, then $\frac{R[x]}{(x^n)} \cong T(R, n)$ is a strongly clean ring.*

Remark 2.4. By [5] ,a ring R is said to satisfy the condition (*) if for each $a \in R$, either a or $1 - a$ is strongly π-regular. by (Remark 2.5 [5]) there exist rings that R not strongly π-regular, but it satisfies (*).

Example 2.5. The condition (*) is suficient for $\frac{R[x]}{(x^n)} \cong T(R,n)$ to be strongly clean, but it is not necessary. Let $R = T(2; \mathbb{Z}_{(2)}))$ and let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \in R.$$

It can be verified easily that either A nor $I - A$ is strongly π-regular. But

$$T(R,n) \cong \frac{R[x]}{(x^n)} \cong \frac{T(2,\mathbb{Z}_{(2)}))[x]}{(x^n)}$$

is strongly clean. Because $\mathbb{Z}_{(2)}$ is local, hence by [10], R is so. Thus $T(R,n) \cong \frac{R[x]}{(x^n)}$ is local.

By Xiao and Tong [18] , an element $a \in R$ is called k-clean if $a = u_1 + \cdots + u_k + e$, where $e^2 = e \in R$ and $u_i \in U(R)$ for each i, where $U(R)$ is the set of all unit elements of R and k is a positive integer. A ring R is called k-clean if every element of R is k-clean. Let $C(R)$ be the center of a ring R and $g(x)$ a fixed polynomial in $C(R)[x]$. Camillo and Simon [16] defined R to be $g(x)$-clean if each $a \in R$ has the form $a = u + b$, where $u \in U(R)$ and $g(b) = 0$. Also by [17], R is $(k, g(x))$-clean if each element $a \in R$ has the form $a = u_1 + \cdots + u_k + b$, where $u_i \in U(R)$ and $g(b) = 0$. Note that clean rings are 1-clean and k-clean rings are $(k, x^2 - x)$-clean. In the following, we show that the $(k, g(x))$-clean property of a ring R and $\frac{R[x]}{(x^n)} \cong T(R,n)$ is equivalent.

Theorem 2.6. *Let R be a ring and $g(x) in C(R)[x]$. Then R is $(k, g(x))$-clean if and only if $\frac{R[x]}{(x^n)} \cong T(R,n)$ is $(k, g(x))$-clean.*

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On subgroups with large relative commutativity degrees

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Abstract
For a finite group G and a subgroup H of G, the relative commutativity degree of H in G, denoted by $d(H,G)$, is the probability that an element of H commutes with an element of G. In the present paper, we characterize the factor group $H/H\cap Z(G)$ when $d(H,G)=d_1,d_2,d_3$ and d_n, where $\mathcal{D}(G)=\{d(H,G)|H\leq G\}=\{d_0,d_1,\dots,d_n\}$ such that $1=d_0>d_1>\cdots>d_n=d(G,G)$.

Keywords: Relative center, relative commutativity degree.
Mathematics Subject Classification [2010]: 20P05, 20E45.

1 Introduction

Let G be a finite group and H be a subgroup of G. Then the relative commutativity degree of H in G is defined as

$$d(H,G)=\frac{|\{(h,g)\in H\times G|[h,g]=1\}|}{|H||G|}.$$

The set of all relative commutativity degrees of G is denoted by $\mathcal{D}(G)$.

In [1], it is shown that a finite group G admits three relative commutativity degrees if and only if $G/Z(G)$ is a non-cyclic group of order pq, where p and q are primes. Moreover, the authors determined all the relative commutativity degrees of some known groups.

Lemma 1.1. ([*1*], *Lemma* 2.1) *Let G be a finite group and $H\leq K$ be subgroups of G. Then $d(K,G)\leq d(H,G)$ and the equality holds if and only if $K=HC_K(g)$ for all $g\in G$.*

Utilizing the above lemma, in what follows, we always assume that $\mathcal{D}(G)=\{d_0,d_1,\dots,d_n\}$ such that $1=d_0>d_1>\cdots>d_n=d(G,G)$.

Definition 1.2. Let G be a group and H be a subgroup of G. The relative center of H in G is defined by $Z(H,G)=H\cap Z(G)$.

In the present paper, we characterize the factor group $H/Z(H,G)$ when $d(H,G)=d_1,d_2,d_3$ and d_n.

*Speaker

2 Main results

Lemma 2.1. *Let G be a finite group. If $H \leq G$ is non-abelian and $K \leq H$ is abelian, then $d(H,G) < d(K,G)$.*

Lemma 2.2. *Let G be a finite group and H be a subgroup of G. If H is not nilpotent of class n, then $d(H,G) < d(Z_n(H),G)$.*

Proposition 2.3. *Let G be a finite group and H be a nilpotent subgroup of G. If $d(H,G) = d_n$, then $|H/Z(H,G)| = p_1 \dots p_m$ for some primes $p_1, \dots, p_m$ and $m \leq n$.*

Theorem 2.4. *Let G be a finite group and H be a subgroup of G such that $d(H,G) = d_1$. Then $H/Z(H,G) \cong \mathbb{Z}_p$ is a cyclic group of prime order.*

Theorem 2.5. *Let G be a finite group and H be a subgroup of G such that $d(H,G) = d_2$. Then $H/Z(H,G)$ is a group of order p or pq, where p and q are primes.*

Theorem 2.6. *Let G be a finite group and H be a subgroup of G such that $d(H,G) = d_3$. Then $H/Z(H,G)$ is a group of order p, pq or pqr, where p, q and r are primes.*

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On the n-c-Nilpotent Groups

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Abstract

In this paper we introduce the notion of n-c-nilpotent group. It is shown that every nilpotent group of class at most c is n-c-nilpotent. Also we find a class of groups that all groups of it are n-c-nilpotent. Finally one equivalent condition for a n-c-nilpotent group to be torsion free is obtained.

Keywords: n-potent, n-center, nilpotent.
Mathematics Subject Classification [2010]: 20E34, 20E36, 20F28.

1 Introduction

In 1979 Fay and Waals [1] introduced the notion of the n-potent and the n-centre subgroups of a group G, for a positive integer n, respectively as follows:

$$G_n = \langle [x, y^n] | x, y \in G \rangle$$

$$Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\}$$

Where $[x, y^n] = x^{-1}y^{-n}xy^n$. It is easy to see that G_n is a fully invariant subgroup and $Z^n(G)$ is a characteristic subgroup of group G. In the case $n = 1$, these subgroups will be G' and $Z(G)$, the drive and center subgroups of G, respectively. In this paper we fix $n \in \mathbf{N}$.

Definition 1.1. A normal series $1 = G_0 \leq G_1 \leq \ldots \leq G_t = G$ of group G is called *n-central series* of length t if and only if

$$\frac{G_{i+1}}{G_i} \leq Z^n(\frac{G}{G_i})$$

Definition 1.2. A group G is called *n-c-nilpotent* if it has at least one n-central series of the length c such that c is the least of the lengths of its n-central series.

Now we introduce upper and lower n-central series of G which give us two examples of n-central series.

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Definition 1.3. The upper n-central series of G is defined to be the series

$$1 = Z_0^n(G) \leq Z_1^n(G) \leq \ldots \leq Z_t^n(G) \leq \ldots$$

where inductively

$$Z_{i+1}^n(G)/Z_i^n(G) = Z^n(G/Z_i^n(G))$$

for $i \geqslant 0$. So $Z_1^n(G) = Z^n(G)$.

Definition 1.4. Put $\gamma_1^n(G) = G$, and let $\gamma_i^n(G)$ be defined inductively for $i \geqslant 1$. Then $\gamma_{i+1}^n(G)$ is defined as the subgroup $[\gamma_i^n(G), G^n]$.

It is immediate from the previous definition that the following series

$$G = \gamma_1^n(G) \geq \gamma_2^n(G) \geq \ldots \geq \gamma_t^n(G) \geq \ldots$$

is an n-central series which is called lower n-central series of G.
Now we make some elementary observations about the properties of $\gamma_{i+1}^n(G)$ and $Z_i^n(G)$ for $i \geq 0$.

Lemma 1.5. *Let G be any group and let i and j be positive integers.*
(1) $\gamma_i^n(G) \lhd^f G$, $Z_i^n(G) \lhd^c G$;
(2) $\gamma_i^n(G) = 1 \Longleftrightarrow Z_{i-1}^n(G) = G$;
(3) $\gamma_i^n(G/N) = (\gamma_i^n(G)N)/N$, $Z_i^n(G/N) \geq (Z_i^n(G)N/N)$;
(4) $\gamma_i^n(G) \leq \gamma_i(G)$, $Z_i(G) \leq Z_i^n(G)$;
(5) $\gamma_i^n(G \times H) = \gamma_i^n(G) \times \gamma_i^n(H)$;
(6) $Z_i^n(G/Z_j^n(G)) = Z_{i+j}^n(G)/Z_j^n(G)$.

Remark 1.6. Of course, if G is nilpotent group of class c, then it is n-c-nilpotent for all positive integer n. But the converse is not hold. For example consider S_3.

Note that the n-c-nilpotency of a group G is equivalent to $Z_c^n(G) = G$. Also by the previous lemma for this group G, $\gamma_{c+1}^n(G) = 1$.
In the sequel we introduce special type of groups such that they are n-c-nilpotent for some c. Also notice that the class of n-c-nilpotent group is closed under subgroups and product.

Definition 1.7. A group G is called n-p-group if $G^n = \langle g^n | g \in G\rangle$ is a p-group.

To close this section we give a result of finit n-p-group about $|Z^n(G)|$.

Proposition 1.8. *Let G be a nontrivial finite n-p-group. Then $|Z^n(G)| > 1$.*

2 Main results

The properties of the center of a nilpotent group are often reflected in the entire group. On such result for n-c-nilpotent group is the following:

Theorem 2.1. *Let G be a n-c-nilpotent group and $1 \neq H \lhd G$. Then $H \cap Z^n(G) \neq 1$.*

Corollary 2.2. *Let G be a n-c-nilpotent group and M normal minimal subgroup of it. Then $M \leq Z^n(G)$.*

Now we say our principal results:

Corollary 2.3. *Let G be a n-c-nilpotent group. G is torsion-free if and only if $Z^n(G)$ is torsion-free.*

In studying the behavior of the maximal subgroups of a group G, Giovani Frattini introduced what he called the Φ-subgroup of G, the intersection of the maximal subgroups of G. Since then, this subgroup is usually known as the frattini subgroup of G. In order to setup clearly the contents of this survey, we mention, the main result of Frattini

$$\text{Finite group } G \text{ is nilpotent} \Longleftrightarrow G' \leq \Phi(G)$$

In the next theorem, we shall consider finitely generated n-c-nilpotent group G, which causes to find a subgroup of $\Phi(G)$.

Theorem 2.4. *Let G be a finitely generated n-c-nilpotent group. Then $G_n \leq \Phi(G)$.*

Our main result is to introduce a class of groups that are n-c-nilpotent for some c.

Theorem 2.5. *Every finite n-p-group is n-c-nilpotent, for some c.*

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On the number of minimal prime ideals

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Abstract

In this paper, we define a new invariant for a commutative ring R, which we call measure of R. Let A be a set as follows

$$\{|\operatorname{Min}(I)| : I \textit{ is an ideal of } R\},$$

where $\operatorname{Min}(I)$ is the set of prime ideals minimal over I. We study A and give an upper bound and a lower bound for the supremum of A.

Keywords: Minimal prime ideals, Semilocal rings
Mathematics Subject Classification [2010]: 13A15, 13C99, 13H99

1 Introduction

Throughout this paper R is a commutative ring with 1. An ideal $\mathfrak{p}$ of R is said to be prime if it is a proper ideal, and if $xy \in \mathfrak{p}$ implies that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. A prime ideal $\mathfrak{p}$ of R is called minimal if there is no prime ideal of R which is properly contained in $\mathfrak{p}$. Thus, for example, if R is an integral domain then o is the only minimal prime ideal of R [4]. Let $I \neq R$ be an ideal of R. Anderson [1] showed that if all the prime ideals minimal over I are finitely generated, then there are only finitely many prime ideals minimal over I. In particular if R is a Noetherian then there are only finitely many prime ideals minimal over I. In fact we only need R to satisfy the ascending chain condition on radical ideals ([3], Theorem 88). Let I be an ideal of R, we denote the set of minimal prime ideals of R by $\operatorname{Min}(R)$, the set of prime ideals of R minimal over I by $\operatorname{Min}(I)$, the set of maximal ideals of R by $\operatorname{Max}(R)$, and the dimension of R by $\dim R$. In this paper we introduce a new invariant for a ring: its $\operatorname{mr}(R)$. We will show, $\operatorname{mr}(R)$ is finite if and only if R satisfies the following two properties:
(1) R is a semilocal ring,
(2) $\dim R \leq 1$.

We will show for a semilocal ring R with $\dim R = 1$ there are the following inequalities

$$\max\big\{|\operatorname{Max}(R)|, |\operatorname{Min}(R)|\big\} \leq \operatorname{mr}(R) \leq |\operatorname{Spec}(R)| - 1.$$

*Speaker

2 The number of minimal prime ideals

We begin with the following known result.

Proposition 2.1. *Let $\mathfrak{p}$, $\mathfrak{q}$ be prime ideals of the Noetherian ring R such that $\mathfrak{p} \subsetneqq \mathfrak{q}$. If there exists one prime ideal of R strictly between $\mathfrak{p}$ and $\mathfrak{q}$ then there are infinity many.*

Proof. (See [5], Ex. 15.3). □

Lemma 2.2. *Let R be a Noetherian ring with* $\dim R \geq 2$ *and n a positive integer. Then there is an ideal I of R such that* $|\mathrm{Min}(I)| = n$.

Proof. Let $\mathfrak{p}$ be a prime ideal of R with $\mathrm{ht}\mathfrak{p} = 2$. Now consider $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \mathfrak{p}$ be a chain of ideals in $\mathrm{Spec}(R)$ such that $\mathrm{ht}\mathfrak{p}_0 = 0$ and $\mathrm{ht}\mathfrak{p}_1 = 1$. So there is a $\{\mathfrak{q}_i\}_{i=1}^{\infty}$ in $\mathrm{Spec}(R)$ such that $\mathrm{ht}(\mathfrak{q}_i) = 1$ for all $i \geq 1$ and $\mathfrak{q}_i \neq \mathfrak{q}_j$ for each $i \neq j$ by Proposition 2.1. Let $I = \mathfrak{q}_1\mathfrak{q}_2...\mathfrak{q}_n$. We show that $\mathrm{Min}(I) = \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n\}$. If $\mathfrak{q} \in \mathrm{Min}(I)$, then $I \subseteq \mathfrak{q}_i \subseteq \mathfrak{q}$ for some $1 \leq i \leq n$, see ([2], page 2). Hence $\mathfrak{q} = \mathfrak{q}_i$ and so $\mathrm{Min}(I) \subseteq \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n\}$. Now it is enough to show that $\mathfrak{q}_i \in \mathrm{Min}(I)$ for each $1 \leq i \leq n$. For this, consider $\acute{\mathfrak{q}} \in \mathrm{Min}(I)$ such that $I \subseteq \acute{\mathfrak{q}} \subseteq \mathfrak{q}_i$. Therefore $\mathfrak{q}_j \subseteq \acute{\mathfrak{q}} \subseteq \mathfrak{q}_i$ for some $1 \leq j \leq n$. Since $\mathrm{ht}\mathfrak{q}_i = \mathrm{ht}\mathfrak{q}_j = 1$, it follows that $\acute{\mathfrak{q}} = \mathfrak{q}_i$. This ends the proof. □

Lemma 2.3. *Let R be a ring with* $|\mathrm{Max}(R)| = \infty$ *and n a positive integer. Then there is an ideal I of R such that* $|\mathrm{Min}(I)| = n$.

Proof. Let $\{\mathfrak{m}_i\}_{i=1}^{\infty}$ be a sequence in $\mathrm{Max}(R)$ such that $\mathfrak{m}_i \neq \mathfrak{m}_j$ for each $i \neq j$. Let $I = \mathfrak{m}_1\mathfrak{m}_2...\mathfrak{m}_n$. We will show $\mathrm{Min}(I) = \{\mathfrak{m}_1, ..., \mathfrak{m}_n\}$. First let $\mathfrak{p} \in \mathrm{Min}(I)$, so $\mathfrak{m}_1\mathfrak{m}_2...\mathfrak{m}_n \subseteq \mathfrak{p}$. Hence $I \subseteq \mathfrak{m}_i \subseteq \mathfrak{p}$ for some $1 \leq i \leq n$ and therefore $\mathfrak{m}_i = \mathfrak{p}$. Now it is enough to show that $\mathfrak{m}_i \in \mathrm{Min}(I)$ for each $1 \leq i \leq n$. For this, consider $\mathfrak{q} \in \mathrm{Min}(I)$ such that $I \subseteq \mathfrak{q} \subseteq \mathfrak{m}_i$. Therefore $\mathfrak{m}_j \subseteq \mathfrak{q} \subseteq \mathfrak{m}_i$ for some $1 \leq j \leq n$. Since $\mathfrak{m}_j$ and $\mathfrak{m}_i$ are in $\mathrm{Max}(R)$, so $\mathfrak{m}_i = \mathfrak{m}_j = \mathfrak{q}$. This ends the proof. □

We are now ready to present our main definition.

Definition 2.4. Let R be a ring. Then

$$\mathrm{mr}(R) = \sup\{|\mathrm{Min}(I)| \ : \ I \textit{ is an ideal of } R\},$$

is called the measure of the ring R.

Corollary 2.5. *Let R be a Noetherian ring such that* $\mathrm{mr}(R)$ *is finite. Then* $\dim(R) \leq 1$.

Proof. It follows from Lemma 2.2. □

Corollary 2.6. *Let R be a ring such that* $\mathrm{mr}(R)$ *is finite. Then* $|\mathrm{Max}(R)|$ *is finite.*

Proof. It follows from Lemma 2.3. □

Proposition 2.7. *Let R be a ring such that* $|\mathrm{Max}(R)|$ *is finite. Then* $|\mathrm{Max}(R)| \leq \mathrm{mr}(R)$.

Proof. If $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n\}$ and $I = \mathfrak{m}_1...\mathfrak{m}_n$, then $\mathrm{Min}(I) = \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n\}$, see the proof of Lemma 2.3. Hence $|\mathrm{Max}(R)| = |\mathrm{Min}(I)|$ and so $|\mathrm{Max}(R)| \leq \mathrm{mr}(R)$. □

Corollary 2.8. *Let R be a ring such that* $\mathrm{Max}(R)$ *finite. Then*

$$\max\big\{|\mathrm{Max}(R)|, |\mathrm{Min}(R)|\big\} \leq \mathrm{mr}\,(R).$$

Proof. It follows from $|\mathrm{Min}(0)| = |\mathrm{Min}(R)| \leq \mathrm{mr}\,(R)$. □

It is clear that if R is an Artinian ring, then $\mathrm{mr}\,(R) = |\mathrm{Spec}(R)| = |\mathrm{Max}(R)|$. The following result shows that the measure of a Noetherian ring of dimension greater than one is infinite.

Theorem 2.9. *Let R be a Noetherian ring. Then* $\mathrm{mr}\,(R) < \infty$ *if and only if R is a semilocal ring and* $\dim R \leq 1$.

Proof. Since $\mathrm{mr}\,(R)$ is finite, $|\mathrm{Max}(R)| \leq \mathrm{mr}\,(R)$ by Corollary 2.6 and Proposition 2.7. So R is semilocal. In view of Lemma 2.2, $\dim R \leq 1$. Conversely, since R is a semilocal ring with $\dim R \leq 1$, it follows that $\mathrm{Spec}(R) = \mathrm{Min}(R) \cup \mathrm{Max}(R)$ and $\mathrm{Max}(R)$ is finite. On the other hand $|\mathrm{Min}(R)| < \infty$, see [1]. Hence $|\mathrm{Spec}(R)| < \infty$ and so $\mathrm{mr}\,(R)$ is finite. □

It is clear that if I is a proper ideal of R and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}(I)$, then $\mathfrak{p} = \mathfrak{q}$. In the sequel we use this fact without notice.

Theorem 2.10. *Let R be a semilocal Noetherian ring with* $\dim R = 1$. *Then*

$$\max\big\{|\mathrm{Max}(R)|, |\mathrm{Min}(R)|\big\} \leq \mathrm{mr}\,(R) \leq |\mathrm{Spec}(R)| - 1.$$

Proof. It is clear that $\mathrm{mr}\,(R) \leq |\mathrm{Spec}(R)|$ and $\mathrm{Spec}(R)$ is finite. In view of Corollary 2.8, it is enough to show that $\mathrm{mr}\,(R) < |\mathrm{Spec}(R)|$. If $\mathrm{mr}\,(R) = |\mathrm{Spec}(R)|$, then there is an ideal I of R such that $|\mathrm{Min}(I)| = |\mathrm{Spec}(R)|$. Since $\dim R = 1$, it follows that there exist $\mathfrak{m} \in \mathrm{Max}(R)$ and $\mathfrak{p} \in \mathrm{Min}(R)$ such that $\mathfrak{p} \subsetneqq \mathfrak{m}$. Hence $\mathfrak{p} \in \mathrm{Min}(I)$ or $\mathfrak{m} \in \mathrm{Min}(I)$. So we will have $\mathrm{mr}\,(R) < |\mathrm{Spec}(R)|$. □

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On Weakly Prime Fuzzy Submodules

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Abstract

In this note we introduce and characterize weakly prime fuzzy submodules of a unitary module M over a commutative ring with identity R, and investigate the Zariski-like topology on the weakly prime Cl-FSpectrum of M, consisting of all weakly prime fuzzy submodules of M.

Keywords: Fuzzy submodule, Weakly prime fuzzy submodule, Zariski like-topology.
Mathematics Subject Classification [2010]: 08A72

1 Introduction

The concept of fuzzy submodules was first introduced by Negoita and Ralescu in 1975 [7] and subsequently studied, among others, by Pan, [8] in 1987. The notion of a fuzzy prime submodules is studied by Ameri and Mahjoob in [1]. Recently, the notion of weakly prime submodules and Zariski Like-Topology on CL.Spec(M), the set of prime submodules of a module M over a commutative ring R, are studied by Behboodi in [4]and [5]. In this paper we introduce the notion of weakly prime fuzzy submodules of a module over a commutative ring with identity. Let R be a commutative ring with identity and M be an unitary R-module. We recall that a submodule N of an R-module M is called weakly prime, if for any elements $a, b \in R$ and $x \in M$, the condition $abx \in N$ implies that $ax \in N$ or $bx \in N$. For more information see [2], [4].

In this paper by fuzzy subset μ of a non-empty set X, we mean a function μ from X to real interval $[0, 1]$. F^X denotes the set of all fuzzy subset of X. For μ, $\nu \in F^X$ we say that μ is contained in ν and we write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$, for all $x \in X$. For $\mu, \nu \in F^M$, the intersection and union, $\mu \cup \nu$, $\mu \cap \nu \in F^X$ are defined by $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$ and $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$, for all $x \in X$. Also for $\mu \in F^X$, $a \in [0, 1]$, μ_a is defined by, $\mu_a = \{x \in M | \mu(x) \geq a\}$, where μ_a is called a-cut or a-level subset of μ.

Let f be a mapping from X into Y and let $\mu \in F^X$, $\nu \in F^Y$. Then $f(\mu) \in F^Y$ and $f^{-1}(\nu) \in F^X$ are defined as follows:

$$f(\mu)(y) = \begin{cases} \bigvee\{\mu(x) | x \in f^{-1}(y)\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$, for all $x \in X$. This is called extension principle. Let M and N be R-modules and $f : M \to N$ be an R-module homomorphism. $\mu \in F^M$ is called

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f-invariant if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$ for all $x, y \in M$.
We recall some definitions and theorems from the book [6], which we need them for development of our paper.

Definition 1.1. Let $\mu \in F^R$. Then μ is called fuzzy ideal of R if for every $x, y \in R$ the following conditions are satisfied:
(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;
(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$
The set of all fuzzy ideals of R is denoted by $FI(R)$.

Definition 1.2. Let $\mu, \nu \in FI(R)$. We define $\mu\nu \in FI(R)$ as follows:
$\mu\nu(x) = \bigvee\{\mu(y) \wedge \nu(z) \mid y, z \in R, x = yz\} \quad \forall x \in R.$

Definition 1.3. Let R be a ring and $\zeta \in FI(R)$. Then ζ is called prime fuzzy ideal of R if ζ is non-constant and for every $\mu, \nu \in FI(R), \mu\nu \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

Theorem 1.4. *Let $\zeta \in F^R$.Then ζ is prime fuzzy ideal of R if and only if $\zeta(0) = 1$ and $\zeta = 1_{\zeta_*} \cup c_R$ such that ζ_* is a prime ideal of R.*

Definition 1.5. A fuzzy subset μ of M is called fuzzy submodule of M if the following hold:
(1)$\mu(0) = 1$;
(2)$\mu(rx) \geq \mu(x)$ for all $r \in R$ and $x \in M$ and
(3)$\mu(x + y) \geq \mu(x) \wedge \mu(y) \quad$ for all $x, y \in M$.
The set of all fuzzy submodules of M is denoted by $F(M)$.

Theorem 1.6. *Let $\mu \in F^M$. Then $\mu \in F(M)$ if and only if each non-empty level subset of μ is a submodule of M. Moreover if $\mu \in F(M)$ then $\mu_* = \{x \in M \mid \mu(x) = 1\}$ is a submodule of M.*

Theorem 1.7. *Let $\zeta \in F^R$ and $\mu \in F^M$. Define $\zeta \cdot \mu \in F^M$ as follows:
$(\zeta \cdot \mu)(x) = \bigvee\{\zeta(r) \wedge \mu(y) \mid r \in R, y \in M, ry = x\}$ for all $x \in M$.*

Definition 1.8. For $\mu, \nu \in F^M$ and $\zeta \in F^R$, define $(\mu : \nu) \in F^R$ and $(\mu : \zeta) \in F^M$ as follows:

$$(\mu : \nu) = \bigcup\{\eta \in F^R \mid \eta \cdot \nu \subseteq \mu\}, \quad (\mu : \zeta) = \bigcup\{\nu \in F^M \mid \zeta \cdot \nu \subseteq \mu\}.$$

Definition 1.9. A fuzzy submodule μ of M is called primary if for $\zeta \in FI(R)$ and $\nu \in F(M)$ such that $\zeta \cdot \nu \subseteq \mu$ then either $\zeta \subseteq \Re(\mu : 1_M)$ or $\nu \subseteq \mu$ where for $\eta \in FI(R)$, $\Re(\eta)(x) = \bigvee_{n \in N} \eta(x^n), \forall x \in R.$.

Theorem 1.10. *Let $c \in [0, 1]$ and N be a submodule of M. Then $(1_N \cup c_M) : 1_M = 1_{(N:M)} \cup c_R$.*

We recall that in [1] a fuzzy submodule μ of M is called prime, if for $\zeta \in FI(R)$ and $\nu \in F(M)$ such that $\zeta \cdot \nu \subseteq \mu$, then either $\nu \subseteq \mu$ or $\zeta \subseteq (\mu : 1_M)$.

Theorem 1.11. *[1] Let μ be a fuzzy submodule of M. Then μ is prime if and only if $\mu = 1_{\mu_*} \cup c_M$ such that, μ_* is a prime submodule of M.*

2 Weakly prime fuzzy submodules

In this section we introduce the notion of weakly prime fuzzy submodules and investigate some basic properties of them.

Definition 2.1. A non-constant fuzzy submodule μ of M is called weakly prime, if for $\zeta, \eta \in FI(R)$ and $\nu \in F(M)$ such that $\zeta \cdot \eta \cdot \nu \subseteq \mu$, then either $\zeta \cdot \nu \subseteq \mu$ or $\eta \cdot \nu \subseteq \mu$.

Theorem 2.2. *Let μ be a fuzzy submodule of M. Then μ is weakly prime if and only if $\mu = 1_{\mu_*} \cup c_M$ such that μ_* is weakly prime submodule of M.*

Remark 2.3. Every prime fuzzy submodule is weakly prime. But the converse, in general is not true. (see the example 2.8).

Theorem 2.4. *If μ is a weakly prime fuzzy submodule of M, then $(\mu : 1_M)$ is a prime fuzzy ideal of R.*

Theorem 2.5. *Let μ be a weakly prime fuzzy submodule of M. Then for all fuzzy submodule ξ, ν of M that are not contained in μ, $(\mu : \nu) \subseteq (\mu : \xi)$ or $(\mu : \xi) \subseteq (\mu : \nu)$.*

Theorem 2.6. *let M be an R-module and μ is a non-constant fuzzy submodule of M. Then μ is a prime fuzzy submodule, if and only if μ is primary and weakly prime fuzzy submodule of M.*

Theorem 2.7. *Let M, N be R-modules and f a homomorphism of M onto N.*
(1) *If μ is a weakly prime fuzzy submodule of M and μ is f-invariant, then $f(\mu)$ is a weakly prime fuzzy submodule of N.*
(2)*If ν is a weakly prime fuzzy submodule of N, then $f^{-1}(\nu)$ is a weakly prime fuzzy submodule of M.*

Example 2.8. Let R be an integral domain and P a non-zero prime ideal of R. Then for the free R-module $M = R \oplus R$, the submodule $(0 \oplus P)$ is a weakly prime submodule, which is not prime. For every element $t \in [0, 1]$, define $\mu \in F(M)$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in (0 \oplus P) \\ t & \text{otherwise} \end{cases}$$

for all $x \in M$.
Then by Theorem 2.2 is weakly prime fuzzy submodule of M.

In [5], the authores have introduced the classical prime spectrume $Cl - Spec(M)$ that is the set of all weakly prime submodule of M. By $Cl - FSpec(M)$ we mean the set of all weakly prime fuzzy submodule of M. Let M be a nonzero R-module. For any $\mu \in F(M)$, we define the fuzzy classical variety of μ by $\mathbb{V}(\mu)$, to be the set of all classical prime fuzzy submodule ν of $F(M)$ such that $\mu \subseteq \nu$. Then
(i) $\mathbb{V}(1_M) = \emptyset$ and $\mathbb{V}(1_{\{0\}}) = CL - FSpec(M)$,
(ii) $\bigcap_{i \in I} \mathbb{V}(\mu_i) = \mathbb{V}(\sum_{i \in I} \mu_i)$,
(iii) $\mathbb{V}(\mu) \cup \mathbb{V}(\xi) \subseteq \mathbb{V}(\mu \cap \xi)$,
where $\mu, \xi, \mu_i \in F(M)$. A fuzzy submodule $\mu \in F(M)$ is called classical semiprime fuzzy submodule if μ is an intersection of classical prime fuzzy submodules.
A classical prime fuzzy submodule $\mu \in F(M)$ is called fuzzy extraordinary if whenever ν, ξ are classical semiprime fuzzy submodule of M with $\nu \cap \xi \subseteq \mu$ then $\nu \subseteq \mu$ or $\xi \subseteq \mu$.

Theorem 2.9. *For an $R-$module M the following statements are equivalent:*
(i) M is classical fuzzy Top module;
(ii) Every classical prime fuzzy submodule of M is fuzzy extraordinary;
(iii) $\mathbb{V}(\mu_1) \cup \mathbb{V}(\mu_2) = \mathbb{V}(\mu_1 \cap \mu_2)$.
For every classical semiprime fuzzy submodule $\mu_1, \mu_2 \in F(M)$.

Theorem 2.10. *Every classical fuzzy Top module is fuzzy Top module.*

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Perfect dimension

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Abstract

In this article, we introduce and study the concept of perfect dimension, which is a Krull like dimension extension of the concept of DCC on finitely generated submodules or being perfect. We show that some of the basic results of Krull dimension is true for perfect dimension.

Keywords: finitely generated module, Krull dimension, Perfect dimension, Distributive module.

MSC(2010): Primary: 16P60; Secondary: 16P40, 16P20.

1 introduction

Lemonnier [6] has introduced the concept of deviation of an arbitrary poset, in particular, when applied to the lattice of all submodules of a module $_RM$, give the concept of Krull dimension (in the sense of Rentschler and Gabriel) see [5, 3, 8]. The Krull dimension of an R-module is denoted by $k\text{-dim}\, M$. It is well known that an R-module M is perfect if and only if it satisfies the descending chain condition (DCC) on finitely generated submodules. Motivated by this fact, one is tempted to extend this for Krull dimension. Let us give a brief outline of this paper. Section 1, is the introduction. In section 2, of this paper we study the concept of perfect dimension of an R-module M, denoted by $p\text{-dim}\, M$, which is the deviation of $F(M)$, the poset of finitely generated submodules of M. It is also denoted by $K(F(M))$ in [1]. We investigate some basic properties of perfect dimension. It is manifest that if $k\text{-dim}\, M$ exists, then $p\text{-dim}\, M \leq k\text{-dim}\, M$, where M is an R-module. We observe that for any ordinal number α, there exists an R-module M such that $p\text{-dim}\, M = \alpha$ but it does not have Krull dimension. It is proved that if M is a perfect R-module and for each small submodule N of M, $\frac{M}{N}$ has finite Goldie dimension, then M is Artinian. Consequently we prove that over perfect rings R, any quotient finite dimensional module M is Artinian. We give another proof for [1, Proposition 1.17]. Consequently we observe that if an R-module M has perfect dimension and for each essential submodule E of M, $\frac{M}{E}$ has finite Goldie dimension, then either M has a non-finitely generated socle or $p\text{-dim}\, M = k\text{-dim}\, M$. We recall that an R-module M

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is called α-critical if k-dim $M = \alpha$ and k-dim $\frac{M}{N} < \alpha$, for each nonzero submodule N of M. M is called critical if it is α-critical for some ordinal number α. We also introduce and study perfect critical modules. Section 3, deals with perfect dimension of distributive modules. We observe that if $\{M_i\}_{i\in I}$ is a family of unrelated distributive modules, see [10], then p-dim $(\sum_{i\in I} \oplus M_i) = \sup\{p\text{-dim}\, M_i : i \in I\}$. Throughout this paper R will always denote an associative ring with a non-zero identity, $1 \neq 0$, and M is a left unital R-module. The notation $N \subseteq M$ (resp., $N \subset M$) means that N is a submodule (resp., proper submodule) of M. The reader is referred to [8, 4, 5], for definitions, concepts, and the necessary background not explicitly given here.

2 Main results

First, we give our definition of perfect dimension.

Definition 2.1. If M is a left R-module, then the perfect dimension of M, denoted by p-dim M, is defined to be the deviation of $F(M)$, the poset of finitely generated submodules of M. It is also denoted by $K(F(M))$ in [1]. In particular p-dim ${}_RR$ is the left perfect dimension of R.

Next, we give our definition of perfect critical modules.

Definition 2.2. An R-module M is called α-perfect critical if p-dim $M = \alpha$ and for any nonzero f.g. submodule N of M, p-dim $\frac{M}{N} < \alpha$. M is said to be perfect critical if it is α-perfect critical for some α.

We have the following interesting results.

Lemma 2.3. Let M be an R-module such that for any small submodule N of M, $\frac{M}{N}$ has finite Goldie dimension. Then k-dim $M = 0$ if and only if p-dim $M = 0$, i.e., M is Artinian if and only if it is perfect.

We should remind the reader that by a quotient finite dimensional module M we mean for each submodule N of M, $\frac{M}{N}$ has finite Goldie dimension.

Theorem 2.4. Let M be a quotient finite dimensional R-module. If p-dim $M = \alpha$, then k-dim $M = \alpha$.

Corollary 2.5. Let R-module M has Krull dimension, then M has perfect dimension and p-dim $M = k$-dim M.

Corollary 2.6. Let R-module M has Krull dimension, then M is α perfect critical if and only if it is α-critical.

Corollary 2.7. The following are equivalent for an R- module M.

1. k-dim $M \leq \alpha$.
2. M is quotient finite dimensional and k-dim $(xR) \leq \alpha$ for any $x \in M$.
3. M is quotient finite dimensional and p-dim $(xR) \leq \alpha$ for any $x \in M$.

4. M is quotient finite dimensional and k-dim $(F) \leq \alpha$ for any finitely generated submodule F of M.

5. M is quotient finite dimensional and p-dim $(F) \leq \alpha$ for any finitely generated submodule F of M.

6. M is quotient finite dimensional and p-dim $M \leq \alpha$.

It is well-know that for each submodule N of an R-module M, $k\text{-dim}\, M = \sup\{k\text{-dim}\, \frac{M}{N}, k\text{-dim}\, N\}$, if either side exists. A slight modification of the proof of this fact gives the following result.

Theorem 2.8. Let M be an R-module and $0 \neq A \subseteq M$ be a submodule of M with Krull dimension, then $p\text{-dim}\, M = \sup\{k\text{-dim}\, A,\ p\text{-dim}\, \frac{M}{A}\}$ if either side exists.

The following result is similar to [5, Corollary 1.5].

Lemma 2.9. Let M be an R-module with finite Goldie dimension if for each essential submodule E of M, $\frac{M}{E}$ has perfect dimension, then M has perfect dimension and $p\text{-dim}\, M \leq \sup\{p\text{-dim}\, \frac{M}{E} + 1\ :\ E \subseteq^e M\}$.

The following result is similar to [5, Proposition 6.1].

Proposition 2.10. Let R be a semiprime left Goldie ring. If for each essential left ideal E of R, $\frac{R}{E}$ has perfect dimension, then R has perfect dimension and $p\text{-dim}\, R = \sup\{p\text{-dim}\, \frac{R}{E} + 1 : E \subseteq^e R\}$.

Recall that an R-module M is said to be a distributive module, written D-module, if the lattice of submodules of M is a distributive lattice. That is: If A, B and C are submodules of M, then $A \cap (B + C) = (A \cap B) + (A \cap C)$. We also recall that two module A and B are said to be unrelated if whenever we have submodules $P' \subseteq P \subseteq A$ and $Q' \subseteq Q \subseteq B$ such that $\frac{P}{P'} \simeq \frac{Q}{Q'}$, then $P = P'$ and $Q = Q'$. For more information about distributive modules, see [10]. We show that if $\{M_i\}_{i\in I}$ is a family of unrelated D-modules, then then $p\text{-dim}\,(\sum_{i\in I} \oplus M_i) = \sup\{p\text{-dim}\, M_i : i \in I\}$, if either side exists.

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Positive implicative filters in triangle algebras

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Abstract

In this paper, we define (MV) positive implicative interval valued residuated lattice -filter (IVRL-filters for short) of triangle algebras. We state and prove some theorems that determine some properties of these filters. Also, we introduce some special triangle algebras, and determine the relationship between them and IVRL-filters.

Keywords: Residuated lattices, Interval-valued structures, Triangle algebras, IVRL-filters
Mathematics Subject Classification [2010]: 08A72, 03G25

1 Preliminaries

Filter theory for logical algebras plays an important role in studying these algebraic structures and the completeness of the corresponding non-classical logics.

Van Gass et al. introduced triangle algebras as a variety of residuated lattices equipped with approximation operators and with a third angular point u, different from 0,1 [5]. They defined some types of filters in triangle algebras and obtained some interesting results [4].

Definition 1.1. [5] A residuated lattice is an algebra $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ with four binary operations and two constants 0,1 such that:
- $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leq z$ iff $x \leq y \rightarrow z$, for all x, y and z in L (residuation principle).

The ordering $\leq$ in a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ is defined as follows, for all x and y in L: $x \leq y$ iff $x \wedge y = x$.

Definition 1.2. [5] Given a lattice $\mathcal{A} = (A, \vee, \wedge)$, its triangularization $\mathbb{T}(\mathcal{A})$ is the structure $\mathbb{T}(\mathcal{A}) = (Int(\mathcal{A}), \vee, \wedge)$ defined by
- $Int(\mathcal{A}) = \{[x_1, x_2] : (x_1, x_2) \in A^2 \text{ and } x_1 \leq x_2\}$,
- $[x_1, x_2] \wedge [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$,
- $[x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$.

The set $D_{\mathcal{A}} = \{[x, x] : x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{A})$.

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Definition 1.3. [5] An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathcal{A}), \vee, \wedge, \odot, \rightarrow_{\odot}, [0,0], [1,1])$ on the triangularization $\mathbb{T}(\mathcal{A})$ of a bounded lattice $\mathcal{A}$, in which the diagonal $D_{\mathcal{A}}$ is closed under $\odot$ and $\rightarrow_{\odot}$, i.e. $[x,x] \odot [y,y] \in D_{\mathcal{A}}$ and $[x,x] \rightarrow_{\odot} [y,y] \in D_{\mathcal{A}}$, for all x, y in L.

Definition 1.4. [5] A triangle algebra is a structure $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ in which $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operations on A, u a constant, and satisfying the following conditions:

$(T.1)\nu x \leq x,$ $(T.1')x \leq \mu x,$

$(T.2)\nu x \leq \nu\nu x,$ $(T.2')\mu\mu x \leq \mu x,$

$(T.3)\nu(x \wedge y) = \nu x \wedge \nu y,$ $(T.3')\mu(x \wedge y) = \mu x \wedge \mu y,$

$(T.4)\nu(x \vee y) = \nu x \vee \nu y,$ $(T.4')\mu(x \vee y) = \mu x \vee \mu y,$

$(T.5)\nu u = 0,$ $(T.5')\mu u = 1,$

$(T.6)\nu\mu x = \mu x,$ $(T.6')\mu\nu x = \nu x,$

$(T.7)\nu(x \rightarrow y) \leq \nu x \rightarrow \nu y,$

$(T.8)(\nu x \leftrightarrow \nu y) * (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y),$

$(T.9)\nu x \rightarrow \nu y \leq \nu(\nu x \rightarrow \nu y).$

Definition 1.5. [4] Let $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. An element x in A is called exact if $\nu x = x$. The set of exact elements of $\mathcal{A}$ is denoted by $E(\mathcal{A})$.

Definition 1.6. [4] Let $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. An IVRL-filter of $\mathcal{A}$ is a non-empty subset F of A satisfying:
$(F.1)$ if $x \in F, y \in A$ and $x \leq y$, then $y \in F$,
$(F.2)$ if $x, y \in F$, then $x * y \in F$,
$(F.3)$ if $x \in F$, then $\nu x \in F$.

For all $x, y \in A$, we write $x \sim_F y$ iff $x \rightarrow y$ and $y \rightarrow x$ are both in F.

Van Gass et al. introduced triangle algebras: a variety of residuated lattice equipped with approximation operators, and with a third angular point u, different from 0,1. They show that these algebras serve as an equational representation of interval-valued residuated lattice (IVRLs) [5]. Van Gass et al. defined some types of filters in triangle algebras as Boolean filters and prime filters. Also, they obtained some interesting results [4].

2 Positive implicative filters in triangle algebras

Definition 2.1. F is an IVRL-extended positive implicative filter if for $x, y \in A$, $(\nu x \rightarrow \nu y) \rightarrow \nu x \in F$, implies $\nu x \in F$.

Definition 2.2. F is a positive implicative IVRL-filter if for $x, y \in A$, $\nu((x \rightarrow y) \rightarrow x) \in F$, implies $\nu x \in F$.

It is clear that every positive implicative IVRL-filter of A is an IVRL-extended positive implicative filter of A, but the converse is not true.

x	νx
0	0
u	0
1	1

x	μx
0	0
u	1
1	1

$\odot$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\Rightarrow$	0	u	1
0	1	1	1
u	0	1	1
1	0	u	1

Example 2.3. Let $A = \{0, u, 1\}$. We define operators $\nu, \mu, \odot, \Rightarrow$ as follow:
$\mathcal{A} = (A, \vee, \wedge, \odot, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra. It is clear that, that $F = \{1\}$ is an IVRL-extended positive implicative filter of A. Let $x = u, y = 0$. Then $\nu((u \Rightarrow 0) \Rightarrow u) = 1 \in F$, but $\nu u = 0 \notin F$. Thus F is not a positive implicative IVRL-filter of A.

Theorem 2.4. *Let F be an IVRL-filter of A. Consider the following assertions:*
(i) *F is an IVRL-extended positive implicative filter of A.*
(ii) *If $x \in A$ and $\neg\nu x \to \nu x \in F$, then $\nu x \in F$.*
(iii) *If $x, y \in A$ and $(\nu x \to \nu y) \to \nu y \in F$, then $(\nu y \to \nu x) \to \nu x \in F$.*
Then:
a) $(i) \Leftrightarrow (ii)$.
b) $(i) \Rightarrow (iii)$.
c) If F is an IVRL-extended implicative filter of A, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Theorem 2.5. *If F,G are two IVRL-filters of A, $F \subseteq G$ and F is an IVRL-extended positive implicative filter(positive implicative IVRL-filter) of A, then G is an IVRL-extended positive implicative filter(positive implicative IVRL-filter) of A.*

Definition 2.6. A triangle algebra A is called a Boolean triangle algebra if $x \vee \neg x = 1$, for all $x \in A$.

Definition 2.7. For a nonempty subset $S \subseteq A$, the smallest IVRL-filter of A which contains S, i.e. $\cap\{F : S \subseteq F\}$, is said to be the IVRL-filter of A generated by S and will be denoted by [S). If $S = \{a\}$, with $a \in A$, we denoted by [a) the IVRL-filter generated by $\{a\}$ ([a) is called principal).

Proposition 2.8. *Let $S \subseteq A$, a nonempty subset of A, $a \in A$. Then $[S) = \{x \in A : s_1 * ... * s_n \leq \nu x$, for some $n \geq 1$ and $s_1, ..., s_n \in S\}$. In particular, $[a) = \{x \in A : a^n \leq \nu x$, for some $n \geq 1\}$.*

Lemma 2.9. *The following conditions are equivalent:*
(i) *$\{1\}$ is an IVRL-extended positive implicative filter of A,*
(ii) *Every IVRL-filter of A is an IVRL-extended positive implicative filter of A,*
(iii) *For every $a \in A$, $[a)$ is an IVRL-extended positive implicative filter of A,*
(iv) *$(\nu x \to \nu y) \to \nu x = \nu x$, for all $x, y \in A$,*

Lemma 2.10. *The following conditions are equivalent:*
(i) *$\{1\}$ is an positive implicative IVRL-filter of A,*
(ii) *Every IVRL-filter of A is an IVRL-extended positive implicative filter of A,*
(iii) *For every $a \in A$, $[a)$ is a positive implicative IVRL-filter of A,*
(iv) *$\nu((x \to y) \to x) = \nu x$, for all $x,y \in A$,*
(v) *A is a Boolean-triangle algebra.*

Proposition 2.11. *Let F be an IVRL-filter of A. A/F is a Boolean triangle algebra if and only if F is a positive implicative IVRL-filter of A.*

Corollary 2.12. *Let A/F be a Boolean triangle algebra. Then F is an IVRL-extended positive implicative filter of A.*

In the following example we show that the converse of above corollary is not true.

Example 2.13. In Example 2.3, it is clear that $F = \{1\}$ is an IVRL-extended positive implicative filter of A. But since $\neg u \vee u = u$, $A/\{1\}$ is not a Boolean triangle algebra.

Definition 2.14. A triangle algebra A is called a BL-triangle algebra if it satisfies the following identities, for all $x, y \in A$:
$(x \to y) \vee (y \to x) = 1$ (prelinearity),
$x \wedge y = x * (x \to y)$(divisibility).
A BL-triangle algebra A is called an MV-triangle algebra if and only if $(x \to y) \to y = (y \to x) \to x$, for all $x, y \in A$.

Definition 2.15. An IVRL-filter F of A will be called IVRL-extended MV-filter if $((\nu x \to \nu y) \to \nu y) \to ((\nu y \to \nu x) \to \nu x) \in F$. And will be called MV-IVRL-filter if $\nu(((x \to y) \to y) \to ((y \to x) \to x)) \in F$, for all $x, y \in A$.

Corollary 2.16. *Let F be an IVRL-extended MV-filter (MV-IVRL-filter)of A. Then* $\neg\neg\nu x \to \nu x \in F$ *(*$\nu(\neg\neg x \to x) \in F$*), for all* $x \in A$.

Theorem 2.17. *F is an MV-IVRL-filter of A if and only if A/F is an MV-triangle algebra.*

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Primary Decomposition of Ideals in MV-algebras

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Abstract

In this paper, we investigate the ideal theory in MV-algebras and we define the notions of implicative MV-algebras and primary (P-primary) ideals in MV-algebras. Then we show that in implicative MV-algebras, if an ideal has a primary decomposition, then it has a reduced primary decomposition.

Keywords: MV-algebra, radical, primary and P-primary ideals, primary decomposition
Mathematics Subject Classification [2010]: 06F35, 06D99, 08A05

1 Introduction

MV-algebras were defined by C.C. Chang [1, 2] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. The notion of prime ideal in an MV-algebra was introduced by Chang. Since the notion of ideal in MV-algebras is important, for completion of study of ideals in MV-algebras, in this paper, we present definitions of radical of an ideal and primary decomposition of an ideal.

Definition 1.1. [3] An *MV-algebra* is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that:
$(MV1)$ $(M, \oplus, 0)$ is an Abelian monoid,
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
If we define the constant $1 = 0'$, then operations $\odot$ and $\ominus$ are defined by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$. Also, operations $\vee$ and $\wedge$ on M are defined by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$. An ideal of MV-algebra M is a subset I of M, satisfying the following condition: $(I1)$ $0 \in I$, $(I2)$ $x \leq y$ and $y \in I$ implies that $x \in I$, $(I3)$ $x \oplus y \in I$, for every $x, y \in I$. We let $\mathcal{I}(M)$ be the set of all ideals of M. A proper ideal P of M is a prime ideal if for $x, y \in M$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. Equivalently, P is prime if and only if $x \ominus y \in P$ or $y \ominus x \in P$, for every $x, y \in M$.

Note: From now on, in this paper, we let M be an MV-algebra and $\mathcal{PI}(M)$ be the set of all prime ideals of M.

*Speaker

2 Primary decomposition of ideals in MV-algebras

Definition 2.1. M is called an implicative MV-algebra if $x \ominus (y \ominus x) = x$, for every $x, y \in M$.

Example 2.2. Let $M_1 = \{0,1,2,3\}$, $M_2 = \{0,1\}$, and operations $\oplus_1$ and $\oplus_2$ be defined by

$\oplus_1$	0	1	2	3
0	0	1	2	3
1	0	1	3	3
2	2	3	2	3
3	3	3	3	3

$\oplus_2$	0	1
0	0	1
1	1	1

If $0' = 3,\ 1' = 2,\ 2' = 1$ and $3' = 0$, then $(M_1, \oplus_1, ', 0, 1)$ is an implicative MV-algebra. Also, if $0' = 1$ and $1' = 0$, then $(M_2, \oplus_2, ', 0, 1)$ is an implicative MV-algebra.

Definition 2.3. Let M be an MV-algebra and $I \in \mathcal{I}(M)$. Then the intersection of all prime ideals of M, including I, is called *radical* of I and it is denoted by $rad_M(I)$ or briefly $rad(I)$. If there is not any prime ideal of M including I, then we let $rad(I) = M$.

Example 2.4. In Example 2.2, $I = \{0,1\}$ and $J = \{0,2\}$ are ideals of M_1. It is easy to show $rad(I) = I$ and $rad(J) = J$.

Lemma 2.5. *Let M be implicative. Then $(x \ominus z) \ominus (y \ominus z) = (x \ominus y) \ominus z$ and $x \ominus (x \ominus y) = y \ominus (y \ominus x)$, for every $x, y, z \in M$.*

Theorem 2.6. *Let $x \oplus x = x$, for every $x \in M$. Then M is a chain if and only if all proper ideals of M are prime.*

Theorem 2.7. *Let M be an implicative chain. Then $rad(I) = I$, for every $I \in \mathcal{I}(M)$.*

Definition 2.8. Let M be an MV-algebra and $\emptyset \neq S \subseteq M$. We say that S is $\wedge$-closed, if $a \wedge b \in S$, for all $a, b \in S$.

Theorem 2.9. *Let M be an MV-algebra, $I \in \mathcal{I}(M)$, $S \subseteq M$ be $\wedge$-closed and $S \cap I = \emptyset$. Then there exists a maximal ideal P of M such that $P \supseteq I$ and $P \cap S = \emptyset$. Furthermore, P is a prime ideal of M.*

Notation. The set of all prime ideals of M that contain $J \in \mathcal{I}(M)$ will be denoted by $\mathcal{PI}_J(M)$.

Lemma 2.10. *Let M be implicative and $a, b, c \in M$. Then $a \wedge (b \ominus c) = (a \wedge b) \ominus c$.*

Theorem 2.11. *Let M be implicative and $I \in \mathcal{I}(M)$. Then*

$$rad(I) = \{x \in M : \forall P \in \mathcal{PI}_I(M), \exists c \in M \setminus P \text{ such that } c \wedge x \in I\}.$$

Proof. Let

$$T = \{x \in M : \forall P \in \mathcal{PI}_I(M), \exists c \in M \setminus P \text{ such that } c \wedge x \in I\}$$

and $x \in rad(I)$. Then $x \in P$, for every $P \in \mathcal{PI}_I(M)$. If $x \in I$, then by considering $c = 1$, we have $x \in T$. Now, let $x \notin I$. If $x \notin T$, then there exists $P_1 \in \mathcal{PI}_I(M)$ such that $c \wedge x \notin I$, for every $c \in M \setminus P_1$. Let $S = \{(c \wedge x) \ominus y : y \in I \text{ and } c \in M \setminus P_1\}$. First, we show that S is $\wedge$-closed. Let $(c_1 \wedge x) \ominus y_1, (c_2 \wedge x) \ominus y_2 \in S$, where $c_1, c_2 \in M \setminus P_1$ and $y_1, y_2 \in I$. By Lemma 2.10, we can show that $((c_1 \wedge x) \ominus y_1) \wedge ((c_2 \wedge x) \ominus y_2) = ((y_2' \wedge c_1 \wedge c_2) \wedge x) \ominus y_1$. Now, we show that $y_2' \wedge c_1 \wedge c_2 \in M \setminus P_1$. Let $y_2' \wedge c_1 \wedge c_2 \in P_1$. Since $c_1 \wedge c_2 \notin P_1$, $y_2' \in P_1$ and so $1 \in P_1$. Since $x \leq 1 \in P_1$, $x \in P_1$, for every $x \in M$ and so $P_1 = M$, which is a contradiction. Hence, $y_2' \wedge c_1 \wedge c_2 \in M \setminus P_1$ and so $((y_2' \wedge c_1 \wedge c_2) \wedge x) \ominus y_1 \in S$. It means that$((c_1 \wedge x) \ominus y_1) \wedge ((c_2 \wedge x) \ominus y_2) \in S$ and so S is $\wedge$-closed. Now, we prove that $S \cap I = \emptyset$. If $S \cap I \neq \emptyset$, then there exist $c' \in M \setminus P_1$ and $y' \in I$ such that $(c' \wedge x) \ominus y' \in I$. It results that $c' \wedge x \in I$. But, by definition of S, $c \wedge x \notin I$, for every $c \in M \setminus P_1$, which is a contradiction. Then $S \cap I = \emptyset$ and so by Theorem 2.9, there exists $P_2 \in \mathcal{PI}_I(M)$ such that $P_2 \cap S = \emptyset$. Since $(c \wedge x) \ominus x = 0 \in P$ and $x \in P$, $c \wedge x \in P$, for every $c \in M \setminus P$ and for every $P \in \mathcal{PI}_I(M)$. Then $(c \wedge x) \in P_2$. On the other hand, $c \wedge x = (c \wedge x) \ominus 0 \in S$. Hence, $c \wedge x \in P_2 \cap S$, which is a contradiction. It implies that $x \in T$. Therefore, $rad(I) \subseteq T$. It is easy to show that $T \subseteq rad(I)$ and so $T = rad(I)$. □

Proposition 2.12. *Let M be implicative and $I \in \mathcal{I}(M)$. If for every $P \in \mathcal{PI}(M)$, $P \cap I \neq \{0\}$ implies that $I \subseteq P$, then*

$$rad(I) = \{x \in X : \ \forall\ P \in \mathcal{PI}(M) \text{ with } P \cap I \neq \{0\},\ \exists c \in M \setminus P \text{ such that } c \wedge x \in I\}.$$

Theorem 2.13. *Let M be an MV-algebra and $I, J, I_1, \cdots, I_n$ be ideals of M. Then*
(i) $I \subseteq rad(I)$,
(ii) $I \subseteq J$ implies $rad(I) \subseteq rad(J)$,
(iii) $rad(I) \cup rad(J) \subseteq rad(I \cup J)$.
Moreover, if M is implicative and $P \cap I_k \neq \{0\}$ implies that $I_k \subseteq P$, for every $P \in \mathcal{PI}(M)$ and $1 \leq k \leq n$, then
(iv) $rad(rad(I)) = rad(I)$,
(v) $rad(\bigcap_{k=1}^{n} I_k) = \bigcap_{k=1}^{n} rad(I_k)$.

Definition 2.14. Let M be an MV-algebra and Q be a proper ideal of M. Then Q is called a *primary* ideal of M if $a \wedge b \in Q$, then there exists $c \in M \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(M)$ and $a, b \in M$.

Example 2.15. In Example 2.2, $I = \{0, 1\}$ and $J = \{0, 2\}$ are primary ideals of M_1.

Proposition 2.16. *Let M be implicative and Q be an ideal of M. Then Q is a primary ideal of M if and only if $a \wedge b \in Q$ implies that $a \in rad(Q)$ or $b \in rad(Q)$, for any $a, b \in M$.*

Proof. ($\Rightarrow$) Let Q be a primary ideal of M and $a \wedge b \in Q$, for $a, b \in M$. If $a \in Q$, then $a \in rad(Q)$. Let $a \notin Q$. Then there exits $c \in M \setminus P$ such that $c \wedge b \in Q$ or $a \wedge c \in Q$, for every $P \in \mathcal{PI}_Q(M)$. If $c \wedge b \in Q$, then $c \wedge b \in P$, for every $P \in \mathcal{PI}_Q(M)$. Since $c \notin P$, $b \in P$, for every $P \in \mathcal{PI}_Q(M)$. It results that $b \in \bigcap_{Q \subseteq P} P = rad(Q)$. Similarly, if $a \wedge c \in Q$, then $a \in rad(Q)$.
($\Leftarrow$) By Theorem 2.11, the result will be obtained. □

Theorem 2.17. *In an MV-algebra, every prime ideal is a primary ideal.*

Theorem 2.18. *Let M be implicative and $I \cap P \neq \{0\}$ implies that $I \subseteq P$, for every $I \in \mathcal{I}(M)$ and $P \in \mathcal{PI}(M)$. Then the radical of every primary ideal of M is a prime ideal of M.*

Definition 2.19. Let M be an MV-algebra and $Q, P \in \mathcal{I}(M)$. Then Q is called a *P-primary* ideal of M if Q is a primary ideal of M and $rad(Q) = P$.

Example 2.20. In Example 2.15, I is a P-primary ideal of M, where $P = \{0, 1\}$.

Definition 2.21. Let M be an MV-algebra, $I \in \mathcal{I}(M)$ and there exist primary ideals $Q_1, Q_2, \cdots, Q_n$ of M such that $I = Q_1 \cap Q_2 \cdots \cap Q_n$. Then we say $Q_1 \cap Q_2 \cdots \cap Q_n$ is a *primary decomposition* of I and I has a primary decomposition. This decomposition is *reduced* if
(i) $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$, for every $1 \leq i, j \leq n$,
(ii) $rad(Q_i) \neq rad(Q_j)$, for every $1 \leq i, j \leq n$.

Lemma 2.22. *Let M be implicative and $Q_1, Q_2, \cdots, Q_n$ be P'-primary ideals of M such that $P \cap Q_i \neq \{0\}$ implies that $Q_i \subseteq P$, for every $P \in \mathcal{PI}(M)$, where $P' \in \mathcal{PI}(M)$. Then $\bigcap_{i=1}^{n} Q_i$ is P'-primary.*

Theorem 2.23. *Let M be implicative, $I = Q_1 \cap \cdots \cap Q_n$ be a primary decomposition of I and $P \cap Q_i \neq \{0\}$ implies that $Q_i \subseteq P$, for every $P \in \mathcal{PI}(M)$ and $1 \leq i \leq n$. Then I has a reduced primary decomposition.*

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Representations of Polygroups Based on Krasner Hypervector Spaces

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Abstract

In this paper we introduce representations of polygroups by Krasner hypervector spaces.The goal of polygroup representation is to study polygroups via their actions on Krasner hypervector spaces. By acting on Krasner hypervector spaces even more detailed information about a polygroup can be obtained.

Keywords: Polygroup, Krasner hypervector space, Representation
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In [8] M. Motameni, R. Ameri and R. Sadeghi studied hypermatrix based on hyperspaces. The goal of repsentation of polygroups is to study polygroups via their actions on hyperspaces. By acting on hyperspaces even more detailed information about a polygroup can be obtained. In this note we introduced and study the representation of polygroups by Krasner hyperspaces and obtain some related basic results.

Recall that for a non-empty set H a *hyperoperation or a join operation* is a map $. : H \times H \longrightarrow P_*(H)$, where $P_*(H)$ is the set of all non-empty subsets of H.

Definition 1.1. [4] A polygroup is a special case of a hypergroup. A polygroup is a system $\mathcal{P} = \langle P, ., e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unary operation on P, . maps $P \times P$ into nonempty subsets of P, and the following axioms hold for all $x, y, z \in P$:
(P_1) $(x.y).z = x.(y.z)$,
(P_2) $x.e = e.x = x$,
(P_3) $x \in y.z$ implies $y \in x.z^{-1}$ and $z \in y^{-1}.x$.

Definition 1.2. [3] A Krasner hyperring is a hyperstructure $(R, \oplus, \star)$ where
(i) $(A, \oplus)$ is a canonical hypergroup;
(ii) $(A, \star)$ is a semigroup endowed with a two-sided absorbing element 0;
(iii) the product distributes from both sides over the sum.

Definition 1.3. [3] Let $(K, \oplus, \star)$ be a hyperfield and $(V, \oplus)$ be a canonical hypergroup. We define a Krasner hyperspace over K to be the quadrupled $(V, \oplus, \cdot, K)$, where $\cdot$ is a single-valued operation

$$\cdot : K \times V \longrightarrow V,$$

*Speaker

such that for all $a \in K$ and $x \in V$ we have $a \cdot x \in V$, and for all $a, b \in K$ and $x, y \in V$ the following conditions are satisfied:
(H_1) $a \cdot (x \oplus y) = a \cdot x \oplus a \cdot y$;
(H_2) $(a \oplus b) \cdot x = a \cdot x \oplus b \cdot x$;
(H_3) $a \cdot (b \cdot x) = (a \star b) \cdot x$;
(H_4) $0 \cdot x = 0$;
(H_5) $1 \cdot x = x$.

Definition 1.4. [8] Let $(V, \oplus, \cdot)$ and $(W, \oplus, \cdot)$ be two $K-$hyperspaces over a hyperfield K. Then the mapping $T : V \longrightarrow P_*(W)$ is called (i) multivalued linear transformation $mv-$transformation if

$$T(x \oplus y) \subseteq T(x) \oplus T(y) \text{ and } T(a \cdot x) = a \cdot T(x).$$

(ii) strong multivalued linear transformation $smv-$transformation if

$$T(x \oplus y) = T(x) \oplus T(y) \text{ and } T(a \cdot x) = a \cdot T(x).$$

where, $P_*(W)$ is the non-empty power set of W.

Definition 1.5. [7] Let $(G, \cdot)$ be a hypergroupoid. The action of $(G, .)$ on a non empty set A is a map $\bullet : G \times A \longrightarrow P_*(A)$ such that for all $(g_1, g_2) \in G \times G, a \in A$:
(i) $\bigcup_{t \in g_1 . g_2} t \bullet a = \bigcup_{s \in g_2 \bullet a} g_1 \bullet s$,
(ii) $\exists e \in G$; $a \in e \bullet a$.

Proposition 1.6. *[7] Let $(G, .)$ be a hypergroupoid and $A^{P_*(A)}$ be the set of all functions from A to $P_*(A)$, endowed with the composition operation $\circ$, then $\varphi : G \longrightarrow A^{P_*(A)}$ defined by $\varphi(g)(a) = g \bullet a$ is a homomorphism.*

The homomorphism $\varphi : G \longrightarrow A^{P_*(A)}$ is called a representation associated with the hypergroupoid action. this process is reversible in the sense that if $\varphi : G \longrightarrow A^{P_*(A)}$ is any homomorphism then the map from $G \times A \longrightarrow P_*(A)$ defined by $g \bullet a = \varphi(g)(a)$ satisfies the properties of a hypergroupoid action of G on A (for more details see [7]).

Definition 1.7. [7] Let $(H, .)$ and $(K, \odot)$ be two hypergroupoids and $\varphi : K \longrightarrow H^{P_*(H)}$ be a representation determined by the hypergroupoid action $\bullet$ of K on H. Let G be the set of ordered pairs (h, k) where $(h, k) \in H \times K$ and define the following hyperoperation on G by

$$(h_1, k_1) * (h_2, k_2) = (h_1.\varphi(k_1)(h_2), k_1 \odot k_2).$$

Clearly this hyperoperation makes G into hypergroupoid which is denoted by $H \int_\varphi K$ or $(H \times K, *)_\varphi$.

Definition 1.8. Let $(V, \oplus, .)$ be a $K-$hyperspace over a hyperfield K and let G be a polygroup. Then V be a $KG-$hypermodule if G act on V, satisfying the following conditions for all $u, v \in V, \lambda \in K$ and $g, h \in G$:
1) $g \bullet (\lambda v) = \lambda (g \bullet v)$;
2) $g \bullet (u \oplus v) \subseteq g \bullet u \oplus g \bullet v$.

Proposition 1.9. *Let $\langle G, ., e, ^{-1}\rangle$ be a polygroup and V is a $KG-$hypermodule, then $\varphi : G \longrightarrow L(V)$, where $L(V) = \{T \mid T : V \longrightarrow P_*(V)$ is a $mv-$transformation$\}$ defined by $\varphi(g)(v) = g \bullet v$ is a good homomorphism.*

Proof. $\varphi(g)$ is a $mv-$transformation. (By the Definition 1.8) $(\varphi(g_1.g_2))(v) = \bigcup_{t\in g_1.g_2} t \bullet v$. From Definition 1.5 (i) obtains

$$\bigcup_{t\in g_1.g_2} t \bullet v = \bigcup_{s\in g_2\bullet v} g_1 \bullet s = g_1 \bullet (g_2 \bullet v) = \varphi(g_1)(\varphi(g_2)(v)) = (\varphi(g_1) \odot \varphi(g_2))(v).$$

□

The homomorphism $\varphi : G \longrightarrow L(V)$ is called a representation associated with the polygroup action.

2 Main results

Note that a $K-$hyperspace V, $L(V)$ by the composition is a monoid, where $(f \circ g)(x) = \bigcup_{t\in g(x)} f(t)$.

Definition 2.1. A representation of a polygroup P is a homomorphism $\varphi : P \longrightarrow L(V)$ for some (finite-dimensional) non-zero $K-$hyperspace V such that $L(V) = \{T : V \longrightarrow P_*(V) \mid T$ is $mv-$transformation$\}$. The dimension of V is called the degree of φ.

If $T : V \longrightarrow P_*(W)$ be a $mv-$transformation, then T induced a map $\overline{T} : P_*(V) \longrightarrow P_*(W)$ by $\overline{T}(A) = \bigcup_{a\in A} T(a)$. Since if $A = B \subseteq V$, then $\overline{T}(A) = \overline{T}(B)$. Thus $\overline{T}$ is well-defined.

Two representations $\varphi : G \to L(V)$ and $\psi : G \to L(W)$ are equivalent if there exists an isomorphism $T : V \to W$ such that $\psi_g = \overline{T}\varphi_g T^{-1}$ for all $g \in P$, i.e., $\psi_g T = \overline{T}\varphi_g$ for all $g \in P$. In this case, we write $\varphi \sim \psi$. In pictures, we have that the diagram

$$\begin{array}{ccc} V & \overset{\varphi_g}{\longrightarrow} & P_*(V) \\ T \downarrow & & \downarrow \overline{T} \\ W & \underset{\psi_g}{\longrightarrow} & P_*(W) \end{array}$$

commutes.

Definition 2.2. Let $\varphi : P \to L(V)$ be a representation. A $K-$subhyperspace $W \leq V$ is $P-$invariant if, for all $g \in P$ and $w \in W$, one has $\varphi_g w \subseteq W$.

Definition 2.3. A representation $\varphi : P \to L(V)$ is said to be irreducible if the only $P-$invariant $K-$subhypervector spaces of V are $\{0\}$ and V.

Definition 2.4. Let P be a polygroup. A representation $\varphi : P \to L(V)$ is said to be completely reducible if $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ where the V_i are non-zero $P-$invariant $K-$subhypervector spaces and $\varphi\mid_{V_i}$ is irreducible for all $i = 1, \ldots, n$.
Equivalently, φ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus \ldots \oplus \varphi^{(n)}$ where the $\varphi^{(i)}$ are irreducible representations.

Definition 2.5. We say that φ is decomposable if $V = V_1 \oplus V_2$ with V_1, V_2 non-zero P-invariant $K-$subhyperspaces. Otherwise, V is called indecomposable.

Proposition 2.6. *Let $\varphi : P \to L(V)$ be equivalent to decomposable representation. Then φ is decomposable.*

Proposition 2.7. *Let $\varphi : P \to L(V)$ be equivalent to an irreducible representation. Then φ is irreducible.*

Proposition 2.8. *Let $\varphi : P \to L(V)$ be equivalent to a completely reducible representation. Then φ is completely reducible.*

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Semi Factorization Structures

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Abstract

In this article the notion of semi factorization structure in a category $\mathcal{X}$ is defined and its properties are investigated. Also conditions under which the semi factorization structure and the factorization structure are equivalent are given.

Keywords: Factorization structure, Semi factorization structure, Category
Mathematics Subject Classification [2010]: 20J99, 18A32

1 Introduction

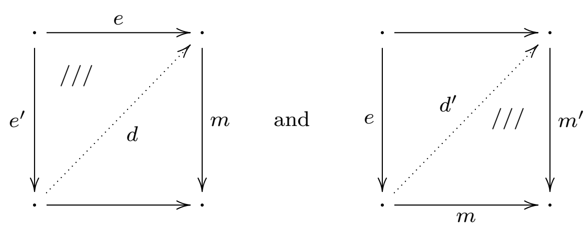
Factorization structures in categories are one of the most studied categorical concepts and weak factorization structures play an important role in homotopy theory(see [2]).

We introduce the notion of semi factorization structure in a category $\mathcal{X}$ and we remark that factorization structures are semi factorization structures. Then we provide an example of a semi factorization structure which is not a factorization structure. Also we analyze some of the properties of semi factorization structures which are similar to those of factorization structures. Finally, we show that if $\mathcal{E}, \mathcal{M}$ are classes of morphisms of $\mathcal{X}$ which are closed under composition and $\mathcal{M} \subseteq Mono(\mathcal{X})$, where $Mono(\mathcal{X})$ is the class of monomorphisms of $\mathcal{X}$, then $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-semi factorization structure if and only if it has $(\mathcal{E}, \mathcal{M})$-factorization structure.

Definition 1.1. Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms in a category $\mathcal{X}$, which are closed under composition with isomorphisms. We say that $\mathcal{X}$ has semi $(\mathcal{E}, \mathcal{M})$-factorizations or $(\mathcal{E}, \mathcal{M})$ is a semi factorization structure in $\mathcal{X}$, whenever:

(i) for all $f : Y \longrightarrow X$ there exist $m \in \mathcal{M}/X$ and $e \in Y/\mathcal{E}$ such that $f = me$; and

(ii) in the unbroken commutative diagrams below, with $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$:



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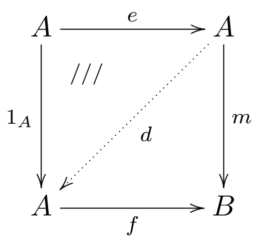
there exist unique morphisms d and d' such that $e = de'$ and $m = m'd'$.

Remark 1.2. Let $(\mathcal{E}, \mathcal{M})$ be a factorization structure in $\mathcal{X}$. Then $(\mathcal{E}, \mathcal{M})$ is a semi factorization structure in $\mathcal{X}$.

Lemma 1.3. *Let $(\mathcal{E}, \mathcal{M})$ be a semi factorization structure in $\mathcal{X}$. If in the following diagram:*

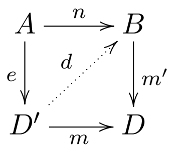


we have $f = me$ and $de = 1$, then $e \in Iso(\mathcal{X})$ and $f \in \mathcal{M}$.

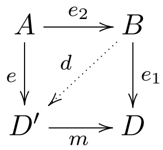
Proposition 1.4. *Let $(\mathcal{E}, \mathcal{M})$ be a semi factorization structure in $\mathcal{X}$. Then:*

(1) *$\mathcal{E} \cap \mathcal{M} = Iso(\mathcal{X})$.*

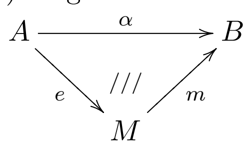
(2) *$\mathcal{M}$ is closed under composition if and only if the following commutative diagram has a diagonal d making both triangles commute, where $e \in \mathcal{E}$ and $m, m', n \in \mathcal{M}$.*



(2) *$\mathcal{E}$ is closed under composition if and only if the following commutative diagram has a diagonal d making both triangles commute, where $e, e_1, e_2 \in \mathcal{E}$ and $m \in \mathcal{M}$.*



Proof. (1) Let $\alpha : A \to B$ in $Iso(\mathcal{X})$ be given and $\alpha = me$ be the semi factorization of α.

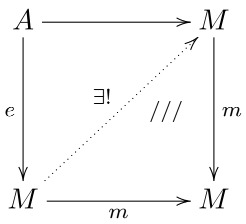


So we have:

$\alpha = me \quad \Rightarrow \quad (\alpha^{-1}m)e = 1$

$\alpha = me \quad \Rightarrow \quad me\alpha^{-1} = 1 \quad \Rightarrow \quad m(e\alpha^{-1}m) = m$

Hence, both morphisms 1_M and $e\alpha^{-1}m$ make the triangle in the following diagram commute:



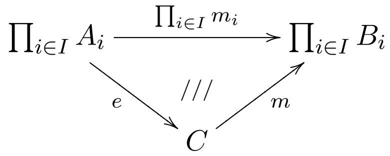
By uniqueness of the diagonal we have, $e(\alpha^{-1}m) = 1_M$. So $e \in Iso(\mathcal{X})$. Therefore $\alpha = me \in \mathcal{M}$ and $Iso(\mathcal{X}) \subseteq \mathcal{M} \cap \mathcal{E}$. Proof of the converse is similar. $\square$

Proposition 1.5. *Let $(\mathcal{E}, \mathcal{M})$ be a semi factorization structure in $\mathcal{X}$. Then,*
(1) $f \circ g \in \mathcal{M} \quad \Rightarrow \quad g \in \mathcal{M}$
(2) $f \circ g \in \mathcal{E} \quad \Rightarrow \quad f \in \mathcal{E}$
(3) *If g is a retraction and $f \circ g \in \mathcal{M}$, then $f \in \mathcal{M}$.*
(4) *If f is a section and $f \circ g \in \mathcal{E}$, then $g \in \mathcal{E}$.*

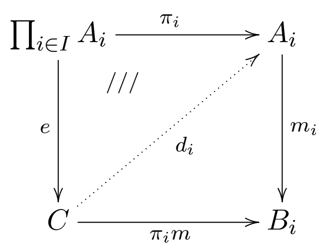
Lemma 1.6. *Let $(\mathcal{E}, \mathcal{M})$ be a semi factorization structure in $\mathcal{X}$ and $Sec(\mathcal{X}), Ret(\mathcal{X})$ be the class of sections and retractions of $\mathcal{X}$, respectively. Then,*
(1) $Sec(\mathcal{X}) \subseteq \mathcal{M}$.
(2) $Ret(\mathcal{X}) \subseteq \mathcal{E}$.

Proposition 1.7. *Let $(\mathcal{E}, \mathcal{M})$ be a semi factorization structure in $\mathcal{X}$. Then, $\mathcal{M}$ is closed under product.*

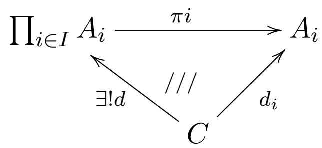
Proof. Let the family $\{m_i : A_i \to B_i\}_{i \in I}$ of morphisms of $\mathcal{M}$ be given. Consider the following $(\mathcal{E}, \mathcal{M})$-semi factorization of $\prod_{i \in I} m_i$:



Now in the following commutative diagram, since for each i, $m_i \in \mathcal{M}, \pi_i, e \in \mathcal{E}$, there exists a unique morphism $d_i : C \to A_i$ such that $d_i e = \pi_i$.

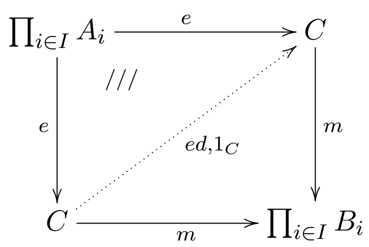


By the property of product we have:



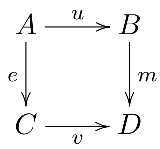
So, $\pi_i de = d_i e = \pi_i$, which implies $de = 1$.

Since ed and 1_C make the triangle in the following diagram commute, by uniqueness of the diagonal we have, $ed = 1$. Therefore $e \in Iso(\mathcal{X})$. Hence $\prod_{i\in I} m_i \in \mathcal{M}$.



□

Proposition 1.8. *Let $(\mathcal{E}, \mathcal{M})$ be a semi factorization structure in $\mathcal{X}$ and $\mathcal{E}, \mathcal{M}$ be closed under composition. Then for all the unbroken commutative diagrams*



with $m \in \mathcal{M}, e \in \mathcal{E}$, there exist morphisms $w, w' : C \to B$ such that $mw = v$ and $w'e = u$.

2 Main result

Theorem 2.1. *Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms of $\mathcal{X}$ that are closed under composition and $\mathcal{M} \subseteq Mono(\mathcal{X})$, where $Mono(\mathcal{X})$ is the class of monomorphisms of $\mathcal{X}$. Then $(\mathcal{E}, \mathcal{M})$ is a semi factorization structure for $\mathcal{X}$ if and only if it is a factorization structure for $\mathcal{X}$.*

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Talk Some Properties Of n-almost Prime Submodules pp.: 1–4

Some Properties Of n-almost Prime Submodules *

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Abstract

Prime ideals have many important properties and so its generalizations have been studied in many papers. The notion of n-almost prime submodules is generalization of prime submodules. In this article we study the behavior of n-almost Prime ideals in unique factorization domains and also we find some properties of n-almost Prime submodules of PI-multiplication modules.

Keywords: n-almost prime submodule, unique factorization domain, PI-multiplication modules
Mathematics Subject Classification [2010]: 13E05, 13C99, 13C13, 13F05, 13F15.

1 Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider $n > 1$ a positive integer. Let N be a submodule of an $R-$module M. The set $\{r \in R | rM \subseteq N\}$ is denoted by $(N : M)$ and particularly we denote $\{r \in R | rN = 0\}$ by $ann(N)$. Also we consider $T(M) = \{m \in M | \exists 0 \neq r \in R,\ rm = 0\}$. A module M is called torsion, if $T(M) = M$. If $T(M) = 0$, it is said that M is a torsion-free module.

An *n-almost prime* ideal was introduced in [1]. The concept of n-almost prime ideals is very strong motivation for the following notion, which is studied in this paper:

Definition 1.1. A proper submodule N of M will be called n-almost prime, if for $r \in R$ and $x \in M$ with $rx \in N \setminus (N : M)^{n-1}N$, either $x \in N$ or $r \in (N : M)$. A 2-almost prime submodule will be called an almost prime submodule.

According to definition, each prime submodule is an n-almost prime submodule, for any integer $n > 1$.

In order to obtain our main results, we use some definitions and lemma such as the following:

Lemma 1.2. *[3, Proposition 3.3] and [4, Proposition 3.1] Let M be a multiplications module. If M is non-torsion or finitely generated and I is an ideal of R containing $ann(M)$, then $(IM : M) = I$.*

*Will be presented in English

Recall that an R-module M is called semi-non-torsion, if M as $\frac{R}{ann(M)}$−module is non-torsion and an R-module M is called a PI-multiplication module, if for any submodule N of M, there exists an element $r \in R$ such that $N = rM$ (see [3]).

Lemma 1.3. *[5, Proposition 2.6] Let I be an ideal of a ring R and N a submodule of an R-module M. If $IM \neq IN$, $IN \neq N$, then $K = IN$ is n-almost prime if and only if $K = (K : M)^{n-1}K$.*

2 Main results

In the following we find some properties of n-almost prime ideal in unique factorization domain.

Proposition 2.1. *Let R be a unique factorization domain and I a proper ideal of R.*

(i) *Suppose that I is an n-almost prime ideal. Then for $x, y \in R$ with $[x, y] \in I \setminus I^n$, either $x \in I$ or $y \in I$.*

(ii) *I is n-almost prime if and only if for any $x \in I \setminus I^n$, there exists a prime element $p \in I$ such that $p \mid x$ and $p^n \nmid x$.*

(iii) *If there exist distinct prime elements $p_1, ..., p_m$ and positive integers $k_1, ..., k_m \geq 2$ such that $p_1^{k_1}...p_m^{k_m} \in I \setminus I^2$, then I is not n-almost prime.*

(iv) *If I is n-almost prime, then the ideal I/I^n of the ring R/I^n can be generated by the set $\{p + I^n \mid p \in I$ and p is a prime element of $R\}$.*

Proof. (i) Note that $x.\frac{y}{(x,y)} = [x, y] \in I \setminus I^n$, where (x, y) is the greatest common divisor of x and y. Then $x \in I$ or $\frac{y}{(x,y)} \in I$. If $\frac{y}{(x,y)} \in I$, then evidently $y \in I$.

(ii) Let I be an n-almost prime ideal and $x \in I \setminus I^n$. If $x = p_1^{k_1}...p_m^{k_m}$ is a prime decomposition for x, then as I is n-almost prime, for some $1 \leq j \leq m$, we have $p_j \in I$. If $k_j \geq n$, then $x \in I^n$, which is a contradiction. Therefore $k_j < n$ and so $p_j | x$, $p_j^n \nmid x$.

Conversely, let $x, y \in R$ with $xy \in I \setminus I^n$ and there exists a prime element $p \in I$ such that $p|xy$. Thus $p \mid x$ or $p \mid y$, and so $x \in I$ or $y \in I$. Consequently I is n-almost prime.

(iii) If I is n-almost prime, then I is almost prime, so for some $1 \leq j \leq m$, we have $p_j \in I$. Since $k_j \geq 2$, then $p_1^{k_1}...p_m^{k_m} \in I^2$, which is impossible. Therefore I is not n-almost prime.

(iv) Let I/I^n be generated by a set X. Then for any $x \in I \setminus I^n$ with $x + I^n \in X$, there exists a prime element p_x of R such that $p_x \in I$ and $p_x \mid x$. This shows that I/I^n is generated by the set $\{p_x + I^n \mid x + I^n \in X,\ x \in I \setminus I^n\}$. □

Proposition 2.2. *Let M be a semi-non-torsion PI-multiplication R-module and I a proper ideal of R containing $ann(M)$. If $N = IM$, then the following are equivalent.*

(i) *N is an n-almost prime submodule of M.*

(ii) *N is a prime submodule of M or $N = I^{n-1}N$.*

(iii) *I is a prime ideal of R or $I = I^n$.*

Proof. Note that $(N : M) = (IM : M) = I$, by Lemma 1.2.

(i) $\Longrightarrow$ (ii) By [3, Theorem 4.18], there exist a positive integer m and prime ideals $P_1, ..., P_m$ of R containing $ann(M)$ such that $N = P_1...P_mM$.

If $N = P_1M$, then since M is non-torsion multiplication $\frac{R}{ann(M)}-$module, by Lemma 1.2, we have $(\frac{P_1}{ann(M)}M : M) = \frac{P_1}{ann(M)}$. Hence by [3, Proposition 4.19], $\frac{P_1}{ann(M)}M$ is a prime $\frac{R}{ann(M)}-$submodule of M, and clearly $N = P_1M$ is a prime R-submodule of M.

Now suppose that $N \neq P_1M$. Without loss of generality, we may suppose that $N \neq P_2P_3 \cdots P_mM$. Then by Lemma 1.3, $N = (N : M)^{n-1}N = I^{n-1}N$.

(ii) $\Longrightarrow$ (iii) If N is a prime submodule, then evidently $I = (N : M)$ is a prime ideal. If $N = I^{n-1}N$, then $I = (IM : M) = (N : M) = (I^nM : M) = I^n$, by Lemma 1.2.

(iii) $\Longrightarrow$ (i) If I is a prime ideal, then by [3, Proposition 4.19], $N = IM$ is a prime submodule. Also note that $I = I^n$ implies that $N = IM = I^nM = I^{n-1}N = (N : M)^{n-1}N$. □

According to ([2]), an endomorphism e of an R-module M is called a scalar multiplication idempotent, if $e^2 = e$ and there exists $r \in R$ with $e(z) = rz$ for all $z \in M$.

The following theorem assert that under some conditions $End_R(\frac{M}{N})$ has a non-trivial scalar multiplication idempotent,for submodule N of M.

Theorem 2.3. *Let N be a non-zero submodule of a multiplication R-module M. Then $End_R(\frac{M}{N})$ has a non-trivial scalar multiplication idempotent if and only if there exist two proper submodules J, K of M such that $N = (K : M)J$, $M = J + K$ and $R = (K : M) + (J : M)$.*

Proof. Assume e is a non-trivial scalar multiplication idempotent of $\frac{M}{N}$. Since e is idempotent, there exist two submodules J, K of M containing N such that $Im\ e = \frac{J}{N}$, $Ker\ e = \frac{K}{N}$ and $\frac{M}{N} = \frac{J}{N} \oplus \frac{K}{N}$ ([1]).

Clearly $N = J \cap K$ and $M = J + K$, so we have $(N : J) = (J \cap K : J) = (K : J) = (K : K + J) = (K : M)$, and similarly $(N : K) = (J : M)$.

Now we claim that $(N : K) + (N : J) = R$, that is $(J : M) + (K : M) = R$. Since e is a non-trivial scalar multiplication idempotent, there exists an element $r \in R$ such that for any $z \in M$, $e(z + N) = rz + N$ and $r, r + 1 \notin (N : M)$. We have $r \in (N : K)$, since $N = e(c+N) = rc+N$, for any $c \in K$. Let $a \in J$. Then there exists an element $b \in M$ such that $a+N = e(b+N) = rb+N$. Then $a+N = rb+N = e(b+N) = e^2(b+N) = r^2b+N$. Thus $(1 - r)(a + N) = (1 - r)rb + N = rb - r^2b + N = N$, and so $1 - r \in (N : J)$. Hence, as $1 = r + (1 - r) \in (N : K) + (N : J)$, then $(N : K) + (N : J) = R$.

Since $(J : M) + (K : M) = R$, it is easy to see that $((J : M)M) \cap ((K : M)M) \subseteq (J : M)(K : M)M$. Note that M is multiplication, then $J = (J : M)M$ and $K = (K : M)M$, consequently $N = J \cap K \subseteq (J : M)(K : M)M = (J : M)K = (K : M)J \subseteq J \cap K = N$ and so $N = (J : M)K = (K : M)J$. Note that K is a proper submodule of M, otherwise $N = (K : M)J = J$, thus $e = 0_{End_R(\frac{M}{N})}$, which is impossible. Also if $J = M$, then $N = (J : M)K = K$, therefore e is a monomorphism and since $e^2 = e$, we have $e = 1_{End_R(\frac{M}{N})}$, which is a contradiction.

Conversely, let there exist two proper submodules J, K of M such that $N = (K : M)J$, $M = J+K$ *and* $R = (K : M)+(J : M)$. Clearly $N \subseteq K\cap J$. Then $(N : M) \subseteq (K\cap J : M)$.

As $(K:M)$ and $(J:M)$ are comaximal ideals, $(N:M) \subseteq (K \cap J:M) = (K:M) \cap (J:M) = (K:M)(J:M)$. Since $(K:M)(J:M) \subseteq ((K:M)J:M) = (N:M)$, we will get $(N:M) = (K:M)(J:M) = (K:M) \cap (J:M)$, and so $\frac{R}{(N:M)} \simeq \frac{R}{(K:M)} \times \frac{R}{(J:M)}$.

If $s+(N:M)$ is a preimage of the element $(1+(K:M), 0+(J:M)) \in \frac{R}{(K:M)} \times \frac{R}{(J:M)}$ in $\frac{R}{(N:M)}$, then $s+(N:M)$ is a non trivial idempotent and so $s^2 - s \in (N:M)$ and $s, s-1 \notin (N:M)$. Define R-homomorphism $h : \frac{M}{N} \longrightarrow \frac{M}{N}$, $h(w+N) = sw+N$, for each $w \in M$. For any $x \in M$, we have $(s^2-s)x+N = h^2(x+N) - h(x+N) = N$, so $h^2(x+N) = h(x+N)$, that is h is idempotent.

If $h = 0$, then for every $z \in M$, $h(z+N) = sz+N = N$, hence $s \in (N:M)$, which is impossible. In case $h = 1$, we have $h(g+N) = sg+N = g+N$, for each $g \in M$, then $s-1 \in (N:M)$, which is a contradiction. Therefore h is a non-trivial scalar multiplication idempotent. □

Corollary 2.4. *Let N be a finitely generated submodule of multiplication torsion-free an R-module M and $End_R(\frac{M}{N})$ has a non-trivial scalar multiplication idempotent. Then N is n-almost prime if and only if $N = 0$.*

Proof. If $N = 0$, then clearly N is $n-$almost prime. Now assume N is n-almost prime. By Theorem 2.3, there exist two proper submodules L, K of M such that $N = (K:M)L$, $M = K + L$ *and* $(K:M) + (L:M) = R$.

Note that $K = (K:M)M \neq (K:M)L = N$, otherwise $M = K$, which is impossible. Also $N = (K:M)L \neq L$, otherwise $L \subseteq N \subseteq K$, and hence $K = M$, a contradiction. Therefore by Lemma 1.3, $N = (N:M)^{n-1}N$, so by Nakayama's lemma for some $t \in (N:M)^{n-1}$, $(t+1)N = 0$, and as M is torsion-free, so $N = 0$. □

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Some properties of the character graph of a solvable group

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Abstract

Let G be a finite solvable group. In this paper we consider the character graph of G and study some parameters of this graph. At first, we answer this question that when is this graph Hamiltonian? Then we obtain conditions which it is a complete graph. Finally, we study the coloring of this graph.

Keywords: Character graph, Solvable group, Hamiltonian graph, Complete graph.
Mathematics Subject Classification [2010]: 20E45, 20C15

1 Introduction

Let G be a finite group, and let $\mathrm{cd}(G)$ be the set of all character degrees of G, that is, $\mathrm{cd}(G) = \{\chi(1) | \ \chi \in \mathrm{Irr}(G)\}$, where $\mathrm{Irr}(G)$ is the set of all complex irreducible characters of G. The set of prime divisors of character degrees of G is denoted by $\rho(G)$. It is well known that the character degree set $\mathrm{cd}(G)$ may be used to provide information on the structure of the group G. For example, Ito-Michler's Theorem [8] states that if a prime p divides no character degree of a finite group G, then G has a normal abelian Sylow p-subgroup. Another result due to J. Thompson [10] says that if a prime p divides every non-linear character degree of a group G, then G has a normal p-complement.

A useful way to study the character degree set of a finite group G is to associate a graph to $\mathrm{cd}(G)$. One of these graphs is the character graph $\Delta(G)$ of G. Its vertex set is $\rho(G)$ and two vertices p and q are joined by an edge if the product pq divides some character degree of G. We refer the readers to a survey by Lewis [5] for results concerning this graph and related topics. When G is a solvable group, some interesting results on the character graph of G have been obtained. For example, Manz in [6] has proved that in this case, $\Delta(G)$ has at most two connected components. Manz, Willems and Wolf in [7] have proved that diameter of $\Delta(G)$ is at most 3. If $\Delta(G)$ is regular with n vertices. Morresi Zuccari in [9] proved that $\Delta(G)$ is either complete or $(n-2)$-regular graph. Moreover, if $\Delta(G)$ is $(n-2)$-regular and G has no normal non-abelian Sylow subgroups, he shown that G is a direct product of groups having disconnected character graph.

Throughout this work all groups are assumed to be finite and all graphs are simple and finite. Here we bring some definitions and notations from [1].

*Speaker

Definition 1.1. Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For a vertex u, the adjacent vertices to u are called the neighbors of u. A complete graph of order n is a graph with n vertices in which any two vertices are adjacent. We denote this graph by K_n. A cycle on n vertices $v_1, \dots, v_n$, $n \geq 3$, is a graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are non-adjacent otherwise. A cycle with n vertices is said to be of length n and is denoted by C_n, i.e., $C_n : v_1, \dots, v_n, v_1$. Let X be a subset of $V(\Gamma)$, the subgraph of Γ whose vertex set is X and whose edge set consists of all edges of Γ which have both ends in X is called the induced subgraph of Γ on X. A cut vertex of a graph Γ is a vertex v such that the number of connected component of $\Gamma - v$ is more than the number of connected component of Γ. A maximal connected subgraph without a cut vertex is called a block. By their maximality, different blocks of Γ overlap in at most one vertex, which is then a cut vertex. Thus, every edge of Γ lies in a unique block and Γ is the union of its blocks. A clique of a graph is a set of mutually adjacent vertices. The clique number of Γ, denoted $\omega(\Gamma)$, is the maximum size of a clique of a graph Γ. If Γ has n vertices, any cycle of Γ of length n is called a Hamilton cycle. We say that Γ is Hamiltonian if it contains a Hamilton cycle. Minimum number of colors needed to color vertices of the graph Γ so that any two adjacent vertices of Γ have different colors, is called the chromatic number of Γ and denoted by $\chi(\Gamma)$. A matching of Γ is a set of pairwise non-adjacent edges of Γ, and that the number of edges in a maximum matching of Γ is said the matching number and denoted by $\alpha'(\Gamma)$. Finally we should mention that throughout this paper, the complement of the graph Γ is denoted by Γ^c. For more details, we refer the reader to basic textbooks on the subject, for instance [1].

2 Main results

When G is a solvable group of Fitting height 2, there is a good result on the structure of $\Delta(G)$ [4].

Lemma 2.1. *Let Γ be a graph with n vertices. There exists a solvable group G of Fitting height 2 with $\Delta(G) = \Gamma$ if and only if the vertices of degree less than $n-1$ can be partitioned into two subsets (X, Y), each of which induces a complete subgraph of Γ and one of which contains only vertices of degree $n-2$.*

In the above lemma, we called the partition (X, Y) as Lewis' partition. Let $\Delta(G)$ be the character graph of a finite solvable group G. Since an important family of graphs is the class of Hamiltonian graphs, in the following, we wish to study Hamiltonian character graphs. For this purpose we gave some results from [3].

Theorem 2.2. *Let G be a solvable group. Then $\Delta(G)$ is Hamiltonian if and only if $\Delta(G)$ is a block with at least 3 vertices.*

Corollary 2.3. *Let G be a solvable group of Fitting height 2 with Lewis' partition (X, Y) such that $|X|, |Y| \geqslant 2$. Then $\Delta(G)$ is Hamiltonian.*

Corollary 2.4. *Let $\Delta(G)$ be the character graph of a finite solvable group G with $n \geqslant 6$ vertices and $\omega(\Delta(G)) = 3$. Then $n \leqslant 9$ and $\Delta(G)$ is Hamiltonian.*

One of the most important classes of finite simple graphs is the class of complete graphs. So in the sequel, we wish to obtain conditions which guarantee the character graph $\Delta(G)$ of a finite solvable group G is complete.

Theorem 2.5. *Let N be a cyclic normal subgroup of G such that $C_G(N)$ is abelian. Then $\Delta(G)$ is a complete graph.*

Corollary 2.6. *Let all Sylow subgroups of G be abelian and G' be cyclic. Then $\Delta(G)$ is a complete graph.*

Finally, in this part we gave some results on coloring of character graphs stated in [2].

Theorem 2.7. *Let G be a finite solvable group. Then $\chi(\Delta(G)) + \alpha'(\Delta(G)^c) = |\rho(G)|$.*

Corollary 2.8. *Suppose G is a finite solvable group and $\Delta(G)^c$ is Hamiltonian. Then $\chi(\Delta(G)) = -[-|\rho(G)|/2]$.*

Corollary 2.9. *Let G be a finite solvable group of Fitting height at most 2. Then $\chi(\Delta(G)) = \omega(\Delta(G))$.*

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Some quotient graphs of the power graphs*

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Abstract
In this paper we define three quotient graphs of the power graphs and study their properties and some relation between them.

Keywords: Power graph, Quotient graph, Quotient power graph, Order graph, Power type graph
Mathematics Subject Classification [2010]: 05C25, 20B30

1 Introduction

Let G be a finite group. The *power graph* $P(G)$ is the graph with vertex set G and edge set E, where there is an edge $\{x, y\} \in E$ between two distinct vertices $x, y \in G$ if one is a positive power of the other (see [2]). Observed that $P(G)$ is 2-connected if and only if $P_0(G)$, the 1_G-cut subgraph of $P(G)$, is connected. Many of results are collected in a survey [1].

In this paper we define quotient power graph, order graph and power type graph of a finite group and study some properties of them, particulary the 2-connectivity of them. Throughout this paper, we use the standard notations of [4]. Also we denote by $c(\Gamma)$, the number of connected components of the graph Γ.

Definition 1.1. Let $\Gamma = (V, E)$ be a graph and $\sim$ is an equivalence relation on the set V. The *quotient graph* $\Gamma/\sim = ([V], [E])$, of Γ with respect to $\sim$ is a graph with vertex set $[V] = V/\sim$ and there is an edge $\{[x], [y]\} \in [E]$ between $[x], [y] \in [V]$ if $[x] \neq [y]$ and there exist $\overline{x}, \overline{y} \in V$ such that $\overline{x} \sim x$, $\overline{y} \sim y$ and $\{\overline{x}, \overline{y}\} \in E$.

Definition 1.2. Define the equivalence relation relation $\sim$ on G as follows: For $x, y \in G$, $x \sim y$ if and only if $\langle x\rangle = \langle y\rangle$. Then $[x] = \{x^m : 1 \leq m \leq o(x),\ (m, o(x)) = 1\}$. The quotient graph $P(G)/\sim = ([G] = G/\sim, [E])$ will be denoted by $\widetilde{P}(G)$ and called the *quotient power graph* of G. We show that $[x] \neq [y]$, $\{[x], [y]\} \in [E]$ if and only if $\{x, y\} \in E$. $\widetilde{P}(G)$ is always connected and it is 2-connected if and only if the 1_G-cut subgraph $\widetilde{P}_0(G)$, of $\widetilde{P}(G)$, is connected.

Definition 1.3. The *order graph* of G is the graph $\mathcal{O}(G)$ with vertex set $O(G) = \{m \in \mathbb{N} : \exists g \in G \text{ with } o(g) = m\}$ and edge set $E_{\mathcal{O}(G)}$, where for each $m, n \in O(G)$, $\{m, n\} \in E_{\mathcal{O}(G)}$ if $m \neq n$ and $m \mid n$ or $n \mid m$. The *proper order graph* $\mathcal{O}_0(G)$ is defined as the 1-cut graph of $\mathcal{O}(G)$. Its vertex set is then $O_0(G) = O(G) \setminus \{1\}$. We set $c(\mathcal{O}_0(G)) = c_0(\mathcal{O}(G))$. $\mathcal{O}(G)$ is always connected and it is 2-connected if and only if $\mathcal{O}_0(G)$ is connected.

*Will be presented in English
†Speaker

Given a permutation $\psi \in S_n$ which decomposes as a product of r pairwise disjoint cycles of lengths $x_1, \ldots, x_r$, we associate with ψ the r-partition $T_\psi = [x_1, \ldots, x_r] \in \mathcal{T}(n)$ which we call the *type* of ψ. Note that the map $t : S_n \to \mathcal{T}(n)$, defined by $t(\psi) = T_\psi$ is surjective, that is, each partition of n may be viewed as the type of some permutation in S_n. If $X \subseteq S_n$, we call $\mathcal{T}(X) = t(X)$ the set of types *admissible* for X.

Definition 1.4. Let $G \leq S_n$. We define the *power type graph* of G, as the graph $P(\mathcal{T}(G))$ with vertex set the set $\mathcal{T}(G)$ of types admissible for G and edge set $E_{\mathcal{T}(G)}$, where for two distinct types $T, T' \in \mathcal{T}(G)$, $\{T, T'\} \in E_{\mathcal{T}(G)}$ if one is the positive power of the other. We define also the *proper power type graph* $P_0(\mathcal{T}(G))$ of G, as the $[1^n]$-cut subgraph of $P(\mathcal{T}(G))$. $P(\mathcal{T}(G))$ is always connected and it is 2-connected if and only if $P_0(\mathcal{T}(G))$ is connected. For short, we put $c_0(\mathcal{T}(G)) = c(P_0(\mathcal{T}(G)))$.

2 Main results

Theorem 2.1. *[3] Let G be a finite group. Then $\widetilde{P}(G)$ is isomorphic to a tree if and only if G is one of the following groups:*
Case 1) G is a p-group of exponent p.
Case 2) G is a nilpotent group of order $p^m q$ as follows:

i) $|G| = p^m q$, where $3 \leq p < q, m \geq 3, |\mathcal{F}(G)| = p^{m-1}$ and $|G : G'| = p$.

ii) $|G| = p^m q$, where $3 \leq q < p, m \geq 1$ and $|\mathcal{F}(G)| = |G'| = p^m$.

iii) $|G| = 2^m p$, where $p \geq 3, m \geq 2$ and $|\mathcal{F}(G)| = |G'| = 2^m$.

iv) $|G| = 2p^m$, where $p \geq 3, m \geq 1, |\mathcal{F}(G)| = |G'| = p^m$ and $\mathcal{F}(G)$ is elementary abelian.

Case 3) $G \cong A_5$.

Theorem 2.2. *[3] Let G be a finite group. Then $\widetilde{P}_0(G)$ is a path if and only if G is isomorphic to one of the groups $\mathbb{Z}_p, \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq}$, where p, q are prime numbers.*

Theorem 2.3. *[3] Let G be a finite group. Then $\widetilde{P}_0(G)$ is a bipartite graph if and only if $\widetilde{P}_0(G)$ is connected and the order of each non-trivial element of G is a prime or a product of two primes,(not necessary distinct).*

Corollary 2.4. *[3] Let G be a finite group. Then the quotient power graph $\widetilde{P}(G)$ is planar if and only if $\pi_e(G) \subseteq \{1, p, p^2, p^3, pq, p^2q\}$, where p, q are distinct prime numbers.*

Theorem 2.5. *[3] Suppose $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}...{p_r}^{\alpha_r}$, where $p_1 < p_2 < ... < p_r$ are prime numbers. Then*
$$\omega(\widetilde{P}(\mathbb{Z}_n)) = \chi(\widetilde{P}(\mathbb{Z}_n)) = 1 + \Sigma_{i=1}^{r}\alpha_i.$$

Proposition 2.6. *For each finite group G, the graph $\mathcal{O}_0(G)$ is a quotient of the graph $\widetilde{P}_0(G)$. Also for every permutation group $G \leq S_n$, $\mathcal{O}_0(G)$ is a quotient of the graph $P(\mathcal{T}(G))$ and $P(\mathcal{T}(G))$ is a quotient of the graph $\widetilde{P}_0(G)$.*

Corollary 2.7. *For every permutation group $G \leq S_n$, we have $c_0(\mathcal{O}(G)) \leq c_0(P(\mathcal{T}(G))) \leq c_0(\widetilde{P}(G))$.*

Theorem 2.8. *The values of $c_0(S_n) = \widetilde{c}_0(S_n), c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$ are given by the following tables:*

Table 1: $c_0(S_n), c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$ for $2 \le n \le 7$.

n	2	3	4	5	6	7
$c_0(S_n)$	1	4	13	31	83	128
$c_0(\mathcal{T}(S_n))$	1	2	3	3	4	3
$c_0(\mathcal{O}(S_n))$	1	2	2	2	2	2

Table 2: $c_0(S_n)$, $c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$ for $n \ge 8$

n	$n \in P$	$n \in P+1$	$n \notin P \cup (P+1)$
$c_0(S_n)$	$(n-2)!+1$	$n(n-3)!+1$	1
$c_0(\mathcal{T}(S_n)) = c_0(\mathcal{O}(S_n))$	2	2	1

Corollary 2.9. *The following facts are equivalent:*

i) $P(S_n)$ *is* 2*-connected;*

ii) $\widetilde{P}_0(S_n)$ *is connected;*

iii) $P(\mathcal{T}(S_n))$ *is* 2*-connected;*

iv) $\mathcal{O}(S_n)$ *is* 2*-connected;*

v) $n \in \mathbb{N} \setminus [P \cup (P+1)]$.

Corollary 2.10. *Apart the trivial case $n = 2$, the minimum $n \in \mathbb{N}$ such that $P(S_n)$ is 2-connected is $n = 9$. There exists infinite $n \in \mathbb{N}$ such that $P(S_n)$ is 2-connected.*

Let P be the set of prime numbers. For $b, c \in \mathbb{N}$, we set $bP + c = \{x \in \mathbb{N} : x = bp + c,$ for some $p \in P\}$ and define $A = P \cup (P+1) \cup (P+2) \cup (2P) \cup (2P+1)$.

Theorem 2.11. *The values of $c_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ are given by the following tables:*

Corollary 2.12. *i) $\mathcal{O}_0(A_n)$ is connected if and only if $n = 3$ or $n, n-1, n-2$ are not prime. The maximum number of connected components of $\mathcal{O}_0(A_n)$ is 3.*

ii) $P_0(A_n)$ is connected if and only if $P_0(\mathcal{T}(A_n))$ is connected, that is, if and only if $n = 3$ or $n \notin A$.

iii) The minimum $n \in \mathbb{N}$ such that $P(A_n)$ is 2-connected and A_n is non-abelian, is $n = 16$. There exists infinite n such that $P(A_n)$ is 2-connected.

Table 3: $c_0(A_n)$ and $c_0(\mathcal{T}(A_n))$, for $3 \leq n \leq 10$.

n	3	4	5	6	7	8	9	10
$c_0(A_n)$	1	7	31	121	421	962	5442	29345
$c_0(\mathcal{T}(A_n))$	1	2	3	4	4	3	4	3
$c_0(\mathcal{O}(A_n))$	1	2	3	3	3	2	2	1

Table 4: $c_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ for $n \geq 11$.

$c_0(A_n)$	$c_0(\mathcal{T}(A_n))$	$c_0(\mathcal{O}(A_n))$	$n \geq 11$
$\frac{n(n-1)(n-4)!}{2} + \frac{4n(n-2)(n-4)!}{n-1} + 1$	3	2	$n-2, \frac{n-1}{2} \in P,\ n \notin P$
$(n-2)! + \frac{4n(n-2)(n-4)!}{n-1} + 1$	3	2	$n, \frac{n-1}{2} \in P,\ n-2 \notin P$
$(n-2)! + \frac{n(n-1)(n-4)!}{2} + 1$	3	3	$n, n-2 \in P,\ \frac{n-1}{2} \notin P$
$\frac{n(n-1)(n-4)!}{2} + 1$	2	2	$n-2 \in P,\ n, \frac{n-1}{2} \notin P$
$\frac{4n(n-2)(n-4)!}{n-1} + 1$	2	1	$\frac{n-1}{2} \in P,\ n, n-2 \notin P$
$(n-2)! + 1$	2	2	$n \in P,\ n-2, \frac{n-1}{2} \notin P$
$n(n-3)! + 1$	2	2	$n-1 \in P,\ \frac{n}{2} \notin P$
$\frac{4(n-1)(n-3)!}{n} + n(n-3)! + 1$	3	2	$n-1, \frac{n}{2} \in P$
$\frac{4(n-1)(n-3)!}{n} + 1$	2	1	$\frac{n}{2} \in P,\ n-1 \notin P$
1	1	1	$n \notin A$

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Some results on complementable semihypergroups

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Abstract

In this paper we first introduce the notion of complementable semihypergroup, proving that the classes of simplifiable semigroups, groups, simplifiable semihypergroups and complete hypergroups are examples of complementable semihypergroups. Then we define when two semihypergroups are disjoint and find examples of such semihypergroups.

Keywords: (semi)hypergroup, complementable semihypergroup, disjoint semihypergroups.
Mathematics Subject Classification [2010]: 20N20

1 Introduction

In this paper we introduce a new type of semihypergroups, called complementable semihypergroups, as semihypergroups having the complement (so the hypergroupoid endowed with the complement hyperoperation) a semihypergroup too. Our first aim is to find several classes of complementable semihypergroups and we prove that the simplifiable semigroups, groups, simplifiable semihypergroups and complete hypergroups have this property.

We recall here some basic notions of hypergroup theory and we fix the notations used in this note. We referee the readers to the following fundamental books Corsini [1], Corsini and Leoreanu [2], Vougiouklis [3].

Let H be a non-empty set and $\mathcal{P}^*(H)$ denote the set of all non-empty subsets of H. Let $\circ$ be a *hyperoperation* (or *join operation*) on H, that is, a function from the chartezian product $H \times H$ into $\mathcal{P}^*(H)$. The image of the pair $(a, b) \in H \times H$ under the hyperoperation $\circ$ in $\mathcal{P}^*(H)$ is denoted by $a \circ b$. The join operation can be extended in a natural way to subsets of H as follows: for non-empty subsets A, B of H, define $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$. The notation $a \circ A$ is used for $\{a\} \circ A$ and $A \circ a$ for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its element a. The hyperstructure $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

*Speaker

2 Complementable semihypergroups

In this section, we firstly introduce the notion of complementable semihypergroup and based on it, we define the class of disjoint (semi)hypergroups of a (semi)hypergroup. We show that the classes of simplifiable semigoups and simplifiable semihypergroups are complementable.

Definition 2.1. A semihypergroup $(H, \circ)$ is called *simplifiable on the left* if the following implication is valid:

$$\forall (x, a, b) \in H^3, x \circ a \cap x \circ b \neq \emptyset \Longrightarrow a = b.$$

Similarly, we can define the simplifiability on the right. The semihypergroup $(H, \circ)$ is called *simplifiable* if it is simplifiable on the left and on the right.

Theorem 2.2. *Let $(H, \circ)$ be a semihypergroup such that, for all $t \in H, t \circ H = H$ and there exists $t_0 \in H$ such that $H \circ t_0 = H$. If H is simplifiable on the left (right), then H is a group.*

Having in mind the concept of the complement of a set, we define the complement hyperoperation and then the complement hypergroupoid of a semihypergroup.

Definition 2.3. Let $(H, \circ)$ be a semihypergroup such that $x \circ y \neq H$, for all $x, y \in H$. We call the *complement* of $(H, \circ)$ the hypergroupoid $(H, \circ^c)$ endowed with the *complement hyperoperation*: $x \circ^c y = H - \{x \circ y\}$. We say that the semihypergroup $(H, \circ)$ is *complementable* if its complement $(H, \circ^c)$ is a semihypergroup too, and in this case $(H, \circ^c)$ is called the *complement semihypergroup* of $(H, \circ)$.

Example 2.4. Suppose that $H = \{e, a, b\}$. Consider the semihypergroup $(H, \circ)$, where the hyperoperation $\circ$ is defined on H by the following table:

$\circ$	e	a	b
e	a, b	b	b
a	b	b	b
b	b	b	b

Notice that H is a complementable semihypergroup, where its complement, defined as follows

$\circ^c$	e	a	b
e	e	e, a	e, a
a	e, a	e, a	e, a
b	e, a	e, a	e, a

is a semihypergroup, too.

Example 2.5. Suppose that $H = \{e, a, b, c\}$. Consider the semihypergroup $(H, \circ)$ endowed with the hyperoperation $\cdot$ defined as follows:

$\circ$	e	a	b	c
e	c	a, b	a, b	c
a	a, b	c	c	a, b
b	a, b	c	c	a, b
c	c	a, b	a, b	c

In this case H is not complementable, since the complement hypergroupoid is not a semihypergroup.

Now we can establish a connection between simplifiable semigroups/ semihypergroups and complementable ones.

Proposition 2.6. *Every simplifiable semigroup of order at least* 2 *is complementable.*

Corollary 2.7. *Every non-trivial group is complementable.*

The following example shows that the above assertion is not true for the class of hypergroups.

Example 2.8. Let $H = \{e, a, b, c\}$ and $(H, \circ)$ be the following hypergroup.

$\circ$	e	a	b	c
e	e, a	e, a	e, b	e, c
a	e, a	e, a	a, b	a, c
b	e, b	a, b	b, c	b, c
c	e, c	a, c	b, c	b, c

Now we can see that the complement $(H, \circ^c)$ is not a hypergroup, so H is not complementable.

Proposition 2.9. *Every simplifiable semihypergroup is complementable.*

We notice that in the above proposition we need the simplifiablity property on the left and on the right. In the following example we show that the left simplifiable semihypergroup (which is not also right simplifiable) in example 3.4 is complementable, too.

Definition 2.10. Let $(H, \circ)$ and $(H, *)$ be two semihypergroups with the same support set. We say that $(H, \circ)$ and $(H, *)$ are *disjoint*, if $x \circ y \cap x * y = \emptyset$, for every $(x, y) \in H^2$ and we write $(H, \circ) \cap (H, *) = \emptyset$.

It is obvious that, if $(H, \circ)$ is a complementable semihypergroup, then $(H, \circ)$ and its complement $(H, \circ^c)$ are disjoint.

Example 2.11. On the set $H = \{e, a, b\}$ consider the semihypergroups $(H, \circ)$ and $(H, \circ')$ defined by the following tables. It is easy to see that $(H, \circ)$ and $(H, \circ')$ are disjoint semihypergroup, while $(H, \circ')$ is not the complement of $(H, \circ)$.

$\circ$	e	a	b
e	a, b	e	e
a	e	a	b
b	e	a, b	a, b

$\circ'$	e	a	b
e	e	a, b	a, b
a	a, b	e	e
b	a, b	e	e

It is easy to see that $(H, \circ)$ and $(H, \circ')$ are disjoint semihypergroups, while $(H, \circ')$ is not the complement of $(H, \circ)$.

Definition 2.12. Let $(H, \circ)$ be a semihypergroup such that $x \circ y \neq H$, for all $x, y \in H$. Denote $D(H, \circ) = \{(H, *) \in \mathcal{SH}(H) | (H, *) \cap (H, \circ) = \emptyset\}$, where $\mathcal{SH}(H)$ is the class of semihypergroups with H as the support set.

Proposition 2.13. *Let $(H, \circ)$ be a semihypergroup such that the quotient $H^* = (H/\beta^*, \cdot)$ is a simplifiable semigroup of order at least 2. Then $(H, \circ_c)$ is a semihypergroup, where the hyperoperation $\circ_c$ is defined by*

$$x \circ_c y = \{t \mid \bar{t} \in H^* - \{\bar{x}\bar{y}\}\}.$$

Moreover, $(H, \circ)$ and $(H, \circ_c)$ are disjoint.

The following consequence follows immediately.

Corollary 2.14. *If $(H, \circ)$ is a hypergroup such that $|H/\beta^*| \geq 2$, then $D(H, \circ) \neq \emptyset$.*

The following example shows that the converse of the above corollary is not always true.

Example 2.15. On the support set $H = \{e, a, b\}$ consider $(H, \circ)$ as the following semihypergroup

$\circ$	e	a	b
e	a, b	b	b
a	b	b	b
b	b	b	b

We have that $(H, \circ)$ is not a hypergroup and $D(H, \circ) \neq \emptyset$. Indeed, the semihypergroup defined by the following table

$\circ^c$	e	a	b
e	e	e, a	e, a
a	e, a	e, a	e, a
b	e, a	e, a	e, a

is disjoint with respect to $(H, \circ)$.

Proposition 2.16. *Let $(H, \circ)$ be a semihypergroup such that $x \circ y \neq H$, for all $x, y \in H$. Then*

(1) $D(H, \circ^c) \cap D(H, \circ) = \emptyset$.

(2) If $(H, *) \in D(H, \circ)$, then $I(H, *) \cap I(H, \circ) = \emptyset$, where $I(H, *)$ is the set of all identities of $(H, *)$.

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Some types of ideals in bounded BCK-algebras.

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Abstract

The aim of this work is to investigate the relationship between ideals in bounded BCK-algebras so we introduce the concepts of involutory and EI-ideals in bounded BCK-algebras and characterise their properties. Also we introduce the concepts of EQI-algebras and EQI-ideals in bounded BCK-algebras and show that EQI-algebras include some important BCK structures such as involutory BCK-algebras, commutative and PC-lattices. The relationships between these ideals and quotient algebras that are constructed via these ideals are described. We clarify that EI, involutory and commutative ideals coincide in PC-lattices, whereas they are not the same in bounded BCK-algebras in general. It is proved that EQI-ideals contain some current ideals such as involutory, commutative, positive implicative and implicative ideals

Keywords: involutory ideal, EI-ideal, EQI-ideal, EQI-algebras
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1 Introduction

This paper by extended view on ideal theory of bounded BCK-algebras introduces concepts of involutory, EI and EQI-ideals in bounded BCK-algebras. By introduce the concept of EQI-algebras, we have a new structure of bounded BCK-algebras that contains some important BCK structures such as PC-lattices, bounded commutative BCK-algebras and involutory BCK-algebras. We describe the relationships between these ideals that mentioned in the abstract.

Definition 1.1. Let X be a set with a binary operation $*$ and a constant 0. Then $(X;*,0)$ is called a BCK-algebra if it satisfies the following axioms:
(BCK-1) $((x*y)*(x*z))*(z*y)=0$,
(BCK-2) $(x*(x*y))*y=0$,
(BCK-3) $x*x=0$,
(BCK-4) $x*y=0$ and $y*x=0$ imply $x=y$.
(BCK-5) $0*x=0$

A partial ordering $\leq$ on X can be defined by $x \leq y$ if only if $x*y=0$.

*Speaker

Definition 1.2. [2, 1] Let X be a BCK-algebra. Then
(i) X is said to be with condition (S), if for any $x, y \in X$, the set $A(x, y) = \{t \in X : t*x \leq y\}$ has the greatest element which is denoted by $x \circ y$.
(ii) $(X, *, \leq)$ is called a BCK-lattice, if $(X, \leq)$ is a lattice, that $\leq$ is a partial BCK-order on X.
(iii) Lattice $(X, \leq)$ is said to be *distributive* if $x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)$, for all $x, y, z \in L$.

Definition 1.3. [3, 5] Let I be a nonempty subset of BCK-algebra X. Then
(i) I is called a *ideal* of X if $0 \in I$, $y \in I$ and $x * y \in I$ imply $x \in I$.
(ii) I is called an *implicative ideal* of X if $0 \in I$, $(x * (y * x)) * z \in I$ and $z \in I$ imply $x \in I$.
(iii) I is called a *positive implicative ideal* of X if $0 \in I$, $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$.
(iv) I is called a *commutative ideal* of X if $0 \in I$, $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$,
for all $x, y, z \in X$.

Definition 1.4. [2, 1] Let X be a BCK-algebra. Then
(i) X is said to be with condition (S), if for any $x, y \in X$, the set $A(x, y) = \{t \in X : t*x \leq y\}$ has the greatest element which is denoted by $x \circ y$.
(ii) $(X, *, \leq)$ is called a BCK-lattice, if $(X, \leq)$ is a lattice, that $\leq$ is a partial BCK-order on X.
(iii) Lattice $(X, \leq)$ is said to be *distributive* if $x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)$, for all $x, y, z \in L$.

Definition 1.5. [4] Let X be a BCK-lattice . Then X is called a PC-lattice if it satisfies in

$$(z * x) * (y * x) = z * (x \vee y)$$

.

2 EI and involutory ideals in bounded BCK-algebras

In this article we suppose that X is a bounded BCK-algebra, unless otherwise is stated.

Definition 2.1. Let I be a nonempty subset of X. Then I is called an *EI-ideal* if $0 \in I$, $NN(x * y) \in I$ and $y \in I$ imply $x \in I$, for all $x, y \in X$.

Lemma 2.2. Let I be an EI-ideal of X. Then
(i) If $x \leq y$ and $y \in I$, then $x \in I$, for $x, y \in X$,
(ii) I is an ideal, but the converse is not true.

Theorem 2.3. Let I be a nonempty subset of X. Then I is an implicative ideal if and only if $x * (x * Nx) \in I$ for all $x \in X$.

Theorem 2.4. Let I be a nonempty subset of X. Then I is an implicative ideal if and only if I is a positive implicative and an EI-ideal.

Definition 2.5. Let I be a nonempty subset of X. Then I is called an *involutory ideal* of X, if $0 \in I$ and $x * NNx \in I$ for all $x \in X$.

Theorem 2.6. Let I be a commutative (implicative) ideal of X. Then I is an involutory ideal of X, but the converse does not hold.

Theorem 2.7. Every involutory ideal of X is an EI-ideal, but the converse does not hold.

Corollary 2.8. Every implicative and commutative ideal of X is an EI-ideal, but the converse does not hold.

Theorem 2.9. Let I be an ideal of X. Then
(i) If X is a bounded BCK-algebra, then I is an EI-ideal if and only if $NNx \in I$ imply $x \in I$, for all $x \in X$.
(ii) If X is a PC-lattice, then the concepts of EI-ideals and involutory ideals coincide.

Theorem 2.10. Let X be a PC-lattice. Then the following are equivalent.
(i) I is a commutative ideal.
(ii) I is an involutory ideal.
(iii) $t * x \in I$ and $t * y \in I$ imply $t * (x * (x * y)) \in I$.
for all $x, y, t \in X$

Corollary 2.11. Let X be a PC-lattice. Then the concepts of EI, commutative and involutory ideals coincide.

3 EQI algebras and EQI- ideals in bounded BCK-algebras

Definition 3.1. Let X be a bounded BCK-algebra. Then X is called an EQI-algebra if

$$N(x * NNx) = 1$$

Theorem 3.2. Every involutory BCK-algebra and PC-lattice is an EQI-algebra . But the converse does not hold in general.

Theorem 3.3. Let X be an EQI-algebra. Then the concepts of EI-ideals and involutory ideals coincide.

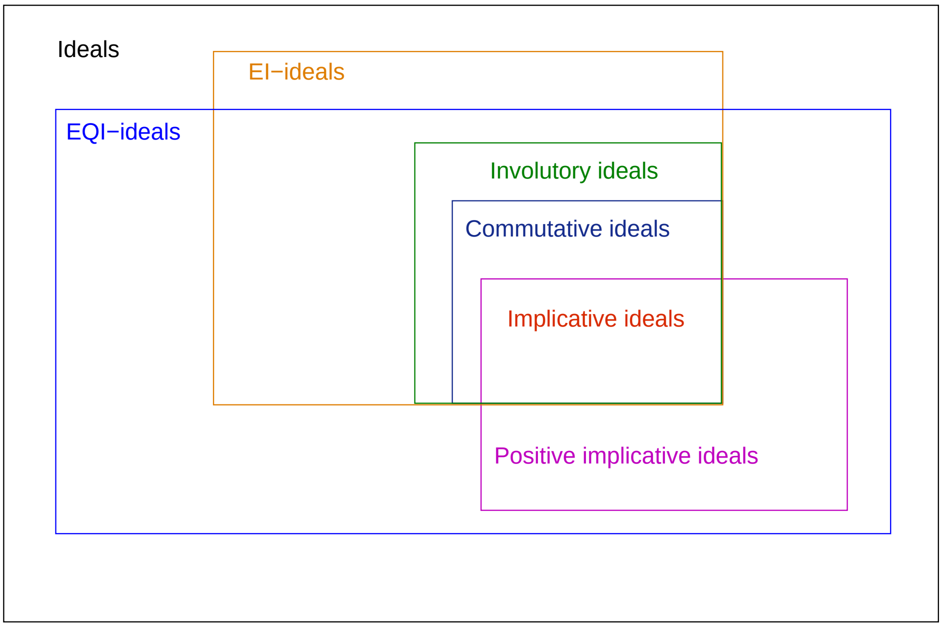
Corollary 3.4. Let I and A be ideals of X and $I \subseteq A$. If I is an EI-ideal, so is E.

Theorem 3.5. Let X be a EQI-algebra. Then the following are equivalent:
(i) $\{0\}$ is an EI-ideal.
(ii) Every ideal of X is an EI-ideal.
(iii) X is an involutory BCK-algebra.

Theorem 3.6. Let X be a bounded BCK-algebras. Then the following are equivalent:
(i) $\{0\}$ is an involutory ideal.
(ii) Every ideal of X is an involutory ideal.
(iii) X is an involutory BCK-algebra.

Theorem 3.7. Let I be an ideal of EQI-algebra X. Then I is an EI-ideal if and only if X/I is an involutory BCK-algebra.

Corollary 3.8. Let X be an EQI-algebra. Then the quotient algebras of X induced by the EI-ideals and induced by involutory ideals coincide.



4 Conclusion

In this research we introduced and studied involutory, EI and EQI-ideals in bounded BCK-algebras. We then established the relationships between these ideals and quotient algebras that are constructed via these ideals. We also introduced EQI-algebras and described the relation between it and other ordered structures. The following figure shows the relations between ideals in bounded BCK-algebras.

Acknowledgment

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The subgroup generated by small conjugacy classes

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Abstract

Let G be a finite group and M(G) be the subgroup generated by all noncentral elements of G that lie in the conjugacy classes of the smallest size. We show some results related to M(G) and direct product of groups.

Keywords: Conjugacy class size, Small subgroup, Direct product

Mathematics Subject Classification [2010]: 20E45, 20K25

1 Introduction

In 2006, A. Mann [1] defined M(G) and showed that for a finite nilpotent group G, M(G) has nilpotency class at most 3. It is generalized by M. Isaacs [2] and M. K. Yadav [3] for some family of groups, particularly solvable groups.
Let H and K be finite groups. The purpose of this paper is to prove that $M(H \times K)$ can be calculated from $M(H)$, $M(K)$ and the centers of H and K.

All groups in this paper are finite.

Definition 1.1. Let G be a finite group and $1 = n_1 < n_2 < \cdots < n_k$ be the sizes of its conjugacy classes. The classes of size n_2 are called minimal or small classes, and their elements are called minimal or small elements.

Let M(G) be the subgroup generated by all small elements of G. In other words, M(G) is the subgroup generated by all noncentral elements that lie in conjugacy classes of the smallest size.

We denote all small elements of G by $Sm(G)$, and n_2 by $n_2(G)$. For an abelian group A, we define $n_2(A) = \infty$ and $Sm(A) = \varnothing$.

Example 1.2.
- For an abelian group G, $M(G)$ is trivial.
- For a nonabelian simple group G, $M(G) = G$.
- $M(A_n) = A_n$ and $M(S_n) = S_n$, for $n \geq 5$.
- $M(S_3) \cong C_3$ and $M(S_4) = M(A_4) \cong C_2 \times C_2$.
- $M(D_{2n}) \cong C_n$ and $[D_{2n} : M(D_{2n})] = 2$, for $n \geq 5$.
- $M(Q_{4n}) \cong C_{2n}$ and $[Q_{4n} : M(Q_{4n})] = 2$, for $n \geq 5$.
- $M(D_8) = D_8$ and $M(Q_8) = Q_8$.

*Speaker

Proposition 1.3. *M(G) is a characteristic subgroup, hence normal, in G.*

Proposition 1.4. *For a nonabelian finite group G, $Z(G) \leq M(G)$, where $Z(G)$ is the center of G. Moreover, M(G) is the subgroup generated by all elements of G that lie in conjugacy classes of the two smallest sizes.*

2 Main results

In this section, H and K are finite groups. For the proofs of the following propositions we need a lemma.

Lemma 2.1. *For any H and K, we have $n_2(H \times K) = \min\{n_2(H), n_2(K)\}$ and*

$$Sm(H \times K) \subseteq \big(Sm(H) \times Z(K)\big) \cup \big(Z(H) \times Sm(K)\big). \tag{1}$$

Proposition 2.2. *If H is a nonabelian and K is an abelian group, then $M(H \times K) = M(H) \times K$.*

Proof. By Lemma 2.1, $n_2(H \times K) = n_2(H)$ and $Sm(H \times K) = Sm(H) \times K$. Since for every group G, $M(G) = \big\langle Sm(G)\big\rangle$, so that

$$M(H \times K) = \big\langle Sm(H \times K)\big\rangle = \big\langle Sm(H) \times K\big\rangle = \big\langle Sm(H)\big\rangle \times K = M(H) \times K.$$

□

Proposition 2.3. *If H and K are nonabelian groups and $n_2(H) < n_2(K)$, then $M(H \times K) = M(H) \times Z(K)$.*

Proof. $n_2(H \times K) = n_2(H)$ and $Sm(H \times K) = Sm(H) \times Z(K)$. The rest of proof is similar to last proposition. □

Proposition 2.4. *If H and K are nonabelian groups and $n_2(H) = n_2(K)$, then $M(H \times K) = M(H) \times M(K)$.*

Proof. In formula (1) equality holds and we have

$$M(H \times K) = \big\langle M(H) \times Z(K), Z(H) \times M(K)\big\rangle.$$

So that $M(H \times K) \subseteq M(H) \times M(K)$. To prove the inverse inclusion, we use the fact that $(a, b) = (a, 1).(1, b)$, for any (a, b) in $H \times K$. □

Now combining the preceding propositions, we have:

Theorem 2.5. *For arbitrary finite groups H and K, we have*

$$M(H \times K) = \begin{cases} 1 & \text{if} \quad H,K \text{ abelian} \\ M(H) \times K & \text{if} \quad H \text{ abelian, } K \text{ nonabelian} \\ M(H) \times Z(K) & \text{if} \quad H,K \text{ nonabelian}, n_2(H) < n_2(K) \\ M(H) \times M(K) & \text{if} \quad H,K \text{ nonabelian}, n_2(H) = n_2(K) \end{cases}$$

Corollary 2.6. *For a nonabelian finite group G, we have $M(G \times G) = M(G) \times M(G)$.*

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Torsion theory cogenerated by a class of modules

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Abstract

We introduce and study a generalization of a class of modules related to radical. The torsion theory cogenerated by this class of modules will be investigated in this paper. We will show that the module $N \in \sigma[M]$ is $M-$radical if and only if For any $M-$ injective module I and any homomorphism $f : N \longrightarrow I$ in $\sigma[M]$, we have $Im(f) \subseteq Rad(I)$. Also we conclude that $N = Re_{Rd[M]}(N)$ if and only if for every nonzero homomorphism $f : N \longrightarrow K$ in $\sigma[M]$, $Im(f) \nsubseteq Rad(K)$, where $Rd[M]$ is the class of all $M-$radicla modules. The relationship between this modules and some other kind of modules will be studied.

Keywords: Torsion theory, Radical modules, Small modules
Mathematics Subject Classification [2010]: 16D60, 16D80

1 Introduction

Throughout this article, all rings are associative and have an identity, and all modules are unitary right modules.

$N \subseteq^{\oplus} M$ means that N is a direct summand of M. A submodule L of M is called *small* in M (denoted by $L \ll M$) if, for every proper submodule K of M, $L+K \neq M$. The sum of all small submodules of M is called the *radical* of M and is denoted by $Rad(M)$.

A submodule N of M is called *essential* in M (denoted by $N \subseteq^{ess} M$) if $N \cap K \neq 0$ for every nonzero submodule K of M.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M. The submodule A of M is called *coclosed* if A has no proper *cosmall* submodule. Also B is called a *coclosure* of A in M if B is a cosmall submodule of A and B is coclosed in M.

For a module M, an injective module E is called an *injective envelope (or injective hull)* of M if, $M \subseteq^{ess} E$. It is well known that for every ring R, every $R-$module has injective envelope. We refer for more information and basic notations to [1].

Let $\mathbb{A}$ be a nonempty class of modules in $\sigma[M]$. Recall the following classes

$$\mathbb{A}^{\circ} = \{B \in \sigma[M] | Hom(B, A) = 0; \forall A \in \mathbb{A}\} = \{B \in \sigma[M] | Re(B, \mathbb{A} = B\}$$

*Speaker

$$\mathbb{A}^{\bullet} = \{B \in \sigma[M] | Hom(A, B) = 0; \forall A \in \mathbb{A}\} = \{B \in \sigma[M] | Tr(\mathbb{A}, B) = 0\}$$
$$\mathbb{A}^{\triangleright} = \{X \in \sigma[M] | Hom(U, A) = 0; \forall U \leq X, A \in \mathbb{A}\} \subseteq \mathbb{A}^{\circ}$$
$$\mathbb{A}^{\blacktriangleright} = \{X \in \sigma[M] | Hom(A, \frac{X}{Y}) = 0; \forall Y \leq X, A \in \mathbb{A}\} \subseteq \mathbb{A}^{\bullet}$$

The class $\mathbb{A}^{\triangleright}$ defines a hereditary pretorsion class of modules and also $\mathbb{A}^{\triangleright} = \{E\}^{\circ}$ for some injective module $E \in \sigma[M]$ (for more details see Proposition 9.5 [6]).

The class $\mathbb{A}^{\blacktriangleright}$ defines a cohereditary class of modules.
It is clear that $\mathbb{A}^{\blacktriangleright}$ is closed under extensions and submodules but is not closed under products.

An ordered pair $(\mathbb{A}, \mathbb{B})$ of classes of modules from $\sigma[M]$ is called a *torsion theory* if $\mathbb{A} = \mathbb{B}^{\circ}$ and $\mathbb{B} = \mathbb{A}^{\bullet}$. In this case $\mathbb{A}$ is called the *torsion class* and it's elements are the torsion modules, while $\mathbb{B}$ is the *torsion free class* and it's elements are the torsion free modules.

2 Main results

In this section we attempt to investigate the torsion theory cogenerated by $M-$radical modules. First we give an proposition that characterize $M-$radical modules.

Proposition 2.1. *Let M be a module and $N \in \sigma[M]$. the following are equivalet*

1. *N is $M-$radical;*
2. *$N \subseteq Rad(\hat{N})$; where $\hat{N}$ is the $M-$injective hull of N;*
3. *For any $M-$ injective module I and any homomorphism $f : N \longrightarrow I$ in $\sigma[M]$, we have $Im(f) \subseteq Rad(I)$.*

Proposition 2.2. *Let R be a ring, M an R-module and $N \in \sigma[M]$. The following are equivalent*

1. *$N = Tr_{Rd[M]}(N)$;*
2. *$N = Tr_{\mathbb{S}}(N)$;*
3. *$N \subseteq Rad(\hat{N})$;*
4. *$xR \ll \hat{N}$ for every $x \in N$;*
5. *$xR \subseteq Rad(\hat{N})$ for every $x \in N$;*
6. *$N \in Gen(\mathbb{S})$;*
7. *$N \in Gen(Rd[M])$.*

Proposition 2.3.

1. $\mathbb{S}^{\bullet} = \{N \in \sigma[M] | Tr_{\mathbb{N}}(N) = 0\} = \{N \in \sigma[M] | Tr_{Rd[M]}(N) = 0\} = Rd[M]^{\bullet}$; *hence the class* $Rd[M]^{\bullet}$ *is cogenerated by simple* M*-injective modules in* $\sigma[M]$.

2. $\mathbb{S}^{\bullet\circ} = \{N \in \sigma[M] | Tr_{\mathbb{S}}(\frac{N}{K}) \neq 0; \forall K \subsetneqq N\} = \{N \in \sigma[M] | Tr_{Rd[M]}(\frac{N}{K}) \neq 0; \forall K \subsetneqq N\} = Rd[M]^{\bullet\circ}$;
hence $Rd[M]^{\bullet\circ} = \{N \in \sigma[M] | N \ has\ no\ simple\ M - injective\ factor\ module\}$.

3. *Let* $N \in \sigma[M]$, *then* $N \in Gen(\mathbb{S})$ *iff* $N = Tr_{\mathbb{S}}(N) = Tr_{Rd[M]}(N)$. *Thus* $N \in Gen(\mathbb{S})$ *iff* $N \in Gen(Rd[M])$. *Now if* M *is* σ*-cohereditary, then* $Gen(Rd[M]) = Rd[M]^{\bullet\circ}$.

Proposition 2.4. *Let* M *be a module and* $N \in \sigma[M]$. *The following conditions are equivalent*

1. $N = Re_{Rd[M]}(N)$;

2. *If* $f : N \longrightarrow K$ *is a nonzero homomorphism in* $\sigma[M]$ *and* L *is a submodule of* $Im(f)$, *then* $\frac{Im(f)}{L} \subseteq Rad(\frac{K}{L})$ *implies* $Im(f) = L$;

3. *For every nonzero homomorphism* $f : N \longrightarrow K$ *in* $\sigma[M]$, $Im(f) \nsubseteq Rad(K)$.

Proof. $1 \Longrightarrow 2$: Suppose that $\frac{Im(f)}{L} \subseteq Rad(\frac{K}{L})$. Consider the map $\pi o f : N \longrightarrow \frac{K}{L}$; where $\pi : K \longrightarrow \frac{K}{L}$ is the natural epimorphism. Then $Im(\pi o f) = \frac{Im(f)}{L}$, and so $\pi o f$ has to be zero. Hence $Im(f) = L$.

$2 \Longrightarrow 3$ is obvious.

$3 \Longrightarrow 1$: Assume $f : N \longrightarrow K$ to be nonzero, where $K \in Rd[M]$. Then the composition map $\iota o f$ is a nonzero homomorphism from N to $\hat{K}$, where $\iota : K \longrightarrow \hat{K}$ is the inclusion map. Now we have $Im(\iota o f) = Im(f) \subseteq K \subseteq Rad(\hat{K})$ a contradiction. Therefore there is no nonzero homomorphism from N to $M-$radical modules; that is $N = Re_{Rd[M]}(N)$. □

In above proposition when condition 2 holds, we say $Im(f)$ is *radical-coclosed* in M.

Proposition 2.5. *Let* M *be a module and* $N \in Rd[M]^{\circ}$. *The following hold*

1. *Every* $M-$*radical proper submodule* $K \subset N$ *is contained in* $Rad(N)$ *and so* $Tr_{Rd[M]}(N) = Rad(N)$.

2. *If* L *is a proper extension module of* N *in* $\sigma[M]$, *then* N *is radical-coclosed in* L.

3. *For any proper submodule* K *of* N, K *is radical-coclosed in* N *iff* $K \in Rd[M]^{\circ}$.

Example 2.6. 1. Let $M = \frac{\mathbb{Z}}{12\mathbb{Z}}$. Then $Rad(M) = \frac{6\mathbb{Z}}{12\mathbb{Z}}$ and so $\mathbb{Z} \notin Rd[M]^{\circ}$.

2. Suppose that M is a divisible $\mathbb{Z}$-module with no nontrivial small submodule. Then every factor module of M is contained in $Rd[M]^{\circ}$.

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Analysis

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A convergence theorem by extragradient method for variational inequalities in Banach spaces

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Abstract

In this paper, we propose a new extragradient method for finding a common element of the set of solutions of a variational inequality for an α-inverse-strongly monotone operator and fixed point of a generalized nonexpansive mapping in Banach spaces. we prove a weak convergence theorem by this method under suitable conditions.

Keywords: Sunny generalized nonexpansive retraction, Variational inequality, weak convergence.
Mathematics Subject Classification [2010]: 47H09, 47H10, 47J05, 47J25

1 Introduction

Let E be a real Banach space and E^* be the dual of E. Let C be a closed convex subset of E. In this paper, we concerned with the following Variational Inequality (VI), which consists in finding a point $u \in C$ such that

$$\langle f(u), y - u\rangle \geq 0, \ \forall \ y \in C, \tag{1}$$

where $f : C \to E^*$ is a given mapping and $\langle ., .\rangle$ denotes the generalized duality pairing. The solution set of (1) denoted by $SOL(C, f)$.

Many algorithms for solving the (VI) are projection algorithms. In 1976, Korpelevich [5] proposed a new algorithm for solving the (VI) in Euclidean space which is known that Extragradient Method putting $x^0 \in H$ arbitrarily, she present her algorithm as follows:

$$\begin{cases} y^k := P_C(x^k - \tau f(x^k)) \\ x^{k+1} := P_C(x^k - \tau f(y^k)) \end{cases}$$

where τ is some positive number and P_C denotes Euclidean least distance projection onto C. Censor et al.[1] presented a modified extragradient algorithm for finding a common element of solution set of a (VI) and the set of fixed points of a nonexpansive mapping.

*Speaker

In recent years, many authors have used extragradient method for finding a common element of solutions set of a (VI) and the set of fixed points of a nonexpansive mapping in the framework of Hilbert spaces and Banach spaces, see for instance [4, 1] and the references there in. In this paper, employing the idea of Censor et al.[1], we propose a new extragradient method. Using this method, we prove a weak convergence theorem under suitable conditions.

2 Preliminaries

We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall\, x \in E.$$

Let $S(E)$ be the unite sphere centered at the origin of E.
A Banach space E is strictly convex if $\|\frac{x+y}{2}\| < 1$, whenever $x, y \in S(E)$ and $x \neq y$. Modulus of convexity of E is defined by

$$\delta_E(\epsilon) = inf\{1 - \frac{1}{2}\|(x+y)\| : \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\}$$

for all $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be p-uniformly convex [9] if there exists a constant $c > 0$ such that $\delta_E \geq c\epsilon^p$ for all $\epsilon \in [0, 2]$. The Banach space E is called smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2}$$

exists for all $x, y \in S(E)$. It is also said to be uniformly smooth if the limit (2) is attained uniformly for all $x, y \in S(E)$ [8]. If a Banach space E uniformly convex, then E is reflexive and strictly convex [7].
We denote the strong convergence and the weak convergence of a sequence $\{x^k\}$ to x in E by $x^k \to x$ and $x^k \rightharpoonup x$, respectively. We also, denote the weak* convergence of a sequence $\{x^{*^k}\}$ to x^* in E^* by $x^{*^k} \rightharpoonup^* x^*$.

Let C be nonempty subset of a Banach space E and $T : C \to E$ be a mapping. Then T is said to be demiclosed at $y \in E$ if for any sequence $\{x^k\}_{k=0}^{\infty}$ in C the following implication holds:

$$x^k \rightharpoonup x \in C \quad and \quad Tx^k \to y \quad imply \quad Tx = y.$$

The duality mapping J is said to be weakly sequentially continuous if $x^k \rightharpoonup x$ implies that $Jx^k \rightharpoonup^* Jx$ [2].
An operator $f : C \to E^*$ is called monotone if $\langle f(x) - f(y), x - y \rangle \geq 0$, for all $x, y \in C$. Also, it is called α-inverse-strongly monotone if there exists a constant $\alpha > 0$ with $\langle f(x) - f(y), x - y \rangle \geqslant \alpha \|f(x) - f(y)\|^2$, for all $x, y \in C$. Let E be a smooth Banach space, we define the function $\phi : E \times E \to \mathbb{R}$ by $\phi(x, y) = \|x\|^2 - 2\langle x, Jx \rangle + \|y\|^2$, for all $x, y \in E$.

Definition 2.1. [3] Let E be a smooth Banach space and Let C be a nonempty subset of E. A mapping $T : C \to C$ is called generalized nonexpansive if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y),$$

for all $x \in C$ and all $y \in F(T)$.

Definition 2.2. [3] Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx,$$

for all $x \in E$ and all $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D.

Lemma 2.3. *[3] Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

1. *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$,*
2. *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

Lemma 2.4. *[10] Let E be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the duality mapping of E and $\frac{1}{c}(0 \leq c \leq 1)$ is the 2-uniformly convex constant of E.

Lemma 2.5. *[4] Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y),$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.6. *[4] Let E be a uniformly convex and smooth Banach space and let $\{x^k\}$ and $\{y^k\}$ be two sequences of E. If $\phi(x^k, y^k) \to 0$ and either $\{x^k\}$ or $\{y^k\}$ is bounded, then $x^k - y^k \to 0$.*

3 Main result

Now, we present an algorithm for finding a solution of the (VI) which is also a fixed point of a generalized nonexpansive mapping. Let $S : C \to C$ be a generalized nonexpansive mapping and denote by $F(S)$ the set of fixed point of S, i.e. $F(S) = \{x \in C \mid S(x) = x\}$.

Let $\{\alpha^k\}_{k=0}^{\infty} \subset [c, d]$ for some $c, d \in (0, 1)$. Let R_C be the sunny generalized nonexpansive retraction from E onto C, where C is nonempty subset of E.
Step 0: Select a arbitrary starting point $x^0 \in C$ and $\tau > 0$, and put $k = 0$.
Step 1: Let x^{k+1} be k *th* iteration, compute

$$\begin{cases} y^k := R_C J^{-1}(Jx^k - \tau f(x^k)), \\ x^{k+1} := J^{-1}(\alpha^k Jx^k + (1 - \alpha^k)JSy^k). \end{cases} \tag{3}$$

Step 2: Set $k \leftarrow (k + 1)$ and return to Step 1.

Theorem 3.1. *Let C be a nonempty closed convex subset of a 2-uniformly convex, uniformly smooth Banach space E. Let $f : C \to E^*$ ba a α-inverse strongly monotone operator such that*

$$\Omega := SOL(C, f) \cap F(S) \neq \emptyset$$

and $\|f(x)\| \leq \|f(x) - f(u)\|$ for all $x \in C$ and $u \in \Omega$, furthermore, assume that
$(i)\liminf_{k\to\infty} \alpha^k > 0$.
$(ii)\alpha \geq \frac{2\tau}{c^2}$*, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.*
If J is weakly sequentially continuous and $I-S$ is demiclosed at 0, then sequences $\{x^k\}_{k=0}^{\infty}$ and $\{y^k\}_{k=0}^{\infty}$ generated by (3) converge weakly to the some solution $u^* \in \Omega$, where

$$u^* = \lim_{k\to\infty} R_\Omega(x^k)$$

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A generalized Hermite-Hadamard type inequality for h-convex functions via fractional integral

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Abstract

An inequality of Hermite-Hadamard type for h-convex functions via Riemann-Liouville fractional integral is studied. Our results generalize and improve the results of other researchers.

Keywords: Hermite-Hadamard's inequality, h-convex function, Riemann-Liouville fractional integral.
Mathematics Subject Classification [2010]: 26A33, 26D15

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined on the interval I of the real numbers and $a, b \in I$, with $a < b$. If f is a convex function, then the Hermite-Hadamard inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Definition 1.1. [4] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function or that f belongs to the class $SX(h, I)$ if f is nonnegative and for all $x, y \in I$ and $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1-\lambda) y) \leq h(\lambda) f(x) + h(1-\lambda) f(y).$$

Notice that the class of h-convex functions generalizes the class of convex functions for $h(x) = x$ for all x.

Definition 1.2. [2] Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $\mathbb{J}_{a^+}^{\alpha} f$ and $\mathbb{J}_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\mathbb{J}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \qquad x > a$$

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and

$$\mathbb{J}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \qquad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathbb{J}_{a+}^{0} f(x) = \mathbb{J}_{b-}^{0} f(x) = f(x)$.

Recently, some generalizations of Hermite-Hadamard inequality for fractional integral have been proved by many researchers [2, 3]. For example, in 2013, Saikaya *et al.*, proved the following inequality for fractional integrals.

Theorem 1.3. *[2] Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\mathbb{J}_{a+}^{\alpha} f(b) + \mathbb{J}_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

In 2013, Tunç [3] proposed the following inequality for fractional integrals based on $h-$convex function.

Theorem 1.4. *[3] Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$ and $f \in L_1 \in [a, b]$. Then one has inequality for h-convex functions for fractional integrals:*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[\mathbb{J}_{a+}^{\alpha} f(b) + \mathbb{J}_{b-}^{\alpha} f(a)\right] \le \left[f(a) + f(b)\right] \int_0^1 t^{\alpha-1} \left[h(t) + h(1-t)\right] dt.$$

In this paper, an inequality of Hermite-Hadamard type for h-convex functions via the Riemann-Liouville fractional integral is studied. Our results generalize and improve the corresponding results of Tunç [3, 2013], Sarikaya *et al.* [2, 2013].

2 Main results

Now, we state and prove the main result of this paper.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a $h-$convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$\frac{\varphi(\alpha,\lambda)}{h\left(\frac{1}{2}\right)} \leqslant \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \Big[\mathbb{J}_{a+}^{\alpha} f(\lambda b + (1-\lambda)a) + \mathbb{J}_{(\lambda b+(1-\lambda)a)^{-}}^{\alpha} f(a)$$

$$+\mathbb{J}_{(\lambda b+(1-\lambda)a)^{+}}^{\alpha} f(b) + \mathbb{J}_{b-}^{\alpha} f(\lambda b + (1-\lambda)a)\Big] \leqslant 2\alpha\Phi(\alpha,\lambda) \int_0^1 (h(t) + h(1-t))\, t^{\alpha-1} dt,$$

for all $\lambda \in [0, 1]$, $t \in (0, 1)$, $\alpha > 0$ where

$$\varphi(\alpha,\lambda) := \lambda^{\alpha} f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda)^{\alpha} f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right),$$

and

$$\Phi(\alpha,\lambda) := \frac{\lambda^{\alpha}}{2} f(a) + \left(\frac{\lambda^{\alpha}}{2} + \frac{1}{2}(1-\lambda)^{\alpha}\right) f(\lambda b + (1-\lambda)a) + \frac{1}{2}(1-\lambda)^{\alpha} f(b).$$

Proof. Since f is a $h-$convex function on $[a,b]$, we have for $x,y\in[a,b]$

$$f\left(\frac{x+y}{2}\right)\leqslant h\left(\frac{1}{2}\right)\left(f\left(x\right)+f\left(y\right)\right). \tag{2}$$

Let $t\in(0,1)$. So, for $x=ta+(1-t)(\lambda b+(1-\lambda)a)$, $y=(1-t)a+t(\lambda b+(1-\lambda)a)$, Ineq. (2) implies that

$$\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\leqslant f\left(ta+(1-t)\left(\lambda b+(1-\lambda)a\right)\right)+f\left((1-t)a+t\left(\lambda b+(1-\lambda)a\right)\right).$$

For $\lambda\neq 0$, multiplying both sides by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)\alpha}f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\leqslant\int_0^1 t^{\alpha-1}f\left(ta+(1-t)\left(\lambda b+(1-\lambda)a\right)\right)dt\\
&+\int_0^1 t^{\alpha-1}f\left((1-t)a+t\left(\lambda b+(1-\lambda)a\right)\right)dt\\
&=\int_{\lambda b+(1-\lambda)a}^{a}\left(\frac{\left(\lambda b+(1-\lambda)a\right)-u}{\lambda(b-a)}\right)^{\alpha-1}f(u)\frac{du}{\lambda(a-b)}\\
&+\int_a^{\lambda b+(1-\lambda)a}\left(\frac{v-a}{\lambda(b-a)}\right)^{\alpha-1}f(v)\frac{dv}{\lambda(b-a)}\\
&=\frac{\Gamma(\alpha)}{\lambda^{\alpha}(b-a)^{\alpha}}\left(\mathbb{J}^{\alpha}_{a^+}f\left(\lambda b+(1-\lambda)a\right)+\mathbb{J}^{\alpha}_{(\lambda b+(1-\lambda)a)^-}f(a)\right).
\end{aligned}$$

So,

$$\frac{1}{h\left(\frac{1}{2}\right)\alpha}f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\leqslant\frac{\Gamma(\alpha)}{\lambda^{\alpha}(b-a)^{\alpha}}\left(\mathbb{J}^{\alpha}_{a^+}f\left(\lambda b+(1-\lambda)a\right)+\mathbb{J}^{\alpha}_{(\lambda b+(1-\lambda)a)^-}f(a)\right). \tag{3}$$

Again for $x=t(\lambda b+(1-\lambda)a)+(1-t)b$, $y=(1-t)(\lambda b+(1-\lambda)a)+tb$ and Ineq. (2), for $\lambda\neq 1$, we have

$$\frac{1}{h\left(\frac{1}{2}\right)\alpha}f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\leq\frac{\Gamma(\alpha)}{(1-\lambda)^{\alpha}(b-a)^{\alpha}}\left(\mathbb{J}^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(b)+\mathbb{J}^{\alpha}_{b^-}f\left(\lambda b+(1-\lambda)a\right)\right).$$

Then

$$\frac{1}{h\left(\frac{1}{2}\right)\alpha}f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\leqslant\frac{\Gamma(\alpha)}{(1-\lambda)^{\alpha}(b-a)^{\alpha}}\left(\mathbb{J}^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(b)+\mathbb{J}^{\alpha}_{b^-}f\left(\lambda b+(1-\lambda)a\right)\right). \tag{4}$$

Multiplying (3) by λ^{α}, (4) by $(1-\lambda)^{\alpha}$, and adding the resulting inequalities, the first inequality is proved. For the proof of the second inequality, since f is a $h-$convex, we have

$$f\left(ta+(1-t)\left(\lambda b+(1-\lambda)a\right)\right)\leqslant h(t)f(a)+h(1-t)f\left(\lambda b+(1-\lambda)a\right).$$

and

$$f\left(\left(1-t\right)a+t\left(\lambda b+\left(1-\lambda\right)a\right)\right)\leqslant\left(1-t\right)f\left(a\right)+tf\left(\lambda b+\left(1-\lambda\right)a\right)$$

By adding these inequalities, then multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}&\frac{\Gamma\left(\alpha\right)}{\lambda^{\alpha}\left(b-a\right)^{\alpha}}\left(\mathbb{J}_{a^{+}}^{\alpha}f\left(\lambda b+\left(1-\lambda\right)a\right)+\mathbb{J}_{\lambda b+\left(1-\lambda\right)a^{-}}^{\alpha}f\left(a\right)\right)\qquad(5)\\&\leqslant\left(f\left(a\right)+f\left(\lambda b+\left(1-\lambda\right)a\right)\right)\int_{0}^{1}t^{\alpha-1}\left(h\left(t\right)+h\left(1-t\right)\right)dt.\end{aligned}$$

Again, the $h-$convexity of f implies that

$$f\left(t\left(\lambda b+\left(1-\lambda\right)a\right)+\left(1-t\right)b\right)\leqslant h\left(t\right)f\left(\lambda b+\left(1-\lambda\right)a\right)+h\left(1-t\right)f\left(b\right)$$

and

$$f\left(\left(1-t\right)\left(\lambda b+\left(1-\lambda\right)a\right)+tb\right)\leqslant h\left(1-t\right)f\left(\lambda b+\left(1-\lambda\right)a\right)+h\left(t\right)f\left(b\right)$$

By adding these inequalities, then multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\frac{\Gamma\left(\alpha\right)}{\left(1-\lambda\right)^{\alpha}\left(b-a\right)^{\alpha}}\left[\mathbb{J}_{\left(\lambda b+\left(1-\lambda\right)a\right)^{+}}^{\alpha}f\left(b\right)+\mathbb{J}_{b^{-}}^{\alpha}f\left(\lambda b+\left(1-\lambda\right)a\right)\right]\qquad(6)\\&\leqslant\left(f\left(\lambda b+\left(1-\lambda\right)a\right)+f\left(b\right)\right)\int_{0}^{1}t^{\alpha-1}\left(h\left(t\right)+h\left(1-t\right)\right)dt.\end{aligned}$$

Multiplying (5) by λ^{α}, (6) by$(1-\lambda)^{\alpha}$ and adding the resulting inequalities, we get to second inequality and the proof is completed.

Remark 2.2. As special cases of Theorem 2.1,
(I) if $\lambda=1$ and $h\left(t\right)=t$ for any $t\in(0,1)$, then we have Theorem 1.3 which obtained by Sarikaya *et al.* (II) If $\lambda=1$, then we have Theorem 1.4 which obtained by Tunç [3, 2013]. (III) If $\alpha=\lambda=1$ and $h\left(t\right)=t$ for any $t\in(0,1)$, then the classical Hermite-Hadamard inequality (1) holds.

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A note on composition operators between weighted Hilbert spaces of analytic functions

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Abstract

In this paper, we consider composition operators on weighted Hilbert spaces of analytic functions and observe that a formula for the essential norm, give a Hilbert-Schmidt characterization and characterize the membership in Schatten-class for these operators. Also, closed range composition operators are investigated.

Keywords: composition operators, essential norm, Hilbert-Schmidt, Schatten-class, closed range
Mathematics Subject Classification [2010]: 30H30, 46E40.

1 Introduction

Let $\mathbb{D}$ denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and φ be an analytic self map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined $C_\varphi f = f \circ \varphi$, for any $f \in H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$. This operator can be generalized to the weighted composition operator uC_φ, $uC_\varphi f(z) = u(z)f(\varphi(z))$, $u \in H(\mathbb{D})$. We consider a *weight* as a positive integrable function $\omega \in C^2[0, 1)$ which is radial, $\omega(z) = \omega(|z|)$. The weighted Hilbert space of analytic functions $\mathcal{H}_\omega$ consists of all analytic functions on $\mathbb{D}$ such that

$$||f'||^2_\omega = \int_{\mathbb{D}} |f'(z)|^2 \omega(z)\, dA(z) < \infty,$$

equipped with the norm $||f||^2_{\mathcal{H}_\omega} = |f(0)|^2 + ||f'||^2_\omega$. Here dA is the normalized area measure on $\mathbb{D}$. Also the weighted Bergman spaces defined by

$$\mathcal{A}^2_\omega = \left\{ f \in H(\mathbb{D}) : ||f||^2_\omega = \int_{\mathbb{D}} |f(z)|^2 \omega(z)\, dA(z) < \infty \right\}.$$

If $f(z) = \sum_{n=0}^\infty a_n z^n$, then $f \in \mathcal{H}_\omega$ if and only if $||f||^2_{\mathcal{H}_\omega} = \sum_{n=0}^\infty |a_n|^2 \omega_n < \infty$, where $\omega_0 = 1$ and for $n \geq 1$

$$\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr,$$

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and $f \in \mathcal{A}_\omega$ if and only if $||f||^2_{\mathcal{A}_\omega} = \sum_{n=0}^{\infty} |a_n|^2 p_n < \infty$, where

$$p_n = 2\int_0^1 r^{2n+1}\omega(r)dr, \quad n \geq 0.$$

By letting $\omega_\alpha(r) = (1-r^2)^\alpha$ (standard weight), $\alpha > -1$, $\mathcal{H}_{\omega_\alpha} = \mathcal{H}_\alpha$. If $0 \leq \alpha < 1$, then $\mathcal{H}_\alpha = \mathcal{D}_\alpha$, the weighted Dirichlet space, and $\mathcal{H}_1 = H^2$, the Hardy space.

There are several papers that studied composition operators on various spaces of analytic functions. The best monographs for these operators are [1, 7]. In [2], Kellay and Lefèvre studied composition operators on weighted Hilbert space of analytic functions by using generalized Nevanlinna counting function. They characterized boundedness and compactness of these operators. Pau and Pérez [6] studied boundedness, essential norm, Schatten-class and closed range properties of these operators acting on weighted Dirichlet spaces.

Our aim in this paper is to generalize the results of [6] to a large class of spaces. Throughout the remainder of this paper, c will denote a positive constant, the exact value of which will vary from one appearance to the next.

2 Preliminaries

In this section we give some notations and lemmas will be used in our work.

Definition 2.1. [2] We assume that ω is a weight function, with the following properties
(W_1): ω is non-increasing,
(W_2): $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$,
(W_3): $\lim_{r\to 1^-} \omega(r) = 0$,
(W_4): One of the two properties of convexity is fulfilled

$$\begin{cases} (W_4^{(I)}): & \omega \textit{ is convex and } \lim_{r\to 1}\omega'(r) = 0, \\ \textit{or} & \\ (W_4^{(II)}): & \omega \textit{ is concave.} \end{cases}$$

Such a weight ω is called admissible.

If ω satisfies conditions (W_1)-(W_3) and $(W_4^{(I)})$ (resp. $(W_4^{(II)})$), we shall say that ω is (I)-admissible (resp. (II)-admissible). Also we use weights satisfy (L1) condition (due to Lusky [5]):

$$(L1) \quad \inf_k \frac{\omega(1-2^{-k-1})}{\omega(1-2^{-k})} > 0.$$

This is equivalent to this condition (see[3]):

There are $0 < r < 1$ and $0 < c < \infty$ with $\frac{\omega(z)}{\omega(a)} \leq c$ for every $a, z \in \Delta(a, r)$, where $\Delta(a, r) = \{z \in \mathbb{D} : |\sigma_a(z)| < r\}$ and $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Mobius transformation on $\mathbb{D}$.
All characterizations in this paper are needed to the generalized counting Nevanlinna function. Let φ be an analytic self map of $\mathbb{D}$ ($\varphi(\mathbb{D}) \subset \mathbb{D}$). The generalized counting Nevanlinna function associated to a weight ω defined as follows

$$N_{\varphi,\omega}(z) = \sum_{a:\varphi(a)=z} \omega(a), \quad z \in \mathbb{D}\backslash\{\varphi(0)\}.$$

By using the change of variables formula we have: If f be a non-negative function on $\mathbb{D}$, then

$$\int_{\mathbb{D}} f(\varphi(z))|\varphi'(z)|^2\omega(z)dA(z) = \int_{\mathbb{D}} f(z)N_{\varphi,\omega}(z)dA(z). \tag{1}$$

Also the generalized counting Nevanlinna function has the sub-mean value property (Lemmas 2.2 and 2.3 [2]). Let ω be an admissible weight. Then for every $r > 0$ and $z \in \mathbb{D}$ such that $D(z,r) \subset \mathbb{D}\backslash D(0,1/2)$

$$N_{\varphi,\omega}(z) \leq \frac{2}{r^2}\int_{|\zeta-z|<r} N_{\varphi,\omega}(\zeta)dA(\zeta). \tag{2}$$

Lemma 2.2. *[2] If ω is a weight satisfying (W_1) and (W_2), then there exists $c > 0$ such that*

$$\frac{1}{c}\omega(z) \leq \omega(\sigma_{\varphi(0)}(z)) \leq c\omega(z), \quad z \in \mathbb{D}.$$

Lemma 2.3. *[2] Let ω be a weight satisfying (W_1) and (W_2). Let $a \in \mathbb{D}$ and*

$$f_a(z) = \frac{1}{\sqrt{\omega(a)}}\frac{(1-|a|^2)^{1+\delta}}{(1-\overline{a}z)^{1+\delta}}.$$

Then $||f_a||_{\mathcal{H}_\omega} \asymp 1$.

3 Hilbert-Schmidt and Schatten-class

For studying Schatten-class we need the Toeplitz operator. For more information about relation between Toeplitz operator and Schatten-class see [8]. Let ψ be positive function in $L^1(\mathbb{D}, dA)$ and ω be a weight. The Toeplitz operator associated to ψ defined by

$$T_\psi f(z) = \frac{1}{\omega(z)}\int_{\mathbb{D}} \frac{f(t)\psi(t)\omega(t)}{(1-\overline{z}t)^2}dA(t).$$

$T_\psi \in S_p(\mathcal{A}^2_\omega)$ if and only if the function

$$\widehat{\psi}_r(z) = \frac{1}{(1-|z|^2)^2\omega(z)}\int_{\Delta(z,r)} \psi(t)\omega(t)dA(t)$$

is in $L^p(\mathbb{D}, d\lambda)$, [4], where $d\lambda = (1-|z|^2)^{-2}dA(z)$ is the hyperbolic measure on $\mathbb{D}$. According to the description of [6] pages 8 and 9, $C_\varphi \in S_p(\mathcal{H}_\omega)$ if and only if $\varphi' C_\varphi \in S_p(\mathcal{A}^2_\omega)$.

Theorem 3.1. *Let ω be an admissible weight satisfy (L1) condition. Then $C_\varphi \in S_p(\mathcal{H}_\omega)$ if and only if*

$$\psi(z) = \frac{N_{\varphi,\omega}(z)}{\omega(z)} \in L^{p/2}(\mathbb{D}, d\lambda).$$

If $p = 2$, then we have a characterization for Hilbert-Schmidt composition operators.

Corollary 3.2. *Let ω be an admissible weight satisfy (L1) condition. Then C_φ is Hilbert-Schmidt on $\mathcal{H}_\omega$ if and only if*

$$\int_{\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\omega(z)(1-|z|^2)^2}dA(z) = \int_{\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\omega(z)}d\lambda(z) < \infty.$$

4 Closed Range

It is well known that having the closed range for a bounded operator acting on a Hilbert space H is equivalent to existing a positive constant c such that for every $f \in H$, $||Tf||_H \geq c||f||_H$. Consider the function

$$\tau_{\varphi,\omega}(z) = \frac{N_{\varphi,\omega}(z)}{\omega(z)}.$$

Proposition 4.1. *Let ω be an admissible weight and C_φ be a bounded operator on $\mathcal{H}_\omega$. Then C_φ has closed range if and only if there exists a constant $c > 0$ such that for all $f \in \mathcal{H}_\omega$*

$$\int_{\mathbb{D}} |f'(z)|^2 \tau_{\varphi,\omega}(z)\omega(z)\; dA(z) \geq c \int_{\mathbb{D}} |f'(z)|^2 \omega(z)\; dA(z). \tag{3}$$

Fredholm composition operator is an example of composition operator with closed range property. Recall that a bonded operator T between two Banach spaces X, Y is called Fredholm if Kernel T and T^* are finite dimensional.

Example 4.2. Suppose that $C_\varphi : \mathcal{H}_\omega \to \mathcal{H}_\omega$ be a Fredholm operator. By Theorem 3.29[1], φ is an authomorphism of $\mathbb{D}$. Then $N_{\varphi,\omega}(z) = \omega(\varphi^{-1}(z))$. If $\varphi(0) = 0$, Schwarz Lemma implies that $|\varphi^{-1}(z)| \leq |z|$. Since ω is non-increasing, $\omega(\varphi^{-1}(z)) = \omega(|\varphi^{-1}(z)|) \geq \omega(|z|) = \omega(z)$. Now (3) holds. If $\varphi(0) \neq 0$, then the same argument can be applied.

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A note on composition operators on Besov type spaces

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Abstract

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. We denote by $H(\mathbb{D})$ the space of all holomorphic function on $\mathbb{D}$. given a holomorphic self map φ on $\mathbb{D}$ the composition operator C_φ on $H(\mathbb{D})$ is defined by

$$(C_\varphi f)(z) = f(\varphi(z))$$

for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

In this article we give some results about the boundednes of the composition operators on Besov type space $B_{p,q}$ for $1 < p < \infty$ and $-1 < q < \infty$.

Keywords: Composition operator, Carleson Measure, Besov Type Space
Mathematics Subject Classification [2010]: 47B33, 30H25

1 Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. We will use the notation $H(\mathbb{D})$ to denote the space of holomorphic functions on the unit disc $\mathbb{D}$. Suppose φ is a holomorphic function defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Each $\psi \in H(\mathbb{D})$ and holomorphic self-map φ of $\mathbb{D}$ induces a linear weighted composition operator $C_{\psi,\varphi} : H(\mathbb{D}) \to H(\mathbb{D})$ defined by

$$C_{\psi,\varphi}(f)(z) = \psi(z) f(\varphi(z))$$

for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

(weighted) Composition operator on various spaces of functions are being studied by many authors. We can refer for example to [4, 5, 6, 7].

Fix any $a \in \mathbb{D}$ and let $\sigma_a(z)$ be the Mobius transform defined by

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}, z \in \mathbb{D}.$$

We denote the set of all Mobius transformations on $\mathbb{D}$ by G. Such a map is its own inverse and satisfies the fundamental identity

$$|\sigma'_a(z)| = \frac{1 - |a|^2}{|1 - \overline{a}z|^2}.$$

see[6, 9].

*Speaker

Definition 1.1. Fix $1 < p < \infty$ and $-1 < q < \infty$. Then f is in the Besov type space $B_{p,q}$ if

$$\|f\|_{p,q} = (\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q dA(z))^{\frac{1}{p}} < \infty, \qquad (1)$$

where $dA(z)$ denotes the Lebesgue area measure on $\mathbb{D}$. Also, if we take $1 < p < \infty$ and $q = p-2$ in (1),then we get the analytic Besov space B_p.

By making a non-univalent change of variables we see that

$$\|C_\varphi f\|_{p,q} = \int_U |f'(w)|^p N_{p,q}(w,\phi)dA(w). \qquad (2)$$

Now consider the restriction C_φ to $B_{p,q}$ .Then C_φ is a bounded operator if and only if there is a positive constant C such that

$$\|C_\varphi f\|_{B_{p,q}} \leq C\|f\|^p_{B_{p,q}}$$

for all $f \in B_{p,q}$ or, equivalently

$$\int_U |f'(w)|^p N_{p,q}(w,\phi)dA(w) \leq C\|f\|^p_{p,q}$$

for all $f \in B_{p,q}$.

Definition 1.2. : Let μ be a positive measure on $\mathbb{D}$ and let $X = B_{p,q}$ $(1 < p < \infty$ $,-1 < q < \infty)$.Then μ is an (X,p)-Carleson measure if there is a constant $A > 0$ such that

$$\int_U |f'(w)|^p \leq A\|f\|^p_X$$

for all $f \in X$.

2 Main results

(Weighted) composition operators on spaces of holomorphic functions on unit disc $\mathbb{D}$ are studied by many authors. See for example [1, 2, 3]. Boundedness and compactness of composition operators on Besov spaces was studied by Sharma and Kumar in [6] and Tjani in [7, 8].

In this section we give some results about the composition operators on Besov type spaces.

Theorem 2.1. *For $1 < p < \infty$, $-1 < q < \infty$,$a \in \mathbb{C}$ then $\sigma_a \in B_{p,q}$.*

In view of (2) we see that C_φ is bounded operator on $B_{p,q}$ if and only if the measure $N_{p,q}(w,\phi)dA(w)$ is a $(B_{p,q}, p)$- Carleson measure

Theorem 2.2. *For $1 < p < \infty$ and $-1 < q < \infty$ if μ is a $(B_{p,q}, p)$- Carleson measure. Then there exists a constant $B > 0$ such that*

$$\int_U |\alpha'_a(z)|^p d\mu(z) \leq B$$

for $a \in \mathbb{D}$.

Theorem (2.2) yields the following:

Theorem 2.3. *Let φ be a holomorphic function on $\mathbb{D}$ and C_φ is a bounded operator on $B_{p,q}$($1 < p < \infty$ and $-1 < q < \infty$).Then*

$$sup_{a\in\mathbb{D}}\|C_\varphi f\|_{B_{p,q}} < \infty.$$

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Talk A note on the transitive groupoid spaces pp.: 1–4

A note on the transitive groupoid spaces

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Abstract

If a group G acts on a set X and H is a subgroup of G, the Frattini argument shows that H acts transitively on X if and only if G acts transitively on X and $G = HStab_x$ for some $x \in X$, where $Stab_x$ is the stabilizer of x in G. There is another useful result in group action which indicates that the action of G on a set X is doubly transitive if and only if, for each $x \in X$, the group $Stab_x$ acts transitively on $X \setminus \{x\}$, where the cardinal number of X is more than two. In this paper if a groupoid acts on a set X, then by using sections, special subsets of X, instead of the points of X in the group case, we will extend these results to the groupoid case.

Keywords: Groupoid; Groupoid space; Frattini argumen
Mathematics Subject Classification [2010]: 18B40, 16W22

1 Introduction

When a group G acts on a set X, the point stabilizer of $x \in X$ is denoted by $Stab_x$ and is a subgroup of G. In the case where G acts transitively on X, then the stabilizers $Stab_x(x \in X)$ form a single conjugacy class of subgroups of G. The Frattini argument indicate that a subgroup H of G acts transitively on X if and only if $G = HStab_x$ for some $x \in X$ [1]. The action of the group G on the set X is naturally extend to an action of G on the cartesian product $X \times X$ by $g.(x, y) = (g.x, g.y)$. The action of G on X is called doubly transitive, if for two pairs $(x_1, x_2), (y_1, y_2)$ in $X \times X$ with $x_1 \neq x_2, y_1 \neq y_2$, there exists $g \in G$ with $g.x_1 = y_1, g.x_2 = y_2$. The action of G on X is doubly transitive if and only if, for each $x \in X$, the group $Stab_x$ acts transitively on $X \setminus \{x\}$ [1].

A groupoid (see definition 1.1 of [4]) is a set G endowed with a product map $(x, y) \mapsto xy : G^2 \to G$ where G^2 as a subset of $G \times G$ is called the set of *composable pairs*, and an inverse map $x \mapsto x^{-1} : G \to G$ such that the following relations are satisfied:

1. For every $x \in G$, $(x^{-1})^{-1} = x$.

2. If $(x, y), (y, z) \in G^2$, then $(xy, z), (x, yz) \in G^2$ and $(xy)z = x(yz)$.

3. For all $x \in G$, $(x^{-1}, x) \in G^2$ and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$. Also for all $x \in G$, $(x, x^{-1}) \in G^2$ and if $(z, x) \in G^2$, then $(zx)x^{-1} = z$.

*Speaker

The maps r and d on G defined by the formulas $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$ are called the *range map* and *domain map*. It follows easily from the definition that they have a common image called the *unit space* of G which is denoted by G^0. The pair (x, y) is composable if and only if the range of y is the domain of x. Condition (3) implies that $r(x)x = x, xd(x) = x$. For $u, v \in G^0$, $G^u = r^{-1}(u), G_v = d^{-l}(v), G^u_v = G^u \cap G_v$ and G^u_u, which is a group, is called the *isotropy group* at u and $G' = \bigcup_{u\in G^0} G^u_u$ is called *stabilizer subgroupoid* of G. A groupoid G is called *transitive* if $G^u_v \neq \emptyset$ for all $u, v \in G^0$.

The notion of groupoid action on a set which is a generalization of group actions is discussed in several places, for example, see [3], [5]. If G is a groupoid and X is a set, we say that G acts (on the left) of X if there is a surjection $\rho : X \to G^0$ and a map $(g, x) \mapsto g.x$ form $G * X = \{(g, x) : d(g) = \rho(x)\}$ to X such that

1) If $(g_1, g_2) \in G^2$ and $(g_2, x) \in G*X$, then $(g_1g_2, x), (g_1, g_2.x) \in G*X$ and $g_1.(g_2.x) = (g_1g_2).x$

2) For all $x \in X, \rho(x).x = x$.

We think of ρ as a "generalized range" map. We see that similar to the groupoid multiplication which is partially defined, a groupoid action on a set is partially defined.

When a groupoid G acts on a set X, then the set X is called *Groupoid-space* or simply $G-$space. If X is a $G-$space, for every $u \in G^0$ we use X^u to denote the set $\rho^{-1}(u)$, the $\rho-$fiber at u. If G is a groupoid and X is a $G-$space, then X is said to be transitive $G-$space if for every pair of point $x, y \in X$ there exists $g \in G$ such that $g.x = y$. It is easy to show that if X is a transitive $G-$space then G must be a transitive groupoid [2].

For a groupoid G and a $G-$space X, if we set $X * X = \{(x, y) : \rho(x) = \rho(y)\}$, then it is easy to check that G acts on $X * X$ by the diagonal action: $g.(x, y) = (g.x, g.y)$ [5]. Obviously Δ, the diagonal in $X \times X$, is a subset of $X * X$ and is an invariant subset of $X * X$ under the diagonal action. Also the diagonal action on Δ is transitive if and only if X is a transitive $G-$space.

In this paper, when G is a groupoid and X is a $G-$space, we obtain a groupoid version of Frattini argument. In this case, instead of points of X in the group case, we use some special subsets of X which are called sections of X. Indeed it is shown that a subgroupoid H of G acts transitively on X if and only if G acts transitively on X and $G = H'G_{\{S\}}$ for some section S of X which H acts transitively on S, where H' is the stabilizer subgroupoid of H. Also we prove that the diagonal action of G on $X * X \setminus \Delta$ is transitive if and only if, for each section S of X, the groupoid $G_{\{S\}}$ acts transitively on $X \setminus S$. As a corollary, we prove that the diagonal action of G on $X * X$ is transitive if and only if the action of G on X is transitive and for a section S_0 of X, the groupoid $G_{\{S_0\}}$ acts transitively on $X \setminus S_0$.

2 Transitive groupoid action

Let G be a groupoid and X be a $G-$space, to avoid trivial misunderstanding, we only consider the $G-$spaces without any singleton fiber. In order to proceed we need the following definition.

Definition 2.1. Suppose that G is a groupoid which acts transitively on a set X. A section of X is a subset S of X where $\rho : S \to G^0$ is a bijection. Therefore $S \subset X$ is a section if and only if, S^u is a singleton $\{s^u\}$ for every $u \in G^0$. The stabilizer of a section S is the set $G_{\{S\}} = \{g \in G : g.s^{d(g)} = s^{r(g)}\}$. By the definition of groupoid action, $G^0 \subset G_{\{S\}}$.

Lemma 2.2. *If G is a groupoid and X is a $G-$space, for every section S of X, the stabilizer of S is a subgroupoid of G, and in the case where X is a transitive $G-$space and T is another section of X, then $G_{\{S\}}$ is isomorphic to $G_{\{T\}}$.*

Proof. Let $g \in G_{\{S\}}$, then $g.s^{d(g)} = s^{r(g)}$, so $g^{-1}.s^{d(g^{-1})} = g^{-1}.s^{r(g)} = s^{d(g)} = s^{r(g^{-1})}$. That is $G_{\{S\}}$ is closed under inversion. To prove that $G_{\{S\}}$ is closed under multiplication, let $g_1, g_2 \in G_{\{S\}}$ and $(g_1, g_2) \in G^2$, then

$$g_1g_2.s^{d(g_1g_2)} = g_1g_2.s^{d(g_2)} = g_1.s^{r(g_2)} = g_1.s^{d(g_1)} = s^{r(g_1)} = s^{r(g_1g_2)},$$

so $g_1g_2 \in G_{\{S\}}$. Now suppose that X is a transitive $G-$space and T is another section of X. Since the action of G on X is transitive, there exists a section K of G' with $k_{d(g)}^{d(g)} s^{d(g)} = t^{d(g)}$ for every $g \in G$, where $\{k_u^u\} = K^u$. It is easy to check that the map $\varphi_K : G_{\{S\}} \to G_{\{T\}}$ by $\varphi_K(g) = \big(k_{r(g)}^{r(g)}\big) g \big(k_{d(g)}^{d(g)}\big)^{-1}$ is well defined and is a groupoid isomorphism. □

In the group case, the Frattini argument indicates that when a group G acts on a set X and H is a subgroup of G, then H acts transitively on X if and only if G acts transitively on X and $G = HStab_x$ for some $x \in X$. In the following we bring the groupoid version of this.

Proposition 2.3. *If a groupoid G acts on a set X and H is a subgroupoid of G, then the following are equivalent:*

1. *H acts transitively on X,*
2. *G acts transitively on X and $G = H'G_{\{S\}}$ for some section S of X which H acts transitively on S.*

Proof. 1) ⇒ 2) Obviously if H acts transitively on X, then G acts too. Let S be a Section of X and $g \in G$. Since H acts transitively on X, for $g.s^{d(g)}, s^{(r(g)} \in X$ there exists $h \in H$ with $g.s^{d(g)} = h.s^{r(g)}$. It is easy to check that $h \in H'$, and $h^{-1}g.s^{d(h^{-1}g)} = h^{-1}g.s^{d(g)} = s^{r(g)} = s^{d(h)} = s^{r(h^{-1}g)}$. That is $h^{-1}g \in G_{\{S\}}$, and so $g \in H'G_{\{S\}}$. To prove the last part of the item 2), let $u, v \in G^0$ and $s^u, s^v \in S$. Since H acts transitively on X, then H is transitive, so there exists $h \in H_v^u$. But $G = H'G_{\{S\}}$ implies that, there exist $h' \in H'$ with $h'h \in G_{\{S\}}$. So $h'h.s^v = h'h.s^{d(h'h)} = s^{r(h'h)} = s^{r(h')} = s^{d(h')} = s^{r(h)} = s^u$. That is H acts transitively on S.

2) ⇒ 1) If G acts transitively on X and $G = H'G_{\{S\}}$ for some section S of X which H acts transitively on S, then $X = G.S = H'G_{\{S\}}.S = H'.S$. Now let $x_1, x_2 \in X$, then there exist two element h_1, h_2 of H' with $x_1 = h_1.s^{d(h_1)}$ and $x_2 = h_2.s^{d(h_2)}$. Since H acts transitively on S, so there exists an element $h_3 \in H$ with $h_3.s^{d(h_1)} = s^{d(h_2)}$ consequently $h_2h_3h_1^{-1}.x_1 = h_2h_3.s^{d(h_1)} = h_2.s^{d(h_2)} = x_2$. □

Corollary 2.4. *If a groupoid G acts on a set X and H is a subgroupoid of G which acts transitively on X and $G_{\{S\}} \subset H$ for some section S of X, then $G = H$.*

For a groupoid G and a $G-$space X, obviously Δ, the diagonal in $X \times X$, is a subset of $X*X$ and is an invariant subset of $X*X$ under the diagonal action, therefore $X*X \setminus \Delta$ is invariant. It is easy to check that the diagonal action of G on Δ is transitive if and only if X is a transitive $G-$space.

Lemma 2.5. *If X is a $G-$space and S is a section of X, then $G_{\{S\}}$ acts on $X \setminus S$.*

Proof. First note that $\rho : X \setminus S \to (G_{\{S\}})^0$ is surjective, since X has no singleton fiber and $G^0 \subset G_{\{S\}}$. It is enough to show that for $g \in G_{\{S\}}$ and $x \in X \setminus S$, $g.x \notin X \setminus S$. If $g.x \in S$, since S is a section, so $g.x = s^{r(g)}$ and therefore $x = g^{-1}.s^{r(g)} = g^{-1}s^{d(g^{-1})} = s^{r(g^{-1})} \in S$, which is a contradiction. □

Proposition 2.6. *The diagonal action of G on $X * X \setminus \Delta$ is transitive if and only if for each section S of X the action of the groupoid $G_{\{S\}}$ on $X \setminus S$ is transitive.*

Proof. Suppose that the diagonal action of G on $X * X \setminus \Delta$ is transitive, by the previous lemma $G_{\{S\}}$ acts on $X \setminus S$. Let $x_1, x_2 \in X \setminus S$, then $(s^{\rho(x_1)}, x_1), (s^{\rho(x_2)}, x_2) \in X*X \setminus \Delta$. So there exists $g \in G$ with $g.(s^{\rho(x_1)}, x_1) = (s^{\rho(x_2)}, x_2)$, hence $g.s^{\rho(x_1)} = s^{\rho(x_2)}$ and $g.x_1 = x_2$, it is enough to show that $g \in G_{\{S\}}$. But $g.x_1 = x_2$ implies that $\rho(x_2) = r(g), \rho(x_1) = d(g)$, so $g.s^{d(g)} = s^{r(g)}$ which means that $g \in G_{\{S\}}$. Conversely, let $(x_1, x_2), (y_1, y_2) \in X*X \setminus \Delta$. Take a section S of X with $x_1, y_1 \in S$. Since S is a section, so $x_2, y_2 \in X \setminus S$ and therefre there exists $g \in G_{\{S\}}$ with $g.x_2 = y_2$. Hence $g.x_1$ is defined and

$$g.x_1 = g.s^{\rho(x_1)} = g.s^{\rho(x_2)} = g.s^{d(g)} = s^{r(g)} = s^{\rho(y_2)} = s^{\rho(y_1)} = y_1.$$

□

Corollary 2.7. *The diagonal action of G on $X * X \setminus \Delta$ is transitive if and only if X is a transitive $G-$space and for one section S_0 of X the subgroupoid $G_{\{S_0\}}$ acts transitively on $X \setminus S_0$.*

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Abstract Convexity of ICR-k Functions

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Abstract

The theory of ICR (increasing and co-radiant) functions defined on ordered topological vector spaces has well been developed. In this paper, we present the theory of ICR-k (increasing and co-radiant of degree k) functions defined on an ordered topological vector space X. We first give a characterization for ICR-k functions and examine abstract convexity of this class of functions. Finally, we characterize support set and subdifferential of ICR-k functions.

Keywords: Abstract convexity, ICR function, ICR-k function, Subdifferential, Support set.
Mathematics Subject Classification [2010]: 26B25, 26A48

1 Introduction

Monotonic analysis is one of the advanced topics in so-called abstract convex analysis which is a natural generalization of classical convex analysis.
Abstract convexity has found many applications in the study of mathematical analysis and optimization problems (see [2, 5]). Functions which can be represented as upper envelopes of subsets of a set H of sufficiently simple (*elementary*) functions, are studied in this theory (for more details see [4, 5, 6]).
It is well-known that some classes of increasing functions are abstract convex. For example, the class of increasing and positively homogeneous (IPH) functions (see [5]) and the class of increasing and convex-along-rays (ICAR) functions are abstract convex (see [4]). The class of increasing and co-radiant (ICR) functions is another class of increasing functions which are abstract convex.
Abstract convexity of ICR functions defined on a topological vector space has been investigated in [1, 3]. In this paper, we study non-negative increasing and co-radiant of degree k (ICR-k) functions defined on an ordered topological vector space X. Finally, we characterize the support set and subdifferential of this functions.

2 Preliminaries

Let X be a topological vector space. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). The increasing property of our

*Speaker

functions will be understood to be with respect to the ordering $\leq$ induced on X by S :

$$x \leq y \iff y - x \in S, \quad (x, y \in X).$$

A function $f : X \longrightarrow [-\infty, +\infty]$ is called co-radiant of degree k ($k > 0$) if $f(\gamma x) \geq \gamma^k f(x)$ for all $x \in X$ and all $\gamma \in (0, 1]$ (a co-radiant of degree 1 function is called co-radiant). It is easy to see that f is co-radiant of degree k if and only if $f(\gamma x) \leq \gamma^k f(x)$ for all $x \in X$ and all $\gamma \geq 1$. The function f is called increasing if $x \geq y \implies f(x) \geq f(y)$. In this paper, we study non-negative ICR-k (increasing and co-radiant of degree k) functions $f : X \longrightarrow [0, +\infty]$ such that

$$0 \in dom f := \{ x \in X : -\infty < f(x) < +\infty \}.$$

Lemma 2.1. *Let $0 < k_1 < k_2$ and $f : X \longrightarrow [0, +\infty]$ be an ICR-k_1 function. Then f is an ICR-k_2 function.*

Lemma 2.2. *Let $f : X \longrightarrow [0, +\infty]$ be an function. Then f is ICR-k if and only if $\sqrt[k]{f}$ is ICR.*

Lemma 2.3. *Let $\{ f_i : i = 1, 2, \dots, k \}$ be a set of non-negative ICR functions defined on X. Then the function $f := f_1 \times f_2 \times \cdots \times f_k$ is ICR-k function.*

Example 2.4. Consider the function $f : \mathbb{R} \longrightarrow [0, +\infty]$ defined as follows

$$f(x) := \begin{cases} a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

for all $x \in \mathbb{R}$ ($a_i \geq 0$ ($i = 1, 2, \dots, k$)). It is clear that f is an ICR-k function.

Definition 2.5. Let X be a non-empty set, H be a non-empty set of functions $h : X \longrightarrow [-\infty, +\infty]$ defined on X and $f : X \longrightarrow [-\infty, +\infty]$ be a function.
1) The support set (or the set of all H-minorants) of f with respect to H is defined by

$$supp(f, H) := \{ h \in H : h(x) \leq f(x), \forall\, x \in X \}. \tag{1}$$

2) The function f is called abstract convex with respect to H (or H-convex) if there exists a subset U of H such that

$$f(x) = \sup_{h \in U} h(x), \quad (x \in X). \tag{2}$$

3) The subdifferential of the function f at a point $x_0 \in dom f$ with respect to H (or H-subdifferential of f) is defined by

$$\partial_H f(x_0) := \{ h \in H : h(x_0) \in \mathbb{R},\ f(x) - f(x_0) \geq h(x) - h(x_0), \forall\, x \in X \}. \tag{3}$$

The set H in Definition 2.5 is called the set of elementary functions. It is worth noting that the support set accumulates global information about the function f in terms of elementary functions H.
Now, consider the function $l^k : X \times X \times \mathbb{R}_{++} \longrightarrow [0, +\infty]$ defined by

$$l^k(x, y, \alpha) := \max\{ 0 \leq \lambda \leq (\alpha)^k : \sqrt[k]{\lambda} y \leq x \}, \quad \forall\, x, y \in X, \forall\, \alpha > 0, \tag{4}$$

(with the convention $\max \emptyset := 0$).
In the following, we give some properties of this function.

Theorem 2.6. *for every $x,\ y,\ x',\ y' \in X;\ \gamma \in (0,1];\ \mu,\ \alpha,\ \alpha' \in \mathbb{R}_{++}$, one has*

$$l^k(\mu x, y, \alpha) = \mu^k l^k(x, y, \frac{\alpha}{\mu}), \tag{5}$$

$$l^k(x, \mu y, \alpha) = \frac{1}{\mu^k} l^k(x, y, \mu\alpha), \tag{6}$$

$$x \leq x' \Longrightarrow l^k(x, y, \alpha) \leq l^k(x', y, \alpha), \tag{7}$$

$$y \leq y' \Longrightarrow l^k(x, y', \alpha) \leq l^k(x, y, \alpha), \tag{8}$$

$$\alpha \leq \alpha' \Longrightarrow l^k(x, y, \alpha) \leq l^k(x, y, \alpha'), \tag{9}$$

$$l^k(\gamma x, y, \alpha) \geq \gamma^k l^k(x, y, \alpha), \tag{10}$$

$$l^k(x, \gamma y, \alpha) \leq \frac{1}{\gamma^k} l^k(x, y, \alpha), \tag{11}$$

$$l^k(x, y, \alpha) = \alpha^k \Longleftrightarrow \alpha y \leq x. \tag{12}$$

Theorem 2.7. *Let $f : X \to [0, +\infty]$ be a function. Then the following assertions are equivalent.*
(i) f is ICR-k.
(ii) $\lambda^k f(y) \leq f(x)$ for all $x,\ y \in X$ and all $\lambda \in (0,1]$ such that $\lambda y \leq x$.
(iii) $l^k(x, y, \alpha) f(\alpha y) \leq \alpha^k f(x)$ for all $x,\ y \in X$ and all $\alpha \in \mathbb{R}_{++}$ with the convention $0 \times (+\infty) = 0$.

3 Main results

Now, we are going to show that each non-negative ICR-k function is supremally generated by a certain class of ICR-k functions.
Assume that $y \in X$ and $\alpha \in \mathbb{R}_{++}$ are arbitrary. Consider the function $l^k_{(y,\alpha)} : X \longrightarrow [0, +\infty]$ defined by $l^k_{(y,\alpha)}(x) := l^k(x, y, \alpha)$ for all $x \in X$. Also, let $L := \{\, l^k_{(y,\alpha)} : y \in X,\ \alpha \in \mathbb{R}_{++} \,\}$ be the set of elementary functions.

Remark 3.1. By (7) and (10), the function $l^k_{(y,\alpha)}$ is an ICR-k function.

Theorem 3.2. *Let $f : X \longrightarrow [0, +\infty]$ be a function. Then f is ICR-k if and only if there exists a set $A \subseteq L$ such that*

$$f(x) = \sup_{l^k_{(y,\alpha)} \in A} l^k_{(y,\alpha)}(x), \quad (x \in X). \tag{13}$$

In this case, one can take $A := \{\, l^k_{(y,\alpha)} \in L : f(\alpha y) \geq \alpha^k \,\}$. Hence, f is ICR-k if and only if f is L-convex.

Theorem 3.3. *Let $f : X \longrightarrow [0, +\infty]$ be an ICR-k function. Then*

$$supp(f, L) = \{\, l^k_{(y,\alpha)} \in L : f(\alpha y) \geq \alpha^k \,\}. \tag{14}$$

In the following, we characterize the L-subdifferential of a non-negative ICR-k function.

Proposition 3.4. *Let $f : X \longrightarrow [0, +\infty]$ be an ICR-k function and $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$. Then*

$$\{\, l^k_{(y,\alpha)} \in L : f(\alpha y) \geq \alpha^k,\; f(x_0) = l^k_{(y,\alpha)}(x_0) \,\} \subseteq \partial_L f(x_0). \tag{15}$$

Moreover, $\partial_L f(x_0) \neq \emptyset$.

Theorem 3.5. *Let $f : X \longrightarrow [0, +\infty]$ be an ICR-k function and $x_0 \in X$ be such that $f(x_0) \neq +\infty$. Then*

$$\{\, l^k_{(y,\alpha)} \in L : f(x_0) \leq l^k_{(y,\alpha)}(x_0),\; \alpha^k - l^k_{(y,\alpha)}(x_0) \leq f(\alpha y) - f(x_0) \,\} \subseteq \partial_L f(x_0). \tag{16}$$

Moreover, the equality holds if and only if $\inf_{x \in X} f(x) = 0$.

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Amenability of Vector Valued Group algebras

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Abstract

Generalizing the notion of amenability for $L^1(G)$, we study the concept of amenability of $L^1(G,A)$. Among the other things, we prove that $L^1(G,A)$ is approximately weakly amenable where A is a unital separable Banach algebra. We investigate the existence of a left invariant mean on various vector valued function spaces. The candidates for the choice of space are $LUC(G,A^*)$, $WAP(G,A^*)$ and $C_0(G,A^*)$.

Keywords: Amenability, Banach algebras, Derivation, Group algebra, Invariant mean.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

It is a well-known theorem of Johnson that a locally compact group G is amenable if and only if $L^1(G)$ is amenable. We now switch from groups to vector-valued Banach algebras. Our references for vector-valued integration theory is [1], [2]. Let G be a locally compact group with a fixed left Haar measure m and A be a unital separable Banach algebra. Let $L^1(G,A)$ be the set of all measurable vector-valued (equivalence classes of) functions $f : G \to A$ such that $\|f\|_1 = \int_G \|f(t)\|dm(t) < \infty$. Equipped with the norm $\|.\|_1$ and the convolution product * specified by

$$f * g(x) = \int f(t)g(t^{-1}x)dm(t) \ (f, g \in L^1(G,A)),$$

$L^1(G,A)$ is a Banach algebra. It is our objective in this paper to demonstrate the corresponding characterization of $L^1(G,A)$. $M(G,A)$ will denote the space of regular A-valued Borel measures of bounded variation on G. $L^1(G,A)$ is a closed two-sided ideal of $M(G,A)$.

Another space considered in this paper is $L^\infty(G,A^*)$, which consists of maps f of G into A^* that are scalarwise measurable and $N_\infty(\|f\|) = \text{loc ess sup}_{t\in G}(\|f(t)\|) < \infty$. The dual of $L^1(G,A)$ may be identified with $L^\infty(G,A^*)$ [2]. We show that every continuous derivation from $L^1(G,A)$ into $L^\infty(G,A^*)$ is approximately inner, that is, of the form

$$D(a) = \lim_\alpha (F_\alpha.a - a.F_\alpha)$$

for some $\{F_\alpha\}_{\alpha\in I} \in L^\infty(G,A^*)$.

*Speaker

As usual we write $C(G,A^*)$ for the bounded continuous functions from G into A^*, $C_0(G,A^*)$ for the continuous functions from G into A^* vanishing at infinity and $C_{00}(G,A^*)$ for the continuous functions from G into A^* with compact support under the norm $\|f\| = \sup_{t\in G}\|f(t)\|$. For $f \in L^\infty(G,A^*)$, set $L_xf(t) = f(xt)(x,\ t \in G)$. Then f is called left uniformly continuous, if the map $x \mapsto L_xf$ from G into $L^\infty(G,A^*)$ is continuous with respect to $N_\infty(\|f\|)$ on $L^\infty(G,A^*)$. The set of uniformly continuous functions is denoted by $LUC(G,A^*)$. A function $f \in C(G,A^*)$ is called weakly almost periodic if the set $\{L_xf :\ x \in G\}$ is relatively compact in the weak-topology on $C(G,A^*)$. The space of these functions are denoted by $WAP(G,A^*)$. In the case $A = \mathbb{C}$, the complex field, these spaces will be denoted by $L^1(G)$, $M(G)$, $C(G)$, $C_0(G)$, $C_{00}(G)$, $LUC(G)$ and $WAP(G)$.

Left invariant means on spaces of vector-valued functions were first considered by Dixmier in [1]. A linear mapping $M : L^\infty(G,A^*) \to A^*$ is called a mean if for each f, $M(f)$ belongs to the weak*-closure of the convex hull of $\{f(x) :\ x \in G\}$ in A^*. A mean M is left invariant if $M(L_af) = M(f)$ for each $a \in G$ and $f \in L^\infty(G,A^*)$. If m is a left invariant mean on $L^\infty(G)$, then m induces a left invariant mean M on $L^\infty(G,A^*)$ such that $\langle M(f),a\rangle = m(\langle f(.),a\rangle)$ for each $a \in A$, here $\langle f(.),a\rangle$ denotes the functions $x \mapsto \langle f(x),a\rangle$. We present some of the properties of left invariant means on $LUC(G,A^*)$, $WAP(G,A^*)$ and $C_0(G,A^*)$.

2 Main results

Theorem 2.1. *Let G be a locally compact group. Then G is amenable if and only if $L^1(G,A)$ is amenable for each unital separable Banach algebra A.*

Theorem 2.2. *Let G be a locally compact group. Then the following statements are equivalent:*

(i) G is amenable.

(ii) For every unital separable Banach algebra A, there exists a bounded net $\{\psi_\alpha\}_{\alpha\in I} \subseteq L^1(G,A)$ such that $\|\delta_x * \psi_\alpha - \psi_\alpha\|_1 \to 0$ whenever $x \in G$.

(iii) For every unital separable Banach algebra A, there exists a bounded net $\{\psi_\alpha\}_{\alpha\in I} \subseteq L^1(G,A)$ such that for every compact set $K \subseteq G$, $\|\psi * \psi_\alpha - \psi_\alpha\|_1 \to 0$ uniformly for all $\psi \in L^1(G,A)$ with $\int_{G\setminus K}\|\psi(t)\|dm(t) = 0$.

It is known that G is amenable if and only if $LUC(G)$ has a left invariant mean. It will be interesting to have a direct proof of this fact. We present a vector version of this characterization.

Theorem 2.3. *Let G be a locally compact group and A be a unital separable Banach algebra. The following statements are hold:*

(i) $L^\infty(G,A^*)L^1(G,A) = LUC(G,A^*)$.

(ii) G is amenable if and only if $LUC(G,A^*)$ has a left invariant mean.

Analogous to the scalar function case, we can easily obtain the following

Theorem 2.4. *Let G be a locally compact group and A be a unital separable Banach algebra. The following statements are hold:*

(i) If $f \in L^\infty(G, A^*)$, then $f \in WAP(L^1(G, A))$ if and only if $\{f\delta_x : \ x \in G\}$ is relatively weakly compact in $L^\infty(G, A^*)$.

(ii) $WAP(L^1(G, A)) = WAP(G, A^*)$.

(iii) $WAP(L^1(G, A))$ has a left invariant mean.

Theorem 2.5. *Let G be a noncompact amenable group and let $f \in C_0(G, A^*)$. If M is left invariant mean on $L^\infty(G, A^*)$, then $|M(f)| = 0$.*

It is known that if two Banach algebras A and B have bounded approximate identities $\{a_\alpha\}_\alpha$ and $\{b_\alpha\}_\alpha$, then $A\hat{\otimes}B$ has a bounded approximate identity $\{a_\alpha \otimes b_\beta\}_{(\alpha,\beta)}$ where $\hat{\otimes}$ denotes the completion of usual tensor product of Banach spaces with respect to the projective tensor norm. Let $\{e_\alpha\}_\alpha$ be a bounded approximate identity for $L^1(G)$ and e_A be an identity in A. Regarding $\{e_\alpha \otimes e_A\}_\alpha$ as an element in $(L^1(G)\hat{\otimes}A)^{**}$, and let $F \in (L^1(G)\hat{\otimes}A)^*$. Using exactly the same notation as in [3], we put $\langle(e_\alpha \otimes e_A), F\rangle = \int F d(e_\alpha \otimes e_A)$. Given a dual Banach space X^* and $F \in B(L^1(G), A; X^*)$, we define $\int F d(e_\alpha \otimes e_A) \in X^*$ by

$$\langle \textstyle\int F d(e_\alpha \otimes e_A), x\rangle = \int \langle F(f, a), x\rangle d(e_\alpha \otimes e_A)(f, a).$$

Theorem 2.6. *Let G be a locally compact group and let A be a unital separable Banach algebra. Then $L^1(G, A)$ is approximately weakly amenable.*

Proof. Consider a continuous derivation $D : L^1(G, A) \to L^1(G, A)^*$. It is well known that the space $L^1(G, A)$ is isometrically isomorphic to $L^1(G)\hat{\otimes}A$. Define $F : L^1(G) \times A \to L^1(G, A)^*$ by $F(f, a) = D(f \otimes a)$. Put $g_\alpha = \int F(f, a)d(e_\alpha \otimes e_A)(f, a)$. For each $F(f, a) \in L^1(G, A)^*$, its image under isometry onto $L^\infty(G, A^*)$ is a map whose values at $x \in G$ is $F(f, a)(x)$. Now put $\overline{F} : L^1(G) \times A \to A^*$ given by $\overline{F}(f, a) = F(f, a)(x)$, $f \in L^1(G)$, $a \in A$ and $x \in G$. So we can define $\int \overline{F}(f, a)d(e_\alpha \otimes e_A)(f, a) \in A^*$ by $\langle \int \overline{F}(f, a)d(e_\alpha \otimes e_A)(f, a), c\rangle = \int \langle \overline{F}(f, a), c\rangle d(e_\alpha \otimes e_A)(f, a)$ for each $c \in A$. Note that $x \mapsto g_\alpha(x) = \int F(f, a)(x)d(e_\alpha \otimes e_A)(f, a)$ is a scalarwise measurable function and $N_\infty(\|g_\alpha(x)\|) < \infty$ for each α. Then there is a map κ_{g_α} from $B(A, L^\infty(G))$ such that $\langle \kappa_{g_\alpha}(a), f\rangle = \int f(x)\langle g_\alpha(x), a\rangle dm(x)$ for each $f \in L^1(G)$ and $a \in A$, where κ_{g_α} is defined by $\kappa_{g_\alpha}(a) = \langle g_\alpha(x), a\rangle$ [2]. For each $F : L^1(G) \times A \to L^1(G, A)^*$ and $f, \ g \in L^1(G)$ and $a, \ b \in A$ we have

$$\begin{aligned}
\lim_\alpha \int F(fg, ab)d(e_\alpha \otimes e_A)(f, a) &= \lim_\alpha \langle \int (fg \otimes ab)d(e_\alpha \otimes e_A)(f, a), F\rangle \\
&= \lim_\alpha \langle \int (gf \otimes ba)d(e_\alpha \otimes e_A)(f, a), F\rangle \\
&= \lim_\alpha \int F(gf, ba)d(e_\alpha \otimes e_A)(f, a).
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_\alpha (g\otimes b)\langle \acute{f}, \kappa_{g_\alpha}(\acute{a})\rangle &= \lim_\alpha \int \acute{f}(x)\langle (g\otimes b).g_\alpha(x), \acute{a}\rangle dm(x)\\
&= \lim_\alpha \int \acute{f}(x)\langle \int (g\otimes b).D(f\otimes a)(x) d(e_\alpha\otimes e_A)(f,a), \acute{a}\rangle dm(x)\\
&= \lim_\alpha \int \acute{f}(x)\langle \int D(gf\otimes ba)(x) d(e_\alpha\otimes e_A)(f,a), \acute{a}\rangle dm(x)\\
&- \lim_\alpha \int \acute{f}(x)\langle \int D(g\otimes b).(f\otimes a)(x) d(e_\alpha\otimes e_A)(f,a), \acute{a}\rangle dm(x)\\
&= \int \acute{f}(x) \lim_\alpha \langle \int F(gf, ba)(x) d(e_\alpha\otimes e_A)(f,a), \acute{a}\rangle dm(x)\\
&- \int \acute{f}(x) \lim_\alpha \langle D(g\otimes b) \int (f\otimes a)(x) d(e_\alpha\otimes e_A)(f,a), \acute{a}\rangle dm(x)\\
&= \int \acute{f}(x) \lim_\alpha \langle \int F(fg, ab)(x) d(e_\alpha\otimes e_A)(f,a), \acute{a}\rangle dm(x)\\
&- \int \acute{f}(x)\langle D(g\otimes b)(x), \acute{a}\rangle dm(x)\\
&= \int \acute{f}(x) \lim_\alpha \langle \int F(f, a)(x) d(e_\alpha\otimes e_A)(f,a), \acute{a}\rangle dm(x)(g\otimes b)\\
&- \int \acute{f}(x)\langle D(g\otimes b)(x), \acute{a}\rangle dm(x)\\
&= \lim_\alpha \int \acute{f}(x)\langle g_\alpha(x), \acute{a}\rangle dm(x)(g\otimes b) - \int \acute{f}(x)\langle D(g\otimes b)(x), \acute{a}\rangle dm(x)\\
&= \lim_\alpha \langle \acute{f}, \kappa_{g_\alpha}(\acute{a})\rangle (g\otimes b) - \langle \acute{f}, \kappa_{D(g\otimes b)}(\acute{a})\rangle
\end{aligned}$$

for all $g\otimes b \in L^1(G)\otimes A$, $\acute{a}\in A$ and $\acute{f}\in L^1(G,A)$. Consequently

$$\begin{aligned}
\lim_\alpha ((g\otimes b)\kappa_{g_\alpha}(\acute{a}) - \kappa_{g_\alpha}(\acute{a})(g\otimes b)) &= -\kappa_{D(g\otimes b)}(\acute{a})\\
\lim_\alpha \langle (g\otimes b).g_\alpha(x), \acute{a}\rangle - \langle g_\alpha(x).(g\otimes b), \acute{a}\rangle &= -\langle D(g\otimes b)(x), \acute{a}\rangle
\end{aligned}$$

for all $g\otimes b \in L^1(G)\otimes A$ and $\acute{a}\in A$. It follows that D is inner. □

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Amenability of weighted semigroup algebras based on a character

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Abstract

In this paper, we study ϕ-amenability and character amenability of weighted semigroup algebra $\ell^1(S,\omega)$. Indeed, we characterize character amenability of weighted semigroup algebras with a zero element. As an application, we give a characterization of character amenability of weighted Brandt semigroup algebras.

Keywords: Semigroup algebras, weight, character amenability
Mathematics Subject Classification [2010]: 43A20, 20M18, 16E40.

1 Introduction

Let A be a Banach algebra and E is a Banach A-bimodule. We regards the dual space E' as a Banach A-bimodule with the following module actions:

$$(a \cdot f)(x) = f(x \cdot a) \quad , \quad (f \cdot a)(x) = f(a \cdot x) \qquad (a \in A, f \in E', x \in E).$$

Kaniuth, Lau and Pym have introduced and studied in [6] and [7] the notion of ϕ-amenability for Banach algebras, where $\phi : A \longrightarrow \mathbb{C}$ is a character. M. S. Monfared in [8] introduced and investigated the notion of character amenability for Banach algebras. Let $\Delta(A)$ be the set of all characters of the Banach algebra A, and let $\phi \in \Delta(A)$. A Banach algebra A is called left ϕ-amenable if for all Banach A-bimodules E for which the right module action is given by

$$x \cdot a = \phi(a)x \qquad (x \in E, a \in A),$$

every continuous derivation $D : A \longrightarrow E'$ is inner. We say that A is left character amenable if A is left ϕ-amenable for all $\phi \in \Delta(A)$ and has a bounded left approximate identity. Similarly, the right and two-sided version of ϕ-amenability and character amenability can be defined. These notions have been studied for various classes of Banach algebras. For more details see, [6], [7], [8].

Recently in [5], the authors studied the notions of ϕ-amenability and character amenability for the semigroup algebra $\ell^1(S)$, where S is a semilattice. Also, they characterized the character amenability of $\ell^1(S)$, where S is a uniformly locally finite inverse semigroup. As

*Speaker

a consequence, they characterized the character amenability of $\ell^1(S)$ for a Brandt semigroup $S = \mathcal{M}^0(G, I)$. For more detailes about semigroup algebras see, [1], [2], [3], [4] and [9].

In this paper, for some classes of semigroups S we study ϕ-amenability and character amenability of weighted semigroup algebra $\ell^1(S,\omega)$. Indeed, we characterize character amenability of weighted semigroup algebras with a zero element. As an application, we give a characterization of character amenability of weighted Brandt semigroup algebras.

2 Main results

First, we establish some notations and define some concepts.

Let S be a semigroup. A weight on S is a function $\omega : S \longrightarrow (0,\infty)$ such that for all $s,t \in S$

$$\omega(st) \leq \omega(s)\omega(t).$$

Now, let S be a semigroup and $\omega : S \longrightarrow (0,\infty)$ be a weight. Then

$$\ell^1(S,\omega) = \{f : S \longrightarrow \mathbb{C} : \|f\|_\omega = \sum_{s\in S} f(s)\omega(s) < \infty\},$$

with $\|\cdot\|_w$ as the norm and the convolution product, specified by the requirement that

$$\delta_s * \delta_t = \delta_{st} \qquad (s,t \in S),$$

is a Banach algebra which is called weighted semigroup algebra.

Let S be a semigroup and $\omega : S \longrightarrow (0,\infty)$ be a weight on S. Denotes by $\widehat{S_\omega}$ the set of all non-zero homomorphism $\phi : S \longrightarrow \mathbb{C}$ such that

$$|\phi(s)| \leq \omega(s) \qquad (s \in S).$$

In the sequel, we characterize character space of weighted semigroup algebras.

Theorem 2.1. *Let S be a semigroup and ω be a weight on S. Then we have*

$$\Delta(\ell^1(S,\omega)) \cong \widehat{S_\omega}.$$

Proof. Define the map $\Psi : \Delta(\ell^1(S,\omega)) \longrightarrow \widehat{S_\omega}$ by

$$\Psi(\phi)(s) = \phi(\delta_s) \qquad (\phi \in \Delta(\ell^1(S,\omega)), s \in S).$$

First, Ψ is well-defined because for each $s \in S$

$$|\Psi(\phi)(s)| = |\phi(\delta_s)| \leq \|\phi\|\|\delta_s\|_\omega = \omega(s).$$

Moreover, it is easy to see that Ψ is a bijection. □

Theorem 2.2. *Let S be a semilattice and ω be a weight on S. Then we have*

$$\Delta(\ell^1(S,\omega)) = \Delta(\ell^1(S)).$$

Proof. It follows by Theorem 2.1. □

Theorem 2.3. *Let S be a semigroup and ω be a weight on S.*

(*i*) *If $\omega \geq 1$ and $\ell^1(S,\omega)$ is character amenable, then $\ell^1(S)$ is character amenable.*

(*ii*) *If $\omega \leq 1$ and $\ell^1(S)$ is character amenable, then $\ell^1(S,\omega)$ is character amenable.*

In the following theorem, we characterize characetr amenability of weighted semigroup algebras with a zero element.

Theorem 2.4. *Let S be a semigroup with a zero element and ω be a weight on S. If $\ell^1(S,\omega)$ is character amenable, then $\ell^1(S)$ is character amenable.*

Let G be a group and let I be a non-empty set. Set

$$\mathcal{M}^0(G,I) = \{(g)_{ij} : g \in G, i,j \in I\} \cup \{0\},$$

where $(g)_{ij}$ denotes the $I \times I$-matrix with entry $g \in G$ in the (i,j) position and zero elsewhere. Then $\mathcal{M}^0(G,I)$ with the multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases} \qquad (g,h \in G,\ i,j,k,l \in I),$$

is an inverse semigroup with $(g)^*_{ij} = (g^{-1})_{ji}$, that is called the *Brandt semigroup* over G with index set I.

In the following, we give a characterization of character amenability of weighted Brandt semigroup algebras.

Corollary 2.5. *Let $S = \mathcal{M}^0(G,I)$ be the Brandt semigroup and ω be a weight on S. Then the following are equivalent:*

(*i*) *$\ell^1(S,\omega)$ is character amenable.*

(*ii*) *$\ell^1(S)$ is character amenable.*

(*iii*) *I is finite and in the case where $|I| = 1$ then G is amenable.*

Proof. It follows by applying Theorem 2.4 and [5, Corollary 2.7]. □

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AN ITERATIVE METHOD FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract

In this paper, with a different iterative method for finding a common fixed point of a countable nonexpansive mappings a strong convergence theorem for a countable family of nonexpansive mappings in a Hilbert space is given. This theorem complete some recent results.

Keywords: Fixed points; Nonexpansive mapping; Iterative method; Variational inequality; Hilbert space.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Moudafi introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 0, \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.
Xu proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1) converges strongly to the unique solution x^* in $Fix(T)$ of the variational inequality:

$$\langle (I - f)x^*, x^* - x\rangle \leq 0, \forall x \in Fix(T). \tag{2}$$

We know iterative methods for nonexpansive mappings can be used to solve a convex minimization problem. See, e.g., [4, 5] and references therein. A typical problem is that of minimizing a quadratic function on the set of the fixed points of nonexpansive mapping on a real Hilbert space

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, a\rangle, \tag{3}$$

*Speaker

where a is a given point in H.
Yamada introduced the following hybrid iterative method for solving the variational inequality

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), n \geq 0, \tag{4}$$

where F is k-Lipschitzian and η-strongly monotone operator with $k, \eta > 0$ and $0 < \mu < 2\eta/k^2$. Let a sequence $\{\lambda_n\}$ in (0, 1) satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (4) converges strongly to the unique solution of the variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \forall x \in Fix(T).$$

Tian [3] combined the following iterative method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, n \geq 0, \tag{5}$$

with the Yamada's method (4) and considered the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, n \geq 0. \tag{6}$$

He proved, if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution $x^* \in Fix(T)$ of the variational inequality

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \forall x \in Fix(T).$$

In this article, under different conditions on γ, and the weaker conditions on f, we prove the strong convergence theorem for a countable family of nonexpansive mappings in a Hilbert space.
On the other hand, if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n S_n y_n, n \geq 1,$$

by means of the technique of measures of noncompactness, converge strongly to $q \in Fix(S)$ which is a solution of the following variational inequality

$$\langle (\mu F - \gamma f)q, q - z \rangle \leq 0,$$

for $\gamma < 0$ and all $z \in Fix(S)$.

2 Preliminaries and Main results

Let E be a Banach space. For a bounded subset $C \subset E$, let

$$\alpha_E(C) = \inf\{\delta > 0 | \exists n : C_i \subset C, C \subseteq \bigcup_i^n C_i, diam(C_i) \leq \delta\}$$

denote the (Kuratowskii) measure of non-compactness, where diam (C_i) denotes the diameter of C_i. Let X, Y be two Banach spaces and Ω be a subset of X. A continuous and bounded map $N : \Omega \to Y$ is k-set contractive if for any bounded set $C \subset \Omega$ we have $\alpha_Y(N(C)) \leq k\alpha_X(C)$. Also, N is strictly k-set contractive if N is k-set contractive and $\alpha_Y(N(C)) < k\alpha_X(C)$ for all bounded sets $A \subset \Omega$ with $\alpha_X(C) \neq 0$. N is a condensing map if N is strictly 1-set contractive.

Theorem 2.1. *[4] Let $\Omega \subset E$ be a bounded open subset and $N : \overline{\Omega} \to E$ is a condensing map and Krasnoselskii condition is satisfied:*
Let H be a Hilbert space, $\theta \in \Omega, \langle Nx, x\rangle \leq \|x\|^2$ for every $x \in \partial\Omega$, then N has at least one fixed point in $\overline{\Omega}$.

Let Ω be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in Ω, denoted by $P_\Omega(x)$, such that

$$\|x - P_\Omega(x)\| \leq \|x - y\|,$$

for all $y \in \Omega$.

Theorem 2.2. *[2] Let H be a real Hilbert space and suppose H. Let $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on H which satisfies $\bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $0 < a < 1$ and F a k-Lipschitzian and η-strongly monotone operator on H with $k, \eta > 0$. Let $0 < \mu < 2\eta/k^2, 0 < \gamma a < \tau = \mu(\eta - \frac{\mu k^2}{2})$ and $\tau < 1$. Define a sequence $\{x_n\} \subset H$ as follows:*
$x_1 = x \in H$ and

$$\begin{aligned} y_n &= \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) S_n x_n, \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n S_n y_n, \quad for \quad n \in \mathbb{N}, \end{aligned} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:

(I) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(II) $\lim_{n\to\infty} \beta_n = 0$ *or* $\beta_n \in [0, b)$ *for some* $b \in (0, 1)$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(III) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Suppose $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ for any bounded subset K of H. Let S be a mapping of H into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in H$ and suppose $Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n)$. Then the sequences $\{x_n\}$ defined by (2.3) converge strongly to $q \in Fix(S)$ which is a unique solution of the following variational inequality

$$\langle(\mu F - \gamma f)q, q - z\rangle \leq 0, \quad \forall z \in Fix(S).$$

Theorem 2.3. *Let H be a Hilbert space. Let $\{S_n\}_{n=1}^{\infty}$ be a family of nonexpansive self-mappings on H which satisfies $\bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$. Let f be a a-Lipschitzian mapping of H into itself and F a k-Lipschitzian and η-strongly monotone operator on H with $k, \eta > 0$. Let $0 < \mu < 2\eta/k^2, -1 - \gamma a < \tau = \mu(\eta - \frac{\mu k^2}{2}) < -\gamma a$ for $\gamma < 0$. Define a sequence $\{x_n\} \subset H$ as follows:*
$x_1 = x \in H$ and

$$\begin{cases} y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) S_n x_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S_n y_n, \qquad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:

(I) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(II) $\lim_{n\to\infty}\beta_n = 0$ *or* $\beta_n \in [0,b)$ *for some* $b \in (0,1)$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;

(III) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *or* $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$.

Let $Q = P_{\bigcap_{n=1}^{\infty}Fix(S_n)}$, $Q(I-\mu F+\gamma f)(x)$ *be condensing mapping from* $\overline{K}$ *to* H *and* $\langle Q(I-\mu F+\gamma f)(x), x\rangle \leq \|x\|^2, \forall x \in \partial K$ *for any open bounded subset* K *of* H *where* $\theta \in K$. *Suppose* $\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_n z\| : z \in \overline{K}\} < \infty$. *Let* S *be a mapping of* H *into itself defined by* $Sz = \lim_{n\to\infty}S_n z$ *for all* $z \in H$ *and suppose* $Fix(S) = \bigcap_{n=1}^{\infty}Fix(S_n)$. *Then the sequences* $\{x_n\}$ *defined by (2.3) converge strongly to* $q \in Fix(S)$ *which is a solution of the following variational inequality*

$$\langle(\mu F - \gamma f)q, q - z\rangle \leq 0, \forall z \in Fix(S).$$

Taking $F = I, \mu = 1, \gamma = -1$ in Theorem 2.3, we get

Corollary 2.4. *We have* $\{x_n\}$ *generated by*

$$\begin{cases} y_n = -\alpha_n f(x_n) + (1-\alpha_n)S_n x_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S_n y_n, \qquad n \geq 1, \end{cases}$$

converges strongly to $q \in Fix(S)$ *which solves the variational inequality* $\langle(I+f)q, q-z\rangle \leq 0$, *for all* $z \in Fix(S)$.

Remark 2.5. When $\gamma > 0$, Theorem 3.1 in [2], cannot help us to finding a fixed point, since $1-(\tau+\gamma a)$ be constant of Lipschitzian in proof of that theorem, and then $Q(I-\mu F+\gamma f)$ cannot be contraction.
The point of this paper is that we replace the parameter of $\gamma > 0$ by condition of $\gamma < 0$ and derive some new results, which complete the corresponding results of [2].

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Best approximation in normed left modules

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Abstract

We introduce a generalized notion of best approximation and also investigate some basic properties of this notion. Some illustrative examples are presented.

Keywords: $A-$best approximation, $A-$proximinal subset, $A-$Chebyshev subset, normed left module.
Mathematics Subject Classification [2010]: 41A50, 46B99

1 Introduction

Suppose that Y is a normed vector space and K is a non-empty subset of Y. An element $k_0 \in K$ is said to be a best approximation for $y \in Y$, if

$$\|y - k_0\| = d(y, K) = inf\{ \ \|y - k\| \mid k \in K \ \}.$$

The set of all best approximations of y in K is denoted by $P_K(y)$. One can easily cheque that if K is closed, then so is $P_K(y)$. The non-empty subset K of Y is said to be proximinal if $P_K(y) \neq \emptyset$ for all $y \in Y$. Also K is said to be Chebyshev, if each point $y \in Y$ has a unique best approximation in K. For the basic results concerning the theory of best approximation, the reader can refer to [1, 3].

Our purpose in this paper is to introduce the module best approximation of the elements of a normed left module and also its module proximinal and module Chebyshev subsets. Also we prove some basic results concerning module best approximation.

For this end we introduce some terminologies. Let A be a non-zero normed algebra, X be a normed left $A-$module and W be a non-empty subset of X. For an element $x \in X$, we say that an element $w_0 \in W$ is an $A-$best approximation for x, if there exists an element $0 \neq a \in A$ such that $ax = x$ and $\|x - aw_0\| = d(x, aW)$. We denote by $(AP)_W(x)$, the set of all $A-$best approximations of $x \in X$ in W. Also we say that W is $A-$proximinal if $(AP)_W(x) \neq \emptyset$ for all $x \in X$, and it is $A-$Chebyshev if each point $x \in X$ has a unique $A-$best approximation in W. The basic properties of the module best approximation in normed left modules are investigated in [2].

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2 Main Results

In this section we introduce a generalized notion of best approximation, that is completely compatible with the previous notion.
From now on, A is a non-zero normed algebra, X is a normed left $A-$module and W is a non-empty subset of X.

Definition 2.1. Let A be a non-zero normed algebra, X be a normed left $A-$module and W be a non-empty subset of X. For an element $x \in X$, we say that an element $w_0 \in W$ is an $A-$best approximation for x, if there exists an element $0 \neq a \in A$ such that $ax = x$ and $\|x - aw_0\| = d(x, aW)$. We denote by $(AP)_W(x)$, the set of all $A-$best approximations of $x \in X$ in W. Also we say that W is $A-$proximinal if $(AP)_W(x) \neq \emptyset$ for all $x \in X$, and it is $A-$Chebyshev if the set $(AP)_W(x)$ is a singleton set for all $x \in X$.

Remark 2.2. This definition is coincide with the usual definition in the best approximation theory. Indeed, let X be a normed vector space and for each $\lambda \in \mathbb{C}$ and $x \in X$ define $\lambda \cdot x = \lambda x$. Clearly with this action, X is a normed left $\mathbb{C}-$module. Let W be a non-empty subset of X, $x \in X$ and w_0 be a $\mathbb{C}-$best approximation of x in W. Then there exists $0 \neq \lambda \in \mathbb{C}$ such that $\lambda x = x$ and $\|x - \lambda w_0\| = d(x, \lambda W)$. If $x = 0$ then $|\lambda|\|w_0\| = |\lambda| d(0, W)$ that implies $\|0 - w_0\| = d(0, W)$. Also in the case where $x \neq 0$ we have $\lambda = 1$ and $\|x - w_0\| = d(x, W)$. It follows that w_0 is a best approximation of x in W. So we can claim that the usual definition of best approximation is a special case of our definition.
Similarly, one can verify that for each normed vector space X with the trivial action $\lambda \cdot x = \lambda x$, the notions of $\mathbb{C}-$proximinality and $\mathbb{C}-$Chebyshevity implies proximinality and Chebyshevity in the usual sense. As one can define a variety of left module actions on a normed vector space, the investigation on the notion of module best approximation, is worthy of consideration.

Example 2.3. Let A be a non-zero normed algebra and X be a normed vector space. For each $a \in A$ and $x \in X$ define $a \cdot x = 0$. One can easily verify that the action " $\cdot$ " turn X into a normed left $A-$module. In this case for every non-empty subset W of X, $(AP)_W(0) = W$ and if $x \neq 0$, $(AP)_W(x) = \emptyset$. This shows that in the case where $X \neq \{0\}$ there is no non-empty $A-$proximinal subset of X.

Let X be a normed vector space and W be a non-empty closed subset of X. It is obvious that X is a faithful normed left $\mathbb{C}-$module with the trivial action. In this case it is well-known that for each $x \in X$, $P_W(x)$ is closed. We conclude a similar result with a mild condition.

Theorem 2.4. *Let A be a non-zero normed algebra, X be a faithful normed left $A-$module and W be a non-empty closed subset of X. Then for each $0 \neq x \in X$, $(AP)_W(x)$ is closed.*

Remark 2.5. We don't know whether the previous theorem is correct in the case where $x = 0$. So in this case we need a new condition.

Theorem 2.6. *Let A be a non-zero normed algebra and X be a normed left $A-$module such that $\|ax\| = \|a\|\|x\|$, $(a \in A,\ x \in X)$. Then for every non-empty closed subset W, the set $(AP)_W(0)$ is closed.*

Let X be a normed vector space and K be a non-empty compact subset of X. It is well-known that for each $x \in X$, $p_K(x) \neq \emptyset$. We extend this result on normed left modules.

Theorem 2.7. *Let A be a non-zero normed algebra, X be a normed left $A-$module and K be a non-empty compact subset of X. Also let $x \in X$ be an element such that there exists $a \in A$ such that $ax = x$. Then $(AP)_W(x) \neq \emptyset$.*

We recall that for a unital normed algebra A, the normed left $A-$module X is unital, if $1_A x = x$ for all $x \in X$.

Corollary 2.8. *Let A be a unital normed algebra, X be a unital normed left $A-$module and K be a non-empty compact subset of X then K is $A-$proximinal.*

Note that because $\mathbb{C}$ is a unital normed algebra and every normed vector space X with the module action $\lambda \cdot x = \lambda x$, is a unital normed left $\mathbb{C}-$module then by applying the previous corollary, every non-empty compact subset of X is proximinal.

In the case where X is a normed vector space it is well-known that each singleton subset of X is Chebyshev. We extend this result on normed left modules.

Proposition 2.9. *Let A be a non-zero normed algebra and X be a normed left $A-$module such that for every $x \in X$ there exists $a \in A$ such that $ax = x$. Then every singleton subset of X is $A-$Chebyshev.*

Corollary 2.10. *Let A be a unital normed algebra and X be a unital normed left $A-$module then every singleton subset of X is $A-$Chebyshev.*

Theorem 2.11. *Let A be a non-zero normed algebra and X be a normed left $A-$module. Also let $x \in X$ and W be a non-empty subset of X such that $(AP)_W(x) \neq \emptyset$. Then,*

$$(AP)_{(AP)_W(x)}(x) = (AP)_W(x).$$

In the sequel we conclude some results. Let A be a non-zero normed algebra and X be a normed left $A-$module. Also let $0 \neq x \in X$ and W be a non-empty subset of X such that $w_0 \in (AP)_W(x)$. Set

$$I_{w_0}(x) = \{a \in A \mid \ ax = x, \ \|x - aw_0\| = d(x, aW)\},$$

so we have the following results. Note that in the case where X is faithful, $I_{w_0}(x)$ has precisely one element.

Proposition 2.12. *Let A be a non-zero normed algebra and X be a normed left $A-$module. Also let $0 \neq x \in X$ and W be a non-empty subset of X such that $w_0 \in (AP)_W(x)$. Then $I_{w_0}(x)$ is non-empty and closed. In particular, if $W = \{w_0\}$ then $I_{w_0}(x) = \{ a \in A \mid ax = x\}$ is a non-empty closed subset of A.*

Proposition 2.13. *Let A be a non-zero normed algebra and X be a faithful normed left $A-$module. Also let $0 \neq x \in X$ and W be a non-empty subset of X such that $(AP)_W(x) \neq \emptyset$. Then,*

$$\cap_{w_0 \in (AP)_W(x)} \ I_{w_0}(x) = \{ \ a \in A \mid ax = x, \ d(x, a(AP)_W(x)) = d(x, aW) \ \}.$$

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Best proximity points for cyclic generalized contractions

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Abstract

In this paper we introduce cyclic generalized contraction maps and the theorems asked about it. Moreover, we obtain existence and convergence of best proximity point for this mappings in uniformly convex Banach space.

Keywords: Best proximity point, Cyclic map, Cyclic contraction, Cyclic generalized contraction map, Uniformly convex Banach space.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Let A and B be two nonempty subsets of a X. A map $T : A\cup B \to A\cup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. Let (X, d) be a metric space and $T : A \cup B \to A \cup B$ a cyclic map. For any two nonempty subsets A and B of X, let

$$d(A, B) = inf\{d(x, y) : x \in A \text{ and } y \in B\}.$$

A point $x \in A \cup B$ is called to be a best proximity point for T if

$$d(x, Tx) = d(A, B).$$

Throughout this paper. We denote by **N** and **R** the sets of positive integers and real numbers, respectively. Recently, the existence, uniqueness and convergence of iterates to the best proximity point were investigated by many authors; see [1-5,8-9]. In 2006, Eldred and Veeramani [4] first gave the concept of cyclic contraction as follows.

Definition 1.1. [4] Let A and B be nonempty subsets of a metric space (X, d). $T : A \cup B \to A \cup B$ is a cyclic contraction map if it satisfies
(1) $T(A) \subseteq B$ and $T(B) \subseteq A$.
(2) there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$$

for all $x \in A$, $y \in B$.

Example 1.2. [4] Given k in $(0, 1)$, let A and B be subsets of $L^p, 1 \leq p \leq \infty$, defined by $A = \{((1 + k^{2n})e_{2n}) : n \in N\}$ and $B = \{((1 + k^{2m-1})e_{2m-1}) : m \in N\}$. Suppose

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$$T((1+k^{2n})e_{2n}) = (1+k^{2n+1})e_{2n+1}$$

and

$$T((1+k^{2m-1})e_{2m-1}) = (1+k^{2m})e_{2m}.$$

Then T is a cyclic contraction on $A \cup B$.

In [3], Amini-Harandi and others introduced following new class of cyclic generalized contraction maps.

Definition 1.3. [3] Let A and B be nonempty subsets of a metric space (X, d). A map $T : A \cup B \to A \cup B$ is a cyclic generalized contraction map if $T(A) \subseteq B$, $T(B) \subseteq A$ and

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + (1 - \alpha(d(x, y))d(A, B)$$

for each $x \in A$ and $y \in B$, where $\alpha : [d(A, B), \infty) \to [0, 1)$ satisfies $limsup_{s \to t^+}\ \alpha(s) < 1$ for each $t \in (d(A, B), \infty)$.

Remark 1.4. If $\alpha(t) = k$ for each $t \in [d(A, B), \infty)$, where $k \in [0, 1)$ is constant, then T is a cyclic contraction.

Example 1.5. [3] Consider the uniformly convex Banach space $X = R^2$ with Euclidean metric. Let $A := \{(0, x) : 0 \leq x\}$ and $B := \{(2, y) : 0 \leq y\}$. Then A and B are nonempty closed and convex subsets of X and $d(A, B) = 2$.
Let $T : A \cup B \to A \cup B$ be defined as

$$T(0, x) = (2, \tfrac{x}{2}) \quad \text{and} \quad T(2, y) = (0, \tfrac{y}{2}) \quad \text{for each } x,y \geq 0.$$

Then T is a cyclic generalized contraction map with $\alpha(t) = \frac{1}{2}$ for $t \in [2, \infty)$.

Lemma 1.6. *[6] Let (X, d) be a complete metric space and let $T : X \to X$ be a map satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \text{for each } x, y \in X,$$

where $\alpha : [0, \infty) \to [0, 1)$ satisfies $limsup_{s \to t^+}\ \alpha(s) < 1$ for each $t \in (0, \infty)$. Then T has a unique fixed point.

2 Main results

In this section, we shall state and prove some results about existence and convergence of best proximity points for cyclic generalized contraction maps in uniformly convex Banach spaces.

Theorem 2.1. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic generalized contraction map. Also, let $x_0 \in A$ and sequence $\{x_n\}$ is generated by*

$$x_{n+1} = Tx_n \qquad \text{for each } n \in \mathbf{N}.$$

Then $\|x_n - x_{n+1}\| \to d(A, B)$.

Theorem 2.2. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic generalized contraction map. Also, let $x_0 \in A$ and sequence $\{x_n\}$ is generated by*

$$x_{n+1} = Tx_n \qquad \text{for each } n \in \mathbf{N}.$$

Then $\|x_{2n+2} - x_{2n}\| \to 0$ and $\|x_{2n+3} - x_{2n+1}\| \to 0$ as $n \to \infty$.

Theorem 2.3. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic generalized contraction map. Also, let $x_0 \in A$ and sequence $\{x_n\}$ is generated by*

$$x_{n+1} = Tx_n \qquad \text{for each } n \in \mathbf{N}.$$

Then, for each $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $m > n \geq N$,

$$\|x_{2m} - x_{2n+1}\| < d(A, B) + \epsilon.$$

Theorem 2.4. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic generalized contraction map. Also, let $x_0 \in A$ and sequence $\{x_n\}$ is generated by*

$$x_{n+1} = Tx_n \qquad \text{for each } n \in \mathbf{N}.$$

Then, $\{x_{2n}\}$ is Cauchy sequence.

Theorem 2.5. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic generalized contraction map. Also, let $x_0 \in A$ and sequence $\{x_n\}$ is generated by*

$$x_{n+1} = Tx_n \qquad \text{for each } n \in \mathbf{N}.$$

Then, there exists unique x in A such that $x_{2n} \to x$ and

$$\|x - Tx\| = d(A, B).$$

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Block matrix operators and p-paranormality

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Abstract

In this paper we introduce a new model of a block matrix operator $M(\gamma, \eta)$ induced by two sequences γ and η. Then by its corresponding composition operator C_T on $\ell^2_+ = L^2(\mathbb{N}_0)$ we characterize p-paranormality the block matrix operator $M(\gamma, \eta)$.

Keywords: p-paranormal operator, composition operator, conditional expectation.
Mathematics Subject Classification [2010]: 47B20, 47B38

1 Introduction

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ and let $T = U|T|$ be the canonical polar decomposition for $T \in \mathcal{L}(\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be p-paranormal if $\||T|^p U|T|^p x\| \geq \||T|^p x\|^2$, for all unit vectors $x \in \mathcal{H}$. By using the property of read quadratic forms T is p-paranormal operator if and only if for all integers $k \geq 0$, $|T|^p U^*|T|^{2p} U|T| - 2k|T|^{2p} + k^2 \geq 0$.

Let (X, Σ, μ) be a complete σ-finite measure space and let $T : X \to X$ be a transformation such that $T^{-1}(\Sigma) \subseteq \Sigma$ and $\mu \circ T^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ T^{-1}/d\mu$ is in $L^\infty(X)$. The composition operator C_T on $L^2(X)$ is defined by $C_T f = f \circ T$. The condition $h \in L^\infty(X)$ assures that C_T is bounded. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. In [3] Jabbarzadeh and Azimi characterize p-paranormality of C_T on $L^2(X)$. A key tool in [3] was the use of the conditional expectation operators for studying p-paranormality of C_T, and this will be the main tool of this note. For a sub-σ-finite algebra $T^{-1}(\Sigma) \subseteq \Sigma$, the conditional expectation operator associated with $T^{-1}(\Sigma)$ is the mapping $f \to E^{T^{-1}(\Sigma)} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma), 1 \leq p \leq \infty$, where $E^{T^{-1}(\Sigma)} f$, by Radon-Nikodym theorem, is the unique $T^{-1}(\Sigma)$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{T^{-1}(\Sigma)} f d\mu, \qquad \forall A \in T^{-1}(\Sigma).$$

Throughout this paper, we assume that $E^{T^{-1}(\Sigma)} = E$. For more details on the properties of the conditional expectation operators see [2, 4].

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In [1] Exner, Jung and Lee introduced the block matrix operator $M(\alpha, \beta)$ and charecterize its p-hyponormality. In section 2 we define a new block matrix operator $M(\gamma, \eta)$ induced by two sequences γ and η such that in the special case its corresponding operator on ℓ^2_+ has the shift operators form, then we obtain its corresponding composition operator C_T on $\ell^2_+ = L^2(\mathbb{N}_0)$ induced by a measurable transformation T on the set of nonnegative integers $\mathbb{N}_0$ with point mass measure. In section 3, we characterize block matrix operator $M(\gamma, \eta)$ for p-paranormality and construct a useful form for some examples.

2 Basic Definitions And Preliminaries

Let $\gamma := \{a_i^n\}_{\substack{1\le i\le r\\ 0\le n<\infty}}$ and $\eta := \{b_j^n\}_{\substack{1\le j\le s\\ 0\le n<\infty}}$ be bounded sequences of positive real numbers. Let $M(\gamma, \eta) := [A_{ij}]_{0\le i,j<\infty}$ be a block matrix operator whose blocks are $(r+s)\times(r+2)$ matrices such that $A_{ij} = 0, i \neq j$, and

$$A_n := A_{nn} = \begin{bmatrix} 0 & a_1^{(n)} & & O & \\ & & \ddots & & \\ & & & a_r^{(n)} & \\ & & & & b_1^{(n)} \\ & O & & & \vdots \\ & & & & b_s^{(n)} \end{bmatrix} \tag{1}$$

where other entries are 0 except a_*^n and b_*^n in (1).

Definition 2.1. For two bounded sequences $\gamma := \{a_i^n\}_{\substack{1\le i\le r\\ 0\le n<\infty}}$ and $\eta := \{b_j^n\}_{\substack{1\le j\le s\\ 0\le n<\infty}}$, the block matrix operator $M := M(\gamma, \eta)$ satisfying in (1) is called a block matrix operator with weight sequence (γ, η).

Let M be a block matrix operator with weight sequence (γ, η) and let $W_{\gamma,\eta}$ be its corresponding operator on ℓ^2_+ relative to some orthornormal basis. Then $W_{\gamma,\eta}$ may provide a repetitive form; for example $r = 2$, $s = 3$ and $a_i^{(n)} = b_j^{(n)} = 1$ for all $i, j, n \in \mathbb{N}$, then the block matrix operator with (γ, η) is unitarily equivalent to the following operator $W_{\gamma,\eta}$ on ℓ^2_+ defined by

$$W_{\gamma,\eta}(x_1, x_2, x_3, x_4.x_5, ...) = (x_2, x_3, x_4, x_4, x_4, x_5, ...).$$

Now we put $X = \mathbb{N}_0$ and the power set $\mathcal{P}(X)$ of X for the σ-algebra Σ. Define a non-singular measurable transformation T on $\mathbb{N}_0$ such that

$$T^{-1}(k(r+1)+r+1) = \{k(r+s)+i+r-1 : 1 \le i \le s\}, \quad k = 0, 1, 2, ..., \tag{2}$$
$$T^{-1}(k(r+1)+i) = k(r+s)+i-1, \quad 1 \le i \le r, \quad k = 0, 1, 2, ...$$

We write $m(\{i\}) := m_i, i \in \mathbb{N}_0$, for the underlying point mass measure on X, and we assume throughout that each m_i is strictly positive.

Proposition 2.2. *With the above notations the bounded composition operator C_T on ℓ^2_+ defined by $C_T f = f \circ T$ is unitarily equivalent to the block matrix operator $M(\gamma,\eta)$, where*

$$\gamma : a_i^{(n)} = \sqrt{\frac{m_{n(r+s)+i-1}}{m_{n(r+1)+i}}} \quad (1 \le i \le r), \quad \eta : b_j^{(n)} = \sqrt{\frac{m_{n(r+s)+j+r-1}}{m_{n(r+1)+r+1}}} \quad (1 \le j \le s),$$

for $n \in \mathbb{N}_0$.

Proposition 2.3. *Let $M(\gamma,\eta)$ be a block matrix operator with weight sequence (γ,η), where $\gamma := \{a_i^n\}_{\substack{1\le i\le r\\ 0\le n<\infty}}$ and $\eta := \{b_j^n\}_{\substack{1\le j\le s\\ 0\le n<\infty}}$. Then there exists a measurable transformation T on a σ-finite measure space $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0), m)$ such that $M(\gamma,\eta)$ is unitarily equivalent to the composition operator C_T on ℓ^2_+.*

3 The Main Results

Theorem 3.1. *Let T be a non-singular measurable transformation on ℓ^2_+ as in (2) and let $p \in (0,\infty)$. Then the following assertions are equivalent*

(i) C_T is p-paranormal on ℓ^2_+;

(ii) the block matrix operator $M(\gamma,\eta)$ as in proposition 2.2 is p-paranormal;

(iii) $h^p \circ T(n) \le E(h^p)(n)$, $n \in \mathbb{N}_0$, where $h = d\mu \circ T^{-1}/d\mu$;

(iv) the following inequality holds

$$\left(\frac{m(T^{-1}(T(n)))}{m_{T(n)}}\right)^p \le \frac{1}{m(T^{-1}(T(n)))} \sum_{l \in T^{-1}(T(n))} \frac{m(T^{-1}(l))^p}{m_l^p} m_l, \quad n \in \mathbb{N}_0. \tag{3}$$

The conditions above simplify considerably if we specialize to the case of a repeated block. Let M be a block matrix operator as follows:

$$\begin{aligned} M(\gamma,\eta) &: A \equiv A_1 = A_2 = ... \\ \gamma &: a_i^{(n)} = a_i, n \in \mathbb{N}_0, 1 \le i \le r; \\ \eta &: b_j^{(n)} = b_j, n \in \mathbb{N}_0, 1 \le j \le s. \end{aligned} \tag{4}$$

Theorem 3.2. *Let $M(\gamma,\eta)$ be as in (4). Then the block matrix operator $M(\gamma,\eta)$ is p-paranormal if and only if the following two conditions hold*

(i) if $n = k(r+s) + i + r - 1 \quad 1 \le i \le s$, then

$$\left(\sum_{1\le i\le s} b_i^2\right)^p \le \sum_{\substack{l\in T^{-1}(T(n))\\ l \equiv r+1 \mod(r+1)}} \left(\sum_{1\le i\le s} b_i^2\right)^p \left(\frac{b_{t_l}^2}{\sum_{1\le i\le s} b_i^2}\right) \tag{5}$$

$$+ \sum_{\substack{l\in T^{-1}(T(n))\\ l \equiv i_l \mod(r+1)}} (a_{i_l})^{2p} \left(\frac{b_{t_l}^2}{\sum_{1\le i\le s} b_i^2}\right), \quad (1 \le i_l \le r \quad \text{and} \quad 1 \le t_l \le s);$$

(ii) if $n = k(r+s) + m - 1$ *for* $1 \leq m \leq r$*, then*

$$(ii-a) \quad a_m^2 \leq \sum_{1\leq i\leq s} b_i^2 \qquad n \equiv r+1 \quad mod(r+1)$$

$$(ii-b) \quad a_m^2 \leq a_{t_n}^2 \qquad n \equiv t_n \quad mod(r+1) \quad (1 \leq t_n \leq r).$$

Corollary 3.3. *Assume that* $M(\gamma, \eta)$ *is as in 3.2, and* $GCD(r+s, r+1) = 1$*. Then* M *is p-paranormal for all* $p \in (0, \infty)$*.*

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C*-Algebras and Dynamical Systems, a Categorical Approach

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Abstract

There are interactions between C*-algebras, essentially minimal dynamical systems, ordered Bratteli diagrams, and dimension groups. We extend these interactions to encompass morphisms of these categories. We show that the category of essentially minimal dynamical systems is equivalent to the category of essentially simple ordered Bratteli diagrams. Especially, one can describe the factors of certain dynamical systems using a graphical approach. The functor K^0 is constructed to distinguish various types of orbit equivalence. Relations with crossed products of C*-algebras are studied.

Keywords: C*-algebra, ordered Bratteli diagram, essentially minimal system, category, dimension group
Mathematics Subject Classification [2010]: 46L05, 37B05, 37A20.

1 Introduction

In 1972, Bratteli in a seminal paper introduced what are now called Bratteli diagrams to study AF algebras [3]. He associated to each AF algebra an infinite directed graph, its Bratteli diagram, and used this very effectively to study AF algebras. In 1976, based on the notion of a Bratteli diagram, Elliott introduced dimension groups and gave a classification of AF algebras using K-theory [4]. In fact, he showed that the functor $K_0 : \mathbf{AF} \to \mathbf{DG}$, from the category of AF algebras to the category of dimension groups is a strong classification functor [4, 5].

In [1], the authors introduced an appropriate notion of morphism between Bratteli diagrams and obtained the category of Bratteli diagrams, **BD**, such that isomorphism of Bratteli diagrams in this category coincides with Bratteli's notion of equivalence. We showed that the map $\mathcal{B} : \mathbf{AF} \to \mathbf{BD}$, defined by Bratteli on objects, is in fact a functor. The fact that this is a strong classification functor [1, Theorem 3.11], is a functorial formulation of Bratteli's classification of AF algebras and completes his work from the classification point of view introduced by Elliott in [5].

In a different direction, Bratteli diagrams were used to study certain dynamical systems. In 1981, A. V. Versik used Bratteli diagrams to construct so-called adic transformations [8]. Based on his work, Herman, Putnam, and Skau introduced the notion of an ordered Bratteli diagram and associated a dynamical system to each (essentially simple) ordered Bratteli diagram [7]. They showed that there is a one-to-one correspondence between essentially simple ordered Bratteli diagrams and essentially minimal dynamical

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systems [7, Theorem 4.7]. This correspondence was used effectively to study Cantor minimal dynamical systems and characterization of various types of orbit equivalence in terms of isomorphism of related C*-crossed products and dimension groups [7, 6].

In this paper, we propose a notion of (ordered) morphism between ordered Bratteli diagrams and obtain the *category* of ordered Bratteli diagrams **OBD** (Theorem 2.1). The isomorphism in this category coincides with the notion of equivalence in the sense of Herman, Putnam, and Skau (Theorem 2.2). Then we show that the correspondence obtained by Herman, Putnam, and Skau in [7] is an equivalence of categories. Denote by **ODS** the category of ordered essentially minimal dynamical systems (see Definition 2.4). We construct the contravariant functor $\mathcal{P}$: **ODS** $\to$ **OBD**. Thus we obtain a kind of *diagram* for a homomorphism between essentially minimal dynamical systems. (This is in particular useful in the study of factors of such systems.)

We show that the contarvariant functor $\mathcal{P}$: **ODS** $\to$ **OBD** is full and faithful. The fact that this functor is full is a tool to obtain homomorphisms between dynamical systems in question by graphically constructing certain arrows (i.e., morphisms) between the associated Bratteli diagrams. The functor $\mathcal{P}$: **ODS** $\to$ $\mathbf{OBD}_{\mathrm{ess}}$ is an equivalence of categories (Theorem 2.6). We construct the inverse of the functor $\mathcal{P}$, i.e., the functor V : $\mathbf{OBD}_{\mathrm{ess}}$ $\to$ **ODS** which is also an equivalence of categories. Therefore, we obtain a functorial formulation of the correspondence between essentially simple ordered Bratteli diagrams and essentially minimal dynamical systems (Theorem 2.9).

Definition 1.1 ([7], Definition 2.1)**.** A *Bratteli diagram* consists of a vertex set V and an edge set E satisfying the following conditions. We have a decomposition of V as a disjoint union $V_0 \cup V_1 \cup \cdots$, where each V_n is finite and non-empty and V_0 has exactly one element, v_0. Similarly, E decomposes as a disjoint union $E_1 \cup E_2 \cup \cdots$, where each E_n is finite and non-empty. Moreover, we have maps $r, s : E \to V$ such that $r(E_n) \subseteq V_n$ and $s(E_n) \subseteq V_{n-1}$, $n = 1, 2, 3, \ldots$ (r = range, s = source). We also assume that $s^{-1}\{v\}$ is non-empty for all v in V and $r^{-1}\{v\}$ is non-empty for all v in $V \setminus V_0$.

We denote such a B by the diagram

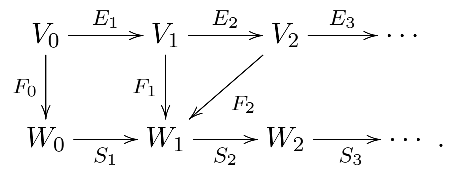
$$V_0 \xrightarrow{E_1} V_1 \xrightarrow{E_2} V_2 \xrightarrow{E_3} \cdots .$$

Definition 1.2 ([7], Definition 2.3)**.** An *ordered Bratteli diagram* is a Bratteli diagram (V, E) together with a partial order $\geq$ on E such that e and e' are comparable if and only if $r(e) = r(e')$. That is, we have a linear order on each set $r^{-1}\{v\}$, $v \in V \setminus V_0$.

Definition 1.3. Let $B = (V, E, \geq)$ and $C = (W, S, \geq)$ be ordered Bratteli diagrams. An *ordered premorphism* $f : B \to C$ is a triple $(F, (f_n)_{n=0}^{\infty}, \geq)$ where $(F, (f_n)_{n=0}^{\infty})$ is a premorphism (see [1, 2]) and $\geq$ is a partial order on F such that:

(1) $e, e' \in F$ are comparable if and only if $r(e) = r(e')$, and $\geq$ is a linear order on $r^{-1}\{v\}$, $v \in W$;

(2) the diagram of $f : B \to C$ commutes:



We define an equivalence relation on ordered premorphisms and we obtain ordered morphisms (see [2] for details).

2 Main Results

Theorem 2.1. *The class* **OBD** *with ordered morphisms, as defined above, is a category.*

Theorem 2.2. *A pair of ordered Bratteli diagrams are isomorphic in the category* **OBD** *with morphisms if, and only if, they are equivalent in the sense of Herman, Putnam, and Skau.*

We refer to **OBD** with ordered morphisms as defined above, as the *category of ordered Bratteli diagrams.*

Definition 2.3 ([7], Definition 1.2)**.** Let X be a compact, totally disconnected metrizable space. Let φ be a homeomorphism on X and $y \in X$. The triple (X, φ, y) is called an *essentially minimal dynamical system* if the dynamical system (X, φ) has a unique minimal set Y and y is in Y.

Definition 2.4. By an *ordered essentially minimal dynamical system* we mean a quadruple $(X, \varphi, y, \mathcal{R})$ where (X, φ, y) is an essentially minimal dynamical system and $\mathcal{R}$ is a system of Kakutani-Rohlin partitions for (X, φ, y). The *category of ordered essentially minimal dynamical systems* **ODS** is the category whose objects is the class of all essentially minimal dynamical systems and its morphism are as follows. Let $(X, \varphi, y, \mathcal{R})$ and $(Y, \psi, z, \mathcal{S})$ be in **ODS**. By a morphism $\alpha : (X, \varphi, y, \mathcal{R}) \to (Y, \psi, z, \mathcal{S})$ we mean a homomorphism from the dynamical system (X, φ) to (Y, ψ) (i.e., a continuous map with $\alpha \circ \varphi = \psi \circ \alpha$) such that $\varphi(y) = z$.

See [2] for the definition of the map $\mathcal{P} : \mathbf{ODS} \to \mathbf{OBD}$.

Theorem 2.5. *The map* $\mathcal{P} : \mathbf{ODS} \to \mathbf{OBD}$ *is a contravariant functor.*

Theorem 2.6. *The functor* $\mathcal{P} : \mathbf{ODS} \to \mathbf{OBD}_{\text{ess}}$ *is an equivalence of categories.*

Corollary 2.7. *Let* $(X, \varphi, y, \mathcal{R})$ *and* $(Y, \psi, z, \mathcal{S})$ *be in* **ODS**. *The following statements are equivalent:*

(1) (X, φ, y) *and* (Y, ψ, z) *are pointed topological conjugate;*

(2) the ordered Bratteli diagrams $\mathcal{P}(X, \varphi, y, \mathcal{R})$ *and* $\mathcal{P}(Y, \psi, z, \mathcal{S})$ *are equivalent;*

(3) $\mathcal{P}(X, \varphi, y, \mathcal{R}) \cong \mathcal{P}(Y, \psi, z, \mathcal{S})$ *in* **OBD**.

See [2] for the definition of the map $\mathrm{V} : \mathbf{OBD}_{\text{ess}} \to \mathbf{ODS}$.

Theorem 2.8. *The map* $\mathrm{V} : \mathbf{OBD}_{\text{ess}} \to \mathbf{ODS}$ *is a contravariant functor which is an equivalence of categories.*

See [2] for the definition of the correspondences σ an τ.

Theorem 2.9. *The functors* $\mathcal{P} : \mathbf{ODS} \to \mathbf{OBD}_{\text{ess}}$ *and* $\mathrm{V} : \mathbf{OBD}_{\text{ess}} \to \mathbf{ODS}$ *are equivalences of categories which are inverse of each other and* $\tau : 1_{\mathbf{OBD}_{\text{ess}}} \cong \mathcal{P}\mathrm{V}$ *and* $\sigma : 1_{\mathbf{ODS}} \cong \mathrm{V}\mathcal{P}$.

See [2] for the definition of the functors $\mathcal{AF} : \mathbf{DS} \to \mathbf{AF}$ and $\mathrm{K}^0 : \mathbf{DS} \to \mathbf{DG}$. That the first three statements in the following theorem are equivalent is a well-known result [6]. A minimal dynamical system (X, φ) is called a *Cantor system* if X is a compact metrizable space with a countable basis of clopen subsets and X has no isolated points.

Theorem 2.10. *Let* (X, φ) *and* (Y, ψ) *be Cantor systems. Let* y *and* z *be arbitrary points in* X *and* Y*, respectively. Then the following are equivalent:*

(1) (X, φ) *and* (Y, ψ) *are strong orbit equivalent;*

(2) $\mathrm{K}^0(X, \varphi)$ *is order isomorphic to* $\mathrm{K}^0(Y, \psi)$ *by a map preserving the distinguished ordered unit;*

(3) $\mathrm{C}(X) \rtimes_\varphi \mathbb{Z} \cong \mathrm{C}(Y) \rtimes_\psi \mathbb{Z}$;

(4) $\mathcal{AF}(X, \varphi, y) \cong \mathcal{AF}(Y, \psi, z)$ *in* $\mathbf{AF}$.

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C^*-algebras of Toeplitz and composition operators

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Abstract

We investigate the unital C^*-algebras generated by an irreducible Toeplitz operator T_ψ and one or more composition operators C_φ induced by linear-fractional self-maps φ of the unit disk acting on the Hardy space H^2, modulo the ideal of compact operators $K(H^2)$. For automorphism symbol φ, we compare this algebra with the one generated by the shift operator T_z and a composition operators.

Keywords: the unilateral shift operator, Toeplitz operator, composition operator, linear-fractional map, automorphism of the unit disk.
Mathematics Subject Classification [2010]: 47B33, 47B32

1 Introduction

The Hardy space $H^2 = H^2(\mathbb{D})$ is the collection of all analytic functions f on the open unit disk $\mathbb{D}$ satisfying the norm condition $\|f\|^2 := \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty$. For any analytic self-map φ of the open unit disk $\mathbb{D}$, a bounded composition operator on H^2 is defined by

$$C_\varphi : H^2 \to H^2, \quad C_\varphi(f) = f \circ \varphi.$$

If $f \in H^2$, then the radial limit $f(e^{i\theta}) := \lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere on the unit circle $\mathbb{T}$. Hence we can consider H^2 as a subspace of $L^2(\mathbb{T})$. Let ϕ is a bounded measurable function on $\mathbb{T}$ and P_{H^2} be the orthogonal projection of $L^2(\mathbb{T})$ (associated with normalized arc-length measure on $\mathbb{T}$) onto H^2. The Toeplitz operator T_ϕ is defined on H^2 by $T_\phi f = P_{H_2}(\phi f)$ for all $f \in H^2$. Coburn in [2] shows that the quotient of the unital C^*-algebra $C^*(T_z)$ generated by the unilateral shift operator T_z on the ideal of compact operators $\mathfrak{K} = K(H^2)$ is $*$-isomorphic to $C(\mathbb{T})$, and determines essential spectrum of Toeplitz operators with continuous symbol. Recently the unital C^*-algebra generated by the shift operator T_z and the composition operator C_φ for a linear-fractional self-map φ of $\mathbb{D}$ is studied. For a linear-fractional self-map φ on $\mathbb{D}$, if $\|\varphi\|_\infty < 1$ then C_φ is a compact operator on H^2. Therefore we consider those linear-fractional self-maps φ which satisfy $\|\varphi\|_\infty = 1$. If moreover φ is an automorphism of $\mathbb{D}$, then $C^*(T_z, C_\varphi)/\mathfrak{K}$ is $*$-isomorphic to the crossed products $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ or $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}_n$ [4]. When φ is not an automorphism there are three different cases:

(i) φ has only one fixed point γ which is on the unit circle $\mathbb{T}$ (i.e. φ is a parabolic map) [7]. In this case, $C^*(T_z, C_\varphi)/\mathfrak{K}$ is a commutative C^*-algebra isomorphic to $C_\gamma(\mathbb{T}) \oplus C_0([0,1])$, where $C_\gamma(\mathbb{T})$ is the set of functions in $C(\mathbb{T})$ vanishing at $\gamma \in \mathbb{T}$.

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(ii) φ has a fixed point $\gamma \in \mathbb{T}$ and fixes another point in $\mathbb{C} \cup \{\infty\}$ (equivalently φ has a fixed point $\gamma \in \mathbb{T}$ and $\varphi'(\gamma) \neq 1$) [7]. In this case, $C^*(T_z, C_\varphi)/\mathfrak{K}$ is $*$-isomorphic to $C_\gamma(\mathbb{T}) \oplus (C_0([0,1]) \rtimes \mathbb{Z})$.

(iii) φ fixes no point of $\mathbb{T}$ but there exist distinct points $\gamma, \eta \in \mathbb{T}$ with $\varphi(\gamma) = \eta$ [5]. In this case, $C^*(T_z, C_\varphi)/\mathfrak{K}$ is a C^*-subalgebra of $C(\mathbb{T}) \oplus M_2(C([0,1]))$.

This paper generalizes the above results by replacing the shift operator T_z by an irreducible Toeplitz operator T_ψ with continuous symbol ψ on $\mathbb{T}$, and a single composition operator with finitely many composition operators on the Hardy space H^2 induced by certain linear-fractional self-maps of $\mathbb{D}$. Moreover we investigate the C^*-algebra generated by a composition operator induced by a rotation and an irreducible Toeplitz operator with a symbol whose range is invariant under this rotation.

2 Main results

Let $\varphi_1, \cdots, \varphi_n$ are linear-fractional non-automorphism self-maps of $\mathbb{D}$ that fix $\gamma \in \mathbb{T}$, and $\ln \varphi_1'(\gamma), ..., \ln \varphi_n'(\gamma)$ are linearly independent over $\mathbb{Z}$. Define the action $\alpha' : \mathbb{Z}^n \to Aut(C_0([0,1]))$ by $\alpha'_{(m_1,...,m_n)}(f)(x) = f(x^{\varphi'(\gamma)^{m_1} \dots \varphi'(\gamma)^{m_n}})$, for $f \in C_0([0,1])$, $(m_1, ..., m_n) \in \mathbb{Z}^n$ and $x \in [0,1]$. First we extend a result of Quertermous in [7] (the case (ii) in the previous section) to finitely many composition operators induced by linear-fractional non-automorphism self-maps of $\mathbb{D}$ with a common fixed point on the unit circle as follows.

Theorem 2.1. *If $\varphi_1, ..., \varphi_n$ are linear-fractional non-automorphism self-maps of $\mathbb{D}$ fixing $\gamma \in \mathbb{T}$ and $\ln \varphi_1'(\gamma), ..., \ln \varphi_n'(\gamma)$ are linearly independent over $\mathbb{Z}$, then $C^*(T_z, C_{\varphi_1}, ..., C_{\varphi_n})/\mathfrak{K}$ $*$-isomorphic to the minimal unitization of the direct sum $C_\gamma(\mathbb{T}) \oplus (C_0([0,1]) \rtimes_{\alpha'} \mathbb{Z}^n)$.*

Let X be a compact Hausdorff space and $\mathcal{A}$ be a C^*-subalgebra of $C(X)$ containing the constants. For $x, y \in X$, put $x \sim y$ if and only if $f(x) = f(y)$ for all f in $\mathcal{A}$. Let $[x]$ denote the equivalence class of x and $[X]$ be the quotient space and equip $[X]$ with the weak topology induced by $\mathcal{A}$. Let $X/\sim$ be the quotient space equipped with the quotient topology. Then $\mathcal{A}$ is $*$-isomorphic to $C([X])$ and a C^*-subalgebra of $C(X/\sim)$ via $f \mapsto \tilde{f}$ where $\tilde{f}([x]) := f(x)$ for $x \in X$.

Note that T_z is irreducible (i.e. the only closed vector subspaces of H^2 reducing for T_z are 0 and H^2) and there are other irreducible Toeplitz operators. If $D = C^*(T_\psi) = C^*(T_z)$ For some continuous function ψ, then $D_0 := \{f \in C(\mathbb{T}) : T_f \in D\} = C(\mathbb{T})$ is generated by ψ and by the Stone-Weierstrass theorem, ψ must be one-to-one on the unit circle. Therefore we are interested in the case that ψ is not one-to-one on $\mathbb{T}$.

The following results are the extension of (i) and (ii) for an arbitrary irreducible Toeplitz operator and finitely many composition operators.

Theorem 2.2. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and ρ is a parabolic non-automorphism self-map of $\mathbb{D}$ fixing $\gamma \in \mathbb{T}$ then, $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is $*$-isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus C_0([0,1])$.*

Theorem 2.3. *If T_ψ is irreducible with symbol ψ in $C(\mathbb{T})$ and $\varphi_1, ..., \varphi_n$ are linear-fractional non-automorphism self-maps of $\mathbb{D}$ fixing $\gamma \in \mathbb{T}$ such that $\ln \varphi_1'(\gamma), ..., \ln \varphi_n'(\gamma)$ are linearly independent over $\mathbb{Z}$, then $C^*(T_\psi, C_{\varphi_1}, ..., C_{\varphi_n})/\mathfrak{K}$ is $*$-isomorphic to the minimal unitization of $C_{[\gamma]}([\mathbb{T}]) \oplus (C_0([0,1]) \rtimes_{\alpha'} \mathbb{Z}^n)$.*

Now consider the case that φ is a linear-fractional non-automorphism self-map of $\mathbb{D}$ such that $\varphi(\gamma) = \eta$ for some $\gamma \neq \eta \in \mathbb{T}$. In the following we extend the work of Kriete, MacCluer and Moorhouse (the case (iii) in the previous section) [5].

Theorem 2.4. *Let φ be a linear-fractional non-automorphism self-map of $\mathbb{D}$ such that $\varphi(\gamma) = \eta$ for distinct points $\gamma, \eta \in \mathbb{T}$ and T_ψ be irreducible with continuous symbol ψ on $\mathbb{T}$. Then every element b in $\mathcal{B} = C^*(T_\psi, C_\varphi)/\mathfrak{K}$ has a unique representation of the form*

$$b = [T_\omega] + f([C_\varphi^* C_\varphi]) + g([C_\varphi C_\varphi^*]) + [U_\varphi]h([C_\varphi^* C_\varphi]) + [U_\varphi^*]k([C_\varphi C_\varphi^*])$$

where $\omega \in C^*(\psi)$ and f, g, h and k are in $C_0([0,1])$. Moreover $\mathcal{B}$ is $*$-isomorphic to the C^*-subalgebra $\mathcal{D}$ of $C([\mathbb{T}]) \oplus M_2(C([0,1]))$ defined by

$$\mathcal{D} = \left\{ (f, S) \in C([\mathbb{T}]) \oplus \mathbb{M}_2(C([0,1])) : S(0) = \begin{bmatrix} f([\gamma]) & 0 \\ 0 & f([\eta]) \end{bmatrix} \right\}.$$

Jury in [4] finds the C^*-algebra $C^*(T_z, C_\varphi)/\mathfrak{K}$, for $\varphi \in Aut(\mathbb{D})$, as a crossed product C^*-algebra. We do the same when the shift operator is replaced by a general irreducible Toeplitz operator T_ψ. If $\varphi \in Aut(\mathbb{D})$ be of the form

$$\varphi(z) = \omega \frac{s - z}{1 - \bar{s}z},$$

for some non-real $\omega \in \mathbb{T}$ and non-zero $s \in \mathbb{D}$, then the quotient $C^*(T_z, C_\varphi)/\mathfrak{K} = C^*(C_\varphi)/\mathfrak{K}$ does not change, if one replaces T_z with T_ψ. Here we check the case $s = 0$.

Theorem 2.5. *Let φ be a rational automorphism $\varphi(z) = \omega z$ for some $\omega \in \mathbb{T}$. If T_ψ is irreducible and $\varphi(\psi(\mathbb{T})) = \psi(\mathbb{T})$, then there is an exact sequence of C^*-algebras*

$$0 \to \mathfrak{K} \to C^*(T_\psi, C_\varphi) \to C(\psi(\mathbb{T}))) \rtimes_\varphi \mathbb{Z} \to 0,$$

if φ has infinite order. In the case that φ has finite order q, in the exact sequence, $\mathbb{Z}$ is replaced by the finite cyclic group $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

Proof. $\mathbb{Z}$ acts on X by

$$\beta : \mathbb{Z} \to Home(X); \quad n \mapsto \beta_n, \quad \beta_n(x) = \varphi^n(x),$$

for $n \in \mathbb{Z}$ and $x \in X$. This induces an action of $\mathbb{Z}$ on $C(X)$ given by

$$\alpha : \mathbb{Z} \to Aut(C(X)); \quad \alpha_n(f)(x) = f(\varphi^{-n}(x)).$$

The C^*-algebra $C^*(T_\psi, C_\varphi)/\mathfrak{K}$ is generated by $C^*(T_\psi)/\mathfrak{K} \cong C(X)$ and unitaries $[C_{\varphi^n}]$. On the other hand the unitary representation $n \to [C_{\varphi^{-n}}]$ satisfies the covariance relation $[C_\varphi] f [C_\varphi^*] = \alpha_n(f)$. Hence there is a surjective $*$-homomorphism from the full crossed product $C(X) \rtimes_\varphi \mathbb{Z}$ to $C^*(T_\psi, C_\varphi)/\mathfrak{K}$. But the action of the amenable group $\mathbb{Z}$ on compact Hausdorff space X is amenable and topologically free (i.e. for each $n \in \mathbb{Z}$, the set of points that are fixed by φ^n has empty interior) thus similar to the proof of Theorem 2.1 in [4], the above $*$-homomorphism is also injective and hence an isometry. □

As a concrete example let the automorphism φ be of the form

$$\varphi(z) = ze^{i\frac{2p}{q}\pi}$$

where p and q are relatively prime integers with q positive. By using the method in [6], we construct a function ψ that satisfies the conditions of the above Theorem, is not one-to-one on the unit circle and

$$\psi(\mathbb{T}) = \mathbb{T} \cup (\bigcup_{n=0}^{q-1} \varphi^n([1/2, 1))).$$

Since the action of finite group $\mathbb{Z}_q$ is free on the compact spaces $\mathbb{T}$ and $\psi(\mathbb{T})$, using the same idea as in the proof of Proposition 5.2 in [8], the spectra of the C^*-algebras $C(\psi(\mathbb{T})) \rtimes_\varphi \mathbb{Z}_q$ and $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}_q$ are $\mathbb{Z}_q\backslash\psi(\mathbb{T})$ and $\mathbb{Z}_q\backslash\mathbb{T}$, respectively. It is easy to see that $\mathbb{Z}_q\backslash\psi(\mathbb{T})$ is homeomorphic to $\mathbb{T}\bigcup[1/2, 1)$ and $\mathbb{Z}_q\backslash\mathbb{T}$ is homeomorphic to $\mathbb{T}$. Therefore the spectra of these C^*-algebras are not homeomorphic, and so they could not be isomorphic.

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Chebyshevity and proximity in quotient spaces

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Abstract

We obtain a sufficient and necessary theorems simple for Chebyshevity of the best approximate sets in quotient spaces. Approximation theory, which mainly consists of theory of nearest points (best approximation) and theory of farthest points (worst approximation), is an old and rich branch of analysis. The theory is as old as Mathematics itself. The ancient Greeks approximated the area of a closed curve by the area of a polygon.

Keywords: Best approximation, ϵ-Proximinality, ϵ-chebyshevity, Quotient spaces
Mathematics Subject Classification [2010]: 41A65, 41A52, 46N10

1 Introduction

In this paper with a new ways we obtain some results on quotient spaces about proximinality, Chebyshevity of approximate sets.

Let W be a non-empty subset of a normed linear space X. For any $x \in X$, the (possibly empty) set of best approximations x from M is defined by

$$P_W(x) = \{y \in W : \|x - y\| = d(x, W)\},$$

where $d(x, W) = \inf\{\|x - y\| : y \in W\}$, and

$$\widehat{W} = \{x \in X : \ \|x\| = d(x, W)\}.$$

The subset W is said to be proximinal if the set $P_W(x)$ is non-empty for every $x \in X$ and the set W is Chebyshev if $P_W(x)$ is a singleton set. The closed unit ball of X is B_X and

$$B_X = \{x \in X : \ \|x\| \leq 1\}$$

Let W be a subspace of a normed space X. We define the quotient space X/W to be the set of all cosets $x + W$ of W together with the following operations:

$$(x + W) + (y + W) = (x + y) + W,$$

and

$$\lambda(x + W) = \lambda x + W,$$

*Speaker

for all $x, y \in X$ and arbitrary scalar λ. Then, the quotient space X/W is a normed space with the norm $\|x + W\| = inf_{w \in W}\|x - w\|$.

The closed unit ball of the quotient space X/W is

$$B_{X/M} = \{x + M : \ \|x + M\| \leq 1\} = \{x + W : \ d(x, M) \leq 1\}.$$

2 Main Results

Theorem 2.1. *Let M be a proximinal subspace of a normed space X, W a subspace of X containing M. Then W/M is Chebyshev if and only if for all $r > 0$, there exists an unique $z \in rB_X$ such that $d(z, W) = r$.*

Corollary 2.2. *Let M be a proximinal subspace of a normed space X, W a subspace of X containing M. Then W/M is Chebyshev if and only if for all $z \in X$ there exists a $f \in X^*$ such that $f|W = 0$ and $f(z) = \|z\|$.*

Lemma 2.3. *If the point $y_0 \in W$ is ϵ-approximation for $x \in X$. Then for $r > 0$, there exists a $z \in \epsilon B_X$ such that $d(z, W) \leq \epsilon$.*

Corollary 2.4. *Let M be a closed subspace of X, $\pi : X \to X/M$ be the canonical map and let W be a proximinal subspace of X containing M. Then, $\pi(P_W(x)) \subseteq P_{W/M}(x + M)$ for all $x \in X$.*

Theorem 2.5. *Let X be a normed linear space, W a linear subspace of X and $r > 0$. If there exists an unique $z \in rB_X$ such that $d(z, W) = r$. Then W is Chebyshev.*

Theorem 2.6. *Let M be a Chebyshev subspace of X and let W be a subspace of X containing M?. If W/M is Chebyshev of X/M. Then W is Chebyshev of X.*

Theorem 2.7. *Let M be a closed subspace of a normed space X and let W be a Chebyshev subspace of X containing M. Then, W/M is Chebyshev of X/M.*

Theorem 2.8. *Let X be a normed linear space, W a linear subspace of X and $r > 0$. If W is Chebyshev, then there exists an unique $z \in rB_X$ such that $d(z, W) = r$.*

Theorem 2.9. *Let M be a f-proximinal subspace of a normed space X and let W be a Proximinal subspace of X containing M. If $\pi : X \to X/M$ is the canonical map. Then,*

$$\pi(P_W(x)) = P_{W/M}(x + M).$$

Theorem 2.10. *If W is a proximinal subspace of a normed space X and $\widehat{W}$ is convex, then W is Chebyshev.*

Theorem 2.11. *Let M be proximinal subspace of a normed space X and let W be a proximinal subspace of X containing M. If $\widehat{W}$ is convex, then W/M is Chebyshev of X/M.*

Theorem 2.12. *Let M be a closed subspace of a normed space X and let W be a coproximinal subspace of X containing M. Then $\pi(\widehat{(W)} \subseteq \widehat{(W/M)}$.*

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Classification of frame graphs by dimension

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Abstract

To each finite frame φ in an inner product space $\mathcal{H}$ we associate a simple graph $G(\varphi)$, called **frame graph**, with the vectors of frame as vertices and there is an edge between vertices f and g provided that $\langle f, g\rangle \neq 0$. In this paper the relation between the order of $G(\varphi)$ and the dimension of $\mathcal{H}$ is investigated for some well-known classes of graphs and their products.

Keywords: Frame Graph, Graph product, Tree, Corona product, inner product space
Mathematics Subject Classification [2010]: 05C50, 42C15, 15A63

1 Introduction

The study of frames, using the properties of graphs, is an exciting research topic and hopefully will become mutually useful for both frame and graph theory. For example, in [1, 3, 4] the relation between equiangular tight frames and graphs was observed. A one-to-one correspondence between a subclass of equiangular tight frames and regular two-graphs was offered in [3] and another one between real equiangular frames of n vectors and graphs of order n was given in [6]. The authors of [5] found some restrictions on the existence of real equiangular tight frames by an equivalence between equiangular tight frames and strongly regular graphs with certain parameters.

To begin with we need to remind the notion of frame.

Definition 1.1. A finite frame for a finite dimensional Hilbert space $\mathcal{H}$ (or inner product space) is a finite sequence $\{f_i\}_{i=1}^n$ in $\mathcal{H}$ such that there exist constants $0 < A \leq B < \infty$ with the property that

$$A \parallel f \parallel^2 \leq \sum_{i=1}^{n} |\langle f, f_i\rangle|^2 \leq B \parallel f \parallel^2$$

holds for all $f \in \mathcal{H}$.

In this work we define another connection between frames and graphs. This connection is made by the zero-nonzero pattern of the correlation between different elements of frame by the following definition.

*Speaker

Definition 1.2. For a finite frame φ in an inner product space $\mathcal{H}$ we associate a simple graph $G(\varphi)$, called **frame graph**, with the elements of frame as vertices and two distinct vertices are adjacent if and only if the respective vectors are non-orthogonal.
It is known and easy to check that each simple graph is a frame graph. Investigating the relation between the dimension of $\mathcal{H}$ and the graph-theoretic properties of G is the main purpose of this paper. Some well known classes of graphs such as trees, cycles, complete and complete bipartite graphs will be characterized as frame graphs. Finally, the relation between $dim(\mathcal{H})$ and the order of graph G will be studied for corona, Cartesian and strong product of some well-known classes of graphs.

2 Main results

Throughout this paper all graphs are non-trivial and connected, and so the associated frames do not include zero vectors.
For a given graph G, we are interested to find all inner product spaces that G is a frame graph in them.

Theorem 2.1. *Let G be a simple graph on n vertices. Then G is a tree if and only if it is just frame graph in inner product spaces of dimension $n-1$ and n.*

Proposition 2.2. *Let G be a bipartite graph which contains $K_{n,m}$ as an induced subgraph and its partite sets U and V are of size m and n where $m \geq n$. Then G is just frame graph in inner product spaces of dimensions $m, m+1, ..., m+n-1, m+n$.*

2.1 Cartesian product

The **Cartesian product** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$ such that (u, v) is adjacent to (u', v') if and only if (1)$u = u'$ and $\{v, v'\} \in E_2$ or (2) $v = v'$ and $\{u, u'\} \in E_1$.

Theorem 2.3. *Let K_m be the compete graph of order m and T be a tree of order n. Then the graph $G = K_m \square T$ is just frame graph in inner product spaces of dimension $mn-m, mn-m+1, ..., mn-1$ and mn.*

2.2 Corona product

The **corona product** of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted $G_1 \circ G_2$, is the graph of order $|V_1||V_2| + |V_1|$ obtained by taking one copy of G_1 and $|V_1|$ copies of G_2, and joining all the vertices in the ith copy of G_2 to the ith vertex of G_1 [2].

Theorem 2.4. *Let T and T' be trees of order m and m', respectively. Then the followings hold.*

(1) The graph $T \circ T'$ is just frame graph in spaces of dimension $mm'-1, mm', ..., mm'+m$.

(2) The graph $T \circ K_n (n \geq 2)$ is just frame graph in spaces of dimension $2m-1, 2m, ..., mn+m$.

(3) The graph $K_n \circ T$ *is just frame graph in spaces of dimension* $nm - n + 1, nm - n + 2, ..., nm + n$.

(4) The graph $K_n \circ K_m$ *is just frame graph in spaces of dimension* $n+1, n+2, ..., nm+n$.

(5) $C_t \circ T$ *is just frame graph in spaces of dimension* $tm - 2, tm - 1, ..., tm + t$.

(6) the graph $T \circ C_t$ *is just frame graph in spaces of dimension* $m(t-1) - 1, m(t-1), ..., mt + m$.

(7) The graph $C_t \circ K_n$ *is just frame graph in spaces of dimension* $2t-2, 2t-1, ..., nt+t$.

(8) The graph $K_n \circ C_t$ *is just frame graph in spaces of dimension* $n(t-2)+1, n(t-2)+2, ..., nt+n$.

(9) The graph $C_t \circ C_{t'}$ *is just frame graph in spaces of dimension* $t(t'-1)-2, t(t'-1)-1, ..., tt'+t$.

2.3 Strong product

The **strong product** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted $G_1 \boxtimes G_2$, is the graph with vertex set $V_1 \times V_2$ such that (u, v) is adjacent to (u', v') if and only if (1)$u = u'$ and $\{v, v'\} \in E_2$ or (2) $v = v'$ and $\{u, u'\} \in E_1$ or (3) $\{u, u'\} \in E_1$ and $\{v, v'\} \in E_2$.

Theorem 2.5. *Let G be the strong product of* P_n *and* P_m*, i.e.,* $G = P_n \boxtimes P_n$*. Then G is just frame graph in inner product spaces of dimension* $(n-1)(m-1), (n-1)(m-1)+1, ..., mn-1$ *and* mn.

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Talk Compact composition operators on real Lipschitz spaces of complex-valued ... pp.: 1–4

Compact composition operators on real Lipschitz spaces of complex-valued bounded functions

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Abstract

We characterize compact composition operators on real Lipschitz spaces of complex-valued bounded functions on metric spaces, not necessarily compact, with Lipschitz involutions.

Keywords: Compact operator, composition operator, Lipschitz function, Lipschitz involution.

Mathematics Subject Classification [2010]: 46J10, 47B48.

1 Introduction and Preliminaries

Let X be a nonempty set, $V_{\mathbb{K}}(X)$ be a vector space over $\mathbb{K}$ of $\mathbb{K}$-valued functions on X and $T : V_{\mathbb{K}}(X) \longrightarrow V_{\mathbb{K}}(X)$ be a linear operator on X. If there exists a self-map $\phi : X \longrightarrow X$ such that $Tf = f \circ \phi$ for all $f \in V_{\mathbb{K}}(X)$, then T is call the composition operator on $V_{\mathbb{K}}(X)$ induded by ϕ.

Let X be a topological space. We denote by $C^b_{\mathbb{K}}(X)$ the set of all $\mathbb{K}$-valued bounded continuous functions on X. Then $C^b_{\mathbb{K}}(X)$ is a unital commutative Banach algebra over $\mathbb{K}$ under the pointwise operations and with the uniform norm

$$\| f \|_X = \sup\{|f(x)| : x \in X\} \quad (f \in C^b_{\mathbb{K}}(X)).$$

We denote by $C_{\mathbb{K}}(X)$ the algebra of all $\mathbb{K}$-valued continuous functions on X. Clearly, $C^b_{\mathbb{K}}(X) = C_{\mathbb{K}}(X)$ whenever X is compact. We write $C^b(X)$ and $C(X)$ instead of $C^b_{\mathbb{C}}(X)$and $C_{\mathbb{C}}(X)$, respectively.

Let (X, d) and (Y, ρ) be metric spaces. A map $\phi : X \longrightarrow Y$ is called a Lipschitz mapping from (X, d) into (Y, ρ) if there exists a constant $M \geq 0$ such that $\rho(\phi(x), \phi(y)) \leq Md(x, y)$ for all $x, y \in X$. A map $\phi : X \longrightarrow Y$ is called supercontractive from (X, d) into (Y, ρ) if

$$\lim_{d(x,y)\to 0} \frac{\rho(\phi(x), \phi(y))}{d(x, y)} = 0.$$

Let (X, d) be a metric space. A function $f : X \longrightarrow \mathbb{K}$ is called a $\mathbb{K}$-valued Lipschitz function on (X, d) if f is a Lipschitz mapping from (X, d) into the Euclidean metric space

*Speaker

$\mathbb{K}$. For a $\mathbb{K}$-valued Lipschitz function f on (X,d), the Lipschitz number of f on (X,d) is denoted by $L_{(X,d)}(f)$ and defined by

$$L_{(X,d)}(f) = \sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in X, x \neq y\}.$$

We denote by $Lip_{\mathbb{K}}(X,d)$ the set of all $\mathbb{K}$-valued bounded Lipschitz functions on (X,d). Clearly, $Lip_{\mathbb{K}}(X,d)$ is a subalgebra of $C^b_{\mathbb{K}}(X)$ and $1_X \in Lip_{\mathbb{K}}(X,d)$, where 1_X is the constant function with value 1 on X. Moreover, $Lip_{\mathbb{K}}(X,d)$ with the norm

$$\|f\|_{X,L} = \max\{\|f\|_X, L_{(X,d)}(f)\}$$

is a Banach space over $\mathbb{K}$. The set of all $f \in Lip_{\mathbb{K}}(X,d)$ for which f is supercontractive on (X,d), is denoted by $lip_{\mathbb{K}}(X,d)$. Clearly, $lip_{\mathbb{K}}(X,d)$ is a subalgebra of $Lip_{\mathbb{K}}(X,d)$ and $1_X \in lip_{\mathbb{K}}(X,d)$. Moreover, $lip_{\mathbb{K}}(X,d)$ is a closed set in $(Lip_{\mathbb{K}}(X,d)\| \cdot \|_{X,L})$ so $(lip_{\mathbb{K}}(X,d), \|\cdot\|_{X,L})$ is a Banach space over $\mathbb{K}$. We write $Lip(X,d)$ and $lip(X,d)$ instead of $Lip_{\mathbb{C}}(X,d)$ and $lip_{\mathbb{C}}(X,d)$, respectively. These algebras were first introduced by Sherbert in [3, 4]. Note that, if $\phi : X \longrightarrow X$ is a Lipschitz mapping then $f \circ \phi \in Lip_{\mathbb{K}}(X,d)$ ($f \circ \phi \in lip_{\mathbb{K}}(X,d)$, respectively) for all f in $Lip_{\mathbb{K}}(X,d)$ ($lip_{\mathbb{K}}(X,d)$, respectively).

Jiménez-Vargas and Villegas-Vallecillos [2] characterized compact composition operators on Banach spaces of Lipschitz functions $Lip_{\mathbb{K}}(X,d)$ with the norm $\|\cdot\|_{X,L}$ and $lip_{\mathbb{K}}(X,d)$ with the norm $\|\cdot\|_{X,L}$, where (X,d) is a metric space, not necessarily compact.

Let X be a topological space. A self-map $\tau : X \longrightarrow X$ is called a topological involution on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$.

Let X be a topological space and τ be a topological involution on X. The map $\sigma : C^b(X) \longrightarrow C^b(X)$ defined by $\sigma(f) = \overline{f} \circ \tau$ is an algebra involution on the complex algebra $C^b(X)$, which is called the algebra involution induced by τ on $C^b(X)$. Note that $\|\sigma(f)\|_X = \|f\|_X$ for all $f \in C^b(X)$. We now define

$$C^b(X,\tau) = \{f \in C^b(X) : \sigma(f) = f\}.$$

Then $C^b(X,\tau)$ is a unital self-adjoint uniformly closed real subalgebra of $C^b(X)$, $i_X \notin C^b(X,\tau)$ where i_X is the constant function with value i on X, $C^b(X) = C^b(X,\tau) \oplus i\,C^b(X,\tau)$ and

$$\max\{\|f\|_X, \|g\|_X\} \leq \|f + ig\|_X \leq 2\max\{\|f\|_X, \|g\|_X\},$$

for all $f, g \in C^b(X,\tau)$. Moreover, $C^b(X,\tau) = C^b_{\mathbb{R}}(X)$ if τ is the identity map on X. Note that if X is compact, then $C^b(X,\tau) = C(X,\tau)$, where $C(X,\tau) = \{f \in C(X) : \overline{f} \circ \tau = f\}$.

In this part we introduce real Lipschitz spaces $Lip(X,d,\tau)$, $lip(X,d,\tau)$ and $Lip_0(X,d,\tau)$.

Definition 1.1. Let (X,d) be a metric space. A self-map $\tau : X \longrightarrow X$ is called a Lipschitz involution on (X,d) if $\tau(\tau(x)) = x$ and τ is a Lipschitz mapping from (X,d) into (X,d).

Note that if τ is a Lipschitz involution on (X,d), then τ is a topological involution on (X,d) and $C \geq 1$ whenever $d(\tau(x), \tau(y)) \leq Cd(x,y)$ for all $x, y \in X$.

Let (X,d) be a metric space, τ be a Lipschitz involution on (X,d) and σ be the algebra involution induced by τ on $C^b(X)$. We can easily show that $\sigma(Lip(X,d)) = Lip(X,d)$, $\sigma(lip(X,d)) = lip(X,d)$, $L_{(X,d)}(\sigma(f)) \leq CL_{(X,d)}(f)$ for all $f \in Lip(X,d)$ and $\|\sigma(f)\|_{X,L} \leq$

$C\|f\|_{X,L}$ for all $f \in Lip(X,d)$, where $C \geq 1$ and $d(\tau(x),\tau(y)) \leq Cd(x,y)$ for all $x, y \in X$. We now define

$$Lip(X,d,\tau) := \{f \in Lip(X,d) : \sigma(f) = f\},$$
$$lip(X,d,\tau) := \{f \in lip(X,d) : \sigma(f) = f\}.$$

In fact, $Lip(X,d,\tau) = Lip(X,d) \cap C^b(X,\tau)$ and $lip(X,d,\tau) = lip(X,d) \cap C^b(X,\tau)$.

In the following result, we give some properties of $Lip(X,d,\tau)$ and $lip(X,d,\tau)$.

Theorem 1.2. *Let (X,d) be a metric space and τ be a Lipschitz involution on (X,d). Suppose that $\mathcal{A} = Lip(X,d,\tau)$ and $\mathcal{B} = Lip(X,d)$ ($\mathcal{A} = lip(X,d,\tau)$ and $\mathcal{B} = lip(X,d)$, respectively). Then:*

(i) $\mathcal{A}$ is a self-adjoint real subalgebra of $C^b(X,\tau)$ and $\mathcal{B}$, $1_X \in \mathcal{A}$ and $i_X \notin \mathcal{A}$.

(ii) $\mathcal{B} = \mathcal{A} \oplus i\,\mathcal{A}$.

(iii) For all $f, g \in \mathcal{A}$ we have

$$\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leq C\|f+ig\|_{X,L} \leq 2C\max\{\|f\|_{X,L}, \|g\|_{X,L}\},$$

where $C \geq 1$ and $d(\tau(x),\tau(y)) \leq Cd(x,y)$ for all $x, y \in X$.

(iv) $\mathcal{A}$ is closed in $(\mathcal{B}, \|\cdot\|_{X,L})$ and so $(\mathcal{A}, \|\cdot\|_{X,L})$ is a real Banach space.

(v) $f \circ \phi \in \mathcal{A}$ for all $f \in \mathcal{A}$ whenever $\phi : X \longrightarrow X$ is a Lipschitz mapping from (X,d) into (X,d) with $\phi \circ \tau = \tau \circ \phi$.

(vi) $\mathcal{A} = Lip_{\mathbb{R}}(X,d)$($\mathcal{A} = lip_{\mathbb{R}}(X,d)$, respectively), if τ is the identity map on X.

Note that $lip(X,d,\tau)$ is a real subalgebra of $Lip(X,d,\tau)$ and a closed set in $(Lip(X,d,\tau), \|\cdot\|_{X,L})$.

Real Lipschitz algebras $Lip(X,d,\tau)$ and $lip(X,d,\tau)$ where first introduced in [1], whenever (X,d) is a compact metric space.

In Section 2, we characterize compact composition operators on real Lipschitz spaces $(Lip(X,d,\tau), \|\cdot\|_{X,L})$ and $(lip(X,d,\tau), \|\cdot\|_{X,L})$ whenever (X,d) is a metric space, not necessarily compact and τ is a Lipschitz involution on (X,d).

2 Compact composition operators

Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space and $\mathfrak{X}_e = \mathfrak{X} \oplus i\mathfrak{X}$ be the complexification of X. We know that, there exists a norm $|||\cdot|||$ on $\mathfrak{X}_{\mathbb{C}}$ such that $|||x+i0||| = \|x\|$ for all $x \in X$, and

$$\max\{\|x\|, \|y\|\} \leq |||x+iy||| \leq 2\max\{\|x\|, \|y\|\},$$

for all $x, y \in \mathfrak{X}$, and so $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$ is a complex Banach space.

Theorem 2.1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space, $\mathfrak{X}_{\mathbb{C}}$ be the complexification of $\mathfrak{X}$ and $|||\cdot|||$ be a norm on $\mathfrak{X}_{\mathbb{C}}$ satisfying $|||f||| = \|f\|$ for all $f \in \mathfrak{X}$ and*

$$\max\{\|f\|, \|g\|\} \leq K_1|||f+ig||| \leq K_2\max\{\|f\|, \|g\|\},$$

for positive contants K_1 and K_2 and for all $f, g \in \mathfrak{X}$. Let $T \in BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X})$ and $T' : \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ defined by $T'(f+ig) = Tf + iTg \quad (f, g \in \mathfrak{X})$. Then:

(i) $T' \in BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ *and* $\|T'\| \leq 2C\|T\|$.

(ii) T' *is compact if and only if* T *is compact.*

(iii) T' *is invertible in* $BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ *if and only if* T *is invertible in* $BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X})$.

(iv) $T' = I_{\mathfrak{X}_{\mathbb{C}}}$ *if and only if* $T = I_{\mathfrak{X}}$.

(v) $\sigma(T') \cap \mathbb{R} = \sigma(T)$.

Compact composition operators on Lipschitz spaces $(Lip_{\mathbb{K}}(X, d), \|\cdot\|_{X,L})$ characterized in [2, Theorem 1.1].

In the following result, we characterize compact composition operators on real lipschitz spaces $(Lip(X, d, \tau), \|\cdot\|_{X,L})$ applying Theorem 2.1 and [2, Theorem 1.1].

Theorem 2.2. *Let* (X, d) *be a metric space,* τ *be a Lipschitz involution on* (X, d) *and* $\phi : X \longrightarrow X$ *be a Lipschitz mapping from* (X, d) *into* (X, d) *such that* $\phi \circ \tau = \tau \circ \phi$. *Then the composition operator* $T : Lip(X, d, \tau) \longrightarrow Lip(X, d, \tau)$ *induced by* ϕ *is compact if and only if* ϕ *is supercontractive and* $\phi(X)$ *is totally bounded in* (X, d).

In [2, Definition 1.1], Jiménez-Vargas and Villegas-Vallecillos obtained the analogous result for compact composition operators on little Lipschitz spaces $(lip_{\mathbb{K}}(X, d), \|\cdot\|_{X,L})$ that satisfy a kind of uniform separation property.

Compact composition operators on Lipschitz space $(Lip_{\mathbb{K}}(X, d), \|\cdot\|_{X,L})$ characterized in [2, Theorem 1.3].

In the following result, we characterize compact composition operators on real little Lipschitz spaces $(lip(X, d, \tau), \|\cdot\|_{X,L})$ when $lip(X, d)$ satisfies aforementioned uniform separation propertyb applying Theorems 2.1 and [2, Theorem 1.3].

Theorem 2.3. *Let* (X, d) *be a metric space,* τ *be a Lipschitz involution on* (X, d) *and* $\phi : X \longrightarrow X$ *be a Lipschitz mapping from* (X, d) *into* (X, d) *with* $\phi \circ \tau = \tau \circ \phi$. *Suppose that* $lip(X, d)$ *separates points uniformly on bounded subsets of* X. *Then the composition operator* $T : lip(X, d, \tau) \longrightarrow lip(X, d, \tau)$*induced by* ϕ *is compact if and only if* ϕ *induced by* ϕ *is supercontractive and* $\phi(X)$ *is totally bounded in* (X, d).

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Complex symmetric weighted composition operators on the weighted Hardy spaces.

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Abstract

Recently many authors have worked on normal weighted composition operators. On the other hand, it is known that every normal operator is a complex symmetric operator. Therefore, in this paper, we study complex symmetric weighted composition operators on the weighted Hardy spaces.

Keywords: Weighted Hardy Space, Weighted Composition Operator, Complex Symmetric.
Mathematics Subject Classification [2010]: 47B33, 47B38

1 Introduction

In 2010, C. C. Cowen and E. Ko obtained an explicit characterization and spectral description of all hermitian weighted composition operators on the classical Hardy space H^2 [5]. This work was later extended to certain weighted Hardy spaces by C. C. Cowen, G. Gunatillake, and E. Ko [4]. Along similar lines, P. Bourdon and S. Narayan have recently studied weighted composition operators on H^2 [1]. Taken together, theses articles have established the existence of several unexpected families of normal weighted composition operators. Then S. R. Garcia and C. Hammond in [11] investigated complex symmetric weighted composition operators on the weighted Hardy spaces.

Definition 1.1. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Let H be a Hilbert space of functions analytic on the unit disk. If the monomials $1, z, z^2, ...$ are an orthogonal set of non-zero vectors with dense span in H, then H is called a weighted Hardy space. We will assume that the norm satisfies the normalization $\|1\| = 1$. The weight sequence for a weighted Hardy space H is defined to be $\beta(n) = \|z^n\|$. The weighted Hardy space with weight sequence $\beta(n)$ will be denoted $H^2(\beta)$. The norm on $H^2(\beta)$ is given by

$$\left\|\sum_{j=0}^{\infty} a_j z^j\right\|^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2.$$

*Speaker

Definition 1.2. Let $w \in \mathbb{D}$ and H be a Hilbert space of analytic functions on $\mathbb{D}$. Let e_w be the point evaluation at w, that is, $e_w(f) = f(w)$ for each $f \in H$. If e_w is a bounded linear functional on H, then the Riesz Representation Theorem implies that there is a function (which is usually called K_w) in H that induces this linear functional, that is, $e_w(f) = \langle f, K_w \rangle$. In this case, the functions K_w are called the reproducing kernels and the functional Hilbert space is also called a reproducing kernel Hilbert space. We know that weighted Hardy spaces are reproducing kernel Hilbert spaces.

Definition 1.3. We say that a bounded operator T on a complex Hilbert space H is complex symmetric if there exits a conjugation (i.e., a conjugate linear, isometric involution) J such that $T = JT^*J$. The general study of such operators was undertaken by S. R. Garcia, M. Putinar and W. Wogen, in various combinations, in [7-10].

Definition 1.4. For any analytic self-map φ of $\mathbb{D}$, the composition operator C_φ on $H^2(\beta)$ is defined by $C_\varphi(f) = f \circ \varphi$. If ψ is a bounded analytic function on $\mathbb{D}$ and φ is an analytic map from $\mathbb{D}$ into itself, the weighted composition operator $C_{\psi,\varphi}$ on $H^2(\beta)$ is defined by $C_{\psi,\varphi}(f)(z) = \psi(z)f(\varphi(z))$.

Definition 1.5. It is well-known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are just the functions

$$\varphi(z) = \lambda \frac{a - z}{1 - \bar{a}z}, \tag{1}$$

where $|\lambda| = 1$ and $|a| < 1$ (see, e.g., [3]). We denote the class of automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$. Also an involutive disk automorphism is an automorphism that $\varphi \circ \varphi = I$.

Definition 1.6. We say that an operator A on a Hilbert space H is hyponormal if $A^*A - AA^* \geq 0$, or equivalently if $\|A^*f\| \leq \|Af\|$ for all $f \in H$ (see [2]).

Definition 1.7. An analytic self-map φ of $\mathbb{D}$ is univalent if it is one-to-one.

Definition 1.8. For any non-constant non-automorphism $\varphi : \mathbb{D} \to \mathbb{D}$ which has a fixed point w_0 in $\mathbb{D}$ and for which $\varphi'(w_0) \neq 0$, there is an analytic $k : \mathbb{D} \to \mathbb{C}$ such that $k \circ \varphi = \varphi'(w_0)k$. This function called the Koenigs eigenfunction for φ, is unique up to scalar multiplication (see [6]).

Definition 1.9. Recall that an operator T on a Hilbert space H is said to be normal if $TT^* = T^*T$.

2 Main results

In this section, we investigate complex symmetric composition and weighted composition operators on $H^2(\beta)$. Also, we show that if $C_{\psi,\varphi}$ is complex symmetric on $H^2(\beta)$, then either ψ is identically zero or ψ is nonvanishing on $\mathbb{D}$. Moreover, if φ is not a constant function and ψ is not identically zero, then φ is univalent (see [11]).

Proposition 2.1. *If φ is either (i) constant, or (ii) an involutive disk automorphism, then C_φ is a complex symetric operator on $H^2(\beta)$.*

Proposition 2.2. *If C_φ is a hyponormal composition operator on $H^2(\beta)$ which is complex symmetric, then $\varphi(z) = az$, where $|a| \leq 1$.*

Proposition 2.3. *Suppose that C_φ is J-symetric on $H^2(\beta)$. If $J(1)$ is a constant multiple of a kernel function K_w, then $\varphi(w) = w$. The converse holds whenever φ is not an automorphism.*

Proposition 2.4. *Suppose that $J : H^2(\beta) \to H^2(\beta)$ is a conjugation, $J(1)$ is a constant multiple of 1, and $J(z)$ is a constant multiple of z^m for some $m \geq 1$. If C_φ is J-symmetric, then $\varphi(z) = az$ for some $|a| \leq 1$.*

Theorem 2.5. *If $C_{\psi,\varphi}$ is complex symmetric on $H^2(\beta)$, then either ψ is identically zero or ψ is nonvanishing on $\mathbb{D}$. Moreover, if φ is not a constant function and ψ is not identically zero, then φ is univalent.*

Theorem 2.6. *Suppose that $C_{\psi,\varphi}$ is a complex symmetric operator on $H^2(\beta)$. If $\varphi(w_0) = w_0$ for some w_0 in $\mathbb{D}$, then $\psi(w_0)\varphi'(w_0)^n$ is an eigenvalue of $C_{\psi,\varphi}$ for every integer $n \geq 0$.*

Proposition 2.7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map which is not an automorphism and suppose that $\varphi(w_0) = w_0$ and $\varphi'(w_0) \neq 0$ for some w_0 in $\mathbb{D}$. If C_φ is a complex symmetric operator on $H^2(\beta)$, then every power k^n of the Koenigs eigenfunction for φ belongs to $H^2(\beta)$.*

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Composition operators on weak vector valued weighted Dirichlet type spaces

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Abstract

In this article we investigate the composition operator C_ϕ on weak vector valued weighted Dirichlet type spaces $w\mathcal{D}_v^p(X)$ for Banach space X and $1 \leq p \leq 2$. This operator is bounded(compact) on those spaces if the related measure $\mu_{p,v}$ is a (compact) Carleson. Also if C_ϕ is bounded(compact) on $w\mathcal{D}_v^p(X)$, then the same behavior holds on $w\mathcal{D}_v^q(X)$ for $1 \leq q < p$.

Keywords: Composition operator, Carleson measure, Compact Carleson measure, Weak vector valued weighted Dirichlet type space.
Mathematics Subject Classification [2010]: 47B33, 47B38, 31C25

1 Introduction

Let X be a complex Banach space and $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. The Lebesgue area measure on $\mathbb{D}$ is defined by $dA(z) = rdrd\theta = dxdy$. Denote by $H(X)$ the class of all analytic functions $f : \mathbb{D} \to X$. The weight function v is a positive function $v(r), 0 \leq r < 1$, which is integrable in $(0, 1)$. We extend v to $\mathbb{D}$ by setting $v(z) = v(|z|), z \in \mathbb{D}$.

For $p \geq 1$, the vector valued weighted Bergman space $A_v^p(X)$ consists of all functions $f \in H(X)$ for which

$$||f||^2_{A_v^p(X)} = \int_{\mathbb{D}} ||f(z)||_X^p v(z) dA(z) < \infty.$$

For $X = \mathbb{C}$ and $v = 1$, the space A^2 is called the (unweighted) Bergman space. Also for $X = \mathbb{C}$ and $v = (1-|z|^2)^\alpha, \alpha > -1$, we have the standard weighted Bergman space $A_\alpha^p(\mathbb{D})$. Note that $A_v^p(X)$ is Banach space for $p \geq 1$ and Hilbert space for $p = 2$ (see [5] for the theory of these spaces).

The vector valued weighted Dirichlet type space $\mathcal{D}_v^p(X)$ is the space of all f in $H(X)$ such that $f' \in A_v^p(X)$, equipped with the norm

$$||f||_{\mathcal{D}_v^p(X)} = ||f(0)|| + ||f'||_{A_v^p(X)}.$$

*Speaker

For $X = \mathbb{C}$ and $v = 1$, the space $\mathcal{D} = \mathcal{D}^2$ is the classical Dirichlet space of analytic functions. Clearly $\mathcal{D}_v^p(X) \subset \mathcal{D}_v^q(X)$ when $1 \leq q < p$.
The weak vector valued weighted Dirichlet space $w\mathcal{D}_v^p(X)$ consists of all analytic functions $f : \mathbb{D} \to X$ for which

$$||f||_{w\mathcal{D}_v^p(X)} = \sup_{||x^*||_{X^*} \leq 1} (||x^* o f||_{\mathcal{D}_v^p(\mathbb{D})})$$

is finite. Here $x^* \in X^*$, the dual space of X. In fact, such kinds of weak version spaces $wE(X)$ can be introduced under more general conditions on any Banach spaces E consisting of analytic functions $f : \mathbb{D} \to \mathbb{C}$. Some strong and weak version spaces are completely different such as Hardy spaces $H^2(X)$ and $wH^2(X)$ by constructing some concrete examples in [1]. Also Dirichlet spaces $w\mathcal{D}_\alpha(X)$ and $\mathcal{D}_\alpha(X)$ are different for any infinite dimensional complex Banach space X as Wang has shown in [10]. Others are the same such as Bloch spaces $\mathcal{B}(X)$ and $w\mathcal{B}(X)$, refer to [1].

Given analytic function ϕ in the unit disc $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, the composition operator C_ϕ defined by $C_\phi f(z) = f(\phi(z))$, for $f \in H(X)$ and $z \in \mathbb{D}$. Clearly this operator is linear.
Let μ be a finite positive Borel measure on $\mathbb{D}$. Then μ is said to be a Carleson measure if there exists a constant C such that $\mu(S(\xi, h) \leq Ch^2$ for all ξ and h, such that $|\xi| = 1$ and $0 < h < 2$. The measure is said to be a compact Carleson measure if $\lim_{h\to 0} \sup_{|\xi|=1} \frac{\mu(S(\xi,h))}{h^2} = 0$. Carleson measures have been useful in the study of composition operators in several settings (see for example [6, 8, 11]). For $w \in \mathbb{D}$, let $N_2(\phi, w)$ denote the number of zeros (counting multiplicities) of $\phi(z) - w$. For $1 \leq p < 2$ and $w \in \mathbb{D}$, we define the modified counting function

$$N_{p,v}(\phi, w) = \sum \frac{v(z)}{|\phi'(z)|^{2-p}}$$

where the sum extends over the zeros of $\phi - w$, repeated by multiplicity. In particular, $N_{p,v}(\phi, w) = 0$ for $w \notin \phi(\mathbb{D})$. Clearly with $v = 1$ and $p = 2$, we have $N_2(\phi, w)$.
Let $\mu_{p,v}$ be the measure defined on $\mathbb{D}$ by $d\mu_{p,v}(w) = N_{p,v}(\phi, w)dA(w)$, $1 \leq p < 2$.

A non negative measure μ on $\mathbb{D}$ is called a Carleson measure for $w\mathcal{D}_v^p(X)$ if there is a constant $C > 0$ such that

$$\int_{\mathbb{D}} ||f(z)||^p d\mu(z) \leq C||f||^p_{w\mathcal{D}_v^p(X)},$$

for all $f \in w\mathcal{D}_v^p(X)$. That is, the inclusion operator i from $w\mathcal{D}_v^p(X)$ into $L^p(X, \mu)$ is bounded. We call the Carleson measure μ, a compact Carleson measure for $w\mathcal{D}_v^p(X)$ if the inclusion operator i from $w\mathcal{D}_v^p(X)$ into $L^p(X, \mu)$ is compact.

2 Boundeness and compactness of composition operator on weak vector valued Dirichlet type spaces

The actions of composition operators and weighted composition operators on analytic function spaces such as Bergman, Hardy, Dirichlet and Dirichlet type spaces have been

studied by many authors, see for example [6, 11].
In [11], Zorboska has studied bounded and compact composition operators on weighted Dirichlet spaces. His method involves integral averages of determining function for the operator. In [8] compactness of composition operator C_ϕ is characterized by MacCluer and Shapiro in term of the angular derivative of the symbol ϕ. Adjoints of rationally induced composition operators on Bergman and Dirichlet spaces were studied in [2] by Ghoshabulaghi and Vaezi. Weighted composition operators on weak vector-valued Bergman and Hardy spaces were studied in [3] by Hassanlou, Vaezi and Wang. We have studied the isometric weighted composition operators on Hardy and Dirichlet spaces in [9]. In this article we study the boundedness and compactness of the composition operators on the weak vector valued weighted Dirichlet type spaces $w\mathcal{D}^p_v(X)$ for $1 \le p \le 2$.

Characterization of Carleson measure has been studied by many authors in the case of scaler and vector valued for different spaces of analytic functions. In [4] Hastings first proved some characterization for Carleson measure in $A^p(\mathbb{D})$, then by Stegenga it has shown for $A^p_\alpha(\mathbb{D})$. Some general methods for this characterization have been proved by Luecking in [7]. Also Kumar, Cima, Wogen, Nevanlinna and many others have worked on it.
Through this facts one can have the following theorem, which characterizes Carleson measure for $\mathcal{D}^p_v(X)$.

Theorem 2.1. *Take $1 < p < q < 2$. Let μ be a positive Borel measure on $\mathbb{D}$. Then*
(a) μ is said to be a Carleson measure for $A^p_v(X)$ if and only if $A^p_v(X) \subset L^p(\mu, X)$. In this case the inclusion operator

$$I : A^p_v(X) \to L^p(\mu, X)$$

is a bounded operator.
(b) μ is said to be a compact Carleson measure for $A^p_v(X)$ if and only if $A^p_v(X) \subset L^p(\mu, X)$ and the inclusion operator I from $A^p_v(X)$ into $L^p(\mu, X)$ is compact.

Remark 2.2. The above theorem is equivalent with the following statement:
There exists a constant C such that

$$\int_{\mathbb{D}} ||f(z)||^p d\mu(z) \le C||f||^p_{A^p_v(X)},$$

for all $f \in A^p_v(X)$.

3 Main results

Our main results are as follows:

Theorem 3.1. *The composition operator C_ϕ is bounded on $w\mathcal{D}^p_v(X)$ if and only if $\mu_{p,v}$ is a Carleson measure.*

Theorem 3.2. *The composition operator C_ϕ is compact on $w\mathcal{D}^p_v(X)$ if and only if $\mu_{p,v}$ is a compact Carleson measure.*

Lemma 3.3. *Suppose that the composition operator C_ϕ is bounded on $wD_v^p(X)$. Then for $1 \le q < p$, $\mu_{q,v}$ is a finite measure on $\mathbb{D}$.*

Theorem 3.4. *Suppose that the composition operator C_ϕ is bounded on $wD_v^p(X)$ and $1 \le q < p$. Then C_ϕ is bounded on $wD_v^q(X)$.*

Theorem 3.5. *If C_ϕ is compact on $wD_v^p(X)$ and $1 \le q < p$, then C_ϕ is compact on $wD_v^q(X)$.*

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Talk Connectivity of idempotent graph of bounded linear operators on a Hilbert . . . pp.: 1–2

Connectivity of Idempotent graph of Bounded Linear Operators on a Hilbert Space

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Abstract

Let H be a complex Hilbert space. The idempotent graph of $B(H)$, the algebra of all bounded linear operators on H, denoted by $I(B(H))$, is a graph whose vertices are all nontrivial idempotents of $B(H)$ and two distinct vertices P and Q are adjacent if and only if $PQ = QP = 0$. In this paper we show if H is a Hilbert space that has not finite dimensional, then $I(B(H))$ is a connected graph and its diameter is at most 4.

Keywords: Idempotent operator, Idempotent Graph, Connected Graph, Diameter
Mathematics Subject Classification [2010]: 47A06

1 Introduction

Throughout this paper H and $B(H)$ denote a complex Hilbert Space and the algebra of all bounded linear operators on H, respectively. If $P \in B(H)$ and $P^2 = P$, we say that P is an idempotent operator.

Let G be a graph. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. A finite non-null sequence $v_0e_1v_1e_2v_2 \cdots e_kv_k$, whose terms are alternatively vertices and edges such that for each i, $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i and for each i and j, $i \neq j$, $v_i \neq v_j$, is a path of length k between v_0 and v_k. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path we define $d(x, y) = \infty$. The diameter of G is $diam(G) = \sup\{d(x, y)|x \,\text{and}y\, \text{are distinct vertices of}\, G\}$. If u and v are two adjacent vertices, then we write $u — v$. The graph G is said connected, if there is a path between every two distinct vertices of G.

Formally, the idempotent graph, $I(B(H))$, of $B(H)$ is a simple (i.e., undirected and loopless) graph whose vertex set consists of all nonscalar idempotents and where two distinct vertices P and Q form an edge $P — Q$ if and only if $PQ = QP = 0$.

In this paper we show if H is a Hilbert space that has not finite dimensional, then $I(B(H))$ is a connected graph and its diameter is at most 4.

*Speaker

2 Main results

Theorem 2.1. *Let H be a complex Hilbert space that has not finite dimensional. Then $I(B(H))$ is a connected graph and furthermore, $diamI(B(H)) \leq 4$.*

Proof. Let A, B be arbitrary non-scalar idempotents in $B(H)$. Suppose that $\alpha \in \ker A$ and $\beta \in \ker B$. First, suppose that $<\alpha>=<\beta>$ and $\gamma \in <\alpha>$. Put $M =<\gamma>$ and define $P_1 = P_M$. Let $x \in H$. Then there are $r \in \mathbb{C}$ and $y \in M^{perp}$ such that $x = r\gamma + y$. Since $\mathrm{Im}A = (\mathrm{kerA})^{\perp}$ and $<\alpha>\subset \mathrm{kerA}$, We have

$$P_M A(x) = P_M A(r\gamma + y) = P_M A(y) = 0.$$

Also $AP_M(x) = A(r\gamma) = 0$. Similarly, $P_M B = BP_M = 0$. Therefore, $d(A, B) \leq 2$.

Now, let $<\alpha>\neq<\beta>$. Put $M =<\alpha>$, $N =<\beta>$, $P_1 = P_M$, and $P_2 = P_N$. As same as we showed in previous case, we can show A is connected to P_1 and B is connected to P_2. Put $S =<\alpha, \beta>^{\perp}$ and $P_3 = P_S$. Suppose that $x \in H$ is arbitrary. Then there are $y \in<\alpha, \beta>$ and $z \in S$ such that $x = y + z$ and there are $r, t \in \mathbb{C}$ such that $y = r\alpha + t\beta$. Since $z \in<\alpha>^{\perp}$ and $r\alpha \in S\alpha$, then

$$P_3 P_1(x) = P_3(P1(r\alpha + t\beta + z)) = P_3(r\alpha) = 0.$$

On the other hand $P_1 P_3(x) = P_1(z) = 0$. Therefore, P_1 is connected to P_3. Also, since $t\beta \in S^{\perp}$, we have

$$P_3 P_2(x) = P_3 P_2(r\alpha + t\beta + z) = P_3(t\beta) = 0.$$

On the other hand, $P_2 P_3(x) = P_2(z) = 0$. Therefore, P_2 is connected to P_3. We have $A - P_1 - P_3 - P_2 - B$ and $d(A, B) \leq 4$. The proof is complete. □

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Constructing dual and approximate dual fusion frames

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Abstract

The main goal of this paper is the construction of dual and approximate dual fusion frames. We introduce the notion of approximate duality for fusion frames, and present some approaches to obtain dual fusion frames. In particular, we characterize all duals of a Riesz decomposition fusion frame.

Keywords: Fusion frames; dual fusion frames; approximate duals; Riesz decomposition
Mathematics Subject Classification [2010]: 42C15

1 Introduction

In this section we review some definitions and primary results of fusion frames and show that, unlike discrete frames, every fusion frame has at least one alternate dual. Throughout this paper, π_V denotes the orthogonal projection from $\mathcal{H}$ onto a closed subspace V.

Definition 1.1. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ be a family of weights, i.e. $\omega_i > 0$, $i \in I$. Then $\{(W_i, \omega_i)\}_{i\in I}$ is called a *fusion frame* for $\mathcal{H}$ if there exist the constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \omega_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1}$$

The constants A and B are called the *fusion frame bounds*. If we only have the upper bound in (1) we call $\{(W_i, \omega_i)\}_{i\in I}$, a *Bessel fusion sequence*. A fusion frame is called A-*tight*, if $A = B$, and *Parseval* if $A = B = 1$. If $\omega_i = \omega$ for all $i \in I$, the collection $\{(W_i, \omega_i)\}_{i\in I}$ is called ω-*uniform* and we abbreviate 1- uniform fusion frames as $\{W_i\}_{i\in I}$. A fusion frame $\{W_i\}_{i\in I}$ is called an orthonormal basis for $\mathcal{H}$ when $\oplus_{i\in I} W_i = \mathcal{H}$ and it is a *Riesz decomposition* of $\mathcal{H}$ if for every $f \in \mathcal{H}$, there is a unique choice of $f_i \in W_i$ such that $f = \sum_{i\in I} f_i$. It is clear that every orthonormal fusion basis is a Riesz decomposition for $\mathcal{H}$, and also every Riesz decomposition is a 1- uniform fusion frame for $\mathcal{H}$.

Let $\{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame, the *fusion frame operator* $S_W : \mathcal{H} \to \mathcal{H}$ is defined by $S_W f = \sum_{i\in I} \omega_i^2 \pi_{W_i} f$ is a bounded, invertible as well as positive. Hence, we have the following reconstruction formula [4]

$$f = \sum_{i\in I} \omega_i^2 S_W^{-1} \pi_{W_i} f, \qquad (f \in \mathcal{H}).$$

*Speaker

The family $\{(S_W^{-1}W_i, \omega_i)\}_{i\in I}$, which is also a fusion frame, is called the *canonical dual* of $\{(W_i, \omega_i)\}_{i\in I}$. In general, every Bessel fusion sequence $\{(V_i, \nu_i)\}_{i\in I}$ is called a *dual* of fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ if

$$f = \sum_{i\in I} \omega_i \nu_i \pi_{V_i} S_W^{-1} \pi_{W_i} f, \qquad (f \in \mathcal{H}). \tag{2}$$

It is proved that a Bessel fusion sequence $\{(V_i, \upsilon_i)\}_{i\in I}$ is a dual of fusion frame $\{(W_i, \omega_i)\}_{i\in I}$, if and only if $T_V \phi_{vw} T_W^* = I_{\mathcal{H}}$, where the bounded operator $\phi_{vw} : \sum_{i\in I} \bigoplus W_i \to \sum_{i\in I} \bigoplus V_i$ is given by

$$\phi_{vw}(\{f_i\}_{i\in I}) = \{\pi_{V_i} S_W^{-1} f_i\}_{i\in I}. \tag{3}$$

Moreover, a Bessel fusion sequence $V = \{(V_i, \omega_i)\}_{i\in I}$ given by $V_i = S_W^{-1} W_i \oplus U_i$, is dual of $\{(W_i, \omega_i)\}_{i\in I}$ in which U_i is a closed subspace of $\mathcal{H}$ for all $i \in I$, [11].

2 Main results- Approximate duals

Dual fusion frames play a key role in fusion frame theory, however their explicit computations seem rather intricate. In this section, we introduce the notion of approximate dual for fusion frames and discuss the existence of dual fusion frames from an approximate dual. Moreover, we present a complete characterization of duals of Riesz decompositions. The notion of approximate dual for discrete frames has been already introduced by Christensen and Laugesen in [6], however many of its results are invalid for fusion frames. Throughout this section we consider a Riesz decomposition as a 1-uniform fusion frame.

First, we recall the notion of approximate dual for discrete frames. Let $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ be Bessel sequences for $\mathcal{H}$ with synthesis operators T and U, respectively. Then F and G are called *approximate dual frames* if $\|I_{\mathcal{H}} - UT^*\| < 1$. In this case $\{(UT^*)^{-1}G\}$ is a dual of F, see [6].

Now we introduce approximate duality for fusion frames.

Definition 2.1. Let $\{(W_i, \omega_i)\}_{i\in I}$ be a Bessel fusion sequence. A Bessel fusion sequence $\{(V_i, \upsilon_i)\}_{i\in I}$ is called an approximate dual of $\{(W_i, \omega_i)\}_{i\in I}$ if

$$\|I_{\mathcal{H}} - T_V \phi_{vw} T_W^*\| < 1.$$

Putting

$$U_{vw} = T_V \phi_{vw} T_W^*. \tag{4}$$

Then, we have the following reconstruction formula

$$f = \sum_{i\in I} (U_{vw})^{-1} \omega_i \upsilon_i \pi_{V_i} S_W^{-1} \pi_{W_i} f = \sum_{n=0}^{\infty} (I - U_{vw})^n U_{vw} f, \qquad (f \in \mathcal{H}).$$

Proposition 2.2. *Let $V = \{(V_i, \upsilon_i)\}_{i\in I}$ be an approximate dual of a Bessel fusion sequence $W = \{(W_i, \omega_i)\}_{i\in I}$. Then W and V are fusion frames.*

The stability of approximate dual of discrete frames can be found in [6]. In the following, we discuss on the stability of approximate dual fusion frames.

Proposition 2.3. *Let $\{e_j\}_{j\in J}$ be an orthonormal basis of $\mathcal{H}$. The Bessel sequence $V = \{(V_i, \upsilon_i)\}_{i\in I}$ is an approximate dual of $W = \{(W_i, \omega_i)\}_{i\in I}$, if and only if $\{\upsilon_i \pi_{V_i} e_j\}_{i\in I, j\in J}$ is an approximate dual of $\{\omega_i \pi_{W_i} S_W^{-1} e_j\}_{i\in I, j\in J}$.*

Theorem 2.4. *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ and $V = \{(V_i, \upsilon_i)\}_{i\in I}$ be Bessel sequences, also $\{g_{i,j}\}_{j\in J_i}$ be a frame for V with bounds A_i and B_i, for every $i \in I$ such that $0 < a = \inf_{i\in I} A_i$. Then V is an approximate dual of W if and only if $G = \{\upsilon_i g_{i,j}\}_{i\in I, j\in J_i}$ is an approximate dual of $F = \{\omega_i \pi_{W_i} S_W^{-1} \widetilde{g}_{i,j}\}_{i\in I, j\in J_i}$, where $\{\widetilde{g}_{i,j}\}_{j\in J_i}$ is the canonical dual of $\{g_{i,j}\}_{j\in J_i}$.*

We know that many concepts of the classical frame theory have not been generalized to the fusion frames. For example in the duality discussion, if $V = \{(V_i, \upsilon_i)\}_{i\in I}$ is a dual of fusion frame $W = \{(W_i, \omega_i)\}_{i\in I}$, then W is not a dual of V, moreover, it is not an approximate dual of V in general. Indeed if

$$W_1 = \overline{span}\{(1,0,0)\}, \quad W_2 = \overline{span}\{(1,1,0)\},$$
$$W_3 = \overline{span}\{(0,1,0)\}, \quad W_4 = \overline{span}\{(0,0,1)\},$$

and $\omega_1 = \omega_3 = \omega_4 = 1$, $\omega_2 = \sqrt{2}$. Then $W = \{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathbb{R}^3$ with an alternate dual as $V = \{(V_i, \upsilon_i)\}_{i\in I}$ where

$$V_1 = \overline{span}\{(0,1,0)\}, \quad V_2 = \mathbb{R}^3, \quad V_3 = \overline{span}\{(1,0,0)\}, \quad V_4 = \overline{span}\{(0,0,1)\},$$

and $\upsilon_1 = \upsilon_3 = 3$, $\upsilon_2 = 3\sqrt{2}$, $\upsilon_4 = 1$, see Example 3.1 of [1]. A straightforward calculation shows that $\|I_{\mathcal{H}} - U_{wv}\| = 1$, hence W is not an approximate dual of V. The next theorem gives sufficient conditions for a fusion frame is approximate dual of its dual.

Theorem 2.5. *Let $\{(V_i, \upsilon_i)\}_{i\in I}$ be a dual of fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ such that*

$$\|S_W^{-1} - S_V^{-1}\| < \|S_W\|^{-1/2} \|S_V\|^{-1/2}.$$

Then $\{(W_i, \omega_i)\}_{i\in I}$ is an approximate dual of $\{(V_i, \upsilon_i)\}_{i\in I}$.

Theorem 2.6. *Let $\{W_i\}_{i\in I}$ be a Riesz decomposition and $\{V_i\}_{i\in I}$ be an approximate dual of $\{W_i\}_{i\in I}$. Then the sequence $\{U_{vw}^{-1} V_i\}_{i\in I}$ is a dual of $\{W_i\}_{i\in I}$.*

Corollary 2.7. *Let $\{W_i\}_{i\in I}$ be a Riesz decomposition. A Bessel sequence $\{V_i\}_{i\in I}$ is an dual of $\{W_i\}_{i\in I}$ if and only if*

$$V_i \supseteq S_W^{-1} W_i, \quad (i \in I). \tag{5}$$

other alternate duals of $\{W_i\}_{i\in I}$ are not Riesz decomposition.

Theorem 2.8. *Let $\{V_i\}_{i\in I}$ be a dual of a Riesz decomposition $\{W_i\}_{i\in I}$. Then $\{V_i\}_{i\in I}$ is Riesz decomposition if and only if, it is the canonical dual of $\{W_i\}_{i\in I}$.*

Corollary 2.9. *Let $\{W_i\}_{i\in I}$ be a Riesz decomposition. A Bessel sequence $\{V_i\}_{i\in I}$ is an dual of $\{W_i\}_{i\in I}$ if and only if*

$$V_i \supseteq S_W^{-1} W_i, \quad (i \in I). \tag{6}$$

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Convergence theorems for a broad class of nonlinear mappings

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Abstract

In this paper, we introduce a new Mann type iteration for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of 2-generalized hybrid mappings in a Hilbert space.

Keywords: Fixed point, Hilbert space, Weak convergence
Mathematics Subject Classification [2010]: 47H10,47H09, 47J25, 47J05

1 Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$, and let E be a nonempty closed convex subset of H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$. The equilibrium problem for $f : E \times E \to \mathbb{R}$ is to find $x \in E$ such that $f(x, y) \geq 0$ for all $y \in E$. The set of solutions of the equilibrium problem for f is denoted by $EP(f)$, i.e., $EP(f) = \{x \in E : \ f(x, y) \geq 0, \ \ \forall y \in E\}$.

A self mapping S of E is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in E$. We denote by $F(S)$ the set of fixed points of S.

In the recent years, many authors studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see for instance, [1, 8] and the references therein.

Let E be a nonempty closed convex subset of a Banach space. In 1953, for a self mapping S of E, Mann [7] defined an iteration procedure by $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n$, where $x_0 \in E$ chosen arbitrarily and $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

In 2007, Tada and Takahashi [8] for finding an element of $EP(f) \cap F(S)$, introduced the following iterative scheme for a nonexpansive self mapping S of a nonempty, closed

*Speaker

convex subset E in a Hilbert space H:

$$\begin{cases} x_1 = x \in H \ \text{ chosen arbitrarily}, \\ u_n \in E \text{ such that } \ \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S u_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $f : E \times E \to \mathbb{R}$ satisfies appropriate conditions, $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. They proved $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(f)$, where $w = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Let E be a nonempty closed convex subset of H. A self mapping S of E is called *generalized hybrid* [6] if there exist $\gamma, \lambda \in \mathbb{R}$ such that

$$\gamma\|Sx - Sy\|^2 + (1-\gamma)\|x - Sy\|^2 \leq \lambda\|Sx - y\|^2 + (1-\lambda)\|x - y\|^2, \tag{1}$$

for all $x, y \in E$. We call such a mapping a (γ, λ)-*generalized hybrid* mapping.

2 Preliminaries

A self mapping S of E is called: (i) *firmly nonexpansive*, if $\|Sx - Sy\|^2 \leq \langle x - y, Sx - Sy\rangle$ for all $x, y \in E$; (ii) *nonspreading*, if $2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|Sy - x\|^2$ for all $x, y \in E$; (iii) *hybrid*, if $3\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|Sx - y\|^2 + \|Sy - x\|^2$ for all $x, y \in E$. Also, a self mapping S of E with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Sy\| \leq \|x - y\|$ for all $x \in F(S)$ and $y \in E$. It is well-known that for a *quasi-nonexpansive* mapping S, $F(S)$ is closed and convex [5].

It easy to see that $(1, 0)$-generalized hybrid mapping is nonexpansive; $(2, 1)$-generalized hybrid mapping is nonspreading; $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping is hybrid.

A self mapping T of C is called 2-*generalized hybrid* [10] if there exist $\gamma_1, \gamma_2, \lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\gamma_1\|T^2x - Ty\|^2 + \gamma_2\|Tx - Ty\|^2 + (1 - \gamma_1 - \gamma_2)\|x - Ty\|^2 \\ &\qquad\qquad \leq \lambda_1\|T^2x - y\|^2 + \lambda_2\|Tx - y\|^2 + (1 - \lambda_1 - \lambda_2)\|x - y\|^2, \end{aligned}$$

for all $x, y \in C$. Such a mapping is called a $(\gamma_1, \gamma_2, \lambda_1, \lambda_2)$-generalized hybrid mapping. It is easy to see that a $(0, \gamma_2, 0, \lambda_2)$-generalized hybrid mapping is an (γ_2, λ_2)-generalized hybrid mapping [4]. Also, one can easily show that a 2-generalized hybrid mapping is quasi-nonexpansive if the set of it's fixed points is nonempty. In [4], Hojo et al. give two examples of 2-generalized hybrid mappings which are not generalized hybrid mappings. So, the class of 2-generalized hybrid mappings is broader than the class of generalized hybrid mappings.

Let K be a closed convex subset of H and let P_K be metric (or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = inf\{\|x - z\| : z \in K\}$). Let $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if $\langle x - z, y - z\rangle \leq 0$, for all $y \in K$. For more details we refer readers to [9].

To study the equilibrium problem, we assume that $f : E \times E \longrightarrow \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in E$;

(A2) f is monotone, i.e., $f(x,y)+f(y,x)\leq 0$ for all $x,y\in E$;
(A3) for each $x,y,z\in E$, $\lim_{t\downarrow 0} f(tz+(1-t)x,y)\leq f(x,y)$;
(A4) for each $x\in E$, $y\mapsto f(x,y)$ is convex and lower semicontinuous.

The following lemma can be found in [2].

Lemma 2.1. *Let E be a nonempty closed convex subset of H, let f be a bifunction from $E\times E$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $r>0$ and $x\in H$. Then, there exists $z\in E$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,$$

for all $y\in E$.

The following lemma is established in [3].

Lemma 2.2. *For $r>0$, $x\in H$, define a mapping $T_r:H\longrightarrow E$ as follows:*

$$T_r(x)=\{z\in E: f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in E\}.$$

Then, the following statements hold:
(i) T_r is singel-valued;
(ii) T_r is firmly nonexpansive, i.e., for all $x,y\in H$ $\|T_rx-T_ry\|^2\leq\langle T_rx-T_ry,x-y\rangle$;
(iii) $F(T_r)=EP(f)$;
(iv) EP(f) is closed and convex.

3 Main Results

In this section, we prove weak convergence theorems for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a 2-generalized hybrid mapping.

Theorem 3.1. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E\times E$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and S be a 2-generalized hybrid self mapping of E with $F(S)\cap EP(f)\neq\phi$ and $\|S^2x-Sx\|\leq\|Sx-x\|$ for all $x\in E$. Assume that $\{r_n\}\subset(0,\infty)$ satisfies $\liminf_{n\to\infty} r_n>0$ and $\{\alpha_n\}$ is sequence in $[a,1]$ for some $a\in(0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$. If $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x=x_1\in H$ and*

$$\begin{cases} u_n\in E \text{ such that } \quad f(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall\, y\in E,\\ x_{n+1}=S((1-\alpha_n)x_n+\alpha_nSu_n),\end{cases}$$

for all $n\in\mathbb{N}$. Then $x_n\rightharpoonup v\in F(S)\cap EP(f)$, where $v=lim_{n\to\infty}P_{F(S)\cap EP(f)}(x_n)$.

Corollary 3.2. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let S be a 2-generalized hybrid self mapping of E with $F(S)\neq\phi$ and $\|S^2x-Sx\|\leq\|Sx-x\|$ for all $x\in E$. Assume that $\{\alpha_n\}$ is sequence in $[a,1]$ for some $a\in(0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$. If $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x=x_1\in H$ and*

$$\begin{cases} u_n\in E \text{ such that } \quad \langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall\, y\in E,\\ x_{n+1}=S((1-\alpha_n)x_n+\alpha_nSu_n),\end{cases}$$

for all $n\in\mathbb{N}$. Then $x_n\rightharpoonup v\in F(S)$, where $v=lim_{n\to\infty}P_{F(S)}(x_n)$.

Corollary 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ with $EP(f) \neq \phi$. Assume that $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $\{\alpha_n\}$ is sequence in $[a, 1]$ for some $a \in (0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. If $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x = x_1 \in H$ and*

$$\begin{cases} u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \end{cases}$$

for all $n \in \mathbb{N}$. Then $x_n \rightharpoonup v \in EP(f)$, where $v = lim_{n\to\infty} P_{EP(f)}(x_n)$.

Remark 3.4. As previously mentioned, the class of 2-generalized hybrid mappings includes the classes of nonexpansive, nonspreading, generalized hybrid and hybrid mappings in a Hilbert space. Hence the Theorems 3.1 and the Corollaries 3.2 and 3.3 hold for these mappings.

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Convolution condition on n-starlike functions

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Abstract

Let $\mathcal{P}(\gamma, \beta)$, $\gamma > 0$, $\beta < 1$ denote the class of analytic function f in the unit disk normalized by $f(0) = 1$, $f'(0) = 1$ and satisfying the condition

$$Re\Big\{e^{i\varphi}\big(f'(z) + \frac{1}{\gamma} z f''(z) - \beta\big)\Big\} > 0, \ |z| < 1,$$

for some $\varphi \in \mathbb{R}$. In this paper consider $S_n(\alpha)$, the class of n-starlike function of order α, defined by G. S. Salagean (1983) [5] and we find condition on γ, β so that $\mathcal{P}(\gamma, \beta) \subseteq S_n(\alpha)$. We take advantage of the Ruscheweh's Duality theory.

Keywords: Univalent functions, Starlike functions, Hadamard product, Salagean differential operator.
Mathematics Subject Classification [2010]: 30C45, 30C50

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n-2}^{\infty} a_n z^n. \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C}; |z| < 1\}$ and let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Let $\mathcal{A}_0$ denote the subclass of analytic functions in the open unit disk $\mathcal{U}$ consisting of functions normalized by $f(0) = 1, f'(0) = 1$. For $0 \le \alpha < 1$, a function $f(z) \in \mathcal{A}$ is said to be starlike of order ρ in $\mathcal{U}$ if it satisfies

$$Re\Big\{\frac{zf'(z)}{f(z)}\Big\} > \alpha, \quad (z \in \mathcal{U}). \tag{2}$$

The set of all starlike functions of order α denote by $\mathcal{ST}(\alpha)$. Note that $\mathcal{ST}(0)$, the class of starlike function denote by $\mathcal{ST}$ (For more details see [1, 2]).

S. Rucheweyh in [3] defined the operator D^n by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \quad n \in \mathbb{N}_0 \ \ f \in \mathcal{A}.$$

If $f \in \mathcal{A}$ is given by (1), then $D^n f(z) = \sum_{k=1}^{\infty} k^n a_k z^k$ (see [5]) .

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Definition 1.1. [5] For $0 \leq \alpha < 1$ and $f \in \mathcal{A}$ the class $S_n(\alpha)$, n-starlike function of order α, is defined by

$$S_n(\alpha) = \Big\{ f \in \mathcal{A} : \quad Re\frac{D^{n+1}f(z)}{D^n f(z)} > \alpha \quad z \in \mathcal{U} \Big\}.$$

Note that $S_n(0)$, the class of n-starlike function, denote by S_n, Further $S_0(\alpha) = \mathcal{ST}(\alpha)$.

For f and g in $\mathcal{A}$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution (Hadamard product) of f and g, denoted by $f * g$, is a function also in $\mathcal{A}$, given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

In convolution theory, the concept of Duality is central. For a set

$$V \subseteq \mathcal{A}_0 = \Big\{ g : g(z) = \frac{f(z)}{z}, \ f \in \mathcal{A} \Big\},$$

the dual set is defined as

$$V^* = \{ g \in \mathcal{A}_0 : (f * g)(z) \neq 0 \ for \ all \ f \in V, z \in \mathcal{U} \}.$$

Further, the second dual, or dual hull, of V is defined as $V^{**} = (V^*)^*$. However, $V^{**} \subseteq (V^*)^*$. The basic reference to this theory is the book by Ruscheweyh [4].

Note that for $f, g \in \mathcal{A}$, $D^n(f * g) = D^n f * g = f * D^n g$. Further, for $n \in \mathbb{N}$, $f \in S_n(\alpha)$ if and only if $D^n f$ is a starlike function.

Theorem 1.2. *[6] The function f is starlike functions of order α in $\mathcal{U}$, if and only if*

$$\frac{1}{z}\Big(f * \frac{z + \frac{x+2\alpha-1}{2(1-\alpha)} z^2}{(1-z)^2} \Big) \neq 0, \quad |x| = 1.$$

Theorem 1.3. *[4] Let*

$$V_\beta = \Big\{ \beta + \frac{(1-\beta)(1+xz)}{1+yz} : \ |x| = |y| = 1, \quad \beta \in \mathbb{R}, \ \beta \neq 1 \Big\}$$

then

$$V_\beta^{**} = \Big\{ g \in \mathcal{A}_0 : \ \exists \varphi \in \mathbb{R} \ such \ that \ Re\big[e^{i\varphi}(g(z) - \beta) \big] > 0, \ z \in \mathcal{U} \ \Big\}.$$

Notice that if $h \in V_\beta$, $h(z) = \beta + (1-\beta)\frac{1+xz}{1+yz}$ with $|x| = |y| = 1$, $\beta \in \mathbb{R}$, $\beta \neq 1$, then $h(z) = 1 + (1-\beta)(1-e^{i\psi}) \sum_{k=1}^{\infty} (yz)^k$, for some $\psi \in \mathbb{R}$.

Theorem 1.4. *(Duality principle, see [4]) Let $V \subseteq \mathcal{A}_0$ be compact and has the following property*

$$f \in V \Rightarrow \forall |x| \leq 1 : f_x \in V$$

where $f_x(z) = f(xz)$. Then $\varphi(V) = \varphi(V^{**})$, for all continuous linear functional φ on $\mathcal{A}$, and $co(V) \subseteq \overline{co} V^{**}$. where $\overline{co}$ stand for the closed convex hull of a set.

In this paper we use the powerful method duality principle in geometric function theory developed by Rucheweyh [4], and try to find condition on γ, β so that $\mathcal{P}(\gamma, \beta) \subseteq S_n(\alpha)$.

2 Main results

Theorem 2.1. *The function f is n-starlike functions of order α in $\mathcal{U}$, if and only if*

$$\frac{1}{z}\Big(f * \frac{z+\frac{x+2\alpha-1}{2(1-\alpha)}z^2}{(1-z)^2} * \frac{z}{(1-z)^{n+1}}\Big) \neq 0, \quad n \in \mathbb{N}_0 |x| = 1.$$

Proof. The function f is n-starlike function of order α for all $n \in \mathbb{N}$ if and only if $D^n f$ is starlike of order α, Hence by applying Theorem 1.2 we have

$$\frac{1}{z}\Big(D^n f * \frac{z+\frac{x+2\alpha-1}{2(1-\alpha)}z^2}{(1-z)^2}\Big) \neq 0, \ \ n \in \mathbb{N}_0, \ |x| = 1, \ \forall z \in \mathcal{U}. \tag{3}$$

Since $D^n f(z) = f * \frac{z}{(1-z)^{n+1}}$, we obtain the inequality (2). Hence, this complete the proof of this theorem. □

Corollary 2.2. *The function f is n-starlike functions in $\mathcal{U}$, if and only if*

$$\frac{1}{z}\Big(f * \frac{z+\frac{x-1}{2}z^2}{(1-z)^2} * \frac{z}{(1-z)^{n+1}}\Big) \neq 0, \quad |x| = 1, \ \ n \in \mathbb{N}_0.$$

Proof. In Theorem 2.1, we set $\alpha = 0$. □

Theorem 2.3. *Suppose that $\gamma > 0$, $\beta < 1$, $\alpha < 1$ and $n \in \mathbb{N}_0$. Then $\mathcal{P}(\gamma, \beta) \subseteq S_n(\alpha)$ if and only if*

$$Re(F(x,z)) > -\frac{1-\alpha}{1-\beta} \tag{4}$$

where

$$H(x,z) = \gamma \sum_{k=1}^{\infty} k^n \frac{k(1+x)+2(1-\alpha)}{(k+1)(k+\gamma)} z^n, \qquad \forall |x| = 1, \ \ n \in \mathbb{N}_0, \quad \forall z \in \mathcal{U}. \tag{5}$$

Proof. Let a function f be in the class $\mathcal{P}(\gamma, \beta)$. If we denote $f'(z) + \frac{z}{\gamma} f''(z) = g_\gamma(z)$ then we have $g_\gamma \in V_\beta^{**}$. If $f(z)\sum_{k=1}^{\infty} a_k z^k$, $a_1 = 1$, then

$$f'(z) + \frac{z}{\gamma} f''(z) = \sum_{k=1}^{\infty} \frac{k(k-1+\gamma)}{\gamma} a_k z^{k-1} = g_\gamma(z).$$

So

$$\frac{f(z)}{z} = \sum_{k=1}^{\infty} a_k z^{k-1} = g_\gamma(z) * \sum_{k=1}^{\infty} \frac{\gamma z^{k-1}}{k(k-1+\gamma)},$$

and we obtain one-to-one correspondence between $\mathcal{P}(\gamma, \beta)$ and V_β^{**}. Thus, by Theorem 2.1, $\mathcal{P}(\gamma, \beta) \subseteq S_n(\alpha)$ if and only if

$$g_\gamma(z) * \sum_{k=1}^{\infty} \frac{\gamma z^{k-1}}{k(k-1+\gamma)} * \frac{1+\frac{x+2\alpha-1}{2(1-\alpha)}z}{(1-z)^2} * \frac{1}{(1-z)^{n+1}} \neq 0, \ \ \forall g_\gamma \in V_\beta^{**} \ \ \forall |x| = 1, \quad \forall z \in \mathcal{U}. \tag{6}$$

Let us consider for $z \in \mathcal{U}$ the continuous linear functional $\lambda_z : \mathcal{A}_0 \longrightarrow \mathbb{C}$, such that

$$\lambda_z(h) = h(z) * \sum_{k=1}^{\infty} \frac{\gamma z^{k-1}}{k(k-1+\gamma)} * \frac{1 + \frac{x+2\alpha-1}{2(1-\alpha)} z}{(1-z)^2} * \frac{1}{(1-z)^{n+1}} \neq 0.$$

By Duality principle we have $\lambda_z(V) = \lambda_z(V_\beta^{**})$. By Theorem 2.3, the inequality (6) holds if and only if

$$\left[1 + (1-\beta)(1-e^{i\psi}) \sum_{k=1}^{\infty} z^k\right] * \left[1 + \sum_{k=1}^{\infty} \frac{\gamma z^{k-1}}{k(k-1+\gamma)}\right] * \left[\frac{1 + \frac{x+2\alpha-1}{2(1-\alpha)} z}{(1-z)^2}\right] * \left[\frac{1}{(1-z)^{n+1}}\right] \neq 0, \tag{7}$$

for all $\psi \in \mathbb{R}$, $n \in \mathbb{N}_0$, $|x| = 1$ and $z \in \mathcal{U}$. Using the properties of convolution we can reformulate (7) as

$$\gamma \sum_{k=1}^{\infty} k^n \frac{k(1+x) + 2(1-\alpha)}{(k+1)(k+\gamma)} z^k \neq -\frac{2(1-\alpha)}{(1-e^{i\psi})(1-\beta)}. \tag{8}$$

For $\psi \in \mathbb{R}$ the quantity on the right side of (8) take its values on the line $Re\ w = -\frac{1-\alpha}{1-\beta}$ so (8) is equivalent to (4). □

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Derivations on the algebra of operators in Hilbert modules over locally C*-algebras

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Abstract

Let E be a Hilbert module over a locally-C*-algebra $\mathcal{A}$ and $\mathcal{L}_{\mathcal{A}}(E)$ be the algebra of all adjointable $\mathcal{A}$-module operators on E. We show that if $\mathcal{A}$ is a unital commutative locally-C*-algebra and $b(E)$, the set of all bounded elements of E, is a full Hilbert $b(A)$-module then every derivation on $\mathcal{L}_{\mathcal{A}}(E)$ is inner. If $\mathcal{A}$ be a commutative σ-C*-algebra with a countable approximate unit and E is full, then every derivation on $\mathcal{L}_{\mathcal{A}}(E)$ is a weakly approximately inner derivation. Moreover, the innerness of derivations on compact operators implies the innerness of derivations on $\mathcal{L}_{\mathcal{A}}(E)$.

Keywords: Hilbert modules, Locally C*-algebras, Derivations
Mathematics Subject Classification [2010]: 46L08, 46L05, 46L57

1 Introduction

Recall that a derivation of an algebra $\mathcal{A}$ is a linear mapping Δ from $\mathcal{A}$ into itself, such that $\Delta(ab) = \Delta(a)b + a\Delta(b)$ for all $a, b \in \mathcal{A}$. We say that Δ is inner if there exists $x \in \mathcal{A}$ such that $\Delta(a) = [a, x] = ax - xa$ for every $a \in \mathcal{A}$. One of the interesting problem in the theory of derivations is to identify those algebras on which all the derivations are inner, i.e. the first cohomology group is trivial. The first result of this problem is probably due to Kaplansky [6] who proved that every derivation of a type I W*-algebra is inner. In 1966, Sakai [8] extended the result of Kaplansky and proved that every derivation of a W*-algebra is inner. Finally Kadison [5] proved the innerness of derivation on von Neumann algebras.

A *locally C*-algebra* is a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C*-seminorm p on $\mathcal{A}$. A *σ-C*-algebra* is a locally C*-algebra whose topology is determined by a countable family of C*-seminorms. These algebras were first introduced by Inoue [3] as a generalization of C*-algebras and appear in the study of certain aspects of C*-algebras such as tangent algebras of C*-algebras, domain of closed $*$-derivations on C*-algebras, multipliers of Pedersen's ideal, noncommutative analogues of classical Lie groups, and K-theory. Let $\mathcal{S}(\mathcal{A})$ be the set of all continuous C*-seminorms on $\mathcal{A}$. For $p \in \mathcal{S}(\mathcal{A})$, $\mathcal{A}_p = \mathcal{A}/N_p$, where $N_p = \{a \in \mathcal{A} : \ p(a) = 0\}$ is a C*-algebra in the norm induced by

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p, and for $p, q \in \mathcal{S}(\mathcal{A})$, $p \geq q$ there is a canonical morphism π_{pq} from $\mathcal{A}_p$ onto $\mathcal{A}_q$ such that $\pi_{pq}(a + N_p) = a + N_q, a \in \mathcal{A}$. Then $\{\mathcal{A}_p; \pi_{pq}\}_{p,q\in\mathcal{S}(\mathcal{A}),p\geq q}$ is an inverse system of C*-algebras and the locally C*-algebras $\mathcal{A}$ and $\varprojlim_p \mathcal{A}_p$ are isomorphic. The canonical map from $\mathcal{A}$ onto $\mathcal{A}_p$ will be denoted by π_p and a_p is reserved to denote $a + N_p$. We denote by $b(\mathcal{A})$ the set of all elements $a \in \mathcal{A}$ such that

$$\|a\|_\infty := \sup\{p(a) : \ p \in \mathcal{S}(\mathcal{A})\} < \infty.$$

Then $b(\mathcal{A})$ is a C*-algebra with respect to the norm $\|.\|_\infty$ and is dense in $\mathcal{A}$. An approximate unit of a locally C*-algebra $\mathcal{A}$ is an increasing net $\{e_i\}_{i\in I}$ of positive elements of $\mathcal{A}$ such that $p(e_i) \leq 1$ for all $i \in I$ and $p \in \mathcal{S}(\mathcal{A})$; $p(ae_i - a) \to 0$ and $p(e_i a - a) \to 0$ for all $p \in \mathcal{S}(\mathcal{A})$ and $a \in \mathcal{A}$. Any locally C*-algebra has an approximate unit.

In 1992, R. Becker [1] proved that if $\mathcal{A}$ be a locally C*-algebra such that every derivation on each C*-quotient of $\mathcal{A}$ is inner, then every derivation Δ on $\mathcal{A}$ is approximately inner, i.e. there exists a net $\{h_i\}_{i\in I}$ in $\mathcal{A}$ such that $\Delta(a) = \mathrm{lim_i[h_i, a]}$ for all $a \in \mathcal{A}$. In 1995, N. C. Phillips [7] improved the previous result of Becker by using interesting techniques. He dropped the assumption of the innerness of the derivations of the C*-quotient algebras of $\mathcal{A}$ and proved that every derivation of a locally C*-algebra is approximately inner. This note is devoted to the study of innerness of derivations on the algebra of operators in Hilbert modules over locally C*-algebras

Let us we present some definitions and basic facts about Hilbert modules over locally C*-algebras. A (right) *pre-Hilbert module* over a locally C*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module E, compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to A \ \ (x, y) \mapsto \langle x, y\rangle$, which is $\mathcal{A}$-linear in the second variable y and has the properties:

$$\langle x, y\rangle = \langle y, x\rangle^*, \text{ and } \langle x, x\rangle \geq 0 \text{ with equality if and only if } x = 0.$$

A pre-Hilbert $\mathcal{A}$-module E is a Hilbert $\mathcal{A}$-module if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p\in\mathcal{S}(\mathcal{A})}$ where $\overline{p}_E(\xi) = \sqrt{p(\langle\xi, \xi\rangle)}$, $\xi \in E$. Hilbert modules over locally C*-algebras have been studied systematically in the book [4]. Denote by $\langle E, E\rangle$ the closure of the linear span of all $\langle x, y\rangle, x, y \in E$. We call E is *full* if $\langle E, E\rangle = \mathcal{A}$. One can always consider any Hilbert module over locally C*-algebra $\mathcal{A}$ as a full Hilbert module over locally C*-algebra $\langle E, E\rangle$. Let $p \in \mathcal{S}(\mathcal{A})$ then $N_p^E = \{\xi \in E;\ \bar{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E/N_p^E$ is a Hilbert $\mathcal{A}_p$-module with $(\xi + N_p^E)\pi_p(a) = \xi a + N_p^E$ and $\langle\xi + N_p^E, \eta + N_p^E\rangle = \pi_p(\langle\xi, \eta\rangle)$. The canonical map from E onto E_p will denote by σ_p^E and ξ_p ie reserved to denote $\sigma_p^E(\xi)$. For $p, q \in \mathcal{S}(\mathcal{A})$ with $p \geq q$, there is a canonical morphism σ_{pq}^E from E_p onto E_q such that $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ for all $\xi \in E$. Then $\{E_p;\ \mathcal{A}_p;\ \sigma_{pq}^E,\ \pi_{pq}\}_{p,q\in\mathcal{S}(\mathcal{A}),p\geq q}$ is an inverse system of Hilbert C*-modules in the following sense:

- $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p)\pi_{pq}(a_p),\ \xi_p \in E_p,\ a_p \in \mathcal{A}_p,\ p, q \in \mathcal{S}(\mathcal{A}),\ p \geq q,$
- $\langle\sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p)\rangle = \pi_{pq}(\langle\xi_p, \eta_p\rangle),\ \xi_p, \eta_p \in E_p, p, q \in \mathcal{S}(\mathcal{A}),\ p \geq q,$
- $\sigma_{qr}^E \ o\ \sigma_{pq}^E = \sigma_{pr}^E$ if $p, q, r \in \mathcal{S}(\mathcal{A}), p \geq q \geq r,$
- $\sigma_{pp}^E(\xi_p) = \xi_p,\ \xi \in E,\ p \in \mathcal{S}(\mathcal{A}).$

In this case $\varprojlim_p E_p$ is a Hilbert $\mathcal{A}$-module which can be identified with E. We denote by $b(E)$ the set of all elements $x \in E$ such that

$$\|x\|_\infty := \sup\{\overline{p}_E(x) : p \in S(\mathcal{A})\} < \infty.$$

Then $b(E)$ is a Hilbert $b(\mathcal{A})$-module and is dense in E. Let E and F be Hilbert $\mathcal{A}$-modules and $T : E \to F$ be an $\mathcal{A}$-module map. The module map T is called bounded if for each $p \in \mathcal{S}(\mathcal{A})$, there is $k_p > 0$ such that $\bar{p}_E(Tx) \leq k_p\ \bar{p}_E(x)$ for all $x \in E$. The module map T is called adjointable if there exists an $\mathcal{A}$-module map $T^* : F \to E$ with the property $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E, y \in F$. It is well-known that every adjointable $\mathcal{A}$-module map is bounded. The set $\mathcal{L}_\mathcal{A}(E, F)$ of all bounded adjointable $\mathcal{A}$-module maps from E into F becomes a locally convex space with topology defined by the family of seminorms $\{\tilde{p}\}_{p\in\mathcal{S}(\mathcal{A})}$, in which, $\tilde{p}(T) = \|(\pi_p)_*(T)\|_{\mathcal{L}_{\mathcal{A}_p}(E_p,F_p)}$ and $(\pi_p)_* : \mathcal{L}_\mathcal{A}(E, F) \to \mathcal{L}_{\mathcal{A}_p}(E_p, F_p)$ is defined by $(\pi_p)_*(T)(\xi + N_p^E) = T\xi + N_p^F$ for all $T \in \mathcal{L}_\mathcal{A}(E, F), \xi \in E$. Let $p, q \in \mathcal{S}(\mathcal{A}), p \geq q$ and $(\pi_{pq})_* : L_{\mathcal{A}_p}(E_p, F_p) \to L_{\mathcal{A}_q}(E_q, F_q)$ is defined by $(\pi_{pq})_*(T_p)(\sigma_q^E(\xi)) = \sigma_{pq}^F(T_p(\sigma_p^E(\xi)))$. Then $\{\mathcal{L}_{\mathcal{A}_p}(E_p, F_p);\ (\pi_{pq})_*\}_{p,q\in\mathcal{S}(\mathcal{A}),p\geq q}$ is an inverse system of Banach spaces and $\varprojlim_p \mathcal{L}_{\mathcal{A}_p}(E_p, F_p)$ can be identified to $\mathcal{L}_\mathcal{A}(E, F)$. In particular, topologizing, $\mathcal{L}_\mathcal{A}(E, E)$ becomes a locally C*-algebra which is abbreviated by $\mathcal{L}_\mathcal{A}(E)$. By definition, the set of all compact operators $\mathcal{K}_\mathcal{A}(E)$ on E is defined as the closure of the set of all finite linear combinations of the operators $\{\theta_{x,y} :\ \theta_{x,y}(\xi) = x\langle y, \xi\rangle,\ x, y, \xi \in E\}$. It is a locally C*-subalgebra and a two sided ideal of $\mathcal{L}_\mathcal{A}(E)$ and moreover $\mathcal{K}_\mathcal{A}(E)$ may be identified to $\varprojlim_p \mathcal{K}_{\mathcal{A}_p}(E_p)$.

Definition 1.1. A derivation $\Delta : \mathcal{L}_\mathcal{A}(E) \to \mathcal{L}_\mathcal{A}(E)$ is called *weakly approximately inner* if there exists a net $\{T_i\}_{i\in I}$ in $\mathcal{L}_\mathcal{A}(E)$ such that $\Delta(A)x = \mathrm{lim_i[T_i, A]x}$ for all $A \in \mathcal{L}_\mathcal{A}(E)$ and $x \in E$.

Since $\mathcal{L}_\mathcal{A}(E)$ is a locally C*-algebra by [7, Theorem 3], every derivation on $\mathcal{L}_\mathcal{A}(E)$ is approximately inner and so is a weakly approximately inner. In this paper, We show that if $\mathcal{A}$ is a unital commutative locally-C*-algebra and $b(E)$ is a full Hilbert $b(\mathcal{A})$-module then every derivation on $\mathcal{L}_\mathcal{A}(E)$ is inner. We use the concept of approximate unit and we construct the net $\{T_i\}_{i\in I}$ in Definition 1.1 for every derivation on $\mathcal{L}_\mathcal{A}(E)$ where $\mathcal{A}$ is a commutative σ-C*-algebra containing a countable approximate unit. Then we extend some results of the paper [2] in the context of locally C*-algebras and Hilbert modules over them. Indeed, we show that the innerness of derivations on $\mathcal{K}_\mathcal{A}(E)$ implies the innerness of derivations on $\mathcal{L}_\mathcal{A}(E)$.

2 Main results

Let $\mathcal{A}$ be a σ-C*-algebra which has a countable approximate unit and E be a Hilbert $\mathcal{A}$-module. If E is full then by [4, lemma 5.2.13], there is a sequence $\{x_n\}$ in E such that $p(\sum_{k=1}^n \langle x_k, x_k\rangle a - a) \to 0$ for all $p \in \mathcal{S}(\mathcal{A})$ and $a \in \mathcal{A}$. Moreover $\| \sum_{k=1}^n \langle x_k, x_k\rangle\|_\infty \leq 1$, for all n and so $\{\sum_{k=1}^n \langle x_k, x_k\rangle\}_n$ can be considered as a sequence in $b(\mathcal{A})$.

Lemma 2.1. *Let $\mathcal{A}$ be a locally C*-algebra and E be a Hilbert $\mathcal{A}$-module. If (a_n) be a sequence in $b(A)$ such that $p(aa_n - a) \to 0$ for all $a \in \mathcal{A}$ and for all $p \in \mathcal{S}(\mathcal{A})$ then $\bar{p}_E(a_nx - x) \to 0$ for all $x \in E$ and for all $p \in \mathcal{S}(\mathcal{A})$.*

Lemma 2.2. *Every derivation of a locally C^*-algebra annihilates its center.*

Theorem 2.3. *Let $\mathcal{A}$ be a unital commutative locally-C^*-algebra and E be a Hilbert $\mathcal{A}$-module such that $b(E)$ is a full $b(\mathcal{A})$-module. Then every derivation on $\mathcal{L}_{\mathcal{A}}(E)$ is inner.*

Theorem 2.4. *Let $\mathcal{A}$ be a commutative locally-C^*-algebra and E be a full Hilbert $\mathcal{A}$-module which contains a sequence $\{x_n\}$ such that $p(\sum_{k=1}^{n}\langle x_k, x_k\rangle a - a) \to 0$ for all $p \in \mathcal{S}(\mathcal{A})$ and $a \in \mathcal{A}$. Then for each positive integer n, the map T_n on E defined by $T_n x = \sum_{k=1}^{n} \Delta(\theta_{x,x_k})x_k$ is an element in $\mathcal{L}_{\mathcal{A}}(E)$ such that for every derivation Δ on $\mathcal{L}_{\mathcal{A}}(E)$,*

$$\Delta(A)x = \mathrm{lim_n}[\mathrm{T_n}, \mathrm{A}]\mathrm{x},$$

for all $A \in \mathcal{L}_{\mathcal{A}}(E)$ and $x \in E$, i.e. Δ is a weakly approximately inner derivation.

Corollary 2.5. *If $\mathcal{A}$ is a commutative σ-C^*-algebra contaning a countable approximate unit and E be a full Hilbert $\mathcal{A}$ module then every derivation on $\mathcal{L}_{\mathcal{A}}(E)$ is weakly approximately inner.*

The following theorem states that the innerness of derivations on $\mathcal{K}_{\mathcal{A}}(E)$ implies the innerness of derivations on $\mathcal{L}_{\mathcal{A}}(E)$.

Theorem 2.6. *Let $\mathcal{A}$ be a commutative σ-C^*-algebra with a countable approximate unit and let E be a full Hilbert $\mathcal{A}$-module. If every derivation on $\mathcal{K}_{\mathcal{A}}(E)$ is inner, than any derivation on $\mathcal{L}_{\mathcal{A}}(E)$ is also inner.*

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Disjoint Hypercyclicity of Composition Operators on the Weighted Dirichlet Spaces

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Abstract

In this paper, we discuss about disjoint hypercyclicity of composition operators on some Weighted Dirichlet spaces.

Keywords: Hypercyclicity, Disjoint hypercyclicity, composition operators, Weighted Dirichlet spaces.
Mathematics Subject Classification [2010]: 47A16, 47B33, 47B38

1 Introduction

Let X be a topological vector space and T a bounded linear operator on X. The T-orbit of a vector $x \in X$ is the set

$$O(x,T) := \{T^n(x) : n \in \mathbb{N} \cup \{0\}\}.$$

Definition 1.1. The operator T is said to be hypercyclic if there exists a vector $x \in X$ such that $O(x,T)$ is dense in X. Such a vector x is said to be hypercyclic vector for T.

It is known that the direct sum of two hypercyclic operators need not be hypercyclic, see [5]. Finitely many hypercyclic operators acting on a common topological vector space are called disjoint if their direct sum has a hypercyclic vector on the diagonal of the product space.

Definition 1.2. For $N \geq 2$, the operators $T_1, T_2, ..., T_N$ are called disjoint hypercyclic or d-hypercyclic if the direct sum $T_1 \oplus T_2 \oplus ... \oplus T_N$ has a hypercyclic vector of the form $(x, x, ..., x) \in X^N$.

Definition 1.3. Let $\{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$. The Weighted Hardy space $H^2(\beta)$ is defined as the space of functions $f = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ analytic on $\mathbb{D}$ such that $\| f \|_{\beta}^2 = \sum_{n=0}^{\infty} | \hat{f}_n |^2 \beta(n)^2 < \infty$. Let $\beta(n) = (n+1)^{\nu}$, where ν is a real number. These spaces are known as weighted Dirichlet spaces or $\mathcal{S}_{\nu}$.

Definition 1.4. Let φ be a holomorphic self map of unit disk $\mathbb{D}$. A composition operator on $\mathcal{S}_{\nu}$, C_{φ}, is defined by $C_{\varphi} f = f o \varphi$ for all $f \in \mathcal{S}_{\nu}$.

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Theorem 1.5. ***(D-Hypercyclicity Criterion)*** *Suppose X is a topological vector space and $T_1, T_2, ..., T_N$ are bounded linear operator on X. If there exist an increasing sequence of positive integers $\{n_k\}$ and dense subsets $X_0, X_1, ..., X_N$ of X and mappings $S_{m,k} : X_m \to X$ where $k \in \mathbb{N}, 1 \le m \le N$, such that*

(i) $T_m^{n_k} \to 0$ point wise on X_0 as $k \to \infty$,

(ii)$S_{m,k} \to 0$ point wise on X_m as $k \to \infty$ and

(iii) $(T_i^{n_k} S_{m,k} - \delta_{i,m} Id_{X_m}) \to 0$ point wise on X_m $(1 \le i \le N)$.

Then $T_1, T_2, ...T_N$ are d-hypercyclic.

Theorem 1.5 was proved in [2]. It is a essential tool for proof of main theorem.

2 Main results

For a positive integer n, the nth iterate of φ is denoted by $\varphi^{[n]}$ and when φ is invertible $\varphi^{[-n]}$ is the nth iterate of φ^{-1}.

The holomorphic self maps of the unit disk are divided into two classes, elliptic and non-elliptic functions. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. The non-elliptic one has a unique fixed point $p \in \overline{\mathbb{D}}$, called the Denjoy-Wolff point of φ, which is known as attractive fixed point, that is the sequence of iterates of φ, $\{\varphi^{[n]}\}_n$ converges to p uniformly on compact subsets of $\mathbb{D}$ (see [4] for more details).

The following lemma that will be proved is useful in the proof of main theorem:

Lemma 2.1. *Let A be a finite set of complex scalars with $A \cap \mathbb{D} = \emptyset$. The set of polynomials that vanishing m times on A is dense in $\mathcal{S}_\nu$, where $m \in \mathbb{N}$ and $\nu < \frac{1}{2}$.*

Theorem 2.2 is the our main theorem:

Theorem 2.2. *Let $C_{\varphi_1}, ...C_{\varphi_N}$ for $N \ge 2$ be hypercyclic composition operators on $\mathcal{S}_\nu$, where $\varphi_1, ..., \varphi_N$ are linear fractional transformations and $\nu < \frac{1}{2}$. Suppose that for each $1 \le l, j \le N$ with $l \ne j$ we have*

$$(\varphi_l^{[-n]} o \varphi_j^{[n]})(z) \to \gamma_l$$

as $n \to \infty$ and for almost all $z \in \mathbb{D}$, where γ_l is a fixed point of φ_l. Then $C_{\varphi_1}, ...C_{\varphi_N}$ are d-hypercyclic.

Corollary 2.3. *Let $C_{\varphi_1}, ...C_{\varphi_N}$ for $N \ge 2$ be hypercyclic composition operators on $\mathcal{S}_\nu$, where $\varphi_1, ..., \varphi_N$ are linear fractional transformations $\nu < \frac{1}{2}$. If the attractive fixed points of $\varphi_1, ..., \varphi_N$ are all distinct, then $C_{\varphi_1}, ...C_{\varphi_N}$ are d-hypercyclic.*

Corollary 2.3 is a direct consequence of Theorem 2.3.

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Eigenvalues of Euclidean Distance Matrices and rs-majorization on $\mathbb{R}^2$

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Abstract

Let D_1 and D_2 be two Euclidean distance matrices (EDMs) with corresponding positive semidefinite matrices B_1 and B_2 respectively. Suppose that $\lambda(A) = ((\lambda(A))_i)_{i=1}^n$ is the vector of eigenvalues of a matrix A such that $(\lambda(A))_1 \geq \ldots \geq (\lambda(A))_n$. In this paper, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to $\prec_{rs}$, on $\mathbb{R}^2$ will be investigated.

Keywords: Euclidean distance matrices, Rs-majorization.
Mathematics Subject Classification [2010]: 34B15, 76A10

1 Introduction

An $n \times n$ nonnegative and symmetric matrix $D = (d_{ij}^2)$ with zero diagonal elements is called a predistance matrix. A predistance matrix D is called Euclidean or a Euclidean distance matrix (EDM) if there exist a positive integer r and a set of n points $\{p_1, \ldots, p_n\}$ such that $p_1, \ldots, p_n \in \mathbb{R}^r$ and $d_{ij}^2 = \|p_i - p_j\|^2$ $(i, j = 1, \ldots, n)$, where $\|.\|$ denotes the usual Euclidean norm. The smallest value of r that satisfies the above condition is called the embedding dimension. As is well known, a predistance matrix D is Euclidean if and only if the matrix $B = \frac{-1}{2}PDP$ with $P = I_n - \frac{1}{n}ee^t$, where I_n is the $n \times n$ identity matrix, and e is the vector of all ones, is positive semidefinite matrix. Let Λ_n be the set of $n \times n$ EDMs, and $\Omega_n(e)$ be the set of $n \times n$ positive semidefinite matrices B such that $Be = 0$. Then the linear mapping $\tau : \Lambda_n \to \Omega_n(e)$ defined by $\tau(D) = \dfrac{-1}{2}PDP$ is invertible, and its inverse mapping, say $\kappa : \Omega_n(e) \to \Lambda_n$ is given by $\kappa(B) = be^t + eb^t - 2B$ with $b = diag(B)$, where $diag(B)$ is the vector consisting of the diagonal elements of B. For general refrence on this topic see, e.g. [1].

Majorization is one of the vital topics in mathematics and statistics. It plays a basic role in matrix theory. One can see some type of majorization in [2]-[13]. In this paper, the relation between the eigenvalues of EDMs and those of the corresponding positive

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semidefinite matrices respect to $\prec_{rs}$ on $\mathbb{R}^2$ will be investigated. An nonnegative matrix R is called row stochastic if the sum of entries of each row of R is equal to one.

The following notation will be fixed throughout the paper.
$Co(A) := \{\sum_{i=1}^{m} \lambda_i a_i \mid m \in \mathbb{N},\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1,\ a_i \in A,\ \ i \in \mathbb{N}_m\}$,
for a subset $A \subset \mathbb{R}^n$;
$Sgn\{\alpha\}$ be 1 if $\alpha > 0$ and be -1 if $\alpha < 0$, $Sgn\{0\}$ can be 1 or -1;
$[T]$ be the matrix representation of a linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis;
r_i be the sum of entries on the ith row of $[T]$.
A linear function T: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be a linear preserver (strong linear preserver) of $\sim$ if $T(x) \sim T(y)$ whenever $x \sim y$ ($T(x) \sim T(y)$ if and only if $x \sim y$).

1.1 Rs-majorization

We introduce the relation $\prec_{rs}$ on $\mathbb{R}^n$ and we state some properties of rs-majorization on $\mathbb{R}^2$.

Definition 1.1. A matrix $R \in \mathbf{M}_n$ with nonnegative entries is called row stochastic if the sum of entries of each row of R is equal to one.

Definition 1.2. For two real vector x and y, we say that x is *rs-majorized* by y (denoted by $x \prec_{rs} y$) if there exists an n-by-n row stochastic matrix R with all its column entries equal such that $x = Ry$.

In this paper, we consider this relation on $\mathbb{R}^2$. The following proposition gives an equivalent condition for rs-majorization on $\mathbb{R}^2$.

Proposition 1.3. *Let $x = (x_1, x_2)^t$, $y = (y_1, y_2)^t \in \mathbb{R}^2$. Then $x \prec_{rs} y$ if and only if $x_1 = x_2 \in \mathcal{C}\{y_1, y_2\}$.*

Here we state all (resp. strong) linear preservers of $\prec_{rs}$ on $\mathbb{R}^2$.

Theorem 1.4. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function, and let $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then T preserves $\prec_{rs}$ if and only if $r_1 = r_2$, $Sgn\{a\} = Sgn\{d\} \neq Sgn\{b\} = Sgn\{c\}$.*

Theorem 1.5. *A linear function $T : \mathbb{R}^2 \to \mathbb{R}^2$ strongly preserves $\prec_{rs}$ if and only if $[T] = \alpha I$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

2 Main results

Till the end of this section, the relation between the eigenvalues of EDMs and those of the corresponding positive semidefinite matrices respect to $\prec_{rs}$ on $\mathbb{R}^2$ will be specify.

Theorem 2.1. *Let $B, \widetilde{B} \in \Omega_2(e)$, and let $D = \kappa(B)$ and $\widetilde{D} = \kappa(\widetilde{B})$. Then*
$\lambda(B) \prec_{rs} \lambda(\widetilde{B}) \Longleftrightarrow \lambda(D) \prec_{rs} \lambda(\widetilde{D})$

Proof. Since $B, \widetilde{B} \in \Omega_2(e)$, there exist $\alpha,\beta \geq 0$ such that $B = \begin{pmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}$, $\widetilde{B} = \begin{pmatrix} \beta & -\beta \\ -\beta & \beta \end{pmatrix}$, and $\{0, 2\alpha\}$ and $\{0, 2\beta\}$ are the set of eigenvalues of B and $\widetilde{B}$, respectively. By the definition of κ, $D = \begin{pmatrix} 0 & 4\alpha \\ 4\alpha & 0 \end{pmatrix}$ and $\widetilde{D} = \begin{pmatrix} 0 & 4\beta \\ 4\beta & 0 \end{pmatrix}$. So $\{-4\alpha, 4\alpha\}$ and $\{-4\beta, 4\beta\}$ are the set of eigenvalues of D and $\widetilde{D}$, respectively.
We see that $\lambda(B) \prec_{rs} \lambda(\widetilde{B})$ if and only if $B = 0$. Also, if $\lambda(D) \prec_{rs} \lambda(\widetilde{D})$ if and only if $D = 0$. Hence $\lambda(B) \prec_{rs} \lambda(\widetilde{B})$ if and only if $\lambda(D) \prec_{rs} \lambda(\widetilde{D})$.

□

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Existence of three solutions for a problem involving the p(x)-Laplacian

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Abstract

In this article, we study $p(x)-$Laplacian problem on a bounded domain and obtain three solutions under appropriate hypotheses. The technical approach is mainly based on the three critical points theorem obtained by Ricceri.

Keywords: three solutions, $p(x)-$Laplacian
Mathematics Subject Classification [2010]: 35J65, 35J60, 47J30, 58E05

1 Introduction

Variational-hemivariational inequalities have been extensively studied in recent years via variational methods: in (cf. [2]), Bonanno and Candito studied a class of variational-hemivariational inequalities; in (cf. [6]), Kristály studied hemivariational inequalities on an unbounded strip-like domain; In (cf. [1]), Alimohammady studied variational-hemivariational inequality on bounded domains by using the mountain pass theorem and the critical point theory for Motreanu-Panagiotopoulos type functionals.

In this paper we study the following nonlinear differential inclusion with $p(x)-$Laplacian

$$\begin{cases} -\Delta_{p(x)}u = -\mu g(x,u) & in\,\Omega \\ -|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} \in -\lambda \partial F(x,u) & on\,\partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $p : \bar{\Omega} \to \mathbb{R}$ is a continuous function satisfying

$$1 < p^- = \min_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ = \max_{x\in\bar{\Omega}} p(x) < +\infty,$$

and $\lambda \in [0, \infty)$. $F : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(\cdot, u)$ is measurable for every $u \in \mathbb{R}$ and $F(x, \cdot)$ is locally Lipschitz for a.e. $x \in \partial\Omega$. Also $\partial F(x, u)$ denotes the generalized Clarke gradient of $F(x, u)$ at $u \in \mathbb{R}$.

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Moreover, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $G(x,u) = \int_0^u g(x,t)dt$.
The generalized Lebesgue-Sobolev space $W^{L,p(x)}(\Omega)$ for $L = 1, 2, ...$ is defined as

$$W^{L,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq L\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$ with $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$.
In this paper, we denote by $X = W^{1,p(x)}(\Omega)$ and $X^\star$ the dual space.
For a locally Lipschitz function $h : X \to \mathbb{R}$ we define the generalized directional derivative of h at $u \in X$ in the direction $\gamma \in X$ by

$$h^0(u;\gamma) = \limsup_{w \to u, t \to 0^+} \frac{h(w + t\gamma) - h(w)}{t}.$$

The generalized gradient of h at $u \in X$ is defined by

$$\partial h(u) = \{x^\star \in X^\star \ : < x^\star, \gamma >_X \leq h^0(u;\gamma), \ \forall \gamma \in X\},$$

which is a nonempty, convex and $w^\star-$compact subset of $X^\star$, where $< \cdot, \cdot >_X$ is the duality pairing between $X^\star$ and X.

Lemma 1.1. *(cf. [3]) Let $h, g : X \to \mathbb{R}$ be a locally Lipschitz function. Then we have:*
(i) $h^0(u;\cdot)$ is subadditive, positively homogeneous.
(ii) $(-h)^0(u;z) = h^0(u;-z), \quad \forall u, z \in X$.
(iii) $h^0(u;v) = \max\{< \xi, v > : \ \xi \in \partial h(u)\}, \ \forall v \in X$.
(iv) $(h+g)^0(u;v) \leq h^0(u;v) + g^0(u;v), \ \forall v \in X$.

Lemma 1.2. *(cf. [4]) For $p, q \in C_+(\overline{\Omega})$ such that $q(x) \leq p_L^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{L,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

Theorem 1.3. *(cf. [5]) Let X be a separable and reflexive Banach space, Λ be a real interval, $\mathcal{B}$ a nonempty, closed, convex subset of X. $\phi \in C^1(X,\mathbb{R})$ a sequentially weakly l.s.c. functional, bounded on any bounded subset of X, such that ϕ' is of type $(S)_+$, $\mathcal{F} : X \to \mathbb{R}$ a locally Lipschitz functional with compact gradient. Assume that:*
(i) $lim_{\|u\| \to +\infty}[\phi - \lambda\mathcal{F}] = +\infty, \quad \forall \lambda \in \Lambda$,
(ii) There exists $\rho_0 \in \mathbb{R}$ such that

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} [\phi + \lambda(\rho_0 - \mathcal{F}(u))] < \inf_{u \in X} \sup_{\lambda \in \Lambda} [\phi + \lambda(\rho_0 - \mathcal{F}(u))].$$

Then, there exist $\lambda_1, \lambda_2 \in \Lambda$ $(\lambda_1 < \lambda_2)$ and $\sigma > 0$ such that, for every $\lambda \in [\lambda_1, \lambda_2]$ and every locally Lipschitz functional $\mathcal{G} : X \to \mathbb{R}$ with with compact gradient, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ the functional $\phi - \lambda\mathcal{F} + \mu\mathcal{G}$ has at least three critical points whose norms are less than σ.

2 Main results

Remark 2.1. (i) By the proposition (1.2) there is a continuous and compact embedding of $W^{1,p(x)}(\Omega)$ into $L^{q(x)}$ where $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.
(ii) Define

$$\|u\| = \inf\{\lambda > 0 \ : \ \int_\Omega |\frac{\nabla u}{\lambda}|^{p(x)} dx \leq 1\},$$

is a norm on $W^{1,p(x)}(\Omega)$.

Theorem 2.2. *Every critical point of the functional $\mathcal{I}$ is a solution of Problem* (1).

Theorem 2.3. *Let Ω, p, F be as mentioned. Then, there exist $\lambda_1, \lambda_2 > 0 (\lambda_1 < \lambda_2)$ and $\sigma > 0$ such that for every $\lambda \in [\lambda_1, \lambda_2]$ and every $\mathcal{G}$ as above, satisfying G, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ problem* (1) *admits at least three solutions whose norms are less than σ.*

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Fekete-Szego Problem for New Subclasses of Univalent Functions with bounded positive real part

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Abstract

In this paper we solve Fekete-Szego problem for $M_\lambda(\alpha,\beta)$ in the open unit disk Δ which maps Δ onto the strip domain ω with $\alpha < Re\omega < \beta$.

Keywords: Univalent functions, Fekete-Szego Problem, Subordination.
Mathematics Subject Classification [2010]: 30C45

1 Introduction

Let A denote the class of functions $f(z)$ of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disck $\Delta = \{z \in C : |z| < 1\}$. The subclass of A, Consisting of all univalent functions $f(z)$ in Δ is denoted by S.
Let f and g be analytic in Δ. The function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function ω such that $\omega(0) = 0$, $|\omega(z)| < 1$, and $f(z) = g(\omega(z))$ on Δ.
Authors in [1,3] proved Fekete-Szego problem for subclasses of univalen functions, In this paper we introdused new subclasses of univalent functions and we solved Fekete-Szego problem for the subclasses. We denoted the subclasses with $M_\lambda(\alpha,\beta)$.

2 Main results

To prove our main results we shall need the following definitions and lemmas.

Definition 2.1. : Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in A$ belongs to the class $\nu(\alpha,\beta)$ satisfies the following inequality;

$$\alpha < Re\{(\frac{z}{f(z)})^2 f'(z)\} < \beta \qquad (z \in \Delta). \tag{2}$$

*Speaker

Definition 2.2. : Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in A$ belongs to the class $\omega(\alpha, \beta)$ satisfies the following inequality;

$$\alpha < Re\{\frac{1}{f'(z)}(\frac{zf''(z)}{f'(z)} + 1)\} < \beta \qquad (z \in \Delta). \tag{3}$$

Definition 2.3. : Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in A$ belongs to the class $M_\lambda(\alpha, \beta)$ satisfies the following inequality;

$$\alpha < Re\{(1-\lambda)(\frac{z}{f(z)})^2 f'(z) + \frac{\lambda}{f'(z)}(\frac{zf''(z)}{f'(z)} + 1)\} < \beta \qquad (z \in \Delta). \tag{4}$$

Remark 2.4. we note that $M_0(\alpha, \beta) = \nu(\alpha, \beta)$ and $M_1(\alpha, \beta) = \omega(\alpha, \beta)$.

Now, we define an analytic function $S_{\alpha,\beta}(z) : \Delta \longrightarrow \mathbf{C}$ by

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} i log(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}) \tag{5}$$

due to Kuroki and Owa[4] and they proved $S_{\alpha,\beta}(z)$ maps Δ onto a convex domain ω with $\alpha < Re(\omega) < \beta$, conformaly. Using this fact and the definition of subordination, we can obtain the following lemmas, directly:

Lemma 2.5. *Let $f \in A$ and $0 \leq \alpha < \alpha < 1 < \beta$, Then $f \in M_\lambda(\alpha, \beta)$ if and only if*

$$(1-\lambda)(\frac{z}{f(z)})^2 f'(z) + \frac{\lambda}{f'(z)}(\frac{zf''(z)}{f'(z)} + 1) \prec 1 + \frac{\beta - \alpha}{\pi} i log(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}) \tag{6}$$

By taking $\lambda = 0$ and $\lambda = 1$, we state the following lemmas respectively:

Lemma 2.6. *Let $f \in A$, Then $f \in \nu(\alpha, \beta)$ if and only if*

$$(\frac{z}{f(z)})^2 f'(z) \prec 1 + \frac{\beta - \alpha}{\pi} i log(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}) \tag{7}$$

where $\alpha < 1, \beta > 1$.

Lemma 2.7. *Let $f \in A$, Then $f \in \omega(\alpha, \beta)$ if and only if*

$$\frac{1}{f'(z)}(\frac{zf''(z)}{f'(z)} + 1) \prec 1 + \frac{\beta - \alpha}{\pi} i log(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}) \tag{8}$$

where $\alpha < 1, \beta > 1$.

We note that

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi} i log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) = 1 + \sum_{n=1}^{\infty} B_n z^n, \quad (9)$$

where

$$B_n = \frac{2(\beta-\alpha)}{n\pi} \sin \frac{n\pi(1-\alpha)}{\beta-\alpha} \qquad (n = 1, 2, 3...). \quad (10)$$

Using the subordination 6 and applying following lemma due to Rogosinski [6] we solve Fekete-szego problem for $f \in M_\lambda(\alpha, \beta)$.

Lemma 2.8. *Let $P(z) = \sum_{n=1}^{\infty} A_n z^n$ and $Q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic in Δ, if $P(z) \prec Q(z) \quad (z \in \Delta)$, then*

$$\sum_{k=1}^{m} |A_k|^2 \leq \sum_{k=1}^{m} |B_k|^2, \qquad (m = 1, 2, 3, ...).$$

Theorem 2.9. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_\lambda(\alpha, \beta)$ then*

$$|(1 + 2\lambda)a_3 - (1 + 3\lambda)a_2^2| \leq \frac{\beta-\alpha}{\pi} \sin \frac{2\pi(1-\alpha)}{\beta-\alpha} \quad (11)$$

Proof. Let

$$\begin{aligned} P(z) &= (1-\lambda)\left(\frac{z}{f(z)}\right)^2 f'(z) + \frac{\lambda}{f'(z)}\left(\frac{zf''(z)}{f'(z)} + 1\right) \\ &= 1 + (a_3 - a_2^2 + 2\lambda a_3 - 3\lambda a_2^2)z^2 + (2a_4 - 4a_2a_3 + 6\lambda a_4 - 18\lambda a_2 a_3 + 18\lambda a_2^3)z^3 + ... \end{aligned}$$

and

$$S_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$$

where B_n is as in 10. Applying lemma 2.8, we can get the results as asserted. □

when $\lambda = 0$ and $\lambda = 1$ we state the following corollaries respectively:

Corollary 2.10. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \nu(\alpha, \beta)$, then*

$$|a_3 - a_2^2| \leq \frac{\beta-\alpha}{\pi} \sin \frac{2\pi(1-\alpha)}{\beta-\alpha}.$$

Corollary 2.11. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \omega(\alpha, \beta)$, then*

$$|a_3 - \frac{4}{3}a_2^2| \leq \frac{1}{3}\frac{\beta-\alpha}{\pi} \sin \frac{2\pi(1-\alpha)}{\beta-\alpha}.$$

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Talk Fixed point theorems in probabilistic metric space and intuitionistic... pp.: 1–4

Fixed point theorems in probabilistic metric space and intuitionistic probabilistic metric space

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Abstract

In this paper, we introduce the non-Archimedean Menger PM-space, Φ-functions, intuitionistic probabilistic metric space and then prove fixed point theorems for family of self-mapping and generalized contraction mapping.

Keywords: non-Archimedean probabilistic Menger space, intuitionistic probabilistic metric space, t-representable
Mathematics Subject Classification [2010]: 47H10, 54H25

1 Introduction

The triangular norm (t-norm) and the triangular conorm (t-conorm) originated from the studies of probabilistic metric spaces [5, 6] in which triangular inequalities were extended using the theory of t-norm and t-conorm. Non-Archimedean probabilistic metric spaces first studied by Isratescu and Crivat [3]. Some fixed point theorems for mappings on non-Archimedean Menger spaces have been proved by Isratescu [1, 2]. Menger [5] initiated the study of probabilistic metric space in 1942 and by now the theory of probabilistic metric spaces has already made a considerable progress in several directions. Kutukcu et. al. [4] introduced the notion of intuitionistic Menger spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger [5].

Definition 1.1. A t-norm is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ which is commutative, associative, nondecreasing for each variable and $a * 1 = a$, for all $a \in [0, 1]$.

Definition 1.2. A distance distribution function is a function $F : [0, \infty] \to [0, 1]$, that is non-decreasing and left continuous on $\mathbb{R}$, moreover, $F(0) = 0$ and $F(\infty) = 1$.
The set of all the distance distribution functions ($d.d.f.$) is denoted by $\triangle^+$. In particular for every $x_0 \geq 0$, ε_{x_0} is the $d.d.f.$ defined by $\varepsilon_{x_0} = \begin{cases} 1 & \text{if } x > x_0, \\ 0 & \text{if } x \leq x_0. \end{cases}$

Definition 1.3. Let X be a non-empty set. A non-Archimedean Menger PM-space is an ordered triple $(X, F, *)$ where $*$ is a t-norm and F is a function from $X \times X$ into $\triangle^+$. satisfying the following conditions: $F_{x,y}(t) = 1$, $t > 0$, if and only if $x = y$; $F_{x,y}(t) = F_{y,x}(t)$; $F_{x,y}(0) = 0$ and $F_{x,y}(\max\{t, s\}) \geq F_{x,z}(t) * F_{z,y}(s)$, for all $x, y, z \in X$, $s, t \geq 0$.

*Speaker

2 Main results

Definition 2.1. We denoted by Φ the class of all Φ-functions $\phi : \mathbb{R} \to \mathbb{R}$ if:
$\phi(t) = 0$ if and only if $t = 0$; $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$; ϕ is left continuous in $(0, \infty)$ and continuous at 0.

Theorem 2.2. *Let $(X, F, *)$ be a G-complete PM-Menger space endowed with minimum t-norm and $\{T_\alpha\}_{\alpha \in J}$ be a family of self-mapping of X. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$*

$$\frac{1}{F_{T_\alpha x, T_\beta y}(\phi(\lambda t))} - 1 \leq \lambda max\{(\frac{1}{F_{x,y}(\phi(t))} - 1), (\frac{1}{F_{x,T_\alpha x}(\phi(t))} - 1) ,(\frac{1}{F_{y,T_\beta y}(\phi(t))} - 1), (\frac{1}{F_{x,T_\beta y}(\phi(t))} - 1), (\frac{1}{F_{y,T_\alpha x}(\phi(2t))} - 1)\} \quad (1)$$

for some $\lambda = \lambda(\alpha)$ and for each $x, y \in X$,$t > 0$. Then all T_α have a unique common fixed point in X and at this point each T_α is continuous.

Theorem 2.3. *Let $(X, F, *)$ be a complete non-Archimedean PM-Menger space endowed with minimum t-norm and $\{T_\alpha\}_{\alpha \in J}$ be a family of self-mapping of X. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$*

$$\frac{1}{F_{T_\alpha x, T_\beta y}(\phi(\lambda t))} - 1 \leq \lambda max\{(\frac{1}{F_{x,y}(\phi(t))} - 1), (\frac{1}{F_{x,T_\alpha x}(\phi(t))} - 1) ,(\frac{1}{F_{y,T_\beta y}(\phi(t))} - 1), (\frac{1}{F_{x,T_\beta y}(\phi(t))} - 1), (\frac{1}{F_{y,T_\alpha x}(\phi(t))} - 1)\} \quad (2)$$

for some $\lambda = \lambda(\alpha)$ and for each $x, y \in X$, $t > 0$. Then all T_α have a unique common fixed point in X and at this point each T_α is continuous.

Theorem 2.4. *Let $(X, F, *)$ be a G-complete PM-Menger space endowed with minimum t-norm. The following property is equivalent to completeness of X :*
If Y is any non-empty closed subset of X and $T : Y \to Y$ is any generalized contraction mapping then T has a fixed point in Y.

Definition 2.5. A binary operation $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-conorm if $\Diamond$ is commutative, associative, nondecreasing for each variable and $a \Diamond 0 = a$ for all $a \in [0, 1]$.

Definition 2.6. A non-distance distribution function is a function $L : [0, \infty] \to [0, 1]$, that is non-increasing and left continuous on $[0, \infty]$, moreover, $L(0) = 1$ and $L(\infty) = 0$. The family of all non-distance distribution functions $(n.d.f.)$ is denoted by Γ^+. In particular for every $x_0 \geq 0$, ζ_{x_0} is the $n.d.f.$ defined by $\zeta_{x_0} = \begin{cases} 0 & \text{if } x > x_0, \\ 1 & \text{if } x \leq x_0. \end{cases}$

The collection of all pairs $(s_1, s_2) \in \triangle^+ \times \Gamma^+$ such that $s_1 + s_2 \leq 1$ will be denoted by Λ. We denote its unit by $1_\Lambda = (\varepsilon_0, \zeta_0)$.

Definition 2.7. An intuitionistic probabilistic metric space (abbreviated, IPM-space) is an ordered pair (X, μ), where X is a non-empty set and $\mu : X \times X \to \Lambda$ is defined by $\mu(p, q) = (F(p, q), L(p, q))(\mu(p, q)$ is denoted by $\mu_{p,q})$, satisfies the conditions:
$\mu_{pq}(t) = 1_\Lambda(t)$, iff $p = q$; $\mu_{pq}(t) = \mu_{qp}(t)$ and if $\mu_{pq}(t) = 1_\Lambda(t)$ and $\mu_{qr}(s) = 1_\Lambda(s)$, then $\mu_{pr}(s + t) = 1_\Lambda(s + t)$ for every $p, q, r \in X$ and $t, s \geq 0$.

Definition 2.8. A triangular norm (briefly, t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions for all $a, b, c, d \in L^*$: $\mathcal{T}(a, 1_{L^*}) = a$; $\mathcal{T}(a, b) = \mathcal{T}(b, a)$; $\mathcal{T}(a, \mathcal{T}(b, c)) = \mathcal{T}(\mathcal{T}(a, b), c)$ and if $a \leq_{L^*} c$ and $b \leq_{L^*} d$, then $\mathcal{T}(a, b) \leq_{L^*} \mathcal{T}(c, d)$, where $L^* = \{(a_1, a_2) : a_1, a_2 \in [0, 1] \quad and \quad a_1 + a_2 \leq 1\}$ and $1_{L^*} = (1, 0)$.

Definition 2.9. A continuous t-norm $\mathcal{T}$ on L^* is called continuous t-representable iff there exist a continuous t-norm T and a continuous t-conorm S on $[0, 1]$ such that, for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$, $\mathcal{T}(a, b) = (T(a_1, b_1), S(a_2, b_2))$.

Definition 2.10. An intuitionistic Menger space is a triple $(X, \mu, \mathcal{T})$, where (X, μ) is IPM-space and $\mathcal{T}$ is a continuous t-representable such that for all $p, q, r \in X$ and for all $t, s \geq 0$, $\mu_{pq}(t + s) \geq_{L^*} \mathcal{T}(\mu_{pr}(t), \mu_{rq}(s))$.

Definition 2.11. A function $\psi(t) : [0, \infty) \to [0, \infty)$ is saied to be a Ψ-function if:
$\psi(t)$ is strictly increasing; $\psi(0) = 0$ and $\lim_{n\to\infty} \psi^n(t) = \infty$ for all $t > 0$.

Theorem 2.12. *Let $(X, \mu, \mathcal{T})$ be a complete IPM-space. Let $T : X \to X$ be a mapping satisfying the following conditions:*
(i) there exists $x_0 \in X$ such that

$$\lim_{t\to\infty} F_{x_0,T^i x_0}(t) = 1 \quad and \quad \lim_{t\to\infty} L_{x_0,T^i x_0}(t) = 0, i = 1, 2, ...; \tag{3}$$

(ii) there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$,

$$F_{T^{m(x)}x,T^{m(x)}y}(t) \geq F_{x,y}(\psi(t)) \quad and \quad L_{T^{m(x)}x,T^{m(x)}y}(t) \leq L_{x,y}(\psi(t)), \tag{4}$$

where the function ψ is a Ψ-function and $\lim_{t\to\infty} [\psi(t) - t] = \infty$.
Then T has a unique fixed point x_*, and the quasi-iterative sequence $\{x_n : T^{m(x_n-1)}x_{n-1}\}$ converges to x_*.

Corollary 2.13. *Let $(X, \mu, \mathcal{T})$ be a complete IPM-space. Let $T : X \to X$ be a mapping satisfying the following conditions:*
(i) there exists $x_0 \in X$ such that

$$\lim_{t\to\infty} F_{x_0,T^i x_0}(t) = 1 \quad and \quad \lim_{t\to\infty} L_{x_0,T^i x_0}(t) = 0, i = 1, 2, ...;$$

(ii) there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$,

$$F_{T^{m(x)}x,T^{m(x)}y}(t) \geq F_{x,y}(\frac{t}{k}) \quad and \quad L_{T^{m(x)}x,T^{m(x)}y}(t) \leq L_{x,y}(\frac{t}{k}),$$

where $0 < k < 1$. Then the conclusion of Theorem 2.11 remains true.

Corollary 2.14. *Let $(X, \mu, \mathcal{T})$ be a complete IPM-space. Let $T : X \to X$ be a mapping. If there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$,*

$$F_{T^{m(x)}x,T^{m(x)}y}(t) \geq F_{x,y}(\psi(t)) \quad and \quad L_{T^{m(x)}x,T^{m(x)}y}(t) \leq L_{x,y}(\psi(t)),$$

where the function ψ is a Ψ-function and $\lim_{t\to\infty} [\psi(t) - t] = \infty$. Then T has a unique fixed point x_*, and the iterative sequence $\{T^n x\}$ converges to x_* for every $x \in X$.

Theorem 2.15. *Let $(X, \mu, \mathcal{T})$ be a complete IPM-space with $t*t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$, and $T : X \to X$ be a continuous mapping satisfying*

$$F_{Tx,Ty}(.) > F_{x,Tx}(.) * F_{y,Ty}(.) * F_{x,y}(.) \quad , \quad L_{Tx,Ty}(.) < L_{x,Tx}(.)\Diamond L_{y,Ty}(.)\Diamond L_{x,y}(.) \qquad (5)$$

for all $x \neq y$. If there exists $x_0 \in X$ such that $\{T^n x_0\}_{n=0}^{\infty}$ has an accumulation point $x_* \in X$, and

$$F_{T^{n-1}x_0,T^n x_0}(t) \leq F_{T^n x_0,T^{n+1}x_0}(t) \quad , \quad L_{T^{n-1}x_0,T^n x_0}(t) \geq L_{T^n x_0,T^{n+1}x_0}(t), \quad \forall t > 0, n = 1, 2, ... \qquad (6)$$

then x_* is the unique fixed point of T , and $\lim_{n\to\infty} T^n x_0 = x_*$.

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Fixed point theory for Ciric-type-generalized φ-probabilistic contraction maps in probabilistic Menger spaces

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Abstract

In this paper, we introduce Ciric-type-generalized φ-probabilistic contraction in probabilistic Menger spaces. We derive some results about existence and uniqueness of a fixed point for this classe of self mappings in probabilistic Menger spaces.

Keywords: Ciric-type-generalized φ-probabilistic contraction, Probabilistic Menger space, Bounded orbit
Mathematics Subject Classification [2010]: 47H10, 47H09

1 Introduction and Preliminaries

Probabilistic metric space (abbreviated, PM space) has been introduced and studied in 1942 by Karl Menger in [4]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a PM space corresponds to the situation when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. In fact the study of such spaces received an impetus with the pioneering works of Schweizer and Sklar [5] and [6]. Recently, the study of fixed point theorems in PM spaces is also a topic of recent interest and forms an active direction of research. Sehgal et al. [7] made the first ever effort in this direction. Since then several authors have already studied fixed point and common fixed point theorems in PM spaces. Next we shall recall some well-known definitions and results in the theory of PM spaces which are used later on in this paper. For more details, we refer the reader to [2] and [5].

Definition 1.1. A probabilistic metric space (abbreviated, PM-space) is an ordered pair (X, F), where X is a nonempty set and $F : X \times X \to D^+$ ($F(p, q)$ is denoted by $F_{p,q}$) where D^+ is the family of all distribution functions on $\mathbb{R}$, satisfies the following conditions: $F_{p,q} = \epsilon_0$, where $\epsilon_0(t) = \begin{cases} 0 & t \leq 0, \\ 1 & t > 0. \end{cases}$, iff $p = q$; $F_{p,q}(t) = F_{q,p}(t)$; if $F_{p,q}(t) = 1$ and $F_{q,r}(s) = 1$, then $F_{p,r}(t + s) = 1$; for every $p, q, r \in X$ and $t, s \geq 0$.

Definition 1.2. A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied: $\Delta(a, b) = \Delta(b, a)$; $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$; $\Delta(a, b) \geq \Delta(c, d)$ whenever $a \geq c$ and $b \geq d$; $\Delta(a, 1) = a$; for every

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$a, b, c, d \in [0, 1]$. Two typical examples of continuous t-norm are $\Delta_p(a, b) = ab$ and $\Delta_m(a, b) = \min\{a, b\}$. It is evident that, as regards the pointwise ordering, $\Delta \leq \Delta_m$, for each t-norm Δ.

Definition 1.3. A t-norm Δ is said to be of Hadžić type (abbreviated, H-type) if the sequence of functions $(\Delta^n(a))$ is equicontinuous at $a = 1$.

The t-norm Δ_m is a trivial example of a t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_m$, see [2]. It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{a\in(0,1)} \Delta(a, a) = 1$.

Definition 1.4. A probabilistic Menger space is a triplet (X, F, Δ), where (X, F) is PM space and Δ is a t-norm such that for all $p, q, r \in X$ and for all $t, s \geq 0$,

$$F_{p,r}(t + s) \geq \Delta(F_{p,q}(t), F_{q,r}(s)).$$

The probabilistic version of the classical Banach contraction principle, was first studied in 1972 by Sehgal and Bharucha-Reid [7].

Theorem 1.5. *[7] Let (X, F, Δ_m) be a complete probabilistic Menger space. If T is a contraction mapping of X into itself, that is*

$$F_{Tp,Tq}(cx) \geq F_{p,q}(x) \qquad \forall x > 0, p, q \in X.$$

Then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.

Definition 1.6. [2] Let (X, F) be a PM space. For every $x_0 \in X$ let $O(x_0, T) = \{T^n x_0 : n \in \mathbb{N}\cup\{0\}\}$. The set $O(x_0, T)$ is the orbit of the mapping $T : X \to X$ at x_0. Let $D_{O(x_0,T)} : \mathbb{R} \to [0, 1]$ be a diameter of $O(x_0, T)$, i.e, $D_{O(x_0,T)}(x) = \sup_{s<x} \inf_{u,v\in O(x_0,T)} F_{u,v}(s)$. If $\sup_{x\in\mathbb{R}} D_{O(x_0,T)}(x) = 1$, then the orbit $O(x_0, T)$ is a probabilistic bounded subset of X. Hence $O(x_0, T)$ is a probabilistic bounded set if and only if $D_{O(x_0,T)} \in D^+$. Also, X is said to be T-orbitally complete if for all $x \in X$, $O(x, T)$ is complete.

In recent years, a number of generalizations of the Banach contraction principle have appeared. Of all these, the following generalization of ciric [1] stands at the top.

Theorem 1.7. *[1] Let (X, F, Δ_m) be a complete probabilistic Menger space. If $T : X \to X$ is generalized contraction mapping on X, that is there exists a constant $0 < c < 1$ such that for every $u, v \in X$*

$$F_{Tu,Tv}(cx) \geq \min\{F_{u,v}(x), F_{u,Tu}(x), F_{v,Tv}(x), F_{u,Tv}(x), F_{Tu,v}(x)\},$$

for all $x > 0$, and X is T-orbitally complete. Then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.

Theorem 1.8. *[3]Let (X, F, Δ) be a complete probabilistic Menger space under a t-norm Δ of H-type. Let $T : X \to X$ be a generalized φ-probabilistic contraction, that is,*

$$F_{Tp,Tq}(\varphi(x)) \geq F_{p,q}(x) \qquad \forall x > 0,\ \forall p, q \in X. \tag{1}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping such that, for any $t > 0$, $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$. Then, there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $\{T^n x_0\}$ converges to x^* for each $x_0 \in X$.

Definition 1.9. Let (X, F, Δ) be a probabilistic Menger space and $T : X \to X$. We say that T is Ciric-type-generalized φ-probabilistic contraction if for every $u, v \in X$ and $x > 0$

$$F_{Tu,Tv}(\varphi(x)) \geq \min\{F_{u,v}(x), F_{u,Tu}(x), F_{v,Tv}(x), F_{u,Tv}(x), F_{Tu,v}(x)\}, \tag{2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping.

The following example shows that a Ciric-type-generalized φ-probabilistic contraction need not be a generalized φ-probabilistic contraction.

Example 1.10. Let $X = [0, \infty)$, $T : X \to X$ be defined by $Tx = x + 1$, and let $\varphi : [0, \infty) \to [0, \infty)$ be defined by

$$\varphi(x) = \begin{cases} \frac{x}{1+x} & 0 \leq x \leq 1, \\ x - 1 & 1 < x. \end{cases}$$

For each $p, q \in X$, let $F_{p,q}(x) = \epsilon_0(x - |p - q|)$ for all $x \in \mathbb{R}$. Then, since $\max\{|p - q - 1|, |q - p - 1|\} = |p - q| + 1$ for all $p, q \in X$, we have $F_{Tp,Tq}(\varphi(x)) \geq \min\{F_{p,Tq}(x), F_{Tp,q}(x)\}$. Thus,

$$F_{Tp,Tq}(\varphi(x)) \geq \min\{F_{p,q}(x), F_{p,Tp}(x), F_{q,Tq}(x), F_{p,Tq}(x), F_{Tp,q}(x)\}.$$

which satisfies (2). If $x = 2, p = 0$ and $q = \frac{3}{2}$, then $F_{T0,T\frac{3}{2}}(\varphi(2)) = 0$ and $F_{0,\frac{3}{2}}(2) = 1$. Thus, $F_{T0,T\frac{3}{2}}(\varphi(2)) < F_{0,\frac{3}{2}}(2)$, which does not satisfy (1).

2 Main results

Now we state and prove our main results about existence and uniqueness of the fixed point for Ciric-type-generalized φ-probabilistic contraction in complete probabilistic Menger space under certain conditions.

Theorem 2.1. *Let (X, F, Δ) be a complete probabilistic Menger space and let $T : X \to X$ be a continuous Ciric-type-generalized φ-probabilistic contraction map such that φ is a bijective mapping, $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$. If there exists $x_0 \in X$ with the bounded orbit, then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $\{T^n x_0\}$ converges to x^*.*

The above theorem has been proved by Ume in 2011 [8], for probabilistic Menger space (X, F, Δ_m) with more conditions.

Theorem 2.2. *Let (X, F, Δ) be a complete Menger space and let the self-maps T and S satisfy the contractive condition*

$$F_{Tu,Tv}(\varphi(x)) \geq \min\{F_{Su,Sv}(x), F_{u,Tu}(x), F_{v,Tv}(x), F_{Su,Tv}(x), F_{Tu,Sv}(x)\}, \quad u, v \in X,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping the same as in Theorem 2.1. If $TX \subseteq SX$ and SX is a complete subset of X, then T and S have a unique concidence point in X. Moreover, if T and S are weakly compatible (i.e, they commut at their concidence points), then T and S have a unique common fixed point.

Example 2.3. Let $X = [-1, 1]$ with the usual metric and $T : X \to X$ and $\varphi : [0, \infty) \to [0, \infty)$ be mappings defined as follows:

$$T(x) = \begin{cases} 0 & -1 \leq x < 0, \\ \frac{x}{1+x} & 0 \leq x \leq \frac{4}{5} \text{ or } \frac{7}{8} < x \leq 1, \\ \frac{-1}{16}x & \frac{4}{5} \leq x \leq \frac{7}{8}, \end{cases} \qquad \varphi(x) = \begin{cases} x - \frac{x^2}{8} & 0 \leq x \leq 1, \\ \frac{7}{8}x & 1 < x, \end{cases}$$

and $F_{p,q}(x) = \epsilon_0(x - |p - q|)$ for all $x \in \mathbb{R}, p, q \in X$. It is easy to see that all of the assumptions of Theorem 2.1 are satisfied, and so T has a unique fixed point ($x = 0$ is a unique fixed point of T). On the other hand, we can show that T does not satisfy (1).

Theorem 2.4. *Let (X, F, Δ) be a complete probabilistic Menger space. Suppose $T : X \to X$ is a mapping satisfying, for all $t > 0$ and $u, v \in X$*

$$F_{Tu,Tv}(\alpha(t)t) \geq \min\{F_{u,v}(x), F_{u,Tu}(x), F_{v,Tv}(x), F_{u,Tv}(x), F_{Tu,v}(x)\},$$

where $\alpha : (0, \infty) \to [0, 1)$ is strictly decreasing function. Assume that there exists $x_0 \in X$ with the bounded orbit. Then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $\{T^n x_0\}$ converges to x^*.

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Fixed points of generalized contractions on intuitionistic fuzzy metric spaces

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Abstract

In this paper, we introduce a new concept of generalized contraction on intuitionistic fuzzy metric spaces and give fixed point results for these classes of contractions.

Keywords: Intuitionistic fuzzy metric space, Generalized contractive mapping, Fixed point.
Mathematics Subject Classification [2010]: 47H10

1 Introduction

Kramosil and Michalek introduced the notion of fuzzy metric spaces [4] and George and Veeramani modified the concept in 1994 [2] in order to obtain a Hausdorff topology in fuzzy metric spaces. In 2014, Park introduced the notion of intuitionistic fuzzy metric spaces [5], and he showed that the topology generated by the intuitionistic fuzzy metric (M, N) coincides with the topology generated by the fuzzy metric M. In [6] Wardowski introduced a new concept of a fuzzy $\mathcal{H}$-contractive mappings and formulated the conditions guaranteeing the convergence of a fuzzy $\mathcal{H}$-contractive sequence to a unique fixed point in a fuzzy M-complete metric space. Recently, Amini-Harandi [1] introduced a new concept of fuzzy generalized contractions as a generalization of the fuzzy $\mathcal{H}$-contractive, by replacing the constant k by a function α and then gave a fixed point result for such mappings in the setting of fuzzy M-complete metric spaces. He also gave an affirmative partial answer to a question posed by Wardowski. In the present paper, we introduce some new classes of generalized contractions in a complete intuitionistic fuzzy metric spaces and give fixed point results for them. Our new result generalized some results obtained by Ionescu et al [3] in the setting of complete intuitionistic fuzzy metric spaces.

Definition 1.1. [5] A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitary set, $*$ a continuous t-norm, $\diamond$ a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:for all $x, y, z \in X, s, t > 0$,

(a) $M(x, y, t) + N(x, y, t) \leq 1$;

(b) $M(x, y, 0) = 0$;

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(c) $M(x,y,t)=1$ for all $t>0$ if and only if $x=y$;

(d) $M(x,y,t)=M(y,x,t)$;

(e) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$ for all $x,y,z\in X$, $s,t>0$;

(f) $M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous;

(g) $\lim_{t\to\infty}M(x,y,t)=1$ for all $x,y\in X$;

(h) $N(x,y,0)=1$;

(i) $N(x,y,t)=0$ for all $t>0$ if and only if $x=y$;

(j) $N(x,y,t)=N(y,x,t)$;

(k) $N(x,y,t)\diamond N(y,z,s)\ge N(x,z,t+s)$ for all $x,y,z\in X$, $s,t>0$;

(l) $N(x,y,\cdot):[0,\infty)\to[0,1]$ is right continuous;

(m) $\lim_{t\to\infty}N(x,y,t)=0$ for all $x,y\in X$.

Then (M,N) is called an intuitionistic fuzzy metric on X. The fuzzy metric (M,N) is called triangular whenever

$$\frac{1}{M(x,y,t)}-1\le\frac{1}{M(x,z,t)}-1+\frac{1}{M(z,y,t)}-1$$

and

$$N(x,y,t)\le N(x,z,t)+N(z,y,t)$$

for all $x,y,z\in X$ and $t>0$.

Definition 1.2. [5] Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon>0$ and $t>0$, there exists a natural number n_0 such that $M(x_n,x_m,t)>1-\epsilon$ and $N(x_n,x_m,t)<\epsilon$ for all $n,m\ge n_0$;

(b) a sequence $\{x_n\}$ in X is said to be converged to x in X (written as $x_n\to x$) if for each $t>0$, $\lim_{n\to\infty}M(x_n,x,t)=1$ and $\lim_{n\to\infty}N(x_n,x,t)=0$.

An intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Remark 1.3. Every fuzzy metric space $(X,M,*)$ is an intuitionistic fuzzy metric space of the form $(X,M,1-M,*,\diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated with $x\diamond y=1-((1-x)*(1-y))$ for all $x,y\in X$.

We denote by $\mathcal{H}$ the family of all onto and strictly decreasing mappings $\eta:(0,1]\to[0,\infty)$, (Note that if $\eta\in\mathcal{H}$, then $\eta(1)=0$, η and η^{-1} are continuous.), and by $\mathcal{S}$ the family of all functions $\alpha:[0,\infty)\to[0,1)$ such that $\limsup_{s\to t}\alpha(s)<1$, for all $t>0$.

In [6] Wardowski proved the following result:

Theorem 1.4. *Let $(X, M, *)$ be an M-complete fuzzy metric space and $T : X \to X$ be a fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$, i.e. there exists $k \in (0, 1)$ satisfying*

$$\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t)),$$

for all $x, y \in X$ and $t > 0$, such that:

(i) $\prod_{i=1}^{k} M(x, Tx, t_i) \neq 0$, *for all $x \in X, k \in \mathbb{N}$ and any sequence $(t_i)_{i\in\mathbb{N}} \subset (0, \infty), t_i \searrow 0$;*

(ii) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, *for all $r, s \in \{M(x, Tx, t) : x \in X, t > 0\}$;*

(iii) $\{\eta(M(x, Tx, t_i)) : i \in \mathbb{N}\}$ *is bounded for all $x \in X$ and any sequence $(t_i)_i \in \mathbb{N} \subset (0, \infty), t_i \searrow 0$.*

Then T has a fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $(T^n x_0)_{n\in\mathbb{N}}$ converges to x^*.

In [1] Amini-Harandi generalized this theorem as the following:

Theorem 1.5. *Let $(X, M, *)$ be an M-complete fuzzy metric space such that $M(x, y, .)$ is continuous unifomly for $x, y \in X$, that is, if for each $t_0 > 0$ and each $\epsilon > 0$ there exists $\delta > 0$ such that $t > 0$, $|t - t_0| \leq \delta$ implies $|M(x, y, t) - M(x, y, t_0)| < \epsilon$, and $T : X \to X$ be a fuzzy generalized $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ and $\alpha \in \mathcal{S}$, i.e.*

$$\eta(M(Tx, Ty, t)) \leq \alpha(\eta(M(x, y, t)))\eta(M(x, y, t)),$$

for all $x, y \in X$ and $t > 0$. Assume that for each $x \in X$, $\mathcal{O}(X) = \{x, Tx, T^2x, ..., T^n(x), ...\}$ is bounded. Then T has a fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $(T^n x_0)_{n\in\mathbb{N}}$ converges to x^*.

In 2013, Ionescu *et al.*[3] introduced new classes of contractive conditions on intuitionistic fuzzy metric space and gave the following fixed point result:

Theorem 1.6. *Let $(X, M, N, *, \diamond)$ be a complete triangular intuitionistic fuzzy metric space, $h \in [0, 1)$ and let $T : X \to X$ be a continuous mapping satisfying the contractive condition*

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq h \max\{\frac{1}{M(x, Tx, t)} - 1, \frac{1}{M(y, Ty, t)} - 1\},$$

for all $x, y \in X$. Then T has a unique fixed point.

In this paper, we intend to generalize this result by weakening the contractive condition to an intuitionistic fuzzy generalized $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ and $\alpha \in \mathcal{S}$.

2 Main results

Definition 2.1. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A mapping $T : X \to X$ is said to be fuzzy quasi-contraction type if there exists $h \in [0, 1)$ satisfying

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq h \max\{\frac{1}{M(x, Tx, t)} - 1, \frac{1}{M(y, Ty, t)} - 1, \frac{1}{M(x, y, t)} - 1, \frac{1}{2}[\frac{1}{M(x, Ty, t)} - 1 + \frac{1}{M(y, Tx, t)} - 1]\},$$

for all $x, y \in X$ and $t > 0$.

Theorem 2.2. *Let $(X, M, N, *, \diamond)$ be a complete triangular intuitionistic fuzzy metric space and let $T : X \to X$ be a continuous mapping satisfying fuzzy quasi-contraction type condition. Then T has a unique fixed point.*

Proof. Put $x_1 = Tx_0$ and $x_{n+1} = T^{n+1}x_0$ for all $n \geq 1$. Assume that $x_{n+1} \neq x_n$ for all n, we obtain

$$\frac{1}{M(Tx_n, Tx_{n-1}, t)} - 1 \leq h \max\{\frac{1}{M(x_n, Tx_n, t)} - 1, \frac{1}{M(x_{n-1}, Tx_{n-1}, t)} - 1\},$$

Put $t_n = \max\{\frac{1}{M(x_n, Tx_n, t)} - 1, \frac{1}{M(x_{n-1}, Tx_{n-1}, t)} - 1\}$.Then $t_n = \frac{1}{M(x_{n-1}, Tx_{n-1}, t)} - 1$ for all n, and so

$$\frac{1}{M(x_{n+1}, x_n, t)} - 1 \leq h(\frac{1}{M(x_{n-1}, Tx_{n-1}, t)} - 1).$$

We can prove that $\{x_n\}$ ia a Cauchy sequence and so there exists $x^* \in X$ such that $x_n \to x^*$. Since T is continuous, $x_{n+1} = Tx_n \to Tx^*$ and so $x^* = Tx^*$. On the contrary, we conclude that x^* is unique fixed point of T. □

Theorem 2.3. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space such that $M(x, y, .)$ is continuous unifomly for $x, y \in X$, and let $T : X \to X$ be an intuitionistic fuzzy generalized $\mathcal{H}$-contractive map with respect to $\eta \in \mathcal{H}$ and $\alpha \in \mathcal{S}$, that is,*

$$\eta(M(Tx, Ty, t)) \leq \alpha(\eta(M(x, y, t))) \max\{\eta(M(x, Tx, t)), \eta(M(y, Ty, t))\},$$

for all $x, y \in X$ and $t > 0$. Assume that for each $x \in X$, $\{x, Tx, T^2x, ..., T^n(x), ...\}$ is bounded. Then T has a fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $(T^n x_0)_{n\in\mathbb{N}}$ converges to x^*.

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Function-valued Gram-Schmidt process in $L_2(0,\infty)$

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Abstract

In this paper, we will look at the Gram-Schmidt process corresponding to a function valued inner product in $L_2(0,\infty)$.

Keywords: function-valued inner product, function-valued norm, function-valued orthogonal, Gram-Schmidt process.
Mathematics Subject Classification [2010]: 42C15

1 Introduction

A function-valued inner product on $L_2(0,\infty)$ by using of the dilation operator and its application in dilation-invariant systems has been introduced in [3]. Fix $a>1$. For each pair $f,g\in L_2(0,\infty)$, the function $\langle f,g\rangle_a$ on $(0,\infty)$ is defined by

$$\langle f,g\rangle_a(x):=\sum_{j\in\mathbb{Z}}a^j f(a^j x)\overline{g(a^j x)}$$

and is called function-valued inner product on $L_2(0,\infty)$ with respect to a. It is easy to show that $\langle f,g\rangle=\int_1^a\langle f,g\rangle_a(x)dx$, where $\langle .,.\rangle$ is the original inner product in $L_2(0,\infty)$. Also, the function-valued norm on $L_2(0,\infty)$ with respect to a is defined by

$$\|f\|_a(x):=\sqrt{\langle f,f\rangle_a(x)},\quad \forall f\in L_2(0,\infty)\quad \textit{and}\quad \forall x\in(0,\infty).$$

The function ϕ on $(0,\infty)$ is called dilation periodic function with period a if $\phi(ax)=\phi(x)$ for all $x\in(0,\infty)$. The set of bounded dilation periodic functions on $(0,\infty)$ is denoted by B_a. For any function ϕ on $[1,a]$, the function $\widetilde{\phi}$ defined by $\widetilde{\phi}(a^j x)=\phi(x)$, for all $j\in\mathbb{Z}$ and $x\in[1,a]$ is dilation periodic. Throughout this paper, let $\widetilde{\phi}$ be the dilation periodic function defined as above for any complex function ϕ on $[1,a]$. A function f defined on $(0,\infty)$ is called function-valued bounded respect to a, or simply function-valued bounded, if there is a $B>0$ such that $\|f\|_a(x)\le B$ for almost all $x\in[1,a]$. The set of function-valued bounded functions denote by $L_a^\infty(0,\infty)$.
The properties of the function-valued inner product are given in the next theorem.

*Speaker

Theorem 1.1. *[3] Let $f,g,h \in L_2(0,\infty)$, $c,d \in \mathbb{C}$, and $b>0$. The following hold:*
1) $\langle f,g\rangle = \int_1^a \langle f,g\rangle_a (x)dx$.
2) $\|f\|_{L_2(0,\infty)} = \|\|f\|_a\|_{L_2[1,a]}$.
3) $\langle cf+dg,h\rangle_a = c\langle f,h\rangle_a + d\langle g,h\rangle_a$.
4) $\langle f,cg+dh\rangle_a = \bar{c}\langle f,g\rangle_a + \bar{d}\langle f,h\rangle_a$.
5) $\langle f,g\rangle_a = \overline{\langle g,f\rangle_a}$.
6) $\langle fg,h\rangle_a = \langle f,\bar{g}h\rangle_a$, *for* $fg, \bar{g}h \in L_2(0,\infty)$.
7) If $\langle f,g\rangle_a = 0$, *then* $\langle f,g\rangle = 0$.
8) $\langle D_b f, D_b g\rangle_a = \frac{1}{\sqrt{b}} D_b \langle f,g\rangle_a$.
9) $\|D_b f\|_a^2 = \frac{1}{\sqrt{b}} D_b \|f\|_a^2$.
10) $\langle D_a f,g\rangle_a = \left\langle f, D_{\frac{1}{a}} g\right\rangle_a$.
11) $D_{\frac{1}{b}} \langle D_b f,g\rangle_a = \frac{1}{\sqrt{b}} \left\langle f, D_{\frac{1}{b}} g\right\rangle_a$.
12) $|\langle f,g\rangle_a| \le \|f\|_a \|g\|_a$.
13) $\|f+g\|_a^2 = \|f\|_a^2 + 2Re\langle f,g\rangle_a + \|g\|_a^2$.
14) $\|f+g\|_a \le \|f\|_a + \|g\|_a$.
15) $\|f+g\|_a^2 + \|f-g\|_a^2 = 2(\|f\|_a^2 + \|g\|_a^2)$.

In function-valued inner products, bounded dilation periodic functions have a behavior similar to scalers.

Proposition 1.2. *[3] Let $f,g \in L_2(0,\infty)$ and $\phi \in B_a$. Then*

$$\langle \phi f,g\rangle_a = \phi\langle f,g\rangle_a \quad \text{and} \quad \langle f,\phi g\rangle_a = \bar{\phi}\langle f,g\rangle_a$$

2 Main results

The definition of orthonormal basis in Hilbert spaces can be found in [1]. Function-valued orthonormal bases are defined similar: For any $f,g \in L_2(0,\infty)$, f and g are function-valued orthogonal with respect to a, or simply function-valued orthogonal if $\langle f,g\rangle_a = 0$ a.e. on $[1,a]$.
A sequence $\{e_n\}_{n\in\mathbb{Z}}$ in $L_2(0,\infty)$ is called function-valued orthogonal with respect to a if $e_n \perp_a e_m$, for all $n \neq m \in \mathbb{Z}$. If also $\|e_n\|_a = 1$ a.e. on $[1,a]$, then $\{e_n\}_{n\in\mathbb{Z}}$ is called a function-valued orthonormal sequence with respect to a, or simply function-valued orthonormal sequence, in $L_2(0,\infty)$.
A sequence $\{e_n\}_{n\in\mathbb{Z}}$ is called function-valued orthonormal basis with respect to a, or simply function-valued orthonormal basis, for $L_2(0,\infty)$ if it is a function-valued orthonormal sequence and $\overline{span}\{\widetilde{\psi_m e_n}\}_{m,n\in\mathbb{Z}} = L_2(0,\infty)$, where ψ_m is defined by $\psi_m(x) = \frac{1}{\sqrt{a-1}} e^{2\pi i \frac{m}{a-1}(a-x)}$ for all $m \in \mathbb{Z}$ and $x \in [1,a]$.

Proposition 2.1. *[3] If $\{e_n\}_{n\in\mathbb{Z}}$ is a function-valued orthonormal basis in $L_2(0,\infty)$, then $\{\widetilde{\psi_m e_n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis in $L_2(0,\infty)$ and $f = \sum_{n\in\mathbb{Z}} \widetilde{\langle f,e_n\rangle_a} e_n$ on $(0,\infty)$.*

For $f \in L_2(0,\infty)$, we define the function valued normalization of f to be

$$N_a(f)(x) = \begin{cases} \frac{f(x)}{\widetilde{\|f\|_a}(x)} & \text{if } \widetilde{\|f\|_a}(x) \neq 0 \\ 0 & \text{if } \widetilde{\|f\|_a}(x) = 0. \end{cases}$$

For any $f,g \in L_2(0,\infty)$ we have

$$\begin{aligned}\langle N_a(f),g\rangle_a(x) &= \sum_{j\in\mathbb{Z}} a^j N_a(f)(a^jx)\overline{g(a^jx)}\\ &= \sum_{j\in\mathbb{Z}} a^j \frac{f(a^jx)}{\widetilde{\|f\|_a}(a^jx)}\overline{g(a^jx)}\\ &= \frac{1}{\widetilde{\|f\|_a}(x)}\sum_{j\in\mathbb{Z}} a^j f(a^jx)\overline{g(a^jx)}\\ &= \frac{\langle f,g\rangle_a(x)}{\widetilde{\|f\|_a}(x)},\end{aligned}$$

where $\widetilde{\|f\|_a}(x)\neq 0$. Thus

$$\langle N_a(f),g\rangle_a = \frac{\langle f,g\rangle_a}{\widetilde{\|f\|_a}} \tag{1}$$

Lemma 2.2. *Let $f,g,h \in L_2(0,\infty)$. we have*
a) $N_a(g) \in span\{\widetilde{\psi_m}g\}_{m\in\mathbb{Z}}$.
b) If any two of f,g,h are in $L_a^\infty(0,\infty)$, then $\langle f,h\rangle_a g \in span\{\widetilde{\psi_m}g\}_{m\in\mathbb{Z}}$.

Definition 2.3. A sequence $\{f_n\}_{n=1}^k$ in $L_2(0,\infty)$ is called function valued linearly independent if for each $n \in \{1,2,3,...,k\}$, $f_n \notin span\{\widetilde{\psi_m}f_i\}_{m\in\mathbb{Z};1\leq i\neq n\leq k}$. A sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L_2(0,\infty)$ is called a function valued linearly independent if every sub-family is function valued linearly independent.

Now we state the Gram-Schmidt process.

Theorem 2.4. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a function valued linearly independent sequence in $L_2(0,\infty)$. Then there existes a function valued orthonrmal sequence $\{e_n\}_{n\in\mathbb{N}}$ such that $span\{\widetilde{\psi_m}f_k\}_{m\in\mathbb{Z};1\leq k\leq n} = span\{\widetilde{\psi_m}e_k\}_{m\in\mathbb{Z};1\leq k\leq n}$, for all $n\in\mathbb{N}$.*

Proof. We proceed by induction. First let $e_1 := N_a(f_1)$. If $\{e_i\}_{i=1}^n$ have been defined to satisfy the theorem, let

$$e_{n+1} := N_a(f_{n+1} - \sum_{i=1}^n \langle f_{n+1},e_i\rangle_a e_i).$$

Let

$$f := f_{n+1} - \sum_{i=1}^n \langle f_{n+1},e_i\rangle_a e_i.$$

Now $f\neq 0$, by the function valued linearly independenty of the sequence $\{f_n\}_{n\in\mathbb{N}}$ and

Lemma 2.2. Using equation 1 for $1 \leq k \leq n$ we have

$$\begin{aligned}
\langle e_{n+1}, e_k \rangle_a &= \left\langle N_a(f_{n+1} - \sum_{i=1}^{n} \langle f_{n+1}, e_i \rangle_a e_i), e_k \right\rangle_a \\
&= \frac{1}{\widetilde{\|f\|_a}} (\langle f_{n+1}, e_k \rangle_a - \sum_{i=1}^{n} \langle f_{n+1}, e_i \rangle_a \langle e_i, e_k \rangle_a) \\
&= \frac{1}{\widetilde{\|f\|_a}} (\langle f_{n+1}, e_k \rangle_a - \langle f_{n+1}, e_k \rangle_a \langle e_k, e_k \rangle_a) \\
&= 0.
\end{aligned}$$

The statement about the linear spans follows from Lemma 2.2. □

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Fusion Riesz basis

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Abstract

In this paper we investigate the equivalence of conditions for fusion Riesz basis and state when a fusion Riesz basis and its canonical dual are dual of each other.

Keywords: fusion frame, canonical dual, fusion Riesz basis
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Frames for Hilbert spaces were first defined by Duffin and Schaeffer in 1952 and introduced in 1986 by Daubechies, Grossmann and Meyer. Fusion frames are a generalization of frmes in Hilbert spaces, were introduced by Casazza and Kutyniok in [1].

In this section we review some definitions and primary results of fusion frames. For more informations see [1]. Throughout this paper, I denotes a countable index set and π_W the orthogonal projection from $\mathcal{H}$ onto a closed subspace W.

Definition 1.1. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ be a family of weights, i.e. $\omega_i > 0, i \in I$. Then $\{(W_i,\omega_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \omega_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}).$$

The constants A, B are called the fusion frame bounds. If we only have the upper bound, we call $\{(W_i,\omega_i)\}_{i\in I}$ a *Bessel fusion sequence*. A fusion frame is called *tight*, if A, B can be chosen to be equal, and *Parseval* if $A = B = 1$. If $\omega_i = \omega$ for all $i \in I$, the collection $\{(W_i,\omega_i)\}_{i\in I}$ is called *ω-uniform* and we abbreviate 1- uniform fusion frams as $\{W_i\}_{i\in I}$. A fusion frame $\{(W_i,\omega_i)\}_{i\in I}$ is said to be an *orthonormal fusion basis* if $\mathcal{H} = \bigoplus_{i\in I} W_i$ and it is called *Riesz decomposition* of $\mathcal{H}$ if for every $f \in \mathcal{H}$, there is a unique choice of $f_i \in W_i$ such that $f = \sum_{i\in I} f_i$. It is clear that every orthonormal fusion basis is a Riesz decomposition for $\mathcal{H}$, and also every Riesz decomposition is a 1-uniform fusiom frame for $\mathcal{H}$.

*Speaker

If $\{(W_i,\omega_i)\}_{i\in I}$ is a fusion frame, the *fusion frame operator* $S_W:\mathcal{H}\to\mathcal{H}$ is defined by $S_W(f)=\sum_{i\in I}\omega_i^2\pi_{W_i}(f)$ is a bounded, invertible and positive. Hence we have the following reconstruction formula [1]

$$f=\sum_{i\in I}\omega_i^2 S_W^{-1}\pi_{W_i}f,\qquad (f\in\mathcal{H}).$$

The family $\{(S_W^{-1}W_i,\omega_i)\}_{i\in I}$, which is also a fusion frame, is called the *canonical dual* of $\{(W_i,\omega_i)\}_{i\in I}$ and satsfies the following reconstruction formula

$$f=\sum_{i\in I}\omega_i^2\pi_{S_W^{-1}W_i}S_W^{-1}\pi_{W_i}f,\quad (f\in\mathcal{H}).$$

In general, every Bessel fusion sequence $\{(V_i,\upsilon_i)\}_{i\in I}$ is called *dual* of fusion frame $\{(W_i,\omega_i)\}_{i\in I}$, if

$$f=\sum_{i\in I}\omega_i\upsilon_i\pi_{V_i}S_W^{-1}\pi_{W_i}f,\quad (f\in\mathcal{H}).$$

In [3], it is shown that a Bessel fusion sequence $\{(V_i,\upsilon_i)\}_{i\in I}$ is a dual of fusion frame $\{(W_i,\omega_i)\}_{i\in I}$, if and only if $T_V\phi_{vw}T_W^*=I_{\mathcal{H}}$, where the bounded operator $\phi_{vw}:(\sum_{i\in I}\bigoplus W_i)_{\ell^2}\to(\sum_{i\in I}\bigoplus V_i)_{\ell^2}$ is given by

$$\phi_{vw}(\{f_i\}_{i\in I})=\{\pi_{V_i}S_W^{-1}f_i\}_{i\in I}$$

If $\{W_i\}_{i\in I}$ is a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ be a family of weights then we say that $\{(W_i,\omega_i)\}_{i\in I}$ is a *fusion Riesz basis* for $\mathcal{H}$ if $\overline{span}_{i\in I}\{W_i\}=\mathcal{H}$ and there exist constants $0<C\leq D<\infty$ such that for each finite subset $J\subseteq I$

$$C(\sum_{j\in J}\|f_j\|^2)^{1/2}\leq\|\sum_{j\in J}\omega_jf_j\|\leq D(\sum_{j\in J}\|f_j\|^2)^{1/2},\qquad (f_j\in W_j).$$

2 Main results

Theorem 2.1. *Let $\{W_i\}_{i\in I}$ be a family of subspaces in $\mathcal{H}$. Then the following are equivalent:*
(1) $\{W_i\}_{i\in I}$ is fusion Riesz basis.
(2) $S_W^{-1}W_i\perp W_j$ for all $i,j\in I, i\neq j$.

Theorem 2.2. *A fusion frame $\{(W_i,1)\}_{i\in I}$ is a fusion Riesz basis if and only if $\pi_{W_i}S_W^{-1}\pi_{W_j}=\delta_{ij}\pi_{W_j}$, for all $i,j\in I$.*

Proposition 2.3. *A fusion Riesz basis $\{W_i\}_{i\in I}$ is dual of $\{S_W^{-1}W_i\}_{i\in I}$ if and only if $\pi_{W_i}\pi_{S_W^{-1}W_i}W_i=W_i$, for all i.*

Proposition 2.4. *If $S_{\widetilde{W}}=S_W^{-1}$ then $\{W_i\}_{i\in I}$ is dual of $\{S_W^{-1}W_i\}_{i\in I}$.*

The reverse of last proposition is not true in general.

Example 2.5. Consider

$$W_1 = span\{(1,1,0)\}, \quad W_2 = span\{(1,0,0),(0,0,1)\}$$

It is easy to see that $\{(W_i,1)\}_{i=1,2}$ is fusion Riesz basis for $\mathbb{R}^3$, with the frame operator

$$S_W = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$\widetilde{W_1} := S_W^{-1} W_1 = span\{(0,1,0)\}, \quad \widetilde{W_2} := S_W^{-1} W_2 = span\{(1,-1,0),(0,0,1)\}.$$

Also

$$S_{\widetilde{W}} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $\{(\widetilde{W_i},1)\}$ and $\{(W_i,1)\}$ are dual of each other and $S_W^{-1} \neq S_{\widetilde{W}}$.

Proposition 2.6. *let $\{V_i\}_{i\in I}$ be alternate dual of fusion frame $\{W_i\}_{i\in I}$. If $S_W = S_V$, then $\{W_i\}_{i\in I}$ is also alternate dual of $\{V_i\}_{i\in I}$.*

In the non- parseval tight fusion frames, the canonical dual of them is themselves, so they are dual of themselves and by Proposition 2.7 the non- parseval tight fusion frames are not fusion Riesz basis. In the next section, we show that by multipliers.

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Fuzzy frame in Fuzzy real inner product space

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Abstract

In this paper we describe the true concept of fuzzy inner product spaces. Then, to clarify the meaning of these spaces look at an example . Below we explain the new concept of alpha frames . A couple of examples of the different spaces with inner frames in classic look .

Keywords: Fuzzy inner product; Fuzzy frame; Inner product;
Mathematics Subject Classification [2010]: 03E72, 15A63

1 Introduction

It was Katsaras[7], who while studying fuzzy topological vector spaces, was the rst to introduced in 1984, the idea of fuzzy norm on a linear space. Later on many other mathematicians like Felbin[5], Cheng & Mordeson[4], Bag & Samanta[3] etc. introduced denition of fuzzy normed linear spaces in dierent approach. studies on fuzzy inner product spaces are relatively recent and few work have been done in fuzzy inner product spaces. Dafyn the first time in 1952 and Scheffer in order to complete his paper on non- harmonic Fourier series theory made frames and frames them as soon as mentioned in that article .But so far nothing has been done about fuzzy frame in fuzzy inner product spaces . In this paper, the definition of a real inner product space that is expressed by A.Hasankhani, A.Nazari, M.Saheli, in[6] .After that, A couple of examples the concept of fuzzy frames in real inner fuzzy spaces between them with frames in real inner product are expressed.

2 Preliminaries

In this section some denitions and preliminary results are given which are used in this paper

Definition 2.1 (6)**.** Let X a linear space over R (the set of real numbers). Then a fuzzy subset $\mu : X \times X \times R \to [0,1]$ is called fuzzy real inner product on X if $\forall x, y, z \in X$ and $t \in R$ the following conditions hold.
(FI-1) $\mu(x, x, t) = 0 \; \forall t < 0$
(FI-2) $\mu(x, x, t) = 1 \; \forall t > 0$ iff $x = \underline{0}$
(FI-3) $\mu(x, y, t)$=$\mu(y, x, t)$
(FI-4)

*Speaker

$$\mu(x,y,t) = \begin{cases} \mu(x,y,\frac{t}{c}) & \text{for } c>0 \\ \text{H(t)} & \text{for } c=0 \\ 1\text{-}\mu(x,y,\frac{t}{c}) & \text{for } c<0 \end{cases}$$

(FI-5) $\mu(x+y,z,t+s) \geq \min\{\ \mu(x,z,t), \mu(y,z,t)\}$
(FI-6) $\lim\limits_{t\to\infty} \mu(\text{x,x,t})=1$

wher

$$\text{H(t)} = \begin{cases} 1 & \text{if } t>0 \\ 0 & \text{if } t\leq 0 \end{cases}$$

Remark 2.2. $\mu(x,y,.)$ is a non-decreasing function of R.

Proof. Let $t_1 > t_2$. Therefore $t_1 - t_2 > 0$ $\mu(0+x,y,t_1-t_2+t_2)$
$\mu(x+y,z,t+s) \geq min\{\ \mu(0,y,t_1-t_2>0), \mu(x,y,t_2)\}$
$\Rightarrow \mu(x,y,t_1) \geq min\{\ 1, \mu(x,y,t_2)\}$[by (FI-4) and since $\text{H}(\text{t}_1 - t_2) = 1$]
$\Rightarrow \mu(x,y,t_1) \geq \mu(x,y,t_2)$ □

Example 2.3. Let $(X,\langle,\rangle)$ be an ordinary inner product space over R. Define $\mu : X \times X \times R \to [0,1]$ by $\mu(cx,y,t) = H(t)$ for c = 0 and for c ≠ 0.

$$\mu(cx,y,t) = \begin{cases} 1 & \text{for } t > c|\langle x,y\rangle| \\ \frac{1}{2} & \text{for } t = c|\langle x,y\rangle| \\ 0 & \text{for } t < c|\langle x,y\rangle| \end{cases}$$

Then (X,μ) is a fuzzy real inner product space.

Proof. see [9] □

Theorem 2.4. *Let (X,μ) be a fuzzy real inner product space. Assume further that (FI-7)$\mu(x,y,st) \geq \mu(x,y,s^2) \wedge \mu(y,y,t^2)$ $\forall s,t \in R$ and $\forall x,y \in X$. $function$ $N : X \times R \to [0,1]$ by*

$$N(x,t) = \begin{cases} \mu(x,x,t^2) & \text{if } t>0 \\ 0 & \text{if } t\leq 0 \end{cases} \quad (1)$$

Then N is a B-S[1] fuzzy norm on X. We call this norm as induced norm ofμ.

Proof. see [9] □

Suppose (X,μ) be a fuzzy real inner product space and $\alpha \in (0,1)$. we Define $\alpha - fuzzy$ real inner products of μ onX. as the following definition

$$\langle x,y\rangle_\alpha = \wedge\{t \in R : \mu(x,y,t) \geq \alpha\},$$

Theorem 2.5. *Let (X,μ) be a fuzzy real inner product space. Assume further that (FI-8) $\wedge\{t \in R : \mu(x,x,t) \geq \alpha\} < \infty, \forall\alpha \in (0,1) and$
$\mu(x,x,t) > 0$ $\forall t > 0 \Rightarrow x = \underline{0}$.
$Then\{\langle,\rangle_\alpha : \alpha \in (0,1)\}$ is an ascending family of inner products on X.*

Proof. see [9] □

Suppose (X, μ) be a fuzzy real inner product space and $\alpha \in (0,1)$. we Define $\alpha - fuzzy$ norm of μ onX as the following definition

$$\|x\|_\alpha = [\langle x, x\rangle_\alpha]^{\frac{1}{2}}$$

Remark 2.6. Let (X, μ) is a fuzzy real inner product space satisfying (FI-7) and (FI-8) and N be its induced fuzzy norm. The α - norms derived from induced fuzzy norm N and from α - inner product are same.

Proof. see [9] □

3 fuzzy frame

Suppose (X, μ) be a fuzzy real inner product space and X has finite-dimensional vector space,$\alpha \in$(0,1).A family of elements $\{f_k\}_{k=1}^m$ in X is a α - frame for X if there exist constants $A, B > 0$ such that

$$A\|f\|_\alpha^2 \leq \sum_{k=1}^m |\langle f, f_k\rangle_\alpha|^2 \leq B\|f\|_\alpha^2 \ \forall f \in X$$

The numbers A,B are called frame bounds. The frame is normalized if $\|f_k\|_\alpha = 1$,$k = 1, 2, ..., m$. A $\alpha - frame$ $\{f_k\}_{k=1}^m$ is tight if we can choose A = B in the definition, if

$$\sum_{k=1}^m |\langle f, f_k\rangle_\alpha|^2 = A_\alpha \|f\|_\alpha^2 \ \forall f \in X \tag{2}$$

Example 3.1. Let $(X, \langle , \rangle)$ be an ordinary inner product space over R. Define μ : X × X× R→ $[0, 1]$ by $\mu(cx, y, t) =$ H(t)$forc = 0andforc \neq 0$,

$$\mu(cx, y, t) = \begin{cases} 1 & \text{for } t > c|\langle x, y\rangle| \\ \frac{1}{2} & \text{for } t = c|\langle x, y\rangle| \\ 0 & \text{for } t < c|\langle x, y\rangle| \end{cases}$$

As shown in the example3.2 (X, μ) is a fuzzy real inner product space $\alpha - fuzzy$ inner products on X defined as follows:

$$\langle x, y\rangle_\alpha = \begin{cases} \langle x, y\rangle| & \text{if } \alpha \geq \frac{1}{2} \\ 0 & \text{if } \alpha < \frac{1}{2} \end{cases}$$

In this example we see for $\alpha \geq \frac{1}{2}$ the concept of a fuzzy frame is the same frame in classic mode Because if for Constant α we put\$ $A_\alpha = A$ and $B_\alpha = B$ we have :

$A_\alpha \|f\|_\alpha^2 \leq \sum_{k=1}^m |\langle f, f_k\rangle_\alpha|^2 \leq B_\alpha \|f\|_\alpha^2 \ \forall f \in X$

$\Leftrightarrow A_\alpha \|f\|^2 \leq \sum_{k=1}^m |\langle f, f_k\rangle|^2 \leq B_\alpha \|f\|^2 \quad \forall f \in X$

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Talk G-ultrametric dynamics and some fixed point theorems pp.: 1–4

G-Ultrametric Dynamics and Some Fixed Point Theorems

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Abstract

This paper is concerned with dynamics in general G-ultrametric spaces, hence we discuss the introduced concepts of these spaces. Also, the fixed point existing results of strictly contractive and non-expansive mappings defined on these spaces by inspiring from the theorem proved by Mustafa and Sims.

Keywords: Fixed point, G-ultrametric space, strictly contractive mapping, non-expansive mapping.
Mathematics Subject Classification [2010]: 47H10, 47H09

1 Introduction

In 2005,Mustafa and Sims introduced a new class of generalized metric spaces (see [4, 5]), which are called G-metric spaces, as generalization of a metric space (X, d). Subsequently, many fixed point results on such spaces appeared (see, for example, [3, 1, 2]). Here, we present the necessary definitions and results in G-metric spaces, which will be useful for the rest of the paper. However, for more details, we refer to [4, 5].

Definition 1.1. [5]. Let X be a nonempty set. Suppose that $G : X \times X \times X \to [0, \infty)$ is a function satisfying the following conditions:

G1) $G(x, y, z) = 0$ if $x = y = z$;

G2) $0 < G(x, x, y)$, for all $x, y, z \in X$ with $x \neq y$;

G3) $G(x, x, y) \leq G(x, y, z)$; for all $x, y, z \in X$ with $z \neq y$;

G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables), and

G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z \in X$, (rectangle inequality),

then the function G is called a generalized metric, or more specifically a G-metric on X, and the pair (X, G) is a G-metric space.

Definition 1.2. [5] Let (X, G) be a G-metric space, then for $x_0 \in X, r > 0$, the G-ball(dressed ball) with center x_0 and radius r is

$$B(x_0, r) = \{y \in X : G(x_0, y, y) < r\},$$

and the stripped ball of radius r and center x_0 is

$$B(x_0, r^+) = \{y \in X : G(x_0, y, y) \leq r\}$$

Proposition 1.3. *[5] Let (X, G) be a G-metric space, then for any $x_0 \in X$ and $r > 0$, we have,*

*Speaker

(1) if $G(x_0, x, y) < r$, then $x, y \in B_G(x_0, r)$;

(2) if $y \in B(x_0, r)$, then there exists $\delta > 0$ such that $B(y, \delta) \subseteq B(x_0, r)$.

Now, first we introduce a new class of G metric spaces which are called G-ultrametric spaces, and in the sequel give examples and results which are required.

Definition 1.4. A G-metric space (X, G) is called a G- ultrametric space if the G-metric G satisfies the strong rectangle inequality, i.e., for all $x, y, z \in X$:

$$G(x, y, z) \leq \max\{G(x, a, a), G(a, y, z)\}, \qquad \text{for all } x, y, z \in X.$$

In this case, G is called to be generalized ultrametric, and the pair (X, G) is a G-ultrametric space.

Examples

(a) Let X be a nonempty set. The following function on X^3 defines a G-ultrametric on X:

$$G(x, y, z) = \begin{cases} 0 & x = y = z, \\ 1 & \text{otherwise.} \end{cases}$$

In this case, (X, G) is called a discrete G-ultrametric space (or trivial G-ultrametric space).

(b) Every G-ultrametric on X defines an ultrametric d_G on X by

$$d_G(x, y) = \max\{G(x, y, y), G(y, x, x)\}, \qquad \text{for all } x, y \in X.$$

Conversely, for any d-ultrameric d on X,

$$G_1(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \qquad \text{for all } x, y \in X.$$

is readily seen to define an G-ulmetric on X^3.

(c) Let $\mathbb{N}$ be the set of positive integer numbers. The mapping $G : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to [0, \infty)$ is defined by

$$G(m, n, l) = \begin{cases} 0 & m = n = l \\ \max\{1 + \frac{1}{m}, 1 + \frac{1}{n}, 1 + \frac{1}{l}\} & \text{otherwise} \end{cases}$$

is a G-ultrametric on $\mathbb{N}^3$.

The G-Ultrametric topology

Proposition 1.5. *Let (X, G) be a G-ultrametric space then:*

(a) any point of a G-ball is a center of the ball.

(b) if two G-balls have a common point, one is contained in the other.

(c) the diameter of a G-ball is less than or equal to its radius.

Proposition 1.6. *Let (X, G) be a G-ultrametric space.*

(a) If $x \in S(x_0, r)$, then $B(x, r) \subseteq S(x_0, r)$ and $S(x_0, r) = \cup_{x \in S(x_0,r)} B(x, r)\}$, which

$$S(x_0, r) = \{y \in X : G(x_0, y, y) = r\}.$$

(b) The spheres $S(x_0, r)$ are open and closed (henceforth we use the word clopen as an abbreviation of " closed and open").

(c) The dressed balls of positive radius are open, and the stripped balls are closed.

Consequently, the G-ultrametric topology $\tau(G)$ is zero-dimensional and coincides with the ultrametric topology arising from d_G. Thus, while isometrically distinct, every G-ultrametric space is topologically equivalent to an ultrametrics space. This allows us to readily transport many concepts and results from ultrametric spaces into the G-ultrametric space setting.

Definition 1.7. A G-ultrametric space (X, G) is said to be spherically complete if every shrinking collection of dressed balls in X has a nonempty intersection.

Definition 1.8. Let (X, G) be a G-ultrametric space. The sequence $\{x_n\} \subseteq X$ is G-convergent to x if it converges to x in the G-ultrametric topology, $\tau(G)$.

Definition 1.9. [5] Let (X, G) be a G-metric space, then a sequence $\{x_n\} \subseteq X$ is said to be G-Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \in \mathbb{N}$.

Proposition 1.10. *In a G-ultrametric space, (X, G), the following statements are equivalent.*

(a) The sequence $\{x_n\}$ is G-Cauchy.

(b) For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_{n+1}, x_{n+1}) < \varepsilon$; for all $n \geq N$.

Remark 1.11. The Proposition 1.10 in G-metric space isn't valid. In fact, if we let $X = \mathbb{R}$ and

$$G : X \times X \times X \to \mathbb{R}^+$$
$$G(x, y, z) = d(x, y) + d(x, z) + d(y, z),$$

which d is Euclidean metric on $\mathbb{R}$, then $G(x_n, x_{n+1}, x_{n+1}) \to 0$, but $\{\ln n\} \nrightarrow o$.

2 The Main Theorem

It is known that a contractive mapping on a G-metric space need not have a fixed point, e.g., let $(\mathbb{R}, G)$ be a G-metric space with

$$G(x, y, z) = |x - y| + |x - z| + |y - z|,$$

the mapping $T : (\mathbb{R}, G) \to (\mathbb{R}, G)$ with $Tx = x + \frac{1}{1+e^x}$ is a strictly contractive mapping, but has no fixed point.

Now, we prove that every contractive mapping $T : X \to X$, where X is G-spherically complete ultrametric space, has a unique fixed point. We give also examples to show that this assertion cannot be extended to include either nonexpansive mappings or nonspherically complete spaces.

Theorem 2.1. *Let (X, G) be G-spherically complete ultrametric space. If $T : X \to X$ is a mapping such that for tree distinct points $x, y, z \in X$,*

$$G(Tx, Ty, Tz) < \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\},$$

then T has a unique fixed point.

Theorem 2.2. *Let (X, G) be G-spherically complete ultrametric space. If $T : X \to X$ is a mapping such that for every $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \le \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\},$$

then either T has at least one fixed point or there exists a sphere B of the radius $r > 0$ such that $T : B \to B$ and for which $G(b, Tb, Tb) = r$ for each $b \in B$.

Example 2.3. Let $\mathbb{Q}_p$ be the p-adic field (i.e., The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the p-adic absolute value $|a|_p = p^{-r}$ if $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p). Also, let $\mathbb{A}_{\mathbb{Q}_p} = \{a : \mathbb{N} \to \mathbb{Q}_p \mid \text{a is bounded map}\}$, where $a \in \mathbb{A}$ is a bounded map if there exists a positive number $M > 0$ such that $\sup\{|a(n)|_p \mid n \in \mathbb{N}\} < M$, and let $\|f\|_p^\infty = \sup\{|a(n)|_p \mid n \in \mathbb{N}\}$, We define the G-ultrametric on $\mathbb{A}_{\mathbb{Q}_p}$ as the following: $G(x, y, z) = \max\{\|x - y\|_p^\infty, \|x - z\|_p^\infty, \|y - z\|_p^\infty\}$. Also, we set $\mathbb{A}^0_{\mathbb{Q}_p} = \{a \in \mathbb{A} \mid G(a(n), 0, 0) \to 0\}$ In this case $\mathbb{A}^0_{\mathbb{Q}_p}$ is spherically complete G-ultrametric space. Suppose $T : \mathbb{A}^0_{\mathbb{Q}_p} \to \mathbb{A}^0_{\mathbb{Q}_p}$ is the mapping defined by $T(x_1, x_2, x_3, \ldots) = (p, x_1, x_2, x_3, \ldots)$ Clearly T is a nonexpansive map, but T has no fixed point in $\mathbb{A}^0$. However, the ball $\{a \in \mathbb{A} \mid \|a\|_p^\infty \le \frac{1}{p}\}$ is minimal T-invariant because for every $a \in \mathbb{Q}_p$ we have $G(a, Ta, Ta) = \frac{2}{p}$.

Example 2.4. Let $\mathbb{C}_p$ be the field of completion of the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{A}_{\mathbb{C}_p}$, $\mathbb{A}^0_{\mathbb{C}_p}$ and G-ultrametric on $\mathbb{A}^0_{\mathbb{C}_p}$ are defined as in Example 2.3. In this case, $\mathbb{A}^0_{\mathbb{C}_p}$ is not spherically complete because the value group $\{|x|_p \mid x \in \mathbb{C}_p\}$ is dense in $[0, \infty)$. Suppose T is the mapping $T : \mathbb{A}^0_{\mathbb{C}_p} \to \mathbb{A}^0_{\mathbb{C}_p}$ defined by

$$T(x_1, x_2, x_3, \ldots) = (1, \pi_1 x_1, \pi_2 x_2, \pi_3 x_3, \ldots, \pi_n x_n, \ldots),$$

where $\{\pi_n\}$ is a sequence in $\mathbb{C}_p$ with $|\pi_n| < 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} |\pi_n| = 1$, and $\lim_{n\to\infty} \prod_{i=1}^n |\pi_n| = 0$. The mapping T is a strongly contractive but it has no fixed points.

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Generalized cyclic contraction and convex structure

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Abstract

In this paper we consider approximate proximity pair for a single map. We apply approximate fixed point for a map and discuss the existence of approximate proximity pair. Approximation theory, which mainly consists of theory of nearest points (best approximation) and theory of farthest points (worst approximation), is an old and rich branch of analysis. The theory is as old as Mathematics itself. The ancient Greeks approximated the area of a closed curve by the area of a polygon. Starting in 1853, Russian mathematician P.L. Chebyshev made significant contributions in the theory of best approximation.

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Keywords: Approximate pair proximity, Best proximity, Generalized cyclic contraction, Approximate fixed point, Convex structure.

1 Introduction

Let (X, d) be a metric space, A, B nonempty subsets of X and $d(A, B)$ is the distance of A and B,

$$d(A, B) = \inf\{d(x, y) : \; x \in A, y \in B\}.$$

If $d(x_0, y_0) = d(A, B)$, then the pair (x_0, y_0) is called a best proximity pair for A and B and put

$$prox(A, B) = \{(x, y) \in A \times B : \; d(x, y) = d(A, B)\} \quad (1.1)$$

as the set of all best proximity pairs for (A, B) (see[1-5]).

Definition 1.1. [3] Let (X, d) be a metric space, $T : X \to X$, $\epsilon > 0$ and $x_0 \in X$. Then x_0 is an ϵ-fixed point (approximate fixed point) of T if

$$d(T(x_0), x_0) < \epsilon$$

.

*Speaker

In this paper we will denote the set of all approximate best proximity of pair (A,B), by

$$P_T^a(A,B) = \{x \in A \cup B : \ d(x,Tx) \leq d(A,B) + \epsilon \ for \ some \ \epsilon > 0\}$$

We say that the pair (A, B) has approximate best proximity pair property if $P_T^a(A, B) \neq \emptyset$.

Definition 1.2. [4] Let A and B be nonempty subsets of a metric space (X, d) and let $T : \ A \cup B \to A \cup B$, $S : \ A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. A point $(x, y) \in A \times B$ is said to be an approximate pair proximity for (T, S) in X if there exists a $\epsilon > 0$

$$d(Tx, Sy) \leq d(A,B) + \epsilon \qquad (2.1)$$

We say that the pair (T, S) has the approximate pair proximity property in X if $P^a_{(T,S)}(A, B) \neq \emptyset$, where

$$P^a_{(T,S)}(A,B) = \{(x,y) \in A \times B : \ d(Tx,Sy) \leq d(A,B) + \epsilon \ for \ some \ \epsilon > 0\}.$$

Theorem 1.3. *[4] Let A and B be nonempty subsets of a metric space (X, d) and let $T : \ A \cup B \to A \cup B$, $S : \ A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If, for every $(x, y) \in A \times B$,*

$$lim_{n\to\infty} d(T^n(x), S^n(y)) = d(A,B), \qquad (2.2)$$

then (T, S) has the approximate pair proximity property.

2 Approximate Best Proximity Pairs

In this section, we will consider the existence of approximate best proximity points for a cyclic map $T : \ X \to X$.

Definition 2.1. Let $T : \ A \cup B \to A \cup B$ be a cyclic map, i.e., a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $x \in A \cup B$. Then x is an approximate best proximity point of the pair (A,B), if

$$d(x,Tx) \leq d(A,B) + \epsilon, \ for \ some \ \epsilon > 0.$$

Theorem 2.2. *Let (X, d) be a metric space. Suppose that the mapping $T : X \to X$ is a cyclic map and*

$$d(Tx,Ty) \leq ad(x,y) + b\{d(x,Tx) + d(y,Ty)\} + c\{d(y,Tx)\}$$

for all $x, y \in X$, where $a, c \geq 0$ and $b < 1$ and $a + \ 2\ b + c < 1$. Then

$$d(T^n x, T^{n+1}x) \leq d(T^{n-1}x, T^n x).$$

Therefore if x ia a ϵ-fixed for T, then x is a ϵ-fixed point for T^n for $n \geq 1$.

Theorem 2.3. *Let A and B be nonempty subsets of a metric space (X, d). Suppose that the mappings $T, S : X \to X$ is a cyclic maps and*

$$d(Tx, Ty) \leq ad(x, y) + b\{d(x, Tx) + d(y, Ty)\} + c\{d(A, B)\}$$

for all $x, y \in X$, where $0 \leq a < 1$ and $c \geq 0$ and $b \geq 0$ and $2a + 2b + c < 1$. If x, y are approximate best proximity of the pair (A, B), then (x, y) is an approximate proximity pair for (T, S).

Theorem 2.4. *Let X be a normed linear space , A and B two nonempty subsets of X. Let $T : A \cup B \to A \cup B$, be a cyclic map satisfying*

$$\| Tx - Ty \| \leq \alpha \| x - y \| + (1 - \alpha)d(A, B), (1.3)$$

for all $x, y \in A \cup B$ and $\alpha \in (0, 1)$. Then

$$\| T^{2n-1}x - T^{2n}x \| \leq \alpha^{2n-1} \| x - Tx \| + (1 - \alpha^{2n-1})d(A, B)$$

for all $x \in A \cup B, n \geq 1$. Therefore for all $x \in A \cup B$ and $n \geq 1$

$$\| T^{2n-1}x - T^{2n}x \| \longrightarrow d(A, B) \qquad as\ n \longrightarrow \infty.$$

Definition 2.5. For a metric space (X, d) ,a continuous mapping $w : X \times X \times [0, 1] \longrightarrow X$ is to be a convex structure on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, w(x, y, \lambda)) \leq \lambda d(u, y) + (1 - \lambda)d(u, y),$$

for all $u \in X$.

Theorem 2.6. *For a metric space (X, d), suppose a $w : X \times X \times [0, 1] \longrightarrow X$ is a convex structure on X and let $T : X \longrightarrow X$ is a map satisfy*

$$d(Tx, Ty) \leq ad(x, y)$$

for all $0 \leq a < 1$ and $x, y \in X$. Then for every $u \in X$

$$d(u, T(w(x, y, \lambda))) \leq \lambda d(u, x) + (1 - \lambda)d(u, y),$$

for all fixed point u of T.

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Generalized weighted composition operators between Zygmund spaces and Bloch spaces

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Abstract

For the analytic selfmap φ and analytic function u on the open unit ball of the complex plane, we investigate generalized weighted composition operators

$$\left(D^k_{\varphi,u}f\right)(z) = u(z)f^{(k)}(\varphi(z)),$$

between weighted Zygmund spaces and weighted Bloch spaces.

Keywords: Generalized weighted composition operators, Weighted composition operators, Weighted Zygmund spaces, Weighted Bloch spaces.
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1 Introduction

Let $\mathbb{D}$ be the open unit ball in $\mathbb{C}$ and u and φ be analytic functions on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. For a nonnegative integer k, the *generalized weighted composition operator* $D^k_{\varphi,u}$ on $H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$, is defined by

$$\left(D^k_{\varphi,u}f\right)(z) = u(z)f^{(k)}(\varphi(z)), \quad z \in \mathbb{D}.$$

Generalized weighted composition operators are generalization of well-known *weighted composition operators* uC_φ defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{D},$$

and also generalization of some other known operators. In this paper, we consider generalized weighted composition operators between *weighted Zygmund spaces* and *weighted Bloch spaces* defined as follows.

By a *weight function* we mean a continuous, strictly positive and bounded function $\nu : \mathbb{D} \to \mathbb{R}_+$. The weight ν is called *radial* if $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. For a weight ν, the *weighted Banach space of analytic functions on* $\mathbb{D}$ is defined as

$$H^\infty_\nu = \left\{ f \in H(\mathbb{D}) \ : \ \|f\|_\nu = \sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty \right\}.$$

*Speaker

For a weight ν, the *associated weight* $\widetilde{\nu}$ is defined by

$$\widetilde{\nu}(z) = \left(\sup\{|f(z)| \ : \ f \in H_\nu^\infty, \|f\|_\nu \leq 1\}\right)^{-1}.$$

It is known that for the *standard weights* $(0 < \alpha < \infty)$

$$\nu_\alpha(z) = (1 - |z|^2)^\alpha, \quad z \in \mathbb{D},$$

and the *logarithmic weight*

$$\nu_{\log}(z) = \left(\log \frac{2}{1-|z|^2}\right)^{-1}, \quad z \in \mathbb{D},$$

the associated weights and the weights are the same.

For $0 < \alpha < \infty$, the *weighted Bloch space* $\mathcal{B}_\alpha$ is the space of all analytic functions $f \in H(\mathbb{D})$ for which

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

Weighted Bloch space $\mathcal{B}_\alpha$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|.$$

In the case of $\alpha = 1$, we get the classical *Bloch space* $\mathcal{B} = \mathcal{B}_1$.

The *Zygmund space* $\mathcal{Z}$ is the class of all functions $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ with

$$\sup_{\substack{e^{i\theta}\in\partial\mathbb{D} \\ h>0}} \frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i\theta}\right)\right|}{h} < \infty.$$

It is known that an analytic function f belongs to $\mathcal{Z}$ if and only if $f' \in \mathcal{B}$, or equivalently $\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) |f''(z)| < \infty$. For $0 < \alpha < \infty$, the *weighted Zygmund space* $\mathcal{Z}_\alpha$ is the space of all analytic functions $f \in H(\mathbb{D})$ for which

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f''(z)| < \infty.$$

Weighted Zygmund space $\mathcal{Z}_\alpha$ is a Banach space if equipped with the norm

$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f''(z)|.$$

Hu and Ye, in 2012, studied boundedness and compactness of weighted composition operators between Zygmund spaces. Boundedness, compactness and essential norms of weighted composition operators between weighted Zygmund spaces and weighted Bloch spaces were investigated by the authors in [5]. Li and Fu, in 2013, studied generalized weighted composition operators from Bloch spaces into Zygmund spaces. In this paper, we investigate generalized weighted composition operators between weighted Zygmund spaces and weighted Bloch spaces.

2 Main results

Before giving the main results, we recall the following lemmas which will be used in the proof of main theorems. The next lemma is due to Montes-Rodríguez [4] and Hyvärinen et al. [2]. We also mention that for the real scalars A and B, the notation $A \asymp B$ means that $cB \leq A \leq CB$ for some positive constants c and C.

Lemma 2.1. *[2, 4] Let ν and ω be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then,*

(i) the weighted composition operator uC_φ maps H_ν^∞ into H_ω^∞ if and only if

$$\sup_{n\geq 0} \frac{\|u\varphi^n\|_\omega}{\|z^n\|_\nu} \asymp \sup_{z\in\mathbb{D}} \frac{\omega(z)}{\widetilde{\nu}(\varphi(z))}|u(z)| < \infty,$$

with norm comparable to the above supremum.

(ii) $\limsup_{n\to\infty} \frac{\|u\varphi^n\|_\omega}{\|z^n\|_\nu} = \limsup_{|\varphi(z)|\to 1} \frac{\omega(z)}{\widetilde{\nu}(\varphi(z))}|u(z)|$.

Lemma 2.2. *[3] For every $0 < \alpha < \infty$ we have*

(i) $\limsup_{n\to\infty}(n+1)^\alpha\|z^n\|_{\nu_\alpha} = (\frac{2\alpha}{e})^\alpha$,

(ii) $\limsup_{n\to\infty}(\log n)\|z^n\|_{\nu_{\log}} = 1$.

In the next theorems we give necessary and sufficient conditions for the boundedness of generalized weighted composition operator $D^k_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ in the case of $k = 1$. The results for the boundedness of $D^1_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ are stated in three different cases of $0 < \alpha < 1$, $\alpha = 1$, and $1 < \alpha < \infty$.

Theorem 2.3. *Suppose that $0 < \alpha < 1$. Then, the generalized weighted composition operator $D^1_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if $u \in \mathcal{B}_\beta$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|u(z)\varphi'(z)| < \infty.$$

Theorem 2.4. *The generalized weighted composition operator $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{B}_\beta$ is bounded if and only if*

(i) $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'(z)|\log\frac{2}{1-|\varphi(z)|^2} < \infty$,

(ii) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| < \infty$.

Theorem 2.5. *Suppose that $1 < \alpha < \infty$. Then, the generalized weighted composition operator $D^1_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if*

(i) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| < \infty$,

(ii) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|u(z)\varphi'(z)| < \infty$.

Regarding Theorems 2.3, 2.4 and 2.5, in the next theorem we consider the case $k > 1$ and give necessary and sufficient conditions for the boundedness of generalized weighted composition operator $D^k_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$.

Theorem 2.6. *Suppose that* $0 < \alpha < \infty$. *Then, the generalized weighted composition operator* $D^k_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ *is bounded if and only if*

(i) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+k-2}} |u'(z)| < \infty$,

(ii) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+k-1}} |u(z)\varphi'(z)| < \infty$.

Remark 2.7. Recall that a linear operator T between Banach spaces X and Y is *compact* if it takes bounded sets to sets with compact closure. The space of all compact operators $T : X \to Y$ is denoted by $\mathcal{K}(X, Y)$. The *essential norm* of a bounded operator $T : X \to Y$ is defined as the distance from T to $\mathcal{K}(X, Y)$. Estimates of essential norms have been extensively studied for different types of operators between many spaces of analytic functions. It is worth mentioning that essential norm estimates of generalized weighted composition operators between weighted Zygmund spaces and weighted Bloch spaces have been studied by the authors [6].

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Hausdorff measure of noncompactness for some paranormed λ-sequence spaces of non-absolute type

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Abstract

Recently some new generalize sequence spaces related to the spaces $l_\infty(p), c(p)$ and $c_0(p)$ have been defined. In this work, we establish estimates for the operator norms and the Hausdorff measure of noncompactness of certain matrix operators on this spaces that are paranormed spaces by the matrix classes (X, Y), where $X \in \{c_0(\lambda, p), c(\lambda, p), l_\infty(\lambda, p)\}$ and $Y \in \{c_0(q), c(q), l_\infty(q)\}$. Further, we apply our results to obtain corresponding subclasses of compact matrix operators.

Keywords: Hausdorff measure of noncompactness; λ-sequence spaces; paranormed spaces
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

We denote W for the space of all real-valued sequences. Any vector subspace of W is called a sequence space.

Definition 1.1. Definitions of K-space, FK-space, BK-space and AK-property are in [2]. If $X \supset \varphi$ is a BK-space and $a = (a_k) \in \mathbb{W}$, then we defined

$$\| a \|_X^* = \sup_{x \in S_X} |\sum_{k=0}^{\infty} a_k x_k|,$$

(1)

provided the expression on the right hand side exist and is finite.

Let X and Y be any two sequence spaces and $A = (a_{nk})$ be any infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$. By (X, Y), we denote the class of all infinite matrices that map X into Y.

*Speaker

Definition 1.2. Assume here and after that $(p_k),(q_k)$ are bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$.
Lemma 1.1([4]). Let X be any of the spaces c_0, c, l_∞ or $l_p (1 \leq p < \infty)$. Then, we have $\| . \|_X^* = \| . \|_{X^\beta}$ on X^β, where $\| . \|_{X^\beta}$ denotes the natural norm on the dual spaces X^β.
Lemma 1.2([2]). Let $X \supset \varphi$ be a BK-space and Y be any of the spaces c_0, c,or l_∞. If $A \in (X, Y)$, then:

$$\| L_A \| = \| A \|_{(X,l_\infty)} = \sup_n \| A_n \|_X^* < \infty.$$

In [1] new sequence spaces have been defined as follows:
$l_\infty(\lambda, p) = \{x = (x_k) \in W : \sup_n |\frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k|^{p_n} < \infty\}$;
$c(\lambda, p) = \{x = (x_k) \in W : \lim_{n\to\infty} |\frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1})(x_k - l)|^{p_n} = 0 \text{ for some } l \in \mathbb{R}\}$;
$c_0(\lambda, p) = \{x = (x_k) \in W : \lim_{n\to\infty} |\frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k|^{p_n} = 0\}$;
$l(\lambda, p) = \{x = (x_k) \in W : \sum_{n=0}^\infty |\frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k|^{p_n} < \infty\}$.
For any $x = (x_k) \in W$, we defined the associated sequence $y = (y_k)$, which will frequently be used, as the Λ-transform of x, i.e, $y = \Lambda(x)$, and hence:

$$y_n = \sum_{k=0}^n (\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}) x_k \quad (n \in \mathbb{N}). \tag{2}$$

Lemma 1.3([1]). The sequence spaces $l_\infty(\lambda, p), c(\lambda, p)$ and $c_0(\lambda, p)$ are BK- spaces with respect to paranorm defined by:

$$f(x) = \sup_n |\frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k|^{\frac{p_n}{M}}.$$

Lemma 1.4. If $a = (a_k) \in \mathbb{X}^\beta$, where $\mathbb{X}$ is any of the spaces $l_\infty(\lambda, p), c(\lambda, p)$or $c_0(\lambda, p)$, then $\hat{a} = (\hat{a}_k) \in l_1$ and the following equality holds for all $x = (x_k) \in X$, where $y = \Lambda(x)$ is given by (2),

$$\sum_{k=0}^\infty a_k x_k = \sum_{k=0}^\infty \hat{a}_k y_k, \tag{3}$$

where,

$$\hat{a}_k = (\frac{a_k}{\lambda_k - \lambda_{k-1}} - \frac{a_{k+1}}{\lambda_{k+1} - \lambda_k}) \lambda_k, \quad (n, k \in \mathbb{N}). \tag{4}$$

Proof. By [Theorem 4, 5] the proof is complete.•

Theorem 1.5. If $a = (a_k) \in X^\beta$, where X is similar to Lemma 1.4, then:

$$\| a \|_{\mathbb{X}}^* = \| \hat{a} \|_{\mathbb{X}^\beta} = \| \hat{a} \|_{l_1} = \sum_{k=0}^{\infty} |\hat{a}_k| < \infty, \tag{5}$$

where $\hat{a} = (\hat{a}_k)$ is a sequence defined by (4).
Proof. By using Lemma 1.4, relations (3) and (2), Lemma 1.3 and relation (1) respectively, we obtain the proof.•
Lemma 1.6. Let X be one of the spaces $l_\infty(\lambda, p), c(\lambda, p)$ or $c_0(\lambda, p)$ and Y be the respective one of the spaces $l_\infty(p), c(p)$ and $c_0(p)$ and Z be a sequence space and $A = (a_{nk})$ an infinite matrix. If $A \in (X, Z)$ then we obtain $\hat{A} \in (Y, Z)$ such that $Ax = \hat{A}y$ for all sequences $x \in X$ and $y \in Y$ which are connected by the relation (2), where $\hat{A} = (\hat{a}_{nk})$ is the associated matrix defined as follows:

$$\hat{a}_{nk} = \begin{cases} s^1 & , 0 \leq k \leq n \\ s^2 & , k = n \\ 0 & , k > n \end{cases}$$

where, $s^1 = \Delta(\frac{a_k}{\lambda_k - \lambda_{k-1}})$ and $s^2 = (\frac{a_k \lambda_k}{\lambda_k - \lambda_{k-1}})$.
Proof. By using relation (2) and Theorem 1.5 the proof is completes.•
Theorem 1.7. Let X is any of the spaces $l_\infty(\lambda, p), c(\lambda, p)$ or $c_0(\lambda, p)$ and $A = (a_{nk})$ be an infinite matrix. If A is in any of the classes $(X, c_0(q)), (X, c(q))$ or $(X, l_\infty(q))$, then

$$\| L_A \| = \| A \|_{(X, l_\infty(q))} = \sup_n (\sum_k |\hat{a}_{nk}| M^{\frac{1}{p_k}})^{q_n} < \infty \quad (\forall n \in \mathbb{N}),$$

where $q = (q_n)$ is a non-decreasing bounded sequence of positive real numbers, and M be natural numbers.
Proof. By combining Lemma 1.2 and Theorem 1.5 the proof is obvious. •
By M_X we denote the collection of all bounded subsets of a metric space (X, d). If $Q \in M_X$, then the Hausdorff measure of noncompactness of the set Q, denoted by $\chi(Q)$, is defined by:

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{−net in } X\}.$$

The function $\chi : M_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness.

2 Main results

Theorem 2.1. Let $A = (a_{nk})$ be an infinite matrix and $\hat{A} = (\hat{a}_{nk})$ the associated matrix defined in Lemma 1.6. Further, assume that X be one of the spaces $l_\infty(\lambda, p), c(\lambda, p)$ or $c_0(\lambda, p)$. Then the following hold:
(a) If $A \in (X, c_0(q))$, then $\| L_A \|_\chi = \limsup_{n\to\infty} (\sum_k |\hat{a}_{nk}| M^{\frac{1}{p_k}})^{q_n} \quad (\forall M)$
and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}(\sum_k |\hat{a}_{nk}|M^{\frac{1}{p_k}})^{q_n} = 0.$$

(b) If $A \in (X, c(q))$, then
$\frac{1}{2}.\limsup_{n\to\infty}(\sum_{k=0}^{\infty} |\hat{a}_{nk}-\hat{a}_k|M^{\frac{1}{p_k}})^{q_n} \leq \| L_A \|_\chi \leq \limsup_{n\to\infty}(\sum_{k=0}^{\infty} |\hat{a}_{nk} - \hat{a}_k|M^{\frac{1}{p_k}})^{q_n}$
and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}(\sum_k |\hat{a}_{nk} - \hat{a}_k|M^{\frac{1}{p_k}})^{q_n} = 0$$

where, $\exists \hat{a}_k, \lim_n |\hat{a}_{nk} - \hat{a}_k|^{q_n} = 0$, $(\forall M)$.

(c) If $A \in (X, l_\infty(q))$, then $0 \leq \| L_A \|_\chi \leq \limsup_{n\to\infty}(\sum_{k=0}^{\infty} |\hat{a}_{nk}|M^{\frac{1}{p_k}})^{q_n} (\forall M)$
and

$$L_A \text{ is compact if } \lim_{n\to\infty}(\sum_{k=0}^{\infty} |\hat{a}_{nk}|M^{\frac{1}{p_k}})^{q_n} = 0.$$

Proof. By using Theorem 1.7, [Theorems 3.1 and 3.2, 2] and [section 2, 7] the proof is completes.•

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Talk Higher nummerical ranges of basic $A-$factor block circulant matrix pp.: 1–4

Higher nummerical ranges of basic $A-$factor block circulant matrix

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Abstract

In this paper, using the notion of $k-$numerical range, the relation between $k-$numerical range of matrix polynomials and the $k-$numerical range of its linearization are investigated. Moreover, the $k-$numerical ranges of basic circulant $A-$factor matrix are studied.

Keywords: $k-$numerical range, matrix polynomial, companion linearization, basic $A-$factor block circulant matrix
Mathematics Subject Classification [2010]: 15A60, 15A18, 47A56

1 Introduction

Let k and n are positive integers, $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices, The set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n\times k}$, i.e., $\mathcal{X}_{n\times k} = \{X \in \mathbb{M}_{n\times k} : X^*X = I_k\}$ and the group of $n \times n$ unitary matrices is denoted by $\mathcal{U}_n$. The $k-$numerical range of $A \in \mathbb{M}_n$ is defined and denoted by $W_k(A) = \{\frac{1}{k} tr\,(X^*AX) \ : \ X \in \mathcal{X}_{n\times k}\}$, where $tr(.)$ denotes the trace. The sets $W_k(A)$, where $k \in \{1, 2, \ldots, n\}$, are generally called *higher numerical ranges* of A. Let $A \in \mathbb{M}_n$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, counting multiplicities. The set of all $k-$averages of eigenvalues of A is denoted by $\sigma^{(k)}(A)$, namely,

$$\sigma^{(k)}(A) = \{\frac{1}{k}\,(\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}) \ : \ 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$

Proposition 1.1. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*
(i) $W_k(A)$ is a compact and convex set in $\mathbb{C}$;
(ii) conv $\left(\sigma^{(k)}(A)\right) \subseteq W_k(A)$, The equality holds if A is normal;
(iii) $\{\frac{1}{n}tr(A)\} = W_n(A) \subseteq W_{n-1}(A) \subseteq \cdots \subseteq W_2(A) \subseteq W_1(A) = W(A)$;
(iv) If $V \in \mathcal{X}_{n\times s}$, where $k \le s \le n$, then $W_k\,(V^*AV) \subseteq W_k(A)$. The equality holds if $s = n$, i.e., $W_k\,(U^*AU) = W_k(A)$, where $U \in \mathcal{U}_n$;
(v) For any $\alpha, \beta \in \mathbb{C}$, $W_k(\alpha A+\beta I_n) = \alpha W_k(A)+\beta$, and for the case $k < n$, $W_k(A) = \{\alpha\}$ if and only if $A = \alpha I_n$;

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Suppose that

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 \tag{1}$$

is a matrix polynomial, where $A_i \in \mathbb{M}_n$ $(i = 0, 1, \ldots, m)$, $A_m \neq 0$ and λ is a complex variable. The numbers m and n are referred as the *degree* and the *order* of $P(\lambda)$, respectively. The *k-numerical range* and the *k-spectrum* of $P(\lambda)$ are, respectively, defined and denoted by

$$W_k[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ tr(X^*P(\mu)X) = 0 \ for\ some\ X \in \mathcal{X}_{n\times k}\}, \tag{2}$$

$$\sigma^{(k)}[P(\lambda)] = \left\{\mu \in \mathbb{C} \ : \ 0 \in \sigma^{(k)}(P(\mu))\right\}. \tag{3}$$

Moreover, if $P(\lambda) = \lambda I_n - A$, where $A \in \mathbb{M}_n$, then $W_k[P(\lambda)] = W_k(A)$ and $\sigma^{(k)}[P(\lambda)] = \sigma^{(k)}(A)$. It is clear that $W_k[P(\lambda)]$ is a closed set in $\mathbb{C}$ which contains $\sigma^{(k)}[P(\lambda)]$.

Consider a matrix polynomial $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ as in (1), in which $m \geq 2$. The *companion linearization* of $P(\lambda)$ is defined, e.g., see [2], as:

$$L(\lambda) = \begin{pmatrix} I_n & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & I_n & 0 \\ 0 & 0 & \cdots & 0 & A_m \end{pmatrix} \lambda - \begin{pmatrix} 0 & I_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_n \\ -A_0 & -A_1 & \cdots & \cdots & \cdots & -A_{m-1} \end{pmatrix} \tag{4}$$

By [2, page 186], there exists unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$ of order mn such that $E(\lambda)L(\lambda)F(\lambda) = \begin{pmatrix} P(\lambda) & 0 \\ 0 & I_{n(m-1)} \end{pmatrix}$. So, $\sigma[P(\lambda)] = \sigma[L(\lambda)]$, and hence, for any integer $1 \leq k \leq mn$, $\sigma^{(k)}[P(\lambda)] = \sigma^{(k)}[L(\lambda)]$.

Theorem 1.2. *Let $1 \leq k \leq n$ be a positive integer, and $P(\lambda)$, as in (1), be a matrix polynomial with the companion linearization $L(\lambda)$ as in (4). Then $W_k[P(\lambda)] \cup \{0\} \subseteq W_k[L(\lambda)]$.*

Corollary 1.3. *If $W_k[L(\lambda)]$ is bounded, then $W_k[P(\lambda)]$ is also bounded.*

2 Main results

In this section, we study the $k-$numerical range of the companion linearization of the matrix polynomial $P(\lambda) = \lambda^m I_n - A$, where $m \geq 2$ and $A \in M_n$. By (4), the companion linearization of $P(\lambda)$ is $L(\lambda) = \lambda I_{mn} - \Pi_A$, where

$$\Pi_A = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n \\ A & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{M}_{mn}, \tag{5}$$

is called the *basic $A-$factor block circulant matrix.* These matrices have important applications in vibration analysis and differential equations. e.g., see [1] and their references.

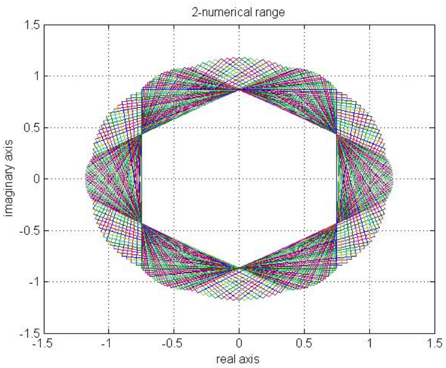

Figure 1: $W_2(\Pi_A)$

Theorem 2.1. *Let $A \in M_n$, $1 \le k \le mn$ be a positive integer. Then $e^{i\frac{2\pi}{m}} W_k(\Pi_A) = W_k(\Pi_A)$. Consequently, if m is even, then $W_k(\Pi_A)$ is symmetric with respect to the origin.*

Theorem 2.2. *Let $1 \le k \le n$ be a positive integer, $A \in \mathbb{M}_n$ and Π_A be the basic $A-$factor block circulant matrix as in (5). Then $conv\left(\sqrt[m]{W_k(A)} \cup \{0\}\right) \subseteq W_k(\Pi_A)$.*

The set equality in Theorem 2.2 does not hold in general, see the following example.

Example 2.3. Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_2$, $k = 2$ *and* $m = 3$. We have $W_2(A) = \{0\}$ and so, $conv\left(\sqrt[3]{W_2(A)}\right) = \{0\}$. Since A is unitary, Π_A is also a unitary matrix. Then $W_2(\Pi_A) = conv\left(\sigma^{(2)}(\Pi_A)\right) = \{0, \pm\frac{1}{2}(1 + e^{i\frac{\pi}{3}}), \pm\frac{1}{2}(1 + e^{i\frac{2\pi}{3}}), \pm\frac{1}{2}(e^{i\frac{\pi}{3}} + e^{i\frac{2\pi}{3}}), \pm\frac{1}{2}(e^{i\frac{\pi}{3}} + e^{i\frac{5\pi}{3}}), \pm\frac{1}{2}(-1 + e^{i\frac{\pi}{3}}), \pm\frac{1}{2}(-1 + e^{i\frac{2\pi}{3}})\} \neq \{0\}$. which is shown in Figure 1.

In the following example, we characterize the $k-$numerical range of Π_{I_n}.

Example 2.4. Let $m \ge 2$ be a positive integer, and $\Pi_{I_n} \in \mathbb{M}_{mn}$ be the companion matrix as in (5). It is clear that the eigenvalues of Π_{I_n}, counting multiplicity, are

$$\underbrace{1, \dots, 1}_{n-times}, \underbrace{\omega, \dots, \omega}_{n-times}, \underbrace{\omega^2, \dots, \omega^2}_{n-times}, \dots, \underbrace{\omega^{m-1}, \dots, \omega^{m-1}}_{n-times},$$

where $\omega = e^{i\frac{2\pi}{m}}$ and $\sigma^{(k)}(\Pi_{I_n})$ contains all points of the following form:

$$\frac{1}{k}(r_0 + r_1\omega + r_2\omega^2 + \cdots + r_{m-1}\omega_{m-1}), \tag{6}$$

where $0 \le r_0, r_1, \dots, r_{m-1} \le k$ are positive integers and $r_0 + r_1 + \cdots + r_{m-1} = k$. Since Π_{I_n} is normal, by Proposition 1.1(ii), we have $W_k(\Pi_{I_n}) = conv(\sigma^{(k)}(\Pi_{I_n}))$. Now, we consider the following cases:
case 1: If $1 \le k \le n$, then $\{1, \omega, \omega^2, \dots, \omega^{m-1}\} \subseteq \sigma^{(k)}(\Pi_{I_n})$ and so,

$$W_k(\Pi_{I_n}) = conv(\sigma^{(k)}(\Pi_{I_n})) = conv(\{1, \omega, \dots, \omega^{m-1}\}).$$

case 2: If $k = tn + l$, where $1 \le t \le m$ and $0 \le l \le n-1$ are integer numbers, then by considering all the points of the form $p_\alpha = \frac{1}{k}\left(n\omega^{\alpha_1} + n\omega^{\alpha_2} + \cdots + n\omega^{\alpha_t} + l\omega^{\alpha_{t+1}}\right)$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{t+1})$ is a $(t+1)$-permutation of $\{0, 1, \ldots, n-1\}$, we have

$$conv\left(\sigma^{(k)}(\Pi_I)\right) = conv\left(\{\, p_\alpha \;:\; \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{t+1}) \text{ is a } (t+1) - \text{permutation of } \{0, 1, \ldots, n-1\}\}\right).$$

For example, if $m = 4$ and $n = 2$, then $W_1(\Pi_{I_2}) = W_2(\Pi_{I_2}) = conv(\{1, i, -1, -i\})$,

$$W_3(\Pi_{I_2}) = conv\left(\left\{\frac{2+i}{3}, \frac{2i+1}{3}, \frac{2i-1}{3}, \frac{i-2}{3}, \frac{-2i-1}{3}, \frac{-i-2}{3}, \frac{2-i}{3}, \frac{1-2i}{3}\right\}\right),$$

$$W_4(\Pi_{I_2}) = conv\left(\left\{\frac{1+i}{2}, \frac{i_1}{2}, \frac{-1-i}{2}, \frac{1-i}{2}\right\}\right),$$

$$W_5(\Pi_{I_2}) = conv\left(\left\{\frac{1+2i}{5}, \frac{2+i}{5}, \frac{2-i}{5}, \frac{1-2i}{5}, \frac{-1+2i}{5}, \frac{-2+i}{5}, \frac{-1-2i}{5}, \frac{-2-i}{5}\right\}\right),$$

$$W_6(\Pi_{I_2}) = conv\left(\left\{\frac{i}{3}, \frac{1}{3}, \frac{-i}{3}, \frac{-1}{3}\right\}\right), \quad W_7(\Pi_{I_2}) = conv\left(\left\{\frac{i}{7}, \frac{1}{7}, \frac{-i}{7}, \frac{-1}{7}\right\}\right),$$

and $W_8(\Pi_{I_2}) = \{\frac{1}{8} tr(\Pi_{I_2})\} = \{0\}$.

At the end of this section, we find a circular disk which contains $W_k(\Pi_A)$.

Theorem 2.5. *Let $1 \le k \le mn$ be a positive integer, $A \in \mathbb{M}_n$ and Π_A be the basic $A-$factor block circulant matrix as in (5). Then $W_k(\Pi_A) \subseteq \{\mu \in \mathbb{C} : |\mu| \le 1 + \|A - I_n\|\}$.*

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Homological properties of certain subspaces of $L^\infty(G)$ on group algebras

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Abstract

Homological properties of several Banach left $L^1(G)$-modules have been studied by Dales and Polyakov and recently by Ramsden. In this paper, we characterize homological properties for some sub-modules of $L^\infty(G)$ as Banach left $L^1(G)$-modules.

Keywords: Banach module, flatness, injectivity and locally compact group.
Mathematics Subject Classification [2010]: 43A15, 43A20, 46H25.

1 Introduction

Throughout this paper, G denotes a locally compact group with the identity element e, the modular function Δ, and a fixed left Haar measure λ. As usual, let $L^1(G)$ denote the group algebra of G as defined in [4] equipped with the norm $\|\cdot\|_1$ and the convolution product $*$ of functions on G defined by

$$(\phi * \psi)(x) = \int_G \phi(y)\psi(y^{-1}x)\, d\lambda(y)$$

for all $\phi, \psi \in L^1(G)$ and locally almost all $x \in G$. Let also $L^\infty(G)$ denote the Banach space as defined in [4] equipped with the essential supremum norm $\|\cdot\|_\infty$. Then $L^\infty(G)$ is the dual bimodule of the Banach $L^1(G)$-bimodule $L^1(G)$ under the pairing

$$\langle f, \phi\rangle = \int_G f(x)\phi(x)\, d\lambda(x).$$

for all $\phi \in L^1(G)$ and $f \in L^\infty(G)$. The left and right module actions of $L^1(G)$ on $L^\infty(G)$ are given by the formulae

$$\phi \cdot f = f * \widetilde{\phi} \quad \text{and} \quad f \cdot \phi = \frac{1}{\Delta}\widetilde{\phi} * f$$

for all $f \in L^\infty(G)$ and $\phi \in L^1(G)$, where $\widetilde{\phi}(x) = \phi(x^{-1})$ for all $x \in G$. We denote by $C_b(G)$ the space of all bounded continuous functions on G, by $LUC(G)$ the space of all bounded left uniformly continuous functions on G and by $C_0(G)$ the space of all

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continuous functions on G vanishing at infinity. Then $C_b(G)$, $LUC(G)$ and $C_0(G)$ are closed submodules of the Banach $L^1(G)$-bimodule $L^\infty(G)$.

Dales and Polyakov [2] have characterized projectivity, injectivity and flatness of certain Banach left $L^1(G)$-modules; see also [1], [9], [8]. In this work, we intend to characterize these homological properties for some sub-modules of $L^\infty(G)$ as Banach left $L^1(G)$-modules in terms of some topological and algebraic properties of G.

2 Main results

Let A be a Banach algebra. A Banach left A-module I is called *injective* if for each Banach left A-modules E and F, each admissible monomorphism $T \in {}_AB(E,F)$, and each $S \in {}_AB(E,I)$, there exists $R \in {}_AB(F,I)$ such that $R \circ T = S$. Similar definitions apply for Banach right A-modules.

For each Banach left A-module E, the space $B(A,E)$ is a Banach left A-module with $(a \cdot T)(b) = T(ba)$ for all $a, b \in A$ and $T \in B(A,E)$. Define the left A-module morphism $\Pi : E \longrightarrow B(A,E)$ by the formula $\Pi(\xi)(a) = a \cdot \xi$ for $\xi \in E$ and $a \in A$. It is shown in [3], Proposition III.1.31, that if A is a Banach algebra, and E is faithful as Banach left A-module (i.e., $A \cdot \xi \neq \{0\}$ for all $\xi \in E \setminus \{0\}$), then E is injective if and only if there exists a left A-module morphism $\rho : B(A,E) \longrightarrow E$ with $\rho \circ \Pi = I_E$.

Theorem 2.1. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *There is a submodule X of $C_b(G)$, $C_0(G) \subset X$, and X is an injective Banach left $L^1(G)$-module.*

(b) *There exists a closed subspace X of $C_b(G)$, $C_0(G) \subset X$, and X is complemented in $L^\infty(G)$.*

(c) *G is discrete.*

Proof. (a)⇒(b). Suppose that a submodule X of $C_b(G)$ is an injective Banach left $L^1(G)$-module such that , $C_0(G) \subset X$. Then there exists a left $L^1(G)$-module morphism

$$\rho_G : B(L^1(G), LUC(G)) \longrightarrow X$$

such that $\rho_G \circ \Pi_G = I_{LUC(G)}$, where $\Pi_G : X \longrightarrow B(L^1(G), LUC(G))$ is the canonical embedding defined by

$$\Pi_G(h)(\phi) = \phi \cdot h$$

for all $h \in X$ and $\phi \in L^1(G)$. Now, consider $Q : L^\infty(G) \longrightarrow B(L^1(G), LUC(G))$ with

$$Q(f)(\phi) = \phi \cdot f$$

for all $f \in L^\infty(G)$ and $\phi \in L^1(G)$. In particular, $Q(h)(\phi) = \Pi_G(h)(\phi)$ for all $h \in X$ and $\phi \in L^1(G)$. The result follows from the fact that $\rho_G \circ Q : L^\infty(G) \longrightarrow X$ is projection on $LUC(G)$.

(b)⇒(c). see [6], Theorem 4.

(c)⇒(a). This follows from facts that $L^\infty(G)$ is always an injective Banach left $L^1(G)$-module and that $C_b(G) = L^\infty(G)$ when G is discrete; see [2], Theorem 2.4. □

As a consequence of Theorem 2.3, we have the following result.

Corollary 2.2. *Let G be a locally compact group. Then $LUC(G)$ is an injective Banach left $L^1(G)$-module if and only if G is discrete.*

Let A be a Banach algebra and let us recall that a Banach left A-module F is called *flat* if F^* is an injective Banach right A-module. Moreover, a locally compact group G is called *amenable* if there is a positive functional $m \in L^\infty(G)^*$ with $\|m\| = 1$ and $m \cdot \delta_x = m$ for all $x \in G$. The class of amenable groups includes all compact groups and all abelian locally compact groups; however, the discrete free group $\mathbb{F}_2$ on two generators is not amenable; see [7] for more details.

Theorem 2.3. *Let G be a locally compact group. Let X be a submodule of $C_b(G)$, $C_0(G) \subset X$. Then the following statements are equivalent.*
(a) *X is a flat Banach left $L^1(G)$-module.*
(b) *G is amenable.*

Proof. (b)$\Leftrightarrow$(a). Suppose that G is amenable. Then by the classical result of Johnson [5], $L^1(G)$ is an amenable Banach algebra; that is, $H^1(L^1(G), E^*) = \{0\}$ for all Banach $L^1(G)$-bimodule E. So, $LUC(G)$ is a flat Banach left $L^1(G)$-module; this follows from the fact that if A is an amenable Banach algebra, then each Banach left or right A-module is flat, see [3], VII.2.29.

For the converse, suppose that X is flat as a Banach left $L^1(G)$-module; that is, X^* is injective as a Banach right $L^1(G)$-module. We will show that the Banach right $L^1(G)$-module $M(G)$ is a retraction of X^*. Thus $M(G)$ is also an injective Banach right $L^1(G)$-module; this is because that each retraction of an injective Banach module is injective; see [3], Proposition III.1.16. Therefore, G is amenable by Crollary 4.7 of [2].

We define $\mathcal{Q} : M(G) \longrightarrow X^*$ to be the map that sends a measure μ in $M(G)$ to the integration functional $h \mapsto \int h \, d\mu$ $(h \in X)$. This is well defined because h is continuous. Clearly, $\mathcal{Q}$ is a right $L^1(G)$-module morphism. Now, let $\mathcal{P} : X^* \longrightarrow M(G)$ be the restriction map, and note that $\mathcal{P}$ is a right $L^1(G)$-module morphism. One can easily check that $\mathcal{Q}$ is a right inverse for $\mathcal{P}$, and thus $M(G)$ is a retraction of X^* . This completes the proof. □

Corollary 2.4. *Let G be a locally compact group. Then $LUC(G)$ is a flat Banach left $L^1(G)$-module if and only if G is amenale.*

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Inequalities for Keronecker product of Matrices

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Abstract

In this paper we present a brief overview on the Kronecker product and ts properties. Triangle and Young inequalities are presented. In particular, the arithmetic-geometric mean inequality for Keronecker product is obtained as special case.

Keywords: Kronecker product, Keronecker sum, Löwner partial order

Mathematics Subject Classification [2010]: 15A69, 47A80

1 Introduction

The kronecker product of two matrices denoted by $A \otimes B$, has been researched since nineteenth century. In fact the kronecker product should be called Zehfuss product because Johann Georg Zehfuss published a paper in 1858 which contained the well-known determinate conclusion $|A \otimes B| = |A|^n |B|^m$, for square matrices $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$. Many propetrties about its trace, determinant, eigenvalues, and other decompositions have been discovered during this time. The Keronecker product has wide applications in system theory [6], matrix calculus [3], and quantum mechanics [2].

Definition 1.1. The Kronecker product of the matrix $A \in M_{mn}(\mathbb{C})$ with the matrix $B \in M_{pq}(\mathbb{C})$ is a matrix in $M_{(mp)(nq)}(\mathbb{C})$ and is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

The Kronecker product of marices has a lot of interesting properties, many of them stated and proven in the basic literature about matrix anlysis (e.g. one can see chapter 4 in [5]). The (relatively few) properties that are used to established the results in this paper are collected in the following theorems.

Theorem 1.2. *Let $A \in M_{mn}(\mathbb{C})$, $B \in M_{qr}(\mathbb{C})$, $C \in M_{np}(\mathbb{C})$, and $D \in M_{rs}(\mathbb{C})$.*

1. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

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2. $A \otimes B = (A \otimes I_q)(I_n \otimes B) = (I_m \otimes B)(A \otimes I_r)$.

3. *If* $A \in M_m(\mathbb{C})$ *and* $B \in M_n(\mathbb{C})$, *then* $A \otimes B = (A \otimes I_n)(I_m \otimes B) = (I_m \otimes B)(A \otimes I_n)$. *This means* $(A \otimes I_n)$ *and* $(I_m \otimes B)$ *are commutative for square matrices* A *and* B.

4. $(A \otimes B)^* = A^* \otimes B^*$.

Theorem 1.3. *Let* $A \in M_m(\mathbb{C})$ *and* $B \in M_n(\mathbb{C})$.

1. $\sigma(A \otimes B) = \{\lambda\mu : \quad \lambda \in \sigma(A), \quad \mu \in \sigma(B)\}$.

2. $tr(A \otimes B) = tr(B \otimes A) = tr(A)tr(B)$.

3. $\det(A \otimes B) = \det(B \otimes A) = (\det A)^n (\det B)^m$.

4. *If* A *and* B *are non-singular, then* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

5. *If* A *and* B *are positive definite matrices, then* $(A \otimes B)^r = A^r \otimes B^r$ *for any real number* r.

Corollary 1.4. *If* $A \in M_M(\mathbb{C})$ *and* $B \in M_n(\mathbb{C})$ *are positive semi-definite matrices, then* $(A \otimes B)$ *is positive semi-definite.*

Corollary 1.5. *If* $A \in M_M(\mathbb{C})$ *and* $B \in M_n(\mathbb{C})$, *then* $|A \otimes B| = |A| \otimes |B|$, *where* $|A|$ *stands for the unique positive square root of* A^*A.

A real $n \times n$ matrix A is called totally positive if determinant of all its minors are positive. The following example shows that Kronecker product does not preserve totally positivity.

Example 1.6. If $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$. Then A and B are totally positive but $A \otimes B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 3 & 4 \\ 2 & 2 & 3 & 3 \\ 6 & 8 & 9 & 12 \end{pmatrix}$ is not totally positive.

Definition 1.7. The Kronecker sum of two square matrices $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$ is a matrix in $M_{mn}(\mathbb{C})$ and is defined as

$$A \oplus B = (I_n \otimes A) + (B \otimes I_m).$$

Note that the definition of the Kronecker sum varies in the literature. Horn and Johnson [5] use the above definition, whereas Graham[3] use $A \oplus B = (A \otimes I_n) + (I_m \otimes B)$. We use Horn and Johnson's version of the Kronecker sum.

2 Triangle and Young Inequalities

Some of the most important inequalities in complex numbers admit generalisations in matrix context. The triangle inequality $|\alpha+\beta| \leq |\alpha| + |\beta|$ and the arithmetic-geometric mean inequality $\sqrt{|\alpha\beta|} \leq \frac{|\alpha|+|\beta|}{2}$, are all in evidence. Another such inequality is the Young inequality that refers to the following elementary, though fundamental, inequality between the moduli of any pair of complex numbers $\alpha, \beta \in \mathbb{C}$:

$$|\alpha\beta| \leq \frac{|\alpha|^p}{p} + \frac{|\beta|^q}{q}, \tag{1}$$

where $p, q \in (1, \infty)$ are conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, it is well known that equality holds if and only if $|\beta|^q = |\alpha|^p$.

The formulation of the triangle inequality for operators with respect to the Löwner partial order (see Theorem 2.1 below) originates with a paper of Thompson [7] for operators acting on finite-dimensional spaces.

Theorem 2.1. *Let A and B be any two matrices in $M_n(\mathbb{C})$. Then there exist unitary matrices U and V such that*

$$|A+B| \leq U|A|U^* + V|B|V^* .$$

The Young inequality was extended to complex matrices in [1] by T. Ando in the following theorem.

Theorem 2.2. *For each complex matrices A and B there exists a unitary matrix U such that for each conjugate exponents p and q,*

$$U^*|AB^*|U \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q. \tag{2}$$

Equality holds in (2) if and only if $|A|^p = |B|^q$ [4].

Since $(|A| \otimes I_n)$ and $(I_m \otimes |B|)$ are commutative for any matrix $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$ (part (3) Theorem 1.2), an almost immediate consequence of the Gelfand theory is that the triangle and Young Inequalities are hold in the following forms.

Theorem 2.3. *Let $A \in M_m(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ be any two matrices. Then*

$$|A \oplus B| \leq |A| \oplus |B| .$$

Theorem 2.4. *Let $A \in M_m(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ be any two matrices. Then*

$$|A \otimes B|^r \leq \frac{1}{p}|A|^p \oplus \frac{1}{q}|B|^q ,$$

where p, q and r are positive real numberssuch such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Moreover, equality holds if and only if $|A|^p \otimes I_n = I_m \otimes |B|^q$.

Corollary 2.5. (Arithmetic-geometric mean inequality) *Let $A \in M_m(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ be any two matrices. Then*

$$\sqrt{|A \otimes B|} \leq \frac{1}{2}(|A| \oplus |B|) .$$

Equality holds if and only if $|A| \otimes I_n = I_m \otimes |B|$.

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∞-Tuples of operators and Hereditarily

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Abstract

In this paper, we introduce for an ∞-tuple of operators on common Ordered Banach space and some conditions to an ∞-tuple to be Hereditary Hypercyclic infinity tuple. The supreme is taken over norm operator defined on the space.

Keywords: Hypercylicity, ∞-tuple, Hereditarily.
Mathematics Subject Classification [2010]: 37A25, 47B37.

1 Introduction

Let $\mathcal{X}$ be an infinite dimensional Banach space and $T_1, T_2, ...$ are commutative bounded linear operators on $\mathcal{X}$. By an ∞-tuple we mean the ∞-component $\mathcal{T} = (T_1, T_2, ...)$. For the ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ the set

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \{ {T_1}^{k_1} {T_2}^{k_2} ... {T_n}^{k_n} : k_i \geq 0, i = 1, 2, ..., n, n \in \mathcal{N} \}$$

is the semigroup generated by $\mathcal{T}$. For $x \in \mathcal{X}$ take

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

In other hand

$$Orb(\mathcal{T}, x) = \bigcup_{n=1}^{\infty} \{ {T_1}^{k_1} {T_2}^{k_2} ... {T_n}^{k_n}(x) : k_i \geq 0, i = 1, 2, ...n \}.$$

Definition 1.1. The set $Orb(\mathcal{T}, x)$ is called, orbit of vector x under $\mathcal{T}$ and ∞-Tuple $\mathcal{T} = (T_1, T_2, ...)$ is called hypercyclic ∞-tuple, if there is a vector $x \in \mathcal{X}$ such that, the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is

$$\overline{Orb(\mathcal{T}, x)} = \overline{\bigcup_{n=1}^{\infty} \{ {T_1}^{k_1} {T_2}^{k_2} ... {T_n}^{k_n}(x) : k_i \geq 0, i = 1, 2, ...n \}} = \mathcal{X}.$$

In this case, the vector x is called a hypercyclic vector for the ∞-tuple $\mathcal{T}$.

*Speaker

Definition 1.2. Let $\{m_{(k,1)}\}_{k=1}^{\infty}, \{m_{(k,2)}\}_{k=1}^{\infty}, ...$ be increasing sequences of non-negative integers. The ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is called hereditarily hypercyclic with respect to $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ..., \{m_{j,n}\}_{j=1}^{\infty}, ...$ if for all subsequences $\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, ...$ of $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ..., \{m_{j,n}\}_{j=1}^{\infty}, ...$ respectively, the sequence

$$\{T_1^{m'_{(k,1)}} T_2^{m'_{(k,2)}} ... T_n^{m'_{(k,n)}}\}_{n=1}^{\infty}$$

is hypercyclic. That is, there exists a vector x in $\mathcal{X}$ such that

$$\overline{\bigcup_{n=1}^{\infty} \{T_1^{k_1} T_2^{k_2} ... T_n^{k_n}(x) : k_i \geq 0, i = 1, 2, ...n\}} = \mathcal{X}.$$

Note 1.3. If $\mathcal{X}$ be an finite dimensional Banach space, then there are no hypercyclic operator on $\mathcal{X}$, also there are no ∞-tuple or n-tuple on $\mathcal{X}$.

Note 1.4. All of operators in this paper are commutative bounded linear operators on a Banach space. Also, note that by $\{j, i\}$ or (j, i) we mean a number, that was showed by this mark and related with this indexes, not a pair of numbers. Also, let $T_1, T_2, ...$ acting on Ordered Banach Space $\mathcal{X}$ and $\mathcal{T} = (T_1, T_2, ...)$ be ∞-tuple of those operators and $x \in \mathcal{X}$

$$\mathcal{T}(x) = Sup_n \bigcup_{h=1}^{n} \{T_1^{k_1} T_2^{k_2} ... T_h^{k_h}(x) : k_i \geq 0, i = 1, 2, ...h\}.$$

Since $\mathcal{X}$ is Ordered Space then the supreme is well fine.

2 Main Results

Theorem 2.1 (The Hypercyclicity Criterion for ∞-Tuples)**.** Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ...)$ is an ∞-tuple of continuous linear mappings on $\mathcal{X}$. If there exist two dense subsets $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathcal{X}$, and strictly increasing sequences $\{m_{j,1}\}_{j=1}^{\infty}$, $\{m_{j,2}\}_{j=1}^{\infty}$, ... such that :
1. $T_1^{m_{j,1}} T_2^{m_{j,2}} ... \to 0$ on $\mathcal{Y}$ as $j \to \infty$,
2. There exist functions $\{S_j : \mathcal{Z} \to \mathcal{X}\}$ such that for every $z \in \mathcal{Z}$, $S_j z \to 0$,
and $T_1^{m_{j,1}} T_2^{m_{j,2}} ... S_j z \to z$, on $\mathcal{Z}$ as $j \to \infty$,
then $\mathcal{T}$ is a hypercyclic ∞-tuple.

We can replace the notation $Sup_n\{T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} S_j z\}$ by $T_1^{m_{j,1}} T_2^{m_{j,2}} ... S_j z$

Theorem 2.2. An ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is hereditarily hypercyclic with respect to increasing sequences of non-negative integers$\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$ if and only if for all given any two open sets $\mathcal{U}$, $\mathcal{V}$, there exist some positive integers $M_i, M_2, ...$ such that

$$(\bigcup_{n=1}^{\infty} \{T_1^{m_{k,1}} T_2^{m_{k,2}} ... T_n^{m_{k,n}}(\mathcal{U}), \forall m_{k,i} > M_i, i = 1, 2, ..., n\}) \bigcap \mathcal{V} \neq \phi$$

Proof. Let $\mathcal{T} = (T_1, T_2, ...)$ be hereditarily hypercyclic ∞-tuple with respect to increasing sequences of non-negative integers

$$\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$$

and suppose that there exist some open sets $\mathcal{U}$, $\mathcal{V}$ such that

$$(\bigcup_{n=1}^{\infty}\{T_1^{m'_{k,1}}T_2^{m'_{k,2}}...T_n^{m'_{k,n}}(\mathcal{U}), \forall m'_{k,i} > M_i, i = 1, 2, ..., n\})\bigcap \mathcal{V} = \phi$$

for some subsequence $\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, ...$ of $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$ respectively.

Since the ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is hereditarily hypercyclic with respect to $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$, thus $\{T_1^{m'_{k,1}}T_2^{m'_{k,2}}...\}$ is hypercyclic, and so we get a contradiction.

Conversely, Suppose that $\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, ...$ are arbitrary subsequences of $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$ respectively, and $\mathcal{U}$, $\mathcal{V}$ are open sets in $\mathcal{X}$, satisfying

$$(\bigcup_{n=1}^{\infty}\{T_1^{m'_{k,1}}T_2^{m'_{k,2}}...T_n^{m'_{k,n}}(\mathcal{U}), \forall m'_{k,i} > M_i, i = 1, 2, ..., n\})\bigcap \mathcal{V} \neq \phi$$

So there exist (i, j), large enough for $j = 1, 2, ...$ such that $m_{(k_i,j)} > M_j$ for $j = 1, 2, ...$ and

$$(\bigcup_{n=1}^{\infty}\{T_1^{m_{k_1,1}}T_2^{m_{k_2,2}}...T_n^{m_{k_n,n}}(\mathcal{U}), \forall m_{k_i,i} > M_i, i = 1, 2, ..., n\})\bigcap \mathcal{V} \neq \phi.$$

This implies that

$$\{T_1^{m_{(k_i,1)}}T_2^{m_{(k_i,2)}}...\}$$

is hypercyclic, so the ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is indeed hereditarily hypercyclic with respect to the sequences

$$\{m_{(k,1)}\}_{k=1}^{\infty}, \{m_{(k,2)}\}_{k=1}^{\infty}, ...$$

By this the proof is complete.

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Integral Operators and Multiplication Operators on $F(p,q,s)$ Spaces

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Abstract

We study integral operators on a large family of analytic function spaces, called $F(p,q,s)$ spaces. Our approach for the study of integral operators is to investigate some related multiplication operators on $F(p,q,s)$ type spaces. As a consequence of this approach, we obtain certain properties of integral operators on $\mathcal{Q}_s$ spaces.

Keywords: Integral operators, Multiplication operators, $F(p,q,s)$ spaces, $\mathcal{Q}_s$ spaces.
Mathematics Subject Classification [2010]: 47B38, 46E15.

1 Introduction

Let $\mathbb{D}$ denote the open unit disc of the complex plane and $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. For $a \in \mathbb{D}$, the Möbius function $\varphi_a : \mathbb{D} \to \mathbb{D}$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z},$$

for all $z \in \mathbb{D}$. Also, the Green's function of $\mathbb{D}$ with logarithmic singularity at a is defined by

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right| = \log\frac{1}{|\varphi_a(z)|},$$

for all $z \in \mathbb{D}$.

For $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the space $F(p,q,s)$, if

$$\|f\|_{p,q,s} = \sup_{a\in\mathbb{D}}\left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\right)^{\frac{1}{p}} < \infty, \tag{1}$$

and $f \in H(\mathbb{D})$ is said to belong to the space $F_0(p,q,s)$, if

$$\lim_{|a|\to 1}\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) = 0, \tag{2}$$

where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. In the case of $s = 0$, a function $f \in H(\mathbb{D})$ is said to belong to the space $F(p,q,0)$, if

$$\|f\|_{p,q,0} = \sup_{a\in\mathbb{D}}\left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^q dA(z)\right)^{\frac{1}{p}} < \infty,$$

and for convenience $F_0(p,q,0)$ is defined to be $F(p,q,0)$.

It is known that for $1 \leq p < \infty$, $-2 < q < \infty$ and $0 \leq s < \infty$, $F(p,q,s)$ is a Banach space if equipped with the norm

$$\|f\| = |f(0)| + \|f\|_{p,q,s}.$$

Moreover, $F_0(p,q,s)$ is a closed subspace of $F(p,q,s)$.

The spaces $F(p,q,s)$, which were first studied by Zhao [5] and Rättyä [3], are also called *"general family of function spaces"* or *"large family of analytic function spaces"* because one can get many well known function spaces by taking special parameters of p, q and s. Two important special cases are *"Bloch type spaces"* and *"$\mathcal{Q}_s$ spaces"*, defined as follows.

For $0 < \alpha < \infty$, the *Bloch type space* $\mathcal{B}^\alpha$ is the space of all analytic functions $f \in H(\mathbb{D})$ for which

$$B^\alpha(f) = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The Bloch type space $\mathcal{B}^\alpha$ is a Banach space if equipped with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + B^\alpha(f).$$

When $\alpha = 1$, we get the classic *Bloch space* $\mathcal{B} = \mathcal{B}^1$.

For any $0 \leq s < \infty$, the $\mathcal{Q}_s$ *space* consists of all analytic functions $f \in H(\mathbb{D})$ such that

$$Q_s(f) = \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^s dA(z) \right)^{\frac{1}{2}} < \infty.$$

The $\mathcal{Q}_s$ space is a Banach space if equipped with the norm

$$\|f\|_{\mathcal{Q}_s} = |f(0)| + Q_s(f).$$

It is known that for $s > 1$, $\mathcal{Q}_s = \mathcal{B}$. Also, when $s = 1$, $\mathcal{Q}_s = BMOA$, the space of all analytic functions of bounded mean oscillation. Moreover, when $s = 0$, the space $\mathcal{Q}_s$ degenerates to the Dirichlet space [4]. Therefore, one may be interested in the study of $\mathcal{Q}_s$ spaces only in the case of $0 < s < 1$.

About the relation between $F(p,q,s)$ spaces and Bloch type spaces $\mathcal{B}^\alpha$, we know that $F(p,q,s) = \mathcal{B}^{\frac{q+2}{p}}$ for $s > 1$, and $F(p,q,s) \subseteq \mathcal{B}^{\frac{q+2}{p}}$ for $0 < s \leq 1$. Also, one can get $\mathcal{Q}_s$ spaces by taking $p = 2$ and $q = 0$ in $F(p,q,s)$ spaces, that is, $F(2,0,s) = \mathcal{Q}_s$ [5].

In this paper, we consider *"integral operators"* and *"multiplication operators"*, on $F(p,q,s)$ spaces, defined as follows.

For $g \in H(\mathbb{D})$, the *integral operator* I_g is given by

$$(I_g f)(z) = \int_0^z g(\xi) f'(\xi) d\xi, \quad (z \in \mathbb{D}),$$

and the *multiplication operator* M_g is given by

$$(M_g f)(z) = g(z) f(z), \quad (z \in \mathbb{D}).$$

Integral operators and multiplication operators acting on various function spaces of analytic functions on $\mathbb{D}$ have been studied by many authors. See, for example, [1, 2] and the references therein.

2 Main results

It is known that for $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$, an analytic function f on $\mathbb{D}$ belongs to $F(p,q,s)$ if and only if [5]

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) < \infty, \tag{3}$$

and belongs to $F_0(p,q,s)$ if and only if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) = 0. \tag{4}$$

Therefore, one may consider (3) and (4) instead of (1) and (2), respectively, in the definition of $F(p,q,s)$ spaces.

Before giving our main results, we next state a useful lemma, Theorem 4.2.2 [3], which will be used in the proof of next theorems.

Lemma 2.1. *Let $f \in H(\mathbb{D})$, $1 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$ such that $-1 < q + s - p$. Then, $f \in F(p,q,s)$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^{-p+q} (1-|\varphi_a(z)|^2)^s dA(z) < \infty.$$

We next give one of the main theorems, giving the idea that study of integral operators I_g between $F(p,q,s)$ type spaces may reduce to the study of multiplication operators M_g between $F(p,q,s)$ type spaces.

Theorem 2.2. *Let $g \in H(\mathbb{D})$, then the integral operator*

$$I_g : F(p,q,s) \to F(p,q,s),$$

is bounded if and only if the multiplication operator

$$M_g : F(p,p+q,s) \to F(p,p+q,s),$$

is bounded.

Applying Theorem 2.2, in the special case of $F(2,0,s)$, leads to the following corollary for the boundedness of integral operator I_g on $\mathcal{Q}_s$ spaces.

Corollary 2.3. *Let $g \in H(\mathbb{D})$, then the integral operator $I_g : \mathcal{Q}_s \to \mathcal{Q}_s$ is bounded if and only if the multiplication operator $M_g : F(2,2,s) \to F(2,2,s)$ is bounded.*

Regarding Theorem 2.2, in the next theorem we apply Lemma 2.1 to characterize boundedness of multiplication operator

$$M_g : F(p,p+q,s) \to F(p,p+q,s),$$

when $1 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$ such that $-1 < q + s$.

Theorem 2.4. *Let $1 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$ such that $-1 < q + s$. Let $g \in H(\mathbb{D})$, then the multiplication operator*

$$M_g : F(p, p+q, s) \to F(p, p+q, s),$$

is bounded if and only if $g \in H^\infty(\mathbb{D})$.

Note that the case of $p = 2$, $q = 0$ and $0 \le s < \infty$ satisfies the conditions of Theorem 2.4. Therefore, Theorem 2.4 implies that the multiplication operator

$$M_g : F(2, 2, s) \to F(2, 2, s),$$

is bounded if and only if $g \in H^\infty(\mathbb{D})$. This, along with Corollary 2.3, leads to the next corollary for the boundedness of integral operator I_g on $\mathcal{Q}_s$ spaces.

Corollary 2.5. *The integral operator $I_g : \mathcal{Q}_s \to \mathcal{Q}_s$ is bounded if and only if $g \in H^\infty(\mathbb{D})$.*

Remark 2.6. As we mentioned before, the main idea of this paper is to give the approach of studying integral operators I_g between $F(p,q,s)$ type spaces by investigating related multiplication operators M_g between $F(p,q,s)$ type spaces. For example, note that the result of Corollary 2.5 has been proved in [2] using a classic approach in the study of integral operators. But, here we obtained Corollary 2.5 as a consequence of our proposed different approach in Theorem 2.2. It is also worth mentioning that, using a similar method as in the proof of Theorem 2.2, one can prove this approach for the compactness of integral operators I_g between $F(p,q,s)$ type spaces. More precisely, we have the following result.

The integral operator

$$I_g : F(p, q, s) \to F(p, q, s),$$

is compact if and only if the multiplication operator

$$M_g : F(p, p+q, s) \to F(p, p+q, s),$$

is compact.

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Mappings under asymptotic pointwise weaker Meir-Keeler-type contractive type conditions

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Abstract

In this paper we first define the asymptotic pointwise weaker Meir-Keeler-type ψ -condition type, $\psi : \mathcal{X} \longrightarrow \mathcal{R}^+$, and present fixed point theorems for mapping such condition in normed and Banach space.

1 Introduction

The notion of asymptotic pointwise mapping was introduction in {[3, 4]}. In this work, we introduce new asymptotic pointwise weaker Meir-Keeler-type ψ -contraction type, $\psi : \mathcal{X} \longrightarrow \mathcal{R}^+$, and present fixed point theorems for mapping such condition in normed and Banach space. In normed spaces, we discuss an asymptotic behavior of a mapping of asymptotic pointwise weaker Meir-Keeler-type ψ-contraction type .Our results extend and improve, for example, the corresponding result of Chi- Ming Chen [1].

Asymptotic contractions are defined as follows. Let Φ denote the class of all mappings $\phi : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ satisfying

(i) ϕ is continuous, (ii) $0 \leq \phi(t) < t$ for all $t \in \mathcal{R}^+ \setminus \{0\}, \phi(0) = 0$.

Definition 1.1. let (M, D) be a metric space. A mapping $T : M \longrightarrow M$ is said to be an asymptotic contraction if

$$d(T^n x, T^n y) \leq \phi_n(d(x, y)) \text{ for all } x, y \in M. \tag{1.1}$$

where $\phi_n \longrightarrow \phi \in \Phi$ uniformly on the range of d.

A function $\psi : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ ([5]) is said to be a Meir-Keeler-type function if for each $\eta \in \mathcal{R}^+$, there exists $\delta > 0$ such that for $t \in R^+$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$.

Definition 1.2. A function $\psi : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ is called a weaker Meir-Keeler-type function if for each $\eta > 0$, there exists $\delta > \eta$ such that for $t \in R^+$ with $\eta \leq t < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

*Speaker

Definition 1.3. Let X be a Banach space, and let $\psi : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ be a weaker Meir-Keeler-type function. Then the mapping $T : X \longrightarrow X$ is said to be asymptotic pointwise weaker Meir-Keeler-type ψ -contraction if for each $n \in \mathbb{N}$,

$$\| T^n x - T^n y \| \leq \psi^n(\| x \|) \| x - y \| \text{ for all } x, y \in X.$$

Theorem 1.4. ([1]). Let A be a weakly compact convex subset of a Banach space X, let $\psi : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ be a weaker Meir-Keeler-type function where for each $t \in \mathcal{R}^+$, $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is nonincreasing, and let $T : A \longrightarrow A$ be an asymptotic pointwise weaker Meir-Keeler-type ψ - contraction. Then T has a unique fixed point $\bar{x} \in A$, and for each $x \in A$, the sequence of Picard iterates, $\{T^n\}$, converges in norm to $\bar{x}$.

2 Asymptotic pointwise weaker Meir-Keeler-type contraction type

Definition 2.1. Let X be a Banach space, and let $\psi : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ be a weaker Meir-Keeler-type function. Then the mapping $T : X \longrightarrow X$ is said to be of asymptotic pointwise weaker Meir-Keeler-type ψ -contraction type (resp. of weak asymptotic pointwise weaker Meir-Keeler-type ψ -contraction type) if T^N is continuous for some integer $N \geq 1$. for each $x \in X$

$$\limsup_{x\to\infty} \sup_{y\in X} \{\| T^n x - T^n y \| - \psi^n(\| x \|) \| x - y \|\} \leq 0 \tag{2.1}$$

$$(\liminf_{x\to\infty} \sup_{y\in X} \{\| T^n x - T^n y \| - \psi^n(\| x \|) \| x - y \|\} \leq 0), \tag{2.2}$$

Taking

$$r_n(x) = \sup_{y\in X} \{\| T^n x - T^n y \| - \psi^n(\| x \|) \| x - y \|\} \in \mathcal{R}^+ \cup \{\infty\} \tag{2.3}$$

it can be easily seen from (2.1) (resp. (2.2)) that

$$\lim_{n\to\infty} r_n(x) = 0 \tag{2.4}$$

$$(resp. \ \liminf_{n\to\infty} r_n(x) \leq 0) \tag{2.5}$$

for all $x \in X$, and

$$\| T^n x - T^n y \| \leq \psi^n(\| x \|) \| x - y \| + r_n(x). \tag{2.6}$$

It is easy to see that an asymptotic pointwise weaker Meir-Keeler-type contraction is of asymptotic pointwise weaker Meir-Keeler-type contraction type; but, the converse is not true:

Example 2.2. Let $X = \Pi_{n\geq 1}[0, \frac{1}{n}] \subseteq C_0(\mathbb{N})$. For aech $x = (x_1, x_2, x_3, ...)$ in X, define

$$T(x_1, x_2, x_3, ...) = (f(x_1), x_2, x_3, ...),$$

where $f : [0, 1] \longrightarrow [0, 1]$ is a nonexpansive mapping. It easy to see that T is a continuous nonlinear mapping from X to X which is of asymptotic pointwise weaker Meir-Keeler-type contraction type. In fact, we notice that for every $x = (x_1, x_2, x_3, ...)$ and $y = (y_1, y_2, y_3, ...)$ in X,

$$\| T^n x - T^n y \| \leq \sup\{| x_i - y_i |: i \geq n+1\} \leq \frac{1}{n+1}.$$

Hence, for $\eta < 1$, we have

$$\sup_{y \in X}(\| T^n x - T^n y \| - \psi^n(\| x \|) \| x - y \|) \leq \frac{1}{n+1} \longrightarrow 0, \ as\ n \longrightarrow \infty.$$

But, T is not an asymptotic pointwise weaker Meir-Keeler-type contraction. Indeed, for any $x = (x_1, x_2, x_3, ...) \in X$ and $n \in \mathbb{N}$,

$$\| T^n x - T^n y \| = \| x - y \|,$$

for every $y = (y_1, y_2, y_3, ...) \in X$ for which $y_i = x_i$, i=1,2,...,n+1.

In this study, we also use the technique of asymptotic centers. Let X be Banach space, A a subset of X, and $\{x_n\}$ a bounded sequence in X. The asymptotic center of $\{x_n\}$ relative to A, denoted as $C_A(x_n)$, is the set of minimizers in A (if any) of the function f given by

$$f(x) = \limsup_{n \to \infty} \| x_n - x \|$$

That is,

$$C_A(x_n) = \{x \in A : f(x) = \inf_A f\}.$$

It is known that $f : X \longrightarrow \mathbb{R}_+$ is convex, nonexpansive and hence weak lower semi-continuous. Moreover, if C is weakly compact, then $A_C(x_n)$ is nonempty (see[2]).

We employ the technique of asymptotic centers to prove the following extension of theorem 1.7.

Theorem 2.3. Let A be a weakly compact convex subset of a Banach space X, let $\psi : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ be a weaker Meir-Keeler-type function where for each $t \in \mathcal{R}^+$, $\{\psi^n(t)\}_{n \in \mathbb{N}}$ is nonincreasing, and let $T : A \longrightarrow A$ be a weak asymptotic pointwise weaker Meir-Keeler-type ψ-contraction type. Then T has a unique fixed point $\bar{x} \in A$, and for each $x \in A$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to $\bar{x}$.

Proof. Fix an $x \in A$ and define a function f by

$$f(y) = \limsup_{n \to \infty} \| T^n x - y \|, \ y \in A.$$

Since A is a weakly compact convex subset of a Banach space X, the asymptotic center of the sequence $\{T^n x\}$ relative to A, $C_A(T^n x) = \{y \in A : f(y) = \min_A f\}$. is a non-empty closed convex subset of A. We now claim that

$$f(T^m y) \leq \psi^m(\| y \|) f(y) + r_m(y), \ y \in A, \ m \geq 1.$$

Indeed, we have

$$\begin{aligned} f(T^m y) &= \limsup_{n \to \infty} \| T^n x - T^m y \| \\ &\leq \limsup_{n \to \infty} \psi^m(\| y \|) \| T^n x - y \| + r_m(y) \\ &= \psi^m(\| y \|) f(y) + r_m(y). \end{aligned}$$

Take an $y \in C_A(T^n x)$, and since $T^m y \in A$, we get, for $m \geq 1$,

$$f(y) \leq f(T^m y) \leq \psi^m(\| y \|)f(y) + r_m(y). \tag{2.7}$$

Since T is of weak asymptotic pointwise weaker Meir-Keeler-type ψ-contraction type, by (2.5), we have $\liminf_{n\to\infty} r_n(y) \leq 0$. Thus, for a subsequence $\{r_{m_k}(y)\}$ of $\{r_m(y)\}$, we have $\liminf_{k\to\infty} r_{m_k}(y) \leq 0$.

On the other hand, since $\{\psi^m(\| y \|)\}_{m\in\mathbb{N}}$ is nonincreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. To the contrary, assume that $\eta > 0$. Then by the definition of the weaker Meir-Keeler-type function, there exists $\delta > \eta$ such that for $y \in A$ with $\eta \leq\| y \|< \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\| y \|) < \eta$. Since $\lim_{m\to\infty} \psi^m(\| y \|) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \leq \psi^m(\| y \|) < \delta$ for all $m \geq m_0$. Thus we conlude that $\psi^{m_0+n_0}(\| y \|) < \eta$. and we get a contraction. So $\lim_{m\to\infty} \psi^m(\| y \|) = 0$.
Taking the limit in (2.7) as $m \to \infty$, we get $f(y) = 0$. This implies that $T^n x \to u$, in norm. From this and the continuity of T^N, for some $N \geq 1$, it follows $T^N y = T^N(\lim_{n\to\infty} T^n x) = \lim_{n\to\infty} T^{n+N} x = y$; namely, y is a fixed point of T^N. Now, repeating the above proof for y instead of x, we deduce that $T^n y$ converges, in norm, to a member of C. But, $T^{kN} y = y$, for all $k \geq 1$. Hence, $T^n y \to u$, in norm. We show that $Ty = y$; for this purpose, consider an arbitrary $\epsilon > 0$. Then, there exists a $K_0 > 0$ such that $\| T^n y - y \|< \epsilon$, for all $n > K_0$. So, by choosing a natural number $k > K_0/N$, we obtain $\| Ty - y \|=\| T(T^{kN} y) - y \|=\| T^{kN+1} y - y \|< \epsilon$. Since the choice of $\epsilon > 0$ is arbitrary, we get $Ty = y$. It is easy to see that T has a unique fixed point. Indeed, if $z \in A$ is also a fixed point of T, then for all $n \in \mathbb{N}$,

$$\| y - z \|=\| T^n y - T^n z \|\leq \psi^n(\| y \|) \| y - z \| +r_n(y).$$

Letting $n \to \infty$, we get $\| y - z \|= 0$, and so $y = z$. □

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Minimal description for the real interpolation in the case of quasi-Banach quaternion

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Abstract

We give a minimal description in the sense of Aronszajn-Gagliardo for the real methods in the case of quasi-Banach quaternion.

Keywords: quasi-Banach spaces, interpolation space, real method of interpolation
Mathematics Subject Classification [2010]: 46M35, 47A60

1 Introduction

Our main reference to the theory of interpolation space is [1]. Let $\bar{A} = (A_0, A_1, A_2, A_3)$ be a quasi-Banach quaternion and $\bar{t} = (t_1, t_2, t_3) \in R^3_+$. The Peetre' K-functional is defined for $a \in A_0 + A_1 + A_2 + A_3 := \sum(\bar{A})$ by
$K(t_1, t_2, t_3, a; \bar{A})$

$$= \inf\{\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + t_2\|a_2\|_{A_2} + t_3\|a_3\|_{A_3} \ : \ a = \sum_{i=0}^{3} a_i, \ a_j \in A_j\}$$

and similarly the J-functional for $a \in A_0 \cap A_1 \cap A_2 \cap A_3 := \triangle(\bar{A})$ by

$$J(t_1, t_2, t_3, a; \bar{A}) = max\{\|a\|_{A_0}, t_1\|a\|_{A_1}, t_2\|a\|_{A_2}, t_3\|a\|_{A_3} \ : \ a \in \triangle(\bar{A})\}.$$

Let $\bar{A} = (A_0, A_1, A_2, A_3)$ be a quaternion of quasi-Banach spaces and $\bar{n} = (n_1, n_2, n_3) \in Z^3$. For $0 < \theta_1, \theta_2, \theta_3 < 1, \theta_1 + \theta_2 + \theta_3 < 1$ and $0 < q \leq \infty$ we define the real interpolation space $\bar{A}_{(\theta_1,\theta_2,\theta_3),q,K}$ as the set of all $a \in \sum(\bar{A})$ which have a finite quasi-norm
$\|a\|_{(\theta_1,\theta_2,\theta_3),q,K}$

$$= \begin{cases} \left(\sum\limits_{\bar{n} \in Z^3} (2^{-n_1\theta_1} 2^{-n_2\theta_2} 2^{-n_3\theta_3} K(2^{n_1}, 2^{n_2}, 2^{n_3}, a; \bar{A}))^q \right)^{1/q} & if \ 0 < q < \infty \\ \sup\limits_{\bar{n} \in Z^3} \{2^{-n_1\theta_1} 2^{-n_2\theta_2} 2^{-n_3\theta_3} K(2^{n_1}, 2^{n_2}, 2^{n_3}, a; \bar{A})\} & if \ q = \infty \end{cases}.$$

Also we define the real interpolation space $\bar{A}_{(\theta_1,\theta_2,\theta_3),q,J}$ as the set of all $a \in \sum(\bar{A})$that may by written as $a = \sum\limits_{\bar{n} \in Z^3} u_{\bar{n}}, \ u_{\bar{n}} \in \triangle(\bar{A})$ (convergence in $\sum(\bar{A})$) and which have a finite

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quasi-norm

$$\|a\|_{(\theta_1,\theta_2,\theta_3),q,J} = \inf_{a=\sum_{\bar{n}\in Z^3} u_{\bar{n}}} \left(\sum_{\bar{n}\in Z^3} (2^{-n_1\theta_1}2^{-n_2\theta_2}2^{-n_3\theta_3} J(2^{n_1},2^{n_2},2^{n_3},u_{\bar{n}};\bar{A}))^q \right)^{1/q}$$

With the usual interpretation when $q=\infty$.
If $\bar{A}=(A_0,A_1,A_2,A_3)$ and $\bar{B}=(B_0,B_1,B_2,B_3)$ are Banach quaternion, we write $T \in \pounds(\bar{A},\bar{B})$ to mean that T is a linear operator from $\sum(\bar{A})$ into $\sum(\bar{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j $(j=0,1,2,3)$. We put

$$\|T\|_{\bar{A},\bar{B}} = \max_{j=0,1,2,3}\{\|T\|_{A_j,B_j}\}.$$

Scalar sequence spaces are defined over Z^3 and given any sequence of positive numbers $(w_{\bar{n}})_{\bar{n}\in Z^3}$ we put

$$l_p(w_{\bar{n}}) = \{(a_{\bar{n}}) : \|a_{\bar{n}}\|_{l_p(w_{\bar{n}})} = \|w_{\bar{n}}a_{\bar{n}}\|_{l_p} < \infty\}.$$

Of special interest for us are the quaternion $\bar{l}_p = \Big(l_p, l_p(2^{-n_1}), l_p(2^{-n_2}), l_p(2^{-n_3})\Big)$, $(0<p\leq 1)$ and $\bar{l}_\infty = \Big(l_\infty, l_\infty(2^{-n_1}), l_\infty(2^{-n_2}), l_\infty(2^{-n_3})\Big)$.

2 Main results

We start this section by introducing the following:

Let T be a mapping from a quasi-Banach space A into a scalar sequence space $\mathcal{M}$. We say that T is quasi-linear with constant $C\geq 1$ if

$$|T(a+b)| \leq C\Big(|Ta|+|Tb|\Big), \quad a,b\in A$$

$$|T(\lambda a)| = |\lambda||Ta|, \;\; a\in A, \quad \lambda\in F(F-scalarfield).$$

Given any quasi-Banach quaternion $\bar{A}=(A_0,A_1,A_2,A_3)$ and $C\geq 1$ we denote by $\pounds_C(\bar{A},\bar{l}_\infty)$ the collection of all those quasi-linear operators $T:\sum(\bar{A})\to\sum(\bar{l}_\infty)$ with the constant C whose restriction to A_i $(i=0,1,2,3)$ defines a bounded operator from A_0,A_1,A_2,A_3 into $l_\infty, l_\infty(2^{-n_1}), l_\infty(2^{-n_2}), l_\infty(2^{-n_3})$ respectively.

Definition 2.1. Let $0<\theta_1,\theta_2,\theta_3<1, \theta_1+\theta_2+\theta_3<1$ and $0<q\leq\infty$.Given any quasi-Banach quaternion $\bar{A}=(A_0,A_1,A_2,A_3)$ we define $H_{(\theta_1,\theta_2,\theta_3),q,C}(\bar{A})$ as the collection of all those $a\in\sum(\bar{A})$ such that $Ta\in l_q(2^{-n_1\theta_1-n_2\theta_2-n_3\theta_3})$ for any $T\in\pounds_C(\bar{A},\bar{l}_\infty)$ and quasi-norm

$$\|a\|_{H_{(\theta_1,\theta_2,\theta_3),q,C}}(\bar{A}) = sup\{\|Ta\|_{l_q(2^{-n_1\theta_1-n_2\theta_2-n_3\theta_3})} : \|T\|_{\bar{A},\bar{l}_\infty}\leq 1\}$$

is finite.

Theorem 2.2. *Let* $\bar{A} = (A_0, A_1, A_2, A_3)$ *be a quasi-Banach quaternion, let* $0 < \theta_1, \theta_2, \theta_3 < 1, \theta_1 + \theta_2 + \theta_3 < 1$ *and* $0 < q \leq \infty$. *Assume that the constant in the triangle inequality of* A_i *is* C_i $(i - 0, 1, 2, 3)$ *and put* $C = max(C_0, C_1, C_2, C_3)$. *Then*

$$(A_0, A_1, A_2, A_3)_{(\theta_1,\theta_2,\theta_3),q,K} = H_{(\theta_1,\theta_2,\theta_3),q,C}(A_0, A_1, A_2, A_3).$$

In the following (A_0, A_1, A_2, A_3) will always denote a quasi-Banach quaternion that A_j is c_j normed with $(c_1 + c_2 + c_3)/3c_0 \leq 1$.

Theorem 2.3. *Let* (A_0, A_1, A_2, A_3) *be a quasi-Banach quaternion and* $a \in A_0 + A_1 + A_2 + A_3$. *Then*

$$K(t_1, t_2, t_3, a; \bar{A}) = K(t_1, t_2, t_3, a; A_0, A_0 + A_1 + A_2 + A_3) \quad (t \geq 1).$$

Theorem 2.4. *Let* (A_0, A_1, A_2, A_3) *be a quasi-Banach quaternion and* $a_0 \in A_0$. *Then*

$$K(t_1, t_2, t_3, a_0; \bar{A}) \leq K(t_1, t_2, t_3, a_0; A_0, A_0 \cap A_1 \cap A_2 \cap A_3) \quad (t > 0).$$

Proposition 2.5. *Let* (A_0, A_1, A_2, A_3) *be a quasi-Banach quaternion. Then the following identities hold.*

$$(A_0 + A_1 + A_2 + A_3, A_0)_{\theta,q} \cap (A_0 + A_1 + A_2 + A_3, A_1 + A_2 + A_3)_{\theta,q}$$
$$= (A_0 + A_1 + A_2 + A_3, A_0 \cap A_1 \cap A_2 \cap A_3)_{\theta,q}. \ (0 < \theta < 1,\ 0 < q \leq \infty)$$

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Monotonicity and dominated best proximity pair in Banach lattices and some applications

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Abstract

In this paper we introduce the dominated Best proximity pair problem in Banach lattices. We give some necessary and sufficiency conditions such that this problem is uniquely solvable in STM space. Also we show that every UM spaces have property UC in Banach Lattices.

Keywords: Banach Lattice, Best proximity pair, STM Space, Property UC.
Mathematics Subject Classification [2010]: 41A65, 46B42

1 Introduction

Let $(X, \leq)$ be a Banach lattice and A, B be two nonempty subset of $X$and T be a mapping from A in to B. $x \in A$ is called a point of best proximity pair if $\|x - Tx\| = d(A, B)$ where

$$d(A, B) = \inf\{\|x - y\| : (x, y) \in A \times B\}.$$

The set of all best proximity points is denoted by T_A^B. T is called a nonexpansive map if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in A$. Best proximity pair also evolves a generalization of the concept of fixed point of mapping. Indeed every best proximity pair is a fixed point of T, whenever $A \cap B \neq \emptyset$. The problem of best proximity pair is discussed by many authors for more information you can refer to [2], [3], [9] and [10]. Eldered and Veeramani in [3] proved that for a cyclic contraction map in a uniformly convex Banach space there exists a unique best proximity pair and Sankar Raj and Veeramani proved similarly results for relatively nonexpansive map. In [10] Suzuki et.al by using Lemma 3.8 in [3] defined property UC and discussed the existence of best proximity pair. In this paper we introduce the concept of dominated best proximity pair and stated some condition to guaranteed the existence of best proximity pair. For general information in Banach lattices we can refer to chapter one of [1] and [7].

Definition 1.1. [6] A Banach lattice X is said to be strictly monotone ($X \in$ STM) if for all $x, y \in X^+$, the conditions $x \geq y$, $y \neq 0$ and $\|x\| = \|y\|$ implies $x = y$.

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Definition 1.2. [6] A Banach lattice X is said to be uniformly monotone ($X \in$ UM) if for all $y_n \geq x_n \geq 0$, such that $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|$, then $\|x_n - y_n\| \to 0$.

Definition 1.3. [7] The norm on a Banach lattice X is called order continuous if $\inf\{\|x\| : x \in A\} = 0$ for every downwards directed system $A \subset X$ such that $\inf(A) = 0$.

Definition 1.4. [10] Let A, B be nonempty subset of a Banach lattice X. Then (A, B) satisfies property UC if the following holds:
• If $\{x_n\}$ and $\{x'_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_n \|x_n - y_n\| = \lim_n \|x'_n - y_n\| = d(A, B)$, then $\lim_n \|x_n - x'_n\| = 0$ holds.

For general definition in Musielak-Orlicz space we can refer to [4], [5], [8].

Definition 1.5. Suppose that (T, Σ, μ) is a σ-finite, complete (non-trivial), positive measure space and $\phi(t, r) : T \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ is a function such that for μ-a.e. $t \in T$, $\phi(t, 0) = 0$, $\phi(t, .)$ is non-trivial (continuous at zero with nonzero values), convex, and lsc. Moreover $\phi(., r)$ is measurable, for all $r > 0$. We call ϕ the Musielak-Orlicz function.

Definition 1.6. Musielak-Orlicz spaces $L^\phi(\mu)$ consist of all μ-measurable functions $f : T \to \overline{\mathbb{R}}$ such that

$$I_\phi(\alpha f) = \int_T \phi(\alpha|f(t)|, t)d\mu < +\infty$$

for some $\alpha > 0$ (depending on f).

Musielak-Orlicz spaces under the natural ordering, it becomes a Banach lattice The function ϕ is said to satisfy a Δ_2, condition ($\phi \in \Delta_2$) if there exist a set T_0 of zero measure, a constant $K > 0$, and an integrable (nonnegative) function h, such that for all $t \in T \setminus T_0$, and $r > 0$, there holds

$$\phi(2r, t) \leq K\phi(r, t) + h(t).$$

2 Main results

In this section we introduce dominated best proximity pair problem and we will state the relationship between monotonicity of Banach lattices and existence and uniqueness of best proximity pair problem. We recall that in this section $A \leq B$ means $x \leq y$ for each $x \in A$ and $y \in B$.

Theorem 2.1. *Let A, B be nonempty sublattice of Banach lattices X such that $A \leq B$. Let $T : A \to B$ is a nonexpansive map. Then X is an STM space if and only if $card(T_A^B) \leq 1$.*

Theorem 2.2. *Let A, B be nonempty closed convex sublattices of Banach lattice X with property UC and $A \leq B$. Let $T : A \to B$ is a nonexpansive map. Then X is an STM space with order continuous norm if and only if $card(T_A^B) = 1$.*

Proposition 2.3. *Let X be a Banach lattice with property UM and A, B be two subsets of X. then (A, B) have property UC.*

Theorem 2.4. *Let A, B be nonempty closed convex sublattices of Banach lattice X such that $A \leq B$. Let $T : A \to B$ is a nonexpansive map. If X is an UM space then $card(T_A^B) = 1$.*

2.1 Some applications of the dominated best proximity pair problem in Musielak-Orlicz spaces

Theorem 2.5. *For the Musielak-Orlicz space $L^{\phi}(\mu)$ the following statement are equivalent*

i) $\phi \in \Delta_2$.

ii) *Let A, B be nonempty closed convex sublattices in $L^{\phi}(\mu)$ with property UC and $A \leq B$. Let $T : A \to B$ is a nonexpansive map. Then $card(T_A^B) \geq 1$.*

Corollary 2.6. *The dominated best proximity pair problem for nonexpansive map in $L^{\phi}(\mu)$ with $\phi < \infty$ with respect to closed bounded sublattices is uniquely solvable if and only if $\phi > 0$ and $\phi \in \Delta_2$.*

Theorem 2.7. *In the Musielak-Orlicz spaces the following statements are equivalent*

i) $\phi \in \Delta_2$

ii) *Let A, B be nonempty closed convex sublattices in $L^{\phi}(\mu)$ with property UC. Let $T : A \to B$ is a nonexpansive map. Then $card(T_A^B) \neq \emptyset$.*

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Multilinear mappings on matrix algebras

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Abstract

We investigate the notion of positive multilinear mappings on matrix algebras. Some matrix inequalities including positive multilinear mappings are introduced.

Keywords: positive multilinear mapping, Jensen inequality, positive matrix, matrix convex function
Mathematics Subject Classification [2010]: Primary 15A69 ; Secondary 47A63,47A64, 47A56.

1 Introduction

Let $\mathcal{M}_n := \mathcal{M}_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ complex matrices with identity matrix I. A linear map $\Phi : \mathcal{M}_q \to \mathcal{M}_p$ is called positive if $\Phi(A) \geq 0$ in $\mathcal{M}_p$, whenever $A \geq 0$ in $\mathcal{M}_q$. Positive linear mapppings on C^*-algebras and their related operator inequalities are well-known and have been studied by many mathematicians; see e.g., [1, 2, 4] and the references therein. Positive linear mappings have been used to characterize matrix convex functions. A continuous real function $f : J \to \mathbb{R}$ is said to be matrix convex if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$ for all $\lambda \in [0, 1]$ and all hermitian matrices A, B with eigenvalues in J. It is well-known that a continuous real function $f : J \to \mathbb{R}$ is matrix convex if and only if

$$f(\Phi(A)) \leq \Phi(f(A)) \tag{1}$$

for every unital positive linear mapping Φ and every hermitian matrix A with spectrum in J. The inequality (1) is known as the Choi-Davis-Jensen inequality, see [2, 4].

The notion of positive linear mappings is introduced also for maps of several variables. Let $\mathcal{A}_k$, $k = 1, \cdots, n$ and $\mathcal{B}$, be C^*-algebras. A map $\Phi : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{B}$ is called to be positive multilinear if, it is linear in each of its variable and for every positive elements $a_k \in \mathcal{A}_k$, $k = 1, \cdots, n$, $\Phi(a_1, \cdots, a_n)$ is positive in $\mathcal{B}$ [5].

It is known that if A and B are positive matrices, then so is their Hadamard (Schur) product, $A \circ B$. The same is true for tensor product, $A \otimes B$. Moreover, the mapping $(A, B) \to A \otimes B$ is also linear in each of its variables. So if we define $\Phi : \mathcal{M}_q^2 \to \mathcal{M}_p$ by $\Phi(A, B) = A \otimes B$, then Φ is multilinear and positive in the sense that $\Phi(A, B)$ is positive, whenever A, B are positive.

*Speaker

However, the Choi-Davis-Jensen type inequality $f\big(\Phi(A,B)\big) \leq \Phi\big((f(A), f(B)\big)$ does not hold in general for a unital positive multilinear mapping Φ and matrix convex functions f. For example, consider the matrix convex function $f(t) = t^2 - t$ and the unital positive multilinear mapping $\Phi(A,B) = A \circ B$. If $A = 2I$ and $B = I$, then $2I = f(\Phi(A,B)) \nleq \Phi(f(A), f(B)) = 0$. This can be a motivation to study operator inequalities via positive multilinear mappings.

We present a version of Choi-Davis-Jensen inequality for positive multilinear mappings. We inquire some matrix inequalities including positive multilinear mappings, which some of them would be generalization of inequalities for the Hadamard product and the tensor product of matrices.

2 Main results

We start by definition of a positive multilinear mapping.

Definition 2.1. A mapping $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is said to be multilinear, if it is linear in each of its variable. It is called positive if $\Phi(A_1, \cdots, A_k) \geq 0$, whenever $A_1, \cdots, A_k \geq 0$. If $\Phi(I, \cdots, I) = I$, then Φ is called unital.

Example 2.2. It is well-known that the Schur product of every two positive matrices is positive again. This ensures that the mapping $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ defined by

$$\Phi(A_1, \cdots, A_k) = A_1 \circ \cdots \circ A_k$$

is positive. Moreover, it is multilinear and unital. The same is true if we define

$$\Phi(A_1, \cdots, A_k) = A_1 \otimes \cdots \otimes A_k.$$

Example 2.3. Assume that $X_i \in \mathcal{M}_q \ \ (i = 1, \cdots, k)$ and $\sum_{i=1}^k X_i^* X_i = I$. The mapping $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ defined by $\Phi(A_1, \cdots, A_k) = \sum_{i=1}^k X_i^* A_i X_i$ is positive and unital. However, it is not multilinear.

Example 2.4. The mappings $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ defined by

$$\Phi(A_1, \cdots, A_k) := \mathrm{Tr}(A_1 \otimes \cdots \otimes A_k) = \mathrm{Tr}(A_1) \cdots \mathrm{Tr}(A_k) I \tag{2}$$

is positive and multilinear.

It is evident that, every positive multilinear mapping $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is adjoint-preserving and monotone [3].

The following theorem can be regarded as a reconstruction of [4, Theorem 1.21] for positive multilinear mappings.

Theorem 2.5. *[3] Let $f : [0, \infty) \to \mathbb{R}$ be a matrix convex and submultiplicative function, i.e., $f(xy) \leq f(x)f(y)$ for all $x, y \in [0, \infty)$ (resp. a super-multiplicative matrix concave function). If $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is a unital positive multilinear mapping, then*

$$f(\Phi(A_1, \ldots, A_k)) \leq \Phi(f(A_1), \ldots, f(A_k))$$
$$(\textit{resp.} \qquad f(\Phi(A_1, \ldots, A_k)) \geq \Phi(f(A_1), \ldots, f(A_k))).$$

Corollary 2.6. *Suppose that $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is a unital positive multilinear mapping.*

(1) If $0 \leq r \leq 1$, then

$$\Phi(A_1^r, \cdots, A_k^r) \leq \Phi(A_1, \cdot, A_k)^r$$

for all positive matrices $A_1, \cdots, A_k$.

(2) If $-1 \leq r \leq 0$ and $1 \leq r \leq 2$, then

$$\Phi(A_1, \cdots, A_k)^r \leq \Phi(A_1^r, \cdots, A_k^r)$$

for all positive matrices $A_1, \cdots, A_k$.

Corollary 2.7. *Let $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ be a uninal positive multilinear mapping. If $1 \leq s < t$, then*

$$\Phi(A_1^s, \cdots, A_k^s)^{\frac{1}{s}} \leq \Phi(A_1^t, \cdots, A_k^t)^{\frac{1}{t}}$$

for all positive matrices $A_1, \cdots, A_k$.

Remark 2.8. It is well known that if $f : [0, \infty) \to [0, \infty)$ is a continuous function, then f is operator monotone if and only if it is operator concave. Suppose that $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is a unital positive multilinear mapping and $f : [0, \infty) \to [0, \infty)$ is a matrix monotone and supermultiplicative function. Then Theorem 2.5 implies that

$$f(\Phi(A_1, A_2, \cdots, A_k)) \geq \Phi(f(A_1), f(A_2), \cdots, f(A_k))$$

for all positive matrices $A_1, \cdots, A_k$.

By a theorem of Ando (see e.g. [1]), if A and B are positive matrices and Φ is a strictly positive linear mapping, then

$$\Phi(A \sharp B) \leq \Phi(A) \sharp \Phi(B), \tag{2}$$

where the geometric matrix mean is defined by $\sharp$, namely

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}}.$$

By aid of Theorem 2.5, we show the positive multilinear mapping version of Ando's inequality (2).

Lemma 2.9. *If $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is a strictly positive multilinear mapping, then*

$$\Phi(A_1 \sharp B_1, \cdots, A_k \sharp B_k) \leq \Phi(A_1, \cdots, A_k) \sharp \Phi(B_1, \cdots, B_k)$$

for all $A_1, \cdots, A_k > 0$ and $B_1, \cdots, B_k \geq 0$.

In [2], Choi generalized Kadison's inequality to normal matrices by showing that if Φ is a unital positive linear mapping, then

$$\Phi(A)\Phi(A^*) \leq \Phi(A^*A) \quad \text{and} \quad \Phi(A^*)\Phi(A) \leq \Phi(A^*A).$$

for every normal matrix A. A similar result holds true for positive multilinear mappings.

Lemma 2.10. *Let $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ be a positive multilinear mapping. Then*

$$\Phi(A_1^*A_1, A_2^*A_2, \cdots, A_k^*A_k) \geq \Phi(A_1, \cdots, A_k)\Phi(A_1, \cdots, A_k)^*$$

for all normal matrices $A_1, A_2, \cdots, A_k$.

A multilinear mapping $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is called completely positive if for every $n \geq 1$, $[\Phi(A_{1,ij}, \cdots, A_{k,ij})]_{ij} \geq 0$ in $\mathcal{M}_{np}$ whenever $[A_{m,ij}]_{ij} \geq 0$, $m = 1, \cdots, k$ in $\mathcal{M}_{nq}$; see e.g. [5]. It is well known that if $f : J \to \mathbb{R}$ is a convex function and Φ is a positive linear mapping, then $f\big(\langle \Phi(A)x, x\rangle\big) \leq \langle \Phi\big(f(A)\big)x, x\rangle$ for all Hermitian matrices A and all unit vector x. We state a similar result for completely positive multilinear mappings.

Lemma 2.11. *Let $A_1, \cdots, A_k$ be positive matrices. If $f : [0, \infty) \to \mathbb{R}$ is a convex and submultiplivative function and $\Phi : \mathcal{M}_q^k \to \mathcal{M}_p$ is a unital completely positive multilinear mapping, then*

$$f\big(\langle \Phi(A_1, \cdots, A_k)x, x\rangle\big) \leq \langle \Phi\big(f(A_1), \cdots, f(A_k)\big)x, x\rangle.$$

for all unit vector x.

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New reverse of continuous triangle inequalities type for Bochner integral in Hilbert C*-modules

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Abstract

In this paper some reverses of continuous triangle inequalities for integrable functions with value in a Hilbert C^*-modules are given.

Keywords: Bochner integral, Reverse of triangle inequality, Hilbert C^*-module.
Mathematics Subject Classification [2010]: 46L08, 26D10, 26D15

1 Introduction

Let $f : [a, b] \to K$, $K = \mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality is the continuous version of the triangle inequality

$$\Big| \int_a^b f(x)dx \Big| \le \int_a^b |f(x)|dx, \tag{1}$$

and plays a fundamental role in mathematical analysis and its applications.
It appears, see [7, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949 [6]:

$$\cos\theta \int_a^b |f(x)|dx \le \Big| \int_a^b f(x)dx \Big| \tag{2}$$

provided

$$-\theta \le arg[f(x)] \le \theta, \qquad x \in [a, b]$$

for given $\theta \in (0, \frac{\pi}{2})$. In [5], S. S. Dragomir has extended the above result for Bochner integrals of vector-valued functions in real or complex Hilbert spaces.
If $(\mathcal{H}; \langle ., .\rangle)$ is a Hilbert space over $K(K = \mathbb{C}, \mathbb{R})$ and $f \in L([a, b]; \mathcal{H})$, this means that $f : [a, b] \to \mathcal{H}$ is strongly measurable on $[a, b]$ and the Lebesgue integral $\int_a^b \|f(t)\|dt$ exists and is finite, and there exist a constant $k \ge 1$ and a vector $e \in H$, $\|e\| = 1$ such that

$$\|f(t)\| \le kRe\big\langle f(t), e\big\rangle \qquad for \qquad a.e. t \in [a, b] \tag{3}$$

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then we have the inequality

$$\int_a^b \|f(t)\|dt \leq k\Big\|\int_a^b f(t)dt\Big\|. \tag{4}$$

This provides a reverse inequality for the well-known result for Bochner integrals and vector-valued functions:

$$\Big\|\int_a^b f(t)dt\Big\| \leq \int_a^b \|f(t)\|dt, \tag{5}$$

for any $f \in L([a,b];\mathcal{H})$. Note that the case of equality holds in (5) (see [5]) if and only if

$$\int_a^b f(t)dt = \frac{1}{k}\Big(\int_a^b \|f(t)\|dt\Big)e. \tag{6}$$

For some particular cases of interest, see [5].

2 Preliminaries

If (Ω, Σ, μ) is a measure space and B is a Banach space, a map $s : \Omega \to B$ is called simple if there exist $b_1, ..., b_n \in B$ and $E_1, ..., E_n \in \Sigma$ which satisfy that $E_i \cap E_j = \emptyset$ for $i \neq j$, such that

$$s(\omega) = \sum_{i=1}^{n} b_i \chi_{E_i}(\omega), \qquad \omega \in \Omega$$

where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $E_i(\omega) = 0$ if $\omega \notin E_i$. A map $f : \Omega \to B$ is called μ-measurable if there exists a sequence of simple maps $\{s_n\}$ from Ω to B with

$$\lim_{n\to\infty} \|f(\omega) - s_n(\omega)\| = 0$$

μ-almost everywhere. A map $f : \Omega \to B$ is called weakly μ-measurable if for each $\phi \in B^*$ the function $\phi(f)$ is μ-measurable, where B^* is the dual space of B. By Pettiss measurability theorem, a μ-measurable map from a measure space to a Banach space is weakly μ-measurable [2].
Let (Ω, Σ, μ) be a measure space, and let B be a Banach space. A μ-measurable map $f : \Omega \to B$ is said to be Bochner integrable if there exists a sequence of simple maps $\{s_n\}$ from Ω to B such that

$$\lim_{n\to\infty} \int_\Omega \|f(\omega) - s_n(\omega)\|d\mu = 0. \tag{7}$$

In this case, for any $E \in \Sigma$, the Bochner integral of f over E is defined by

$$\int_E f(\omega)d\mu = \lim_{n\to\infty} \int_E s_n(\omega)d\mu,$$

in the sense of strong convergence in B, where $\int_E s_n(\omega)d\mu$ is defined in an obvious way [2]. By [2, Chapter II, Theorem 2], a μ-measurable function $f : \Omega \to B$ is Bochner integrable if and only if $\int_X \|f\|d\mu < \infty$. Hence in the case where (Ω, Σ, μ) is a finite measure space, if a measurable function $f : \Omega \to B$ is bounded, then it is integrable. We can see that the sequence $\{s_n\}_{n\in\mathbb{N}}$ satisfying (7) may be chosen so that it converges everywhere on Ω to f and $\|s_n(\omega)\| \leq \|f(\omega)\|$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$.

3 Main results

Theorem 3.1. *Let X be a Hilbert C^*-module over a unital C^*-algebra A and $f \in L([a,b];X)$. If there exist a constant $k \geq 1$ with*

$$|f(t)| \leq kRe\langle f(t), e\rangle \tag{8}$$

for some $e \in X$ with $|e| = 1$ and all $t \in [a,b]$, then

$$\int_a^b |f(t)|dt \leq k\Big\| \int_a^b f(t)dt\Big\|. \tag{9}$$

If the case of equality holds in (9)*, then*

$$\int_a^b f(t)dt = \frac{1}{k}\Big(\int_a^b |f(t)|dt\Big)e. \tag{10}$$

Corollary 3.2. *Let X be a Hilbert C^*-modules, $e \in X$ with $|e| = 1$, $\rho \in (0,1)$ and $f \in L([a,b];X)$ such that for a.e $t \in [a,b]$,*

$$|f(t) - e| \leq \rho. \tag{11}$$

Then we have the inequality

$$\sqrt{1-\rho^2}\int_a^b |f(t)|dt \leq \Big\| \int_a^b f(t)dt\Big\|. \tag{12}$$

If the case of equality holds in (12)*, then*

$$\int_a^b f(t)dt = \sqrt{1-\rho^2}\Big(\int_a^b |f(t)|dt\Big)e. \tag{13}$$

Corollary 3.3. *Let X be a Hilbert C^*-module on C^*-algebra A, $e \in X$ with $|e| = 1$ and $M \geq m > 0$. If $f \in L([a,b];X)$ is such that*

$$Re\langle Me - f(t), f(t) - me\rangle \geq 0; \quad for \quad a.e. \quad t \in [a,b] \tag{14}$$

or, equivalently,

$$\Big|f(t) - \frac{M+m}{2}e\Big| \leq \frac{1}{2}(M-m); \quad for \quad a.e. \quad t \in [a,b], \tag{15}$$

then we have the inequality

$$\frac{2\sqrt{mM}}{M+m}\int_a^b |f(t)|dt \leq \Big\| \int_a^b f(t)dt\Big\|. \tag{16}$$

If the case of equality holds in (16)*, then*

$$\int_a^b f(t)dt = \frac{2\sqrt{mM}}{M+m}\Big(\int_a^b |f(t)|dt\Big)e. \tag{17}$$

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Talk Non-linear semigroups in Hadamard spaces pp.: 1–4

Non-linear Semigroups in Hadamard Spaces

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Abstract

There are at least two methods to generate a non-linear semigroup of non-expansive operators in Hadamard spaces: gradient flows of convex maps and semigroups generated by m-co-accretive operators. Using an inner product-like notion of quasilinearization, we have established a *link* between these two approaches. We prove that in each geodesically unbounded Hadamard space X, each convex map $f : X \to (-\infty, +\infty]$ induces a co-accretive operator $T_f : X \to 2^X$ such that it generates a nonlinear semigroup which coincides the gradient flow of f.

Keywords: Hadamard space, non-linear semigroup, co-accretive operator, gradient flow, quasilinearization.
Mathematics Subject Classification [2010]: 47H20, 53C23.

1 Introduction

1.1 Hadamard Space

A $CAT(0)$ space is a metric space (X, d) such that for each two points $x_0, x_1 \in X$ and for each $0 < t < 1$ there exists some $x_t \in X$ such that

$$d^2(y, x_t) \leq (1-t)d^2(y, x_0) + td^2(y, x_1) - t(1-t)d^2(x_0, x_1) \quad (y \in X). \tag{1}$$

It can be seen that such x_t must be unique, so one can write $(1-t)x_0 \oplus tx_1 = x_t$. A complete $CAT(0)$ space is called a *Hadamard* space. These spaces are well-studied by many authors; we refer the reader to the standard texts such as [5, 6]. There are many various examples of Hadamard spaces: Hilbert spaces, Hadamard manifolds (i.e., simply-connected complete Riemannian manifolds with nonpositive sectional curvature which can be of infinite dimension), any bounded domain in a complex Banach space with Carathéodory metric, e.g., open unit ball of a complex Hilbert space with Poincaré metric, $\mathbb{R}$-trees as well as examples that have been built out of given Hadamard spaces as: closed convex subsets, direct products, warped products, L^2-spaces, direct limits and Reshetnyak's gluing.

1.2 Co-accretive Operator

A Hadamard space (X, d) is called *geodesically unbounded* if for each $x, y \in X$ there exists a geodesic line $c : \mathbb{R} \to X$ passing through x, y, i.e., $d(c(t), c(s)) = |t-s|d(x, y)$ for $t, s \in \mathbb{R}$, $c(0) = x$ and $c(1) = y$. Every geodesically unbounded Hadamard space is a hyperbolic space in the sense of Reich and Shafrir [8, (2.1)] and we can consider the notion of co-accretive operators on it. Let (X, d) be a geodesically unbounded Hadamard space. For $x, y \in X$ and $r \geq 0$, the point $(1+r)x \ominus ry$ is the unique point $z \in X$ such that $x = \frac{1}{1+r}z \oplus \frac{r}{1+r}y$, see [8, p. 539]. Following [8, (3.1),(3.5)and (7.2)]; a set-valued operator $T : X \to 2^X$ with domain $\mathcal{D}(T) = \{x \in X | Tx \neq \emptyset\}$ and range $\mathcal{R}(T) = \cup\{Tx | x \in X\}$ is called co-accretive if

$$d(x_1, x_2) \leq d((1+r)x_1 \ominus ry_1, (1+r)x_2 \ominus ry_2) \quad (y_i \in Tx_i,\ i = 1, 2,\ r > 0) \tag{2}$$

and is called m-co-accretive if in addition

$$\mathcal{R}((1+r)I \ominus rT) = X \quad (r > 0) \tag{3}$$

If T is co-accretive and $r > 0$, the resolvent $J_r(T) : \mathcal{R}((1+r)I \ominus rT) \to \mathcal{D}(T)$ of T is a single-valued nonexpansive mapping which is defined by

$$J_r(T)((1+r)x \ominus ry) = x \quad (x \in \mathcal{D}(T), y \in Tx). \tag{4}$$

The following is a direct consequence of Theorem 8.1 in [8].

Theorem 1.1. *Let (X, d) be a geodesically unbounded Hadamard space and $T : X \to 2^X$ be an m-co-accretive operator. Then T generates a continuous semigroup of nonlinear nonexpansive maps on $cl\mathcal{D}(T)$ via the exponential formula*

$$S_t x = \lim_{n \to +\infty} J^n_{t/n}(T)x \quad (x \in cl\mathcal{D}(T), t \geq 0) \tag{5}$$

1.3 Gradient Flow

Let $f : X \longrightarrow (-\infty, +\infty]$ be a lower semicontinuous convex function which is proper, i.e., its efficient domain $\mathcal{D}(f) = \{x \in X | \, f(x) < +\infty\}$ is non-empty. For each $r > 0$, the resolvent $J_r(f) : X \to X$ of f is a single-valued nonexpansive mapping which is defined by

$$J_r(f)(x) = argmin\{y \mapsto f(y) + \frac{1}{2r}d^2(x, y)\} \tag{6}$$

The following is deduced from Theorems 1.13, 2.1 and 2.5 in [7].

Theorem 1.2. *Let (X, d) be a Hadamard space and $f : X \longrightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then f generates a continuous semigroup of nonlinear nonexpansive maps on $cl\mathcal{D}(f)$ via the exponential formula*

$$S_t x = \lim_{n \to +\infty} J^n_{t/n}(f)x \quad (x \in cl\mathcal{D}(f), t \geq 0). \tag{7}$$

More details about non-linear semigroups on Hadamard spaces and their properties can be found in [1].

1.4 Quasilinearization and Dual Metric Space

Berg and Nikolaev in [4] have introduced the concept of quasilinearization along this lines. Let us formally denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then the quasilinearization map $\langle\, ,\, \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}\,,\, \overrightarrow{uv} \rangle = \frac{1}{2}(d^2(a,v) + d^2(b,u) - d^2(a,u) - d^2(b,v)) \quad (a,b,u,v \in X). \tag{8}$$

Berg and Nikolaev have then proved [4, Corollary 3] that a geodesically connected metric space (X,d) is a $CAT(0)$space if and only if it satisfies the Cauchy-Schwartz inequality, i.e.,

$$\langle \overrightarrow{ab}\,,\, \overrightarrow{uv} \rangle \le d(a,b)d(u,v) \quad (a,b,u,v \in X).$$

Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X;\mathbb{R})$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}\,,\, \overrightarrow{ax} \rangle \quad (t \in \mathbb{R}, a, b, x \in X)$$

where $C(X;\mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwartz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz seminorm $L(\Theta(t,a,b)) = t\, d(a,b) \quad (t \in \mathbb{R}\,,\, a,b \in X)$. Now, we introduce a pseudometric D on $\mathbb{R} \times X \times X$ by

$$D\big((t,a,b)\,,\,(s,u,v)\big) = L\big(\Theta(t,a,b) - \Theta(s,u,v)\big) \quad (t,s \in \mathbb{R}, a,b,u,v \in X).$$

The pseudometric space $(\mathbb{R} \times X \times X\,,\, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X,\mathbb{R})\,,\, L)$. Also, D imposes an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t,a,b) is

$$[t\overrightarrow{ab}] = \{\, s\overrightarrow{uv} \mid t\,\langle \overrightarrow{ab}\,,\, \overrightarrow{xy} \rangle = s\,\langle \overrightarrow{uv}\,,\, \overrightarrow{xy} \rangle \} \quad (x,y \in X).$$

The set $X^* := \{[t\,\overrightarrow{ab}] \mid (t,a,b) \in \mathbb{R} \times X \times X\,\}$ is a metric space with metric D, which is called the *dual metric space* of (X,d).

For example if X is a closed and convex subset of a Hilbert space $\mathcal{H}$ with non-empty interior, then $X^* = \mathcal{H}$; see [3, p.3451]. Among other properties, we have a separation property of dual metric space [3, Proposition 2.3] and a new characterization of the so-called Δ-convergence in terms of this duality; see [2, Theorem 2.6].

1.5 Subdifferential

The subdifferential of each $f \in \Gamma_0(X)$ is a set-valued operator $\partial f : X \to 2^{X^*}$ with definition

$$\partial f(x) = \{x^* \mid f(z) - f(x) \ge \langle x^*\,,\, \overrightarrow{xz} \rangle \quad \forall z \in X\,\} \tag{9}$$

when $x \in \mathcal{D}(f)$ and $\partial f(x) = \emptyset$ otherwise [3, Definition 4.1].

Theorem 1.3. [3, Theorem4.2] *Let $f \in \Gamma_0(X)$ then*

i) The subdifferential map ∂f is a monotone operator, i.e.,

$$\langle x_2^* - x_1^*\,,\, \overrightarrow{x_1x_2} \rangle \ge 0 \quad (x_i \in X,\ x_i^* \in \partial f(x_i),\, i = 1,2) \tag{10}$$

here, we have used the notation (2.10).

ii) For each $y \in X$ there exists a point $x \in X$, such that $[\overrightarrow{xy}] \in \partial f(x)$.

When X is a Hilbert space, this theorem asserts that the subdifferential of f is a maximal monotone operator.

2 Main results

Let (X, d) be a geodesically unbounded Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous convex map.
We introduce the set-valued operator $T_f : X \to 2^X$ by

$$T_f(x) = \{2x \ominus y \,|\, y \in X, [\overrightarrow{xy}] \in \partial f(x)\} \quad (x \in X), \tag{11}$$

Proposition 2.1. *For each proper lower semicontinuous convex map f the set $\mathcal{D}(\partial f)$ is dense in the set $\mathcal{D}(f)$.*

Theorem 2.2. *The operator T_f is a co-accretive operator and $cl\mathcal{D}(T_f) = cl\mathcal{D}(f)$. Moreover $\mathcal{R}((1 + r)I \ominus rT_f) = X$ and $J_r(T_f) = J_r(f)$ for each $0 < r \leq 1$.*

Corollary 2.3. *We have*

$$S_t x = \lim_{n\to+\infty} J^n_{t/n}(f)x = \lim_{n\to+\infty} J^n_{t/n}(T_f)x \quad (x \in cl\mathcal{D}(f) = cl\mathcal{D}(T_f), t \geq 0).$$

This means that the semigroup generated by the operator T_f and the gradient flow generated by the mapping f are the same.

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On a new notion of injectivity of Banach modules

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Abstract

In this paper, we introduce a new homological properties of Banach modules. It is shown that for a locally compact group G, the dual space of all bounded left uniformly continuous functions $LUC(G)'$ is 0-injective in the category of left Banach $M(G)$-modules.

Keywords: Banach algebra, injective module, character, ϕ-injective module, locally compact group.
Mathematics Subject Classification [2010]: 46M10, 43A20, 46H25

1 preliminaries

Let A be a Banach algebra and $\Delta(A)$ denote the character space of A, i.e., the space of all non-zero homomorphisms from A onto $\mathbb{C}$. We denote by **A-mod** and **mod-A** the category of all Banach left A-modules and all Banach right A-modules respectively. In the case that A has an identity we denote by **A-unmod** the category of all Banach left unital modules. For $E, F \in$ **A-mod**, let ${}_AB(E, F)$ be the space of all bounded linear left A-module morphisms from E into F.

Let $E, F \in$ **A-mod**. Suppose that $Z^1(A \times E, F)$ denotes the Banach space of all continuous bilinear maps $B : A \times E \longrightarrow F$ satisfying

$$a \cdot B(b, \xi) - B(ab, \xi) + B(a, b \cdot \xi) = 0 \quad (a, b \in A, \xi \in E).$$

Define $\delta_0 : B(E, F) \longrightarrow Z^1(A \times E, F)$ by $(\delta_0 T)(a, \xi) = a \cdot T(\xi) - T(a \cdot \xi)$ for all $a \in A$ and $\xi \in E$. Then we have

$$\mathrm{Ext}^1_A(E, F) = Z^1(A \times E, F)/\mathrm{Im}\delta_0.$$

By [6, Proposition VII.3.19], we know that $\mathrm{Ext}^1_A(E, F)$ is topologically isomorphic to $H^1(A, B(E, F))$ where $B(E, F)$ is a Banach A-bimodule with the following module actions:

$$(a \cdot T)(\xi) = a \cdot T(\xi), \quad (T \cdot a)(\xi) = T(a \cdot \xi) \quad (a \in A, \xi \in E, T \in B(E, F)).$$

To see further details about $\mathrm{Ext}^1_A(E, F)$; see [7].

Definition 1.1. Let A be a Banach algebra and $J \in$ **A-mod**. We say that J is injective if for each $F, E \in$ **A-mod** and admissible monomorphism $T : F \to E$ the induced map $T_J : {}_AB(E, J) \to {}_AB(F, J)$ defined by $T_J(R) = R \circ T$ is onto.

*Speaker

Suppose that $\phi \in \Delta(A)$. For $E \in$ **A-mod**, put

$$I(\phi, E) = \text{span}\{a \cdot \xi - \phi(\xi)a : a \in A, \xi \in E\},$$

and

$${}_{\phi}B(A^{\sharp}, E) = \{T \in B(A^{\sharp}, E) : T(ab - \phi(b)a) = a \cdot T(b - \phi(b)e^{\sharp}), \quad (a, b \in A)\}.$$

Obviously, ${}_{\phi}B(A^{\sharp}, E)$ is a Banach subspace of $B(A^{\sharp}, E)$. On the other hand, for each $b \in \ker(\phi)$, if $T \in {}_{\phi}B(A^{\sharp}, E)$, then $T(ab) = a \cdot T(b)$ for all $a \in A$. Therefore, we conclude that ${}_{\phi}B(A^{\sharp}, E)$ is a Banach left A-submodule of $B(A^{\sharp}, E)$.

Note that if $E, F \in$ **A-mod** and $\rho : E \to F$ is a left A-module homomorphism, we can extend the module actions of E and F from A into $A^{\sharp}$ and ρ to a left $A^{\sharp}$-module homomorphism in a natural way. For Banach spaces E and F, $T \in B(E, F)$ is admissible if and only if there exists $S \in B(F, E)$ such that $T \circ S \circ T = T$.

The following definition of a *ϕ-injective* Banach module, introduced by Nasr-Isfahani and Soltani Renani in [10].

Definition 1.2. Let A be a Banach algebra, $\phi \in \Delta(A)$ and $J \in$ **A-mod**. We say that J is ϕ-injective if for each $F, E \in$ **A-mod** and admissible monomorphism $T : F \to E$ with $I(\phi, E) \subseteq \text{Im}T$, the induced map T_J is onto.

By Definition 1.1 and 1.2, one can easily check that each injective module is ϕ-injective, although by [10, Example 2.5], the converse is not valid. In [3], the authors with use of the semigroup algebras, gave two good examples of ϕ-injective Banach modules which they are not injective.

Now, we give our new concept of injectivity as follows.

Definition 1.3. Let A be a Banach algebra and $E \in$ **A-mod**. We say that E is (left) 0-injective if for each $F, K \in$ **A-mod** and admissible monomorphism $T : F \to K$ for which $A \cdot K = \text{span}\{a \cdot k : a \in A, k \in K\} \subseteq \text{Im}T$, the induced map T_J is onto.

Clearly, every injective module is 0-injective but the converse is not valid in general; see [5, Example 3.4].

In this paper we provide a wide range of non-injective 0-injective Banach modules. Indeed, for each locally compact group G, we prove that $LUC(G)' \in$ **M(G)-mod** is 0-injective, while we know that $LUC(G)' \in$ **M(G)-mod** is injective if and only if G is amenable.

2 Main Results

We start this section with the following Lemma which is an essential tool in the sequel.

Lemma 2.1. *Let $E \in$ **A-mod**. If $\text{Ext}^1_A(F, E) = \{0\}$ for all $F \in$ **A-mod** with $A \cdot F = 0$, then $E \in$ **A-mod** is 0-injective.*

Proof. To show this, let $K, W \in$ **A-mod** and $T : K \to W$ be an admissible monomorphism with $A \cdot W \subseteq \text{Im}T$. We claim that the induced map T_E is onto.

We know that the short complex $0 \to K \xrightarrow{T} W \xrightarrow{q} \frac{W}{\mathrm{Im}T} \to 0$ is admissible where q is the quotient map. But for all $a \in A$ and $x \in W$, $a \cdot (x + \mathrm{Im}T) = \mathrm{Im}T$, because $A \cdot W \subseteq \mathrm{Im}T$. Therefore, by assumption $\mathrm{Ext}^1_A(\frac{W}{\mathrm{Im}T}, E) = \{0\}$. Now, by [7, III Theorem 4.4], the complex

$$0 \to {}_AB(\frac{W}{\mathrm{Im}T}, E) \to {}_AB(W, E) \xrightarrow{T_E} {}_AB(K, E) \to \mathrm{Ext}^1_A(\frac{W}{\mathrm{Im}T}, E) \to \cdots ,$$

is exact. Therefore, T_E is onto.

□

Recall that if E, F be two Banach spaces and $E\widehat{\otimes}F$ denotes the projective tensor product space, then $(E\widehat{\otimes}F)^*$ is isomorphic to $B(E, F^*)$ as two Banach spaces with the pairing

$$< Tx, y >= T(x \otimes y) \quad (x \in E, y \in F, T \in (E\widehat{\otimes}F)^*).$$

Also, note that $E\widehat{\otimes}F$ is isometrically isomorphic to $F\widehat{\otimes}E$ as two Banach spaces.

Theorem 2.2. *Let A be a Banach algebra. Then A is left 0-amenable if and only if each $J \in$ **mod-A** is 0-flat.*

Proof. Suppose that A is left 0-amenable. We show that $\mathrm{Ext}^1_A(E, J^*) = \{0\}$ for all $E \in$ **A-mod** with $A \cdot E = 0$. We have

$$\mathrm{Ext}^1_A(E, J^*) = H^1(A, B(E, J^*)) = H^1(A, (E\widehat{\otimes}J)^*) = \{0\},$$

because $E\widehat{\otimes}J \in$ **mod-A** has the module action, $a \cdot z = 0$ for all $z \in E\widehat{\otimes}J$. Therefore, by Lemma 2.1, $J^* \in$ **A-mod** is 0-injective.

Conversely, let $J \in$ **mod-A** be 0-flat. So, for Banach right A-module $\mathbb{C}$ with module action $\lambda \cdot a = 0$ for all $a \in A$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} H^1(A, J^*) = H^1(A, B(J, \mathbb{C})) &= H^1(A, B(J, \mathbb{C}^*)) \\ &= H^1(A, (J\widehat{\otimes}\mathbb{C})^*) \\ &= H^1(A, (\mathbb{C}\widehat{\otimes}J)^*) \\ &= H^1(A, B(C, J^*)) \\ &= \mathrm{Ext}^1_A(\mathbb{C}, J^*) \\ &= 0. \end{aligned}$$

Hence, if we take J a left A module with module action $a \cdot x = 0$ for all $a \in A$ and $x \in J$, then the above relation implies that A is 0-amenable. □

Corollary 2.3. *If A is a Banach algebra with a bounded approximate identity, then each $E \in$ **mod-A** is 0-flat.*

For a locally compact group G, the space of all bounded left uniformly continuous functions $LUC(G)$, is a closed submodule of $L^\infty(G)$ as a Banach $M(G)$-bimodule. Thus, we can regard $LUC(G)'$ as a Banach $M(G)$-bimodule with the dual module actions; for more details see [1] and [9]. It is shown in [9, Theorem 2.6] that $LUC(G)'$ as the Banach left (right) $M(G)$-module is injective if and only if G is amenable.

Now, with using Corollary 2.3 we give the following generalization of the aforementioned theorem which also provide for us a good source of 0-injective Banach modules. Note that it is well-known that $L^1(G)$ has a bounded approximate identity and $M(G)$ is unital.

Corollary 2.4. *Let G be a locally compact group. Then we have*

(i) $LUC(G)' \in \mathbf{L^1(G)}$**-*mod*** *is 0-injective.*

(ii) $LUC(G)' \in \mathbf{M(G)}$**-mod** *is 0-injective.*

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Talk On a one-dimensional Laplacian-like problem via a local minimization principle pp.: 1–4

On a one-dimensional Laplacian-like problem via a local minimization principle

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Abstract

A critical point theorem (local minimum result) for differentiable functionals is exploited in order to prove that a one-dimensional Laplacian-like problem admits at least one non-trivial and non-negative weak solution.

Keywords: One-dimensional Laplacian-like problem, Existence results, Critical method theorem.
Mathematics Subject Classification [2010]: 34B15, 49Q20.

1 Introduction

The aim of this paper is to study the following one-dimensional Laplacian-like problem:

$$\begin{cases} -\left(\left(1+\frac{u'^2}{\sqrt{1+u'^4}}\right)u'\right)' = \lambda f(t,u) \quad \text{in } (0,1), \\ u(0)=u(1)=0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e. the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e. the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as motion of drops, bubbles and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems. Existence, non-existence and multiplicity of positive solutions of problem (1) have been discussed by several authors in the last decades. See, for instance, the papers [1, 2, 4, 5, 6].

If we recall that *weak solution* of problem (1) is a function $u \in W_0^{1,2}(]0,1[)$ such that

$$\int_0^1 \left(u'(t)v'(t) + \frac{u'(t)^3 v'(t)}{\sqrt{1+u'(t)^4}}\right)dt - \lambda \int_0^1 f(t,u(t))v(t)\,dt = 0$$

for all $v \in W_0^{1,2}(]0,1[)$.

*Speaker

2 Main results

Now, we present our main result.

Theorem 2.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that*

(i) $f(t,0) = 0$ *for a.e.* $t \in [0,1]$,

(ii) *there are two real positive constants τ and k such that for a.e. $t \in [0,1]$ and every $x \in [0,\tau]$ one has $|f(t,x)| \le k$.*

In addition, assume that there are a non-empty open set $D \subseteq (0,1)$ and $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi \to 0^+} \frac{\operatorname{ess\,inf}_{t\in B} F(t,\xi)}{\xi^2} = +\infty, \qquad \liminf_{\xi \to 0^+} \frac{\operatorname{ess\,inf}_{t\in D} F(t,\xi)}{\xi^2} > -\infty,$$

where $F(t,\xi) := \int_0^\xi f(t,x)\,dx$ for all $t \in [0,1]$ and $\xi \in \mathbb{R}$. Then, there exists an open interval $\Lambda \subseteq (0,+\infty)$ such that for each parameter $\lambda \in \Lambda$, problem (1) admits at least one non-trivial and non-negative weak solution $u_\lambda \in C^{1,\beta}([0,1])$ for some $\beta \in (0,1]$. Moreover, we have

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{C^1([0,1])} = 0,$$

and the real function

$$\lambda \mapsto \frac{1}{2}\int_0^1 \left(|u_\lambda'(t)|^2 + \sqrt{1 + |u_\lambda'(t)|^4} \right) dt - \lambda \int_0^1 \left(\int_0^{u_\lambda(t)} f(t,x)dx \right) dt - \frac{1}{2}$$

is negative and strictly decreasing in the open interval Λ.

Proof. Let $a : [0,+\infty) \to (0,+\infty)$ be the $C^{1,1}$ function defined by

$$a(s) := \begin{cases} 1 + \frac{s}{\sqrt{1+s^2}} & \text{if } s \in [0,1), \\ \frac{2+\sqrt{2}}{16}(s-2)^2 + \frac{14+7\sqrt{2}}{16} & \text{if } s \in [1,2), \\ \frac{14+7\sqrt{2}}{16} & \text{if } s \in [2,+\infty). \end{cases}$$

Set, for every $s \ge 0$, $A(s) := \int_0^s a(t)\,dt$. We have

$$1 \le a(s) \le \frac{2+\sqrt{2}}{2} \Rightarrow s \le A(s) \le \frac{2+\sqrt{2}}{2} s \tag{2}$$

for every $s \ge 0$. Further, as the function $s \mapsto sa(s^2)$ is increasing, the function $s \mapsto A(s^2)$ is convex in $[0,+\infty)$. Note that a satisfies the structure and the regularity conditions assumed in [3]. For a.e. $t \in [0,1]$, we truncate f as follows:

$$g(t,x) := \begin{cases} 0, & x \in (-\infty,0), \\ f(t,x), & x \in [0,\tau), \\ f(t,\tau), & x \in [\tau,+\infty). \end{cases}$$

By (i) the function g is L^1-Carathéodory function and $G : [0,1] \times \mathbb{R} \to \mathbb{R}$ denotes its primitive, that is, $G(t,\xi) := \int_0^\xi g(t,x)dx$ for all $(t,\xi) \in [0,1] \times \mathbb{R}$, g and G satisfy the assumptions of the theorem. Let us consider the auxiliary truncated problem

$$\begin{cases} -(a(|u'|^2)u')' = \lambda g(t,u) & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\int_0^1 A(|u'(t)|^2)dt, \qquad \Psi(u) := \int_0^1 G(t,u(t))\,dt, \quad I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$$

for every $u \in X$. From now on, we divide the proof in several steps.

Step 1. The local minimization technique for the truncated problem.

Due to (2), Φ is well defined on X, continuous and coercive. Moreover, by the convexity of the function $s \mapsto A(s^2)$ in $[0,+\infty)$, Φ is convex and then sequentially weakly lower semicontinuous. The functional Ψ is well defined and sequentially weakly (upper) continuous. Moreover, Φ and Ψ are continuously Gâteaux differentiable with derivative given by

$$\Phi'(u)(v) = \int_0^1 a(|u'(t)|^2)u'(t)v'(t)\,dt, \qquad \Psi'(u)(v) = \int_0^1 g(t,u(t))v(t)\,dt$$

for any u, $v \in X$. Now, thanks to Theorem [7, Theorem 2.5], for every $\lambda \in (0,\lambda^\star) \subseteq (0, 1/\varphi(r))$, the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$.

Step 2. For every fixed $\lambda \in (0,\lambda^\star)$ we prove that $u_\lambda \neq 0$ and the map $(0,\lambda^\star) \ni \lambda \mapsto I_\lambda(u_\lambda)$ is negative.

To this end, we easily see that $\lim_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty$.

Step 3. We claim that $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$.

Bearing in mind that Φ is coercive and that for every $\lambda \in (0,\lambda^\star)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \leq L$ for every $\lambda \in (0,\lambda^\star)$. Then, since $0 \leq \|u_\lambda\|^2 \leq \Phi'(u_\lambda)(u_\lambda)$, we have $0 \leq \|u_\lambda\|^2 \leq \Phi'(u_\lambda)(u_\lambda) = \lambda\int_0^1 g(t,u_\lambda(t))u_\lambda(t)\,dt$ for any $\lambda \in (0,\lambda^\star)$.

Step 4. The map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0,\lambda^\star)$.

For our goal we observe that for any $u \in X$, one has $I_\lambda(u_\lambda) = \lambda\Big(\frac{\Phi(u)}{\lambda} - \Psi(u)\Big)$. Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^\star$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi^{-1}(-\infty, r)$ for $i = 1,2$. Also, let $m_{\lambda_i} = \Big(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i})\Big) = \inf_{v\in\Phi^{-1}(-\infty,r)}\Big(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\Big)$ for $i = 1,2$. we get that $I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1})$.

Step 5. Let us prove that the critical points of the energy functional I_λ are non-negative.

Arguing by a contradiction, assume that u is a critical point of I_λ and that the open set $A := \{t \in [0,1] : u(t) < 0\}$ is of positive Lebesgue measure. Put $v := \min\{0, u\}$. Clearly, $v \in X$ and, taking into account that u is a critical point, one has

$$\begin{aligned} 0 &= \Phi'(u)(v) - \lambda\Psi'(u)(v) = \int_0^1 a(|u'(t)|^2)u'(t)v'(t)\,dt - \lambda\int_0^1 g(t,u(t))v(t)\,dt \\ &= \int_A a(|u'(t)|^2)|u'(t)|^2\,dt \geq \int_A |u'(t)|^2 dt, \end{aligned}$$

since $a(s) \geq 1$ for all $s \geq 0$ and $g(t,s) = 0$ for a.e $t \in [0,1]$ and every $s < 0$. Hence, since $u|_A \in W_0^{1,2}(A)$, one has $u \equiv 0$ on A which is a contradiction.

Step 6. There is a $\Lambda \subseteq (0,+\infty)$ *such that, for every* $\lambda \in \Lambda$*, problem* (1) *has a non-negative solution* $u_\lambda \in C^{1,\beta}([0,1])$ *for some* $\beta \in (0,1]$ *satisfying* $\lim_{\lambda\to 0^+} \|u_\lambda\|_{C^1([0,1])} = 0$.

For $\lambda \in (0,\lambda^\star)$, if u_λ is a critical point of I_λ, then it is a weak solution of the auxiliary problem (3) and it is non-negative. Moreover, since $X \hookrightarrow C^0([0,1])$, there exists a $\lambda^\star$ such that $\|u_\lambda\|_\infty \leq \tau$ for every $\lambda \in (0,\lambda^\star)$. On the other hand, by (ii) and bearing in mind the definition of g, it follows that $|g(t,x)| \leq k$, for a.e. $t \in [0,1]$ and $x \in \mathbb{R}$. there are constants $\beta \in (0,1]$ and $\kappa > 0$ such that $u_\lambda \in X \cap C^{1,\beta}([0,1])$ and $\|u_\lambda\|_{C^{1,\beta}([0,1])} \leq \kappa$. Pick any sequence $\{\lambda_n\}$ with $\lambda_n \in (0,\lambda^\star)$ and $\lim_{n\to\infty}\lambda_n = 0$, and let $\{u_{\lambda_n}\}$ be the corresponding sequence of the truncated problem (3). Arzelà-Ascoli Theorem yields the existence of a subsequence, still denoted by $\{u_{\lambda_n}\}$, converging to zero in $C^1([0,1])$. So, we conclude that $\lim_{\lambda\to 0^+} \|u_\lambda\|_{C^1([0,1])} = 0$. □

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ON BEST APPROXIMATION IN KM FUZZY METRIC SPACES

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Abstract

In this paper we introduce the notation of t-best approximatively compact sets, t-best approximation points, t-proximinal sets, t-boundedly compact sets and t-best proximity pair in fuzzy metric spaces. The results derived in this paper are more general than the corresponding results of metric spaces, fuzzy metric spaces, fuzzy normed spaces and probabilistic metric spaces.

Keywords: best approximation, topology, fuzzy metric spaces
Mathematics Subject Classification [2010]: 54A40, 41A50

1 Introduction

Kramosil and Michálek [5] introduced the fuzzy metric space by generalizing the concept of probabilistic metric space to the fuzzy situation with the help of continuous t-norm. Best approximation has important applications in diverse disciplines of mathematics, engineering and economics in dealing with problems arising in: Fixed point theory, Approximation theory, game theory, mathematical economics, best proximity pairs, Equilibrium pairs, etc. Many authors have studied best approximation and best proximity pair in the both metric and fuzzy metric spaces. Also Best approximation has important applications in diverse disciplines of mathematics, engineering and economics in dealing with problems arising in: Fixed point theory, Approximation theory, game theory, mathematical economics, best proximity pairs, Equilibrium pairs, etc. Many authors have studied best approximation and best proximity pair in the both metric and fuzzy metric spaces (e. g. see [1,6,7,9–11]). Best proximity pair theorems in the metric space (X, d) are consider to expound the sufficient conditions that ensure the existence of $x \in A$ such that $d(x, Tx) = d(A, B) := \inf\{d(a, b); a \in A, b \in B\}$, where $T : A \to 2^B$ is a multifunction defined on suitable subsets A, B of X. Also, a best proximity pair theorem evolves as a generalization of the problem, considered by Beer and Pai [1], Sahney and Singh [6], Singer [8] and Xu [11], of exploring the sufficient conditions for the non-emptiness of the set $\mathrm{Prox}(A, B) = \{(a, b) \in A \times B : d(a, b) = d(A, B)\}$, where A, B are suitable subsets of metric or linear normed space X. In this paper, we generalize some notions, definitions and results in [4, 7–10] such as set of best approximation points, proximinal sets

*Speaker

and approximatively compact sets for the fuzzy metric space in the sense of Kramosil and Michálek [5]. In addition, some examples and applications are presented.

Recall that a continuous t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], \leq, *)$ is an ordered Abelian topological monoid with unit 1.

Definition 1.1. (Kramosil and Michálek [5]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times [0, \infty)$ satisfying the following properties, for all $x, y, z \in X, s, t > 0$:

(KM1) $M(x, y, 0) = 0$;

(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM3) $M(x, y, t) = M(y, x, t)$;

(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(KM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

Example 1.2. Let $X = \mathbb{R}$. For every $x, y \in X, t > 0$ define the metric d_t on $X \times X$ by $d_t(x, y) = \min\{| x - y |, t\}$, and the map $M : \mathbb{R}^2 \times [0, \infty) \to [0, 1]$ by $M(x, y, 0) = 0$ and

$$M(x, y, t) = \frac{t}{t + d_t(x, y)},$$

then $(X, M, \cdot)$ is a fuzzy metric space, wherein $\cdot$ is the product t-norm.

2 Best approximation and Generalization

Definition 2.1. Let A be a non-empty subset of fuzzy metric space $(X, M, *)$. For each $x \in X$ and $t > 0$, define

$$M(A, x, t) = \sup\{M(x, y, t) : y \in A\}.$$

An element $y_0 \in A$ is said to be a t-best approximation point to x from A if

$$M(y_0, x, t) = M(A, x, t).$$

We denote by $P_A^M(x, t)$ the set of t-best approximation points to x. For $t > 0$ a subset A of a fuzzy metric space $(X, M, *)$ is called t-proximinal if for every point $x \in X$, $P_A^M(x, t) \neq \emptyset$.

By a slight modification in the definitions and the results in [7, 9, 10] we can extend those results to the fuzzy metric spaces, e. g., the following is given for fuzzy normed spaces in [9].

Definition 2.2. Let A be a non-empty subset of a fuzzy metric space $(X, M, *)$. An element $y_0 \in A$ is said to be an F-best approximation of $x \in X$ from A if it is a t-best approximation of x from A, for every $t > 0$. The set of all elements of F-best approximations of X from A is denoted by

$$FP_A^M(x) = \bigcap_{t \in (0,\infty)} P_A^M(x, t).$$

If each $x \in X$ has at least one F-best approximation in A, then A is called a F-proximinal set.

Remark 2.3. Let $(X, M_d, *)$ be a standard fuzzy metric space in [3] and $A \subseteq X$ and $x \in X$, then for every $t_1, t_2 > 0$, $P_A^M(x, t_1) = P_A^M(x, t_2)$, thus, $FP_A^M(x) = P_A^M(x, t_1) = P_A^M(x, 1)$. Also this property holds for Example 2.15 of [9] and other known examples in the literature, the following shows that the above property is not true in general and the definition of best approximation point in fuzzy metric spaces is related to parameter t in its definition, so it is different from the classical theory of metric spaces.

Example 2.4. Consider Example 1.2, take $A = [0, 1]$ and $y_0 = 2$ then one can easily shows that if $t \geq 1$ then $P_A^M(y_0, t) = \{1\}$ and if $0 < t < 1$ then $P_A^M(y_0, t) = A$.

Following the approach of Kainen [4] we introduce a new definition to generalize t-approximatively compact set, then, we introduce t-best approximation point, t-proximinal set and t-boundedly compact set relative to set in fuzzy metric spaces.

Definition 2.5. Let $(X, M, *)$ be a fuzzy metric space and A, B are non-empty subsets of X and $t > 0$, let

$$M(A, B, t) = \sup\{M(a, b, t); a \in A, b \in B\}.$$

We say a sequence $x_n \in A$, t-converges in distance to B if

$$M(x_n, B, t) \to M(A, B, t).$$

If $B = \{b\}$ is singleton then we use b instead of $\{b\}$. Let $\mathfrak{B}$ denote the family of non-empty subsets of X, we say the subset A is t-approximatively compact relative to $\mathfrak{B}$ if for every $B \in \mathfrak{B}$ and every sequence $x_n \in A$ which converges in distance to B, then there exists a subsequence y_{n_k} of y_n and $y_0 \in A$ such that $y_{n_k} \to y_0$. If $\mathfrak{B} = \{B\}$ is singleton then we use B instead of $\{B\}$.

Definition 2.6. For $t > 0$, an element $y_0 \in A$ is said to be a t-best approximation point to B from A if

$$M(y_0, B, t) = M(A, B, t).$$

We denote by $P_A^M(B, t)$ the set of t-best approximation points to B. A subset A is called t-proximinal relative to $\mathfrak{B}$ if for every $B \in \mathfrak{B}$, $P_A^M(B, t) \neq \emptyset$ and A is called t-quasi Chebyshev relative to $\mathfrak{B}$ if for every $B \in \mathfrak{B}$, $P_A^M(B, t)$ be a compact set.

Let $(X, M, *)$ be a fuzzy metric spaces. In the sequel for arbitrary $t > 0$, let $\mathcal{C}(X), \mathcal{A}(X)$ and $\mathcal{B}(X)$ denote the set of compact, t-approximatively compact and t-boundedly compact subsets of X respectively. Also we denote by $(\mathcal{A}(X), \mathfrak{B})$ the set of t-approximatively compact subsets of X relative to $\mathfrak{B}$ and for non-empty subsets A, B of X, denote by $\text{Prox}(A, B, t)$ the set of t-best proximity pairs, i.e. $(a, b) \in A \times B$ such that $M(a, b, t) = M(A, B, t)$.

The following main result shows that the notion of t-approximatively compact set can be applied to compact sets.

Theorem 2.7. *Let $t > 0$. A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. If $A \in \mathcal{A}(X)$ and $B \in \mathcal{C}(X)$ then $A \in (\mathcal{A}(X), B)$.*

The following investigates the above notions for product of fuzzy metric spaces.

Theorem 2.8. *Let A and B be non-empty subsets of a fuzzy metric space $(X_1, M_1, *)$ and $(X_2, M_2, *)$, respectively. Suppose $B \in \mathcal{C}(X_2)$, if $A \in \mathcal{B}(X_1)$ or $A \in \mathcal{A}(X_1)$ then $A \times B \in \mathcal{B}(X_1 \times X_2)$ or $A \times B \in \mathcal{A}(X_1 \times X_2)$, respectively.*

The following generalizes [7, Theorem 2.19] and shows that the metric projection $P_A^M(x,t)$ also preserves compactness.

Theorem 2.9. *Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. Suppose $B \in \mathcal{C}(X)$, if $A \in \mathcal{B}(X)$ or $A \in \mathcal{A}(X)$ then A is t-quasi Chebyshev relative to B.*

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On Chatterjea Contractions in Metric Space with a Graph

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Abstract

In this talk, we introduce Chatterjea contractions using directed graphs in metric spaces with a graph and investigate the existence of fixed points for Chatterjea contractions under two different conditions and discuss the main theorem. We also discuss the uniqueness of the fixed point.

Keywords: G-Chatterjea mapping, Fixed point, Orbitally G-continuous mapping.
Mathematics Subject Classification [2010]: 47H10, 05C20

1 Introduction

Let (X, d) be a metric space. In [3], Chatterjea introduced the notion of Chatterjea contraction on a metric space X as follows:

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big] \tag{1}$$

for all $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$. He also investigated the existence and uniqueness of fixed points for self-map T and proved that such mappings have a unique fixed point in complete metric spaces.

Recently in 2008, Jachymski [4] proved some fixed point results in metric spaces endowed with a graph and generalized simultaneously the Banach contraction principle from metric and partially ordered metric spaces. Recently in 2013, Bojor [1] followed Jachymski's idea for Kannan contractions using a new assumption called the weak T-connectedness of the graph.

The aim of this paper is to study Chatterjea contractions in metric spaces endowed with a graph by standard iterative techniques and avoid imposing the assumption of weak T-connectedness on the graph. Our main result generalizes Chatterjea's fixed point theorem in metric spaces and also in metric spaces equipped with a partial order.

We next review some basic notions of graph theory in relation to uniform spaces that we need in the sequel. For more details on the theory of graphs, see, [2, 4].

An edge of an arbitrary graph with identical ends is called a loop and an edge with distinct ends is called a link. Two or more links with the same pairs of ends are said to be parallel edges.

Let (X, d) be a metric space and G be a directed graph with vertex set $V(G) = X$ such that the set $E(G)$ consisting of the edges of G contains all loops, that is, $(x, x) \in E(G)$

*Speaker

for all $x \in X$. Assume further that G has no parallel edges. Then G can be denoted by the ordered pair $(V(G), E(G))$, and also it is said that the metric space (X, d) is endowed with the graph G.

We denote by G^{-1} the conversion of the graph G, that is, $V(G^{-1}) = V(G)$ and

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

The metric space (X, d) can also be endowed with the graph $\widetilde{G}$, where the former is the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words, $V(G) = V(\widetilde{G})$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

It should be remarked that if both (x, y) and (y, x) belong to $E(G)$, then we will face with parallel edges in the graph $\widetilde{G}$. A graph $G = (V(G), E(G))$ is said to be transitive if $(x, y), (y, z) \in E(G)$ implies $(x, z) \in E(G)$ for all $x, y, z \in V(G)$.

By a subgraph of G, we mean a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that $V(H)$ contains the vertices of all edges of $E(H)$, i.e., $(x, y) \in E(H)$ implies $x, y \in V(H)$ for all $x, y \in V(G)$.

Definition 1.1 ([6])**.** A self-map T on X is called a Picard operator if T has a unique fixed point $\widehat{x} \in X$ and $T^n x \to \widehat{x}$ for all $x \in X$ and is called weakly Picard operator if the sequence $\{T^n x\}$ converges to a fixed point of T for all $x \in X$. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then

It is clear that a Picard operator is a weakly Picard one but the identity mapping of any metric space with more that one point shows that the converse is not generally true.

Definition 1.2 ([4])**.** Self-map T on metric space (X, d) endowed with a graph $G$is called orbitally G-continuous on X if for each $x, y \in X$ and each sequences $\{b_n\}$ of positive integers with $(T^{b_n} x, T^{b_{n+1}} x) \in E(G)$ for all $n \geq 1$, the convergence $T^{b_n} x \to y$ implies $T(T^{b_n} x) \to Ty$.

It is clear that, a continuous mapping on a metric space is orbitally G-continuous for all graphs G. But the converse of these relations is not true in general as the next example shows.

2 Main results

Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be an arbitrary mapping. Throughout this section, we denote the set $\{x \in X : (x, Tx) \in E(G)\}$ by the X_T and the set $\{x \in X : Tx = x\}$ by the Fix(T). Since $E(G)$ contains all loops, it follows that Fix$(T) \subseteq X_T$.

Motivated by [4, Definition 2.1] and [1, Definition 4], we introduce G-Chatterjea mappings in metric spaces endowed with a graph as follows:

Definition 2.1 ([5])**.** Let (X, d) be a metric space endowed with a graph G. We say that a mapping $T : X \to X$ is a G-Chatterjea mapping if

C1) T preserves the egdes of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

C2) there exists an $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big]$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

If $T : X \to X$ is a G-Chatterjea mapping, then we call the number α in (C2) the constant of T.

We now give some examples of G-Chatterjea mappings in metric spaces endowed with a graph.

Example 2.2. Let $(X, \preceq)$ be a poset and d be a metric on X. Consider the poset graphs G_1 and G_2 by

$$V(G_1) = X \quad \text{and} \quad E(G_1) = \big\{(x, y) \in X \times X : x \preceq y\big\}$$

and $G_2 = \widetilde{G_1}$. Since $\preceq$ is reflexive, it follows that both $E(G_1)$ and $E(G_2)$ contain all loops. Assume that (X, d) is endowed with one of the graphs G_1 and G_2. Then a mapping $T : X \to X$ preserves the edges of G_1 if and only if T is nondecreasing, and T satisfies (C2) for the graph G_1 if and only if

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big] \tag{2}$$

for all comparable elements $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$. Moreover, T preserves the edges of G_2 if and only if T maps the comparable elements of $(X, \preceq)$ onto comparable elements, and T satisfies (C2) for the graph G_2 if and only if (2) holds. Thus, each G_1-Chatterjea mapping is a G_2-Chatterjea one.

In order to prove our main theorem, we begin with an interesting and important property of G-Chatterjea mappings which is needed in the sequel.

Proposition 2.3 ([5])**.** *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping. Then* $\mathrm{Fix}(T)$ *does not contain both ends of any link of G.*

The next useful lemma shows that in a metric space (X, d) endowed with a graph G, two successive iterates of any point of X_T under a G-Chatterjea mapping $T : X \to X$ are getting arbitrarily closer whenever the numbers of the iterates are getting sufficiently large.

Lemma 2.4 ([5])**.** *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping with constant α. Then*

$$d(T^n x, T^{n+1} x) \leq \Big(\frac{\alpha}{1-\alpha}\Big)^n \cdot d(x, Tx) \tag{3}$$

for all $x \in X_T$ and all $n \geq 0$. In particular, $d(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$, for all $x \in X_T$.

Our main theorem shows that a G-Chatterjea mapping T defined on a complete metric space (X, d) endowed with a graph G has a fixed point in X whenever T is orbitally G-continuous on X or the triple (X, d, G) has a suitable property.

Theorem 2.5 ([5]). *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping. Then the restriction of T to the set X_T is a weakly Picard operator if one of the following statements holds:*

1) *T is orbitally G-continuous on X;*

2) *The triple (X, d, G) has the following property:*

   ($*$) *If $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 1$.*

In particular, whenever (1) *or* (2) *holds, then* $\mathrm{Fix}(T) \neq \emptyset$ *if and only if* $X_T \neq \emptyset$.

Combining Theorem 2.5 and Proposition 2.4 yields Chatterjea's fixed point theorem [3] in complete metric spaces as follows:

Corollary 2.6 ([5]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping which satisfies* (1). *Then T is a Picard operator.*

Theorem 2.7 ([5]). *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping. Then T has at most one fixed point in X if one of the following statements holds:*

a) *For all $x, y \in X$, there exists a path in G from x to y of length 2;*

b) *The subgraph of G with the vertices* $\mathrm{Fix}(T)$ *is weakly connected.*

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On linear operators from a Banach space to analytic Lipschitz spaces

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Abstract

In this note, we characterize boundedness and (weak) compactness of linear operators from a Banach space into analytic Lipschitz spaces $\mathrm{lip}_A(X,\alpha)$. We also obtain a lower bound for the essential norm of such operators.

Keywords: Analytic Lipschitz algebra;compact linear operator; weakly compact linear operator; essential norm.

Mathematics Subject Classification [2010]: 47B38, 46E15

1 Introduction

Let E be a Banach space, (X,d) be a compact metric space, and $\alpha \in (0,1]$. The space $\mathrm{Lip}_\alpha(X,E)$ consist of E-valued functions f on X that

$$p_\alpha(f) = \sup\left\{\frac{\|f(x)-f(y)\|_E}{d^\alpha(x,y)} : x,y \in X, x \neq y\right\} < \infty,$$

and $\mathrm{lip}_\alpha(X,E)$ is the subspace of those functions f for which

$$\lim_{d(x,y)\to 0} \frac{\|f(x)-f(y)\|_E}{d^\alpha(x,y)} = 0.$$

The spaces $\mathrm{Lip}_\alpha(X,E)$ and $\mathrm{lip}_\alpha(X,E)$ are Banach spaces with the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup_{x\in X}\|f(x)\|_E$. In the case that E is the scalar field of the complex numbers $\mathbb{C}$, we have classic Lipschitz algebras $\mathrm{Lip}(X,\alpha) = \mathrm{Lip}_\alpha(X,\mathbb{C})$ and $\mathrm{lip}(X,\alpha) =$

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$\mathrm{lip}_\alpha(X,\mathbb{C})$. It is known that $\mathrm{Lip}(X,\alpha)$ for $0<\alpha\leq 1$ and $\mathrm{lip}(X,\alpha)$ for $0<\alpha<1$ are Banach function algebras and their character spaces (maximal ideal spaces) coincide with X. These algebras first studied by de Leeuw [2] and Sherbert [3, 4]. When X is a compact plane set, analytic Lipschitz algebras are subalgebras of Lipschitz algebras consist of analytic functions on the interior of X and denoted by $\mathrm{Lip}_A(X,\alpha)$ and $\mathrm{lip}_A(X,\alpha)$, that is

$$\mathrm{Lip}_A(X,\alpha)=\mathrm{Lip}(X,\alpha)\cap A(X) \quad \text{and} \quad \mathrm{lip}_A(X,\alpha)=\mathrm{lip}(X,\alpha)\cap A(X),$$

where $A(X)$ is the algebra of all continuous functions on X which are analytic on $\mathrm{int}X$.

Let G be an open set in $\mathbb{C}$ and E be a complex topological vector space. A function $f:G\to E$ is said to be analytic if Λf is analytic in the ordinary sense for every Λ in E^*, the dual space of E.

In this paper we study the properties of linear operators from a Banach space B into $\mathrm{lip}_A(X,\alpha)$ and we provide some results analogue to the results obtained in [1].

2 Main results

For convenience, we recall some notions which we require in the sequel. Suppose that B is a Banach space. We denote by $B^\times$ the algebraic dual space of B, the space of all linear functionals on B. The topological dual space of B is the Banach space B^* whose elements are the bounded linear functionals on B.

For a linear operator T (not necessarily bounded) from a Banach space B into $\mathrm{lip}_A(X,\alpha)$, we denote the restriction of the algebraic adjoint $T^\times:\mathrm{lip}_A(X,\alpha)^*\to B^\times$ of T to the space X by ψ. Therefore, by the definition of adjoint, we have $\psi=T^\times|_X:X\to B^\times$, $\psi(x)=T^\times(e_x)=e_x\circ T$, that $e_x\in\mathrm{lip}_A(X,\alpha)^*$ is the evaluation functional at point $x\in X$ defined by $e_x(f)=f(x)$ for every $f\in\mathrm{lip}_A(X,\alpha)$. In this case, one can say that the linear operator T is induced by the function ψ or that ψ induces T by means of $\psi(x)=e_x\circ T$ or equivalently, $(Tb)(x)=\psi(x)(b)$ for each $b\in B$ and $x\in X$. If T is bounded, then the function ψ maps X into B^*. In fact, ψ is the restriction of the topological adjoint $T^*:\mathrm{lip}_\alpha(X)^*\to B^*$ of T to the space X, and it is continuous with the weak*-topology on B^*. It is interesting to know when a function $\psi:X\to B^\times$ induces linear operator $T:B\to\mathrm{lip}_A(X,\alpha)$. In other words, under what conditions the function $Tb:X\to\mathbb{C}$ defined by $Tb(x) = \psi(x)b$ belongs to $\mathrm{lip}_A(X, \alpha)$ whenever $b \in B$. In the following theorem, we give conditions on a function $\psi : X \to B^\times$ to induce a linear operator $T : B \to \mathrm{lip}_A(X, \alpha)$.

Theorem 2.1. Let B be a Banach space. If T is a linear operator from B into $\mathrm{lip}_A(X, \alpha)$, then the function $\psi = T^\times|_X$ is analytic in $\mathrm{int}X$ and satisfies

$$\lim_{d(x,y)\to 0} \frac{\psi(x) - \psi(y)}{d^\alpha(x, y)} = 0, \tag{2.1}$$

in the weak*-topology of $B^\times$. Conversely, if a function $\psi : X \to B^\times$ is analytic in $\mathrm{int}X$ and satisfies (2.1) in the weak*-topology of $B^\times$, then the linear operator T defined by $Tb(x) = \psi(x)b$ maps B into $\mathrm{lip}_A(X, \alpha)$.

The following results concerning with the problem to describe when such linear operators are bounded, compact or weakly compact in terms of function theoretic properties of the induced function. For the boundedness, we have the following result.

Theorem 2.2. Suppose that B is a Banach space and $T : B \to \mathrm{lip}_A(X, \alpha)$ is a linear operator induced by $\psi : X \to B^\times$. Then T is bounded if and only if $\psi \in \mathrm{Lip}_\alpha(X, B^*)$. Moreover, $\|T\| \le \|\psi\|_\alpha \le 2\|T\|$.

Note that as shown in Theorem 2.1, a function $\psi : X \to B^*$ may not, in general, induce a linear operator $T : B \to \mathrm{lip}_A(X, \alpha)$. Even if $\psi \in \mathrm{Lip}_\alpha(X, B^*)$ is analytic in $\mathrm{int}X$, the operator T defined by $(Tb)(x) = \psi(x)b$ does not, in general, map B into $\mathrm{lip}_A(X, \alpha)$. For example, set $B = \mathrm{lip}_A(X, \alpha)$ and let $\lambda_0 \in \mathrm{lip}_A(X, \alpha)^*$, $f_0 \in \mathrm{Lip}_A(X, \alpha) \setminus \mathrm{lip}_A(X, \alpha)$ and define $\psi : X \to \mathrm{lip}_A(X, \alpha)^*$ by $\psi(x) = f_0(x)\lambda_0$. Note that $\psi \in \mathrm{Lip}_\alpha(X, \mathrm{lip}_A(X, \alpha)^*)$. Let T be the induced operator by ψ. Then $(Tf)(x) = \psi(x)f = f_0(x)\lambda_0(f)$ for each $f \in \mathrm{lip}_A(X, \alpha)$ and $x \in X$. Thus, $Tf = \lambda_0(f)f_0$ is not in $\mathrm{lip}_A(X, \alpha)$ for any $f \in \mathrm{lip}_A(X, \alpha)$ with $\lambda_0(f) \neq 0$. Therefore, T does not map $B = \mathrm{lip}_A(X, \alpha)$ into $\mathrm{lip}_A(X, \alpha)$.

In the following theorem, we characterize the compactness of these operators.

Theorem 2.3. Let B be a Banach space. Then a linear operator $T : B \to \mathrm{lip}_A(X, \alpha)$ induced by ψ is compact if and only if $\psi \in \mathrm{lip}_\alpha(X, B^*)$.

Using the above theorem, we determine a lower bound for the essential norm of a bounded linear operator $T : B \to \mathrm{lip}_A(X, \alpha)$. The essential norm $\|T\|_e$ of a bounded linear operator T, is defined as

$$\|T\|_e = \inf_K \|T - K\|,$$

where the infimum is taken over all compact operators $K : B \to \mathrm{lip}_A(X, \alpha)$. Note that $\|T\|_e = 0$ if and only if T is compact.

Theorem 2.4. If B is a Banach space and $T : B \to \mathrm{lip}_A(X, \alpha)$ is a bounded linear operator induced by a function $\psi : X \to B^*$, then

$$\limsup_{d(x,y)\to 0} \frac{\|\psi(x) - \psi(y)\|}{d^\alpha(x, y)} \leq \|T\|_e.$$

We next characterize weak compactness of a bounded linear operator $T : B \to \mathrm{lip}_A(X, \alpha)$.

Theorem 2.5. Let B be a Banach space. Then a linear operator $T : B \to \mathrm{lip}_A(X, \alpha)$ induced by a function $\psi : X \to B^*$ is weakly compact if and only if

$$\lim_{d(x,y)\to 0} \frac{\psi(x) - \psi(y)}{d^\alpha(x, y)} = 0,$$

in the weak topology of B^*.

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On pseudospectrum of matrix polynomials

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Abstract

In this paper, some algebraic and geometrical properties of the pseudospectrum of matrix polynomials are investigated. The notion of pseudonumerical range of matrix polynomials is also introduced, and some properties of this notion are studied.

Keywords: Matrix polynomial, pseudospectrum, numerical range
Mathematics Subject Classification [2010]: 15A18, 15A60, 47A56

1 Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices, $A \in \mathbb{M}_n$, and $\epsilon > 0$. The pseudospectrum of A is defined and denoted, e.g., see [5] and [1] by

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} \ : \ \exists E \in \mathbb{M}_n \text{ s.t. } \|E\| < \epsilon \text{ and } z \in \sigma(A + E)\}, \tag{1}$$

where $\sigma(.)$ denotes the spectrum and $\|.\|$ is the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm). It is known that

$$\sigma(A) = \bigcap_{\epsilon>0} \sigma_\epsilon(A).$$

The spectrum of a matrix provides a fundamental tool for understanding the behavior of it. For instance, if $\sigma(A) \subseteq \{z \in \mathbb{C} \ : \mid z \mid < 1\}$, then $\sum_{i=0}^{\infty} A^i$ is convergent.
Pseudospectra provide an analytical and graphical alternative for investigating nonnormal matrices and operators, give a quantitative estimate of departure from non-normality and give information about stability. There are many interesting results concerning the pseudospectrum and its application; See [5]. In the following proposition, we list some known properties of pseudospectrum of matrices.

*Speaker

Proposition 1.1. *Let $A \in M_n$ and $\epsilon > 0$. Then the following assertions are true:*
(a) $\sigma(A) + D(0, \epsilon) \subseteq \sigma_\epsilon(A) \subseteq W(A) + D(0, \epsilon)$,
where $D(0, \epsilon) = \{z \in \mathbb{C} : |z| < \epsilon\}$ and $W(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\| = 1\}$ is the numerical range of A;
(b) $\sigma_\epsilon(\alpha I + \beta A) = \alpha + \beta\sigma_{\epsilon/|\beta|}(A)$ where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus \{0\}$.
(c) $\sigma_\epsilon(A) = \sigma_\epsilon(U^*AU)$, where $U \in M_n$ is a unitary matrix.

Consider a matrix polynomial

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{2}$$

where $A_i \in M_n, A_m \neq 0$ and λ is a complex variable. The numbers m and n are, respectively, called the degree and the order of $P(\lambda)$. The matrix polynomial $P(\lambda)$, as in (2), is called a monic matrix polynomial if $A_m = I_n$.
For the case $m = 1$, $P(\lambda)$ is said to be a linear pencil. A scalar $\lambda_0 \in \mathbb{C}$ is called an eigenvalue of $P(\lambda)$ if the system $p(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution is known as an eigenvalue of $P(\lambda)$ corresponding to λ_0. The set of all eigenvalues of $P(\lambda)$ is called the spectrum of $P(\lambda)$; namely,

$$\sigma[P(\lambda)] = \{\mu \in \mathbb{C} : \det P(\mu) = 0\}. \tag{3}$$

In this paper, we are going to study some algebraic and geometric properties of the pseudospectrum of matrix polynomials. For this, in section 2, we state definitions and general properties of the pseudonumerical range of matrix polynomials and we investigate some algebraic properties of this notion.

2 Main results

Let $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a matrix polynomial as in (2). We begin our discussion by introducing the pseudospectrum of $P(\lambda)$.

Definition 2.1. Let $\epsilon > 0$ and $P(\lambda)$ be a matrix polynomial as in (2). The pseudospectrum of $P(\lambda)$ is defined and denoted by

$$\sigma_\epsilon[P(\lambda)] = \{\mu \in \mathbb{C} : 0 \in \sigma_\epsilon(P(\mu))\}$$

where for every $\mu \in \mathbb{C}$, $P(\mu) \in M_n$ is a matrix and $\sigma_\epsilon(P(\mu))$ is as in (1).
In view of Define 2.1 and (1), we have the following proposition.

Proposition 2.2. *Let $\epsilon > 0$ and $P(\lambda)$ be a matrix polynomial as in (2). Then*

$$\sigma_\epsilon[P(\lambda)] = \{\mu \in \mathbb{C} : \exists Q(\lambda) \text{ of degree } m \text{ and order } n, \text{ s.t. } \|Q(\mu)\| < \epsilon \text{ and } \det(P(\mu)+Q(\mu)) = 0\}$$

By Define 2.1 and (1) and Proposition 2.2, we have the following proposition.

Proposition 2.3. *Let $\epsilon > 0$ and $P(\lambda) = \lambda I_n - A$, where $A \in M_n$, (i.e., $P(\lambda)$ is a monic linear pencil). Then $\sigma_\epsilon[P(\lambda)] = \sigma_\epsilon(A)$.*

In view of Proposition 2.3, we conclude that the pseudospectrum of matrix polynomials is a generalization of the pseudospectrum of matrices.
By a result in [4], we have the following theorem.

Theorem 2.4. *Let $\epsilon > 0$ and $P(\lambda)$ be a matrix polynomial as in (2). Then (a) $\sigma_\epsilon[P(\lambda)]$ is bounded if and only if $0 \in \sigma_\epsilon(A_m)$. (b) If $\sigma[P(\lambda)]$ has k element(s), then $\sigma_\epsilon[P(\lambda)]$ has at most k connected component(s).*

At the end of this section, we introduce and study the pseudonumerical range of matrix polynomials.

Definition 2.5. Let $\epsilon > 0$ and $P(\lambda)$ be a matrix polynomial as in (2). The pseudonumerical range of $P(\lambda)$ is defined and denoted by

$$W_\epsilon[P(\lambda)] = \{\mu \in \mathbb{C} : 0 \in W(P(\mu)) + D(0, \epsilon)\}.$$

By Definition 2.5, it is clear that $\sigma_\epsilon[P(\lambda)] \subseteq W_\epsilon[P(\lambda)]$. In the following proposition, we characterize the pseudonumerical range of monic linear pencils.

Proposition 2.6. *Let $\epsilon > 0$ and $P(\lambda) = \lambda I_n - A$, where $A \in M_n$,be a monic linear pencil. Then $W_\epsilon[P(\lambda)] = W(A) + D(0, \epsilon)$.*

Proof. Since for every $S \subseteq \mathbb{C}, S + D(0, \epsilon) = S - D(0, \epsilon)$, the result follows from Definition 2.5. □

Remark 2.7. Let $\epsilon > 0$ and $A \in M_n$. The set $W(A) + D(0, \epsilon)$, as in Proposition 2.6, is called the augmented numerical range of A; See [3] for more information.

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On some means inequalities in matrix spases

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Abstract

In this paper, we state some recent results on non-commutative version of refinements and reverses of ν-weighted arthimetic-geometric-harmonic mean inequality, which is a fundamental relation between two nonnegative real numbers, in the frame work of matrices.

Keywords: Mean value, positive definite matrix, Young inequality
Mathematics Subject Classification [2010]: 15A42, 15A60

1 Introduction

The well-known Young inequality, states that if a, b are two positive numbers and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and equality holds if and only if $a = b$. Equivalently, for distinct positive numbers a, b and $0 < \nu < 1$, we have

$$a^\nu b^{1-\nu} < \nu a + (1 - \nu)b.$$

By defining weighted arithmetic and geometric means as $A_\nu(a, b) = \nu a + (1 - \nu)b$ and $G_\nu(a, b) = a^\nu b^{1-\nu}$, respectively, the Young inequality can be written as $G_\nu(a, b) < A_\nu(a, b)$, which is known as the arithmetic-geometric mean inequality. A similar inequality, known as geometric-harmonic mean inequality, states that $H_\nu(a, b) < G_{nu}(a, b)$ where $H_\nu(a, b) = (\nu a^{-1} + (1 - \nu)b^{-1})^{-1}$ is the harmonic mean of a, b.

One can consider these inequalities on the complex matrix space.

Definition 1.1. For two positive definite matrices A, B, we define

- arithmetic mean of A, B:

$$A\nabla_\nu B = \nu A + (1 - \nu)B,$$

- geometric mean of A, B:

$$A\sharp_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-\nu}A^{1/2},$$

- harmonic mean of A, B:

$$A!_\nu B = (\nu A^{-1} + (1 - \nu)B^{-1})^{-1}.$$

Some mathematicians investigated on the above inequalities and found different refinements of them. They found some sharper upper and lower bounds for the difference and the ratio of these two means.

In this paper, we focus on the matrix inequalities which compare the difference of weighted arithmetic and geometric means and also arithmetic and harmonic means with respect to two different weights ν, μ.

2 Arithmetic-geometric mean type inequalities

Theorem 2.1. *[1] Let $0 < \nu \leq \mu < 1$. If A, B are two positive definite matrices, then*

$$\frac{\nu}{\mu}(A\nabla_\mu B - A\sharp_\mu B) \leq A\nabla_\nu B - A\sharp_\nu B \leq \frac{1-\nu}{1-\mu}(A\nabla_\mu B - A\sharp_\mu B). \tag{1}$$

The following special case was proved by Kittaneh and Manasrah [2, 3], independently.

Corollary 2.2. *Let $0 < \nu < 1$ and A, B be two positive definite matrices. Then*

$$r_0(A\nabla_{1/2}B - A\sharp_{1/2}B) \leq A\nabla_\nu B - A\sharp_\nu B \leq R_0(A\nabla_{1/2}B - A\sharp_{1/2}B),$$

where $r_0 = 2\min\{\nu, 1-\nu\}$ and $R_0 = 2\max\{\nu, 1-\nu\}$.

A similar version of (3) for positive numbers can be stated as follow.

Theorem 2.3. *Let $0 < \nu < \mu < 1$ and $n \geq 1$. Then*

$$\left(\frac{\nu}{\mu}\right)^n \leq \frac{A_\nu(a,b)^n - G_\nu(a,b)^n}{A_\mu(a,b)^n - G_\mu(a,b)^n} \leq \left(\frac{1-\nu}{1-\mu}\right)^2. \tag{2}$$

for two distinct positive numbers a, b.

Since every two commuting matrices are simultaneously diagonalizable, so we have the following result.

Corollary 2.4. *Let A, B be two positive definite commuting matrices and $0 < \nu < \mu < 1$ and $n \geq 1$. Then*

$$\begin{aligned}\left(\frac{\nu}{\mu}\right)^n [A\nabla_\mu B^n - A\sharp_\mu B^n] &\leq A\nabla_\nu B^n - A\sharp_\nu B^n \\ &\leq \left(\frac{1-\nu}{1-\mu}\right)^n [A\nabla_\mu B^n - A\sharp_\mu B^n\end{aligned}$$

3 Arithmetic-harmonic mean type inequalities

Theorem 3.1. *[5] Let $0 < \nu \leq \mu < 1$. If A, B are two positive definite matrices, then*

$$\frac{\nu}{\mu}(A\nabla_\mu B - A!_\mu B) \leq A\nabla_\nu B - A!_\nu B \leq \frac{1-\nu}{1-\mu}(A\nabla_\mu B - A!_\mu B). \tag{3}$$

As an special case was have the following result which is proved in [4] and [6].

Corollary 3.2. *Let $0 < \nu < 1$ and A, B be two positive definite matrices. Then*

$$r_0(A\nabla_{1/2}B - A\sharp_{1/2}B) \leq A\nabla_\nu B - A\sharp_\nu B \leq R_0(A\nabla_{1/2}B - A\sharp_{1/2}B),$$

where $r_0 = 2\min\{\nu, 1-\nu\}$ and $R_0 = 2\max\{\nu, 1-\nu\}$.

Theorem 3.3. *Let $A, B \in M_n(\mathbb{C})$ be positive definite matrices satisfy $0 < mI \leq A \leq B \leq MI$. If ν is real number with $0 \leq \nu \leq 1$, then*

$$A\nabla_\nu B - A!_\nu B \leq \nu(1-\nu)\left(1 - \frac{M}{m}\right)^2 B.$$

Also we state and prove a generalization of these result for weighted power mean of operators.

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On the stability of Szász-Mirakjan operators

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Abstract

A linear operator T from normed space A into normed space B is said to be HU-stable if there exists a constant K such that for any $g \in T(A)$, $\epsilon > 0$ and $f \in A$ with $\|Tf - g\| \leq \epsilon$, there exists an $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\epsilon$. We present the modified Szász-Mirakjan operators and prove that this operators are HU-unstable.

Keywords: Hyers–Ulam stability, approximation, Szász-Mirakjan operators
Mathematics Subject Classification [2010]: 39B82, 41A35

1 Introduction

The Hyers-Ulam stability of linear operators was considered for the first time in the papers by Miura and Takahasi et al. (see [1, 2]).

Definition 1.1. Let A and B be normed spaces and $T : A \to B$ be a linear operator. We say that T is HU-stable if there exists a constant K such that for any $g \in T(A)$, $\epsilon > 0$ and $f \in A$ with $\|Tf - g\| \leq \epsilon$, there exists an $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\epsilon$ [5]. The number K is called a HUS constant of T and the infimum of all HUS constants of T is denoted by K_T.

Theorem 1.2. *[5] Let A and B be Banach spaces and $T : A \to B$ be a bounded linear operator. Then the following statements are equivalent:*

1. *T is HU-stable;*
2. *The range $R(T)$ of T is closed in B;*
3. *The linear operator $\widetilde{T}^{-1}$ from $R(T)$ onto the quotient space $\frac{A}{N(T)}$ is bounded, where $N(T)$ is the kernel of T and $\widetilde{T} : \frac{A}{N(T)} \to B$ is defined by*

$$\widetilde{T}(f + N(T)) = T(f) \quad (f \in A).$$

Moreover, if one of the conditions (1),(2),(3) is satisfied, then $K_T = \|\widetilde{T}^{-1}\|$.

*Speaker

Remark 1.3. It is easy to see that a bounded linear operator $T : A \to B$ is HU-stable if and only if there exists a constant K such that for any $f \in A$ with $\|Tf\| \leq 1$ there exists an $f_0 \in N(T)$ such that $\|f - f_0\| \leq K$.

Popa and Rasa obtained some important results on the HU-stability of some classical operators from approximation theory in [3]. In particular, it is shown that the Szász-Mirakjan operators are HU-unstable. In this talk, we present the modified Szász-Mirakjan operators $S_{n;r}$ and prove that this operators are HU-unstable.

2 Main result

Let $C_b[0, \infty)$ be the space of all continuous, bounded, real-valued functions on $[0, \infty)$. This space with the supremum norm is a Banach space. The nth Szász-Mirakjan operator $L_n : C_b[0, \infty) \to C_b[0, \infty)$ is defined by

$$S_n(f; x) = e^{-nx} \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right) \frac{n^i}{i!} x^i \qquad x \in [0, \infty). \tag{1}$$

In [4], it is introduced the following modified Szász-Mirakjan operators

$$S_{n;r}(f; x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} f\left(\frac{rk}{n}\right) \frac{(nx)^{rk}}{(rk)!} \qquad x \in [0, \infty), \tag{2}$$

for every $f \in C_b[0, \infty)$ and every fixed $r \in \mathbb{N}$, where

$$A_r(t) := \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} \qquad (t \in [0, \infty)).$$

clearly, $S_{n;1}(f; x) = S_n(f; x)$. Now we prove that the modified Szász-Mirakjan operators are HU-unstable.

Lemma 2.1. *[4] For every fixed $r \in \mathbb{N}$, there exists a positive constant M such that*

$$1 \leq \frac{e^{nx}}{A_r(nx)} \leq M \qquad (x \in [0, \infty), n \in \mathbb{N}).$$

Theorem 2.2. *For each $n \in \mathbb{N}$ and $r \in \mathbb{N}$ the operator $S_{n;r}$ is HU-unstable.*

Proof. Suppose that there exist $n \in \mathbb{N}$ and $r \in \mathbb{N}$ such that $S_{n;r}$ is HU-stable. Then there exists a constant K such that for any $f \in C_b[0, \infty)$ with $\|S_{n;r} f\| \leq 1$ there exists a $g \in N(S_{n;r})$ such that $\|f - g\| \leq K$. By Lemma 2.1, there exists $M > 0$ such that

$$\frac{1}{A_r(nx)} \leq \frac{M}{e^{nx}} \qquad (x \in [0, \infty)).$$

By Stirling's formula, we have $\lim_{i\to\infty} \frac{i^i}{i! e^i} = 0$. Hence there exists $j \in \mathbb{N}$ such that $\frac{j}{r} \in \mathbb{N}$ and $M(K + 1) \frac{j^j}{e^j j!} \leq 1$. Define the function f by $f(x) = 0$ for $x \in [0, \frac{j-r}{n}] \cup [\frac{j+r}{n}, \infty)$;

$f\left(\frac{j}{n}\right) = K+1$; f linear on $[\frac{j-r}{n}, \frac{j}{n}]$ and on $[\frac{j}{n}, \frac{j+r}{n}]$. Then it is proved that $\|S_{n;r}f\| \leq 1$. Hence there exists a $g \in N(S_{n;r})$ such that $\|f-g\| \leq K$. Now $g(\frac{j}{n}) = 0$ and so we have

$$K+1 = \left|f\left(\frac{j}{n}\right) - g\left(\frac{j}{n}\right)\right| \leq \|f-g\| \leq K$$

which is a contradiction.

□

Corollary 2.3. *For each $n \in \mathbb{N}$ and $r \in \mathbb{N}$, the range of the operator $S_{n;r}$ is not closed in $C_b[0,\infty)$.*

Proof. By Theorem 2.2, $S_{n;r}$ is HU-unstable. Hence by Theorem 1.2, $R(S_{n;r})$ is not closed. □

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On the Zeros of the Elliptic Operator

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Abstract

In this note we discuss about the problem of existence and uniqueness of local extremum points of multi variable zeros of the elliptic operator defined on an open or compact subset of the Euclidean space. We also obtain some results on the theory of partial differential equations.

Keywords: Boundary behavior, Harmonic, Subharmonic.
Mathematics Subject Classification [2010]: 31A05, 31A20.

1 Preliminaries

Let $U \subseteq R^n$ be an open set, $f : U \to R$ be a map of class C^2 and $\nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$. The function f is called Harmonic if $\nabla^2 f = 0$. For the function ε defined on U, the C^2 function $f : U \to R$ is called $\varepsilon-$Subharmonic if $\nabla^2 f = \varepsilon$. For a non empty set $D \subseteq R^n$ the map $f : D \to R$ is called Harmonic on D if there exists an open set U containing D and a map $g : U \to R$ of class C^2 such that $g_{|D} = f$ and $\nabla^2 g = 0$ on D. The set of all Harmonic (res. $\varepsilon-$Subharmonic) functions on D is denoted by $H(D)$ (res. $S(\varepsilon, D)$). Let $A = [a_{ij}]$ be an $n \times n$ positive definite symmetric matrix and $L = (\frac{\partial}{\partial X})A(\frac{\partial}{\partial X})^t$, then L is called an Elliptic operator and the C^2 function $f : U \to R$ is called $L-$Harmonic (res. $\varepsilon L-$Subharmonic) on U if $L(f) = 0$(res. $L(f) = \varepsilon$). The set of all $L-$Harmonic (res. $\varepsilon L-$Subharmonic) functions on U is denoted by $H(L, U)$ (res. $S(\varepsilon, L, U)$). Similarly if $D \subseteq R^n$ be a nonempty set, then the sets $H(L, D)$ and $S(\varepsilon, L, D)$ defined as above.

2 Introduction

The problem of existence of local extremum points of Holomorphic functions is discussed in [1]. Also the similar problem for two variable Harmonic functions defined on a compact subset of the Euclidean plane is proposed in [2] and [5] as the real part of some Holomorphic functions. Dowling [3] showed that an extension of maximum principle for vector valued harmonic functions defined on the open unit disc to a complex Banach space is hold. A new method for finding the extremum points of smooth functions is discussed in [6, 7]. In this note we generalize the similar results for multi variable generalized Harmonic and Subharmonic functions defined on a compact set $D \subseteq R^n$, i.e., the elements f of $H(D)$, $S(\varepsilon, D)$, $H(L, D)$ and $S(\varepsilon, L, D)$ for which L is an elliptic operator defined on $C^2(R^n, R)$. Then we deduce some uniqueness theorems on the theory of Boundary Value Problem $LT = \varepsilon$.

*Speaker

3 Main results

Theorem 3.1. *Let $\varepsilon \in R$ be a constant function. Then $H(U) \approx S(\varepsilon, U)$.*

Proof. Let $\varphi : H(U) \to S(\varepsilon, U)$ defined by $\varphi(f) = f + h$ in which $h : R^n \to R$ is the map $h(x_1, ..., x_n) = \frac{\varepsilon}{2n} \sum_{i=1}^{n} x_i^2$. Then φ is well defined, one to one and surjective. □

Theorem 3.2. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ and $\varepsilon > 0$ (res. $\varepsilon < 0$) on D. Then every $f \in S(\varepsilon, D)$ has not a local maximum (res. minimum) on U.*

Proof. Let $x_0 \in U$ be the local maximum point of f. Then $\frac{\partial f}{\partial x_i}(x_0) = 0$ and $\frac{\partial^2 f}{\partial x_i^2}(x_0) \leq 0$ for all $i = 1, ..., n$. So $\nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x_0) = \varepsilon \leq 0$ which is a contradiction. □

Corollary 3.3. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ be a bounded set and $\varepsilon > 0$ (res. $\varepsilon < 0$). Then every $f \in S(\varepsilon, D)$ has an absolute maximum (res. minimum) point on ∂U.*

Theorem 3.4. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ be a bounded set. If $f \in H(D)$ has an absolute maximum (res. minimum) point on U, then it has an absolute maximum (res. minimum) point on ∂U with the same value.*

Proof. Let $x_0 \in U$ be the absolute maximum point of f and define the sequence of functions $\{f_n\}_{n \in N}$ by $f_n(x) = f(x) + \frac{1}{n} exp \circ p_1(x)$ in which p_1 is the first projection map $p_1 : R^n \to R$. Then

$$\nabla^2 f_n(x) = \frac{exp \circ p_1(x)}{n} > 0$$

Let $x_n \in \partial U$ be the absolute maximum point of f_n, then $f_n(x_n) \geq f_n(x_0)$ and

$$f(x_0) \geq f(x_n) \geq f(x_0) + \frac{1}{n}[exp \circ p_1(x_0) - exp \circ p_1(x_n)]$$

Therefore $p_1(x_0) \leq p_1(x_n)$. Let $lim_{n \to +\infty} x_n = x_\infty \in \partial U$, then

$$\begin{aligned} f(x_0) &\geq lim_{n \to +\infty} f(x_n) \\ &\geq f(x_0) + lim_{n \to +\infty} \frac{1}{n}[exp \circ p_1(x_0) - exp \circ p_1(x_n)] \end{aligned}$$

and so $f(x_0) \geq f(x_\infty) \geq f(x_0)$, therefore $f(x_0) = f(x_\infty)$. □

Lemma 3.5. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ be a bounded set and $\varepsilon : \partial U \to R$ be a continuous map. If $f : D \to R$ be a zero of the Laplace equation $\nabla^2 T = \varepsilon$ on D and there exists a continuous map $\phi : \partial U \to R$ such that $f = \phi$ for all $x \in \partial U$, then f is unique on $U \cup \partial U$.*

Any symmetric positive definite matrix is orthogonally similar to the diagonal matrix $\Lambda = diag(\lambda_1, ..., \lambda_n)$ of its eigenvalues. So there exists an invertible matrix C such that $\Lambda = C^{-1}AC$ and $C^{-1} = C^t$. Let $\frac{\partial}{\partial X}$ be the $1 \times n$ matrix $\frac{\partial}{\partial X} = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$, X be the $1 \times n$ matrix $X = (x_1, ..., x_n)$, and $\frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} = \frac{\partial^2}{\partial x_i \partial x_j}$ symbolically. Define the new matrix

$Y = (y_1, ..., y_n)$ by $Y^t = CX^t$. A simple calculation shows that $\lambda_i > 0$ for all $1 \leq i \leq n$, $A = C\Lambda C^t$ and $\frac{\partial}{\partial X} = \frac{\partial}{\partial Y}C$, therefore

$$L = \frac{\partial}{\partial X}A(\frac{\partial}{\partial X})^t = (\frac{\partial}{\partial Y}C)AC^t\left(\frac{\partial}{\partial Y}\right)^t = \frac{\partial}{\partial Y}\Lambda\left(\frac{\partial}{\partial Y}\right)^t = \Sigma_{i=1}^n \lambda_i \frac{\partial^2}{\partial y_i^2}$$

The following theorems are immediate consequences of the preceding discussion,

Theorem 3.6. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$, L be an elliptic operator and $\varepsilon > 0$ (res. $\varepsilon < 0$) on D. Then every $f \in S(\varepsilon, L, D)$ has not a local maximum (res. minimum) point on U.*

Theorem 3.7. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ be a bounded set, L be an elliptic operator and $\varepsilon > 0$ (res. $\varepsilon < 0$) on D. Then every $f \in S(\varepsilon, L, D)$ has an absolute maximum (res. minimum) on ∂U.*

Theorem 3.8. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ be a bounded set, and L be an elliptic operator. If $f \in H(L, D)$ has an absolute maximum (res. minimum) point on U, then it has an absolute maximum (res. minimum) point on ∂U with the same value.*

4 Application

Theorem 4.1. *Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ be a bounded set, L be an elliptic operator and $w, \phi : \partial U \to R$ are continuous maps. If $f : D \to R$ be a zero of the equation $LT = w$ on D and $f = \phi$ for all $x \in \partial U$, then f is unique on $U \cup \partial U$.*

Theorem 4.2. *There exists a C^2 function ε such that for any non empty open subset $U \subseteq R^n$ and any $n \times n$ positive definite symmetric matrix $A = [a_{ij}]$, and the elliptic operator $L = (\frac{\partial}{\partial X})A(\frac{\partial}{\partial X})^t$, the equation $Lf = \varepsilon$ has a unique zero on $U \cup \partial U$.*

Proof. If $\varepsilon = exp(\Sigma_{i=1}^n x_i)$ and $M = \Sigma_{i=1}^n a_{ii} + 2\Sigma_{i<j} a_{ij}$, then a simple argument shows that $M \neq 0$ and $f = M^{-1}\varepsilon$ is a unique zero of the equation. □

Remark 4.3. Let $U \subseteq R^n$ be an open set, $D = U \cup \partial U$ be a bounded set. Any affine function defined on D takes its extremums on ∂U. This, generalizes the theorem on the existence of best feasible solution in OR [8].

Remark 4.4. An elliptic operator has not a zero in general. Let $U = \{(x, y)||x| < 1, |y| < 1\} - \{(0, 0)\}$ and $\phi(x, y) = \varepsilon(x, y) = -\frac{1}{x^2} - \frac{1}{y^2}$. Then $D = \{(x, y)||x| \leq 1, |y| \leq 1\}$ and the equation $Lf = \varepsilon$ for the operator $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ has not a C^2 zero on $U \cup \partial U$.

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On two types of approximate identities

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Abstract

Let A be a Banach algebra with non-empty character space. We study two types of approximate identities of A depending on its character space and with using of the generalized Fourier algebra and the disc algebra, we give examples which show the difference between these notions.

Keywords: Banach algebra, approximate identity, character space, locally compact group.
Mathematics Subject Classification [2010]: 22D05, 43A15, 43A07

1 preliminaries

Let A be a Banach algebra and let $\Delta(A)$ be the character space of A, that is, the space consisting of all non-zero homomorphisms from A into $\mathbb{C}$. The notion of a bounded approximate identity first arose in Harmonic analysis; see [1, Section 2.9] for a full discussion of approximate identities and its applications.

A net $\{u_\alpha\}$ in A is called a bounded weak approximate identity if there exists a non-negative constant $C < \infty$ such that $||u_\alpha|| < C$ for each α and

$$\lim_\alpha |\phi(au_\alpha) - \phi(a)| = 0,$$

for all $a \in A$ and $\phi \in \Delta(A)$; see [3] for more details. In the case that A is a natural Banach function algebra, a bounded weak approximate identity $\{u_\alpha\}$ is called a bounded pointwise approximate identity; see [2, Definition 2.11]

Let G be a locally compact group. If $1 < p < \infty$, let $A_p(G)$ denote the Figà-Talamanca Herz algebra introduced by A. Figà-Talamanca in the case that G is Abelian and in the general case by C. Herz; see [4]. For each $u \in A_p(G)$ we know that $||u|| \leq ||u||_{A_p(G)}$ where $||u||$ is the norm of u in $C_0(G)$. Also, we know that $\Delta(A_p(G)) = G$, that is, each character of $A_p(G)$ is an evaluation function at some $x \in G$ [4, Theorem 3].

The group G is said to be amenable if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$ where $L_x f(y) = f(x^{-1}y)$ [6, Definition 4.2].

There are many characterizations of amenability of a group G that can be found in the literature. One of these characterizations is the following theorem. Here $C_c^+(G)$ denotes the space of all positive continuous functions from G into $\mathbb{C}$ with compact support.

*Speaker

Theorem 1.1. [6, Theorem 9.6] *Let G be a locally compact group. The group G is amenable if and only if for one $q \in (1,\infty)$ and each $f \in C_c^+(G)$*

$$\begin{aligned} ||f||_1 = ||L_f||_{CV_q(G)} &= \sup\{||L_f(g)|| : g \in L^q(G), ||g||_q \leq 1\} \\ &= \sup\{||f * g|| : g \in L^q(G), ||g||_q \leq 1\}. \end{aligned}$$

It is easy to verify that $C_c(G) * C_c(G) \subseteq C_c(G)$ and for each $n \in \mathbb{N}$ and $\psi \in C_c^+(G)$, $||\psi||_1^n = ||(\psi^*)^n||_1$ which $\psi^*(x) = \Lambda(x^{-1})\overline{\psi(x^{-1})}$ and Λ shows the modular function of group G. If $\phi \in C_c(G) \subseteq M(G)$, the function $F_\phi : A_p(G) \to \mathbb{C}$ defined by

$$F_\phi(u) =< u, \phi > \qquad (u \in A_p(G)),$$

is an element of $A_p(G)^*$. Here $< \cdot, \cdot >$ denotes the pairing between $M(G)$ and $C_0(G)$. In view of [6, Proposition 10.3] we have $||F_\phi|| = ||L_\phi||_{CV_q(G)}\cdot$

It is easy to see that for each $\psi \in C_c^+(G)$, $n \in \mathbb{N}$ and $p \in (1,\infty)$,

$$||L_{(\psi^*)^n}||_{CV_p(G)} \leq ||L_\psi||^n_{CV_p(G)}\cdot$$

2 Main Results

Let A be a Banach algebra with $\Delta(A) \neq \emptyset$. Recall that for each $a \in A$, $\widehat{a}$, denotes the Gel'fand transform of a and $\mathcal{K}(\Delta(A))$ denotes the collection of all compact subset of $\Delta(A)$.

Definition 2.1. A *cw-approximate identity* for A is a net $\{e_\alpha\}$ in A such that for each $a \in A$ and $K \in \mathcal{K}(\Delta(A))$

$$\lim_\alpha ||\widehat{ae_\alpha} - \widehat{a}||_K = \lim_\alpha \sup_{\phi\in K} |\phi(ae_\alpha) - \phi(a)| = 0\cdot$$

If the net $\{e_\alpha\}$ is bounded, we say that it is a bounded cw-approximate identity (b.cw-a.i) for Banach algebra A.

Definition 2.2. A net (a_λ) in A is a *weakly bounded cw-approximate identity* (w.b.cw-a.i) if the net $(\widehat{a_\lambda})$ is a b.a.i for the topological algebra $\widehat{A}[\tau_{co}]$, that is, for each $a \in A$ and $K \in \mathcal{K}(\Delta(A))$, $\lim_\alpha ||\widehat{ae_\alpha} - \widehat{a}||_K = 0$ and there exists a constant $M > 0$ such that

$$P_K(\widehat{a_\lambda}) = \sup\{|\phi(a_\lambda)| : \phi \in K\} < M, \qquad \text{for all } \lambda \text{ and } K \in \mathcal{K}(\Delta(A)).$$

Since for each $\phi \in \Delta(A)$, $||\phi|| \leq 1$, it is a routine calculation that each b.cw-a.i is a w.b.cw-a.i. But we will show in the sequel that this two concepts are different.

A classical theorem due to Leptin and Herz, characterize the amenability of a group G through the existence of a bounded approximate identity for the Figà-Talamanca-Herz algebra.

Now, we give the following theorem that is a variant of Leptin-Herz theorem.

Theorem 2.3. *Let G be a locally compact group and $1 < p < \infty$. Then the following are equivalent.*

1. *G is an amenable group,*

2. $A_p(G)$ has a b.cw-a.i,

Proof. Let (2) holds and let $\{u_i\}$ be a b.cw-a.i for $A_p(G)$ bounded by C. Suppose that q is the conjugate exponent of p, that is, $1/p + 1/q = 1$. By Theorem 1.1, it is enough to show that for q and each $\psi \in C_c^+(G)$, $||L_\psi||_{CV^q} = ||\psi||_1$.

If K is any compact subset of G, we choose V an arbitrary compact neighborhood of G containing the identity of G and put $f = |V|^{-1}\chi_V * \chi_{V^{-1}K}$.

A routine verification shows that if $x \in K$, $f(x) = 1$ and otherwise $f(x) = 0$. Since $\{u_i\}$ is a b.cw-a.i and $f \in A_p(G)$ for $K \subseteq G = \Delta(A_p(G))$ we have

$$||\widehat{u_i f} - \widehat{f}||_K = \sup_{t\in K} |u_i(t)f(t) - f(t)| = \sup_{t\in K} |u_i(t) - 1| \to 0.$$

Hence, for $\epsilon > 0$, there exists i_0 such that $\sup_{t\in K} |\mathrm{Re}(u_{i_0}(t)) - 1| < \epsilon$. Therefore, $\inf\{\mathrm{Re}(u_{i_0}(t)) : t \in K\} \geq 1 - \epsilon$.

Let $\phi \in C_c^+(G)$ and $K = \mathrm{supp}(\phi)$. By the discussion after Theorem 1.1, we have

$$| < u_{i_0}, \phi > | = |F_\phi(u_{i_0})| \leq ||L_\phi||_{CV_q(G)} ||u_{i_0}|| \leq C||L_\phi||_{CV_q(G)}.$$

But

$$\mathrm{Re} < u_{i_0}, \phi >= \int_K \mathrm{Re}(u_{i_0}(x))\phi(x)dx \geq (1 - \epsilon)||\phi||_1.$$

Hence, if ϵ tends to 0, we have $||\phi||_1 \leq C||L_\phi||_{CV_q(G)}$.

Let $\psi \in C_c^+(G)$ be arbitrary and $n \in \mathbb{N}$. Thus we have

$$||\psi||_1^n = ||(\psi^*)^n||_1 \leq C||L_{(\psi^*)^n}||_{CV_q(G)} \leq C||L_\psi||^n_{CV_q(G)}.$$

Therefore, $||\psi||_1 \leq ||L_\psi||_{CV_q(G)}$. Hence, $||L_\psi||_{CV_q(G)} = ||\psi||_1$ and this completes the proof. □

Remark 2.4. By using [5, Lemma 4.1], we can give another proof for Theorem 2.3 but the above proof is direct. Indeed, we adopt the proof of [6, Theorem 10.4].

The following example provide for us an example of a Banach algebra with a w.b.cw-a.i such that has no b.cw-a.i.

Example 2.5. Let $1 < p < \infty$ and G be a non-amenable locally compact group. By Theorem 2.3, $A_p(G)$ does not have any b.cw-a.i. Now, we construct a w.b.cw-a.i for $A_p(G)$. Put $\Lambda = \{K \subseteq G : K \text{ is compact and } |K| > 0\}$. It is obvious that Λ with inclusion is a directed set. For each $K \in \Lambda$ define u_K as follows,

$$u_K := |K|^{-1}\chi_{KK} * \check{\chi}_K.$$

Clearly, (u_K) is a net in $A_p(G)$. For each $x \in G$ we have

$$\begin{aligned} u_K(x) = |K|^{-1}\int_G \chi_{KK}(y)\check{\chi}_K(y^{-1}x)dy &= |K|^{-1}\int_{KK} \chi_K(x^{-1}y)dy \\ &= |K|^{-1}\int_{KK} \chi_{xK}(y)dy \\ &= \frac{|KK \bigcap xK|}{|K|}. \end{aligned}$$

If $x \in K$, $KK \bigcap xK = xK$. Therefore, $u_K(x) = 1$ and otherwise since $KK \bigcap xK \subseteq xK$, $0 \leq u_K(x) \leq 1$. Hence, for each compact set $K^{'}$ of G and K of Λ we have,

$$P_{K'}(\widehat{u_K}) = \sup\{|u_K(x)| : x \in K^{'}\} \leq 1.$$

So, $\{\widehat{u_K}\}$ is a bounded net in $\widehat{A_p(G)}[\tau_{co}]$.

Now, let f be an arbitrary element of $A_p(G)$ and $K^{'}$ be a compact subset of G. Since G is a locally compact group, for each $x \in K^{'}$ there exists a compact neighborhood V_x of x. On the other hand, we know that $K^{'} \subseteq \bigcup_{x \in K'} V_x$ and for each x, $|V_x| > 0$. But $K^{'}$ is compact, so there are points $x_1, \ldots, x_n$ in $K^{'}$ such that $K^{'} \subseteq \bigcup_{i=1}^{n} V_{x_i}$. Therefore, by putting $K^{''} = \bigcup_{i=1}^{n} V_{x_i}$ we have an element $K^{''}$ of Λ such that $K^{'} \subseteq K^{''}$.

Now, it is obvious that $\lim_{K \in \Lambda} ||\widehat{u_K f} - \widehat{f}||_{K'} = 0$ and this completes the proof.

It is worth noting that there exists Banach algebras without any w.b.cw-a.i as the following example shows.

Example 2.6. Let $A = A(\mathbb{D})$ be the disc algebra and for $z_0 \in \text{int}\mathbb{D}$, let $B = M_{z_0} = \{f \in A : f(z_0) = 0\}$. Clearly, $\mathbb{D} \setminus \{z_0\} \subseteq \Delta(B)$. So, if B has a w.b.cw-a.i, then B has a bounded pointwise approximate identity which is in contradiction with [2, Example 4.8(i)].

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Talk Orthogonality preserving mappings in inner product C^*-modules pp.: 1–4

Orthogonality preserving mappings in inner product C^*-modules

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Abstract

We investigate orthogonality preserving mappings in the setting of inner product C^*-modules to obtain their general structure. We also give some characterizations of orthogonality preserving mappings between inner product C^*-modules.

Keywords: Orthogonality preserving mapping, inner product C^*-module, local map
Mathematics Subject Classification [2010]: 46L08, 46C05

1 Introduction

The set of all orthogonality preserving bounded linear mappings on Hilbert spaces is fairly easy to describe, and it coincides with the set of all conformal linear mappings there: a linear map T between two Hilbert spaces is orthogonality preserving if and only if T is the scalar multiple of an isometry. As a natural generalization of the described situation one may change the algebra of coefficients to arbitrary C^*-algebras A and the Hilbert spaces to C^*-valued inner product A-modules, the Hilbert C^*-modules. Hilbert C^*-modules are an often used tool in the study of locally compact quantum groups and their representations, in noncommutative geometry, in KK-theory, and in the study of completely positive maps between C^*-algebras, among other research fields. To be more precise, an inner product A-module is a complex linear space E which is a right A-module with a compatible scalar multiplication and equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \longrightarrow A$ satisfying
(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$,
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$,
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$,
(iv) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
for all $x, y, z \in E, a \in A, \alpha, \beta \in \mathbb{C}$.

The mapping $\|.\| : E \longrightarrow \mathbb{R}$ defined by $\|x\| = \|\langle x, x \rangle\|_A^{\frac{1}{2}}$ is a norm on E. If E is complete with respect to this norm, then it is called a Hilbert A-module, or a Hilbert C^*-module over A. Complex Hilbert spaces are Hilbert $\mathbb{C}$-modules. Any C^*-algebra A can be regarded as a Hilbert C^*-module over itself via $\langle a, b \rangle := a^* b$. For every $x \in E$ the positive square root of $\langle x, x \rangle$ is denoted by $|x|$. In the case of a C^*-algebra we get the usual modulus of a, that is $|a| = (a^* a)^{\frac{1}{2}}$. Although the definition of $|x|$ has the same form as that of the norm

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of elements of inner product spaces, there are some significant differences. For instance, it does not satisfy the triangle inequality in general. Note that the theory of inner product C^*-modules is quite different from that of inner product spaces. For example, not any closed submodule of an inner product C^*-module is complemented; a bounded C^*-linear operator on an inner product C^*-module may not have an adjoint operator. We refer the reader to [4] for more information on the basic theory of C^*-algebras and Hilbert C^*-modules.

Orthogonality preserving mappings in the framework of Hilbert C^*-modules have been recently treated in [1, 2, 3, 5]. In the next section we investigate orthogonality preserving mappings in the setting of inner product C^*-modules to obtain their general structure. We also give some characterizations of orthogonality preserving mappings between inner product C^*-modules.

2 Main results

Recall that a linear mapping $T : E \longrightarrow F$, where E and F are inner product A-modules, is said to be orthogonality preserving if $\langle x, y\rangle = 0 \implies \langle Tx, Ty\rangle = 0$ for all $x, y \in E$. Also, T is called A-linear if it is linear and $T(xa) = (Tx)a$ for all $x \in E$, $a \in A$.

Theorem 2.1. *[5] Let E and F be two inner product A-modules. For a nonzero A-linear mapping $T : E \longrightarrow F$ the following statements are equivalent:*

(i) there exists $\gamma > 0$ such that $\|Tx\| = \gamma\|x\|$ for all $x \in E$, i.e., T is a similarity;

(ii) T is injective and $\frac{\langle Tx,Ty\rangle}{\|Tx\|\|Ty\|} = \frac{\langle x,y\rangle}{\|x\|\|y\|}$ for all $x, y \in E \setminus \{0\}$.

Furthermore, each one of the assertions above implies:

(iii) $\langle x, y\rangle = 0 \iff \langle Tx, Ty\rangle = 0$ for all $x, y \in E$, i.e., T is strongly orthogonality preserving;

(iv) $|x| = |y| \iff |Tx| = |Ty|$ for all $x, y \in E$;

(v) $|x| \le |y| \iff |Tx| \le |Ty|$ for all $x, y \in E$.

The following example shows that conditions (iii)-(v) are not equivalent to conditions (i)-(ii) in general.

Example 2.2. [5] Let Ω be a locally compact Hausdorff space. Let us take $E = F = C_0(\Omega)$, the C^*-algebra of all continuous complex-valued functions vanishing at infinity on Ω. For a nonzero function $f_0 \in C_0(\Omega)$, suppose that $T : C_0(\Omega) \longrightarrow C_0(\Omega)$ is given by $T(g) = f_0 g$. Obviously T is $C_0(\Omega)$-linear and satisfies conditions (iii)-(v) but need not satisfies conditions (i)-(ii). Indeed, if there exists $\gamma > 0$ such that $\|T(g)\| = \gamma\|g\|$ for all $g \in C_0(\Omega)$, then $\frac{1}{\gamma^2}\overline{f_0} f_0 h = h$ for all $h \in C_0(\Omega)$ and hence, $\frac{1}{\gamma^2}\overline{f_0} f_0$ is the identity in $C_0(\Omega)$, which is a contradiction.

Recall that a linear mapping $T : E \longrightarrow F$, where E and F are inner product A-modules, is called local if

$$xa = 0 \implies (Tx)a = 0 \qquad (a \in A, x \in E).$$

Examples of local mappings include multiplication and differential operators. Note that every A-linear mapping is local, but the converse is not true, in general (take linear differential operators into account). Moreover, every bounded local mapping between inner product modules is A-linear.

Theorem 2.3. *[2] Let E and F be two inner product A-modules such that $\mathbb{K}(H) \subseteq A \subseteq \mathbb{B}(H)$. Suppose that $T : E \longrightarrow F$ is a nonzero orthogonality preserving A-linear map. Then there exists a positive number γ such that*

$$\langle Tx, Ty \rangle = \gamma \langle x, y \rangle \tag{1}$$

for all $x, y \in E$.

Note that the assumption of A-linearity, even in the case $A = \mathbb{K}(H)$, is necessary in Theorem 2.3 as can be seen from the following example.

Example 2.4. [5] Let H be a Hilbert space such that $\dim H = \infty$ and $H_* = H$ as an additive group, but define a new scalar multiplication on H_* by setting $\lambda \cdot x = \overline{\lambda} x$, and a new inner product by setting $\langle x|y \rangle_* = \langle y|x \rangle$. Then H_* equipped with the operations

$$\langle x, y \rangle := x \otimes y \qquad \text{and} \qquad x \cdot S := S^* x \quad (x, y \in H_*, S \in \mathbb{K}(H))$$

is an inner product $\mathbb{K}(H)$-module. If $T : H_* \longrightarrow H_*$ is any unbounded linear map, then T preserves orthogonality (namely, if $\langle x, y \rangle = x \otimes y = 0$, then $x = 0$ or $y = 0$. So $\langle Tx, Ty \rangle = Tx \otimes Ty = 0$), but T obviously does not satisfy (1).

Theorem 2.5. *[5] Let E and F be two inner product A-modules such that $\mathbb{K}(H) \subseteq A \subseteq \mathbb{B}(H)$. Suppose that $T : E \longrightarrow F$ is a local and nonzero orthogonality preserving map. Then*

(i) $|x| = |y| \iff |Tx| = |Ty|$ *for all $x, y \in E$;*

(ii) $|x| \leq |y| \iff |Tx| \leq |Ty|$ *for all $x, y \in E$.*

Corollary 2.6. *[5] Let E and F be two inner product A-modules and $\mathbb{K}(H) \subseteq A \subseteq \mathbb{B}(H)$. Suppose that $T : E \longrightarrow F$ is a nonzero A-linear mapping between inner product A-modules. Then T is orthogonality preserving if and only if*

$$|x| \leq |y| \implies |Tx| \leq |Ty|$$

for all $x, y \in E$.

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Periodic point for the generalized (ψ, ϕ)-contractive mapping in right complete generalized quasimetric spaces

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Abstract

In this paper, we introduce concept of generalized (ψ, ϕ)-contractive mappings of type I and II for generalized quasimetric spaces. We show that if f is a (ψ, ϕ)-contractive map of type I or II, then f has a periodic point.

Keywords: Contractive mapping, generalized quasimetric spaces, periodic point.
Mathematics Subject Classification [2010]: 47H10, 47H09

1 Introduction

The concept of metric space represented in 1906 by Frechet [4]. The metric space and its generalizations are important in many branches of mathematics, particular fixed point theory. This theory is one of the old theory in mathematics that it has wide range of applications. Banach contraction principle creates simple and suitable conditions to guarantee existance and uniquencess of solution of operator equation $Tx = x$. This principle is the most essential theorem of classical functional analysis. Over the past few decades, with the change in contraction's condition or change in the definition of the metric space and or both, the generalization of this theorem is obtained [1, 5]. For example, Branciari [1] has introduced the concept of generalized metric by replacing the triangle inequality to overall inequality is called a quadrilateral inequality. Branciari proved the fixed point theorem in this space and claimed that a generalized metric is a continuous function, generalized metric space is Hausdorff and any convergent sequance is Cauchy sequence in generalized metric space. Sarma and et al. [8] and Samet [7] provide an example showed that some features claimed by Branciari are not true, especially Hausdorffness. Note that in the proof of uniquensess of the fixed point, the condition is necessary Hausdorff space. Despite the weakness in generalized metric space, several authors have been proposed some of techinques to ablian a unique fixed point [2, 3].

Recently, quasimetric space have been one of intersting issues for the researchers in the field of fixed point theory, because the assumption of quasimetric are weaker than the standard metric, thus fixed point results obtained in this space is very public. So it also covers the corresponding results in the metric space. Very recently Lin and et al. [6] introduced the concept of generalized quasimetric space and examine the existence of determined operator on such space.

*Speaker

In this paper supposed that the generalized quasimetric space is Hausdorff and obtain some periodic point theorems on generalized (ψ,ϕ)-contractive mappings on generalized quasimetric spaces.

Definition 1.1. Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a mapping. Then d is called a metric on X and (X,d) is a metric space if for every $x,y,z \in X$, it satisfies
(1) $d(x,y) = d(y,x) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$.
d is called a quasimetric on X and (X,d) is a quasimetric space, if conditions (1) and (3) hold. d is called a generalized metric on X and (X,d) is a generalized metric space if conditions (1) and (2) hold and for every $x,y \in X$ and every distinict $u,v \in X$ each of which is different from x,y
(4) $d(x,z) \leq d(x,y) + d(u,v) + d(v,z)$.
Finally, d is called generalized quasimetric and (X,d) is a generalized quasimetric space if conditions (1) and (4) hold.

Definition 1.2. Let (X,d) be a generalized quasimetric space, $\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ is called generalized quasimetric convergent to x if and only if

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

(ii) $\{x_n\}$ is called right Cauchy if for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n,x_m) < \epsilon$ for all $mn > k$.

Definition 1.3. Generalized quasimetric space (X,d) is called right complete if each right Cauchy sequence in X is convergent.

In the following, let Ψ,Φ be the family of continuous and nondecresing functions $\psi,\phi : [0,\infty] \to [0,\infty]$ such that
(i) $\psi(t) = 0$ if and only if $t = 0$.
(ii) $\phi(t) = 0$ if and only if $t = 0$.
Now, let (X,d) be a generalized quasimetric space, $f : X \to X$ be a self mapping, $\psi \in \Psi$ and $\phi \in \Phi$. Then
(i) f is called a (ψ,ϕ)-contractive mapping

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

for all $x,y \in X$.
(ii) f is called a (ψ,ϕ)-contractive mapping type of I if

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y))$$

for all $x,y \in X$, where

$$M(x,y) = max\{d(x,y), d(x,fx), d(y,fy)\}.$$

(iii) f is called a (ψ, ϕ)-contractive mapping type of II if

$$\psi(d(Tx, Ty)) \leq \psi(N(x, y)) - \phi(N(x, y))$$

for all $x, y \in X$, where

$$N(x, y) = max\{d(x, y), \frac{d(x, fx) + d(y, fy)}{2}\}.$$

2 Main results

We commence this section with the main result of the paper.

Theorem 2.1. *Let (X, d) be a right complete generalized quasimetric space and let $f : X \to X$ be a continuous (ψ, ϕ)-contractive mapping of type I. Then f has periodic point.*

Theorem 2.2. *Let (X, d) be a right complete generalized quasimetric space and let $f : X \to X$ be a continuous (ψ, ϕ)-contractive mapping of type II. Then f has periodic point.*

Denote by Λ the set of functions $\alpha : [0, \infty] \to [0, \infty]$ satisfying following hypothesis:
(i) α is a Lebesgue integrable mapping on each compact subset of $[0, \infty]$;
(ii) for every $\epsilon > 0$, we have $\int_0^\epsilon \alpha(s)ds > 0$.
In the following, let $P(x, y)$ be either $M(x, y)$ or $N(x, y)$.

Theorem 2.3. *Let (X, d) be a Hausdorff and right complete generalized quasimetric space and let $f : X \to X$ be a continuous self-mapping satisfying*

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq \int_0^{P(x,y)} \alpha(s)ds - \int_0^{P(x,y)} \beta(s)ds.$$

for all $x, y \in X$, where $\alpha, \beta \in \Lambda$ Then f has a periodic point.

Taking $\beta(s) = (1-k)\alpha(s)$ for $k \in [0, 1)$ in Theorem 2.1, we obtain the following result.

Corollary 2.4. *Let (X, d) be a Hausdorff and right complete generalized quasimetric space and let $T : X \to X$ be a continuous self - mapping satisfying*

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq k \int_0^{P(x,y)} \alpha(s)ds.$$

for all $x, y \in X$, where $\alpha \in \Lambda$ and $k \in [0, 1)$. Then f has a periodic point.

Taking $\alpha \equiv 1$ in pervious corollary, we obtain the following result.

Corollary 2.5. *Let (X, d) be a Hausdorff and right-complete generalized quasimetric space and let $T : X \to X$ be a continuous self-mapping satisfying*

$$d(Tx, Ty) \leq kP(x, y).$$

for all $x, y \in X$, where $\alpha \in \Lambda$ and $k \in [0, 1)$. Then f has a periodic point.

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φ-means of some Banach subspaces on a Banach algebra

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Abstract

In this paper, among the other things, we study the concept of φ-amenability of a Banach algebra A, where φ is a nonzero multiplicative linear functional on A. We present a few results in the theory of φ-amenable Banach algebras, and we obtain necessary and sufficient conditions for A^{**} to have a left invariant φ-mean on Banach subspaces of A^*. The candidates for the choice of space are A_*, $WAP(A)$ and $S(G)$.

Keywords: Banach algebra, φ-amenability, φ-means, weak* topology.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In [3], Lau introduced and investigated a large class of Banach algebras which he called F-algebras. Later, F-algebras were termed Lau algebras. They are Banach algebras A such that the dual A^* is a von Neumann algebra and the identity of A^* is a multiplicative linear functional on A. The concept of left amenability for a Lau algebra has been extensively extended for an arbitrary Banach algebra by introducing the notion of φ-amenability (see [2]). Let A be an arbitrary Banach algebra and φ a character of A, that is a homomorphism from A onto $\mathbb{C}$. A is called φ-amenable if there exists a bounded linear functional m on A^* satisfying $\langle m, \varphi\rangle = 1$ and $\langle m, f.a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in A$ and $f \in A^*$. This concept considerably generalizes the notion of left amenability for Lau algebras.

The main purpose of this paper is to investigate the φ-amenability for certain Banach subspaces of dual Banach algebras. We continue [1] in the study of amenability of a Banach algebra A defined with respect to a character φ of A. Various necessary and sufficient conditions are found for a Banach algebra to possess a left invariant φ-mean. Throughout the paper, $\Delta(A)$ will denote the set of all homomorphisms from A onto $\mathbb{C}$.

We prove that A^{**} has a left invariant φ-mean on A_* if and only if for every normal φ-bimodule E, every bounded weak*-continuous derivation $D : A \to E$ is inner. Other results in this direction are also obtained. Our second purpose in this paper is to present several characterizations of the existence of a left (right) invariant φ-mean on $Wap(A)$. Finally we obtain sufficient conditions and some necessary conditions about $S(G)$ to have a left invariant 1-mean.

*Speaker

2 Main results

Let A be a dual Banach algebra with predual A_*, and let $\varphi \in \Delta(A) \cap A_*$. The Banach A-bimodules E that are relevant to us are those where the right action is of the form $x.a = \varphi(a)x$. For sake of brevity, such E will occasionally be called Banach φ-bimodule. A dual φ-bimodule E is called normal φ-bimodule if for each $x \in E$, the map $a \mapsto a.x$ is weak*-continuous. Note that φ is taken to be in a closed submodule A_* of A^*. For an element φ in A^*, the map $a \mapsto x.a = \varphi(a)x$ is in general not weak* continuous unless φ_* and E is not normal.

Definition 2.1. Let A be a Banach algebra and let X be a closed subspace of A^* with $\varphi \in X$ that is invariant. A continuous functional m on X is called left invariant φ-mean on X if the following properties holds:

$$\langle m, \varphi \rangle = 1, \quad \langle m, f.a \rangle = \varphi(a)\langle m, f \rangle \quad (f \in X,\ a \in A)$$

Theorem 2.2. *Let A be a dual Banach algebra with predual A_*, and let $\varphi \in \Delta(A) \cap A_*$. Then A^{**} has a left invariant φ-mean on A_* if and only if for every normal φ-bimodule E, every bounded weak*-continuous derivation $D : A \to E$ is inner.*

Theorem 2.3. *Let A be a dual Banach algebra with predual A_*, and let $\varphi \in \Delta(A) \cap A_*$. Let A has a bounded approximate identity. Then A^{**} has a left invariant φ-mean on A_* if and only if A is φ-amenable.*

Let A and B be commutative dual Banach algebras and let $f \in \mathcal{A}^*$ and $g \in B^*$. Let $f \otimes g$ denote the element of $(A\hat{\otimes}B)^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ for all a and b. Note that with this notation

$$\Delta(A\hat{\otimes}B) = \{\varphi \otimes \psi;\ \varphi \in \Delta(A),\ \psi \in \Delta(B)\}.$$

We use $\otimes_w$ to denote the injective tensor product of two Banach spaces and $\hat{\otimes}$ to denote the projective tensor product of two dual Banach algebras.

Theorem 2.4. *Let A_* and B_* be the preduals of commutative dual Banach algebras A and B, respectively. Let $\varphi \in \Delta(A) \cap A_*$ and $\psi \in \Delta(B) \cap B_*$. If $A\hat{\otimes}B$ is a dual Banach algebra with predual $A_* \otimes_w B_*$, then $(A\hat{\otimes}B)^{**}$ has a left invariant $(\varphi \otimes \psi)$-mean on $A_* \otimes_w B_*$ if and only if A^{**} has a left invariant φ-mean mean on A_* and B^{**} has a left invariant ψ-mean on B_**

We write A^*A for the closed linear span in A^* of $\{f.a;\ f \in A^*,\ a \in A\}$. When A has a bounded right approximate identity the Cohen-Hewitt factorization theorem shows that in fact $A^*A = \{f.a;\ f \in A^*,\ a \in A\}$.

Theorem 2.5. *Let A be a Banach algebra and let $\varphi \in \Delta(A)$. Suppose that A has a bounded approximate identity. Then A is φ-amenable if and only if $(A^*A)^*$ has a left invariant φ-mean.*

A special interesting case is that there exists a left invariant φ-mean on $WAP(A)$. A functional $f \in A^*$ for which $\{f.a;\ \|a\| \leq 1\}$ is relatively compact in the weak topology of A^* is said to be weakly almost periodic. The set of weakly almost periodic functionals on A is denoted by $WAP(A)$. We put

$$\|a\|_{WAP(A)} = \sup\{|\langle f, a \rangle| : f \in WAP(A), \|f\| \leq 1\} \quad (a \in A)$$

Theorem 2.6. *Let A be a Banach algebra with a bounded approximate identity and $\varphi \in \Delta(A)$. Then the following statements are equivalent:*

(i) There exists a left invariant φ-mean on $WAP(A)$;

(ii) There exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\|_{WAP(A)} \to 0$ for each $a \in A$;

(iii) There exists a bounded net $\{a_\alpha\}_{\alpha\in I}$ in $\{a \in A;\ \varphi(a) = 1\}$ such that for each weakly compact subset $C \subseteq A$, $\|aa_\alpha - \varphi(a)a_\alpha\|_{WAP(A)} \to 0$ uniformly for all $a \in C$.

If $\mathcal{S}$ is a semigroup of operators on $WAP(A)$, the orbit $O(f)$ of an element f of $WAP(A)$ is defined to be $\{T(f);\ T \in \mathcal{S}\}$. $\mathcal{S}$ will be called weakly almost periodic if each orbit has compact closure in the weak topology of $WAP(A)$.

We say that an element a of A is φ-maximal if it satisfies $\|a\| = \varphi(a) = 1$. Let $P_1(A,\varphi)$ denote the collection of all φ-maximal elements of A. Let $X(A,\varphi)$ denote the closed linear span of $P_1(A,\varphi)$. If $f \in A^*$ and $a \in A$, we also consider $\lambda_a(f) = f.a$.

Theorem 2.7. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. The closure $\overline{\mathcal{S}}$ of $\mathcal{S} = \{\lambda_a;\ a \in P_1(A,\varphi)\}$ in the weak operator topology is a compact convex semitopological semigroup in the same topology. Moreover, among the following two properties, the implication (i) $\to$ (ii) hold. If $X(A,\varphi) = A$, then (ii) $\to$ (i).*

(i) $WAP(A)^*$ has a left invariant φ-mean $m \in \overline{P_1(A,\varphi)}^{w^*}$;

(ii) The semigroup $\mathcal{S}$ has a left zero, that is, there exists some $S \in \mathcal{S}$ such that $SoT = S$ for any $T \in \mathcal{S}$.

A linear functional $m \in WAP(A)^*$ is called a right invariant φ-mean on $WAP(A)$ if $\langle m,\varphi\rangle = 1$ and $\langle m, af\rangle = \varphi(a)\langle m, f\rangle$ whenever $f \in WAP(A)$ and $a \in A$. A left invariant and right invariant φ-mean on $WAP(A)$ is called invariant φ-mean.

Theorem 2.8. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. If $WAP(A)^*$ has a left invariant φ-mean $m \in \overline{P_1(A,\varphi)}^{w^*}$ and a right invariant φ-mean $n \in \overline{P_1(A,\varphi)}^{w^*}$, then the compact semitopological semigroup $\overline{\mathcal{S}}$ contains a left zero and a right zero. Moreover, $m = n$ and it is the unique invariant φ-mean on $WAP(A)$.*

Recall that a Segal algebra $S(G)$ on a locally compact group G, is a dense left ideal of $L^1(G)$ that satisfies the following conditions:

(i) $S(G)$ is a Banach space with respect to a norm $\|.\|_S$, called a Segal norm, satisfying $\|\psi\|_1 \le \|\psi\|_S$ for $\psi \in S(G)$, where $\|.\|_1$ denotes the L^1-norm.
(ii) For $\psi \in S(G)$ and $y \in G$, $L_y\psi \in S(G)$, where L_y is the left translation operator defined by $L_y\psi(x) = \psi(y^{-1}x)$, $x \in G$. Moreover, the left translation $L_y\psi$, $y \in G$, is continuous in y for each $\psi \in S(G)$.
(iii) The equality $\|L_y\psi\|_S = \|\psi\|_S$ holds for $\psi \in S(G)$, $y \in G$.

Equipped with the norm $\|.\|_S$ and the convolution product, denoted by $*$, $S(G)$ is a Banach algebra. The inequality $\|h * \psi\|_S \le \|h\|_1\|\psi\|_S$ holds for all $h \in L^1(G)$, and $\psi \in S(G)$.

In the following theorem, we obtain necessary and sufficient conditions for $S(G)$ to have a left invariant 1-mean. $P_1((S(G), \|.\|_1), 1)$ denotes the collection of all 1-maximal elements of a Segal algebra $S(G)$ with respect to L^1-norm.

Theorem 2.9. *Let G be a locally compact group. Then the following statements are equivalent:*

(i) There is a left invariant 1*-mean* $m \in \overline{P_1((S(G), \|.\|_1), 1)}^{w^*}$ *on* $P_1((S(G), \|.\|_1), 1)$.

(ii) There is a net $\psi_\alpha \in P_1((S(G), \|.\|_1), 1)$ *such that* $\|\psi * \psi_\alpha - \psi_\alpha\|_S \to 0$ *for each* $\psi \in P_1((S(G), \|.\|_1), 1)$.

(iii) There is a net $\psi_\alpha \in P_1((S(G), \|.\|_1), 1)$ *such that for each weakly compact subset* $C \subseteq P_1((S(G), \|.\|_1), 1)$, $\|\psi * \psi_\alpha - \psi_\alpha\|_S \to 0$ *uniformly for all* $\psi \in C$.

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PPF dependent fixed point results for α_c-admissible integral type mappings in Banach spaces

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Abstract

In this paper, we prove some new PPF dependent fixed point theorems in the Razumikhin class for some integral type mappings involving α_c-admissible mappings where the domain and range of the mappings are not the same. Our results extend and generalize some results in the literature.

Keywords: Fixed point, Complete metric space, PPF dependent fixed point, α_c-admissible mapping, integral typ mapping, Banach space.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Fixed point theory plays an important role in Banach spaces. In 1997, Bernfeld *et al.* [2] introduced the concept of PPF dependent fixed point. They also proved the existence of PPF dependent fixed point in the Razumikhin class for Banach type contraction mappings. Very recently, some authors established the existence and uniqueness of PPF dependent fixed point for different types of contractive mappings and generalized some results of Bernfeld *et al.* [2].

During last four decades, the Banach contraction principle has been widely generalized and extended. In 2002, Branciari [7], proved the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space, $c \in (0, 1)$, and let $f : X \to X$ be a mapping such that for each $x, y \in X$*

$$\int_0^{d(fx,fy)} \Phi(t)dt \le c \int_0^{d(x,y)} \Phi(t)dt$$

where $\Phi : [0, 1) \to [0, 1)$ is a nonnegative Lebesgue-integrable map which is summable, (i.e., with finite integral) on each compact subset of $[0, \infty)$, and for each $\epsilon > 0$, $\int_0^{\epsilon} \Phi(t)dt > 0$. Then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

*Speaker

Throughout this paper, let $(E, \|\cdot\|_E)$ be a Banach space, I denotes a closed interval $[a, b]$ in $\mathbb{R}$ and $E_0 = C(I, E)$ denotes the set of all continuous E-valued functions on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t \in I} \|\phi(t)\|_E.$$

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in E_0 is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \ \|\phi\|_{E_0} = \|\phi(c)\|_E\}.$$

Clearly, every constant function from I to E is a member of $\mathcal{R}_c$. It is easy to see that the class $\mathcal{R}_c$ is algebraically closed with respect to difference, *i.e.*, $\phi - \xi \in \mathcal{R}_c$ when $\phi, \xi \in \mathcal{R}_c$. Also the class $\mathcal{R}_c$ is topologically closed if it is closed with respect to the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Definition 1.2. [2] A mapping $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

Definition 1.3. [2] The mapping $T : E_0 \to E$ is called a Banach type contraction if there exists $k \in [0, 1)$ such that,

$$\|T\phi - T\xi\|_E \leq k\|\phi - \xi\|_{E_0}$$

for all $\phi, \xi \in E_0$.

The concept of α_c-admissible mapping was introduced by Agarwal [1] in 2013.

Definition 1.4. Let $c \in I$, $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$. We say that T is an α_c-admissible mapping if for all $\phi, \xi \in E_0$

$$\alpha(\phi(c), \xi(c)) \geq 1 \Longrightarrow \alpha(T\phi, T\xi) \geq 1. \tag{1}$$

Definition 1.5. [4] Let $c \in I$, $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$. We say that T is a triangular α_c-admissible mapping if

(T1) $\alpha(\phi(c), \xi(c)) \geq 1$ implies $\alpha(T\phi, T\xi) \geq 1$,

(T2) $\alpha(\phi(c), \mu(c)) \geq 1$ and $\alpha(\mu(c), \xi(c)) \geq 1$ implies $\alpha(\phi(c), \xi(c)) \geq 1$,

for $\phi, \xi, \mu \in E_0$.

Let Φ be the collection of all mappings $\Phi : [0, 1) \to [0, 1)$ which are Lebesgue-integrable, summable on each compact subset of $[0, 1)$ and satisfying the following condition:

$$\int_0^{\epsilon} \Phi(t)dt > 0 \ \ for\ each \ \ \epsilon > 0.$$

2 Main results

Let $\mathcal{F}$ denotes the class of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to +\infty. \tag{2}$$

β is called a Geraghty [5] mapping.

Definition 2.1. Let $T : E_0 \to E$ be a nonself-mapping and $\alpha : E \times E \to [0, \infty)$ be a function. We say that T is a integral type rational Geraghty contraction if there exists $\beta \in \mathcal{F}$ and $c \in I$ such that,

$$\int_0^{\alpha(\phi(c),T\phi)\alpha(\xi(c),T\xi)\|T\phi-T\xi\|_E} \Phi(t)dt \leq \int_0^{\beta(M(\phi(c),\xi(c)))M(\phi(c),\xi(c))} \Phi(t)dt$$

for all $\phi, \xi \in E_0$, where

$$M(\phi(c), \xi(c)) = \max\left\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}.$$

Theorem 2.2. *Let $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following assertions:*

(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference,

(b) T is α_c-admissible,

(c) T is a integral type rational Geraghty contractive mapping,

(d) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq 1$, then $\alpha(\phi(c), T\phi) \geq 1$ for all $n \in \mathbb{N}$,

(e) There exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$.

Then, T has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, the sequence $\{\phi_n\}$ of iterates of T defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, converges to $\phi^* \in \mathcal{R}_c$.

Let Ψ be the family of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for all $t > 0$.

As an example $\psi_1(t) = kt$ for all $t \geq 0$, where $k \in [0, 1)$ and $\psi_2(t) = \ln(t + 1)$ for all $t \geq 0$, are in Ψ.

Theorem 2.3. *Let $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following assertions:*

(a) There exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference,

(b) T is triangular α_c-admissible,

(c) Suppose that there exists $\psi \in \Psi$ such that,

$$\int_0^{\alpha(\phi(c),\xi(c))\|T\phi - T\xi\|_E} \Phi(t)dt \le \psi\Big(\int_0^{(M(\phi(c),\xi(c)))} \Phi(t)dt\Big), \tag{3}$$

where

$$M(\phi(c),\xi(c)) = \max\left\{\|\phi-\xi\|_{E_0}, \frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}}, \frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|T\phi-T\xi\|_E}\right\}$$

for all $\phi,\xi \in E_0$,

(d) if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \ge 1$, then $\alpha(\phi(c), T\phi) \ge 1$ for all $n \in \mathbb{N}$,

(e) there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \ge 1$.

Then, T has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, the sequence $\{\phi_n\}$ of iterates of T defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to the PPF dependent fixed point of T in $\mathcal{R}_c$.

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Pseudonumerical range of matrices

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Abstract

In this paper for a given $\epsilon > 0$ and an $n \times n$ complex matrix A, the notion of pseudonumerical range of A is introduced. Also, some algebraic and geometrical properties of this notion are investigated moreover the relationship between this notion and the pseudospectrum of A is stated.

Keywords: Spectrum, Pseudospectrum, Numerical range, Pseudonumerical range, Pseudonumerical radius.
Mathematics Subject Classification [2010]: 15A60, 47A10, 65F15

1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex equipped with the operator norm $\|.\|$ induced by the usual vector norm $\|x\| = (x^*x)^{1/2}$ on $\mathbb{C}^n$, i.e.,

$$\|A\| = max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

In our discussion we assume that $D(a, r) = \{\mu \in \mathbb{C} : |\mu - a| < r\}$, where $a \in \mathbb{C}$ and $r > 0$. Also, we use the convention that if z is an eigenvalue of $A \in \mathbb{M}_n(\mathbb{C})$, then $\|(A - zI)^{-1}\| := \infty$. For $\epsilon > 0$ and a matrix $A \in \mathbb{M}_n(\mathbb{C})$, the pseudospectrum of A is defined and denoted, e.g., see [4], by

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} : \|(A - zI)^{-1}\| > 1/\epsilon\}. \tag{1}$$

It is known that

$$\begin{aligned} \sigma_\epsilon(A) &= \{z \in \sigma(A + E) : E \in \mathbb{M}_n \text{ and } \|E\| < \epsilon\} \\ &= \{z \in \mathbb{C} : s_n(zI - A) < \epsilon\}, \end{aligned} \tag{2}$$

where $s_n(.)$ denotes the smallest singular value.

Pseudospectrum provides an analytical and graphical alternative for investigating nonnormal matrices and operators, gives a quantitative estimate of departure from non-normality

*Speaker

and gives information about stability; See [4], [1] and their references. Like the spectrum, the numerical range is a set of complex numbers naturally associated with a given $A \in \mathbb{M}_n$, namely,

$$W(A) = \{x^*Ax : \ x \in \mathbb{C}^n, \|x\| = 1\}. \tag{3}$$

The spectrum of a matrix is a discrete point set; While the numerical range can be a continuum set, it is always a compact convex set. It is a set that can be used to learn something about the matrix, and it can often give information that the spectrum alone cannot give; For instance, $W(A) \subseteq \mathbb{R}$ if and only if A is Hermitian. For more information about the numerical range of matrices, see [2] and [3]. In this paper we are going to introduce the notion of pseudonumerical range of matrices. We also investigate some algebraic and geometrical of this notion.

2 Main results

We begin this section by introducing the notion of pseudonumerical range of square complex matrices.

Definition 2.1. Let $\epsilon > 0$ and $A \in \mathbb{M}_n(\mathbb{C})$. The $\epsilon-$pseudonumerical range of A is defined and denoted by

$$W_\epsilon(A) = \{\lambda \in \mathbb{C} : \exists E \in \mathbb{M}_n(\mathbb{C}) \text{ with } \|E\| < \epsilon \text{ and } \exists x \in \mathbb{C}^n \text{ with } x^*x = 1 \text{ s.t. } \lambda = x^*(A+E)x\}.$$

Let $A \in \mathbb{M}_n(\mathbb{C})$. From Definition 2.1, it follows that the pseudonumerical ranges associated with various ϵ are nested sets, i.e.,

$$W_{\epsilon_1}(A) \subseteq W_{\epsilon_2}(A), \qquad 0 < \epsilon_1 \leqslant \epsilon_2.$$

Also, for $\epsilon > 0$, we obtain that:

$$W_\epsilon(A) = \bigcup_{\|E\|<\epsilon} W(A+E). \tag{4}$$

From Definition 2.1 , it follows that the intersection of all the pseudonumerical ranges is the numerical range; namely,

Proposition 2.2. *Let $\epsilon > 0$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$W(A) = \bigcap_{\epsilon>0} W_\epsilon(A). \tag{5}$$

In view of Proposition 2.2 and relation (4), we have the following result.

Corollary 2.3. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$W(A) = \bigcap_{\epsilon>0} \bigcup_{\|E\|<\epsilon} W(A+E) := \limsup_{\epsilon>0,\, \|E\|<\epsilon} W(A+E).$$

We know that the numerical range contains the spectrum.That is also verified for the pseudonumerical range.

Proposition 2.4. *Let $\epsilon > 0$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sigma_\epsilon(A) \subseteq W_\epsilon(A)$$

Proof. Let $\lambda \in \sigma_\epsilon(A)$be given. Then by (2), there exists a $E \in \mathbb{M}_n(\mathbb{C})$ such that $\lambda \in \sigma(A+E)$. Since $\sigma(A+E) \subseteq W(A+E)$, $\lambda \in W(A+E)$, and hence by (4), we have $\lambda \in W_\epsilon(A)$. So, the proof is complete. □

Theorem 2.5. *Let $\epsilon > 0, \|D\| = \delta < \epsilon$ and $A, D \in \mathbb{M}_n(\mathbb{C})$. Then*

(a) $W_{\epsilon-\delta}(A+D) \subset W_\epsilon(A) \subset W_{\epsilon+\delta}(A+D)$
(b) $W_{\epsilon-\delta}(A) \subset W_\epsilon(A+D) \subset W_{\epsilon+\delta}(A)$

In the following theorem, we state some algebraic properties of $\epsilon-$pseudonumerical range of matrices.

Theorem 2.6. *Let $\epsilon > 0$, $0 \neq \alpha, \beta \in \mathbb{C}$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then the following assertions are true:*
(a) $W_\epsilon(\alpha A) = \alpha W_{\epsilon/|\alpha|}(A)$;
(b) $W_\epsilon(A + \beta I) = W_\epsilon(A) + \beta$;
(c) $W_\epsilon(\alpha A + \beta I) = \alpha W_{\epsilon/|\alpha|}(A) + \beta$.

We define the $\epsilon-$pseudonumerical radius of $A \in \mathbb{M}_n(\mathbb{C})$ as

$$r_\epsilon(A) = \sup_{z \in W_\epsilon(A)} |z|.$$

The following result follows from Theorem 2.6.

Corollary 2.7. *Let $\epsilon > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$r_\epsilon(\alpha A) = |\alpha| r_{\epsilon/|\alpha|}(A)$$

Lemma 2.8. *Let $\epsilon > 0$. Then*

$$\bigcup_{\|E\|<\epsilon} W(E) = D(0,\epsilon) \quad , \qquad E \in \mathbb{M}_n(\mathbb{C}).$$

By the above lemma, we can characterize the $\epsilon-$pseudonumerical range of matrices.

Theorem 2.9. *Let $\epsilon > 0$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$W_\epsilon(A) = W(A) + D(0,\epsilon)$$

Corollary 2.10. *Let $\epsilon > 0$,$\alpha \in \mathbb{C} \setminus \{0\}$ and $A \in \mathbb{M}_n(\mathbb{C})$. Then*
(a) $r_\epsilon(A) = r(A) + \epsilon$; *where* $r(A) = \max_{z \in W(A)} |z|$ *is the numerical radius of* A;
(b) $r_\epsilon(\alpha A) = |\alpha| r_{\epsilon/|\alpha|}(A)$;
(c) $r_\epsilon(\alpha A) = |\alpha| r(A) + \epsilon$.

We illustrate Theorem 2.9 by the following example.

Example 2.11. Let $\epsilon = 2$ and $A =$

$$\begin{bmatrix} -4i & 10i & 0 & 0 \\ i & 5i & 0 & 0 \\ 0 & 0 & 5-5i & 10 \\ 0 & 0 & 5 & -5i \end{bmatrix}.$$

In the Figure 1, the red region is the numerical range of A and the red and blue region are $2-$pseudonumerical range of A.

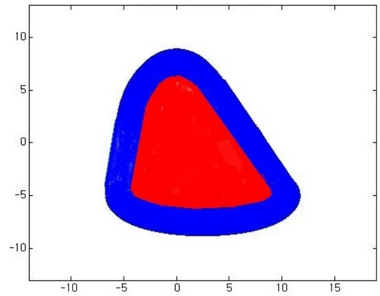

Figure 1: numerical range and augmented numerical range of matrix A

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Real interpolation method of martingale spaces

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Abstract

We describe the real interpolation spaces with a function parameter when we apply the real K-method of LionsPeetre to martingale Hardy spaces. As application we get interpolation spaces of the martingale Hardy-Lorentz spaces $\Lambda_q^s(\varphi)$.

Keywords: Martingale Hardy–Lorentz spaces, Lorentz spaces, Interpolation.
Mathematics Subject Classification [2010]: 60G42 and 46E30, 46B70.

1 Introduction

The family of martingale Hardy spaces is one of the important martingale function spaces. The study of the martingale Hardy spaces is extended to the martingale Hardy– Lorentz spaces [7, 4, 5].These spaces play an important role in the theory of Banach spaces since they have been defined are the objects of extensive investigations, results of which are contained among others in the papers [2, 6] and in probability theory and in statistics [3, 1]. Moreover, interpolation of martingale Hardy spaces is one of the main topics in martingale H_p theory, and its theory has been applied to Fourier analysis. Here the interpolation spaces with a function parameter between martingale Hardy–Lorentz spaces are identified. Some results due to [8] are extended to interpolation with a function parameter.

2 preliminaries

To achieve our goal we first fix our notations and terminology. Let us denote the set of integers and the set of non–negative integers, by $\mathbf{Z}$ and $\mathbf{N}$, respectively.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration $(\mathcal{F}_n)_{n\in\mathbf{N}}$ is a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\cup_{n\in\mathbf{N}}\mathcal{F}_n)$. We denote by E and E_n the expectation and the conditional expectation operators with respect to $(\mathcal{F}_n)_{n\in\mathbf{N}}$. For simplicity, we assume that $E_n f = 0$ if $n = 0$.

For a martingale $f = (f_n, n \in \mathbf{N})$ relative to $(\Omega, \mathcal{F}, P)$, denote the martingale differences by $d_n f := f_n - f_{n-1}$ with convention $d_0 f = 0$. The conditional square function of f is defined by

$$s_m(f) := \left(\sum_{n\leq m} E_{n-1} \mid d_n f \mid^2\right)^{1/2} \quad , \quad s(f) := \left(\sum_{n\in\mathbf{N}} E_{n-1} \mid d_n f \mid^2\right)^{1/2} .$$

*Speaker

Let us recall briefly the construction of Lorentz spaces and the real interpolation method. For measurable function f, we define a distribution function $m(r, f)$ by setting $m(r, f) = P(\{w \in \Omega : |f(w)| > r\})$. The function

$$f^*(t) = \inf\{r > 0 : m(r, f) \leq t\}, \quad (t \geq 0)$$

is called the decreasing rearrangement of f.

Let $\varphi > 0$ be a non-negative and local integrable function on $[0, \infty)$. The classical Lorentz spaces $\Lambda_q(\varphi)$ is defined to be the collection of all measurable functions f for which the quantity

$$\|f\|_{\Lambda_q(\varphi)} := \begin{cases} \left(\int_0^\infty (f^*(t)\varphi(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & 0 < q < \infty, \\ \sup_t f^*(t)\varphi(t) & (q = \infty) \end{cases}$$

is finite. Recall that for $0 < q \leq \infty$, $\|.\|_{\Lambda_q(\varphi)}$ is only a quasi-norm.

For $0 < q \leq \infty$, martingale Hardy-Lorentz spaces $\Lambda^s_q(\varphi)$ is defined by:

$$\Lambda^s_q(\varphi) = \left\{ f = (f_n)_{n\in\mathbf{N}} : \|f\|_{\Lambda^s_q(\varphi)} := \|s(f)\|_{\Lambda_q(\varphi)} < \infty \right\}.$$

Note that if $\varphi(t) = t^{\frac{1}{p}}$, then $\Lambda_q(\varphi) = L_{p,q}$ and $\Lambda^s_q(\varphi) = H^s_{p,q}$. In particular, if $\varphi(t) = t^{\frac{1}{q}}$, then $\Lambda_q(\varphi) = L_q$ and $\Lambda^s_q(\varphi) = H^s_q$.

Let (A_0, A_1) be a quasi–Banach couple ,that is, two quasi-Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. The K–functional is defined by

$$K(t, f, A_0, A_1) = K(t, f) := \inf_{f_0+f_1=f} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\}$$

for $t > 0$ and $f \in A_0 + A_1$, where $f_i \in A_i$, $i = 0, 1$.
For $0 < q \leq \infty$ and each measurable function ϱ, the real interpolation space $(A_0, A_1)_{\varrho,q}$ consists of all elements of $f \in A_0 + A_1$ such that the quantity

$$\|f\|_{(A_0,A_1)_{\varrho,q}} := \begin{cases} \left(\int_0^\infty \left(\frac{K(t,f)}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & (0 < q < \infty), \\ \sup_{t>0} \frac{K(t,f)}{\varrho(t)} & (q = \infty) \end{cases}$$

is finite. Let a and b be real numbers such that $a < b$. The notation $\varphi(t) \in Q[a, b]$ means that $\varphi(t)t^{-a}$ is non–decreasing and $\varphi(t)t^{-b}$ is non-increasing for all $t > 0$. Moreover, we say that $\varphi(t) \in Q(a, b)$, wherever $\varphi(t) \in Q[a + \epsilon, b - \epsilon]$ for some $\epsilon > 0$. The notation $\varphi(t) \in Q(a, -)$ (or $\varphi(t) \in Q(-, b)$) means that $\varphi(t) \in Q(a, c)$ (or $\varphi(t) \in Q(c, b)$) for some real number c.

In what follows, $a \lesssim b$ means that $a \leq Cb$ for some positive constant C independent of the quantities a and b. If both $a \lesssim b$ and $b \lesssim a$ are satisfied (with possibly different constants), we write $a \approx b$.

3 interpolation

In this section, some interpolation theorems for martingale–Hardy spaces are formulated and these results will be extended to interpolation of martingale Hardy–Lorentz spaces. First, the following Lemmas, which will be used in the proof of Theorem 3.3 are given.

Lemma 3.1. *Let $f \in \Lambda_q^s(\varphi)$, $0 < q \leq \infty$, $y > 0$ and fix $0 < p \leq 1$. Then f can be decomposed into the some of two martingales g and h such that*

$$\|g\|_{H^s_\infty} \leq 6y$$

and

$$\|h\|_{H^s_p} \lesssim \left(\int_{\{s(f)>y\}} s(f)^p dP\right)^{\frac{1}{p}}.$$

Lemma 3.2. *If $0 < p \leq 1$ then*

$$K(t, f, H^s_p, H^s_\infty) \lesssim \left(\int_0^{t^p} s(f)^*(x)^p dx\right)^{\frac{1}{p}}, \qquad t > 0.$$

Theorem 3.3. *Let $0 < p \leq 1$, $0 < q \leq \infty$ and $\varrho \in Q(0,1)$ be a parameter function. Then*

$$(H^s_p, H^s_\infty)_{\varrho,q} = \Lambda^s_q(t^{\frac{1}{p}}/\varrho(t^{\frac{1}{p}})).$$

If we take $\varrho(t) = t^\theta$ in Theorem 3.3, then we get the following result, which has proved by Weisz [8].

Corollary 3.4. *If $0 < \theta < 1, 0 < p_0 \leq 1$ and $0 < q \leq \infty$, then*

$$(H^s_{p_0}, H^s_\infty)_{\theta,q} = H^s_{p,q} \qquad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

Applying the Theorem 3.3 we get the next theorem.

Theorem 3.5. *Let $\varphi_i(t) \in Q(0, -), i = 0, 1$, $0 < p \leq 1$, $0 < q_0, q_1, q \leq \infty$ and $\varrho \in Q(0, 1)$. Then*

1. $$\left(\Lambda^s_{q_0}(\varphi_0), H^s_\infty\right)_{\varrho,q} = \Lambda^s_q(\varphi),$$

   where $\varphi(t) = \varphi_0(t)/\varrho(\varphi_0(t))$;

2. *If, in addition $\varphi_1(t) \in Q(0, 1/p)$. then*

   $$\left(H^s_p, \Lambda^s_{q_1}(\varphi_1)\right)_{\varrho.q} = \Lambda^s_q(\varphi),$$

   where $\varphi(t) = t^{1/p}/\varrho(t^{1/p}/\varphi_1(t))$;

3. If, in addition $\varphi_0(t)/\varphi_1(t) \in Q(0,-)$ or $\varphi_0(t)/\varphi_1(t) \in Q(-,0)$, then

$$\left(\Lambda^s_{q_0}(\varphi_0), \Lambda^s_{q_1}(\varphi_1)\right)_{\varrho,q} = \Lambda^s_q(\varphi),$$

where $\varphi(t) = \varphi_0(t)/\rho(\varphi_0(t)/\varphi_1(t))$.

The following result is a simple application of Theorem 3.5, if we take $\varphi_i(t) = t^{\frac{1}{p_i}}, i = 0, 1$.

Corollary 3.6. *Let $0 < p_i < \infty, 0 < q_i, q \leq \infty, i = 0,1$ and $\varrho \in Q(0,1)$. If $p_0 \neq p_1$, then*

$$\left(H^s_{p_0,q_0}, H^s_{p_1,q_1}\right)_{\varrho,q} = \Lambda^s_q(t^{\frac{1}{p_0}}/\varrho(t^{\frac{1}{p_0}-\frac{1}{p_1}})).$$

and

$$\left(H^s_{p_0}, H^s_{p_1}\right)_{\varrho,q} = \Lambda^s_q(t^{\frac{1}{p_0}}/\varrho(t^{\frac{1}{p_0}-\frac{1}{p_1}})).$$

In particular, if $\varrho(t) = t^\theta$, then

$$\left(H^s_{p_0}, H^s_{p_1}\right)_{\theta,q} = H^s_{p,q}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

According to Theorem 3.5 we have the following corollary.

Corollary 3.7. *Under the hypothesis of (3) in Theorem 3.5, we have*

$$\left(\Lambda^s_q(\varphi_0), \Lambda^s_q(\varphi_1)\right)_{\theta,q} = \Lambda^s_q(\varphi_0^{1-\theta}\varphi_1^\theta).$$

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Real interpolation of quasi-Banach spaces

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Abstract

We inter relate the real interpolation space with the quasi-Banach couple (A_0, A_1), (A_0+A_1, A_1) and $(A_0, A_0 \cap A_1)$ that A_j is c_j normed. Proving among others the identities

$$(A_0 + A_1, A_1)_{\theta,q} \cap A_0 = (A_0, A_1)_{\theta,q} \cap A_0 = (A_0, A_0 \cap A_1)_{\theta,q}.$$

$$(A_0 \cap A_1, A_1)_{\theta,q} + A_0 = (A_0, A_1)_{\theta,q} + A_0 = (A_0, A_0 + A_1)_{\theta,q}.$$

for all $0 < q \leq \infty$, $0 < \theta < 1$, and $c_1/c_0 \leq 1$.

Keywords: quasi-Banach spaces, interpolation space, real method of interpolation
Mathematics Subject Classification [2010]: 46M35, 47A60

1 Introduction

Our main reference to the theory of interpolation space is [1]. Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple, let $0 < \theta < 1$ and $0 < q \leq \infty$. The real interpolation space $(A_0, A_1)_{\theta,q}$ consist of all elements $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\theta,q,} = \begin{cases} (\sum_{\nu \in Z} (2^{-\nu\theta} K(2^\nu, a))^q)^{1/q} & if\ 0 < q < \infty \\ sup_{\nu \in Z} \{2^{-\nu\theta} K(2^\nu, a)\} & if\ q = \infty \end{cases} .$$

Here, for $0 < t < \infty$, we put

$$K(t, a) = K(t, a; A_0, A_1) = inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \ :\ a = a_0 + a_1, a_j \in A_j\}$$

and similarly the J-functional for $a \in A_0 \cap A_1 := \triangle(\bar{A})$ by

$$J(t, a; \bar{A}) = max\{\|a\|_{A_0}, t\|a\|_{A_1} \ :\ a \in \triangle(\bar{A})\}.$$

For $0 < \theta < 1$ we abbreviate $\bar{\theta} = max(\theta, 1-\theta)$ and $\underline{\theta} = min(\theta, 1-\theta)$.

*Speaker

2 Main results

We start this section by introducing the following:

In the following (A_0, A_1) will always denote a quasi-Banach couple that A_j is c_j normed with $c_1/c_0 \leq 1$.

Theorem 2.1. *Let (A_0, A_1) be a quasi-Banach couple and $a \in A_0 + A_1$. Then*

$$J(t, a; A_0, A_1) = J(t, a; A_0, A_0 \cap A_1) \quad (t \geq 1).$$

Theorem 2.2. *Let (A_0, A_1) be a quasi-Banach couple. Then*

$$(A_0, A_0 \cap A_1)_{\theta,q} = \{a \in A_0 \cap A_1 | (\sum_{\nu \in Z} (2^{-\nu\theta} J(2^\nu, a))^q)^{1/q},\ \nu \leq 0\}$$

$$(A_0 \cap A_1, A_1)_{\theta,q} = \{a \in A_0 \cap A_1 | (\sum_{\nu \in Z} (2^{-\nu\theta} J(2^\nu, a))^q)^{1/q},\ \nu \geq 0\}.$$

Proposition 2.3. *Let (A_0, A_1) be a quasi-Banach couple. Then the following identities hold.*

$$(A_0 + A_1, A_1)_{\theta,q} \cap A_0 = (A_0, A_1)_{\theta,q} \cap A_0 = (A_0, A_0 \cap A_1)_{\theta,q}. \quad (1)$$

$$(A_0 \cap A_1, A_1)_{\theta,q} + A_0 = (A_0, A_1)_{\theta,q} + A_0 = (A_0, A_0 + A_1)_{\theta,q}. \quad (2)$$

Proof.
Let us prove the identity (1). The chain of inclusions " $\supset$ " is clear, whence we have to show $(A_0 + A_1, A_1)_{\theta,q} \cap A_0 \subset (A_0, A_0 \cap A_1)_{\theta,q}$. Take $a_0 \in (A_0 + A_1, A_1)_{\theta,q} \cap A_0$. Since $a_0 \in A_0$, only the behaviour of $K(t, a_0; A_0, A_0 \cap A_1)$ on $(0, 1)$ matters. According to theorem 2.3 and theorem 2.1

$$\begin{aligned} K(t, a_0; A_0, A_0 \cap A_1) &\leq (c_0 + 1)K(t, a_0; A_0, A_1) + c_0 t\|a_0\|_{A_0} \\ &= (c_0 + 1)tK(t^{-1}, a_0; A_1, A_0) + c_0 t\|a_0\|_{A_0} \\ &= (c_0 + 1)tK(t^{-1}, a_0; A_1, A_0 + A_1) + c_0 t\|a_0\|_{A_0} \\ &= (c_0 + 1)K(t, a_0; A_0 + A_1, A_1) + c_0 t\|a_0\|_{A_0} \end{aligned}$$

also

$$\begin{aligned} \|a_0\|_{A_0, A_0 \cap A_1} &\leq (\sum_{\nu \in Z} (2^{-\nu\theta} K(2^\nu, a))^q)^{1/q}) \\ &\leq (\sum_{\nu \leq 0} ((C_0 + 1)2^{-\nu\theta} K(2^\nu, a))^q)^{1/q} + (\sum_{\nu \leq 0} (c_0 2^{-\nu\theta} \|a_0\|_{A_0})^q)^{1/q} \\ &\leq (c_0 + 1)[\|a_0\|_{A_0 + A_1, A_1} + \|a_0\|_{A_0}] \end{aligned}$$

Now, the identity (1)follows.

To prove the identity (2), we note as before that one chain of inclusions is trivial. Take $a \in (A_0, A_0 + A_1)_{\theta,q}$ and write $a = a_0 + a_1$ with $a_0 \in A_0, a_1 \in A_1$. Then by theorem 2.1

we have

$$K(t, a_1; A_0, A_1) \leq c_0[K(c_1t/c_0, a; A_0, A_1) + K(c_1t/c_0, a_0; A_0, A_1)]$$
$$\leq c_0[K(c_1t/c_0, a; A_0, A_1) + \|a_0\|_{A_0}]$$

for $t \geq 1, c_1/c_0 \leq 1$

$$\leq c_0[K(t, a; A_0, A_1) + \|a_0\|_{A_0}]$$
$$\leq c_0[K(t, a; A_0, A_0 + A_1) + \|a_0\|_{A_0}]$$

Then

$$\|a_1\|_{A_0,A_1} \leq c_0[\|a\|_{A_0,A_0+A_1} + \|a_0\|_{A_0}]$$

And $K(t, a_1; A_0, A_1) \leq t\|a_1\|_{A_1}$ for $t \leq 1$. then $\|a_1\|A_0, A_1 \leq \|a_1\|_{A_1}$. Hence we have $a_1 \in (A_0, A_1)_{\theta,q}$. □

Proposition 2.4. *Let* (A_0, A_1) *be a quasi-Banach couple. Then the following identities hold.*

$$(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{1-\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\bar{\theta},q}. \quad (3)$$
$$(A_0, A_1)_{\theta,q} + (A_0, A_1)_{1-\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\underline{\theta},q}. \quad (4)$$

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Talk Sobolev embedding theorem for weighted variable exponent Lebesgue space pp.: 1–4

Sobolev Embedding theorem for weighted variable exponent Lebesgue space

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Abstract

This paper gives some Sobolev type embedding theorems for generalized weighted Lebesgue- Sobolev space $W^{1,p(x)}_{a(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$ $(N \geq 2)$ with $p \in C(\overline{\Omega})$ and $a(x)$ is a measurable, nonnegative real valued function. The main result can be stated as follows, under some conditions we show the compact Sobolev embedding

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega).$$

Keywords: variable exponent Lebesgue space, variable exponent Sobolev space, compact embedding.
Mathematics Subject Classification [2010]: 46E35

1 Introduction

The Sobolev space $W^{m,p}(\Omega)$, where p is constant, is suitable for studding of many problems in physics and mechanics. Whereas, by introducing the problems with p(x)- growth conditions that arising by studding some materials with inhomogeneities such as Electrorheological fluids, which was due to Willis Winslow in 1949, the classical Sobolev spaces do not work and so the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and Sobolev space $W^{m,p(\cdot)}(\Omega)$ are defined, where $p(.)$ is some appropriate function; [7]. Despite the sufficient reasons for developing the Lebesgue and so the Sobolev space, the variable exponent Lebesgue and Sobolev spaces can be seen as a mathematical generalization of the classical space which are with constant exponent.

Hence the considerable attentions of mathematicians be involved in problems with p(x) growth conditions since the idea of generalizing the results has always been the incentive factor in Development of mathematics. We refer to [1] for the basic information about variable exponent Lebesgue and Sobolev spaces. Let Ω be an open subset of $\mathbb{R}^N$, $p \in L^\infty(\Omega)$ and

$$p^- := ess \inf_{x \in \Omega} p(x) \geq 1.$$

Moreover $a(x)$ is a measurable, nonnegative real valued function for $x \in \Omega$. The variable exponent Lebesgue space $\mathbf{L}^{p(\cdot)}(\Omega)$ is defined by

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$$\mathbf{L}_{a(.)}^{p(.)}(\Omega) = \{u : \ u : \Omega \longrightarrow \mathbb{R} \ \ \textit{is measurable}, \int_\Omega a(x)|u|^{p(x)}dx < \infty\}$$

which is equipped with the norm

$$|u|_{\mathbf{L}_{a(.)}^{p(.)}(\Omega)} = \inf\{\sigma > 0 : \ \int_\Omega a(x)|\frac{u}{\sigma}|^{p(x)}dx \leq 1\}.$$

The Sobolev space $\mathbf{W}_{a(.)}^{1,p(.)}(\Omega)$ which is defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm, $\|u\| = |\nabla u|_{L_{a(.)}^{p(.)}(\Omega)} + |u|_{L^{p(.)}(\Omega)}$. $\mathbf{W}_{a(.)}^{1,p(.)}(\Omega)$ is named weighted variable exponent Sobolev space which introduced in [5].

We refer to [2, 3, 4, 5, 6] for simialr disscution and interesting results in this issue.

2 Main results

Theorem 2.1. *Let $p, s \in C(\overline{\Omega})$, $1 < p(x)$, $1 < s(x)$ for all $x \in \overline{\Omega}$ and $a(x)$ be a measurable positive and a.e. finite function in $\mathbb{R}^N$ satisfying*

(a_1) $0 < a \in \mathbf{L}_{Loc}^1(\Omega)$, $a(x)^{-\frac{1}{p(x)-1}} \in \mathbf{L}_{Loc}^1(\Omega)$.

(a_2) $a(x)^{-s(x)} \in \mathbf{L}^1(\Omega)$ *where* $s(x) \in C(\overline{\Omega})$ *and* $s(x) > \frac{1}{p(x)-1}$.

(b_1) $0 < b \in \mathbf{L}^{\beta(x)}(\Omega)$, $1 < \beta(x) \in C(\overline{\Omega})$.

(q) $q \in C(\overline{\Omega})$ *and* $1 < q(x) < \frac{p_s^*(x)}{\beta'(x)}$ *for all* $x \in \overline{\Omega}$; *where*

$$p_s^*(x) = \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)}, & N > p_s(x) := \frac{p(x)s(x)}{1+s(x)}; \\ \infty, & N \leq p_s(x). \end{cases}$$

.

Then we have the following compact embedding,

$$W_{a(x)}^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L_{b(x)}^{q(x)}(\Omega);$$

when $1 < q(x) < \frac{p_s^*(x)}{\beta'(x)}$ *in* $\overline{\Omega}$.

Theorem 2.2. *Assume $p \in C(\overline{\Omega})$, $1 < p(x)$ for all $x \in \overline{\Omega}$, (a_1), (b_1) are satisfied and moreover*

(a_3) $a(x)^{-\frac{\xi(x)}{p(x)-\xi(x)}} \in \mathbf{L}^1(\Omega)$ *where* $\xi(x) \in C(\overline{\Omega})$ *and* $1 < \xi(x) < p(x)$.

Then we have the following compact embedding,

$$W_{a(x)}^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L_{b(x)}^{q(x)}(\Omega).$$

for every $q \in C(\overline{\Omega})$ *and* $1 < q(x) < \frac{\xi^*(x)}{\beta'(x)}$

Proof. First, we show that $W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow W^{1,\xi(x)}(\Omega)$ continuously. Let $u \in W^{1,p(x)}_{a(x)}(\Omega)$ we have

$$\int_\Omega |\nabla u|^{\xi(x)} dx = \int_\Omega |\nabla u|^{\xi(x)} a(x)^{\frac{\xi(x)}{p(x)}} a(x)^{-\frac{\xi(x)}{p(x)}} dx$$
$$\leq C|a(x)^{-\frac{\xi(x)}{p(x)}}|_{L^{\frac{p(x)}{p(x)-\xi(x)}}(\Omega)} |a(x)^{\frac{\xi(x)}{p(x)}} |\nabla u|^{\xi(x)}|_{L^{\frac{p(x)}{\xi(x)}}(\Omega)}.$$

By (iii) of the main properties that recalled in the first part of preliminaries we deduce

$$|a(x)^{-\frac{\xi(x)}{p(x)}}|_{L^{\frac{p(x)}{p(x)-\xi(x)}}(\Omega)} \leq (\int_\Omega a(x)^{-\frac{\xi(x)}{p(x)-\xi(x)}} dx + 1)^{\frac{p^+-\xi^-}{p^-}}.$$

So, by assumption (a_3), there exists $C > 0$ such that

$$\int_\Omega |\nabla u|^{\xi(x)} dx \leq C|a(x)^{\frac{\xi(x)}{p(x)}} |\nabla u|^{\xi(x)}|_{L^{\frac{p(x)}{\xi(x)}}(\Omega)}. \tag{1}$$

Without loss of generality, we can assume that $\int_\Omega |\nabla u|^{\xi(x)} > 1$. By applying (iii) when $\int_\Omega a(x)|\nabla u|^{p(x)} < 1$, from (1) we obtain

$$|\nabla u|_{L^{\xi(x)}(\Omega)} \leq C|\nabla u|^{\frac{p^-}{p^+}}_{L^{p(x)}_{a(x)}(\Omega)}.$$

Moreover, if $\int_\Omega a(x)|\nabla u|^{p(x)} > 1$ we deduce,

$$|\nabla u|_{L^{\xi(x)}(\Omega)} \leq C|\nabla u|^{\beta}_{L^{p(x)}_{a(x)}(\Omega)};$$

where $\beta = \frac{p^+\xi^+}{p^-\xi^-}$. So we get $\nabla u \in L^{\xi(x)}(\Omega)$. On the other hand, $L^{p(x)}(\Omega) \hookrightarrow L^{\xi(x)}(\Omega)$; hence

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow W^{1,\xi(x)}(\Omega). \tag{2}$$

Now by classical Sobolev embedding (iv) we have,

$$W^{1,\xi(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega) \tag{3}$$

for $r(x) < \xi^*(x)$. Let $r(x) = q(x)\beta'(x)$. So if $u \in W^{1,p(x)}_{a(x)}(\Omega)$ then

$$\int_\Omega b(x)|u|^{q(x)} dx \leq C|b|_{L^{\beta(x)}(\Omega)} ||u|^{q(x)}|_{L^{\beta'(x)}(\Omega)} \leq C|b|_{L^{\beta(x)}(\Omega)} \min(|u|^{q^+}_{L^{r(x)}(\Omega)}, |u|^{q^-}_{L^{r(x)}(\Omega)});$$

and since $u \in L^{r(x)}(\Omega)$, $u \in L^{q(x)}_{b(x)}(\Omega)$. Moreover if $u_n \rightharpoonup 0$ in $W^{1,p(x)}_{a(x)}(\Omega)$ then by (2) $u_n \rightharpoonup 0$ in $W^{1,\xi(x)}(\Omega)$ and by (3) $u_n \longrightarrow 0$ in $L^{r(x)}(\Omega)$. Then we have

$$\int_\Omega b(x)|u_n|^{q(x)} dx \leq C|b|_{L^{\beta(x)}} ||u_n|^{q(x)}|_{L^{\beta'(x)}} \longrightarrow 0,$$

which implies $|u_n|_{L^{q(x)}_{b(x)}} \longrightarrow 0$ and hence we can deduce

$$W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{b(x)}(\Omega).$$

□

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Some C^*- algebraic results on expansion of semigroups

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Abstract

In this work, we present a definition of an inverse semigroup, $\Pr(S)$, which is associated to an inverse semigroup S. Also, we show the existence of a kind of correspondence between partial representations of S and representations of $\Pr(S)$, on a Hilbert space. Some results of graded C^*-algebras over a group are extended to pre-grading C^*-algebras over inverse semigroups.

Keywords: Inverse emigroup, Partial action, Partial representation, Partial homomorphism.
Mathematics Subject Classification [2010]: 20M18, 16W22

1 Introduction

During the last two decades, partial actions of groups and actions of semigroups on C^*-algebras have played a major role in constructing some mathematical constructions. R. Exel [2] introduced the concept of *pre-grading* C^*-algebra for an inverse semigroup S. Here, we use this concept to show that what is the maximum number of subspaces of a pre-grading C^*-algebra that we can obtain by taking the closure of its finite products?

2 The inverse semigroup associated to an inverse semigroup

The major new result of this section is Theorem 2.5. Throughout this work, by S we mean an inverse semigroup.

Definition 2.1. By $\Pr(S)$ we mean the universal semigroup which is defined via generators and relations, that is, we associate a generator, $[s]$, to each $s \in S$. The generator $[s]$ comes from any fixed set having as many elements as S such that, for every s, t in S, the following conditions hold
(i) $[s^*]\,[s]\,[t] = [s^*]\,[st]$,
(ii) $[s]\,[t]\,[t^*] = [st]\,[t^*]$,
(iii) $[s]\,[s^*]\,[s] = [s]$.

Following [3, Proposition 2.4] we have the next Lemma.

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Lemma 2.2. *For given $t \in S$, let $\varepsilon_t = [t]\,[t^*]$. For each $s, t \in S$ the following statements hold*
(i) ε_t is idempotent,
(ii) $[t]\varepsilon_s = \varepsilon_{ts}[t]$
(iii) ε_s and ε_t commute.

Proposition 2.3. *Let $E(S)$ be the idempotent semilattice of S.For given $e \in E(S)$ and $s \in S$ the following statement hold*
(i) $\varepsilon_e = [e]$, that is, $[e]$ is idempotent,
(ii) $[e]\,[s] = [es]$, and $[s]\,[e] = [se]$,
(iii) $\varepsilon_e\varepsilon_s = \varepsilon_{es}$.

By [1, Proposition 2.14] each element of $Pr(S)$ can be written as a certain product. For the definitions of concepts of *partial homomorphism* of an inverse semigroup in a semigroup and *partial representation* of S on a Hilbert space $\mathbf{H}$ we will refer the reader to [1]

Proposition 2.4. *Let H be an inverse semigroup and $\pi : S \to H$ be a partial homomorphism. There exists a unique semigroup homomorphism $\tilde{\pi} : \mathrm{Pr}(S) \to H$ such that $\tilde{\pi} \circ i_S = \pi$, [1, Proposition 2.20].*

In the above Proposition, let π be the identity map on S. Obviously, this map is a partial homomorphism of S in itself. By Proposition 2.4 there exists a semigroup homomorphism $\partial : \mathrm{Pr}(S) \to S$ such that $\partial([s]) = s$, for all s in S. This ∂ is called the d*egree map.*

For a partial representation π, it should be noted that since $\alpha\alpha^*\alpha = \alpha$ we have $\pi(\alpha)\pi(\alpha^*)\pi(\alpha) = \pi(\alpha)$, that is, $\pi(\alpha)$ is a partial isometry on $\mathbf{H}$. Also, if ε is an idempotent element of S, since $\varepsilon = \varepsilon^*$ we have $\pi(\varepsilon^*) = \pi(\varepsilon)^*$,that is, $\pi(\varepsilon)$ is a self adjoint operator in B($\mathbf{H}$). Now, we are ready to state and prove the main Theorem of this section.

Theorem 2.5. *(a) If π is a representation of S on a Hilbert space $\boldsymbol{H}$, then $\tilde{\pi}$ such that $\tilde{\pi}([s]) = \pi(s)$ is a representation of $\mathrm{Pr}(S)$ on $\boldsymbol{H}$.*
(b) If ρ is a representation of $\mathrm{Pr}(S)$ on $\boldsymbol{H}$, then π such that $\pi(t) = \rho([t])$ is a partial representation of S on $\boldsymbol{H}$.

3 On pre-graded C^*-algebras

This section starts with the definition of pre-grading of a C^*-algebra, say A, over an inverse semigroup, then we shall show that $\mathrm{Pr}(S)$ plays an important role in describing certain subspaces of A. After introducing the semigroup associated to the C^*-algebra A, we state and prove Lemma 3.3 and Theorem 3.4. With the aid of Theorem 3.4 we are able to provide an answer for the question arise in the Remark 3.1. Let M and N be two subspaces of a C^*-algebra A. By MN we mean the closed linear span of the set of all products xy such that $x \in M$ and $y \in N$. Let A be any C^*-algebra. A *pre-grading* of A over S is a family of closed linear subspaces $\{A_s\}_{s\in S}$ of A such that for every s, t in S, the following statements hold
(i) $A_sA_t \subseteq A_{st}$,
(ii) $A_s^* = A_{s^*}$,

(iii) if $s \leq t$, then $A_s \subseteq A_t$,
(iv) A is the closed linear span of the union of all A_s.
In this case, each A_t is called a *pre-grading subspace* of A. If in addition A_sA_t is dense in A_{st}, the pre-grading is called *full*. Obviously, for given s, t in S, the product A_sA_t is contained in A_{st}. This means that, not only A_sA_t need not coincide with A_{st}, but it also may not be dense there. Hence, one could ask the following question.

Remark 3.1. What is the maximum number of subspaces of a pre-grading C^*-algebra that we can obtain by taking the closure of its finite products? Before we proceed to answer the above question, we should keep in mind the definition of A_sA_t.

Now, let A_t be a pre-grading subspace of A. If we define $D_t := A_tA_{t^*}$, then A_t is a $D_t - D_{t^*}$-Hilbert bimodule. To show this, it suffices to show that A_t is a left D_t-module and a right D_{t^*}-module. Let multiplication maps $A_t \times D_{t^*} \to A_t$ and $D_t \times A_t \to A_t$ be the multiplication of A. Now, let $a \in A_t, r \in D_t$, where $r = xy$ for some $x \in A_t$ and $y \in A_{t^*}$. Then $ra = xya \in A_tA_{t^*}A_t \subseteq A_{tt^*t} = A_t$, that is, the multiplication is well-defined. Since A is a C^*-algebra, we conclude that the above multiplication is associative. That is, A_t is a left D_t-module. On the other hand if $r \in D_{t^*}$ we assume that $r = xy$ where $x \in A_{t^*}$ and $y \in A_t$. Now for given $a \in A_t$ we have

$$ar = axy \in A_tA_{t^*}A_t \subseteq A_{tt^*t} = A_t.$$

We see that A_t is a right D_{t^*}-module since the associativity inherits from C^*-algebra A. Now, we would like to show that A_t is a D_{t^*}-Hilbert module. For given $a, b \in A_t$ let

$$< a, b >:= a^*b$$

be the inner product on A_t. For $a \in A_t$ since$A_{t^*} = A_t^*$ we have $a^*b \in A_{t^*}A_t = D_{t^*}$, therefore,
$<,>$ $maps\, A_t \times A_t$ into D_{t^*}. Obviously, this map is conjugate linear on first component and linear on the second one. If $< a, a >= 0$, then

$$0 = || < a, a > || = ||a^*a|| = ||a||^2,$$

that is, $a = 0$ and A_t is a D_{t^*}-Hilbert module. Now, let $B_a(A_t)$ be the C^*-algebra of all adjointable operators on A_{t^*}, and define $\lambda : D_t \to B_a(A_t)$ such that $\lambda(a)b = ab$ for $b \in A_t$. Obviously, $\lambda(a)$is linear and $\lambda(a^*) = \lambda(a)^*$, simply because, if $b, c \in A_t$ and $a \in D_t$ then

$$< c, \lambda(a)b >= c^*ab = (a^*c)^*b =< \lambda(a^*)c, b > .$$

That is λ is a *-homomorphism. Therefore, A_t is a $D_t - D_{t^*}$-Hilbert bimodule. Also, as we have seen for all right modules the product $A_tD_{t^*}$ is coincide with A_t [6, 1.1.4]. This shows that $A_tA_{t^*}A_t = A_t$.

Definition 3.2. For a given C^*-algebra A, let $B_l(A) = \{X : X$ is a closed linear subspace of $A\}$. Given X, Y in $B_l(A)$, define the product of X, Y as mentioned before. $B_l(A)$ with this multiplication is a semigroup called the semigroup associated to A.

Here, we take steps to provide an answer for the question posed in the above Remark.

Lemma 3.3. *Let $A = \overline{span}(\bigcup_{s\in S} A_s)$ be a pre-graded C^*-algebra over S, then for every s, t in S we have*
(i) $A_{s^*}A_sA_t = A_{s^*}A_{st}$,
(ii) $A_sA_tA_{t^*} = A_{st}A_{t^*}$,
(iii) $A_sA_{s^*}A_s = A_s$.

Theorem 3.4. *For a given pre-graded C^*-algebra, $A = \overline{span}(\bigcup_{s\in S} A_s)$, there exists a correspondence which assigns to each α in $\Pr(S)$ a closed subspace A^α of A such that for all α, β in $\Pr(S)$ and all s, t in S the following hold*
(i) $A^{[t]} = A_t$,
(ii) if $\partial(\alpha) = t$ then A^α is contained in A_t,
(iii) the closed linear span of the product of A^α by A^β is exactly equal to $A^{\alpha\beta}$.

The above Theorem shows that the collection $\{A^\alpha\}_{\alpha\in\Pr(S)}$ is closed under multiplication, that is, this collection is a subsemigroup of $B_l(A)$ Since it contains the A_t's we see that the maximum number of different pre-grading subspaces of $A = \overline{span}(\bigcup_{s\in S} A_s)$ that we can obtain by finite product is at most the order of $\Pr(S)$, when S is finite. By [1, Proposition 5.14] if S is a finite inverse semigroup, $e \in E(S)$, $S^e := \{s \in S : ss^* = e\}$, and $|S^e| = p_e$ then $|\Pr(S)| = \sum_{e\in E(S)} 2^{p_e-2}(p_e + 1)$, where by $|\Pr(S)|$ we mean the order of $\Pr(S)$. We close this section by the following conjecture.

Conjecture 3.5. There is a one-to-one correspondence between
(a) partial representations of S on $\mathbf{H}$,
(b) representations of $\Pr(S)$ on $\mathbf{H}$, and
(c) C^*- algebra representations of $C^*_p(S)$ on $\mathbf{H}$,
where by $C^*_p(S)$ we mean the partial inverse semigroup C^*-algebra [7].

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Some equivalent conditions to strong uniqueness in normed linear space

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Abstract

In this work we investigate equivalent condition for strong unique best approximation and its uniqness and also strongly unique. Also, for finite dimensional subspace of $C(X,\mathbb{R})$, Lipschitz continuity of order 1 and strong uniqueness of order 1 are essentially equivalent.

Keywords: Best approximation, Haar space, Strongly unique, Unicity space, Lipschitz condition
Mathematics Subject Classification [2010]: 41A50, 41A65

1 Introduction

Let X be a finite set with the discrete topology and $C(X,\mathbb{R}^k)$ be the space of vector-valued functions from X to k-dimensional Euclidean space $\mathbb{R}^k$. A norm for functions in $C(X,\mathbb{R}^k)$ is defined as follows:

$$\|f\| := \max_{x\in X} \|f(x)\|_2,$$

where $\|.\|_2$ denotes the Euclidean norm on $\mathbb{R}^k$.

Definition 1.1. Let G be a nonempty subset of a normed linear space X and let $x \in X$. An element $y_0 \in G$ is called a best approximation, or nearest point to x from G, if

$$\|x - y_0\| = \mathrm{d}(x, G),$$

where $\mathrm{d}(x,G) = \inf_{y\in G} \|x - y\|$. The number $\mathrm{d}(x,G)$ is called the distance from x to G, or the error in approximating x by G.

The set (possibly empty) of all best approximation from x to G is denoted by $P_G(x)$, i.e.

$$P_G(x) := \{y \in G |\ \mathrm{d}(x,G) = \|x - y\|\}.$$

This defines a mapping P_G from X into the subsets of G called the metric projection onto G.

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Definition 1.2. Let G be a nonempty subset of a normed linear space X. If for any $x \in X$, the set $P_G(x)$ is a singlton (for any $x \in X$, there is unique best approximation to x from G), then G is called a Chebyshev subset of X.

If G is a Chebyshev subspace of a normed linear space X, then G is called unicity space.

Definition 1.3. Let $f \in C(X, \mathbb{R}^k)$ and G be a Chebyshev subset of $C(X, \mathbb{R}^k)$. The unique best approximation $g_0 \in P_G(f)$ is called strongly unique (or strong unicity) of order α if there exists a positive constant γ (depending on f, α and G) such that

$$\|f - g\|^\alpha \geq \|f - g_0\|^\alpha + \gamma\|g - g_0\| \quad \text{for } g \in G.$$

It might be conjectured that uniqueness and strong uniqueness are equivalent properties in $C(X, \mathbb{R})$, where X is a compact Hausdorff space. This is not true. The following theorem has proved by Nürnberger, Singer [5].

Theorem 1.4. *Let G be a finite dimensional subspace of $C(X, \mathbb{R})$. Then the set of functions with a strongly unique best approximant is dense in the set of functions with a unique best approximant.*

Definition 1.5. Let X be a compact Hausdorff space and let G be an n-dimensional subspace in $C(X, \mathbb{R}^k)$ with dim$G \geq 1$ and basis $\{g_1, \ldots, g_n\}$. We say G is satisfied the Haar condition (Haar space), if any $g \in G,\ g \not\equiv 0$, has at most $n - 1$ zeros in X.

The following result was proved by Haar [1918].

Theorem 1.6. *An n-dimensional subspace G of $C(X, \mathbb{R})$ is a unicity space if and only if it is a Haar space.*

Definition 1.7. Let G be a nonempty Chebyshev subset of a normed linear space X. We say that, the best approximation operator P_G is satisfied in Lipschitz condition of order α at f if there exists a positive constant λ such that

$$\|P_G(f) - P_G(h)\| \leq \lambda\|f - h\|^\alpha, \qquad \text{for any } h \in X.$$

2 Main results

In $C(X, \mathbb{R}^k)$ the best approximation operator from a Haar subspace has Lipschitz continuity of order 1 when X is finite and in space $C(X, \mathbb{R}^k)$, Chebyshev subspace and unicity subspace are equivalent and are used interchangeably [1, 2, 3, 4].

Theorem 2.1. *Let X be a compact Hausdorff space and G a finite dimensional subspace of $C(X, \mathbb{R})$. For given $f \in C(X, \mathbb{R})$, the following are equivalent.*

(i) There exists a $\lambda > 0$ such that

$$\|f - g\| - \|f - P_G(f)\| \geq \lambda\|g - P_G(f)\|, \qquad \textit{for all } g \in G.$$

(ii) There exists a $\gamma > 0$ *such that*

$$\|P_G(f) - P_G(g)\| \leq \gamma \|f - g\|, \qquad \textit{for all } g \in C(X, \mathbb{R}).$$

In the assumptions of Theorem 2.1, the metric projection, P_G, is said to be Lipschitz continuous of order 1 at f if there is a positive constant λ such that

$$H(P_G(f), P_G(g)) \leq \lambda \|f - g\|, \quad \text{for all } g \in C(X, \mathbb{R}),$$

where H denotes the Hausdorff metric. Theorem 2.1 implies that, Lipschitz continuity of order 1 and strong uniqueness of order 1 are essentially equivalent.

Theorem 2.2. *Let* G *be a subset of a normed linear space* X, $f \in X$, *and for some* $\lambda > 0$, *we have*

$$\|f - g\| - \|f - P_G(f)\| \geq \lambda \|g - P_G(f)\|, \qquad \textit{for all } g \in G.$$

Then for any $g \in X$ *and any element of* $P_G(g)$,

$$\|P_G(f) - P_G(g)\| \leq \frac{2}{\lambda} \|f - g\|.$$

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Some fixed point results for the sum of two mappings

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Abstract

In this paper, we obtain some new fixed point theorems for the sum of two weakly sequentialy continuous mappings T_1 and T_2 on an $L-$embedded convex subset C in a Banach space X, in which $T_1 : C \to X$ is nonexpansive and $T_2 : C \to X$ is continuous with $T_2(C)$ being contained in a compact set. As a result, we derive fixed point theorems on weak* compact convex subsets of the continuous dual X^* of an $M-$embedded Banach space X.

Keywords: nonexpansive, fixed point, $L-$embedded, $M-$embedded, weakly sequentialy continuous
Mathematics Subject Classification [2010]: 37C25 ,46B25

1 Introduction

Let X be a Banach space and C be a subset of X. A mapping $T : C \to X$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in X$ is called a fixed point of T, if $Tx = x$. A mapping $T : C \to X$ is called compact continuous if T is compact and continuous on C. In [4] O'Regan studied the fixed points of the sum of a nonexpansive mapping with a compact continuous on a weakly compact subset C of X and in [2] and [3] Krasnoselskii combined two well-known fixed point theorems (Schauder's fixed point Theorem and the contraction mapping principle) to gain the fixed points of the sum of two mappings T_1 and T_2 on a closed convex subset C in a Banach space X, in which $T_1 : C \to X$ is a contraction and $T_2 : C \to X$ is continuous with $T_2(C)$ being contained in a compact set . In this paper, among other things we study the fixed point of the sum of two such mapings on an $L-$embedded convex subset of X allowing T_1 to be a nonexpansive mapping instead of a contraction(Theorem 2.2). In [1], Lau and Zhang called a nonempty subset C of a Banach space X, $L-$embedded if there is a subspace X_s of X^{**} such that $X + X_s = X \oplus_1 X_s$ in X^{**} and $\overline{C}^{w^*} \subset C \oplus_1 X_s$. That is, for each $x \in \overline{C}^{w^*}$ there are $c \in C$ and $\xi \in X_s$ such that $x = c + \xi$ and $\|x\| = \|c\| + \|\xi\|$. As remarked in the same paper, (by taking $X_s = 0$) it is readily seen that every $L-$embedded subset C of a Banach space X is weak$^*-$closed and hence closed. Also every weakly compact subset of Banach space is $L-$embedded, but not vice-versa, [1].

Next, we use our results to derive fixed point theorems on weak* compact convex subsets of the dual space X^* of an $M-$embedded Banach space X (Theorem 2.4). As in [5], a Banach space X is $M-$embedded if X is an M-ideal in its bidual X^{**}, i.e. $X^{\perp} = \{\varphi \in X^{***} : \varphi(x) = 0 \text{ for all } x \in X\}$ is an l_1-summand in X^{***}.

*Speaker

2 Main results

Before going through our main theorems, let us recall some results from [1], [2] and [3]. Let C and B be two nonempty subsets of a Banach space X with B bounded.

$$r_C(B) = \inf\{r \geqslant 0 : \exists x \in C, \sup_{b\in B} \|x - b\| \leqslant r\}$$

and

$$W_C(B) = \{x \in C : \sup_{b\in B} \|x - b\| \leq r_C(B)\}$$

$$K_C(B) = \{x \in C : \|x - b\| \leq r_C(B), for\ some\ \ b \in B\}.$$

The number $r_C(B)$ and the set $W_C(B)$ are, respectively, called the Chebyshev radius and Chebyshev center of B in C and we have $W_C(B) \subseteq K_C(B)$. It is proved that if C is a nonempty convex $L-$embedded subset of a Banach space X and B is a nonempty bounded subset of X, then the Chebyshev center $W_C(B)$ of B in C and $K_C(B)$ of B in C is nonempty convex and weakly compact. It is also proved in the same paper that if C is a weak* closed subset of the dual space X^* of an $M-$embedded Banach space X. Then C is $L-$embedded, [1, Lemma 3.2]. As a consequence of Krasnoselskii's result [2] we arrive at the next one which we need in the sequel.

Proposition 2.1. *Let $\alpha, \beta \in (0, 1)$ and C be an $L-$embedded, convex subset of a Banach space X. Suppose that T_1 and T_2 map C into X such that*

(i) *T_1 is nonexpansive,*

(ii) *T_2 is continuous and $T_2(C)$ is contained in a compact set or T_2 is compact continuous with C bounded,*

(iii) *$\alpha T_1x + \beta T_2y \in C$, for all $x, y \in C$.*

Then $\alpha T_1 + \beta T_2$ has a fixed point in C.

The next theorem, which is one of our main results, asserts the existence of a fixed point for the sum of two mappings on an $L-$embedded convex subset of a Banach space.

Theorem 2.2. *Let C be an $L-$embedded, convex subset of a Banach space X. Suppose that $0 \in C$, T_1 and T_2 map C into X such that*

(i) *T_1 is norm nonexpansive and weakly sequentialy continuous,*

(ii) *T_2 is continuous and $T_2(C)$ is contained in a compact set and T_2 is weakly sequentialy continuous,*

(iii) *$T_1x + T_2y \in C$ for all $x, y \in C$,*

(iv) *$\{x \in C : (1 - \frac{1}{n})T_1x + (1 - \frac{1}{n})T_2x = x, for\ some\ n \in \mathbb{N}\} \subseteq K_C(B)$ for some bounded subset B.*

Then $T_1 + T_2$ has a fixed point in C.

Corollary 2.3. *Let C be a weakly compact, convex subset of a Banach space X. Suppose that T_1 and T_2 map C into X such that*

(i) T_1 *is norm nonexpansive and weakly sequentialy continuous,*

(ii) T_2 *is compact and continuous and weakly sequentialy continuous,*

(iii) $(1-\frac{1}{n})T_1x+(1-\frac{1}{n})T_2y \in C$ *for all* $x, y \in C, n \in \mathbb{N}$.

Then there exists a point $x \in C$ *with* $T_1x + T_2x = x$.

Theorem 2.4. *Let C be a weak* compact convex subset of the dual space X^* of an $M-$embedded Banach space X. Suppose that $0 \in C$ and T_1, T_2 map C into X^* such that*

(i) T_1 *is norm nonexpansive and weak* continuous,*

(ii) T_2 *is continuous and* $T_2(C)$ *is contained in a compact set and* T_2 *is weakly sequentialy continuous,*

(iii) $T_1x + T_2y \in C$ *for all* $x, y \in C$,

(iv) $\{x \in C : (1-\frac{1}{n})T_1x+(1-\frac{1}{n})T_2x = x, \text{ for some } n \in \mathbb{N}\} \subseteq K_C(B)$ *for some bounded subset B of C.*

Then $T_1 + T_2$ *has a fixed point in* C.

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Talk Some fixed point results in non-Archimedean probabilistic Menger space pp.: 1–4

Some fixed point results in non-Archimedean probabilistic Menger space

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Abstract

In this paper, we introduce the notions of (α, β, φ)-contractive mapping, (α, ϕ, ψ)-contractive mapping and establish some results of fixed point for this class of mappings in the setting of non-Archimedean probabilistic Menger spaces. Also, some examples are given to support the usability of our results.

Keywords: Continuous t-norm, non-Archimedean probabilistic Menger space, contractive mapping
Mathematics Subject Classification [2010]: 47H10, 54H25

1 Introduction

In 1972, Menger [1] introduced the concept of a probabilistic metric space, and a large number of authors have done considerable work in such field [5, 6]. The notion of non-Archimedean Menger space has been established by Istratescu and Crivat [2]. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istratescu [3]. In this paper, we give some fixed point results for some new classes of contractive mappings in probabilistic Menger space. We first bring notion, definitions and known results, which are related to our work. For more details, we refer the reader to [4].

Definition 1.1. A $t-$norm is a function $T : [0, 1]^2 \to [0, 1]$ which is associative, commutative, nondecreasing in each cordinate and $T(a, 1) = a$ for every $a \in [0, 1]$.

Definition 1.2. Let X be a non-empty set and D be the set of all left-continuous distribution functions. An ordered pair (X, F) is called a non-Archimedean probabilistic metric space (briefly a N.A PM-space) if F is a mapping from $X \times X \to D$ satisfying the following conditions:

(i) $F_{x,y}(t) = 1$, for all $t > 0$ if and only if $x = y$,

(ii) $F_{x,y}(t) = F_{y,x}(t)$,

(iii) $F_{x,y}(0) = 0$,

(iv) If $F_{x,y}(t) = F_{y,z}(s) = 1$ then $F_{x,z}(max\{t, s\}) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

*Speaker

Definition 1.3. A N.A Menger PM-space is an ordered triple (X, F, T) where (X, F) is a non-Archimedean PM-space and T is a $t-$norm satisfying the following condition:

(iiv) $F_{x,y}(max\{t,s\}) \geq T(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $s, t \geq 0$.

2 Main results

Definition 2.1. Let $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then f is an $\alpha-$admissible mapping if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1, \quad x, y \in X.$$

Definition 2.2. Let $f : X \to X$ and $\beta : X \times (0, \infty) \to [0, \infty)$ and $K : (0, \infty) \to (0, 1)$. Then f is a $(K, \beta)-$admissible mapping if

$$\beta(x, t) \leq \sqrt{K(t)} \Rightarrow \beta(fx, t) \leq \sqrt{K(t)}, \quad x \in X, t > 0.$$

We denote by ϕ the class of all functions $\varphi : [0, 1] \to [0, 1]$ such that satisfying the following conditions:

(i) φ is decreasing and continuous,

(ii) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

Definition 2.3. Let (X, F, T) be a non-Archimedean Menger PM-space and f be an $\alpha-$admissible and $(K, \beta)-$admissible mapping. If there exists $\varphi \in \phi$ such that :

$$\alpha(x, fx)\ \alpha(y, fy)\ \varphi(F_{fx,fy}(t)) \leq \beta(x, t)\ \beta(y, t)\ \varphi(F_{x,y}(t)), \tag{1}$$

holds for all $x, y \in X$ with $x \neq y$ and $t > 0$, then f is called a $(\alpha, \beta, \varphi)-$ contractive mapping.

Theorem 2.4. *Let (X, F, T) be a complete non-Archimedean Menger PM-space, $\alpha : X \times X \to [0, \infty)$, $\beta : X \times (0, \infty) \to [0, \infty)$ and $K : (0, \infty) \to (0, 1)$. Assume the following conditions hold:*

(i) f is (α, β, φ)-contractive mapping,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\beta(x_0, t) \leq \sqrt{K(t)}$ for all $t > 0$,

(iii) if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x, fx) \geq 1$. Then f has a fixed point. Moreover if $y = fy$ implies $\alpha(y, fy) \geq 1$ and for all $x \in X$ and all $t > 0$, $\beta(x, t) < 1$, then f has a unique fixed point.

Definition 2.5. Let (X, F, T) be a non-Archimedean Menger PM-space and $f : X \to X$ be an α-admissible mapping. Also, suppose that $\psi, \varphi : [0, 1] \to [0, 1]$ are two continuous functions such that ψ is decreasing, $\psi(t) > \psi(1) - \varphi(1)$ and $\varphi(t) > 0$ for all $t \in (0, 1)$. We say, f is a $(\alpha - \varphi - \psi)$-contractive mapping if

$$\alpha(x, fx)\alpha(y, fy)\psi(F_{fx,fy}(t)) \leq \psi(F_{x,y}(t)) - \varphi(F_{x,y}(t)), \tag{2}$$

holds for all $x, y \in X$ and $t > 0$.

Theorem 2.6. *Let (X, F, T) be a complete non-Archimedean Menger PM-space, $\alpha : X \to [0, \infty]$ and $\psi, \varphi : [0, 1] \to [0, 1]$ as in definition and f be a $(\alpha - \varphi - \psi)$-contractive mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(ii) if x_n is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x, fx) \geq 1$. Then f has a fixed point. Moreover, if $y = fy$ implies $\alpha(f, fy) \geq 1$, then f has a unique fixed point.

Example 2.7. Let $X = [0, \infty)$, $T(a, b) = \min\{a, b\}$,

$$F_{x,y}(t) = \begin{cases} \dfrac{1}{1 + \max\{x, y\}} & \text{if } x \neq y, \\ 1 & \text{if } x = y, \end{cases}$$

for all $t > 0$, $fx = \dfrac{x}{(2(x+2))}$, $\beta^2(x, t) = k(t) = \dfrac{1}{2}$, $\alpha(x, y) = 1$ for all $x, y \in X$ and $t > 0$. Also define $\varphi(t) = 1 - t$ for all $t \in [0, 1]$.

Solution. Clearly (X, F, T) is a non-Archimedean Menger PM-space. Without loss of generality we assume that $x > y$. We have

$$fx = \frac{x}{(2(x+2))} \leq \frac{x}{x+2} \Longrightarrow xfx + 2fx \leq x.$$

Thus

$$\max\{x, y\} \max\{fx, fy\} + 2\max\{fx, fy\} \leq \max\{x, y\}.$$

Therefore

$$\max\{x, y\} \max\{fx, fy\} + \max\{fx, fy\} + \max\{x, y\} \leq 2\max\{x, y\} - \max\{fx, fy\},$$

and so

$$\begin{aligned} &(1 + \max\{fx, fy\})(1 + \max\{x, y\}) \\ &\leq 2\max\{x, y\} - \max\{fx, fy\} + 1 \\ &= 2(1 + \max\{x, y\}) - (1 + \max\{fx, fy\}). \end{aligned}$$

Hence, we have

$$1 \leq \frac{2(1 + \max\{x, y\}) - (1 + \max\{fx, fy\})}{(1 + \max\{fx, fy\})(1 + \max\{x, y\})} = 2F_{fx,fy}(t) - F_{x,y}(t).$$

Which implies

$$1 - F_{fx,fy}(t) \leq \frac{1}{2}(1 - F_{x,y}(t)).$$

That is

$$\alpha(x, fx)\alpha(y, fy)\varphi(F_{fx,fy}(t)) \leq \beta(x, t)\beta(y, t)\varphi(F_{x,y}(t)),$$

for all x, y with $x \neq y$ and hence f is a $(\alpha, \beta, \varphi)-$contractive mapping. Then all the conditions of Theorem (2.4) hold and f has a fixed point $x = 0$. Moreover, for all $x \in X$, we have $\alpha(x, fx) \geq 1$ and so the fixed point of f is unique.

Example 2.8. Let $X = [1, \infty)$, $T(a,b) = \min\{a,b\}$ and $F_{x,y}(t) = \dfrac{\min\{x,y\}}{\max\{x,y\}}$ for all $t > 0$. Define

$$fx = \begin{cases} \dfrac{\pi}{3} & \text{if } x \in [1,3], \\ \sqrt{1+x^2+e^x} & \text{if } x \in (3,+\infty). \end{cases}$$

Also define

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [1,3], \\ 0 & \text{otherewise,} \end{cases}$$

$\psi(t) = \dfrac{1}{2} - \dfrac{t}{2}$ and $\varphi(t) = 1 - t$ for all $t \in [0,1]$.

Solution. Clearly, (X, F, T) is a non-Archimedead Menger PM-space, $\psi, \varphi : [0,1] \to [0,1]$ are continuous, ψ is decreasing, $\psi(t) > \psi(1) - \varphi(1)$ and $\varphi(t) > 0$ for all $t \in (0,1)$.
Let $x, y \in [1,3]$. Then $\psi(F_{fx,fy}(t)) = 0$ and hence

$$\alpha(x,fx)\alpha(y,fy)\psi(F_{fx,fy}(t)) = 0 \leq \psi(F_{x,y}(t)) - \varphi(F_{x,y}(t)).$$

Otherewise, $\alpha(x,fx)\alpha(y,fy) = 0$ and so

$$\alpha(x,fx)\alpha(y,fy)\psi(F_{fx,fy}(t)) = 0 \leq \psi(F_{x,y}(t)) - \varphi(F_{x,y}(t)).$$

Since f is $\alpha-$admissible we obtain that f is a $(\alpha - \varphi - \psi)$-contractive mapping. Also conditions (i) and (ii) of Theorem(2.6) hold. Then f has a unique fixed point.

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Some fixed point theorems for C^*-algebra-valued α-contractive mappings

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Abstract

In this paper we introduce the concept of C^*-algebra-valued α-contractive mappings and then we give some fixed point theorems for these kind of mappings.

Keywords: C^*-algebra, contractive mapping, expansive mapping, fixed point, α-admissible.
Mathematics Subject Classification [2010]: 46L07; 47H70; 54H25.

1 Introduction

The notion of C^*-algebra-valued metric spaces has been investigated by Z. Ma, L. Jiang and H. Sun [1]. They presented some fixed point theorems for self-maps with contractive or expansive conditions on such spaces. Taking some ideas from [1, 3] we introduce the concept of C^*-algebra-valued α-contractive mappings and C^*-algebra-valued α-expansion mappings and then we deal with some fixed point theorems for these new ones.

We provide some notations, definitions and auxiliary facts which will be used later in this paper.
Let $\mathbb{A}$ be a unital algebra with unit I. An involution on $\mathbb{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathbb{A}$, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. An assign to each $*$-algebra is $(\mathbb{A}, *)$. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ $(a \in \mathbb{A})$. For more details we refer the reader to [2].
Throughout this manuscript, $\mathbb{A}$ stands for a unital C^*-algebra with unit I. We say an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x = x^*$ and $\sigma(x) \subseteq \mathbb{R}_+ = [0, \infty)$, where θ means the zero element in $\mathbb{A}$ and $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ as follows: $x \preceq y$ if and only if $y - x \succeq \theta$ $(x, y \in \mathbb{A})$. From now on, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

Remark 1.1. When $\mathbb{A}$ is a unital C^*-algebra, then for any $x \in \mathbb{A}_+$, $x \preceq I$ if and only if $\|x\| \leq 1$ ([2]).

Definition 1.2. ([1]) Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:

*Speaker

1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y)=\theta$ if and only if $x=y$;
2) $d(x,y)=d(y,x)$ for all $x,y \in X$;
3) $d(x,y) \preceq d(x,z)+d(z,y)$ for all $x,y,z \in X$.
Then d is called a C^*-algebra-valued metric on X and $(X,\mathbb{A},d)$ is called a C^*-algebra-valued metric space.

Lemma 1.3. *([2]) Suppose that $\mathbb{A}$ is a unital C^*-algebra with unit I.*
1) If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I-a$ is invertible and $\|a(I-a)^{-1}\| < 1$;
2) suppose that $a,b \in \mathbb{A}$ with $a,b \succeq \theta$ and $ab=ba$, then $ab \succeq \theta$;
3) by $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab=ba,$ for all $b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$, if $b,c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $I-a \in \mathbb{A}'_+$ is an invertible element, then $(I-a)^{-1}b \succeq (I-a)^{-1}c$.

Lemma 1.4. *([2]) Let $a,b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$.*

2 Main results

Definition 2.1. Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α-admissible if

$$\alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx,Ty) \geq 1.$$

Definition 2.2. Let $\{x_n\}$ be a sequence in a C^*-algebra-valued-metric space $(X,\mathbb{A},d)$.
1. $\{x_n\}$ is said to be a convergent to $x \in X$ with respect to $\mathbb{A}$, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \|d(x_n,x)\| = 0$.
2. $\{x_n\}$ is said to be a Cauchy sequence with respect to $\mathbb{A}$ in X, if $\lim_{n,m\to\infty} \|d(x_n,x_m)\| = 0$.
3. $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Definition 2.3. Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued metric space. We call a mapping $T : X \to X$ is a C^*-algebra-valued α-contractive mapping on X, if T is a α-admissible and there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that:
$\alpha(x,y)d(Tx,Ty) \preceq A^*d(x,y)A$, for each $x,y \in X$.

Theorem 2.4. *Assume that $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued metric space and $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ are two mappings. Suppose that the following conditions hold:*
(a) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$,
(b) T is a C^*-algebra-valued α-contractive mapping on X,
(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \geq 1$, then $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}$,
then T has a fixed point x^* in X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$, we define the sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$. Since T is α-admissible and $\alpha(x_0,x_1) = \alpha(x_0,Tx_0) \geq 1$ we deduce that

$$\alpha(x_1,x_2) = \alpha(Tx_0,Tx_1) \geq 1.$$

By induction, for all $n \in \mathbb{N}$ we get

$$\alpha(x_{n-1}, x_n) \geq 1.$$

Next we will show that $\{x_n\}$ is a Cauchy sequence in X. For each $n \in \mathbb{N}$ we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\preceq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \\ &\preceq A^* d(x_{n-1}, x_n) A. \end{aligned}$$

By repeating the process above, we get

$$d(x_n, x_{n+1}) \preceq (A^*)^n d(x_0, x_1) A^n = (A^*)^n B A^n, \quad (1)$$

where $B = d(x_0, x_1)$. Hence

$$\begin{aligned} \|d(x_n, x_{n+1})\| &\leq \|B^{\frac{1}{2}} A^n\|^2 \\ &\leq \|B\| \|A\|^n. \end{aligned}$$

Letting $n \to \infty$, one observes that $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Using condition (d) we get $\alpha(x_n, x^*)$ we have

$$\begin{aligned} d(x^*, Tx^*) &\preceq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &\preceq d(x^*, x_{n+1}) + \alpha(x_n, x^*) d(Tx_n, Tx^*) \\ &\preceq d(x^*, x_{n+1}) + A^* d(x_n, x^*) A. \end{aligned}$$

For all $n \in N$, letting $n \to \infty$, we obtain

$$d(Tx^*, x^*) = 0,$$

hence $Tx^* = x^*$, i.e., x^* is a fixed point of T.

□

Definition 2.5. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. We call a mapping $T : X \to X$ is a C^*-algebra-valued α-expansion mapping on X, if T is an α-admissible and satisfies the following conditions:
(E1) $T(X) = X$;
(E2) $d(Tx, Ty) \succeq \alpha(Tx, Ty) A^* d(x, y) A$, for each $x, y \in X$,
where A is an invertible element in $\mathbb{A}$ such that $\|A^{-1}\| < 1$.

Theorem 2.6. *Assume that $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space and $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ are two mappings. Suppose that the following conditions hold:*
(a) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,
(b) T is a C^*-algebra-valued α-expansion mapping on X,
(c) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$,
then T has a fixed point x^* in X.

Theorem 2.7. *Assume that* $(X, \mathbb{A}, d)$ *is a complete* C^**-algebra-valued metric space and* $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$ *are two mappings. Suppose that the following conditions hold: (a)* T *is* α*-admissible,*
(b) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,
(c) for all $x, y \in X$, *we have*

$$\alpha(x, y)d(Tx, Ty) \leq A[d(Tx, y) + d(Ty, x)],$$

where $A \in \mathbb{A}$ *and* $\|A\| < 1$,
(d) if $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in N$,
then T *has a fixed point* x^* *in* X.

Theorem 2.8. *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra-valued metric space. Suppose the mapping* $T : X \to X$ *satisfies the following condition for all* $x, y \in \mathbb{A}$

$$d(Tx, Ty) \preceq A[d(Tx, x) + d(Ty, y)],$$

where $A \in \mathbb{A}'_+$ *and* $\|A\| < \frac{1}{2}$. *Then* T *has a unique fixed point in* X.

Theorem 2.9. *Assume that* $(X, \mathbb{A}, d)$ *is a complete* C^**-algebra-valued metric space and* $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$ *are two mappings. Suppose that the following conditions hold: (a)* T *is* α*-admissible,*
(b) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,
(c) there exists $B \in \mathbb{A}$ *such that* $\sigma(B) \subseteq [1, \infty)$ *and*

$$[d(Tx, Ty) + B]^{\alpha(x,y)} \preceq A^* d(x, y)A + B$$

(d) if $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$,
then T *has a fixed point* x^* *in* X.

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SOME INEQUALITIES FOR THE NUMERICAL RADIUS OF OPERATORS

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Abstract

In this talk, we provide a generalization of a numerical radius inequality including product of two operators on a Hilbert space which is sharper than original inequality in a particular position. An application of this inequality to prove a numerical radius inequality that involves the generalized Aluthge transform is also given. In addition, our results generalize some known inequalities. For any $A, B, X \in \mathcal{B}(H)$ such that $A, B \geq 0$, we prepare new estimation for the numerical radius of two terms $A^\alpha X B^\alpha$, $A^\alpha X B^{1-\alpha}$ $(0 \leq \alpha \leq 1)$ and Heinz means. Other related inequalities are also discussed.

Keywords: Positive operator, numerical radius, Heinz means, Aluthge transform.
Mathematics Subject Classification [2010]: 47A12, 47A30, 47A63 47B47.

1 Introduction

Recall that an operator $A \in \mathcal{B}(H)$ is called positive, denote by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in$ H. The numerical radius of $A \in \mathcal{B}(H)$ is defined by

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\},$$

where $W(A)$ is the numerical range of A defined by $W(A) = \{\langle Ax, x\rangle : x \in \mathrm{H}, \|x\| = 1\}$. For a comprehensive account of theory of the numerical range and numerical radius we refer the reader to [2].

It is well known that $w(\cdot)$ defines a norm on $\mathcal{B}(H)$ such that for all $A \in \mathcal{B}(H)$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1}$$

On the second inequality in (1), Kittaneh [3] has shown that if $A \in \mathcal{B}(H)$, then

$$w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}). \tag{2}$$

Obviously, inequality (2) is sharper than the second inequality of (1).
Inequalities (1) are sharp. If $A^2 = 0$, then $w(A) = \frac{1}{2}\|A\|$, while if A is normal, then $w(A) = \|A\|$. For $A \in \mathcal{B}(H)$, let $A = U|A|$ be the polar decomposition of A, the Aluthge

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transform of A is defined by $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. Here U is partial isometry and $|A| = (A^*A)^{\frac{1}{2}}$. Yamazaki [6] has established an improvement of inequality (2) as follows:

$$w(A) \leq \frac{1}{2}(\|A\| + w(\tilde{A})). \tag{3}$$

The Euclidean operator radius of two bounded linear operators in a Hilbert space denoted by

$$w_e(B, C) = \sup_{\|x\|=1} \left(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\right)^{\frac{1}{2}}.$$

Several investigation on Euclidean operator radius and its extension to $n-$tuples of operators can be found in [4]. Dragomir [1] proved that for any $A, B \in \mathcal{B}(H)$ and for all $r \geq 1$,

$$w^r(B^*A) \leq \frac{1}{2}\|(A^*A)^r + (B^*B)^r\|, \tag{4}$$

$$w^2(A) \leq \frac{1}{2}\big(w(A^2) + \|A\|^2\big). \tag{5}$$

Some interesting numerical radius inequalities improving inequalities in (1) have been obtained in [3, 5, 6]. In this note, we first generalize inequality (4). Our generalization of inequality (4) in a special case is sharper than this inequality. Moreover, we apply our results to prove an extension of inequality (3) that contains the generalized Aluthge transform. A generalization of inequality (5) for any $r \geq 1$, is also obtained. Next we present two different versions of numerical radius inequality for Heinz means. Furthermore, upper bounds for two terms $A^\alpha X B^\alpha$ and $A^\alpha X B^{1-\alpha}$ under conditions $A, B \geq 0$ and $0 \leq \alpha \leq 1$ are given.

2 Main Results

We start this section to give an upper bound for $w(B^*A)$. This estimation is better than inequality (4) in a particular case when both A and B are normal operators.

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathrm{H})$. Then*

$$w^r(B^*A) \leq \frac{1}{4}\|(AA^*)^r + (BB^*)^r\| + \frac{1}{2}w^r(AB^*).$$

for all $r \geq 1$.

By Theorem 2.1 and inequality (4), we have

$$w^r(B^*A) \leq \frac{1}{4}\|(AA^*)^r + (BB^*)^r\| + \frac{1}{2}w^r(AB^*) \leq \frac{1}{2}\|(AA^*)^r + (BB^*)^r\|.$$

Utilizing Theorem 2.1, we obtain an extension of inequality (3).

Corollary 2.2. *Let $A \in \mathcal{B}(H)$ and $A = U|A|$ be the polar decomposition of A, and let $\tilde{A}(\alpha) = |A|^\alpha U|A|^{1-\alpha}$ be the generalized Aluthge transformation of A. Then we have*

$$w^r(A) \leq \frac{1}{4}\||A|^{2r\alpha} + |A|^{2r(1-\alpha)}\| + \frac{1}{2}w^r(\tilde{A}(\alpha)).$$

holds for $r \geq 1$.

The following proposition gives to us other bound for the numerical radius.

Proposition 2.3. *Let $A \in \mathcal{B}(\mathrm{H})$ and f, g be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t,\ (t \geq 0)$. Then*

$$w^{2r}(A) \leq \frac{1}{2}\left(\|A\|^{2r} + \left\| \frac{1}{p} f^{pr}(|A^2|) + \frac{1}{q} g^{qr}(|(A^2)^*|) \right\| \right).$$

for all $r \geq 1$, $p \geq q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $qr \geq 2$.

The next result is a generalization of inequality (5) for any $r \geq 1$.

Theorem 2.4. *If $A \in \mathcal{B}(\mathrm{H})$, then*

$$w^{2r}(A) \leq \frac{1}{2}\big(w^r(A^2) + \|A\|^{2r}\big).$$

for any $r \geq 1$.

In the rest of this section, we are going to obtain upper bounds for $A^\alpha X B^\alpha$ and $A^\alpha X B^{1-\alpha}$ $(0 \leq \alpha \leq 1)$.

The next result detect an upper bound for power of the numerical radius of $A^\alpha X B^{1-\alpha}$ under assumption $0 \leq \alpha \leq 1$.

Theorem 2.5. *Suppose $A, B, X \in \mathcal{B}(\mathrm{H})$ such that A, B are positive. Then*

$$w^r(A^\alpha X B^{1-\alpha}) \leq \|X\|^r \|\alpha A^r + (1-\alpha) B^r\|.$$

for all $r \geq 2$ and $0 \leq \alpha \leq 1$.

The Heinz means for matrices are defined by

$$H_\alpha(A, B) = \frac{A^\alpha X B^{1-\alpha} + A^{1-\alpha} X B^\alpha}{2}$$

For any $A, B, X \in \mathcal{B}(\mathrm{H})$ in which $0 \leq \alpha \leq 1$ and $A, B \geq 0$.

The following lemma is an essential item for proving the numerical radius of Heinz means.

Lemma 2.6. *Let $A, B \in \mathcal{B}(\mathrm{H})$ be invertible self-adjoint operators and $X \in \mathcal{B}(\mathrm{H})$. Then*

$$w(X) \leq w\left(\frac{AXB^{-1} + A^{-1}XB}{2} \right).$$

One of our main results is to find a numerical radius inequality for Heinz means. For this purpose, we use Theorem 2.5, the convexsity of function $f(t) = t^r (r \geq 1)$ and Lemma 2.6.

Theorem 2.7. *Suppose $A, B, X \in \mathcal{B}(\mathrm{H})$ such that A, B are positive. Then*

$$\begin{aligned} w^r(A^{\frac{1}{2}} X B^{\frac{1}{2}}) &\leq w^r\big(H_\alpha(A, B)\big) \\ &\leq \|X\|^r w\left(\frac{A^r + B^r}{2} \right) \\ &\leq \frac{\|X\|^r}{2}\Big(\big\|\alpha A^r + (1-\alpha) B^r\big\| + \big\|(1-\alpha) A^r + \alpha B^r\big\| \Big). \end{aligned}$$

for all $r \geq 2$ and $0 \leq \alpha \leq 1$.

In the next result we give an additional upper bound for norm of Heinz means. Applying this norm inequality then we find an another numerical radius inequality for Heinz means.

Theorem 2.8. *Suppose $A, B, X \in \mathcal{B}(\mathrm{H})$ such that A, B are positive. Then*

$$\left\|\frac{A^{\alpha}XB^{1-\alpha}+A^{1-\alpha}XB^{\alpha}}{2}\right\|^{2} \leq \|X\|^{2}\left\|\frac{A^{2\alpha r}+A^{2(1-\alpha)r}}{2}\right\|^{\frac{1}{r}}\left\|\frac{B^{2\alpha s}+B^{2(1-\alpha)s}}{2}\right\|^{\frac{1}{s}}$$

for all $r, s \geq 1$ and $0 \leq \alpha \leq 1$.

By putting $s = r$ and the second inequality of (1), we reach the following result as follows.

Corollary 2.9. *Assume $A, B, X \in \mathcal{B}(\mathrm{H})$ such that A, B are positive. Then*

$$w^{2r}\Big(\frac{A^{\alpha}XB^{1-\alpha}+A^{1-\alpha}XB^{\alpha}}{2}\Big) \leq \|X\|^{2r}\left\|\frac{A^{2\alpha r}+A^{2(1-\alpha)r}}{2}\right\|\left\|\frac{B^{2\alpha r}+B^{2(1-\alpha)r}}{2}\right\|.$$

for all $0 \leq \alpha \leq 1$ and $r \geq 1$.

Our final result in this section provide a new bound for powers of the numerical radius.

Theorem 2.10. *Suppose $A, B, X \in \mathcal{B}(\mathrm{H})$ such that A, B are positive. Then*

$$w^{r}(A^{\alpha}XB^{\alpha}) \leq \|X\|^{r}\|\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\|^{\alpha}.$$

for all $0 \leq \alpha \leq 1$, $r \geq 0$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr, qr \geq 2$.

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Some new singular value inequalities for compact operators

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Abstract

In this paper, by applying the concept of operator h-convex functions we prove several singular value inequalities for operators which provide refinements of previous results.

Keywords: Hermite-Hadamard inequality, Operator h-convex function, Singular value inequality
Mathematics Subject Classification [2010]: 47A63, 47B05, 26D15

1 Introduction

Let $B(H)$ stand for the C^*-algebra of all bounded linear operators on a complex separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in B(H)$ is positive and write $A \geq 0$ if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. Let $B(H)^+$ stand for all positive operators in $B(H)$.

If A is a self-adjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{1}$$

A real valued continuous function f on an interval I is said to be *operator convex* if

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)f(A) + \lambda f(B), \tag{2}$$

in the operator order, for all $\lambda \in [0,1]$ and for every self-adjoint operator A and B on a Hilbert space H whose spectra are contained in I (see [3]).

As an example of such functions, we note that $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$ (see [1, p.147]).

In [3], Dragomir investigated the operator version of the Hermite-Hadamard inequality for operator convex functions asserts that if $f : I \to \mathbb{R}$ is an operator convex function on the interval I then, for any self-adjoint operators A and B with spectra in I the following inequalities hold

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\left((1-t)A + tB\right) dt \leq \frac{f(A)+f(B)}{2}. \tag{3}$$

*Speaker

2 Inequalities for operator h-convex function

In this section, we give Hermite-Hadamard type inequalities for operator h-convex functions.

Let $I, J \subseteq \mathbb{R}$, $(0,1) \subseteq J$ and functions f, h are real non-negative on I and J.

Definition 2.1. [8] Let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\lambda \in (0,1)$ we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{4}$$

The following inequalities due to Sarikaya [7], gives the Hermite-Hadamard type inequalities for h-convex functions. Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L^1([a,b])$. Then

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (f(a)+f(b))\int_0^1 h(t)dt. \tag{5}$$

Here, we define operator h-convex function.

Definition 2.2. A continuous function $f : I \to \mathbb{R}$ is said to be operator h-convex on I if

$$f(\lambda A + (1-\lambda)B) \leq h(\lambda)f(A) + h(1-\lambda)f(B), \tag{6}$$

for all $\lambda \in (0,1)$ and self-adjoint $A, B \in B(H)$ whose spectra are contained in I.

Theorem 2.3. *Let f be an operator h-convex function. Then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B)dt \leq (f(A)+f(B))\int_0^1 h(t)dt. \tag{7}$$

Let $h(t) = t^s$ for $s \in (0,1)$ and $h(t) = t$ in (7) respectively, then we have

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B)dt \leq \frac{f(A)+f(B)}{s+1}, \tag{8}$$

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B)dt \leq \left(\frac{f(A)+f(B)}{2}\right). \tag{9}$$

Example 2.4. [5] Let $AB + BA \geq 0$ for $A, B \in B(H)^+$, (AB+BA is called symmetrized product of A and B) then the continuous function $f(t) = t^s$, $0 < s \leq 1$ is an operator s-convex function on $[0, \infty)$.

3 Some singular value inequalities for operators

In this section we give some inequalities for singular values of operators. First we recall some preliminaries.

Let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$

is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T and all unitary operators U and V in $B(H)$.

We denote the singular values of an operator $A \in K(H)$ as $s_1(A) \geq s_2(A) \geq \dots$ are the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$ which repeated accordingly to multiplicity.

The following inequality is due to Hirzallah and Kittaneh [6, Corollary 2.2] asserts that if $A, B \in K(H)$, then

$$s_j\left(\frac{A+B}{2}\right) \leq s_j(A \oplus B), \tag{10}$$

for $j = 1, 2, \dots$.

We give a refinement of above inequality for positive operators.

Theorem 3.1. *Let X be an arbitrary operator in $B(H)$. Then,*

1. *We have*

$$\begin{aligned} \frac{1}{2}s_j\left((A+B)^{1/2}X\right)^{2r} &\leq s_j\left(\int_0^1 (X^*(tA+(1-t)B)X)^r dt\right) \\ &\leq \frac{2}{r+1}\|X\|^{2r} s_j^r(A \oplus B), \end{aligned}$$

for $j = 1, 2, \dots$ where $r \in [0, \frac{1}{2}]$ and positive operators $A, B \in K(H)$ such that $AB + BA \geq 0$.

2. *We also have*

$$\begin{aligned} \frac{1}{2^r}s_j((A+B)^{1/2}X)^{2r} &\leq s_j\left(\int_0^1 (X^*(tA+(1-t)B)X)^r dt\right) \\ &\leq s_j\left(|A^{1/2}X|^{2r} \oplus |B^{1/2}X|^{2r}\right), \end{aligned}$$

for $j = 1, 2, \dots$ where $r \in [-1, 0] \cup [1, 2]$ and positive operators $A, B \in K(H)$.

Theorem 3.2. *Let $A, B \in K(H)$ such that $A^*AB^*B + B^*BA^*A \geq 0$. Then*

$$\frac{3}{2\sqrt{2}}s_j^{\frac{1}{2}}(AB^*) \leq \frac{3}{2}s_j\left(\int_0^1 (tA^*A + (1-t)B^*B)^{\frac{1}{2}} dt\right) \leq s_j(|A| + |B|),$$

for $j = 1, 2, \dots$.

Above inequality gives a generalization the main inequality in [4].

The following inequality due to Bhatia and Kittaneh [2] asserts that if $A, B \in B(H)$ are positive operators and m is any positive integer, then

$$|||A^m + B^m||| \leq |||(A+B)^m|||.$$

We obtain several singular value and unitarily invariant inequalities motivated by above inequality.

Theorem 3.3. *Let $A, B \in K(H)^+$ and $r \in [-1, 0] \cup [1, 2]$, then*

$$s_j(A+B)^r \leq 2^r s_j\left(\int_0^1 (tA+(1-t)B)^r dt\right) \leq 2^{r-1} s_j(A^r+B^r), \tag{11}$$

for $j = 1, 2, \ldots$.

Corollary 3.4. *Let $A, B \in K(H)^+$ then*

$$|||(A+B)^r||| \leq 2^r \left|\left|\left|\int_0^1 (tA+(1-t)B)^r dt\right|\right|\right| \leq 2^{r-1}|||A^r+B^r|||.$$

for $r \in [-1, 0] \cup [1, 2]$ and

$$|||A^r+B^r||| \leq 2\left|\left|\left|\int_0^1 (tA^r+(1-t)B^r)^{\frac{1}{r}} dt\right|\right|\right|^r \leq 2^{1-r}|||(A+B)^r|||.$$

for $r \in [\frac{1}{2}, 1]$.

Remark 3.5. Let a and b be positive real numbers. Then,

$$(a+b)^r \leq 2^{r-1}(a^r+b^r) \tag{12}$$

for $r \geq 1$.

The following inequality

$$s_j(A+B)^r \leq 2^{r-1} s_j(A^r+B^r),$$

which obtained in (11), gives an operator version of (12) for $r \in [1, 2]$.

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Some properties of λ-spirallike functions with respect to $2k$-symmetric conjugate points

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Abstract

In the present paper, we introduce and investigate some new subclass of λ-spirallike functions with respect to $2k$-symmetric conjugate points. Also we obtain some integral representations for functions belonging to this classes.

Keywords: λ-Spirallike functions, Differential subordination, $2k$-Symmetric Points
Mathematics Subject Classification [2010]: 30C45, 30C50

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C}; |z| < 1\}$. Let $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{SP}(\lambda)$ denote the familiar subclass of $\mathcal{A}$ consisting of functions which are, respectively, univalent, starlike and λ-spirallike in $\mathcal{U}$ (See for details, [2, 3]).

We also let $\mathcal{P}$ denote the class of analytic function of the form

$$p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m, \quad (z \in \mathcal{U}),$$

which satisfy the condition that $Re(p(z)) > 0$.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. The function f is said to be subordinate to g if there exists a function h analytic in $\mathcal{U}$ such that $|h(z)| \leq |z|$ and $f(z) = g(h(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$ (see [2]).

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^k_{sc}(\phi)$ if $f(z)$ satisfies the condition

$$\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} \prec \phi(z), \quad z \in \mathcal{U},$$

where $\phi(z) \in \mathcal{P}$. The classes $\mathcal{S}^k_{sc}(\phi)$ of functions starlike with respect to symmetric conjugate point is considered recently by Ravichandran [4]. We refer to the monographs [1], [6] for more details.

*Speaker

Definition 1.1. [5] The function $f \in \mathcal{A}$ is in the class $\mathcal{SP}^*(\lambda, \phi)$ if it is satisfies the subordination condition

$$e^{-i\lambda}\frac{zf'(z)}{f(z)} \prec \cos\lambda\phi(z) + i\sin\lambda,$$

where $\phi \in \mathcal{P}$ and λ real with $|\lambda| < \pi/2$.

In this paper, we give the definition of λ-spirallike functions with respect to k-conjugate point and obtain the integral representation for the function belonging to this classes.

2 Main results

Definition 2.1. A function $f \in \mathcal{A}$ is said to be λ-spirallike with respect $2k$-symmetric conjugate point which satisfy the inequality

$$Re\Big(e^{i\lambda}\frac{zf'(z)}{f_{2k}(z)}\Big) > 0,$$

where $k \geq 1$ is a fixed positive integer and f_{2k} is defined by

$$f_{2k}(z) = \frac{1}{2k}\sum_{v=0}^{k-1}\Big(\epsilon^{-v}f(\epsilon^v z) + \epsilon^v\overline{f(\epsilon^v\overline{z})}\Big), \quad \epsilon = exp(\frac{2\pi i}{k}). \tag{2}$$

Definition 2.2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{SP}^k_{sc}(\lambda, \phi)$ if it satisfies the subordination condition

$$e^{-i\lambda}\frac{zf'(z)}{f(z)} \prec \cos\lambda\phi(z) + i\sin\lambda, \tag{3}$$

where $\lambda \in \mathbb{R}$ with $|\lambda| < \pi/2$, $\phi \in \mathcal{P}$ and f_{2k} is defined by (2). Also a function $f \in \mathcal{A}$ is said to be in the $\mathcal{KSP}^k_{sc}(\lambda, \phi)$ if and only if

$$zf'(z) \in \mathcal{SP}^k_{sc}(\lambda, \phi) \quad z \in \mathcal{U}.$$

Theorem 2.3. *Let $\phi(z) \in \mathcal{P}$, then we have $\mathcal{SP}^k_{sc}(\lambda, \phi) \subset \mathcal{SP}(\lambda) \subset \mathcal{S}$.*

Proof. Suppose that $f(z) \in \mathcal{SP}^k_{sc}(\lambda, \phi)$, it suffices to show that $f_{2k} \in \mathcal{SP}(\lambda)$. From the condition 3 , we have

$$Re\Big\{e^{-i\lambda}\frac{zf'(z)}{f(z)}\Big\} > 0, \quad z \in \mathcal{U},$$

since $Re\{\phi(z)\} > 0$. Substituting z by $\epsilon^\mu z$, $(\mu = 0, 1, ..., k-1)$, we have

$$Re\Big\{e^{-i\lambda}\frac{\epsilon^\mu zf'(\epsilon^\mu z)}{f_{2k}(\epsilon^\mu z)}\Big\} > 0, \quad z \in \mathcal{U}. \tag{4}$$

From the inequality (4), we obtain

$$Re\Big\{e^{-i\lambda}\frac{\overline{\epsilon^\mu\overline{z}}\,\overline{f'(\epsilon^\mu\overline{z})}}{\overline{f_{2k}(\epsilon^\mu\overline{z})}}\Big\} > 0, \quad z \in \mathcal{U}. \tag{5}$$

Note that $f_{2k}(\epsilon^{\mu}z)=\epsilon^{\mu}f_{2k}(z)$ and $\overline{f_{2k}(\epsilon^{\mu}z)}=\epsilon^{-\mu}f_{2k}(z)$ the inequality (4) and (5) can be written as

$$Re\Big\{e^{-i\lambda}\frac{zf'(\epsilon^{\mu}z)}{f_{2k}(\epsilon^{\mu}z)}\Big\}>0,\quad z\in\mathcal{U},\tag{6}$$

and

$$Re\Big\{e^{-i\lambda}\frac{z\overline{f'(\epsilon^{\mu}\overline{z})}}{f_{2k}(\epsilon^{\mu}z)}\Big\}>0,\quad z\in\mathcal{U}.\tag{7}$$

From the inequalities (6) and (8), we can get

$$Re\Big\{e^{-i\lambda}\frac{z(f'(\epsilon^{\mu}z)+\overline{f'(\epsilon^{\mu}\overline{z})})}{f_{2k}(z)}\Big\}>0,\quad z\in\mathcal{U}.\tag{8}$$

Let $\mu=0,1,2,...,k-1$ in (8) respectively, and summing them we have

$$Re\Bigg\{e^{-i\lambda}\frac{z\Big(\frac{1}{2k}\sum_{\mu=0}^{k-1}f'(\epsilon^{\mu}z)+\overline{f'(\epsilon^{\mu}\overline{z})}\Big)}{f_{2k}(z)}\Bigg\}>0,\quad z\in\mathcal{U},$$

or equivalently

$$Re\Big\{e^{-i\lambda}\frac{zf'_{2k}(z)}{f_{2k}(z)}\Big\}>0,\quad z\in\mathcal{U},$$

that is $f_{2k}(z)\in\mathcal{SP}(\lambda)\subset\mathcal{S}$. □

Theorem 2.4. *Let $f(z)\in\mathcal{SP}^{k}_{sc}(\lambda,\phi)$, then we have*

$$f_{2k}(z)=(e^{-i\lambda}z)^{e^{i\lambda}}exp\Bigg\{\frac{\cos\lambda}{2k}\sum_{\mu=0}^{k-1}\int_0^{\xi}\frac{1}{e^{i\lambda}\xi}\Big(\phi(w(\epsilon^{\mu}\xi))+\overline{\phi(w(\epsilon^{\mu}\overline{\xi}))}\Big)d\xi\Bigg\}.\tag{9}$$

where $f_{2k}(z)$ is defined by (2) *and $w(z)$ is analytic in $\mathcal{U}$ and $w(0)=0,\ |w(z)|\le 1$.*

Proof. Let $f(z)\in\mathcal{SP}^{k}_{sc}(\lambda,\phi)$. From the definition of $\mathcal{SP}^{k}_{sc}(\lambda,\phi)$, we have

$$e^{-i\lambda}\frac{zf'(z)}{f(z)}=\cos\lambda\phi(w(z))+i\sin\lambda,\quad z\in\mathcal{U},\tag{10}$$

where $w(z)$ is analytic in $\mathcal{U}$, $w(0)=0$ and $|w(z)|<1$. Substituting by $\varepsilon^{\mu}z$, $(\mu=0,1,2,...,k-1)$ in (10), we have

$$e^{-i\lambda}\frac{\varepsilon^{\mu}zf'(\varepsilon^{\mu}z)}{f_{2k}(\varepsilon^{\mu}z)}=\cos\lambda\phi(w(\varepsilon^{\mu}z))+i\sin\lambda,\quad z\in\mathcal{U}.\tag{11}$$

From the above inequality, we have

$$e^{-i\lambda}\frac{\overline{\varepsilon^{\mu}\overline{z}}\,\overline{f'(\varepsilon^{\mu}\overline{z})}}{\overline{f_{2k}(\varepsilon^{\mu}\overline{z})}}=\cos\lambda\overline{\phi(w(\varepsilon^{\mu}\overline{z}))}-i\sin\lambda,\quad z\in\mathcal{U}.\tag{12}$$

Summing equalities (11) and (12), and making use of the same method as in theorem 2.3, we have

$$e^{-i\lambda}\frac{f'_{2k}(z)}{f_{2k}(z)}=\frac{\cos\lambda}{2k}\sum_{\mu=0}^{k-1}\frac{1}{e^{-i\lambda}z}\Big(\phi(w(\epsilon^{\mu}z))+\overline{\phi(w(\epsilon^{\mu}\overline{z}))}\Big).$$

From the above equality, we have

$$\frac{f'_{2k}(z)}{f_{2k}(z)} - \frac{1}{e^{-i\lambda}z} = \frac{\cos\lambda}{2k}\sum_{\mu=0}^{k-1}\frac{1}{e^{-i\lambda}z}\Big(\phi(w(\epsilon^{\mu}z)) + \overline{\phi(w(\epsilon^{\mu}\overline{z}))} - 2\Big).$$

By integrating this equality, we have

$$\log\frac{f_{2k}(z)}{(e^{-i\lambda}z)^{e^{i\lambda}}} = \frac{\cos\lambda}{2k}\sum_{\mu=0}^{k-1}\int_0^{\xi}\frac{1}{e^{i\lambda}\xi}\Big(\phi(w(\epsilon^{\mu}\xi)) + \overline{\phi(w(\epsilon^{\mu}\overline{\xi}))}\Big)d\xi.$$

From this equality we can get (9). Hence the proof is complete. □

Corollary 2.5. *Let $f(z) \in \mathcal{SP}^k_{sc}(\lambda,\phi)$, then we have*

$$f(z) = \int_0^z (e^{-i\lambda}z)^{e^{i\lambda}} exp\Big\{\frac{\cos\lambda}{2k}\sum_{\mu=0}^{k-1}\int_0^{\xi}\frac{1}{e^{i\lambda}\xi}\Big(\phi(w(\epsilon^{\mu}\xi))+\overline{\phi(w(\epsilon^{\mu}\overline{\xi}))}\Big)\Big(\cos\lambda\phi(w(\xi))+i\sin\lambda\Big)d\xi\Big\}.$$

where $f_{2k}(z)$ is defined by (2) *and $w(z)$ is analytic in $\mathcal{U}$, $w(0) = 0$ and $|w(z)| \leq 1$.*

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Some Results concerning 2-frames

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Abstract

In this paper, we show that a finite sequence of vectors in 2-Hilbert space can be a 2-frames for the linear span of their elements, and introduce the optimal 2-frame bounds according to the frame operators.

Keywords: 2-inner product space, 2-frame, 2-frame bounds
Mathematics Subject Classification [2010]: 46C50, 42C15

1 Introduction

Let H be a Hilbert space and I a set which is finite or countable. A collection $\{f_i\}_{i\in I}$ $\subseteq H$ is called a frame for H if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |< f, f_i >|^2 \leq B\|f\|^2$$

for all $f \in H$. The constants A and B are called frame bounds. Frames have many applications in mathematics and engineering including wavelet theory, signal and image processing, operator theory, harmonic analysis and so on [5, 7]. A sequence satisfying the upper frame condition is called a Bessel sequence. For a frame $\{f_i\}_{i\in I}$ of H, the operator $T : \ell^2(\mathbb{N}) \to H$ defined by $Te_i = f_i$, $i \in \mathbf{N}$ is called the pre-frame operator. The frame operator $S = TT^*$ is defined by $S(f) = \sum_{i\in I} < f, f_i f_i$. A technique for representing the elements of a Hilbert space introduced by Duffin and Schaeffer [6] by frame theory. Nowadays frames work an alternative to orthonormal bases in Hilbert spaces which has many advantages [7]. In [1] A. A. Arefijamaal and Gh. Sadeghi have also introduced definition of 2-frame for a 2-inner product space and described some properties of them. First of all we recall the concept of 2-inner product space was first introduced by Y. J. Cho, et al, in [3].

Definition 1.1. Let X be a linear space of dimension greater than 1 over the field $\mathbf{F}$. Suppose that $(.,.|.)$ is a function from $X \times X \times X$ into $\mathbf{F}$ satisfying the following conditions:

(i) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ iff x and z are linearly dependent; (ii) $(x, x|z) = (z, z|x)$;

(iii) $(y, x|z) = \overline{(x, y|z)}$;

(iv) $(\alpha x, x|z) = \alpha(x, x|z)$ for all every $\alpha \in \mathbf{F}$;

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(v) $(x_1 + x_2, y|z) = (x_1, y|x) + (x_2, y|z)$.

$(.,. \mid .)$ is called a 2-inner product on X and $(X, (.,. \mid .))$ is called a 2-inner product space.

For more details see [8]. A 2-inner product space X is called a 2-Hilbert space if it is complete, with respect to the 2-metric defined by 2-inner product. A sequence $\{x_n\}$ of X is said to be convergent if there exists an element $a \in X$ such that $\lim \parallel x_n - a, x \parallel = 0$ for all $x \in X$. A subset B of a 2-normed space X is said to be compact if every sequence $\{x_n\}$ of B has a convergent subsequence in B. For a 2-norm space X, consider the subsets

$$B_\omega(a, r) = \{x, \parallel x - a, \omega \parallel < r\}$$

and

$$B_\omega[a, r] = \{x, \parallel x - a, \omega \parallel \leq r\},$$

of X.

Definition 1.2. [2] Let X be a linear space of dimension greater than 2 over the field $\mathbf{R}$. Suppose that $(.,.|.,.)$ is a function from X^4 into $\mathbf{R}$ satisfying the following conditions:

(1) $(a_1, a_2|a_1, a_2) > 0$ if a_1, a_2 are linearly independent vectors;

(2) $(a_1, a_2|b_1, b_2) = (b_1, b_2|a_1, a_2)$ for any $a_1, a_2, b_1, b_2 \in X$;

(3) $(\lambda a_1, a_2|b_1, b_2)=\lambda(a_1, a_2|b_1, b_2)$ for any scalar $\lambda \in \mathbf{R}$, and any $a_1, a_2, b_1, b_2 \in X$;

(4) $(a_1, a_2|b_1, b_2)=-(a_{\sigma(1)}, a_{\sigma(2)}|b_1, b_2)$ for any odd permutation σ in the set $\{1, 2\}$, and any $a_1, a_2, b_1, b_2 \in X$;

(5) $(a_1 + c_1, a_2|b_1, b_2) = (a_1, a_2|b_1, b_2) + (c_1, a_2|b_1, b_2)$ for any $a_1, a_2, c_1, b_1, b_2 \in X$; (5) $(a_1 + c_1, a_2|b_1, b_2) = (a_1, a_2|b_1, b_2) + (c_1, a_2|b_1, b_2)$ for any $a_1, a_2, c_1, b_1, b_2 \in X$;

(6) $(a_1, b_1, ..., b_{i-1}, b_{i+1}, ..., b_2|b_1, b_2) = 0$ for each $i \in \{1, 2\}$, then $(a_1, a_2|b_1, b_2) = 0$ for arbitary vector a_2.

Then the function $(.,.|.,.)$ is called an generelized 2-inner product and the pair $(X, (.,.|.,.))$is called a generelized 2-inner product space. Also know that generelized 2-inner product is a continupous map. For more details see [2].

Remark 1.3. [2] In the special case of definition 1.2 if we consider such pairs of sets a_1, a_2, b_1, b_2 which differ from at most one vectors for example $a_1 = a, b_1 = b, a_2 = b_2 = x_1$, then by putting $(a, b|x_1) = (a, x_1|b, x_1)$ we obtain a 2-inner product.

Definition 1.4. [1] Let $(X, (.,. \mid .))$ be a 2-Hilbert space and $\omega \in X$. A sequence $\{f_i\}_{i=1}^{\infty}$ of elements in X is called a 2-frame (associated to ω) if there exists $A, B > 0$ such that

$$A \parallel f, \omega \parallel^2 \leq \sum_{i=1}^{\infty} \mid (f, f_i \mid \omega) \mid^2 \leq B \parallel f, \omega \parallel^2$$

for all $f \in X$. A sequence satisfying the upper 2-frame condition is called a 2-Bessel sequence.

2 Main results

Let $(X, (.,.|.))$ a 2-Hilbert space and $\omega \in X$. Let $\{f_i\}_{i=1}^k$ be a sequence in X be such that $f_i \neq 0$ for all $i = 1, ..., k$. So by the Cauchy-Schwarz inequality, we observe that

$$\sum_{i=1}^{k} \mid (f, f_i \mid \omega) \mid^2 \leq \sum_{i=1}^{k} \parallel f_i, \omega \parallel^2 \parallel f, \omega \parallel^2,$$

for all $f \in X$. Now by considering the set $F = \{\sum_{i=1}^k \mid (f, f_i \mid \omega) \mid^2, f \in span\{f_i\}_{i=1}^k\}$, it can be seen that F is a compact subset of the real line and contains its infimum. In fact F is the range of the continuouse function from $span\{f_i\}_{i=1}^k$ into $\mathbf{R}$. So we can find $g \in span\{f_i\}_{i=1}^k$ with $\parallel g, \omega \parallel = 1$ such that

$$A = \sum_{i=1}^{k} \mid (g, f_i \mid \omega) \mid^2 = inf\{\sum_{i=1}^{k} \mid (f, f_i \mid \omega) \mid^2 : f \in W, \parallel f, \omega \parallel = 1\} > 0.$$

Then for any $f \in span\{f_i\}_{i=1}^k$, $f \neq 0$, we have

$$\begin{aligned} \sum_{i=1}^{k} \mid (f, f_i \mid \omega) \mid^2 &= \sum_{i=1}^{k} \mid (\frac{f}{\parallel f, \omega \parallel}, f_i \mid \omega) \mid^2 \parallel f, \omega \parallel^2 \\ &\geq A \parallel f, \omega \parallel^2 . \end{aligned}$$

Then we have proved the following theorem.

Theorem 2.1. *Any finite subset of a 2-Hilbert space is a 2-frame for its span.*

Clearly a family of elements $\{f_i\}_{i=1}^k$ in 2-Hilbert space H is a frame for H if and only if $H = span\{f_i\}_{i=1}^k$. So the frame might contains more elements than needed to be a basis. Now if $\{f_i\}_{i=1}^\infty$ is a 2-frame (associated to ω), and

$$A \parallel f, \omega \parallel^2 \leq \sum_{i=1}^{\infty} \mid (f, f_i \mid \omega) \mid^2 \leq B \parallel f, \omega \parallel^2$$

for all $f \in X$, then for optimal upper 2-frame bound B we have

$$\begin{aligned} B = sup_{\parallel f, \omega \parallel = 1} \sum_{i=1}^{\infty} \mid (f, f_i \mid \omega) \mid^2 &= sup_{\parallel f, \omega \parallel = 1} (S_\omega f, f \mid \omega) \\ &= sup_{\parallel f \parallel_\omega = 1} < S_\omega f, f >_\omega \\ &= \parallel S_\omega \parallel . \end{aligned}$$

On the other hand

$$\parallel S_\omega \parallel = \parallel T_\omega T_\omega^* \parallel = \parallel T_\omega \parallel^2,$$

where S_ω is a 2-frame operator and T_ω is a 2-pre frame operator [1]. Since $\{S_\omega^{-1} f_i\}_{i=1}^\infty$ is also a frame for X with upper 2-frame bound A^{-1} and 2-frame operator S_ω^{-1}, then

$A^{-1} = \| S_\omega^{-1} \|$. Finally, via Theorem 5.3.7 of [4]

$$\begin{aligned} \| S_\omega^{-1} \| &= sup_{\|f,\omega\|=1} \sum_{i=1}^{\infty} \mid (f, S_\omega^{-1} f_i \mid \omega) \mid^2 \\ &= sup_{\|f,\omega\|=1} \sum_{i=1}^{\infty} |< f, S_\omega^{-1} f_i >_\omega|^2 \\ &= Sup_{\|f\|_\omega=1} \| T_\omega^\dagger f \|^2 \\ &= \| T_\omega^\dagger \|^2, \end{aligned}$$

where $T_\omega^\dagger$ is the pseudo inverse 2-pre frame operator T_ω. Now we are ready to give the following theorem.

Theorem 2.2. *Let $(X, (.,.|.))$ be a 2-Hilbert space and $\omega \in X$. Also $\{f_i\}_{i=1}^\infty$ is a 2-frame (associated to ω) with the optimal 2-frame bounds A, B. Then A, B are given by*

$$A = \| S_\omega^{-1} \|^{-1} = \| T_\omega^\dagger \|^{-2}, B = \| S_\omega \| = \| T_\omega \|^2 .$$

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Some results on t-remotest points and t-approximate remotest points in fuzzy normed spaces

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Abstract

In this paper, we define t-remotest points and t-approximate remotest points in fuzzy normed spaces and prove some theorems on theses concepts. In particular, we find t-remotest points and t-approximate remotest points by considering a cyclic map.

Keywords: t-remotest point, t-approximate remotest point, t-remotest fuzzy set.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

The theory of fuzzy sets was introduced by L. Zadeh [7] in 1965. Many authors have introduced the concept of fuzzy metric in different ways ([1]-[7]). George and Veeramani ([3], [4]) modified the concept of fuzzy metric space intoduced by Kramosil and Michálek [5] and defined a Housdorff topology on this fuzzy metric space. In this paper we obtain the t-remotest points and the t-approximate remotest points of the non-empty f-bounded subsets A and B of a fuzzy normed space $(X, N, *)$, by considering a cyclic map $T : A \cup B \longrightarrow A \cup B$ i.e. $T(A) \subseteq B$ and $T(B) \subseteq A$.
First, we recall the basic definitions and preliminaries that is need for main results.

Definition 1.1. [3] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-norm if $([0,1], *)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 1.2. [6] The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$,

$(i) N(x,t) > 0$,
$(ii) N(x,t) = 1 \Leftrightarrow x = 0$,
$(iii) N(\alpha x, t) = N(x, t/|\alpha|)$, for all $\alpha \neq 0$,
$(iv) N(x,t) * N(y,s) \leq N(x+y, t+s)$,
$(v) N(x, .) : (0,\infty) \longrightarrow [0,1]$ is continuous,
$(vi) \lim_{t\to\infty} N(x,t) = 1$.

*Speaker

Lemma 1.3. *[6] Let $(X, N, *)$ be a fuzzy normed space. Then*
(i) $N(x, t)$ is nondecreasing with respect to t for each $x \in X$,
(ii)$N(x - y, t) = N(y - x, t)$.

Definition 1.4. [3] Let $(X, N, *)$ be a fuzzy normed space. A subset X is called fuzzy bounded (f-bounded), if there exists $t > 0$ and $0 < r < 1$ such that $N(x, t) > 1 - r$ for all $x \in X$.

Definition 1.5. [3] Let $(X, N, *)$ be a fuzzy normed space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said convergent to $x \in X$ if for each $0 < \epsilon < 1$ and $t \in (0, \infty)$ there existe N_0 such that $N(x_n - x, t) > 1 - \epsilon$ for each $n \geq N_0$.

Definition 1.6. Let A be a non-empty f-bounded subset of a fuzzy normed space $(X, N, *)$ for $x \in X, t > 0$, let

$$\delta(A, x, t) = \bigwedge_{y \in A} N(y - x, t).$$

An element $y_0 \in A$ is said to be a t-farthest point of x from A if

$$N(y_0 - x, t) = \delta(A, x, t).$$

We shall denote the set of all elements of t-farthest points of x from A by $F^t_A(x)$; i.e.,

$$F^t_A(x) = \{y \in A : \delta(A, x, t) = N(y - x, t)\}.$$

If each $x \in X$ has at least one t-farthest in A, then A is called a t-remotest fuzzy set.

Let $(X, N, *)$ be a fuzzy normed space, A and B, f-bounded subsets of X. If there is a pair $(x_0, y_0) \in A \times B$ for which $N(x_0 - y_0, t) = \delta(A, B, t)$, that $\delta(A, B, t)$ is t-remotest fuzzy distance of A and B, define by

$$\delta(A, B, t) = \bigwedge_{x \in A} \delta(B, x, t).$$

Then the pair (x_0, y_0) is called a t-remotest pair for A and B and put

$$F^t(A, B) := \{(x, y) \in A \times B : \ N(x - y, t) = \delta(A, B, t)\}$$

as the set of all remotest pairs.

2 Main results

In this section we prove existence of the t-remotest points and the t-approximate remotest points by considering the cyclic map T on $A \cup B$.

Definition 2.1. Let A and B be non-empty f-bounded subsets of a fuzzy normed space $(X, N, *)$ and $T : A \cup B \longrightarrow A \cup B$ a cyclic map. The point $x \in A \cup B$ is a t-remotest point of the map T, if $N(x - Tx, t) = \delta(A, B, t)$.

Definition 2.2. Let A and B be non-empty f-bounded subsets of a fuzzy normed space $(X, N, *)$ and $T : A\cup B \longrightarrow A\cup B$ be a cyclic map. The point $x \in A\cup B$ is a t-approximate remotest point of the map T, if $N(x - Tx, t) \le \delta(A, B, t) + \epsilon$, for some $0 < \epsilon < 1$. Put

$$F_T^{a,t}(A, B) = \{x \in A \cup B : \ N(x - Tx, t) \le \delta(A, B, t) + \epsilon \ \ for\ some\ 0 < \epsilon < 1\}.$$

We say that the pair (A, B) is a t-approximate remotest pair.

Example 2.3. Suppose $X = \mathbb{R}^2$ with usual metric, $A = \{(x, y) : (x - 2)^2 + (y - 2)^2 \le 1\}$ and $B = \{(x, y) : (x + 2)^2 + (y - 2)^2 \le 1\}$. We define $T(x, y) = (-x, y)$ for $(x, y) \in A \cup B$. Let $x := (x_1, y_1)$ and $y := (x_2, y_2)$, define

$$N(x - y, t) = \frac{t}{t + d(x, y)}.$$

So

$$N((2.9, 1.9) - (-2.9, 1.9), t) = \frac{t}{t + 5.8} \le \frac{t}{t + 6} + \frac{t}{(t + 6)^2}.$$

Therefore

$$N((2.9, 1.9) - (-2.9, 1.9), t) \le \delta(A, B, t) + \epsilon,$$

for $\epsilon = \frac{t}{(t+6)^2}$. Hence the pair (A, B) is a t-approximate remotest pair.

Theorem 2.4. *Let A and B be non-empty f-bounded subsets of a fuzzy normed space $(X, N, *)$. Suppose that the continuous cyclic mapping $T : A \cup B \longrightarrow A \cup B$ satisfying*

$$N(Tx - Ty, t) \le \alpha N(x - y, t) + \beta[N(x - Tx, t) + N(y - Ty, t)] + \gamma\delta(A, B, t) \qquad (2.1)$$

for all $x, y \in A \cup B$, where $\alpha, \beta, \gamma > 0$ and $\alpha + 2\beta + \gamma < 1$. For x_0 is an arbitrary point in A, define $x_{n+1} = Tx_n$ for every $n \ge 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists a $x \in A$ with $N(x - Tx, t) = \delta(A, B, t)$.

Theorem 2.5. *Let A and B be non-empty f-bounded subsets of a fuzzy normed space $(X, N, *)$. Suppose that the cyclic mapping $T : A \cup B \longrightarrow A \cup B$ satisfying*

$$\lim_{n\to\infty} N(T^n x - T^{n+1}x, t) = \delta(A, B, t) \ \ for\ some \ \ x \in A \cup B.$$

Then the pair (A, B) is a t-approximate remotest pair.

Theorem 2.6. *Let A and B be non-empty f-bounded subsets of a fuzzy normed space $(X, N, *)$. Suppose that the cyclic mapping $T : A \cup B \longrightarrow A \cup B$ satisfying inequality (2.1), for all $x, y \in A \cup B$, where $\alpha, \beta, \gamma > 0$ and $\alpha + 2\beta + \gamma < 1$. Then the pair (A, B) is a t-approximate remotest pair.*

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Some results on almost L-Dunford–Pettis sets in Banach lattices

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Abstract

Following the concept of L–limited sets in dual Banach spaces, we introduce the concept of almost L–Dunford–Pettis sets in dual Banach lattices. Then by a class of operators on Banach lattices, so called disjoint Dunford–Pettis completely continuous operators, we characterize Banach lattices with the positive relatively compact Dunford–Pettis property.

Keywords: Dunford–Pettis set, relatively compact Dunford–Pettis property, Dunford–Pettis completely continuous operator.
Mathematics Subject Classification [2010]: 46A40, 46B42

1 Introduction

A subset A of a Banach space X is called limited (resp. Dunford–Pettis (DP)), if every weak* null (resp. weak null) sequence (x_n^*) in X^* converges uniformly on A, that is

$$\lim_{n\to\infty}\sup_{a\in A}|\langle a, x_n^*\rangle| = 0.$$

Also if $A \subseteq X^*$ and every weak null sequence (x_n) in X converges uniformly on A, we say that A is an L–set.
Every relatively compact subset of E is DP. If every DP subset of a Banach space X is relatively compact, then X has the relatively compact DP property (abb. DP$_{rc}$P). For example, dual Banach spaces with the weak Radon-Nikodym property (abb. $WRNP$) and Schur spaces (i.e., weak and norm convergence of sequences in X coincide) have the DP$_{rc}$P [4] and [5]. Also we recall that a Banach space X has the DP$_{rc}$P if and only if every DP and weakly null sequence (x_n) in X is norm null.
Recently, the authors in [7] and [8], introduced the class of L–limited sets and Dunford–Pettis completely continuous (abb. $DPcc$) operators on Banach spaces. In fact, a bounded linear operator $T : X \to Y$ between two Banach spaces is $DPcc$ if it carries DP and weakly null sequences in X to norm null ones in Y. The class of all $DPcc$ operators from X to Y is denoted by $DPcc(X, Y)$. A norm bounded subset B of a dual Banach space X^* is said to be an L–limited set if every weakly null and limited sequence (x_n) of X converges uniformly to zero on the set B, that is $\sup_{f\in B}|f(x_n)| \to 0$. We use some techniques to those in [2] for L–sets and almost L– sets in Banach lattices.
We refer the reader for undefined terminologies, to the classical references [1]

*Speaker

2 Almost L-DP sets in Banach lattices

In this section we introduce a new class of sets and operators. Recall that a sequence (x_n) in a Banach lattice E is (pairwise) disjoint, if for each $i \neq j$, $|x_i| \wedge |x_j| = 0$.

Definition 2.1. Let E be a Banach lattice and X be a Banach space. Then

(a) A norm bounded subset B of a dual Banach lattice E^* is said to be an almost L–DP set if every disjoint weakly null and DP sequence (x_n) of E converges uniformly to zero on the set B, that is $\sup_{f \in B} |f(x_n)| \to 0$.

(b) An operator T from a Banach lattice E into a Banach space X is a disjoint DP completely continuous (abb. $DP^d cc$) operator if the sequence $(\|Tx_n\|)$ converges to zero for every weakly null and DP sequence of pairwise disjoint elements in E.

Note that every L–DP set of a dual Banach lattice, is an almost L–DP set, but the converse is false, in general. In fact for many Banach lattices E with the positive $\mathrm{DP}_{rc}\mathrm{P}$ and without the $\mathrm{DP}_{rc}\mathrm{P}$, the closed unit ball of the dual Banach lattice E^* is an almost L–DP set, but it is not L–DP set. As an example, the closed unit ball B_{ℓ_∞} of ℓ_∞ is an almost L–DP set in ℓ_∞, but the closed unit ball $B_{(\ell_\infty)^*}$ is not an almost L–DP set in $(\ell_\infty)^*$.

Proposition 2.2. *Let E be a Banach lattice and B be a norm bounded set in E^*. Then the following are equivalent:*

(a) B is an almost L–DP set,

(b) For each sequence (f_n) in B, $f_n(x_n) \to 0$, for every disjoint weakly null and DP sequence (x_n) of E.

Now, similar [2] we show that an order interval of a dual Banach lattice E^* is an almost L–DP set.

Proposition 2.3. *Let E be a Banach lattice, then $[-f, f]$ is an almost L–DP set in E^*, for each $f \in (E^*)^+$.*

From [1], an operator T from a Banach lattice E into another F is said to be order bounded if for each $x \in E^+$, the subset $T([-x, x])$ is order bounded in F.

Proposition 2.4. *Let T be an order bounded operator from a Banach lattice E into a Banach lattice F. Then $T^*([-f, f])$ is an almost L–DP set, for each $f \in (E^*)^+$.*

Theorem 2.5. *Let T be an order bounded operator from a Banach lattice E into a Banach lattice F and B be a norm bounded solid subset of F^*. Then the following are equivalent:*

(a) $T^*(B)$ is an almost L–DP set in E^*,

(b) $\{T^* f_n : n \in N\}$ is an almost L–DP set, for each $f \in B^+$ and for each disjoint sequence (f_n) in B^+.

Corollary 2.6. *Let T be an order bounded operator from a Banach lattice E into another Banach lattice F and B be a norm bounded solid subset of F^*. Then the following are equivalent:*

(a) $T^*(B)$ *is an almost L–DP set in* E^*,

(b) $f_n(Tx_n) \to 0$, *for every disjoint weakly null and DP sequence* (x_n) *of* E^+ *and for each disjoint sequence* (f_n) *in* B^+.

Corollary 2.7. *Let* E *be a Banach lattice and* B *be a norm bounded solid subset of* E^*. *Then the following are equivalent:*

(a) B *is an almost L–DP set,*

(b) $\{f_n : n \in N\}$ *is an almost L–DP set for each disjoint sequence* (f_n) *in* B^+.

The next result characterizes the class of DP^dcc operators by almost L–limited sets.

Corollary 2.8. *For an order bounded operator* T *from a Banach lattice* E *into another Banach lattice* F, *the following are equivalent:*

(a) T *is* DP^dcc,

(b) $T^*(B_{F^*})$ *is an almost L– DP set, where* B_{F^*} *is the closed unit ball of* F^*,

(c) $\{T^*(f_n) : n \in N\}$ *is an almost L–DP set for each disjoint sequence* (f_n) *in* $(B_{F^*})^+$,

(d) $f_n(T(x_n)) \to 0$, *for every disjoint weakly null and DP sequence* (x_n) *of* E^+ *and for each disjoint sequence* (f_n) *in* $(B_{F^*})^+$.

Definition 2.9. A Banach lattice E has the positive $\mathrm{DP}_{rc}\mathrm{P}$ if each weakly null and DP sequence with the positive terms is norm null.

It is clear that the $\mathrm{DP}_{rc}\mathrm{P}$ implies the positive $\mathrm{DP}_{rc}\mathrm{P}$.

Theorem 2.10. *Let* E *be a Banach lattice and* E^* *has the weakly sequentially continuous lattice operations. Then the following are equivalent:*

(a) E *has the positive* $DP_{rc}P$,

(b) *Every weakly null and disjoint DP sequence in* E *converges to zero in norm.*

Corollary 2.11. *Let* E *be a Banach lattice and* E^* *has the weakly sequentially continuous lattice operations. Then the following are equivalent:*

(a) E *has the positive* $DP_{rc}P$,

(b) *For each Banach lattice* F, $DP^dcc(E, F) = L(E, F)$,

(c) $DP^dcc(E, \ell_\infty) = L(E, \ell_\infty)$.

Theorem 2.12. *A Banach lattice* E *such that* E^* *has the weakly sequentially continuous lattice operations has the positive* $DP_{rc}P$ *iff every bounded set in* E^* *is an almost L–DP set.*

In the following Theorem 2.13, we show that the positive $\mathrm{DP}_{rc}\mathrm{P}$ and the $\mathrm{DP}_{rc}\mathrm{P}$, coincide in the class of discrete Banach lattices. We know that, every weakly null sequence in ℓ_∞ and c_0 is DP.

Theorem 2.13. *Let E be a discrete Banach lattice. Then E has the positive $DP_{rc}P$, if and only if, it has the $DP_{rc}P$.*

Proof. Since the positive DP$_{rc}$P is inherited by closed Riesz subspaces and c_0 does not have the positive DP$_{rc}$P, then E does not contain any order copy of c_0. According to [6, Corollary 2.4.12], E is KB space, and so it possesses the DP$_{rc}$P by [3]. □

As an application of the above Theorem 2.13, $L^1[0,1]$ does not have the DP$_{rc}$P, but it has positive DP$_{rc}$P.

Lemma 2.14. *Let $T : E \to X$ from a Banach lattice E such that E^* has the weakly sequentially continuous lattice operations to a Banach space be an operator. Then the following are equivalent:*

(a) T is $DP^d cc$,

(b) if the sequence $(\|Tx_n\|)$ converges to zero for every weakly null and DP sequence in E^+,

(c) if the sequence $(\|Tx_n\|)$ converges to zero for every weakly null and DP sequence of pairwise disjoint elements in E^+

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Some Results on Best Proximity Pairs in Banach lattice spaces

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Abstract

We are going to study best proximity pair in a lattice Banach space X with a strong unit **1**. Also we develop a theory of best pair proximity for closed upward sets. By the way, give efcient algorithm for finding distance between two sets.

Keywords: Best proximity pair, Lattice Banach space, Upward set.
Mathematics Subject Classification [2010]: 46B42, 41A65.

1 Introduction

A pair $(x_0, y_0) \in A \times B$ for which $\|x_0 - y_0\| = dist(A, B)$ is called a best proximity pair for A B, in this case the pair (A, B) is said to have the best proximity pair in X. Now

$$Prox(A, B) = \{(x, y) \in A \times B : \|x - y\| = dist(A, B)\}$$

is the set of all best proximity pairs for the pair (A, B).

A best proximity pair evolves as a generalization of the best approximation considered by Beer, Pai and Veeramani [1, 2], Kima and Lee [3, 4], Sahney and Singh [5], Singer [6] and Xu [7], of exploring some of the sufficient conditions for the non-empty of the set $Prox(A, B)$.

In this paper we discuss the concepts of best proximity pair on lattice Banach with strong unit **1**; Also we are intend to find an algorithm for the distance of two sets by best proximity pair.

2 Main results

Recall that the set X endowed with partially ordered relation $\leq$ is said to be lattice if for every $x, y \in X$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in X which is denoted by $(X, \leq)$. Also vector lattice $(X, \leq, +, .)$ is a lattice $(X, \leq)$, with a binary operation $+$ and scalar product . such that $(X, +, .)$ is a vector space (see in [2]).

*Speaker

Recall that an element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists a $0 < \lambda \in \mathbf{R}$ such that $x \leq \lambda \mathbf{1}$. Using a strong unit $\mathbf{1}$, then

$$\|x\| = \inf\{\lambda > 0 : \ |x| \leq \lambda \mathbf{1}\}$$

for every $x \in X$, is a norm on X. In this case for every $x \in X$, $|x| \leq \|x\|\mathbf{1}$. If vector lattice $(X, \leq)$ induced by this normed that is complete is said to be Banach lattice. There are well-know examples of vector lattices with the strong units, the lattice of all bounded functions defined on a set X and also the lattice $L^\infty(S, \Sigma, \mu)$ of all essentially bounded functions on a space S with a σ-algebra of Σ and measure μ.

In the following, we suppose that the Banach lattice $(X, \leq)$ with a strong unit $\mathbf{1}$ satisfies in one of the following equivalent conditions:

(1) Every non-empty lower bounded set admits an infimum,

(2) Every non-empty upper bounded set admits a supremum,

also

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|,$$

for every $x, y \in X$. For every subset B of X and for every positive real r, define

$$\begin{aligned} U(B,r) &= \{y \in X : \inf_{b \in B} \|b - y\| \leq r\} \\ &= \{y \in X : \inf B - r\mathbf{1} \leq y \leq \sup B + r\mathbf{1}\} \end{aligned}$$

Recall that $U \subseteq X$ is an upward set if $u \in U$ and $u \leq x$, then $x \in U$.

For instance suppose $x \in X$ and $U = \{y \in X : \ x \leq y\}$. Then U is an upward set of X.

Let X be a conditionally complete lattice Banach space with a strong unit $\mathbf{1}$. We start with the following result which has proved in [8]. In the rest of the paper we shall assume that S is a non-empty bounded set in X.

Lemma 2.1. *([8], Proposition 3.1). Let W be a upward subset of X and $x \in X$. Then the following are true:*
(1) If $x \in W$, then $x - \epsilon\mathbf{1} \in intW \ for \ all \ \epsilon > 0$.
(2) We have

$$intW = \{x \in X : x + \epsilon\mathbf{1} \in intW \ for \ some \ \epsilon > 0\}.$$

Proposition 2.2. *Let A, B be closed subsets of X such that $A \cap B = \emptyset$. Then $Prox(A, B) \subset \partial A \times \partial B$.*

In the following we show that if X do not be a vector space previous theorem is incorrect.

Example 2.3. Suppose $X := \{(x, y) \in R^2 : |x| \geq 1\}$ in the Euclidean plane, endowed with the metric induced by the Euclidean metric, let

$$A := \{(x, y) \in X : |x - 2| \leq 1 \ and \ |y| \leq 1\}$$

$$B := \{(x, y) \in X : |x + 2| \leq 1 \ and \ |y| \leq 1\}.$$

Then $Prox(A,B) = \{((-1,a),(1,a)) : |a| \leq 1\}$, which $\in IntA \times IntB$.

$$\begin{aligned} U(S,r) &= \{y \in X : \inf_{s\in S} \|s-y\| \leq r\} \\ &= \{y \in X : \inf S - r\mathbf{1} \leq y \leq \sup S + r\mathbf{1}\} \end{aligned}$$

Theorem 2.4. *Let A be a closed upward subset of X and B a compact set in X. Then the pair (A, B) has best proximity pair in X.*

Example 2.5. Let $X := R^2$ endowed by the Euclidean norm and natural order. If

$$A = \{(x,0) : x \in R\} \quad , \quad B = \{(x,y) : x \in R^+, y \geq \frac{1}{x}\}.$$

Therefore B is a upward set and since A is not compact, $Prox(A,B) = \emptyset$.

Theorem 2.6. *Let A be a closed upward subset of X and B a compact set in X. Then there exists the largest element $(a_0,b_0) := max(Prox(A,B))$.*

In continue we want to find an algorithm for distance of two sets A, B of normed space X by best proximity pair.

Lemma 2.7. *Let A, B be closed subsets of normed space X. If $(a_0,b_0) \in A\times B$ is a unique best proximity pair, then there exist a decreasing sequence $\{\alpha_k\}_{k\geq 1}$ and an increasing sequence $\{\beta_k\}_{k\geq 1}$ of positive real numbers, such that*

$$\beta_k < \|a_0 - b_0\| < \alpha_k$$

and

$$\alpha_k - \beta_k = \frac{1}{2^{k-1}}(\alpha_1 - \beta_1)$$

for every $k \in N$.

This theorem suggests an "algorithm" for computing the dist(A,B).

Theorem 2.8. *Let X be a normed space, A and B compact subsets of X. If $(a_0,b_0) \in A \times B$ is a unique best proximity pair for $A \times B$, then there exists a sequence $\{a_n\}_{n\geq 1}$ in ∂A and $\{b_n\}_{n\geq 1}$ in ∂B such that*

$$\lim_{n\to\infty} a_n = a_0 \quad \textit{and} \quad \lim_{n\to\infty} b_n = b_0$$

and

$$\|a_0 - a_{n+1}\| \leq \|a_0 - a_n\| \quad \textit{and} \quad \|b_0 - b_{n+1}\| \leq \|b_0 - b_n\|$$

for every $n \in N$.

Remark 2.9. *If X is finite-dimensional, then in Theorem (2.8), we can omit the compactness of A and B.*

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Some Sufficient Conditions for Subspace-hypercyclicity

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Abstract

In this paper we state some sufficient conditions for an operator to be subspace-hypercyclic. We also Costruct some interesting examples of subspace-hypercyclic operators with special properties.

Keywords: Subspace-hypercyclic operators, Subspace-mixing operators, Subspace-transitive operators.
Mathematics Subject Classification [2010]: 47A16, 47B37

1 Introduction

Recently Madore and Martinez-Avendano in [3] introduced the concept of subspace-hypercyclicity for an operator as follows:

Definition 1.1. Let $T \in B(X)$ and let M be a closed subspace of X. We say that T is M-hypercyclic, if there exists $x \in X$ such that $orb(T, x) \cap M$ is dense in M. Such a vector x is called an M-hypercyclic vector for T.

Definition 1.2. Let $T \in B(X)$ and let M be a closed subspace of X. We say that T is M-transitive, if for any non-empty open sets $U \subseteq M$ and $V \subseteq M$, there exists $n \in N_0$ such that $T^{-n}(U) \cap V$ contains a relatively open nonempty subset of M.

Theorem 1.3. *([3])Let $T \in B(X)$ and let M be a nonzero closed subspace of X. If T is M-transitive, then T is M-hypercyclic.*

It is proved in [3] by Madore and Martinez-Avendano that the converse of Theorem1.3 is not always true. So there are subspace-hypercyclic operators that are not subspace-transitive.
In [1], [2] and [5] one can find more results about subspace-hypercyclic operators.

In this paper we state some sufficient conditions for an operator to be subspace-hypercyclic. Also we construct various examples of subspace-hypercyclic operators by using these conditions.

*Speaker

2 Main results

Throughout this paper X always is an F-space, a complete metrizable topological vector space and $B(X)$ is the space of bounded linear operators on X. We also denote by M a closed nonempty subspace of X. We can also assume that M is separable, since subspace-hypercyclicity can only occur with respect to separable and infinite dimensional subspaces ([3]).

Lemma 2.1. *Let $T \in B(X)$ and let M be a closed subspace of X. Suppose that for any nonempty open subsets $U \subseteq M$ and $V \subseteq M$, there is $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \phi$. Then $\cup_{n \geq n_0} T^n(cB_M)$ is dense in M for any $c > 0$ and any $n_0 \in \mathbb{N}$, where B_M is the open unit ball of M.*

By using Lemma 2.1, we state our first sufficient condition for subspace-hypercyclicity.

Theorem 2.2. *([4]) Let $T \in B(X)$ and let M be a closed subspace of X such that T satisfies the following conditions:*

(i) For any nonempty open subsets $U \subseteq M$ and $V \subseteq M$, there is $n \in N_0$ such that $T^n(U) \cap V \neq \phi$.

(ii) There exists a dense subset X_0 of M such that $T^n x \to 0$ as $n \to \infty$, for any $x \in X_0$.

Then T is M-hypercyclic.

Let $T \in B(X)$. We say that T is an M-mixing operator, if for any non-empty open sets $U \subseteq M$ and $V \subseteq M$, there exists a positive integer N such that $T^n(U) \cap V$ is non-empty for any $n > N$ ([6]). If an operator be M-mixing, it satisfies condition (i) of Theorem 2.2. So we have the following corollary:

Corollary 2.3. *Let $T \in B(X)$ be M-mixing. If there exists a dense subset X_0 of M such that for any $x \in X_0$, $T^n(x) \to 0$ as $n \to \infty$, then T is M-hypercyclic.*

In the next theorem we give a sufficient condition for an operator to be subspace-mixing that also is a sufficient condition for subspace-hypercyclicity.

Theorem 2.4. *([4]) Let $T \in B(X)$ and let M be a closed subspace of X. If there are dense subsets X_0 and Y_0 of M and there is a map $S : Y_0 \to Y_0$ such that:*

(i) $T^n x \to 0$ for any $x \in X_0$.

(ii) $S^n y \to 0$ for any $y \in Y_0$.

(iii) $TSy = y$ for any $y \in Y_0$.

Then T is M-mixing. Specially T is M-hypercyclic.

Proof. Let U and V be nonempty open subsets of M. Suppose that $x \in U \cap X_0$ and $y \in V \cap Y_0$. If we define $u_n = S^n y$, then $u_n \in Y_0$ by hypothesis. Also:

$$u_n \to 0 \quad and \quad x + u_n \to x$$

as $n \to \infty$. So

$$T^n(x+u_n) = T^n(x) + T^n(u_n) \to y \quad as \quad n \to \infty$$

Therefore if we choose N large enough, for any $n \geq N$;

$$x + u_n \in U \quad and \quad T^n(x+u_n) \in V$$

That means for any $n \geq N$, we have $T^n(U) \cap V \neq \phi$. Hence T is an M-mixing operator. Now since $T^n x \to 0$ for every $x \in X_0$, by Theorem 2.2, T is M-hypercyclic. □

By using Theorem 2.4, we construct an operator such that for any $m \in N$, T^m is subspace-hypercyclic.

Example 2.5. Let B be the backward shift on l^2, that is for $(x_1, x_2, x_3, ...) \in l^2$ defined as

$$B(x_1, x_2, x_3, ...) = (x_2, x_3, ...).$$

Let λ be a scalar with $|\lambda| > 1$ and let $T = \lambda B$. Then for any $m \in N$, $T^m = (\lambda B)^m$ is subspace-mixing with respect to

$$M = \{\{a_n\}_{n=1}^{\infty} : a_{2k+1} = 0 \quad for \quad all \quad k \in N\}.$$

Specially T^m is M-hypercyclic for any $m \in N$.

Proof. Consider a natural number m. Let $X_0 = Y_0$ be the subsets of M, that consist all finite sequences. That is not hard to see that $(T^m)^n(x) = T^{mn}(x) \to 0$ as $n \to \infty$. If we define $S = (\frac{1}{\lambda}F)^m$, where F is the forward shift on l^2, that is defined as:

$$F(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...),$$

then the conditions (ii) and (iii) of Theorem 2.4 are satisfied. So T^m is M-mixing. Moreover by Theorem 2.2 T^m is M-hypercyclic. □

In the next example you see an operator T, that both T and T^*, the adjoint of T, are subspace-hypercyclic with respect to same subspace.

Example 2.6. Let $K = B$ be the backward shift on $l^1(N, v) = \{\{x_n\}_{n \in N}; ||x_n|| = \sum |x_n| v_n < \infty\}$, where for every $n \in N$, $v_n = \frac{1}{n+1}$ and

$$M = \{\{x_n\}_{n \in N} \in l^1(N, v); x_n = 0 \quad for \quad n < m\}.$$

Similar to Example 2.5, K is M-hypercyclic.

The adjoint of K is the forward shift F. Let $T = K^* = F$. If we consider $X_0 = Y_0$, the set of finite sequence and consider $S = B$, then X_0, Y_0 and S satisfies three conditions of Theorem 2.4. So $K^* = F$ is M-mixing. Clearly for every $x \in X_0$, $T^n(x) \to 0$ as $n \to \infty$. So by Theorem 2.2 K^* is M-hypercyclic too.

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Talk Space of operators pp.: 1–4

Space of Operators

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Abstract

The purpose of this work is to give some new results concerning space of operators in terms of some subsets of Banach space X. We will give equivalent characterization of Banach spaces X in which every V^*-subset of X is relatively compact. We also discuss some applications of these results to the subspaces of bounded linear operators.

Keywords: L-set, DP set, V-set, V^*-set, completely continuous operator, unconditionally converging operator
Mathematics Subject Classification [2010]: Primary 46B20; Secondary 46B25, 46B28.

1 Introduction

Throughout this talk, X and Y will denote real Banach spaces. A bounded subset A of X is called a *Dunford-Pettis* (DP) (resp. *limited*) subset of X if

$$\lim_n(\sup\{| x_n^*(x) |: x \in A\}) = 0$$

for each weakly null (resp. w^*-null) sequence (x_n^*) in X^*.

A bounded subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. The unit ball of a Banach space X is weakly precompact if and only if X does not contain copies of ℓ_1 (by Rosentlal's ℓ_1 theorem). Every Dunford-Pettis set is weakly precompact , e.g., see [12], p. 377, [1], [8]. We note that every relatively compact subset of X is limited and every limited subset of X is Dunford-Pettis. Thus every relatively compact subset of X is DP.

A Banach space X has the *Gelfand-Phillips* (GP) *property* if every limited subset of X is relatively compact. The Banach space X has the *Dunford-Pettis relatively compact property* ($DPrcP$) (resp. the RDP^* *property*) if every Dunford-Pettis subset of X is relatively compact (resp. relatively weakly compact) [3], [7] . Certainly, if a Banach space X has the $DPrcP$, then X has the (GP) property (since any limited set is a DP set). Note that every Schur space has the $DPrcP$.

Closely related to the notions of DP sets and limited sets is the idea of an L-set, e.g., see Bator [2] and Emmanuele [5], [6]. A Bounded subset A of X^* is called an *L-subset* of X^* if

$$\lim_n(\sup\{| x^*(x_n) |: x^* \in A\}) = 0$$

for each weakly null sequence (x_n) in X. Emmanuele and Bator [5], [2] showed that $\ell_1 \not\hookrightarrow X$ iff any L-subset of X^* is relatively compact iff X^* has the $DPrcP$.

A bounded subset A of X (resp. A of X^*) is called a V^*-*subset* (resp. V-*subset of* X^*) of X if

$$\lim_n(\sup\{|\, x_n^*(x)\, | : x \in A\}) = 0$$

$$(\text{resp.} \lim_n(\sup\{|\, x^*(x_n)\, | : x^* \in A\}) = 0)$$

for each wuc series $\sum x_n^*$ in X^* (resp. $\sum x_n$ in X).

2 Main results

The following proposition can easily be derived directly from definitions.

Proposition 2.1. *Let X be a Banach space. Then we have the following:*
1. i) Every DP subset of X^* is an L-subset of X^*.
ii) Every V^*-subset of X^* is a V-subset of X^*.
2. i) Every L-subset of X^* is a V-subset of X^*.
ii) Every DP subset of X is a V^*-subset of X.

Proposition 2.2. *Let X be a Banach space. If every V-subset of X^{**} is an L-subset of X^{**}, then every V^*-subset of X is a DP subset of X.*

The next proposition plays a consistent and important role in this study.

Proposition 2.3. *i) (Theorem 3.1 (ii), [3]) $T : X \to Y$ is completely continuous if and only if $T^*(B_{Y^*})$ is an L-subset of X^*.*
ii) $T^* : Y^* \to X^*$ is completely continuous if and only if $T(B_X)$ is a DP subset of Y.
iii) (Theorem 36.1, [9]) $T : X \to Y$ is unconditionally converging if and only if $T^*(B_{Y^*})$ is a V-subset of X^*.
iv) (Theorem 36.2, [9]) $T^* : Y^* \to X^*$ is unconditionally converging if and only if $T(B_X)$ is a V^*-subset of

Theorem 2.4. *Suppose X is a Banach space X. Then every V^*-subset of X is a DP subset of X if and only if for every Banach space Y, every unconditionally converging adjoint operator $T^* : X^* \to \ell_\infty$ is completely continuous.*

Recall that a Banach space X has the *Dunford-Pettis property* (DPP) if every weakly compact operator $T : X \to Y$ is completely continuous, for each Banach space Y. A bounded subset S of X is said to be *weakly sequentially compact* provided that every sequence from S has a subsequence weakly converging to an element of X. A Banach space X has *property* (V) (resp. V^*) if every V-subset of X^* (resp. V^*-subset of X) is weakly sequentially compact in the weak topology of X^* (resp. X). Equivalently, X has *property* (V) if for every Banach space Y, every unconditionally converging operator $T : X \to Y$ is weakly compact [10].

Corollary 2.5. *Let Y be a Banach space. Then we have the following:*
(i) If X has property (V) and the Dunford-Pettis property, then every V-subset of X^* is an L-subset of X^*.
(ii) If every V-subset of X^* is an L-subset of X^*, then X has the Dunford-Pettis property.

Corollary 2.6. *Let Y be a Banach space. Then we have the following:*
(i) If X has property (V^*) and the Dunford-Pettis property, then every V^*-subset of X is a DP subset of X.
(ii) If every unconditionally converging adjoint operator $T^* : X^* \to \ell_\infty$ is completely continuous, then X has the Dunford-Pettis property .

A Banach space X is said to have the *Reciprocal Dunford-Pettis property* $(RDPP)$ if for every Banach space Y, every completely continuous operator $T : X \to Y$ is weakly compact. Banach spaces with property (V) have the $RDPP$ [10]. Also, Banach spaces which which do not contain ℓ_1 have property $RDPP$.

Corollary 2.7. *If every V-subset of X^* is an L-subset of X^* and the RDPP, then X has property (V).*

Corollary 2.8. *Suppose that every V-subset of X^* is an L-subset of X^* and $\ell_1 \not\hookrightarrow X$. Then $UC(X,Y) = K(X,Y)$.*

Theorem 2.9. *Suppose that X and Y are Banach spaces. Then every V^*-subset of X is relatively compact if and only if X has the $DPrcP$ and every unconditionally converging adjoint operator $T^* : X^* \to \ell_\infty$ is completely continuous.*

Proposition 2.10. *Let X is a Banach space. Then B_{X^*} is a V-subset of X^* if and only if $L(X,Y) = UC(X,Y)$ for any Banach space Y.*

Corollary 2.11. *Suppose that X is a Banach space in which every V-subset of X^* is an L-subset of X^* and Y is a Banach space. If B_{X^*} is a V-subset of X^*, then $L(X,Y) = CC(X,Y)$.*

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Talk Spectrum and eigenvalues of quaternion matrices pp.: 1–4

Spectrum and Eigenvalues of Quaternion Matrices

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Abstract

In this paper we introduce left and right eigenvalues for quaternion-valued matrix Q. Also, we will show that the spectrum of Q is not the set of its eigenvalues.

Keywords: Quaternion, Quaternion matrix, Right eigenvalue, Real representation

Mathematics Subject Classification [2010]: 15A45, 15A42

1 Introduction

The study of inequalities for compact operators, especially operators acting upon finite-dimensional spaces, is frequently carried out through an analysis of the eigenvalues or singular values. For matrices with entries in a general ring $\mathcal{R}$ there is no theory of eigenvalues. However, if the ring $\mathcal{R}$ is an algebra over algebraically closed field, then existance of eigenvalues can be proved.

The real quaternion algebra $\mathbb{H}$ is known as a four dimensional vector space over the real number field $\mathbb{R}$ with its basis $\{1, i, j, k\}$ satisfying the multiplication laws

$$\begin{array}{lll} i^2 = j^2 = k^2 = -1 & , \quad ijk = -1 & \\ ij = -ji = k & , \quad jk = -kj = i & , \quad ki = -ik = j \end{array}$$

and 1 acting as unity element. In this case any element in $\mathbb{H}$ can be written as $q = a_0 + a_1 i + a_2 j + a_3 k$ where $a_j' s$ are all real numbers.
We shall always write every quaternion q in the form $q = z_1 + z_2 j$ where $z_1 = a_0 + a_1 i$ and $z_2 = a_2 + a_3 i$ are complex numbers.
A quaternion matrix Q therefore can be written $Q = A_1 + A_2 j$, where $A_1$and A_2 are unique complex matrices. The function $\phi : M_n(\mathbb{H}) \to M_{2n}(C)$ then defined by

$$\phi(Q) = \left[\begin{array}{cc} A_1 & -A_2 \\ \overline{A_2} & \overline{A_1} \end{array} \right]$$

is an injective $*$-homomorphism. The matrix $\phi(Q)$ is called the complex representation of Q.
Various operation properties on complex representation of quaternion matrices can easily be proved:

*Speaker

Theorem 1.1. *Let $A, B, C \in M_n(\mathbb{H})$ and $r \in R$ be given then*

a). $A = B$ *if and only if* $\phi(A) = \phi(B)$,

b). $\phi(A + B) = \phi(A) + \phi(B)$, $\phi(AC) = \phi(A)\phi(C)$, $\phi(rA) = \phi(Ar) = r\phi(A)$,

c). $\phi(A^*) = (\phi(A))^*$,

d). A *is invertible if and only if* $\phi(A)$ *is invertible and* $\phi(A^{-1}) = (\phi(A))^{-1}$,

e). A *is Hermitian if and only if* $\phi(A)$ *is Hermitian,*

f). A *is unitary if and only if* $\phi(A)$ *is unitary.*

2 Eigenvalues and eigenvectors of quaternion matrices

The spectrum $\sigma(T)$ of a linear transformation T acting on a finite-dimensional complex vector space is the set of eigenvalues of T. For $Q \in M_n(\mathbb{H})$, the spectrum of Q is not generally the set of its eigenvalues. Because $\mathbb{H}$ is noncommutative, we have left and right eigenvalues for quaternion matrices.

Definition 2.1. Let $q \in \mathbb{H}$ and ξ be a nonzero vector in $\mathbb{H}^n$.

1. If $Q\xi = \xi q$, then q is a right eigenvalue and ξ is a right eigenvector associated with q of Q.
2. If $Q\xi = q\xi$, then q is a left eigenvalue and ξ is a left eigenvector associated with q of Q.

De Leo and Scolarici in [3], argue that in quaternionic quantum mechanics left eigenvalues do not represent the same physical quantities as those represented by right eigenvalues. For each $q \in \mathbb{H}$ we denote the similarity orbit $\theta(q)$ of q by

$$\theta(q) = \{w^{-1}qw : w \in \mathbb{H} \setminus \{0\}\}.$$

Proposition 2.2. *If ξ is a right eigenvector of Q associated with q then for each $w \in \mathbb{H} \setminus \{0\}$, $w\xi$ is a right eigenvector associated with $w^{-1}qw$.*

Proof. $Q(\xi w) = (Q\xi)w = (\xi q)w = (\xi w)(w^{-1}qw)$. □

Thus if Q has a nonreal eigenvalue, then it has infinitely many non real right eigenvalues, note that the similarity orbit $\theta(r)$ of a real number r is the singleton $\{r\}$.

The following Lemma of Caylay [2] however shows that only two of these elements are complex numbers.

Lemma 2.3. *If q is a nonreal quaternion then there is a nonreal $\lambda \in (C)$ such that*

$$\theta(q) \cap C = \{\lambda, \overline{\lambda}\}.$$

Consequently for each $q \in \mathbb{H} \setminus R$, the similarity orbit $\theta(q)$ contains exactly one complex number in the closed upper halfplane C^+. The following theorem characterizes the complex eigenvalues of a quaternion matrix Q.

Theorem 2.4. *Let $Q \in M_n(\mathbb{H})$ then complex right eigenvalues of Q are exactly eigenvalues of $\phi(Q)$.*

Corollary 2.5. *Every quaternion matrix Q has at least one right eigenvalue of rank* 1.

In [7], Zhang discus the canonical forms, determinants, and numerical ranges of matrices over quaternions. $M_n(\mathbb{H})$ is a real algebra of finite dimension, hence, every $Q \in M_n(\mathbb{H})$ satisfies a polynomial equation $f(Q) = 0$ for some $f \in \mathbb{R}[x]$, where $\mathbb{R}[x]$ is the set of all polynomial functions of an unknown x with coefficients in $\mathbb{R}$. There is a unique monic polynomial $m_Q \in \mathbb{R}[x]$ of minimal degree for which $f(Q) = 0$ if and only if $f = g.m_Q$ for some $g \in \mathbb{R}[x]$. The polynomial m_Q is called the minimal annihilating polynomial of Q. We may define therefore the spectrum $\sigma(Q)$ of Q to be the set of all roots of m_Q:

$$\sigma(Q) = \{\lambda \in C : \lambda \textit{ is a root of } m_Q\}.$$

One of the advantages of the spectrum so defined is the polynomial spectral mapping theorem (See [4],Theorem 2.26):

$$\sigma(f(Q)) = \{f(\lambda) : \lambda \in \sigma(Q)\}$$

for every $f \in \mathbb{R}[x]$. However this definition of the spectrum does not completely matches the spectrum of a complex matrix (see example below)

Example 2.6. Let $Q = \begin{bmatrix} j & 0 \\ 0 & i \end{bmatrix}$. We have $Q^2 = -I$ which implies that $m_Q(x) = x^2 + 1$ and therefore $\sigma(Q) = \{i, -i\}$ but the diagonal entry j, of the diagonal matrix Q do not appears as an elements of the spectrum.

Among the results on the complex matrices, an important one, is that normal matrices are unitarily similar to the diagonal ones. The following theorem shows that the same result is valid for quaternion matrices.

Theorem 2.7. *If $Q \in M_n(\mathbb{H})$ is normal then there is a unitary matrix $U \in M_n(\mathbb{H})$ and a diagonal matrix $D \in M_n(C^+)$ such that $U^*QU = D$ and $q \in \mathbb{H}$ is a right eigenvalue of Q if and only if $q \in \theta(\lambda)$ for some diagonal element λ of D.*

Example 2.8. The diagonal matrix $Q = \begin{bmatrix} j & 0 \\ 0 & k \end{bmatrix}$ is normal and its diagonal form is

$$U^*QU = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = Ii$$

where the unitary $U \in M_2(\mathbb{H})$ is

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1-k & 0 \\ 0 & 1-j \end{bmatrix}$$

This example shows another unattractive feature: Unlike the situation of complex matrices the diagonal form of a quaternion diagonal normal matrix may not be the matrix itself.

Proposition 2.9. *If $Q \in M_n(\mathbb{H})$ is normal then $Q^* = W^*QW$ for some unitary $W \in M_n(\mathbb{H})$.*

The following proposition says that the spectra of Hermitian matrices are eigenvalues and, the same as the complex matrices, they are all real numbers.

Proposition 2.10. *If $Q \in M_n(\mathbb{H})$ is Hermitian then every right eigenvalue of Q is real and $\lambda \in \sigma(Q)$ if and only if λ is real and λ is a right eigenvalue of Q.*

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Starlikeness Of A General Integral Operator On Meromorphic Multivalent Functions

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Abstract

We define a new integral operator $\mathcal{F}^{p}_{\delta_0,...,\delta_m}(f_1, ..., f_n)$ for meromorphic multivalent functions in the punctured open unit disk. The starlikeness condition for this integral operator is determined. Several special cases are also discussed in the form of Corollaries.

Keywords: Meromorphic functions, Integral operator, Meromorphic starlike functions, Meromorphic convex functions.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Let Σ_p denote the class of all meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and p-valent in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} \ : \ 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \ : \ |z| < 1\}$. In particular, we set $\Sigma_1 = \Sigma$.

A function $f \in \Sigma_p$ is said to be meromorphic p-valent starlike and belongs to the class $\mathcal{MS}^*_p$, if it satisfies the inequality:

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

A function $f \in \Sigma_p$ is said to be meromorphic p-valent convex and belongs to the class $\mathcal{MC}_p$, if it satisfies the inequality:

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

*Speaker

We note that

$$f \in \mathcal{MC}_p \Longleftrightarrow -\frac{zf'}{p} \in \mathcal{MS}_p^*.$$

In particular, we set

$$\mathcal{MS}_1^* = \mathcal{MS}^* \ , \ \mathcal{MC}_1 = \mathcal{MC}.$$

Definition 1.1. Let $n \in \mathbb{N}, m \in \{0,1,2,...\}, \delta_{ij} \in \mathbb{R}_+ \cup \{0\},\ for\ all\ i = 0,1,2,...,m\ and\ j = 1,2,...,n$ and also $\delta_i = (\delta_{i1},...,\delta_{in})\ for\ all\ i = 0,1,2,...,m$, we introduce a new general integral operator

$$\mathcal{F}^p_{\delta_0,...,\delta_m}(f_1,...,f_n) : \Sigma_p^n \longrightarrow \Sigma_p,$$

$$\mathcal{F}^p_{\delta_0,...,\delta_m}(f_1,...,f_n)(z) = \frac{1}{z^{p+1}} \int_0^z \prod_{j=1}^n \prod_{i=0}^m \left((-1)^i \frac{u^{p+i}}{p^i} f_j^{(i)}(u) \right)^{\delta_{ij}} du, \tag{2}$$

where $f_j^{(i)}$ is the derivative of the function f_j of the order i .

Remark 1.2. The integral operator introduced here generalizes the integral operators defined and studied in [1, 2, 4, 5, 6].

In order to prove the main result, we will need the following Lemma:

Lemma 1.3. *(see [3]) Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ satisfy the following condition:*

$$\Re\{\psi(is,t)\} \leq 0, \quad \left(s,t \in \mathbb{R}; t \leq -\frac{|a+is|^2}{2}\right). \tag{3}$$

If the function $h(z) = a + h_1 z + h_2 z^2 + \ldots$, where $\Re(a) > 0$, is analytic in $\mathbb{U}$ and

$$\Re\left\{\psi\left(h(z), zh'(z)\right)\right\} > 0 \quad (z \in \mathbb{U}), \tag{4}$$

then $\Re\{h(z)\} > 0$.

2 Main results

Theorem 2.1. *Let $f_j \in \Sigma_p\ \delta_{ij} \in \mathbb{R}_+ \cup \{0\},\ for\ all\ i = 0,1,2,...,m$ and $j = 1,2,...,n$ and also $\delta_i = (\delta_{i1},...,\delta_{in})\ for\ all\ i = 0,1,2,...,m$. If*

$$\Re\left\{\sum_{t=0}^m \left(-\delta_{tj} \frac{zf_j^{(t+1)}(z)}{f_j^{(t)}(z)}\right)\right\} > -\frac{p}{n} + \sum_{t=0}^m (p+t)\delta_{tj}, \tag{5}$$

for all $j = 1,2,...,n$, then the general integral operator $\mathcal{F}^P_{\delta_0,...,\delta_m}(f_1,...,f_n)$ defined in Definition 1.1 belongs to the meromorphic starlike function class $\mathcal{MS}_p^*$.

Several special cases are also discussed in the form of Corollaries.

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Sublinear operators on two-parameter martingale spaces

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Abstract

we prove atomic decomposition theorem for the two-parameter martingale weighted Lorentz spaces. With the help of atomic decomposition we obtain a sufficient condition for sublinear operators defined martingale weighted Lorentz spaces to be bounded.

Keywords: Atomic decompositions, Two-parameter martingales, Lorentz spaces, Sublinear operators.
Mathematics Subject Classification [2010]: 60G46, 60G42 and 46E30.

1 Introduction and preliminaries

Atomic decompositions of Lorentz martingales are first studied by Jiao et al. in [2], and in [1] Ho investigated the atomic decomposition of Lorentz-Karamata martingale spaces similarly to the idea of [2]. Riyan and Shixin [5] obtained atomic decomposition for B-valued martingales in two-parameter case and in [3] Li and Liu proved atomic decomposition theorems for two-parameter B-valued martingales in weak Hardy spaces. The technique of stopping times used in the case of one–parameter is usually unsuitable for the case of two–parameter, but the method of atomic decompositions can deal with them in the same way. In this paper, by using some ideas of [6, 4] we prove atomic decomposition theorem for the martingale weighted Lorentz spaces. As an application, of atomic decomposition, we obtain a sufficient condition for sublinear operator defined on martingale weighted Lorentz spaces to be bounded.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The distribution function λ_f of a measurable function f on Ω is given by

$$\lambda_f(t) = P(\{w \in \Omega : |f(w)| > t\}), \quad (t \geq 0)$$

and its decreasing rearrangement of f is the function $\tilde{f}$ defined on $[0, \infty)$ by

$$\tilde{f}(s) = \inf\{t > 0 : \lambda_f(t) \leq s\}, \quad (s \geq 0).$$

Let $\varphi > 0$ be non-negative and local integrable function on $[0, \infty)$. The classical Lorentz spaces $\Lambda_q(\varphi)$ is defined to be the collection of all measurable functions f for which the quantity

$$\|f\|_{\Lambda_q(\varphi)} := \begin{cases} \left(\int_0^\infty \left(\tilde{f}(t)\varphi(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & 0 < q < \infty, \\ \sup_s \tilde{f}(s)\varphi(s) & (q = \infty) \end{cases}$$

*Speaker

is finite.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n, n \in \mathbf{N}^2\}$ be an increasing family of sub-σ–algebras of $\mathcal{F}$ and $f = (f_n, n \in \mathbf{N}^2)$ be an integrable process. Then f is a martingale if

- f is adapted to the filtration $(\mathcal{F}_n, n \in \mathbf{N}^2)$, i.e. each f_n is $\mathcal{F}_n$–measurable,
- $E[f_m|\mathcal{F}_n] = f_n$ for all $n \leq m$.

The maximal function of a martingale $f = (f_n, n \in \mathbf{N}^2)$ is denoted by

$$f_n^* := \sup_{m \leq n} |f_m|, \qquad f^* := \sup_{m \in \mathbf{N}^2} |f_m|.$$

For a martingale $f = (f_n, n \in \mathbf{N}^2)$ relative to $(\Omega, \mathcal{F}, P)$, denote the martingale differences by

$$d_m f := f_{m_1,m_2} - f_{m_1-1,m_2} - f_{m_1,m_2-1} + f_{m_1-1,m_2-1},$$

and $d_m f := 0$ if $m_1 = 0$ or $m_2 = 0$.

We define the square function and the conditional square function of f as follows:

$$S_m(f) := \left(\sum_{n \leq m} | \, d_n f \, |^2 \right)^{1/2}, \qquad S(f) := \left(\sum_{n \in \mathbf{N}^2} | \, d_n f \, |^2 \right)^{1/2},$$

$$s_m(f) := \left(\sum_{n \leq m} E_{n-1} | \, d_n f \, |^2 \right)^{1/2}, \qquad s(f) := \left(\sum_{n \in \mathbf{N}^2} E_{n-1} | \, d_n f \, |^2 \right)^{1/2}.$$

For $0 < q \leq \infty$, martingale weighted Lorentz spaces as follows are defined by

$$\Lambda_q^*(\varphi) = \left\{ f = (f_n)_{n \in \mathbf{N}^2} : \|f\|_{\Lambda_q^*(\varphi)} := \|f^*\|_{\Lambda_q(\varphi)} < \infty \right\},$$

$$\Lambda_q^s(\varphi) = \left\{ f = (f_n)_{n \in \mathbf{N}^2} : \|f\|_{\Lambda_q^s(\varphi)} := \|s(f)\|_{\Lambda_q(\varphi)} < \infty \right\},$$

$$\Lambda_q^S(\varphi) = \left\{ f = (f_n)_{n \in \mathbf{N}^2} : \|f\|_{\Lambda_q^S(\varphi)} := \|S(f)\|_{\Lambda_q(\varphi)} < \infty \right\}.$$

Note that if $\varphi(t) = t^{\frac{1}{p}}$, then $\Lambda_q(\varphi) = L_{p,q}$ and $\Lambda_q^s(\varphi) = H_{p,q}^s$. In particular, if $\varphi(t) = t^{\frac{1}{q}}$, then $\Lambda_q(\varphi) = L_q$, $\Lambda_q^*(\varphi) = H_q^*$, $\Lambda_q^s(\varphi) = H_q^s$ and $\Lambda_q^S(\varphi) = H_q^S$. For two non–negative quantities A and B by $A \lesssim B$ we mean that there exists a constant $C > 0$ such that $A \leq CB$, and by $A \approx B$ that $A \lesssim B$ and $B \lesssim A$.

2 Atomic decomposition

In this section, we establish atomic decomposition theorem of martingale weighted Lorentz spaces.

Definition 2.1. A function $a \in L_r$ is called a (p, r) atom if there exists a stopping time ν such that

1. $a_n := E_n a = 0 \quad if \quad \nu \not\leq n$
2. $\| a^* \|_r \leq P(\nu \neq \infty)^{1/r-1/p} \qquad (0 < p \leq r, 1 < r \leq \infty).$

Theorem 2.2. *If $f = (f_n, n \in \mathbf{N}^2) \in \Lambda_q^s(\varphi)$, $0 < q \leq \infty$, then there exist a sequence $\{(a^k, \nu_k)\}_{k\in\mathbf{Z}}$ of $(p, 2)$ atoms $(0 < p \leq 2)$ such that*

$$\sum_{k=-\infty}^{\infty} \mu_k E_n a^k = f_n$$

where $\mu_k = 2^{k+1}\sqrt{2}P(\nu_k \neq \infty)^{1/p}$ and

$$\|\{2^k w(P(\nu_k \neq \infty))\}_{k\in\mathbf{Z}}\|_{l_q} \lesssim \|f\|_{\Lambda_q^s(\varphi)}. \tag{1}$$

Moreover, if $0 < q \leq 1$, then

$$\|f\|_{\Lambda_q^s(\varphi)} \approx \inf \|\{2^k w(P(\nu_k \neq \infty))\}_{k\in\mathbf{Z}}\|_{l_q}$$

where the infimum is taken over all the preceding decompositions of f.

Applying the Theorem 2.2 for $\varphi(t) = t^{1/q}$ we get the next theorem

Corollary 2.3. *If the martingale $f \in H^s_{p,q}, 0 < q \leq \infty, 0 < p \leq 2$ then there exist a sequence a^k of $(p, 2)$ atoms and a sequence $\mu \in l_p$ such that*

$$f_n = \sum_{k=-\infty}^{\infty} \mu_k a_n^k, n \in \mathbf{N}^2$$

and

$$\|(\mu_k)_{k\in\mathbf{Z}}\|_{l_q} \lesssim \|f\|_{H^s_{p,q}}.$$

Conversely if $0 < q \leq 1, q \leq p \leq 2$, and the martingale f has the above decomposition, then $f \in H^s_{p,q}$ and

$$\|f\|_{H^s_{p,q}} \approx \inf \|(\mu_k)_{k\in\mathbf{Z}}\|_{l_q}.$$

If we take $\varphi(t) = t^{1/p}$ in Theorem 2.2, then we get the following result, which has proved by Weisz [6]

Corollary 2.4. *If the martingale $f \in H^s_p, 0 < p \leq 2$ then there exist a sequence a^k of $(p, 2)$ atoms and a sequence $\mu \in l_q$ such that for all $n \in \mathbf{N}^2$*

$$f_n = \sum_{k=-\infty}^{\infty} \mu_k a_n^k, n \in \mathbf{N}^2$$

and

$$\Big(\sum_{k=-\infty}^{\infty} | \mu_k |^p\Big)^{1/p} \lesssim \|f\|_{H^s_p}.$$

Conversely if $0 < p \leq 1$, and the martingale f has the above decomposition, then $f \in H^s_p$ and

$$\|f\|_{H^s_p} \approx \inf\Big(\sum_{k=-\infty}^{\infty} | \mu_k |^p\Big)^{1/p}.$$

3 Sublinear operator on martingale spaces

As an application of atomic decompositions, we get some sufficient conditions which make the sublinear operator to be bounded from the martingale weighted Lorentz spaces to weighted Lorentz spaces.

An operator $T : X \to Y$ is called a sublinear operator if it satisfies

$$|T(f+g)| \leq |Tf| + |Tg|, \quad |T(\alpha f)| \leq |\alpha||Tf|, \quad (\alpha \in \mathbf{R})$$

where X is a martingale spaces, Y is a measurable function space.

Theorem 3.1. *Let $T : H_2^s \to L_2$ be a bounded sublinear operator. For every atom a of $(p, 2)$ $(0 < p < 2)$, if $Ta = 0$ on $\{\nu_k = \infty\}$, where ν is the stopping time associated with a, then*

$$\|Tf\|_{\Lambda_\infty(\varphi)} \leq \|f\|_{\Lambda^s_\infty(\varphi)}, \quad (f \in \Lambda^s_\infty(\varphi)).$$

Corollary 3.2. *The following imbeddings hold:*

$$\Lambda^s_\infty(\varphi) \hookrightarrow \Lambda^*_\infty(\varphi), \qquad \Lambda^s_\infty(\varphi) \hookrightarrow \Lambda^S_\infty(\varphi).$$

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Ternary (σ, τ, ξ)-Derivations on Banach Ternary Algebras

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Abstract

Let A be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and X be a Banach ternary A-module. Let σ, τ and ξ be linear mappings on A. We define a ternary (σ, τ, ξ)-derivation and a Lie ternary (σ, τ, ξ)-derivation. Moreover, we prove the generalized Hyers-Ulam-Rassias stability of ternary and lie ternary (σ, τ, ξ)-derivations on Banach ternary algebras.

Keywords: Banach ternary A-module, Ternary (σ, τ, ξ)-derivation, Hyers–Ulam–Rassias stability.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Ternary algebraic operations were considered in the 19 th century by several mathematicians such as A. Cayley [3] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([4]).
A ternary (associative) algebra $(A, [\ ])$ is a linear space A over a scalar field $\mathbb{F} = (\mathbb{R}$ or $\mathbb{C})$ equipped with a linear mapping, the so-called ternary product, $[\ ]\colon A \times A \times A \to A$ such that $[[abc]de] = [a[bcd]e]$ for all $a, b, c, d, e \in A$. This notion is a natural generalization of the binary case. It is known that unital ternary algebras are trivial and finitely generated ternary algebras are ternary subalgebras of trivial ternary algebras [1].

By a Banach ternary algebra we mean a ternary algebra equipped with a complete norm $\|.\|$ such that $\|[abc]\| \leq \|a\|\|b\|\|c\|$.

Let A be a Banach ternary algebra and X be a Banach space. Then X is called a ternary Banach A-module, if module operations $A \times A \times X \to X$, $A \times X \times A \to X$, and $X \times A \times A \to X$ are $\mathbb{C}$-linear in every variable. Moreover satisfy:

$$\max\{\|[xab]_X\|, \|[axb]_X\|, \|[abx]_X\|\} \leq \|a\|\|b\|\|x\|$$

for all $x \in X$ and all $a, b \in A$.

Let σ, τ and ξ be linear mappings on A. A linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a ternary (σ, τ, ξ)-derivation, if

$$D([abc]_A) = [D(a)\tau(b)\xi(c)]_X + [\sigma(a)D(b)\xi(c)]_X + [\sigma(a)\tau(b)D(c)]_X \tag{1}$$

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for all $a, b, c \in A$.

The stability of functional equations was first introduced by S. M. Ulam [11] in 1940. More precisely, let G_1, be a group, (G_2, d) be a metric group and ϵ be a positive number, S. M. Ulam asked, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$?. When this problem has a solution, we say that the homomorphism from G_1 to G_2 is stable.

This phenomenon of stability that was introduced by Th. M. Rassias [8] is called the Hyers-Ulam-Rassias stability, according to J. M. Rassias Theorem, as follows:

Theorem 1.1. *Let $f : V \longrightarrow W$ be a mapping from a norm vector space V into a Banach space W subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{2}$$

for all $x, y \in V$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : V \longrightarrow W$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{3}$$

for all $x \in V$. If $p < 0$ then inequality (2) holds for all $x, y \neq 0$, and (3) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into W is continuous for each fixed $x \in V$, then T is linear.

On the other hand J. M. Rassias ([7]) generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias Theorem [9]:

Theorem 1.2. *If it is assumed that there exist constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : V \to W$ is a mapping from a norm space V into a Banach space W such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \Theta\|x\|^{p_1}\|y\|^{p_2}$$

for all $x, y \in V$ holds, then there exists a unique additive mapping $T : V \to W$ such that

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p}\|x\|^p,$$

for all $x \in V$. If in addition for every $x \in V$, $f(tx)$ is continuous in real t for each fixed x, then T is linear (see [6]).

2 Ternary (σ, τ, ξ)-derivations on Banach ternary algebras

Throughout this section, assume that $(A, [\ ]_A)$ is a Banach ternary algebra and $(X, [\ ]_X)$ is a ternary Banach A-module.

Lemma 2.1. *Let V and W be linear spaces and let $f : V \to W$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in V$ and all $\mu \in \mathbb{T}^1 (:= \{\lambda \in \mathbb{C} ; |\lambda| = 1\})$. Then the mapping f is $\mathbb{C}$-linear. [5]*

Lemma 2.2. *Let $f : A \to X$ be a mapping such that*

$$f(\frac{\mu x+y+z}{4})+f(\frac{3\mu x-y-4z}{4})+f(\frac{4\mu x+3z}{4})=2\mu f(x), \tag{4}$$

for all $x, y, z \in A$ and $\mu \in \mathbb{T}^1$. Then f is $\mathbb{C}$-linear. [2]

The first result is as follows:

Theorem 2.3. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f : A \to X$ be a mapping and σ, τ, and ξ be linear mappings on A such that*

$$f(\frac{\mu x+y+z}{4})+f(\frac{3\mu x-y-4z}{4})+f(\frac{4\mu x+3z}{4})=2\mu f(x), \tag{5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|f([xyz]_A)-[f(x)\tau(y)\xi(z)]_X-[\sigma(x)f(y)\xi(z)]_X-[\sigma(x)\tau(y)f(z)]_X\| \leq \theta\|x\|^p\|y\|^p\|z\|^p \tag{6}$$

for all $x, y, z \in A$. Then the mapping $f : A \to X$ is a ternary (σ,τ,ξ)-derivation.

We prove the following Ulam stability problem for functional equation $f(\frac{x+y+z}{4}) + f(\frac{3x-y-4z}{4}) + f(\frac{4x+3z}{4}) = 2f(x)$ controlled by the mixed type product-sum function

$$(x, y) \to \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p)$$

introduced by J. M. Rassias (see [10]).

Theorem 2.4. *Let p, p_1, p_2, p_3 be real numbers such that $p \neq 1$, $p_1 + p_2 + p_3 \neq 1$, and $\theta > 0$. Suppose $f : A \to X$ is a mapping for which there exist mappings $g, h, k : A \to A$ whit $g(0) = h(0) = k(0) = 0$ such that*

$$\begin{aligned}\|f(\frac{\mu x+y+z}{4})+f(\frac{3\mu x-y-4z}{4})+f(\frac{4\mu x+3z}{4})-2\mu f(x)\| \\ \leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3}+\|x\|^p+\|y\|^p+\|z\|^p),\end{aligned} \tag{7}$$

$$\|g(\lambda x+\lambda y)-\lambda g(x)-\lambda g(y)\| \leq \theta(\|x\|^p+\|y\|^p) \tag{8}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$.
Also, the above equation holds for h and k.

$$\|f([xyz]_A)-[f(x)h(y)k(z)]_X-[g(x)f(y)k(z)]_X-[g(x)h(y)f(z)]_X\| \leq \theta\|x\|^p\|y\|^p\|z\|^p \tag{9}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique ternary (σ,τ,ξ)-derivation $D : A \to X$ satisfying

$$\|g(x)-\sigma(x)\| \leq \theta\frac{2}{|\ 2-2^p\ |}\|x\|^p \tag{10}$$

Also, the above equation holds for h and k.

$$\|f(x)-D(x)\| \leq 2\theta\frac{2^p}{|\ 2-2^p\ |}\|x\|^p \tag{11}$$

for all $x \in A$.

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The BSE property of semigroup algebras

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Abstract

The concepts of BSE property and BSE algebras were introduced and studied by Takahasi and Hatori in 1990 and later by Kaniuth and Ülger. This abbreviation refers to a famous theorem proved by Bochner and Schoenberg for $L^1(\mathbb{R})$, where $\mathbb{R}$ is the additive group of real numbers, and by Eberlein for $L^1(G)$ of a locally compact abelian group G. In this paper we investigate the BSE property for certain semigroup algebras.

Keywords: Representation algebra, BSE algebra, Foundation semigroup, Reflexive semigroup
Mathematics Subject Classification [2010]: 46Jxx, 22A20

1 Introduction

Let A be a commutative Banach algebra. Denote by $\Delta(A)$ and $\mathcal{M}(A)$ the Gelfand spectrum and the multiplier algebra of A, respectively. A bounded continuous function σ on $\Delta(A)$ is called a *BSE-function* if there exists a constant $C > 0$ such that for every finite number of $\varphi_1, ..., \varphi_n$ in $\Delta(A)$ and complex numbers $c_1, ..., c_n$, the inequality

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \leq C. \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{A^*}$$

holds. The BSE-norm of σ ($\|\sigma\|_{BSE}$) is defined to be the infimum of all such C. The set of all BSE-functions is denoted by $C_{BSE}(\Delta(A))$. Takahasi and Hatori [9] showed that under the norm $\|.\|_{BSE}$, $C_{BSE}(\Delta(A))$ is a commutative semisimple Banach algebra.

A bounded linear operator on A is called a *multiplier* if it satisfies $xT(y) = T(xy)$ for all $x, y \in A$. The set $\mathcal{M}(A)$ of all multipliers of A is a unital commutative Banach algebra, called the *multiplier algebra* of A.

For each $T \in \mathcal{M}(A)$ there exists a unique continuous function $\widehat{T}$ on $\Delta(A)$ such that $\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\hat{a}(\varphi)$ for all $a \in A$ and $\varphi \in \Delta(A)$. See [6] for a proof.

Define

$$\widehat{\mathcal{M}(A)} = \{\widehat{T} : T \in \mathcal{M}(A)\}.$$

A commutative Banach algebra A is called without order if $aA = \{0\}$ implies $a = 0$ ($a \in A$).

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A commutative and without order Banach algebra A is called a BSE-algebra (or has *BSE-property*) if it satisfies the condition

$$C_{BSE}(\Delta(A)) = \widehat{\mathcal{M}(A)}.$$

The abbreviation BSE stands for Bochner-Schoenberg-Eberlin and refers to a famous theorem, proved by Bochner and Schoenberg [1, 8] for the additive group of real numbers and in general by Eberlein [3] for a locally compact abelian group G, saying that, in the above terminology, the group algebra $L^1(G)$ is a BSE-algebra (See [7] for a proof).

It worths to note that the semigroup algebra $l^1(\mathbb{Z}^+)$ (where $\mathbb{Z}^+$ is the additive semigroup of nonnegative integers) is a BSE algebra [10], but for $k \geq 1$, $l^1(\mathbb{N}_k)$ ($\mathbb{N}_k = \{k, k+1, k+2, ...\}$) is not a BSE algebra.

In [4], we established affirmatively a question raised by Takahasi and Hatori [9] that whether $L^1(\mathbb{R}^+)$ is a BSE-algebra.

In this paper we investigate the BSE property for certain semigroup algebras. To this aim, we first give a characterization of the $L^\infty-$representation algebra $\mathfrak{R}(S)$ of a foundation semigroup S with identity and then we apply this characterization in order to prove that $M_a(S)$, for a reflexive foundation semigroup S, is a BSE algebra. We present examples which show that the hypothesis 'reflexive' cannot be dropped.

we also prove that for a compact foundation semigroup S, the semigroup algebra $M_a(S)$ is BSE if and only if it has a Δ-weak bounded approximate identity.

2 Main results

We start this section with the following theorem which characterizes the $L^\infty-$representaion $\mathfrak{R}(S)$ of a foundation semigroup S.

Theorem 2.1. *Let S be an abelian foundation semigroup with identity. Then the following statements about a continuous function φ defined on S, are equivalent:*
(a) $\varphi \in \mathfrak{R}(S)$ and $\|\varphi\|_{\mathfrak{R}} \leq \beta$.
(b) For every function f on $\widehat{S}$ of the form

$$f(\gamma) = \sum_{i=1}^{n} c_i \gamma(x_i) \quad (\gamma \in \widehat{S}),$$

where $c_1, ...c_n$ are complex numbers and $x_1, ..., x_n \in S$, we have

$$\left| \sum_{i=1}^{n} c_i \varphi(x_i) \right| \leq \beta \|f\|_\infty. \qquad (II)$$

Remark 2.2. Note that in previous theorem ,(b) implies (a) for an arbitrary commutative topological (not necessarily Foundation) semigroup.

As an application of the above result, in the following theorem we prove that for any reflexive foundation semigroup S, the Banach algebra $M_a(S)$ is a BSE algebra.

Theorem 2.3. *Suppose that S is a reflexive foundation semigroup, then $M_a(S)$ is a BSE-algebra.*

Example 2.4. (a) For any discrete inverse semigroup S with identity, $l^1(S)$ is a BSE algebra. For instance, if $S = (\mathbb{Z}^+, \max)$, where $\mathbb{Z}^+$ is the discrete semigroup of non negative integers, then S is a reflexive semigroup and so $l^1(S)$ is a BSE algebra.

(b) Let

$$T = \{-\frac{1}{2n} : n \in \mathbb{N}\} \cup \{0\} \cup \{\frac{1}{2n+1} : n \in \mathbb{N}\}$$

with the operation

$$xy = yx = x \text{ if } |x| \geq |y| \quad (x, y \in T),$$

and the topology of T coincides with the restriction of the line topology on $T = \{-\frac{1}{2n} : n \in \mathbb{N}\} \cup \{0\}$ while its restriction on $\{\frac{1}{2n+1} : n \in \mathbb{N}\}$ is discrete. Then T defines a compact inverse foundation semigroup with identity (P. 65 of [2]). So by Remark 2.2 and Theorem 2.3, $M_a(T)$ is BSE.

If we set $S := G \times T$, where G is an abelian topological group, then S is a reflexive foundation semigroup and again by Theorem 2.3, $M_a(S)$ is BSE.

(c)Let $S := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with the relative topology of the line and multiplication given by $xy = \max\{x, y\}$. Then S is a compact foundation semigroup with identity 0 (P. 34 of [2]). For any abelian locally compact group G, $T = S \times G$ is a reflexive foundation semigroup and by Theorem 2.3, $M_a(T)$ is BSE.

Theorem 2.5. *Let S be a compact foundation semigroup. Then $M_a(S)$ is a BSE-algebra if and only if $M_a(S)$ has a Δ-weak approximate identity.*

Example 2.6. (a) Consider the semigroup $S = [0, 1]^n$, $n \in \mathbb{N}$ with ordinary multiplication and restriction topology of $\mathbb{R}^n$. Since $[0, 1]^n$ is a compact semigroup and $L^1([0, 1]^n)$ has a bounded approximate identity, then $L^1([0, 1]^n)$ is a BSE algebra, for all $n \in \mathbb{N}$.

(b) Let T be as in part (b) and S be as in part (c) of Example 2.4. Then by Theorem 2.5, $M_a(T)$ and $M_a(S)$ are BSE algebras.

(C) $S = [0, 1]$ with the restriction topology of $\mathbb{R}$ and multiplication defined by $xy := \min\{x + y, 1\}$. Then S is a compact foundation semigroup with identity (page 48 of [2]) and $M_a(S)$ is BSE.

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Talk The existence of efficient solutions for generalized systems and the... pp.: 1–4

the existence of efficient solutions for generalized systems and the properties of their solution sets

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Abstract

In this paper, we first give a density theorem. We will see that, under some suitable conditions, the set of positive proper efficient solutions is dense in the set of the efficient solutions. Finally, we discuss about the connectedness for the set of the efficient solutions of a generalized system.

Keywords: Equilibrium problem, Efficient solution, connnectedness

1 Introduction

Throughout this paper, let X be a real Hausdorff topological vector space and let Y be a real Hausdorff topological vector space. Let Y^* be the topological dual space of Y. Let C be a closed convex pointed cone in Y. The cone C induces a partial ordering in Y defined by

$$x \leq y, \text{ if and only if } y - x \in C.$$

Let

$$C^* = \{f \in Y^* : f(y) \geq 0, \text{ for all } y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e.

$$C^\sharp := \{f \in Y^* : f(y) > 0 \text{ for all } y \in C \setminus \{0\}\}.$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\text{cone}(D) = \{td : t \geq 0, d \in D\}.$$

Denote the closure of D by $\text{cl}(D)$. A nonempty convex subset M of the convex cone C is called a base of C if $C = \text{cone}(M)$. It is easy to see that $C^\sharp \neq \emptyset$ if and only if C has a base.

Let A be a nonempty subset of X and $F : A \times A \to 2^Y \setminus \{\emptyset\}$ be a set-valued mapping. A vector $x \in A$ is called an efficient solution if

$$F(x, y) \notin -C \setminus \{0\}, \text{ for all } y \in A.$$

The set of efficient solutions is denoted by $V(A, F)$. If int $C \neq \emptyset$, a vector $x \in A$ is called a weakly efficient solution if

$$F(x, y) \notin -\text{int } C, \text{ for all } y \in A.$$

The set of weakly efficient solutions is denoted by $V_W(A, F)$. Let $f \in C^* \setminus \{0\}$. A vector $x \in A$ is called an $f-$efficient solution if

$$f(F(x, y)) \geq 0, \text{ for all } y \in A.$$

The set of $f-$efficient solutions is denoted by $V_f(A, F)$.

Definition 1.1. A vector $x \in A$ is called a positive proper efficient solution if there exists $f \in C^\sharp$ such that

$$f(F(x, y)) \geq 0, \text{ for all } y \in A.$$

By definitions, we can get easily the followin Proposition.

Proposition 1.2. *If int $C \neq \emptyset$, then*

$$V(A, F) \subset V_W(A, F)$$

and

$$\bigcup_{f \in C^* \setminus \{0\}} V_f(A, F) \subset V_W(A, F).$$

Lemma 1.3. *Suppose that int $C \neq \emptyset$ and for each $x \in A$, $F(x, A) = \bigcup_{y \in A} F(x, y)$ is $C-$convex, that is $F(x, A) + C$ is a convex set. Then*

$$V_W(A, F) = \bigcup_{f \in C^* \setminus \{0\}} V_f(A, F).$$

2 Main results

In this section, we first give a density theorem. We will see that, under some suitable conditions, the set of positive proper efficient solutions is dense in the set of the efficient solutions. Finally, we discuss about the connectedness for the set of the efficient solutions.

Lemma 2.1. *(See Theorem 3.1 of [4]) Let $A \subset X$ be a nonempty compact convex set. Let $\psi : A \to Y$ and $\varphi : A \times A \to Y$ be two mappings. Assume that the following conditions are satisfied:*

1. *ψ is $C-$lower semicontinuous;*
2. *$\varphi(x, x) \geq 0$ for all $x \in A$ and φ is $C-$monotone;*
3. *for each $x \in A, \varphi(x, y)$ is $C-$lower semicontinuous in y and for each $y \in A, \varphi(x, y)$ is $C-$upper semicontinuous in x;*
4. *for each $x \in A, \psi(y) + \varphi(x, y)$ is $C-$convex mapping in y.*

Then, for each $f \in C^* \setminus \{0_{Y^*}\}, V_f(A, F)$ is a nonempty compact convex set, where

$$F(x, y) = \psi(y) + \varphi(x, y) - \psi(x), \quad \text{for} \quad x, y \in A.$$

The following result establishes an existence and uniqueness theorem for an efficient solution for bifunctions which one can consider it as an extension of Lemma 2.8 and Theorem 3.1 in [4] by relaxing the $C-$lower semicontinuity of the mapping φ in the second variable and compactness of the set as well extending the result for the mapping ψ is a bifunction, that is from one variable to two variables in the setting of topological vector spaces (more exact, we replace the locally convex topological vector space Y by topological vector space). Further, the coercivity (that is condition (5) in the next result is more general than the coercivity condition used in Theorem 3.1 of [9].

Lemma 2.2. *Let $A \subset X$ be a nonempty convex set. Let $\psi : A\times A \to Y$ and $\varphi : A\times A \to Y$ be two mappings. Assume that the following conditions are satisfied:*

1. *for each $y \in A, \psi(x, y) + \varphi(x, y)$ is $C-$ upper semicontinuous (or $(-C)-$ lower semicontinuous) in x;*
2. $$\psi(x, x) + \varphi(x, x) = 0, \text{ for all } x \in A;$$
3. *for each $x \in A, \psi(x, y) + \varphi(x, y)$ is $C-$convex mapping in y.*
4. *φ, ψ are $C-$strongly monotone on $A \times A$.*
5. *There exist a nonempty compact convex subset B and a compact subset D of A such that*
$$\forall y \in A\backslash D, \exists x \in B : \quad \psi(x, y) + \varphi(x, y) \in -intC.$$

Then, for each $f \in C^* \setminus \{0_{Y^*}\}$, the set of $f-$ efficient solutions, that is $V_f(A, F)$ is singleton and so convex and compact, where

$$F(x, y) = \psi(x, y) + \varphi(x, y), \text{ for all } x, y \in A.$$

The following result is the main goal of the paper that provides a density theorem between the solution set of efficient solutions and properly $f-$ efficient solutions.

Theorem 2.3. *Let $A \subset X$ be a nonempty compact convex set. Let $\psi : A \times A \to Y$ and $\varphi : A \times A \to Y$ be two mappings. Assume that the following conditions are satisfied:*

1. *for each $y \in A, \psi(x, y) + \varphi(x, y)$ is $C-$ upper semicontinuous (or $(-C)-$ lower semicontinuous) in x;*
2. $$\psi(x, x) + \varphi(x, x) = 0, \text{ for all } x \in A;$$
3. *for each $x \in A, \psi(x, y) + \varphi(x, y)$ is $C-$convex mapping in y.*
4. *φ, ψ are $C-$strongly monotone on $A \times A$.*
5. *$\Psi(A \times A)$ and $D = \{\varphi(x, y) : x, y \in A\}$ are bounded subsets of Y.*
6. *$C^\sharp \neq \emptyset$ and $intC \neq \emptyset$*

Then,

$$\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset cl(\bigcup_{f\in C^\sharp} V_f(A,F))$$

where

$$F(x,y) = \psi(x,y) + \varphi(x,y), \text{ for all } x,y \in A.$$

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The spectra of endomorphisms of analytic Lipschitz algebras

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Abstract

In this paper the spectra of certain endomorphisms of the analytic Lipschitz algebras $Lip_A(\bar{\mathbb{D}}, \alpha)$ are determined. We consider endomorphisms T of $Lip_A(\bar{\mathbb{D}}, \alpha)$ defined by $T(f) = f \circ \varphi$ for some $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha)$ for the case where φ has an interior fixed point.

Keywords: Spectra, Endomorphism, Analytic Lipschitz algebra
Mathematics Subject Classification [2010]: 47A10,46J15

1 Introduction

An endomorphism of an algebra B is a linear operator T of B into itself satisfying $T(ab) = (Ta)(Tb)$ for all $a, b \in B$. If a Banach function algebra B on a compact Hausdorff space X is natural, then every nonzero endomorphism T of B has the form $Tf = f \circ \varphi$ for a self-map φ of X. We call T the endomorphism of B induced by φ. The spectrum of an operator T on an algebra B is the set of complex numbers λ for which $\lambda - T$ is not invertible. We denote the spectrum of an operator T by $\sigma(T)$.

Let (X, d) be a metric space and $0 < \alpha \leq 1$. The complex valued function f on X is said to satisfy the Lipschitz condition of order α on X, if there exists a constant $K > 0$ such that $|f(x) - f(y)| \leq Kd(x, y)^\alpha$, for all $x, y \in X$. In this case we write

$$p_\alpha(f) = \sup\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y\}.$$

Suppose that $\mathbb{D}$ is the open unit disc in the complex plane $\mathbb{C}$. The analytic Lipschitz algebra on the closed unit disc $\bar{\mathbb{D}}$, $Lip_A(\bar{\mathbb{D}}, \alpha)$ is the algebra of functions f analytic in the open unit disc $\mathbb{D}$ that satisfy a Lipschitz condition of order α on $\bar{\mathbb{D}}$. It is well known that the analytic Lipschitz algebra $Lip_A(\bar{\mathbb{D}}, \alpha)$ is a natural Banach function algebra with the norm

$$\|f\| = |f|_{\bar{\mathbb{D}}} + p_\alpha(f) \qquad (f \in Lip_A(\bar{\mathbb{D}}, \alpha)),$$

where $|f|_{\bar{\mathbb{D}}} = \sup_{z \in \bar{\mathbb{D}}} |f(z)|$.

Kamowitz in [2] determined the spectra of a class of endomorphisms of the disc algebra $A(\bar{\mathbb{D}})$, the uniform algebra of functions analytic on the open unit disc $\mathbb{D}$ and continuous on

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$\bar{\mathbb{D}}$. In [3] and [4], other algebras of analytic functions were considered and the techniques and results of [2] were used to prove a generalization of theorems in there. In [1], the spectra of compact endomorphisms of analytic Lipshictz algebras on certain compact plane sets have been determined. In this paper, we determine the spectra of endomorphisms (not necessarily compact) of analytic Lipschitz algebras.

We remark that it follows from Schwarz's Lemma that if a continuous self-map $\varphi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ that is analytic on $\mathbb{D}$ has more than one fixed point in the open unit disc, then φ is identity function z. However, such φ can have infinitely many fixed points on the unit circle and yet it need not be equal to the identity function z. It is worth to mention that by Denjoy-Wolf's Theorem, every such self-map on $\bar{\mathbb{D}}$ has a fixed point in $\bar{\mathbb{D}}$.

We begin by showing that if φ has a fixed point z_0 in the open unit disc, it is no restriction to assume that $z_0 = 0$.

Lemma 1.1. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1$ and T be the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ. Suppose $|z_0| < 1$ and $\varphi(z_0) = z_0$. Let g be the linear fractional transformation $g(z) = \frac{z_0 - z}{1 - \bar{z}_0 z}$ and S the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by $\psi = g \circ \varphi \circ g$. Then $\psi(0) = 0, \psi'(0) = \varphi'(z_0)$ and $\sigma(S) = \sigma(T)$.*

2 Main results

At first we consider special case that operator defined on $Lip_A(\bar{\mathbb{D}}, \alpha)$ is an automorphism (a one to one and onto endomorphism).

Theorem 2.1. *If T is an automorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$, then $\sigma(T) = \{\lambda : |\lambda| = 1\}$.*

The main subject is to describe the spectra of endomorphisms of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ in terms of function theoretic properties of φ.

Definition 2.2. Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha)$ with $|\varphi|_{\bar{\mathbb{D}}} \leq 1$. For each nonnegative integer k, we denote the k^{th} iterate of φ by φ_k. That is, $\varphi_0(z) = z$ and $\varphi_k(z) = \varphi(\varphi_{k-1}(z)), |z| \leq 1$. The fixed set of φ is $\cap_k \varphi_k(\bar{\mathbb{D}})$.

It is not hard to show that the fixed set of φ is a compact, connected subset of the unit disc and that φ maps its fixed set onto itself. The spectra of the endomorphisms which we are considering, depend on the fixed set of the inducing maps. We follow by stating some useful lemmas.

Lemma 2.3. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1, \varphi(0) = 0$ and T be the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ. Then $\{(\varphi'(0))^n : n$ is a positive integer$\} \subset \sigma(T)$.*

We now try to investigate whether the converse of the inclusion in the above lemma holds.

Lemma 2.4. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1, \varphi(0) = 0$ and T be the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ. Assume $\lambda \neq (\varphi'(0))^n$ for all positive integers n, and $\lambda \neq 0, 1$. If m is a positive integer, $f, g \in Lip_A(\bar{\mathbb{D}}, \alpha)$ with $(\lambda - T)f = g$ and $g(0) = g'(0) = \cdots = g^m(0) = 0$, then $f(0) = f'(0) = \cdots = f^m(0) = 0$.*

To investigate the spectra of these operators we require the following result which can be easily deduced from the above lemma.

Corollary 2.5. *If $\lambda \neq (\varphi'(0))^n$ for all positive integers n, and $\lambda \neq 0, 1$, then λ is not an eigenvalue.*

Lemma 2.6. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1$ and T be the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ. If $f, g \in Lip_A(\bar{\mathbb{D}}, \alpha)$ with $(\lambda - T)f = g$, then*

$$\lambda^n f = f \circ \varphi_n + \lambda^{n-1} g + \lambda^{n-2} g \circ \varphi + \cdots + \lambda g \circ \varphi_{n-2} + g \circ \varphi_{n-1}$$

Lemma 2.7. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1, \varphi(0) = 0$. If $|z| < 1$ (or in fact, if $|\varphi_j(z)| < 1$ for some positive integer j), then, $\limsup_k |\varphi_k(z)|^{\frac{1}{k}} \leq |\varphi'(0)|$. Furthermore, (1) if $\varphi'(0) = 0$, then given $\epsilon > 0$, and $r \in [0, 1)$, there exists $C > 0$ so that for each positive integer m, $|\varphi_m(z)| \leq C\epsilon^m$ for all z, $|z| \leq r$. (2) If $0 < |\varphi'(0)| < 1$, then given $\epsilon > 0$, and $r \in [0, 1)$, there exists $C > 0$ so that for each positive integer m, $|\varphi_m(z)| \leq C((1 + \epsilon)|\varphi'(0)|)^m$ for all z, $|z| \leq r$.*

Theorem 2.8. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1$ and T be the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ. Suppose φ has a fixed point in the open unit disc and that the fixed set of φ is infinite. If T is not an automorphism, then $\sigma(T) = \{\lambda : |\lambda| \leq 1\}$..*

Lemma 2.9. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1, \varphi(0) = 0$ and T be the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ. Let m be a positive integer. Suppose every function in $Lip_A(\bar{\mathbb{D}}, \alpha)$ with a zero of order at least $(m+1)$ at 0 is in the range of $(\lambda - T)$, where $\lambda \neq 0, 1, (\varphi'(0))^n, n$ a positive integer. Then $1, z, z^2, , z^m$ are in the range of $(\lambda - T)$.*

We are now ready to show that the converse of the inclusion stated in Lemma 2.3 may be true.

Theorem 2.10. *Let $\varphi \in Lip_A(\bar{\mathbb{D}}, \alpha), |\varphi|_{\bar{\mathbb{D}}} \leq 1$ and T be the endomorphism of $Lip_A(\bar{\mathbb{D}}, \alpha)$ induced by φ. Let z_0 be a fixed point of φ in the open unit disc and suppose $\{z_0\}$ is the fixed set of φ. Then $\sigma(T) = \{(\varphi'(0))^n : n$ is a positive integer$\} \cup \{0, 1\}$.*

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Two Modes of Limit in Probabilistic Normed Spaces

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Abstract

In this paper, we study the concept of statistical limit superior and statistical limit inferior in probabilistic normed spaces. Our results are analogous to the results of Fridy and Orhan [Proc. Amer. Math. Soc. 125(1997), 3625-3631] but proofs are somewhat different and interesting.

Keywords: probabilistic normed space; statistical convergence; statistical limit superior; statistical limit inferior.
Mathematics Subject Classification [2010]: 40C05, 46S40.

1 Introduction

In [3] Menger introduced the notion of statistical metric space, now called probabilistic metric space, which is an interesting and important generalization of the notion of a metric space. Later on this notion was developed by many authors. The notion of probabilistic metric space gives rise to the concept of probabilistic normed space [5] which is an important and useful generalization of the concept of normed space. These two concepts of PM and PN-spaces the theory of statistical conhelp us to deal with the fuzzy like situations. The concept of statistical convergence studied by many authors. This idea was extended for double sequences by Mursaleen and Edely [4]. The idea of statistical convergence in probabilistic normed space has been studied by Karakus [2]. Many of the results in the theory of ordinary convergence have been extended to convergence. In this paper, we study the concept of statistical limit superior and statistical limit inferior in probabilistic normed space.

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}_\circ^+$ is called a *distributionfunction* if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$ We will denote the set of all distribution funtions by$D$.

Definition 1.2. A binary operationis $* : [0,1] \times [0,1] \to [0,1]$ said to be a continuous t-norm if it satisfies the following conditions:
(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) a $*$ 1 = a for all a$\in [0,1]$,
(d) a $*$ b $\leq$ c $*$ d whenever a $\leq$ c and b $\leq$ d for each $a, b, c, d \in [0,1]$.

*Speaker

Definition 1.3. A triplet $(X, N, *)$ is called a probabilistic normed space (in short PN-space) if X is a real vector space, $N : X \to D$ (for $x \in X$, the distribution funtion $N(x)$ is denoted by N_x, and $N_x(t)$ is the value of N_x at $t \in R$) and $*$ a continuous t-norm satisfying the following conditions:
(i) $N_x(0) = 0$,
(ii) $N_x(t) = 1$ for all $t > 0$ if and only if $x = 0$,
(iii) $N_{\alpha x}(t) = N_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R} - \{0\}$,
(iv) $N_{x+y}(s+t) \geq N_x(s) * N_y(t)$ for all $x, y \in X and\ s, t \in \mathbb{R}^+_\circ$.

Definition 1.4. Let $(X, N, *)$ be a PN-space. Then a sequence $x = (x_n)$ is said to be convergent to L with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $k_\circ$ such that $N_{x_n - L}(\varepsilon) > 1 - \lambda$ whenever $n \geq k_\circ$. It is denoted by N-lim x = L or $x_n \underset{\to}{N} L$ as $n \to \infty$.

Definition 1.5. Let$(X, N, *)$ be a PN-space. Then a sequence $x = (x_n)$ is said to be a Cauchy sequence with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists a positive integer $k\circ$ such that $N_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n, m \geq k_\circ$.

Definition 1.6. If K is a subset of $\mathbb{N}$, then the natural density of K denoted by $\delta(K)$, is dfined by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists. The natural density $\overline{\delta}$ may not exist for each set K. But the upper density $\overline{\delta}$ always exists for each set K identified as follows:

$$\overline{\delta}(K) := \limsup_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

.

Definition 1.7. A sequence $x = (x_n)$ of numbers is said to be statistically convergent to L if

$$\delta(\{k \in \mathbb{N} : | x_n - L | \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $st - lim\ x = L$.

Definition 1.8. A sequence $x = (x_n)$ of numbers is said to be statistically bounded if there is a number B such that

$$\delta(\{k \in \mathbb{N} : |x_n| \geq B\}) = 0$$

.

Definition 1.9. The real number sequence x is said to be statistically bounded with respect to the probabilistic norm N if there exists some $t_\circ \in \mathbb{R}$ and $b \in (0, 1)$ such that $\delta(\{k : N_{x_k}(t_\circ) \leq 1 - b\}) = 0$.

2 Main results

In this section we define the concept of statistical limit superior and statistical limit inferior in probabilistic normed spaces and demonstrate through an example how to compute these points in a PN-space.

Definition 2.1. Let$(X, N, *)$ be a PN-space. We say that a sequencein $x = (x_k)$ is statistically convergent to $L \in X$ with respect to the probabilistic norm N provided that for every $\varepsilon > 0$ and $b \in (0, 1)$

$$\delta(\{k \in \mathbb{N} : N_{x_k - L}(\varepsilon) \leq 1 - b\}) = 0,$$

In this case we write $st_N - lim\ x = L$, where $L = st_N - limx$.

Definition 2.2. Let $(X, N, *)$ be a PN-space. $l \in X$ is called a limit point of the sequence $x = (x_k)$with respect to the probabilistic norm N provided that there is a subsequence of x that converges to l with respect to the probabilistic norm N. Let $L_N(x)$ denote the set of all limit points of the sequence x with respect to the probabilistic norm N.

Definition 2.3. If $\{x_{k_{(j)}}\}$ is a subsequence of $x = (x_k)$ and $K := \{k(j) : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k_{(j)}}\}$ by $\{x\}_K$. If $\delta(K) = 0$ then $\{x\}_K$ is called a subsequence of density zero or a thin subsequence. On the other hand, $\{x\}_K$ is a nonthin subsequence of x if K does not have density zero.

Definition 2.4. Let $(X, N, *)$be a PN-space. Then $\xi \in X$ is called a statistical limit point of the sequence $x = (x_k)$ with respect to the probabilistic norm N provided that there is a nonthin subsequence of x that converges to ξ with respect to the probabilistic norm N. In this case we say ξ is an st_N-limit point of sequence $x = (x_k)$.

Let $\Lambda_N(x)$ denote the set of all st_N-limit points of the sequence x.

Definition 2.5. Let $(X, N, *)$ be a PN-space. Then $\eta \in X$ is called a statistical cluster point of the sequence $x = (x_k)$ with respect to the probabilistic norm N provided that for every $\varepsilon > 0$ and a $\in (0, 1)$,

$$\overline{\delta}(\{k \in \mathbb{N} : N_{x_k - \eta}(\varepsilon) > 1 - a\}) > 0.$$

In this case we say η is an st_N-cluster point of the sequence x. Let $\Gamma_N(x)$ denote the set of all st_N-cluster points of the sequence x.

Definition 2.6. The real number sequence x is said to be bounded with respect to the probabilistic norm N if there exists some $t_\circ \in \mathbb{R}$ and for every $b \in (0, 1)$ such that $N_{xk}(t_\circ) > 1 - b$ for all k. For a real sequence x let us define the sets B_x^N and A_x^N by

$$B_x^N := \{b \in (0, 1) : \delta(\{k : N_{xk}(\varepsilon) < 1 - b\}) \neq 0\}$$

$$A_x^N := \{a \in (0, 1) : \delta(\{k : N_{xk}(\varepsilon) > 1 - a\}) \neq 0\}$$

Note that throughout this paper the statement $\delta(\{K\}) \neq 0$ means that either $\delta(\{K\}) > 0$ or K does not have natural density.

Definition 2.7. If x is a real number sequence then the statistical limit superior of x with respect to the probabilistic norm N is defined by

$$st_N - \limsup x := \begin{cases} \sup B_x^N & \text{if } B_x^N \neq 0 \\ 0 & \text{if } B_x^N = 0 \end{cases}$$

Also, the statistical limit inferior of x with respect to the probabilistic norm N is defined by

$$st_N - \liminf x := \begin{cases} \inf A_x^N & \text{if } A_x^N \neq 0 \\ 1 & \text{if } A_x^N = 0 \end{cases}$$

Theorem 2.8. *If $b = st_N - \limsup x$ is finite, then for every positive numbers ε and γ*

$$\delta(\{k : N_{x_k}(\varepsilon) < 1 - b + \gamma\}) \neq 0\} \quad and \quad \delta(\{k : N_{xk}(\varepsilon) < 1 - b - \gamma\}) = 0\} \tag{1}$$

Conversely, if (1) holds for every positive ε and γ then $b = st_N - \limsup x$.

Theorem 2.9. *If $a = st_N - \liminf x$ is finite, then for every positive numbers ε and γ*

$$\delta(\{k : N_{x_k}(\varepsilon) > 1 - a - \gamma\}) \neq 0\} \quad and \quad \delta(\{k : N_{xk}(\varepsilon) > 1 - a + \gamma\}) = 0\} \tag{2}$$

Conversely, if (2) holds for every positive ε and γ then $a = st_N - \liminf x$.

Remark 2.10. From the definition of statistical cluster points in [1] we see that Theorems 1.17 and 1.18 can be interpreted as saying that $st_N - \limsup x$ and $st_N - \liminf x$ are the greatest and the least statistical cluster points of x, respectively.

Theorem 2.11. *For any sequence $x, st_N - \liminf x \leq st_N - \limsup x$.*

Theorem 2.12. *In PN-space $(X, N, *)$ the statistically bounded sequence x is statistically convergent if and only if*

$$st_N - \liminf x = st_N - \limsup x$$

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Universal metric space of dimension n and its application in clustering

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Abstract

In this paper we introduce an n-dimensional ($n \geq 2$) distance metric over a given space to define a universal metric space. This distance metric measures how separated every n points of the space. One goal of this paper suggest a possible application of this theory is clustering.

Keywords: Universal metric spaces, G-metric spaces, Clustering
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

The theory of metric spaces plays a major role in different fields of mathematics and applied sciences. Gähler [1] introduced the notion of a 2-metric space. In 1992, Dhage [2] proposed the notion of a $D-$metric space. They introduced a new class of generalized metric spaces called G-metric spaces. In 2014, Dr. Dehghan Nezhad proposed the notion of a metric spaces called U_n-metric spaces as follows.

2 Universal metric spaces of dimension n

For $n \geq 2$, let X^n denotes the cartesian product $X \times \ldots \times X$. We begin with the following definition.

Definition 2.1. Let X be a non-empty set. Let $U : X^n \longrightarrow \mathbb{R}^+$ be a function that satisfies the following conditions:

(U1) $U(x_1, \ldots, x_n) = 0$ if $x_1 = \ldots = x_n$.

(U2) $U(x_1, \ldots, x_n) > 0$ for all $x_1, ..., x_n$ with $x_i \neq x_j$, for some $i, j \in \{1, ..., n\}$.

(U3) $U(x_1, \ldots, x_n) = U_n(x_{\pi_1}, \ldots, x_{\pi_n})$, for every permutation $(\pi_{(1)}, ..., \pi_{(n)})$ of $(1, 2, ..., n)$.

(U4) $U(x_1, x_2, \ldots, x_{n-1}, x_{n-1}) \leq U(x_1, x_2, \ldots, x_{n-1}, x_n)$ for all $x_1, \ldots, x_n \in X$.

(U5) $U(x_1, x_2, \ldots, x_n) \leq c(U(x_1, a, ..., a) + U(a, x_2, ..., x_n))$, for all $x_1, \ldots, x_n, a \in X,\ 0 < c \leq 1$.

*Speaker

The function U is called a universal metric of dimension n, or more specifically a U_n-metric on X, and the pair (X, U) is called a U_n-metric space.

In the sequel, for simplicity we assume that $c = 1$. The following useful properties of a U_n-metric are easily derived from the axioms.

Example 2.2. Let (X, d) be a usual metric space, then (X, S_n) and (X, M_n) are U_n-metric spaces, where

$$S_n(x_1, \dots, x_n) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} d(x_i, x_j), \tag{1}$$

$$M_n(x_1, \dots, x_n) = \max\{d(x_i, x_j) : 1 \le i < j \le n\}. \tag{2}$$

Proposition 2.3. *Let (X, U) be a U_n-metric space, then for $x_0 \in X$, $r > 0$,*
(i) If $U(x_0, x_2, ..., x_n) < r$, then $x_2, ..., x_n \in B_U(x_0, r)$;
(ii) If $y \in B_U(x_0, r)$, then there exists, $\delta > 0$ such that $B_U(y, \delta) \subseteq B_U(x_0, r)$.

Fixed point theorems are the basic mathematical tools used in showing the existence of solution concepts in game theory and economics [3].
In this section, we consider U_n-approximate fixed point for the map $T : X \longrightarrow X$.

Definition 2.4. Let (X, U) be a U_n-metric space. We say that the map $T : X \longrightarrow X$ has a U_n-approximate fixed point, if for every $\epsilon > 0$, there exists $x_0 \in X$ such that $U(x_0, Tx_0, Tx_0, \dots, Tx_0) < \epsilon$.

Theorem 2.5. *Let (X, U) be a U_n-metric space and $T : X \longrightarrow X$ be a map. If for all $x \in X$,*

$$\lim_{n \longrightarrow \infty} U(T^n x, T^{n+1} x, T^{n+1} x, \dots, T^{n+1} x) = 0,$$

then the map T has a U_n-approximate fixed point.

Theorem 2.6. *Let (X, U) be a U_n-metric space and $T : X \longrightarrow X$ be a map. If for all $x_1, x_2, \dots, x_n \in X$,*

$$\begin{aligned} U(Tx_1, Tx_2, \dots, Tx_n) \le\ & k \ \max\{U(x_1, x_2, \dots, x_n), U(x_1, Tx_1, Tx_1, \dots, Tx_1), \\ & U(x_2, Tx_2, Tx_2, \dots, Tx_2), \dots, U(x_n, Tx_n, Tx_n, \dots, Tx_n)\}, \end{aligned}$$

where $k \in [0, 1/2)$. Then T has a U_n-approximate fixed point.

3 Generalized K-means clustering

Clustering is a division of data into groups of similar objects. One of the most widely used clustering algorithm which is based on minimizing a formal objective function is k-means clustering. It was designed to cluster numerical data in which each cluster has a center called the mean. In this algorithm, the number of clusters k is assumed to be fixed. There is an error function in this algorithm. The conventional k-means algorithm is briefly described below [4]. Let D be a data set with m instances $x_1, ..., x_m$, and let $C_1, C_2, ..., C_k$ be the k disjoint clusters of D. Then the error function is defined as

$$E = \sum_{j=1}^{k} \sum_{x \in C_j} d(x, \mu(C_j)),$$

where $\mu(C_j)$ is the centroid of cluster (calculated by averaging the observations of each cluster), and $d(x, \mu(C_j))$ denotes the an ordinary distance between the point x and $\mu(C_j)$. Here we propose a k-means algorithm that picks $n-1$ points at a time and calculates the U_n distance between this points and center of clusters. Then, these $n-1$ points are assigned to that cluster having least distance between the center and $n-1$ data points. The proposed algorithm is given below:
(1) Choose integer k, the number of clusters.
(2) Assume k number of initial seed points.
(3) Randomly assign the data into k initial cluster C_1,...,C_k and determine $\mu(C_1)$,...,$\mu(C_k)$.
(4) Consider a subset $\{x_{i1}, ..., x_{i(n-1)}\}$ from the data set $\{x_1, ..., x_m\}$, with $n < m$, and calculate the $d_{ij} = U_n(x_{i1}, ..., x_{i(n-1)}, \mu(C_j))$.
(5) Let $L_j = \text{argmin}_{1 \leq j \leq k} d_{ij}$ and assign the $n-1$ points to the cluster L_j.
(6) Compute the new centriods after assigning all data points to k clusters.
(7) Repeat steps (4) to (6) until the difference between the previous and current centriods is less than the specified threshold value.
(8) Repeat steps (2) to (7) with different initial seed points until the algorithm reaches the minimum objective function.

3.1 Experimental results

A generalized K-means clustering of distances in miles between some Italian cities. The name of this cities is Rome, Neples, Potenza, Milan, Venice, Trento, Florence, Turin. In this method, we use triple-linkage clustering using U_n-metric spaces. The symbol of cluster is k_i and E represents the error rate. For case $n = 3$, first the nearest triplet of cities are merged into a single cluster. Then we compute the distance (space) from this new compound object to all other pairs of objects. In triple-linkage clustering the rule is that the space of the triangle formed by the compound object with another pair of objects is equal to the smallest space values of the triangles formed by each member of the compound cluster with the pair of outside objects.

	Ro	Ne	Po	Mi	Ve	Tr	Fl	Tu
Ro	0	136	225	358	330	368	173	419
Ne	136	0	98	489	461	499	304	550
Po	225	98	0	578	550	588	393	639
Mi	358	489	578	0	170	150	186	86
Ve	330	468	550	170	0	98	159	250
Tr	368	499	588	150	98	0	195	221
Fl	173	304	393	186	159	195	0	246
Tu	419	550	639	86	250	221	246	0

Members of each cluster in beginning is: $k_1 = [Ve, Fl, Ne, Tu]$ and $k_2 = [Tr, Po, Mi, Ro]$. After clustering, the results are summarized in the table below:

	k_1	k_2	E
n=2	[Ro,Ve,Fl,Tu]	[Ne,Tr,Po,Mi]	2039
n=3	[Ne,Fi,Ve,Po]	[Tr,Mi,Ro,Tu]	2014
n=4	[Mi,Tu,Po,Ne]	[Ve,Tr,Fl,Ro]	1938

Now we compare the results (clustering of eight cities) of the case $n = 2$ (conventional $k - means$ clestering) and case $n = 3$ (generalized K-means clustering). Following table show the results of applying the conventional $k - means$ clestering and generalized K-means clustering to our example data of eight points. U_n distance for $n = 3$ is much better (faster) than Euclidian distance (case $n = 2$). As you seen, there is minimal error in $n = 4$ and its value is equal 1938. so, in this case we have the best clustering.

4 Conclusion

The principal conclusion from the research in this paper that generalized G-metric spaces into U_n-metric spaces. This conclusion is justified for the following reasons. We have shown that U_n-metric spaces as an extension of classical G-metric spaces have been more considered in recently decade. In this article we prove some U_n-approximate fixed point results for mappings that satisfy certain conditions on a U_n-metric space. The primary motivation for this work has been to develop metric based tools for applications in program verification in theoretical computer science.

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Weak fixed point property in closed subspaces of some compact operator spaces

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Abstract

For suitable Banach spaces X and Y with Schauder decompositions and closed subspace M of some compact operator spaces from X to Y, it is shown that the complete continuity of all evaluation operators on M, is a sufficient condition for the weak fixed point property of M; where for each $y^* \in Y^*$, the evaluation operator on M is defined by $\psi_{y^*}(T) = T^*y^*$, $T \in M$.

Keywords: weak fixed point property, evaluation operator, compact operator, completely continuous operator
Mathematics Subject Classification [2010]: 47H10, 47L05

1 Introduction

If C is a subset of a Banach space X, a mapping $T : C \to X$ is called a nonexpansive map if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We say that X has the fixed point property (fpp) if every nonexpansive self map $T : C \to C$ of each nonempty, closed, bounded and convex subset C of X has a fixed point. But when the same holds for every nonempty weakly compact convex subset of X, we say that X has the weak fixed point property (wfpp). It is evident that fpp implies the wfpp and for reflexive Banach spaces, both properties are the same.

For example, every uniformly convex Banach space and every Banach space with uniform normal structure have the fpp [8], every Banach space with weak normal structure and every Banach space with the Schur property (i.e. the weak and norm convergence of sequences are the same), have the wfpp [12, 8].

Following the work of Maurey [10] and Dowling-Lennard [7], which proved that a closed subspace M of the Bochner integrable function space $L^1([0, 1])$, has the fpp if and only if M is reflexive; it is natural to ask for a given Banach space X, what closed subspaces of it have the (weak) fpp.

There are a few works on fpp and wfpp in operator spaces. In 1999, Dowling and Randrianantoanina [6] along with a result of Besbes [4], have shown that a closed subspace of $K(H)$, of all compact operators on the Hilbert space H, has the fixed point property if and only if it is reflexive. Also, the Banach space $K(l^2)$, and then all it's closed subspaces has the wfpp [4].

*Speaker

On the other hands, in [5], [11] and [13] the authors proved that for some closed subspace M of some compact operator spaces between Hilbert or Banach spaces X and Y, compactness of all evaluation operators $\phi_x : M \to Y$ and $\psi_{y^*} : M \to X^*$ on M is a necessary and/or sufficient condition for the Schur property of the dual M^* of M, where for each $x \in X$ and $y^* \in Y^*$, the evaluation operators on M are defined by

$$\phi_x(T) = Tx \ , \ \psi_{y^*}(T) = T^*y^* \ , \ T \in M.$$

Since the Schur property implies the wfpp, it is natural to ask under what conditions, a closed subspace M of an operator space has the wfpp. Here, we obtain some sufficient conditions for the wfpp of a closed subspace of some compact operator spaces relative to complete continuity of all evaluation operators.

Now we remember the following Lemma of Goebel and Karlovitz [9] and elementary ultraproduct techniques in the fixed point theory [2].

Lemma 1.1. *Suppose that X is a Banach space and $T : X \to X$ is a nonexpansive map. If K is a minimal T-invariant, weakly compact and convex subset of X and (x_n) is an approximate fixed point sequence in K, then for all $x \in X$,*

$$\lim_n \|x - x_n\| = diam(K).$$

Let $\mathcal{U}$ be a nontrivial ultrafilter on the natural numbers $\mathbb{N}$, and X be a Banach space. The ultrapower space $\widetilde{X}$ of X is the quotient space

$$l^\infty(X) = \{(x_n) : x_n \in X \quad for \quad all \quad n \in \mathbb{N}, \quad \|(x_n)\| = \sup_n \|x_n\| < \infty\},$$

by $N = \{(x_n) \in l^\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$, where $\lim_{\mathcal{U}} \|x_n\|$ denotes the ultraproduct limit of the sequence $(\|x_n\|)$, [2]. We will denote the coset $(x_n) + N \in \widetilde{X}$ by $\widetilde{(x_n)}$. Clearly

$$\|\widetilde{(x_n)}\|_{\widetilde{X}} = \lim_{\mathcal{U}} \|x_n\|.$$

We want to use the ultraproduct techniques to prove the main results of the article. These techniques was originally used by Alspach and Maurey ([3] and [10]). Maurey proved that reflexive subspaces of $L^1([0,1])$ have the fpp while the fact that $L^1([0,1])$ does not have the fpp, is due to Alspach.

We need now some definitions and lemmas to prove the main theorem. At first, we remember the WORTH property.

Definition 1.2. A Banach space X has the WORTH property if whenever $x \in X$ and (x_n) is a weakly null sequence in X then we have

$$\lim_{n\to\infty} \Big| \|x + x_n\| - \|x - x_n\| \Big| = 0,$$

For example, every Schur space has the WORTH property. The following Lemma 1.5, establish some sufficient conditions for the WORTH property of some closed subspaces of compact operator spaces. In order to prove this lemma, we need the following lemma which can be proved by [1, Remark 2.3].

Definition 1.3. An operator T between two Banach spaces is completely continuous if T takes weakly convergent sequences into norm convergent sequences.

Lemma 1.4. *Let X and Y be two Banach spaces and M be a closed subspace of $L(X,Y)$.*
(a) If all evaluation operators ϕ_x are completely continuous, then for all compact operator $S \in K(X)$, the operator $K \mapsto KS$ from M into $L(X,Y)$ is completely continuous.
(b) If all evaluation operators ψ_{y^*} are completely continuous, then for all compact operator $T \in K(Y)$, the operator $K \mapsto TK$ from M into $L(X,Y)$ is completely continuous.

In the following, for each complemented subspace V of a Banach space X, the projection of X onto V is denoted by P_V.

Lemma 1.5. *Let X and Y be two Banach spaces such that Y has finite dimensional Schauder decomposition $\sum_{n=1}^{\infty} \bigoplus Y_n$. Let M be a closed subspace of $K(X,Y)$ and $\limsup_m \|I-2P_{W_m}\| \leq 1$ whenever $W_m = \sum_{i=1}^{m} \bigoplus Y_i$ for all $m \in \mathbb{N}$. If all of the evaluation operators ψ_{y^*} are completely continuous, then M has the WORTH property.*

2 Main results

Now for the proof of the main Theorem 2.2, we need the following Lemma.

Lemma 2.1. *Suppose X and Y are two Banach spaces which have Schauder decompositions $\sum_{n=1}^{\infty} \bigoplus X_n$ and $\sum_{n=1}^{\infty} \bigoplus Y_n$ respectively, such that the decomposition of X is shrinking, decomposition of Y is finite dimensional and M is a closed subspace of $K(X,Y)$. If (K_n) is a weakly null sequence in M, then there is a subsequence (K_{n_i}) of (K_n) and a sequence (U_i) of $K(X,Y)$ such that $\lim_i \|U_i - K_{n_i}\| = 0$.*

Now, we give some sufficient conditions of wfpp of some closed subspace M of compact operators with respect to complete continuity of all evaluation operators.

Theorem 2.2. *Suppose X and Y are two Banach spaces which have Schauder decompositions $\sum_{n=1}^{\infty} \bigoplus X_n$ and $\sum_{n=1}^{\infty} \bigoplus Y_n$ respectively, such that the decomposition of X is shrinking, the decomposition of Y is monotone and finite dimensional and $\limsup_m \|I-2P_{W_m}\| \leq 1$ whenever $W_m = \sum_{i=1}^{m} \bigoplus Y_i$ for all $m \in \mathbb{N}$. Let M be a closed subspace of $K(X,Y)$ such that all evaluation operators ψ_{y^*} are completely continuous. Then M has the weak fixed point property.*

There are several Banach spaces that are embedded into $K(X,Y)$, and one can obtain the wfpp for these spaces. In the following corollaries we give some classes of Banach spaces such that the space of compact operators between them has the property $\limsup_m \|I - 2P_{W_m}\| \leq 1$.

Corollary 2.3. *Let X be a Banach space with shrinking Schauder decomposition $\sum_{n=1}^{\infty} \bigoplus X_n$ and $Y = \sum_{n=1}^{\infty} \bigoplus Y_n$ be a c_0-direct sum of finite dimensional Banach spaces Y_n. Let M be a closed subspace of $K(X,Y)$ such that all evaluation operators ψ_{y^*} are completely continuous. Then M has the weak fixed point property.*

Corollary 2.4. *Let X be a Banach space with shrinking Schauder decomposition and Y be an l^p-direct sum of finite dimensional Banach spaces, where $1 \leq p < \infty$. Let M be a closed subspace of $K(X,Y)$ such that all evaluation operators ψ_{y^*} are completely continuous. Then M has the weak fixed point property.*

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Weighted Composition Operators on Spaces of Analytic Vector-valued Lipschitz Functions

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Abstract

Let φ be an analytic self-map of $\mathbb{D}$ and ψ be an analytic operator-valued function on $\mathbb{D}$, where $\mathbb{D}$ is the unit disk. We discuss the boundedness and compactness of weighted composition operators $W_{\psi,\varphi} : f \mapsto \psi(f \circ \varphi)$ on $\mathrm{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$, the space of analytic X-valued Lipschitz functions f, where X is a complex Banach space and $\alpha \in (0, 1]$.

Keywords: Analytic vector-valued Lipschitz functions, vector-valued Bloch spaces, weighted composition operator, compactness.
Mathematics Subject Classification [2010]: 46E40, 47A56, 47B33.

1 Introduction

Given X and Y two complex Banach spaces, let $H(\mathbb{D}, X)$ be the space of analytic X-valued functions $f : \mathbb{D} \to X$ and $S(\mathbb{D}, X)$ be any subspace of $H(\mathbb{D}, X)$, where $\mathbb{D}$ is the unit disk in the complex plane. If φ is an analytic self-map of $\mathbb{D}$ and $\psi : \mathbb{D} \to L(X, Y)$ is an analytic operator-valued function, where $L(X, Y)$ is the Banach space of all bounded linear operators from X to Y, then the weighted composition operator $W_{\psi,\varphi}$ from $S(\mathbb{D}, X)$ to $S(\mathbb{D}, Y)$ is defined to be the linear operator of the form $W_{\psi,\varphi}(f)(z) = \psi_z(f(\varphi(z)))$ for every $f \in S(\mathbb{D}, X)$ and $z \in \mathbb{D}$, where ψ_z is $\psi(z)$.

Let (S, d) be a metric space and $\alpha \in (0, 1]$. The space of all functions $f : S \to X$ for which

$$p_\alpha(f) = \sup\left\{\frac{\|f(s_1) - f(s_2)\|}{d^\alpha(s_1, s_2)} : s_1, s_2 \in S, s_1 \neq s_2\right\} < \infty$$

and

$$\|f\|_S = \sup_{s\in S}\{\|f(s)\| : s \in S\} < \infty,$$

is denoted by $\mathrm{Lip}_\alpha(S, X)$. The subspace of functions f for which

$$\lim_{d(s_1,s_2)\to 0} \frac{\|f(s_1) - f(s_2)\|}{d^\alpha(s_1, s_2)} = 0,$$

is denoted by $\mathrm{lip}_\alpha(S, X)$. The spaces $\mathrm{Lip}_\alpha(S, X)$ and $\mathrm{lip}_\alpha(S, X)$ equipped with the norm $\|f\|_\alpha = \|f\|_S + p_\alpha(f)$ are Banach spaces. These are called vector-valued Lipschitz spaces, see e.g. [4, 3].

If S is a compact subset of the complex plane $\mathbb{C}$ with nonempty interior, then the space of all continuous X-valued functions on S which are analytic on the interior of S is denoted by $A(S,X)$. For $\alpha \in (0,1]$, we define the analytic vector-valued Lipschitz spaces as

$$\mathrm{Lip}_A(S,X,\alpha) = A(S,X) \cap \mathrm{Lip}_\alpha(S,X), \quad \mathrm{lip}_A(S,X,\alpha) = A(S,X) \cap \mathrm{lip}_\alpha(S,X).$$

Clearly, $\mathrm{Lip}_A(S,X,\alpha)$ and $\mathrm{lip}_A(S,X,\alpha)$ are closed subspaces of $\mathrm{Lip}_\alpha(S,X)$. In the case that $X = \mathbb{C}$, we omit X in the notation.

Composition operators and weighted composition operators between vector-valued Lipschitz spaces and analytic vector-valued functions have been studied in [4, 2, 5]. The composition operators on analytic Lipschitz spaces in the scalar-valued case have been investigated in [1, 6]. This work was motivated by finding an essential norm estimate of weighted composition operators between analytic vector-valued Lipschitz spaces $\mathrm{Lip}_A(\overline{\mathbb{D}},X,\alpha)$, whenever $\alpha \in (0,1]$. However, for now, we discuss boundedness and compactness of these operators.

For a positive real number α and a Banach space X, the vector-valued Bloch space $B_\alpha(X)$, denotes the Banach space of all analytic functions $f : \mathbb{D} \to X$ for which

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \|f'(z)\| < \infty,$$

endowed with the norm $\|f\|_{B_\alpha(X)} = \|f(0)\| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\|f'(z)\|$. Let $\Lambda_\alpha(X) = \mathrm{Lip}_\alpha(\mathbb{D},X) \cap H(\mathbb{D},X)$ for $\alpha \in (0,1]$ and $\Lambda^0_\alpha(X) = \mathrm{lip}_\alpha(\mathbb{D},X) \cap H(\mathbb{D},X)$ for $\alpha \in (0,1)$.

Here we adopt the notation of [2, Section 5]. Let E be a Banach subspace of $H(\mathbb{D})$ which contains the constant functions and its closed unit ball $U(E)$ is compact for the compact open topology. The space

$$^*E := \{u \in E^* : u|U(E) \quad is \quad co-continuous\}$$

endowed with the norm induced by E^* is a Banach space and the evaluation map $E \to (^*E)^*$, $f \mapsto [u \mapsto u(f)]$ is an isometric isomorphism. In particular, $^*E$ is a predual of E. For a Banach space X, the vector-valued space $E[X]$ defined as

$$E[X] := \{f \in H(\mathbb{D},X) : x^* \circ f \in E, \quad x^* \in X^*\},$$

by the norm $\|f\|_{E[X]} = \sup_{\|x^*\|\le 1} \|x^* \circ f\|$, is a Banach space. The map $\Delta : \mathbb{D} \to^* E$, $\Delta(z) = \delta_z$, where δ_z is the evaluation map on E, is analytic and the linear operator $\chi : L(^*E,X) \to E[X]$, $\chi(T) = T \circ \Delta$ is bounded. For $g \in E[X]$ and $u \in^* E$, consider the map $\psi(g)(u) : X^* \to \mathbb{C}$ by $\psi(g)(u)(x^*) = u(x^* \circ g)$. Clearly, $\psi(g) \in L(^*E, X^{**})$ and $\psi(g)(\delta_z) \in L(^*E,X)$. Hence $\psi : E[X] \to L(^*E,X)$ is a bounded linear operator. Using ψ and χ, Bonet, et al. in [2, Lemma 10] showed that the space $E[X]$ is isomorphic to $L(^*E,X)$. We use this result for the spaces $\Lambda_\alpha[X]$ and $B_\alpha[X]$.

Let $\alpha \in (0,1)$. By Hardy-Littlewood theorem, $\Lambda_\alpha = B_{1-\alpha}$ and $\|\cdot\|_\alpha \asymp \|\cdot\|_{B_{1-\alpha}}$. That is, there are strictly positive constants a,b such that $a\|\cdot\|_\alpha \le \|\cdot\|_{B_{1-\alpha}} \le b\|\cdot\|_\alpha$. Hence, $^*\Lambda_\alpha =^* B_{1-\alpha}$, where $^*\Lambda_\alpha$ and $^*B_{1-\alpha}$ are the preduals of Λ_α and $B_{1-\alpha}$, respectively. Therefore, $\Lambda_\alpha[X] = B_{1-\alpha}[X]$ and

$$\mathrm{id} : \Lambda_\alpha[X] \xrightarrow{\psi} L(^*\Lambda_\alpha, X) = L(^*B_{1-\alpha},X) \xrightarrow{\chi} B_{1-\alpha}[X]$$

$$\mathrm{id} : B_{1-\alpha}[X] \xrightarrow{\psi} L(^*B_{1-\alpha}, X) = L(^*\Lambda_\alpha,X) \xrightarrow{\chi} \Lambda_\alpha[X]$$

are bounded. Hence, $\|\cdot\|_{\Lambda_\alpha[X]} \asymp \|\cdot\|_{B_{1-\alpha}[X]}$. Since $\Lambda_\alpha(X) = \Lambda_\alpha[X]$ and $B_{1-\alpha}(X) = B_{1-\alpha}[X]$, we conclude that

$$\|\cdot\|_{\Lambda_\alpha(X)} = \|\cdot\|_{\Lambda_\alpha[X]} \asymp \|\cdot\|_{B_{1-\alpha}[X]} = \|\cdot\|_{B_{1-\alpha}(X)}.$$

Moreover, $f \in \Lambda_1(X)$ if and only if $f' \in H^\infty(X)$ (the space of bounded X-valued analytic functions on $\mathbb{D}$) and $\|f\|_{\Lambda_1(X)} = \|f'\|_{\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}}$. Hence the norm

$$\|f\|_{\Lambda_\alpha(X)} = \|f\|_{\overline{\mathbb{D}}} + \sup_{z\in\mathbb{D}}(1-|z|^2)^{(1-\alpha)}\|f'(z)\|, \quad (f \in \Lambda_\alpha(X))$$

defines an equivalent norm on $\Lambda_\alpha(X)$. In the sequel we use this norm for $\Lambda_\alpha(X)$.

Since every function in $\Lambda_\alpha(X)$ has a unique extension to a Lipschitz function on $\overline{\mathbb{D}}$, to show the boundedness and compactness of $W_{\psi,\varphi} : \mathrm{Lip}_A(\overline{\mathbb{D}}, X, \alpha) \longrightarrow \mathrm{Lip}_A(\overline{\mathbb{D}}, Y, \beta)$, we characterize the boundedness and compactness of $W_{\psi,\varphi} : \Lambda_\alpha(X) \longrightarrow \Lambda_\beta(Y)$ for $\alpha \in (0, 1]$.

2 Main Results

For every $f \in H(\mathbb{D}, X)$ and $z \in \mathbb{D}$ we have

$$(W_{\psi,\varphi}(f))'(z) = \varphi'(z)\psi_z(f'(\varphi(z))) + \psi'_z(f(\varphi(z))). \tag{1}$$

Identifying each $x \in X$ with the constant function $1_x(z) = x$ for $z \in \mathbb{D}$, the Banach space X can be considered as a subspace of $\Lambda_\alpha(X)$. For every $x \in X$ and $f \in \Lambda_\alpha$, the function f_x defined by $f_x(z) = f(z)x$ belongs to $\Lambda_\alpha(X)$. Moreover, $\|f_x\|_{\Lambda_\alpha(X)} = \|f\|_{\Lambda_\alpha}\|x\|$ and

$$(W_{\psi,\varphi}(f_x))'(z) = \varphi'(z)f'(\varphi(z))\psi_z(x) + f(\varphi(z))\psi'_z(x). \tag{2}$$

In the next theorem, we characterize bounded weighted composition operators between analytic vector-valued Lipschitz spaces.

Theorem 2.1. *For $0 < \alpha \leq 1$ the operator $W_{\psi,\varphi}$ maps $\Lambda_\alpha(X)$ boundedly into $\Lambda_\beta(Y)$ if and only if $\psi \in \Lambda_\beta(L(X, Y))$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{1-\beta}}{(1-|\varphi(z)|^2)^{1-\alpha}}|\varphi'(z)|\|\psi_z\| < \infty. \tag{3}$$

Here, we characterize compact weighted composition operators from $\Lambda_\alpha(X)$ into $\Lambda_\beta(Y)$, where $\alpha \in (0, 1]$. For this, we use the idea of [5] and define $T_\psi : X \to B_\beta(Y)$, by $T_\psi(x)(z) = \psi_z(x)$. In the case that $W_{\psi,\varphi}$ is bounded, T_ψ is a bounded linear operator and $\|T_\psi\|_{X\to B_\beta(Y)} \leq \|W_{\psi,\varphi}\|_{B_\alpha(X)\to B_\beta(Y)}$.

Theorem 2.2. *Let $0 < \alpha, \beta \leq 1$ and $W_{\psi,\varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)$ be a bounded weighted composition operator. Then $W_{\psi,\varphi}$ is compact if and only if T_ψ is compact and*

$$\limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^{1-\beta}}{(1-|\varphi(z)|^2)^{1-\alpha}}|\varphi'(z)|\|\psi_z\| = 0. \tag{4}$$

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3. *For $0 < \alpha, \beta \leq 1$, the bounded weighted composition operator $W_{\psi,\varphi}$: $\mathrm{Lip}_A(\overline{\mathbb{D}}, X, \alpha) \longrightarrow \mathrm{Lip}_A(\overline{\mathbb{D}}, Y, \beta)$ is compact if and only if T_ψ is compact and (4) holds.*

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Which commutators of composition operators with adjoints of composition operators on weighted Bergman spaces are compact?

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Abstract

For two linear-fractional self-maps of the unit disk, where at least one of them is a non-automorphism, we show that the commutator of composition operator with the adjoints of another composition operator is non-trivially compact on the weighted Bergman spaces if and only if either these functions are both parabolic or both hyperbolic, with associated conclusions about their fixed points in each case.

Keywords: weighted Bergman spaces, composition operator, essential normality.
Mathematics Subject Classification [2010]: 47B33, 47B38

1 Introduction

In [1], Bourdon, Levi, Narayan and Shapiro determined when C_φ is essentially normal on the Hardy space H^2 in the case when φ is a linear-fractional self-map of $\mathbb{D}$. Here, we say that C_φ is essentially normal if the commutator $[C_\varphi^*, C_\varphi]$ is compact, which will be trivially true when C_φ is either normal or compact. Recent work of Clifford, Levi and Narayan [2] extended this line of investigation by considering the question of when, for linear-fractional self-maps φ and ψ of $\mathbb{D}$, the commutator $[C_\psi^*, C_\varphi]$ is non-trivially compact on H^2. After that MacCluer, Narayan, and Weir in [5] investigated this problem on the weighted Bergman spaces.

Definition 1.1. For any analytic self-map φ of $\mathbb{D}$, we define the composition operator C_φ by $C_\varphi(f) = f \circ \varphi$, where f is analytic in $\mathbb{D}$.

Definition 1.2. Recall that for $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D}) = A^2_\alpha$, is the set of functions f analytic on the unit disk, satisfying the norm condition

$$\|f\|_\alpha^2 = \int_{\mathbb{D}} |f(z)|^2 w_\alpha(z) dA(z) < \infty,$$

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where $w_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$ and dA is the normalized area measure. When $\alpha = 0$, this gives the Bergman space $A^2(\mathbb{D}) = A^2$. If $\hat{f}(n)$ is the nth coefficient of f in its Maclaurin series, then we have another representation for the norm of f on A^2 as follows:

$$\|f\|_0^2 = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} < \infty.$$

The formula above defines a norm that turns A^2 into a Hilbert space whose inner product is given by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \frac{\hat{f}(n)\overline{\hat{g}(n)}}{n+1}$$

for each $f, g \in A^2$ (see [4]).

Definition 1.3. The Hardy space $H^2(\mathbb{D}) = H^2$ is defined by

$$H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : \ \|f\|_{H^2}^2 = \lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty\}.$$

Definition 1.4. We write H^∞ for the space of bounded analytic functions on $\mathbb{D}$, and denote its natural norm by $\|.\|_\infty$, i.e.

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| \qquad (f \in H^\infty).$$

Definition 1.5. A linear-fractional self-map of $\mathbb{D}$ is a map of the form

$$\varphi(z) = \frac{az+b}{cz+d} \tag{1}$$

with $ad - bc \neq 0$, with the property that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. We denote the set of those maps by LFT($\mathbb{D}$).

Definition 1.6. It is well-known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are just the functions

$$\varphi(z) = \lambda \frac{a-z}{1-\bar{a}z}, \tag{2}$$

where $|\lambda| = 1$ and $|a| < 1$ (see, e.g., [3]). We denote the class of automorphisms of $\mathbb{D}$ by Aut($\mathbb{D}$).

Definition 1.7. If $\varphi(z) = \frac{az+b}{cz+d}$ is a linear-fractional self-map of $\mathbb{D}$, then the adjoint of any linear-fractional composition operator C_φ, acting on H^2 and A^2_α, is given by

$$C_\varphi^* = T_g C_{\sigma_\varphi} T_h^*,$$

where $\sigma_\varphi(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$ is a self-map of $\mathbb{D}$, $g(z) = (-\bar{b}z+\bar{d})^{-\gamma}$, $h(z) = (cz+d)^\gamma$, with $\gamma = 1$ for H^2 and $\gamma = \alpha+2$ for A^2_α. Note that g and h are in H^∞ (see [4]). If $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma_\varphi(\eta) = \zeta$. The map σ_φ is called the Krein adjoint of φ, we will write σ for σ_φ except when confusion could arise. We will refer to g and h as the Cowen auxiliary functions for φ. We know that φ is an automorphism if and only if σ is, and in this case $\sigma = \varphi^{-1}$. From now on, unless otherwise stated, we assume that σ, h and g are given as above.

Definition 1.8. For a bounded operators S and T on a Hilbert space, the commutator of S and T, denoted $[S,T]$ is $ST - TS$.

Remark 1.9. Suppose that $\varphi \in \mathrm{LFT}(\mathbb{D})$ is not an automorphism with $\|\varphi\|_\infty = 1$. We classify φ as follows:

• Hyperbolic non-automorphism of $\mathbb{D}$ which has a fixed point in $\partial\mathbb{D}$ of multiplicity 1. Also it has another fixed point in the complement of $\partial\mathbb{D}$.

• Parabolic non-automorphism of $\mathbb{D}$ with a fixed point in $\partial\mathbb{D}$ of multiplicity two.

• Non-automorphism with sup-norm equal to 1 such that it does not have a fixed point in $\partial\mathbb{D}$. It necessarily has a fixed point in $\mathbb{D}$ (see [4]).

Definition 1.10. We say that the commutator $[C_\psi^*, C_\varphi]$ is non-trivially compact if $[C_\psi^*, C_\varphi]$ is compact but nonzero, and $C_\psi^* C_\varphi$ and $C_\varphi C_\psi^*$ are not compact.

Remark 1.11. For each $f \in H^\infty$, the radial limit

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}),$$

exists for almost all θ (see [4]).

2 Main results

In this section, we consider the case when φ or ψ is non-automorphism and we find all φ and ψ such that $[C_\psi^*, C_\varphi]$ is non-trivially compact.

Proposition 2.1. *Suppose that $\varphi \in LFT(\mathbb{D})$ is not an automorphism and that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Let $\alpha > -1$ and $s = ((\overline{c}\overline{\zeta} + \overline{d})/(-\overline{b}\eta + \overline{d}))^{\alpha+2}$. Then there exists a compact operator K on A_α^2 so that*

$$C_\varphi^* = sC_\sigma + K = |\varphi'(\zeta)|^{-(\alpha+2)} C_\sigma + K.$$

Proposition 2.2. *Let φ and ψ be linear-fractional self-maps of $\mathbb{D}$, at least one of which is a non-automorphism. Then*
(a) $C_\psi^* C_\varphi$ is not compact on A_α^2 if and only if there exists points w_1 and w_2 in $\partial\mathbb{D}$ such that $\varphi^{-1}(w_1) = \psi^{-1}(w_2)$, and
(b) $C_\varphi C_\psi^*$ is not compact on A_α^2 if and only if there exists points ζ_1 and ζ_2 in $\partial\mathbb{D}$ such that $\varphi(\zeta_1) = \psi(\zeta_2) \in \partial\mathbb{D}$.

Corollary 2.3. *Let φ and ψ be linear-fractional self-maps of $\mathbb{D}$, at least one of which is a non-automorphism. Suppose that $\varphi(\zeta) = \psi(\zeta) = w$ for some $\zeta, w \in \partial\mathbb{D}$ with $\zeta \neq w$. Then $[C_\psi^*, C_\varphi]$ is not compact on A_α^2.*

Proposition 2.4. *Let φ and ψ be linear-fractional self-maps of $\mathbb{D}$, at least one of which is a non-automorphism. If $[C_\psi^*, C_\varphi]$ is non-trivially compact on A_α^2, then φ and ψ have a common boundary fixed point.*

Theorem 2.5. *Let φ and ψ be linear-fractional self-maps of $\mathbb{D}$, at least one of which is a non-automorphism. The commutator $[C_\psi^*, C_\varphi]$ is non-trivially compact on A_α^2 if and only if either*
(1) φ and ψ are both parabolic with the same boundary fixed point, or
(2) φ and ψ are both hyperbolic with the same boundary fixed point and with non-boundary fixed points which are conjugate reciprocals.

Theorem 2.6. *Suppose that φ is a parabolic non-automorphism of $\mathbb{D}$. Then $[C_\psi^*, C_\varphi]$ is compact on H^2 or A_α^2 if and only if ψ is also parabolic, with the same fixed point as φ.*

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d-self Center Graphs and Graph Operations

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Abstract

Let G be a simple connected graph. The graph G is called d-self center if it's vertices are of eccentricity d. In this paper, some self center composite graphs are investigated. Some mathematical properties of self center graphs is investigated. It is proved that a self center graph is 2-connected. Some infinite family of asymmetric self center graphs is constructed.

Keywords: eccentricity, d-self center graph, composite graphs
Mathematics Subject Classification [2010]: 05C12, 05C76, 05C90

1 Introduction

All considered graphs are simple and connected. Distance between two vertices is defined as usual length of shortest path connecting them. Eccentricity of vertex v is denoted by $\varepsilon(v)$ is the maximum distance between v and other vertices. The maximum and minimum eccentricity among all vertices of G are called diameter of G, diam(G) and radius of G, rad(G) respectively. The Center of G, $C(G)$ is the set of vertices of rad(G). Let G be a simple connected graph. The graph G is called d-self center if it's vertices are of eccentricity d. Center of graph G is the set of vertices of minimum eccentricity. Then the Center of a self center graph contains all vertices of the graph. In a series of paper, topological indices based on eccentricity of vertices were studied and for some family of molecular graphs such indices were calculated. For more information, we refer the reader to [1, 2, 3, 4, 5, 6, 8, 7, 10]. In this paper, self center graphs under some graph operations is studied. It is proved that a self center graph is 2-connected. By graph operations, some asymmetric self center graphs is constructed.

2 Main Results

As example complete graphs K_n, cycles C_n and sierpinski graphs S_k^n are three well-known family of self center graphs. A graph G is called vertex- transitive if for given vertices u and v there is an auto-morphism of G, f such that $f(u) = v$. For example the complete graphs and cycle graphs and Petersen graph are vertex transitive graphs. Since distance between vertices and eccentricity are invariant under auto morphism of graphs, then the

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vertex transitive graphs are self center but the reverse is not true. The Sierpinski graphs S_k^n are a family of self center graphs but non vertex transitive. A regular graph that is edge-transitive but not vertex-transitive is called a semi-symmetric graph. The Gray graph (with 54 vertices), the Tutte 12-cage (with 126 vertices) are two namely semi-symmetric and self center graph and the Folkman graph (with 20 vertices) and the Ljubljana graph (with 112 vertices) are other semi-symmetric graph but non self center graph. It seems an interesting problem to characterize the self center semi-symmetric graphs. A self center graph with $n \geq 3$ vertices is a block graph or 2-connected graph.

Theorem 2.1. *Let G be a self center graph with $n \geq 3$ vertices. Then G is 2-connected.*

There are some self center graphs such as cycles that are not 3-connected graph. Let $Aut(G)$ be the group automorphism of graph G. Orbit of vertex v is denoted by $Orb(v)$ and defined as $Orb(v) = \{f(v)| \ f \in Aut(G)\}$. The vertices of $Orb\,(v)$ have the same eccentricity. A graph G is vertex transitive if and only if G has exactly one orbit. The following example illustrated in Figure 1. is a graph with 7 orbits but all vertices have a same eccentricity then the graph is self center.

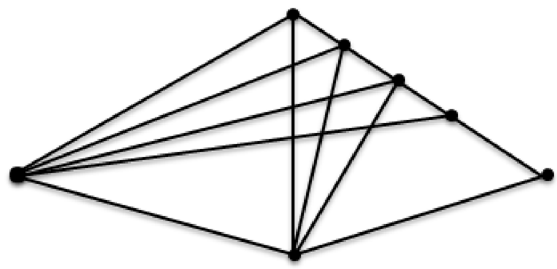

Figure 1: asymmetric graph with 7 orbits

Proposition 2.2. *[9] If G is a 2-self center graph on $n \geq 5$ vertices then G has at least $2n - 5$ edges.*

3 Composite graphs

In this section some self centered graph arised from graph operation are presented. We start by join of graphs.

Theorem 3.1. *Let G_1 and G_2 be two simple connected graphs. Then $G_1 + G_2$ is self center graph if and only if G_1 and G_2 are self center graphs.*

Proposition 3.2. *For any $n \geq 4$ there is a family of 2-self center graphs on n vertices.*

It is enough to consider $\overline{K_m} + \overline{K_p}$ where $m + p = n$ and $m, p \geq 2$. We have a similar statement about cartesian product of graphs.

Theorem 3.3. *Let G and H be two simple connected graphs. Then $G_1 \times G_2$ is self center graph if and only if G_1 and G_2 are self center graphs.*

Corollary 3.4. $\prod_{i=1}^{n} G_i$ *is self center iff each* G_i *is self center for* $1 \leq i \leq n$.

Corollary 3.5. *Ther is an infinite family of non vertex transitive self centered .*

Consider the powers of a non-vertex transitive self center graph such as the Gray graph or the Tutte 12-cage graph.
For any two simple connected graph with at least two vertices, we can construct a self center graph by symmetric difference operation of the graphs.

Theorem 3.6. *Let G and H be two simple connected graph with at least two vertices. Then the symmetric difference $G \oplus H$ is 2-self center.*

It is easy to see that the disjunction of two complete graphs is a complete graph and vice versa. In the case radius of both G and H is at least 2 the disjunction $G \vee H$ is 2-self center.

Theorem 3.7. *Let G and H be two simple connected graph with radius at least 2. Then the disjunction $G \vee H$ is 2-self center.*

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Talk Bounds on some variants of clique cover numbers pp.: 1–4

Bounds on Some Variants of Clique Cover Numbers

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Abstract

A clique covering of G is defined as a family of cliques of G such that every edge of G lies in at least one of the cliques. The weight of a clique covering is defined as the sum of the number of vertices of the cliques. The sigma clique cover number (resp. sigma clique partition number) of graph G, denoted by $\mathrm{scc}(G)$ (resp. $\mathrm{scp}(G)$), is defined as the smallest integer k for which there exists a clique covering (resp. clique partition) for G of weight k. In this paper, among some results we prove an upper bound on scc. Also, we provide a new lower bound on scp that improves a result of Erdős as a corollary. Then, we explore scc and scp for complete multipartite graphs as well as the product of graphs.

Keywords: Clique covering, Clique partition, Sigma clique covering, Sigma clique partition
Mathematics Subject Classification [2010]: 05C70,05C62,05D05

1 introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph G, we mean a subset of mutually adjacent vertices of G as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices.

A *clique covering* of G is defined as a family of cliques of G such that every edge of G lies in at least one of the cliques comprising this family. The minimum size of a clique covering of G is called *clique cover number* of G and is denoted by $\mathrm{cc}(G)$.

A clique covering in which each edge belongs to exactly one clique, is called a *clique partition*. The minimum size of a clique partition of G is called *clique partition number* of G and is denoted by $\mathrm{cp}(G)$.

Chung et al. in [2] and independently Tuza in [10] defined the concept of *weight* for a clique covering. Let $\mathcal{C}$ be a clique covering for graph G. The weight of $\mathcal{C}$ is defined as $\sum_{C\in\mathcal{C}} |V(C)|$.

The *sigma clique cover number* of G, denoted by $\mathrm{scc}(G)$, is defined as the minimum integer k for which there exists a clique covering $\mathcal{C}$ for G of weight k. In fact,

$$\mathrm{scc}(G) = \min_{\mathcal{C}} \sum_{C\in\mathcal{C}} |C|,$$

*Speaker

where the minimum is taken over all clique coverings of G.

Analogously, one can define *sigma clique partition number* of G, denoted by $\mathrm{scp}(G)$. As a general upper bound, in [1, 6, 7] it was proved that for every graph G on n vertices, $\mathrm{scc}(G) \leq \mathrm{scp}(G) \leq n^2/2$.

Clique covering parameters have close relation to other combinatorial concepts such as *set representations*, *line hypergraph* and *pairwise balanced designed.* For a survey of the classical results on the clique coverings see [8, 9].

2 General Bounds

2.1 Upper Bound for scc

Let G be a graph on n vertices. The only known general upper bound on $\mathrm{scc}(G)$ is $n^2/2$ [1, 7, 6]. In the following theorem, using the probabilistic methods, we stablish an upper bound for $\mathrm{scc}(G)$.

Theorem 2.1. *If G is a graph on n vertices with no isolated vertex and $\Delta(\overline{G}) = d - 1$, then*

$$\mathrm{scc}(G) \leq (e^2 + 1)nd \left\lceil \ln \left(\frac{n-1}{d-1} \right) \right\rceil .$$

Sketch of proof. Let $0 < p < 1$ be a fixed number and let S be a random subset of $V(G)$ defined by choosing every vertex u independently with probability p. For every vertex $u \in S$, if there exists a non-neighbour of u in S, then remove u from S. The resulting set is a clique of G. Repeat this procedure t times, independently, to get t cliques $C_1, C_2, \ldots, C_t$ of G.

Let F be the set of all the edges which are not covered by the cliques $C_1, \ldots, C_t$. The cliques $C_1, \ldots, C_t$ along with all edges in F comprise a clique covering of G. Hence,

$$\begin{aligned} \mathrm{scc}(G) &\leq \mathbf{E}\left(\sum_{i=1}^{t} |C_i| + 2|F| \right) \\ &\leq npt + 2\binom{n}{2} e^{-tp^2(1-p)^{2(d-1)}}. \end{aligned}$$

Finally, set $p := 1/d$ and $t := \lceil e^2 d^2 \ln(\frac{n-1}{d-1}) \rceil > 0$ to get the desired corollary. □

2.2 Lower Bound for scp

Theorem 2.2. *Let U and V be a partition of vertices of G into the two sets. If G has t edges between parts U and V, then* $\mathrm{scp}(G) \geq 2(t - (p + q))$, *in which p and q are number of edges of G with both ends in U and V, respectively. Moreover, equality holds if and only if there exists a clique partition of edges of G, say $\mathcal{C}$, such that for each $C_i \in \mathcal{C}$, $|C_i \cap U| = |C_i \cap V|$.*

Remark 2.3. Without loss of generality assume that $p \leq q$. Erdős et al. in [5] proved that $cp(G) \geq t - 2p - q$. On the other hand, by Theorem 2 (ii) in [4], $\mathrm{cp}(G) \geq \mathrm{scp}^2(G)/(2m + \mathrm{scp}(G))$, where m is the number of edges of G. Since $x^2/(2m + x)$ is increasing for $x > 0$, Theorem 2.2 concludes that $\mathrm{cp}(G) \geq (t - (p + q))^2/t$ which improves Erdős bound if and only if $t \leq (p + q)^2/q$.

3 Clique Covering of Special Graphs

In this section, our focus is on determining scc and scp for some well-known families of graphs. First, we consider the Turan graphs because of their importance in covering problems. Then, by determining the value of scc and scp for *Cartesian product* of graphs, we give a tight lower bound for scp of *tensor product* of complete graphs and study its asymptotic behaviour.

The complement of the union of complete graphs is the s-partite complete graph $K_{t_1,t_2,\ldots,t_s}$, whose parts are of size $t_1, t_2, \ldots, t_s$, respectively. If each part has the same size, $t_1 = t_2 = \cdots = t_s = t > 1$, then we denote the graph by $K_s(t)$.

Theorem 3.1. *Let $N(t)$ be the maximum number of mutually orthogonal Latin squares of order t. If $N(t) \geq s-2$, then* $\mathrm{scc}(K_s(t)) = \mathrm{scp}(K_s(t)) = st^2$.

Theorem 3.2. *If $G\Box H$ is the Cartesian product of G and H, then*

$$\mathrm{scc}(G\Box H) = n(G)\,\mathrm{scc}(H) + n(H)\,\mathrm{scc}(G)$$
$$\mathrm{scp}(G\Box H) = n(G)\,\mathrm{scp}(H) + n(H)\,\mathrm{scp}(G).$$

For the tensor product of complete graphs, $K_n \times K_n$, we have the following theorem.

Theorem 3.3. $\mathrm{scp}(K_n \times K_n) \geq n^3 - n^2$. *If n is a prime power, then equality holds.*

Sketch of proof. By Theorem 2.5 in [3], for a graph G on n vertices, if $\max\{\omega(G), \omega(\overline{G})\} \leq \lfloor\sqrt{n}\rfloor$, then $\mathrm{scp}(G) + \mathrm{scp}(\overline{G}) \geq n(\sqrt{n}+1)$.

First note that complement of $K_n \times K_n$ is $K_n \Box K_n$. Since $\omega(K_n \times K_n) = \omega(K_n \Box K_n) = n$, we conclude that $\mathrm{scp}(K_n \times K_n) \geq n^2(n+1) - scp(K_n \Box K_n)$. Thus, the lower bound is proved by Theorem 3.2.

Now, let n be a prime power. Thus, there exist $(n-2)$ idempotent MOLS(n) and equvallently an (n,n)-orthogonal array. Consider each row of the (n,n)-orthogonal array as a clique except the row $in + (i+1)$, for $0 \leq i \leq n-1$. These $n^2 - n$ cliques of size n, form a clique partition for the edges of $K_n \times K_n$. □

Theorem 3.4. *For large enough n,* $\mathrm{scp}(K_n \times K_n) \sim n^3$.

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Cospectral Regular graphs

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Abstract

Graphs G and H are called cospectral if they have the same characteristic polynomial, equivalently, if they have the same eigenvalues considering multiplicities. Generalizing the construction of $G_4(a,b)$ and $G_5(a,b)$ due to Wang and Hao, we define graphs $G_4^r(a,b)$ and $G_5^r(a,b)$ and show that they are cospectral only if $r=1$ and $a+2=b$.

Keywords: eigenvalue, cospectral graphs, adjacency matrix, integral graphs.
Mathematics Subject Classification [2010]: 05C50

1 Introduction

We consider simple graphs, that is, graphs without loops or parallel edges. For basic notions in graph theory we refer to [4], whereas for preliminaries on graphs and matrices, see [1]. By the eigenvalues of a graph G, we mean the eigenvalues of its adjacency matrix $A(G)$. Graphs G and H are said to be cospectral if they have the same eigenvalues, counting multiplicities, or equivalently, they have the same characteristic polynomial. There is considerable literature on construction of cospectral graphs.

This paper is motivated by [3]. Bussemaker and Cvetković [2] introduced connected integral cubic graphs, denoted $G_1, G_2, \ldots, G_{13}$, among which G_4 and G_5 are cospectral. Wang and Hao [3] constructed graphs $G_4(a,b)$ and $G_5(a,b)$ based on G_4 and G_5. They showed that for any positive integer a, $G_4(a,a+2)$ and $G_5(a,a+2)$ form a pair of integral cospectral $(a+2)-$regular graphs, and concluded that there exist infinitely many pairs of cospectral integral graphs. We first give a generalization of $G_4(a,b)$ and $G_5(a,b)$ based on the method used in Lemma 1.1. We determine the characteristic polynomial of the resulting graphs. We also show that $G_4(a,b)$ and $G_5(a,b)$ are cospectral if and only if $a+2=b$.

Lemma 1.1. *Suppose that X and Y are square matrices of the same order. Let*

$$T=\begin{pmatrix} X & Y & \ldots & Y \\ Y & X & \ldots & Y \\ \vdots & \vdots & \ddots & \vdots \\ Y & Y & \ldots & X \end{pmatrix} \tag{1}$$

*Speaker

be an $r \times r$ block matrix. Then the eigenvalues of T are the eigenvalues of $X - Y$, $r - 1$ times, and the eigenvalues of $X + (r-1)Y$.

We first recall the adjacency matrices of $G_4(a,b)$ and $G_5(a,b)$. The adjacency matrix of $G_4(a,b)$ is $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$, where

$$A_0 = \begin{pmatrix} 0_{a\times a} & J_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ J_{b\times a} & 0_{b\times b} & I_b & 0_{b\times b} \\ 0_{b\times a} & I_b & 0_{b\times b} & B \\ 0_{b\times a} & 0_{b\times b} & B & 0_{b\times b} \end{pmatrix},\ A_1 = \begin{pmatrix} 0_{a\times a} & 0_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ 0_{b\times a} & I_b & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & I_b \end{pmatrix} \tag{2}$$

and

$$B = \begin{pmatrix} 1 & J_{1\times(b-2)} & 0 \\ J_{(b-2)\times 1} & J_{(b-2)\times(b-2)} - I_{b-2} & J_{(b-2)\times 1} \\ 0 & J_{1\times(b-2)} & 1 \end{pmatrix}. \tag{3}$$

The adjacency matrix of $G_5(a,b)$ is $\begin{pmatrix} M_0 & M_1 \\ M_1 & M_0 \end{pmatrix}$, where

$$M_0 = \begin{pmatrix} 0_{a\times a} & J_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ J_{b\times a} & 0_{b\times b} & I_b & I_b \\ 0_{b\times a} & I_b & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & I_b & 0_{b\times b} & 0_{b\times b} \end{pmatrix},\ M_1 = \begin{pmatrix} 0_{a\times a} & 0_{a\times b} & 0_{a\times b} & 0_{a\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & B & 0_{b\times b} \\ 0_{b\times a} & 0_{b\times b} & 0_{b\times b} & B \end{pmatrix} \tag{4}$$

and B is the same as in (3).

Lemma 1.2. *Let $r \geq 1$ be an integer. Then*

$$\begin{aligned} \det(A_0 + rA_1 - \lambda I) &= (-1)^{a-b}\lambda^{a-1}(\lambda - r)^{b-1}(\lambda^2 - r\lambda - 2)^{b-1} \\ &\times [\lambda^4 - 2\lambda^3 r + (-b^2 + (-a+2)b - 2 + r^2)\lambda^2 + (2 + b^2 + (a-2)b)r\lambda + ba(b-1)^2]. \end{aligned}$$

Theorem 1.3. *For a positive integer $r \geq 1$, suppose that $G_4^r(a,b)$ denotes the graph whose adjacency matrix is the $(r+1)\times(r+1)$ block matrix*

$$A(G_4^r(a,b)) = \begin{pmatrix} A_0 & A_1 & \dots & A_1 \\ A_1 & A_0 & \dots & A_1 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_1 & \dots & A_0 \end{pmatrix}, \tag{5}$$

where A_0 and A_1 are the same as in (2). Then the characteristic polynomial of $G_4^r(a,b)$ is

$$\begin{aligned} \det(A(G_4^r(a,b)) - \lambda I) &= \lambda^{(r+1)(a-1)}(\lambda - r)^{b-1}(\lambda^2 - r\lambda - 2)^{b-1}(\lambda+2)^{r(b-1)}(\lambda-1)^{r(b-1)}(\lambda+1)^{r(b-1)} \\ &\times [\lambda^4 - 2\lambda^3 r - (b^2 - 2b + ab - r^2 + 2)\lambda^2 + (b^2 + ab - 2b + 2)r\lambda + ba(b-1)^2] \\ &\times [\lambda^4 + 2\lambda^3 - (b^2 - 2b + ab + 1)\lambda^2 - (b^2 + ab - 2b + 2)\lambda + ba(b-1)^2]^r. \end{aligned}$$

Lemma 1.4. *Suppose that x and y are scalars and let B be the matrix as in (3). Then*

$$\det(rB + xJ + yI) = (y+r)(y-r)^{b-2}(bx + y + rb - r).$$

Lemma 1.5. *If* $T = \begin{pmatrix} xI_a & J_{a\times b} \\ J_{b\times a} & xI_b \end{pmatrix}$ *is invertible, then*

$$T^{-1} = \begin{pmatrix} \frac{1}{x}(I_a + \frac{b}{x^2-ab}J_a) & -J_{a\times b} \\ -J_{b\times a} & \frac{1}{x}(I_b + \frac{a}{x^2-ab}J_b) \end{pmatrix}$$

Theorem 1.6. *The characteristic polynomial of* $A(G_5^r(a,b))$ *is*

$$\begin{aligned}\det(A(G_5^r(a,b) - \lambda I) &= \lambda^{(r+1)(a-1)}(\lambda+2)^r(\lambda+1)^{r(b-1)}(\lambda-1)^{r(b-1)}(\lambda-2)^{r(b-2)}(\lambda+b-1)^r \\ &\times (\lambda-r)(\lambda+r)^{b-2}(\lambda-rb+r)(\lambda^2-r\lambda-2)(\lambda^2+r\lambda-2)^{b-2} \\ &\times [\lambda^3 - r\lambda^2(b-1) - \lambda(2+ab) + rab(b-1)][\lambda^3+(b-1)\lambda^2-(2+ab)\lambda-(b-1)ab]^r\end{aligned}$$

Corollary 1.7. *Let* a *and* b *be positive integers. Then* $G_4^r(a,b)$ *and* $G_5^r(a,b)$ *are cospectral if and only if* $r = 1$ *and* $b = a + 2$.

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Diameter of $\Gamma(M_1 \oplus M_2)$

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Abstract

Let M_1 and M_2 be finietly generated multiplication R-modules such that

$$(0 : M_1) + (0 : M_2) = R.$$

We compare $diam\Gamma(M_1 \oplus M_2)$ with $diam\Gamma(M_1)$ and $diam\Gamma(M_2)$.

Keywords: Zero-divisor graphs, Diameter of graphs, Multiplication modules.
Mathematics Subject Classification [2010]: 13C12, 13A15

1 Introduction

The notion of multiplication modules introduced by Barnard in 1981 [4], and then E-Bast and Smith found various properties of multiplication modules to hold in 1988[6]. On the other hand, Beck first introduced the notion of a zero-divisor graph of a ring in 1988 [5] from the view of colorings. Since then, others, such as in [1], [3] have studied and modified these graphs, whose vertices are the zero-divisors of R, and found various properties to hold. Multiplication modules are natural generalizations of commutative rings, and hence it is natural for us to generalize zero-divisor graphs of commutative rings to those of multiplication modules.

An R-module M is called a *multiplication module* provided that for each submodule N of M there exists an ideal I of R such that $N = IM$. We say that I is a *presentation ideal* of N. Let N and K be submodules of a multiplication R-module M. Then there exist ideals I and J of R such that $N = IM$ and $K = JM$. The *product* of N and K, denoted by $N * K$, is defined to be $(IJ)M$. By [2], the product of N and K is independent of presentation ideals of N and K. An element x of M is called a *zero-divisor element* of M if there exists a nonzero element y of M such that $Rx * Ry = 0$ in M.

Proposition 1.1. *Let M be a multiplication R-module with $|M| \geq 3$. Let x, y and z be distinct vertices of $\Gamma(_RM)$ such that x is adjacent to y and y is not adjacent to z. Then there exists a nonzero element m in $Ry * Rz$ such that $Rx * Rm = 0$.*

Proposition 1.2. *Let M be a multiplication module. Let x, x_1, y_1 and y be vertices of $\Gamma(_RM)$ such that $x \neq x_1$, $y \neq y_1$, and $x_1 \neq y_1$. Assume that x is not adjacent to y and x_1 is not adjacent to y_1. If x is adjacent to x_1 and y is adjacent to y_1, then $(Rx_1 * Ry_1)^* \subseteq Z(_RM)^*$ and there exists an element z in $(Rx_1 * Ry_1)^*$ such that x is adjacent to z and z is adjacent to y.*

*Speaker

2 Main result

Theorem 2.1. *Let M_1, M_2 be finitely generated multiplication R-modules such that $(0 :_R M_1) + (0 :_R M_2) = R$. Then the following statements are true.*

1. *If $\mathcal{P}(M_1) = \{0\}$ and $\mathcal{P}(M_2) = \{0\}$, then $\Gamma(M_1 \oplus M_2)$ is complete.*
2. *$max\{diam(\Gamma(M_1)), diam(\Gamma(M_2))\} \leq diam(\Gamma(M_1 \oplus M_2)) \leq 3$*

Let $M_1 = \mathbb{Z}_{12}$, $M_2 = \mathbb{Z}_5$. Then $(9, 4) - (4, 0) - (6, 0) - (2, 3)$ is a shortest path (of length 3) between $(9, 4)$ and $(2, 3)$. Therefore, $dim(M_1 \oplus M_2) = 3$.

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Domination polynomial of generalized friendship graphs

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Abstract

Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x)=\sum_{i=0}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. Let n and $q \geq 3$ be any positive integer and $F_{q,n}$ be the *generalized friendship graph* formed by a collection of n cycles (all of order q), meeting at a common vertex. We study the domination polynomials of some generalized friendship graphs. In particular we examine the domination roots of these families, and find the limiting curve for the roots.

Keywords: Domination polynomial; friendship graph; flower graphs.
Mathematics Subject Classification [2010]: 05C60

1 Introduction

Let $G=(V,E)$ be a simple graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v)=\{u \in V(G)|\{u,v\} \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v]=N(v)\cup\{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S)=\bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S]=N(S)\cup S$. A set $S \subseteq V(G)$ is a *dominating set* if $N[S]=V$ or equivalently, every vertex in $V(G)\backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. Let $\mathcal{D}(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i)=|\mathcal{D}(G,i)|$. The *domination polynomial* $D(G,x)$ of G is defined as $D(G,x)=\sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G (see [1, 2]). A root of $D(G,x)$ is called a domination root of G. The set of distinct roots of $D(G,x)$ is denoted by $Z(D(G,x))$.

Calculating the domination polynomial of a graph G is difficult in general, as the smallest power of a non-zero term is the domination number $\gamma(G)$ of the graph, and determining whether $\gamma(G) \leq k$ is known to be NP-complete [6]. But for certain classes of graphs, we can find a closed form expression for the domination polynomial. The domination polynomial of friendship graphs and its limiting curve for their domination roots studied recently [3]. In this paper we consider *generalized friendship graph (or flower graphs)*, calculate their domination polynomials, exploring the nature and location of their roots.

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2 Main results

Let us consider the graphs F_n obtained by selecting one vertex in each of n triangles and identifying them (Figure 1). Some call them Dutch-Windmill graphs and friendship graphs.

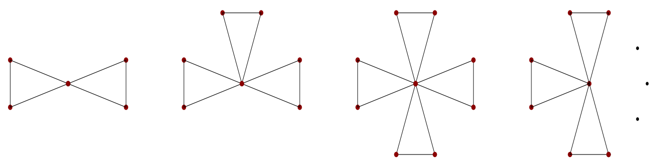


Figure 1: Friendship graphs F_2, F_3, F_4 and F_n, respectively.

The generalized friendship graph $F_{q,n}$ is a collection of n cycles (all of order q), meeting at a common vertex (see Figure 2). The generalized friendship graph may also be referred to as a flower [7].

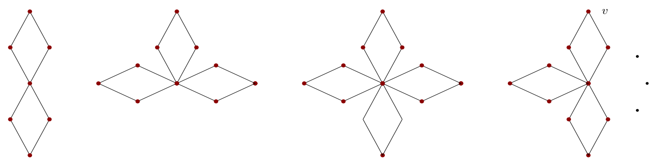


Figure 2: The flowers $F_{4,2}$, $F_{4,3}$, $F_{4,4}$ and $F_{4,n}$, respectively.

The following theorem gives formula for the domination polynomial of F_n.

Theorem 2.1. [3] *For every $n \in \mathbb{N}$,*

$$D(F_n, x) = (2x + x^2)^n + x(1 + x)^{2n}.$$

The following theorem gives recurrence relation for the domination polynomial of $F_{4,n}$.

Theorem 2.2. *For every $n \geq 2$,*

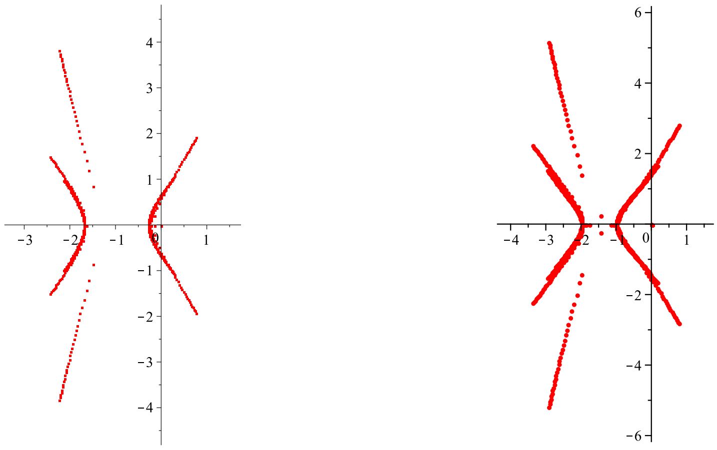
$$\begin{aligned} D(F_{4,n}, x) &= ((1+x)^3 + x)D(F_{4,n-1}, x) - (1+3x)(x+3x^2+x^3)^{n-1} \\ &\quad +(1+x)^3x^{n-1} - (x^2+x)(x^3+3x^2+3x)^{n-1}, \end{aligned}$$

where $D(F_{4,1}, x) = x^4 + 4x^3 + 6x^2$.

The domination roots of F_n and $F_{4,n}$ exhibit a number of interesting properties (see Figure 3).

If we can find an explicit formula for the domination polynomial of a graph, there are still interesting, difficult problems concerning the roots. We have the following result:

Theorem 2.3. (*i*) *For every odd natural number n, no nonzero real number is a domination root of F_n and $F_{4,n}$.*
(*ii*) *For even natural number n, F_n has exactly three real domination roots.*
(*iii*) *For even $n \geq 4$, $F_{4,n}$ has exactly three real domination roots.*



(a) Domination roots of graphs F_n (b) Domination roots of graphs $F_{4,n}$

Figure 3: Domination roots of graphs F_n and $F_{4,n}$, for $1 \leq n \leq 30$.

It is natural to ask about the complex domination roots of F_n and $F_{4,n}$. The plots in Figure 3 suggest that the roots tend to lie on some curves. In order to find the limiting curve, we need a definition and a well known result.

Definition 2.4. If $f_n(x)$ is a family of (complex) polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{R}(f_n(x))$, where $\mathbb{R}(f_n(x))$ is the union of the roots of the $f_n(x)$.

The following theorem is the Beraha-Kahane-Weiss theorem [4].

Theorem 2.5. *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + ... + \alpha_k(x)\lambda_k(x)^n$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either

(i) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or

(ii) for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.

Theorem 2.6. [3] *The limit of dominiation roots of frienship graphs is -1 together with the hyperbola*

$$(\Re x + 1)^2 - (\Im x)^2 = \frac{1}{2}.$$

Figure 4 shows the limiting curve. We see that this curve meet the real axis at $-1 - \frac{1}{\sqrt{2}}$ ≈ -1.7071 and $-1 + \frac{1}{\sqrt{2}} \approx -0.2929$. Also, in [5] a family of graphs was produced with roots just barely in the right-half plane (showing that not all domination polynomials are stable), but Theorem 2.6 provides an explicit family (namely the friendship graphs) whose domination roots have unbounded positive real part. Also we think that this is true for $F_{4,n}$.

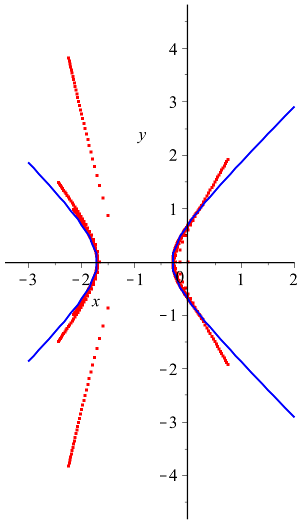


Figure 4: Domination roots of graphs F_n, for $1 \leq n \leq 30$ along with limiting curve.

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Notes on STP Number of a Graph

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Abstract

Spanning tree packing number of a graph G is the maximum number of edge disjoint spanning trees contained in G. This quantity is one of the connectivity measure of a graph. We give two main theorems for to compute this parameter in some cases of graphs. In particular for a positive integer n we prove that when H is a forest subgraph of the complete graph K_{2n+1} with at most n edges, then the spanning tree packing number of $K_{2n+1} - H$ is equal to n. In another result we prove that when H is a forest of at least $n + 1$ edges, then the spanning tree number of $K_{2n+1} - H$, may vary depending the maximum degree vertex of the spanning tree that may be obtained by extending H in K_{2n+1}.

Keywords: Spanning Tree, Complete Graph, STP Number
Mathematics Subject Classification [2010]: 05C05, 05C80

1 Introduction

Let G be a graph. The spanning tree packing number of G denoted by $STP(G)$ is defined to be the maximum number of edge disjoint spanning trees contained in G. This concept has an interconnection with the robustness concept of a network since any spanning tree is merely a complete and least connection routs between all nodes. So clearlly a network has more robustness when it has more STP number. Clearly a graph with more STP number has more possible alternative connections whenever there is a treat over a revealed connection. In other words a network with more STP number is a more secure network and deserve more investments.

One of the main issues in the topic of spanning trees is to compute the STP number of a given or known graphs. In what follows some classical results are recalled [2] (note that $\lfloor x \rfloor$ denotes the greatest integer not more than x):

1. $STP(K_n) = \lfloor n/2 \rfloor$, where K_n is the complete graph with n vertices,
2. $STP(K_{m,n}) = \lfloor \frac{mn}{m+n-1} \rfloor$, where $K_{m,n}$ is the complete bipartite graph,
3. $STP(Q_n) = \lfloor n/2 \rfloor$, Where Q_n is the n-cube graph [1, p.33],
4. $STP(K_m \times K_n) = \lfloor \frac{m+n-2}{2} \rfloor; (2 \leq m \leq n)$.

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5. $STP(K_m \times C_n) = \lfloor \frac{m+1}{2} \rfloor$, where C_n is the cycle with n vertices,

6. $STP(P_q) = STP(C_m \times C_n) = 2$, where P_q is the paley graph with q vertices [1, p.221].

One of the most famous and basic results in this context is obtained by Nash-Williams and Tutte, independently[3]:

Theorem 1.1. *Let G be a connected graph. Then $STP(G) = k$ if and only if $|F| \geq k(\omega(G - F) - 1))$ for every $F \subseteq G$, where $\omega(G - F)$ denotes the number of connected components of $G - F$.*

By this theorem we obtain a sufficient condition for a graph to be $2k$-edge connected. The following basic theorem of Catlin [2] improved this idea:

Theorem 1.2. *Let G be a connected graph. Then G is $2k$ edge connected if and only if $G - F$ has k edge disjoint spanning trees for any $F \subseteq G$ with $|F| = k$.*

2 Main results

In this note we pay our attention to the following question: By the formula 1 above, for any positive integer n we have $STP(K_{2n+1}) = n$. So one may ask how much this number is stable within removing the subgraphs. In what follows we give a particular response to this question showing that one can remove any forest subgraph of K_{2n+1} with at most n edges while the number of edge disjoint spanning trees does not change. In other words we show that:

Theorem 2.1. *Let H be a forest subgraph of K_{2n+1} with at most n edges, then we have:*

$$STP(K_{2n+1} - H) = n.$$

Proof. The proof is by induction on n. For $n = 2$ the claim is true since H is a one edge subgraph and $K_5 - H$ is connected. Now let the claim be true for all m, where $m \leq n$. Let u be a leaf in H and suppose that u is connected to a vertex, say w in H. Also let v be a vertex not incident with edges in H (there exist at least $(2n + 1) - 2n = 1$ vertex of this kind). Consider $K_{2n-1} = K_{2n+1} - \{u, v\}$ and put $H' = H \cap K_{2n-1}$, which is a forest subgraph of K_{2n-1} with at most $n - 1$ edges. Now by induction hypothesis $STP(K_{2n-1} - H') = n - 1$. Let $T'_1, \cdots, T'_{n-1}$ be a set of $n - 1$ edge disjoint spanning trees in $K_{2n-1} - H'$. Partition all $2n - 1$ vertices of K_{2n-1} in two sets of n and $n - 1$ sizes, as $A = \{a_1, \cdots, a_n\}$ and $B = \{b_1, \cdots, b_{n-1}\}$ such that $a_n = w$. Consider T'_i and join a_i to u and b_i to v and call the new tree T_i, in other words let $T_i = T'_i \cup \{a_i u, b_i v\}$. The set of trees $\{T_1, \cdots, T_{n-1}\}$ are $n - 1$ edge-disjoint spanning trees in K_{2n+1}. By the following method we make another tree. Let $G = K_{2n+1} - H - \bigcup_{1 \leq i \leq n-1} T_i$. In G we have $deg_G(u) = n + 1$ and $deg_G(v) = n + 2$. The last tree obtained by joining u to all vertices of B and joining v to all vertices of A plus the edge uv. This tree is the n-th edged disjoint spanning tree in $K_{2n+1} - H$ as we liked. □

Note that the converse of this theorem is not true as one can see in case $n = 3$. In other words one can find three edge disjoint spanning trees in $K_7 - C_3$. Now we consider the case when a subgraph with more than n edges is removed. As above, the following theorem gives a particular response to this question showing that one can remove any forest subgraph of K_{2n+1} with at least $n+1$ edges, while the number of edge disjoint spanning tree may vary depending the maximum degree vertex of the spanning tree that may obtained by extending the given forest. Note that clearly any forest in a graph can be extended to a spanning tree. Now we have:

Theorem 2.2. *Let H be a forest subgraph of K_{2n+1} with at least $n+1$ edges, then $STP(K_{2n+1} - H) = m - 1$ if and only if H can be extended to a spanning tree T_H such that $\Delta(T_H) = 2n + 1 - m$.*

The following is a different approach in identifying STP number. An ear of a graph G is a maximal path whose internal vertices have degree 2 in G. For the definition of an *ear decomposition of G started from a subgraph H* see [4, p.163]. An $K_{1,r}$-ear decomposition of G started from a subgraph H is defined similarly. Now we have:

Theorem 2.3. *Let G be a connected simple graph without leaves. Then $STP(G) \geq 2$ if and only if G has a P_2-ear decomposition started from a subgraph H, where $STP(H) \geq 2$.*

More generally we have:

Theorem 2.4. *let G be a connected simple graph with no leaves. Then $STP(G) \geq r$ if and only if G has an $K_{1,r}$-ear decomposition started from a subgraph H, where $STP(H) \geq r$.*

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On the Biclique Cover of Graphs

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Abstract

The *biclique cover number* $bc(G)$ of a graph G is the smallest number of bicliques of G such that every edge of G belongs to at least one of these bicliques. A *k-clique covering* of a graph G, is an edge covering of G by its cliques such that each vertex is contained in at most k cliques. The smallest k for which G admits a k-clique covering is called *local clique cover number* of G and is denoted by $lcc(G)$. In this paper, we find the relation between $bc(G)$ and $lcc(\overline{G})$ of the graphs. As a consequence, we show that if G is a graph with m edges such that $\overline{G}$ is a line graph then $bc(G) \leq 8 \ln m$.

Keywords: Biclique Cover, Clique Cover, Local Biclique Cover, Local Clique Cover, Intersection Representation.
Mathematics Subject Classification [2010]: 05B40

1 Introduction

Throughout the paper, all graphs are finite and simple graph. Let $V(G)$ denote the vertex set of the graph G and $E(G)$ denote its edge set. The complement $\overline{G}$ of the graph G is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. The term clique stands for the complete graph and biclique for the complete bipartite graph. The *biclique* (resp. *clique*) *cover number* $bc(G)$ (resp. $cc(G)$) of a graph G is the smallest number of bicliques (resp. cliques) of G such that every edge of G belongs to at least one of these bicliques (resp. cliques). A *k-biclique* (resp. *k-clique*)*covering* of a graph G, is an edge covering of G by its bicliques (resp. cliques) such that each vertex is contained in at most k bicliques (resp. cliques). The smallest k for which G admits a k-biclique (resp. clique) covering is called *local biclique* (resp. *clique*) *cover number* of G and is denoted by $lbc(G)$ (resp. $lcc(G)$). In the same manner, we can define biclique partition number $bp(G)$ and local biclique partition number $lbp(G)$, if we use partition instead of cover. These measures and its applications have been studied extensively throughout the literature; see [2, 3, 4, 5, 6]. Finding the relation between these parameters are also interesting and have been studied in the literature; see [8]. In [8], it has been shown that $bp(G)$ can be bounded in term of $bc(G)$, in particular, they have shown that $bp(G) \leq \frac{1}{2}(3^{bc(G)} - 1)$. However, they showed that the analogous result does not hold for the local measures. In this paper, we find a relation between $bc(G)$ and $lcc(\overline{G})$. In particular, we show that if G is a graph with m edges then $bc(G) \leq \frac{1}{2}4^{lcc(\overline{G})} \ln m$. Finding

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the bounds for the biclique cover of graphs is interesting and have been investigated extensively; see [1, 2, 3, 5]. One of the most important results in this direction is the degree bound proved by Alon in [1]. Alon has shown that if G is a graph with n vertices such that the maximum degree of its complement is Δ then $O(\Delta^2 \ln n)$ complete bipartite graphs can cover the edges of the graph G. In this paper, we introduce some graphs such that its complement have constant local clique cover and large maximum degree. Hence, obtained upper bound in this paper improved the existing upper bound of the biclique cover for these graphs.

2 Main results

Definition 2.1. An *intersection representation* for graph $G = (V, E)$ is an assignment of sets A_x of labels L to vertices x so that any two vertices x and y are adjacent if and only if $A_x \cap A_y \neq \emptyset$. A *k-representation* is an intersection representation such that for each $x \in V$, $|A_x| \leq k$.

Theorem 2.2. *If G is a graph with m edges such that $\overline{G}$ has a k-representation, then*

$$bc(G) \leq \frac{\ln m}{-\ln p},$$

where $p = 1 - (\frac{1}{2})^{2k-1}$.

Assume that G has a k-representation. For each $i \in L$, let V_i be the vertices of the graph G such that the corresponding set in this intersection representation containing i. The induced graph G_i on V_i is a clique of the graph G. It is not difficult to see that the collection $\{G_i \mid i \in L\}$ form a clique cover for the graph G such that each vertex is contained in at most k cliques. On the other hand, let $\mathcal{C} = \{G_1, \ldots, G_t\}$ be a clique covering such that each vertex is contained in at most k cliques. Assign to each vertex x the set $A_x = \{i \mid x \in V(G_i)\}$. Easily one can see that with this assignment we have a k-representation. This sets up a one-to-one correspondence between the clique coverings of G and the intersection representations for G. (see e.g. [7]). By the aforementioned discussion, if we define the *representation dimension* of a graph G to be the minimum number k such that G has a k-representation then the representation dimension of G is equal to $lcc(G)$.

Corollary 2.3. *Let G be a graph with m edges then*

$$bc(G) \leq \frac{1}{2} 4^{lcc(\overline{G})} \ln m.$$

Proof. Let $p = 1 - \frac{1}{2^{2k-1}}$. Using the approximation $e^{-x} \approx 1 - x$, if we set $x = \frac{1}{2^{2k-1}}$, then we can see that

$$\frac{1}{-\ln p} \approx 2^{2k-1}.$$

□

The *Line graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$ in which two vertices are joint just as corresponding edges are adjacent as edges in the graph G.

Corollary 2.4. *If G is a graph with m edges in which $\overline{G}$ is a line graph, then*

$$bc(G) \leq 8 \ln m.$$

Proof. Let $\overline{G}$ be the line graph of a graph H. Let the vertices of the graph H have distinct labels. Assign to each vertex x of the graph $\overline{G}$ that is an edge of the graph H a set A_x containing labels of its vertices. This assignment is a 2-representation for the graph $\overline{G}$. □

By a *bipartite complement* of a bipartite graph $G = (X \cup Y, E)$ we will mean the bipartite graph $G^c = (X \cup Y, E^c)$ where $E^c = X \times Y \setminus E$. Jukna, in [5], proved the following theorem.

Theorem 2.5. [5] *For every bipartite $n \times n$ graph G of maximal degree Δ, $bc(G^c) \leq 2e\Delta \ln n$. Where G^c is the bipartite complement of the graph G.*

Let G be a bipartite graph in which (X, Y) is its bipartition. We denote the maximum degree in parts X and Y by Δ_X and Δ_Y, respectively.

Proposition 2.6. *Let G be a bipartite graph such that $k = \min\{\Delta_X, \Delta_Y\}$ then G has a k-representation.*

Proof. Without loss of generality, assume that $k = \Delta_X$. We assign to the vertices of Y, the distinct 1-sets. Then for each vertices of the part X assign the union of the sets of its neighbours. It is not difficult to see that this assignment is a k-representation. □

Remark 2.7. By a discussion similar to the proof of Theorem 2.2, one can obtain the following result. If G is a bipartite graph with m edges such that G^c has a k-representation, then

$$bc(G) \leq \frac{1}{2} 4^{2k} \ln m. \tag{1}$$

Assume that G is a bipartite graph in which $\min\{\Delta_X, \Delta_Y\}$ for the graph G^c is 2 (or a constant). But the maximum degree of G^c is nearly equal in size with $\max\{|X|, |Y|\}$. By (1) and Proposition 2.6 we have $bc(G) \leq 8 \ln m$ (or $bc(G) \leq l \ln m$, where l is a constant number). It will be an improvement of Theorem 2.5 for these graphs.

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On the construction of 3-way 3-homogeneous Steiner trades

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Abstract

A μ-way d-homogeneous (v, k, t) Steiner trade $T = \{T_1, T_2, ..., T_\mu\}$ of volume m consists of μ disjoint collections $T_1, T_2, ..., T_\mu$, each of m blocks of size k, such that every t-subset of v-set V occurs at most once in T_1 $(T_j,\ j \geq 2)$ and each element of V occurs in precisely d blocks of T_1 $(T_j,\ j \geq 2)$. In this paper we characterize the 3-way 3-homogeneous $(v, 3, 2)$ Steiner trades of volume v.

Keywords: Steiner trade, μ-way trade, Homogeneous trade
Mathematics Subject Classification [2010]: 05B05

1 Introduction

Let V be a set of v elements and k and t be two positive integers such that $t < k < v$. A (v, k, t) trade $T = \{T_1, T_2\}$ of volume m consists of two disjoint collections T_1 and T_2, each of containing m, k-subsets of V, called blocks, such that every t-subset of V is contained in the same number of blocks in T_1 and T_2. A (v, k, t) trade is called (v, k, t) Steiner trade if any t-subset of V occurs in at most once in $T_1(T_2)$. In a (v, k, t) trade, both collections of blocks must cover the same set of elements.

The concept of μ-way (v, k, t) trade, was defined recently in [3].

Definition 1.1. A μ-way (v, k, t) trade $T = \{T_1, T_2, ..., T_\mu\}$ of volume m consists of μ disjoint collections $T_1, T_2, ..., T_\mu$, each of m blocks of size k, such that for every t-subset of v-set V the number of blocks containing this t-subset is the same in each T_i (for $1 \leq i \leq \mu$). In other words any pair of collections $\{T_i, T_j\}$, $1 \leq i < j \leq \mu$ is a (v, k, t) trade of volume m. It is clear by the definition that a trade is a 2-way trade. A μ-way (v, k, t) trade is called μ-way (v, k, t) Steiner trade if any t-subset of found(T) occurs at most once in T_1 $(T_j,\ j \geq 2)$.

Definition 1.2. A μ-way (v, k, t) trade is called d-homogeneous if each element of V occurs in precisely d blocks of T_1 $(T_j,\ j \geq 2)$.

Definition 1.3. A trade $T' = \{T'_1, T'_2, ..., T'_\mu\}$ is called a subtrade of $T = \{T_1, T_2, ..., T_\mu\}$, if $T'_i \subseteq T_i$ for $1 \leq i \leq \mu$.

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For $\mu = 2$, Cavenagh et al. [2], constructed minimal d-homogeneous $(v, 3, 2)$ Steiner trades for sufficiently large values of v, (specifically, $v > 3(1.75d^2 + 3)$ if v is divisible by 3 and $v > d(4^{d/3+1} + 1)$ otherwise). In this paper, we aim to construct 3-way 3-homogeneous $(v, 3, 2)$ Steiner trades. The Latin trades are a useful tool for building these trades when $v \equiv 0 \pmod 3$, so we use some obtained results on 3-way 3-homogeneous Latin trades which proved by Bagheri et al. [1].

A Latin square of order n is an $n \times n$ array $L = (\ell_{ij})$ usually on the set $N = \{1, 2, ..., n\}$ where each element of N appears exactly once in each row and exactly once in each column. We can represent each Latin square as a subset of $N \times N \times N$, $L = \{(i, j; k)|$ element k is located in position $(i, j)\}$. A partial Latin square of order n is an $n \times n$ array $P = (p_{ij})$ of elements from the set N where each element of N appears at most once in each row and at most once in each column.

Definition 1.4. A μ-way Latin trade, $(L_1, L_2, ..., L_\mu)$, of volume s is a collection of μ partial Latin squares $L_1, L_2, ..., L_\mu$ containing exactly the same s filled cells, such that if cell (i, j) is filled, it contains a different entry in each of the μ partial Latin squares, and such that row i in each of the μ partial Latin squares contains, set-wise, the same symbols and column j, likewise. A μ-way Latin trade which is obtained from another one by deleting its empty rows and empty columns, is called a μ-way d-homogeneous Latin trade (for $\mu \leq d$) or briefly a (μ, d, m) Latin trade, if it has m rows and in each row and each column L_r for $1 \leq r \leq \mu$, contains exactly d elements, and each element appears in L_r exactly d times.

Lemma 1.5. *If there exist two 3-way d-homogeneous $(v_1, 3, 2)$ and $(v_2, 3, 2)$ Steiner trades of volume m_1 and m_2, respectively, then we have a 3-way d-homogeneous $(v_1 + v_2, 3, 2)$ Steiner trade of volume $m_1 + m_2$.*

Lemma 1.6. *Let (L_1, L_2, L_3) be a 3-way d-homogeneous Latin trade of order m. For each $\alpha \in \{1, 2, 3\}$, define $T_\alpha = \{\{i, j, k\}| (i, j; k) \in L_\alpha\}$. Then $T = \{T_1, T_2, T_3\}$ is a 3-way d-homogeneous $(3m, 3, 2)$ Steiner trade.*

2 3-way 3-homogeneous $(v, 3, 2)$ Steiner trades

In this section we construct and characterize 3-way 3-homogeneous $(v, 3, 2)$ Steiner trades.

Lemma 2.1. *For every $v = 8\ell$ or $v = 9\ell$ where $\ell \in \{1, 2, 3, \cdots\}$, there exists a 3-way 3-homogeneous $(v, 3, 2)$ Steiner trade of volume v.*

Lemmas 2.1 and 1.5 yields the following theorem.

Theorem 2.2. *For every non-zero $v = 9\ell + 8\ell'$, where $\ell, \ell' \in \{0, 1, 2, 3, \cdots\}$, there exists a 3-way 3-homogeneous $(v, 3, 2)$ Steiner trade of volume v.*

The following lemmas can be used for characterizing 3-way 3-homogeneous $(v, 3, 2)$ Steiner trades of volume v.

Lemma 2.3. *There exist only four non-isomorphic 3-way $(v, 2, 1)$ Steiner trade of volume 3.*

Lemma 2.4. *Every* 3*-way* 3*-homogeneous* $(v, 3, 2)$ *Steiner trade of volume* v *contains a* 3*-way* 3*-homogeneous* $(u, 3, 2)$ *Steiner trade of volume* 8 *or* 9*, as a subtrade.*

Theorem 2.5. *If there exists a* 3*-way* 3*-homogeneous* $(v, 3, 2)$ *Steiner trade of volume* v*, then it can be represented as a union of disjoint* 3*-way* 3*-homogeneous* $(8, 3, 2)$ *or* $(9, 3, 2)$ *Steiner trades of volume* 8 *or* 9*, respectively.*

Define $[a, b] = \{c \in Z | \ a \leq c \leq b\}$.

Theorem 2.6. *The* 3*-way* 3*-homogeneous* $(v, 3, 2)$ *Steiner trade of volume* v *does not exist for* $v \in [1, 7] \cup [10, 15] \cup [19, 23] \cup [28, 31] \cup [37, 39] \cup \{46, 47, 55\}$.

Theorem 2.7. *For every* $v \geq 8$*, there exists a* 3*-way* 3*-homogeneous* $(v, 3, 2)$ *Steiner trade of volume* v*, except for* $v \in [10, 15] \cup [19, 23] \cup [28, 31] \cup [37, 39] \cup \{46, 47, 55\}$.

Proof. According to Theorem 2.2, it is enough to represent every $v \geq 8$ in the form $9\ell+8\ell'$, where $\ell, \ell' \geq 0$ as follows:
$v = 9k$, where $k \geq 1$
$v = 9k + 1 = 9(k - 7) + 64$, where $k - 7 \geq 0$
$v = 9k + 2 = 9(k - 6) + 56$, where $k - 6 \geq 0$
$v = 9k + 3 = 9(k - 5) + 48$, where $k - 5 \geq 0$
$v = 9k + 4 = 9(k - 4) + 40$, where $k - 4 \geq 0$
$v = 9k + 5 = 9(k - 3) + 32$, where $k - 3 \geq 0$
$v = 9k + 6 = 9(k - 2) + 24$, where $k - 2 \geq 0$
$v = 9k + 7 = 9(k - 1) + 16$, where $k - 1 \geq 0$
$v = 9k + 8$, where $k \geq 0$
Using Theorem 2.6 completes the proof. □

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On the cospectrality of graphs

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Abstract

Richard Brualdi proposed in [Research problems from the Aveiro workshop on graph spectra, *Linear Algebra and its Applications*, **423** (2007) 172-181.] the following problem:
(Problem AWGS.4) Let G_n and G'_n be two nonisomorphic graphs on n vertices with spectra

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,$$

respectively. Define the distance between the spectra of G_n and G'_n as

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2 \quad \Big(\text{or use } \sum_{i=1}^{n} |\lambda_i - \lambda'_i|\Big).$$

Define the cospectrality of G_n by

$$\text{cs}(G_n) = \min\{\lambda(G_n, G'_n) \ : \ G'_n \ \text{not isomorphic to } G_n\}.$$

Let

$$\text{cs}_n = \max\{\text{cs}(G_n) \ : \ G_n \ \text{a graph on } n \text{ vertices}\}.$$

Problem A. Investigate $\text{cs}(G_n)$ for special classes of graphs.
Problem B. Find a good upper bound on cs_n.
In this paper we study Problem A and determine the cospectrality of all complete bipartite graphs by the Euclidian distance. Let $K_{p,q}$ be the complete bipartite graphs with parts of sizes p and q. We prove that for every positive integers p and q there are some positive integers p', q' and a non-negative integer r such that $\text{cs}(K_{p,q}) = \lambda(K_{p,q}, K_{p',q'} + rK_1)$. As a consequence we determine the cospectrality of stars.

Keywords: Spectra of graphs, Cospectrality of graphs, Measures on spectra of graphs, Adjacency matrix of a graph
Mathematics Subject Classification [2010]: 05C50, 05C31

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1 Introduction

Throughout the paper all graphs are simple, that is finite and undirected without loops and multiple edges. By the spectrum of a graph G, we mean the multiset of eigenvalues of adjacency matrix of G.
Richard Brualdi proposed in [9] the following problem:
(Problem AWGS.4) Let G_n and G'_n be two nonisomorphic graphs on n vertices with spectra

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,$$

respectively. Define the distance between the spectra of G_n and G'_n as

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2 \quad \Big(\text{or use } \sum_{i=1}^{n} |\lambda_i - \lambda'_i|\Big).$$

Define the cospectrality of G_n by

$$cs(G_n) = \min\{\lambda(G_n, G'_n) \ : \ G'_n \ \text{not isomorphic to } G_n\}.$$

Thus $\text{cs}(G_n) = 0$ if and only if G_n has a cospectral mate. Let

$$\text{cs}_n = \max\{\text{cs}(G_n) \ : \ G_n \ \text{a graph on } n \text{ vertices}\}.$$

This function measures how far apart the spectrum of a graph with n vertices can be from the spectrum of any other graph with n vertices.

Problem A. Investigate $\text{cs}(G_n)$ for special classes of graphs.
Problem B. Find a good upper bound on cs_n.

In this paper we study Problem A and determine the cospectrality of complete bipartite graphs by the Euclidian distance, that is

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2.$$

For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. By *order* of G we mean the number of vertices of G; $\overline{G}$ denotes the complement of G and $A(G)$ denotes the adjacency matrix of G. For two graphs G and H with disjoint vertex sets, $G + H$ denotes the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$, i.e. the disjoint union of two graphs G and H. The complete product (join) $G \nabla H$ of graphs G and H is the graph obtained from $G + H$ by joining every vertex of G with every vertex of H. In particular, nG denotes $\underbrace{G + \cdots + G}_{n}$ and $\nabla_n G$ denotes $\underbrace{G \nabla G \nabla \ldots \nabla G}_{n}$.
We denote by $Spec(G)$ the multiset of the eigenvalues of the graph G. For positive integers $n_1, \ldots, n_\ell$, $K_{n_1,\ldots,n_\ell}$ denotes the complete multipartite graph with ℓ parts of sizes $n_1, \ldots, n_\ell$. Let K_n denote the complete graph on n vertices, $nK_1 = \overline{K_n}$ denote the null graph on n vertices and P_n denote the path with n vertices.

Recently the authors study the cospectrality of graphs, see [1, 2, 3] for more details. In [3] the authors find $\mathrm{cs}(K_n)$, $\mathrm{cs}(nK_1)$, $\mathrm{cs}(K_2 + (n-2)K_1)$ $(n \geq 2)$ and $\mathrm{cs}(K_{n,n})$, for every $n \geq 1$. In particular, they find that there exists a unique graph G_H such that $\lambda(H, G_H) = \mathrm{cs}(H)$ if $H \in \{K_n, nK_1, K_2+(n-2)K_1), K_{n,n}\}$. In [1] the authors completely answered Problem B. Also they show that if m and n are some positive integers such that $m+2 \leq n < m-1+2\sqrt{m-1}$, then $\mathrm{cs}(K_{m,n}) = \lambda(K_{m,n}, H)$ if and only if $H \cong K_{m+1,n-1}$. In this paper we generalize this result. In fact we show that for every positive integers m and n there are some positive integers r and s and a non-negative integer t such that $\mathrm{cs}(K_{m,n}) = \lambda(K_{m,n}, K_{r,s} + tK_1)$.

2 Main results

In this section we show that for every positive integers m and n, the minimum value of $\lambda(K_{m,n}, G)$ is attained at a complete bipartite graph with some isolated vertices, say G. We need the following results.

Theorem 2.1 (Theorem 9.1.1 of [6])**.** *Let G be a graph of order n and H be an induced subgraph of G with order m. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of G and H, respectively. Then for every i, $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.*

Theorem 2.2 ([8], see also Theorem 6.7 of [5])**.** *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Theorem 2.3. [7] *Let G be a graph without isolated vertices and let $\lambda_2(G)$ be the second largest eigenvalue of G. Then $0 < \lambda_2(G) \leq \sqrt{2} - 1$ if and only if one of the following holds:*

1. *$G \cong (\nabla_t(K_1 + K_2))\nabla K_{n_1,\ldots,n_m}$.*
2. *$G \cong (K_1 + K_{r,s})\nabla \overline{K_q}$.*
3. *$G \cong (K_1 + K_{r,s})\nabla K_{p,q}$.*

First we prove some lemmas that are essential to prove the main result of this paper.

Lemma 2.4. *Let m and n be two positive integers and G be a graph of order $n + m$. If G has $K_{1,1,2}$ or $(K_1 + K_2)\nabla K_1$ as an induced subgraph, then $\lambda(G, K_{m,n}) \geq 1$.*

Lemma 2.5. *Let m and n be two positive integers and G be a graph of order $n + m$. Suppose that there is no positive integers r, s and a non-negative integer t such that $G \cong K_{r,s} + tK_1$. If $\lambda_2(G) \leq \sqrt{2} - 1$, then $\lambda(G, K_{m,n}) \geq 1$.*

The following theorem is the main result of the paper.

Theorem 2.6. *Let m and n be two positive integers such that $(m, n) \neq (1, 1)$. Then*

$$cs(K_{m,n}) = \lambda(K_{m,n}, K_{r,s} + tK_1),$$

for some integers $r, s \geq 1$ and $t \geq 0$ such that $r + s + t = m + n$ and $\{r, s\} \neq \{m, n\}$. Moreover if $cs(K_{m,n}) = \lambda(K_{m,n}, H)$ for some graph H, then $H \cong K_{r,s} + tK_1$, where $r, s \geq 1$ and $t \geq 0$ are some integers so that $r + s + t = m + n$.

As a consequence we determine $cs(K_{1,n})$ for any n.

Theorem 2.7. *Let $n \geq 1$ be an integer. Then $cs(K_{1,n})$ is the following:*

1. *If $n \leq 2$, then $cs(K_{1,1}) = \lambda(K_{1,1}, 2K_1)$ and $cs(K_{1,2}) = \lambda(K_{1,2}, K_{1,1})$.*
2. *If $n \geq 3$ is a prime number, then $cs(K_{1,n}) = \lambda(K_{1,n}, K_{2,\frac{n+1}{2}} + \frac{n-3}{2}K_1)$.*
3. *If $n \geq 3$ is not a prime number, then $cs(K_{1,n}) = \lambda(K_{1,n}, K_{r,s}) = 0$, where r and s are some positive integers such that $r, s < n$ and $n = rs$.*

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On the signed Roman domination number of graphs

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Abstract

A signed Roman dominating function (simply, a "SRDF") on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that (i) the sum of its function values over each closed neighborhood is at least one and (ii) each vertex x for which $f(x) = -1$ is adjacent to at least one vertex y for which $f(v) = 2$. The weight of a SRDF is the sum of its function values over all vertices. The signed Roman domination number of G, denoted by $\gamma_{sR}(G)$, is the minimum weight of a SRDF on G. In this paper we determine γ_{sR} for some important families of graphs.

Keywords: Domination, Signed Roman domination, SRDF.
Mathematics Subject Classification [2010]: 05C69, 05C78

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order $n = |V(G)|$ and of size $m = |E(G)|$. When x is a vertex of G, the open neighborhood of x in G is the set $N_G(x) = \{y : xy \in E(G)\}$ and the closed neighborhood of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. The degree of vertex x is the number of edges adjacent to x and is denoted by $\deg_G(x)$. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set $D \subseteq V(G)$ is called a **dominating set** of G if each vertex outside D has at least one neighbor in D. The minimum cardinality of a dominating set of G is the **domination number** of G and is denoted by $\gamma(G)$. For example, the domination numbers of the n-vertex complete graph, path, and cycle are given by $\gamma(K_n) = 1$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ and $\gamma(C_n) = \lceil \frac{n}{3} \rceil$, respectively [5]. Domination is a rapidly developing area of research in graph theory, and its various applications to ad hoc networks, distributed computing, social networks, biological networks and web graphs partly explain the increased interest. The concept of domination has existed and studied for a long time and early discussions on the topic can be found in the works of Berge [3] and Ore [8]. At present, domination is considered to be one of the fundamental concepts in graph theory with an extensive research activity. Determining the domination number of an arbitrary graph is an NP-complete problem. The domination number can be defined equivalently by means of a function, which can be considered as a characteristic function of a dominating set, see [5]. A function $f : V(G) \to \{0, 1\}$ is called a **dominating function** on G if for each vertex

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$x \in V(G)$, $\sum_{y\in N_G[x]} f(y) \geq 1$. The value $w(f) = \sum_{x\in V(G)} f(x)$ is called the **weight** of f. Now, the domination number of G can be defined as

$$\gamma(G) = \min\{w(f) : \ f \text{ is a dominating function on } G\}.$$

Analogously, a **signed dominating function** of G is a labeling of the vertices of G with $+1$ and -1 such that the closed neighborhood of each vertex contains more $+1$'s than -1's. The **signed domination number** of G is the minimum value of the sum of vertex labels, taken over all signed dominating functions of G. This concept is closely related to combinatorial discrepancy theory as shown by Füredi and Mubayi in [4]. In general, many domination parameters are defined by combining domination with other graph theoretical properties.

Definition 1.1. [1] Let $G = (V, E)$ be a graph. A **signed Roman dominating function** (simply, a "SRDF") on the graph G is a function $f : V \to \{-1, 1, 2\}$ which satisfies two following conditions:

(a) For each $x \in V$, $\sum_{y\in N_G[x]} f(y) \geq 1$,

(b) Each vertex x for which $f(x) = -1$ is adjacent to at least one vertex y for which $f(y) = 2$.

The value $f(V) = \sum_{x\in V} f(x)$ is called the **weight** of the function f and is denoted by $w(f)$. The **signed Roman domination number** of G, $\gamma_{sR}(G)$, is the minimum weight of a SRDF on G.

This concept is introduced by Ahangar, Henning, et al. in [1]. They described the usefulness of this concept in various applicative areas like "defending the Roman empire" (see [1], [6] and [10] for more details). It is obvious that for every graph G of order n we have $\gamma_{sR}(G) \leq n$, because assigning $+1$ to each vertex yields a SRDF. In [1] Ahangar et al. present various lower and upper bounds on the signed Roman domination number of a graph in terms of it's order, size and vertex degrees. Moreover, they characterized all graphs which attain these bounds. Also, they investigate the relation between γ_{sR} and some other graphical parameters, and the signed Roman domination number of some special bipartite graphs. It is proved in [1] that $\gamma_{sR}(K_n) = 1$ for each $n \neq 3$, $\gamma_{sR}(K_3) = 2$, $\gamma_{sR}(C_n) = \lceil \frac{2n}{3} \rceil$, $\gamma_{sR}(P_n) = \lfloor \frac{2n}{3} \rfloor$, and that the only n-vertex graph G with $\gamma_{sR}(G) = n$ is the empty graph $\overline{K}_n$. The Signed Roman Dominiation Number of the join of graphs is considered in [2].

Henning and Volkmann investigate the signed Roman domination number of trees in [7]. Also, the signed Roman domination number of directed graphs is considered in [9].

Note that each signed Roman dominating function f on G is uniquely determined by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$, where $V_i = \{x \in V(G) : \ f(x) = i\}$ for each $i \in \{-1, 1, 2\}$. Specially, $w(f) = 2|V_2| + |V_1| - |V_{-1}|$. For convenience, we usually write $f = (V_{-1}, V_1, V_2)$ and, when $S \subseteq V$ we denote the summation $\sum_{x\in S} f(x)$ by $f(S)$. If $w(f) = \gamma_{sR}(G)$, then f is called a **$\gamma_{sR}(G)$-function** or an **optimal SRDF** on G.

In this paper we determine γ_{sR} for some important families of graphs.

2 Main results

For investigating the signed Roman domination number of the complete multipartite graphs, the following two technical lemmas are useful.

Lemma 2.1. *If G is a graph with $\Delta(G) = |V(G)| - 1$, then $\gamma_{sR}(G) \geq 1$.*

Proof. Let f be an optimal signed Roman dominating function on G and let $x \in V(G)$ be a vertex of maximum degree $\Delta(G)$. Since $N_G(x) = V(G) \setminus \{x\}$, using the definition of a $SRDF$ we have

$$\gamma_{sR}(G) = w(f) = \sum_{v \in V(G)} f(v) = f(x) + \sum_{v \in N_G(x)} f(v) = f(N_G[x]) \geq 1.$$

□

Lemma 2.2. *For each signed Roman domination function f of the complete multipartite graph $G = K_{n_1,n_2,...,n_k}$, $k \geq 3$, we have*

$$w(f) \geq \min\{2 + \frac{2}{k-1}, n_1, n_2, ..., n_k, n_1 + 1, ..., n_k + 1, 2n_1 - 1, ..., 2n_k - 1\}.$$

Proof. Let f be a SRDF on G and let X_j be the partite set of G of size n_j, $1 \leq j \leq k$. Since the label of each vertex $x \in X_j$ is at most 2 and $f(N_G[x]) \geq 1$, we should have $w(f) - f(X_j) \geq -1$. If $w(f) - f(X_j) = -1$, then the label of each vertex $x \in X_j$ is 2 and this implies that $w(f) = f(X_j) - 1 = 2n_j - 1$. If $w(f) - f(X_j) = 0$, then none of the vertices of X_j has label -1. This means that $w(f) \geq |X_j| = n_j$. If $w(f) - f(X_j) = 1$, then the label of each vertex in X_j is 1 or 2. Thus, $w(f) \geq |X_j| + 1 = n_j + 1$. In all of these cases we have

$$w(f) \geq \min\{n_j, n_j + 1, 2n_j - 1\}.$$

Therefore, when such a situation occures for a partite set, then the result follows. Otherwise, $w(f) - f(X_j) \geq 2$ for each $j \in \{1, 2, ..., k\}$. This implies that

$$(k-1)\ w(f) = k\ w(f) - w(f) = \sum_{j=1}^{k} (w(f) - f(X_j)) \geq 2k,$$

and hence, $w(f) \geq \frac{2k}{k-1} > 2$, which completes the proof. □

Corollary 2.3. *If $n_i \geq 3$ for each $i \in \{1, 2, ..., k\}$, then $\gamma_{sR}(K_{n_1,n_2,...,n_k}) \geq 3$.*

Proposition 2.4. *The signed Roman domination number of the complete 3-partite graph $K_{m,m,m}$ is given as follows:*

$$\gamma_{sR}(K_{m,m,m}) = \begin{cases} 3 & m \neq 1 \\ 2 & m = 1. \end{cases}$$

Theorem 2.5. *Let $k \geq 3$ be an integer. Then, for each complete multipartite graph $G = K_{n_1,n_2,...,n_k}$ we have $1 \leq \gamma_{sR}(G) \leq 7$.*

The following theorem shows that the signed Roman domination number of almost all of complete multipartite graphs is three.

Theorem 2.6. *Let $G = K_{n_1,n_2,\ldots,n_k}$, $k \geq 3$, be a complete multipartite graph such that $n_j \geq 5$ for each $1 \leq j \leq k$. Then, $\gamma_{sR}(G) = 3$.*

Theorem 2.7. *Let $G = K_{n_1,n_2,\ldots,n_k}$, $k \geq 3$, be an n-vertex complete multipartite graph such that $\gamma_{sR}(G) \neq 1$ and $p_2 \neq 0$, where $p_j = |\{i :\ n_i = j\}|$ for each $j \in \{1, 2, ..., n-2\}$. Then we have $\gamma_{sR}(G) = 2$ if and only if one of the following coditions holds.*

a) $p_1 \geq 1$.

b) $k - p_1 - p_2 - p_4 \geq 2$.

c) $p_2 \geq 2$ *and* $p_4 \geq 2$.

d) $p_2 \geq 2$, $p_4 = 1$ *and there exists* $j \geq 6$ *such that* $p_j \geq 1$.

Theorem 2.8. *Let $G = K_{n_1,n_2,\ldots,n_k}$, $k \geq 3$, be an n-vertex complete multipartite graph such that $p_1 \neq k$, where $p_j = |\{i :\ n_i = j\}|$ for each $j \in \{1, 2, ..., n-2\}$. Then we have $\gamma_{sR}(G) = 1$ if and only if $k < 3p_1 - p_2$.*

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Talk
On the Wiener index of Sierpiński graphs
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On the Wiener index of Sierpiński graphs

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Abstract

Wiener index of graph G is defined as sum of distances of all pairs of vertices. In this paper, the Wiener index of Sierpiński graphs is computed and explicit formula is obtained.

Keywords: Wiener index, Sierpiński graphs, Total distance
Mathematics Subject Classification [2010]: 05C12, 05C76, 05C90

1 Introduction

Sierpiński graphs S_k^n were introduced by S. Klavzar and Milutinovic in [2] The graph S_k^1 is the complete graph in k vertices and S_3^n are isomorphic to the tower of Hanoi graphs. Mathematical properties of the graph S_k^n have been well studied. For example a classification of their covering codes is given in [1] metric properties of Sierpiński graphs were studied in [3] and [4] . The S_k^n can be defined recursively with the following process: S_k^1 is a complete graph. To construct S_k^{n+1}, consider S_k^n and adding exactly one edge between each pair of copies. When $k = 2$ then S_k^n is isomorphic to P_{2^n} and in the case $k = 3$ these graphs are exactly tower of Hanoi graphs. The structure of tower of Hanoi graph is illustrated in Fig 1. The vertices of S_k^n can be identified with words of size n on alphabet

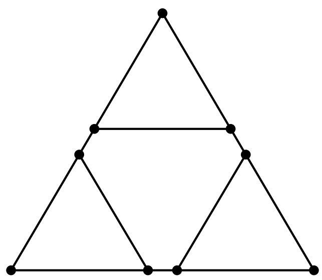

Figure 1: Structure of Sierpiński graph S_3^n

$\{1, 2, \cdots, k\}$. Let $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ be two different vertices. u and v are adjacent if and only if there exists $i \in \{1, 2, \cdots, k\}$ such that

- $u_t = v_t$ for $1 \leq t \leq i - 1$

*Speaker

- $u_i \neq v_i$
- $u_t = v_i$ and $v_t = u_i$ for $i + 1 \leq t \leq n$.

A vertex of the form $(t, t, \cdots, t)$ is called an extreme vertex. S_k^n contains k extreme vertices.

Let G be a simple connected graph. Distance between two vertices u, v, $\mathrm{d}(u, v)$ is length of shortest path connecting them. Let $n \geq 2$, then for $i = 1, \cdots, k$ let iS_k^{n-1} be the subgraph of S_k^ninduced by the vertices of the form $(i, v_2, v_3, \cdots, v_n)$. Let $i \neq j$, then the edge $(i, j, j, \cdots, j)(j, i, i, \cdots, i)$ is the unique edge between iS_k^{n-1} and jS_k^{n-1}.

The Wiener index of graph G is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} \mathrm{d}_G(u, v)$$

. Let $u \in V(G)$, then distance of u is $\mathrm{d}_G(u) = \sum_{v \in V(G)} \mathrm{d}_G(u, v)$

It is easy to see that $W(G) = \frac{1}{2} \sum_{u \in V(G)} \mathrm{d}_G(u)$. The Wiener index is the first topological index bases on distance and this graph invariant has been extensively investigated. We refer the reader to see [5, 6, 7, 8]

In this paper, the Wiener index of Sierpiński graphs S_k^n is computed and explicit formula is presented.

2 Main results

To find the Wiener index of S_k^n, we partition the pair of vertices into two sets: pairs of vertices in one copy of iS_k^{n-1} and pairs of vertices that are in two different copy of S_k^{n-1}. We have

$$W(S_k^n) = kW(S_k^{n-1}) + \sum_{1 \leq i \leq n, x \in iS_k^{n-1}} \sum_{1 \leq j \leq n, y \in jS_k^{n-1}} \mathrm{d}(x, y)$$

Since there are $\binom{k}{2}$ copies of each pair of vertices in the sets. Then

$$W(S_k^n) = kW(S_k^{n-1}) + \binom{k}{2} \sum_{x \in iS_k^{n-1}} \sum_{y \in jS_k^{n-1}} \mathrm{d}(x, y).$$

Let $W_n = W(S_k^n)$. It is clear that $W_1 = \binom{k}{2}$. Let $v_{ij} \in iS_k^{n-1}$ and $v_{ji} \in jS_k^{n-1}$ be two adjacent vertices. Since each path connecting two vertices $a \in iS_k^{n-1}$ and $b \in jS_k^{n-1}$ contains edge $v_{ij}v_{ji}$, then $\mathrm{d}(a, b) = \mathrm{d}(a, v_{ij}) + \mathrm{d}(v_{ji}, b)$ and we have

$$\begin{aligned}
\sum_{a \in iS_k^{n-1}} \sum_{b \in jS_k^{n-1}} \mathrm{d}(a, b) &= \sum_{a \in iS_k^{n-1}} \sum_{b \in jS_k^{n-1}} (\mathrm{d}(a, v_{ij}) + 1 + \mathrm{d}(v_{ji}, b)) \\
&= |jS_k^{n-1}||\mathrm{d}_{S_k^{n-1}}(v_{ij})| + |jS_k^{n-1}||iS_k^{n-1}| + |\mathrm{d}_{jS_k^{n-1}}(v_{ji})||iS_k^{n-1}| \\
&= k^{n-1}\mathrm{d}_{iS_k^{n-1}}(v_{ij}) + k^{2(n-1)} + k^{n-1}\mathrm{d}_{jS_k^{n-1}}(v_{ji})
\end{aligned}$$

There for

$$W_n = kW_{n-1} + \binom{k}{2}(k^{n-1}\mathrm{d}_{S_k^{n-1}}(v_{ij}) + k^{2(n-1)} + k^{n-1}\mathrm{d}_{S_k^{n-1}}(v_{ji})) \tag{1}$$

Now we show that $\mathrm{d}_{iS_k^n}(v_{ij}) = \mathrm{d}_{tS_k^n}(v_{ts})$.

Theorem 2.1. $\mathrm{d}_{iS_k^{n-1}}(v_{ij}) = \mathrm{d}_{jS_k^{n-1}}(v_{ji})$

Proof. By induction on n. When $n = 2$, it is clear that $\mathrm{d}_{iS_k^1}(v_{ij}) = \mathrm{d}_{tS_k^1}(v_{ts}) = k - 1$. Let $v_i \in iS_k^{n-1}$ be an extreme vertex of S_k^n. $\mathrm{d}_{S_k^n}(v) = \sum_{a\in jS_k^{n-1}} \mathrm{d}(v,a)$. It was proved that $\mathrm{diam}(S_k^n) = 2^n - 1$, therefore $\mathrm{d}_{iS_k^{n_1}}(v_i, v_{ij}) = 2^{n-1} - 1 = \mathrm{diam}(S_k^{n-1})$. For vertex $a \in jS_k^{n-1}$, where $\mathrm{d}(v_i,a) = \mathrm{d}_{iS_k^{n-1}}(v_i, v_{ij}) + 1 + \mathrm{d}_{jS_k^{n-1}}(v_{ji}, a)$. Then

$$\begin{aligned}\mathrm{d}_{S_k^n}(v_i) &= \mathrm{d}_{iS_k^{n-1}}(v_i) + \sum_{a\in iS_k^{n-1}, j\neq i}((2^{n-1} - 1) + 1 + \mathrm{d}(v_{ji}, a))\\ &= \mathrm{d}_{iS_k^{n-1}}(v_i) + |k-1||S_k^{n-1}|2^{n-1} + \sum_{j\neq i}\mathrm{d}_{jS_k^{n-1}}(v_{ji})\end{aligned}$$

Since $\mathrm{d}_{S_k^{n-1}}(v_i) = \mathrm{d}_{jS_k^{n-1}}(v_{ij})$ where $\mathrm{d}_{S_k^n}(v_i) = k\mathrm{d}_{S_k^{n-1}}(v_i) + 2^{n-1}k^{n-1}(k-1)$ and it conclude that $\mathrm{d}_{iS_k^{n-1}}(v_{ij}) = \mathrm{d}_{tS_k^{n-1}}(v_{st})$.

Now, we find an explicit formula for the distance an extreme vertex v_i of iS_k^{n-1}. Let $d_i = d_{S_k^n}(v_i)$. Then $d_n = kd_{n-1} + (k-1)k^{n-1}2^{n-1}$, $d_0 = 0$ and $d_1 = k-1$. The following formula is obtained for d_n,

$$d_n = k^{n-1}(k-1)(2^n - 1). \tag{2}$$

Relations 1 and 2 concludes

$$\begin{aligned}W_n &= kW_{n-1} + \binom{k}{2}(2k^{n-1}d_{n-1} + k^{2(n-1)})\\ &= kW_{n-1} + k^{2(n-1)}(k-1)^2(2_1^{n-1}) + (k^{2n-1}(k-1))/2.\end{aligned}$$

$W_1 = \binom{k}{2}$. By solving the above reduction relations,

$$\begin{aligned}W_n &= k^{n-1}(k-1)^2(k + k(2^2-1) + \cdots + k^{n-2}(2^{n-1}-1)) + 1/2k^n(k-1)(1 + k + \ldots + k^{n-1})\\ &= k^{n-1}(k-1)^2(2k\frac{(2k)^{n-1}-1}{2k-1} - k\frac{k^{n-1}-1}{k-1}) + \frac{1}{2}k^n(k^n-1)\end{aligned} \tag{3}$$

□

Example 2.2. The following figure is the Sierpiński graph S_5^2. Also by above formula the Wiener index of S_5^n could be obtained by, $W(S_5^n) = \frac{32}{9}5^n(10^{n-1}-1) - 4\cdot 5^n(5^{n-1}-1) + \frac{1}{2}5^n(5^n-1)$

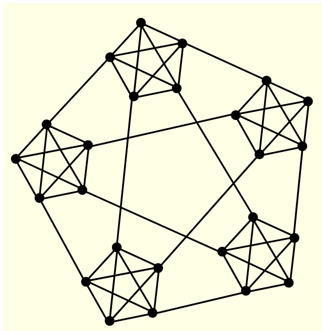

Figure 2: The Sierpiński graph S_5^2

Example 2.3. In the case $k = 2$, we will have $W(S_2^n) = W(P_{2^n})$. It was proved that $W(P_{2^n}) = \binom{2^n+1}{3} = \frac{1}{3}2^{n-1}(2^{2n} - 1)$. Now by the relation 3, $W(S_2^n) = 2^{n-1}(\frac{4}{3}(4^{n-1} - 1)) - 2(2^{n-1} - 1) + \frac{1}{2}2^n(2^n - 1) = \frac{1}{3}2^{n-1}(2^{2n} - 1)$. Which verifies our formula for Wiener index of Sierpiński graph S_k^n.

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One-solely balanced sets and related Steiner trades

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Abstract

A μ-way t-solely balanced set is a μ-way (v, k, t) Steiner trade $T = \{T_1, T_2, \ldots, T_\mu\}$ such that T_i and T_j $(1 \le i < j \le \mu)$ Contains no common $(t + 1)$-subset. The one-solely sets are the most important tool for building Steiner trades. In this article we introduce some techniques for construction the one-solely sets and related Steiner trades.

Keywords: One-Solely, 3-way (v, k, t) Steiner trade,
Mathematics Subject Classification [2010]: 05B30; 05B05

1 Introduction

The concept of trade has represented in the graph theory, design theory and latin square. In this paper we investigate this concept in design theory. This subject is originated in the 1960s by Hedayat [3]. The concept of trade was introduced in 1916 by Cole and Cumming in other forms. This concept have been generated in [4] recently, as μ-way (v, k, t) trade $\mu \ge 2$.

Definition 1.1. A μ-way (v, k, t) trade of volume m consists of μ disjoint collections $\{T_1, T_2, \ldots, T_\mu\}$ each of m blocks, such that for every t-subset of v-set V, the number of blocks containing this t-subset is the same in each $T_i (1 \le i \le \mu)$. In the other words any pair of T_i's is a (v, k, t) trade of volume m.

A μ-way (v, k, t) trade is called μ-way (v, k, t) Steiner trade if any t-subset of found(T) occurs at most once in $T_1 (T_j, j \ge 2)$.

Definition 1.2. Let $T = \{T_1, T_2, \ldots, T_\mu\}$ be a μ-way (v, k, t) Steiner trade. We say T is μ-way (v, k) t-solely balanced if T_i and $T_j (1 \le i < j \le \mu)$ contain no common $(t + 1)$ subset.

*Speaker

In section two, we state some (v, k) one-solely sets and their related 2-way $(v+3, k+1, 2)$ Steiner trade from [1] and [2]. We Construct some new one-solely sets and their related trade in section three.

2 Preliminary Results

Following theorem will be used repeatedly in the sequel.

Theorem 2.1. *[4] (i) Let $T = \{T_1, T_2, \ldots, T_\mu\}$ be a μ-way (v, k, t) trade of volume m. Then, based on T, a μ-way $(v + \mu, k + 1, t + 1)$ trade T^* of volume μm can be constructed. (ii) If T is t-solely balanced, then T^* is a Steiner trade.*

Theorem 2.2. *There exist a μ-way $(2m + \mu, 3, 2)$ Steiner trade of volume μm for $2 \leq \mu \leq 2m + 1$.*

Proof. We know the complete graph K_{2m} has $2m - 1$ disjoint 1-factors. If we take μ 1-factors F_1, $F_2, \ldots, F_\mu$ as T_1, $T_2, \ldots, T_\mu$ respectively, then $T = \{T_1, T_2, \ldots, T_\mu\}$ is a μ-way $(2m, 2)$ one-solely set of volume m. Now, we can apply Theorem 2.1 □

A 3-way one-solely set can be constructed from an array $A(k)$ of size $k-1$, Let S_1, S_2 and S_3 be the collections of elements of each of the rows, columns and forward diagonals of $A(k)$ respectively. We can see S_1, S_2 and S_3 together construct a 3-way $((k-1)^2, k-1)$ one-solely set.

Example 2.3. $A(3)$:

1	2	3
4	5	6
7	8	9

One solely set:

S_1	S_2	S_3
123	147	159
456	258	267
789	369	348

Denote $A'(k, r)$ to be an array of size $(k - 1)$ with each of the elements a_{ij} of the first r rows of $A(k)$ replaced by a'_{ij}

Example 2.4. $A'(3, 1)$:

$1'$	$2'$	$3'$
4	5	6
7	8	9

One solely set:

S_1	S_2	S_3
$1'2'3'$	$1'47$	$1'59$
456	$2'58$	$2'67$
789	$3'69$	$3'48$

In the following example we can see the construction of a 2-way $(v, k, 2)$ Steiner trade for the above one-solely sets.

Example 2.5. T :

T_1	T_2
$x123$	$x159$
$x456$	$x267$
$x789$	$x348$
$y159$	$y123$
$y267$	$y456$
$y348$	$y789$

T^* :

T_1	T_2
$z1'2'3'$	$z1'47$
$z456$	$z2'58$
$z789$	$z3'69$
$x1'47$	$x1'2'3'$
$x2'58$	$x456$
$x3'69$	$x789$

T and T^* are two 2-way $(11, 4, 2)$ Steiner trade of volume 6. Now $T + T^*$ is a 2-way $(15, 4, 2)$ Steiner trade of volume 10.

3 New Constructions

In this section, we construct some new one-solely sets. Then we apply Theorem 2.1 and obtain some new 3-way Steiner trades. We can construct a 4-way one-solely set as follows.

Example 3.1. Consider the following table.

1	2	3	4	5
6	7	8	9	a
b	c	d	e	f
g	h	i	j	k
l	m	n	o	p
q	r	s	t	u
x	y	z	w	v
A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T

Now consider the following four classes.

12345	1QMIE	16bgl	28ekP
6789a	26RNJ	q27ch	39fKQ
bcdef	37bOS	mr38d	4aFLR
ghijk	48cgT	ins49	5AGMS
lmnop	59dhl	ejot5	17djp
qrstu	aeimq	afkpu	6ciou
xyzwv	fjnrx	xAFKP	bhnt7
ABCDE	kosyA	yBGLQ	gmswE
FGHIJ	ptzBF	zCHMR	lrzDJ
KLMNO	uwCGK	wDINS	qyCIO
PQRST	vDHLP	vEJOT	xBHNT

Now, we can apply the Theorem 2.1 to obtain a 3-way $(v, 6, 2)$ Steiner trade. In the next example we generalized the idea which stated in the previous section.

Example 3.2. Consider the following three matrices.

$$A : \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \qquad B : \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \mathrm{x} & \mathrm{y} & \mathrm{z} \\ \hline \mathrm{w} & \mathrm{u} & \mathrm{v} \\ \hline \end{array} \qquad C : \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \hline \end{array}$$

Now consider the following 3-way one solely sets:

$$S_A : \begin{array}{c|c|c} 123 & 147 & 159 \\ 456 & 258 & 267 \\ 789 & 369 & 348 \end{array} \quad S_B : \begin{array}{c|c|c} 123 & 1\mathrm{xw} & 1\mathrm{yv} \\ \mathrm{xyz} & 2\mathrm{yu} & 2\mathrm{zw} \\ \mathrm{wuv} & 3\mathrm{zv} & 3\mathrm{xu} \end{array} \quad S_C : \begin{array}{c|c|c} 123 & 1\mathrm{ad} & 1\mathrm{bf} \\ \mathrm{abc} & 2\mathrm{bc} & 2\mathrm{cd} \\ \mathrm{def} & 3\mathrm{ef} & 3\mathrm{ae} \end{array}$$

Then we construct three 3-way $(12, 4, 2)$ Steiner trades of volume 9. The following trades have the common block $\tilde{\mathrm{x}}123$. By adding these trades, we have a 3-way $(12, 4, 2)$

Steiner trade of volume $9+9+9-1=27$.

T_A :

T_1	T_2	T_3
x̃123	x̃147	x̃159
x̃456	x̃258	x̃267
x̃789	x̃369	x̃348
ỹ147	ỹ159	ỹ123
ỹ258	ỹ267	ỹ456
ỹ369	ỹ348	ỹ789
z̃159	z̃123	z̃147
z̃267	z̃456	z̃258
z̃348	z̃789	z̃369

T_B :

T_1	T_2	T_3
ŷ123	ŷ1xw	ŷ1yv
ŷxyz	ŷ2yu	ŷ2zw
ŷwuv	ŷ3zv	ŷ3xu
x̃1xw	x̃1yv	x̃123
x̃2yu	x̃2zw	x̃xyz
x̃3zv	x̃3xu	x̃wuv
ẑ1yv	ẑ123	ẑ1xw
ẑ2zw	ẑxyz	ẑ2yu
ẑ3xu	ẑwuv	ẑ3zv

T_C :

T_1	T_2	T_3
ý123	ý1ad	ý1bf
ýabc	ý2bc	ý2cd
ýdef	ý3ef	ý3ae
ź1ad	ź1bf	ź123
ź2bc	ź2cd	źabc
ź3ef	ź3ae	źdef
x̃1bf	x̃123	x̃1ad
x̃2cd	x̃abc	x̃2bc
x̃3ae	x̃def	x̃3ef

Theorem 3.3. *There exist a μ-way $(q^2+\mu, q+1, 2)$ Steiner trade of volume $m = q\mu$, $\mu = 2, ...q+1$ When q is a prime power.*

Proof. We know, there exists a $(q^2, q, 1)$ resolvable block design with $q+1$ parallel classes for q be a prime power. Let $P_1, P_2, ..., P_{q+1}$, be (q+1) parallel classes of $(q^2, q, 1)$ resolvable block design. We can construct a 3-way $(q^2, q, 1)$ one-solely set of volume q as follows.

T_1	....	T_{q+1}
P_1	....	P_{q+1}

Now we can apply Theorem 2.1 to construct the μ-way $(q^2+\mu, q+1, 2)$ Steiner trade of volume $m = q\mu, \mu = 3, ...q+1$. □

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Talk Permutation representation of graphs pp.: 1–2

Permutation Representation of Graphs

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Abstract

There are many geometric and algebraic representations of graphs. Recently, we introduce a new representation of graphs by use of permutations and present some results about this representation and related parameter.
Let G be a graph. A k-permutation representation of G is a map π of $V(G)$ to symmetric group S_k, such that for any two vertices v and u, $v \sim u$ if and only if for each $i \in \{1, 2, 3, \cdots, k\}$ we have $\pi(v)(i) \neq \pi(u)(i)$. In other words, $\pi(v) \circ \pi(u)^{-1} \in D_k$ where D_k denote the set of all derangements of S_k. We define the permutation representation number $pr(G)$ to be the minimum of k such that G has a k-permutation representation. In addition, we find upper and lower bounds for this parameter of graphs.

Keywords: Permutation, Representation of graph, Cayley graph
Mathematics Subject Classification [2010]: 05C62,

1 Introduction

There are many geometric and algebraic representations of graphs. In this paper, we introduce a new representation of graphs by use of permutations and present some results about this representation and related parameters.

2 Main results

Definition 2.1. Let G be a graph. A k-permutation representation of G is a map π of $V(G)$ to symmetric group S_k, such that for any two vertices v and u, $v \sim u$ if and only if for each $i \in \{1, 2, 3, \cdots, k\}$ we have $\pi(v)(i) \neq \pi(u)(i)$. In other words, $\pi(v) \circ \pi(u)^{-1} \in D_k$ where D_k denotes the set of all derangements of S_k.

In this representation, we have a common adjacency rule and so for defining a graph, we only need to introduce the vertex set of graph.

Definition 2.2. The permutation representation number, $pr(G)$, is the minimum of k such that G has a k-permutation representation.

Example 2.3. Consider graph K_n. We have $pr(K_n) = n$. In fact any permutation representation of K_n give us a latin square of order n.

*Speaker

As you know, Cayley theorem is one of the main theorems in group theory.

Theorem 2.4. [1854] *Every group is isomorphic to a subgroup of S_n for some n.*

Definition 2.5. [1971] For every finite group $\mu(G) = \min\{n|\ G \cong H \leq S_n\}$.

Theorem 2.6. $f(|G|) \leq \mu(G) \leq |G|$, *where* $f(n) = \max\{k|\ k! \leq n\}$.

Example 2.7. $\mu(Z_6) = 5$. In fact $Z_6 \cong < (12)(345) >$ that is a subgroup of S_5.

Definition 2.8. For a subset S of a group G such that the identity $e \notin S$ and $S = S^{-1}$ (where $S^{-1} = \{s^{-1}|s \in S\}$), the *Cayley graph* $\Gamma = Cay(G, S)$ is the graph with vertex set G such that $x \sim y$ if and only if $xy^{-1} \in S$.

We call the following theorem *Cayley type theorem for graphs.*

Theorem 2.9. *Every graph is an induced subgraph of $Cay(S_n, D_n)$ for some n where D_n is the set of all derangements of S_n.*

Theorem 2.10. *Let G be a finite group with $\mu(G) = m$ and φ be an isomorphism from G to a subgroup of S_m. Then*

$$\mu(G) \geq pr(Cay(G, G \cap \varphi^{-1}(D_m))).$$

Theorem 2.11. *Let G be a graph of order n. Then* $\chi(G) \leq pr(G) \leq \frac{n(n-1)}{2}$.

Remark 2.12. The lower bound in Theorem 2.11 is sharp. For example consider K_n. We have $pr(K_n) = \chi(K_n) = n$.

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Talk Relations between some packing and covering parameters of graphs pp.: 1–3

Relations between some packing and covering parameters of graphs*

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Abstract

Many packing and covering parameters have been associated to an arbitrary graph $G = (V, E)$ which studying relations between them is very interesting problem in graph theory. In this paper we consider some of well-known packing and covering parameters such as matching ,vertex covering, domination and irredundance number and find interesting relations between them.

Keywords: Total domination number, Irredundance number, Matching number
Mathematics Subject Classification [2010]: 05C69

1 Introduction

Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a *dominating set* of G if every vertex in $V - D$ has a neighbor in D. The cardinality of a minimum dominating set of G is denoted by $\gamma(G)$. If, in addition, the induced subgraph $\langle D \rangle$ has no isolated vertex, then D is called a *total dominating set.* The cardinality of a minimum total dominating set of G is denoted by $\gamma_t(G)$. for more details about domination parameters you can see [1] or [4].

Definition 1.1. If every vertex of $V - D$ has exactly one neighbor in D and $\langle D \rangle$ is an empty induced subgraph of G, then we call D a perfect code or efficient dominating set.

Definition 1.2. If every vertex of $V - D$ is adjacent to exactly one vertex of D and induced subgraph $\langle D \rangle$ is also a matching, then we call D a total perfect code or efficient open dominating set.

Definition 1.3. The set $X \subseteq V$ is an OO-irredundant set if and only if for each $v \in X$, $N(v) - N(X \setminus \{v\}) \neq \emptyset$. The minimum cardinality among all maximal OO-irredundant set denoted by $ooir(G)$ and called OO-irredundance number of the graph G.

*Will be presented in English
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A set $M \subseteq E$ of graph G is called a *matching* if no vertex is incident to more than one edge in M. We use $\nu(G)$ to denote the size of a maximum matching of the graph G. A dual pair of matching problem is called *vertex covering.* A set $U \subseteq V$ is a vertex cover if each edge has at least one endpoint in S. We use $\tau(G)$ to denote the size of a minimum vertex cover of the graph G.

We can see matching as a disjoint subgraph, isomorphic to K_2. So it is possible to generalize it as follows:

Definition 1.4. For a graph G a set of edge disjoint induced subgraphs isomorphism to K_r is called K_r-packing and K_r-packing of maximum size is denoted by $\nu_r(G)$. Its dual parameters is minimum number of edges cover all induced subgraph isomorphism to K_r in G which is denoted by $\tau_r(G)$ and called K_r-covering number of G.

By Definitions it is easy to see that $\nu_r(G) \leq \tau_r(G) \leq \binom{r}{2}\nu_r(G)$.
All graph parameters can be modeled by linear programming which their real relaxations are fractional parameters. For example fractional matching and fractional dominations are defined as follows:

$$\nu^*(G) = max\{1^T x \;:\; x(\delta(v)) \leqslant 1 \;\; \forall v \in V; x \geqslant 0\}$$

(where $\delta(v)$ denotes the set of edges incident to v.)

$$\gamma^*(G) = min\{1^T x \;\sum_{u \in N[v]} x_u \geq 1 \; \forall v \in V; \; x \geq 0\}$$

and

$$\gamma_t^*(G) = min\{1^T x \;\sum_{u \in N(v)} x_u \geq 1 \; \forall v \in V; \; x \geq 0\}$$

2 Main results

2.1 Domination parameters

In [3] it is proved that for the family of claw-free graphs with minimum degree at least three and for the family of k-regular graphs when $k \geq 3$, $\gamma_t(G) \leq \nu(G)$. We can prove similar result for relevant fractional parameters in almost all graphs.

Theorem 2.1. *For almost all graphs* $\gamma_t^*(G) \leq \nu^*(G)$.

In the following theorems we can see very interesting relation between irredundance and domination parameters of some class of graphs which domination numbers is easy to determine but their irredundance number is still NP-hard.

Theorem 2.2. *If* G *has a total perfect code then* $\gamma_t(G) = ooir(G)$.

Also we can see that:

Theorem 2.3. *If* G *has a total perfect code then* $\gamma_t(G) \leq \nu(G)$.

2.2 K_r-packing and covering[1]

A famous König theorem says that for a bipartite graph G, $\tau(G) = \nu(G)$. Tuza's conjecture is a famous conjecture about relations between $\tau_3(G)$ and $\nu_3(G)$ [5].
Tuza's Conjecture: For a graph G, $\tau_3(G) \leq 2\nu_3(G)$.
This conjecture is proved for tripartite graphs in [2], in that paper authors conjectured it may possible to improve this bound to a constant close to 1. By using Maxflow-Mincut theorem we can prove their conjecture for special tripartite graphs in the following lemma.

Lemma 2.4. *We call a graph G* purple of order $k \in \mathbb{N}$ *if and only if there is a bipartite graph H with bipartition (X, Y) such that*

$$V(G) = X \cup Y \cup \{u_1, \dots u_k\}$$

and

$$E(G) = E(H) \cup \{zu_1, \dots zu_k \mid z \in X \cup Y\}.$$

If G is a purple graph of order k, then

$$\tau_3(G) = \nu_3(G).$$

Also by algebraic topology methods we can prove the following for K_4-packing and covering.

Theorem 2.5. *If G is a 4-partite graph then $\tau_4(G) \leq 5\nu_4(G)$.*

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Roman k-Domination Number Upon Vertex and Edge Removal

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Abstract

Let $k \geq 1$ be an integer. A *Roman k-dominating function* on a graph G with vertex set V is a function $f : V \to \{0, 1, 2\}$ such that every vertex $v \in V$ with $f(v) = 0$ has at least k neighbors $u_1, u_2, \cdots, u_k$ with $f(u_i) = 2$ for $i = 1, 2, \cdots, k$. The weight of a Roman k-dominating function is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of Roman k-dominating functions on a graph G is called the *Roman k-domination number*, denoted by $\gamma_{kR}(G)$. In this paper, we consider the effects of vertex and edge removal on the Roman k-domination number of a graph. Some of our results improve these one given by Kämmerling and Volkmann in [6] for the Roman k-domination number.

Keywords: Roman domination, Roman k-domination number, Roman k-dominating function.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to [5, 10]. In this paper, G is a simple graph with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The *order* $|V|$ and the *size* $|E|$ are denoted by $n = n(G)$ and $m = m(G)$. For disjoint subsets A and B of vertices we denote by $E(A, B)$ the set of edges between A and B. The *open* and *closed neighborhoods* of a vertex $v \in V$ are $N_G(v) = \{u \in V | uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Also the *open* and *closed neighborhoods* of a subset $S \subseteq V(G)$ are $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$, respectively. The *degree* of a vertex $v \in V$ is $deg_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $S \subseteq V(G)$, the *induced subgraph* $G[S]$ is the subgraph of G with the vertex set S and for two vertices $u, v \in S$, $uv \in E(G[S])$ if and only if $uv \in E(G)$. We write $K_{p,q}$ for the *complete bipartite graph* with bipartition X and Y such $|X| = p$ and $|Y| = q$. If $\omega(G)$ is the number of components of G, then $c(G) = m - n + \omega(G)$ is the well-known *cyclomatic number* of G.

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A subset $S \subseteq V(G)$ is a *k-dominating set* of G if $|N_G(v) \cap S| \geq k$ for every vertex in $V \setminus S$. The *k-domination number* $\gamma_k(G)$ is minimum cardinality among the k-dominating sets of G. The concept of k-domination was introduced by Fink and Jacobson in [4]. If $k = 1$, then the k-domination number is the classical *domination number.*

Let $f : V(G) \to \{0, 1, 2\}$ be a function and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) | f(v) = i\}$ for $i = 0, 1, 2$. We notice that there is an obvious one-to-one correspondence between f and the ordered partition (V_0, V_1, V_2) of $V(G)$. Therefore one can write $f = (V_0, V_1, V_2)$. Let $k \geq 1$ be an integer. The function $f = (V_0, V_1, V_2)$ is a *Roman k-dominating function*, abbreviated RkDF, on G, if $|N_G(v) \cap V_2| \geq k$ for every $v \in V_0$. The *weight* of f is the value $f(V(G)) = \sum_{v \in V(G)} f(v) = |V_1| + 2|V_2|$. The *Roman k-domination number* $\gamma_{kR}(G)$ is the minimum weight of an RkDF on G, and we say that a function $f = (V_0, V_1, V_2)$ is a *$\gamma_{kR}(G)$-function* if it is an RkDF on G and $f(V(G)) = \gamma_{kR}(G)$. The Roman k-domination number was introduced by Kämmerling and Volkmann in [6], and it has been studied, for example in [1, 7].

If $k = 1$, then the Roman k-domination number is called *Roman domination number* denoted by $\gamma_R(G)$, which was given implicitly by Steward in [9] and by ReVelle and Rosing in [8]. More details on Roman domination have been given in many papers, see for example [2, 3, 9].

In this paper, we are interested in studying the effects that a graph modification has on the Roman k-domination number. More precisely, we first study the changes of the Roman k-domination number upon the removal of any vertex. Then, we study the changes of the Roman k-domination number upon the removal of any edge.

2 Main results

In this section, we investigate the effects of vertex and edge removal on the Roman k-domination number and we present lower and upper bounds on the Roman k-domination number in graphs.

Lemma 2.1. *Let G be a graph of order $n \geq 2$. If v is a vertex of G, then*

$$\gamma_{kR}(G) \leq \gamma_{kR}(G - v) + 1.$$

Proof. If $f = (V_0, V_1, V_2)$ is a $\gamma_{kR}(G - v)$-function, then $g = (V_0, V_1 \cup \{v\}, V_2)$ is an RkDF on G and therefore $\gamma_{kR}(G) \leq \gamma_{kR}(G - v) + 1$. □

Corollary 2.2. *Let G be a graph of order $n \geq 2$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{kR}(G)$-function. If $v \in V_1$, then*

$$\gamma_{kR}(G - v) = \gamma_{kR}(G) - 1.$$

Proof. Since $g = (V_0, V_1 - \{v\}, V_2)$ is an RkDF on $G - v$, we deduce that $\gamma_{kR}(G - v) \leq |V_1 - \{v\}| + 2|V_2| = \gamma_{kR}(G) - 1$. According to Lemma 2.1, $\gamma_{kR}(G) \leq \gamma_{kR}(G - v) + 1$ and thus $\gamma_{kR}(G - v) = \gamma_{kR}(G) - 1$. □

Proposition 2.3. *Let G be a graph of order $n \geq 2$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_{kR}(G)$-function. If $v \in V_0$, then*

$$\gamma_{kR}(G - v) \leq \gamma_{kR}(G).$$

Proof. If we define $g = (V_0 - \{v\}, V_1, V_2)$, then g is an RkDF on $G - v$, and thus $\gamma_{kR}(G-v) \leq g(V(G-v)) = |V_1| + 2|V_2| = \gamma_{kR}(G)$. □

Theorem 2.4. *Let G be a graph of order n and $uv \in E(G)$. Then*

$$\gamma_{kR}(G) \leq \gamma_{kR}(G - uv) \leq \gamma_{kR}(G) + 1.$$

Proof. If g is a $\gamma_{kR}(G-uv)$-function, then g is an RkDF on G and thus $\gamma_{kR}(G) \leq \gamma_{kR}(G-uv)$. Now let $f = (V_0, V_1, V_2)$ be a $\gamma_{kR}(G)$-function. If $uv \in E(G[V_0])$, $uv \in E(G[V_1])$, $uv \in E(G[V_2])$, $uv \in E(V_1, V_2)$ or $uv \in E(V_0, V_1)$, then f is an RkDF on $G - uv$ and hence $\gamma_{kR}(G-uv) \leq \gamma_{kR}(G)$. Thus $\gamma_{kR}(G-uv) = \gamma_{kR}(G)$ in these cases. Let now $uv \in E(V_0, V_2)$. Without loss of generality, suppose that $f(u) = 0$. Then $g = (V_0 \setminus \{u\}, V_1 \cup \{u\}, V_2)$ is an RkDF on $G - uv$, and so $\gamma_{kR}(G - uv) \leq \gamma_{kR}(G) + 1$. □

Theorem 2.5. *[6] Let G be a graph of order n. If $k \geq 2$, then*

$$\gamma_{kR}(G) \geq \min\{n, n + 1 - c(G)\}.$$

Next we improve the lower bound in Theorem 2.5 for any graph of order n and $k \geq 3$.

Theorem 2.6. *Let G be a graph of order n. If $k \geq 2$ is an integer, then*

$$\gamma_{kR}(G) \geq \min\{n, n + k^2 - k - 1 - c(G)\}.$$

Theorem 2.7. *Let $k \geq 2$ be an integer, and let G be a graph of order n. If $\gamma_{kR}(G) \leq n-1$, then*

$$\gamma_{kR}(G) \leq \gamma_R(G) + (k-1)\left(n - \frac{3k}{2}\right).$$

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Roman entire domination in graphs

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Abstract

A *Roman entire dominating function* on a graph $G = (V, E)$ is a function $h : Z = V \cup E \to \{0, 1, 2\}$ satisfying the condition that each element $x \in Z$ for which $h(x) = 0$ is either adjacent to or incident with at least one element $y \in Z$ with $h(y) = 2$. The weight of a Roman entire dominating function is the value $w(h) = \sum_{x \in Z} h(x)$. The *Roman entire domination number* of a graph G, denoted by $\gamma_{ren}(G)$, is the minimum weight of a Roman entire dominating function on G. In this paper, we obtain several bounds for $\gamma_{ren}(G)$. We also investigate the behavior of $\gamma_{ren}(G)$ when a vertex or an edge is deleted.

Keywords: Dominating set, Entire dominating set, Roman dominating function, Roman entire dominating function.
Mathematics Subject Classification [2010]: 05C69.

1 Introduction

Cockayne et al. [3] introduced the concept of Roman dominating function (RDF) (See also [2, 4, 6]). A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v such that $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_r(G)$, is the minimum weight of a Roman dominating function on G.
A Roman edge dominating function (REDF) on a graph $G = (V, E)$ is a function $g : E \to \{0, 1, 2\}$ satisfying the condition that every edge e_1 for which $g(e_1) = 0$ is adjacent to at least one edge e_2 such that $g(e_2) = 2$. The weight of a Roman edge dominating function is the value $w(g) = \sum_{e \in E} g(e)$. The Roman edge domination number of a graph G, denoted by $\gamma_{re}(G)$, is the minimum weight of a Roman edge dominating function on G. This concept was studied by Soner et al. in [7].
In this paper, we introduce the concept of Roman entire dominating function and initiate a study of the Roman entire domination number.

*Speaker

2 Basic Results

Theorem 2.1. *Let $h = (Z_0, Z_1, Z_2)$ be a γ_{ren}-function of G. Then the following are true.*

(i) No element of Z_1 is adjacent to an element of Z_2.

(ii) The set $V \cap Z_1$ is independent.

(iii) Each element of Z_0 is adjacent to at most two elements of Z_1.

(iv) Z_2 is a γ_{en}-set of the induced subgraph $H = \langle Z_0 \cup Z_2 \rangle$.

(v) Each $x \in Z_2$ has at least two Z_2-private neighbors in H.

(vi) If x is isolated in $\langle Z_2 \rangle$ and has precisely one external Z_2-private neighbor $y \in Z_0$ in H, then $N(y) \cap Z_1 = \emptyset$.

Proposition 2.2. *Let G be a graph without isolated vertices and let $h = (Z_0, Z_1, Z_2)$ be a γ_{ren}-function of G such that $|Z_1|$ is minimum. Then*

(i) Z_1 is independent,

(ii) $Z_0 \succ Z_1$ and

(iii) each element of Z_0 is adjacent to at most one element of Z_1.

Theorem 2.3. *Let G be a graph. Then $\gamma_{en} \leq \gamma_{ren} \leq 2\gamma_{en}$. Further, $\gamma_{en} = \gamma_{ren}$ if and only if $G = K_p^c$. Also, $\gamma_{ren} = 2\gamma_{en}$ if and only if there exist a γ_{ren}-function $h = (Z_0, Z_1, Z_2)$ with $Z_1 = \emptyset$.*

Theorem 2.4. *Let G be any graph. Then $max\{\gamma_r(G), \gamma_{re}(G)\} \leq \gamma_{ren}(G) \leq \gamma_r(G) + \gamma_{re}(G)$.*

Remark 2.5. For the star $G = K_{1,n}$, $n \geq 2$, we have $\gamma_r(G) = \gamma_{re}(G) = \gamma_{ren}(G) = 2$ and hence $\gamma_{ren}(G) = max\{\gamma_r(G), \gamma_{re}(G)\}$. Also for the graph $G = K_4 - e$, we have $\gamma_r(G) = \gamma_{re}(G) = 2$, $\gamma_{ren}(G) = 4$ and hence $\gamma_{ren}(G) = \gamma_r(G) + \gamma_{re}(G)$. Thus the bounds given in Theorem 2.4 are sharp.

The following theorem gives the effect of the removal of a vertex or an edge on $\gamma_{ren}(G)$.

Theorem 2.6. *Let G be any graph with $\gamma_{ren}(G) = k$. Let $v \in V(G)$ and $e \in E(G)$. Then*

(i) $k - 1 \leq \gamma_{ren}(G - e) \leq k + 2$ and

(ii) $k - 2 \leq \gamma_{ren}(G - v) \leq \max\{k, k - 2 + deg(v)\}$.

Proposition 2.7. *Let G be any graph with $\gamma_{ren}(G) = k$, $e \in E(G^c)$. Then $k - 2 \leq \gamma_{ren}(G + e) \leq k + 1$.*

We give sharp lower and upper bounds for the Roman entire domination function of a graph.

Theorem 2.8. *For any graph G with maximum degree $\Delta(G) \geq 1$,*

$$\left\lceil \frac{p+q+\gamma_{en}(G)}{\Delta(G)+1} \right\rceil \leq \gamma_{ren}(G).$$

The bound of Theorem 2.8 is sharp for P_p such that $p \not\equiv 4 \pmod 5$, $K_{1,p-1}$, $(p \geq 2)$, C_p, $3 \leq p \leq 5$ and mK_2.

Theorem 2.9. *For any graph G, $\gamma_{ren}(G) \leq p$ and the bound is sharp.*

3 Roman Entire Domination Number

In this section we determine the value of $\gamma_{ren}(G)$ for several classes of graphs.

Proposition 3.1. *For the path P_p with $p \geq 2$,*

$$\gamma_{ren}(P_p) = \begin{cases} 2\lfloor \frac{2p-1}{5} \rfloor & \textit{if } r = 0\ , \\ 2\lfloor \frac{2p-1}{5} \rfloor + 1 & \textit{if } r = 1\ , \\ 2\lfloor \frac{2p-1}{5} \rfloor + 2 & \text{otherwise.} \end{cases}$$

where $2p - 1 \equiv r \pmod{5}$, $0 \leq r \leq 4$.

Proposition 3.2. *For cycle C_p with $p \geq 3$,*

$$\gamma_{ren}(C_p) = \begin{cases} 2\lfloor \frac{2p}{5} \rfloor & \textit{if } r = 0\ , \\ 2\lfloor \frac{2p}{5} \rfloor + 1 & \textit{if } r = 1\ , \\ 2\lfloor \frac{2p}{5} \rfloor + 2 & \text{otherwise.} \end{cases}$$

where $2p \equiv r \pmod{5}$, $0 \leq r \leq 4$.

Proposition 3.3. *For wheel W_p with $p \geq 4$,*

$$\gamma_{ren}(W_p) = \begin{cases} 4 & \textit{if } p = 4 \text{ or } 5, \\ 2 + \lceil \frac{2(p-1)}{3} \rceil & \text{otherwise.} \end{cases}$$

Proposition 3.4. *For complete bipartite graph $G = K_{m,n}$ with $m \leq n$, $\gamma_{ren}(G) = 2m$.*

Lemma 3.5. *Let $h = (Z_0, Z_1, Z_2)$ be any γ_{ren}-function of the complete graph K_p. Then* $|Z_2 \cap V(K_p)| \leq 1$.

Proposition 3.6. *For the complete graph K_p, $\gamma_{ren}(K_p) = p$.*

Proposition 3.7. *For any graph G of order $p \geq 2$, $\gamma_{ren}(G) = 2$ if and only if G is a star or $G = K_2^c$.*

Theorem 3.8. *Let T be a tree with $p \geq 2$, then $\gamma_{ren}(T) \leq 2\beta_1(T)$. And this bound is sharp for $K_{1,p-1}$, P_4, P_5, P_7, P_9.*

Proposition 3.9. *Let G be any unicyclic graph. Then $\gamma_{ren}(G) \leq 2\beta_1(G) + 1$. Further, the equality holds for K_3.*

Theorem 3.10. *For given any integer $k \geq 0$, there exist a tree T for which $2\beta_1(T) - \gamma_{ren}(T) = k$.*

Theorem 3.11. *Let G be any graph. Then $\gamma_{ren}(G) \leq 2(p - \beta_0(G))$. And this bound is sharp for $K_{1,p-1}$, C_4, P_4, P_5, P_7, P_9.*

Proposition 3.12. *For any graph G of order $p \geq 3$, $\gamma_{ren}(G) = 3$ if and only if G is isomorphic to one of the graphs: K_3^c, $K_{1,p-2} \cup K_1$ or $K_{1,p-1} + \{e\}$.*

Lemma 3.13. *If G is a connected graph and $\gamma_{ren}(G) = \gamma_{en}(G)+1$, then $1 \leq diam(G) \leq 2$.*

Theorem 3.14. *For any connected graph G, $\gamma_{ren}(G) = \gamma_{en}(G) + 1$ if and only if there is a vertex $v \in V(G)$ of degree $p - 1$ and the remaining vertices of degree at most 2.*

Lemma 3.15. *If T is a tree and $\gamma_{ren}(T) = \gamma_{en}(T) + 2$, then $3 \leq diam(T) \leq 5$.*

Theorem 3.16. *If T is a tree of order $p \geq 4$, then $\gamma_{ren}(T) = \gamma_{en}(T) + 2$ if and only if either (i) T is a double star (ii) T is obtained by subdividing the center edge of double star at most twice.*

The following theorem gives the bound of $|Z_0|$, $|Z_1|$ and $|Z_2|$ for a $\gamma_{ren}(G)$-function $h = (Z_0, Z_1, Z_2)$.

Theorem 3.17. *Let $h = (Z_0, Z_1, Z_2)$ be any $\gamma_{ren}(G)$-function of a connected graph G of order p greater than or equal to three. Then*

(i) $1 \leq |Z_2| \leq \frac{p}{2}$.

(ii) $0 \leq |Z_1| \leq p - 2$.

(iii) $q + 1 \leq |Z_0| \leq p + q - 1$.

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Some new families of 2-regular self-complementary k-hypergraphs for $k = 4, 5$

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Abstract

A k-hypergraph with vertex set V and edge set E is called t-regular if every t-element subset of V lies in the same number of elements of E. In this note, we prove the existence of some new families of 2-regular self-complementary k-hypergraphs for k=4,5.

Keywords: k-hypergraph, self-complementary hypergraph, large sets of t-designs
Mathematics Subject Classification [2010]: 05C65, 05B05, 05E20

1 Introduction

A k-*uniform hypergraph* of order v is an ordered pair $H = (V, E)$, where $V = V(H)$ is a v-set (called *vertex set*) and $E = E(H)$ (called *edge set*) is a subset of the set of all k-subsets of V ($P_k(V)$). We call a k-*uniform hypergraph* simply a k-*hypergraph* [4]. A k-hypergraph H of order v is t-*subset-regular* (for short t-*regular*) if there exists a positive integer λ (called the t-*valence* of H), such that each element of $P_t(V)$ is a subset of exactly λ elements of $E(H)$. Henceforth, we denote such a structure by RHG(t, k, v). Two k-hypergraphs H_1 and H_2 are isomorphic, if there is a bijection $\theta : V(H_1) \to V(H_2)$, such that θ induces a bijection from $E(E_1)$ into $E(H_2)$. A k-hypergraph H is called *self-complementary* if H is isomorphic to $H' = (V, P_k(V) \setminus E(H))$. An *antimorphism* of self complementary hypergraph H, is an isomorphism between H and H'. Henceforth, we denote this structure by SRHG(t, k, v). An easy counting argument shows that an SRHG(t, k, v) is also an SRHG(i, k, v) for $0 \le i \le t$. Hence a set of necessary conditions for the existence of an SRHG(t, k, v) is that $\binom{v-i}{k-i}$ is an even integer for all $i = 0, 1, ..., t$. The following theorem gives the necessary conditions in terms of some congruence relations. Let p be a prime number and r and m be positive integers. Then by $r_{[m]}$ we denote the remainder of division r by m and by $r_{(p)}$ we denote the largest integer i such that p^i divides r.

Theorem 1.1. *[2] If there exists an SRHG(t, k, v), then there exists an integer q, where $k_{(2)} < q \le \min\{i \;:\; 2^i \;>\; k\}$ such that $v_{[2^q]} \in \{t, t+1, ..., k_{[2^q]} - 1\}$.*

It should be noted that in [2] the above theorem is stated for large sets of t-designs. We may obtain more hypergraphs from a given hypergraph as the following theorem suggests (see [4]). The proof is clear by successive applying of the above remark.

*Speaker

Theorem 1.2. *If there exists an SRHG(t, k, v) with an antimorphism having at least t fixed points, then there exists SRHG$(t - i, k - j, v - l)$ for all $0 \leq j \leq l \leq i \leq t$.*

2 Some New Partitionable Sets

A powerful method in constructing large sets is obtained from the notion of partitionable sets [1]. In what follows we generalize this method to construct hypergraphs with different parameters.

Let H_1, $H_2 \subseteq P_k(V)$. We say that H_1 and H_2 are *t-equivalent* if every t-subset of V appears in the same number of members of H_1 and H_2. If there exists a partition of $H \subseteq P_k(V)$ into N mutually t-equivalent subsets, then H is called an *(N, t)-partitionable set*. If $H = \{H_1, H_2\}$ is a $(2, t)$-partitionable set such that there is a permutation σ on V which maps H_1 onto H_2, then H is called a *$(\sigma, 2, t)$- partitionable set*.

Let V_1 and V_2 be two disjoint sets and $H_i \subseteq P_{k_i}(V_i)$ for $i = 1, 2$. In what follows we need the following definition:

$$H_1 * H_2 = \{h_1 \cup h_2 | h_1 \in H_1, h_2 \in H_2\}.$$

The following Lemma is a minor revision of a lemma given in [3] in terms of large sets of t-designs. We only need to prove the existence of their corresponding permutations.

Lemma 2.1. *Let V_1 and V_2 be two disjoint sets and let $H_i \subseteq P_{k_i}(V_i)$ for $i = 1, 2$. Also let σ_1 and σ_2 be permutations on V_1 and V_2, respectively. Suppose that H_1 is a $(\sigma_1, 2, t_1)$-patitionable set.*

(i) If H_2 is a $(\sigma_2, 2, t_1)$-patitionable set, then there is a permutation (say σ) on $V_1 \cup V_2$ such that $H_1 * H_2$ is a $(\sigma, 2, t_1)$-partitionable set.

(ii) If H_2 has a partition into two t_2-equivalent sets and σ_2 induces a permutation on each part, then there is a permutation (say σ) on $V_1 \cup V_2$, such that $H_1 * H_2$ is a $(\sigma, 2, t_1 + t_2 + 1)$-partitionable set.

(iii) If H_2 is a $(\sigma_2, 2, t_1)$-patitionable set, then there is a permutation (say σ) on $V_1 \cup V_2$ such that the union of H_1 and H_2 is also a $(\sigma, 2, t_1)$-partitionable set.

3 A Recursive Method

In this section, we present a recursive method to construct SRHG(t, k, v) using $(\sigma, 2, t)$-partitionable sets.

Theorem 3.1. *Assume that there exist SRHG(t, i, v_1) for all $t + 1 \leq i \leq k$ with θ_1 as an antimorphism and also suppose there exists SRHG(t, i, v_2) such that θ_2 be an antimorphism, then an SRHG$(t, k, v_1 + v_2 - t)$ exists.*

Let θ be a permutation on a v-set with at least t fixed points.

Corollary 3.2. *If there exist an SRHG*(t, i, v) *for* $t+1 \leq i \leq k$ *with* θ *as an antimorphism and also there exist SRHG*(t, k, u) *with an antimorphism having at least t fixed points, then there exist SRHG*$(t, k, u + l(v - t))$ *for all* $l \geq 1$.

Corollary 3.3. *If there exist an SRHG*$(t, t + 1, v + t)$ *with an antimorphism having at least t fixed points, then there exist SRHG*$(t, t + 1, lv + t)$ *for all* $l \geq 1$.

4 The existence

In this section we give some existence results on SRHG$(2, k, v)$. At first step note to the following corollary of Theorem 1.1. This corollary presents a necessary condition to the existence of SRHG$(2, k, v)$.

Corollary 4.1. *Suppose that there exists an SRHG*$(2, k, v)$*. Then*

(i) If $k = 4$*, then* $v \equiv 2, 3 \pmod 8$*;*

(ii) If $k = 5$*, then* $v \equiv 2, 3, 4 \pmod 8$*;*

Now we show that the necessary conditions for the existence of SRHG$(2, k, v)$ for $k = 4, 5$ are sufficient.

Theorem 4.2. *There exist an SRHG*$(2, 4, v)$ *if and only if* $v \equiv 2, 3 \pmod 8$.

Theorem 4.3. *There exist an SRHG*$(2, 5, v)$ *if and only if* $v \equiv 2, 3, 4 \pmod 8$.

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Some Remarks of bipolar fuzzy graphs

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Abstract

In this paper, we discussed some properties of the $\mu-$complement of bipolar fuzzy graphs. Busy vertices and free vertices in bipolar fuzzy graphs are introduced and their image under an isomorphism are studied. Finally, we investigated some properties of isomorphism on bipolar fuzzy graphs.

Keywords: Bipolar fuzzy graphs, $\mu-$complement, busy vertex and free vertex
Mathematics Subject Classification [2010]: 05C99

1 Introduction

Presently, science and technology is featured with complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models is based on an extension of the ordinary set theory, namely, fuzzy sets. Graph theory has numerous application to problem in computer science, electrical engineering, system analysis, operations research, economics, networking routing, and transportation. In 1965 Zadeh [10] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. In 1975, Rosenfeld [4] introduced the notion of fuzzy graphs and proposed another definitions including paths, cycles, connectedness and etc. The complement of a fuzzy graph was defined by Mordeson and Nair [3] and further studied by Sunitha and Kumar [9].

In 1994, Zhang initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. The first definition of bipolar fuzzy graphs was proposed by Akram [1]. Rashmanlou et al. [2, 5, 6, 7] investigated bipolar fuzzy graphs with categorical properties, product of bipolar fuzzy graphs and their degree, domination in vague graphs and a study on bipolar fuzzy graphs.

*Speaker

2 Main result

Let X be a non-empty set. A bipolar fuzzy set B in X is an object having the form $B = \{(x, \mu_B^P(x), \mu_B^N(x)) \mid x \in X\}$, where $\mu_B^P : X \to [0,1]$ and $\mu_B^N : X \to [-1,0]$ are mappings. Let X be a non-empty set. Then we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0,1] \times [-1,0]$ a bipolar fuzzy relation on X such that $\mu_A^P(x,y) \in [0,1]$ and $\mu_A^N(x,y) \in [-1,0]$.

Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be bipolar fuzzy sets on a set X. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy relation on a set X, then $A = (\mu_A^P, \mu_A^N)$ is called a bipolar fuzzy relation on $B = (\mu_B^P, \mu_B^N)$ if $\mu_A^P(x,y) \le min(\mu_B^P(x), \mu_B^P(y))$ and $\mu_A^N(x,y) \ge max(\mu_B^N(x), \mu_B^N(y))$ for all $x, y \in X$.

Definition 2.1. By a bipolar fuzzy graph $G = \langle V, E, A, B \rangle$ of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set on V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on E such that
$\mu_B^P(xy) \le \min(\mu_A^P(x), \mu_A^P(y))$ and $\mu_B^N(xy) \ge \max(\mu_A^N(x), \mu_A^N(y))$ for all $xy \in E$.

Definition 2.2. Let G_1 and G_2 be two bipolar fuzzy graphs. A homomorphism f from G_1 to G_2 is a mapping $f : V_1 \to V_2$ which satisfies the following conditions:
(a) $\mu_{A_1}^P(x_1) \le \mu_{A_2}^P(f(x_1)), \mu_{A_1}^N(x_1) \ge \mu_{A_2}^N(f(x_1))$,
(b) $\mu_{B_1}^P(x_1y_1) \le \mu_{B_2}^P(f(x_1)f(y_1)), \mu_{B_1}^N(x_1y_1) \ge \mu_{B_2}^N(f(x_1)f(y_1))$ for all $x_1, y_1 \in V_1, x_1y_1 \in E_1$.

Definition 2.3. Let G_1 and G_2 be two bipolar fuzzy graphs. An isomorphism f from G_1 to G_2 is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:
(c) $\mu_{A_1}^P(x_1) = \mu_{A_2}^P(f(x_1)), \mu_{A_1}^N(x_1) = \mu_{A_2}^N(f(x_1))$,
(d) $\mu_{B_1}^P(x_1y_1) = \mu_{B_2}^P(f(x_1)f(y_1)), \mu_{B_1}^N(x_1y_1) = \mu_{B_2}^N(f(x_1)f(y_1))$
for all $x_1, y_1 \in V_1, x_1y_1 \in E_1$.

Definition 2.4. Let G_1 and G_2 be two bipolar fuzzy graphs. Then, a weak isomorphism f from G_1 to G_2 is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:
(e) f is homomorphism
(f) $\mu_{A_1}^P(x_1) = \mu_{A_2}^P(f(x_1)), \mu_{A_1}^N(x_1) = \mu_{A_2}^N(f(x_1))$, for all $x_1 \in V_1$. Thus a weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.

Theorem 2.5. *Let G_1 and G_2 be bipolar fuzzy graphs. If $G_1 \cong G_2$, an arc in G_1 is strong if and only if the corresponding image arc in G_2 is also strong.*

Theorem 2.6. *Let G_1 and G_2 be bipolar fuzzy graphs and G_1 be isomorphic to G_2. Then G_1 is connected if and only if G_2 is connected.*

Definition 2.7. Let $G = (A, B)$ be a bipolar fuzzy graph. The $\mu-$complement of G is denoted by $G^\mu = (A^\mu, B^\mu)$, where $A^\mu = A$, $B^\mu = (\mu^\mu_{B^P}, \mu^\mu_{B^N})$ and

$$\mu^\mu_{B^P}(xy) = \begin{cases} \mu_A^P(x) \wedge \mu_A^P(y) - \mu_{B^P}(xy) & \text{if } \mu_{B^P}(xy) > 0 \\ 0 & \text{if } \mu_{B^P}(xy) = 0, \end{cases}$$

$$\mu^\mu_{B^N}(xy) = \begin{cases} \mu_A^N(x) \vee \mu_A^N(y) - \mu_{B^N}(xy) & \text{if } \mu_{B^N}(xy) < 0 \\ 0 & \text{if } \mu_{B^N}(xy) = 0. \end{cases}$$

Several properties have been investigated for this graph.

Proposition 2.8. *Let G_1 and G_2 be bipolar fuzzy graphs, if G_1 and G_2 are isomorphic, then their $\mu-$complements, G_1^{μ} and G_2^{μ}, are also isomorphic.*

Theorem 2.9. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 \cap V_2 = \phi$. Then, $(G_1 + G_2)^{\mu} \cong G_1^{\mu} \cup G_2^{\mu}$.*

Theorem 2.10. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 \cap V_2 = \phi$. Then, $(G_1 \cup G_2)^{\mu} \cong G_1^{\mu} \cup G_2^{\mu}$.*

Definition 2.11. The busy value of a node v of a bipolar fuzzy graph $G = (A, B)$ is defined to be $D(v) = (D_P(v), D_N(v))$ where $D_P(v) = \sum_i \mu_{A^P}(v) \wedge \mu_{A^P}(v_i)$ and $D_N(v) = \sum_i \mu_{A^N}(v) \vee \mu_{A^N}(v_i)$ which v_i are neighbors of v and the busy value of a bipolar fuzzy graph G is defined to be the sum of the busy values of all vertices of G, i.e. $D(G) = \sum_i D(v_i)$ where v_i are vertices of G.

Definition 2.12. A vertex v of a bipolar fuzzy graph $G = (A, B)$ is said to be
(i) a partial free vertex if it is a free vertex in both G and G^{μ}.
(ii) a fully free node if it is a free vertex in G, but it is a busy vertex in G^{μ}.
(iii) a partial busy vertex if it is a busy vertex in both G and G^{μ}.
(iv) a fully busy vertex if it is a busy vertex in G, but it is a free vertex in G^{μ}.

Lemma 2.13. *Let $G_1 \cong G_2$ and h be an isomorphism from G_1 to G_2. Then* $\deg(x) = \deg(h(x))$ *for all $x \in V$.*

Theorem 2.14. *If $G_1 \cong G_2$ and if v is a busy vertex in G_1, then it is a busy vertex in G_2 also.*

Theorem 2.15. *Let a bipolar fuzzy graph G_1 be weak isomorphism to G_2. If $u \in V_1$ is a busy vertex in G_1, then its image under a weak isomorphism in G_2 is also busy.*

Theorem 2.16. *For any two isomorphism bipolar fuzzy graphs, their order and size are same.*

Theorem 2.17. *If the bipolar fuzzy graphs be co-weak isomorphism then, their size are same. But, if the bipolar fuzzy graphs are of same size need not to be co-weak isomorphic.*

Theorem 2.18. *If G_1 and G_2 be isomorphic bipolar fuzzy graphs then, the degrees of their vertices are preserved*

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SOME RESULT ABOUT RELATIVE NON-COMMUTING GRAPH

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Abstract

In this paper we define the relative non-commuting graph $\Gamma_{H,G}$ where G is a non-ablian group and H a subgroup of G . We obtain upper bounds for diameter and girth of this graph. We discuss about dominating set and planarity of $\Gamma_{H,G}$. Moreover, we explain a connection between $\Gamma_{H,G}$ and the commutativity degree of G. Furthermore, we prove that if (H_1, G_1) and (H_2, G_2), are relative isoclinic then $\Gamma_{H_1,G_1} \cong \Gamma_{H_2,G_2}$ under special condition. consequent, we discuss about the energy of $\Gamma_{H,G}$ in some special cases. Finally we compute the number of spanning trees for some certain groups .

Keywords: non-commuting graph; non-ablian group; commutativity degree ;relative isoclinism

1 Introduction

Study of algebraic structures, by using the properties of graphs, becomes an exciting research topic in the last twenty years. This fact leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and investigation of algebraic properties of ring or group using the associated graph, for instance see [1, 3]. A simple graph Γ_G is associated to a group G, whose vertex set is $G \setminus Z(G)$ and the edge set is all pairs (x, y), where x and y are distinct non-central elements such that $[x, y] = x^{-1}y^{-1}xy \neq 1$. This graph the non-commuting graph of G and was introduced by Erdös and by asking whether there is a finite bound for the cardinalities of cliques in Γ_G, if Γ_G has no infinite clique. This problem was posed by Neumann in [8] and a positive answer was given to Erdös question. In the next section, after introducing the relative non-commuting graph $\Gamma_{H,G}$, we state some of basic graph theoretical properties of $\Gamma_{H,G}$ which are mostly new or a generalization of some results in [2],

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2 THE RELATIVE NON COMMUTING GRAPH

Definition 2.1. the relative non-commuting graph $\Gamma_{H,G}$ where G is a non-ablian group and H a subgroup of G . Take $G \backslash C_G(H)$ as the vertices of the graph and two distinct vertices x and y join,whenever x or y in H and $[x, y] \neq 1$.

Theorem 2.2. *For non-abelian group G ,and its subgroup H with trivial center,* $\text{diam}(\Gamma_{H,G}) = 2$. *Also* $\text{girth}(\Gamma_{H,G}) = 3$

Theorem 2.3. *Let H be a subgroup of non-abelian group G. If x is a dominating set for $\Gamma_{H,G}$, then $C_G(H) = 1$, $x^2 = 1$ and $C_G(x) =< x >$, where x is a non-trivial element of H.*

Lemma 2.4. *Let H be a subgroup of non-abelian group G then $S = HC_G(H) - C_H(G)$ is a dominating set for $\Gamma_{H,G}$.*

For any finite group G, the commutativity degree of G, denoted by $d(G)$ is the probability that two randomly chosen elements of G commute with each other [6]. It can be defined az the following ratio:

$$d(G) = \frac{1}{|G|^2} |\{(x, y) \in G \times G : [x, y] = 1\}|.$$

Similarly if H is the subgroup of G then the relative commutativity degree of H in G is defined as follows

$$d(H, G) = \frac{1}{|H||G|} |\{(h, g) \in H \times G : [h, g] = 1\}|.$$

It is clear that if G is abelian or H is central subgroup then $d(H, G) = 1$. There are many results concerning the above degrees in series of papers for instance see [8]. What we would like to mention in this section is to establish some relations between commutativity degrees $d(G)$, $d(H, G)$ and the graphs Γ_G and $\Gamma_{H,G}$ for non-abelian group G.

Lemma 2.5. *Let H be a subgroup of non-abelian group G . Then the number of edges for the relative non-commuting graph is obtained by,*

$$|E(\Gamma_{H,G})| = |H||G|(1 - d(H, G)) - \frac{|H|^2}{2}(1 - d(H)). \quad (1)$$

Example 2.6. (i) Suppose $G = D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1} \rangle$ is the dihedral group of order 8 and $H = \{1, a, a^2, a^3\}$ Obviously $V(\Gamma_{H,G}) = \{a, a^3, b, ab, a^2b, a^3b\}$, $d(H, G) = \frac{3}{4}$ and $d(H) = 1$, $|E(\Gamma_{H,G})| = 8$.

(ii) Suppose $G = D_{10} = \langle a, b : a^5 = b^2 = 1, a^b = a^{-1} \rangle$ is the dihedral group of order 10,and $H = \{1, b\}$ by a simple computing we have $V(\Gamma_{H,G}) = \{a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$, $d(H, G) = \frac{3}{5}$, $d(H) = 1$ and $|E(\Gamma_{H,G})| = 8$.

(iii) Let $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ be the symmetric group on 3 symbols and $H = \{1, (1\ 2)\}$ $V(\Gamma_{H,G}) = \{(1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. It is clear that again $d(H, G) = \frac{2}{3}$, $d(H) = 1$ and $|E(\Gamma_{H,G})| = 4$.

Theorem 2.7. *If $\Gamma_{H_1,G_1} \cong \Gamma_{H_2,G_2}$, $|H_1\backslash Z(H_1)| = |H_2\backslash Z(H_2)|$ and Γ_{H_1,G_1} has a vertex of degree p, where p is an odd prime, then $H_1 \cong H_2$ or $|H_1| = |H_2|$.*

We convent that, if (H_1, G_1) and (H_2, G_2) are relative 1-isoclinic, then denote it by abbreviate form $(H_1, G_1) \sim (H_2, G_2)$ and called relative isoclinism. Furthermore, if $H_i = G_i$ and $n = 1$ then we obtain isoclinism.

Theorem 2.8. *Let $H_i \leq G_i, (i = 1, 2)$ and $(H_1, G_1) \sim (H_2, G_2)$ be relative isoclinic. If $|Z(G_1) \cap H_1| = |Z(G_2) \cap H_2|$ and $|Z(G_1)| = |Z(G_2)|$ then $\Gamma_{H_1,G_1} \cong \Gamma_{H_2,G_2}$.*

Now,let us start to discuss about the concept of energy graph [4, Section 3.4] and adjacency matrix [4, chapter 3] of the $\Gamma_{H,G}$ in special case.

Remark: For any graph G the energy of the graph is defined as $\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of the adjacency matrix of G

Example 2.9. (i) Let $D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1}\rangle$ be dihedral group of order 8. $V(\Gamma_{H,G}) = \{a, a^3, b, ab, a^2b, a^3b\}$, Similarly, $|E(\Gamma_{H,G})| = 8$. The following matrix is the adjacency matrix of $\Gamma_{H,G}$,

$$A=\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, we obtain the eigenvalues of the adjacency matrix $\lambda_1 = 2.82$, $\lambda_2 = -2.82$, $\lambda_3 = 0$, $\lambda_4 = 0$, $\lambda_5 = 0 and$ $\lambda_6 = 0$. Hence $\varepsilon(G) = \sum_{i=1}^{6} |\lambda_i| = 5.64$.

(ii) Suppose $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ be the symmetric of order 6 , it is clear that $V(\Gamma_{H,G}) = \{(1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$,which implies that , $|E(\Gamma_{H,G})| = 4$. The following matrix is the adjacency matrix of $\Gamma_{H,G}$,

$$A=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, we obtain the eigenvalues of the adjacency matrix $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 0$, $\lambda_4 = 0$ and $\lambda_5 = 0$. Hence $\varepsilon(G) = \sum_{i=1}^{5} |\lambda_i| = 4$.

Now,let us start to discuss about the concept of spaning tree [4, Theorem 4.11] and laplacian matrix [4, chapter 4] of the $\Gamma_{H,G}$ in special case.

Example 2.10. (i) Let $D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1}\rangle$ be dihedral group of order 8. Similarly $V(\Gamma_{H,G}) = \{a, a^3, b, ab, a^2b, a^3b\}$, The following matrix is the laplacian matrix of $\Gamma_{H,G}$,

$$L=\begin{pmatrix} 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Now, we obtain the eigenvalues of the laplacian matrix $\lambda_1 = 6, \quad \lambda_2 = 4 \quad , \quad \lambda_3 = 2, \quad \lambda_4 = 2, \quad \lambda_5 = 2 and \quad \lambda_6 = 0$. Thus the number of spaning tree equal 192.

(ii) Suppose $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ be the symmetric of order 6 , more over $V(\Gamma_{H,G}) = \{(1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. The following matrix is the laplacian matrix of $\Gamma_{H,G}$,

$$L=\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, we obtain the eigenvalues of the laplacian matrix $\lambda_1 = 5, \quad \lambda_2 = 1, \quad \lambda_3 = 1, \quad \lambda_4 = 1 and \quad \lambda_5 = 0$. Thus the number of spaning tree equal 5.

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Talk Some results on the annihilator graph of a commutative ring pp.: 1–4

Some results on the annihilator graph of a commutative ring

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Abstract

Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The annihilator graph of R is defined as the undirected graph $AG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. In this talk, some relations between annihilator graph and zero-divisor graph associated with a commutative ring are studied. Moreover, we give some conditions under which the annihilator graph and the zero-divisor graph associated with a ring are identical.

Keywords: Annihilator graph, Zero-divisor graph, Associated prime ideal
Mathematics Subject Classification [2010]: 13A15, 13B99, 05C99

1 Introduction

Recently, a major part of research in algebraic combinatorics has been devoted to the application of graph theory and combinatorics in abstract algebra. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory, see for example [1, 2, 3, 5] and [7].

Throughout this talk, all rings are assumed to be non-domain commutative rings with identity. We denote by $Nil(R)$ and $Z(R)$, the set of all nilpotent elements and the set of zero-divisors elements of R, respectively. Let $A \subseteq R$. The set of annihilators of A is denoted by $ann_R(A)$ and by A^*, we mean $A \setminus \{0\}$. The ring R is said to be *reduced*, if $Nil(R) = 0$. A prime ideal P of R is called an *associated prime ideal*, if $ann_R(x) = P$, for some non-zero element $x \in R$. The set of all associated prime ideals of R is denoted by $Ass(R)$.

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. The girth of a graph G is denoted by $gr(G)$. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Also G is called a *null graph* if it has no edge. For a vertex x in G,

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we denote the set of all vertices adjacent to x by $N_G(x)$. A complete bipartite graph of part sizes m, n is denoted by $K^{m,n}$. If $m = 1$, then the complete bipartite graph is called *star graph*. Also, a complete graph of n vertices is denoted by K^n.

Any undefined notation or terminology which we use in this talk may be found in [4, 8, 9].

The *annihilator graph* of a ring R is defined as the graph $AG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. This graph was first introduced and investigated in [5] and many of interesting properties of annihilator graph were studied. For example, it was proved the annihilator graph is a connected graph of diameter at most 2. Also, the author in [5], studied some relations between two graphs $AG(R)$ and $\Gamma(R)$, where $\Gamma(R)$ is the zero-divisor graph of a ring R. The *zero- divisor graph* of a ring R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this talk, we continue the study of annihilator graphs associated with commutative rings. Especially, we focus on the conditions under which the annihilator graph is identical to the zero-divisor graph. For instance, for a non-reduced ring R, it is proved that the annihilator graph and the zero-divisor graph of R are identical to the join of a complete graph and a null graph if and only if $ann_R(Z(R))$ is a prime ideal if and only if R has at most two associated primes.

2 Main results

We begin with the following lemma.

Lemma 2.1. *Let R be a ring.*

(1) *Let x, y be distinct elements of $Z(R)^*$, and suppose that $Z(R) = ann_R(x) \cup ann_R(y)$. Then $x - y$ is an edge of $\Gamma(R)$ if and only if $x - y$ is an edge of $AG(R)$.*

(2) *Let x, y, z be elements of $Z(R)^*$, and suppose that $ann_R(x) = ann_R(y)$. Then $x - z$ is an edge of $AG(R)$ if and only if $y - z$ is an edge of $AG(R)$.*

(3) *Let $\Gamma(R) = K^{1,n}$ for some $n \geq 1$ such that x is adjacent to every other vertex. If $ann_R(x) = ann_R(y)$ for some $y \in Z(R)^*$, then either $x = y$, or $\Gamma(R) = AG(R) = K^{1,1}$.*

By using Lemma 2.1, we provide a simple proof of [5, Theorem 3.17].

Theorem 2.2. ([5, Theorem 3.17]) *Let R be a commutative ring such that $AG(R) \neq \Gamma(R)$. Then the following statements are equivalent:*

(1) *$\Gamma(R)$ is a star graph;*

(2) *$\Gamma(R) = K^{1,2}$;*

(3) *$AG(R) = K^3$.*

Proof. Since $AG(R) \neq \Gamma(R)$, (3) $\Rightarrow$ (1) and (3) $\Leftrightarrow$ (2) are obvious. We have only to prove (1) $\Rightarrow$ (3). Let a be the center of the star graph $\Gamma(R)$. Since $\Gamma(R)$ is a star graph and $AG(R) \neq \Gamma(R)$, we deduce that $|Z(R)^*| \geq 3$ and $ann_R(x) = ann_R(y) = \{0, a\}$, for every $x, y \in Z(R) \setminus \{0, a\}$. Furthermore, by [3, Theorem 2.5] and [5, Theorem 3.6], $Z(R) = ann_R(a)$ for a non-zero element $a \in R$. To complete the proof, we show that $|Z(R)^*| = 3$. Suppose to the contrary, a, b, c, x are distinct elements of $Z(R)^*$. With no loss of generality, one may assume that $b - x$ is an edge of $AG(R)$ ($AG(R) \neq \Gamma(R)$). Since

$ann_R(b) = ann_R(c)$, Part (2) of Lemma 2.1 implies that $c - x$ is also an edge of $AG(R)$. Similarly, the equality $ann_R(c) = ann_R(x)$ shows that $c - b$ is an edge of $AG(R)$. Since $bx \neq 0$ and $ann_R(bx) \neq ann_R(b) \cup ann_R(x)$, we have $ann_R(bx) = ann_R(a)$. By Part (3) of Lemma 2.1, $bx = a$. Similarly, $cx = a$ and $cb = a$. Hance $x(b-c) = b(c-x) = c(b-x) = 0$ and so $b - x = c - x = b - c = a$, a contradiction.

□

To prove Theorem 2.5, the following lemma is needed.

Lemma 2.3. *Let R be a ring and $x \in Z(R)^*$. Then*
(1) *If $ann_R(x)$ is a prime ideal of R, then $N_{\Gamma(R)}(x) = N_{AG(R)}(x)$.*
(2) *If $x \in Nil(R)^*$ and $N_{\Gamma(R)}(x) = N_{AG(R)}(x)$, then $ann_R(x)$ is a prime ideal of R.*

In light of Lemma 2.3, we have the following corollary.

Corollary 2.4. *Let R be a ring. If $\Gamma(R) = AG(R)$, then for every $x \in Nil(R)^*$, $ann_R(x) \in Ass(R)$.*

Theorem 2.5. *Let R be a ring such that for every edge of $AG(R)$, say $x - y$, either $ann_R(x) \in Ass(R)$ or $ann_R(y) \in Ass(R)$. Then $\Gamma(R) = AG(R)$.*

Let R be a Noetherian ring and $\Sigma = \{ann_R(x) \,|\, 0 \neq x \in R\}$. Recall that the set of all maximal elements of Σ (under $\subseteq$) is a subset of $Ass(R)$. We set $\Sigma^* = \Sigma \setminus \{(0)\}$. Now, we are ready to present the following result.

Corollary 2.6. *Let R be a ring. If $\Sigma^* = Ass(R)$, then $\Gamma(R) = AG(R)$.*

We finish this talk with the following result.

Theorem 2.7. *Let R be a non-reduced ring. Then the following statements are equivalent:*
(1) *$\Gamma(R) = AG(R) = K^n \vee \overline{K}^m$, where $n = |Nil(R)^*|$ and $m = |Z(R) \setminus Nil(R)|$;*
(2) *$ann_R(Z(R))$ is a prime ideal of R;*
(3) *$\Sigma^* = Ass(R)$ and $|\Sigma^*| \leq 2$.*

Proof. (1) ⇒ (2) With no loss of generality, one may assume that $m \neq 0$. Since $\Gamma(R) = K^n \vee \overline{K}^m$, every vertex of K^n is adjacent to all other vertices of $\Gamma(R)$ and there is no edge between vertices of $\overline{K}^m$. Thus $ann_R(Z(R)) = V(K^n) \cup \{0\}$, $xy \neq 0$ and $ann_R(x) = ann_R(y) = ann_R(Z(R))$, for every $x, y \in V(\overline{K}^m)$. Now, we show that $ann_R(Z(R))$ is a prime ideal of R. To see this, let $xy \in ann_R(Z(R))$, $x \notin ann_R(Z(R))$ and $y \notin ann_R(Z(R))$. Thus $x \neq y$, and hence $Z(R) = ann_R(xy) \neq ann_R(x) \cup ann_R(y) = ann_R(Z(R))$. Therefore, $x - y$ is an edge of $AG(R)$, a contradiction. So, $ann_R(Z(R))$ is a prime ideal of R.

(2) ⇒ (1) Assume that $ann_R(Z(R))$ is a prime ideal of R. Thus $xy = 0$, for all $x, y \in ann_R(Z(R))$, and $xy \neq 0$, for all $x, y \in Z(R) \setminus ann_R(Z(R))$. Now, it is not hard to see that $\Gamma(R)[ann_R(Z(R))^*]$ and $\Gamma(R)[Z(R) \setminus ann_R(Z(R))]$ are two subgraph of $\Gamma(R)$ such that $\Gamma(R)[ann_R(Z(R))^*]$ is complete, $\Gamma(R)[Z(R) \setminus ann_R(Z(R))]$ is null and $\Gamma(R) = \Gamma(R)[ann_R(Z(R))^*] \vee \Gamma(R)[Z(R) \setminus ann_R(Z(R))]$. To complete the proof, we have only to show that $\Gamma(R) = AG(R)$. Let x, y be non-adjacent vertices of $\Gamma(R)$. Then $x, y, xy \in Z(R) \setminus ann_R(Z(R))$. Since $ann_R(Z(R))$ is a prime ideal of R, we conclude that

$\text{ann}(x) = \text{ann}(y) = ann_R(xy) = ann_R(Z(R))$, i.e., x, y are not adjacent in $AG(R)$, as desired.

(2) $\Rightarrow$ (3) Since $ann_R(Z(R))$ is a prime ideal of R, for every $x \in Z(R)^*$, either $ann_R(x) = ann_R(Z(R))$ or $ann_R(x) = Z(R)$. Hence $\Sigma^* = \{ann_R(Z(R)), Z(R)\}$ and so $\Sigma^* = Ass(R)$ and $|\Sigma^*| \leq 2$.

(3) $\Rightarrow$ (2) Let $ann_R(x)$ and $ann_R(y)$ be elements of Σ^*. Since $\Sigma^* = Ass(R)$, by Corollary 2.6, $\Gamma(R) = AG(R)$ and hence it follows from [5, Theorem 3.15] that $Z(R)$ is an ideal of R. This, together with the fact $Z(R) = ann_R(x) \cup ann_R(y)$ imply that either $ann_R(x) \subseteq ann_R(y)$ or $ann_R(y) \subseteq ann_R(x)$. With no loss of generality, suppose that $ann_R(x) \subseteq ann_R(y)$. Thus $Z(R) = ann_R(y)$. Now, we have only to show that $ann_R(x) = ann_R(Z(R))$. We consider the following two cases:

Case 1. Let $a, b \in ann_R(x)$. Then either $ann_R(a) = ann_R(x)$ or $ann_R(a) = Z(R)$. Thus $ab = 0$.

Case 2. Let $a \in ann_R(x)$ and $b \notin ann_R(x)$. Then it is easily seen that $ann_R(b) = ann_R(x)$ and so $ab = 0$.

The proof is complete. $\square$

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Total domination number of a family of graph product

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Abstract

Let $G = (V, E)$ be a simple finite graph and $\gamma_t(G)$ shows the cardinality of the smallest total dominating set, when a total dominating set is a vertex subset such that every vertex is adjacent to at least one vertex of it. In this paper, we study the total domination number of the Cartesian product $P_m \Box C_n$.

Keywords: Cartesian product graph, total domination number, cylindrical grid graphs
Mathematics Subject Classification [2010]: 05C69.

1 Introduction

Let $G = (V, E)$ be a graph with *vertex set* V of *order* $n(G)$ and *edge set* E of *size* $m(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex v is also $deg_G(v) =\mid N_G(v) \mid$. The *minimum* and *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write K_n, P_n and C_n for the *complete graph*, the *path* and the *cycle* of order n, respectively.

The *Cartesian product* $G \Box H$ of two graphs G and H is a graph with $V(G \Box H) = V(G) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or $h_1 = h_2$ and $(g_1, g_2) \in E(G)$. The Cartesian product graph $P_m \Box C_n$ is known as *cylindrical grid graph.* Here, we assume that

$$V(P_m \Box C_n) = \{(i, j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\},$$

and

$$\begin{aligned} E(P_m \Box C_n) &= \{((i, j), (i, j + 1)) \mid 1 \leq i \leq m,\ 1 \leq j \leq n \text{ (to modulo } n)\} \\ &\cup \{((i, j), (i + 1, j)) \mid 1 \leq i \leq m - 1,\ 1 \leq j \leq n\}. \end{aligned}$$

The study of total domination number of graphs was initiated by Cokayne, Dawes and Hedetemini [1]. The literature on this subject has been surveyed in [2]. A subset D of V is called a *total dominating set*, abbreviated TDS, of G if every vertex $x \in V$ is adjacent to at least one vertex of D. The *total domination number* $\gamma_t(G)$ of G is the cardinality of the smallest total dominating set.

Total domination number of Cartesian products of two paths were intensively investigated (see [4, 5, 6]). Here, we study the total domination number of cylindrical grid graphs. The next known results are useful for our investigations.

*Speaker

Proposition 1.1. (**Henning, Kazemi [3] 2010**) *If G is a graph of order n with no isolated vertices, then $\gamma_t(G) \geq \lceil \frac{n}{\Delta(G)} \rceil$.*

Proposition 1.2. (**Klobučar [5] 2004**) *Let $n \neq 6$ be an integer at least 2. Then $\gamma_t(P_5 \Box P_n) = \lfloor \frac{3n+4}{2} \rfloor$.*

2 total domination number of $P_m \Box C_n$, when $m = 2, 3, 4, 5$

Proposition 2.1. *For any integer $n \geq 3$, we have*

$$\gamma_t(P_2 \Box C_n) = \begin{cases} \lceil \frac{2n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod 3 \text{ and } n \neq 7, \\ \lceil \frac{2n}{3} \rceil & \text{otherwise.} \end{cases}$$

Proposition 2.2. *For any $n \geq 3$, we have $\gamma_t(P_3 \Box C_n) = n$.*

Proposition 2.3. *For any integer $n \geq 3$, we have*

$$\gamma_t(P_4 \Box C_n) = \begin{cases} 6\lceil \frac{n}{5} \rceil & \text{if } n \equiv 0, 4, \pmod 5, \\ 6\lceil \frac{n}{5} \rceil - 4 & \text{if } n \equiv 1 \pmod 5, \\ 6\lceil \frac{n}{5} \rceil & \text{if } n \equiv 3 \pmod 5, \text{ and } n \text{ is even}, \\ 6\lceil \frac{n}{5} \rceil - 2 & \text{if } n \equiv 3 \pmod 5, \text{ and } n \text{ is odd}, \\ 6\lceil \frac{n}{5} \rceil - 3 & \text{if } n \equiv 2 \pmod 5 \text{ and } n \text{ is odd}, \\ 6\lceil \frac{n}{5} \rceil - 2 & \text{if } n \equiv 2 \pmod 5 \text{ and } n \text{ is even.} \end{cases}$$

Proposition 2.4. *Let $n \geq 3$ be a positive integer. Then*

$$\gamma_t(P_5 \Box C_n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \pmod 4, \\ \frac{5n}{3} & n = 3, 6. \end{cases}$$

Proposition 2.5. *Let $n \neq 3, 6$ be a positive integer. If $n \equiv r \pmod 4$ and $r \neq 0$, then*

$$6\lfloor \tfrac{n}{4} \rfloor + r - \alpha \leq \gamma_t(P_5 \Box C_n) \leq 6\lfloor \tfrac{n}{4} \rfloor + r + 2,$$

where $\alpha = 0$ if $r = 3$ and $\alpha = 1$ otherwise.

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Twin 2-rainbow dominating sets in graphs

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Abstract

A 2-rainbow dominating function ($2RDF$) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V(G)$ with $f(v)=\emptyset$ the condition $\sum_{u\in N(v)} f(u)=\{1,2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. The weight of a $2RDF$ is the value $w(f)=\sum_{v\in V(G)}|f(v)|$. The 2-rainbow domination number of a graph G, denoted by $\gamma_{r2}(G)$, is the minimum weight of a $2RDF$ of G. In this paper, for a directed graph D we define twin 2-rainbow dominating function in which a vertex of label $\emptyset$ has $\{1,2\}$ both in its in-neighbourhood and its out-neighbourhood. We investigate it for some well-known graphs and then obtain a Nordhaus Gaddum inequality for the twin 2-rainbow domination number. Also, we provide upper bounds on this parameter in terms of the diameter of the graph.

Keywords: 2-rainbow domination, cartesian product, Harary graphs, Petersen graphs, Nordhaus Gaddum inequality
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

For the basic terminology on graphs and digraphs (directed graphs) we refer the reader to [2]. Rainbow domination and other related concepts have been widely studied for undirected graphs, see [1] and [6]. The respective analogues on directed graphs however have not received the same amount of interest.
A function $f : V(G)(\{1,...,k\})$ is called a k-rainbow dominating function (for short $kRDF$) of G if $\sum_{u\in N(v)} f(u)=\{1,...,k\}$ for each vertex $v\in V(G)$ with $f(v)=\emptyset$. By $w(f)$ we mean $\sum_{v\in N(v)}|f(v)|$ and we call it the weight of a k-rainbow dominating function f in G. The minimum weight of a $kRDF$ of G is called the k-rainbow domination number of G and it is designated by $\gamma_{rk}(G)$. An assignment f is called a γ_{rk}-function if it is a $kRDF$ of G and $w(f)=\gamma_{rk}(G)$. For more information about k-rainbow dominating functions consult [3] and [5].
We consider the case $k=2$ in this paper. The 2-rainbow dominating functions are extensively studied in recent literature. Here we define twin 2-rainbow dominating function and study the parameter for complete graphs, paths, cycles, Harary graphs and Petersen graphs. A similar definition, so-called twin dominating function, has been already offered for graphs. Refer to [4].

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Definition 1.1. A twin 2-rainbow dominating function is an assignment of subsets of $\{1,2\}$ to the vertices of G in which a vertex of label $\emptyset$ has $\{1,2\}$ both in its in-neighbourhood and its out-neighbourhood.

In the following some preliminary results are provided to better understand the concept.

Proposition 1.2. *For an arbitrary graph G, $max\{\gamma^+_{r2}, \gamma^-_{r2}\} \leq \gamma^*_{r2} \leq \gamma^+ + \gamma^-$.*

Theorem 1.3. *For directed paths and cycles, $\gamma^*_{r2} = n$.*

Theorem 1.4. *for any graph G, $\gamma^*_{r2} \leq \gamma^*_R$.*

2 Main results

Proposition 2.1. *There is an orientation of a complete graph for which $\gamma^*_{r2} = 4$.*

Proposition 2.2. *For the joint graph of G and K_2, say $G \circ K_2$, there is an orientation for which $\gamma^*_{r2} = 4$.*

Theorem 2.3. *For a graph G of order $n \geq 3$ there exists an orientation D for which $\gamma^*_{r2}(D) = 4$ if and only if G contains $K_{2,n-2}$, $K_{3,n-3}$ or $K_{4,n-4}$ as a spanning subgraph.*

Theorem 2.4. *There exists an orientation of a Petersen graph $P(m,s)$ such that $\gamma^*_{r2} \leq \frac{3}{2}m$ whenever $(m,s) = 1$ and m is even.*

Proposition 2.5. *For a bipartite graph with a minimum degree $\delta \geq 2$ the twin rainbow domination number $\gamma^*_{r2} \leq 8$.*

Theorem 2.6. *Consider a directed Harary graph $H_{4,n}$. Then $\gamma^*_{r2} \geq \lceil \frac{n}{2} \rceil$. Also, there is an orientation of $H_{4,n}$ for which $\gamma^*_{r2} \leq 2\lceil \frac{n}{3} \rceil$.*

Lemma 2.7. *Consider a graph G. Let u and v be two vertices in G that have the maximum number of common neighbours, say k. Then, there exists an orientation D for G such that $\gamma^*_{r2}(D) \leq n - k + 2$.*

Proof. Assign to u and v label $\{1,2\}$ and to their common neighbours $\emptyset$. To all other vertices assign $\{1\}$ or $\{2\}$ and call this function D. Adding up the weights over all vertices gives $\gamma^*_{r2}(D) \leq n - (k+2) + 4 = n - k + 2$. □

In the following a Nordhaus Gaddum inequality is obtained for an arbitrary graph G using the lemma above.

Theorem 2.8. *Assume a graph G. Let u and v be two vertices that have the maximum number k of common neighbours. Let s be the number of non-common neighbours of u and v (except u and v if they are adjacent). Then*

$$\gamma^*_{r2}(G) + \gamma^*_{r2}(\bar{G}) \leq n + s + 6$$

Proof. Assume first that u and v are not adjacent. Then they will have $n-2-(N(u)+N(v)-k)$ common neighbours in $\bar{G}$. So, according to Lemma 2.7,

$\gamma^*_{r2}(\bar{G}) \leq n-(n-2-N(u)-N(v)+k)+2 = n-n+2+N(u)+N(v)-k+2 = N(u)+N(v)-k+4.$

Hence,

$\gamma^*_{r2}(G)+\gamma^*_{r2}(\bar{G}) \leq n-k+2+N(u)+N(v)-k+4 = n+N(u)+N(v)-2k+6 = n+s+6.$

In case that u and v are adjacent, the number of their common neighbours in $\bar{G}$ is $n-2-(N(u)+N(v)-k-2)$. Using Lemma 2.7 again, we obtain

$\gamma^*_{r2}(\bar{G}) \leq n-(n-2-(N(u)+N(v)-k-2)+2 =$
$n-n+2+N(u)+N(v)-k-2+2 = N(u)+N(v)-k+2.$

Replacing this in our inequality gives

$\gamma^*_{r2}(G)+\gamma^*_{r2}(\bar{G}) \leq n-k+2+N(u)+N(v)-k+2 = n+N(u)+N(v)-2k-2+2+4 = n+s+6.$

□

Theorem 2.9. *There exists an orientation of $C_m \Box C_n$ for which the twin rainbow domination number is $\gamma^*_{r2}(C_m \Box C_n) = \frac{mn}{4}$ if m and n are even and $\frac{(m-1)(n-1)}{4}+m+n-1$ if they are odd.*

Proof. Assume that n is an even. Orient every edges on each row forward and every edges on each column downward. Assign sets $\emptyset$ and $\{1\}$ alternatively in odd rows and sets $\{2\}$ and $\emptyset$ alternatively in even rows. If n is odd, we do the same for the first $n-1$ rows unless for the last vertices of odd rows for which we assign $\{1\}$. For the last row, we assign alternatively $\{2\}$ and $\{1\}$ to the first $n-1$ vertices and to the last vertex we assign $\emptyset$. Then this will be a twin rainbow dominating function. □

Theorem 2.10. *For an arbitrary graph whose background is k-regular, $\gamma^*_{r2} \geq \frac{4n}{k+4}$.*

Proof. Let D be a directed graph whose background, G is k-regular. Assume that f is a twin $2RDF$ for D. Also, set $S = \{x \in V(G) | f(x) \neq \emptyset\}$. Obviously, $\forall u \in V(G) \setminus S$, $f(N^+(u)) \geq 2$ and $f(N^-(u)) \geq 2$. Summing up these two inequalities over all vertices out of S gives

$$\sum_{u \in V(G) \setminus S} f(N^+(u)) \geq 2(|V(G)| \setminus |S|) \geq 2(n-\gamma^*_{r2})$$

and

$$\sum_{u \in V(G) \setminus S} f(N^-(u)) \geq 2(|V(G)| \setminus |S|) \geq 2(n-\gamma^*_{r2}).$$

Every vertex in S is adjacent to k vertices of $V(G) \setminus S$. So,

$$k\gamma^*_{r2}(G) \geq \sum_{u \in V(G) \setminus S} f(N^+(u)) + \sum_{u \in V(G) \setminus S} f(N^-(u)) = \sum_{u \in V(G) \setminus S} f(N^+(u)) + f(N^-(u)) \geq 4(n-\gamma^*_{r2})$$

which results in $(k+4)\gamma^*_{r2} \geq 4n$ or $\gamma^*_{r2} \geq \dfrac{4n}{k+4}$. □

Proposition 2.11. *For a caterpillar all of whose vertices are of degree 4, there exists an orientation for which* $\gamma^*_{r2}(D) \leq n - \lfloor \dfrac{diam(G)-1}{2} \rfloor$.

Proof. Orient the edges of the diameter forwardly and assign the sets $\{1\}$ and $\emptyset$ alternatively to its vertices. To the leaves other than the two corresponding to the diameter assign $\{2\}$ and $\{1\}$ alternatively from left to right and orient all of them downward. □

Proposition 2.12. *If any vertex in a caterpillar be of degree 3 then there exists an orientation for which* $\gamma^*_{r2}(D) \leq n - \lfloor \dfrac{diam(G)-1}{4} \rfloor$.

Proof. Assign to the vertices of the diameter $\{1\}$, $\{\emptyset\}$, $\{1,2\}$, $\emptyset$, $\{2\}$, $\{\emptyset\}$, $\{1,2\}$, $\emptyset$ successively and orient them to the forward. To all other vertices assign $\{1\}$ and orient them downward. □

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When the annihilator graphs are ring graph and outerplanner

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Abstract

Let R be a commutative ring. The annihilator graph of R, denoted by $\mathrm{AG}(R)$, is an undirected graph with all nonzero zero-divisors of R as vertex set, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$, where for $z \in R$, $\mathrm{ann}_R(z) = \{r \in R \mid rz = 0\}$. In this paper, we characterize all finite commutative rings R with planar, outerplanar or ring graph annihilator graphs. We also characterize all finite commutative rings R whose annihilator graphs have clique number $1, 2$ or 3.

Keywords: Annihilator graph, Planar graph, Ring graph, Clique number
Mathematics Subject Classification [2010]: 05C75, 13A99, 05C99

1 Introduction

Let R be a commutative ring with nonzero identity. We denote the sets of all zero-divisors and nilpotent elements of R by $Z(R)$ and $\mathrm{Nil}(R)$, respectively. In 1999, Anderson and Livingston introduced the zero-divisor graph of R, denoted by $\Gamma(R)$, that is the graph with vertices $Z(R)^* = Z(R)\backslash\{0\}$ and distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. Beck introduced this concept in 1988 but he allowed all the elements of R as vertices and was mainly interested in colorings. Recently, in [4], the concept of the annihilator graph is defined and studied. The annihilator graph of R, denoted by $\mathrm{AG}(R)$, is an undirected graph with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$, where for $z \in R$, $\mathrm{ann}_R(z) = \{r \in R \mid rz = 0\}$. By [4, Lemma 2.1], zero-divisor graph $\Gamma(R)$ is a (spanning) subgraph of the annihilator graph $\mathrm{AG}(R)$. In [2], the authors studied the situations that the unit, unitary and total graphs are ring graph or outerplanar. Also, in [1], they studied the ring graph and outerplanarity for comaximal and zero-divisor graphs. In the second section of this paper, we completely characterize all finite commutative rings with planar, outerplanar or ring graph annihilator graphs. Also we characterize all finite commutative rings R, whose annihilator graphs have clique number $1, 2$ or 3.

Now, we recall some definitions and notations on graphs. Let G be a simple graph with vertex set $V(G)$ and C be a cycle of G. A chord in G is any edge joining two nonadjacent

*Speaker

vertices in C. A primitive cycle is a cycle without chord. Moreover, if any two primitive cycles intersect in at most one edge, then we say G has the primitive cycle property (PCP). The number of primitive cycles of G is the free rank of G and is denoted by $\mathrm{frank}(G)$. We have $\mathrm{rank}(G) := q - n + r$, where q, n and r are the number of egdes of G, the number of vertices of G and the number of connected components of G, respectively.

A graph G is called planar if it can be drawn in the plane without crossing edges. A graph G is an outerplanar graph if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. The precise definition of a ring graph can be found in section 2 of [6]. Also, in [6], the authors showed that the following conditions are equivalent:

(i) G is a ring graph,

(ii) $\mathrm{rank}(G) = \mathrm{frank}(G)$,

(iii) G satisfies PCP and G does not contain a subdivision of K^4 as a subgraph.

So every ring graph is planar. Moreover, in [6], authors showed that every outerplanar graph is a ring graph. Also we denote the complete graph with n vertices by K^n and we denote the complete bipartite graph by $K^{m,n}$. We denote the star graph by $K^{1,n}$. Let k be a positive integer. For a graph G, a k-coloring of the vertices of G is an assignment of k colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The chromatic number of G, denoted by $\chi(G)$, is the smallest number k such that G admits a k-coloring. Any subgraph of G is called a clique if it is complete and the size of a largest clique in a graph G is denoted by $cl(G)$. A graph G is called weakly perfect provided $\chi(G) = cl(G)$.

2 Ring graphs and outerplanar annihilator graphs

In this section, we investigate all finite commutative rings R such that their annihilator graphs are planar, outerplanar or ring graph. Throughout this section, R is a finite commutative ring with nonzero identity and $\mathbb{F}$ is a finite field. Specially, $\mathbb{F}_4$ is a field with four elements.

Theorem 2.1. *The annihilator graph* $\mathrm{AG}(R)$ *is planar if and only if* R *is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,

(ii) $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$,

(iii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$.

In the following theorem, we characterize all rings with ring graph annihilator graphs.

Theorem 2.2. *The annihilator graph* $\mathrm{AG}(R)$ *is a ring graph if and only if* R *is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$,

(ii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$.

In the next theorem, by using the fact that every outerplanar graph is a ring graph in conjunction with Theorem 2.2, we determine all rings R with outerplanar annihilator graphs.

Theorem 2.3. *The annihilator graph* $\mathrm{AG}(R)$ *is outerplanar if and only if* R *is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$,

(ii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$.

Now we show that the annihilator graph of the product of three fields is weakly perfect.

Lemma 2.4. *Let* K_1, K_2 *and* K_3 *be fields. Then* $cl(\mathrm{AG}(K_1 \times K_2 \times K_3)) = \chi(\mathrm{AG}(K_1 \times K_2 \times K_3)) = 3$.

In following theorem we charaterize all finite rings R whose annihilator graphs have clique number 1, 2 or 3.

Theorem 2.5. *Let* R *be a finite commutative ring and let* K_1, K_2 *and* K_3 *be finite fields. Also let* $\mathbb{F}_4$ *be a field with four elements. Then the following statements hold.*

(a) $cl(\mathrm{AG}(R)) = 1$ *if and only if* R *is isomorphic to* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$.

(b) $cl(\mathrm{AG}(R)) = 2$ *if and only if* R *is isomorphic to one of the following rings:*

$$K_1 \times K_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_9, \mathbb{Z}_3[x]/(x^2).$$

(c) $cl(\mathrm{AG}(R)) = 3$ *if and only if* R *is isomorphic to one of the following rings:*

$$K_1 \times K_2 \times K_3, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2,x)^2,$$
$$\mathbb{F}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2+x+1), \mathbb{Z}_2[x,y]/(x,y)^2, \mathbb{Z}_4[x]/(2x, x^2-2).$$

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Talk A generalization of α-dominating set and its complexity pp.: 1–4

A generalization of α-dominating set and its complexity

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Abstract

Let $G = (V, E)$ be a simple and undirected graph. For some real number α with $0 < \alpha \leq 1$, a set $D \subseteq V$ is called an *α-dominating set* in G if every vertex v outside D has at least $\alpha \cdot d_v$ neighbor(s) in S where d_v is the degree of v. The cardinality of a minimum α-dominating set in a graph G is called the *α-domination number* of G and denoted by $\gamma_\alpha(G)$. In this paper, we introduce a generalization of α-dominating set, that we call it *f_{deg}-dominating set.* Given a function f_{deg} where f_{deg} is as $f_{deg} : \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$, and f_{deg} may not be an integer-value function. A set $D \subseteq V$ is called an f_{deg}-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \geq f_{deg}(d_v)$. In this paper, for this new concept, we will present some results on the its NP-completeness, APX-completeness and inapproximability.

Keywords: Domination, α-Domination, k-Domination, APX-Complete, NP-Complete
Mathematics Subject Classification [2010]: 05C69, 11Y16

1 Introduction

Let $G = (V, E)$ be an undirected and simple graph. A set $D \subseteq V$ is called a *dominating* set if every vertex outside D has at least one neighbor in D. The cardinality of a minimum dominating set is called the *domination number* of G denoted by $\gamma(G)$. In 2000, Dunbar et al. [5], introduced the concept of α-domination. Let α be a real number with $0 < \alpha \leq 1$. A set $D \subseteq V$ is called an α-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \geq \alpha \times d_v$ where $N(v)$ is the set of all neighbors of v in G, and $d_v := |N(v)|$ is the degree of v. Also, let k be a real number with $k \geq 1$. A set $D \subseteq V$ is called a *k-dominating* set in G if for every vertex v outside D, $|N(v) \cap D| \geq k$.

Now consider the definition of α-dominating. One generalization of this concept is that instead of having at least $\alpha \times d_v$ neighbors in D for each vertex $v \notin D$, we have at least $f(d_v)$ neighbors in D, for some special function f. By selecting $f(x) = \alpha x$, the definition match the α-dominating. It seems that this generalization is much near to the reality. Hence, in this paper, we define the *f_{deg}-dominating* set. Given a function f_{deg} where f_{deg} is as $f_{deg} : \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$, and f_{deg} may not be an integer-value function. A set $D \subseteq V$ is called an f_{deg}-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \geq f_{deg}(d_v)$. In this paper, we consider the graphs with no isolated vertices. We can easily extend the results for the graphs with isolated vertices. In this

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paper, we prove the NP-completeness of the following problem: given a graph G and a positive integer k, decide whether G has an f_{deg}-dominating set S with $|S| \leq k$. Moreover, we prove that the problem of finding a minimum f_{deg}-dominating set when $f_{deg}(x) = k$ (in the other words, the k-dominating set) for any integer $k \geq 1$ is APX-complete (there is no PTAS). Also, we present some inapproximability result for the problem of finding a minimum f_{deg}-dominating set for constant function $f_{deg}(x) = k$.

2 NP-completeness result

In this section, we will prove that the problem of finding the f_{deg}-domination number of a graph is NP-complete, for every given function f_{deg} with some special properties. It is well known that the following decision problem, denoted by 3-REGULAR DOMINATION (3RDM), is NP-complete [6]: given a 3-regular graph $G = (V, E)$ and a positive integer k, does G has a dominating set S with $|S| \leq k$? Now, consider the following decision problem, denoted by f-DOMINATION (fDM): given a graph $G = (V, E)$ without isolated vertices and a positive integer k, does G has an f_{deg}-dominating set S with $|S| \leq k$?

We will show that fDM is NP-complete for some special functions. We will extend the proof of the result in which that α-domination is NP-complete (see [5]).

Theorem 2.1. *If an increasing function f_{deg} with domain $\mathbb{N}$ satisfies*
a. $\forall x \in \mathbb{N}, 0 < f_{deg}(x) \leq x$,
b. $\exists x_0 > 0$ *such that* $\forall x \geq x_0,\ x + 1 \geq f_{deg}(x + 3)$.
c. *For every two integers x and y, $f_{deg}(y + x) \leq f_{deg}(y) + f_{deg}(x)$,*
d. *For a given $x \in \mathbb{N}$, there is $y \in \mathbb{N}$, such that $y > x$ and $f_{deg}(y) \leq x$,*
then, the problem fDM is an NP-complete problem.

Sketch of Proof. Let f_{deg} be an arbitrary function that has the conditions of the theorem. We fix the function f. We can easily see that $fDM \in NP$. Now, we proof the completeness. We make a transformation from 3RDM to fDM. Suppose that x is the smallest integer such that $(x + 1) \geq f_{deg}(x + 3)$, and y is the largest integer with $y > x$ and $x \geq f_{deg}(y)$. Consider the complete graph K_{y+1} and assume that $U = \{v_1, v_2, \ldots, v_x\}$ is a subset of vertices of K_{y+1} with x elements. We call the vertex set of K_{y+1} by W.

We transform a 3-regular graph G to a graph denoted by $\hat{G}$ by joining each vertex of set U to all vertices of G. Assume that S is a dominating set in G such that $|S| \leq k$. Consider the set $D = S \cup U$. Using the conditions **b** and **d**, it is easy to see that D is an f_{deg}-dominating set in $\hat{G}$ with $|D| \leq x + k$.

Now, we assume that D is an f_{deg}-dominating set in $\hat{G}$ with $|D| \leq x + k$. Among all f_{deg}-dominating set in $\hat{G}$ with $|D| \leq x + k$, we suppose that D is the one with maximum $|D \cap U|$. Also, without loss of generality we can suppose that there is a vertex in $W - U$ that is outside D. Using conditions **a, b, c,** and **d**, it is not hard to prove that the set $D \cap V(G)$ is a dominating set in G with $|D \cap V(G)| \leq k$. Because 3RDM is NP-complete [6], fDM is also NP-complete for the function f that satisfies the conditions of Theorem 2.1. □

There are many functions that satisfy the conditions of Theorem 2.1, such as $\sqrt{x}$, $\ln x$ and $\frac{x}{2}$.

3 APX-completeness result

In this section, we prove that the problem of finding a minimum f_{deg}-dominating set of a graph with maximum degree $k+2$ and $f_{deg}(x) = k$ for any $k \geq 1$ is APX-complete (there is no PTAS). We denote the problem of finding a minimum f_{deg}-dominating set of a graph where $f_{deg}(x) = k$ by Min k-Dom Set, and when the problem is restricted to the graphs with maximum degree $k+2$, we call it Min k-Dom Set-$(k+2)$.

At first, we recall the L-reduction.

Definition 3.1. (L-reduction)[2]. Given two NP optimization problems F and G and a polynomial transformation f from instances of F to instances of G, we say that f is an L-reduction if there are two positive constants α and β such that for every instance x of F

1. $opt_G(f(x)) \leq \alpha opt_F(x)$
2. for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ we can, in polynomial time, find a solution y' of x with $m_F(f(x), y') = c_1$ such that $|opt_F(x) - c_1| \leq \beta |opt_G(f(x)) - c_2|$.

To prove that a problem F is APX-complete, it is sufficient to prove that $F \in$APX and there is an L-reduction from some APX-complete problem to problem F.

Theorem 3.2 ([4])**.** *For a graph G,* Min k-Dom Set *can be approximated in polynomial time by a factor of* $\ln(2\Delta(G)) + 1$ *where* $\Delta(G)$ *is the maximum degree of* G.

Theorem 3.3. Min k-Dom Set-$(k+2)$ *is an APX-complete problem for any* $k \geq 1$.

Sketch of Proof. The case $k = 1$ proved in [1]. Consider $k > 1$. Clearly, by Theorem 3.2, if the degree of vertices of the graph is bounded by a constant then the approximation ratio is constant. Thus the problem Min k-Dom Set-$(k+2)$ is in APX. Suppose that $G = (V, E)$ is a graph of bounded degree 3. Construct a graph $G_k = (V_k, E_k)$ of bounded degree $k+2$ as follows. Create a set S_v of $k-1$ new vertices for each vertex v. Join each vertex $v \in V$ to $k-1$ vertices of S_v. Given a k-dominating set D_k of $G_k = f_k(G)$ (f_k is a transformation from G to G_k. Recall Definition 3.1), we can find a dominating set D in G as $D = D_k - \left(\bigcup_{v \in V(G)} S_v\right)$. So $\gamma(G) \leq |D| = |D_k| - (k-1)n$, where $n = |V|$. Also, given a dominating set D of G, clearly the set $D_k = \left(\bigcup_{v \in V(G)} S_v\right) \cup D$ is a k-dominating set in G_k. So $\gamma_k(G_k) \leq |D_k| = |D| + (k-1)n$. Hence, we can easily conclude that $\gamma_k(G_k) = \gamma(G) + (k-1)n$.

Finally, using the above argument, we can find an L-reduction with parameters $\alpha = 4k-3$ and $\beta = 1$. So, the problem Min k-Dom Set-$(k+2)$ is APX-complete. □

4 Inapproximability result on Min k-Dom Set

In this section, we presents some inapproxmabillity result for Min k-Dom Set.

Theorem 4.1 ([3])**.** *For any constant $\epsilon > 0$ there is no polynomial time algorithm approximating* Min 1-Dom Set *within a factor of* $(1-\epsilon)\ln n$ *unless* $NP \subseteq DTIME(n^{O(\log\log n)})$*. The same result holds for bipartite graphs.*

Theorem 4.2. *For every $k \geq 1$ and every $\epsilon > 0$, there is no polynomial time algorithm approximating* Min k-Dom Set *for bipartite graphs within a factor of $(1-\epsilon)\ln n$, unless $NP \subseteq DTIME(n^{O(\log\log n)})$.*

Sketch of Proof. It is sufficient that, we make some modifications in the proof of Theorem 4.1. We make a reduction from domination on a bipartite graph G with n vertices such that $n + 2k - 2 \leq n^{1+\epsilon}$ and $\gamma(G) \geq \frac{2(k-1)(1+2\epsilon)}{\epsilon^2}$. Then we transform the bipartite graph $G = (V_1, V_2, E)$ into a bipartite graph G' by adding to it two sets K_1 and K_2 each have $k-1$ new vertices inducing a graph with no edges. Join each vertex of V_1 to each vertex of K_2 and join each vertex of V_2 to each vertex of K_1. We can easily prove that $\gamma_k(G') \leq \gamma(G) + 2k - 2$. Now, suppose that there is a polynomial time approximation algorithm that computes a k-dominating set D' for G' such that $|D'| \leq (1-\epsilon)\ln(|V(G')|)\gamma_k(G')$. It is easy to see that $D := D' \cap V(G)$ is a dominating set in G. So,

$$\begin{aligned}
|D| &\leq |D'| \\
&\leq (1-\epsilon)(\ln|V(G')|)\gamma_k(G') \text{ (suppose that } n := |V(G')|) \\
&\leq (1-\epsilon)(\ln n)(1+\epsilon+\epsilon^2)\gamma(G) \\
&= (1-\epsilon')(\ln n)\gamma(G),
\end{aligned}$$

where $\epsilon' = \epsilon^3 > 0$. Therefore, the set D approximates a minimum dominating set in G within factor $(1-\epsilon')\ln n$. But this contradicts Theorem 4.1. This completes the proof. □

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An Approximation Algorithm for a Heterogeneous Capacitated Vehicle Routing Problem

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Abstract

The capacitated vehicle routing problems with heterogeneous vehicles (HCVRP) arise in many logistics and distribution problems. The vehicles in these problems can be variant in their capacities or per unit distance costs. In this paper, we present an approximation algorithm for the HCVRP where there exist a fixed number of heterogeneous vehicles at the depot and the fleet of vehicles is non-uniform in their capacity and per unit distance cost and the objective is to minimize the total cost of travel. We have assumed that the distance between two locations/customers is symmetric and satisfies the triangle inequality.

Keywords: Heterogeneous Capacitated Vehicle Routing Problem (HCVRP), Approximation Algorithms, Generalized assignment problem
Mathematics Subject Classification [2010]: 68W25, 90B99, 05C99.

1 Introduction

The vehicle routing problem (VRP) is one of the most important and more studied combinatorial optimization problems. It calls for the determination of the optimal set of routes to be performed by a fleet of vehicles to serve a given set of customers. Logistics management and distribution are two central places for variants of these problems, specially capacitated VRP (CVRP). In logistical and transportation problems the company uses multiple vehicles in parallel for the distribution. The objective in this case is to minimize the number of tours or the overall cost of travel. The vehicles may be identical (i.e. have same capacity and cost) or heterogeneous (have different capacity or different per unit distance cost). The routes have to be designed according to the characteristics (i.e. capacity and cost) of vehicles. In this article, vehicles are considered to be heterogeneous if they differ in capacity and per unit cost of distance travel. There exists a good survey for vehicle routing problems in [4], [1]. A related work is heterogeneous traveling salesman problem with 2 depots and the objective function of minimizing the total cost of travel [2]. The vehicles at the depots differ in per unit cost of distance travel. In that manuscript Bae and Rathinam obtained a 2-approximation for HTSP by the use of primal-Dual technique. The first result in this area is related to Yadlapalli et al. which obtains a 8-approximation ratio [6]. Recently they have improved this result to a 3-approximation ratio [7]. The main

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goal of this article is to develop an approximation algorithm for the HCVRP to measure the performance of heuristics. Γ is an $\alpha-$ approximation algorithm for a minimization problem if it runs in polynomial time and on every instance, the cost of the solution obtained by Γ is at most $\alpha-$ times the cost of an optimal solution [5].

1.1 Problem Formulation

The demand locations are assumed as vertices of the graph $G = (V, E)$ in a finite metric space (V, d) . The edge set $E = \{(i, j) : i, j \in V, i \neq j\}$ represents the distance between two locations. The distance function $d : V \times V \to R^+$ is symmetric and satisfies the triangle inequality. There exists a depot $r \in V$, a set of heterogeneous vehicles indexed by $\{1, 2, ..., N\}$ stationed at the depot with capacities $\{Q_1, Q_2, ..., Q_N\}$ and per unit costs of distance travel $\{C_1, C_2, ..., C_N\}$. The required item is identical for all the demands and we have an infinite amount of it at the depot. The request function $q : V \to N$ is un-splittable in the sense that each request must be served entirely by a vehicle. The objective is to find a set of tours starts and ends at the depot that covers all the vertices in V such that the overall cost of travel become minimum. The only constraint on the tours is the capacity of the corresponding vehicle. $Q_{\max}$is the maximum capacity of the vehicles and $q_{\min}$ is the minimum request.

2 Approximation Algorithm

Heterogeneous CVRP is a generalization of both variable bin packing and travelling salesman problem. The important problem in our HCVRP is the existence of a feasible solution. This problem is NP-complete by a reduction from the bin-packing problem. One can observed that if there exists a feasible solution for the variable bin packing, a simple algorithm obtains a $3\,/2.n-$ approximation ratio for vehicles with uniform per distance unit cost and $(3C_{\max}/2C_{\min})n-$ approximation for non-uniform vehicles. In this paper, we show that if the problem satisfies some conditions then there exists a feasible solution of $4\,\lceil Q_{\max}/q_{\min}\rceil -$ approximation ratio for HCVRP which in some ways obtains a better approximation ratio than the other one. The last ratio depends on the number of vehicles denoted by n but the main algorithm ratio depends on the maximum number of items which could be located in a vehicle. Indeed, if the number of items serviced by a vehicle is bounded, our main algorithm gives a constant factor approximation ratio for HCVRP. An algorithm for a generalized assignment problem is used as a sub-routine algorithm. we have assumed each vehicle as a machine whose capacity is at most the capacity of the vehicle and each location as a job that needs time at most the demand of the related customer. There exists an edge from each vehicle to each location/customer where the cost of the edges is the multiplication of the per unit distance travel cost of the vehicles and the total distance from depot to the corresponding demand location. The Generalized Assignment Problem (GAP) is the problem of minimizing the cost of assigning n different items to m agents, such that each item is assigned to precisely one agent, subject to a capacity constraint for each agent. This problem can be formulated as

$$\begin{aligned}
&\min \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{c_{ij}x_{ij}} \\
&\text{s.t.} \sum_{j=1}^{n} \mathrm{a_{ij}x_{ij}} \le \mathrm{b_i}, \qquad \mathrm{i = 1,..., m} \\
&\sum_{i=1}^{m} \mathrm{x_{ij}} = 1, \qquad \mathrm{j = 1,..., n} \\
&\mathrm{x_{ij}} \in \{0,1\}, \qquad \mathrm{i = 1,..., m, j = 1,..., n.} \qquad (1)
\end{aligned}$$

Here, c_{ij} is the cost of assigning item j to agent i, a_{ij} is the claim on the capacity of agent i by item j , and b_i is the capacity of agent i . An approximation algorithm of $(2, 1)$ -approximation ratio is given for this problem in [3]. By this algorithm if there is a feasible solution for the linear relaxed problem then there exists a solution for the integer problem (1) of cost at most the optimal cost where the capacity of the vehicles is twice the capacity of the vehicles used in the related linear problem. Furthermore, by deleting one item from each vehicle, one obtains a feasible solution for GAP. We have used this algorithm as sub-routine algorithm to approximate our HCVRP. Now, we present our approximation algorithm for HCVRP. Let $M = \{(v, u)|\ v \in \{vehicles\}, u \in \{demands\}, x_{vu} = 1\}$ be the obtained solution of the generalized assignment problem. We order the demands corresponding to each vehicle increasingly according to their distance from the depot(vehicle): $d(u_1, v) \le d(u_2, v) \le ... \le d(u_{h(v)}, v)$, and construct the tours as follows:

$$D = \{(v, u_1, u_2, ..., u_{h(v)}, v)|v \in \{vehicles\}\}$$

Theorem 2.1. $\cos t(D) \le 2. \cos t(M)$.

Lemma 2.2. *If the set of vehicles could be splitted into two sets V_1, V_2 in a way that $\sum_{1\le i\le n} q_i \le \sum_{Q_i\in V_j} Q_i$ and the number of same vehicles are equal in each set, then there is a feasible solution for (1) of cost at most $2OPT(GAP)$ (i.e. $cost(M) \le 2OPT(GAP)$).*

Theorem 2.3. *Under the assumptions of lemma 2.2,* $\cos t(M) \le 2\lceil Q_{\max}/q_{\min}\rceil . \cos t(OPT_{HCVRP})$.

Proof. Let $D^* = \{(v, u^*_{v,1}, u^*_{v,2}, ..., u^*_{v,h^*(v)}, v)|v \in \{vehicles\}\}$ be the optimal solution for the heterogeneous capacitated vehicle routing problem. Here, $h^*(v)$ denotes the number of requests served by the vehicle v. We construct M^* as follows: $M^* = \{(v, u)|\ v \in \{vehicles\},\ u \in \{demands\}\}$.

$$\begin{aligned}
&\cos t(M) \le\ 2cost(OPT_{GAP}) \le 2\cos t(M^*) \le 2 \sum_{v\in\{vehicles\}} [\sum_{j=1}^{h^*(v)} d(v, u^*_{v,j})] \\
&\le 2 \sum_{v\in\{vehicles\}} [\sum_{j=1}^{h^*(v)} d(v, u^*_{v,1}) + \sum_{i=2}^{j} d(u^*_{v,i-1}, u^*_{v,i}) + d(u_{h^*(v)}, v)] \\
&\le 2 \sum_{v\in\{vehicles\}} h^*(v).[d(v, u^*_{v,1}) + \sum_{i=2}^{h^*(v)} d(u^*_{v,i-1}, u^*_{v,i}) + d(u_{h^*(v)}, v)] \\
&\le 2\lceil Q_{\max}/q_{\min}\rceil . \sum_{v\in\{vehicles\}} [d(v, u^*_{v,1}) + \sum_{i=2}^{h^*(v)} d(u^*_{v,i-1}, u^*_{v,i}) + d(u_{h^*(v)}, v)] \\
&= 2\lceil Q_{\max}/q_{\min}\rceil . \cos t(OPT_{HCVRP}).
\end{aligned}$$

The first inequality arises from lemma 2.2. The fourth inequality is due the triangle inequality.

Lemma 2.4. *The main algorithm under the assumptions of lemma 2.2 gives* $4\lceil Q_{\max}/q_{\min}\rceil-$ *approximation algorithm for the HCVRP.*

Lemma 2.5. *The main algorithm without the assumptions of lemma2.2 gives a solution of* $(2, 2\lceil Q_{\max}/q_{\min}\rceil)-$ *approximation ratio for the HCVRP.*

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On the number of Doche-Icart-Kohel Curves over finite fields

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Abstract

We give explicit formulas for the number of distinct elliptic curves over a finite field in the family of Doche-Icart-Kohel curves of cryptographic interest.

Keywords: Elliptic curves, , Isomorphism, Cryptography, Number theory
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1 Introduction

An elliptic curve is a smooth projective genus 1 curve, with a given rational point. Traditionally, an elliptic curve E over a filed $\mathbb{F}$ is represented by the Weierstrass equation

$$E: \quad Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6, \tag{1}$$

where the coefficients $a_1, a_2, a_3, a_4, a_6 \in \mathbb{F}$. Elliptic curves can be represented by several other models (see [3, Chapter 2]). In the past few years, these alternative models have been revisited duo to cryptoraphic applications. Moreover, some new families of elliptic curves have been proposed following the cryptographic interests. In the cryptographic settings the curves are usually considered over finite fields $\mathbb{F}_q$ of q element.

In this work, we consider the family of elliptic curves introduced by C.Doche, T.Icart and D.R.Kohel (DIK) over finite fields $\mathbb{F}_q$ of characteristic $p \geq 3$

$$E_{D,u} : Y^2 = X^3 + uX^2 + 16uX, \tag{2}$$

where $u \in \mathbb{F}_q$ and $u \neq 0, 64$, since the curve is nonsingular. Doche et. al. have build this family of elliptic curves for which the isogeny of doubling splits into 2 isogenies of degree 2 and proposed more efficient doubling formulas leading to a fast scalar multiplication algorithm. Notice, an elliptic curve defined over $\mathbb{F}_q$ with a rational 2-torsion subgroup can be expressed in the latter form (up to twists). Accordingly a natural question arises about the number of distinct (up to isomorphism) elliptic curves over $\mathbb{F}_q$ in the family (2).

Throughout the paper, for a field $\mathbb{F}$, we denote its algebraic closure by $\overline{\mathbb{F}}$. The letter p always denotes a prime number and the letter q always denotes a prime p power. As usual, $\mathbb{F}_q$ is a finite field of size q. Let χ_2 denote the quadratic character in $\mathbb{F}_q$, where $p \geq 3$. Then, for any q where $p \geq 3$, we have $u = w^2$ for some $w \in \mathbb{F}_q^*$ if and only if $\chi_2(u) = 1$. The cardinality of a finite set $\mathcal{S}$ is denoted by $\#\mathcal{S}$, and the cardinality of the set of projective points on an elliptic curve E over a field $\mathbb{F}$ is denoted by $E(\mathbb{F})$.

*Speaker

2 Main results

2.1 Preliminaries

Here, we briefly recall some words on isomorphisms between elliptic curves, see [2, 3] for a general background on isomorphisms and elliptic curves. Two elliptic curves given by Weierstrass equations (1) are isomorphic over a field $\mathbb{F}$ if and only if there is a change of variables between them of the form:

$$(x, y) \to (\alpha^2 x + r, \alpha^3 y + \alpha^2 sx + t),$$

where $\alpha \neq 0$, and $\alpha, r, s, t \in \mathbb{F}$. We use $E_1 \cong_{\mathbb{F}} E_2$ to denote the elliptic curves E_1 and E_2 are $\mathbb{F}$-*isomorphic*. If $\alpha, r, s, t \in \overline{\mathbb{F}}$, the two elliptic curves are called *isomorphic* over $\overline{\mathbb{F}}$ or *twists* of each other.

The elliptic curve E over $\mathbb{F}$ given by the Weierstrass equation (1) has the non-zero discriminant

$$\Delta_E = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6,$$

$$b_2 = a_1^2 + 4a_2,\ b_4 = a_1 a_3 + 2a_4,\ b_6 = a_3^2 + 4a_6,\ b_8 = a_1^2 a_6 - a_1 a_3 a_4 + 4a_2 a_6 + a_2 a_3^2 - a_4^2.$$

And, the j-invariant of E is explicitly defined as

$$j(E) = (b_2^2 - 24b_4)^3/\Delta_E.$$

It is known that two elliptic curves E_1, E_2 over a field $\mathbb{F}$ are isomorphic over $\overline{\mathbb{F}}$ if and only if $j(E_1) = j(E_2)$, see [2, Proposition III.1.4(b)]. However, two elliptic curves with the same j-invariant need not be isomorphic over $\mathbb{F}$.

In this work, we count the number of distinct $\mathbb{F}_q$-isomorphism classes of DIK curves given by (2) over a finite field $\mathbb{F}_q$. The j-invariant of $E_{D,u}$ is obtained as $j(E_{D,u}) = F(u)$ where $F(U) = \frac{(U-48)^3}{(U-64)}$. The number of distinct elliptic curves $E_{D,u}$ up to isomorphism over $\overline{\mathbb{F}}_q$ equals the number of distinct values $F(u)$, for all $u \in \mathbb{F}_q \backslash \{0, 64\}$. To compute this number, we consider the bivariate rational function $F(U) - F(V) = g(U, V)/h(U, V)$ with two relatively prime polynomials g and h. For $u \in \mathbb{F}_q \backslash \{0, 64\}$ we let the polynomial g_u to be $g_u(V) = g(u, V)$. We have

$$g_u(V) = (u - 64)V^2 + (u^2 - 208u + 9216)V + (-64u^2 + 9216u - 331776).$$

Therefore, for $v \neq u$, two curves $E_{D,u}$ and $E_{D,v}$ are isomorphic if and only if $g_u(v) = g(u, v) = 0$. We denote the discriminant of g_u by Δ_u and the set of its $\mathbb{F}_q$ roots by $\mathcal{Z}_u$. So, we have

$$\Delta_u = u(u - 64)(u - 48)^2 \qquad \text{and} \qquad \mathcal{Z}_u = \{v : v \in \mathbb{F}_q \backslash \{u, 0, 64\},\ g_u(v) = 0\}.$$

For all $u \in \mathbb{F}_q \backslash \{0, 64\}$, one can easily show that

$$\#\mathcal{Z}_{48} = 1, \qquad \#\mathcal{Z}_{72} = 1, \qquad \#\mathcal{Z}_u = 1 + \chi_2(u/(u - 64)),\ u \neq 48, 72. \tag{3}$$

For $u \in \mathbb{F}_q \backslash \{0, 64\}$, also let

$$\mathcal{J}_u = \{E_{D,v} : v \in \mathbb{F}_q \backslash \{0, 64\}, E_{D,v} \cong_{\overline{\mathbb{F}}_q} E_{D,u}\},\ \mathcal{I}_u = \{E_{D,v} : v \in \mathbb{F}_q \backslash \{0, 64\}, E_{D,v} \cong_{\mathbb{F}_q} E_{D,u}\}.$$

Let $\mathcal{N}_q$ and $\overline{\mathcal{N}}_q$ denote the number of isomorphisms between distinct elliptic curves in the family (2) over $\mathbb{F}_q$ and $\overline{\mathbb{F}}_q$ respectively, and let

$$\overline{n}_q = \#\overline{\mathcal{N}}_q,\ n_q = \#\mathcal{N}_q,\ c_i = \#\{\mathcal{J}_u : u \in \mathbb{F}_q \backslash \{0, 64\},\ \#\mathcal{J}_u = i\} \text{ for } i = 1, 2, 3.$$

2.2 Number of $\overline{\mathbb{F}}_q$-isomorphism classes

Here, we give the number of distinct doubling Doche-Icart-Kohel curves up to $\overline{\mathbb{F}}_q$-isomorphism classes.

Theorem 2.1. *For any finite field $\mathbb{F}_q$ of characteristic $p \geq 3$, for the number $J_D(q)$ of distinct values of the j-invariant of family* (2)*, we have*

$$J_D(q) = \begin{cases} (2q-3)/3 & \text{if } q \equiv 0 \pmod 3, \\ (2q+1)/3 & \text{if } q \equiv 1 \pmod 3, \\ (2q-1)/3 & \text{if } q \equiv 2 \pmod 3. \end{cases}$$

Proof. According to (3), if we define $C := \{(x,y) \in \mathbb{F}_q^2 : x(x-64) = y^2\}$, then

$$c_3 = \frac{1}{3}\#\left\{x : x \in \mathbb{F}_q \backslash \{0, 64, 48, 72\}, (x,y) \in C\right\} = \begin{cases} \dfrac{\#C-2}{6} & \text{if } q \equiv 0 \pmod 3, \\ \dfrac{\#C-5-\chi_2(-3)}{6} & \text{if } q \not\equiv 0 \pmod 3. \end{cases}$$

We have $\chi_2(-3) = 1$ if $q \equiv 1 \pmod{}$, $\chi_2(-3) = -1$ if $q \equiv 2 \pmod{}$ and $\#C = q-1$. Since $J_D(q) = c_1 + c_2 + c_3 = q - 2 - 3c_3 + 0 + c_3 = q - 2 - 2c_3$, the proof is complete. □

2.3 Number of $\mathbb{F}_q$-isomorphism classes

In order to compute $I_D(q)$, we need to know how many $\mathbb{F}_q$-isomorphisms there are between distinct curves of family (2).

Lemma 2.2. *Suppose $p \geq 3$. For every $u, v \in \mathbb{F}_q \backslash \{0, 64\}$ such that $u \neq v$ and $E_{D,u} \cong_{\overline{\mathbb{F}}_q} E_{D,v}$, we have $E_{D,u} \cong_{\mathbb{F}_q} E_{D,v}$ iff there are $a, b \in \mathbb{F}_q$ so that u, v, a and b lie in the following equations*

$$L_1 : 16a^2 = \frac{b(b+32)}{2(b+24)}, \qquad \begin{cases} a^2 v = u + 3b, \\ u = (-b^2)/(b+16). \end{cases}$$

In fact b always exists and (u,v) uniquely determines (a^2, b) and vice versa.

Proof. Suppose $E_{D,u}$ and $E_{D,v}$ are $\overline{\mathbb{F}}_q$-isomorphic. According to [2] the curves are $\mathbb{F}_q$-isomorphic iff there are elliptic curve isomorphism ψ and elements $a, b \in \mathbb{F}_q$ such that

$$\begin{cases} \psi : E_{D,u} \to E_{D,v}, \\ \psi(x,y) = (a^2 x + b, a^3 y), \end{cases} \quad \text{and} \quad \begin{cases} a^2 v = u + 3b, \\ 16a^4 v = 16u + 2bu + 3b^2, \\ b(16u + bu + b^2) = 0. \end{cases}$$

A simple computation completes the proof. □

One can check that if $q \equiv 0 \pmod 3$, then there are $q-2$ points on L_1. In other cases, except for a few points, all of the other points on L_1 and the ellipic curve L, defined as bellow, are one to one correspondent to each other.

$$L : a'^2 = b'(b'+1)(b'+3/4) \tag{4}$$

Since points on L, except for a few points, are one to one correspondent to $\mathbb{F}_q$-isomorphisms in family (2), subtracting the exceptional points on L, we have

$$n_q = \begin{cases} q-5 & \text{if } q = 3^{2k}, \\ q-3 & \text{if } q = 3^{2k+1}, \\ \#L(\mathbb{F}_q) - 12 & \text{if } q \equiv 1 \pmod{12}, \\ \#L(\mathbb{F}_q) - 8 & \text{if } q \equiv 5, 7 \pmod{12}, \\ \#L(\mathbb{F}_q) - 4 & \text{if } q \equiv 11 \pmod{12}. \end{cases}$$

where $\#L(\mathbb{F}_q)$ denotes the number of $\mathbb{F}_q$-rational points on elliptic curve (4).

Since there are c_3 classes of cardinality three and each $\overline{\mathbb{F}}_q$-isomorphism class has twelve $\overline{\mathbb{F}}_q$-isomorphisms, $\overline{n}_q = 12c_3$. For each $\overline{\mathbb{F}}_q$-isomorphism class of cardinality three like $\mathcal{J}_u = \{E_{D,u}, E_{D,v_1}, E_{D,v_2}\}$, at least two of the curves are $\mathbb{F}_q$-isomorphic, for each curve in $\overline{\mathbb{F}}_q$-isomorphism class is either $\mathbb{F}_q$-isomorphic to another curve in the class or to its nontrivial quadratic twist. Hence $\mathcal{J}_u = \mathcal{I}_u$ or $\mathcal{J}_u = \mathcal{I}_u \cup \mathcal{I}_{v_1}$ or $\mathcal{J}_u = \mathcal{I}_u \cup \mathcal{I}_{v_2}$. This shows that some of $\overline{\mathbb{F}}_q$-isomorphism classes have eight $\overline{\mathbb{F}}_q$-isomorphisms more than $\mathbb{F}_q$-isomorphisms. Therefore $I_D(q) - J_D(q) = \frac{\overline{n}_q - n_q}{8}$, which gives us the following theorem.

Theorem 2.3. *For any finite field $\mathbb{F}_q$ of characteristic $p \geq 3$, for the number $I_D(q)$ of $\mathbb{F}_q$-isomorphism classes of the family* (2), *we have*

$$I_D(q) = \begin{cases} (19q-27)/24 & \text{if } q = 3^{2k}, \\ (19q-33)/24 & \text{if } q = 3^{2k+1}, \\ (11q+1)/12 - N/8 & \text{if } q \equiv 1 \pmod{12}, \\ (11q-7)/12 - N/8 & \text{if } q \equiv 5 \pmod{12}, \\ (11q-5)/12 - N/8 & \text{if } q \equiv 7 \pmod{12}, \\ (11q-13)/12 - N/8 & \text{if } q \equiv 11 \pmod{12}. \end{cases}$$

where $N = \#L(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-rational points on elliptic curve (4) *(including infinity).*

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Projection method combining preconditioners for solving large and sparse linear systems

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Abstract

Solving the sparse and large size linear systems is an important problem in linear algebra and have so many complex applications. One of the iterative methods for solving linear systems is Full Orthogonalization Method (FOM). In this paper, the iterative FOM method is described and for faster convergence some Incomplete preconditioners and Incremental Incomplete preconditioners are Combined with this method and results show converage rate of this preconditioners are faster.

Keywords: Preconditioning, Incomplete LU factorizations, Incremental Incomplete LU factorizations, Full Orthogonalization Method.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

One of the most important problems in linear algebra is solving the linear system Ax=b. Two types of methods for solving linear systems are Direct methods and Iterative methods.

The direct methods consist of a finite number of steps that all must be performed for any given instance before the solution is obtained, on the other hand, iterative methods are by choosing initial solution x and computing a sequence of approximations to the solution x and computation stops whenever a certain desired accuracy is obtained or after certain number of iterations [3].

The iterative methods are used primarily for large and sparse systems and should write the system Ax=b in an equivalent form:

$$x = Bx + r \tag{1}$$

then, starting with an initial approximation $x^{(1)}$ of the solution vector x and generate a sequence of approximation $\{x^{(k)}\}$ iteratively defined by

$$x^{(k+1)} = Bx^{(k)} + r \qquad k = 1, 2, \cdots . \tag{2}$$

One of these methods is Full Orthogonalization Method(FOM) .

*Speaker

1.1 Full Orthogonalization Method

For the original linear system Ax=b if we have initial guess vector x_0 , an orthogonal projection method takes $\kappa = \kappa_m(A, r_0)$, where

$\kappa_m(A, r_0) = Span\left\{r_0, Ar_0, A^2r_0, \cdots, A^{m-1}r_0\right\}$

and $r_0 = b - Ax_0$. This method seeks an approximate solution x_m from the affine subspace $x_0 + \kappa_m$ of dimension m by imposing the Galerkin condition $b - Ax_m \perp \kappa_m$.

If in Arnoldi's method $v_1 = \frac{r_0}{\|r_0\|_2}$ and we set $\beta = \|r_0\|_2$, then $V_m^T AV_m = H_m$ and $V_m^T r_0 = V_m^T(\beta v_1) = \beta e_1$. So the approximate solution using the above m-dimensional subspaces is $x_m = x_0 + V_m y_m$ where $y_m = H_m^{-1}(\beta e_1)$.

A method based on this approach is called the Full Orthogonalization Method(FOM) [4].

1.2 Preconditioners

Preconditioning transforms the problem conditions into a form that is more suitable for numerical solution, and solving the problem mathematically be more easy. Preconditioning typically reduces condition number of the problem. Preconditioners are also useful in iterative methods to solve a linear system $Ax = b$ for x since the rate of convergence for most iterative linear solvers increases as the lower condition number of a matrix. Preconditioned iterative solvers are typically used for large and especially sparse matrices. Some of the preconditioners are based on LU factorization. In the following some examples of these methods are described.

1.2.1 Incomplete LU factorization(ILU)

Incomplete LU (ILU) factorization process computes a sparse lower triangular matrix L and a sparse upper triangular matrix U such that the residual matrix R = LU - A satisfies certain constraints(for example having zero entries in some locations). A general algorithm for Incomplete LU factorizations can be derived by performing Gaussian elimination and dropping some elements in predetermined nondiagonal positions. The simplest form of the ILU preconditioners is ILU(0).

The $ILU(0)$ factorization is any pair of unit lower triangular matrices L and upper triangular matrices U so that the elements of $A - LU$ are zero in the locations of $NZ(A)$ where NZ(A) is the set of pairs $(i, j) \in A, 1 \leq i, j \leq nsuchthata_{i,j} \neq 0$. In general, infinitely many pairs of matrices L and U are exist which satisfy these requirements, for more detail see [1].

1.2.2 Alternating L–U descent methods (MERLU)

Suppose we have an approximate factorization in the form of $A = LU + R$, where R is an error matrice for the approximate factorization. The best factorization is when R=0, so our goal is to minimize error matrix R or equivalently finding sparse matrices X_L and X_U, such that $L + X_L$ and $U + X_U$ be a better pair of factors than L, U. Now we can write R as:

$$A - (L + X_L)(U + X_U) = (A - LU) - X_L U - LX_U - X_L X_U \tag{3}$$

We would like to make the right-hand side equal to zero. By replacing the matrix (A-LU) by R, we have:

$$X_L U + L X_U + X_L X_U - R = 0 \tag{4}$$

Now we would like to approximately solve nonlinear system 4 because of pair of unknowns X_L and X_U. In equation 4, we choice $X_L = 0$ and update U while L is kept frozen. X_U should minimizes $F(X_U) = \|A - L(U + X_U)\|_F^2 = \|R - LX_U\|_F^2$.

the optimum X_U is equals to $L^{-1}R$. Here, we seek an approximation only to this exact solution. A method based on this approach is Alternating L–U descent methods (MERLU).

1.2.3 ALTERNATING SPARSE-SPARSE ITERATION(ITALU)

In Equation 4 we set $X_U = 0$. If U is nonsingular we obtain:

$$X_L U = R \rightarrow X_L = RU^{-1} \tag{5}$$

Thus, the correction to L can be obtained by solving a sparse triangular linear system with a sparse right-hand side matrix, i.e. the system $U^T X_L^T = R^T$. However, as was noted before, the updated matrix $L + X_L$ obtained in this way is not necessarily unit lower triangular so we use lower triangular of $L + X_L$ or unit lower triangular of X_L. This procedure can be repeated by freezing U and updating L and vice versa alternatively. A method based on this approach is Alternating Sparse-Sparse Iteration(ITALU).

1.2.4 Left-Preconditioned FOM

The left preconditioned FOM algorithm defines as the FOM algorithm applied to the system,

$$M^{-1}Ax = M^{-1}b$$

Where M is a matrice that derived by using preconditioners to matrice A.

2 Main results

For each matrice described in table.1 we run FOM method with left preconditioner Algorithm for all described preconditioners and table.2 shows performance and coverage rate of FOM method with and without pereconditioners.

Relative tolerance is set to $droptol = 0.2$ and Matlab's estimated condition number yields $cond(A) \approx 4.03E + 05$. In preconditioner MERLU(1) we run MERLU algorithm two times and the output of first ruing time is used for L_0 and U_0 in next iteration. The preconditioner ITALU(1) also have the same definition.

The results show the FOM method with preconditioner is faster than without preconditioner and more than it, the FOM method with preconditioners ITALU(1) and MERLU(1) converges faster than preconditioner ILU(0).

Table 1: Matrices Properties

No	Name	n_x	n_y	n_z	a_x	a_y	a_z	shift	size	Condest
1	Fd3d-2250	15	15	10	0.1	0.2	0.1	0.3	2250×2250	4.6906e+003
2	Fd3d-2000	20	10	10	0.1	-0.2	0.1	0.1	2000×2000	189.1547
3	Fd3d-1800	15	15	8	-0.3	0.5	1	0.5	1800×1800	193.2077
4	Fd3d-1000a	10	10	10	-0.8	2	0.1	-0.5	1000×1000	28.6727
5	Fd3d-1000b	10	10	10	0.1	-0.2	-1.2	0.1	1000×1000	126.9967
6	Fd3d-500a	10	10	5	0.1	0.1	0.1	0.3	500×500	142.8448
7	Fd3d-500b	10	10	5	0.1	-0.2	0.1	0.1	500×500	55.3150
8	Fd3d-500c	10	10	5	-0.9	2	1.1	0.3	500×500	52.4005

Table 2: Performance of FOM method

	FOM without pre		FOM with ILU(0)		FOM with MERELU(1)		FOM with ITALU(1)	
No	iter	residual	iter	residual	iter	residual	iter	residual
1	106	7.433130e-007	80	7.838e-007	31	5.588294e-007	16	5.921434e-007
2	64	6.894008e-007	27	3.563e-007	16	4.297365e-007	10	1.566921e-007
3	45	3.396037e-007	20	1.240e-007	12	8.271740e-008	5	4.630272e-008
4	40	7.213034e-007	14	2.318e-007	11	3.029245e-007	5	8.964088e-009
5	48	4.198608e-007	19	2.995e-007	14	6.115026e-007	8	5.584333e-007
6	44	2.406152e-007	20	1.537e-007	14	3.278705e-007	9	1.581423e-008
7	41	4.036493e-007	16	1.814e-007	13	6.504595e-008	7	6.949785e-008
8	38	4.182589e-007	13	1.782e-007	12	4.871453e-008	5	5.435241e-008

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Talk A neurodynamic model for solving invex optimization problems pp.: 1–4

A Neurodynamic model for solving invex optimization problems

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Abstract

In this paper, a neural network model is constructed to solve general invex programming problems. Based on the Saddle point theorem, the equilibrium point of the proposed neural network is proved to be equivalent to the optimal solution of the invex programming problem. By employing Lyapunov function approach, it is also shown that this model is globally convergent and stable in the sense of Lyapunov at each equilibrium points. The simulation result shows that the proposed neural network is efficient.

Keywords: Invex function, Neural network, Nonconvex optimization, Global optimality conditions
Mathematics Subject Classification [2010]: 90C26, 90C30

1 Introduction and Preliminaries

Most of the theory and computational procedures in mathematical programming have been developed in which the various functions are convex. This is a severe limitation in practical applications and much effort has been devoted to removing this limitation. Usually, generalized convex functions have been introduced in order to weaken as much as possible the convexity requirements for results related to optimization theory, to optimal control problems, to variational inequalities, etc. A very broad generalization of convexity, now known as invexity, was introduced by Hanson [3].

Definition 1.1. Assume $X \subseteq \mathbb{R}^n$ is an open set.The differentiable function $f : X \to \mathbb{R}$ is invex function if there exists some function $\eta : X \times X \to \mathbb{R}^n$ such that for each $\boldsymbol{x}_1,\ \boldsymbol{x}_2 \in X$,

$$f(\boldsymbol{x}_2) \geq f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x})^{\mathrm{T}} \eta(\boldsymbol{x}_1, \boldsymbol{x}_2).$$

Consider the following optimization problem:

$$\begin{aligned} &\min\ f(\boldsymbol{x}) \\ &\text{s.t. } G(\boldsymbol{x}) \leq \mathbf{0}, \end{aligned} \tag{1}$$

where $G(\boldsymbol{x}) = [g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), ..., g_m(\boldsymbol{x})]$, f and g_i, $i = 1, ..., m$ are continuously differentiable functions. If f and g_i, $i = 1, ..., m$ be invex, then problem (1) is called invex programming problem.

*Speaker

Definition 1.2. f is said to be a pseudoinvex function if $\nabla f(\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{x}_1, \boldsymbol{x}_2) \geq 0$ implies that $f(\boldsymbol{x}_2) \geq f(\boldsymbol{x}_1)$. Similarly f is said to be a quasi-invex if $f(\boldsymbol{x}_2) \leq f(\boldsymbol{x}_1)$ implies that $\nabla f(\boldsymbol{x})^{\mathrm{T}}\eta(\boldsymbol{x}_1, \boldsymbol{x}_2) \leq 0$.

We say that f and g are η-invex, if a common η, with respect to which both f and g are invex, exists.

Theorem 1.3. *[2] Let $f : X \to \mathbb{R}$ be differentiable. Then f is invex if and only if every stationary point is a global minimizer.*

Corollary 1.4. *[2] If f has no stationary points, then f is invex.*

Theorem 1.5. *[7] Let $f_1\ , f_2\ , ..., f_m\ : X \to \mathbb{R}$ are all η-invex on the open set $X \subseteq \mathbf{R}^n$. Then:*

1. *For each $\alpha \in \mathbb{R}$, $\alpha > 0$, the function αf_i, $i = 1, ..., m$, is η-invex.*
2. *The linear combination of $f_1\ , f_2\ , ..., f_m$, with nonnegative coefficients is η-invex.*

Theorem 1.6. *[7] Let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be invex. f and g are η-invex if and only if $\forall \boldsymbol{x},\ \boldsymbol{y} \in X$ either*

(i) $\nabla f(\boldsymbol{x}) \neq \lambda \nabla g(\boldsymbol{x})$ for any $\lambda > 0$ or

(ii) $\nabla f(\boldsymbol{x}) = -\lambda \nabla g(\boldsymbol{x})$ for some $\lambda > 0$ and $f(\boldsymbol{y}) - f(\boldsymbol{x}) \geq -\lambda[g(\boldsymbol{y}) - g(\boldsymbol{x})]$.

Theorem 1.7. *[7] Let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be invex. f and g are η-invex if and only if $f + \lambda g$ is invex for all $\lambda > 0$.*

Consider problem (1) and let $\boldsymbol{x}^*$ be a feasible solution and $\mathcal{I} = \{i : g_i(\boldsymbol{x}^*) = 0\}$. Suppose that there exists scalar $\lambda^* \in \mathbb{R}^m_+$ such that $(\boldsymbol{x}^*, \lambda^*)$ satisfies the Karush-Kuhn-Tucker (KKT) conditions for problem (1).

Theorem 1.8. *[1] Let $\boldsymbol{x}^*$ be a KKT point. Then $\boldsymbol{x}^*$ is a optimal solution if one of the following conditions hold:*

(i) f and g_i for $i \in \mathcal{I}$ are all η-invex.

(ii) f is η-pseudoinvex and g_i, $i \in \mathcal{I}$, are η-quasi-invex.

2 Neurodynamic model

Let $\boldsymbol{x}(.)$, $\lambda(.)$ and $\boldsymbol{y}(.)$ be some time dependent variables. We propose a recurrent neural network model for solving (1), whose dynamical system for initial point $(\boldsymbol{x}_0^{\mathrm{T}}, \lambda_0^{\mathrm{T}})^{\mathrm{T}}$ is defined as follows:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -(\nabla f(\boldsymbol{x}) + \nabla G(\boldsymbol{x})^{\mathrm{T}}\lambda),$$
$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = (\lambda_i + g_i(\boldsymbol{x}))^+ - \lambda_i, \qquad i = 1, ..., m,$$

where $(\boldsymbol{z})^+ = [(z_1)^+, ..., (z_n)^+]^T$ and $(z_i)^+ = max\{0, z_i\}$. Define

$$H(\boldsymbol{y}) \quad = \quad \begin{bmatrix} -(\nabla f(\boldsymbol{x}) + \nabla G(\boldsymbol{x})^{\mathrm{T}}\lambda) \\ (\lambda + g(\boldsymbol{x}))^+ - \lambda \end{bmatrix}. \tag{2}$$

We propose the following neural network model:

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = MH(\boldsymbol{y}), \\ \boldsymbol{x}(t) = (I_n, \boldsymbol{0})\boldsymbol{y}(t) \end{cases} \tag{3}$$

where $\boldsymbol{y}(t) = (\boldsymbol{x}(t)^{\mathrm{T}}, \lambda(t)^{\mathrm{T}})^{\mathrm{T}}$ is the state vector, $\boldsymbol{x}(t)$ is the output vector and M is a nonsingular matrix.

Proposition 2.1. *Let Ω^* be a set of equilibrium points of the recurrent neural model (3) in $\mathbf{R}^{n+m}$. Then $\boldsymbol{y}^* \in \Omega^*$ if and only if the KKT conditions hold at $\boldsymbol{x}^*$ with multiplier λ^*.*

Lemma 2.2. *[6] For any initial point $\boldsymbol{y}_0$ there exists a unique continuous solution $\boldsymbol{y}(t)$ for model (3).*

Theorem 2.3. *Assume that there exists $\aleph \subseteq \mathbf{R}^{n+m}$ such that for any $\boldsymbol{y} = (\boldsymbol{x}^{\mathrm{T}}, \lambda^{\mathrm{T}})^{\mathrm{T}} \in \aleph$ we have the Jacobian matrix $\nabla H(\boldsymbol{y})$ of the mapping H defined in (2) is a negative semi-definite matrix. Let $\Omega^* \subseteq \aleph$, then*
(i) the equilibrium point of the proposed neural network model (3) is stable in the sense of Lyapunov,
(ii) the proposed neural network model (3) is globally convergent to the stationary point $\boldsymbol{y}^* = ((\boldsymbol{x}^*)^{\mathrm{T}}, \lambda^{*\mathrm{T}})^{\mathrm{T}}$ of (3), where $\boldsymbol{x}^*$ is the local optimal solution of the problem (1).

3 Simulation result

Example 3.1. [4] Consider an invex optimization problem as follows

$$\begin{aligned} &\min \;\; f(\boldsymbol{x}) = 1 + x_1^2 - \mathrm{e}^{x_2^2} \\ &\text{s.t.} \;\;\; x_1^2 - x_2 + 0.5 \le 0, \;\;\; 2x_2 - x_1^2 - 3 \le 0. \end{aligned}$$

The objective function f has one stationary point, namely $\boldsymbol{x}^* = (0,0)$, and $\boldsymbol{x}^*$ is a global minimizer of f, so f is invex as depicted in Figure 1. The constraint function $g_1(\boldsymbol{x}) = x_1^2 - x_2 + 0.5$ is convex and consequently is invex and $g_2(\boldsymbol{x}) = 2x_2 - x_1^2 - 3$ has no stationary point thus is invex. The feasible region $S = \{\boldsymbol{x} \in \mathbb{R}^2 | g_1(\boldsymbol{x}) \le 0, \;\; g_2(\boldsymbol{x}) \le 0\}$ is not a convex set. This invex optimization problem has a unique KKT point $(0, 0.5)$, which is the global minimum solution by Theorem 1.8. We solve this problem by using proposed model (3) with $M = I_{n+m}$. Simulation results show that the proposed neural network model can globally convergent to the global optimal solution to the invex optimization problem. Figure 2 (a) illustrates the transient behaviors of the proposed neural network from 10 random initial states. Figure 2 (b) shows the 2-dimensional phase plot from 10 random initial states.

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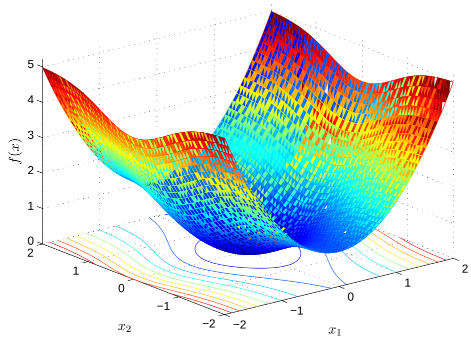
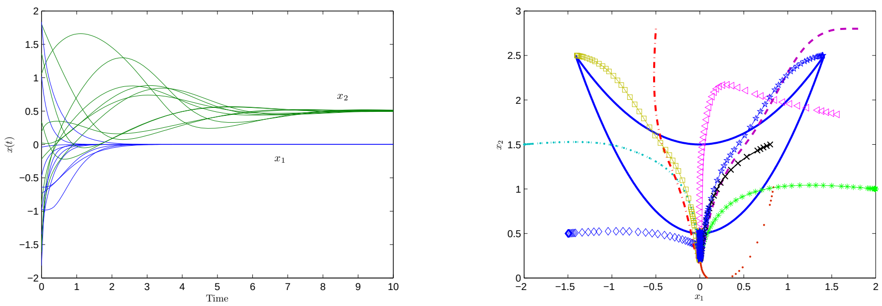


Figure 1: Objective function in Example 3.1



(a) Output trajectories (b) The feasible region

Figure 2: Transient behaviors of the proposed neural network for Example 3.1

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A New Nonstandard Finite Difference Scheme for Burger Equation

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Abstract

In this paper for the numerical solution of Burger equation, a nonstandard finite difference (NSFD) scheme is constructed. In continuation the main properties of NSFD schemes, i.e., positivity and boundedness, are established for proposed NSFD scheme. The efficiency of our scheme are demonstrated by presenting some numerical results.

Keywords: Boundedness, Burger equation, Nonstandard finite difference scheme, Positivity.
Mathematics Subject Classification [2010]: 39A14, 39A70, 65M06, 65M22

1 Introduction

The non-linear partial differential equation plays an important role in physical science and engineering. Recently, the non-linear equations have attracted much attention of researchers. There are various powerful mathematical methods, including the first integral method, the variational iteration method, the homotopic mapping method, the tanh method and the other methods have been proposed to obtain exact or approximate analytic solutions for the non-linear equations [1, 3]. In this paper we consider Burger equation of the form

$$u_t + uu_x = \mu u_{xx}, \tag{1}$$

where μ is the diffusion coefficient. Analytical solution of this equation is given by

$$u(x,t) = \frac{1}{1 + e^{\frac{1}{2\mu}(x - \frac{1}{2}t)}}. \tag{2}$$

In order to solve Burger equation numerically, many researchers have proposed various numerical methods. Among various techniques for solving partial differential equations, the NSFD schemes have been proved to be one of the most efficient approaches in recent years.

*Speaker

2 Analysis of NSFD scheme

NSFD schemes were firstly proposed by Mickens [2] for partial differential equations and, successively, their use have been investigated in several fields.

In the classical sense, the first derivative approximation can be represented as $u_t \cong (u_j^{n+1} - u_j^n)/\Delta t$ and $u_x \cong (u_{j+1}^n - u_j^n)/\Delta x$. In our sense, the discrete derivative is generalized as follows [2].

$$u_t \cong \frac{u_j^{n+1} - u_j^n}{\psi(\Delta t, \lambda)}, \qquad \psi(\Delta t, \lambda) = \Delta t + O((\Delta t)^2),$$
$$u_x \cong \frac{u_{j+1}^n - u_j^n}{\phi(\Delta x, \xi)}, \qquad \phi(\Delta x, \xi) = \Delta x + O((\Delta x)^2),$$

where λ, ξ are parameters that may be appeared in the differential equation and u_j^n is an approximation to $u(x_j, t_n)$. Similar to the classical difference scheme, we can obtain a NSFD scheme for the Burger equation as follows:

$$\frac{u_j^{n+1} - u_j^n}{\Psi} + \frac{1}{2}\frac{u_j^{n+1}(u_j^n - u_{j-1}^n) + u_j^n(u_j^n - u_{j-1}^n)}{\Phi} = \mu\frac{(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{\Gamma}. \tag{3}$$

Comparing equation (3) with equation (1), we note the non-linear on the left hand side of equation (1) is in the form

$$u(x_j, t_n)u_x(x_j, t_n) = \frac{1}{2}(2u(x_j, t_n)u_x(x_j, t_n)) \approx \frac{1}{2}\frac{u_j^{n+1}(u_j^n - u_{j-1}^n) + u_j^n(u_j^n - u_{j-1}^n)}{\Phi}.$$

By setting $R_1 = \Psi/\Phi$ and $R_2 = \Psi/\Gamma$, equation (3) can be rewritten in the following form

$$u_j^{n+1} = -\frac{(R_1u_j^n - R_1u_{j-1}^n + 4\mu R_2 - 2)u_j^n - 2\mu R_2u_{j+1}^n - 2R_2u_{j-1}^n}{R_1u_j^n - R_1u_{j-1}^n + 2}. \tag{4}$$

We can write the following theorem to ensure the positivity and boundedness.

Theorem 2.1. *If $1 - R_1 > 0$ and $2\mu R_2 < 1 - R_1$, the numerical solution (4) satisfies*

$$0 \leqslant u_j^n \leqslant 1 \Longrightarrow 0 \leqslant u_j^{n+1} \leqslant 1,$$

for all relevant values of n and j.

Proof. It is obvious that $-R_1(u_j^n)^2 - 4\mu R_2u_j^n + 2u_j^n < -R_1(u_j^n)^2 + R_1u_{j-1}^nu_j^n - 4\mu R_2u_j^n + 2u_j^n$ therefore if $R_1 < 2$ and $R_2 < \frac{1}{4}\frac{2-R_1}{\mu}$, then the discrete-time solution (4) is positive. Using the upside of (4) minus downside, we get

$$\begin{aligned}&-R_1(u_j^n)^2 + R_1u_j^nu_{j-1}^n - 4\mu R_2u_j^n + 2\mu R_2u_{j-1}^n + 2\mu R_2u_{j+1}^n\\&+2u_j^n - R_1u_j^n + R_1u_{j-1}^n - 2 \le -R_1(u_j^n)^2 + R_1u_j^n - 4\mu R_2u_j^n\\&+4\mu R_2 + 2u_j^n - R_1u_j^n + R_1 - 2\end{aligned}$$

Now, assumptions of Theorem imply

$$-R_1(u_j^n)^2 + R_1u_j^n - 4\mu R_2u_j^n + 4\mu R_2 + 2u_j^n - R_1u_j^n + R_1 - 2 < 0.$$

□

There are numerous choices for the stepsize functions Ψ, Φ and Γ, but according to the analytical solution of problem and Theorem 2.1 the stepsize functions $\Psi = 4\mu(1 - e^{-\frac{1}{4\mu}\Delta t})$, $\Phi = 2\mu(1 - e^{-\frac{1}{2\mu}\Delta x})$ and $\Gamma = 2\mu\Phi(e^{\frac{1}{2\mu}\Delta x} - 1)$ are good choices. With these choices of the stepsize functions and $\Delta t = \frac{1}{5}(\Delta x)^2$ the conditions of Theorem 2.1 still hold.

3 Numerical results

To verify the efficiency of proposed NSFD scheme we simulate the initial-boundary value problem:

$$\begin{array}{lll} u_t + uu_x = \mu u_{xx}, & 0 \le x \le 1, & t \ge 0, \\ u(x,0) = \frac{1}{1+e^{\frac{1}{2\mu}x}}, & 0 \le x \le 1, & \\ u(0,t) = \frac{1}{1+e^{-\frac{1}{4\mu}t}}, & t \ge 0, & \\ u(0,t) = \frac{1}{1+e^{\frac{1}{2\mu}-\frac{1}{4\mu}t}}, & t \ge 0. & \end{array} \tag{5}$$

The numerical results of the problem (5) are shown in Figs 1 and 2. The Figs. 1 and 2 compare the numerical results with the exact one for $\mu = 1.5$ and $\mu = 0.2$, respectively in 3D form up to time $t = 5$.

4 Conclusion

In this paper, we present a NSFD scheme for Burger equation based on the analytical solution. The proposed step function depends on Δx, Δt and NSFD scheme for Burger equation can be constructed using the method in Mickens papers. Numerical experiments for a particular example are given. The results show that the numerical solutions of our scheme meet the properties that the relevant solutions should have in their physical manner.

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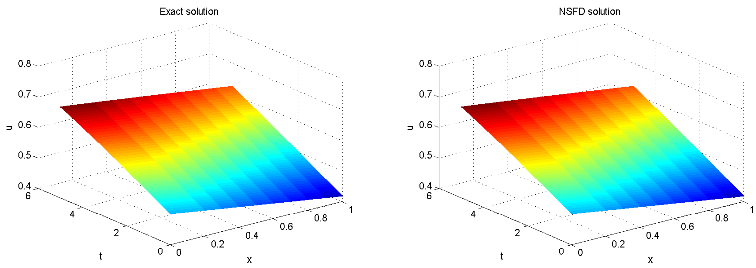
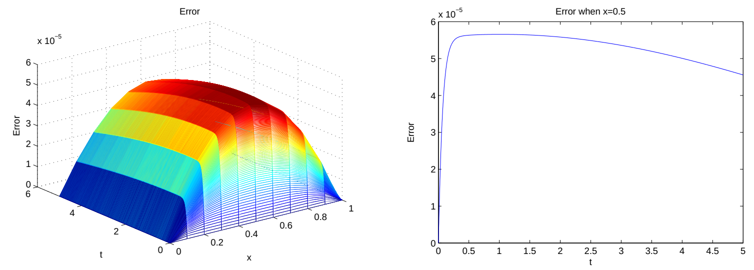
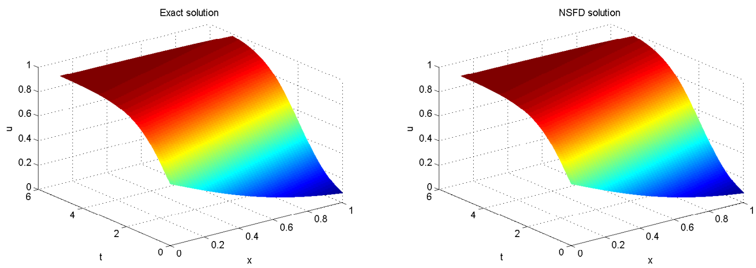


(a) Exact and NSFD solutions.

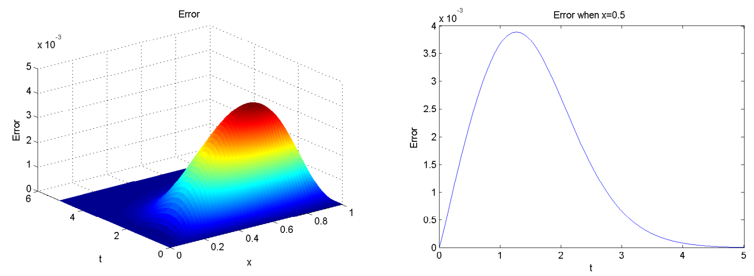


(b) Error between exact and NSFD solutions.

Figure 1: NSFD and exact solutions of the Burger equation for $\mu = 1.5$ with $\Delta x = 0.1$.



(a) Exact and NSFD solutions.



(b) Error between exact and NSFD solutions.

Figure 2: NSFD and exact solutions of the Burger equation for $\mu = 0.2$ with $\Delta x = 0.02$.

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A Numerical Method for Discrete Fractional–Order Chen System Derived from Nonstandard Numerical Scheme

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Abstract

In this paper, the nonstandard finite difference (NSFD) scheme is implemented to study the dynamic behaviors in the fractional–order Chen chaotic system. The Grünwald–Letnikov method is used to approximate the fractional derivatives. Numerical results show that the NSFD approach is easy to implement and accurate when applied to fractional-order Chen chaotic system.

Keywords: Chaos, Fractional calculus, Fractional–order Chen system, Nonstandard finite difference scheme
Mathematics Subject Classification [2010]: 37M05, 34A08, 34H10

1 Introduction

In the recent years there is increasing interest in fractional calculus which deals with integration/differentiation of arbitrary orders. The list of applications of fractional calculus has been evergrowing and includes control theory, viscoelasticity, diffusion, turbulence, electromagnetism and many other physical processes. An exhaustive treatment of fractional calculus in this respect can be found in references [2]. Recently, most of the dynamical systems based on the integer–order calculus have been modified into the fractional order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer–order modeling. The study of chaotic systems is an important aspect of dynamical systems that finds applications in different areas ranging from engineering to ecology. Although more than three decades have passed since the existence of "chaotic solutions" was demonstrated, still we do not have a theory of chaos from which the existence of chaotic solutions can be predicted. Extensive numerical work has been carried out in order to understand chaos in dynamical systems. Lu and Chen [1] have studied the dynamic of the fractional–order generalization of the well–known Chen system.

This paper is devoted to the construction of a nonstandard discretization scheme given by Mickens to the Grünwald–Letnikov (GL) discretization process for solving the fractional–order Chen chaotic system.

*Speaker

2 Preliminaries

Derivatives of fractional–order have been introduced in several ways. In this paper we consider GL approach. The GL method of approximation for the one–dimensional fractional derivative takes the following form [2]

$$D^q x(t) = f(t, x(t)), \quad x(0) = x_0, \quad t \in [0, t_f], \tag{1}$$

$$D^q x(t) = \lim_{h \to 0} h^{-q} \sum_{j=0}^{[t/h]} (-1)^j \binom{q}{j} x(t - jh),$$

where $0 < q \leq 1$, D^q denotes the fractional derivative, h is the step size and $[\frac{t}{h}]$ represents the integer part of $\frac{t}{h}$. Therefore, Eq. (1) is discretized in the next form

$$\sum_{j=0}^{n} c_j^q x(t_{n-j}) = f(t_n, x(t_n)), \qquad n = 1, 2, 3, ...$$

where $t_n = nh$ and c_j^q are the GL coefficients defined as

$$c_j^q = (1 - \frac{1+q}{j}) c_{j-1}^q, \qquad c_0^q = h^{-q}, \qquad j = 1, 2, 3, ...$$

The nonstandard discretization technique is a general scheme where we replace the step size h by a function $\phi(h)$ [3]. By applying this technique and using the GL discretization method, it yields the following relations

$$x(t_{n+1}) = c_0^{-q} \Big(- \sum_{j=1}^{n+1} c_j^q x(t_{n+1-j}) + f(t_{n+1}, x(t_{n+1})) \Big), \qquad n = 0, 1, \dots$$

where $c_0^q = \phi(h)^{-q}$.

3 NSFD scheme for fractional–order Chen chaotic system

Consider a fractional–order generalization of the Chen system [1]. In this system, the integer–order derivatives are replaced by fractional–order derivatives, as follows:

$$\begin{aligned} D^{q_1} x(t) &= a(y(t) - x(t)), \\ D^{q_2} y(t) &= (c - a)x(t) - x(t)z(t) + cy(t), \\ D^{q_3} z(t) &= x(t)y(t) - bz(t), \end{aligned}$$

where $0 < q_i \leq 1$, for $i = 1, 2, 3$ and x, y, z are the state variables and $(a, b, c) \in \mathbb{R}^3$. If $q_1 = q_2 = q_3 = q$ then the Chen system is called commensurate otherwise incommensurate, a minimal order q for chaotic behavior can be determinned [1] and it is $q \geq 0.8244$. This system is equivalent to the classical integer–order Chen system when $q = 1$, which is chaotic at $(a, b, c) = (35, 3, 28)$. The stability analysis of such kind of system have been studied in [1].

Applying Mickens scheme by replacing the step size h by a function $\phi(h)$ and using the GL discretization method, it can be seen that

$$\begin{aligned}
\sum_{j=0}^{n+1} c_j^{q_1} x(t_{n+1-j}) &= a(y(t_n) - x(t_{n+1})), \\
\sum_{j=0}^{n+1} c_j^{q_2} y(t_{n+1-j}) &= (c-a)x(t_{n+1}) - x(t_{n+1})z(t_n) + cy(t_n), \qquad (2)\\
\sum_{j=0}^{n+1} c_j^{q_3} z(t_{n+1-j}) &= x(t_{n+1})y(t_{n+1}) - bz(t_{n+1}).
\end{aligned}$$

Invoking some algebraic manipulations to Eqs. (2), the following relations are obtained

$$x(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{q_1} x(t_{n+1-j}) + ay(t_n)}{c_0^{q_1} + a},$$

$$y(t_{n+1}) = c_0^{-q_2}\Big(- \sum_{j=1}^{n+1} c_j^{q_2} y(t_{n+1-j}) + (c-a)x(t_{n+1}) - x(t_{n+1})z(t_n) + cy(t_n)\Big),$$

$$z(t_{n+1}) = \frac{-\sum_{j=1}^{n+1} c_j^{q_3} z(t_{n+1-j}) + x(t_{n+1})y(t_{n+1})}{c_0^{q_3} + b},$$

where

$$c_0^{q_1} = \phi_1(h)^{-q_1}, \qquad c_0^{q_2} = \phi_2(h)^{-q_2}, \qquad c_0^{q_3} = \phi_3(h)^{-q_3},$$

with

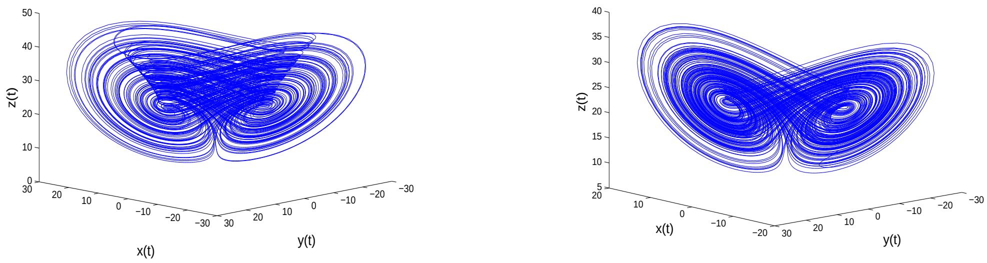
$$\phi_1(h) = \frac{e^{ah-1}}{a}, \qquad \phi_2(h) = \frac{e^{bh-1}}{b}, \qquad \phi_3(h) = \frac{e^{ch}-1}{c}.$$

4 Numerical results and conclusion

In this section, numerical results from the implementation of NSFD scheme for the fractional–order Chen chaotic system are presented.

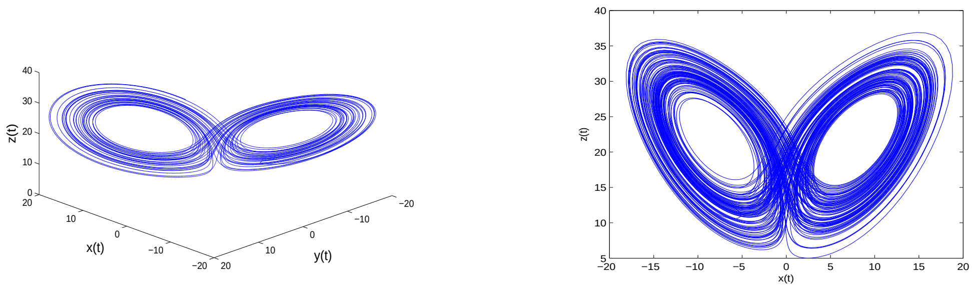
In Fig. 1a and Fig. 1b are depicted the simulation results of the Chen system, where system parameters are $a = 35$, $b = 3$ and $c = 28$ commensurate orders of the derivatives are $q = 1$ and $q = 0.9$ with the initial conditions are $(x_0, y_0, z_0) = (-9, -5, 14)$ for the simulation time $t = 100$s and time step $h = 0.005$.

In Fig. 2a and Fig. 2b are depicted the simulation results of the Chen system, where system parameters are $a = 35$, $b = 3$ and $c = 28$ incommensurate orders of the derivatives are $q_1 = 0.8$, $q_2 = 1$, $q_3 = 0.9$ with the initial conditions are $(x_0, y_0, z_0) = (-9, -5, 14)$ for the simulation time $t = 30$s and time step $h = 0.001$.



(a) $q = 1$ (b) $q = 0.95$

Figure 1: Simulation result of the Chen system in state space for parameters: $a = 35$, $b = 3$, $c = 28$ with initial conditions $(x_0, y_0, z_0) = (-9, -5, 14)$.



(a) $q_1 = 0.8$, $q_2 = 1$, $q_3 = 0.9$ (b) $q_1 = 0.8$, $q_2 = 1$, $q_3 = 0.9$

Figure 2: Chaotic attractor of the Chen system projected into 3D state space and 2D phase planes parameters: $a = 35$, $b = 3$, $c = 28$ with initial conditions $(x_0, y_0, z_0) = (-9, -5, 14)$.

From the graphical results in Figs. 1 and 2, it is concluded that the approximate solutions obtained using Mickens nonstandard discretization method is in good agreement with the approximate solutions obtained in [1].

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A reliable algorithm based on the Sumudu transform for solving partial differential equations

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Abstract

In this paper, a new combination of the Adomian decomposition method and the Sumudu transform (ADST) is introduced for solving nonlinear partial differential equations (PDEs). The main objective of this paper is to present a reliable approach to compute an approximate solution of PDEs.

Keywords: Sumudu transform, Adomian decomposition method
Mathematics Subject Classification [2010]: 65Mxx, 34A34

1 Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in many fields of applied sciences, such as chemistry, physics, fluid dynamics, plasma physics, hydrodynamics and engineering disciplines. The application of the Adomian decomposition method (ADM) [1], in nonlinear problems has been used by scientists and engineers, since this method continuously deform the under study nonlinear equation into a simple problem which is easy to solve. In recent years, Wazwaz etc., [2], improved the ADM and expanded fields of its application. Recently, Watugala introduced a new transform and named it as Sumudu transform. This transform is used to find the solution of ordinary differential equations and control engineering problems, [3]. Very recently, Singh et al. [4], have proposed a new approach named homotopy perturbation Sumudu transform method (HPSTM) to solve the nonlinear partial differential equations. The homotopy perturbation Sumudu transform method (HPSTM) is a combination of Sumudu transform method, HPM and Hes polynomials and is mainly due to Ghorbani [5] Singh and Shishodia [6]. The basic motivation of this paper is to propose a new modification of ADM and Sumudu transform algorithm. By using this new method, which is a combination of the Adomian decomposition method and Sumudu transform ADST, all conditions will be satisfied.

*Speaker

2 Methodology: Analysis of this method

In this section, the basic idea of the Adomian decomposition Sumudu transform method ADST will be given. Consider a nonlinear non-homogenous partial differential equation of the form

$$[L + R + N]\, f\,(t_1, ..., t_m) = g\,(t_1, ..., t_m)\,, \tag{1}$$

where the differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, N expresses the nonlinear terms, f is an unknown function and $g(t_1, ..., t_m)$ is an inhomogeneous term. We can rewrite Eq. (1) down a correction functional as follows

$$L_{t_i}\,[f\,(t_1, ..., t_m)] = g\,(t_1, ..., t_m) - [R + N]\, f\,(t_1, ..., t_m)\,, \tag{2}$$

where $L_{t_i}\,[f\,(t_1, ..., t_m)] = \frac{\partial^h}{\partial {t_i}^h}\,[f\,(t_1, ..., t_m)]$ and h is the order of differential operator L. Applying Sumudu transform on both sides of Eq. (2) one get

$$F\,(u\,(t_1, ..., t_m)) = \sum_{k=0}^{h-1} u^k f^{(k)}\,(t_1, ..., t_{i-1}, 0, t_{i+1}, ..., t_m) + u^h S\,[g, t_i; u] - u^h S\,[[R + N]\, f, t_i; u]\,. \tag{3}$$

Now applying the inverse Sumudu transform on both sides of Eq. (3) and also by using the convolution theorem, we have

$$\begin{aligned} f\,(t_1, ..., t_m) = {} & S^{-1}\left[\sum_{k=0}^{h-1} u^k f^{(k)}\,(t_1, ..., t_{i-1}, 0, t_{i+1}, ..., t_m) + u^h S\,[g, t_i; u]\,, u; t_i\right] \\ & - \int_0^{t_i} [R + N]\, f\,(t_1, ..., t_{i-1}, \xi, t_{i+1}, ..., t_m)\, w\,(t_i - \xi) d\xi, \end{aligned} \tag{4}$$

for simplicity put $B = S^{-1}\left[\sum\limits_{k=0}^{h-1} u^k f^{(k)}\,(t_1, ..., t_{i-1}, 0, t_{i+1}, ..., t_m) + u^h S\,[g, t_i; u]\,, u; t_i\right]$. Let $f\,(t_1, ..., t_m) = \sum\limits_{n=0}^{\infty} f_n\,(t_1, ..., t_m)$ and $Nf\,(t_1, ..., t_m) = \sum\limits_{n=0}^{\infty} A_n\,[(f_0, ..., f_n)\,(t_1, ..., t_m)]$ where for every k, A_k is the Adomian polynomial. Substituting these relations to Eq. (4), one obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} f_n\,(t_1, ..., t_m) = B \\ & - \int_0^{t_i} \left\{ R\left[\sum_{n=0}^{\infty} f_n\,(t_1, ..., \xi, ..., t_m)\right] + \sum_{n=0}^{\infty} A_n\,(f_0, ..., f_n)\,(t_1, ..., t_m) \right\} w\,(t_i - \xi) d\xi. \end{aligned} \tag{5}$$

Accordingly, by the Adomian decomposition method we can obtain the following recursively formula

$$\begin{aligned} & f_0\,(t_1, ..., t_m) = B, \\ & f_{n+1}\,(t_1, ..., t_m) = - \int_0^{t_i} \{R\,[f_n\,(t_1, ..., \xi, ..., t_m)] + A_n\,[(f_0, ..., f_n)\,(t_1, ..., \xi, ..., t_m)]\}\, w\,(t_i - \xi) d\xi, \end{aligned}$$

Example 2.1. Consider the nonlinear PDE of the form

$$f_{yy}+f_x^2+f-f^2=ye^{-x},\quad f(x,0)=0,\quad f_y(x,0)=e^{-x}.$$

Applying Adomian decomposition Sumudu transform method and using the initial conditions result in

$$f_0\left(x,y\right)=B=S^{-1}\left[f(x,0)+uf_x(x,0)+u^2S\left[ye^{-x},y;u\right],u;y\right]=ye^{-x}+\frac{1}{3!}y^3e^{-x},$$
$$f_{n+1}\left(x,y\right)=-\int_0^y\left\{f_n\left(x,\xi\right)+A_n\left[\left(f_0,...,f_n\right)\left(x,\xi\right)\right]\right\}(y-\xi)d\xi=-\int_0^y\psi[n;\xi]\left(y-\xi\right)d\xi,$$

where $\psi[n;\xi]=f_n\left(x,\xi\right)+A_n\left[\left(f_0,...,f_n\right)\left(x,\xi\right)\right]$ and A_n are Adomian polynomials that represent the nonlinear term $f_x^2-f^2$, and given by

$$A_0=f_{0_x}^2-f_0^2,\quad \psi[0;\xi]=f_0+f_{0_x}^2-f_0^2,$$
$$A_1=2f_{0_x}f_{1_x}-2f_0f_1,\quad \psi[1;\xi]=f_1+2f_{0_x}f_{1_x}-2f_0f_1,$$
$$A_2=f_{1_x}^2+2f_{0_x}f_{2_x}-f_1^2-2f_0f_2,\quad \psi[2;\xi]=f_2+f_{1_x}^2+2f_{0_x}f_{2_x}-f_1^2-2f_0f_2,$$

Applying the recursive relation, we obtain

$$f_1=-\int_0^y\frac{1}{6}\xi e^{-x}\left(6+\xi^2\right)d\xi=-\frac{1}{3!}y^3e^{-x}-\frac{1}{5!}y^5e^{-x},$$
$$f_2=-\int_0^y-\frac{1}{5!}\xi^3e^{-x}\left(20+\xi^2\right)d\xi=\frac{1}{5!}y^5e^{-x}+\frac{1}{7!}y^7e^{-x},$$
$$f_3=-\int_0^y\frac{1}{7!}\xi^5e^{-x}\left(42+\xi^2\right)d\xi=-\frac{1}{7!}y^7e^{-x}-\frac{1}{9!}y^9e^{-x},$$

Therefore, the solution in a series form is given by

$$f\left(x,y\right)=ye^{-x}+\frac{1}{3!}y^3e^{-x}-\frac{1}{3!}y^3e^{-x}-\frac{1}{5!}y^5e^{-x}+\frac{1}{5!}y^5e^{-x}+\frac{1}{7!}y^7e^{-x}+\cdots.$$

Which is converge to closed form solution $f\left(x,y\right)=ye^{-x}$.

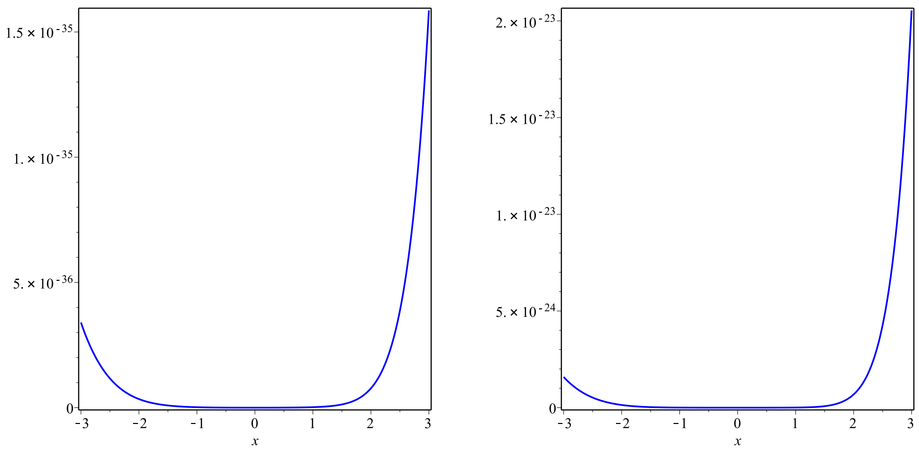
Example 2.2. Consider the following nonlinear partial differential equation

$$f_{yy}=f_{xx}+f+f^2-xy\left(1+xy\right),\quad 0\le x\le\pi,\ 0\le y<1, \tag{6}$$

subjected to the following boundary and initial conditions

$$BC:\begin{cases}f(0,y)=0,\\ f(\pi,y)=\pi y,\end{cases},\quad IC:\begin{cases}f(x,0)=0,\\ f_y(x,0)=x.\end{cases}$$

By using the Adomian decomposition Sumudu transform method, we use fifteen terms approximation and hence, $f(x,y)=\sum_{i=0}^{14}f_i(x,y)$, to examine the accuracy of the ADST. The absolute errors of the 15-terms approximate solutions are plotted in figures 1(a)-(b).



(a) $y = 0.25$ (b) $y = 0.50$

Figure 1: Graphs of the absolute error functions for different values of y

3 Conclusion

This paper is about a new combination of the Sumudu transform technique and the Adomian decomposition method for solving nonlinear partial differential equations. The capabilities of the proposed method were demonstrated by some tested problems. It is concluded from the given figures that the ADST is an accurate and efficient algorithm to solve the nonlinear differential equations.

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Talk A spectral method for the solution of KdV equation via orthogonal rational . . . pp.: 1–4

A spectral method for the solution of KdV equation via orthogonal rational basis functions

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Abstract

In this paper, a set of orthogonal rational Chebyshev functions in $L^2(0,+\infty)$ is generated by the orthogonal Chebyshev polynomials. Moreover a new computational method based on these new basis functions is proposed for solving KdV equations on the semi–infinite interval with initial–boundary conditions. In this way, a weak formulation for the above mentioned problems is obtained, and also a Galerkin method using these basis functions is applied. Some numerical examples are included for demonstrating the efficiency of the method.

Keywords: Partial differential equations, KdV equation, Spectral methods, Chebyshev polynomials, Orthogonal rational Chebyshev functions.
Mathematics Subject Classification [2010]: 35Q53, 65N12

1 Introduction

In 1895 Kurteweg and de Vries proposed the equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

as a model for water waves in shallow regions. This equation, which has been known as KdV equation [8], is a well-known equation in the field of nonlinear waves. In 1965 Zabusky and Kruskal used the leap-frog method for discretizing the KdV equation [9].Two years later, Gardner, Greene, Kruskal and Miura discovered that assuming the solutions decay at infinity with sufficient rates, equation (1) can be efficiently solved via a method called the Inverse Scattering Method [5]. In 1982, Christov [4] and Boyd [3, 2] developed some spectral methods on infinite intervals by using orthogonal systems of rational functions. In 2000, Guo [7] developed a rational spectral method based on a weighted orthogonal system consisting of rational function built from Legendre polynomials with a rational transformation. Recently, Zhang and Ma [10] proposed a combined Petrov-Galerkin scheme using orthogonal Legendre rational functions for solution of the following problem:

$$\begin{cases} u_t + uu_x + u_{xxx} = f(x,t)\ ,\ x \in [0,+\infty]\ ,\ t \in (0,T] \\ u(0,t) = \lim\limits_{x\to+\infty} u(x,t) = \lim\limits_{x\to+\infty} u_x(x,t) = 0\ , \\ u(x,0) = u_0(x). \end{cases} \tag{2}$$

*Speaker

In this paper, a set of orthogonal rational Chebyshev functions in $L^2(0,+\infty)$ is generated by using the orthogonal Chebyshev polynomials. Moreover a new computational method based on these new basis functions is proposed for solving KdV equations on the semi–infinite interval with initial–boundary conditions. In this method, a weak formulation for the above mentioned problems is obtained, and also a Galerkin method using these basis functions is applied.

2 Rational Chebyshev functions

The rational Chebyshev function of order n is defined on $[0,+\infty)$ by the formula:

$$R_n(x) = \frac{1}{x+L} T_n\left(\frac{x-L}{x+L}\right), \quad n \in \mathbb{N} \cup \{0\} \tag{3}$$

where the parameter L sets the length scale of the mapping and $T_n(x)$ is the Chebyshev polynomial. The rational Chebyshev functions are orthogonal on $[0,+\infty)$ with respect to the weight function $w_R(x) = (x+L)\sqrt{\frac{L}{x}}$ and we have

$$\int_0^\infty R_m(x)R_n(x)w_R(x)dx = \begin{cases} \pi\ , & m=n=0 \\ \frac{\pi}{2}\ , & m=n\neq 0 \\ 0\ , & m\neq n \end{cases} \tag{4}$$

3 Explanation of the method

We consider the inhomogeneous KdV equation

$$u_t + uu_x + u_{xxx} = f(x,t)\ ,\ x\in[0,+\infty]\ ,\ t\in(0,T] \tag{5}$$

accompanied with the initial-boundary conditions:

$$u(0,t) = \lim_{x\to+\infty} u(x,t) = \lim_{x\to+\infty} u_x(x,t) = 0,\ u(x,0) = u_0(x). \tag{6}$$

For non-negative integer N, we define $R_N = \text{span}\{R_n(x)|n=0,1,...,N\}$ where $R_n(x)$ is the rational Chebyshev function introduced in (3). Also we put $R_N^0 = R_N \cap H_0^1(0,+\infty)$ where $H_0^1(0,+\infty) = W_0^{1,2}(0,+\infty)$ and $W_0^{1,2}(0,+\infty)$ is a special case of $W_0^{s,p}(0,+\infty)$, that is the closure of the space $C_0^\infty(0,+\infty)$ in the Sobolev space $W^{s,p}(0,+\infty)$ [1].

Now we first obtain the weak formulation for problem 5. In this direction we consider the test functions space:

$$T = \{v\in H_0^1[0,+\infty)|v(0,t) = v_x(0,t) = \lim_{x\to+\infty} v(x,t) = \lim_{x\to+\infty} v_x(x,t) = 0\}. \tag{7}$$

Then for any $v\in T$ we have:

$$\int_0^\infty u_t v dx + \int_0^\infty uu_x v dx + \int_0^\infty u_{xxx} v dx = \int_0^\infty fvdx. \tag{8}$$

Integrating $\int_0^\infty uu_x vdx$ and $\int_0^\infty u_{xxx}vdx$ by parts we have:

$$\int_0^\infty u_t v dx - \frac{1}{2}\int_0^\infty u^2 v_x dx - \int_0^\infty uv_{xxx}dx = \int_0^\infty fvdx \tag{9}$$

Finally in the weak formulation for problem (5) we are seeking a function $u \in H^2(0,+\infty) \cap H_0^1(0,+\infty)$ such that for any $v \in H_0^1(0,+\infty)$, we have:

$$\begin{cases} (u_t, v) - \frac{1}{2}(u^2, v_x) - (u, v_{xxx}) = (f, v) \ , \quad t \in (0, T] \\ (u(0), v) = (u_0, v) \end{cases} \tag{10}$$

where $(u, v) = \int_0^\infty u(x)v(x)w_R(x)dx$.
A discrete spectral method for solution of problem (5) is to find $u_N \in R_N^0$ such that for any $v \in R_N^0$ we have:

$$\begin{cases} (\partial_t u_N, v) - \frac{1}{2}(P_N^C u_N^2, \partial_x v) - (u, \partial_x^3 v) = (P_N^C f, v) \ , \quad t \in (0, T] \\ (u(0), v) = (P_N^C u_0, v) \end{cases} \tag{11}$$

where $P_N^C u(x) = (1-y) I_N^C \frac{v(y)}{1-y}$ where I_N^C is the Chebyshev–Gauss interpolation operator on $(-1, 1)$, and $y = \frac{x-L}{x+L}$ such that $v(y) = u(x)$. Now, we propose a numerical scheme for the solution of (11) by discretizing the KdV equation. Dividing the interval $(0, T]$ in n equal parts with lengths $\Delta t = \frac{T}{n}$ and putting $t_k = k\Delta t$, $k = 0, 1, ..., n$ and using the symbols:

$$u^k = u^k(x) = u(x, t_k), \ \overline{u}^k = \frac{u^{k+1} + u^{k-1}}{2}, \ \overline{u}_t^k = \frac{u^{k+1} - u^{k-1}}{2\Delta t}, \tag{12}$$

we apply the Crank-Nicolson and leap-frog schemes on discrete KdV equation:

$$\begin{cases} \left(\overline{u}_t^k, v\right) - \frac{1}{2}\left(P_N^C(\overline{u}_N^k)^2, \partial_x v\right) - \left(\overline{u}_N^k, \partial_x^3 v\right) = \left(P_N^C \overline{f}^k, v\right) \ , \quad t \in (0, T] \\ (u(0), v) = (P_N^C u_0, v). \end{cases} \tag{13}$$

To access a more efficient algorithm, we choose a suitable set of basis functions, and put [6]: $\phi_n(x) = R_n(x) + R_{n+1}(x)$ and $\psi_n(x) = \frac{2}{1+x}\phi_n(x)$ Then we can write u_N^k in terms of ψ_n's:

$$u_N^k(x) = \sum_{n=0}^{N-2} \overline{u}_n^k \psi_n(x) \tag{14}$$

putting $v = \phi_m(x)$, $0 \le m \le N-2$ in relation (13), we obtain the following linear system $(A + \Delta t B)\overline{u}^{k+1} = g^k$ where $A_{mn} = (\psi_n, \phi_m)$, $B_{mn} = -\left(\psi_n, \partial_x^3 \phi_m\right)$ and

$$g^k = \sum_{n=0}^{N-2} \overline{u}_n^{k-1}\left(\psi_n - \Delta t\, \partial_x^3 \phi_n\right) + \Delta t\, \partial_x P_N^C \left(u_N^k\right)^2 + 2\Delta t\, P_N^C \overline{f}^k \tag{15}$$

4 Main results

In this section, two test problems will be solved by using the above method.

Example 4.1. [10] We consider the following KdV equation

$$u_t + uu_x + u_{xxx} = f(x, t) \ , \ x \in [0, +\infty] \ , \ t \in (0, T] \tag{16}$$

$$u(0, t) = \lim_{x \to +\infty} u(x, t) = \lim_{x \to +\infty} u_x(x, t) = 0,$$

$$u(x, 0) = sech^2(ax - c),$$

$$f(x, t) = \frac{-2\sinh(ax - bt - c)\left(4a^3\cosh^2(ax - bt - c) - b\cosh^2(ax - bt - c) - 12a^3 + a\right)}{\cosh^5(ax - bt - c)}$$

with the exact solution $u(x, t) = sech^2(ax - bt - c)$ where $a = b = 1$, $c = 0$.

Table 1: Errors for test problem 1

N	τ	L^∞ error	L^2 error
16	1E-3	9.8652E-3	1.2654E-2
32	1E-3	4.1536E-6	8.2874E-6
48	1E-3	1.0356E-7	3.9851E-7
64	1E-3	1.0852E-7	6.8523E-7

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An approximation of a two-dimensional Volterra-Fredholm integral equations via Inverse Multiquadric RBFs

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Abstract

The main purpose of this article is to present an approximate solution for the mixed two-dimensional nonlinear Volterra-Fredholm integral equations using inverse multiquadric (IMQ) functions as two-dimensional RBFs. In this method, we interpolate the given function by these RBFs. Also we obtain good results for error by different shape parameters in comparison with the approximation by multiquadric (MQ) RBFs. The numerical results are compared with MQ method to display efficiency of the proposed method.

Keywords: Mixed volterra-Fredholm integral equation, Inverse Multiquadric, Multiquadric, Radial basis function.
Mathematics Subject Classification [2010]: 65R20, 45D05, 45B05

1 Introduction

Integral equations have recieved considerable interest in the mathematical applications in different areas of sciences. RBFs interpolations were evaluated as the most accurate techniques. This method allows scattered data to be easily used in computation. There are many works on developing and analyzing numerical methods for solving Volterra-Frdeholm integral equations (IE) in [5, 6]. Alipanah et. al. [1], used RBFs method for solving a nonlinear integral equation in the one-dimensional case. Here we want to propose a method to approximate a class of mixed two-dimensional nonlinear Volterra-Fredholm integral equations on the interval $[-1, 1]$ by using IMQs radial basis functin.
The outline of this paper is as follows: At first we introduce the Volterra-Fredholm IEs, and IMQs interpolation. Next we describe the Legendre-Gauss-Lobatto quadrature, briefly. In the next section we discuss how to solve the integral equation by using the suggested method. In section 3 one numerical example shows the accuracy of the method.

1.1 Preliminaries and notaions

In this paper, we consider a mixed Volterra-Fredholm integral equation

$$f(s,t) = g(s,t) + \int_0^s \int_0^1 U\Big(s,t,x,y,f(x,y)\Big)dydx, \qquad (1)$$

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where $f(s,t)$ is an unknown function, $g(s,t)$ is a continuous function defined on $[0,T] \times [0,1]$, and $U\big(s,t,x,y,f(x,y)\big)$ is defined on $S = \{(s,t,x,y,f) : 0 \leq x \leq s \leq T, t,y \in [0,1]\}$. We transform the interval $[0,T]$ to $[-1,1]$ and assume
$U\Big(s,t,x,y,f\Big) = K(s,t,x,y)[f(s,t)]^p$ for the given positive integer p.
Suppose $\phi(x,y)$ be the IMQ radial function. Then we can interpolate $f(x,y)$ as

$$f(x,y) \simeq \sum_{i=0}^{N} \sum_{j=0}^{M} C_{ij}\phi_{ij}(x,y) = C^T \psi(x,y), \tag{2}$$

where

$$\phi_{ij} = \phi_{ij}(x,y) = \phi(\|(x,y)-(x_i,y_j)\|) = \frac{1}{\sqrt{(\|(x,y)-(x_i,y_j)\|)^2 + c^2}}, \tag{3}$$

$$i = 0,\cdots,N, \qquad j = 0,\cdots,M.$$

where (x_i, y_i) are the Legendre-Gauss-Lobatto nodes [2]. Let p_N be the Legendre polynomials of order N on the interval $[-1,1]$. The Legendre-Gauss-Lobatto nodes L-G-L are

$$x_0 = -1 < x_1 < \cdots < x_{N-1} < x_N = 1, \tag{4}$$

where x_m, $1 \leq m \leq N-1$ are the zeros of $\dot{p}_N(x)$ and $\dot{p}_N(x)$ is derivative it. There is no explicit formula for calculating the nodes x_m , but they are computed numerically using the existing subroutines. Also $\int_{-1}^{1} f(x)dx = \sum_{i=0}^{N} w_i f(x_i)$ where $w_i = \frac{2}{N(N+1)} \cdot \frac{1}{(p_N(x_i))^2}$ and x_i, are the L-G-L weights and nodes, respectively [2].

2 Solving the problem

Consider the above Volterra-fredholm IEs (1). We produce

$$C^T\psi(s,t) = g(s,t) + \int_0^s \int_0^1 U\Big(s,t,x,y,C^T\psi(x,y)\Big)dydx. \tag{5}$$

We transform the the region to $[-1,1]$ taking the change of variables $\eta_1 = \frac{2}{s_i}x - 1$ and $\eta_2 = 2y - 1$. Next by using the L-G-L quadrature we have

$$C^T\psi(s_i,t_j) = g(s_i,t_j) + \frac{s_i}{4}\sum_{k=0}^{r_1}\sum_{l=0}^{r_2} w_k w_l U\Big(s_i,t_j,\frac{s_i}{2}(\eta_1+1),\frac{\eta_2+1}{2},C^T\psi(\frac{s_i}{2}(\eta_1+1),\frac{\eta_2+1}{2})\Big), \tag{6}$$

$$i = 0,\cdots,N, \qquad j = 0,\cdots,M.$$

Eq (6) generates a nonlinear system of equations that can be solved by the Newton's iteration method.

3 Illustrative example

In this section, one numerical example is included to demonstrate the validity and efficiency of the proposed technique. In order to demonstrate the error of method, we introduce the notation $e(x, y) = |f(x, y) - \bar{f}(x, y)|$ on the interval $[0, 1] \times [0, 1]$ where $f(x, y)$ and $\bar{f}(x, y)$ are the exact and approximate solutions, respectively.

Example 3.1. consider a nonlinear Volterra-Fredholm IE [6]
f(s,t)=s$^2 e^{2t} - 1/5s^5 + t^2 + \int_0^s \int_0^1 t^2 e^{-4x}[f(x, y)]^2 dydx, \quad 0 \le s < 1$.
We apply the presented method and solve the Eq (3.1). Numerical results are presented in Tables (1), (2) and Figure (1). Table (2) shows the error $e(x, y)$ at L-G-L points together with the obtained results by the method of [3].

Table 1: Errors for example (3.1) with $c = 0.4, 1.4$ for $N = 2, 3$.

(s, t)	$N = 2, \quad c = 0.4$	$N = 2, \quad c = 1.4$	$N = 3, \quad c = 0.4$	$N = 3, \quad c = 1.4$
$(0, 0)$	$6.6407743E-02$	$3.3231627E-02$	$2.8317723E-02$	$4.5141715E-03$
$(0.1, 0.1)$	$1.6736290E-02$	$6.5679240E-03$	$2.4730265E-03$	$8.8969007E-03$
$(0.2, 0.2)$	$5.6443366E-03$	$6.0700325E-02$	$3.0909447E-02$	$3.8696836E-02$
$(0.3, 0.3)$	$2.8808737E-02$	$1.1655871E-01$	$9.4390555E-02$	$9.3426757E-02$
$(0.4, 0.4)$	$9.2085286E-02$	$1.7603972E-01$	$1.5690233E-01$	$1.7043149E-01$
$(0.5, 0.5)$	$2.6640362E-01$	$2.6737589E-01$	$2.0808878E-01$	$2.6555843E-01$
$(0.6, 0.6)$	$6.4390002E-01$	$4.3144954E-01$	$3.0491358E-01$	$3.8220809E-01$
$(0.7, 0.7)$	$1.1866665E+00$	$6.7102704E-01$	$6.1011560E-01$	$5.4014225E-01$
$(0.8, 0.8)$	$1.4910882E+00$	$8.8205118E-01$	$1.1707949E+00$	$7.4794170E-01$
$(0.9, 0.9)$	$6.5492677E-01$	$8.0901973E-01$	$1.2352508E+00$	$9.1729544E-01$

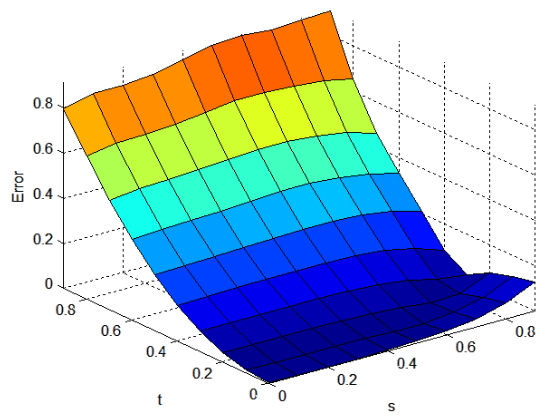

Figure 1: Errors for example (3.1) with $c = 0.7$ for $N = 5$.

Table 2: Errors for example (3.1) with L-G-L as points for $N = 4$, $c = 0.4$ and $N = 5$, $c = 0.6$.

(s, t)	IMQ $N = 4$	IMQ $N = 5$	MQ$N = 4$	MQ $N = 5$
$(0.9869533, 0.9869533)$	$4.6647953E-01$	$9.2278386E-01$	$8.094732E-01$	$9.908731E-01$
$(0.9325317, 0.9325317)$	$1.1242569E+00$	$9.9628259E-01$	$9.958854E-01$	$9.476697E-01$
$(0.8397048, 0.8397048)$	$9.8781039E-01$	$7.6975636E-01$	$8.241093E-01$	$7.636676E-01$
$(0.7166977, 0.7166977)$	$4.7715018E-01$	$5.3442056E-01$	$5.489545E-01$	$5.587436E-01$
$(0.5744372, 0.5744372)$	$3.1408675E-01$	$3.6335572E-01$	$3.617686E-01$	$3.530029E-01$
$(0.4255628, 0.4255628)$	$2.1037328E-01$	$1.9398081E-01$	$1.853911E-01$	$1.815405E-01$
$(0.2833023, 0.2833023)$	$9.0310392E-02$	$8.4702606E-02$	$7.947958E-02$	$8.810770E-02$
$(0.1602952, 0.1602952)$	$1.9001185E-02$	$2.6175707E-02$	$2.752501E-02$	$2.581437E-02$
$(0.6746832, 0.6746832)$	$1.6014702E-04$	$3.5617186E-03$	$2.861469E-03$	$1.743795E-03$
$(0.1304674, 0.1304674)$	$1.0960145E-02$	$1.3724430E-03$	$7.603924E-03$	$3.630280E-03$

4 Conclusion

In this paper we apply the IMQ method for the numerical solution a class of mixed two-dimensional nonlinear Volterra-Fredholm integral equations, by different shape parameters and compare the results with MQ method. The results show validity and good accurate of the proposed method.

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Comparison between the Direct and local discontinuous Galerkin methods for the third order kdv equation

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Abstract

In this article, a class of Discontinuous Galerkin method(DG) for solving KdV equation containing the third derivative term in one space dimention, which is called Direct Discontinuous Galerkin (DDG) method has been mainly discussed. Numerical examples are shown to illustrate the accuracy and capability of the method in comparison with Local Discontinuous Galerkin method.

Keywords: Direct discontinuous Galerkin method, Korteweg de Vries, Stability.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

This work is concerned with the numerical approximation for the one dimensional generalized Kortewegde Vries (KdV) [1] equation

$$u_t + f(u)_x + \epsilon u_{xxx} = 0, \tag{1}$$

where ϵ is a given constant, and f is a smooth function. This equation is a nonlinear dispersive partial differential equation for u with two real variables x space and time t. The original form of the KdV equation is corresponds to $\epsilon = 1$ and $f = -3u^2$.

In this paper, we discuss about a class of finite element method using completely discontinuous piecewise-polynomial spaces for the numerical solution of problems. These DG methods have several attractive properties, for instance it can be easily designed for any order of accuracy. The method at first is performed for diffusion problems by Liu and Yan [2]. In this study we apply the direct discontinuous Galerkin method for the KdV equation and then we compare the numerical results with the Local Discontinuous Galerkin (LDG) method. Also the nonlinear L^2-norm stability of the method is illustrated. It has been shown that the numerical results for the KdV equation have high accuracy in comparison with LDG method.

*Speaker

2 DDG method

In this section, we consider the general form of the KdV equation as follows:

$$u_t + f(u)_x + (r'(u)g(r(u)_x)_x)_x = 0, \ x \in (L, R), t > 0. \tag{2}$$

This equation will be considered in the mesh $Ij = [x_{j-1/2}, x_{j+1/2}]$ $(j = 1, \cdots, N)$, with the center of the cell denoted by $\frac{1}{2}(x_{j-1/2}+x_{j+1/2})$ and the size of each cell $\Delta_x = x_{j+1/2} - x_{j-1/2}$. Denote $v^-_{j+1/2}$ and $v^+_{j+1/2}$ for the value of the left and right limit of v, respectively at the interface where v is discontinuous. We replace test functions v, w and z with piecewise polynomials of degree at most k. This means that v, w and z are belong to V_{Δ_x}, where:

$$V_{\Delta_x} = \{v : v \in P^k(I_j), j = 1 : N\}. \tag{3}$$

This method will be achvied through multiplying Eq (2) by three test functions v, w and z respectively, integrate over the interval I_j, and integrate by parts.Thus we seek piecewise polynomial solutions u, p and $q \in V_{\Delta_x}$ where V_{Δ_x} is defined in (3), such that for all test functions v, w and z we have, for $j = 1, \cdots, N$, the following relations:

$$\begin{aligned} \int u_t v dx - \int f(u) + r'(u)p)v_x dx + (\hat{f} + \hat{r'}\hat{p})_j + 1/2 v^-_{j+1/2} \\ -\hat{f} + \hat{r'}\hat{p}_{j-1/2} v^+_{j-1/2} = 0, \\ \int pw dx + \int g(q) w_x dx - \hat{g}_{j+1/2} w^-_{j+1/2} + \hat{g}_{j-1/2} w^+_{j-1/2} = 0, \\ \int qz dx + \int r(u) z_x dx - \hat{r}_{j+1/2} z^-_{j+1/2} + \hat{r}_{j-1/2} z^+_{j-1/2} = 0, \end{aligned} \tag{4}$$

where all the integrals will be taken over interval I_j. In order to design the DDG method, the following notations will be define as follows:

$$u^{\pm} = u(x \pm 0, t), [u] = u^+ - u^-, \overline{u} = \frac{u^+ - u^-}{2}.$$

Now we introduce a numerical flux formula at the cell interface $x_{j\pm1/2}$ as follows:

$$\hat{f} = \beta_0 \frac{[u]}{\Delta x} + \beta_1(\Delta x)[u_{xx}] + \beta_2(\Delta x)^3[u_{xxxx}] + \cdots . \tag{5}$$

Choosing $\beta_0 = \frac{7}{6}$, $\beta_1 = \frac{1}{12}$, the numerical flux (5) enables us to obtain the optimal $3rd$ orders of accuracy. It should be noted that in the LDG method the flux is:

$$\hat{f}(u^-, u^+) = \frac{1}{2}(f(u^-) + f(u^+) - \alpha(u^+ - u^-)), \tag{6}$$

where $\alpha = \max_u |f'(u)|$ [3].

Definition 2.1. The L^2 norm stability of DDG method for the KdV equation is defined as:

$$\frac{d}{dt}\int_{I_j} \frac{(u^2(x,t))}{2} dx + (\hat{H}_{j+1/2} - \hat{H}_{j-1/2}), \tag{7}$$

where, $\hat{H}_{j+1/2}$, $\hat{H}_{j-1/2}$ are numerical entropy fluxes.

Proposition 2.2. *(cell entropy inequality)There exist numerical entropy fluxes $\hat{H}_{j+1/2}$ such that the solution of the Eq (4) is*

$$\frac{d}{dt}\int_{I_j}(\frac{u^2(x,t)}{2})dx + (\hat{H}_{j+1/2} - \hat{H}_{j-1/2}) \leq 0. \tag{8}$$

Proof. See [3]. □

Example 2.3. In order to see the accuracy of DDG method for nonlinear problems, we compute the classical soliton solution for the KdV equation

$$u_t - 3(u^2)_x + u_{xxx} = 0, \tag{9}$$

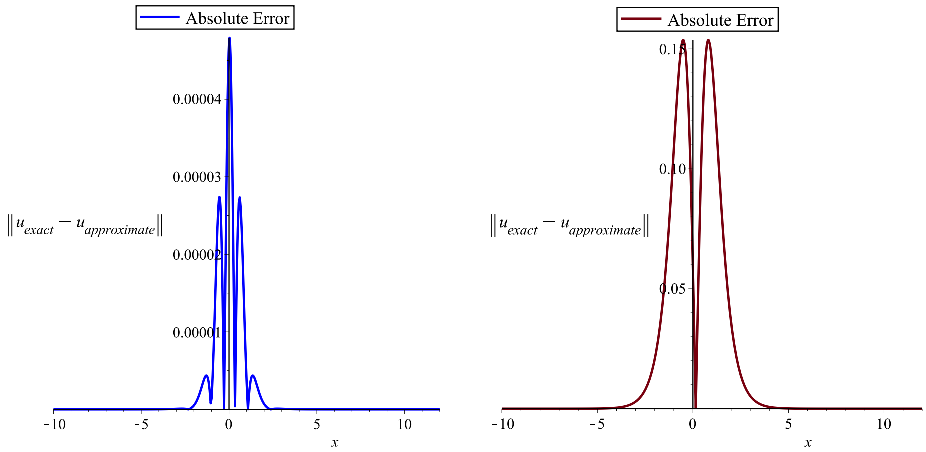
with the given initial condition $u(x,0) = -2sech^2(x)$, where $-10 \leq x \leq 12$. Using this method the exact solution of this problem is $u(x,t) = -2sech^2(x-4t)$.

Table 1: The comparison of L^2-Error between LDG and DDG methods for k =2, 3, t=1 in Example (2.3).

uniform.mesh	DDG	LDG	order
N=40	2.7071e-02	1.7869e-01	2.50
	2.0350e-03	1.78692e-01	3.00
	3.2212e-04	8.73187e-03	3.50
N=80	4.9216e-03	1.20167e-02	2.50
	2.0344e-04	1.20205e-02	3.00
	1.8451e-05	5.3600e-04	3.50
N=160	7.5751e-04	1.0681e-03	2.50
	2.4988e-05	7.5839e-04	3.00
	1.1715e-06	7.5751e-04	3.50

Table 2: The comparison of L^2-Error between LDG and DDG methods for k =3, t=0.5 in Example (2.3).

uniform.mesh	DDG	LDG	order
N=40	0.1e-04	0.5e-01	2.50
	0.2e-04	1.0e-01	3.00
	0.3e-04	1.5e-01	3.50
N=80	0.5e-05	0.4e-01	2.50
	1.5e-05	0.9e-01	3.00
	2.5e-05	1.4e-01	3.50
N=160	2.2e-06	1.2e-01	2.50
	1.2e-06	1.2e-01	3.00
	2.5e-06	0.6e-01	3.50



(a) DDG-Method (b) LDG-Method

Figure 1: Graph of L^2-Error with 40 mesh for $t = 0.5$ and $k = 3$ in example 2.3.

3 conclusion

In this study, we have designed a class of direct discontinuous Galerkin method and local discontinuous Galerkin method for solving KdV type equations containing the third derivatives. Results revealed that these methods seemed to have a reasonabl proficiency for solving the nonlinear equations. Numerical example by means of selecting suitable numerical fluxes appeared to illustrate the accuracy and capability of the DDG method compared to LDG method in the 3rd order for solving the KdV equation.

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Confidence interval for number of population in stochastic exponential population growth models with mixture noise

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Abstract

we consider the stochastic exponential population growth model. We suppose the noise in the population growth model be the mixture noise. The expectations and variances of solutions are obtained. However, the confidence interval for the solution of stochastic exponential population growth model where the so-called parameter, population growth rate is not completely definite and it depends on some random environmental effects is obtained.

Keywords: Stochastic differential equation, Ito integral, Mixture noise, Population growth model, Confidence interval
Mathematics Subject Classification [2010]: 60H10, 60H05

1 Introduction

Population growth is the change in population over time. Environmental scientists use two models to describe how populations grow over time, the exponential growth model and the logistic growth model. In exponential growth, the population size increases at an exponential rate over time. As such as, the growth rate at time t is not completely definite and it depends on some random environment effects. Braumann[1] proposed the applications of stochastic differential equations to population growth. Matisa and Kiffe[2], Andreis and Ricci[3] used of the stochastic exponential population growth model in their studies. We know, the growth rate is depended to many different random environment effect. So, in this here, we let that the this random effects were to the linear combination of some white noise[5]. Then, we consider the perturbation effects the mixture noise on the growth rate of population model. The organization of this paper is as follows: In this next section, we will define the calculus stochastic and mixture noise. In section 3, we will consider the stochastic exponential population growth model with mixture noise. In section 4, We construct a confidence interval for number of population obtained.

2 Preliminaries

There are two main stochastic calculus, Ito and Stratonovich calculus. They yield different solutions and even qualitatively different predictions. In this here, we consider the Ito

*Speaker

calculus for random population growth rate model. The goal of this section is to recall notations and definition of the Ito integral and stochastic differential equation that are important for this paper.

Definition 2.1. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a grid of points on the interval [0, T]. The Ito integral is the limit:

$$\int_0^T f(t)dW(t) = \lim_{\triangle t \to 0} \Sigma_{i=1}^n f(t_{i-1}) \triangle W_i$$

where $\triangle W_i = W(t_i) - W(t_{i-1})$, a step of Brownian motion across the interval. The differential is a notional convenience, thus, $I = \int_0^T f(t)dW(t)$ is expressed in differential form as $dI = f dW_t$ The differential dW_t of Brownian motion W_t is called white noise.

Definition 2.2. A diffusion process is modeled as a differential equation involving deterministic, or drift terms, and stochastic, or diffusion terms, the latter represented by a wiener process, as in the equation:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \tag{1}$$

or the form integral equation is,

$$X_t = X_0 + \int_0^T f(s, X_s)ds + \int_0^T g(s, X_s)dW_s. \tag{2}$$

The equation (1) is the stochastic differential equation (SDE) and the meaning of the last integral in (2) is called the Ito integral.

Definition 2.3. A mixture noise may be interpreted as any linear combination of Wiener processes. The process X_t is a mixture noise if it satisfy the linear additive SDE:

$$dX_t = \sum_{k=1}^{n} \alpha_k W_k(t), \ \Sigma_{k=1}^n \alpha_k = 1, \tag{3}$$

where, $W_k(t) = \frac{dB_k(t)}{dt}$ are one dimensional white noise processes, $B_k(t)$ are the one dimensional Brownian motion and α_k are constants.

3 Stochastic exponential population growth model with mixture noise

Let $N = N(t)$ be the size at time $t \geq 0$ of a population. However, we assume $\frac{dN}{dt}$ be the total growth rate and to the per capita growth rate $a_t = \frac{1}{N}\frac{dN}{dt}$ simply by growth rate.

Consider the following simple population growth model:

$$\frac{dN(t)}{dt} = a(t)N(t),\ N(0) = N_0, \tag{4}$$

where, N_0 is the initial number at time $t = 0$ and $a(t)$ is the growth rate at time. If $a(t) = r(t)$ be the nonrandom function, then $N(t) = N_0 \exp(\int_0^t r(s)ds)$. Now, suppose that $a(t)$ depends on some random environment effects, i.e. $a(t) = r(t) +$ "*mixturenoise*", where $r(t)$ is a nonrandom function, " mixture noise"$= \sum_{k=1}^{n} \alpha_k W_k(t)$.

Theorem 3.1. *Let*

$$\frac{dN(t)}{dt} = (r(t) + \sum_{k=1}^{n} \alpha_k \frac{dB_k(t)}{dt})N(t), N(0) = N_0 \tag{5}$$

be stochastic exponential model, then the solution is given by

$$N(t) = N_0 \exp(\int_0^t [r(s) - \frac{1}{2}\sum_{k=1}^{n}\int_0^t \alpha_k^2(s)]ds + \sum_{k=1}^{n}\int_0^t \alpha_k(s)dB(s)) \tag{6}$$

Proof. See [5]. □

Theorem 3.2. *In (5), if N_0 and $B_k(t)$ $(k = 1, 2, \cdots, n)$ be independent random variables, then the expected value and variance of is:* $E(N_t) = E(N_0)\exp(\int_0^t r(s)ds)$

$$Var(N_t) = \exp(2\int_0^t r(s)ds)\{(Var(N_0) + E^2(N_0))\exp(\sum_{k=1}^{n}\int_0^t \alpha^2(s)ds) - E^2(N_0)\} \tag{7}$$

Proof. See [5]. □

4 Confidence interval

Since N(t) is a random process, we can construct an confidence interval for it.

Theorem 4.1. *Let $\alpha(t)$ be non-random such that $\int_0^t \alpha^2(s)ds < \infty$.then $(1-\epsilon)$ confidence interval for N(t) is given by:*

$$D(t)\exp(-Z_{\frac{\epsilon}{2}}\sqrt{\sum_{k=1}^{n}\int_0^t \alpha_k^2(s)ds}) \le N(t) \le D(t)\exp(Z_{\frac{\epsilon}{2}}\sqrt{\sum_{k=1}^{n}\int_0^t \alpha_k^2(s)ds})$$

$D(t) = N_0\exp(\int_0^t [r(s) - \frac{1}{2}\sum\int \alpha_k^2(s)]ds)$.

Proof. It is easy to see that if $\alpha(t)$ is non-random such that $\int_0^t \alpha^2(s)ds < \infty$ then its Ito integral $Y(t) = \sum_{k=1}^n \int_0^t \alpha_k(s)dB(s)$ is a Gaussian process with zero mean and variance given by $\sum_{k=1}^n \int_0^t \alpha_k^2(s)ds$. So we can rewrite (6) as

$$N(t) = N_0 \exp(\int_0^t [r(s) - \frac{1}{2}\sum \int \alpha_k^2(s)]ds. \exp(\sum_{k=1}^n \int_0^t \alpha_k(s)dB(s)).$$

$N(t) = D(t)\exp(\sum_{k=1}^n \int_0^t \alpha_k(s)dB(s))$, $D(t) = N_0 \exp(\int_0^t [r(s) - \frac{1}{2}\sum \int \alpha_k^2(s)]ds)$.
Thus, $\sum_{k=1}^n \int_0^t \alpha_k(s)dB(s) = \ln \frac{N(t)}{D(t)} \to N(0, \sum_{k=1}^n \int_0^t \alpha_k^2(s)ds)$.
So we can put:

$$-Z_{\frac{\epsilon}{2}}\sqrt{\sum_{k=1}^n \int_0^t \alpha_k^2(s)ds} \leq \ln \frac{N(t)}{D(t)} \leq Z_{\frac{\epsilon}{2}}\sqrt{\sum_{k=1}^n \int_0^t \alpha_k^2(s)ds}$$

We know Z_ϵ is the area under the standard normal curve to its right equals ϵ. So, the critical region for testing the null hypothesis $\mu = E(N(t)) = \mu_0$ against the alternative hypothesis $\mu \neq \mu_0 = E(N(0))$ is $|Z_{\frac{\epsilon}{2}}|$ where $Z = \frac{E(N(t))-\mu_0}{\sqrt{Var(N(t))}}$.
If $\epsilon = 0.05$, the dividing lines, or critical values, of the criteria are -1.96 and 1.96 for the two-sided alternatives hypothesis. □

Conclusion

we considered the stochastic exponential population growth model. We supposed the noise in the population growth model be the mixture noise. The expectations and variances of solutions obtained. However, the confidence interval for the solution of stochastic exponential population growth model obtained.

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Continuous Single-Species Population Model with Delay

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Abstract

In this paper, the logistic equation with two different delay times is considered. Firstly, consider time delay depends on food resources in population and stability at equilibrium point is investigated. Secondly, consider delay distributed over time and the stability conditions at equilibrium point is determined.

Keywords: Dynamical system, logistic equation, Time Delay, Population dynamic
Mathematics Subject Classification [2010]: 37C75, 34D05

1 Introduction

Time delay have been incorporated into biological models to represent resource regeneration times. By many researchers such as, Cushing(1977), Gopalsamy(1992) and Kuang (1993) time delay differential equations in Biology have investigated [3, 4].
Delay differential equations exhibit much more complicated dynamics than ODEs. Since a time delay could cause a stable equilibrium to become unstable.
In this paper, consider logistic equation for population model. Let $r(>0)$ be intrinsic growth rate and $K(>0)$ be the carry capicity of the population. The logistic model is

$$\frac{dX}{dt} = rX(t)(1 - \frac{X(t)}{K}) \tag{1}$$

where $X(t)$ is the population size. Set $\frac{X(t)}{K} = x(t)$, so

$$\frac{dx}{dt} = rx(t)(1 - x(t)) \tag{2}$$

In model 2, when x is small, the population grows and when x is large the number of the species compete with each other for the limit resources. In the above logistic equation, it is assumed that the growth rate of a population at any time t depends on the relative number of individuals at time t. But in fact, the population size at time t is not only dependent at that time but also at time $(t-\tau)$, where τ is time delay. Thus the model is

$$\frac{dx}{dt} = rx(t)(1 - x(t-\tau)) \tag{3}$$

*Speaker

In the model 3, τ is constant. But in many species the time delay depends on the rate of available food. Therefore, we introduce the following model

$$\frac{dx}{dt} = rx(t)(1 - x(t - \tau(a))) \tag{4}$$

where a is the rate of available food in population.
in the next section, stability conditions at equilibrium points are investigated.
In the last section, consider delay distributed over time and stability conditions at equilibrium point is determined.

2 Stability of equilibrium points

Consider, the time delay, τ, is constant, so the logistic equation with discrete delay is

$$\frac{dx}{dt} = rx(t)(1 - x(t - \tau)) \tag{5}$$

Notice that equation 5 has equilibrium $x = 0$ and $x = 1$. Small perturbation from $x = 0$ satisfy the linear equation $\frac{dx}{dt} = rx$, which shows $x = 0$ is unstable with exponential growth. Hence we consider the stability of the equilibrium point $x = 1$.

Theorem 2.1. *[1] i) If $0 \leq r\tau < \frac{\pi}{2}$, then the equilibrium point $x = 1$ of equation 5 is asymptotically stable.*
ii) If $r\tau > \frac{\pi}{2}$, then $x = 1$ is unstable.

As follow, consider τ is a function of parameter a (a is the rate of food available in the population). The logistic equation is

$$\frac{dx}{dt} = rx(t)(1 - x(t - \tau(a))) \tag{6}$$

where $\tau(a)$ satisfies the following conditions
i) $0 \leqslant \tau(a) \leqslant \tau_0$ for some $\tau_0 > 0$ (i.e. τ is a bounded map).
ii) τ is deceasing function.
By above explains, we can see easily $x = 0$ is unstable. Now consider the stability of equilibrium point $x = 1$. Let $X = x - 1$. Then

$$\frac{dX}{dt} = -rX(t)X(t - \tau(a)) - rX(t - \tau(a)). \tag{7}$$

The linearized equation is

$$\frac{dX}{dt} = -rX(t - \tau(a)). \tag{8}$$

We look for solutions of the form $X(t) = ce^{\lambda t}$, where c is constant and the eigenvalues λ are solutions of the characteristic equation

$$\lambda + re^{-\lambda\tau(a)} = 0. \tag{9}$$

Set $\lambda = \mu + i\nu$. Separating the real and imaginary parts of characteristic equation, obtain

$$\mu + re^{-\mu\tau(a)} cos\nu\tau(a) = 0 \tag{10}$$

$$\nu - re^{-\mu\tau(a)} sin\nu\tau(a) = 0 \tag{11}$$

If there is a_0 such that $\tau(a_0) = 0$, so $\nu = 0$ and $\mu = -r < 0$. Hence $x = 1$ is asymptotically stable. In this case, there is not time delay in the population. Therefore the model is the original logistic equation.
We seek conditions on a such that $Re\lambda$ changes from negative to positive. By continuity, there must be some value of a, say a_1 at which $\mu(\tau(a_1)) = 0$.
Set $Re\ \lambda = \mu(\tau(a)) = 0$, so using equation 10, $cos\nu\tau(a) = 0$. Therefore $\tau(a_k) = \frac{1}{r}(2k\pi + \frac{\pi}{2}) \qquad k = 0, 1, 2, ...$
Also by equation 11, $\nu = re^{-\mu\tau(a_k)} sin\nu\tau(a_k) = r$. τ is decreasing function, so $\frac{d\mu}{da}|_{a=a_1} = \frac{4r^2}{4+\pi^2}\frac{d\tau}{da}|_{a=a_1} < 0$. . Hence $\mu(a) < 0$ for all $a > a_1$.

Theorem 2.2. *i) If $a > a_1$, then $x = 1$ is asymptotically stable.*
ii) If $a < a_1$, then $x = 1$ is unstable.

3 Delay distributed over time

More generally, we could assume a delay distributed over time. If the probability that the delay is between u and $u + \Delta u$ is approximately $p(u)\Delta u$, when $p(u)$ is nonnegative function with $\int_0^\infty p(u)du = 1$[1, 2]. Then we are led to the integrodifferential equation

$$\frac{dx}{dt} = x(t)\int_0^\infty (1 - x(t-u))p(u)du. \tag{12}$$

Which is transferred by the change of variable $t - u = s$ to the equivalent form

$$\frac{dx}{dt} = x(t)\int_0^\infty (1 - x(s))p(t-s)ds. \tag{13}$$

The average time delay will then be $\int_0^\infty up(u)du$. One form of continuous delay frequently used in population models is

$$p(u) = \frac{u}{T^2}e^{\frac{-u}{T}} \tag{14}$$

which is not difficult to verify that $\int_0^\infty p(u)du = 1$ and $\int_0^\infty up(u)du = 2T$. $p(u)$ has a maximum for $u = T$.

Definition 3.1. An equilibrium of the differential equation

$$\frac{dx}{dt} = x(t)(1 - x(t-T)) \tag{15}$$

is a value x_∞ such that $x_\infty(1 - x_\infty) = 0$, so that $x(t) = x_\infty$ is constant solution of differential equation.

Let $u(t) = x(t) - x_\infty$, so

$$\frac{du}{dt} = (x_\infty + u(t))\int_0^\infty (1 - x_\infty - u(t-s))p(s)ds. \tag{16}$$

Set $g(x_\infty + u(t-s)) = 1 - x_\infty - u(t-s)$ and using Taylor's theorem, obtaining

$$\frac{du}{dt} = x_\infty g(x_\infty) + g(x_\infty)u(t) + x_\infty g^{'}(x_\infty)\int_0^\infty u(t-s)p(s)ds \tag{17}$$

we know $x_\infty g(x_\infty) = 0$. Set $a = g(x_\infty) = 1 - x_\infty$ and $b = x_\infty g^{'}(x_\infty) = -x_\infty$. Therefore

$$\frac{du}{dt} = au(t) + b\int_0^\infty u(t-s)p(s)ds. \tag{18}$$

The solution is $u(t) = ce^{\lambda t}$, so the equation 18 is transferred $\lambda = a + b\int_0^\infty e^{-\lambda s}p(s)ds = a + bL\{p(s)\}$ and, we know $p(s) = \frac{s}{T^2}e^{-\frac{s}{T}}$, so $L\{p(s)\} = \frac{1}{(T\lambda+1)^2}$.
Thus, $\lambda = a + \frac{b}{(T\lambda+1)^2}$ and the characteristic equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda - a_0 = 0$$

where $a_0 = -\frac{a+b}{T^2}$, $a_1 = \frac{1-2aT}{T^2}$ and $a_2 = \frac{2-aT}{T}$.
By above discussions, the following theorem is true

Theorem 3.2. *If $a < \frac{2}{5T}$, then x_∞ is asymptotically stable.*

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Direct meshless local Petrov-Galerkin (DMLPG) method for numerical solution of 2D nonlinear Klein-Gordon equation

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Abstract

In this paper, we propose a direct meshless local Petrov-Galerkin (DMLPG) method for solving the 2D nonlinear Klein-Gordon equation. This method is based on a generalized moving least square and a local weak form of the Klein-Gordon equation.

Keywords: Local weak form, Direct meshless local Petrov-Galerkin (DMLPG) method, Klein-Gordon equation, Generalized moving least square approximation
Mathematics Subject Classification [2010]: 35Q55, 35J66

1 Introduction

The nonlinear Klein-Gordon (KG) equation is used to model many nonlinear phenomena such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics [1]. The 2D nonlinear KG is given by

$$\frac{\partial^2 u}{\partial t^2} + \alpha\nabla^2 u + \psi(u) = f(x,t), \qquad x \in \Omega \subset \mathbb{R}^2, \quad t \geq 2 \tag{1}$$

with the initial and boundary conditions

$$u(x,t) = g_1(x), \quad u_t(x,0) = g_2(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = u_D(x,t), \quad x \in \Gamma_D, t > 0, \tag{3}$$

$$n(x).\nabla u = u_N(x,t), \quad x \in \Gamma_N, t > 0, \tag{4}$$

where $u = u(x,t)$ shows the wave movement at position x and time t, α is known constant and ψ is the nonlinear force.The nonlinear KG equation has been solved by several methods like radial basis functions (RBFs) [1], the boundary integral equation (BIE) and the dual reciprocity boundary elemement method (DRBEM) [4].

There have been many meshless techniques based on the MLS approximation for the numerical solution of differential equations in recent years. The Meshless Local Petrov–Galerkin (MLPG) method is one of the popular meshless methods that has been used very successfully to solve several types of boundary value problems since the late nineties (see [2] and refernces therein).

The direct MLPG (DMLPG) technique, using a generalized moving least squares (GMLS) approximation, was first introduced by Mirzaei and Schaback [2]. In the following, we recall the GMLS approximation in a form very similar to [2].

*Speaker

2 GMLS approximation

Let $u \in \mathbb{C}^m(\Omega)$, for some $m \geq 0$, and let $\{\lambda_j(u)\}_{j=1}^N$ be a set of continuous linear functionals from the dual $\mathbb{C}^m(\Omega)^*$ of $\mathbb{C}^m(\Omega)$. For a fixed functional $\lambda \in \mathbb{C}^m(\Omega)^*$, the GMLS method approximates the value of $\lambda(u)$ from the values $\{\lambda_j(u)\}_{j=1}^N$. The approximation $\widehat{\lambda(u)}$ of $\lambda(u)$ is a linear function of $\lambda_j(u)$ as follows

$$\widehat{\lambda(u)} = \sum_{j=1}^{N} a_j(\lambda)\lambda_j(u), \tag{5}$$

and the coefficients a_j should be linear in λ. As in the classical MLS, we assume the approximation equation (5) to be exact for a finite dimensional subspace $\mathcal{P} = \text{span}\{p_1, p_2, \ldots, p_Q\}$, in which $\mathcal{P}$ is the space of d-variate polynomials of degree at most m.

The GMLS approximation $\widehat{\lambda(u)}$ of $\lambda(u)$ can also be obtained as $\widehat{\lambda(u)} = \lambda(p^*)$, where $p^* \in \mathcal{P}$ is minimizing the weighted least-squares error functional

$$\sum_{j=1}^{N} \big(\lambda_j(u) - \lambda_j(p)\big)^2 \omega_j, \tag{6}$$

among all $p \in \mathcal{P}$ and ω_j are given non-negative weights. Even if a different numerical method is used to minimize (6), the optimal solution $a^*(\lambda) \in \mathbb{R}^N$ can be written as

$$a^*(\lambda) = WP^T(PWP^T)^{-1}\lambda(\mathbf{p}), \tag{7}$$

where W is the diagonal matrix carrying the weights ω_j on its diagonal, P is the $N \times Q$ matrix of values $\lambda_j(p_k)$, $1 \leq j \leq N$, $1 \leq k \leq Q$, and $\lambda(\mathbf{p}) \in \mathbb{R}^Q$ is the vector with values $\lambda(p_1), \ldots, \lambda(p_Q)$ of λ on the basis of $\mathcal{P}$. It should be noted that the weight function ω in GMLS depends on functional λ and since all our functionals are finally considered as point evaluation functionals at point x, we can choose the same $\omega(x)$ for all. Moreover, a small domain Ω_j containing x_j is associated with node j such that $\omega(x, x_j)$ equals zero outside Ω_j. In this paper, the Gaussian weight function is used for all computations, which is

$$\omega(x, x_j) = \begin{cases} \dfrac{exp\left(-(\parallel x - x_j \parallel_2 /c)^2\right) - exp\left(-(\delta/c)^2\right)}{1 - exp\left(-(\delta/c)^2\right)}, & 0 \leq \parallel x - x_j \parallel_2 \leq \delta, \\ 0, & elsewhere \end{cases} \tag{8}$$

where c is a constant controlling the shape of the weight function and δ is the size of the support domains.

3 Weak form and DMLPG formulation

In meshless methods, everything write entirely in terms of scattered nodes as $X = \{x_1, x_2, \ldots, x_N\}$ located in the spatial domain Ω and its boundary Γ. In every type of MLPG methods, a small sub-domain $\Omega_s \subset \Omega \cup \Gamma$ is chosen around each node such that integrations over Ω_s are comparatively cheap. On these sub-domains, the KG equation (1) including boundary conditions is stated in the following weak form

$$\int_{\Omega_s} \nu\left(\frac{\partial^2 u}{\partial t^2} + \alpha\nabla^2 u + \psi(u) - f\right) d\Omega = 0, \tag{9}$$

where ν is an appropriate test function. Applying integration by parts, and using Divergence theorem, we get

$$\frac{\partial^2}{\partial t^2}\int_{\Omega_s} u\nu d\Omega + \int_{\partial\Omega_s}\alpha\frac{\partial u}{\partial n}\nu d\Gamma - \int_{\Omega_s}\alpha\nabla u\nu d\Omega = \int_{\Omega_s} f\nu d\Omega - \int_{\Omega_s}\psi(u)\nu d\Omega. \tag{10}$$

The DMLPG method is based on the local weak form (10). All integrals in (10) can be approximated by GMLS method as

$$\lambda_{1,k}(u) := \int_{\Omega_s} uvd\Omega \quad \approx \quad \widehat{\lambda_{1,k}(u)} = \textstyle\sum_{j=1}^N a_{1,j}(x_k)u(x_j), \tag{11}$$

$$\lambda_{2,k}(u) := -\int_{\Omega_s} k\nabla u.\nabla v\Omega \quad \approx \quad \widehat{\lambda_{2,k}(u)} = \textstyle\sum_{j=1}^N a_{2,j}(x_k)u(x_j), \tag{12}$$

$$\lambda_{3,k}(u) := \int_{\partial\Omega_s} \alpha\frac{\partial u}{\partial n}vd\Gamma \quad \approx \quad \widehat{\lambda_{3,k}(u)} = \textstyle\sum_{j=1}^N a_{3,j}(x_k)u(x_j). \tag{13}$$

Now, we have the following time-dependent system

$$A^{(1)}\frac{\partial^2}{\partial t^2}\mathbf{u}(t) + A^{(\ell)}\mathbf{u}(t) = \mathbf{b}(t), \quad \ell = 2 \text{ or } 3 \tag{14}$$

where $\mathbf{u}(t) = \big(u(x_1,t),\dots,u(x_N,t)\big)^T \in \mathbb{R}^N$ is the time-dependent vector of nodal values, and $\mathbf{b}(t)$ is the collection of right-hand sides with components

$$b_k(t) = \int_{\Omega_s} f(x,t)\nu d\Omega - \int_{\Omega_s}\psi(u)\nu d\Omega - \int_{\partial\Omega_s\cap\Gamma_N}\alpha u_N(x,t)\nu d\Gamma, \tag{15}$$

and $A^{(\ell)}_{kj} = a_{\ell,j}(x_k), \ell = 1,2,3$. The k-th row of $A^{(\ell)}$ is

$$a^{(\ell)}_k = WP^T(PWP^T)^{-1}\lambda_{\ell,k}(\mathbf{p}) \tag{16}$$

where

$$\lambda_{1,k}(\mathcal{P}) = \Big[\int_{\Omega_s} p_1vd\Omega, \int_{\Omega_s} p_2vd\Omega, \cdots, \int_{\Omega_s} p_Qvd\Omega\Big]^T, \tag{17}$$

$$\lambda_{2,k}(\mathcal{P}) = -\Big[\int_{\Omega_s}\alpha\nabla p_1.\nabla vd\Omega, \int_{\Omega_s}\alpha\nabla p_2.\nabla vd\Omega, \cdots, \int_{\Omega_s}\alpha\nabla p_Q.\nabla vd\Omega\Big]^T, \tag{18}$$

$$\lambda_{3,k}(\mathcal{P}) = \Big[\int_{\partial\Omega_s}\alpha\frac{\partial p_1}{\partial n}vd\Gamma, \int_{\partial\Omega_s}\alpha\frac{\partial p_2}{\partial n}vd\Gamma, \cdots, \int_{\partial\Omega_s}\alpha\frac{\partial p_Q}{\partial n}vd\Gamma\Big]^T. \tag{19}$$

In DMLPG1, where we used in this paper, the (19) can be omitted because of the test function ν vanishes on the $\partial\Omega_s$. To discretize the time derivative in (14), we consider the following finite difference approximations

$$\frac{\partial^2}{\partial t^2}\mathbf{u}(t) \simeq \frac{1}{(dt)^2}\big[\mathbf{u}^{(k+1)} - 2\mathbf{u}^{(k)} + \mathbf{u}^{(k-1)}\big], \tag{20}$$

$$\mathbf{u}(t) \simeq \frac{1}{3}\big[u^{(k+1)} + u^{(k)} + u^{(k-1)}\big], \quad \mathbf{b}(t) \simeq \frac{1}{2}\big[b^{(k+1)} + b^{(k)}\big] \tag{21}$$

where $\mathbf{u}^{(k)} = \mathbf{u}(kdt)$. By using of (20)-(21), system (14) can be written as

$$\big(A^{(1)} + \xi A^{(\ell)}\big)u^{(k+1)} = \big(2A^{(1)} - \xi A^{(\ell)}\big)u^{(k)} - \big(A^{(1)} + \xi A^{(\ell)}\big)u^{(k-1)} + \frac{1}{2}\big(b^{(k+1)} + B^{(k)}\big), \tag{22}$$

where $\xi = (dt)^2/3$.

4 Numerical results

Consider the nonlinear 2D KG equation with $\psi = u^2 - 2u$ and $\alpha = 1$ in the domain $\Omega = [0, \pi] \times [0, \pi]$. The exact solution is

$$u(x, y, t) = sin(x)sin(y)cosh(t). \tag{23}$$

The initial conditions and right-hand side function f are obtained from the exact solution and the boundary conditions are chosen as Dirichlet type. The L_∞ and RMS errors and CPU times are obtained in Table 1 at $t = 1, 2, 3s$ with $dt = 0.01$ on the 33×33 nodes. The numerical results demonstrate the good accuracy of this scheme.

Table 1: L_∞ and RMS errors, and CPU times.

t	$L_\infty - error$	$RMSerror$	CPU time
1	3.1505×10^{-3}	1.4085×10^{-3}	$9.8s$
2	1.9693×10^{-2}	9.6031×10^{-3}	$18.4s$
3	4.9913×10^{-2}	2.4856×10^{-2}	$27.7s$

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Discrete mollification method and its application to solving backward nonlinear cauchy problem

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Abstract

In this article a nonlinear backward cauchy problem consisting of two unknown functions is considered. A space marching algorithm based on discrete mollification method is presented to solve this problem. Finally we illustrate some numerical examples to show efficiency of the proposed method.

Keywords: Nonlinear backward cauchy problem, Space marching algorithm, Discrete mollification
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Consider a nonlinear backward inverse problem governed by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}((a(x) + b(x)u^2)\frac{\partial u}{\partial x}) + f(x,t); \quad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

$$u(x,T) = \varphi(x); \quad 0 \leq x \leq 1, \tag{2}$$

$$u(0,t) = g_1(t); \qquad 0 \leq t \leq T, \tag{3}$$

$$u_x(0,t) = g_2(t); \qquad 0 \leq t \leq T, \tag{4}$$

where f(x,t), $a(x) > 0, b(x), \varphi(x), g_1(t)$ and $g_2(t)$ are known.We are going to determine $u(x,t)$ and $u(x,0)$ satisfying (1)-(4). Now, we add random noise ,with maximum level of ε, in the initial data $\varphi(x), g_1(t)$ and $g_2(t)$. These noisy data are represented by $\varphi^\varepsilon(x), g_1^\varepsilon(t)$ and $g_2^\varepsilon(t)$,respectively. The particular difficulty of the backward problem is its ill-possedness, on the other hand since we have noise in the problem's data so should first regularize this problem by discrete mollification method [2]. The stabilized problem is described as

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}[(a(x) + b(x)v^2)\frac{\partial v}{\partial x}] + f(x,t); \quad 0 < x < 1, \quad 0 < t < T, \tag{5}$$

$$v(x,T) = J_{\delta_1}\varphi^\varepsilon(x); \quad 0 \leq x \leq 1, \tag{6}$$

$$v(0,t) = J_{\delta_2}g_1^\varepsilon(t); \quad 0 \leq t \leq T, \tag{7}$$

*Speaker

$$v_x(0,t) = J_{\delta_3} g_2^\varepsilon(t); \qquad 0 \le t \le T. \tag{8}$$

Where $J_\delta g_1^\varepsilon(t)$ is discrete mollification of $g_1^\varepsilon(t)$ with respect to t,which is defined by [2]

$$J_\delta g_1^\varepsilon(t) = \sum_{j=0}^{N} (\int_{s_j}^{s_{j+1}} \rho_\delta(t-s)ds) g_1^\varepsilon(t_j),$$

$\rho_{\delta,p}(t)$ and s_j are defined as follows

$$\rho_{\delta,p}(t) = \begin{cases} A_p \delta^{-1} \exp(-\frac{t^2}{\delta^2}), & |t| \le p\delta \\ 0 & |t| > p\delta \end{cases}$$

$$s_j = \frac{t_j + t_{j+1}}{2} \ , \quad j = 1, .., N-1$$

$$s_0 = 0, \qquad s_N = 1$$

such that $A_p = (\int_{-p}^{p} \exp(-s^2)ds)^{-1}$. We usually take p=3.
The mollification parameters δ_1, δ_2 and δ_3 are selected automatically by GCV method [4]. Stability and consistency properties of the discrete mollification are stated and proved in [2]. Now, we implement a space marching finite difference method on problem (5)-(8) to find $v(x,t)$ which satisfy in this problem. Let $h = 1/M, k = 1/N$ be the parameters of finite difference discretization, $x_j = jh, j = 0, .., M$ and $t_n = nk, n = 0, .., N$. The computed approximations of the $v(jh, nk), v_t(jh, nk), v_x(jh, nk), f(jh, nk), a(jh), b(jh)$ are denoted by $U_j^n, W_j^n, R_j^n, f_j^n, a_j, b_j$ respectively. The space marching scheme for this problem is

$$U_{j+1}^n = U_j^n + hR_j^n, \tag{9}$$

$$R_{j+1}^n = \frac{1}{a_{j+1} + b_{j+1}(U_{j+1}^n)^2}((a_j + b_j(U_j^n)^2)R_j^n + h(W_j^n - F_j^n)), \tag{10}$$

$$W_{j+1}^n = W_j^n + hD_t(J_\delta R_j^n). \tag{11}$$

2 Main results

Example 2.1. In this section by illustrating a numerical example, the role of mollification in stabilization of the problem is investigated. Consider the function

$$u(x,t) = x(t+1)e^x$$

as exact solution of problem (1)-(4) with

$$a(x) = 3x^2 e^{2x},$$

$$b(x) = 1,$$

$$\varphi(x) = xe^x,$$

$$g_1(t) = 2t,$$

and

$$g_2(t) = e^t.$$

In this example we take $\varepsilon = 0.1$,h=1/50, k=1/50, p=3.

Table 1: Numerical result for u(x,t)

t	Exact solution u(x,0.5)	computed solution u(x,0.5) with mollification
0.2	0.989233	0.945873
0.4	1.15460	1.08744
0.6	1.31898	1.23065
0.8	1.48385	1.37885

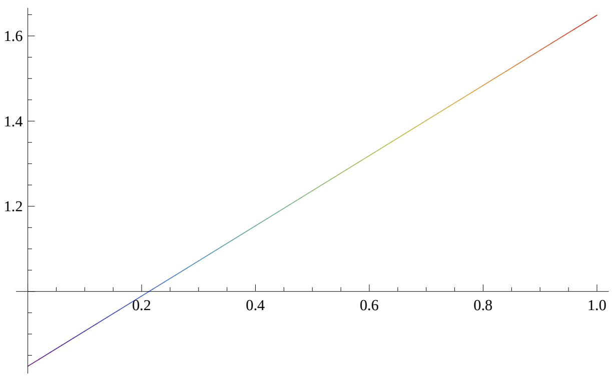
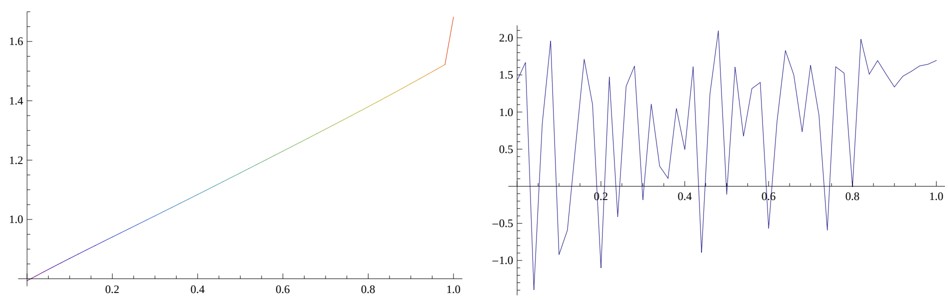


Figure 1: Exact solution u(x,t) at x=0.5



(a) Computed solution with mollification (b) computed solution without mollification

Figure 2: Computed solution u(x,t) at x=0.5 with and without mollification with space marching algorithm.

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Dynamic analysis of a fractional-order prey-predator model

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Abstract

In this paper, we introduce a fractional-order prey-predator model. First we obtain equilibrium point of the system, and determine stability and dynamical behaviors of the equilibria of this system. Dynamical behaviors is investigated from the point of view of local stability. Further by numerical solution of the fractional system and numerical simulation, we reveal more dynamical behaviors of the model.

Keywords: Fractional Prey-predator model, Stability of equilibrium, Dynamical behavior
Mathematics Subject Classification [2010]: 34A08

1 Introduction and Preliminaries

In this paper we consider a planar autonomous differential equation introduced in [3]. This model which is a prey-predator interaction is define as follows:

$$\begin{cases} \dfrac{dx}{dt} = (1 - \dfrac{x}{k}) - \dfrac{\beta yx}{1 + ax} \\ \dfrac{dy}{dt} = -\gamma y + \dfrac{c\beta yx}{1 + ax} \end{cases} \tag{1}$$

Where x, y denote prey and predator population respectively at any time t, and α, k, γ, β, a, c are all positive constants. α represent the intrinsic growth rate and k the carrying capacity of the prey; γ is death rate of the predator; β/a is the maximum number of prey that can eaten by each predator in unit time; 1/a is the density of prey necessary to achieve one half that rate; c is the conversion factor denoting the number of newly born predator for each captured prey.

This paper extends the above model by incorporating a refuge protecting mx of the prey, where $m \in [0, 1)$ is constant. This leaves $(1 - m)x$ of the prey available to the predator, and modifying system(1) as follow:

$$\begin{cases} \dfrac{dx}{dt} = (1 - \dfrac{x}{k}) - \dfrac{\beta(1-m)yx}{1 + a(1-m)x} \\ \dfrac{dy}{dt} = -\gamma y + \dfrac{c\beta(1-m)yx}{1 + a(1-m)x} \end{cases} \tag{2}$$

*Speaker

We introduce the fractional order derivative of this model by Caputo-type derivative to obtain the following fractional order system:

$$\begin{cases} \frac{d^n x}{dt^n} = (1 - \frac{x}{k}) - \frac{\beta(1-m)yx}{1+a(1-m)x} \\ \frac{d^n y}{dt^n} = -\gamma y + \frac{c\beta(1-m)yx}{1+a(1-m)x} \end{cases} \tag{3}$$

2 Dynamic behavior

Theorem 2.1. *[1]. The autonomous system* $\frac{d^n x}{dt^n} = Ax$, $x(0) = x_0$, *with* $0 < n \leq 1$, $x \in \mathbb{R}^n$, *is asymptotically stable if and only if* $|\, arg(\lambda_i)| > \frac{\theta\pi}{2}$ *is satisfied for all eigenvalues of matrix A. Also this system is stable if and only if* $|\, arg(\lambda_i)| \geq \frac{\theta\pi}{2}$ *is satisfied for all eigenvalues of matrix A whit those critical eigenvalues satisfying* $|\, arg(\lambda_i)| = \frac{n\pi}{2}$ *having geometric multiplicity of one.*

Theorem 2.2. *[2]. consider the following commensurate fractional-order system:* $\frac{d^n x}{dt^n} = f(x)$, *x(0)=0, with* $0 < n \leq 1$, $x \in \mathbb{R}^n$, *the equilibrium of the system (3) are calculated by solving the following equation:* $f(x) = 0$ *.this point are locally asymptotically stable if all eigenvalues of the jacobian matrix* $J = \frac{\partial f}{\partial x}$ *evaluated at the equilibrium point satisfy:* $|\, arg(\lambda_i)| > \frac{n\pi}{2}$.

The system has three equilibrium; $P_0(0,0)$, $P_1(k,0)$, $P_2(x^*, y^*)$ where:

$$x^* = \frac{\gamma}{(c\beta - \gamma a)(1-m)}, \qquad y^* = \frac{ac}{k}[\frac{k(c\beta - \gamma a)(1-m) - \gamma}{(c\beta - \gamma a)(1-m)^2}].$$

3 Numerical simulation

In order to solve (3), we use a numerical method introduce by Atanackovic and Stankovic [4] to solve the linear fractional differential equation. For a function $f(t)$, the Caputo derivative of order n with $0 < n \leq 1$ may be expressed as follow:

$$D^n f(t) \simeq \frac{1}{\Gamma(2-n)} \times$$

$$\{ \frac{f^{(1)}(t)}{t^{n-1}}[1+\sum_{p=1}^{M} \frac{\Gamma(p-1+n)}{\Gamma(n-1)p}] - [\frac{n-1}{t^n} f(t) + \sum_{p=2}^{M} \frac{\Gamma(p-1+n)}{\Gamma(n-1)(p-1)!}(\frac{f(t)}{t^n} + \frac{v_p(f)(t)}{t^{p-1+n}})]\}, \tag{4}$$

Where

$$\frac{d}{dt} v_p(f) = -(p-1)t^{p-2} f(t), p = 2, ..., M. \tag{5}$$

We can rewrite Eq.(4) as follow:

$$D^n f(t) \simeq \Omega(n,t,M) f^{(1)}(t) + \Phi(n,t,M) f(t) + \sum_{p=2}^{M} A(n,t,M) \frac{v_p(f)(t)}{t^{p-1+n}}, \tag{6}$$

Where

$$\Omega(n,t,M) = \frac{1 + \sum_{p=1}^{M} \frac{\Gamma(p-1+n)}{\Gamma(n-1)p!}}{\gamma(2-n)t^{n-1}}, \qquad R(n,t) = \frac{1-n}{t^n \Gamma(2-n)}, \tag{7}$$

$$A(n,t,p) = -\frac{\Gamma(p-1+n)}{\Gamma(2-n)\Gamma(n-1)p!}, \qquad \Phi(n,t,m) = R(n,t) + \sum_{p=2}^{M} \frac{A(n,t,M)}{t^n}. \qquad (8)$$

We set $v_p(x)(t) = w_p(t), v_p(y)(t) = u_p(t), p = 2, 3....$

We use (4) and (6) and rewrite system (3) as a system of ordinary differential equation and solve this system by Rung-Kutta method of order fourth.

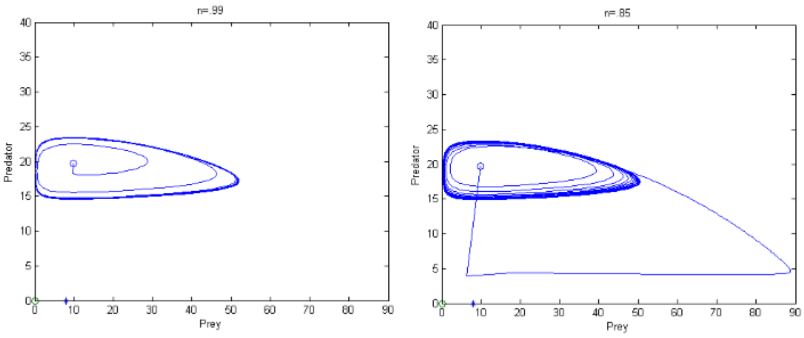

Figure 1: Phase portrait of system (3), for n=0.99, 0.85 and m=0.1.

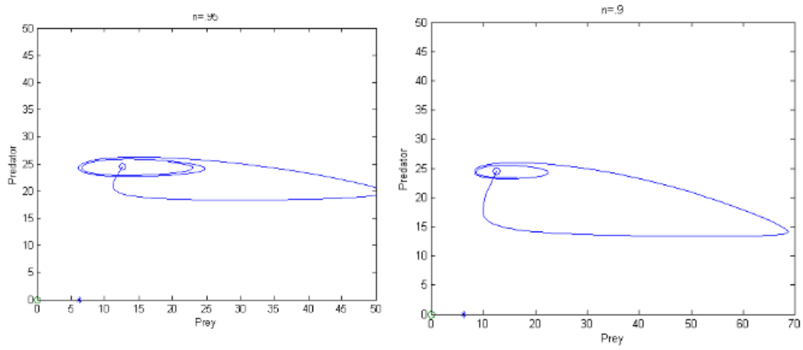

Figure 2: Phase portrait of system (3), for n=0.95, 0.9 and m=0.3.

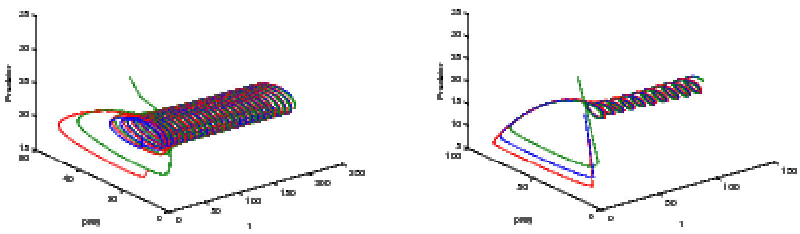

Figure 3: Phase portrait of system (3), for n=.99 and n=.91.

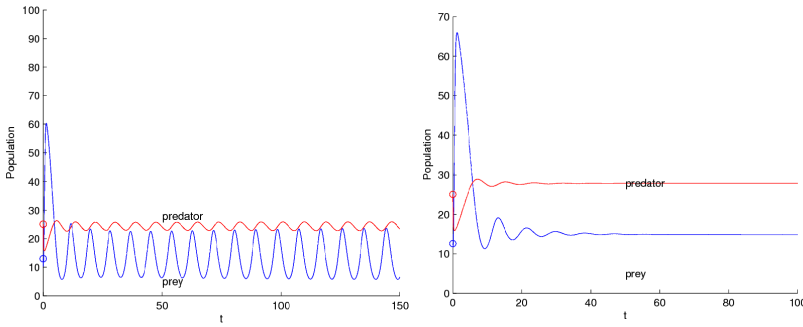

Figure 4: Numerical values x(t), y(t) of system(3) for n=0.95 and m=0.4, 0.3.

The values of constant parameters are $M = 5$, $\delta = 0.01$, $\alpha = 10$, $k = 100$, $a = 0.02$, $\gamma = 0.09$, $\beta = 0.6$, $c = 0.02$ and in fig.1 (I) $m = 0.1$, $x_0 = 9.8$, $y_0 = 19.65$ free parameters are $n = 0.99, .85$. In fig.1 (II) $m = 0.3$, $x_0 = 12.6$,$y_0 = 24.5$ and free parameters are $n = 0.95, .9$. In fig.2 initial conditions are $x_0 = 12.6$, $y_0 = 24.5$, $x_0 = 17.65$, $y_0 = 32.3$, $x_0 = 9.8$, $y_0 = 19.65$, $x_0 = 13$, $y_0 = 25.09$ and free parameters are (a) $m = 0.3$, $n = 0.99$. (b) $m = 0.3$, $n = 0.91$. In fig.4 $n = 0.95$, $x_0 = 12.6$, $y_0 = 24.5$ and free parameters are $m = 0.3, 0.4$.

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Existence and uniqueness of the mild solution for fuzzy fractional semilinear initial value problems

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Abstract

In this paper we will study the existence and uniqueness of mild solution for the fuzzy fractional semilinear initial value problem:

$$\begin{cases} u^{\eta}(t) & = Au(t) + f(t, u(t), Gu(t), Su(t)), t > t_0, \eta \in (0, 1], \\ u(t_0) & = u_0, \end{cases}$$

where $f(t, u(t), Gu(t), Su(t))$ is a given function that is satisfied in Lipschitz condition and fuzziness in this fractional problem occurs as a result of fuzzy initial value. To this aim, we introduce Caputo-differentiability concept and purpose mild solution for fuzzy fractional differential equation.

Keywords: Fuzzy fractional differential equations, Existence and uniqueness, Caputo-differentiability, Fuzzy mild solution, Fuzzy-valued function
Mathematics Subject Classification [2010]: 34A12, 94D05, 34A08

1 Introduction

The importance and popularity of fractional differential equations have been increased during the recent decades, mainly due to its widespread use in numerous variety fields of science and engineering. The existence and uniqueness of the crisp mild solution for the fractional semilinear initial value functions have been studied before, [3], [4]. Since a little uncertainty in data such as uncertainty in the initial value or ambiguity in function as a result of vagueness in one of its constant elements, can change the crisp case of fractional differential equation to fuzzy one, recently fuzzy fractional differential equation has been also regarded, so the existence and uniqueness of solution for this type of equations must be considered. In this paper, we study the existence and uniqueness of mild solution for the fuzzy fractional semilinear initial value problem. To this regards, the uniqueness and existence of the mild solution for fuzzy fractional semilinear initial value problems is proved.
The fuzzy semilinear initial value problem of non-integer order which is considered here is

$$\begin{cases} u^{\eta}(t) & = Au(t) + f(t, u(t), Gu(t), Su(t)), t > t_0, \eta \in (0, 1], \\ u(t_0) & = u_0, \end{cases} \tag{1}$$

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where, A is the generator of strongly semigroup $\{T(t), t \geq 0\}$ on Banach space E and $f : [t_0, T] \times E \times E \times E \to E$ is continuous in t and f satisfies the following condition:
$d(f(t, u(t), Gu(t), Su(t), f(t, v(t), Gv(t), Sv(t))) \leq L_1(t)d(u, v)+L_2(t)d(Gu, Gv)+L_3(t)d(Su, Sv)$
and

$$Gu(t) = \int_{t_0}^{t} K(t, s)u(s)\,\mathrm{d}s, \qquad K \in C[D, \mathbb{R}^+], \qquad Su(t) = \int_{t_0}^{t} H(t, s)u(s)\,\mathrm{d}s, \qquad H \in C[D, \mathbb{R}^+]$$

where

$$D = \{(t, s) \in \mathbb{R}^2; 0 \leq s \leq T\} \qquad D_0 = \{(t, s) \in \mathbb{R}^2 : 0 \leq t, s \leq T\}$$

2 Fuzzy fractional semilinear initial value problem

Consider the fuzzy semilinear initial value problem (1), here we purpose the mild solution for fuzzy fractional semilinear initial value problem by using the definition of its crisp case [3], [4].

Definition 2.1. A continuous fuzzy solution $u(t)$ of the integral equation

$$u(t) = T(t - t_0)u_0 + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t - s)^{\eta-1} T(t - s)f(s, u(s), Gu(s), Su(s))\,\mathrm{d}s \tag{2}$$

will be called a mild solution of the initial value problem (1), if u is ${}^C[(i) - \eta]$-differentiable and if u is ${}^C[(ii) - \eta]$-differentiable the mild solution is

$$u(t) = T(t - t_0)u_0 \ominus (-1)\frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t - s)^{\eta-1} T(t - s)f(s, u(s), Gu(s), Su(s))\,\mathrm{d}s \tag{3}$$

Here we need to use the Caputo-differentiability, so the following theorem is presented.

Theorem 2.2. *Let $f : (a, b) \to E$ and $x_0 \in (a, b)$, $0 < \eta < 1$ such that for all $0 \leq \alpha < 1$ then*
(1) If $f(x)$ be a ${}^C[(i)-\eta)]$-differentiable then $\left({}^cD^{\eta}_{a^+} f\right)(x_0, \alpha) = \left[{}^cD^{\eta}_{a^+} \underline{f}(x_0, \alpha), {}^cD^{\eta}_{a^+} \bar{f}(x_0, \alpha)\right]$
(2) If $f(x)$ be a ${}^C[(ii)-\eta)]$-differentiable then $\left({}^cD^{\eta}_{a^+} f\right)(x_0, \alpha) = \left[{}^cD^{\eta}_{a^+} \bar{f}(x_0, \alpha), {}^cD^{\eta}_{a^+} \underline{f}(x_0, \alpha)\right]$
where ${}^cD^{\eta}_{a^+} \underline{f}(x_0, \alpha) = \frac{1}{\Gamma(1-\eta)} \int_a^{x_0} (x_0 - t)^{-\eta} \underline{f}'(t, \alpha)\,\mathrm{d}t$
and ${}^cD^{\eta}_{a^+} \bar{f}(x_0, \alpha) = \frac{1}{\Gamma(1-\eta)} \int_a^{x_0} (x_0 - t)^{-\eta} \bar{f}'(t, \alpha)\,\mathrm{d}t$

Proof. See [2]. □

Lemma 2.3. *The initial value problem* (1) *is equivalent to the nonlinear integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} Au(s)\,\mathrm{d}s + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} T(t-s)f(s, u(s), Gu(s), Su(s))\,\mathrm{d}s \tag{4}$$

for case ${}^C[(i) - \eta)]$-differentiability, and we have

$$u(t) = u_0 \ominus (-1) \left(\frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t - s)^{\eta-1} Au(s)\,\mathrm{d}s + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t - s)^{\eta-1} T(t - s)f(s, u(s), Gu(s), Su(s))\,\mathrm{d}s \right) \tag{5}$$

for case $^C[(ii)-\eta]$*-differentiablity, where* $0 \leq t_0 < t \leq t_0 + a$ *and provided that the mentioned Hukuhara differences exists. In other words, every solution of the integral equation* (4) *or* (5) *is also solution of our original initial value problem* (1) *and vice versa.*

Proof. The above theorem can be proved using the definition (2.1) and theorem (2.2). □

3 The main result

In this section we shall prove our main result. We prove a theorem concerned with the existence and uniqueness of mild solution for the semilinear initial value problem (1), which was proved for non-fuzzy case [3].

Lemma 3.1. *let* $u(t)$ *and* $v(t)$ *be fuzzy functions and* $f(t)$ *be a crisp function, then we have* $d(f(t)u(t), f(t)v(t) = \|f(t)\|d(u(t), v(t))$

Proof. By defining the differential value it will be proved easily. □

Theorem 3.2. *Let* $f : [t_0, T] \times E \times E \times E \to E$ *be continuous in* $t \in [t_0, T]$ *and uniformly Lipschitz continuous (with constant* L*) on* E_0 *if* A *is generator of strongly continuous semigroup* $T(t); t \geq 0$ *on* E *then for every* $u_0 \in E$*, the initial value problem* (1) *has a unique mild solution* $u \in C([t_0, T], E)$*.*

Proof. Let $u_0 \in E$ be fixed. We define a mapping $F : C([t_0, T], E) \to C([t_0, T], E)$ by

$$(Fu)(t) = T(t-t_0)u_0 + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} T(t-s) f(s, u(s), Gu(s), Su(s))\, \mathrm{d}s, \qquad t_0 \leq t \leq T.$$

Now we show that F is contraction. For $u, v \in C([t_0, T], E)$ it follows from the definition of F that
$d((Fu)(t), (Fv)(t)) = d(T(t-t_0)u_0 + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} T(t-s) f(s, u(s), Gu(s), Su(s))\, \mathrm{d}s =$
$d(T(t-t_0)u_0 + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} T(t-s) f(s, u, Gu, Su\, \mathrm{d}s, T(t-t_0)u_0 + \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} T(t-$
$s) f(s, v, Gv, Sv)\, \mathrm{d}s)$
$= d(\frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} T(t-s) f(s, u(s), Gu(s), Su(s))\, \mathrm{d}s, \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t-s)^{\eta-1} T(t-s) f(s, v(s), Gv(s), Sv(s))\, \mathrm{d}s)$
$\leq \frac{M}{\Gamma(\eta)} \int_{t_0}^{t} \|(t-s)^{\eta-1}\| (L_1(s) d(u, v) + L_2(s) d(Gu(s), Gv(s)) + L_3(s) d(Su(s), Sv(s)))\, \mathrm{d}s$
Now
$\frac{M}{\Gamma(\eta)} \int_{t_0}^{t} \|(t-s)^{\eta-1}\| L_2(s) d(Gu(s), Gv(s))\, \mathrm{d}s \leq \frac{M}{\Gamma(\eta)} \int_{t_0}^{t} \|(t-s)^{\eta-1}\| L_2(s) \int_{t_0}^{t} \|K(s,z)\| d(u(z), v(z))\, \mathrm{d}z\, \mathrm{d}s$
$\leq \frac{M}{\Gamma(\eta)} \int_{t_0}^{t} \|(t-s)^{\eta-1}\| L_2(s) d(u(s), v(s)) \int_{t_0}^{t} \|K(s,z)\|\, \mathrm{d}z\, \mathrm{d}s$
$\leq \frac{M}{\Gamma(\eta)} \int_{t_0}^{t} \|(t-s)^{\eta-1}\| L_2(s) d(u(s), v(s)) K^*\, \mathrm{d}s \leq MK^* I^\eta L_2(t) d(u, v)$
Similarly

$$\frac{M}{\Gamma(\eta)} \int_{t_0}^{t} \|(t-s)^{\eta-1}\| L_3(s) d(Su(s), Sv(s))\, \mathrm{d}s \leq MH^* I^\eta L_3(t) d(u(s), v(s)),$$

$$\frac{M}{\Gamma(\eta)} \int_{t_0}^{t} \|(t-s)^{\eta-1}\| L_1(s) d(u(s), v(s))\, \mathrm{d}s \leq MI^\eta L_1(t) d(u(s), v(s))$$

then we have $d((Fu)(t), (Fv)(t)) \leq MI^\eta L_1(t) d(u(s), v(s)) + MK^* I^\eta L_2(t) d(u(s), v(s)) +$ $MHI^\eta L_3(t) d(u(s), v(s)) \leq MI^\eta L(t) d(u(s), v(s))(1 + K^* + H^*) \leq d(u(s), v(s))$

Therefore, F is a contraction operator on $C([t_0, T], E)$ and has a fixed point $Fu(t) = u(t)$. Hence the initial value problem (1) has a solution.
This fixed point is the desired solution of the integral equation

$$u(t) = T(t - t_0)u_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\eta-1} T(t - s) F(s, u(s), Gu(s), Su(s))\, \mathrm{d}s$$

To prove the uniqueness of $u(t)$ let $v(t)$ be another mild solution of (1) with the initial value v_0 then

$$\begin{aligned} d(u(t), v(t)) &\leq d(T(t - t_0)u_0, T(t - t_0)v_0) \\ &+ \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} \|(t - s)^{\eta-1} T(t - s)\| d(f(s, u(s), Gu(s), Su(s)), f(s, v(s), Gv(s), Sv(s)))\, \mathrm{d}s \end{aligned}$$

and based on Gronwalls inequality we get $d(u(t), v(t)) \leq Me^{ML(T-t_0)} d(u_0, v_0)$ which yields the uniqueness of $u(t)$. We proved this for case ${}^{C}[(i) - \eta)]$-differentiability of u, if u be ${}^{C}[(i) - \eta)]$-differentiable we define mapping F as follow:

$$(Fu)(t) = T(t - t_0)u_0 \ominus (-1)(\frac{1}{\Gamma(\eta)} \int_{t_0}^{t} (t - s)^{\eta-1} T(t - s) f(s, u(s), Gu(s), Su(s))\, \mathrm{d}s)$$

and the proof is similar to the pervious case. □

4 Conclusion

For solving real problems, which is formulated with fuzzy fractional differential equation, with numerical methods, we need to know the existence of solution. To this regard, in this paper we study the existence and uniqueness of the mild solution for fuzzy fractional semilinear initial value problems, which is proved in crisp case later.

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Existence of infinitely many solutions for coupled system of Schrӧdinger-Maxwell's equations

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Abstract

In this paper we study the existence of infinitely many large energy solutions for the coupled system of Schrӧdinger-Maxwell's equations via the Fountain theorem under Cerami condition. More precisely, we consider the More general case and weaken conditions with respect to [2].

Keywords: SchrӧdingerMaxwell system ,Cerami condition,Variational methods, Strongly indefinite functionals.
Mathematics Subject Classification [2010]: 35Pxx, 46Txx

1 Introduction

In this paper, we study the nonlinear coupled system of Schrӧdinger-Maxwell's equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = H_v(x,u,v) & \text{in } \mathbb{R}^3 \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3 \\ & \\ -\Delta v + V(x)v + \psi v = H_u(x,u,v) & \text{in } \mathbb{R}^3 \\ -\Delta \psi = v^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$ and $H \in C^1(\mathbb{R}^3, \mathbb{R})$ which are satisfied in some suitable conditions. In the classical model, the interaction of a charge particle with an electromagnetic field can be described by the nonlinear Schrӧdinger-Maxwell's equations. In this article, we want to study the interaction of two charge particles Simultaneously with same potential function $V(x)$ and different scalar potential ϕ and ψ which are satisfied in suitable conditions. More precisely, we have to solve the system 1 if we want to find electrostatic-type solutions.

Existence of solutions are obtained via Fountain theorem in critical point theory. More precisely, in this paper we consider the more general case and weaken the condition of V_1 in [2] and we assume that the potential V is non-periodic and sing changing. We assume the following conditions :

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V_1^*) $V \in C(\mathbb{R}^3, \mathbb{R})$ and there exists some $M > 0$ such that the set $\Omega_M = \{x \in \mathbb{R}^3 \quad ; \quad V(x) \leq M\}$ is not nonempty and has finite Lebesgue measure.

H_1) $H \in C^1(\mathbb{R}^3 \times \mathbb{R}^2, \mathbb{R})$ and for some $2 < p < 2^* = 6$, and $M_1, M_2 > 0$,

$$|H_u(x,u,v)| \leq M_1|u| + M_1|u|^{p-1} \qquad \textit{and} \qquad |H_v(x,u,v)| \leq M_2|v| + M_2|v|^{p-1},$$

for a.e $x \in \mathbb{R}^3$ and $u, v : \mathbb{R}^3 \to \mathbb{R}$, and also

$$\lim_{u\to 0} \frac{H_u(x,u,v)}{u} = 0 \qquad \textit{and} \qquad \lim_{u\to 0} \frac{H_v(x,u,v)}{v} = 0,$$

uniformly for $x \in \mathbb{R}^3$ and $u, v \in \mathbb{R}$.

H_2) $\lim\limits_{|(u,v)|\to\infty} \frac{H(x,u,v)}{|(u,v)|^4} = +\infty$, uniformly in $x \in \mathbb{R}^3$ and $(u,v) \in \mathbb{R}^2$ and

$$H(x,0,0) = 0, H(x,u,v) \geq 0$$

for all $(x,u,v) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$.

H_3) There exists a constant $\theta \geq 1$ such that

$$\theta \hat{H}(x,u,v) \geq \hat{H}(x,su,sv)$$

for all $x \in \mathbb{R}^3, (u,v) \in \mathbb{R}^2$ and $t, s \in [0,1]$, where

$$\hat{H}(x,u,v) = H_u(x,tu,v)tu + H_v(x,u,sv)sv - 4H(x,tu,sv).$$

H_4) $H(x,-u,v) = H(x,u,v)$ and $H(x,u,-v) = H(x,u,v)$ for all $x \in \mathbb{R}^3$ and $(u,v) \in \mathbb{R}^2$.

Here, we express the Cerami condition which was established by G. Cerami in [1]

Definition 1.1. Suppose that functional I is C^1 and $c \in \mathbb{R}$, if any sequence $\{u_n\}$ satisfying $I(u_n) \to c$ and $(1 + \|u_n\|)I'(u_n) \to 0$ has a convergence subsequence, we say the I satisfies Cerami condition at the level c.

To approach the main result, we need the following critical point theorem.

Theorem 1.2. *(Fountain theorem under Cerami condition) Let X be a Banach space with the norm $\|.\|$ and let X_j be a sequence of subspace of X with $dim X_j < \infty$ for any $j \in \mathbb{N}$. Further, $X = \overline{\oplus_{j\in\mathbb{N}} Xj}$, the closure of the direct sum of all X_j. Set $W_k = \oplus_{j=0}^{k} X_j$, $Z_k \overline{\oplus_{j=k}^{\infty} X_j}$. Consider an even functional $I \in C^1(X, \mathbb{R})$, that is , $I(-u) = I(u)$ for any $u \in X$. Also suppose that for any $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

I_1) $a_k := \max_{u\in W_k, \|u\|=\rho_k} I(u) \leq 0$,

I_2) $b_k := \inf_{u\in Z_k, \|u\|=r_k} I(u) \to +\infty$ *as* $k \to \infty$,

I_3) *the Cerami condition holds at any level $c > 0$. Then the functional I has an unbounded sequence of critical values.*

2 Main results

Now, we consider the function space

$$E := \{u \in H^1(\mathbb{R}^3) \quad | \quad \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx < \infty\}.$$

Then E is Hilbert space with the inner product

$$(u, v)_E := \int_{\mathbb{R}^3} (\nabla u.\nabla v + V(x)uv)dx \tag{2}$$

and the norm $\|u\|_E := (u, u)_E^{\frac{1}{2}}$. We set

$$X_E := E \times E, Y_{HD} := H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \quad and \quad Z_{ED} := E \times D^{1,2}(\mathbb{R}^3).$$

Hence , we can define an inner product on X_E as

$$((u, v)(w, z))_{X_E} := (u, w)_E + (v, z)_E \tag{3}$$

and the corresponding norm on X_E by this inner product as following

$$\|(u, v)\|_{X_E} := (\|u\|_E^2 + \|v\|_E^2)^{\frac{1}{2}} = ((u, u)_E + (v, v)_E)^{\frac{1}{2}}. \tag{4}$$

Proposition 2.1. *The following statements are equivalent :*
i) $((u, \phi_u), (v, \psi_v)) \in Z_{ED} \times Z_{ED}$ *is a critical point of* J*;*
ii) (u, v) *is a critical point of functional* I *and* $(\phi, \psi) = (\phi_u, \psi_v)$*.*

Proposition 2.2. *under the conditions* $H_1 - H_3$*, the functional* $I(u, v)$ *satisfies the Cerami condition at any positive level.*

Now, our main result is the following :

Theorem 2.3. *Let* V_1^**,* $H_1 - H_4$ *be satisfied. Then the system 1 has infinitely many solutions* $\{((u_k, \phi_k), (v_k, \psi_k))\}$ *in product space* $Y_{HD} \times Y_{HD}$ *(see section 2)which satisfies in*

$$\frac{1}{2}\int_{\mathbb{R}^3} [\ |\nabla u_k|^2 + |\nabla v_k|^2 + V(x)(u_k^2 + v_k^2)\ ]dx - \frac{1}{4}\int_{\mathbb{R}^3} [\ |\nabla \phi_k|^2 \ + \ |\nabla \psi_k|^2\ ]dx$$
$$+ \frac{1}{2}\int_{\mathbb{R}^3} [\ \phi_k u_k^2 + \ \psi_k v_k^2\ ]dx - \int_{\mathbb{R}^3} H(x, u, v)dx \to +\infty.$$

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Existence results for a k-dimensional system of multi-term fractional integro-differential equations with anti-periodic boundary value problems

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Abstract

In this paper, we establish the existence and uniqueness of solutions for a k-dimensional system of multi-term fractional integro-differential equations with antiperiodic boundary conditions by applying some standard fixed point results. We include an example to show the applicability of our results.

Keywords: Caputo fractional derivative, k-dimensional system, fractional integro-differential equations, Fixed point
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Fractional differential and integro-differential equations have been proved that they are very valued tools in the modeling of many phenomena in various fields of science and engineering, such as, viscoelasticity, electrochemistry, electromagnetism, economics, optimal control and so forth. Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes (see for example, [1], [2]). The study of a coupled system of fractional differential equations is also very significant because this kind of system can often occur in applications (see for example, [3], [4]).

Let $T > 0$ and $I = [0, T]$. In this paper, we study the existence and uniqueness of so-

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lutions for the k-dimensional system of multi-term fractional integro-differential equations

$$\begin{cases} {}^cD^{\alpha_1}x_1(t) = f_1\Big(t, x_1(t), x_2(t), \dots, x_k(t), \phi_{11}x_1(t), \phi_{12}x_2(t), \dots, \phi_{1k}x_k(t), \\ {}^cD^{\mu_{11}}x_1(t), {}^cD^{\mu_{12}}x_2(t), \dots, {}^cD^{\mu_{1k}}x_k(t), {}^cD^{\beta_{11}}x_1(t), {}^cD^{\beta_{12}}x_2(t), \dots, {}^cD^{\beta_{1k}}x_k(t)\Big), \\ \\ {}^cD^{\alpha_2}x_2(t) = f_2\Big(t, x_1(t), x_2(t), \dots, x_k(t), \phi_{21}x_1(t), \phi_{22}x_2(t), \dots, \phi_{2k}x_k(t), \\ {}^cD^{\mu_{21}}x_1(t), {}^cD^{\mu_{22}}x_2(t), \dots, {}^cD^{\mu_{2k}}x_k(t), {}^cD^{\beta_{21}}x_1(t), {}^cD^{\beta_{22}}x_2(t), \dots, {}^cD^{\beta_{2k}}x_k(t)\Big), \\ \vdots \\ {}^cD^{\alpha_k}x_k(t) = f_k\Big(t, x_1(t), x_2(t), \dots, x_k(t), \phi_{k1}x_1(t), \phi_{k2}x_2(t), \dots, \phi_{kk}x_k(t), \\ {}^cD^{\mu_{k1}}x_1(t), {}^cD^{\mu_{k2}}x_2(t), \dots, {}^cD^{\mu_{kk}}x_k(t), {}^cD^{\beta_{k1}}x_1(t), {}^cD^{\beta_{k2}}x_2(t), \dots, {}^cD^{\beta_{kk}}x_k(t)\Big), \ (t \in I), \end{cases} \tag{1}$$

with anti-periodic boundary conditions $x_i(0) = -x_i(T)$, ${}^cD^{p_i}x_i(0) = -{}^cD^{p_i}x_i(T)$ and ${}^cD^{q_i}x_i(0) = -{}^cD^{q_i}x_i(T)$ for $i = 1, 2, \dots, k$, where ${}^cD$ denotes the Caputo fractional derivative, $\alpha_i \in (2, 3]$, $p_i, \mu_{ij} \in (0, 1)$, $q_i, \beta_{ij} \in (1, 2)$ for $i, j = 1, 2, \dots, k$, $(\phi_{ij}x_j)(t) = \int_0^t \lambda_{ij}(t, s)x_j(s)ds$ and $f_j \in C(I \times \mathbb{R}^{4k}, \mathbb{R})$, $\lambda_{ij} : I \times I \to [0, \infty)$ are continuous functions for all $i, j = 1, 2, \dots, k$.

2 Preliminaries

In this section we introduce preliminary facts and some basic results, which are used throughout this paper.

Lemma 2.1. *For each $y \in C([0, T])$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^{\alpha}x(t) = y(t), \quad (t \in [0, T],\ T > 0,\ 2 < \alpha \le 3) \\ x(0) = -x(T),\ {}^cD^{p}x(0) = -{}^cD^{p}x(T),\ {}^cD^{q}x(0) = -{}^cD^{q}x(T), \quad (0 < p < 1,\ 1 < q < 2), \end{cases}$$

is given by $x(t) = \int_0^T G_\alpha(t, s)y(s)ds$, where $G_\alpha(t, s)$ is the Green's function defined as

$$G_\alpha(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1} - \frac{1}{2}(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2-p)(T-2t)(T-s)^{\alpha-p-1}}{2\Gamma(\alpha-p)T^{1-p}} \\ - \frac{[pT^2 - 4Tt + 2(2-p)t^2]\Gamma(3-q)(T-s)^{\alpha-q-1}}{4(2-p)\Gamma(\alpha-q)T^{2-q}}, & s \le t, \\ \\ -\frac{(T-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{\Gamma(2-p)(T-2t)(T-s)^{\alpha-p-1}}{2\Gamma(\alpha-p)T^{1-p}} \\ - \frac{[pT^2 - 4Tt + 2(2-p)t^2]\Gamma(3-q)(T-s)^{\alpha-q-1}}{4(2-p)\Gamma(\alpha-q)T^{2-q}}, & t \le s. \end{cases}$$

Theorem 2.2. *Let E be a Banach space, $T : E \longrightarrow E$ a completely continuous operator. Suppose that the set $V = \{u \in E : u = \mu Tu,\ 0 \le \mu \le 1\}$ is bounded. Then T has a fixed point in E.*

We shall use the last two results for solving the problem (1).

3 Main results

Let us introduce the space $X = \{u(t) :\ u(t) \in C^2(I)\}$ endowed with the norm $\|x\|_X = \sup_{t\in I}|x(t)| + \sup_{t\in I}|x'(t)| + \sup_{t\in I}|x''(t)|$. In fact, $(X, \|.\|_X)$ and the product space $(X^k = \underbrace{X \times X \times \cdots \times X}_{k}, \|.\|_*)$ endowed with the norm $\|(x_1, x_2, \ldots, x_k)\|_* = \|x_1\|_X + \|x_2\|_X + \cdots + \|x_k\|_X$ are Banach spaces.

For each $i = 1, 2, \ldots, k$, put

$$M_i = \left(\frac{3}{2\Gamma(\alpha_i+1)} + \frac{\Gamma(2-p_i)}{2\Gamma(\alpha_i-p_i+1)} + \frac{(4-p_i)\Gamma(3-q_i)}{4(2-p_i)\Gamma(\alpha_i-q_i+1)}\right)T^{\alpha_i}$$

$$+\left(\frac{1}{\Gamma(\alpha_i)}+\frac{\Gamma(2-p_i)}{\Gamma(\alpha_i-p_i+1)}+\frac{\Gamma(3-q_i)}{(2-p_i)\Gamma(\alpha_i-q_i+1)}\right)T^{\alpha_i-1}+\left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(3-q_i)}{\Gamma(\alpha_i-q_i+1)}\right)T^{\alpha_i-2}$$

and

$$M = \min_{1\le j\le k}\left\{1 - \sum_{i=1}^{k} M_i\left(b_{ij} + c_{ij}\lambda_{ij}^0 + d_{ij}\frac{T^{1-\mu_{ij}}}{\Gamma(2-\mu_{ij})} + e_{ij}\frac{T^{2-\beta_{ij}}}{\Gamma(3-\beta_{ij})}\right)\right\},$$

where $\lambda_{ij}^0 = \sup_{t\in I}\left|\int_0^t \lambda_{ij}(t,s)ds\right|$ for all $i, j = 1, 2, \ldots, k$.
Define the operator $T : X^k \longrightarrow X^k$ by

$$T(x)(t) = \begin{pmatrix} T_1(x)(t) \\ T_2(x)(t) \\ \vdots \\ T_k(x)(t) \end{pmatrix},$$

where $x = (x_1, x_2, \ldots, x_k)$ and

$$T_i(x)(t) = \int_0^T G_{\alpha_i}(t,s)\widetilde{f}_i(s, x(s))ds$$

for $i = 1, 2, \ldots, k$, where

$$\widetilde{f}_i(s, x(s)) = f_i(s, x_1(s), x_2(s), \ldots, x_k(s), \phi_{i1}x_1(s), \phi_{i2}x_2(s), \ldots, \phi_{ik}x_k(s),$$

$${}^cD^{\mu_{i1}}x_1(s), {}^cD^{\mu_{i2}}x_2(s), \ldots, {}^cD^{\mu_{ik}}x_k(s), {}^cD^{\beta_{i1}}x_1(s), {}^cD^{\beta_{i2}}x_2(s), \ldots, {}^cD^{\beta_{ik}}x_k(s)).$$

Theorem 3.1. *The operator $T : X^k \longrightarrow X^k$ is completely continuous.*

Theorem 3.2. *Assume that there exist positive constants $a_i > 0$, $b_{ij} \geq 0$, $c_{ij} \geq 0$, $d_{ij} \geq 0$, $e_{ij} \geq 0$ $(i, j = 1, 2, \ldots, k)$ such that*

$$|f_i(t, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k, z_1, z_2, \ldots, z_k, w_1, w_2, \ldots, w_k)|$$
$$\leq a_i + \sum_{j=1}^{k} b_{ij}|x_j| + \sum_{j=1}^{k} c_{ij}|y_j| + \sum_{j=1}^{k} d_{ij}|z_j| + \sum_{j=1}^{k} e_{ij}|w_j|$$

and $\sum_{i=1}^{k} M_i\left(b_{ij} + c_{ij}\lambda_{ij}^0 + d_{ij}\frac{T^{1-\mu_{ij}}}{\Gamma(2-\mu_{ij})} + e_{ij}\frac{T^{2-\beta_{ij}}}{\Gamma(3-\beta_{ij})}\right) < 1$ *for all* $x_i, y_i, z_i, w_i \in \mathbb{R}$, $t \in I$ *and* $i, j = 1, 2, \ldots, k$. *Then problem* (1) *has at least one solution.*

Theorem 3.3. *Suppose that there exist non-negative constants $\eta_{ij} \geq 0$, $\theta_{ij} \geq 0$, $\nu_{ij} \geq 0$, $\xi_{ij} \geq 0$ for $i, j = 1, 2, \ldots, k$ such that*

$$|f_i(t, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k, z_1, z_2, \ldots, z_k, w_1, w_2, \ldots, w_k)$$
$$-f_i(t, x'_1, x'_2, \ldots, x'_k, y'_1, y'_2, \ldots, y'_k, z'_1, z'_2, \ldots, z'_k, w'_1, w'_2, \ldots, w'_k)|$$
$$\leq \sum_{j=1}^{k} \eta_{ij}|x_j - x'_j| + \sum_{j=1}^{k} \theta_{ij}|y_j - y'_j| + \sum_{j=1}^{k} \nu_{ij}|z_j - z'_j| + \sum_{j=1}^{k} \xi_{ij}|w_j - w'_j|$$

and

$$\sum_{j=1}^{k} \eta_{ij} + \sum_{j=1}^{k} \theta_{ij}\lambda_{ij}^0 + \sum_{j=1}^{k} \nu_{ij}\frac{T^{1-\mu_{ij}}}{\Gamma(2-\mu_{ij})} + \sum_{j=1}^{k} \xi_{ij}\frac{T^{2-\beta_{ij}}}{\Gamma(3-\beta_{ij})} \leq \frac{1}{2kM_i}$$

for all $t \in I$, $x_i, y_i, z_i, w_i, x'_i, y'_i, z'_i, w'_i \in \mathbb{R}$ and $i = 1, 2, \ldots, k$. Then the problem (1) *has a unique solution.*

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Green's function for fractional differential equation with Hilfer derivative

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Abstract

In this article, as an interpolation between the Reimann-Liouville and Caputo fractional derivatives (Hilfer fractional derivative), we obtain the associated Green's function for the fractional boundary value problem. We use the Laplace transform to derive the associated Green's function.

Keywords: Green's function, Hifer fractional derivative, Caputo fractional derivative, Reimann-Liouville fractional integral and derivative, Laplace transform.
Mathematics Subject Classification [2010]: 26A33, 44A15, 65M80.

1 Introduction and Preliminaries

Recently Ferreira has obtained the Green's functions for the fractional boundary value problems with the Caputo and Reimann-Liouville fractional derivatives and used these functions for obtaining Lyapunov type inequalities for these problems [1, 2]. In this paper as generalization, we consider the following fractional boundary value problem including the Hilfer fractional derivative

$$D_{a+}^{\alpha,\beta}y(t) + q(t)y(t) = 0, \tag{1}$$

with the boundary conditions

$$y(a) = y(b) = 0, \tag{2}$$

where $1 < \alpha < 2$, $0 \leq \beta \leq 1$ and $q : [a, b] \longrightarrow \mathbb{R}$ is a continuous function. We intend to change our boundary value problem as an equivalent integral equation. At first, we consider the Hilfer fractional derivatives and some properties of it.

Definition 1.1. For $n-1 < \alpha < n$, the fractional Caputo derivative of order α is defined as [5]

$${}^{C}D_{a+}^{\alpha}y(t) = I_{a+}^{n-\alpha}D_{a+}^{n}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-1-\alpha}\frac{d^n}{dt^n}y(u)du, \tag{3}$$

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where I^{α}_{a+} and D^{α}_{a+} are the Reimann-Liouville fractional integral and derivative of order α, respectively, that are

$$\big(I^{\alpha}_{a+}y\big)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(u)}{(t-u)^{1-\alpha}}dt, \qquad \alpha \in \mathbb{C}, \Re(\alpha) > 0, \tag{4}$$

and

$$\big(D^{\alpha}_{a+}y\big)(t) = \Big(\frac{d}{dt}\Big)^n \big(I^{n-\alpha}_{a+}y\big)(t), \qquad \alpha \in \mathbb{C}, \Re(\alpha) > 0, n = [\Re(\alpha)] + 1. \tag{5}$$

Remark 1.2. The fractional Caputo derivative has the Laplace transform

$$\mathcal{L}\{{}^{C}D^{\alpha}_{a+}y(t); s\} = s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}y^{(k)}(0), \qquad n-1 < \alpha \leq n, \tag{6}$$

and the Laplace transform of the fractional Riemann-Liouville integral is

$$\mathcal{L}\{I^{\alpha}_{a+}y(t); s\} = \frac{1}{s^{\alpha}}Y(s), \tag{7}$$

where $Y(s)$ is the Laplace transform $y(t)$.

Lemma 1.3. *If $y(t) \in C(a,b) \cap L(a,b)$, then*

$${}^{C}D^{\alpha}_{a+}I^{\alpha}_{a+}y(t) = y(t). \tag{8}$$

Also, if $y(t)$ and its fractional derivative of order $\alpha > 0$ belong to $C(a,b) \cap L(a,b)$, then for $c_j \in \mathbb{R}$ we have

$$I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+}y(t) = y(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_n(t-a)^{n-1}, \qquad n-1 < \alpha \leq n. \tag{9}$$

Definition 1.4. (Hilfer derivative) The right-sided fractional derivative $D^{\alpha,\beta}_{a+}$ and the left-sided fractional derivative $D^{\alpha,\beta}_{a-}$ of order α and type β are defined by [3, 4]

$$\Big(D^{\alpha,\beta}_{a\pm}f\Big)(x) = \Big(\pm I^{\beta(1-\alpha)}_{a\pm}\frac{d}{dx}\big(I^{(1-\beta)(1-\alpha)}_{a\pm}f\big)\Big)(x), \qquad -\infty \leq a < t < b \leq \infty. \tag{10}$$

The generalization (10), for $\beta = 0$ coincides with the Riemann-Liouville derivative (5) and for $\beta = 1$ coincides with the Caputo derivative (3). From relation (10) we deduce the following lemma.

Lemma 1.5. *Let $-\infty \leq a < t < b \leq \infty$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then the relation*

$$D^{\alpha,\beta}_{a+}y(t) = {}^{C}D^{\alpha}_{a+}y(t) + \frac{t^{-\alpha}y(a+)}{\Gamma(1-\alpha)}, \tag{11}$$

is valid between the Hilfer and Caputo fractional derivatives and shows that it is independent of parameter β.

Proof. By using the relation (10) and applying the following relation

$$I_{a+}^{\alpha} I_{a+}^{\mu} = I_{a+}^{\alpha+\mu} = I_{a+}^{\mu} I_{a+}^{\alpha}, \tag{12}$$

we obtain

$$\begin{aligned} D_{a+}^{\alpha,\beta} y(t) &= \Big(I_{a+}^{\beta(1-\alpha)} \frac{d}{dt}\big(I_{a+}^{(1-\beta)(1-\alpha)} y\big)\Big)(t) \\ &= \Big(I_{a+}^{\beta(1-\alpha)} I_{a+}^{(1-\beta)(1-\alpha)} \frac{d}{dt} y\Big)(t) + I_{a+}^{\beta(1-\alpha)} \frac{t^{\beta\alpha-\beta-\alpha} y(a+)}{\Gamma(1-\beta-\alpha+\beta\alpha)} \\ &= I_{a+}^{1-\alpha} \frac{d}{dt} f(t) + \frac{t^{-\alpha} y(a+)}{\Gamma(1-\alpha)} = {}^{C}D_{a+}^{\alpha} y(t) + \frac{t^{-\alpha} y(a+)}{\Gamma(1-\alpha)}. \end{aligned}$$

□

2 Main Theorem

Theorem 2.1. *The fractional boundary value problem*

$$D_{a+}^{\alpha,\beta} y(t) + q(t) y(t) = 0, \qquad y(a) = y(b) = 0, \tag{13}$$

is equivalent to the integral equation

$$y(t) = \int_a^b G(t,u) q(u) y(u) du + \frac{y(a+)}{\Gamma(1-\alpha)} \int_a^b G(t,u) u^{-\alpha} du, \tag{14}$$

where the Green's function G is given by

$$G(t,u) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{t-a}{b-a}(b-u)^{\alpha-1} - (t-u)^{\alpha-1}, & a \le u \le t \le b, \\ \frac{t-a}{b-a}(b-u)^{\alpha-1}, & a \le t \le u \le b. \end{cases} \tag{15}$$

Proof. Applying the relation (11) and using the Lemma (1.3), we have

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} q(u) y(u) du \\ &- \frac{y(a+)}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^t (t-u)^{\alpha-1} u^{-\alpha} du + c_0 + c_1(t-a), \end{aligned} \tag{16}$$

where c_0 and c_1 are real constants. Now, by employing the boundary conditions we can obtain the coefficients c_0 and c_1 as follows

$$y(a) = 0 \Leftrightarrow c_0 = 0,$$

$$\begin{aligned} y(b) = 0 \Leftrightarrow c_1 = &\frac{1}{(b-a)\Gamma(\alpha)} \int_a^b (b-u)^{\alpha-1} q(u) y(u) du \\ &+ \frac{y(a+)}{(b-a)\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^b (b-u)^{\alpha-1} u^{-\alpha} du. \end{aligned}$$

Therefore, the unique solution of (13) is

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1} q(u)y(u)du - \frac{y(a+)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_a^t (t-u)^{\alpha-1}u^{-\alpha}du \\ &+ \frac{1}{(b-a)\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1}(t-a)q(u)y(u)du \\ &+ \frac{y(a+)}{(b-a)\Gamma(\alpha)\Gamma(1-\alpha)}\int_a^b (b-u)^{\alpha-1}u^{-\alpha}(t-a)du, \end{aligned}$$

or equivalently

$$\begin{aligned} y(t) = &\frac{1}{\Gamma(\alpha)}\int_a^t \Big(\frac{(b-u)^{\alpha-1}}{b-a}(t-a) - (t-u)^{\alpha-1}\Big)q(u)y(u)du \\ &+ \frac{1}{\Gamma(\alpha)}\int_t^b \frac{(b-u)^{\alpha-1}}{b-a}(t-a)q(u)y(u)du \\ &+ \frac{y(a+)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_a^t \Big(\frac{(b-u)^{\alpha-1}}{b-a}(t-a) - (t-u)^{\alpha-1}\Big)u^{-\alpha}du \\ &+ \frac{y(a+)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_t^b \frac{(b-u)^{\alpha-1}}{b-a}(t-a)u^{-\alpha}du \\ = &\int_a^b G(t,u)q(u)y(u)du + \frac{y(a+)}{\Gamma(1-\alpha)}\int_a^b G(t,u)u^{-\alpha}du. \end{aligned}$$

□

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Talk Hopf bifurcation in a general class of delayed BAM neural networks pp.: 1–4

Hopf bifurcation in a general class of delayed BAM neural networks

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Abstract

In this paper, Hopf bifurcation analysis of delayed BAM neural networks, which consist of one neuron in the X-layer and other neurons in the Y-layer, will be discussed. Here, the number of neurons can be chosen arbitrarily. The associated characteristic equation is studied by classification according to the number of neurons. Numerical examples are also presented.

Keywords: Hopf bifurcation, Time delay, Characteristic equation
Mathematics Subject Classification [2010]: 34C23, 34K18, 37C75

1 Introduction

Since Hopfield constructed a simplified neural network (NN) model [1], the dynamical characteristics of artificial neural networks have been applied in many sciences such as mathematics, physics and computer sciences. As time delays always occur in the signal transmission, Marcus and Westervelt proposed an NN model with delay [2].

The bidirectional associative memory (BAM) networks were first introduced by Kasko (see [3]). It is well known that BAM NNs are able to store multiple patterns, but most of NNs have only one storage pattern or memory pattern. BAM NNs have practical applications in storing paired patterns or memories and possess the ability of searching the desired patterns through both forward and backward directions. It should be noted that periodic solutions can be resulted from the Hopf bifurcation in delay differential equations. In fact, various local periodic solutions can arise from the different equilibrium points of BAM NNs by applying Hopf bifurcation technique.

The delayed BAM neural network is described as follows:

$$\begin{cases} \dot{x_i}(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_{ji})) + I_i & (i = 1, 2, \ldots, n) \\ \dot{y_j}(t) = -\upsilon_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-\sigma_{ij})) + J_j & (j = 1, 2, \ldots, m) \end{cases} \tag{1}$$

where c_{ji} and d_{ij} are the connection weights through the neurons in two layers: the X-layer and the Y-layer. The stability of internal neuron processes on the X-layer and Y-layer are described by μ_i and υ_j, respectively. On the X-layer, the neurons whose states are denoted by $x_i(t)$ receive the input I_i and the inputs outputted by those neurons in the

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Y-layer via activation function f_i, while the similar process happens on the Y-layer. Also, τ_{ji} and σ_{ij} correspond to the finite time delays of neural processing and delivery of signals. For further details, see [3].

Since the exhaustive analysis of the dynamics of such a large system is complicated, some authors have studied the dynamical behaviors of simplified forms of (1). For example, the simplified three-neuron and six-neuron BAM NNs with multiple delays have been studied in [5, 6]. In [4], we studied a five-neuron model with two neurons in the X-layer and three neurons in the Y-layer.

In this paper, Hopf bifurcation analysis of the n-neuron BAM neural network with two time delays will be discussed. In fact, the number of neurons is arbitrary. However, in the previous works, the authors considered models with a determined number of neurons. To be more precise, we consider the following general class of delayed BAM neural network:

$$\begin{cases} \dot{x_1}(t) = -\mu_1 x_1(t) + \sum_{j=1}^{n-1} c_{j1} f_1(y_j(t-\tau_2)) + I_1 \\ \dot{y_j}(t) = -\upsilon_j y_j(t) + d_{1j} g_j(x_1(t-\tau_1)) + J_j \qquad (j = 1, 2, \dots, n-1) \end{cases} \tag{2}$$

where$\mu_1 > 0$, $\upsilon_j > 0 (j = 1, 2, \dots, n-1)$ and $c_{j1}, d_{1j} (j = 1, 2, \dots, n-1)$ are real constants. The time delay from the X-layer to another Y-layer is τ_1, while the time delay from the Y-layer back to the X-layer is τ_2, and there are one neuron in the X-layer and other n-1 neurons in the Y-layer. In the next section, we study Hopf bifurcation on the system (2). To illustrate our theoretical results, numerical examples are also given.

2 Main results

System (2) can be rewritten as the following equivalent system:

$$\begin{cases} \dot{u_1}(t) = -\mu_1 u_1(t) + c_{11} f_1(u_2(t-\tau)) + c_{21} f_1(u_3(t-\tau)) \\ \qquad\qquad + \dots + c_{(n-1)1} f_1(u_n(t-\tau)) \\ \dot{u_2}(t) = -\upsilon_1 u_2(t) + d_{11} g_1(u_1(t)) \\ \dot{u_3}(t) = -\upsilon_2 u_3(t) + d_{12} g_2(u_1(t)) \\ \quad . \\ \quad . \\ \quad . \\ \dot{u_n}(t) = -\upsilon_{n-1} u_n(t) + d_{1(n-1)} g_{n-1}(u_1(t)) \end{cases} \tag{3}$$

where $u_1(t) = x_1(t-\tau_1)$, $u_2(t) = y_1(t)$, $u_3(t) = y_2(t), \dots, u_n(t) = y_{n-1}(t)$ and $\tau = \tau_1 + \tau_2$.

Under the hypothesis

$$\text{(H1) } f_1, g_j \in C^1,\ f_1(0) = g_j(0) = 0,\ (j = 1, 2, \dots, n-1)$$

the associated characteristic equation is as follows:

$$det \begin{pmatrix} \lambda + \mu_1 & -\alpha_{21} e^{-\lambda\tau} & -\alpha_{31} e^{-\lambda\tau} & \dots & -\alpha_{n1} e^{-\lambda\tau} \\ -\alpha_{12} & \lambda + \upsilon_1 & 0 & \dots & 0 \\ -\alpha_{13} & 0 & \lambda + \upsilon_2 & \dots & 0 \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ -\alpha_{1n} & 0 & 0 & \dots & \lambda + \upsilon_{n-1} \end{pmatrix} = 0,$$

where $\alpha_{i1} = c_{(i-1)1} f_1'(0)$, $\alpha_{1i} = d_{1(i-1)} g_{i-1}'(0)$ for $i = 2, \ldots, n$. It can be rewritten as the following equation:

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n + (b_1\lambda^{n-1} + b_2\lambda^{n-2} + \ldots + b_n)e^{-\lambda\tau} = 0. \quad (4)$$

In order to have Hopf bifurcation, we need to study the existence of pure imaginary roots of (4). Letting $\lambda = i\omega$ and substituting this into (4), we have the following four cases: **case I:** $n = 4k\,(k \in N)$, **case II:** $n = 4k - 2\,(k \in N)$, **case III:** $n = 4k - 1\,(k \in N)$ and **case IV:** $n = 4k - 3\,(k \in N)$. In each of the above cases, seperating the real and imaginary parts of (4) and doing some simplifications such as squaring both sides and adding them up leads to:

$$z^n + p_1 z^{n-1} + p_2 z^{n-2} + \ldots + p_{n-1} z + p_n = 0 \quad (5)$$

where $z = \omega^2$.

Lemma 2.1. *If $p_n < 0$, then equation (5) has at least one positive root.*

Proof. Let $h(z) = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \ldots + p_{n-1} z + p_n$. Since $h(0) = p_n < 0$ and $\lim_{z\to+\infty} h(z) = +\infty$, it can be resulted that there exists at least one $z_0 > 0$ such that $h(z_0) = 0$. □

Now, we can state the following main theorem:

Theorem 2.2. *If $p_n < 0$, then at $\tau = \tau_0$, Hopf bifurcation occurs in (3) and a family of periodic solutions bifurcate from the origin.*

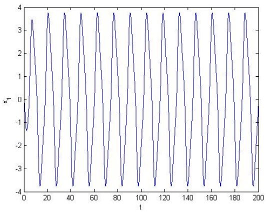
Proof. By using Lemma 2.1, we are sure that (5) has at least one positive root. Let $\omega_0 = \sqrt{z_0}$ where z_0 is the positive root of (5). Then, by substituting $sin\omega_0\tau = \pm\sqrt{1 - cos^2\omega_0\tau}$, we get an equation that all the coeficients are known except $cos\omega_0\tau$. Thus, τ_0 can be computed. Therefore, by using the Hopf bifurcation theory, the proof is complete. □

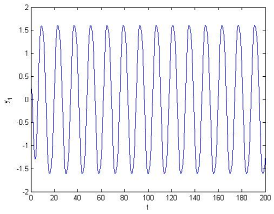
To illustrate our theoretical results, we consider the following example:

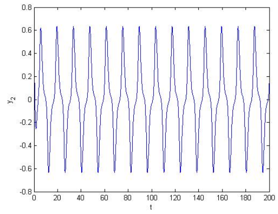
Example 2.3. Consider the following five-neuron BAM neural network model:

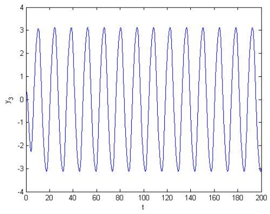
$$\begin{cases} \dot{x_1}(t) = -2x_1(t) + tanh(y_1(t-\tau_2)) - tanh(y_2(t-\tau_2)) \\ \qquad\quad + tanh(y_3(t-\tau_2)) + tanh(y_4(t-\tau_2)) \\ \dot{y_1}(t) = -2y_1(t) + tanh(x_1(t-\tau_1)) \\ \dot{y_2}(t) = -y_2(t) + tanh(x_1(t-\tau_1)) \\ \dot{y_3}(t) = -0.5y_3(t) + tanh(x_1(t-\tau_1)) \\ \dot{y_4}(t) = -y_4(t) + tanh(x_1(t-\tau_1)) \end{cases} \quad (6)$$

In fact, here, n=5 and case IV happens. When $\tau = \tau_1 + \tau_2$ passes through the critical value τ_0, Hopf bifurcation occurs and a family of periodic solutions bifurcates from the origin. See Figure 1, where periodic solutions are given with respect to the five neurons namely, x_1, y_1, y_2, y_3 and y_4.


(a)


(b)


(c)


(d)

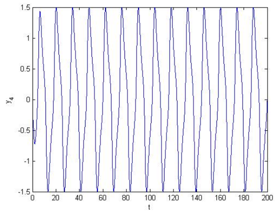

(e)

Figure 1: A family of periodic solutions bifurcate from the origin.

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Irreducible Smale spaces

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Abstract

Irreducible spaces play an important role in topological dynamical systems. There exists several equivalent definition for irreducible shift of finit type spaces which are the simplest Smale spaces. In this paper, we generalize them to Smale spaces and then get some results about the degree of factor maps on Smale spaces.

Keywords: Degree of factor maps, Irreducible spaces, Shift of finite type, Smale spaces
Mathematics Subject Classification [2010]: 37B10, 37D99

1 Introduction

1.1 Smale spaces

Definition 1.1. [1] A dynamical system is a pair (X, φ) where X is a topological space and φ is a homeomorphism of X.

Definition 1.2. [3] A dynamical system (X, φ) is said to be irreducible if, for every (ordered) pair of non-empty open sets U, V, there is a positive integer N such that $\varphi^N(U) \cap V$ is non-empty.

Definition 1.3. [3, 4] Suppose that (X, φ) is a compact metric space and φ is a homeomorphism of X. Then (X, φ) is called a Smale space if there exist constants ε_X and $0 < \lambda < 1$ and a continuous map from

$$\triangle_{\varepsilon_X} = \{(x, y) \in X \times X \quad | \quad d(x, y) \leq \varepsilon_X \quad \}$$

to X (denoted with $[,]$) such that:

$B\,1$ $[x, x] = x$,
$B\,2$ $[x, [y, z]] = [x, z]$,
$B\,3$ $[[x, y], z] = [x, z]$,
$B\,4$ $[\varphi(x), \varphi(y)] = [x, y]$,
$C\,1$ $d(\varphi(x), \varphi(y)) \leq \lambda\, d(x, y)$, whenever $[x, y] = y$,
$C\,2$ $d(\varphi^{-1}(x), \varphi^{-1}(y) \leq \lambda\, d(x, y)$, whenever $[x, y] = x$, whenever both sides of an equation are defined.

*Speaker

Examples of Smale spaces include solenoids, substitution tiling spaces, the basic sets for Smale's Axiom A systems and shifts of finite type.[3]

Definition 1.4. [3] Two points x and y in X are stably (or unstably) equivalent if

$$\lim_{n\to+\infty} d(\varphi^n(x), \varphi^n(y)) = 0 \qquad (or \lim_{n\to-\infty} d(\varphi^n(x), \varphi^n(y)) = 0, \text{resp.}).$$

Let $X^s(x)$ and $X^u(x)$ denote the stable and unstable equivalence classes of x, respectively.

We recall that a factor map between two Smale spaces (Y, ψ) and (X, φ) is a continuous function $\pi : Y \to X$ such that $\pi \circ \psi = \varphi \circ \pi$. Of particular importance in this paper are factor maps which are s-bijective: that is, for each y in Y, the restriction of π to $Y^s(y)$ is a bijection to $X^s(\pi(y))$. There is obviously an analogous definition of a u-bijective factor map, which will not be needed here.[3]

1.2 Shifts of finite type

[2, 3] A graph G consists of finite sets G^0 and G^1 and maps $i, t : G^1 \to G^0$. The elements of G^0 are called vertices and the elements of G^1 are called edges. The notation for the maps is meant to suggest initial and terminal and the graph is drawn by depicting each vertex as a dot and each edge e as an arrow from $i(e)$ to $t(e)$. To any graph G, we associate the following dynamical system:

$$\Sigma_G = \{ \ (e_n)_{n\in\mathbb{Z}} \mid e_n \in G^1, \ t(e_n) = i(e_{n+1}) \ for \ all \ n \in \mathbb{Z}\},$$

$$(\sigma(e))_n = e_{n+1}.$$

For any e in Σ_G and $K \le L$, we let $e_{[K,L]} = (e_K, e_{K+1}, \ldots, e_L)$. It is also convenient to define $e_{[K+1,K]} = t(e_K) = i(e_{K+1})$. We use the metric

$$d(e, f) = \inf\{1, 2^{-K-1} \mid K \ge 0, e_{[1-K,K]} = f_{[1-K,K]}\}$$

on Σ_G. It is then easy to see that (Σ_G, σ) is a Smale space with constants $\varepsilon_X = \lambda = \frac{1}{2}$ and

$$[e, f]_k = \begin{cases} f_k & k \le 0 \\ e_k & k \ge 1. \end{cases}$$

Definition 1.5. [2] A point x in a shift of finite type space X is doubly transitive if every block in X appeara in x infinitely often to the left and to the right.

Theorem 1.6. *[2] The set of doubly transitive points of a shift of finite type space X is nonempty if and only if X is irreducible.*

2 Main results

Theorem 2.1. *Let (X, φ) be a dynamical system with X compact, metric. If it is irreducible, then the set of all points x with dense forward orbit is a dense G_δ subset of X.*

Proof. Take a finite open cover of X. Look at the set of all points whose forward orbit meets each element of the cover. It is pretty easy to see this set is open. A short argument using irreducibilty implies that it is dense. Finally, intersect these sets over a sequence of finite open covers that generate the topology of X. □

Theorem 2.2. *Let (X, φ) be a Smale space. Suppose x is a point whose forward orbit limits on every periodic point of X. Then (X, φ) is irreducible.*

Proof. Let y be an accumulation point of the backward orbit of x. It is clearly non-wandering and so there are periodic points arbitrarily close. It follows that y is also a limit point of the forward orbit of x. By patching the forward orbit of x that gets close to y with part of the backward orbit of x that begins close to y we can form pseudo-orbits from x to itself. It follows then that x is in the non-wandering set and lies in one irreducible component. The orbit of x will remain in the same irreducible component of the non-wandering set and for this forward orbit to limit on every periodic point, X has only a single irreducible component. □

Theorem 2.3. *Let (Y, ψ) and (X, φ) be Smale spaces and let $\pi : (Y, \psi) \to (X, \varphi)$ be an s-bijective factor map. Assume that x, x' are in X and x has a dense forward orbit. (This implies that (X, φ) is irreducible.) Then we have $\sharp\pi^{-1}(x) \leq \sharp\pi^{-1}(x')$ which $\sharp$ denote the number of the finite set.*

Proof. List $\sharp\pi^{-1}(x) = \{y_1, ..., y_I\}$. Since the orbit of x is dense, we may find an increasing sequence of positive integers n_k such that $\varphi^{n_k}(x)$ converges to x. Passing to a subsequence, we may assume that for each $1 \leq i \leq I$, the sequence $\psi^{n_k}(yi)$ converges to some point of y and by continuity these points must all lie in $\pi^{-1}(x')$. We claim that no two sequences can have the same limit. This will complete the proof. If they do, then for some i, j we have $d(\psi^{n_k}(y_i), \psi^{n_k}(y_j))$ tends to zero as k goes to infinity. Notice that

$$\pi(\psi^{n_k}(y_i)) = \psi^{n_k}(\pi(y_i)) = \phi^{n_k}(x) = \psi^{n_k}(\pi(y_j)) = \pi(\psi^{n_k}(y_i))$$

By Prop. 2.5.2 of [3], for k suciently large, we have

$$\psi^{n_k}(y_i) \in Y^u(\psi^{n_k}(y_j), \varepsilon_\pi).$$

and this implies that $y_i \in Y^u(y_j, \lambda^{n_k}\varepsilon_\pi)$ Since this is true for all k, $y_i = y_j$ and we are done. □

Definition 2.4. If $\pi : (Y, \psi) \to (X, \varphi)$ is an s-bijective factor map and that (X, φ) is irreducible. We define the degree of π denoted $\deg(\pi)$ to be $\sharp\pi^{-1}\{x\}$, where x is any point of X with a dense forward orbit.

Lemma 2.5. *Let $\pi : Y \to X$ be a finite-to-one continuous function. The set $\{x \in X | \sharp\pi^{-1}\} = 1\}$ is a G_δ subset of X. (Of course, the set might be empty.)*

Proof. It follows from Lemma 2.5.9 of [3] that for any positive integer n,
$\{x \in X | diam(\pi^{-1}\{x\}) < \dfrac{1}{n}\}$ is open. Intersecting over all n yields the result. □

Remark 2.6. Notice that this result combines nicely with Theorem 2.1 : for a degree one factor map onto an irreducible Smale space, the points with a dense forward orbit and a one-point pre-image are a dense G_δ.

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Talk Isospectral matrix flows and numerical integrators on Lie groups pp.: 1–4

Isospectral Matrix Flows and Numerical Integrators on Lie Groups*

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Abstract

This paper illustrates how classical integration methods for differential equations on manifolds can be modified in order to preserve certain geometric properties of the exact flow. Runge-Kutta-Munthe-Kass method is considered and some examples are shown to verify the efficiency of the method.

Keywords: Isospectral matrix flow, Lie group, Geometric integration, Differential equation on manifold.
Mathematics Subject Classification [2010]: 58J53, 15A18, 15B35, 15A24

1 Introduction

Isospectral matrix flows on the space of real $n \times n$ matrices M_n are characterized by the matrix differential equation

$$\frac{dA}{dt} = [A, F(A)], \quad A(0) = A_0, \tag{1}$$

where $A \in M_n$, $F : [0, \infty) \times M_n \to M_n$ is a matrix operator, $[X, Y] = XY - YX$ is the matrix commutator (also known as the Lie bracket) and A_0 is a given $n \times n$ matrix. The function A and F that obey the differential equation (1) are usually called a Lax pair. Many interesting problems can be written in this form. We just mention the Toda system, the continuous realization of QR-type algorithms, projected gradient flows, and inverse eigenvalue problems, see Chu [2] and Calvo, Iserles and Zanna [1].

Lemma 1.1. *Consider a matrix differential equation (1). Then, all eigenvalues of $A(t)$, the solution of (1), are independent of t, so that the flow (1) is isospectral flow.*

Proof. To prove the isospectrality of the flow, we define $U(t)$ by

$$\frac{dU}{dt} = -F(A(t))U(t), \quad U(0) = I_n, \tag{2}$$

*Will be presented in English
†Speaker

where I_n is the identity matrix. Then, we have

$$\frac{d}{dt}(U(t)^{-1}A(t)U(t)) = U^{-1}(\dot{A} - AF + FA)U = 0,$$

and hence the matrix function $U(t)^{-1}A(t)U(t)$ is time independent. Hence $U(t)^{-1}A(t)U(t) = const = A_0$ and $A(t) = U(t)A_0U(t)^{-1}$. This proves the result. □

In many important examples, the matrix function F is skew-symmetric. Then the equation (2) is an orthogonal flow, since its solution is an orthogonal matrix, i.e. $UU^T = U^TU = I_n$, hence we have $A(t) = U(t)A_0U(t)^T$. It is easy to see that if $A_0 \in S_n$, then $A(t) \in S_n$. Then S_n is invariant under (1). Moreover, $A(t)$ has the same spectrum as A_0, so that the flow (1) is an isospectral matrix flow.

While isospectral flows are interesting from a theoretical point of view, sooner or later you'll probably want to solve one numerically. Standard numerical methods such as linear multistep method and Runge-Kutta (RK) do not preserve the eigenvalue of an isospectral flow in general. This was proven by Calvo, Iserles and Zanna in [1].

The proof of Lemma 1.1 suggests an interesting approach for the numerical solution of (1). For $n = 1, 2, ...,$ we solve numerically

$$\frac{dU}{dt} = -F(UA_nU^T)U(t), \quad U(0) = I_n, \tag{3}$$

and we put $A_{n+1} = \widehat{U}A_n\widehat{U}^T$, where $\widehat{U}$ is the numerical approximation $\widehat{U} \approx U(h)$ after one step. If $F(A)$ is skew-symmetric for all matrices A, then U^TU is a quadratic invariant of (3) and some methods such as Runge-Kutta with some conditions will produce an orthogonal $\widehat{U}$. Consequently, A_{n+1} and A_n have the same eigenvalues, and they remain symmetric.

In this paper, we use one class of method called Runge-Kutta-Munthe-Kaas method, which is guaranteed to preserve the eigenvalue of an isospectral flow for solving isospectral flows. This method is based on geometric interpretation.

2 Main results

Isospecrality is a geometric constraint on the flow. An isospectral flow evolves on a smooth subset of M_n, for each initial value A_0, which is called the isospectral manifold for A_0. This manifold is naturally parameterized by a Lie group, which has an associated algebra. All the manifolds we are interested in are manifold of matrices. This manifolds exist naturally as surfaces embedded in $\mathbb{R}^{n^2}$, then the following definition will suffice.

Definition 2.1. A d-dimensional manifold $\mathcal{M}$ is a d-dimensional smooth surface $\mathcal{M} \subseteq \mathbb{R}^n$ for some $n \geq d$.

Many manifolds of interest can be described as the zero set of a smooth function $g : \mathbb{R}^n \to \mathbb{R}^m$. For example, the group of orthogonal matrices $O(n)$ is the zero set of $g(X) =\parallel XX^T - I \parallel_F^2$.

Definition 2.2. Let $\mathcal{M}$ be a d-dimensional manifold. The tangent space at $X \in \mathcal{M}$, denoted by $T_X\mathcal{M}$, is vector space of vectors $V \in \mathbb{R}^n$ such that

$$V = \frac{d\mu(s)}{ds}\Big|_{s=0}$$

for some smooth path μ in $\mathcal{M}$ such that $\mu(0) = X$.

Definition 2.3. Consider a differential equation

$$\dot{y} = f(y), \quad y(0) = y_0. \tag{4}$$

We say that the differential equation is on the manifold $\mathcal{M}$, if $y_0 \in \mathcal{M}$ implies $y(t) \in \mathcal{M}$ for all t.

Theorem 2.4. *The problem (4) is a differential equation on the manifold $\mathcal{M}$ if and only if*

$$f(y) \in T_y\mathcal{M} \qquad for\ all\ y \in \mathcal{M}.$$

For more details, see [4].

Definition 2.5. A Lie group is a group $\mathcal{G}$ which is a manifold. Additionally, the group action must be a smooth map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$. A matrix Lie group is a Lie group whose elements are matrices, with matrix multiplication as the group operation.

Definition 2.6. Let $\mathcal{G}$ be a matrix Lie group and let $\mathfrak{g} = T_I\mathcal{G}$ (Lie algebra) be the tangent space at the identity. The Lie bracket $[A, B] = AB - BA$ defines an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear, skew-symmetric ($[A, B] = -[B, A]$), and satisfies the Jacobi identity

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

Definition 2.7. Let $\mathcal{G}$ be a Lie group and $\mathcal{M}$ be a manifold. Then a Lie group action Λ is a smooth map $\Lambda : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that the following two properties hold:

1. $\Lambda(I, X) = X$,
2. $\Lambda(P, \Lambda(Q, X)) = \Lambda(PQ, X)$.

Theorem 2.8. *For a Lie group action $\Lambda : \mathcal{G}\times\mathcal{M} \to \mathcal{M}$, C^1 function $A : [0,\infty)\times\mathcal{M} \to \mathcal{M}$, smooth map $\phi : \mathfrak{g} \to \mathcal{G}$ such that $\phi(O) = I$, and $X_0 \in \mathcal{M}$, the solution X of*

$$\dot{X}(t) = \lambda(A(t, X(t)))(X(t)), \quad X(0) = X_0,$$

which evolves in $\mathcal{M}$, can be expressed as

$$X(t) = \Lambda(\phi(\Omega(t)), X_0),$$

where $\Omega : [0, \infty) \to \mathfrak{g}$ satisfies

$$\dot{\Omega}(t) = d\phi_{\Omega}^{-1} A(t, \Lambda(\phi(\Omega(t)), X_0)), \quad \Omega = O. \tag{5}$$

Remark 2.9. Let $\mathcal{G} = GL(n) = \{Y | \det Y \neq 0\}$, $\mathcal{M} = \mathfrak{gl}(n) = \{A|arbitrary\ matrix\}$ and group action $\Lambda(P, X) = PXP^{-1}$. Then $\lambda(A)(X) = [A, X]$.

Remark 2.10. The primary example for a smooth function ϕ with property $\phi(O) = I$ is *exp*. Then we have

$$dexp_{\Omega}^{-1}(A) = \sum_{k\geq 0} \frac{B_k}{k!} ad_{\Omega}^k(A), \tag{6}$$

where B_k are the Bernoulli numbers $1, -1/2, 1/6, 0, ...$ and $ad_{\Omega}^0(A) = A$, $ad_{\Omega}(A) = [\Omega, A]$, $ad_{\Omega}^2(A) = [\Omega, [\Omega, A]]$ and so on. For more details, see [4].

2.1 Runge-Kutta-Munthe-Kaas method

The idea of the Munthe-Kaas method [5] consists of solving the differential equation (5) with an arbitrary RK scheme (truncating $dexp^{-1}$ to appropriate order), so that, once $\Omega_1 = \Omega(h)$ is known, one can approximate

$$X_{n+1} = \Lambda(exp(\Omega_1), X_n).$$

Lie algebra is a linear space that is closed under the Lie bracket, so any one step method involving only linear operation and Lie brackets is guaranteed to stay in the Lie algebra, which is then mapped back to the manifold. Therefore, this method is guaranteed to evolve on $\mathcal{M}$.

Example 2.11. Consider the Toda flow

$$\frac{dA}{dt} = [A, S] = AS - SA, \quad A(0) = A_0,$$

for $A \in S_n$, where $S = {A^+}^T - A^+$, and A^+ is the upper triangular part of A. It is well known that the Toda flow is an isospectral flow. Gladwell showed that, if A_0 be TP (all the minors are (strictly) positive), then $A(t)$, the solution of the Toda flow, has the same property [3]. Take initial matrix A_0 in 3×3 case as

$$A_0 = \begin{pmatrix} 5 & 4 & 1 \\ 4 & 6 & 4 \\ 1 & 4 & 5 \end{pmatrix}.$$

We can easily check that A_0 is TP. The eigenvalues of A_0 are 4, 11.6568 and 0.3431. We applied the Runge-Kutta-Munthe-Kaas method on the Toda flow with given A_0. Numerical results confirms the analytic properties.

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Talk Lie group classification of the Kuramoto-Sivashinsky equation pp.: 1–4

Lie group classification of the Kuramoto-Sivashinsky equation

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Abstract

In this paper, the Lie symmetry analysis is performed for Kuramoto-Sivashinsky equation(KS). The exact solutions and similarity reductions generated from the symmetry transformations are provided. Furthermore, the all exact explicit solutions and similarity reductions based on the Lie group method are obtained, some new method and techniques are employed simultaneously. Such exact explicit solutions and similarity reductions are important in both applications and the theory of nonlinear science.

Keywords: Similarity solutions, Lie symmetry, Kuramoto-Sivashinsky equation, Invariant solution, Optimal system.
Mathematics Subject Classification [2010]: 22E70, 81R05, 70G65, 34C14.

1 Introduction

Symmetry is one of the most important concepts in the area of partial differential equations, especially in integrable systems, which exist infinitely many symmetries. To find the Lie point symmetry of a nonlinear equation, some effective methods have been introduced, such as the nonclassical method and the direct method . In this paper we will consider the following variable coefficients KuramotoSivashinsky equation (KS) by using the compatibility method.
The Kuramoto-Sivashinsky (KS) equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0 \tag{1}$$

is a simple nonlinear PDE which exhibits complex spatio-temporal dynamics. It has been derived in the context of plasma ion mode instabilities by LaQuey et al. reaction-diffusion systems by Kuramoto and Tsuzuki, laminar flame fronts by Sivashinsky and viscous liquid flows on an inclined plane by Sivashinsky and Michelson.

2 Main results

In this section, we will perform Lie symmetry analysis for Eq.(1) firstly.The vector field associated with the group of transformations can be written as

$$V = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{2}$$

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The symmetry group of Eq.(1) will be generated by the vector field of the form Eq.(2). Applying the fourth prolongation $pr^{(4)}V$ to Eq.(1), we find that the coefficient functions $\xi_1(x,t,u)$, $\xi_2(x,t,u)$ and $\phi(x,t,u)$.We obtain the vector field of Eq.(1) is:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{3}$$

It is easy to check that the two vector fields V_1, V_2, V_3 are closed under the Lie bracket, respectively. For example, for Eq.(1), we have:

Table 1: Commutator of the Lie algebra of the Eq. (1)

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$-V_1$
V_3	0	V_1	0

Remark 2.1. V_1 is the casimir operator.

Theorem 2.2. *The Lie algebra of Eq.* (1)

1. *is solvable,*
2. *is nilpotent,*
3. *is not semi-simple.*

Then from the the commutation Table 1, we will obtain the following Table 2:

Table 2: Adjoint representation of the Lie algebra of the Eq. (1)

	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 + \varepsilon V_1$
V_3	V_1	$V_2 - \varepsilon V_1$	V_3

Theorem 2.3. *The optimal system of one-dimensional subalgebras corresponds to Eq.*(1)*is expressed by*

1. $\alpha V_2 + V_3$*, where* $\alpha \in \{-1, 0, 1\}$.
2. $\alpha V_1 + V_2$*,where* $\alpha \in \{-1, 0, 1\}$
3. V_1*,*

Using a straightforward analysis, the characteristic equations used to find similarity variables are:

$$\frac{dx}{\xi_1} = \frac{dt}{\xi_2} = \frac{du}{\phi} = d\varepsilon. \tag{4}$$

Integration of first order differential equations corresponding to pairs of equations involving only independent variables of (4) leads to similarity variables. We distinguish three cases:

1. For the linear combination $V = \alpha V_2 + V_3$, we have:

$$\zeta = \alpha x - \frac{t^2}{2}, \qquad S(\zeta) = \alpha u - t$$

By substituting above equations into the Eq. (1) we obtain:

$$1 + SS' + \alpha^2 S'' + \alpha^4 S^{(4)} = 0 \tag{5}$$

by numerical solution we obtain Fig.1.

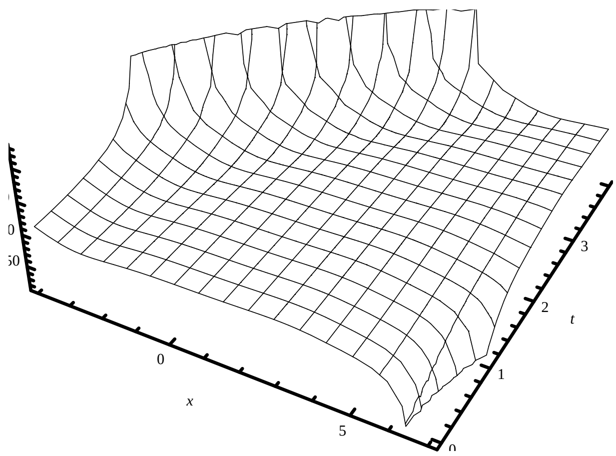

Figure 1: Solution curves of the nODE (5) generated by different initial values, $S(1) = 1, S'(1) = 1, S''(1) = 1, S^{(3)}(1) = 1, \alpha = 1$

2. For the linear combination $\alpha V_1 + V_2$, we have:

$$\zeta = \alpha t - x, \qquad S(\zeta) = u$$

By substituting above equations into the Eq. (1) we obtain:

$$\alpha S' - SS' + S'' + S^{(4)} = 0 \tag{6}$$

by numerical solution we obtain Fig.2.

3. For the generator $V = V_1$,the invariants are:

$$\zeta = t, \qquad S(\zeta) = u$$

We reduce Eq. (1) to the following ODE:

$$S' = 0 \tag{7}$$

therefore, $S(\zeta) = c$, where c is arbitrary constant.

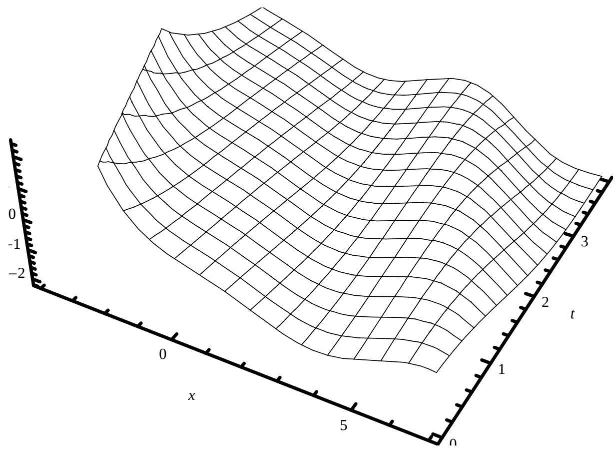

Figure 2: Solution curves of the nODE (6) generated by different initial values, $S(1) = 1, S'(1) = 1, S''(1) = 1, S^{(3)}(1) = 1, \alpha = -1$

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A mathematical model of hepatitis E virus transmission and its application for vaccination strategy in a displaced persons camp in Uganda

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Abstract

Hepatitis E virus is an enterically transmitted disease that mainly effects people in developing countries. The dynamics and the factors causing outbreaks of these diseases can be better understood using mathematical models, which are fit to data. Here we investigate the dynamics of a Hepatitis E outbreak in internally displaced persons (IDP) camps in Sudan and Uganda during 2007 to 2009. We use the data to determine that R_0 is approximately 2.25 for the outbreak. Secondly, we use a model to estimate that the critical level of latrine and bore hole coverages needed to eradicate the epidemic is at least 16% and 17% respectively. Lastly, we further investigate the relationship between the co-infection factor for Malaria and Hepatitis E on the value of R_0 for Hepatitis E. Taken together, these results provide us with a better understanding of the dynamics and possible causes of Hepatitis E outbreaks.

Keywords: Mechanistic models, Dynamic models, Reproduction number
Mathematics Subject Classification [2010]: 37N25, 92B05

1 Introduction

HEV is classified in the genus Hepevirus of the family Hepeviridae. Outbreaks of diseases such as Avian Influenza, SARS and West Nile Virus have alerted us to the potentially grave public health threat from emerging and re-emerging pathogens [2, 3]. The recent outbreak of Hepatitis E in northern Uganda, has left many dead and a number of infectives that continue to spread the infection . Hepatitis E is caused by infection with the Hepatitis E virus (HEV) which has a fecal-oral transmission route. The Kitgum outbreak, which we study here, has been linked to contaminated water or food supplies . Another possible factor that could be implicated in the outbreak of Hepatitis E is its possible relationship with Malaria. Malaria has been shown to disarm the immune system and increase susceptibility to viral infections such as HIV . Recently, in a 3-month follow-up study the pattern of co-infection of Plasmodium falciparum Malaria and acute Hepatitis

*Speaker

A (HAV), in 222 Kenyan children under the age of 5 years was observed [1]. In this paper, mathematical models are used to study the effects of both environmental conditions and Malaria on Hepatitis E infections. The models designed are fit to data from the Kitgum outbreak, to estimate the basic reproduction number and to relate them to the level of contamination of the environment.

2 Formulation of the Model Equations

2.1 Variables and Parameters of the Existing Model

Definition 2.1. The simplest compartmental disease transmission model that includes an environmental reservoir needs four compartments. Three of these compartments relate to the disease status in an individual with Susceptible (S), Infective (I) and Recovered (R) classes. After successful infection, the individual is now exposed to HEV and moves to the exposed class E.
In the human population, susceptibles, S, are recruited at a rate μ that equals to the per capita natural mortality rate for each group. This assumption is made to keep the population constant, while keeping a turnover of individuals in the population. We assume that a fraction b of the population has access to clean bore hole water and cannot become infected, β is the transmission rate of HEV, Individuals recover from the disease and move into the recovered class at a rate γ, Of the total infected individuals, a fraction of them die due to the infection, and recover to join the immune group, The incubation period takes a mean period of days.

2.2 The Equations of the Existing Model

Theorem 2.2. *The dynamics of the population is governed by the following system of ordinary differential equations:*

$$\begin{aligned}
\frac{dS}{dt} &= \mu - \beta\rho(1-N)(1-b)IS - \mu S,\\
\frac{dE}{dt} &= \beta\rho(1-N)(1-b)IS - (\mu+\sigma)E,\\
\frac{dI}{dt} &= \sigma E - (\mu+g)I,\\
\frac{dR}{dt} &= (1-P)\gamma I - \mu R,
\end{aligned} \tag{1}$$

where $S+E+I+R=N$.

3 Model Analysis

In this section we consider the existence of equilibrium states, the effective reproduction number and the stability of the equilibrium states.

3.1 Endemic Steady State (EEP)

The endemic stationary state is given by

$$\begin{aligned} S^* &= \frac{1}{R_0}, \\ E^* &= \frac{\mu(\mu+\gamma)}{\sigma\beta\rho(1-N)(1-b)}(R_0-1), \\ I^* &= \frac{\mu}{\beta\rho(1-N)(1-b)}(R_0-1), \\ R^* &= N - S^* - E^* - I^*. \end{aligned} \tag{2}$$

where

$$R_0 = \frac{\sigma\beta\rho(1-N)(1-b)}{(\mu+\sigma)(\mu+\gamma)}. \tag{3}$$

is the basic reproduction number for HEV. The term $\frac{\sigma}{\mu+\sigma}$ is the proportion of the exposed humans that survive the incubation period. The other fraction, $\frac{\beta\rho(1-N)(1-b)}{(\mu+\gamma)}$ is transmission rate of HEV during the infectious period of the human.

Theorem 3.1. *If $R_0 < 1$, then The disease-free equilibrium point is stable and When $R_0 > 1$ the endemic equilibrium point in equation (1) exists and is stable.*

3.2 The Co-infection Model

In addition to Hepatitis E, individuals in the Kitgum region were at a risk of acquiring Malaria which is endemic to Uganda. To model possible co-infection we adopt the model to include a susceptible group which comprises both those with and without Malaria. That is, the total susceptible population $S' = S + M$ where M is the proportion of individuals infected with Malaria.

Proposition 3.2. *The Malaria dynamics will not be modelled in detail here but an assumption is made that Malaria continuously invades the population, and individuals move back and forth between infection and recovery from the disease. This implies that*

$$\begin{aligned} \frac{dS}{dt} &= -fS + rM, \\ \frac{dM}{dt} &= -rM + fS, \end{aligned} \tag{4}$$

The equilibrium state for this model is given by

$$\begin{aligned} S^* &= \frac{r}{f+r}, \\ M^* & \frac{f}{f+r}, \end{aligned} \tag{5}$$

Our concern here, however, is how background levels of Malaria effect transmission dynamics of HEV.

Lemma 3.3. *Using the next generation method [4], the basic reproduction number for Hepatitis E in presence of Malaria is given by*

$$R_c = \frac{\beta\rho\sigma(1-N)(1-b)[S+\zeta M]}{(\mu+\sigma)(\mu+\gamma)} = [S(t)+\zeta M(t)]R_0, \tag{6}$$

where R_0 is as defined in equation (2) When $R_c < 1$ infected individuals will have more chances of recovery than of transmitting the disease further hence the epidemic will die out. When $R_c > 1$, there exists an endemic equilibrium point as shown in Supporting Information S2 given by

$$I^* = \frac{\zeta(\mu+f)+(\mu+r)}{2\zeta\beta\rho(1-N)(1-b)}$$
$$\Big[-1 \pm \frac{2\sqrt{\zeta m(\mu+Nf+Nr)}}{\zeta(\mu+f)+(\mu+r)}\sqrt{\frac{[\zeta(\mu+Nf)+(\mu+Nr)]^2}{4\zeta\mu(\mu+Nf+Nr)}-1}\Big], \tag{7}$$

If $\frac{[\zeta(\mu+Nf)+(\mu+Nr)]^2}{4\zeta\mu(\mu+Nf+Nr)} < 1$, then the roots of the quadratic equation in I^* are complex conjugates and of the form $a+bi$, where $i=\sqrt{-1}$.

4 Main results

This paper provides a case study of how a simple epidemic model can be fit to such an outbreak disease. Two fitting methods have been used; the first, an analytical method and the other based on a freely available fitting tool. Using these methods, a reliable estimate of $R_0 \approx 2.2$ has been provided.

We then use the model to find the measures to keep $R_0 < 1$. The necessary levels of latrine and bore hole coverages needed to eradicate the epidemic are both around 16 to 18%. Although the cost of construction of the required number of latrines is a one off cost, the benefits are large.

We have also considered co-infection with Malaria. If we assume that presence of Malaria during a Hepatitis E outbreak increases persistence infection, then we estimate that a Malaria infective can be infected with Hepatitis E up to 16 times more than one without Malaria.

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Nehari Manifold approach to p- Laplacian eigenvalue problem with variable exponent terms

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Abstract

The multiplicity of positive solutions for problem

$$(\mathbf{P}) \begin{cases} -\Delta_p u = \lambda a(x)|u|^{q(x)-2}u + b(x)|u|^{r(x)-2}u; & in \quad \Omega \\ u \equiv 0; & on \quad \partial\Omega. \end{cases}$$

is discussed. This investigation is based on Nehari manifold technique and variational argument.

Keywords: Nehari Manifold, fibering map, variable exponent Lebesgue space, variable exponent Sobolev space.
Mathematics Subject Classification [2010]: 35J20, 35R01

1 Introduction

The classes of problems dealing with variable exponent Lebesgue and Sobolev space have attracted steadily increased interest over the last ten years, although their history goes back to W. Orlicz (see for example [5]). We mention briefly, some of the basic definition and refer to [2, 3, 4, 5, 6] for the fundamental properties of these spaces. The basic definition of variable exponent Lebesgue space is mentioned in the following. Let Ω be an open subset of $\mathbb{R}^N$, $q \in L^\infty(\Omega)$ and

$$q^- := ess \inf_{x\in\Omega} p(x) \geq 1.$$

The variable exponent Lebesgue space $\mathbf{L}^{q(.)}(\Omega)$ is defined by

$$\mathbf{L}^{q(.)}(\Omega) = \{u : \ u : \Omega \longrightarrow \mathbb{R} \ \ is\ measurable, \int_\Omega |u|^{q(x)} dx < \infty\};$$

which is a considered by the norm

$$|u|_{\mathbf{L}^{q(.)}(\Omega)} = \inf\{\sigma > 0 : \ \int_\Omega |\frac{u}{\sigma}|^{q(x)} dx \leq 1\}.$$

We have consider problem $(\mathbf{P})$ with the following conditions:

*Speaker

(I) Ω is a bounded subset of $\mathbb{R}^N$ with sufficiently smooth boundary and $N \geq 3$.

(II) $1 < p < N$ and q, r are Lipschitz continuous functions which belongs to $L^\infty(\Omega)$ with $1 < q^- \leq q^+ < p < r^- \leq r^+ < p^* := \frac{NP}{N-p}$ in which $q^+ := ess\sup_{x\in\Omega} q(x)$.

(III) $0 < a, b \in \mathbf{L}^\infty(\Omega)$

The appropriate Sobolev space to study the problem (P) is the space $\mathbf{W}_0^{1,p}(\Omega)$, defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = |\nabla u|_{L^p}$.

The Euler functional associated with problem (**P**) is

$$E_\lambda(u) = \int_\Omega \frac{1}{p}|\nabla u|^p dx - \lambda \int_\Omega \frac{a(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega \frac{b(x)}{r(x)}|u|^{r(x)}dx.$$

It is well known that the weak solutions of **P** corresponds to critical points of E_λ on $X = \mathbf{W}^{1,p}(\Omega)$.
In many problems, such as **P**, E_λ is not bounded below on X, but it is bounded below on an appropriate subset of X and there is a minimizer on this set (if it exists), and is usually a critical point of E_λ, thus the weak solution of the corresponding elliptic equation.
A good coordinate for an appropriate subset of X is called Nehari Manifold, which is introduced by

$$M(\lambda) = \{u \in X \setminus \{0\}; < E'_\lambda(u), u >= 0\}.$$

The Nehari Manifold is closely linked to the behavior of the functions of the form $\phi_{\lambda,u} : t \longrightarrow E_\lambda(tu); (t > 0)$. It is easy to see that for $t > 0$, $tu \in M(\lambda)$ if and only if $\phi'_{\lambda,u}(t) = 0$. It is natural to divide $M(\lambda)$ in to tree subset $M^+(\lambda)$, $M^-(\lambda)$ and $M^0(\lambda)$ corresponding to local minima, local maxima and points of inflection of Fibering maps. Hence, we define $M^+(\lambda)$, $M^-(\lambda)$ and $M^0(\lambda)$ with $u \in M(\lambda)$ where $\phi''_{\lambda,u}(1) > (<, =)0$ respectively. Also It can be shown that

Lemma 1.1. *Suppose that u_0 is a local minimizer of E_λ on $M(\lambda)$ and $u_0 \notin M^0(\lambda)$ then u_0 is a critical point of E_λ.*

Here we refer to [1] for application of intuitive insight about fibering map approach which is used by Kenneth Brown and Tsung-Fang Wu.

2 Main results

We shall now describe the nature of the Fibering maps for all possible situations. Let $A_u := \int_\Omega a(x)|u|^{q(x)}dx$, $B_u := \int_\Omega b(x)|u|^{r(x)}dx$, $\mu_{\lambda,u}(t) = t^p\|u\|^p - \lambda t^{q^+}A_u - t^{r^+}B_u$ and $\nu_{\lambda,u}(t) = t^p\|u\|^p - \lambda t^{q^-}A_u - t^{r^-}B_u$. Hence,

$$\mu_{\lambda,u}(t)\chi_{[1,+\infty)}(t) + \nu_{\lambda,u}(t)\chi_{(0,1)}(t) \leq \phi'_{\lambda,u}(t) \leq \nu_{\lambda,u}(t)\chi_{[1,+\infty)}(t) + \mu_{\lambda,u}(t)\chi_{(0,1)}(t). \quad (1)$$

For sufficiently small λ the graph of $\mu_{\lambda,u}$ and $\nu_{\lambda,u}$ can be described as it shown in the Figure 2-4.

By inequalities (1), we obtain the graph of $\phi'_{\lambda,u}(t)$ is between two graphs $\mu_{\lambda,u}(t)$ and $\nu_{\lambda,u}(t)$. Hence for λ sufficiently small the graphs of $\mu_{\lambda,u}$ and $\nu_{\lambda,u}$ for $u \in M^+(\lambda)$ and

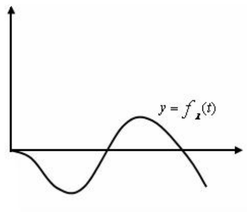

Figure 2-4

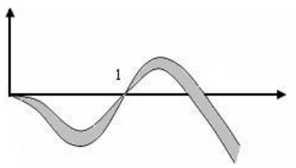

Figure 2-5

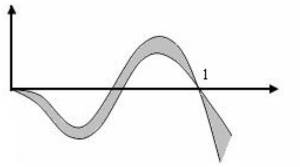

Figure 2-6

$u \in M^-(\lambda)$ are shown in Figure 2-5 and Figure 2-6, respectively, and so $\phi'_{\lambda,u}$ would be placed in the gray space between them.

It follows that $\phi_{\lambda,u}$ has at least two critical points; a local minimum at $t_1 = t_1(u)$ and a local maximum at $t_2 = t_2(u)$ which for $u \in M^+(\lambda)$, $t_1 = 1 < t_2$ and $t_2 u \in M^-(\lambda)$ and for $u \in M^-(\lambda)$, $t_1 < t_2 = 1$ and $t_1 u \in M^+(\lambda)$.

Moreover, $\phi_{\lambda,u}$ is decreasing in $(0, t_1)$, increasing in (t_1, t_2) and deceasing in $(t_2, +\infty)$. It follows from the last argument that there exist $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$ we have when $\phi'_{\lambda,u}(t) = 0$ i.e $tu \in M(\lambda)$, then $tu \notin M^0(\lambda)$ and so we have the following lemma.

Lemma 2.1. *There exist $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$, we have $M^0(\lambda) = \emptyset$. Moreover λ_1 is positive and independent of u.*

Theorem 2.2. *If $\lambda < \lambda_1$, there exist a minimizer of E_λ on $M^+(\lambda)$.*

Proof. Since E_λ is bounded below on $M(\lambda)$ and so on $M^+(\lambda)$, there exists a minimizing sequence $\{u_n\} \subseteq M^+(\lambda)$ such that $\lim_{n\to\infty} E_\lambda(u_n) = \inf_{u \in M^+(\lambda)} E_\lambda(u)$. Since E_λ is coercive, $\{u_n\}$ is bounded in X. Thus, we may assume that, without loos of generality $u_n \rightharpoonup u_0$ in X and by the compact embedding, we have $u_n \to u_0$ in $L^{p(x)(\Omega)}$ and in $L^{r(x)}(\Omega)$. Now, we shall prove $u_n \to u_0$ in X. Otherwise, suppose $u_n \not\to u_0$ in X, then

$$\int_\Omega \nabla u_0|^p dx < \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p dx. \tag{2}$$

$$\phi'_{\lambda,u_n}(t) = \int_\Omega t^{p-1} |\nabla u_n|^p dx - \lambda \int_\Omega a(x) t^{q(x)-1} |u_n|^{q(x)} dx - \int_\Omega b(x) t^{r(x)-1} |u_n|^{r(x)} dx.$$

By arguments of the previous section, we know there exists $t_0 = t_0(u_0)$ such that $t_0 u_0 \in M^+(\lambda)$, and hence, $\phi'_{\lambda,u_0}(t_0) = 0$ and by (2), we deduce,

$$\lim_{n\to\infty} \phi'_{\lambda,u_n}(t_0) = t_0^{p-1} \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^p - |\nabla u_0|^p) dx > 0.$$

Hence, $\phi'_{\lambda,u_n}(t_0) > 0$, for sufficiently large n. Since $\{u_n\} \subseteq M^+(\lambda)$, by taking notice to the possible maps for $\phi'_{\lambda,u}$ when $u \in M^+(\lambda)$, as is shown in Figure 2-5, it is easy to see that $\phi'_{\lambda,u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{\lambda,u_n}(1) = 0$; for all n; so, we must have $t_0 > 1$. But by considering the possible form of the Fibering maps, we deduce,

$$\phi_{\lambda,t_0u_0}(1) < \phi_{\lambda,t_0u_0}(t); \quad t < 1.$$

Let $t = \frac{1}{t_0}$, hence $E_\lambda(t_0u_0) = \phi_{\lambda,t_0u_0}(1) < \phi_{\lambda,t_0u_0}(\frac{1}{t_0}) = E_\lambda(u_0)$. So $E_\lambda(t_0u_0) < E_\lambda(u_0) < \lim_{n\to\infty} E_\lambda(u_n) = \inf_{u\in M^+(\lambda)} E_\lambda(u)$, which is contradicted by $t_0u_0 \in M^+(\lambda)$. Hence, $u_n \longrightarrow u_0$ in X and

$$E_\lambda(u_0) = \lim_{n\to\infty} E_\lambda(u_n) = \inf_{u\in M^+(\lambda)} E_\lambda(u).$$

Since $u_n \longrightarrow u_0$ in X, $u_n \subset M^+(\lambda)$ and $X \hookrightarrow\hookrightarrow L^{q(x)}, L^{r(x)}$ hence

$$\int_\Omega |\nabla u_0|^p dx - \lambda \int_\Omega a(x)|u_0|^{q(x)}dx = \int_\Omega b(x)|u_0|^{r(x)}dx.$$

and since $M^0(\lambda) = \emptyset$ we obtain

$$\int_\Omega p|\nabla u_o|^p dx > \lambda \int_\Omega a(x)q(x)|u_0|^{q(x)}dx - \int_\Omega b(x)r(x)|u_0|^{r(x)}dx.$$

Thus $u_0 \neq 0$.

□

By the same arguments the following theorem can be proved, in which we omit its proof.

Theorem 2.3. *If $\lambda < \lambda_1$, there exists a minimizer of E_λ on $M^-(\lambda)$.*

Corollary 2.4. *Equation (P) has at least two positive solutions for $0 < \lambda < \lambda_1$.*

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Talk Positive solutions of nonlinear fractional differential inclusions pp.: 1–4

Positive solutions of nonlinear fractional differential inclusions

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Abstract

In this paper, we study fractional differential inclusions with integral boundary value conditions. We prove the existence of a solution under both convexity and nonconvexity conditions on the multi-valued right-hand side. The proofs rely on Bohnenblust-Karlin's fixed point theorem, and Covitz and Nadlers fixed point theorem for multivalued contractions.

Keywords: Fractional differential inclusions; Fractional derivative; Fractional integral; Fixed point
Mathematics Subject Classification [2010]: 34A60, 34B18, 34B15

1 Introduction

The purpose of this paper is to study a fractional differential inclusions with multi-point boundary conditions given by

$$\begin{cases} {}^cD_{0+}^{\alpha} u(t) \in F(t, u(t)), & t \in (0,1),\ 2 < \alpha < 3, \\ u(0) = u''(0), \quad u(1) = \lambda \int_0^1 u(s)ds \end{cases} \tag{1}$$

where ${}^cD_{0+}^{q}$ is the Caputo's fractional derivative, $2 < \alpha < 3$, and $0 < \lambda < 2$, $F : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map, $P(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

We establish existence results for the problem (1), when the right-hand side is convex as well as non-convex valued. The first result relies on Bohnenblust-Karlin's fixed point theorem. In the second result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

In this section we sum up some basic facts that we are going to use later.

For a normed space $(X, \|\cdot\|)$, let

$P(X) = \{Y \subset X : Y \neq \emptyset\}$
$P_{cp}(X) = \{Y \in P(X) : Y \textit{ is compact}\}$
$P_c(X) = \{Y \in P(X) : Y \textit{ is convex}\}$
$P_{cl}(X) = \{Y \in P(X) : Y \textit{ is closed}\}$
$P_b(X) = \{Y \in P(X) : Y \textit{ is bounded}\}$
$P_{cp,c}(X) = \{Y \in P(X) : Y \textit{ is compact and convex}\}$

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A multi-valued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$.

Let $C(J)$ denote the Banach space of all continuous mapping $u : J \longrightarrow \mathbb{R}$ with norm

$$\|u\| = sup\{|u(t)| : t \in J\}$$

.

Let $L^1(J, \mathbb{R})$ be the Banach space of measurable functions $x : J \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_0^1 |x(t)|dt.$$

Let (X, d) be a metric space induced from the normed space $(X; \|.\|)$. Consider $H_d : P(X) \times P(X) \to \mathbb{R} \cup \{+\infty\}$ given by

$$H_d(A, B) = max\{sup_{a\in A} d(a, B), sup_{b\in B} d(A, b)\},$$

where $d(A, b) = inf_{a\in A} d(a; b)$ and $d(a, B) = inf_{b\in B} d(a; b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition 1.1. A multivalued operator $N : X \to P_{cl}(X)$ is called:
$(a)\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y) \; for \; each \; x, y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Definition 1.2. ([5])The Riemann-Liouville fractional integral of order q is defined as

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \quad \alpha > 0$$

provided the integral exists.

Definition 1.3. ([5]).For at least n-times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds, \quad n-1 < \alpha < n, n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 1.4. *([Bohnenblust-Karlin])[2].Let X be a Banach space, D a nonempty subset of X, which is bounded, closed, and convex. Suppose $F : D \to P(D)$ is u.s.c. with closed, convex values, and such that $F(D) \subset D$ and $\overline{F(D)}$ compact. Then F has a fixed point.*

Lemma 1.5. *([Covitz and Nadler])[4]. Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \infty$.*

2 Main results

2.1 Convex case

Let us introduce the following hypotheses:

(H1) $F : J \times \mathbb{R} \to P_{b,cl,c}(\mathbb{R})$ is measurable with respect to t for each $y \in \mathbb{R}$, u.s.c. with respect to y for a.e. $t \in J$, and for each fixed $y \in \mathbb{R}$ the set

$$S_{F,y} = \Big\{ f(t) \in L^1(J, \mathbb{R}) : f(t) \in F(t, y) \; for \; a.e. \; t \in J \Big\}$$

is nonempty.

(H2) For each $r > 0$, there exists a function $m_r \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\| = \sup\{|v| : v(t) \in F(t, y)\} \leq m_r(t)$$

for each $(t, y) \in J \times \mathbb{R}$ with $|y| \leq r$, and

$$lim \inf_{r\to\infty} \frac{\int_0^1 m_r(t)dt}{r} = \gamma < \infty.$$

Theorem 2.1. *Suppose that (H1) and (H2) are satisfied. Then the problem (1) has at least one solution on J, provided that*

$$\gamma < \frac{\Gamma(\alpha)(2 - \lambda)}{2}. \tag{2}$$

proof.We transform the problem (1) into a fixed point problem. Consider the multi-valued map $N : C(J) \to P(C(J))$ defined by

$$N(y) := \{h \in C(J) : h(t) = \int_0^1 G(t, s)f(s) : f \in S_{F,y}\}$$

Next we shall show that N satisfies all the assumptions of Lemma 1.4, and thus N has a fixed point which is a solution of the problem (1). For the sake of convenience, we subdivide the proof into several steps.

Step 1. $N(y)$ is convex for each $y \in C(J)$.

Step 2. For each constant$r > 0$, let $B_r = \{y \in C(J) : \|y\| \leq r\}$. Then B_r is a bounded closed convex set in $C(J)$.

Step 3. $N(B_r)$ is equi-continuous.

Step 4. N has closed graph.

Therefore, N is a compact multi-valued map, u.s.c. with convex closed values. As a consequence of Lemma 1.4, we deduce that N has a fixed point y which is a solution of the problem (1). □

2.2 The nonconvex case

Now we prove the existence of solutions for the problem (1) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [4].

Theorem 2.2. *Assume that the following conditions hold:*
(H3)$F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(., x)$ *is measurable for each* $x \in \mathbb{R}$.
(H4) $H_d(F(t, x(t)) - F(t, y(t)) \leq m(t)\|x - y\|$ *for almost all* $t \in J$*, and* $x, y \in \mathbb{R}$ *whit* $m \in L^1(J, \mathbb{R}^+), d(0, F(t, 0)) \leq m(t)$ *for almost all* $t \in J$.
Then the boundary value problem (1) has at least one solution on J *if*

$$\frac{2}{(2-\lambda)\Gamma(\alpha)}\|m\|_{L^1} < 1$$

proof. We show that the operator $N(y)$, defined in the beginning of proof of Theorem 2.1, satisfies the assumptions of Lemma 1.5. Firstly we show that $N(y) \in P_{cl}(C(J))$, Finally, we show that $N(y)$ is a contraction on $C^1(J, \mathbb{R})$.

Since $N(y)$ is a contraction, it follows by Lemma 1.5 that $N(y)$ has a fixed point y which is a solution of (1). This completes the proof.□

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Product Integration Method for numerical solution of a heat conduction problem

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Abstract

In this paper we reduce a heat conduction problem to a weakly singular Volterra integral equation of the second kind. The integral equation is solved by the product integration technique, which is explained in Section3. Numerical implementation of the method is illustrated by benchmark problem originated from heat conduction.

Keywords: Heat equation, Weakly singular Volterra integral equation, Product integration method
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In this work we consider the following heat conduction problem in one spatial dimension

$$u_t = u_{xx}, \qquad 0 < x < \infty, \quad 0 < t, \tag{1}$$

$$u(x,0) = f(x), \quad 0 < x < \infty, \tag{2}$$

$$u_t(0,t) + \alpha(t)u_x(0,t) + \beta(t)u(0,t) = g(t), \qquad 0 < t, \tag{3}$$

and

$$|u(x,t)| \le C_1 \exp\left\{C_2 x^2\right\}. \tag{4}$$

Here $u(x,t)$ is the temperature and is unknown, $C_i, i = 1,2$, are positive constants, and the known functions f, α, β, g, are explained in theorem 2.4.

2 Equivalent Integral Equation

We give some definitions, lemmas and theorems associated with this section

Definition 2.1. The fundamental solution of heat equation is denoted by $K(x,t)$, the Neumann's function is denoted by $N(x,\xi,t)$ and the Green's function is denoted by $G(x,\xi,t)$,

$$K(x,t) := \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}, N(x,\xi,t) := K(x-\xi,t)+K(x+\xi,t), G(x,\xi,t) := K(x-\xi,t)-K(x+\xi,t).$$

*Speaker

Lemma 2.2. *For any integrable function f that satisfies $|f(x)| \leq C_1 \exp\{C_2 x^2\}$, where C_1 and C_2 are positive constants, $\lim_{t\downarrow 0} \int_{-\infty}^{\infty} K(x-\xi,t) f(\xi) d\xi = f(x), \quad 0 < t$, at the point x of continuity of f.*

Proof. See Lemma 3.4.3 of [8]. □

Lemma 2.3. *At a point of continuity of g, $\lim_{x\downarrow 0} -2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau) g(\tau) d\tau = g(t)$.*

Proof. See Lemma 4.2.1 of [8]. □

Theorem 2.4. *The problem of determining the unique bounded solution u that satisfies (1)- (4), where $C_i, i = 1,2$, are positive constants, f, is twice continuously differentiable, and α, β, g are continuous, is equivalent to the problem of determining the unique continuous solution ϕ to the integral equation,$\phi(t) + \alpha(t) \int_0^\infty N(0,\xi,t) f'(\xi) d\xi - 2\alpha(t) \int_0^t K(0, t-\tau)\phi(\tau) d\tau + \beta(t) f(0) + \beta(t) \int_0^t \phi(\tau) d\tau = g(t), \quad 0 < t$. And the solution u has the representation, $u(x,t) = -2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau) \left(\int_0^\tau \phi(s) ds + f(0)\right) d\tau + \int_0^\infty G(x,\xi,t) f(\xi) d\xi$.*

Proof. We are going to search $u(x,t) = u_1(x,t) + u_2(x,t)$, such that u_1, u_2 satisfy heat equation and each of them establish one of the equations (2), (3). For this aim define $\phi(\tau) = u_t(0,\tau)$, and hence $u(0,\tau) = \int_0^\tau \phi(s) ds + f(0)$. For $0 < x < \infty$, $0 < t$, let $u_1(x,t) = -2\int_0^t \frac{\partial K}{\partial x}(x, t-\tau)\left(\int_0^\tau \phi(s) ds + f(0)\right) d\tau$, $u_2(x,t) = \int_0^\infty G(x,\xi,t) f(\xi) d\xi$. From [8], chapter one, both of u_1 and u_2 are solutions of equation (1). Lemma 2.2 leads $u(x,0) = u_2(x,0) = \lim_{t\downarrow 0} \int_0^\infty G(x,\xi,t) f(\xi) d\xi = \lim_{t\downarrow 0} \int_{-\infty}^\infty K(x-\xi,t) f_o(\xi) d\xi = f(x)$, where f_o is the odd extension of f to $-\infty < x < \infty$. Equation (3) equivalent with $\phi(t) + \alpha(t) u_x(0,t) + \beta(t) \int_0^t \phi(\tau) d\tau + \beta(t) f(0) = g(t)$. Now we evaluate $u_x(0,t)$, for this purpose we have

$$
\begin{aligned}
u_x(x,t) &= -2\int_0^t \frac{\partial^2 K}{\partial x^2}(x,t-\tau)\left(\int_0^\tau \phi(s)ds + f(0)\right)d\tau + \int_0^\infty G_x(x,\xi,t) f(\xi) d\xi \\
&= -2\int_0^t \frac{\partial K}{\partial t}(x,t-\tau)\left(\int_0^\tau \phi(s)ds + f(0)\right)d\tau + \int_0^\infty \left[\frac{\partial K}{\partial x}(x-\xi,t) - \frac{\partial K}{\partial x}(x+\xi,t)\right] f(\xi) d\xi \\
&= -2\int_0^t -\frac{\partial K}{\partial \tau}(x,t-\tau)\left(\int_0^\tau \phi(s)ds + f(0)\right)d\tau + \int_0^\infty \left[-\frac{\partial K}{\partial \xi}(x-\xi,t) - \frac{\partial K}{\partial \xi}(x+\xi,t)\right] f(\xi) d\xi \\
&= 2\left[K(x,t-\tau)\left(\int_0^\tau \phi(s)ds + f(0)\right)\Bigg|_{\tau=0}^{\tau=t} - \int_0^t K(x,t-\tau)\phi(\tau)d\tau\right] - \int_0^\infty \frac{\partial N}{\partial \xi}(x,\xi,t) f(\xi) d\xi \\
&= -2K(x,t)f(0) - 2\int_0^t K(x,t-\tau)\phi(\tau)d\tau - \left[N(x,\xi,t)f(\xi)\Bigg|_{\xi=0}^{\xi=\infty} - \int_0^\infty N(x,\xi,t) f'(\xi) d\xi\right] \\
&= -2K(x,t)f(0) + N(x,0,t)f(0) + \int_0^\infty N(x,\xi,t) f'(\xi) d\xi - 2\int_0^t K(x,t-\tau)\phi(\tau)d\tau \\
&= \int_0^\infty N(x,\xi,t) f'(\xi) d\xi - 2\int_0^t K(x,t-\tau)\phi(\tau)d\tau, \qquad (5)
\end{aligned}
$$

where the following implementations are used

1. K is a solution of heat equation (1), and this is used in row 2,

2. the chain rule is applied in row 3,
3. integration by parts is used in row 4,
4. $\lim_{\xi\to+\infty} N(x,\xi,t) = \lim_{\xi\to+\infty} \frac{1}{\sqrt{4\pi t}} \exp(-\xi^2) = 0$ is used in row 5,
5. $N(x,0,t)f(0) = 2K(x,t)f(0)$, is used in row 6.

For $x = 0$, $u_x(0,t) = \int_0^\infty N(0,\xi,t)f'(\xi)d\xi - 2\int_0^t K(0,t-\tau)\phi(\tau)d\tau$. By substitution $u_x(0,t)$ we obtain $\phi(t)+\alpha(t)\int_0^\infty N(0,\xi,t)f'(\xi)d\xi - 2\alpha(t)\int_0^t K(0,t-\tau)\phi(\tau)d\tau + \beta(t)f(0) + \beta(t)\int_0^t \phi(\tau)d\tau = g(t)$. By consideration of chapter3 of [8] the solution u in the class (4) is unique, and hence the proof is completed. □

3 Product integration technique

For development of the method we consider the following integral equation

$$\phi(t) = g(t) + \int_0^t p(t,\tau)k(t,\tau,\phi(\tau))d\tau, \quad t \in [0,b]. \tag{6}$$

Here p is weakly singular and k is smooth. Suppose $0 \le t_0 < t_1 < ... < t_N \le b$ be the $N+1$ nodal points in $[0,b]$. We are going to evaluate $\phi(t)$ at the nodal points, and for this purpose let the numerical approximation to $\phi(t_n)$ is written as ϕ_n. Algorithm of the product integration method is as follow

step1 Put $t = t_n$ in (6); i.e., $\phi(t_n) = g(t_n) + \int_0^{t_n} p(t_n,\tau)k(t_n,\tau,\phi(\tau))d\tau$.

step2 substitute $L_N(k,t_n;\tau) = \sum_{j=0}^N l_{N,j}(\tau)k(t_n,t_j,\phi(t_j))$ instead of $k(t_n,\tau,\phi(\tau))$ in step1 and get $\phi_n = g(t_n)+\sum_{j=0}^N \omega_j(t_n)k(t_n,t_j,\phi_j)$, where $\omega_j(t) = \int_0^t p(t,\tau)l_{N,j}(\tau)d\tau$.

step3 compute ϕ_j from Step 2. and obtain $\phi_N(t) = g(t) + \sum_{j=0}^N \omega_j(t)k(t,t_j,\phi_j)$, as a Nystrom approximation for $\phi(t)$.

For more details about the method and its convergence analysis, see [7, 6, 1, 2]. Another convergence analysis is obtained in [3], for the linear Volterra integral systems. Some applications of the method are obtained in [4, 5].

4 Numerical results

In the problem (1)-(4), for $f(x) = \cos x, \alpha(t) = 1, \beta(t) = 0, g(t) = -\exp\{-t\}$, the exact solution is $u(x,t) = \exp\{-t\}\cos x$. The integral equation associated with this problem is, $\phi(t) - \frac{2}{\sqrt{\pi}}DawsonF(\sqrt{t}) - \frac{1}{\sqrt{\pi}}\int_0^t \frac{\phi(\tau)}{\sqrt{t-\tau}}d\tau = -e^{-t}$, which has the exact solution $\phi(t) = -e^{-t}$.
In Table 1, column2 shows absolute errors of $\widetilde{\phi}$ at $t = 0.02i, i = 1,2,3,4,5$ with $b = 0.1$, ϕ is exact solution and $\widetilde{\phi}$ is evaluated by product integration technique.
In Table 1, columns 3, 4, 5, 6, 7, shows absolute errors of $\widetilde{u}$ at $(x,t) = (0.02i, 0.02j), i,j = 1,2,3,4,5$ with $b = 0.1$, u is exact solution and $\widetilde{u}$ is the approximated solution evaluated numerically by substitute of $\widetilde{\phi}$, instead of ϕ in u representation formula. Here $e_{ij}, i,j = 1,2,3,4,5$ is the absolute error of $\widetilde{u}$ at $(0.02i, 0.02j)$, and for example $3.36D-12$ means 3.36×10^{-12}. As we see in the error of $\widetilde{u}$ the bad behavior, is near $t = 0$, and it improve as t keeps aloof from zero. All of programs written by Mathematica programming.

Table 1: absolute errors of $\widetilde{\phi}$ and $\widetilde{u}$

i	$\lvert\phi-\widetilde{\phi}\rvert_i$	e_{i1}	e_{i2}	e_{i3}	e_{i4}	e_{i5}
1	$3.36D-12$	$1.62D-4$	$1.55D-6$	$6.66D-9$	$3.27D-10$	$7.82D-12$
2	$1.37D-11$	$9.59D-4$	$1.06D-5$	$1.01D-8$	$3.98D-8$	$2.08D-9$
3	$2.08D-10$	$4.24D-3$	$1.13D-4$	$2.19D-7$	$4.44D-7$	$1.91D-8$
4	$1.48D-9$	$1.16D-3$	$1.53D-4$	$1.82D-5$	$1.46D-6$	$4.08D-8$
5	$1.99D-9$	$6.54D-3$	$5.72D-4$	$1.53D-5$	$2.05D-6$	$4.69D-7$

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Ratio-dependent functional response predator-prey model with threshold harvesting

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Abstract
This paper deals with a ratio-dependent functional response predator-prey model, with a threshold harvesting in the predator equation. We study the equilibria of the system before and after the threshold. Furthermore, we show that the threshold harvesting can improve the undesirable behavior, such as nonexistence of interior equilibria. Finally, some numerical simulations are performed to support our analytic results.

Keywords: Predator-prey model, functional response, threshold harvesting
Mathematics Subject Classification [2010]: 37N25, 92D25

1 Introduction

Classically a predator-prey model is defined as the following system

$$\begin{cases} \dot{x} &= rx(1-\frac{x}{k}) - F(x,y)y \\ \dot{y} &= \beta\, F(x,y)y - \delta y, \end{cases} \tag{1}$$

where x and y are the number of prey and predator, respectively. In this model, in the prey equation, the parameter $r > 0$ is the prey intrinsic growth rate and k represents the environmental carrying capacity. The function $F(x,y)$ describes predation and is called the *functional response*. In the predator equation, the parameter β accounts for conversion rate to change prey biomass into predator reproduction, and δ is the predator's death rate. Moreover, from the point of view of human needs, it is necessary to consider the harvesting of populations in some models [5]. An important harvesting policy for the predator-prey model is the threshold harvesting function. It works as follows:

when population is above of certain level or threshold T, harvesting occurs; when the population falls below that level, harvesting stops. The policy was first studied by Collie and Spencer [2], and additional analysis has been done since then [1]. So the continuous threshold function proposed as the following

$$H(z) = \begin{cases} 0 & z \le T \\ \frac{h(z-T)}{h+z-T} & z > T, \end{cases} \tag{2}$$

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for $z = x$ or $z = y$ [3]. In (2), T is the threshold population size that determines when harvesting starts or stops and h is the rate of harvesting limit. The model allows managers to smoothly increase the harvesting rate as the population increases. So in this paper, we consider the ratio-dependent functional response model with a predator threshold harvesting policy in the predator equation. Some numerical simulations has been done in the final section to support the analytic results.

2 Equilibria and the stability

In this section, we consider the following ratio-dependent functional response predator-prey model, with a predator threshold harvesting policy and some time delay in predator equation

$$\begin{cases} \dot{x} &= x(1-x) - \frac{\alpha xy}{x+y} \\ \dot{y} &= y\left(-\delta + \frac{\beta x}{x+y}\right) - H(y), \end{cases} \tag{3}$$

where

$$H(y) = \begin{cases} 0 & y \leq T \\ \frac{h(y-T)}{h+y-T} & y > T, \end{cases} \tag{4}$$

and the initial conditions $x(0) > 0, y(0) > .$

Theorem 2.1. *The boundary equilibria of the system (3) in the first quadrant, are the co-extinction point $O = (0,0)$ and the predator-free point $E = (1,0)$. If $\beta > \delta$ and $\beta - \alpha\beta + \alpha\delta > 0$, then the unharvested model has a co-existence equilibrium $E^* = (x^*, y^*)$ defined by*

$$x^* = \frac{\beta-\alpha\beta+\alpha\delta}{\beta}, y^* = \frac{\beta-\delta}{\delta}x^* = \frac{x^*(x^*-1)}{1-\alpha-x^*} = \frac{\beta^2-\alpha\beta^2+2\alpha\delta\beta-\beta\delta-\alpha\delta^2}{\beta\delta}. \tag{5}$$

Furthermore if $y^* \leq T$, then E^* is an equilibrium of the harvested model too. If $y^* > T$ then the harvested model has a co-existence equilibrium $E^{**} = (x^{**}, y^{**})$ defined by

$$\begin{cases} y &= \frac{x(x-1)}{1-\alpha-x} \\ x &= \frac{\delta(h+y-T)y^2+h(y-T)y}{(\beta-\delta)(h+y-T)y-h(y-T)}. \end{cases} \tag{6}$$

and we have $x^{**} > x^*$, $T < y^{**} < y^*$.

The general jacobian matrix of system (3) around an arbitrary point (x, y) equals

$$J = \begin{pmatrix} 1 - 2x - \frac{\alpha y^2}{(x+y)^2} & -\frac{\alpha x^2}{(x+y)^2} \\ \frac{\beta y^2}{(x+y)^2} & -\delta + \frac{\beta x^2}{(x+y)^2} - \frac{dH(y)}{dy} \end{pmatrix}. \tag{7}$$

By Hartman-Grobman theorem for hyperbolic equilibria and the following outcome we prove our stability results. For the proof of the following result see for instance [4].

Theorem 2.2. *Consider the linear system $\dot{x} = Ax$.*

1. *If* $Det(A) < 0$*, then the system has a saddle at the origin.*

2. *If* $Det(A) > 0$*,* $Tr^2(A) - 4Det(A) \geq 0$*, then the system has a node at the origin; it is stable if* $Tr(A) < 0$ *and unstable if* $Tr(A) > 0$*.*

3. *If* $Det(A) > 0$*,* $Tr^2(A) - 4Det(A) < 0$*, then the system has a focus at the origin; it is stable if* $Tr(A) < 0$ *and unstable if* $Tr(A) > 0$*.*

4. *If* $\det(A) > 0$*,* $Tr(A) = 0$*, then the system has a center at the origin.*

Theorem 2.3. *The extinction point* $(0, 0)$ *is a saddle for the system (3).*

Theorem 2.4. *At the point* $E = (1, 0)$*,the trace and the determinant of (7) are* $Tr(J)_{(1,0)} = -1 - \delta + \beta$ *and* $Det(J)_{(1,0)} = \delta - \beta$*. Therefore*

1. *if* $\delta - \beta < 0$ *then* E *is a saddle.*

2. *If* $\delta - \beta > 0$ *then* E *is a stable node.*

3. *If* $\delta - \beta = 0$ *then* E *remains a stable node.*

Theorem 2.5. *Let*

$$M = \frac{(\beta - \delta)(-\alpha\delta\beta^2 + \alpha\delta^2\beta + \beta^2\delta)}{\beta^3}, N = \frac{-\beta^2 + \alpha(\beta^2 - \delta^2) - \beta\delta(\beta - \delta)}{\beta^2}.$$

1. *If* $M < 0$*, then* E^* *is a saddle.*

2. *If* $M > 0$ *and* $N < 0$*, then* E^* *is a stable node or focus.*

3. *If* $M > 0$ *and* $N > 0$*, then* E^* *is an unstable node or focus.*

Note that at the equilibrium (x^{**}, y^{**}) the trace and the determinant of the jacobian matrix equals

$$Tr(J) = C - \frac{B^2}{\alpha} - \delta + \frac{\beta A^2}{\alpha^2} - \phi, Det(J) = C(\frac{\beta}{\alpha^2}A^2 - \phi - \delta) + \frac{1}{\alpha}B^2(\delta + \phi),$$

where $\phi = \frac{h^2}{h - T - \frac{x^{**}B}{A}}$, $A = 1 - \alpha - x^{**}$, $B = x^{**} - 1$, $C = 1 - 2x^{**}$.

Theorem 2.6.

1. *If* $C - \frac{1}{\alpha}B^2 > \frac{\beta CA^2}{\alpha^2(\phi + \delta)}$*, then* E^{**} *is a saddle point.*

2. *If* $C - \frac{1}{\alpha}B^2 < \frac{\beta CA^2}{\alpha^2(\phi + \delta)}$ *and* $C - \frac{1}{\alpha}B^2 < \delta + \phi - \frac{\beta A^2}{\alpha^2}$*, then* E^{**} *is a stable node or focus.*

3. *If* $\delta + \phi - \frac{\beta A^2}{\alpha^2} < C - \frac{1}{\alpha}B^2 < \frac{\beta CA^2}{\alpha^2(\phi + \delta)}$*, then* E^{**} *is a unstable node or focus.*

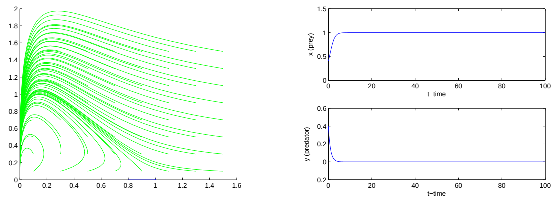

Figure 1: $\alpha = 1.3, \beta = 0.8, \delta = 0.1$, without harvesting.

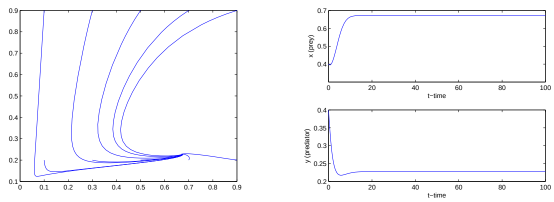

Figure 2: $\alpha = 1.3, \beta = 0.8, \delta = 0.1, T = 0.1, h = 1$.

3 Numerical simulations

In this section, we present some numerical simulations to illustrate our theoretical analysis. In the following example the harvesting create a co-existence equilibrium when it does not exits in unharvested model. By Theorem 2.1 we know that if $\beta > \delta$, $\beta - \alpha\beta + \alpha\delta \leq 0$, then the system (3) has no interior equilibria and the extinction of the species is inevitable.

Example 3.1. In Fig. 1, the phase portrait of the system with the parameter values $\alpha = 1.3, \beta = 0.8, \delta = 0.1$ without harvesting has been shown. The system has no co-existence equilibria since $\beta - \alpha\beta + \alpha\delta < 0$. Then in Fig. 2, the threshold harvesting function with the parameter values $h = 1, T = 0.1$ is added to the system. In this case the system has a stable interior equilibrium.

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Regularized Sinc-Galerkin Method for Solving a Two-Dimensional Nonlinear Inverse Parabolic Problem

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Abstract

In this paper, Sinc-Galerkin method is used to solve a two-dimensional nonlinear inverse parabolic problem and a stable numerical solution is determined. To do this, the Levenberg-Marquardt method is applied to deal with the ill-posedness of the discretized system. The accuracy and reliability of the proposed method is demonstrated by a test problem.

Keywords: Sinc-Galerkin method, Inverse parabolic problem, Levenberg-Marquardt method.
Mathematics Subject Classification [2010]: 35R30, 35K55

1 Introduction

In this paper, a two-dimensional nonlinear inverse parabolic problem of the form

$$\begin{aligned} u_t - \Delta u = G(x,y,t,u), &\quad \text{(x,y)} \in \Omega \subset \mathbb{R}^2,\ t>0,\ n>1,\ (n \in \mathbb{N}), \\ u(x,y,0) = 0, &\quad \text{(x,y)} \in \Omega \subset \mathbb{R}^2, \\ u(x,y,t) = 0, &\quad \text{(x,y)} \in \partial\Omega \subset \mathbb{R}^2,\ t \geqslant 0, \end{aligned} \tag{1}$$

is considered, where $\partial\Omega$ is the boundary of $\Omega = [0,1] \times [0,1]$, $G(x,y,t,u) = f(x,y) + H(x,y,t) - u^n$ such that $H(x,y,t)$ is known a function and the functions $f(x,y)$ and $u(x,y,t)$ are unknown. If $f = f(x,y)$ is given, then the problem (1) is called the *direct problem* (DP). The existence and uniqueness of the DP (1) have been investigated in [1]. To find the pair (u,f), we use the overposed measured data

$$u(x^*,y^*,t_i) = E(t_i), \qquad 0 < x^*,\ y^* < 1,\ i = 1,2,\dots,I. \tag{2}$$

Let us denote by the notation $u[x,y,t;f]$ the solution of the DP (1). Then from the additional condition (2) it is seen that the nonlinear inverse parabolic problem (1) consists of solving the following nonlinear functional equation

$$u[x^*,y^*,t_i;f] = E(t_i), \qquad 0 < x^*,\ y^* < 1,\ i = 1,2,\dots,I. \tag{3}$$

In general, instead of solving the functional equation (3), an optimization problem is solved, where objective function is minimized by an effective regularization method. This objective function is defined by

*Speaker

$$S(f)=\sum_{i=1}^{I}(u[x^*,y^*,t_i;f]-E(t_i))^2. \tag{4}$$

In this paper, we attempt to obtain an approximate solution for the unknown function $f(x,y)$. For this purpose, first let

$$f(x,y)\simeq \bar{f}(x,y)=\sum_{i=1}^{n}\sum_{j=1}^{m}e_{i,j}Sinc(\frac{x-ih}{h})Sinc(\frac{y-jh}{h}), \tag{5}$$

be a linear combination of Sinc functions, where h is increment of x and y variables and $e_{i,j}$'s are unknown parameters that should be derived. In other words, the nonlinear inverse parabolic problem is reduced to a parameter approximation problem. These parameters are determined by minimizing the objective function (4). Due to this the Levenberg-Marquardt method is used. This method is applied to the solution of linear problems that are too ill-conditioned [2].

2 Mathematical formulation

The Sinc function is defined on the whole real line $-\infty<x<\infty$ by

$$Sinc(x)=\begin{cases}\frac{\sin(\pi x)}{\pi x} & x\neq 0\\ 1 & x=0.\end{cases}$$

For $h_x,h_y,h_t>0$, the translated Sinc functions with evenly spaced nodes for space and time variables are given as $S(k,h_x)(x)=Sinc(\frac{x-kh_x}{h_x})$, $S(k,h_y)(y)=Sinc(\frac{y-kh_y}{h_y})$ and $S(k,h_t)(t)=Sinc(\frac{t-kh_t}{h_t})$, $k=0,\pm1,\pm2,\ldots$. To construct approximations on the intervals $(0,1)$ and $(0,\infty)$, which are used in this paper, we should apply $\varphi(z)=\ln\left(\frac{z}{1-z}\right)$ and $\Upsilon(t)=\ln(t)$, respectively. In other words, the compositions $S_j(x)=S(j,h_x)o\varphi(x)$, $S_j(y)=S(j,h_y)o\varphi(y)$ and $S_j^*(t)=S(j,h_t)o\Upsilon(t)$ define the basis elements on the intervals $(0,1)$ and $(0,\infty)$, respectively. Now, to find the unknown function $f(x,y)$ of the problem (1), a computational algorithm is provided.

Algorithm: Identification of the unknown function $f(x,y)$

Step 1. Let (5) ba an approximation of the unknown function $f(x,y)$.

Step 2. Using Sinc-Galerkin method, obtain an approximate solution for $u[x,y,t,\bar{f}]$. Due to this, set $u_{m_x,m_y,m_t}(x,y,t)=\sum_{i=-M_x}^{N_x}\sum_{j=-M_y}^{N_y}\sum_{k=-M_t}^{N_t}u_{i,j,k}S_i(x)S_j(y)S_k^*(t)$ be an approximate solution of the DP (1), where $m_x=M_x+N_x+1$, $m_y=M_y+N_y+1$ and $m_t=M_t+N_t+1$ and $u_{i,j,k}=u(x_i,y_j,t_k)$ are unknown coefficients. These unknown coefficients are determined by orthogonalizing the residual with respect to the functions $S_{l,\gamma,\lambda}$. This yields the discrete system $\left(u_t-\Delta u+u^n-\bar{f}(x,y)-H(x,y,t),S_{l,\gamma,\lambda}\right)=0$, $-M_x\leqslant l\leqslant N_x$, $-M_y\leqslant\gamma\leqslant N_y$, $-M_t\leqslant\lambda\leqslant N_t$, where $S_{l,\gamma,\lambda}=S_l(x)S_\gamma(y)S_\lambda^*(t)$. The weighted inner product is defined by $(f,g)=\int_0^\infty\int_0^1\int_0^1 f(x,y,t)g(x,y,t)v(x)w(y)\tau(t)dxdydt$, where $v(x)w(y)\tau(t)$ is a product weight function. The method of approximating the integrals, begins by integrating by parts to transfer all derivatives from u to $S_{l,\gamma,\lambda}$. Then, by choosing $v(x)=\frac{1}{\sqrt{\varphi'(x)}}$, $w(y)=\frac{1}{\sqrt{\varphi'(y)}}$ and $\tau(t)=\sqrt{\Upsilon'(t)}$ and the Sinc trapezoidal quadrature rule we have

$$\begin{aligned}
&(u_t-(u_{xx}+u_{yy})+u^n-F(x,y,t),S_{l,\gamma,\lambda})\simeq\\
&-h_xh_yh_t\sum_{p=-M_x}^{N}\sum_{q=-M_y}^{N_y}\sum_{r=-M_t}^{N_t}u(x_p,y_q,t_r)S_l(x_p)S_\gamma(y_q)\frac{v(x_p)w(y_q)\frac{\partial}{\partial t}(S_\lambda(t)\tau(t))\big|_{t=t_r}}{\varphi'(x_p)\varphi'(y_q)\Upsilon'(t_r)}\\
&-h_xh_yh_t\sum_{p=-M_x}^{N}\sum_{q=-M_y}^{N_y}\sum_{r=-M_t}^{N_t}\frac{u(x_p,y_q,t_r)S_\gamma(y_q)w(y_q)S_\lambda(t_r)\tau(t_r)\frac{\partial^2}{\partial x^2}(S_l(x)v(x))\big|_{x=x_p}}{\varphi'(x_p)\varphi'(y_q)\Upsilon'(t_r)}\\
&-h_xh_yh_t\sum_{p=-M_x}^{N}\sum_{q=-M_y}^{N_y}\sum_{r=-M_t}^{N_t}\frac{u(x_p,y_q,t_r)S_l(x_p)v(x_p)S_\lambda(t_r)\tau(t_r)\frac{\partial^2}{\partial y^2}(S_\gamma(y)w(y))\big|_{y=y_q}}{\varphi'(x_p)\varphi'(y_q)\Upsilon'(t_r)}\\
&-h_xh_yh_t\sum_{p=-M_x}^{N}\sum_{q=-M_y}^{N_y}\sum_{r=-M_t}^{N_t}\frac{(u(x_p,y_q,t_r))^2S_l(x_p)v(x_p)S_\gamma(y_q)w(y_q)S_\lambda(t_r)\tau(t_r)}{\varphi'(x_p)\varphi'(y_q)\Upsilon'(t_r)}\\
&-h_xh_yh_t\sum_{p=-M_x}^{N}\sum_{q=-M_y}^{N_y}\sum_{r=-M_t}^{N_t}\frac{F(x_p,y_q,t_r)S_l(x_p)v(x_p)S_\gamma(y_q)w(y_q)S_\lambda(t_r)\tau(t_r)}{\varphi'(x_p)\varphi'(y_q)\Upsilon'(t_r)}=0,
\end{aligned}$$

where $F(x,y,t)=\bar{f}(x,y)+H(x,y,t)$ and $x_p=\varphi^{-1}(ph_x)$, $y_q=\varphi^{-1}(qh_y)$ and $t_r=\Upsilon^{-1}(rh_t)$. We note that [3], $[S(i,h_x)o\varphi(x)]\Big|_{x=x_k}=\delta^{(0)}_{i,k}$, $\frac{d}{d\varphi}[S(i,h_x)o\varphi(x)]\Big|_{x=x_k}=\frac{1}{h_x}\delta^{(1)}_{i,k}$ and $\frac{d^2}{d\varphi^2}[S(i,h_x)o\varphi(x)]\Big|_{x=x_k}=\frac{1}{h_x^2}\delta^{(2)}_{i,k}$. The similar formulas are satisfied for $S_j(y)=S(j,h_y)o\varphi(y)$ and $S^*_j(t)=S(j,h_t)o\Upsilon(t)$. Thus, we have a nonlinear system of $m_x\times m_y\times m_t$ equations of the $m_x\times m_y\times m_t$ unknown coefficients $u_{i,j,k}$. These coefficients are obtained for example by using Newton's method. [3].

Step 3. Obtain the $m\times n$ unknown parameters $e_{i,j}$, based on the minimization of the least squares norm $S(f)=\sum_{i=1}^{I}(u_{m_x,m_y,m_t}(x^*,y^*,t_i)-E(t_i))^2$. Since, the obtained system of algebraic equations is ill-conditioned, therefore the Levenberg-Marquardt method according to step 4 is used.

Step 4. Levenberg-Marquardt regularization [2]. Suppose that,

$$\begin{aligned}
U_{m_x,m_y,m_t}(f) &= [U_1,U_2,\dots,U_I]^T,\\
E &= [E_1,E_2,\dots,E_I]^T,
\end{aligned}$$

and $f=[e_{1,1},e_{2,1},\dots,e_{n,1},e_{1,2},e_{2,2},\dots,e_{n,2},\dots,e_{1,m},e_{2,m},\dots,e_{n,m}]^T$, where $E_i=E(t_i)$ and $U_i=u_{m_x,m_y,m_t}(x^*,y^*,t_i)$, $i=1,2,\dots,I$. Then the matrix form of the functional is given by $S(f)=\left[E-U_{m_x,m_y,m_t}(f)\right]^T\left[E-U_{m_x,m_y,m_t}(f)\right]$, in which $\left[E-U_{m_x,m_y,m_t}(f)\right]^T\equiv[E_1-U_1,E_2-U_2,\dots,E_I-U_I]$. The superscript T denotes the transpose and I is the total number of measurements. To minimize the least squares norm, the derivatives of $S(f)$ with respect to each unknown parameters $\{e_{i,j}\}_{i=1,j=1}^{i=n,j=m}$ are equated to zero. That is

$$\nabla S(f)=2\left[-\frac{\partial U^T_{m_x,m_y,m_t}(f)}{\partial f}\right]\left[E-U_{m_x,m_y,m_t}(f)\right]=0.$$

The sensitivity matrix is defined by $J(f)=\left[\frac{\partial U^T_{m_x,m_y,m_t}(f)}{\partial f}\right]^T$ (see [2]). Now, in the following the computational algorithm for the Levenberg-Marquardt regularization is provided

[2]. Suppose an initial guess for the vector of unknown coefficients f is available. Denote it with $f^{(0)}$.
1. Set μ_0 be an arbitrary regularization parameter (for example 0.001) and $k = 0$.
2. Compute $U_{m_x,m_y,m_t}(f^{(0)})$ and $S(f^{(0)})$.
3. Compute the sensitivity matrix J^k and $\Omega^k = diag[(J^k)^T J^k]$, by using the current values of $f^{(k)}$.
4. Solve the following linear system of algebraic equations

$$\left[(J^k)^T J^k + \mu^k \Omega^k\right] \Delta f^k = (J^k)^T \left[E - U_{m_x,m_y,m_t}(f^k)\right],$$

in order to compute $\Delta f^k = f^{k+1} - f^k$.
5. Compute $f^{k+1} = \Delta f^k + f^k$.
6. If $S(f^{k+1}) \geq S(f^k)$ replace μ^k by $10\mu^k$ and go to 4.
7. If $S(f^{k+1}) < S(f^k)$ accept f^{k+1} and replace μ^k by $0.1\mu^k$.
8. Assume that tol (tolerance) is given. If $\left\|f^{k+1} - f^k\right\| \leq tol$, then an acceptable approximation is obtained. Otherwise, replace k by $k + 1$ and go to 3.

3 Numerical example

In this section, to show the validation of the introduced method a numerical example is given. In this example, we put $n = 2$ and $H(x, y, t) = 2t(y - y^2)\cos(x) + (2t(1 - x) + ty + (1+t)(-xy - y^2 + xy^2))\sin(x) + t^2(1 - x)^2(1 - y)^2 y^2 \sin^2(x)$. Thus, the exact solutions are $u(x, y, t) = ty(1 - x)(1 - y)\sin(x)$ and $f(x, y) = y\sin(x)$. Also, the additional condition is considered as $u(0.5, 0.5, t) = E(t) = 0.6t$.
Table 1 shows the L_1-norm error of the introduced method. As we observe, the results show the efficiency and accuracy of the method. Also, Fig. 1 shows the exact and approximate solutions of $f(x, y)$. These results are obtained by using $M_x = N_x = 6$, $M_y = N_y = 3$, $M_t = N_t = 2$, $h = \sqrt{\pi}$, $h_x = \sqrt{\frac{\pi}{6}}$, $h_y = \sqrt{\frac{2\pi}{3}}$, $h_t = \sqrt{\pi}$ and

$$f(x, y) \simeq \bar{f}(x, y) = e_{1,1}Sinc(\frac{x - h}{h})Sinc(\frac{y - h}{h}) + e_{22}Sinc(\frac{x - 2h}{h})Sinc(\frac{y - 2h}{h}).$$

Table 1: The L_1-norm error of the introduced method

$\left\|f(x,y) - \bar{f}(x,y)\right\|_1$	0.07257	0.12183	0.00297	0.03906	0.05702	0.00406
x	0.1	0.1	0.4	0.4	0.7	0.7
y	0.3	0.9	0.1	0.6	0.3	0.9

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Singular normal forms and computational algebraic geometry

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Abstract

In this talk we discuss the possible applications of techniques from computational algebraic geometry in germ computations of smooth local singular normal forms. Due to the algebraic nature of these techniques, we need to address the ideal and module membership problem. Here, as a part of our ongoing project, we briefly describe how we utilize concepts from algebraic geometry like *local rings*, *Mora normal form*, and *standard bases* to obtain an algorithm for computing the normal forms for such bifurcation problems. This work contributes into enhancement of our developed Maple library, called "*Singularity*".

Keywords: Normal form; Standard basis; Singularity theory; Local ring.
Mathematics Subject Classification [2010]: 34C20; 13P10; 14H20.

1 Introduction

The bifurcation theory of singular smooth vector and scalar valued germs is an important subject and has many applications in engineering problems; see [5–9]. The applications include bifurcation control and designing effective controllers for uncontrollable singular systems; see [8]. In order to achieve this, we need to compute certain ideals and modules. Hence, a convenient answer to the ideal and module membership problem is a desirable goal. There are various ways to answer the ideal membership problem: the first method is to find a convenient representation of the ideals or module structures such as the use of intrinsic ideals and module representations. The other approach is by the use of efficient algorithms through a symbolic computer algebra system. Given the local nature of our problem, our rings constitute a local ring and therefore, techniques such as Gröbner basis does not properly work here. Thereby, we shall use Standard basis and Mora remainder instead of the usual Gröbner remainder. Mora normal form is a more common terminology in the literature than Mora remainder, yet we prefer Mora remainder since it does not confuse with the normal form of a singular germ. The results presented in this talk has some contributions in *Singularity*. *Singularity* is an end-user friendly symbolic library for bifurcation analysis of singularities. We hereby announce that the first version of *Singularity* will soon be released for public use and it will be enhanced and updated as our research progresses.

*Speaker

The aim of this conference paper is to compute the normal form of a smooth germ given by $g(x, \lambda)$. Here, x is a state variable and λ is treated as a control parameter. The normal form of a map g is a *simple* representative of the class of all germs equivalent to g. The equivalence relation depends on its applications. Here we use *contact equivalence* as it is the most natural equivalence relation that preserves the zero structures of smooth maps. Denote $\mathscr{E}$ for the ring of smooth germs whose base point is the origin and $g(x, \lambda) \in \mathscr{E}$. Additional parameters may have adverse effects on the qualitative type of the associated bifurcation diagrams. The study of these is important and is usually dealt with them through the universal unfolding which is not discussed here.

The rest of this conference paper is summarized as follows. Section 2 is devoted to a brief long history of the subject. We then present algebraic formulation needed for computations in Section 3. Finally, our approach is proposed in Section 4.

2 Literature Review

Armbruster and Kredel proposed to study the universal unfolding of singular germs by using Gröbner basis; see [1]. However, Gröbner basis is inappropriate tool for computations of singularities as it leads to wrong results in certain circumstances; also see [3]. Gatermann and Lauterbach [4] used the standard basis for study of equivariant bifurcation problems. There are two important local rings that they are contained in $\mathscr{E}$ and are of our central attention. The first one is the local ring of fractional germ maps, i.e.,

$$K[x, \lambda]_{\langle x,\lambda\rangle} = \{\frac{f}{g} \mid f, g \in K[x, \lambda],\ g(0, 0) \neq 0\}$$

and the second one is the ring of formal power series denoted by $K[[x, \lambda]]$. The following Lemma provides an example of an efficient approach for computations, when its hypothesis is satisfied. This is because it allows us to use smaller rings rather than using the ring of smooth germs.

Lemma 2.1. *Let* $g_i \in K[x, \lambda]_{\langle x,\lambda\rangle} \subset \mathscr{E}$, $I = \langle g_1, \ldots, g_n\rangle_{\mathscr{E}}$, $J = \langle g_1, \ldots, g_n\rangle_{K[x,\lambda]_{\langle x,\lambda\rangle}}$, *and*

$$\mathcal{M}^k := {\langle x, \lambda\rangle_{K[x,\lambda]_{\langle x,\lambda\rangle}}}^k \subseteq J.$$

Then for $f \in K[x, \lambda]_{\langle x,\lambda\rangle}$,

$$f \in J \quad \textit{if and only if} \quad f \in I.$$

Proof. Let $f \in I$. We claim that $I \cap K[x, \lambda]_{\langle x,\lambda\rangle} \subseteq J$. Let $I = \langle g_1, \ldots, g_n\rangle_{\mathscr{E}}$ and $J = \langle g_1, \ldots, g_n\rangle_{K[x,\lambda]_{\langle x,\lambda\rangle}}$. Choose

$$\sum_{i=1}^{n} a_i p_i \in I \cap K[x, \lambda]_{\langle x,\lambda\rangle}$$

where $a_i \in \mathscr{E}$ and $p_i \in K[x, \lambda]_{\langle x,\lambda\rangle}$. Therefore we may write

$$\sum_{i=1}^{n} a_i p_i = \frac{f}{g}, \tag{1}$$

with

$$f,\, g \in k[x, \lambda], \quad \text{and} \quad g(0, 0) \neq 0.$$

If we substitute a_i by its $(k-1)$-jet, i.e., $J^{k-1}a_i$, Equation (1) is valid modulo $\mathcal{M}^k$. Since $\mathcal{M}^k \subseteq J$, the germ f/g belongs to J. Now consider $f \in I$. By hypothesis $f \in K[x, \lambda]_{\langle x,\lambda\rangle}$, we have

$$f \in I \cap K[x, \lambda]_{\langle x,\lambda\rangle} \subset J \text{ and } f \in J.$$

The *if* part is trivial since $J \subseteq I$. □

3 Algebraic Objects

In this section we recall some basic concepts from singularity theory and algebraic geometry; see [2, 10]. An ideal I is called *intrinsic* if for any $f \in I$,

$$g \sim_s f \Longrightarrow g \in I.$$

Here, $\sim_s$ denotes *strongly equivalence* relation. For a given singularity g, $\mathscr{P}(g)$ and $\mathscr{S}(g)$ are defined by

$$\mathscr{P}(g) = \mathrm{Itr}(\mathscr{J}(g)) \quad \text{for} \quad \mathscr{J}(g) = \langle xg, \lambda g, x^2 g_x, \lambda g_x\rangle,$$

and

$$\mathscr{S}(g) = \Sigma_{(\alpha_1,\alpha_2)}\{\mathscr{M}^{\alpha_1}\langle\lambda^{\alpha_2}\rangle \mid \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial \lambda^{\alpha_2}} g(0, 0) \neq 0\}.$$

Here $\mathrm{Itr}(I)$ is the largest intrinsic ideal in I. The ideal $\mathscr{P}(g)$ represents the ideal of higher order terms while $\mathscr{S}(g)$ is the smallest intrinsic ideal containing g. Monomials of the form $x^{\alpha_1}\lambda^{\alpha_2}$ are called *intrinsic generators* of $\mathscr{S}(g)$.

4 Computational approach

The most challenging part of normal form computation is to give a suitable presentation for ideal $\mathscr{P}(g)$. Intrinsic ideals admit the following form

$$\mathscr{M}^k + \mathscr{M}^{k_1}\langle\lambda^{l_1}\rangle + \cdots + \mathscr{M}^{k_s}\langle\lambda^{l_s}\rangle, \tag{2}$$

satisfying $k > k_1 + l_1 > \cdots > k_s + l_s$ and $0 < l_1 < \cdots < l_s$; see [10]. The intrinsic representation (2) is computed by the standard basis computation and follows the Hironaka's lemma. Details of intrinsic computations shall be discussed in our talk presentation while they are skipped in this page-limited extended abstract conference paper.

To obtain normal form of germ g, we first omit higher order terms from g. This leads to an alternative equivalent germ f. Next, we detect the intermediate order terms. This is performed via the vector space

$$\mathscr{P}(f)^\perp - \mathscr{S}(f)^\perp.$$

Applying all possible effective transformation generators on f, we find the maximal solvable associated subsystem. This gives rise to the desirable simplest normal form.

We apply our above mentioned algorithmic approach on

$$g(x, \lambda) = \lambda^2 - \cos(x) + x\lambda + 1.$$

Therefore, $\mathscr{P}(g) = \mathscr{M}^3$ and this concludes that

$$f(x, \lambda) = \lambda^2 + x^2 + x\lambda$$

is contact-equivalent to g. Then, algebraic computations lead to

$$\mathscr{P}(f) = \mathscr{M}^3,\ \mathscr{S}(f) = \mathscr{M}^2,\ \text{and}\ \mathscr{P}(f)^\perp - \mathscr{S}(f)^\perp = \mathbb{R}\{x\lambda\}.$$

One may find the simplest normal form

$$x^2 + \frac{3}{4}\lambda^2$$

by applying $x \to ax + b\lambda$ (for $a > 0$ and arbitrary b) in f and solving the corresponding maximal solvable subsystem, that is, to remove the intermediate order term $x\lambda$ from f.

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Solving Linear Fuzzy Fredholm Integral Equations System by Triangular Functions

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Abstract

In this paper we intend to offer a numerical method to solve linear fuzzy fredholm integral equations system of the second kind. This method converts the given fuzzy system into a linear system of algebraic equations by using triangular orthogonal functions. The proposed method is illustrated by an example and also results are compared with the exact solution by using computer simulations.

Keywords: Fuzzy number, Fuzzy Fredholm integral equations system, Triangular functions
Mathematics Subject Classification [2010]: 45D05, 03E72

1 Introduction

There are many numerical methods which have been focused on the solution of fuzzy integral equations. Recently, introduced a new set of triangular orthogonal functions have been applied for solving integral equation by Babolian et al. [1]. Mr Mirzaee et al. [2] have used the triangular functions for solving fuzzy Fredholm integral equation of second kind (FFIE-2). The aim of this paper is to apply the triangular functions for the linear fuzzy Fredholm integral equations system of the second kind (FFIES-2).

2 Preliminaries

Definition 2.1. ([1]) Two m-sets of triangular functions (TFs) are defined over the interval [0,T] as:

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & o.w \end{cases}, \quad T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & o.w \end{cases},$$

where $i = 0, 1, \cdots, m-1, h = \frac{T}{m}$, with a positive integer value for m.

In this paper, it is assumed that $T = 1$. Consider the first m terms of $T1_i$ and $T2_i$,we can write them concisely as m-vectors:

$$T1(t) = [T1_0(t), T1_1(t), \cdots, T1_{m-1}(t)]^T, \quad T2(t) = [T2_0(t), T2_1(t), \cdots, T2_{m-1}(t)]^T.$$

*Speaker

We denote the TF vector $T(t)$ as $T(t) = [T1(t) \quad T2(t)]^T$. So, we have

$$\int_0^1 T(t)T^T(t)\mathrm{d}t \simeq \begin{pmatrix} \frac{h}{3}I_m & \frac{h}{6}I_m \\ \frac{h}{6}I_m & \frac{h}{3}I_m \end{pmatrix} = D_{2m\times 2m}. \tag{1}$$

Also, let $f(s,t) \in L^2([0,1)\times[0,1))$, the expansion of $f(s,t)$ with respect to TFs, can be defined as follows

$$f(t,s) \simeq T1^T(t).F11.T1(s) + T1^T(t).F12.T2(s) + T2^T(t).F21.T1(s) + T2^T(t).F22.T2(s),$$

or

$$f(t,s) \simeq T^T(t).F.T(s) \tag{2}$$

where $F11, F12, F21$ and $F22$ are $m\times m$ matrices and $(F11)_{ij} = f(ih, jh), (F12)_{ij} = f(ih,(j+1)h), (F21)_{ij} = f((i+1)h, jh)$ and $(F22)_{ij} = f((i+1)h,(j+1)h)$, for $i,j = 0,1,\dots,m-1$, and $T(t)$, $T(s)$ are $2m_1$ and $2m_2$ dimensional TFs and F is a $2m_1 \times 2m_2$ TFs coefficient matrix [1]. For convenience, we put $m_1 = m_2 = m$, so we can write

$$F = \begin{pmatrix} (F11)_{m\times m} & (F12)_{m\times m} \\ (F21)_{m\times m} & (F22)_{m\times m} \end{pmatrix}. \tag{3}$$

Definition 2.2. ([2]) A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \to [0,1]$ such that (1): u is upper semi-continuous, (2): $u(x) = 0$ outside some interval [a, d], (3): There are real numbers b, c such as $a \le b \le c \le d$ and (i) $u(x)$ is monotonically increasing on [a, b], (ii) $u(x)$ is monotonically decreasing on [c, d], (iii) $u(x) = 1, b \le x \le c$. The set of all fuzzy numbers is denoted by E^1 and is a convex cone.

Definition 2.3. ([2]) A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, such that (1): $\underline{u}(r)$ is abounded monotonic increasing left continuous function, (2): $\overline{u}(r)$ is abounded monotonic decreasing left continuous function, (3): $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k \ge 0$, we define
(1): addition, $u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,
(2): scalar multiplicationand, $k \odot u = (k\underline{u}(r), k\overline{u}(r))$,
(3): $D(u,v) = max\{\sup_{0\le r\le 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0\le r\le 1} |\overline{u}(r) - \overline{v}(r)|\}$, is distance between u and v. (For More details about the properties of the fuzzy integral see [2])

3 Solving linear fuzzy Fredholm integral equations system

In this section, we present a TFs method to solve linear FFIES-2. The FFIES-2 is in the form

$$U(x) = G(x) \oplus \Lambda \odot \mathcal{K}U(x) \tag{4}$$

where

$$U(x) = [u_1(x), u_2(x), \dots, u_n(x)]^T, \quad G(x) = [g_1(x), g_2(x), \dots, g_n(x)]^T,$$

$$\mathcal{K}U(x) = \int_0^1 K(x,t) \odot U(t)\mathrm{d}t, \qquad K(x,t) = [k_{ij}(x,t)], \quad \Lambda = [\lambda_{ij}],$$

where $k_{ij}(x,t)$ is an orbitary kernel function over the square $0 \leq x,t \leq 1$ and $\lambda_{ij} \neq 0$ for $i,j = 1,2,\ldots,n$ are real constants. In system (4), the fuzzy function $g_i(x)$ and kernel $k_{ij}(x,t)$ are given and assumed to be sufficiently differentiable with respect to all their arguments on the interval $0 < x,t < 1$. Also $u_i(x)$ is a fuzzy real valued function, and $U(x) = [u_1(x), u_2(x), \ldots, u_n(x)]^T$ is the solution to be determined. For convenience, we consider the ith equation of Eq.(4) as

$$u_i(x) = g_i(x) \oplus \sum_{j=1}^{n} \lambda_{ij} \odot \int_0^1 k_{ij}(x,t) \odot u_j(t)\mathrm{d}t, \tag{5}$$

Let $(\underline{g}_i(x,r), \overline{g}_i(x,r))$ and $(\underline{u}_i(x,r), \overline{u}_i(x,r))$, $0 \leq r \leq 1$ and $x \in [0,1)$ be parametric forms of $g_i(x)$ and $u_i(x)$, respectively. In this paper, we assumed that $k_{ij}(x,t) \geq 0$. Now, for solving (4) we write the parametric form of the given fuzzy integral equations system as follows

$$\underline{u}_i(x,r) = \underline{g}_i(x,r) + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t)\underline{u}_j(t,r)\mathrm{d}t, \tag{6}$$

$$\overline{u}_i(x,r) = \overline{g}_i(x,r) + \sum_{j=1}^{n} \lambda_{ij} \int_0^1 k_{ij}(x,t)\overline{u}_j(t,r)\mathrm{d}t, \tag{7}$$

for $i,j = 1,2,\ldots,n$. Let us expand $\underline{u}_i(x,r)$, $\underline{g}_i(x,r)$ and $k_{ij}(x,t)$ by using Eq.(2) as follows

$$\underline{u}_i(x,r) \simeq T^T(x).U_i.T(r), \quad \underline{g}_i(x,r) \simeq T^T(x).G_i.T(r), \quad \underline{k}_{ij}(x,t) \simeq T^T(x).K_{ij}.T(t), \tag{8}$$

where U_i, G_i and K_{ij} are in the form of eq.(3). substituting the Eqs.(8) in Eq.(6):

$$T^T(x)U_iT(r) \simeq T^T(x)G_iT(r) + \sum_{j=1}^{n} \lambda_{ij} T^T(x)K_{ij} \left(\int_0^1 T(t)T^T(t)\mathrm{d}t \right) U_jT(r) \tag{9}$$

substituting the Eqs. (1) in Eq. (9), we can write

$$T^T(x)U_iT(r) \simeq T^T(x)G_iT(r) + T^T(x)(\sum_{j=1}^{n} \lambda_{ij}K_{ij}DU_j)T(r) \Rightarrow U_i = G_i + \sum_{j=1}^{n} \lambda_{ij}K_{ij}DU_j$$

Then we get the following system

$$\sum_{j=1}^{n} (\Delta_{ij} - \lambda_{ij}K_{ij}D)\, U_j = G_i, \quad \Delta_{ij} = \begin{cases} I & i = j \\ 0 & i \neq j \end{cases} \tag{10}$$

for $i,j = 1,2,\ldots,n$, and I is a $2m \times 2m$ identity matrix. By solving this matrix system, we can find matrix U_i, for $i = 1,\ldots,n$. So $\underline{u}_i(x,r) \simeq T^T(x)U_iT(r)$. The same trend hold for $\overline{u}_i(x,r)$. Therefore, for solving system (4), we need to solve two systems of (10).

Theorem 3.1. *(Convergence Analysis) Let $k_{ij}(x,t)$, $i,j = 1,2,\ldots,n$ and $0 \leq x,t \leq 1$ are bounded and continuous, then approximate solution of system (4), converges to the exact solution.*

Proof. suppose that $\tilde{u}_i(x)$ is approximate solution of exact solution $u_i(x)$. Therefore $\tilde{u}_i(x) \simeq \mathcal{U}_i^T T(x)$ [1], and by using the properties of the fuzzy integral [2], we can write

$$\lim_{m\to\infty} D(u_i(x), \tilde{u}_i(x)) \leq M \sum_{j=1}^{n} \int_0^1 \lim_{m\to\infty} D\left(u_j(t), \mathcal{U}_j^T T(t)\right) \mathrm{d}t \to 0,$$

where $M = \max_{0\leq x,t\leq 1} |\lambda_{ij} k_{ij}(x,t)| < \infty$. So the proof is completed. □

Example 3.2. Consider the system of fuzzy linear Fredholm integral equations with

$$(\underline{g}_1(x,r), \overline{g}_1(x,r)) = x^2(r^2+2r+2, 7-2r) + \frac{x}{3}(r^2+r+1, 4-r)$$
$$(\underline{g}_2(x,r), \overline{g}_2(x,r)) = x(r^2+3r+3, 10-3r), \quad 0 \leq x,t \leq 1, for \quad 0 \leq r \leq 1,$$
$$k_{11}(x,t) = x, k_{12}(x,t) = 2x^2, k_{21}(x,t) = 4xt, k_{22}(x,t) = 2x, \quad \lambda_{ij} = -1, i,j = 1,2$$

The exact solution in this case is given by $(\underline{u}_1(x,r), \overline{u}_1(x,r)) = x^2(r^2+r+1, 4-r)$, $(\underline{u}_2(x,r), \overline{u}_2(x,r)) = x(r+1, 3-r)$. After solving this system by the proposed method, we see that the absolute error is zero. (see Table 1)

Table 1: Numerical results for Example 3.2, with $x = 0.5, m = 32$.

r	Absolute error $\underline{u}_1(x,r)$	Absolute error $\overline{u}_1(x,r))$	Absolute error $\underline{u}_2(x,r)$	Absolute error $\overline{u}_2(x,r)$
0.1	3.3792e-07	1.3604e-04	7.7449e-05	2.7208e-04
0.3	1.0099e-05	1.2906e-04	9.6989e-05	2.5813e-04
0.5	6.1044e-05	1.2209e-04	1.2209e-04	2.4417e-04
0.7	1.7806e-05	1.1511e-04	1.5280e-04	2.3022e-04
0.9	5.5473e-05	1.0813e-04	1.8907e-04	2.1627e-04

4 Conclusion

In this paper, we introduce TFs method for approximating the solution of linear FFIES-2. The structural properties of TFs are utilized to reduce the FFIES-2 to a system of algebraic equations, without using any integration. In the above presented numerical example we see that the proposed method well performs for linear FFIES-2.

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Talk Some properties Sturm-Liouville problem with fractional derivative pp.: 1–4

Some properties Sturm-Liouville problem with fractional derivative

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Abstract

In this paper we establish the properties of Fractional singular Sturm-Liouville problem. Our main issue is to investigate the spectral properties for the operator. Furthermore, we prove new results according to the fractional Sturm-Liouville problem.

Keywords: Fractional Sturm-Liouville problem,Riemann-Liouville derivatives, eigenvalues and eigenfunctions
Mathematics Subject Classification [2010]: 34B24 ,34B40

1 Introduction

We consider the following SturmLiouville problem with factional derivative in the leading term

$$\begin{cases} -{}^cD_{0^+}^{\alpha}u(t) + q(t)u(t) = \lambda u(t), & 0 < t < 1, \\ u(0) = u(1) = 0, \quad \alpha \in (1,2) \end{cases} \tag{1}$$

Definition 1.1. [2] (RiemannLiouville fractional integrals)We define the left and the RiemannLiouville fractional integrals by

$$I_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 1.2. [2] The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$${}^cD_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

*Speaker

Proposition 1.3. *[2] Let $\alpha, \beta > 0$ and $\in L^p(a,b)(1 \leq p \leq \infty)$ Then, equations*

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} f(t) = I^{\alpha+\beta}_{a^+} f(t)$$

are satisfied.

Proposition 1.4. *[2] Let $\alpha > 0$ and $f \in L^p(a,b)(1 \leq p \leq \infty)$ then the following is true:*

$$D^{\alpha}_{a^+} I^{\alpha}_{a^+} f(t) = f(t)$$

$${}^{c}D^{\alpha}_{0^+} I^{\alpha}_{0^+} f(t) = f(t)$$

for almost all $t \in [0,1]$. If function f is continuous, then the composition rules hold for all $t \in [0,1]$.

Proposition 1.5. *[2] Let $\beta \in \mathbb{R}_+$ and $p \geq 1$ The fractional integral operator $I^{\beta}_{a^+}$ is bounded in $L^p(a,b)$:*

$$\|I^{\beta}_{a^+}\|_{L^p} \leq K_\beta \|f\|_{L^p}, \quad K_\beta = \frac{(b-a)^\beta}{\Gamma(\beta+1)}.$$

2 Main results

We shall replace the analysis of the unbounded Sturm- Liouville operator from 1 (denoted as L) with the inverse integral and bounded operator (denoted as T) The following is a displayed formula with a number to being able to refer to it, like formula

$$Lu(x) = (-{}^{c}D^{\alpha}_{0^+} + q(x))u(x) = \lambda u(x)$$

$$\frac{1}{\lambda} u(x) = -I^{\alpha}_{a^+} F_\lambda(u(x)) + I^{\alpha}_{a^+} F_\lambda(u(x))|_{x=0} x = Tu(x)$$

Where

$$F_\lambda u(x) = (q(x) - \lambda)u(x)$$

Let us observe that operator T can be expressed as the following integral operator with kernel K

$$u(x) = \int_0^1 K(x,s)u(s)ds$$

where the form of the kernel is:

$$K(x,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -(x-s)^{\alpha-1} + x(1-s)^{\alpha-1}, & s < x \\ (1-s)^{\alpha-1} x, & x < s \end{cases} \tag{2}$$

Theorem 2.1. *Fractional Sturm-Liouville operator T is compact.*

Proof. First, let us note that operator T is correctly defined as an operator mapping $L^2(a,b) \to L^2(a,b)$. In order to prove its compactness, it is enough to show that

$$\int_0^1 \int_0^1 K^2(x,s)dxds < \infty \tag{3}$$

This integral can be rewritten as

$$\int_0^1 \int_0^1 K^2(x,s)dxds = \int_0^1 \Big[\int_0^x K^2(x,s)ds + \int_x^1 K^2(x,s)ds\Big]dx \tag{4}$$

For the first of the above integrals, we have the following valid inequality:

$$\int_0^x K^2(x,s)ds < \frac{2}{[\Gamma(\alpha+1)]^2}$$

and

$$\int_x^1 K^2(x,s)ds < \frac{2}{[\Gamma(\alpha+1)]^2}$$

By applying the above derived estimations for parts of integral 4, we obtain for the kernel of operator T the inequality:

$$\int_0^1 \int_0^1 K^2(x,s)dxds < \infty$$

which implies that T is indeed a compact operator on $L^2(a,b)$. □

Remark 2.2. Let us observe that in the case $\alpha > 1$, operator T defined using the kernel given in 2 is also compact. This fact results from the fact that the kernel is then a function continuous in $[0,1] \times [0,1]$. Thus 3 is fulfilled.

Theorem 2.3. *The unique continuous eigen-function y_λ for fractional Sturm-Liouville problem with potential 1 corresponding to each eigenvalue obeying*

$$\|q-\lambda\| \leq \frac{1}{M_\varphi + \varphi(1)} \tag{5}$$

exists and such an eigenvalue is simple. where

$$\varphi(x) = I_{0^+}^{\alpha} 1 = \frac{-(x-t)^\alpha}{\Gamma(\alpha+1)}\Big|_0^x = \frac{x^\alpha}{\Gamma(\alpha+1)}, M_\varphi = \|\varphi(x)\|$$

Proof. We have to say that equation

$$y_\lambda = -I_{a^+}^{\alpha} F_\lambda(y) + I_{a^+}^{\alpha} F_\lambda(y)|_{x=0} x \tag{6}$$

can be interpreted as a fixed point condition on the function space $C[0,1]$,

$$y_\lambda = Ty_\lambda,$$

where the mapping on the right-hand side for any continuous function $g \in C[0,1]$ is defined as

$$g_\lambda = -I^{\alpha}_{a^+}F_\lambda(g) + I^{\alpha}_{a^+}F_\lambda(g)|_{x=0}x$$

The following inequality will be useful in further estimations:

$$\|F_\lambda(g) - F_\lambda(r)\| = \|(q-\lambda)g - (q-\lambda)r\| \leq \|q-\lambda\|\|g-r\|$$

By performing necessary operations for the distance between images Tg and Tr for a pair of arbitrary continuous functions $g, r \in C[0,1]$,

$$\begin{aligned}\|Tg - Tr\| &= \Big\| - I^{\alpha}_{0+}F_\lambda(g) + I^{\alpha}_{0+}F_\lambda(r)\Big|_{x=1} x + I^{\alpha}_{0+}F_\lambda(r) - I^{\alpha}_{0+}F_\lambda(r)\Big|_{x=1} x\Big\| \\ &\leq \Big\|I^{\alpha}_{0+}\Big(F_\lambda(g) - F_\lambda(r)\Big) - I^{\alpha}_{0+}\Big(F_\lambda(g) - F_\lambda(r)\Big)\Big|_{x=1} x\Big\| \\ &\leq \|g-r\|\|q-\lambda\|\big\|\varphi(x) - \varphi(1)\big\| \\ &\leq \|g-r\|\|q-\lambda\|\Big(M_\varphi + \varphi(1)\Big) \leq L\|g-r\|\end{aligned}$$

where constant $L = \|q-\lambda\|\Big(M_\varphi + \varphi(1)\Big)$. By means of 5, we state that mapping T is a contraction on the space $(C[0,1], \|.\|)$

$$\|Tg - Tr\| \leq L\|g-r\|, \quad L \in (0,1)$$

Hence, a unique fixed point enounced as$y_\lambda \in C[0,1]$ exists that solves equation 6, 1 provided 5 is applied. In that case, such eigenvalues are simple. The proof is completed. □

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Talk Spectral solutions of time fractional telegraph equations pp.: 1–4

Spectral solutions of time fractional telegraph equations

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Abstract

In this paper, A spectral scheme is proposed to approximate the solution of time fractional telegraph equations. Eigenfunctions of second order self-adjoint differential operator are used for discretization of spatial variable and Shifted Legendre polynomials are applied to discretization of time variable. Numerical results are presented for some problems to demonstrate the usefulness and accuracy of this approach. The method is easy to apply and produces very accurate numerical results.

Keywords: Fractional telegraph equation, Spectral method, Fractional differential operational matrix, Shifted Legendre polynomial
Mathematics Subject Classification [2010]: 34A08, 65M70

1 Introduction

Consider time fractional telegraph equation as

$$D_c^{\beta}U(x,t)+k_1D_c^{\alpha}U(x,t)+k_2U(x,t)=\frac{\partial^2 U}{\partial x^2}(x,t)+f(x,t), \tag{1}$$
$$(x,t)\in[0,1]\times[0,1],\quad 0<\alpha\le 1<\beta\le 2,$$

subject to the homogeneous boundary condition

$$U(x,0)=U(x,1)=0, \tag{2}$$

and the initial condition

$$U(x,0)=f_0(x),\quad U_t(x,0)=f_1(x), \tag{3}$$

which k_1 and k_2 are constant and D_c^{α} is the Caputo-type fractional derivative of order α. These equations , when $\beta=2$ and $\alpha=1$, are commonly used in the study of wave propagation of electric signals in a cable transmission line and also in wave phenomena. And also they have been used in modeling the reactiondiffusion processes in various branches of engineering sciences and biological sciences by many researchers (see [1] and references therein).
The advantage of fractional derivatives [2, 3] become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties

*Speaker

of rocks, and in many other fields. So we consider the equation with fractional derivative on time variable.
Spectral methods play an important role in recent researches for numerical solution of differential equations in regular domains. These methods have shown their efficiency and convergence in solving numerous problems [4]. In [5], Saadatmandi and Dehghan proposed an operational matrix of derivatives with fractional order for Legendre polynomials and used it for solving ordinary fractional differential equation.
In this article, first, eigenfunctions of second order self-adjoint differential operator are used for discretization of spatial variable and reduce the problem to a system of fractional differential equation then shifted Lengendre tau method is applied to solve this system.

2 Description of the proposed scheme

As we know, the operator $L = \frac{\partial^2}{\partial x^2}$, on the defined domain as $D = \{v \in L^2([0,1]) | v \text{ satisfy } (3)\}$, is self-adjoint which lead to a countable infinite set of real eigenvalues $\{\lambda_m = -(m\pi)^2\}$ and corresponds to the set of orthonormal eigenfunctions $v_m(x) = \sqrt{2}\sin(m\pi x)$.

By expanding the function $U(x,t)$ and $f(x,t)$ in terms of the finite eigenfunctions $v_m(x)$ of the operator L as

$$U(x,t) = \sum_{m=1}^{M} u_m(t)v_m(x) = \vec{U}^T(t)\vec{V(x)}, \tag{4}$$

$$f(x,t) = \sum_{m=1}^{M} < f(x,t), v_m(x) > v_m(x) = \vec{F}^T(t)\vec{V(x)}, \tag{5}$$

where

$$\vec{U(t)}^T(t) = [u_1(t), u_2(t), \cdots, u_M(t)], \tag{6}$$

$$\vec{V(x)}^T(x) = [v_1(x), v_2(x), \cdots, v_m(x)], \tag{7}$$

$$\vec{F(t)}^T(t) = [< f, v_1 >, < f, v_2 >, \cdots, < f, v_M >], \tag{8}$$

which $< f, v_m >= \int_0^1 f(x,t)v_m(x)dx$ and T stands for a vector transpose and the definition of eigenfunctions, the Eq. (1) will be transformed to

$$D_c^\beta \vec{U}^T(t)\vec{V(x)} + k_1 D_c^\alpha \vec{U}^T(t)\vec{V(x)} + k_2\vec{U}^T(t)\vec{V(x)} = \vec{U}^T(t)\Lambda_M\vec{V(x)} + \vec{F}^T(t)\vec{V(x)}, \tag{9}$$

which Λ_M is a $M \times M$ diagonal matrix that is obtained from eigenfunction definition with corresponding eigenvalues of $\frac{\partial^2}{\partial x^2}$ on its diagonal i. e.

$$\Lambda_M = \text{ diag } [\lambda_m = -(m\pi)^2], \quad m = 1, 2, \cdots, M.$$

Taking the dot product of resulting expression (9) by $\vec{V(x)}$, and integrating with respect to x over the $[0,1]$, and imposing transpose operator on the result follows

$$D_c^\beta \vec{U}(t) + k_1 D_c^\alpha \vec{U}(t) + k_2\vec{U}(t) = \Lambda_M\vec{U}(t) + \vec{F}(t). \tag{10}$$

Up to Now, we have a system of fractional differential equations with M equations that we want to solve it by shifted Legendre tau method. Shifted Legendre polynomials and their operational matrix of fractional differential equation are presented in [5] so we refer enthusiastic reader to it for more details. They define on the interval $[0,1]$ and can be obtained as follows:

$$p_{i+1}(t) = \frac{(2i+1)(2t-1)}{(i+1)} p_i(t) - \frac{i}{i+1} p_{i-1}(t), \quad i = 1, 2, ..., \tag{11}$$

where $p_0(t) = 1$ and $p_1(x) = 2t - 1$.
Suppose

$$\vec{U}(t) = A.L(t), \tag{12}$$

which A is a $M \times (N+1)$ unknown matrix and $L(t) = [p_0(t), p_1(t), ..., p_N(t)]^T$ is vector of shifted Legendre polynomials (11). Substituting (12) in (10) and using fractional operational matrix of shifted Legendre polynomials [5] lead to the following residual

$$R \simeq (A.\mathbf{D}^{(\beta)} + k_1 A.\mathbf{D}^{(\alpha)} + k_2 A - \Lambda_M.A)L(t) - F(t), \tag{13}$$

where $\mathbf{D}^{(\alpha)}$ and $\mathbf{D}^{(\beta)}$ are operational matrix of fractional differential operator of shifted Legendre polynomial proposed in [5].
As in a typical tau method [4, 5] we generate $M(N-1)$ linear equations, and with a similar process from initial conditions we obtain $2M$ equations. So we reach to a linear system that can be easily solved. Now , numerical results are presented for some problems to demonstrate the usefulness and accuracy of this approach.
Note: We use the first (N-1) shifted Legendre polynomials as test function.

Example 2.1. Consider the problem Eqs. (1)-(3) with following assumption

$$\begin{aligned} &\alpha = \frac{3}{4}, \quad \beta = \frac{7}{4}, \\ &k_1 = 1 = k_2, \\ &f_0(x) = \sin(\pi x), \quad f_1(x) = 0, \end{aligned} \tag{14}$$

and exact solution as $U(x,t) = \cos(t) sin(\pi x)$.

Solution. The absolute error of founded approximated solution of problem by our scheme with $N = 6$ and $M = 10$ for some point shown in Table 1. And also 3D plot of the error $[U(x,t) - U_{M,N}(x,t)]$ is presented in Figure 1. This typical example and other examples clearly imply the accuracy and effectiveness of our scheme.

Table 1: The absolute errors of solution for M = 10, N = 7 for Example 2.1

x	t=0.25	t=0.5	t=0.75	t=1
0.25	$2.1 * 10^{-10}$	$8.9 * 10^{-9}$	$1.4 * 10^{-8}$	$1.6 * 10^{-8}$
0.5	$3 * 10^{-10}$	$1.3 * 10^{-8}$	$2 * 10^{-8}$	$2.3 * 10^{-8}$
0.75	$2.1 * 10^{-10}$	$8.9 * 10^{-9}$	$1.4 * 10^{-8}$	$1.6 * 10^{-8}$

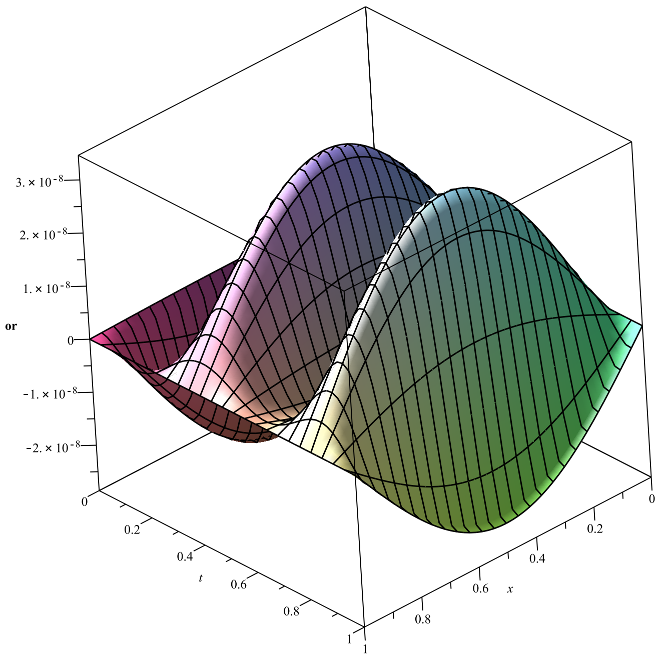

Figure 1: 3D plot of the error $[U(x,t) - U_{M,N}(x,t)]$ for M = 10, N = 7

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Steklov problem for a three-dimensional Helmholtz equation in bounded domain

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Abstract

This paper is devoted to study of solutions of a Steklov problem for a three-dimensional Helmholtz equation with an eigenvalue parameter λ in the non-local boundary conditions on the two-party smooth boundary of a connected bounded domain.The derived necessary conditions construct a system of second kind Fredholm integral equations with multi-dimensional singular integrals. Finally,a new method for regularization of these singularities is represented.

Keywords: Steklov problem, Fundamental solution, Fredholm integral equation
Mathematics Subject Classification [2010]: 45C99, 45B05

1 Introduction

Our method for investigation of Steklov problem has been used for the first and second order elliptic equations such as Cauchy-Riemann and Laplace equations, respectively, in a two-dimensional bounded domain [1], [2] and we apply this method for a three-dimensional Helmholtz equation

$$Lu(x) = (\Delta + k^2)u(x) = 0 \qquad in \quad \Omega \subset \mathbb{R}^3, \tag{1}$$

with the non-local boundary conditions

$$l_j u(x) = \sum_{k=1}^{3} [\alpha_{jk}(x') \frac{\partial u(x)}{\partial x_k}|_{x_3=\gamma_1(x')} + \beta_{jk}(x') \frac{\partial u(x)}{\partial x_k}|_{x_3=\gamma_2(x')}] = \lambda u(x', \gamma_j(x')) \quad x' \in S,$$
$$u(x', \gamma_j(x')) = 0 \quad j = 1, 2, \; x' \in \partial S. \tag{2}$$

where Ω is a simply connected bounded domain in $\mathbb{R}^3$ and its boundary Γ, is in the form of Lyapunov surface which contains two parts; $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 : \; x_3 = \gamma_1(x')$, $\Gamma_2 : \; x_3 = \gamma_2(x')$ such that $\gamma_2(x') < \gamma_1(x')$, $x' \in S$ and S is the projection of the domain Ω on the plane Ox_1x_2.
Here Δ is the Laplace operator in $\mathbb{R}^3$, $\lambda \in \mathbb{C}$ is a spectral parameter and $\alpha_{jk}(x')$, $\beta_{jk}(x')$ $j = 1, 2$, $k = 1, 2, 3$, are given sufficiently smooth functions.

*Speaker

1.1 Necessary conditions

By means of the fundamental solution $U(x-\xi)$ of (1) which is given as follows [3],

$$U(x-\xi) = -\frac{1}{4\pi|x-\xi|}e^{ik|x-\xi|}. \tag{3}$$

we get the first necessary condition:

$$\begin{aligned}\int_\Gamma u(x)\frac{\partial U(x-\xi)}{\partial n}\,dx - \int_\Gamma U(x-\xi)\frac{\partial u(x)}{\partial n}\,dx &= \int_\Omega (\Delta + k^2)U(x-\xi)u(x)\,dx \\ &= \int_\Omega \delta(x-\xi)u(x)\,dx = \begin{cases} u(\xi), & \xi\in\Omega, \\ 1/2\,u(\xi), & \xi\in\Gamma, \\ 0, & \xi\notin\bar{\Omega}, \end{cases}\end{aligned} \tag{4}$$

In a similar way, the rest of three necessary conditions are obtained;

$$\begin{aligned}&\int_\Gamma \frac{\partial u(x)}{\partial x_j}\frac{\partial U(x-\xi)}{\partial n}\,dx + \int_\Gamma \frac{\partial U(x-\xi)}{\partial x_j}\frac{\partial u(x)}{\partial n}\,dx + k^2\int_\Gamma u(x)U(x-\xi)\cos(n,x_j)dx \\ &-\int_\Gamma \cos(n,x_j)\nabla u(x)\cdot\nabla U(x-\xi)\,dx = \int_\Omega \delta(x-\xi)\frac{\partial u(x)}{\partial x_j}\,dx = \begin{cases} \frac{\partial u(\xi)}{\partial \xi_j}, & \xi\in\Omega, \\ 1/2\,\frac{\partial u(\xi)}{\partial \xi_j}, & \xi\in\Gamma, \\ 0, & \xi\notin\bar{\Omega}, \end{cases}\end{aligned} \tag{5}$$

where n is the outer unit normal vector on Γ and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain along the direction x_3 with its boundary Γ which is a Lyapunov's surface. Then any solution of (1) in Ω satisfies in the necessary conditions (4) and (5).*

1.2 Separation of singularities and regularization

Computing of the first order derivatives of (3) we obtain

$$\begin{aligned}u(\xi) &= \frac{1}{2\pi}\int_\Gamma e^{ik|x-\xi|}\left(\frac{1}{|x-\xi|^2} - \frac{ik}{|x-\xi|}\right)u(x)\cos(x-\xi,n)\,dx \\ &+ \frac{1}{2\pi}\int_\Gamma \frac{e^{ik|x-\xi|}}{|x-\xi|}\frac{\partial u(x)}{\partial n}\,dx, \quad \xi\in\Gamma.\end{aligned} \tag{6}$$

Theorem 1.2. *On the conditions of theorem 1.1,the obtained first necessary (6) is regular.*

On the other hand

$$\frac{\partial u(\xi)}{\partial \xi_j} = \frac{1}{2\pi}\int_\Gamma \frac{e^{ik|x-\xi|}}{|x-\xi|^2}\left(\sum_{m=1}^{3}\frac{\partial u(x)}{\partial x_m}K_{jm}(x,\xi)\right)dx + \cdots. \tag{7}$$

where

$$K_{jm}(x,\xi) = \cos(x-\xi,x_j)\cos(n,x_m) - \cos(x-\xi,x_m)\cos(n,x_j); \quad j,m = 1,2,3.$$

and "$\cdots$" denotes all of integrals with weak singularities.

Theorem 1.3. *On the conditions of theorem 1.1, the necessary conditions (7) are singular.*

Opening the surface integrals in (7), we obtain

$$\frac{\partial u(\xi)}{\partial \xi_j}|_{\xi_3=\gamma_p(\xi')} = \frac{(-1)^{p-1}}{2\pi}\int_S \frac{e^{ik|x-\xi|_p}}{|x'-\xi'|^2 L_p(x',\xi')} K_{jm}^{(p)}(x',\xi')\frac{\partial u(x)}{\partial x_m}|_{x_3=\gamma_p(x')}\frac{dx'}{\cos(n_p,x_3)}+ \frac{(-1)^{p-1}}{2\pi}\int_S \frac{e^{ik|x-\xi|_p}}{|x'-\xi'|^2 L_p(x',\xi')} K_{jn}^{(p)}(x',\xi')\frac{\partial u(x)}{\partial x_n}|_{x_3=\gamma_p(x')}\frac{dx'}{\cos(n_p,x_3)}+\cdots, \quad (8)$$

where

$$K_{jm}^{(p)}(x',\xi') = K_{jm}(x,\xi)|_{\substack{x_3=\gamma_p(x')\\ \xi_3=\gamma_p(\xi')}} \qquad p=1,2, \quad m,n,j=1,2,3;\ m,n\neq j$$

$$|x-\xi|_p = |(x-\xi)_p|; (x-\xi)_p = (x_1-\xi_1, x_2-\xi_2, \gamma_p(x')-\gamma_p(\xi')),$$

$$L_p(x',\xi') = 1+\sum_{m=1}^{2}(\frac{\partial\gamma_p(x')}{\partial x_m})^2\cos^2(x'-\xi',x_m)+O(|x'-\xi'|).$$

Constructing special linear combinations of necessary conditions (8), applying the boundary conditions (2) and finally regularization them by a new method, we get the following theorem:

Theorem 1.4. *In boundary problem (1)-(2),if the coefficients $\alpha_{jk}(x');\ j=1,2,\ \ k=1,2,3$ belong to some Holder's class, then the obtained linear combinations of (8) are regular.*

2 Main results

Under conditions theorem 1.4, we obtain a system of six second order Fredholm integral equations for the boundary values of first order derivatives which are regular. Finally, we combine the system with regular necessary conditions (6) and get the system of second kind Fredholm integral equations with respect to the eight unknowns

$$u(\xi_1,\xi_2,\gamma_p(\xi_1,\xi_2)), \quad \frac{\partial u(\xi)}{\partial \xi_1}|_{\xi_3=\gamma_p(\xi_1,\xi_2)}, \quad \frac{\partial u(\xi)}{\partial \xi_2}|_{\xi_3=\gamma_p(\xi_1,\xi_2)}; \quad p=1,2. \quad (9)$$

So, the boundary problem (1)-(2) is reduced to the system of second Fredholm integral equations with unknowns (9) which has no singularity in the kernel.

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Using Chebyshev polynomials zeros as mesh points for numerical solution of linear and nonlinear PDEs by differential quadrature method- based RBFs

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Abstract

In this paper Differential Quadrature (DQ) method- based Radial Basis Functions (RBFs) is applied to find the numerical solution of the linear and nonlinear Partial Differential Equations (PDEs). The Multiquadric (MQ) RBF as basis function will introduce and applied to discretized PDEs. DQ method will introduce briefly and then we obtain the numerical solution of the PDEs by propose DQ method.

Keywords: Radial basis function, Differential quadrature, Partial differential equation
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Definition 1.1. If f be a real value function that defined on the real line $\mathbf{R}$, then the function $\varphi : \mathbf{R^d} \longrightarrow \mathbf{R}$ that $\varphi(r_j) = f(r_j)$ and $r_j = \|x - x_j\|$, $x, x_j \in \mathbf{R^d}$ is said a radial function.

$\|.\|$ Is the Euclidian norm and $x_j \in \mathbf{R^d}$ is a special mesh point and called the center of radial function. Some of RBFs has a shape parameter c and we named them parametric RBFs. Parametric RBFs are smooth and infinitely differentiable. In interpolation or solving PDEs with RBFs, their system matrix is nonsingular and hence the problem of solving PDE with RBFs has a unique solution. It is well known that the value of c strongly influences the accuracy of approximation solution. Thus, there exists a problem of how to select a "good" value of c so that the numerical solution of PDEs can achieve satisfactory accuracy. In general, there are three main factors that could affect the optimal shape parameter c for giving the most accurate results. These three factors are the scale of supporting region, the number of supporting nodes, and the distribution of supporting nodes [1]. Most popular RBFs are shown in Table 1.

*Speaker

Table 1: Most popular RBFs

RBF Name	formula
Multiquadric	$\varphi(r_j) = \sqrt{r_j^2 + c^2}$
Inverse multiquadric	$\varphi(r_j) = \frac{1}{\sqrt{r_j^2+c^2}}$
Thin plate spline	$\varphi(r_j) = r_j^2 \ln(r_j^2 + c^2)$
Gaussian	$\varphi(r_j) = e^{-cr_j^2}$

2 Differential Quadrature (DQ) method

The DQ method was introduced by Richard Bellman and his associates in the early of 1970s [3,4]. The basic idea of the DQ method is that any derivative at a mesh point can be approximated by a weighted linear sum of all the functional values along a mesh line [4]. Currently, the DQ method has been extensively applied in engineering. DQ method is a numerical method for solving PDEs or ODEs. In this method, we approximate the spatial derivatives of the function f at mesh points $x_j \in \mathbf{R^d}$ using linear weighted sum of all the functional values at points in the domain of the problem. We assume N grid points on the real axis with step length. The discretization of the nth and the mth order derivatives by DQ method at a point (x_i, y_i) with respect to x and y, respectively, is given by the below equations that $f_x^{(n)}$ is nth order derivative of f with respect to x and $f_y^{(m)}$ is mth order derivative of f with respect to y.

$$f_x^{(n)}(x_i, y_i) = \sum_{j=1}^{N} w_{ij}^{(n)} f(x_j, y_j) \qquad , \qquad i = 1, 2, \ldots, N \tag{1}$$

$$f_y^{(m)}(x_i, y_i) = \sum_{j=1}^{N} v_{ij}^{(m)} f(x_j, y_j) \qquad , \qquad i = 1, 2, \ldots, N \tag{2}$$

Where $w_{ij}^{(n)}$ and $v_{ij}^{(m)}$ are unknown weighting coefficients. There are many approaches to find these coefficients such as Bellmans approaches [5] and Shu's approach [1]. From these approaches, Shu's approach is very general approach in the recent years. The function $f(x, y)$ in above equations is called test functions and for obtain the weighting coefficients we need a suitable test function. Some of the most general test functions are: Legendre polynomials, Lagrange interpolation polynomials, Lagrange interpolated cosine and RBFs. We use RBFs and in particular Multiquadric (MQ) as test functions. For obtaining the coefficients $w_{ij}^{(n)}$ and $v_{ij}^{(m)}$ we substitute the function MQ with equation

$$\varphi_k(x, y) = \sqrt{(x - x_k)^2 + (y - y_k)^2 + c^2}$$

In the equations (1) and (2) and obtain the below equations:

$$\varphi_{kx}^{(n)}(x_i, y_i) = \sum_{j=1}^{N} w_{ij}^{(n)} \varphi_k(x_j, y_j) \qquad , \qquad i = 1, 2, \ldots, N \tag{3}$$

$$\varphi_{ky}^{(n)}(x_i, y_i) = \sum_{j=1}^{N} v_{ij}^{(m)} \varphi_k(x_j, y_j) \qquad , \qquad i = 1, 2, \ldots, N \tag{4}$$

That $\varphi_{kx}^{(n)}$ and $\varphi_{ky}^{(m)}$ are nth and mth order derivatives of φ_k with respect to x and y respectively. For the any given i, any of equation systems of (3) and (4) has N unknowns with N equations. So, with solving this equation system, we can obtain the weighting coefficients.

3 Numerical Examples

Example 3.1. Consider the 2-dimansion Poisson equation in a square domain $[-1, 1] \times [-1, 1]$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2\pi^2 \sin(\pi x) \sin(\pi y)$$

With below exact solution

$$u(x, y) = \sin \pi x \sin \pi y$$

The numerical solution by propose method is evaluate for this example, and existence of analytical solution helps to measure the accuracy of numerical method. The numerical computations have been done with the help of Matlab software. In numerical experiments L_2 error is calculated by formula $L_2 = \sqrt{\Sigma_{i=1}^{N}(u_i - \overline{u}_i)^2}$, where u_i is exact solution and $\overline{u}_i$ is numerical solution. The numerical results are shown in Table 2.

Table 2: Numerical result for linear PDE with DQ method

N	L_2	Optimal c
4	6.9902×10^{-2}	0.1066
9	3.4423×10^{-3}	0.4233
49	5.02182×10^{-5}	0.9218
64	1.2641×10^{-4}	1.0315
100	9.5520×10^{-3}	1.4252

From Tables 2, It can be seen that within the certain number of mesh points, the accuracy of numerical results can be improved by increasing the number of mesh points. However, when the number of mesh points is further increased after a critical value, the accuracy of numerical results is decreased. The reason may be due to the fact that, when the number of mesh points is increased, the condition number of the matrix becomes very large and the matrix tends to be ill-conditioned.

Example 3.2. Consider the below 2-dimansion nonlinear PDE that we suppose that $[-1, 1] \times [-1, 1]$ be the domain of this problem.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}) - 2(x + y)u = 4 \tag{5}$$

With the below Dirichlet boundary condition for the four edges of the square domain

$$\begin{cases} u(x = -1) = 1 + y^2 \quad , \quad u(x = 1) = 1 + y^2 \\ u(y = -1) = 1 + x^2 \quad , \quad u(y = 1) = 1 + x^2 \end{cases} \tag{6}$$

The exact solution of this problem is $u(x, y) = x^2 + y^2$. This problem is nonlinear and hence, the system of equations that we obtain from discretization of (5) is nonlinear. we have discretized the equation (5) as follow

$$\frac{\partial^2 u(x_i, y_i)}{\partial x^2} + \frac{\partial^2 u(x_i, y_i)}{\partial y^2} + u(\frac{\partial u(x_i, y_i)}{\partial x} + \frac{\partial u(x_i, y_i)}{\partial y}) - 2(x_i + y_i)u(x_i, y_i) = 4 \quad (7)$$

That, in (7) we have $i = 1, 2, \ldots, N$. Now with applying DQ method we have:

$$\sum_{j=1}^{N} a_{ij}u_j + u_i \sum_{j=1}^{N} b_{ij}u_j - 2(x_i + y_i)u_i = 4 \quad , \quad i = 1, 2, \ldots, N \quad (8)$$

And in above we have $a_{ij} = w_{ij}^{(2)} + v_{ij}^{(2)}$ and $b_{ij} = w_{ij}^{(1)} + v_{ij}^{(1)}$. However, (8) is a nonlinear system of equations and we solved it with Jacobi iteration method and obtained the numerical results in Table 3.

Table 3: Numerical result for nonlinear PDE with DQ method

N	L_2	Optimal c
4	3.2504×10^{-2}	0.1066
9	9.0273×10^{-3}	0.4233
49	1.0883×10^{-4}	0.9218
64	5.6075×10^{-2}	1.0315
100	9.0471×10^{-2}	1.4252

From Table 2, we see that the optimal shape parameter c achieved in the example (3.1) works very well for the nonlinear case.

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Mathematical Finance

Arbitrage and curvature

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Abstract

In this article we describe the relation between the financial concept of arbitrage and the geometric concept of curvature. To this end we construct the projective space corresponding to the financial market. Using This space as a manifold we study parallel transports along the paths on this manifold which has a nice relatioship with both curvatures of paths on the manifold and arbitrage on the market. It is shown that existence of arbitrage corresponds to the non-zero curvature of a path on the manifold.

Keywords: Arbitrage, Curvature, relative price, projective market
Mathematics Subject Classification [2010]: 91B70, 91B24, 91B25

1 Introduction

The field of mathematical finance is an active field of research that unifies subjects from other fields such as economics, calculus, differential equations, stochastic processes, differential geometry, and physics. The idea of linking the concepts arbitrage and curvature is due to Iliniski who introduced the topic "Gauge theory of arbitrage"[8]. This topic has grown up to a branch in mathematical finance in the last decade due to the works of Morisawa [9], Farinelli [5] and many others. Also, there is a bridge between arbitrage theory and classical Kirchhoff theory of electrical circuits introduced by Ellerman [4].

2 Arbitrage

Arbitrage is one of the keywords of the theory of finance. It is a situation in market that makes it possible to earn money without taking any risk or even without any real investment. Mispricing, misdistribution of information, political events, inefficiency of the market are some origions of arbitrage possibility. Arbitrage possibility is the possibility of instants. Arbitrage is it's own enemy and disappears soonly by market arbitraguers and speculators. They buy for low prices (so rise the prices) and sell in high prices(and so lower the prices), and soforce the market in equilibrium. They short sell, i.e. they first sell the asset (get loan) then deliver the asset(get money), after all pay the loan and get

*Speaker

the profit. The exess money they get as profit prevents the cycle to be closed. But partly we have:

$$get - loan \to get - money \to pay - loan$$

A real cycle can occur in exchange market. Let the spot rate of changing one USD into 1 PS at time t be $r(t)$, and the rate of interest of these moneies be r_1 and r_2 respectively, then after 1 unit of time we have the following partial cycle. We observe that they begin by 1 USD, do they end with the same PS's?

$$1(D) \to 1 + r_1 \to (1 + r_1)r(t)(P),$$

$$1(D) \to r(t) \to (1 + r_2)r(t + 1)(P).$$

Therefore the no-arbitrage (arbitrage-free) conditions in the market are very important to investigate. In fact our main question is: In terms of cycles when does there occur arbitrage possibility?

3 Geometry

In this section we will connect the notion of cycle above with the notion of connection, and hence parallel translation, and hence curvature. To do so let us fix a filtered probability space $(\Omega, F, \mathcal{F}, p)$, the so called objective probability space. Consider a market consisting of n risky assets and one non-risky asset, say bank account. Let $S_i(t), i = 1, \ldots, n$ be the price of the $i-$th asset and $S_0(t) > 0$ the price of the non-risky asset at time $t, 0 \leq t \leq T$. Let $h(t) = (h_1(t), \ldots, h_n(t)), 0 \leq t \leq T$ be a self financing portfolio of risky assets and $S(t) = (S_1(t), \ldots, S_n(t)), 0 \leq t \leq T$. Consider the curve $\alpha(t) = (h(t), S(t))$ and let $\sigma(h, S) = \sum_1^n f_i(h, S)e_i$, which associates to each vector-price an other portfolio. Now define the total differential of σ, the total differential of σ in the direction of $\dot{\alpha}$ as follows:

$$\nabla^0 \sigma = \sum_1^n df_i(h, S)e_i, \tag{1}$$

$$\nabla^0_{\dot{\alpha}} \sigma = \sum_{i=1}^n \{\sum_{j=1}^n \frac{\partial f_i}{\partial h_j}\frac{dh_j}{dt} + \frac{\partial f_i}{\partial S_j}\frac{dS_j}{dt}\}e_i. \tag{2}$$

Using these notions we will define the notion of economical covariant derivative of σ, and then proceed to study parallel translation and curvature properly.

4 Projective Market

By definition the price vector of the risky assets with respect to the numeraire $S_0(t) > 0$ is $Z = (Z_1(t), \ldots, Z_n(t))$, where $Z_i(t) = \frac{S_i(t)}{S_0(t)}$. The projective conterpart of this vector is:

$$Y = \frac{Z_1(t)}{||Z||}e_1 + \ldots + \frac{Z_n(t)}{||Z||}e_n, \tag{3}$$

where

$$||Z|| = \sqrt{Z_1(t)^2 + \ldots + Z_n(t)^2},$$

$e_i \in R^n$ is the unit vector with zero components every where except 1 at the $i-$th place.

Definition 4.1. The projective market consists of all traded assets whose projective price vector is an element of the projective space RP^n, such as Y in (1).

As the following proposition shows the projective market($P-$market) has the main properties of the $S-$market.

Theorem 4.2. *The $P-$market is*

- *arbitrage free if and only if the $S-$market is.*
- *self-financing if and only if the $S-$market is.*

Proof. The first assertion is the result of the $S-$ value process of a portfolio is positive if and only if it's $P-$ value process is positive.
To prove the second assertion we assume the $S-$ prices follow Ito diffusions and use stochastic differentiation formula. □

In $S-$market the price vector of the portfolio $H = (h_1, \dots, h_n)$ is $(h_1S_1, \dots, h_nS_n)$ which corresponds to the price vector

$$\frac{1}{||H.Z||}(h_1Z_1, \dots, h_nZ_n)$$

in $P-$market.

Definition 4.3. The distance between two price vectors $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ of the projective market is

$$d(X, Y) = cos^{-1}| < X, Y > |,$$

where $< X, Y >$ is the usual inner product in R^n.

We see that $d(X, Y)$ is the small arc length between X and Y in a plane passing through X , Y and the origion that intersects the unit sphere in a great circle part of which is the small arc mentioned above.

Definition 4.4. The transition probability passing from X to Y or vise versa is

$$p(X, Y) = Cos^2 d(X, Y) = \frac{1}{2}(1 + 2cos2d(X, Y)). \tag{4}$$

We observe that $p(X, Y) = 0$ if and only if the angle between X and Y is $\frac{\pi}{2}$, i.e. it is impossible to reach from X to Y in $P-$ market.
$p(X, Y) = 1$ if and only if $Y = X$ or $Y = -X$ i.e. (X, Y) is only one point in the projective market.
In the case $0 < p(X, Y) < 1$ one can reach from the portfolio X to portfolio Y with a positive probability.
consider a smooth curve γ on the projective space and let V_A be the tangent vector to γ at $A \in \gamma$. Parallel transport of V_A along γ to the point $B \in \gamma$ may not coincide to V_A, i.e. they have non-zero angle. This is called the angle of holonomy.

5 Main result

One of the most important notions in mathematical finance is the notion of arbitrage opportunity or in short arbitrage, which means getting profit without investing. It is nice to notice that due to it's nature arbitrage opportunity occur's in short time intervals and cannot live for a long time. Arbitraguers in the market buy low and sell high rapidly untill they bring the goods in equilibrium. On the contrary in a complete arbitrage free market every asset has a unique price. In the market it is not important which numeraire is used, this is termed as the prices are gague invariant. Mathematically this means that the transformations like

$$X(t) \to C(t)X(t)$$

do not affect the prices, they may change Rials to Tomans, but do not chan Choosing suitably the notions from manifold geometry one can formulate the relation between existence of arbitrage and non-zero curvature as follows.

Theorem 5.1. *The arbitrage opportunity in the market occurs if and only if some paths in the projctive market have non-zero curvature.*

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Numerical solution of stochastic optimal control problems: experiences from Merton portfolio selection model

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Abstract

In this paper, the variational iteration method (VIM), is applied for solving stochastic optimal control(SOC) problems. First, SOC problems are transferred to Hamilton-JacobiBellman (HJB) equation. Then, the basic VIM is applied to construct the value function and the corresponding optimal strategy. Also, we solve Merton's portfolio selection model as a problem of portfolio optimization to highlight the applications of SOC problems. Convergence of the method is proved by using Banach's fixed point theorem and some illustrative examples are presented to show the efficiency and reliability of the presented method.

Keywords: Stochastic optimal control(SOC) problems, Hamilton-Jacobi-Bellman (HJB) equation, Variational iteration method (VIM), Banach's fixed point theorem
Mathematics Subject Classification [2010]: 91G80, 93E20, 97M30

1 Introduction

Optimal controls models play a prominent role in a range of application areas, including aerospace, chemical engineering, robotic, economics and finance. It deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A controlled process is the solution to an ordinary differential equation which some parameters of the ordinary differential equation can be chosen. Hence, the trajectory of the solution is obtained. Each trajectory has an associated cost, and the optimal control problem is to minimize this cost over all choices of the control parameter. Stochastic optimal control is the stochastic extension of this; In fact, a stochastic differential equation with a control parameter is given. Each choice of the control parameter yields a different stochastic process as a solution to the stochastic differential equation. Each path wise trajectory of this stochastic process has an associated cost, and we seek to minimize the expected cost over all choices of the control parameter. Recently, Kushner has presented a survey of the early development of selected areas in nonlinear continuous-time stochastic control [1].

*Speaker

2 SOC problems and HJB Equation

Consider the following stochastic controlled system with initial condition:

$$\begin{cases} dX(t) = f(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dw(t), \\ X(s) = y, \end{cases} \tag{1}$$

where y is a given vector in $\mathbb{R}^n$. Also, $X(t)$ is the state process, $u(t)$ is the control process, $w(t)$ is a Wiener process, f is defined as a drift, and σ is diffusion. The optimal control rule μ, that determines the control u, is Markovian and is presented by $u(t) = \mu(t, x(t))$ and is chosen so as to minimize $J(s, y; u)$ where,

$$J(s, y; u) = \mathbb{E}_{sy}\left[\int_s^T L(\tau, X(\tau), u(\tau))d\tau + \psi(X(T))\right],$$

here, L is running cost and $\psi(x)$ is terminal cost. Principle of optimality, dynamic programming, was first proposed by Bellman; for details, see [2]. This lead to derive an equation for solving optimal control problems. In fact, a family of fixed initial point control problems is considered in dynamic programming. We can shown that V solves the HJB equation:

$$(HJB) \qquad \begin{cases} \frac{\partial V}{\partial t} + L(s, y, \phi) + f(s, y, \phi).D_yV + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(s, y, \phi)V_{y_iy_j} = 0, \\ V(T, x) = \psi(x), \end{cases} \tag{2}$$

Now it is clear how we might use this to solve a SOC problem; we first solve HJB equation to obtain $V(s, y)$ and hopefully in the process divine the optimal control $u^*(.)$ is found. HJB equation is a sufficient condition for optimality and it is not possible to solve this equation analytically. Thus finding an approximate solution is at least the most logical way to solve it. Here, solutions for the value function and the corresponding optimal strategies of a SOC are obtained by using VIM. Correction functional for equation (2) can be written as:

$$\begin{aligned} V_{n+1}(t, x) = V_n(t, x) + \int_t^T \lambda(\xi)\Big(\tfrac{\partial V_n(\xi,x)}{\partial \xi} + L(s, y, \phi) \\ +f(\xi, x, \phi).\tfrac{\partial \tilde{V}_n(\xi,x)}{\partial x} + \tfrac{1}{2}\sum_{i,j=1}^n a_{ij}(\xi, x, \phi)\tfrac{\partial^2 \tilde{V}_n(\xi,x)}{\partial x_i x_j})\Big)d\xi, \end{aligned} \tag{3}$$

Note that, this is a modified general Lagrange's multiplier method, presented by Inokuti [4]. This technique was proposed by He [5] and was successfully applied for solving deterministic optimal control problems [6]. In equation (3), $\lambda(\xi)$ is the Lagrange multiplier, here it may be a constant or a function of ξ, and $\tilde{V}_n$ is a restricted value with $\delta\tilde{V}_n = 0$. Taking the variation of both sides of (3) with respect to the independent variable V_n. After some detailed calculations, we obtain:

$$\delta V_{n+1}(t, x) = \delta V_n(t, x) + \delta\Big(\int_t^T \lambda(\xi)\frac{\partial V_n(\xi,x)}{\partial \xi}d\xi\Big), \tag{4}$$

by using $\delta H = 0$ where

$$H(s, y, D_yV, D_y^2V) = \min_{v\in U}\left[L(s, y, v) + f(s, y, v).D_yV + \tfrac{1}{2}Tr\left(a(s, y, v)D_y^2V\right)\right],$$

Integrating the integral of (4) by parts we obtain:

$$\delta V_{n+1} = \delta\Big(1 - \lambda(\xi)\Big|_{\xi=t}\Big)V_n - \delta\Big(\int_t^T \lambda'(\xi)V_n(\xi, x)d\xi\Big). \tag{5}$$

The extremum condition of V_{n+1} requires that $\delta V_{n+1} = 0$ then the left hand side of (5) is 0, and as a result the right hand side should be 0. This yields the stationary conditions

$(1-\lambda(\xi))|_{\xi=t} = 0$ and $(-\lambda(\xi)')|_{\xi=t} = 0$. This in turn gives $\lambda(\xi) = 1$. Substituting this value of the Lagrange multiplier into the functional (3) gives the iteration formula:

$$\begin{aligned} V_{n+1}(t,x) &= V_n(t,x) + \int_t^T \Big(\tfrac{\partial V_n(\xi,x)}{\partial \xi} + L(s,y,\phi) \\ &+ f(\xi,x,\phi).\tfrac{\partial \tilde{V}_n(\xi,x)}{\partial x} + \tfrac{1}{2}\textstyle\sum_{i,j=1}^n a_{ij}(\xi,x,\phi)\tfrac{\partial^2 \tilde{V}_n(\xi,x)}{\partial x_i x_j})\Big) d\xi, \end{aligned} \tag{6}$$

Now, with final condition $V(T,x) = \psi(x)$ of HJB equation. Considering the given condition, we can select the zeroth approximation $V_0(t,x) = \psi(x)$. The successive approximations $V_{n+1}(t,x), n \geq 0$ of the solution $V(t,x)$ will be obtained readily upon using correction functional (6) and by using any selective function V_0. With $\lambda(\xi)$ determined, then several approximations $V_n, n \geq 0$ follow immediately. Consequently, the exact solution may be obtained as $V(t,x) = \lim_{n\to\infty} V_n(t,x)$. Note that, theoretical treatment of the convergence of the approximated solution of the VIM has been considered in [6].

3 Merton's portfolio selection problem: Application of SOC problem in Financial Mathematics

Suppose we are an investor with two investment options. We can either invest money in a back with a fixed rate of return r, or we can invest money in a risky stock with an expected rate of return $\mu > r$ but with volatility σ. Let $u(s)$ be the proportion of our money invested in the stock at time s. Letting $x(s)$ be our money at time s, we have that $x(.)$ satisfies the following stochastic differential equation [7]:

$$\begin{cases} dX(t) = X(t)(r + u(t)(b-r))dt + X(t)u(t)\sigma dw(t), \\ X(0) = x_0 \end{cases}$$

Suppose we wish to maximize $F(X(T))$ where F is some concave utility function. For this we take $F(x) = \dfrac{1}{\gamma}X^\gamma, 0 < \gamma < 1$. That is, in our standard framework we seek to minimize the cost functional:

$$J(t,x;u) = \mathbb{E}_{sy}\{F(X(T))\}.$$

Here we take $U = L^\infty([0,T];[0,1])$. We compute,

$$H(t,x,V_x,V_{xx}) = \min_{v\in U}\big\{V_x\big(rx(t) + (b-r)x(t)v\big) + \frac{1}{2}\sigma^2 x^2 v^2 V_{xx}\big\},$$

which optimal control is as $u^* = \frac{V_x(r-b)}{x\sigma^2 V_{xx}}$, then, the HJB equation is as follow:

$$HJB \quad \begin{cases} V_t - \frac{(r-b)^2 V_x^2}{2\sigma^2 V_{xx}} + xrV_x = 0, \\ V(T,x) = \dfrac{1}{\gamma}x^\gamma. \end{cases} \tag{7}$$

The correction functional for this equation leads to the iteration formula,

$$V_{n+1}(t,x) = V_n(t,x) + \int_t^T \left(\frac{\partial V_n(\xi,x)}{\partial \xi} - \frac{(r-b)^2 (\frac{\partial V_n(\xi,x)}{\partial x})^2}{2\sigma^2 \frac{\partial^2 V_n(\xi,x)}{\partial x^2}} + xr\frac{\partial V_n(\xi,x)}{\partial x}\right) d\xi.$$

For the purpose of illustration, the following parameters have been chosen: $r = 0.05$, $b = 0.11$, $\sigma = 0.1$ and $\gamma = \frac{1}{2}$. In this case, we have selected $V_0(t,x) = 2\sqrt{x}$ from the given initial condition yields the successive approximations:

$$V_0(t,x) = 2\sqrt{x},$$
$$V_1(t,x) = (2.0725 - 0.0725\,t)\,\sqrt{x},$$
$$V_2(t,x) = \big(2.073814062 - 0.075128125\,t + 0.0013140625\,t^2\big)\sqrt{x},$$
$$V_3(t,x) = \big(2.073829940 - 0.07517575977\,t + 0.001361697266\,t^2 - 1.587825521 \times 10^{-5}\,t^3\big)\sqrt{x},$$
$$\vdots$$

We can calculate control variable approximately after choosing of an approximation for $V(t,x)$. The approximated solution for the performance index is $J = 2.073830086$ which is exact solution of J. The results show the advantage using proposed method for this problem.

4 Conclusion

A classical financial problem is the modeling of optimal investment-consumption decisions under uncertainty. This was solved in the pioneering work of Merton as an application of dynamic programming. In the Merton dynamic programming result a nonlinear differential equation is derived on the optimal controls. Here, stochastic optimal control problems are transferred to HJB equation as a nonlinear first order hyperbolic partial differential equation. Then, the basic VIM is applied to construct a nonlinear optimal feedback control law.

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Risk measure in a financial market

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Abstract

In this paper we extend the definition of risk measure from L^∞ to an arbitrary Polish space with special conditions. For this purpose we present a measure preserving transformation between two Polish spaces with special conditions.

Keywords: Polish space, Risk measure, Risk management, Transformation
Mathematics Subject Classification [2010]: 60Hxx, 60Bxx, 60Gxx

1 Introduction

Risk management is a very important concept in financial mathematics and specially in a financial market.
For managing risk in a financial market we need to compute risk measure in a financial market which in [1, 2, 4, 5] is defined on $\mathbb{L}^\infty$. In this paper we extend the definition of risk measure from $\mathbb{L}^\infty$ to an arbitrary uncountable Polish space. For this purpose we construct a measure preserving transformation between two Polish spaces which have special conditions.

2 Risk Measure

Risk measure is widely used as instrument to control risk. In fact risk measures assign a real number to a risk in a financial market. As usual in actuarial sciences we assume that X describes a potential loss, but we allow X to assume negative values. Let $(\Omega, \mathscr{F}, P)$ be a probability space and expectation of a random variable X with respect to P is denoted by $E[X]$.

Definition 2.1. [2, 3] Let X be the set of all functions $f : \Omega \to \mathbb{R}$. A mapping $\rho : X \to \mathbb{R}$ is called a risk measure if it has the following conditions.

- Monotonicity: If $X \leq Y$ then $\rho(X) \leq \rho(Y)$;
- Translation invariance: if $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) + m$;
- Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$;
- Positive homogeneity: if $\lambda > 0$, then $\rho(\lambda X) = \lambda\rho(X)$;

*Speaker

- Convexity: $\rho(\lambda X + (1-\lambda)Y) = \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $\lambda \in [0,1]$;
- Law invariance: If $P_X = P_Y$, then $\rho(X) = \rho(Y)$.

According to Artzner et al. [2] a functional is called a coherent risk measure, if it is monotone, translation invariant, subadditive and positively homogeneous. They show that any coherent risk measure has a representation

$$\rho(X) = \sup_{Q\in\mathcal{Q}} E_Q(X), \tag{1}$$

where $\mathcal{Q}$ is some set of probability measures. This means that $\rho(X)$ is the worst expected loss under Q, where Q varies over some set of probability measures. Follmer and Schied [6] introduced the weaker concept of ρ being a convex risk measure if it satisfies the condition of monotonicity, translation invariance and convexity. They show that any convex risk measure is of the form

$$\rho(X) = \sup_{Q\in\mathcal{Q}} (E_Q(X) - \alpha(Q)), \tag{2}$$

where is a penalty function, which can be chosen to be convex and lower semi-continuous with $\alpha(Q) \geq -\rho(0)$.

Definition 2.2. Let $(\Omega, \mathscr{F}, \mu)$ be a probability space. Call for a partition $\mathcal{P}$ of Ω consisting of elements of $\mathscr{F}$, $\sup_{I\in\mathcal{P}} \mu(I)$ the norm of $\mathcal{P}$, w.r.t. μ and denote it by $|\mathcal{P}|_\mu$.

Definition 2.3. [7] For a probability space $(\Omega, \mathscr{F}, \mu)$ a sequence $\{\triangle_n\}_{n\geq 1}$ of partitions of Ω is called a system of partitions if:

1. for each $n \geq 1$, $\triangle_n$ is a countable collection of elements of $\mathscr{F}$;
2. the collection $\cup_{n\geq 1}\triangle_n$ of subsets of Ω generates $\mathscr{F}$;
3. $\lim_{n\to\infty} |\triangle_n|_\mu = 0$.

Call a system of partitions decreasing if for each $n \geq 1$, $\triangle_{n+1}$ is a refinement of $\triangle_n$. Henceforth $\triangle_n$, $n \geq 1$, denotes a system of partitions of Ω.

Definition 2.4. For $\omega \in \Omega$, $n \geq 1$, let $In(\omega)$ be the unique element of $\triangle_n$ containing ω. Call the sequence $In(\omega)$, $n \geq 1$, the $\omega-$ tower in the system.

Remark 2.5. Euclidean spaces and more generally, locally compact second countable Hausdorff topological spaces and hence complete separable, i.e. Polish, metric spaces, with Borel $\sigma-$ algebras and diffuse probability measures, when they admit such measures, yield decreasing systems of partitions which generate the Borel $\sigma-$ algebra.

3 Main results

In this section we extend the definition 2.1. For this purpose we present some theorems.

Let $[0,1]$ be equipped by the Borel $\sigma-$ algebra $\mathbf{B}$ and the Lebesgue measure m. Let Ω be a Polish space and μ a non atomic probability measure on $\mathscr{F}$. Consider Ω and $[0,1]$ equipped by the system of partition Δ and Δ', respectively.

Theorem 3.1. *[7] There is a transformation* $\widehat{X} : \Omega \to [0,1]$ *which has the following properties:*

1. $\widehat{X}$ *yields a natural one to one correspondence between the collection of towers of* $\triangle$ *and* $\triangle'$*;*
2. $\widehat{X}$ *is* $\mathscr{F}$*-B measurable and in fact* $\widehat{X}^{-1}(B) = \mathscr{F}$ *;*
3. $\widehat{X}$ *transforms the measure* μ *on* Ω *to the Lebesgue measure* m *on* I_0*.*

Theorem 3.2. *Let* Ω_1 *and* Ω_2 *be uncountable and Polish spaces. Then there is a measure preserving transformation between them.*

Theorem 3.3. *By above theorem, the definition of risk measure is extendable from* L^∞ *to an arbitrary uncountable Polish space.*

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Stochastic Terminal Times in G-Backward Stochastic Differential Equations

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Abstract

In this paper, we study G-backward stochastic differential equations with random terminal time . We explain how to extend the results of the case of fixed terminal time to the case of a random terminal time. We present the existence and uniqueness of a solutions for G-backward stochastic differential equations with a random terminal time.

Keywords: G-expectation, G-Brownian motion, G-Backward stochastic differential equations, Random terminal time.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

We consider the G-backward stochastic differential equations with the random terminal time τ in the following form:

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)ds + \int_{t\wedge\tau}^{\tau} g(s, Y_s, Z_s)d\langle B\rangle_s - \int_{t\wedge\tau}^{\tau} Z_s dB_s - (K_\tau - K_{t\wedge\tau}), \tag{1}$$

where τ is a stopping time with respect to natural filtration $\mathbb{F}$, the processes Y, Z and K are unknown and the random functions f and g , said generators, and the random variable ξ, said terminal value, are given. We present the existence and uniqueness of a solution (Y, Z, K) for G-BSDE (1).

2 Preliminaries

Let Ω be a given set and let $\mathcal{H}$ be a linear space of random variables defined on Ω. We assume the functions on $\mathcal{H}$ are all bounded. Let $(\Omega, \mathcal{H}, \mathbb{E})$ be the G-expectation space. We denote by $lip(\mathbb{R}^n)$ the space of all bounded and Lipschitz real functions on $\mathbb{R}^n$.
In this paper we set $G(a) = \frac{1}{2}(a^+ - \sigma_0^2 a^-)$, where $a \in \mathbb{R}$ and $\sigma_0 \in [0, 1]$ is fixed. We extend some notations and conditions of the case of fixed terminal time to the case of a random terminal time.

*Speaker

Definition 2.1. [1] Let $\Omega = \mathbb{R}$ and $\mathcal{H} = lip(\mathbb{R})$, $X \in \mathcal{H}$ with the G**-normal distribution** (with mean $x \in \mathbb{R}$ and variance $t > 0$) is characterized by its G-expectation defined by

$$\mathbb{E}[\varphi(x + \sqrt{t}X)] = P_G^t(\varphi(x)) := u(t,x),$$

Where $\varphi \in lip(\mathbb{R})$ and $u = u(t,x)$ is a bounded continuous function on $[0,\infty) \times \mathbb{R}$ which is the solution of the following G-heat equation

$$\partial_t u - G(\partial^2_{xx} u) = 0, \quad u(0,x) = \varphi(x).$$

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$with $\omega_0 = 0$. We set, for each $t \in [0,\infty)$

$$\begin{aligned} W_t &:= \{\omega_{.\wedge t} : \omega \in \Omega\}, \\ F_t &:= \mathcal{B}_t(W) = \mathcal{B}(W_t), \\ F_{t^+} &:= \mathcal{B}_{t^+}(W) = \bigcap_{s>t} \mathcal{B}_s(W), \\ F &:= \bigvee_{s>t} F_s. \end{aligned}$$

Then (Ω, F) is the canonical space. Let $\mathbb{F}$ be the natural filtration generated by $\omega = (\omega_t)_{t\geq 0}$. This space is used throughout the rest of this paper.
Let τ be a stopping time with respect to $\mathbb{F}$ and let us assume that τ is finite. We consider the following space of random variables

$$l^0_{ip}(F_\tau) := \{X(\omega) = \varphi(\omega_{t_1}, \dots, \omega_{t_m}), \quad \forall m \geq 1,\ t_1, \dots, t_m \in [0, \tau(\omega)], \forall \varphi \in lip(\mathbb{R}^m)\}.$$

We further define $l^0_{ip}(F) := \bigcup_{n=1}^\infty l^0_{ip}(F_{n\wedge\tau})$.

Definition 2.2. [2] The canonical process $B_t(\omega) = \omega_t$ is called a G-Brownian motion under a nonlinear expectation $\mathbb{E}$ defined on $l^0_{ip}(F)$ if

1. For each $s, t \geq 0$ and $\psi \in lip(\mathbb{R})$, B_t and $B_{t+s} - B_s$ are identically distributed:

$$\mathbb{E}[\psi(B_{t+s} - B_s)] = \mathbb{E}[\psi(B_t)] = P^t_G(\psi).$$

2. For each $m = 1, 2, \cdots$, $0 \leq t_1 < t_2 < \cdots < t_m < \infty$, the increment $B_{t_m} - B_{t_{m-1}}$ is backwardly independent from $B_{t_1}, \cdots, B_{t_{m-1}}$ in the following sense: for each $\psi \in lip(\mathbb{R}^m)$,

$$\mathbb{E}[\psi(B_{t_1}, \cdots, B_{t_{m-1}}, B_{t_m})] = \mathbb{E}[\psi_1(B_{t_1}, \cdots, B_{t_{m-1}})],$$

where $\psi_1(x_1, \cdots, x_{m-1}) = \mathbb{E}[\psi(x_1, \cdots, x_{m-1}, B_{t_m} - B_{t_{m-1}} + x_{m-1})]$ and $x_1, \cdots, x_{m-1} \in \mathbb{R}$.

It is easy to check that $\mathbb{E}[.]$ defines a nonlinear expectation on the vector lattice $l^0_{ip}(F_\tau)$ as well as on $l^0_{ip}(F)$, It follows that $\mathbb{E}[|X|]$ where $X \in l^0_{ip}(F_\tau)$ (resp. $l^0_{ip}(F)$) forms a norm and that $l^0_{ip}(F_\tau)$ (resp. $l^0_{ip}(F)$) can be continuously extended to a Banach space, denoted by $L^1_G(F_\tau)$ (resp. $L^1_G(F)$). For a given $p > 1$, we also denote $L^p_G(F) = \{X \in L^1_G(F), |X|^p \in L^1_G(F)\}$. $L^p_G(F)$ is also a Banach space under the norm $||X||_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$.

Definition 2.3. Let $M_G^{p,0}(0,\tau)$ be the collection of processes in the following form: for a given partition $\pi_\tau = \{t_0,\dots,t_N\}$ of $[0,\tau(\omega)]$

$$\mu_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t),$$

Where $p \geq 1$ and $\xi_j \in L_G^p(F_{t_j})$, are given.

We need to introduce further notation. Let us consider $p > 1$ and $\alpha \in \mathbb{R}$. We set $||\eta||_{H_G^{p,\alpha}} = \left[\mathbb{E}\left[\left(\int_0^\infty e^{\alpha s}|\eta_s|^2 ds\right)^{\frac{p}{2}}\right]\right]^{\frac{1}{p}}$, $||\eta||_{M_G^p} = \left[\mathbb{E}\left[\int_0^\tau |\eta_s|^p ds\right]\right]^{\frac{1}{p}}$ and denote by $H_G^{p,\alpha}(\mathbb{R})$, $M_G^p(0,\tau)$ the completions of $M_G^{p,0}(0,\tau)$ under the norms $||\eta||_{H_G^{p,\alpha}}$, $||\eta||_{M_G^p}$ respectively.

Let $S_G^{p,0}(0,\tau) = \left\{h(t, B_{t_1\wedge t},\dots,B_{t_n\wedge t}) : t_1,\dots,t_n \in [0,\tau(\omega)], h \in lip(\mathbb{R}^{n+1})\right\}$. For $\eta \in S_G^{p,0}(0,\tau)$, set $||\eta||_{S_G^{p,\alpha,\tau}} = \left[\mathbb{E}\left[sup_{t\geq 0}\ e^{(p/2)\alpha(t\wedge\tau)}|\eta_t|^p\right]\right]^{\frac{1}{p}}$. Denote by $S_G^{p,\alpha,\tau}(\mathbb{R})$ the completion of $S_G^{p,0}(0,\tau)$ under the norm $||\eta||_{S_G^{p,\alpha,\tau}}$.

Definition 2.4. For each $\eta \in M_G^{2,0}(0,\tau)$ with the form $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t)$, we define

$$\mathcal{I}(\eta) = \int_0^\tau \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

The mapping $\mathcal{I} : M_G^{2,0}(0,\tau) \to L_G^2(F_\tau)$ is a linear continuous mapping and thus can be continuously extended to $\mathcal{I} : M_G^2(0,\tau) \to L_G^2(F_\tau)$.

Definition 2.5. We define, for a fixed $\eta \in M_G^2(0,\tau)$, the stochastic integral

$$\int_0^\tau \eta(s)dB_s := \mathcal{I}(\eta).$$

We consider the following type of G-BSDEs for simplicity

$$Y_t = \xi + \int_{t\wedge\tau}^\tau f(s,Y_s,Z_s)ds - \int_{t\wedge\tau}^\tau Z_s dB_s - (K_\tau - K_{t\wedge\tau}), \tag{2}$$

Where $f(t,\omega,y,z) : \mathbb{R}^+ \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$. It is clear that $Z_t = 0$ if $t > \tau$. Moreover since τ is finite, (2) implies that $Y_t = \xi$ if $t \geq \tau$.

We present an existence and uniqueness result for the G-BSDE (2) under assumptions which are very similar to the case of G-BSDEs with fixed terminal times. We make the following assumptions:

H1. there exist constants $\gamma \geq 0$, $\mu \in \mathbb{R}$, $c \geq 0$, $p > 1$ and $\kappa \in \{0,1\}$ such that

1. $\forall t, y, (z,z'), \quad |f(t,y,z) - f(t,y,z')| \leq \gamma|z - z'|$,
2. $\forall t, z, (y,y'), \quad (y - y').\,(f(t,y,z) - f(t,y',z)) \leq -\mu|y - y'|^2$,
3. $\forall y, z, \quad f(.,.,y,z) \in M_G^p(0,\tau)$,
4. $\forall t, y, z, \quad |f(t,y,z)| \leq |f(t,0,z)| + c(\kappa + |y|^p)$,
5. $\forall t, z, \quad y \to f(t,y,z)$ is continuous.

H2. $\xi \in L_G^{2p}(F_\tau)$ and there exists a real number ρ such that $\rho > \gamma^2 - 2\mu$ and

$$\mathbb{E}\Bigg[\kappa e^{\rho\tau} + \{e^{\rho\tau} + e^{p\rho\tau}\}|\xi|^{2p} + \left(\int_0^\tau e^{\rho s}|f(s,0,0)|^2 ds\right)^p + \left(\int_0^\tau e^{(\rho/2)s}|f(s,0,0)|\ ds\right)^{2p}\Bigg] < \infty.$$

Remark 2.6. In the case $\rho < 0$, which may occure if τ is an unbounded stopping time, our integrability conditions are fulfilled if we assume that

$$\mathbb{E}\left[e^{\rho\tau}|\xi|^{2p} + \left(\int_0^\tau e^{(\rho/2)s}|f(s,0,0)|^2\ ds\right)^p\right] < \infty.$$

3 Main results

In this section, we deal with the existence and uniqueness of the solutions of the G-BSDE (2) with random terminal time τ, under the assumptions (H1) and (H2).

3.1 Existence and Uniqueness of the solutions

Theorem 3.1. *Assume that $\xi \in L_G^2(\mathcal{F}_\tau)$ and (H1) and (H2) are satisfied by f. Then the G-BSDE* (2) *has at most one solution $(Y, Z, K) \in S_G^{2,\alpha,\tau}(\mathbb{R}) \times H_G^{2,\alpha}(\mathbb{R}) \times L_G^2(\mathcal{F}_\tau)$.*

Theorem 3.2. *Under the assumptions (H1) and (H2), the G-BSDE* (2) *has a unique solution (Y, Z, K) in the space $S_G^{2,\alpha,\tau}(\mathbb{R}) \times H_G^{2,\alpha}(\mathbb{R}) \times L_G^2(\mathcal{F}_\tau)$.*

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The application of game theory in the real option (bond and convertible bond financing)

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Abstract

In this study an optimal investment policy of a firm which is finance by issuing bond and convertible bond was examined by means of real option framework by using of stopping game. The interaction between bondholder and shareholder was studied and the effect of each bonds on investment timing and optimal bankruptcy, convert and call threshold were investigated. Also the impact of volatility on these thresholds was investigated.

Keywords: bond, convertible bond,stopping game, real option
Mathematics Subject Classification [2010]: 91Gxx, 91Axx

1 Introduction

One of the most important topics in firms is the optimal investment strategies. Deciding to investment composes of two parts: when and how much to invest. First part is decision for investment time and second is decision for asset allocation. To decide the investment timing a standard framework called real options approach is used. On the other side financing can done via share and bond or other financial instruments. One of the financial instruments is hybrid security that is a compound of debt and equity. An example this instrument is convertible bonds that embodied the characteristics of both straight bond and equities. The bondholder receives coupons periodically and has right to convert the bonds to previously defined equity[1]. Bonds contract can include put option (for bondholder) and call option (for investor) or without any extra option. Interaction between bondholder and shareholder can affect the value of this bond considerably. In this study financing by bond and convertible bond after investment by means of stopping game was investigated.

2 Model

We consider a firm with an option to invest at any time by paying a fixed investment cost. The firm partially finances the cost of investment with bond and convertible bond. According to feature of convertible bond, issuer and bondholder performance after the

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investment can be consider as a two player game. We suppose that block conversion that is all bondholders exercise the conversion option simultaneously. Also assume that x_t is the firms instantaneous EBIT. Suppose that x_t is given by a geometric Brownian motion [2]:

$$\frac{dx_t}{x_t} = \mu dt + \sigma dB_t.$$

where μ and σ are the risk-adjusted expected growth rate and the volatility of x_t. Maturity of bond is infinity and bondholders will receive coupon payment amounting cdt in every time interval (t, dt) before convert, call or firm bankruptcy. Also bondholder has right to convert their bonds for some amount common shares. The bond can be convert λ percentage of the firm value. After convert bondholder will obtain λx amount. Another characteristics is the call option with strike price K for issuer. When issuer calling, the bondholder must select the strik price or exercise the conversion right immediately by force. There for the bond value will be $max\{k, \lambda x\}$. The firm enjoys tax credit κcdt by serving coupon payment. In the case of asset reduction ρx of the asset value is lost. If $1-\rho > \lambda$ bondholder can convert the bond. When $1-\rho < \lambda$ bankruptcy will happened and $(1-\rho)x$ return to bondholder. When x is EBIT and denote the conversion, bankruptcy and call time by τ_{con}, τ_b and τ_{cal} respectively. $E(x)$ and $D(x)$ bond and equity values. Therefore according to the consider strategy we have following revised equation [1, 3]:

$$E(x) = \sup_{\tau_d, \tau_{cal} > 0} \mathbb{E}[\int_{\tau}^{\tau_d \wedge \tau_{cal} \wedge \tau_{con}} e^{-r(u-\tau)}(1-\kappa)(x_u - c)\,\mathrm{d}u + 1_{\{\tau_{con} < \tau_d \wedge \tau_{cal}\}} \tag{1}$$

$$(1-\lambda)\int_{\tau_{con}}^{\infty} e^{-r(u-\tau)}(1-\kappa)x_u\,\mathrm{d}u + 1_{\{\tau_{cal} < \tau_d \wedge \tau_{con}\}}\{\int_{\tau_{cal}}^{\infty} e^{-r(u-\tau)}(1-\kappa)x_u\,\mathrm{d}u$$

$$-e^{-r(\tau_{cal}-\tau)} max(K, \lambda \int_{\tau_{con}}^{\infty} e^{-r(u-\tau)}(1-\kappa)x_u\,\mathrm{d}u)\} \mid x_0 = x]$$

$$D_c(x) = \sup_{\tau_{con} > 0} \mathbb{E}[\int_{\tau}^{\tau_d \wedge \tau_{cal} \wedge \tau_{con}} e^{-r(u-\tau)} c\,\mathrm{d}u + 1_{\{\tau_{con} < \tau_d \wedge \tau_{cal}\}}\lambda \int_{\tau_{con}}^{\infty} e^{-r(u-\tau)}(1-\kappa)x_u\,\mathrm{d}u$$

$$+1_{\{\tau_d < \tau_{con} \wedge \tau_{cal}\}} e^{-r(\tau_d - \tau)}(1-\rho)\epsilon(x_{\tau_d}) + 1_{\{\tau_{cal} < \tau_{con} \wedge \tau_d\}} e^{-r(\tau_{cal}-\tau)} \tag{2}$$

$$.max(K, \lambda \int_{\tau_{con}}^{\infty} e^{-r(u-\tau)}(1-\kappa)x_u\,\mathrm{d}u)\} \mid x_0 = x]$$

The aim of bond holder and issuer is to maximize their benefit. Bondholder chooses when to convert and the issuer selects both bankruptcy and call time. This creates a two player-game as each stopping time is equilibrium. Stopping problem is an important and well developed class of stochastic control problem and is used when there are several deciders whit different aims. For obtaining stopping points problems (1) and (2) must be solved at the same time. Optimal bankruptcy, convert and call time is defined as following:

$$\tau_d^* = inf\{\tau_d \in [0, \infty) \mid x_{\tau_d} \leq x_d\} \tag{3}$$

$$\tau_{con}^* = inf\{\tau_{con} \in [0, \infty) \mid x_{\tau_{con}} \geq x_{con}\} \tag{4}$$

$$\tau_d^* = inf\{\tau_{cal} \in [0, \infty) \mid x_{\tau_{cal}} \geq x_{cal}\} \tag{5}$$

As x_d, c_{con} and x_{cal} are bankruptcy, convert and call thresholds respectively. According to following differential equations[3]:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 E}{\partial x^2} + \mu x \frac{\partial E}{\partial x} - rE + (1-\kappa)(x-c) = 0 \tag{6}$$

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 D_c}{\partial x^2} + \mu x \frac{\partial D_c}{\partial x} - rD_c + c = 0 \tag{7}$$

Problems were solved numerically. Next a firm was considered which has an option of the investment that is financed straight debt whit coupon payment s. Once the investment option has been exercised, the optimal bankruptcy policy is established from the issue of debt. The optimal equity value $E(x)$ is given by:

$$E(x) = \sup_{\tau_d > 0} \mathbb{E}[\int_{\tau}^{\tau_d} e^{-r(u-\tau)}(1-\kappa)(x_u - s)\,\mathrm{d}u \tag{8}$$

And the debt value $D_s(x)$ is according to following. Notably debt holder has no right to stop the game [4].

$$D_s(x) = \mathbb{E}[\int_{\tau}^{\tau_d} e^{-r(u-\tau)} s\,\mathrm{d}u + e^{-r(\tau_d - \tau)}(1-\rho)\epsilon(x_d) \tag{9}$$

3 Main results

In this research the effect of financing by bond and convertible bond on investment timing and bondholders and shareholders strategy after investment was studied and thresholds were calculated. Numerical value was obtained via newton method by $fsolve$ function in MATLAB software. For this purpose primary parameters $\mu = 0.01$, $\sigma = 0.2$, $r = 0.05$, $I = 5$, $\rho = 0.3$, $s = 0.4$, $c = 0.4$ and $\kappa = 0.3$ was used. convertible bond investment threshold is $X^* = 0.74$. Figure (1) and (2) showed the behavior of bond, convertible bond and equity. Also the optimal thresholds after investment were observable. According to the figure bond in comparison whit convertible bond has stable behavior and this is because of the presence of conversion right. Table (1) shows investment and bankruptcy bond threshold for different amount σ and I. also figure (3) illustrates the effect of volatility on convertible bond thresholds. According to the figure, by increasing risk, bankruptcy will happen earlier.

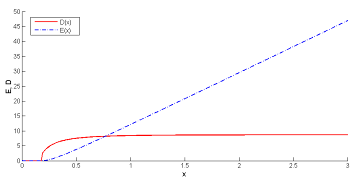
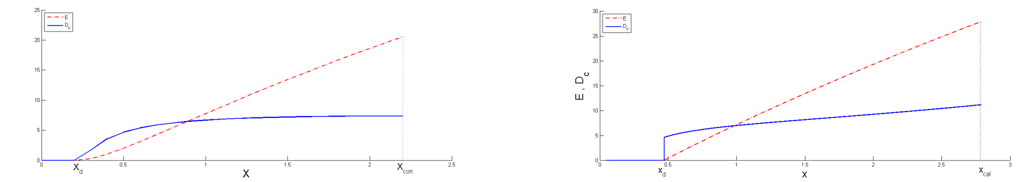


Figure 1: Equity and bond value, and stopping point

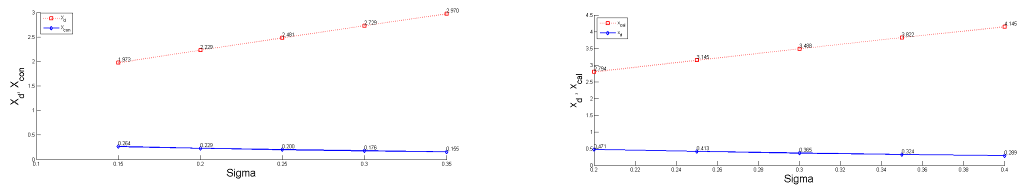


(a) Convert before call (b) Call before convert

Figure 2: Equity and convertible bond value after investment, and stopping points

Table 1: Effect of investment cost and volatility on investment and bankruptcy threshold

I=5	I=3	I=1	
$X^* = 0.558$	$X^* = 0.384$	$X^* = 0.263$	$\sigma = 0.2$
$X_d = 0.183$	$X_d = 0.183$	$X_d = 0.183$	
$X^* = 0.732$	$X^* = 0.464$	$X^* = 0.258$	$\sigma = 0.3$
$X_d = 0.135$	$X_d = 0.135$	$X_d = 0.135$	



(a) Convert before call (b) Call before convert

Figure 3: Effect of volatility on convertible bond thresholds

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A generalization of contact metric manifolds

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Abstract

We consider quasi contact metric manifolds and give a necessary and sufficient condition for a quasi contact metric manifold, to be contact metric manifold and K-contact, then we prove that a quasi contact metric manifold is not nearly cosymplectic.

Keywords: Almost contact metric manifold, Quasi contact metric manifold, Kähler manifold.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

A quasi Kähler manifold (see[2]) is an almost Hermitian manifold (M, J, g), that the Levi-Civita connection satisfies:

$$(\nabla_X^J)Y + (\nabla_{JX}^J)JY = 0, \qquad X, Y \in \tau(M).$$

A quasi contact metric manifold was primary introduced by Y. Tashiro ([4]) as hypersurface of a quasi Kähler manifold, and named $O^* - manifold$ by him. Then J. H. Kim and his colleagues gived a characterization of a contact metric manifold as a special almost contact metric manifold and discussed an almost contact metric manifold which is a natural generalization of the contact metric manifolds introduced by Y. Tashiro and proved that ([3]) an almost contact metric manifold (ϕ, ξ, η, g) is a quasi contact metric manifold if and only if it satisfies the following relation:

$$(\nabla_X^\varphi)Y + (\nabla_{\varphi X}^\varphi)\varphi Y = 2g(X, Y)\xi - \eta(Y)(X + \eta(X)\xi + hX)$$

in which $h = \frac{1}{2}L_\xi\varphi$.
In this paper we consider conditions on quasi contact metric manifolds, endowed with which, being contact and $K-$contact. Also we show that quasi contact metric manifolds can not be nearly cosymplectic.

*Speaker

2 Main results

An almost contact metric manifold $M = (\varphi, \xi, \eta, g)$ satisfying the followin relation for every X and Y in $\tau(M)$ is called a quasi contact metric manifold([3]):

$$(\nabla^{\varphi}_{X})Y + (\nabla^{\varphi}_{\varphi X})\varphi Y = 2g(X,Y)\xi - \eta(Y)(X + \eta(X)\xi + hX) \tag{1}$$

The relation (1), holds in every contact metric manifold([1], page116) , thus contact metric manifolds are quasi contact metric manifold, and quasi contact metric manifolds can be regarded as a generalization of contact metric manifolds. We can show easily that the following properties all satisfy in the quasi contact metric manifold.

Theorem 2.1. *In a quasi contact metric manifold $M = (\varphi, \xi, \eta, g)$, the following relations hold:*

(a) $\nabla^{\varphi}_{\xi} = 0$

(b) $\nabla^{\xi}_{\xi} = 0$

(c) $\nabla^{\xi}_{X} = -\varphi X - \varphi hX$

(d) $\eta o h = 0$

(e) $(\nabla^{\eta}_{X})Y + (\nabla^{\eta}_{\varphi X})\varphi Y = 2g(X, \varphi Y)$

(f) $\varphi h + h\varphi = 0.$

Let $M = (\varphi, \xi, \eta, g)$ be a $(2n+1)-$dimensional almost contact metric manifold and $\overline{M} = M \times \mathbb{R}$ be the product manifold of M and the real line $\mathbb{R}$. It is proved that ([1]) $\overline{M}$ can be equipped by an almost Hermitian structure $(\bar{J}, \bar{g})$. $\bar{J}$ is said to be integrable if its Nijenhuis torsion,

$$N_{\bar{J}}(X,Y) := [\bar{J}X, \bar{J}Y] - [X,Y] - \bar{J}[\bar{J}X, Y] - \bar{J}[X, \bar{J}Y]$$

vanishes. Computing the Nijenhuis torsion of $\bar{J}$, leads to define four tensors :

$$\begin{aligned}
N^{(1)}(X,Y) &:= [\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi \\
N^{(2)}(X,Y) &:= (L_{\varphi X}\eta)Y - (L_{\varphi Y}\eta)X \\
N^{(3)} &:= L_{\xi}\varphi \\
N^{(4)} &:= L_{\xi}\eta.
\end{aligned}$$

It is proved that in an almost contact manifold, vanishing of $N^{(1)}$, implies the vanishing of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ and if M is contact, then $N^{(2)}$ and $N^{(4)}$ vanish , and moreover if M is $K-$contact, say ξ is a killing vector field. Then $N^{(3)}$ vanishes. Now we have the following theorem for quasi contact metric manifolds:

Theorem 2.2. *For a quasi contact metric manifold $M = (\varphi, \xi, \eta, g)$, $N^{(4)}$ vanishes. Moreover, $N^{(2)}$ vanishes if and only if M is contact , and $N^{(3)}$ vanishes if and only if M is $K-$contact.*

Proof. Let $M = (\varphi, \xi, \eta, g)$ be a quasi contact metric manifold. Then we have:

$$\begin{aligned} N^{(4)}(X) &= (L_\xi \eta)(X) \\ &= \xi(\eta(X)) - \eta([\xi, X]) \\ &= (\nabla^\eta_\xi)X - g(\nabla^\xi_X, \xi) \\ &= (\nabla^\eta_\xi)X \\ &= 0, \end{aligned}$$

the last equality is obtained by theorem 2.1(b). It is evident that every contact metric manifold is quasi contact metric manifold. Now let $M = (\varphi, \xi, \eta, g)$ be a quasi contact manifold in which $N^{(2)}$ vanishes, so

$$\begin{aligned} 0 &= N^{(2)}(X, Y) \\ &= (L_{\varphi X}\eta)(Y) - (L_{\varphi Y}\eta)(X) \\ &= \varphi X(\eta(Y)) - \eta([\varphi X, Y]) - \varphi Y(\eta(X)) + \eta([\varphi Y, X]) \\ &= (\nabla^\eta_{\varphi X})Y - (\nabla^\eta_Y)\varphi X - (\nabla^\eta_{\varphi Y})X + (\nabla^\eta_X)\varphi Y \\ &= 2g(\varphi X, \varphi Y) + (\nabla^\eta_X)\varphi Y - (\nabla^\eta_Y)\varphi X - 2g(\varphi X, \varphi Y) - (\nabla^\eta_Y)\varphi X + (\nabla^\eta_X)\varphi Y \\ &= 2(\nabla^\eta_X)\varphi Y - 2(\nabla^\eta_Y)\varphi X \\ &= 2(g(\nabla^\xi_X, \varphi Y) - g(\nabla^\xi_Y, \varphi X)). \end{aligned}$$

Substituting Y by φY, we get

$$-g(\nabla^\xi_X, Y) = g(\nabla^\xi_{\varphi Y}, \varphi X).$$

By the above equality we have

$$\begin{aligned} d\eta(X, Y) &= \frac{1}{2}[g(\nabla^\xi_X, Y) - g(\nabla^\xi_Y, X)] \\ &= \frac{-1}{2}[g(\nabla^\xi_{\varphi Y}, \varphi X) + g(\nabla^\xi_Y, X)] \\ &= \frac{-1}{2}((\nabla^\eta_{\varphi Y})\varphi X + (\nabla^\eta_Y)X) \\ &= -g(Y, \varphi X) \\ &= \Phi(X, Y). \end{aligned}$$

Thus the manifold is contact.

Now let $N^{(3)}$ vanishes, then:

$$0 = N^{(3)} = -2h,$$

thus by (c) of theorem 2.1 we have

$$\nabla^\xi_X = -\varphi X.$$

Considering the above equality we have

$$\begin{aligned} d\eta(X,Y) &= \frac{1}{2}[g(\nabla^{\xi}_{X},Y) - g(\nabla^{\xi}_{Y},X)] \\ &= \frac{1}{2}[g(-\varphi X,Y) - g(-\varphi Y,X)] \\ &= \Phi(X,Y). \end{aligned}$$

Thus the manifold is contact and we know that in a contact manifold, $N^{(3)}$ vanishes if and only if it is $K-$contact. □

Theorem 2.3. *A quasi contact structure can not be cosymplectic.*

Proof. We know that a cosymplectic manifold is a normal almost contact manifold in which $d\eta = 0$. But by the above theorem it is evident that a normal quasi contact manifold is contact and thus $d\eta \neq 0$ and it is a contradiction. □

Proposition 2.4. *In a quasi contact metric manifold $M = (\phi, \xi, \eta, g)$ we have $d\eta \neq 0$.*

Corollary 2.5. *Nearly cosymplectic manifolds which was defined by ([1]), is an almost contact manifold (ϕ, ξ, η) that satisfies $d\eta = 0$ and $d\phi = 0$. By the above Proposition it is convenient that a quasi contact manifold can not be nearly cosymplectic.*

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A note on an ideal of C(X) with λ- compact support

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Abstract

We introduce and investigate some properties of the set of functions in $C(X)$ with λ-compact support which is denoted by $C_K^\lambda(X)$, where λ is an infinite regular cardinal number. We extend some of the basic results concerning $C_K(X)$ (i.e., the family of all elements of $C(X)$ having compact support) for $C_K^\lambda(X)$. For instance, the purity of $C_K^\lambda(X)$ is studied and characterized through P_λ-spaces and λ-locally compact spaces which are not λ-compact. Finally some relations between topological properties of the space X and algebraic properties of the ideal $C_K^\lambda(X)$ are investigated.

Keywords: λ-compact, support, purity , λ-locally compact.
Mathematics Subject Classification [2010]: Primary: 54C30, 54C40, 54C05, 54G12; Secondary: 13C11, 16H20.

1 Introduction

Let $C(X)$ be the ring of all continuous real-valued functions on a completely regular Hausdorff space X. Throughout this article ideals are assumed to be proper ideals. For each $f \in C(X)$, let $Z(f) = \{x \in X : f(x) = 0\}$ and $cozf = X \setminus Z(f)$. If I is an ideal of $C(X)$, we put $cozI = \bigcup_{f\in I} cozf$. The support of f is the closure of $X \setminus Z(f)$ and $C_K(X)$ is the set of functions in $C(X)$ with compact support, see [4]. The concept λ-compact in [5] and [7], motivates us to introduce $C_K^\lambda(X)$. Our main purpose in this article is the study of the ideal structure of $C_K^\lambda(X)$ and of the relation between topological properties of the subspaces of X and algebraic properties of the ideal $C_K^\lambda(X)$. The space X is called λ-compact whenever each open cover of X can be reduced to an open cover of X whose cardinality is less than λ, where λ is the least infinite cardinal number with this property. We remind that the space X is P_λ-spaces if and only if every intersection with cardinality less than λ of open sets (i.e., G_λ-set) be open. The space X is called λ-locally compact space whenever every element of X has a λ-compact neighborhood, see [7]. For undefined terms and notations the reader is referred to [3] and [4].

2 Functions in $C(X)$ with λ-compact support

We need the following definition.

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Definition 2.1. $C_K^\lambda(X)$ denote the family of all functions in $C(X)$ having λ-compact support.

We investigate certain properties of $C_K^\lambda(X)$ compared with $C_K(X)$.

Lemma 2.2. *$C_K^\lambda(X)$ is a z-ideal of $C(X)$.*

Remark 2.3. $C_K(X) \subseteq C_K^\lambda(X)$. If X is compact, $C_K(X) = C_K^\lambda(X) = C(X)$ and also if X is λ-locally compact, $C_K^\lambda(X) = C(X)$. We note that if X is P_λ-space, then $suppf = cozf$. Hence $C_K^\lambda(X) = \{f \in C(X) : cozf \quad \text{is} \quad \lambda - compact\}$.

Example 2.4. It is known that $C_K(\mathbb{Q}) = (0)$, since $\mathbb{Q}$ has not a compact neighborhood but $C_K^{\aleph_1}(\mathbb{Q}) = \{f \in C(\mathbb{Q}) : suppf \quad is \quad \aleph_1 - compact\} = C(\mathbb{Q})$.

The following properties and corollary are proved in [7]. They will be used in the following discussion.

Proposition 2.5. *If $\mu \leq \lambda$ is a cardinal number, then every μ-compact subspace of a Hausdorff P_λ-space is closed.*

Notice that for every subset F of a space X, $d_c(F) \leq d_c(X)$, see[7, proposition 6.1.2]. Also if X is P-space then $suppf$ is finite for each $f \in C_K(\lambda(X)$, since every pseudo compact P-space is finite.

Corollary 2.6. *Every subspace A of a P_λ-space of X is closed and discrete, where $|A| < \lambda$.*

Now, by previous corollary we have the following proposition.

Proposition 2.7. *Let X be P_{λ^+}-space and A is a λ-compact closed subset of X, then A has the cardinality less than λ.*

Proof. If $|A| \geq \lambda$, then we get a contradiction. At first, we suppose $|A| = \lambda$ by Corollary 2.6, A is discrete and closed. So A is λ^+-compact, which is impossible. Now, let $|A| > \lambda$, in this case there exists a subspace of A with cardinality λ, say B, see [7, Proposition 5.2.3]. Consequently, B is λ^+-compact which is absurd. □

Corollary 2.8. *If X is a P_{λ^+}-space, then cardinality of suppf is less than λ for every $f \in C_K^\lambda(X)$.*

Theorem 2.9. *Let X be a P_{λ^+}-space. Then $C_K^\lambda(X)$ is a free proper ideal if and only if X is λ-Locally compact but not λ-compact.*

Theorem 2.10. *Let X be a P_λ-space. Then $C_K^\lambda(X)$ is free in $C(X)$ if and only if for every λ-compact set A there exists $f \in I$ having no zero in A.*

Recall that an ideal I of $C(X)$ is called P-ideal if $cozI$ is a P-space and I is pure(i.e., for every $f \in I$ there exists $g \in I$ such that $f = fg$ and in the case $g = 1$ on $suppf$). P-ideal is a concept which was originally defined and characterize by *David Rudd* , see [6]. The motivation, we extend the P-ideal to P_λ-ideal. In this paper we extend this concept and define the P_λ-ideals.

Definition 2.11. An ideal I of $C(X)$ is called P_λ-ideal if $cozI = \bigcup_{f\in I} cozf$ is a P_λ-space.

If X is a P_λ-space then every ideal of $C(X)$ is a P_λ-ideal. But the converse is not true. Note the following example:

Example 2.12. Let $X = \mathbb{N}^* = \mathbb{N} \cap \{\omega\}$ be the one-point compactification of the discrete space of the natural numbers and I be the ideal of functions which are eventually zero (i.e., $I = \{f \in C(X) : f = 0, \text{except on a finite set}\}$). Since $Z(f)$ is open for each $f \in I$, we conclude that I is a $P_{\aleph_1}$-ideal but X is not a $P_{\aleph_1}$-space.

Theorem 2.13. *Let X be an arbitrary topological space and I is a P_λ- pure ideal of $C(X)$. The following holds:*

1. *$Z(f)$ is open for each $f \in I$.*
2. *Every ideal of I is pure.*
3. *I is a regular ring.*

Proof. Since every P_λ-space is a P-space, the above statements hold, see[2]. □

For our the other results, we need the following lemma in [2] and the concept of λ-discrete.

Lemma 2.14. *If I is a pure ideal, then $suppf \subseteq cozI$ for each $f \in I$.*

Definition 2.15. An element $x \in X$ is called a λ-isolated point if x has been a neighborhood with cardinality less than λ.

If every point of topological space of X is λ-isolated, then$X$ is called a λ-discrete space.

Theorem 2.16. *If $C_K^\lambda(X)$ is a P_{λ^+}-ideal then $coz(C_K^\lambda(X))$ is λ-discrete.*

3 Relation between purity $C_K^\lambda(X)$ and the subspace λ-locally compact of X

In trying to characterize the properties of $C_K^\lambda(X)$, we introduce the subspace of all points with λ-compact neighborhoods which we will denote by X_L^λ. X is nowhere λ-locally compact if and only if $X_L^\lambda = \phi$.

Lemma 3.1. *Let X is P_λ- space, then $X_L^\lambda = coz(C_K^\lambda(X))$.*

Corollary 3.2. *X_L^λ is a open λ-locally compact subspace of X.*

Our main purpose, investigate purity of $C_K(X)$ using the subspace X_L^λ.

Lemma 3.3. *If I is a pure ideal of $C(X)$, then $cozI = \bigcup_{f\in I} suppf$.*

Proof. see lemma, 2.14 and [1, Lemma 3.1]. □

Theorem 3.4. *Let $I = C_K^\lambda(X)$ is P_{λ^+}-ideal, then I is pure ideal if and only if $cozI = \bigcup_{f\in I} suppf$.*

The following theorem generalize the results in [1] which it was proved for $C_K(X)$. At first, we give the following lemma which is needed in the sequel, see [7].

Lemma 3.5. *Let X and Y be two topology spaces and $f : X \longrightarrow Y$ is a continuous function. If $A \subseteq X$ in X, is λ-compact then $f(A)$ is β-compact in Y, where $\beta \leq \lambda$, see[7, lemma4.1.2].*

Theorem 3.6. *Let $C_K^\lambda(X)$ and $C_K^\lambda(Y)$ be pure ideals. Then X_L^λ is homeomorphic to Y_L^λ if and only if $C_K^\lambda(X)$ is isomorphic to $C_K^\lambda(Y)$.*

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An extension of $C_F(X)$

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Abstract

Let $C_F(X)$ be the socle of $C(X)$ (i.e., the sum of minimal ideals of $C(X)$). We define $LC_F(X)=\left\{f\in C(X):\overline{S_f}=X\right\}$, where S_f is the union of all open subsets U in X such that $|U\backslash Z(f)|<\infty$, $LC_F(X)$ is called the locally socle of $C(X)$ and it is a z-ideal of $C(X)$ containing $C_F(X)$. We characterize spaces X for which the equality in the relation $C_F(X)\subseteq LC_F(X)\subseteq C(X)$ is hold. We determine the conditions such that $LC_F(X)$ is not prime in any subrings of $C(X)$ which contains the idempotents of X. We investigate the primness of $LC_F(X)$ in some subrings of $C(X)$.

Keywords: Socle, Locally socle, Compact space, Prime ideal, Scattered space.
Mathematics Subject Classification [2010]: Primary: 54C30, 54C40, 54C05, 54G12; Secondary: 13C11, 16H20.

1 Introduction

$C(X)$ denotes the ring of all real valued continuous functions on a topological space X. We recall that a nonzero ideal E in a commutative ring R is called essential if it intersects every nonzero ideal nontrivially. Let I be an ideal in $C(X)$, then $Z[I]=\{Z(f):f\in I\}$ and $Z(X)=\{Z(f):f\in C(X)\}$. If $Z^{-1}[Z[I]]=I$, then I is called a z-ideal. Let $C_c(X)=\{f\in C(X):|f(X)|\leq\aleph_0\}$ and $C^F(X)=\{f\in C(X):|f(X)|<\infty\}$, see [6] and [7]. The socle of $C(X)$ (i.e., $C_F(X)$) which is in fact a direct sum of minimal ideals of $C(X)$ is characterized topologically in [10, Proposition 3.3], and it turns out that $C_F(X)=\{f\in C(X):|X\backslash Z(f)|<\infty\}$ is a useful object in the context of $C(X)$, see [10], [1], [5], [2], and [3]. This motivates us to investigate the locally socle of $C(X)$. We define $LC_F(X)=\left\{f\in C(X):\overline{S_f}=X\right\}$, where S_f is the union of all open subsets U in X such that $|U\backslash Z(f)|<\infty$, $LC_F(X)$ is called the locally socle of $C(X)$ and it is a z-ideal of $C(X)$ containing $C_F(X)$. We characterize spaces X for which the equality in the relation $C_F(X)\subseteq LC_F(X)\subseteq C(X)$ holds. In fact, we show that X is an almost discrete space if and only if $LC_F(X)=C(X)$. We note that if X is an infinite space, then $C_F(X)\subsetneq C(X)$. We also observe that $|I(X)|<\infty$ if and only if $C_F(X)=LC_F(X)$. Moreover, it is shown that if $|I(X)|<\infty$, then $LC_F(X)$ is never essential in any subring of $C(X)$, while $LC_F(X)$ is an intersection of essential ideals of $C(X)$. We determine the conditions such that $LC_F(X)$ is not prime in any subrings of $C(X)$ which contains the idempotents of X. We investigate the primness of $LC_F(X)$ in some subrings of $C(X)$. All

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topological spaces that appear in this article are assumed to be infinite completely regular Hausdorff, unless otherwise mentioned. For undefined terms and notations the reader is referred to [8] and [4].

2 Locally socle

Definition 2.1. Let $f \in C(X)$ and S_f be the union of all open subsets $U \subseteq X$ such that $U \backslash Z(f)$ is finite. We denote the locally socle of $C(X)$ by $LC_F(X)$ and define it to be the set of all $f \in C(X)$ such that S_f is dense in X. i.e.,

$$S_f = \bigcup_{\substack{U \subseteq X \\ |U \backslash Z(f)| < \infty}} U$$

$$LC_F(X) = \{f \in C(X) : \overline{S_f} = X\}$$

Lemma 2.2. *$\overline{S_f} = X$ if and only if for each open subset $G \subseteq X$, there exists an open subset $U \subseteq X$ such that $|U \backslash Z(f)| < \infty$ and $U \cap G \neq \emptyset$ if and only if for each open subset $G \subseteq X$, there exists an open subset $U \subseteq X$ such that $|U \backslash Z(f)| < \infty$ and $U \subseteq G$.*

We note that if U is a finite open subset in a Hausdorff space X and $x \in U$, then x is isolated and $\overline{\bigcup_{|U|<\infty} U} = X$ if and only if $\overline{\mathrm{I}(X)} = X$.

Proposition 2.3. *Let U, V be open in X. Then*

$$S_f = \bigcup_{\substack{U \subseteq X \\ |U \backslash Z(f)| < \infty}} U = \bigcup_{\substack{V \subseteq X \\ |V \backslash Z(f)| \leq 1}} V$$

Lemma 2.4. *If $f, g \in C(X)$, then the following statements hold.*

1. $S_{f+g} \supseteq S_f \cap S_g$.
2. $S_{fg} \supseteq S_f \cup S_g$.
3. $S_{|f|} = S_f$.
4. *If $f, g \in LC_F(X)$, then $\overline{S_f \cap S_g} = X$.*

Proposition 2.5. *$LC_F(X) \subseteq L_F(X)$.*

Proposition 2.6. *$LC_F(X)$ is an ideal of $C(X)$.*

Proposition 2.7. *$LC_F(X)$ is a z-ideal in $C(X)$.*

Proposition 2.8. *If X is a connected space, then $C_F(X) = LC_F(X) = (0)$.*

3 The equality in the relation $C_F(X) \subseteq LC_F(X) \subseteq C(X)$

If X is an uncountable scattered space, then $C_F(X) \subsetneq LC_F(X) = C(X)$. If X is a connected space, then $(0) = LC_F(X) \subsetneq C(X)$.

Proposition 3.1. $|\mathrm{I}(X)| < \infty$ *if and only if* $C_F(X) = LC_F(X)$.

Proposition 3.2. *If* X *is discrete, then* $LC_F(X) = C(X)$.

Theorem 3.3. X *is an almost discrete space if and only if* $LC_F(X) = C(X)$.

Corollary 3.4. *If* X *is an scattered space, then* $LC_F(X) = C(X)$.

The converse of the above corollary does not hold. For instance, let at each point $x \in \mathbb{Q}$, the basic neighborhood of x be the singleton $\{x\}$, and for $x \in \mathbb{Q}^c$, the basic neighborhood of x be the usual open interval containing x. This constitutes a topology on $\mathbb{R}$ and clearly $\mathbb{R}$ with this topology is Hausdorff normal which is almost discrete for, $\mathrm{I}(X) = \mathbb{Q}$. Hence $LC_F(X) = C(X)$, but $\mathbb{R}$ is not scattered.

4 The primeness of $LC_F(X)$ in some subrings of $C(X)$

Theorem 4.1. *Let* X *has finite components and at least two of them are infinite, then* $LC_F(X)$ *is never prime in any subring of* $C(X)$ *which contains the idempotents of* $C(X)$.

Theorem 4.2. *If* $|\mathrm{I}(X)| < \infty$ *and* $X \setminus \mathrm{I}(X)$ *is disconected, then* $LC_F(X)$ *is never prime in any subring of* $C(X)$ *which contains the idempotents of* $C(X)$.

Theorem 4.3. *Let* $|\mathrm{I}(X)| < \infty$ *and* R *be a subring of* $C(X)$. $LC_F(X)$ *is prime in* R*, if every* $f \in R$ *is constant on* $X \setminus \mathrm{I}(X)$. *Conversely, if* $LC_F(X)$ *is prime in* R *and* R *contains the idempotents of* $C(X)$*, then* $X \setminus \mathrm{I}(X)$ *is connected.*

Corollary 4.4. *If* $|\mathrm{I}(X)| < \infty$ *and* $X \setminus \mathrm{I}(X)$ *is connected, then* $LC_F(X)$ *is prime in* $C_c(X)$ *and* $C^F(X)$.

Theorem 4.5. *Let* A *be a submodule of the ring* C*, then* A *is an intersection of essential submodules of* C *if and only if* $\mathrm{Soc}(C) \leq A$.

Proof. See [9]. □

Proposition 4.6. $LC_F(X)$ *is an intersection of essential ideals.*

Finally, we investigate the essentiality of the locally socle of $C(X)$ and the socle of $C(X)$, whenever the space X has finite isolated points.

Proposition 4.7. *If* $|\mathrm{I}(X)| < \infty$*, then* $LC_F(X)$ *is never essential in any subring of* $C(X)$ *containing* $LC_F(X)$.

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Classification pseudosymmetric (κ, μ)-contact metric manifolds

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Abstract

This paper deals with a classification of the pseudosymmetric contact metric manifolds under the condition that the characteristic vector field ξ belong to the (κ, μ)-nullity distribution in the R. Deszcz sense.

Keywords: Pseudosymmetric, Semisymmetric, (κ, μ)-nullity distribution, Contact manifold
Mathematics Subject Classification [2010]: 53D10, 53C35

1 Introduction

Chaki [3]and Deszcz [4] introduced two difierent concept of a pseudosymmetric manifold. In both senses various properties of pseudosymmetric manifolds have been studied. We shall study properties of pseudosymmetric manifolds in the Deszcz sense. A Riemannian manifold is called semisymmetric if $R(X,Y)\ .\ R = 0$. Deszcz [4] generalized the concept of semisymmetry and introduced pseudosymmetric manifolds. Let (M^n, g), $n \geq 3$ be a Riemannian manifold. Let ∇ and R denote the Levi-Civita connection and the curvature tensor of (M, g). We define endomorphism $X \wedge Y$ by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{1}$$

For a $(0, k)$-tensor field T, the $(0, k+2)$ tensor fields $R.T$ and $Q(g, T)$ are defined by [4]

$$\begin{aligned}&(R.T)(X_1, ..., X_k; X, Y) = (R(X,Y).T)(X_1, ..., X_k)\\&= -T(R(X,Y)X_1, X_2, ..., X_k) - ... - T(X_1, ..., X_{k-1}, R(X,Y)X_k),\end{aligned} \tag{2}$$

$$\begin{aligned}&Q(g,T)(X_1, ..., X_k; X, Y) = ((X \wedge Y).T)(X_1, ..., X_k)\\&= -T((X \wedge Y)X_1, X_2, ..., X_k) - ... - T(X_1, ..., X_{k-1}, (X \wedge Y)X_k),\end{aligned} \tag{3}$$

A Riemannian manifold M is said to be pseudosymmetric if the tensors $R.R$ and $Q(g, R)$ are linearly dependent at every point of M, i.e. $R.R = L_R Q(g, R)$. This is equivalent to

$$(R(X,Y).R)(U,V,W) = L_R[((X \wedge Y).R)(U,V,W)] \tag{4}$$

holding on the set $U_R = \{x \in M : Q(g, R) \neq 0\ at\ x\}$, where L_R is some function on U_R [4]. The manifold M is called a pseudosymmetric of constant type if L is constant. Particularly if $L_R = 0$ then M is a semisymmetric manifold. Papantoniou classified semisymmetric (κ, μ)-contact metric manifolds [5]. As a generalization, in this paper, we study pseudosymmetric (κ, μ)-contact metric manifolds.

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2 Preliminaries

A contact manifold is an odd-dimensional C^∞ manifold M^{2n+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Since $d\eta$ is of rank $2n$, there exists a unique vector field ξ on M^{2n+1} satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any $X \in \chi(M)$ is called characteristic vector field of η. A Riemannian metric g is said to be an associated metric if there exists a (1,1) tensor field φ such that $d\eta(X,Y) = g(X,\varphi Y)$, $\eta(X) = g(X,\xi)$, $\varphi^2 = -I + \eta \otimes \xi$. The structure (φ,ξ,η,g) is called a contact metric structure and a manifold $M^{2n+1}(\varphi,\xi,\eta,g)$ is said to be a contact metric manifold. Given a contact metric structure (φ,ξ,η,g), we define a tensor field h by $h = (1/2)\mathcal{L}_\xi\varphi$ where $\mathcal{L}$ denotes the operator of Lie differentiation. A contact metric manifold is said to be a Sasakian manifold if

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X \quad and \quad R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{5}$$

The (κ,μ)-nullity distribution of a contact metric manifold M is a distribution [2]

$$\begin{aligned} N(\kappa,\mu) : p \longrightarrow N_p(\kappa,\mu) = \{W \in T_pM | R(X,Y)W \; &= \kappa[g(Y,W)X - g(X,W)Y] \\ &+\mu[g(Y,W)hX - g(X,W)hY]\}, \end{aligned}$$

where κ,μ are real constants. Hence if the characteristic vector field ξ belongs to the (κ,μ)-nullity distribution, then we have

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{6}$$

We easily check that Sasakian manifolds are (κ,μ)-manifolds with $\kappa = 1$ and $h = 0$ [2].

3 Pseudosymmetric (κ,μ)-manifolds

We know that [1] if M^{2n+1} be a contact metric manifold and $R_{XY}\xi = 0$ for all vector fields X and Y, then M^{2n+1} is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature 4. In [2] Blair et al. studied the condition of (κ,μ)-nullity distribution on a contact manifold and obtain the following Theorem.

Theorem 3.1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a contact manifold with ξ belonging to the (κ,μ)-nullity distribution. If $\kappa < 1$ then for any X orthogonal to ξ; the sectional curvature of a plan section $\{X,Y\}$ normal to ξ is given by*

$$K(X,Y) = \begin{cases} i) & 2(1+\lambda) - \mu \quad if\ X,Y \in D(\lambda) \\ ii) & -(\kappa+\mu)[g(X,\varphi Y)]^2 \quad for\ any\ unit\ vectors\ X \in D(\lambda), X \in D(-\lambda) \\ iii) & 2(1-\lambda) - \mu \quad if\ X,Y \in D(-\lambda),\ n > 1 \end{cases} \tag{7}$$

When $\kappa < 1$, the nonzero eigenvalues of h are $\pm\sqrt{1-\kappa}$ each with multiplicity n. Let λ be the positive eigenvalue. Then M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0), D(\lambda)$ and $D(-\lambda)$ defined by the eigenspaces of h [2]. Firstly we give the following propositions.

Proposition 3.2. *Let M^{2n+1} be a (κ,μ)-contact metric pseudosymmetric manifold. Then for any unit vector fields $X, Y \in \chi(M)$ orthogonal to ξ and $g(X,Y)=0$ we have:*

$$\{(\kappa - L_R)K(X,Y) + \mu g(hX, R(X,Y)Y) - \kappa(\kappa - L_R) - \mu(\kappa - L_R)g(hY,Y) - \mu^2 g(hX,X)\, g(hY,Y) + \mu^2 g^2(hX,Y)\}\xi - (\kappa - L_R)\eta(R(X,Y)Y)X - \mu\eta(R(X,Y)Y)hX = 0. \tag{8}$$

Proposition 3.3. *Every pseudosymmetric Sasakian manifold with $L_R \neq 1$ is of constant curvature 1.*

Theorem 3.4. *Let M^{2n+1}, $n > 1$ be a (κ,μ)-contact metric pseudosymmetric manifold. Then M^{2n+1} is either*

1) *A Sasakian manifold of constant sectional curvature 1 if $L_R \neq 1$ or*
2) *Locally isometric to the product of a flat $(n+1)$-dimensional Euclidean manifold and an n-dimensional manifold of constant curvature 4.*

Proof. If $\kappa = 1$ then M is a Sasakian manifold and result get from Proposition 2. Let $\kappa < 1$ and X, Y are orthonormal vectors of the distribution $D(\lambda)$. Applying the relation (8) for $hX = \lambda X,\ hY = \lambda Y$ and taking inner product with ξ we get

$$i)\ \ K(X,Y) = \kappa + \lambda\mu \quad \text{or} \quad ii)\ \ \kappa = -\lambda\mu + L_R \tag{9}$$

Comparing part (i) of equations (7) and (9) gives

$$\mu = 1 + \lambda. \tag{10}$$

Similarly for $X, Y \in D(-\lambda)$ and $g(X,Y) = 0$ we have

$$i)\ \ K(X,Y) = \kappa - \lambda\mu \quad \text{or} \quad ii)\ \ \kappa = \lambda\mu + L_R \tag{11}$$

Comparing the equations (7)(iii) and (11)(i) we have

$$i)\ \ \mu = 1 - \lambda \quad \text{or} \quad ii)\ \ \lambda = 1. \tag{12}$$

In the case $X \in D(\lambda), Y \in D(-\lambda)$ and $X \in D(-\lambda), Y \in D(\lambda)$ we prove that

$$i)\ K(X,Y) = \kappa - \lambda\mu \quad \text{or} \quad \kappa = -\lambda\mu + L_R \tag{13}$$

$$i)\ K(X,Y) = \kappa + \lambda\mu \quad \text{or} \quad \kappa = \lambda\mu + L_R. \tag{14}$$

By the combination now of the equation (9)(ii), (10), (11)(ii), (12), (13) and (14) we establish the following nine systems among the unknowns κ, λ, μ and L_R.

1) $\{\mu = 1 - \lambda,\ \mu = 1 + \lambda,\ \lambda = 0\}$
2) $\{\kappa = -\lambda\mu + L_R,\ \kappa = \lambda\mu + L_R,\ \mu = 0,\ \lambda > 0\}$
3) $\{\kappa = -\lambda\mu + L_R,\ \lambda = 1,\ \mu = 0\}$
4) $\{\kappa = -\lambda\mu + L_R,\ \lambda = 1,\ \mu = L_R\}$

5) $\{K(X,Y) = \kappa + \lambda\mu,\ K(X,Y) = \kappa - \lambda\mu,\ \mu = 1 - \lambda,\ \kappa = -\lambda\mu + L_R\}$

6) $\{\mu = 1 + \lambda,\ \lambda = 1,\ L_R = \pm 2\}$

7) $\{\mu = 1 + \lambda,\ K(X,Y) = \kappa - \lambda\mu,\ K(X,Y) = \kappa + \lambda\mu\}$

8) $\{\kappa = -\lambda\mu + L_R,\ \mu = 1 - \lambda,\ K(X,Y) = \kappa + \lambda\mu\}$

9) $\{\mu = 1 + \lambda,\ \kappa = \lambda\mu + L_R,\ K(X,Y) = \kappa - \lambda\mu\}$

From the first system we get easily $\mu = 1$ and since $\lambda^2 = 1 - \kappa$ we have $\kappa = 1$, which is a contradiction, since we required that $\kappa < 1$. The systems 2, 3, 4 and 5 have as the only solution $\kappa = 0,\ \mu = 0, \lambda = 1,\ L_R = 0$. Then $R_{XY}\xi = 0$ for any $X, Y \in \chi(M)$ and M is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ [1]. We show that remainder systems can not occur. In system 6, from $\lambda = 1$ we have $\mu = 0$ and $\kappa = 0$. Using equation (13) (or (14)) and (7)(ii) we have $[g(X, \varphi Y)]^2 = -1$ and this is a contradiction. From system 7, one can get easily $\lambda\mu = 0$. But $\lambda \neq 0$ (since $\kappa < 1$) and then $\mu = 0$. Therefore $\lambda = \mu - 1 = -1$ and this is a contradiction with $\lambda > 0$. In two last systems for all $X, Y \in \chi(M)$ we have

$$K(X,Y) = L_R, \tag{15}$$

Let $Y = \varphi X$ in (15) and comparing it with equation (7)(ii) we get

$$L_R = -(\kappa + \mu), \tag{16}$$

Replacing κ and μ of two last systems in (16) we get

$$i)(1-\lambda)^2 = -2L_R, \qquad ii)(1+\lambda)^2 = -2L_R. \tag{17}$$

Then in systems 8 and 9 $L_R \leq 0$. In system 8, by virtue of $\kappa = -\lambda\mu + L_R$ and $\kappa = 1 - \lambda^2$, we have $2\lambda^2 - \lambda + (L_R - 1) = 0$. This quadratic equation has two roots $\lambda = 1 \pm \sqrt{9 - 8L_R}$. If $\lambda = 1 + \sqrt{9 - 8L_R}$ and replacing it in (17)(i) we get $L_R = 1.5$ and if $\lambda = 1 - \sqrt{9 - 8L_R}$, since λ is positive, we get $L_R > 1$. Then in the both case we get contradction whit $L_R \leq 0$. The roots of equation (17)(ii) in last system are $\lambda = -1 \pm \sqrt{-2L_R}$ and since $\lambda > 0$ then $\lambda = -1 + \sqrt{-2L_R}$ and hence $\mu = \sqrt{-2L_R}$. Substituting λ and μ in $\kappa = \lambda\mu + L_R$ and $\kappa = 1 - \lambda^2$ we get $L_R = -2$ and then $\lambda = 1, \mu = 2$ and $\kappa = 0$ which are not acceptable since from (13) (or (14)) we get a contradiction from (7)(ii) and this complete the proof. □

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Talk Complete CGC hypersurfaces in hyperbolic space pp.: 1–4

COMPLETE CGC HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract

In this paper, we deal with complete and connected Hypersurfaces immersed in the Hyperbolic space with constant scalar curvature, and constant Gauss- kronecker curvature. Let $\phi : M^n \to \mathbb{H}^{n+1}$ be an orientable hypersurface with constant scalar curvature R , which has zero Gauss Kronecker curvature.Then M^n is a totally geodesic hypersurface.

Keywords: Complete hypersurfaces, Gauss-Kronecker curvature, Hyperbolic space, Scalar curvature, Totally umblicial hypersurfaces
Mathematics Subject Classification [2010]: 53B30, 53C21, 53C17

1 Introduction

In this paper, we are interested in the study of the geometry of a complete hypersurface isometrically immersed in $\mathbb{H}^{n+1}$, and the correlation between r-th elementary symmetric functions of M^n, Scalar curvature, and r-th mean curvature H_r of M^n. Also, we give some correction to a mistake happend in a paper about Gauss mapping of hypersurfaces with constant scalar curvature in $\mathbb{H}^{n+1}$ [2].

We deal with Minkowski space $\mathbb{R}^{n+2}_1$ as the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric g given by

$$g(u, v) = \sum_{i=1}^{n+1} u_i v_i - u_{n+2} v_{n+2},$$

for $u, v \in \mathbb{R}^{n+2}_1$.

The $(n + 1)$-dimensional hyperbolic space

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}_1; < x, x >= -1, x_{n+2} \geqslant 1\}.$$

is a spacelike hypersurface in $\mathbb{R}^{n+2}_1$.

Let M^n be a connected and oriented isometrically immersed hypersurface $\phi : M^n \to \mathbb{H}^{n+1}$, and denote $A : T_pM^n \to T_pM^n$ as the shape operatore of the immersion ϕ at the point $p \in M$.

At each point the linear operator A on $T_p(M^n)$ is self-adjoint $(c.f[4], chapter4)$, so the real eigenvalues of the operator A are called the Principal curvatures, and we will be denoted by $k_1, \ldots, k_n$.

For a suitably chosen local field of orthonormal frames $\{e_1, \ldots, e_n\}$ on M^n, we have

$$Ae_i = k_i e_i \quad i = 1, \cdots, n$$

Definition 1.1. Associated to the shape operator A one has, for each $0 \leqslant r \leqslant n$ algebraic invariants S_r given by

$$S_r = \sigma_r(k_1, \ldots, k_n) = \sum_{i_1 < \cdots < i_r} k_{i_1} \ldots k_{i_r}.$$

Where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

Definition 1.2. The r-th mean curvature H_r of the hypersurface is then defined by

$$\begin{aligned} H_r &= \frac{1}{\binom{n}{r}} \sum_{i_1 < \cdots < i_r} k_{i_1} \ldots k_{i_r} \\ &= \binom{n}{r}^{-1} S_r. \end{aligned}$$

Remark 1.3. From above equality and using definition (1.1), we have

$$S_0 = 1,$$

$$S_1 = \sum_{i=1}^{n} k_i = tr(A) = nH_1, \tag{1}$$

Nothe that when $r = 1$, H_1 is the mean curvature, and when $r = n$, H_r is the *Gauss-Kronecker* curvature. Therefore, $det(A) = k_1 \ldots k_n$, is called *Gauss-Kronecker* curvature.

Also, we consider the traceless operator $T : T_pM \to T_pM$, which is given by

$$T(X_p) = A(X_p) - H_1(X_p),$$

For all $X_p \in T_pM^n$.

And the Hilbert-schmidt norm of oprator T is given by

$$|T|^2 = \frac{1}{n}(k_i - k_j)^2.$$

The next results, due to previous definition, and Gauss Weingarten equation, to use them for proving the rigidity theorem.

Lemma 1.4. *Let $k_1, \ldots, k_n$ be real eginvalues of operator A, suppose that S_1, S_2 are the first, and the second elementary symmetric functions of M^n, we have*

$$|A|^2 + 2S_2 = S_1^2. \tag{2}$$

Now we are ready to correct the mistake in [2].

Proposition 1.5. *S_2 is a constant function on M^n if, and only if the scalar curvature R of M^n is constant.*

Proof. from Gauss equation we have that the Ricci curvature tensor of M^n, denoted by Ric_M, is given by

$$Ric_M(X,Y) = (1-n) < X,Y > +nH_1 < AX,Y > - < AX,AY >, \tag{3}$$

for all $X, Y \in TM^n$. on the other hand, from 3 we have that the scalar curvature R of M^n satisfies

$$R = \sum_{i=1}^{n} Ric(e_i, e_i) = n(1-n) + n^2H_1^2 - |A|^2. \tag{4}$$

we use 2 to conclude that

$$\begin{aligned} R &= n(1-n) + S_1^2 - |A|^2 \\ &= n(1-n) + 2S_2. \end{aligned}$$

□

2 Main result

Now, we are finally in position to prove the following theorem.

Theorem 2.1. *Let $\phi : M^n \to \mathbb{H}^{n+1}$, $n \geqslant 3$, be a hypersurface immersed in $\mathbb{H}^{n+1}$. suppose that scalar curvature of M^n is $R = n(1-n)$, and the* Gauss-Kronecker *curvature is zero. Then M^n is a totally geodesic hypersurface.*

Proof. We observe that our hypothesis under the scalar curvature of M^n, and our proposition, amounts to the fact that, S_2 is zero on M^n, so by using 2 we have that $|A|^2 = S_1^2 = n^2H_1^2$. where A is the wiengarten operator of M^n. Since the Hilbert- schmidt norm of T satisfies $|T|^2 = |A|^2 - nH_1^2$, we get that

$$|\phi|^2 = n(n-1)H_1^2, \tag{5}$$

Also, we have

$$tr(T^3) = (n-2)H_1|T|^2,$$

Therfore, from 5 we have

$$|tr(T^3)| = \frac{n-2}{\sqrt{n(n-1)}}|T|^3.$$

And it follows from lemma (c.f [1]) that at least $n-1$ of eigenvalues of T are equal, and hence at least $n-1$ of eigenvalues of A are equal. Denoted by

$$k_1 = \cdots = k_{n-1} = a \quad k_n = b$$

On the other hand, $H_n = 0$. so, $S_n = k_1 \dots k_n = (a)^{n-1}b = 0$. Therfore, $ab = 0$ on M^n, and we have that

$$S_2 = (n-2)^2a^2 + (n-2)ab = 0.$$

Hence,$a = 0$.

Let θ_{ij} is the 2D subspace of T_xM^n generated by e_i and e_j. Therfore, from Gauss equation (c.f[4]) we have

$$K(\theta_{ij}) = -1 + k_ik_j,$$

From S_2 and S_n, we get that

$$K = -1.$$

For all $1 \leqslant i, j \leqslant n$. Thus, since M^n is a hypersurface of constant sectional curvature -1, we have from [4] that M^n is isometric to $\mathbb{H}^n$.Therfore, M^n must be a totally umblicial hypersurface immersed in hyperbolic space.

Moreover, since$S_2 = 0$ we can conclude that M^n must be a totally geodesic hypersurface in hyperbolic space.

□

Example 2.2. Consider an integer λ satisfying $0 \leqslant \lambda < n$. Let

$$\phi : M^n = S^\lambda(\tau) \times \mathbb{H}^{n-\lambda}(\sqrt{1+\tau^2}) \to \mathbb{H}^{n+1},$$

Be a hypersurface immersed in hyperbolic space. [2] showed that M^n is a totally umblical hypersurface, but the scalar curvature $R = \frac{1}{\tau^2}n(n-1) \neq n(1-n)$, and we have $H_n \neq 0$. Consequently, our theorem is not a biconditional theorem. There are quite a few examples to show the same result.

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Containment problem for the ideal of fatted almost collinear closed points in $\mathbb{P}^2$

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Abstract

In this paper, we study the containment problem for the ideal of a zero dimensional closed subscheme $Z = cp_0 + p_1 + \cdots + p_n$ of $\mathbb{P}^2$, where all points p_i except p_0, lie on a line and p_0 is considered with multiplicity c. We determine some numerical invariants of the ideal of this type of configuration, that is, the least degree of the generators of $I(Z)^{(r)}$, the resurgence of $I(Z)$ as well as the Waldschmit's constant of $I(Z)$.

Keywords: Symbolic power, Resurgence, Fat point
Mathematics Subject Classification [2010]: 14N20, 13F20

1 Introduction

Let $R = \mathbb{K}[\mathbb{P}^N] = \mathbb{K}[x_0, x_1, \ldots, x_N]$ be the homogeneous coordinate ring of the projective space $\mathbb{P}^N$, where $\mathbb{K}$ is an algebraically closed field of arbitrary characteristic. Let I be a nontrivial homogeneous ideal of R. The r^{th} symbolic power of I is defined to be the ideal

$$I^{(r)} = \cap_{P \in \operatorname{Ass}(I)}(R \cap I^r R_P).$$

Equivalently, $I^{(r)}$ is the contraction of the ideal $I^r R_U$ to R, i.e.,

$$I^{(r)} = R \cap I^r R_U,$$

where U is the multiplicative closed set $R - \cup_{P \in \operatorname{Ass}(I)} P$.
A natural algebraic operation for investigating the algebraic structure of I is to study the behavior of its ordinary power I^r, for each positive integer r, i.e., the ideal generated by products of r elements of I. On the other hand, I^r determines a closed subscheme of $\mathbb{P}^N$, a geometric object that is defined by the intersection of those primary components of I^r which their radical are strictly contained in $\langle x_0, x_1, \ldots, x_N \rangle$, denoted by $I^{(r)}$. But contrary to I^r, the generators of $I^{(r)}$ can not be obtained easily. A natural way to obtain information about the generators of $I^{(r)}$, is to compare its generators with the generators of different ordinary powers of I. In this direction, it can be easily proved that $I^m \subseteq I^{(r)}$

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if and only if $m \geq r$. If $I^{(r)} \subseteq I^m$ then it follows $m \leq r$. However, nearly less is known about pairs of integers (r, m), where the containment $I^{(r)} \subseteq I^m$ holds. Since this question is of interest for both commutative algebraist and algebraic geometers it has motivated a lot of research on this topic and the related subjects in both these communities. The recent the survey article [5] reflects the current status of the question.
To determine how much the symbolic power $I^{(r)}$ deviates from containment in ordinary power I^m, Bocci and Harbourne introduced an asymptotic measure, so called *resurgence* of I, which is defined for a non-trivial homogeneous ideal of R as:

$$\rho(I) = \sup\{\frac{r}{m} \mid I^{(r)} \nsubseteq I^m\}.$$

If $r/m \geq \rho(I)$, as an immediate consequence of this definition, it follows $I^{(r)} \subseteq I^m$. Moreover, by [4], the containment $I^{(mN)} \subseteq I^m$ always holds, which implies $\rho(I) \leq N$. In addition, by the definition of $\rho(I)$, it follows $\rho(I) \geq 1$.
Computing the symbolic power of a nontrivial homogeneous ideal of R is not so straightforward. However, in some cases one can progress toward it further. For example if the ideal I, can be represented as $I_1^{m_1} \cap I_2^{m_2} \cap \cdots \cap I_s^{m_s}$, where for each $1 \leq j \leq s$, m_j is a positive integer and the ideal I_j is a complete intersection, then by unmixedness theorem, $I^{(r)} = I_1^{rm_1} \cap I_2^{rm_2} \cap \cdots \cap I_s^{rm_s}$. In particular, since the ideal of forms which vanish on a closed point $p \in \mathbb{P}^N$ is a complete intersection, if I is the ideal of forms which vanish with multiplicity at least $m_i, 1 \leq i \leq s$, at each point of the set $\{p_1, \ldots, p_n\}$ in $\mathbb{P}^N$, then $I = \cap_{i=1}^n I(p_i)^{m_i}$ and

$$I^{(r)} = \cap_{i=1}^n I(p_i)^{rm_i}. \tag{1}$$

In this case, I is known as the ideal of a fat points subscheme of $\mathbb{P}^N$, which is denoted by $Z = m_1p_1 + m_2p_2 + \cdots + m_np_n$, and $I^{(r)}$ is the ideal of the fat points scheme $rZ = rm_1p_1 + \cdots + rm_np_n$. Due to this simple description of symbolic power these type of ideals, it is natural to restrict to these ideals to understand about the structure of pairs (r, m) such that $I^{(r)} \subseteq I^m$.

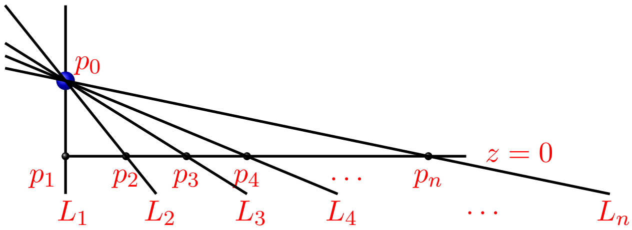
2 Main results

The closed subscheme of points of $\mathbb{P}^2$ which we will consider have a special structure, which are defined as follows.

Definition 2.1. Let $Z = p_0 + p_1 + \cdots + p_n$, where $n \geq 3$, be a zero dimensional subscheme of $\mathbb{P}^2$. Z is called an almost collinear subscheme of $n+1$ points if all these points except p_0, lie on a line L. Moreover, if $1 \leq c \leq n$ is an integer, the subscheme $Z = cp_0 + p_1 + \cdots + p_n$ is called a fatted almost collinear points with multiplicity c subscheme.

Let Z be a fatted almost collinear subscheme of $\mathbb{P}^2$, and let $I = I(Z)$ be the ideal of forms vanishing on Z. Without loss of generality, we may assume the collinear points $p_1, \ldots, p_n$ all lie on the line $z = 0$ and let p_1 be the intersection point of the lines $x = 0$ and $z = 0$. For each $2 \leq i \leq n$, let p_i be the intersection point of the lines $z = 0$ and $x - \ell_i y = 0$, where ℓ_is are non zero distinct elements of $\mathbb{K}$. Moreover, we may assume p_0 is the intersection point of the lines $x = 0$ and $y = 0$. Then

$I = I(Z) = (x, y)^c \cap (z, F)$, where $F = x(x-\ell_2 y)\dots(x-\ell_n y)$ is a homogeneous polynomial in x, y of degree n. Since the ideals (x, y) and (z, F) are complete intersection, by (1), $I^{(r)} = ((x, y)^c)^{(r)} \cap (z, F)^{(r)} = (x, y)^{cr} \cap (z, F)^r$.
Let i be a nonnegative integer, then by division algorithm $i = an + e$, with $0 \leq e < n$. We show the polynomial $x^e F^a$ by H_i.



Fatted almost collinear point configuration

Now let I be the ideal of a $(n+1)$ fatted almost collinear points. One of the main results of this note is the computation of resurgence of this I. For this purpose, we use [3, Lemma 2.2], which gives a $\mathbb{K}$-vector space basis for the ring $R = \mathbb{K}[x, y, z]$ consisting of elements in the form $H_i y^j z^l$.

Lemma 2.2. *A $\mathbb{K}$-basis of R is given by $\mathcal{B}_R = \cup_{i\geq 0} B_i$, where*

$$B_i = \{H_i y^j z^l \mid i = an + e,\ 0 \leq e < n,\ H_i = x^e F^a,\ \text{and } i, j, l \geq 0\}.$$

Lemma 2.3. *Let $m \geq 1$. Then*

(a) $H_i y^j z^l \in I^{(r)}$ if and only if $i, j, l \geq 0, i + ln \geq rn$, and $i + j \geq cr$.

(b) Moreover, $I^{(r)}$ is the $\mathbb{K}$-vector space span of the elements of the form $H_i y^j z^l$, contained in $I^{(r)}$.

Lemma 2.4. *Let $m \geq 1$. Then*

(a) The ideal I^m is $\mathbb{K}$-vector space span of the elements of the form $H_i y^j z^l \in I^m$. In addition, if $H_i y^j z^l \in I^m$, then $H_i y^j z^l$ is a product of m elements of I.

(b) Moreover, $H_i y^j z^l \in I^m$ if and only if $i, j, l \geq 0$, and either;

(1) $l < \frac{j}{c}$ and $i + nl \geq mn$, or
(2) $\frac{j}{c} \leq l < \frac{i+j}{c}$ and $i + j + (n-c)l \geq mn$, or
(3) $\frac{i+j}{c} \leq l$ and $m \leq \frac{i+j}{c}$.

Theorem 2.5. *Let I be the ideal of an $n+1$ fatted almost collinear points. Then $I^{(r)} \nsubseteq I^m$ if and only if $r \leq \frac{n^2 m - n}{n^2 - nc + c^2}$. In particular, $\rho(I) = \frac{n^2}{n^2 - nc - c^2}$.*

Remark 2.6. In the Definition 2.1, we assumed that the multiplicity of p_0 to be $1 \leq c \leq n$. In fact, since $c \leq n$, we have $(x, y)^n \subset (x, y)^c$, and since $F = x(x-\ell_2)\dots(x-\ell_n y) \in (x, y)^n$, we have $F \in (x, y)^c$ and therefore,

$$I = (x, y)^c \cap (z, F) = (zx^c, zx^{c-1}y, \dots, zxy^{c-1}, zy^c, F).$$

We need this description of I for computational purposes. Moreover, if we assume $c > n$, we can not use the above theorem to compute the resurgence of I, because with this assumption, we obtain $\rho(I) < 1$, which is impossible.
However, if we consider the case $c > n$, then by a computer algebra system (such as Singular [2]), one can easily check $I^{(r)} = I^r$.

For a homogeneous ideal I of the ring $R = \mathbb{K}[\mathbb{P}^N]$, let $\alpha(I)$ be the least degree of the of a non-zero homogeneous element of I. It is trivial that $\alpha(I)$ is an invariant of I. Moreover, for any r, $\alpha(I^r) = r\alpha(I)$. On the other hand, the behavior of $\alpha(I^{(r)})$ is not similar to the behavior of the ordinary power of I. In fact, since $I^r \subseteq I^{(r)}$, we have $\alpha(I^{(r)}) \leq r\alpha(I)$.
For any homogeneous ideal I of R, by [1, Lemma 2.3.1], the limit

$$\gamma(I) = \lim_{r\to\infty} \frac{\alpha(I^{(r)})}{r} \tag{2}$$

exits and is called the Waldschmit constant of I. This is also one of the invariants of I.
For a general homogeneous ideal, computing $\alpha(I^{(r)})$ as well as its resurgence is not so easy. However, if I is the ideal of a fatted almost collinear points, then in the following theorems, $\alpha(I^{(r)})$ and $\gamma(I)$ are given explicitly.

Theorem 2.7. *Let I be the ideal of a fatted almost collinear points with multiplicity c. Then*

$$\alpha(I^{(m)}) = \lceil \frac{m(1+c)n - c}{n} \rceil.$$

Theorem 2.8. *Let I be the ideal of a fatted almost collinear points with multiplicity c. Then*

$$\gamma(I) = (1+c) - \frac{c}{n}.$$

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Curvature of multisymplectic connections of order 3

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Abstract

In this paper we show that on multisymplectic manifold (m, ω) of order 3 there is at leat a multisymplectic connection. Then we study the curvature tensor of a multisymplectic connection of order 3.

Keywords: Multisymplectic manifold, Multisymplectic connection, Curvature tensor
Mathematics Subject Classification [2010]: 53D05, 53C05

1 multisymplectic connection of order 3

Multisymplectic structures in field theory play a role similar to that of symplectic structures in classical mechanics. A multisymplectic manifold (M, ω) of order 3 is a manifold M endowed with a closed 3-form ω on M which is nondegenerate. Nondegeneracy of ω meas that for a vector field X on M

$$i_X\omega = 0 \text{ if and only if } X = 0.$$

A connection ∇ on (M, ω) is called multisymplectic connection it is both symmetric $(\nabla_X Y - \nabla_Y X = [X, Y])$ and compatible to the ω $(\nabla\omega = 0)$. If ∇ be a connection on M then $\nabla\omega = 0$ if and only if

$$V(\omega(X, Y, Z)) = \omega(\nabla_V^X, Y, Z) + \omega(X, \nabla_V^Y, Z) + \omega(X, Y, \nabla_V^Z), \tag{1}$$

for any vector field X, Y, Z, V.
Also ω is closed if and only if

$$X(\omega(Y, Z, V)) - Y(\omega(X, Z, V)) + Z(\omega(X, Y, V)) - V(\omega(X, Y, Z)) - \tag{2}$$

$$\omega([X, Y], Z, V) + \omega([X, Z], Y, V) - \omega([X, V], Y, Z) - \omega([Y, Z], X, Y) + \omega([Y, V], X, Z) - \omega([Z, V], X, Y) = 0$$

for any vector field X, Y, Z, V.
Let (M, ω) be a multisymplectic manifold of order 3 and ∇ be a connection on M. If $x^1, ..., x^n$ are local coordinates, introduce the Christoffel symbols Γ_{ij}^k by $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k$. The components of ω in these coordinates are $\omega_{ijk} = \omega(\partial_i, \partial_j, \partial_k)$. It is sufficient to write (1) for $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$ and $V = \partial_l$. This gives

$$\partial_l\omega_{ijk} = \omega(\nabla_{\partial_l}\partial_i, \partial_j, \partial_k) + \omega(\partial_i, \nabla_{\partial_l}\partial_j, \partial_k) + \omega(\partial_i, \partial_j, \nabla_{\partial_l}\partial_k)$$

*Speaker

$$= \omega_{\lambda jk}\Gamma^{\lambda}_{li} + \omega_{i\lambda k}\Gamma^{\lambda}_{lj} + \omega_{ij\lambda}\Gamma^{\lambda}_{lj} = \Gamma_{jkli} - \Gamma_{iklj} + \Gamma_{ijlk},$$

where $\Gamma_{ijlk} = \omega_{ij\lambda}\Gamma^{\lambda}_{lk}$.
The equality $d\omega = 0$ means

$$\partial_i\omega_{jkl} - \partial_j\omega_{ikl} + \partial_k\omega_{ijl} - \partial_l\omega_{ijk} = 0.$$

Consider Π be another symmetric connection on M. We have $\Pi_{ijkl} = \Pi_{ijlk} = -\Pi_{jikl}$.

Proposition 1.1. *Let Π be a symmetric connection. If we define $\Gamma_{ijkl} = \partial_l\omega_{kij} + \Pi_{ijkl} - \Pi_{jilk} - \Pi_{likj} + \Pi_{ljik}$ then Γ compatible to the ω.*

Proof. Since $d\omega = 0$, we have $\partial_l\omega_{ijk} = \partial_i\omega_{ljk} - \partial_j\omega_{lik} + \partial_k\omega_{lij}$. It is easy to show that $\Gamma_{jkli} - \Gamma_{iklj} + \Gamma_{ijlk} = \partial_l\omega_{ijk}$. So $\nabla\omega = 0$. □

2 Curvature of multisymplectic connections of order 3

If ∇ be a multisymplectic connection of order 3 on M. The curvature ∇ is defined by usual formula

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The components of the curvature tensor are introduce by

$$R(\partial_i,\partial_k)\partial_j = R^m_{ijk}\partial_m.$$

The curvature R^m_{klt} satisfies the tensor equations

$$R^m_{klt} + R^m_{ltk} + R^m_{tkl} = 0.$$

And

$$\nabla_s R^m_{klt} + \nabla_l R^m_{kts} + \nabla_t R^m_{ksl} = 0.$$

Denote also

$$R_{ijklt} = \omega_{ijm}R^m_{klt} = \omega(\partial_i,\partial_j,R(\partial_l,\partial_t)\partial_k).$$

The components of the curvature tensor in terms of the Christoffel symbols has the standard form;

$$R^l_{ijk} = \partial_j\Gamma^l_{ki} - \partial_k\Gamma^l_{ij} + \Gamma^m_{ki}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{km}.$$

Instead of R_{ijklt} we can also consider $R(X,Y,Z,V,W)$ which is a multilinear function on any tangent space T_xM:

$$R(X,Y,Z,V,W) = \omega(X,Y,R(V,W)Z).$$

So that

$$R_{ijklt} = R(\partial_i,\partial_j,\partial_k,\partial_l,\partial_t).$$

It is obvious that

$$R_{ijklt} = -R_{ijktl}.$$

And

$$R_{ij(klt)} = R_{ijklt} + R_{ijltk} + R_{ijtkl} = 0.$$

Proposition 2.1. *For any multisymplectic connection of order* 3

$$R_{jkilt} - R_{ikjlt} + R_{ijklt} = 0.$$

Proof. Let us consider

$$\begin{aligned}
\partial_t \partial_l \omega_{ijk} &= \partial_t(\omega(\nabla_{\partial_l}\partial_i, \partial_j, \partial_k) + \omega(\partial_i, \nabla_{\partial_l}\partial_j, \partial_k) + \omega(\partial_i, \partial_j, \nabla_{\partial_l}\partial_k) \\
&= \omega(\nabla_t \nabla_l \partial_i, \partial_j, \partial_k) + \omega(\nabla_l \partial_i, \nabla_t \partial_j, \partial_k) + \omega(\nabla_l \partial_i, \partial_j, \nabla_t \partial_k) \\
&= \omega(\nabla_t \partial_i, \nabla_l \partial_j, \partial_k) + \omega(\partial_i, \nabla_t \nabla_l \partial_j, \partial_k) + \omega(\partial_i, \nabla_l \partial_j, \nabla_t \partial_k) \\
&= \omega(\nabla_t \partial_i, \partial_j, \nabla_l \partial_k) + \omega(\partial_i, \nabla_t \partial_j, \nabla_l \partial_k) + \omega(\partial_i, \partial_j, \nabla_t \nabla_l \partial_k).
\end{aligned}$$

Changing places t, l and subtracting the result, we obtain

$$\begin{aligned}
0 &= \omega([\nabla_{\partial_l}, \nabla_{\partial_t}]\partial_i, \partial_j, \partial_k) + \omega(\partial_i, [\nabla_{\partial_l}, \nabla_{\partial_t}]\partial_j, \partial_k)\omega(\partial_i, \partial_j, [\nabla_{\partial_l}, \nabla_{\partial_t}]\partial_k) \\
&= \omega(R^m_{ilt}\partial_m, \partial_j, \partial_k) + \omega(\partial_i, R^m_{jlt}\partial_m, \partial_k) + \omega(\partial_i, \partial_j, R^m_{klt}\partial_m) \\
&= \omega_{mjk} R^m_{ilt} + \omega_{imk} R^m_{jlt} + \omega_{ijm} R^m_{klt} = R_{jkilt} - R_{ikjlt} + R_{ijklt}.
\end{aligned}$$

□

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Curvature properties and totally geodesic hypersurfaces of some para-hypercomplex Lie groups*

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Abstract

In this paper we study some geometrical properties of four-dimensional para- hypercomplex Lie groups. In fact we first explicitly give all totally geodesic hypersurfaces on four types of these homogeneous spaces. Then we investigate Einstein like metrics on these spaces. The existence of four-dimensional para-hypercomplex Lie groups with parallel or cyclic Ricci tensor are also proved.

Keywords: Totally geodesic hypersurfaces, Para-hypercomplex Lie groups, Parallel Ricci tensor
Mathematics Subject Classification [2010]: 53C42, 53C30.

1 Introduction

Hypercomplex and para-hypercomplex structures are interesting structures in mathematics which have many important applications in physics. In [3] Barberis studied four dimensional Lie groups which admit hypercomplex structures and gave a classification for these spaces. Four dimensional real Lie algebras which admit para- hypercomplex structures are classified in [4] by Blazic and Vukmirovic. Then in [7] Salimi Moghaddam considered connected Lie groups corresponding to some of these Lie algebras and gave the exact form of their Levi-Civita connections and sectional curvatures. Also in [1] we have studied harmonicity of invariant vector fields and left-invariant Ricci solitons on these homogeneous spaces. Our aim in this paper is to describe explicitly totally geodesic hypersurfaces on these homogeneous spaces. We also prove the existence of four- dimensional para-hypercomplex Lie groups whose Ricci tensor is parallel or cyclic.

2 Four-dimensional para-hypercomplex Lie groups

Here we report the following classification which is given in [4].

Theorem 2.1. *Up to an isomorphism the only four-dimensional Lie algebras $\mathcal{G}$ admitting an integrable para-hypercomplex structure are either abelian or isomorphic to one of the following Lie algebras*

*Will be presented in English
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(A_1) $[X_1, X_2] = X_3$, (A_2) $[X_1, X_2] = X_1$, (A_3) $[X_1, X_3] = X_1$, $[X_2, X_4] = X_2$,
(A_4) $[X_1, X_2] = X_4$, $[X_2, X_4] = -X_1$, $[X_4, X_1] = X_2$, (A_5) $[X_1, X_2] = X_2$, $[X_1, X_4] = X_4$,
(A_6) $[X_1, X_3] = X_1$, $[X_1, X_4] = X_2$, $[X_2, X_3] = X_2$, $[X_2, X_4] = aX_1 + bX_2$,
(A_7) $[X_1, X_2] = X_3$, $[X_1, X_4] = X_1 + aX_2 + bX_3$, $[X_4, X_2] = X_2$,
(A_8) $[X_3, X_4] = X_3$, $[X_2, X_4] = X_2$, $[X_1, X_4] = cX_1 + aX_2 + bX_3$,
(A_9) $[X_2, X_1] = X_3$, $[X_4, X_3] = cX_3$, $[X_4, X_1] = \frac{1}{2}X_1 + aX_2 + bX_3$, $[X_4, X_2] = (c - \frac{1}{2})X_2$.
where $c \neq 0, a, b \in \mathbb{R}$ and $\{X_1, X_2, X_3, X_4\}$ is an orthonormal basis.

Here we consider connected Lie groups which correspond to Lie algebras of this classification and by using the results from [7] which are on the cases $(A_1), \cdots, (A_6)$, we obtain the exact form of totally geodesic hypersurfaces on four types of these spaces. For this purpose we first recall the following definition and for more details we refer to [2] and [5]. Let $F : M^n \to N^{n+1}$ be an isometric immersion of Riemannian manifolds $(M, <, >)$ and $(N, <, >)$ with the Levi-Civita connections ∇^M and ∇. Also let ξ be a unit normal vector field on the hypersurface M, h be the second fundamental form $h(X, Y) =< SX, Y >$ and S be the shape operator $SX = -\nabla_X \xi$. Then the Gauss formula is given by

$$\nabla_X Y = \nabla_X^M Y + h(X, Y)\xi, \tag{1}$$

which yields the following Codazzi equation

$$< R(X, Y)Z, \xi >= (\nabla^M h)(Y, X, Z) - (\nabla^M h)(X, Y, Z), \tag{2}$$

where R is the curvature tensor of M and $(\nabla^M h)$ is defined by $(\nabla^M h)(X, Y, Z) = X(h(Y, Z)) - h(\nabla_X^M Y, Z) - h(Y, \nabla_X^M Z)$. The hypersurface M is said to be totally geodesic, if the second fundamental form vanishes identically i.e., $h = 0$.

Lemma 2.2. *Let $F : M^3 \to G$ be a non-degenerate hypersurface of a four-dimensional para-hypercomplex Lie group G. Also let ξ be a unit normal vector field on M and $\{X_1, \ldots, X_4\}$ be the orthonormal frame field on G. The second fundamental form of this immersion is a Codazzi tensor if and only if there exists a local function θ on M^3 such that ξ has one of the following forms*
(a) For the type (A_1), $\xi = \pm X_1, \pm X_2, \pm X_3, \pm X_4$, or $cos\theta X_1 + sin\theta X_2$.
(b) For the type (A_2), $\xi = \pm X_1, \pm X_2, \pm X_4, cos\theta X_1 + sin\theta X_2$, or $cos\theta X_3 + sin\theta X_4$.
(c) For the type (A_3), $\xi = \pm X_1, \pm X_3, \pm X_4, cos\theta X_1 + sin\theta X_3$, or $cos\theta X_2 + sin\theta X_4$.
(d) For the type (A_4), $\xi = \pm X_1, \pm X_2, \pm X_3, \pm X_4$ or $cos\theta X_2 + sin\theta X_4$.

Proof. Assume that $\xi = \sum_{i=1}^{4} b_i X_i$ is a unite normal vector field on the hypersurface M, where $b_i : U \subseteq M \to \mathbb{R}$ are some functions. Then $V_1 = b_1X_2 - b_2X_1$, $V_2 = b_1X_3 - b_3X_1$, $V_3 = b_1X_4 - b_4X_1$, $V_4 = b_3X_2 - b_2X_3$, $V_5 = b_4X_2 - b_2X_4$, and $V_6 = b_4X_3 - b_3X_4$ are tangent to the hypersurface M^3. First we assume that h is a Codazzi tensor. Then by (2) we have $< R(V_i, V_j)V_k, \xi >= 0$, where $i, j, k \in \{1, \ldots, 6\}$. In particular for the type (A_2) from $< R(V_1, V_2)V_1, \xi >= 0$ we have $b_1b_3(b_1^2 + b_2^2) = 0$, which implies the following three cases $b_1 = 0$, $b_3 = 0$ and $b_2 = b_1 = 0$.
Case 1: $b_1 = 0$. In this case from $< R(V_1, V_4)V_1, \xi >= 0$ we have $b_2^3 b_3 = 0$, which gives us $b_3 = 0$ and $b_2 = 0$. If $b_3 = 0$, then from $< R(V_1, V_5)V_1, \xi >= 0$ we have $b_2^3 b_4 = 0$ which yields that $\xi = \pm X_2$ and $\xi = \pm X_4$. If $b_2 = 0$, then for all i, j, k we have

$< R(V_i, V_j)V_k, \xi >= 0$, which implies that $\xi = cos\theta X_3 + sin\theta X_4$.
Case 2: $b_3 = 0$. In this case from $< R(V_1, V_5)V_1, \xi >= 0$ we have $b_2 b_4(b_2^2 + b_1^2) = 0$ which gives us three subcases: $b_2 = 0$, $b_4 = 0$ and $b_1 = b_2 = 0$. If $b_2 = 0$, then from $< R(V_1, V_3)V_1, \xi >= 0$ we have $b_1^3 b_4 = 0$ which gives us $\xi = \pm X_1$ and $\xi = \pm X_4$. If $b_4 = 0$, then $\xi = cos\theta X_1 + sin\theta X_2$. Also if $b_1 = b_2 = 0$, it yields that $\xi = \pm X_4$.
Case 3: $b_1 = b_2 = 0$. In this case we have $\xi = cos\theta X_3 + sin\theta X_4$.
Conversely, If ξ has one of the forms given in the case (b), then for all i, j and k we have $< R(V_i, V_j)V_k, \xi >= 0$, which gives us that h is totally geodesic.
Types (A_1), (A_3) and (A_4) have a similar proof. □

Theorem 2.3. *Let $F : M^3 \to G$ be a totally geodesic hypersurface of a simply connected four-dimensional para-hypercomplex Lie group G with the Lie algebra $\mathcal{G}$. If $\mathcal{G}$ has one of the types $(A_1), \ldots, (A_4)$, then there exists a local coordinate (w_1, w_2, w_3) on M^3 such that, the immersion with respect to these coordinates, up to isometrics is given by one of the following expressions:*

$$\begin{aligned} &F(w_1, w_2, w_3) = (w_1, w_2, w_3, 0), && F(w_1, w_2, w_3) = (0, w_1, w_2, w_3),\\ &F(w_1, w_2, w_3) = (A, B, w_2, w_3), && F(w_1, w_2, w_3) = (w_1, C, w_3, D),\\ &F(w_1, w_2, w_3) = (w_1, w_2, 0, w_3), && F(w_1, w_2, w_3) = (w_1, w_2, -sin\theta w_3, cos\theta w_3),\\ &F(w_1, w_2, w_3) = (A, w_2, B, w_3), && \end{aligned} \tag{3}$$

where $A = -\int sin(2tan^{-1}(e^{w_1 - k_1}))dw_1$, $B = \int cos(2tan^{-1}(e^{-w_1 - k_1}))dw_1$, $C = -\int sin (2tan^{-1}(e^{w_1 - k_1}))dw_2$, $D = \int cos(2tan^{-1}(e^{-w_1 - k_1}))dw_2$ and k_1 is a real constant.

Proof. Assume that M is a totally geodesic hypersurface. Then ξ has one of the forms $(a), (b), (c)$ and (d) which are given in the lemma 2.2. Let us consider the case (b). If $\xi = cos\theta X_3 + sin\theta X_4$, then $Y_3 = -sin\theta X_3 + cos\theta X_4$, $Y_1 = X_1$ and $Y_2 = X_2$ span the tangent space to M at each point and the non-zero Levi-Civita connections of M are

$$\nabla_{Y_1} Y_1 = -Y_2,\ \nabla_{Y_1} Y_2 = Y_1,\ \nabla_{Y_1} Y_3 = -Y_1(\theta)\xi,\ \nabla_{Y_2} Y_3 = -Y_2(\theta)\xi,\ \nabla_{Y_3} Y_3 = -Y_3(\theta)\xi.$$

Then by the Gauss formula (1) $h = 0$ gives us θ is constant. If we put $Y_i = \partial_{w_i}$ with $i = 1, \ldots, 3$ and denote the immersion of the hypersurface M by $F : M \to G : (w_1, w_2, w_3) \mapsto (F_1(w_1, w_2, w_3), \ldots, F_4(w_1, w_2, w_3))$, then we have

$$\begin{aligned} (\partial_{w_1} F_1, \partial_{w_1} F_2, \partial_{w_1} F_3, \partial_{w_1} F_1) &= (1, 0, 0, 0)\\ (\partial_{w_2} F_1, \partial_{w_2} F_2, \partial_{w_2} F_3, \partial_{w_2} F_1) &= (0, 1, 0, 0)\\ (\partial_{w_3} F_1, \partial_{w_3} F_2, \partial_{w_3} F_3, \partial_{w_3} F_1) &= (0, 0, -sin\theta, cos\theta). \end{aligned}$$

From these equations, we have $F(w_1, w_2, w_3) = (w_1, w_2, -sin\theta w_3, cos\theta w_3)$, where θ is constant. If $\xi = \pm X_4$, then $Y_1 = X_1, Y_2 = X_2, Y_3 = X_3$ span the tangent space to M at each point and the non-zero Levi-Civita connections of M are $\nabla_{Y_1} Y_1 = -Y_2$ and $\nabla_{Y_1} Y_2 = Y_1$. Then by the Gauss formula (1) for $i, j = 1, \ldots, 3$ we have $h(Y_i, Y_j) = 0$ which gives us the totally geodesic hypersurface $F(w_1, w_2, w_3) = (w_1, w_2, w_3, 0)$. If $\xi = \pm X_1$, then by a similar way $F(w_1, w_2, w_3) = (0, w_1, w_2, w_3)$ is a totally geodesic hypersurface. If $\xi = cos\theta X_1 + sin\theta X_2$, then $Y_1 = -sin\theta X_1 + cos\theta X_2$, $Y_2 = X_3$, $Y_3 = X_4$ span the tangent space to M at each point and $h(Y_i, Y_j) = 0$ which gives us $Y_1(\theta) = -sin\theta$ and

$Y_2(\theta) = Y_3(\theta) = 0$. Then by considering the coordinate system $\frac{\partial}{\partial w_i} = Y_i$, $i = 1, 2, 3$, the hypersurface $F(w_1, w_2, w_3) = (A, B, w_2, w_3)$ is totally geodesic, where for a real constant k_1, $A = -\int sin(2tan^{-1}(e^{w_1-k_1}))dw_1$ and $B = \int cos(2tan^{-1}(e^{-w_1-k_1}))dw_1$. The cases $(a), (c)$ and (d) have a similar proof. □

Einstein like metrics are defined through conditions on the Ricci tensor, as follows. A Riemannian manifold (M, g) belongs to the class $\mathcal{A}$ if and only if its Ricci tensor is cyclic-parallel, more exactly $\nabla_{X_i}\rho_{X_jX_k} + \nabla_j\rho_{X_kX_i} + \nabla_k\rho_{X_iX_j} = 0$, and it belongs to the class $\mathcal{B}$ if and only if its Ricci tensor is Codazzi tensor i.e., $\nabla_{X_i}\rho_{X_jX_k} = \nabla_{X_j}\rho_{X_iX_k}$. Also it belongs to $\mathcal{P}$ if and only if its Ricci tensor is parallel that is $\nabla_{X_i}\rho_{X_jX_k} = 0$, where X_i, X_j and X_k are tangent vectors on M(see [6]).

Theorem 2.4. *Let G be a simply connected four-dimensional para-hypercomplex Lie group with the Lie algebra $\mathcal{G}$. If $\mathcal{G}$ has one of the types (A_1) and (A_4), then G is cyclic and if it has the types (A_2) and (A_3), it is parallel.*

Proof. For the type (A_4) the non-zero components are

$$\nabla_{X_2}\rho_{X_1X_4} = 1,\ \nabla_{X_2}\rho_{X_4X_1} = 1,\ \nabla_{X_4}\rho_{X_1X_2} = -1,\ \nabla_{X_4}\rho_{X_2X_1} = -1. \tag{4}$$

Since $\nabla_{X_1}\rho_{X_2X_4} \neq \nabla_{X_2}\rho_{X_1X_4}$ it is not Codazzi. The other types have a similar proof. □

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Talk Existence of extensions for generalized Lie groups pp.: 1–4

Existence of extensions for generalized Lie groups

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Abstract

In this paper introducing the cohomology of generalized Lie groups, we characterize the extensions for generalized Lie groups by elements of the second cohomology group. Moreover we identify a cohomological obstruction to the existence of extensions in non-Abelian case.

Keywords: Generalized Lie groups, Cohomology, Group extensions
Mathematics Subject Classification [2010]: 22N99, 57T10, 22E99

1 Introduction

The problem of extending a group in terms of cohomology can be found in [2]. This problem can be generalized to Lie groups and their generalizations. A special generalization of Lie groups is called generalized Lie groups or top spaces which was introduced by M. R. Molaei in 1998, [4]. In this generalized field, several authors (Araujo, Molaei, Mehrabi, Oloomi, Tahmoresi, Ebrahimi, etc.) have studied different aspects of generalized groups and top spaces [4], [3], [5].

Definition 1.1. [3] A top space T is a non-empty Hausdorff smooth d-dimensional differentiable manifold which is endowed with an operation "." called multiplication such that:

i. $(t_1.t_2).t_3 = t_1.(t_2.t_3)$, for all $t_1, t_2, t_3 \in T$.

ii. For each $t \in T$, there exists a unique $e(t)$ in T such that $t.e(t) = e(t).t = t$.

iii. For each $t \in T$, there exists $s \in T$ such that $t.s = s.t = e(t)$.

iv. $e(t_1.t_2) = e(t_1).e(t_2)$, for all $t_1, t_2 \in T$.

v. The mappings

$$. : T \times T \to T, (t_1, t_2) \mapsto t_1.t_2,$$
$$^{-1} : T \to T, t \mapsto t^{-1},$$

are smooth.

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Throughout this paper by T_a we mean $T \cap e^{-1}(e(a))$.

Definition 1.2. [5] If T and S are two top spaces, then a homomorphism $f : T \to S$ is called a morphism if it is also a C^∞ map.

By f_a we mean $f|_{e^{-1}(e(a))}$, where f is a morphism of top spaces. There exist a correspondence between an action σ of a top space on a manifold M and partial actions $\{\sigma_i\}_{i\in e(T)}$ of $e^{-1}(i)$ on M, for any $i \in e(T)$[6]. Also, in the same reference, we get that if T is a top space, then there is an isomorphism between T and $e(T) \ltimes \{T_i\}_{i\in e(T)}$, where $T_i = e^{-1}(i)$, for all $i \in e(T)$ and $e(T) \ltimes \{T_i\}_{i\in e(T)} = \{(i,t)|t \in T_i\}$, by the production rule

$$(i_1,t_1) \ltimes (i_2,t_2) = (i_1 i_2, t_1 t_2), \quad i_1, i_2 \in e(T), t_1 \in T_{i_1}, t_2 \in T_{i_2}.$$

2 Extensions of top spaces and cohomology

Definition 2.1. Let T, K be top spaces. A top space $\tilde{T}$ is said to be an extension of T by K if K is a top generalized normal subgroup of $\tilde{T}$, i.e $K \prec \tilde{T}$, and $\tilde{T}/K = T$.

Lemma 2.2. In terms of exact sequences, adapting the notations of [5], [6] and the last remark of the previous section, definition (2.1) is equivalent to saying that

$$e(a) \longrightarrow K_a \longrightarrow \tilde{T}_a \longrightarrow T_a \longrightarrow e(a)$$

is exact for all $a \in e(T)$; thus K_a is injected into $\tilde{T}_a$ and $\tilde{T}_a$ projected onto T_a by the canonical homomorphism so that $T_a = \tilde{T}_a/K_a$.

Let Aut$K$ be the group of all automorphisms of K. Then there exist functions $f_a : \tilde{T}_a \to AutK_a$, such that $t \mapsto [\tilde{t}]$ where $[\tilde{t}]$ is defined by

$$[\tilde{t}] : k \in K_a \mapsto \tilde{t}k\tilde{t}^{-1} \in K_a\ .$$

The kernel of f_a is the centeralizer $C_{\tilde{T}_a}(K_a)$ of K_a in $\tilde{T}_a$. Thus, we have the following exact sequence of top space homomorphisms:

$$e(a) \longrightarrow C_{\tilde{T}_a}(K_a) \longrightarrow \tilde{T}_a \xrightarrow{f_a} AutK_a\ .$$

The center C_{K_a} of K_a is top generalized normal subgroup in $C_{\tilde{T}}(K)$; if H_a denotes the quotient group $H_a = C_{\tilde{T}_a}(K_a)/C_{K_a}$,

$$e(a) \longrightarrow C_{K_a} \longrightarrow C_{\tilde{T}_a}(K_a) \longrightarrow H_a \longrightarrow e(a)$$

is exact.

Let T be a top space, and A a Lie group, on which T operates through the homomorphism $\sigma : T \to$Aut$A$ = Out A (since A is Abelian, Int A is reduced to the trivial automorphism). So there exist a family of partial actions

$$\sigma_a : T_a \to AutA_a,\ a \in e(T).$$

To construct a cohomology for top spaces, we need to introduce the n-cochain maps. A mapping $\alpha_n : T \times \ldots \times T \to A$ is a n-cochain, i.e.

$$\alpha_n : (t_1, \ldots, t_n) \mapsto \alpha_n(t_1, \ldots, t_n) \in A \ .$$

The n-cochains form an Abelian top space, i.e. a Lie group, which will be denoted $C^n(T, A)$. Note that α_a for all $a \in e(T)$ is defined on $T_a \times \ldots \times T_a$ with values in A_a. So we may consider $C^n(T_a, A_a)$. The operator $\delta_a : C^n :\to C^{n+1}$ (the coboundary operator) can be defined according to the way that the action $\sigma(t) \in \mathrm{Aut}A$ of the elements t of T is defined on $A[1]$. Consider the following sequence of Abelian top spaces:

$$C_a^0 \xrightarrow{\delta_a^0} C_a^1 \xrightarrow{\delta_a^1} C_a^2 \longrightarrow \ldots \xrightarrow{\delta_a^n} C_a^{n+1} \longrightarrow \ldots \ .$$

for every $a \in e(T)$, we define

$$Z^n_{\sigma_a} := ker\delta^n_a \equiv \{cocycles\},$$

$$B^n_{\sigma_a} := range\delta^{n-1}_a \equiv \{coboundaries\}.$$

Both $Z^n_{\sigma_a}$ and $B^n_{\sigma_a}$ are Lie subgroups of $C^n(T_a, A_a)$.

The quotient group

$$H^n_{\sigma_a}(T_a, A_a) := Z^n_{\sigma_a}(T_a, A_a)/B^n_{\sigma_a}(T_a, A_a)$$

is called the n-th cohomology of T_a with values on A_a for every $a \in e(T)$.

As in the case of abstract groups, the elements of the second cohomology group, characterize the extensions $\tilde{T}$ of the top space T by the Abelian top space A for the given action σ of T on $A[1]$.

We are here concerning about extensions of the top space T by K in the case where K is not Abelian. The main difference from the Abelian case is that not every top space is associated with one or more extensions, i.e. not every top space is extendible. In fact, one of the aims of this chapter is to show that the top space K determines an obstruction to the extension in the form of a certain three-cocycle; the top space T is extendible if this cocycle is, by an abuse of language, trivial.

For every $a \in e(T)$, consider the following exact sequence

$$e(a) \longrightarrow IntK_a \longrightarrow AutK_a \longrightarrow OutK_a \longrightarrow e(a),$$

which makes $AutK_a$ as an extension of $OutK_a$ by $IntK_a$. Let g_a be a trivializing section and let $\alpha_a = g_a \circ \sigma_a$ be defined by

$$\begin{array}{ccccccccc} & & & & & & T_a & & \\ & & & & & \swarrow & \downarrow \sigma & & \\ e(a) & \longrightarrow & IntK_a & \longrightarrow & AutK_a & \underset{g_a}{\rightleftarrows} & OutK_a & \longrightarrow & e(a). \end{array}$$

It is clear that there exists an element $h_a(t', t) \in K_a$ such that

$$\alpha_a(t')\alpha_a(t) = [h_a(t', t)]\alpha_a(t't). \tag{2.1}$$

Consequently, (2.1) defines a mapping

$$[h]_a : T_a \times T_a \to IntK_a,\ [h]_a : (t', t) \mapsto [h_a(t', t)],$$
$$[h_a(t', t)]k := h_a(t', t)kh_a^{-1}(t', t).$$

The associative property in $AutK_a$, leads to the two-cocycle property for $[h_a(t', t)] \in Z^2_{\alpha_a}(T_a, IntK_a)$, where

$$[(\alpha_a(t'')h_a(t', t))h_a(t'', t't)] = [h_a(t'', t')h_a(t''t', t)]. \tag{2.2}$$

The above equation implies that the elements

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t', t),\ h_a(t'', t')h_a(t''t', t)$$

of K_a determine the same element of $IntK_a$. Thus they differ by an element of the center C_{K_a}. Therefore the equality (2.2) in $IntK_a$ leads to an equality in K_a,

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t't) = f_a(t'', t', t)h_a(t'', t')h_a(t''t', t); \tag{2.3}$$

note that $h_a(t', t)$ would itself be a two-cocycle for $f_a = e(a)$. Equation (2.3) determines a mapping $f_a : T_a \times T_a \times T_a \to C_{K_a}$, i.e. a three-cochain on T_a with values in the Abelian top space C_{K_a}.

Theorem 2.3. The map $f_a \in Z^3_{(\sigma_0)_a}(T_a, C_{K_a})$ for $(\sigma_0)_a(t) = \sigma_a(t)$ acting on C_{K_a}, where it coincides with $\alpha_a(t)$ for all $a \in e(T)$.

Theorem 2.4. Non-Abelian top space K together with the action σ characterize an element of the third cohomology group $H^3_{(\sigma_0)_a}(T_a, C_{K_a})$ for every $a \in e(T)$.

Theorem 2.5. A top space T is extendible if and only if the cocycle f_a which it determines for every $a \in e(T)$ is a three-coboundary for such a.

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New results on induced almost contact structure on product manifolds

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Abstract

In this paper, first, we investigate some new results on relations between the structures J (on almost Hermitian manifold M) and Σ (on almost contact metric manifold N) with the induced almost contact metric structure $\overline{\Sigma}$ on $M \times N$ by the mentioned structures.

Keywords: Almost complex structure (Hermitian, Kählerian), Almost contact structures (Cosymplectic, Kenmotsu, Sasakian), Product manifolds
Mathematics Subject Classification [2010]: 53C15, 53D15

1 Preliminaries

1.1 Almost Hermitian and almost hypercomplex structures

Let M be an even-diminational differentiable manifold. An almost Hermitian structure on M is by definition a pair (J, g) on almost complex structure J and a Riemannian metric g satisfying

$$J^2X = -X, \quad g(JX, JY) = g(X, Y) \tag{1}$$

for any vector fields X, Y on M.

The fundamental form Ω of an almost Hermitian structure is defined by

$$\Omega(X, Y) = g(JX, Y)$$

for any vector fields X, Y and is skew-symmetric. An almost Hermitian manifold is called an almost Kähler manifold if its fundamental form Ω is closed, that is, $d\Omega = 0$.

The Neijenhuis (or the torsion) tensor of an almost complex structure J is defined dy

$$\mathcal{N}(X, Y) = [X, Y] - [JX, JY] + J[X, JY] + J[JX, Y] \tag{2}$$

for any vector fields X , Y on M. An almost complex structure is said to be integrable if it has no torsion. It is well known that an almost complex structure is a complex structure if and only if it is integrable ([6]). A complex manifold with a Hermitian structure (J, g) is said to be Kählerian if its fundamental form is closed, which is equvalent to

$$\nabla J = 0. \tag{3}$$

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1.2 Almost contact metric structure

Let M be an odd-dimensional differentiable manifold. An almost contact structure on M is by definition a pair (Σ, g) of an almost contact structure $\Sigma = (\phi, \xi, \eta)$ and a Riemannian metric g, where ϕ is a tensor field of type $(1,1)$, ξ is a vector field and η is a $1-$form, satisfying the following conditions

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi. \tag{4}$$

for any vector field X on M ([2]). A Riemannian metric g is called compatible with this structure if

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y). \tag{5}$$

for any vector fields A and Y on M and (Σ, g) is called an almost contact metric structure. Also we have $g(X,\xi) = \eta(X)$. If it satisfies

$$d\eta(X,Y) = g(\phi X, Y). \tag{6}$$

for any vector fields X and Y on M, then (M, Σ, g) is called a contact Riemannian manifold. If $\nabla_X \xi = -\phi X$, for any X in TM, M is called a $k-$contact manifold.

Let M be an almost contact manifold and define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X + f\frac{d}{dt}) = \phi X - f\xi + \eta(X)\frac{d}{dt}. \tag{7}$$

for any vector field X on M, where f is a C^∞ function on $M \times \mathbb{R}$. An almost contact structure is called to be normal if J is integrable.

A cosympletic structure is a normal almost contact metric structure (Σ, g) with both η and Φ closed, given by $\Phi(X,Y) = g(\phi X, Y)$ for any vector fields X, Y on M ([2]).

1.3 Induced almost contact structure on product manifolds

Let (M, J) be an almost complex manifold and $(N, \Sigma) = (\phi, \xi, \eta)$ an almost contact manifold. In [9], Oubiña has defined an almost contact structure $\overline{\Sigma} = (\overline{\phi}, \overline{\xi}, \overline{\eta})$ on $M \times N$ as follows

$$\overline{\phi}(X + Y) = JX + \phi Y, \quad \overline{\eta}(X + Y) = \eta(Y), \quad \overline{\xi} = \xi \tag{8}$$

for any vector fields $X \in TM$ and $Y \in TN$.

2 Main Results

Theorem 2.1. *Let (M, J) and (N, Σ) be an almost complex manifold and an almost contact manifold resp. Then by product metric, $(M \times N, \overline{\Sigma})$ can not be a contact metric manifold.*

Theorem 2.2. *For the above mentioned structures, the following statements are equivalent:*
(i) *$M \times N$ is normal.*
(ii) *M and N are Kähler and normal respectively.*

Theorem 2.3. *By the above assumptions, the following statements hold:*
(i) *$M \times N$ is cosymplectic if and only if M be Kähler and N cosymlectic.*
(ii) *$M \times N$ is almost cosymplectic if and only if M be almost Kähler and N almost cosymlectic.*

Theorem 2.4. *$M \times N$ can not be a $k-$contact manifold.*

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On a subalgebra of $C(X)$ containing $C_c(X)$

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Abstract

Let $C_c(X) = \{f \in C(X) : |f(X)| \leq \aleph_0\}$, $C^F(X) = \{f \in C(X) : |f(X)| < \infty\}$, and $L_c(X) = \{f \in C(X) : \overline{C_f} = X\}$, where C_f is the union of all open subsets $U \subseteq X$ such that $|f(U)| \leq \aleph_0$, and $C_F(X)$ be the socle of $C(X)$ (i.e., the sum of minimal ideals of $C(X)$). It is shown that if X is a locally compact space, then $L_c(X) = C(X)$ if and only if X is locally scattered. We observe that $L_c(X)$ enjoys most of the important properties which are shared by $C(X)$ and $C_c(X)$.

Keywords: Functionally countable space, Zero-dimensional space, Locally scattered space.
Mathematics Subject Classification [2010]: Primary: 54C30, 54C40, 54C05, 54G12; Secondary: 13C11, 16H20.

1 Introduction

$C(X)$ denotes the ring of all real valued continuous functions on a topological space X. In [4] and [5], $C_c(X)$, the subalgebra of $C(X)$, consisting of functions with countable image are introduced and studied. It turns out that $C_c(X)$, although not isomorphic to any $C(Y)$ in general, enjoys most of the important properties of $C(X)$. This subalgebra has recently received some attention, see [4], [1], and [5]. Since $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, this motivates us to consider a natural subring of $C(X)$, namely $L_c(X)$, which lies between $C_c(X)$ and $C(X)$. Our aim in this article, similarly to the main objective of working in the context of $C(X)$, is to investigate the relations between topological properties of X and the algebraic properties of $L_c(X)$. In particular, we are interested in finding topological spaces X for which $L_c(X) = C(X)$. An outline of this paper is as follows: We show that if X is a locally compact space, then $L_c(X) = C(X)$ if and only if X is locally scattered, which is somewhat similar to a classical result due to Rudin in [10], and Pelczynski and Semadeni in [8] (of course, by no means as significant). This classical result says that a compact space X is scattered if and only if $C(X) = C_c(X)$. Let us for the sake of the brevity, call the latter classical result, RPS-Theorem. If X is an almost discrete space or a P-space, then $L_1(X) = L_F(X) = L_c(X) = C(X)$, where $L_F(X)$ and $L_1(X)$ are the locally functionally finite (resp., constant) subalgebra of $C(X)$, see Definition 2.3.

All topological spaces that appear in this article are assumed to be infinite completely regular Hausdorff, unless otherwise mentioned. For undefined terms and notations the reader is referred to [6], [3].

*Speaker

2 The subalgebra $L_c(X)$ of $C(X)$

Definition 2.1. Let $f \in C(X)$ and C_f be the union of all open sets $U \subseteq X$ such that $f(U)$ is countable. We define $L_c(X)$ to be the set of all $f \in C(X)$ such that C_f is dense in X, i.e.,

$$C_f = \bigcup_{\substack{U \subseteq X \\ |f(U)| \leq \aleph_0}} U$$

$$L_c(X) = \{f \in C(X) : \overline{C_f} = X\}$$

We shall briefly and easily notice that, $L_c(X)$ is a subalgebra as well as a sublattice of $C(X)$ containing $C_c(X)$, and we call it the *locally functionally countable subalgebra of* $C(X)$.

It is manifest that $C_F(X) \subseteq C^F(X) \subseteq C_c(X) \subseteq L_c(X) \subseteq C(X)$, where $C^F(X) = \{f \in C(X) : |f(X)| < \infty\}$, see [4].

Corollary 2.2. *$L_c(X)$ is a sublattice of $C(X)$.*

Definition 2.3. Let $f \in C(X)$ and C_f^F be the union of all open sets $U \subseteq X$ such that $f(U)$ is finite. We define $L_F(X)$ to be the set of all $f \in C(X)$ such that C_f^F is dense in X, and call it *locally functionally finite subalgebra of* $C(X)$. In particular, let $f \in C(X)$ and C_f^c be the union of all open sets $U \subseteq X$ such that $f(U)$ is constant. We define $L_1(X)$ to be the set of all $f \in C(X)$ such that C_f^c is dense in X, and we call it *locally functionally constant subalgebra of* $C(X)$. Clearly, $L_F(X)$ and $L_1(X)$ are subalgebras of $L_c(X)$. It is evident that $C^F(X) \subseteq L_F(X)$.

Remark 2.4. We note that Corollary 2.2 are also valid for $L_F(X)$ and $L_1(X)$.

Remark 2.5. It is manifest that $C_c(X) = \mathbb{R}$, where $X = [0,1]$. But the Cantor function f is a monotonic nonconstant continuous function, and $\overline{C_f^c} = \overline{[0,1] \backslash C} = [0,1]$, where C is the Cantor set, see [2]. Therefore $\mathbb{R} \subsetneq L_1([0,1])$, hence $\mathbb{R} \subsetneq L_c([0,1])$. We emphasize that $C_c(X) = \mathbb{R}$, but $\mathbb{R} \subsetneq L_c(X)$, and this can be considered as an advantage of $L_c(X)$ over $C_c(X)$, in this case.

3 The equality between $C(X)$ an $L_c(X)$

We are interested in characterizing topological spaces X for which $L_c(X) = C(X)$. In the following proposition we have a simple result, which is similar to RPS-Theorem. Let us recall that in a commutative ring R by an annihilator ideal I, we mean $I = \mathrm{Ann}(S) = \{r \in R : rS = 0\}$, where $S \neq \{0\}$ is a nonempty subset of R.

Proposition 3.1. *If X is an almost discrete space (i.e., $I(X)$, the set of isolated points of X, is dense in X), then $L_1(X) = L_F(X) = L_c(X) = C(X)$. In particular, if every annihilator ideal of $C(X)$, where X is any space, contains a nonzero minimal ideal, then the latter equalities hold.*

Proposition 3.2. *If X is a scattered space, then $L_1(X) = L_F(X) = L_c(X) = C(X)$. In particular, if X is a compact scattered space, then the latter rings coincide with $C_c(X)$.*

In view of RPS-Theorem we may naturally define a compact space X to be scattered if given any $f \in C(X)$ and any $x \in X$, there exists a compact neighborhood V_f of x such that $|f(V_f^\circ)| \leq \aleph_0$. Motivated by this we give the following definition.

Definition 3.3. A space X is called *locally scattered* if given any $f \in C(X)$ and a nonempty open set G, there exists a compact subset V_f of X in G, with $\emptyset \neq V_f^\circ \subseteq G$ and $|f(V_f^\circ)| \leq \aleph_0$.

The space βX where X is discrete is locally scattered. Clearly, every scattered space is a locally scattered space, but the converse is not true. For example, $\beta\mathbb{N}$ is a locally scattered space which is not scattered, for $\beta\mathbb{N}\setminus\mathbb{N}$ has no isolated point (note, each clopen subset of $\beta\mathbb{N}\setminus\mathbb{N}$ has the same cardinality as $\beta\mathbb{N}\setminus\mathbb{N}$, see [6, 6S(4)]).

Lemma 3.4. *Let X be a locally scattered space. Then every open C-embedded subset of X (e.g., any clopen subset) is also locally scattered.*

Let us recall that a Hausdorff space X is locally compact if and only if each point in X has a compact neighborhood. Clearly, every compact Hausdorff space is locally compact. The following result is somewhat similar to RPS-Theorem.

Theorem 3.5. *Let X be a compact space. Then $L_c(X) = C(X)$ if and only if X is locally scattered. In particular, if X is a discrete space and Y is a non-scattered clopen subset of βX (e.g., $X = \mathbb{N}$ and $Y = \beta\mathbb{N}$), then $L_c(Y) = C(Y) = C^*(Y) \neq C_c(Y)$.*

The previous proof immediately yields the following fact, too.

Corollary 3.6. *Let X be a locally compact space. Then $L_c(X) = C(X)$ if and only if X is locally scattered.*

An interesting result due to A. W. Hager asserts that a P-space X is functionally countable (i.e., $C(X) = C_c(X)$) if and only if it is pseudo-$\aleph_1$-compact (i.e., each locally finite family of open sets is countable), see [7, Proposition 3.2]. This result is extended to $C_c(X) = C^F(X)$ in [5, Proposition 4.1]. The following is also a counterpart of the latter result.

Proposition 3.7. *If $\overline{\mathcal{P}_X} = X$ (in particular, if X is a P-space), then $L_1(X) = L_F(X) = L_c(X) = C(X)$.*

We note that $\beta\mathbb{N}$ is not a P-space while $L_1(\beta\mathbb{N}) = L_F(\beta\mathbb{N}) = L_c(\beta\mathbb{N}) = C(\beta\mathbb{N})$. By [6, 6V(6)], $\beta\mathbb{N}\setminus\mathbb{N}$ has a dense set of P-points, hence $L_1(\beta\mathbb{N}\setminus\mathbb{N}) = L_F(\beta\mathbb{N}\setminus\mathbb{N}) = L_c(\beta\mathbb{N}\setminus\mathbb{N}) = C(\beta\mathbb{N}\setminus\mathbb{N})$.

Definition 3.8. A topological space X is called *locally functionally countable* if every point $x \in X$ is *countably P-point*, in the sense that there exists an open neighborhood U_x of x such that $C(U_x) = C_c(U_x)$.

The following result implies that if a space X is second countable or a compact space, then X is locally functionally countable if and only if it is functionally countable (i.e., $C(X) = C_c(X)$).

Proposition 3.9. *Let X be a Lindelöf space. Then X is locally functionally countable if and only if it is functionally countable.*

Proposition 3.10. *If X is a locally functionally countable space, then $L_c(X) = C(X)$.*

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On conservative generalized recurrent structures

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Abstract

In the present paper we study conservative generalized recurrent manifolds. We investigate their Ricci tensor and show these manifolds are quasi Einstein manifolds.

Keywords: Generalized recurrent, Quasi Einstein
Mathematics Subject Classification [2010]: 53C25, 53D15

1 Introduction

The notion of generalized recurrent manifolds was introduced by U.C.De and N.Guha [4].

Definition 1.1. A Riemannian manifold (M^n, g) is called generalized recurrent manifold if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y,Z) = A(W)R(X,Y)Z + B(W)(g(Y,Z)X - g(X,Z)Y),\ \forall X,Y,Z \in TM,$$

where A and B are two none zero 1-forms such that $A(W) = g(\rho, W),\ B(W) = g(\acute{\rho}, W)$ and $\rho, \acute{\rho}$ are two none zero vector fields associated with the 1-forms A and B, respectively.

This type of manifolds are denoted by $(GK)_n$ and it is obvious that if $B = 0$, then $(GK)_n$ reduces to a recurrent manifold.

Definition 1.2. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be quasi Einstein manifold ($(QE)_n$), if its Ricci tensor S is not zero identically and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y),\ \forall X,Y \in TM, \tag{1}$$

where a and $b \neq 0$ are scalars and A is none zero 1-form such that $g(X,U) = A(X)$, $\forall$ X $\in TM$ and U is a unit vector field.

The conformal curvature (Weyl) tensor of M is said to be conservative if the divergence of C be zero, i.e. $divC = 0$. It is well known [3] that M is conservative if and only if

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z),\ \forall X,Y,Z \in TM. \tag{2}$$

*Speaker

2 On the conservative generalized recurrent manifold

Proposition 2.1. *Let (M^n, g) be a conservative generalized recurrent manifold. Then the Ricci tensor of M satisfies*

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cA(X)B(Y) + A(X)C(W). \tag{3}$$

Proof. Since M is a $(GK)_n$, we have

$$(\nabla_X \tilde{R})(Z,Y,W,U) = A(X)\tilde{R}(Z,Y,W,U) + B(X)(g(Y,W)g(Z,U) - g(Y,U)g(Z,W)). \tag{4}$$

By contracting (4) on Z and U, we get

$$\nabla_X S(Y,W) = A(X)S(Y,W) + (n-1)B(X)(g(Y,W) - g(Y,W)), \tag{5}$$

and

$$\nabla_Y S(X,W) = A(Y)S(X,W) + (n-1)B(Y)(g(X,W) - g(X,W)). \tag{6}$$

Equations (5) and (6), imply

$$\nabla_X S(Y,W) - \nabla_Y S(X,W) = A(X)S(Y,W) + \tag{7}$$

$$(n-1)B(X)(g(Y,W) - g(Y,W) - A(Y)S(X,W) + (n-1)B(Y)(g(X,W) - g(X,W)).$$

On the other hand, M is conservative, thus

$$\nabla_X S(Y,W) - \nabla_Y S(X,W) = \frac{1}{2(n-1)}(g(Y,W)dr(X) - g(X,W)dr(Y)). \tag{8}$$

Comparing (7) and (8) we obtain

$$A(X)S(Y,W) + (n-1)B(X)(g(Y,W) - g(Y,W) - A(Y)S(X,W) + \tag{9}$$

$$(n-1)B(Y)(g(X,W) - g(X,W)) = \frac{1}{2(n-1)}(g(Y,W)dr(X) - g(X,W)dr(Y)).$$

Replacing X and ρ in the latest equation, we get

$$S(Y,W) = [-(n-1)B(\rho) + \frac{dr(\rho)}{2(n-1)}]g(Y,W) + A(W)[(n-1)B(Y) - \frac{dr(Y)}{2(n-1)}] + A(Y)S(\rho,W). \tag{10}$$

By contracting (6) on X and W we have

$$dr(Y) = A(Y)r + n(n-1)B(Y). \tag{11}$$

So from (10) and (11), we get

$$S(Y,W) = [-(n-1)B(\rho) + \frac{dr(\rho)}{2(n-1)}]g(Y,W) - \tag{12}$$

$$\frac{r}{2(n-1)}A(W)A(Y) + \frac{n-2}{2}A(W)B(Y) + A(Y)S(\rho,W).$$

□

Proposition 2.2. *let M be a conservative $(GK)_n$ which admit a unit concircular vector field ρ then its Ricci tensor satisfies*

$$S(Y,W) = ag(Y,W) + bA(Y)A(W) + cA(W)B(Y).$$

Proof. Since ρ is a unit concircular vector field so

$$(\nabla_X A)(Y) = \alpha[g(X,Y) - A(X)A(Y)]. \tag{13}$$

Applying Ricci identity on (13) we get

$$A(R(X,Y)Z) = -\alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)]. \tag{14}$$

Contracting this equation on Y, Z we obtain

$$S(\rho, X) = (n-1)\alpha^2 A(X). \tag{15}$$

Using (15) in (12) follows

$$S(Y,W) = [-(n-1)B(\rho) + \frac{dr(\rho)}{2(n-1)}]g(Y,W) + \tag{16}$$

$$[-\frac{r}{2(n-1)} + (n-1)\alpha^2]A(W)A(Y) + \frac{n-2}{2}A(W)B(Y).$$

□

Theorem 2.3. *Let M be a $(Gk)_n$ manifold. If $\nabla C = 0$ and $C \neq 0$ then M is a quasi Einstein manifold.*

Proof. Since $\nabla C = 0$ and $C \neq 0$, M is locally symmetric ($\nabla R = 0$), so (4) implies

$$A(X)\tilde{R}(Z,Y,W,U) = -B(X)(g(Y,W)g(Z,U) - g(Y,U)g(Z,W)).$$

Contracting on Z, U and putting $X = \rho$ imply

$$A(Y)S(\rho, W) = (1-n)B(Y)A(W). \tag{17}$$

By using (17) in (12), it follows

$$S(Y,W) = [-(n-1)B(\rho) + \frac{dr(\rho)}{2(n-1)}]g(Y,W) - \frac{r}{2(n-1)}A(W)A(Y) + \frac{-n}{2}A(W)B(Y). \tag{18}$$

Moreover, $\nabla R = 0$, so the scalar curvature is constant, it means that $dr = 0$ and from (11), we have

$$B(Y) = \frac{-r}{n(n-1)}A(Y), \tag{19}$$

by putting (19) in (18), it follows

$$S(Y,W) = [(1-n)B(\rho)]g(Y,W) - (\frac{r}{2(n-1)})(\frac{n-r-1}{n-1})A(Y)A(W). \tag{20}$$

□

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On Generalized Covering Subgroups of a Fundamental Group

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Abstract

In this talk, after reviewing concepts of covering, semicovering and generalized covering subgroups introduced by J. Brazas, we give a new criterion for a subgroup $H \leq \pi_1(X, x_0)$ to be a generalized covering subgroup.

Keywords: Genertalized covering subgroup, Fundamental group, covering map, semicovering map
Mathematics Subject Classification [2010]: 55Q05, 57M05, 57M10

1 Introduction

Recently, the notion of covering space has been extended using eliminating some of its properties and keeping some others [1,2,3,5]. For instance, semicoverings are introduced by eliminating the evenly covered property and keeping local homeomorphismness and unique path lifting property [2]. In the case of generalized coverings, local homeomorphismness has been replaced with unique lifting property [1,3,5]. It is well-khown that for connected and locally path connected spaces every covering is a semicovering and every semicovering is a generalized covering. Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a map and $H = p_*\pi_1(\tilde{X}, \tilde{x}_0) \leq \pi_1(X, x_0)$. Then H is called a covering, a semicovering or a generalized covering subgroup if p is covering, semicovering or generalized covering map, respectively. It is shown that H is a covering subgroup if and only if it contains an open normal subgroup of $\pi_1^{qtop}(X, x_0)$ [2,6]. Brazas showed that H is a semicovering subgroup if and only if it is an open subgroup of $\pi_1^{qtop}(X, x_0)$. He also proved that H is a generalized covering subgroup if and only if $p_H : \tilde{X}_H \to X$ has the uniqe path lifting property, where $p_H : \tilde{X}_H \to X$ is the well-known endpoint projection [3]. Now in this talk, we show that for a connected and locally path connected space X, a subgroup H of $\pi_1(X, x_0)$ is a generalized covering subgroup if and only if $(p_H)_*\pi_1\left(\tilde{X}_H, e_H\right) = H$.

2 Notations and Preliminaries

Definition 2.1. A pointed continuous map $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ has **UL (unique lifting)** property if for every connected, locally path connected space (Y, y_0) and every continuous map $f : (Y, y_0) \longrightarrow (X, x_0)$ with $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$, there exists a

*Speaker

unique continuous lifting $\tilde{f}$ with $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = \tilde{x}_0$. If $\tilde{X}$ is a connected, locally path connected space and $p : \tilde{X} \longrightarrow X$ is surjective with UL property, then $\tilde{X}$ is called a **generalized covering space** for X. A subgroup $H \leq \pi_1(X, x_0)$ is called a **generalized covering subgroup** of $\pi_1(X, x_0)$ if there is a generalized covering map $p : \left(\tilde{X}, \tilde{x}_0\right) \longrightarrow (X, x_0)$ such that $H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

Definition 2.2. A map $f : Y \longrightarrow X$ has **UPL (unique path lifting)** property if it has UL property for the closed interval $I = [0, 1]$. A map $f : Y \longrightarrow X$ has $\mathbf{U}PL^{'}$ **(only unique path lifting)** property if any two paths $\alpha, \beta : [0, 1] \to Y$ are equal whenever $f \circ \alpha = f \circ \beta$ and $\alpha(0) = \beta(0)$.

Definition 2.3. Let H be a subgroup of $\pi_1(X, x_0)$ and $P(X, x_0) = \{\alpha : (I, 0) \to (X, x_0) | \, \alpha \textit{ is a path}\}$ be a path space. Then $\alpha_1 \sim \alpha_2 \; mod \, H$ if both $\alpha_1(1) = \alpha_2(1)$ and $\left[\alpha_1 * \alpha_2{}^{-1}\right] \in H$. It is easy to check that this is an equivalence relation on $P(X, x_0)$. The equivalence class of α is denoted by $\langle\alpha\rangle_H$. Now one can define the quotient space $\tilde{X}_H = \frac{P(X,x_0)}{\sim}$ and the map $p_H : (\tilde{X}_H, e_H) \to (X, x_0)$ by $p_H(\langle\alpha\rangle_H) = \alpha(1)$, where e_H is the class of constant path at x_0.

For $\alpha \in P(X, x_0)$ and an open neighborhood U of $\alpha(1)$, a continuation of α in U is a path $\beta \in P(X, x_0)$ of the form $\beta = \alpha * \gamma$, where $\gamma(0) = \alpha(1)$ and $\gamma(I) \subseteq U$. Thus we can define a set $\langle U, \langle\alpha\rangle_H\rangle = \{\langle\beta\rangle_H \in X_H | \; \beta \textit{ is a continuation of } \alpha \textit{ in } U\}$. It is shown that the subsets $\langle U, \langle\alpha\rangle_H\rangle$ as defined above form a basis for a topology on $\tilde{X}_H$ for which the function $p_H : (\tilde{X}_H) \to X$ is continuous [7, Theorem 10.31]. Moreover, if X is path connected, then p_H is surjective. This topology on $\tilde{X}_H$ is called the Whisker topology [4].

Some properties of the space $\tilde{X}_H$ and the map p_H are as follows: The map $p_H : \tilde{X}_H \to X$ has the path lifting property. Moreover, every path α in X beginning at x_0 can be lifted to a path $\tilde{\alpha}$ in $\tilde{X}_H$ beginning at e_H and end at $\langle\alpha\rangle_H$ [7, Theorem 10.32]. For every $H \leq \pi_1(X, x_0)$ the space $\tilde{X}_H$ is path connected [7, Corollary 10.33].

Brazas [3, theorem 24] showed that a subgroup $H \leq \pi_1(X, x_0)$ is a generalized covering subgroup of $\pi_1(X, x_0)$ if and only if $p_H : \tilde{X}_H \longrightarrow X$ has $\mathbf{U}PL^{'}$property.

3 Main results

In the trivial case $H = 1$, clearly $H \leq (p_H)_* \pi_1(\tilde{X}_H, e_H)$. Fischer and Zastrow [5] using this fact found an equivalent condition for UPL property in $p_e : \tilde{X}_e \to X$. They also showed that a space X admits a generalized universal covering if and only if $p_e : \tilde{X}_e \to X$ has $UPL^{'}$ property [5, Lemma 2.8]. Then Brazas extended the result for every generalized covering subgroup [3, Lemma 21] and showed that for any subgroup $H \leq \pi_1(X, x_0)$, $H \leq (p_H)_* \pi_1(\tilde{X}_H, e_H)$ [3, corollary 20]. Moreover, he showed that if $p_H : \tilde{X}_H \to X$ has UPL property, then $H = (p_H)_* \pi_1(\tilde{X}_H, e_H)$ [3, Lemma 21]. In the following theorem we investigate the convers of the above result.

Theorem 3.1. *For any $H \leq \pi_1(X, x_0)$, if $(p_H)_* \pi_1(\tilde{X}_H, e_H) \leq H$, then $p_H : \tilde{X}_H \to X$ has UPL property.*

The following corollary is the main result of this talk.

Corollary 3.2. *Let $H \leq \pi_1(X, x_0)$. Then the end point projection $p_H : \tilde{X}_H \to X$ is a generalized covering map if and only if $(p_H)_* \pi_1 \left(\tilde{X}_H, e_H\right) = H$.*

Proof. Brazas showed that $H \leq (p_H)_*\pi_1(\tilde{X}_H, e_H)$ for any subgroup H of $\pi_1(X, x_0)$ [3, Corollary 20]. Combining this fact with Theorem 3.1 implies that if $(p_H)_*\pi_1\left(\tilde{X}_H, e_H\right) = H$, then $p_H : \tilde{X}_H \to X$ has **UPL (unique path lifting)** property. The convers holds using [3, Lemma 21].

Brazas [3, Theorem 15] showed that for any collection of generalized covering subgroups of $\pi_1(X, x_0)$, the intersection of them is also a generalized covering subgroup. But its proof is too long and need to use pullbacks. We will give a simple proof using Corollary 3.2.

Corollary 3.3. *If $\{H_j \mid j \in J\}$ is any set of generalized covering subgroups of $\pi_1(X, x_0)$, then $H = \cap_{j\in J} H_j$ is a generalized covering subgroup.*

Proof. At first, we show that $(p_H)_*\pi_1\left(\tilde{X}_H, e_H\right) \leq \cap\left(p_{H_j}\right)_*\pi_1\left(\tilde{X}_{H_j}, e_{H_j}\right) = H$ then, use Theorem 3.1 and assume that $[\alpha] = [p_H \circ \widetilde{\alpha}] = (p_H)_*[\widetilde{\alpha}] \in (p_H)_*\pi_1\left(\tilde{X}_H, e_H\right)$ where $\widetilde{\alpha}: I \to \tilde{X}_H$ is a loop in $\tilde{X}_H$ at e_H with $\widetilde{\alpha}(t) = \langle\beta_t\rangle_H$. We define for every $j \in J$, $\widetilde{\alpha}_j : I \to \tilde{X}_{H_j}$ by $\widetilde{\alpha}_j(t) = \langle\beta_t\rangle_{H_j}$. It is clear that $\widetilde{\alpha}_j$ is a loop at e_{H_j}, $p_H \circ \widetilde{\alpha} = p_{H_j} \circ \widetilde{\alpha}_j$ and so $[p_H o \widetilde{\alpha}] = \left[p_{H_j} o \widetilde{\alpha}_j\right] = [\alpha]$ for every $j \in J$. Therefore, $(p_H)_* \leq H$. Now using Theorem 3.1 the result holds.

For a pointed space (X, x_0) we define: $\pi_1^{gc}(X, x_0) = \bigcap\{H \leq \pi_1(X, x_0) | H \text{ is a generalized covering subgroup}\}$.

Corollary 3.4. *For a pointed space (X, x_0), $\pi_1^{gc}(X, x_0)$ is a generalized covering subgroup.*

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Talk On the flag curvature of bi-invariant Randers metrics pp.: 1–4

On the flag curvature of bi-invariant Randers metrics*

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Abstract

In this paper we study the flag curvature of bi-invariant Randers metrics. We first correct a minor error which occurred for the flag curvature formula of a bi-invariant Randers metric. Then we improve this formula on a connected Lie group G and as an application we explicitly give this formula for the Lie groups $SO(4)$ and $U(3)$ which show that these spaces are of non-negative flag curvatures. Some results on the flag curvature formula of a naturally reductive Randers metric are also improved.

Keywords: Flag curvature, Bi-invariant Randers metrics, Connected Lie groups
Mathematics Subject Classification [2010]: 53C60, 53C30.

1 Introduction

The study of invariant structures on Lie groups and homogeneous manifolds is an interesting subject in differential geometry. In the last decade a generalization of these concepts from the Riemannian geometry into the Finsler geometry, specially Randers metrics have been done [1, 2, 3, 4, 5, 6]. One of these invariant structures are bi-invariant metrics and the study of the flag curvature of bi-invariant metrics as a generalization of sectional curvatures in the Riemannian geometry has absorbed a special attention of the mathematics scientists. In particular in [6] an explicit formula for the flag curvature of bi-invariant Randers metrics is given which has a minor error. Our aim in this paper is to correct this formula. We also improve this formula and apply it for calculating the flag curvature of the compact Lie groups $SO(4)$ and $U(3)$. Some interesting results for the flag curvature of naturally reductive are also proved.

2 The flag curvature of a bi-invariant Randers metric

The following formula

$$K(P,y) = \frac{<[y,[u,y]],V>_0 . <V,u>_0 + <[y,[u,y]],u>_0 (1+<V,y>_0)}{4(1+<V,y>_0)^2(1-<V,y>_0)}, \qquad (1)$$

is given in [6] for the flag curvature of a Randers metric which is defined by a bi-invariant Riemannian metric g_0 and a left-invariant vector field V which is parallel with respect to g_0. In the correct way it can be written as

*Will be presented in English
†Speaker

Theorem 2.1. *Suppose that g_0 is a bi-invariant Riemannian metric on a Lie group G and $\tilde{V}$ is a left invariant vector field on G such that $g_0(\tilde{V},\tilde{V}) < 1$ and $\tilde{V}$ is parallel with respect to g_0. Then we can define a left invariant Randers metric F as follows: $F(x,y) = \sqrt{g_0(x)(y,y)} + g_0(x)(\tilde{V}_x, y)$. Assume that (P,y) is a flag in T_eG such that $\{y,u\}$ is an orthonormal basis of P with respect to $<.,.>_0$. Then the flag curvature of the flag (P,y) in T_eG is given by*

$$K(P,y) = \frac{<[y,[u,y]],V>_0 . <V,u>_0 + <[y,[u,y]],u>_0 (1+ <V,y>_0)}{4(1+ <V,y>_0)^3}. \qquad (2)$$

Proof. Since for a Randers metric we have $g_y(u,v) = <u,v>_0 + <V,u>_0 <V,v>_0 + \frac{<u,v>_0 <V,y>_0}{\sqrt{<y,y>_0}} - \frac{<v,y>_0 <u,y>_0 <V,y>_0}{<y,y>_0 \sqrt{<y,y>_0}} + \frac{<V,v>_0 <u,y>_0}{\sqrt{<y,y>_0}} + \frac{<V,u>_0 <v,y>_0}{<y,y>_0}$, where $0 \neq y,u,v$ are tangent vectors in T_xG, then $g_y(u,u) = 1 + <V,y>_0 + <V,u>_0^2$, $g_y(y,y) = (1+ <V,y>_0)^2$ and $g_y(y,u) = <V,u>_0 + <V,y>_0 <V,u>_0$, where $\{y,u\}$ is an orthonormal basis of P, which yields that $g_y(u,u)g_y(y,y) - g_y(y,u)^2 = (1+ <V,y>_0)^3$. By replacing this equation in the flag curvature formula $K(P,y) = \frac{g_y(R(u,y)y,u)}{g_y(u,u)g_y(y,y) - g_y(y,u)^2}$ and using the equation (9) from [6] we obtain the equation (2). □

To improve theorem 2.1 for a connected Lie group, we prove the following result.

Proposition 2.2. *Let the connected Lie group G equipped with a left invariant Randers metric F defined by the left invariant Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the left invariant vector field V. Then the following conditions are equivalent.*
(1) F is bi-invariant.
(2) F is naturally reductive.
(3) $\tilde{a}$ is bi-invariant and V is parallel with respect to $\tilde{a}$.

Proof. By theorem 3.5 in [2] and theorem 3.2 in [3] F is naturally reductive if and only if F is bi-invariant. In order to prove that (2) is equivalent with (3), we suppose that F is naturally reductive. Then F has the following cases simultaneously
Case 1: By theorem 4.1 in [4] $(G,\tilde{a})$ is naturally reductive, and $\tilde{a}$ is bi-invariant.
Case 2: By theorem 3.2 in [2] (G,F) is of Berwald type, and V is parallel with respect to $\tilde{a}$. Cases 1 and 2 give us the condition (3). Conversely, since $\tilde{a}$ is bi-invariant, it is naturally reductive and since V is parallel with respect to $\tilde{a}$, F is of Berwald type. Then by theorem 4.2 in [4], (G,F) is naturally reductive which implies (2). □

By proposition 2.2 to improve the formula (2) it is sufficient to obtain the flag curvature of a naturally reductive homogeneous Randers space. So we have

Lemma 2.3. *Let $(\frac{G}{H}, F)$ be a naturally reductive homogeneous Randers space with F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field V. Let (P,y) be a flag in $\mathcal{M}$ such that $\{y,u\}$ is an orthonormal basis of P with respect to $\tilde{a}$. Then the flag curvature of the flag (P,y) in $\mathcal{M}$ is given by*

$$K(P,y) = \frac{1}{(1+\tilde{a}(V,y))^2}\left(\frac{1}{4} \parallel [u,y]_{\mathcal{M}} \parallel^2 + \tilde{a}([[u,y]_{\mathcal{H}}, u]_{\mathcal{M}}, y)\right). \qquad (3)$$

Proof. It is proved in [2] that for this case we have

$$K(P,y)=\frac{1}{(1+\tilde{a}(V,y))^2}\Big(\frac{1}{4}\parallel[u,y]_{\mathcal{M}}\parallel^2+\tilde{a}([[u,y]_{\mathcal{H}},u]_{\mathcal{M}},y)\Big)$$
$$+\frac{\tilde{a}(V,u)}{(1+\tilde{a}(V,y))^3}\Big(\frac{1}{4}\tilde{a}([u,y]_{\mathcal{M}},[V,y]_{\mathcal{M}})+\tilde{a}([u,y]_{\mathcal{H}},[V,y]_{\mathcal{M}})\Big). \tag{4}$$

Since $(\frac{G}{H},F)$ is naturally reductive, by theorem 4.1 in [4] $(M,\tilde{a})$ is naturally reductive. i.e., for all $x,y,z\in\mathcal{M}$ we have $\tilde{a}(z,[y,x]_{\mathcal{M}})+\tilde{a}(x,[y,z])=0$, which implies that

$$\tilde{a}([u,y]_{\mathcal{M}},[y,V]_{\mathcal{M}})+\tilde{a}([y,[u,y]_{\mathcal{M}}]_{\mathcal{M}},V)=0. \tag{5}$$

Also, since $(\frac{G}{H},F)$ is naturally reductive, by theorem 3.5 in [2] we have $\tilde{a}([y,[u,y]_{\mathcal{M}}]_{\mathcal{M}},V)=0$. If we replace this equation in the equation (5) we get

$$-\tilde{a}([u,y]_{\mathcal{M}},[V,y]_{\mathcal{M}})=\tilde{a}([u,y]_{\mathcal{M}},[y,V]_{\mathcal{M}})=0. \tag{6}$$

Also by using the fact that $\mathcal{M}$ is orthogonal to $\mathcal{H}$ with respect to the inner product $\tilde{a}(,)$, we have $\tilde{a}([u,y]_{\mathcal{H}},[x,y]_{\mathcal{M}})=0$. So by replacing this equation and the equation (6) in the equation (4) we have the equation (3). □

Proposition 2.2 and Lemma 2.3 imply the following result.

Corollary 2.4. *Let G be a connected Lie group with a left invariant Randers metric F defined by the left invariant Riemannian metric $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$ and the left invariant vector field V. If F has one of the three cases given in proposition 2.2, then the flag curvature formula is given by $K(P,y)=\frac{1}{4(1+\tilde{a}(V,y))^2}\parallel[u,y]\parallel^2$.*

Corollary 2.4 improved the flag curvature formula 2 in theorem 2.1 for a connected Lie group. So theorem 2.1 for a connected Lie group G can be expressed as follows.

Theorem 2.5. *Suppose that g_0 is a bi-invariant Riemannian metric on a connected Lie group G and $\tilde{V}$ is a left invariant vector field on G such that $g_0(\tilde{V},\tilde{V})<1$ and $\tilde{V}$ is parallel with respect to g_0. Then we can define a left invariant Randers metric F as follows: $F(x,y)=\sqrt{g_0(x)(y,y)}+g_0(x)(\tilde{V}_x,y)$. Assume that (P,y) is a flag in T_eG such that $\{y,u\}$ is an orthonormal basis of P with respect to $<.,.>_0$. Then the flag curvature of the flag (P,y) in T_eG is given by $K(P,y)=\frac{<[u,y],[u,y]>_0}{4(1+<V,y>_0)^2}$.*

Also by corollary 2.4, the flag curvature formula given in the corollary 3.5 in [3] for a connected Lie group G can be improved as follows.

Theorem 2.6. *Let G be a connected Lie group with a bi-invariant Randers metric F defined by the Riemannian metric $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$ and the vector field V. Also let (P,y) be a flag in $\mathcal{G}$ such that $\{y,u\}$ is an orthonormal basis of P with respect to $\tilde{a}=a_{ij}dx^i\otimes dx^j$. Then the flag curvature of the flag (P,y) in $\mathcal{G}$ is given by $K(P,y)=\frac{1}{4(1+\tilde{a}(V,y))^2}\parallel[u,y]\parallel^2$, where $\parallel[u,y]\parallel^2$ denotes the norm of $[u,y]$ with respect to $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$.*

Here as an application we explicitly give the flag curvature formula for the compact Lie groups $SO(4)$ and $U(3)$. By theorem 1 in [5] every connected Lie group admits a bi-invariant metric, so we have the following result.

Theorem 2.7. *Let G be one of the compact Lie groups $So(4)$ and $U(3)$ with a bi-invariant Randers metric F. Then for the orthonormal basis $\{y, u\}$ of P the flag curvatures of the flag (P, y) in $\mathcal{G}$ for $So(4)$ and $U(3)$ are given respectively by*

$$K(P,y) = \frac{1}{4}\{(b_1^2 + b_6^2)(\sum_{i=2,3,4,5} a_i^2) + (b_3^2 + b_4^2)(\sum_{i=1,2,5,6} a_i^2) + (b_2^2 + b_5^2)(\sum_{i=1,3,4,6} a_i^2)\},$$

where $u = \sum_{i=1}^{6} a_i e_i$, $v = \sum_{i=1}^{6} b_i e_i$ and $\{e_1, \ldots, e_6\}$ is an orthonormal base for $\mathcal{SO}(4)$.

$$\begin{aligned} K(P,y) = \frac{1}{4}\{ & b_1^2(\sum_{i=2,3,4,6} a_i^2) + b_2^2(\sum_{i=1,4,5,6,7,8} a_i^2 + 8a_3^2) + b_3^2(\sum_{i=1,4,5,6,7,8} a_i^2 + 8a_2^2) \\ & + b_4^2(\sum_{i=1,2,3,7,8,9} a_i^2 + 8a_6^2) + b_5^2(\sum_{i=2,3,7,8} a_i^2) + b_6^2(\sum_{i=1,2,3,7,8,9} a_i^2 + 8a_4^2) \\ & + b_7^2(\sum_{i=2,3,4,5,6,9} a_i^2 + 8a_8^2) + b_8^2(\sum_{i=2,3,4,5,6,9} a_i^2 + 8a_7^2) + b_9^2(\sum_{i=4,6,7,8} a_i^2)\}. \end{aligned}$$

where $u = \sum_{i=1}^{9} a_i e_i$, $v = \sum_{i=1}^{9} b_i e_i$ and $\{e_1, \ldots, e_9\}$ is an orthonormal base for $\mathcal{U}(3)$.

Proof. Since for $SO(4)$ the non-zero Lie brackets are

$$[e_1, e_2] = e_4, \ [e_1, e_3] = e_5, \ [e_1, e_4] = -e_2, \ [e_1, e_5] = -e_3, \ [e_2, e_3] = e_6, \ [e_2, e_4] = e_1,$$
$$[e_2, e_6] = e_4, \ [e_1, e_3] = e_5, \ [e_1, e_4] = -e_2, \ [e_1, e_5] = -e_3, \ [e_2, e_3] = e_6, \ [e_2, e_4] = e_1,$$

Then by calculating the non-zero Levi-Civita connection ∇ the parallel vector field is $V = 0$. So by using the flag curvature formula which is given in theorem 2.6 we have the result. For $U(3)$ we have a similar proof. □

Theorem 2.7 shows that flag curvature formulae in both cases are non-negative.

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On the fundamental group of Yamabe solitons

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Abstract

In the present work a generalization of Riemannian Yamabe solitons for inequalities is studied and among the others it is shown that every Riemannian complete nontrivial shrinking Yamabe soliton has finite fundamental group and its first cohomology group vanishes, whenever the scalar curvature is bounded. As well the fundamental group of the sphere bundle, and its cohomology group vanishes.

Keywords: Yamabe soliton, shrinking, fundamental group.
Mathematics Subject Classification [2010]: 53C20; 53C25

1 Introduction

In recent decades the geometric flows are studied by many mathematicians specially the Fields Medalists. A geometric flow is the gradient flow associated to a functional on a manifold which has a geometric interpretation, usually related to some curvatures. Geometric flows are of fundamental interest in the calculus of variations, and include several famous problems and theories. Among them Yamabe flow is introduced by R.S. Hamilton in order to study Yamabe's conjecture, stating that any metric is conformally equivalent to a metric with constant scalar curvature, cf., [3]. Yamabe flow is an evolution equation on a Riemannian manifold (M, g) defined by

$$\frac{\partial g}{\partial t} = -Rg, \qquad g(t = 0) := g_0,$$

where R is the scalar curvature. Under Yamabe flow, the conformal class of a metric does not change and is expected to evolve a manifold toward one with constant scalar curvature. Yamabe solitons are special solutions of the Yamabe flow and naturally arise as limits of dilations of singularities in the Yamabe flow. Let (M, g) be a Riemannian manifold, a quad (M, g, X, λ) is said to be a *Yamabe soliton* if g satisfies the equation

$$\mathcal{L}_X g = (\lambda - R)g, \tag{1}$$

where X is a smooth vector field on M, $\mathcal{L}_X$ the Lie derivative along X and λ a real constant. A Yamabe soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$

*Speaker

or $\lambda < 0$, respectively. If the vector field X is gradient of a potential function f, then (M, g, X) is said to be *gradient* and (1) takes the familiar form

$$\nabla\nabla f = (\lambda - R)g.$$

The Yamabe soliton is said to be compact (resp. complete) if (M, g) is compact (resp. complete). It is well known the scalar curvature of any compact gradient Yamabe soliton is constant, cf., [2, 4]. A complete shrinking gradient Yamabe solitons under suitable scalar curvature assumptions have finite topological type, cf., [7]. We note that the Yamabe flow has some similarities to Ricci flow. Moreover, as Ricci solitons are self similar solutions of Ricci flow, Yamabe solitons are self similar solutions of Yamabe flow. It is natural to ask whether classical results for Ricci solitons remain valid for Yamabe solitons.

2 Main results

In the present work an extension of Riemannian Yamabe solitons for inequalities is studied and the following theorems are proved. First we obtain an estimation for the distance function of complete Yamabe solitons for inequalities as follows.

Theorem 2.1. *Let (M, g) be a complete Riemannian manifold satisfying*

$$\mathcal{L}_X g \geqslant (\lambda - R)g, \tag{2}$$

and $R \leqslant \Lambda < \lambda$, where, $\lambda > 0$, Λ is a constant and $V = v^i(x)\frac{\partial}{\partial x^i}$ is a vector field on M. Then, for any $p, q \in M$

$$d(p, q) \leqslant \max\left\{1, \frac{1}{\lambda - \Lambda}\Big(\|X_p\| + \|X_q\|\Big)\right\}. \tag{3}$$

It is well known a compact shrinking Yamabe soliton satisfying (1), for $\lambda > 0$, has the constant scalar curvature $R = \lambda$. This shows that the Theorem 2.1 can not be for compact shrinking Yamabe solitons. Therefore, we discuss the complete non-compact cases in the following theorem.

Theorem 2.2. *Let (M, g) be a complete non-compact Riemannian manifold with bounded above scalar curvature satisfying (2). Then the fundamental group $\pi_1(M)$ of M is finite and its first cohomology group vanishes, i.e., $H^1_{\mathrm{dR}}(M) = 0$.*

We illustrate an example for Theorem 2.2 and show that the inequality $R \leqslant \Lambda < \lambda$ is sharp. Let us denote by SM the sphere bundle defined by $SM := \bigcup_{x\in M} S_xM$ where, $S_xM := \{v \in T_xM | g(v, v) = 1\}$. SM is a subbundle of the tangent bundle TM which has some applications in extension of Riemannian geometry.

Corollary 2.3. *Let (M, g, X, λ) be a complete non-compact shrinking Yamabe soliton with the bounded above scalar curvature $R \leqslant \Lambda < \lambda$. Then the fundamental group $\pi_1(SM)$ of the sphere bundle SM is finite and therefore the de Rham cohomology group $H^1_{\mathrm{dR}}(SM)$ vanishes.*

The following example illustrates Theorem 2.2 and shows that the inequality $R \leqslant \Lambda < \lambda$ is necessary for this result.

Example 2.4. Let $(\mathbb{R}^n, \delta_{ij})$ be the Euclidean space with the standard metric. Assuming $\lambda = 1$ and $f = \frac{1}{2}|x|^2$ we have a shrinking gradient Yamabe soliton. On the other hand, we know that the fundamental group of $\mathbb{R}^n$, i.e., $\pi_1(\mathbb{R}^n)$ vanishes.

Note that the condition $R \leqslant \Lambda < \lambda$ is necessary in Theorem 2.2. In fact being $X = 0$ in (M, g) chosen to be the Riemannian product of a hyperbolic manifold and a standard sphere, with factor metrics scaled so that the resulting (constant) scalar curvature is positive. As the equality version of (2) is preserved, in an obvious sense, under Riemannian products, and one can use a factor of the type described earlier to make the fundamental group infinite.

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Talk On the space of Finslerian metrics pp.: 1–4

On the space of Finslerian metrics

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Abstract

In the present paper, we first prove that the space of Finslerian metrics is an infinite dimensional manifold. Next, we introduce some inner products in the space of Finslerian metrics. Then it is given decomposition for the tangent space of this infinite dimensional manifold by means of Riemannian metric and the Berger-Ebin theorem.

Keywords: Berger-Ebin theorem, Differential operator, Finite type PDE
Mathematics Subject Classification [2010]: 53B40, 58B20, 58E11

1 Introduction

Let (M, g) be a connected, compact Finsler manifold. That is, there is a function F on the tangent bundle TM satisfying the following conditions:

- F is a smooth function on the entire slit tangent bundle TM_o.
- F is a positive homogeneous function on the second variable, y.
- The matrix (g_{ij}), $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is non-degenerate.

Geodesics of a Finsler structure F are characterized locally by $\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$, where $G^i = \frac{1}{4}g^{ih}(\frac{\partial^2 F^2}{\partial y^h \partial x^j}y^j - \frac{\partial F^2}{\partial x^h})$ are called geodesic spray coefficients. Let $G^i_j = \frac{\partial G^i}{\partial y^j}$ be the coefficients of a nonlinear connection on TM. By means of this nonlinear connection, the tangent space TM_o splits into horizontal and vertical subspaces. TTM_0 spanned by $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j}$ are called Berwald bases and their dual bases are denoted by $\{dx^i, \delta y^i\}$, where $\delta y^i := dy^i + G^i_j dx^j$. Furthermore, this nonlinear connection can be used to define a linear connection called the Berwald connection and its connection 1-forms are defined locally by $\pi^i_j = G^i_{jk}dx^k$ where $G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}$. The connection 1-forms of the Cartan connection are defined by $\tilde{\nabla}\frac{\partial}{\partial x^i} = \omega^j_i \frac{\partial}{\partial x^j}$, where $\omega^i_j = \Gamma^i_{jk}dx^k + C^i_{jk}\delta y^k$ such that

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^m}) - (C^i_{js}G^s_k + C^i_{ks}G^s_j - C_{kjs}G^{si}),$$

and

$$C^i_{jk} = \frac{1}{2}g^{im}(\frac{\partial g_{mj}}{\partial y^k} + \frac{\partial g_{mk}}{\partial y^j} - \frac{\partial g_{kj}}{\partial y^m}), \tag{1}$$

Hence we have $\tilde{\nabla} = \nabla + \dot{\nabla}$ where, ∇ is the horizontal coeffiecients of the Cartan connection and $\dot{\nabla}$ is the vertical coeffiecients of the Finslerian(Cartan) connection.

*Speaker

2 Main results

The space of Riemannian metrics on a given manifold is an infinite dimensional manifold. It is easy to see this property since the Riemannian metrics space is the open and convex set of the space of all sections of S^2T^*M. Ebin used the manifold structure in [1] and gave a Riemannian structure to the manifold of Riemannian metrics on a compact manifold M. The aim of this section is to consider the geometry of the space of Finslerian metrics. Dealing with Finslerian case is not as easy as Riemannian case because of PDEs and integrability conditions for defining the Finsler metrics. The outline of the proof is to start by the generalized Lagrange metrics and restricted it to find a suitable PDE for introducing Finsler metric space. The generalized Lagrange metric is a metric structure on π^*TM or VTM and is defined as follows:

Definition 2.1. A generalized Lagrange metric, briefly a GL-metric on an n-dimensional manifold M, is a $(0,2)$ d-type tensor field $g_{ij}(x,y)$ on TM satisfying the following

- $g_{ij}(x,y) = g_{ji}(x,y)$, i.e. it is symmetric,
- $\det g_{ij}(x,y) \neq 0$, i.e. it is regular,
- The quadratic form $g_{ij}(x,y)X^iX^j, X \in \mathbb{R}^n$ has a constant signature.

If we only consider positive signature, then $g(x,y)$ is a Euclidean product of the vector space $\pi^*|_zTM$ for each $z=(x,y) \in U \subset TM$. So π^*TM is a Riemann vector bundle over TM. A GL-metric is called a Lagrange metric, if there is a potential function $L : TM \to \mathbb{R}$ such that

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}(x,y), \tag{2}$$

are components of a positive definite matrix. A GL-metric is reducible to a Lagrange metric if and only if the Cartan tensor (1) is symmetric in all three indices. This condition is equivalent to the integrability condition of the system (2) i.e. $\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}$ is satisfied. It signifies that the equation (1) is reduced to the form $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{2}\partial_k g_{ij}$. Furthermore, the coefficients of a Finslerian metric are zero homogeneous, so they are lying on SM. Hence a Lagrange metric is reduced to a Finsler metric if and only if the coefficients of the metric are satisfied with a system of the linear partial differential equations, $y^k\frac{\partial g_{ij}}{\partial y^k} = 0$, see [4] for more details. So the problem of introducing the space of Finsler metrics is reduced to finding the solution space of the following system:

$$\begin{cases} y^i\partial_i g_{jk} = 0; & i,j,k = 1,\dots,n \\ g(X,X) > 0; & X \in \Gamma(\pi^*TM_0) \end{cases} \tag{3}$$

We note that since these equations are defined in L-metrics space so the potential function is always defined by $L(x,y) = g_{ij}(x,y)y^iy^j$. for the solutions of (3). It means that the integrability condition is satisfied for these solutions. Now, the procedure is to define another system of equations which is equivalent to (3). A GL-metric is a field of cones on $S^2\pi^*T^*M$, that is

$$\begin{aligned} k : TM &\to S^2\pi^*T^*M \\ z &\to k(z) \subset E_z \end{aligned} \tag{4}$$

where $k(z) = \{g_{ij} \in S^2\pi^*T^*M | detg_{ij} > 0\} \cup \{g_{ij} \in S^2\pi^*T^*M | detg_{ij} < 0\}$. So the space of GL-metrics is a symmetric 2-forms bundle over TM endowed with a field of cones which is denoted by $E := [S^2\pi^*T^*M; K]$. Let F be the subbundle of J^1E which is spanned at each point $z \in TM$ by $(u^{ij}, u^{ij}_k, u^{ij}_\alpha, u^{ij}_\xi)$ where, $\xi = y^k\frac{\partial}{\partial y^k}$ is the vertical Liouville vector field. Suppose that $P : \Gamma(E) \to \Gamma(F)$ is a linear first order differential operator which is defined by $P(g) := \Phi o j^1(g) = y^k\partial_k g_{ij}$. The symbol of P is defined by:

$$\sigma(P) : T^*(TM) \otimes E \to F$$
$$\sigma_t(P) = P(fg),$$

where $t = df$. In local coordinate, we have $P(fg) = y^k\partial_k(fg_{ij})$. So by means of the integrability condition for system (3), the kernel of this symbole is $\{fg | f \in C^\infty(M)\}$. For any $s \geq 3$, the vector space

$$V_s := (T^*TM \otimes E) \cap (S^{s-1}T^*TM \otimes ker(\sigma(P))),$$

is vanish. Therefore, the system (P, E, F) is of finite type. So the equation $P(g) = 0$ is equivalent to the closed system of PDEs of the form $\partial_k g_{ij} = \psi_k(ij)$ where, $\psi_k(ij)$ are a combination of the homogeneous functions of order -1 of y^i, $L(x, y)$, $\frac{\partial L}{\partial y^i}(x, y)$ and $\frac{\partial^2 L}{\partial y^i \partial y^j}(x, y)$. Hence the system of equations (3) is equivalent to the following system:

$$\begin{cases} \partial_k g_{ij} = \psi_k(ij) \quad ; \quad i, j, k = 1, \dots, n \\ g(X, X) > 0 \quad ; \quad X \in \Gamma(\pi^*TM_0). \end{cases} \tag{5}$$

It will thus be sufficient to prove that the system (5) has a solution. Since this system is of finite type, i.e. the higher order derivatives can be written in lower order derivatives, the integrability condition is always true for this system.

Proposition 2.2. *The system of PDEs (5) has a solution.*

Theorem 2.3. *The completion space of all Finsler metrics on a compact manifold M has a Riemannian structure.*

It is well known that π^*TM is isomorphic to VTM. Let us consider a section $s : M \to TM$. The pullback bundle s^*VTM is a vector bundle over M and for all $x \in M$ there is an isomorphism $\Pi_x : (VTM)_{s(x)} \to (s^*VTM)_x \cong (s^*\pi^*TM)_x$. We use this isomorphism frequently without notification in this work . Consider a vector field $V \in \Gamma(TM)$ and denote by η_t the 1-parameter local flow of V. Let $\tilde{\eta}$ be the natural extension of η on TM defined by $\tilde{\eta}_t : (x^i, y^i) \to (x^i + tv^i, y^i + ty^m\frac{\partial v^i}{\partial x^m})$. Clearly, $\hat{V} := \frac{d}{dt}|_{t=0}\tilde{\eta}_t$ is the complete lift of the vector field V on TM. Let $X = X^i\frac{\partial}{\partial x^i}$ be a section of π_s^*TM where $\pi_s : SM \to M$. Consider the canonical linear mapping $\varrho : T_zTM \to \pi_s^*T_xM$ which is defined by $\varrho_z(\frac{\delta}{\delta x^i}) = \frac{\partial}{\partial x^i}|_z$ and $\varrho_z(\frac{\partial}{\partial y^i}) = 0$ in local coordinates. Now, Let $\hat{X}$ be the complete lift of a vector field X on M. The Lie derivative of metric g in local coordinates is

$$L_{\hat{X}}g_{ij} = \nabla_i X_j + \nabla_j X_i + 2y^m\nabla_m X^k C_{kij}. \tag{6}$$

Lemma 2.4. *Let (M, g) be a compact Finslerian manifold and h an arbitrary symmetric 2-form in $S^2\pi_s^*T^*M$. Then the adjoint of Lie derivative of h in local coordinates is given by*

$$\delta h = -(\nabla^i h_{ik} - h_{kj}\nabla_0 C^j + \dot{C}_{kij}h^{ij} + C_{kij}\nabla_o h^{ij}), \tag{7}$$

Theorem 2.5. *The Berger-Ebin decomposition of $T_g\mathcal{M}_F \subset S^2\pi_s^*T^*M$ is $T_g\mathcal{M}_F = \{h|h = L_{\hat{X}}g\} \oplus S^T$ where $S^T := \{h|\delta_g h = 0\}$.*

The point-wise conformal deformation of a Finslerian metric g is defined $\tilde{g}(x, y) = f(x)g(x, y)$ where, f is a smooth positive function on M. Since there is a one to one correspondence between the space of positive functions and space of exponential functions by $f \to e^f$, we can write $\tilde{g} = e^f g$. Let $\mathcal{P}$ be the product group of positive functions on M that acts on $\mathcal{M}_F$ as follows:

$$A : \mathcal{P} \times \mathcal{M}_F \to \mathcal{M}_F$$
$$A(f, g) := fg,$$

This action is free and smooth. The orbit of this action at $g \in \mathcal{M}_F$ is defined by $A_g = \{fg|f \in \mathcal{P}\}$ which is a submanifold of $\mathcal{M}_F$. The tangent space of this submanifold at g is defined by $\mathcal{F}g = \{h = kg|k \in C^\infty(M)\}$ which is a subbundle of $S^2\pi_s^*T^*M$ at each point $g \in \mathcal{M}_F$. The orthogonal subspace of $\mathcal{F}g$ with respect to the global inner product is $S^T := \{h \in S^2\pi_s^*T^*M| \int_{SM} kgh\eta = 0\} = \{h \in S^2\pi_s^*T^*M|tr(h) = 0\}$. On the other hand, by means of the variation of volume forms [3], $tr(h) = 0$ if and only if SM has constant volume. So the orthogonal space of $\mathcal{F}g$ is the space of 2-forms which preserve volume SM through metric variations. Thus, there is a point-wise decomposition like

$$T_g\mathcal{M}_F = \mathcal{F}g \oplus S^T. \tag{8}$$

Theorem 2.6. *The York decomposition of $\mathcal{B} \subset T_g\mathcal{M}_F$ is $\mathcal{B} = \mathcal{F}g \oplus S^{TT} \oplus (S^T \cap Im\tau_g)$, where $\mathcal{B}$ is defined as the solution space of the system $\frac{\partial h^i_j}{\partial y^k} = 0$, where $S^{TT} = \{h \in T_g\mathcal{M}_F|tr(h) = 0, div(h) = 0\}$, and τ_g is a map from $\Gamma(TM)$ to $T_g\mathcal{M}_F$.*

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Talk On topologies generated by subrings of the algebra of all real-valued functions pp.: 1–4

On topologies generated by subrings of the algebra of all real-valued functions

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Abstract

Let X be a topological space and R be a subring of $\mathbb{R}^X$. Associated with the subring R, we generalize the separation axioms on X. Moreover, we specify three topologies on X, namely $Z(R)$-topology, $Coz(R)$-topology and the weak topology induced by R. Comparsion and coincidence of each pair of these topologies are investigated. Using these topologies, a one-one correspondence between points of X and fixed maximal ideals of R is given

Keywords: $Z(R)$-topology, $Coz(R)$-topology, weak-topology, maximal fixed ideal.
Mathematics Subject Classification [2010]: 54C30, 46E25.

1 Introduction

Throughout this article, $\mathbb{R}^X$ denotes the algebra of all real-valued functions on X and $C(X)$ (resp., $C^*(X)$) denotes the subalgebra of $\mathbb{R}^X$ consisting of all continuous functions (resp., bounded continuous functions). Note that X is not necessarily a Tychonoff space. For each $f \in \mathbb{R}^X$, $Z(f) = \{x \in X : f(x) = 0\}$ denotes the zero-set of f and $Coz(f)$ denotes the complement of $Z(f)$ with respect to X. For a subring R of $\mathbb{R}^X$, $Z(R)$ denotes $\{Z(f) : f \in R\}$, cleary $Z(C(X)) = Z(X) = \{Z(f) : f \in C(X)\}$. Also, we use $M_x(R)$ to denote $\{f \in R, x \in Z(f)\}$. An ideal I in R is called free, if $\bigcap_{f\in I} Z(f) = \emptyset$. Otherwise, it is called fixed. By a maximal fixed ideal of R, we mean a fixed ideal that is maximal in the set of all fixed ideals of R. Clearly, fixed maximal ideals in $C(X)$ coincide with maximal fixed ideals and have the form $M_x = \{f \in C(X) : x \in Z(f)\}$, for $x \in X$. Note that for a subset A of X, M_A denotes $\{f \in C(X) : A \subseteq Z(f)\}$. The intersection of all the free ideals in $C(X)$ is denoted by $C_K(X)$. It is well-known that $C_K(X)$ is the subset of $C(X)$ consisting of all functions with compact support. Note that $cl_X Coz(f)$ is called the support of f for every $f \in C(X)$. The annihilator of $f \in R$ is defined by $Ann_R(f) = \{g \in R : fg = 0\}$. Assume that P and Q are partially ordered sets, then a function $f : P \longrightarrow Q$ is called an order-homomorphism if whenver $a \leq b$, then $f(a) \leq f(b)$. The function f is called an order-isomorphism if it is moreover bijective and $f^{-1} : Q \longrightarrow P$ is also an order-homomorphism. For terms and notations not defined here we follow the standard text of [4].

*Speaker

2 Main results

Definition 2.1. Let $\mathcal{P}$ be a family of subsets of a set X and $x \in X$. Then x is called a $\mathcal{P} - T_0$ point, if for each $y \in X$ with $x \neq y$, there exists $A \in \mathcal{P}$ which contains only one of the points. Similarly, we can define $\mathcal{P} - T_1$ and $\mathcal{P} - T_2$ (or $\mathcal{P}$-Hausdorff) points of X. Also, if X is $\mathcal{P} - T_i$ at each point, then X is called a $\mathcal{P} - T_i$ space, for each $0 \leq i \leq 2$.

The implications $(\mathcal{P} - T_2) \Longrightarrow (\mathcal{P} - T_1) \Longrightarrow (\mathcal{P} - T_0)$ are clear. Evidently, the converse of these implications may be true for some special $\mathcal{P}$, but this not true for the case $\mathcal{P} = Z(R)$. The following examples shows these facts. Note that in these examples S is a subring of $\mathbb{R}$ such that $S \cap \mathbb{Z} = \{0\}$, also, $F(X, S)$ denotes the collection of all real-valued functions on X with values in S. Moerover, for each $D \subseteq X$ we set $\kappa_D = 1 - \chi_D$ in which χ_D is the characteristic function of D.

Example 2.2.
(1) Let $A, B \subseteq X$ are such that $X = A \cup B$ and $A \cap B$, $A \setminus B$ and $B \setminus A$ have more than one point. Set $R = \{n + r\kappa_A + s\kappa_B : \ n \in \mathbb{Z},\ r, s \in S\}$. Then R is a subring of $\mathbb{R}^X$ and no point of X is $Z(R) - T_0$.
(2) Let $X = N^* = \mathbb{N} \cup \{a\}$ be the one-point compactification of $\mathbb{N}$ and let $A_n = \{a, n, n+1, ...\}$, also put $R = \mathbb{Z} + [M_a + (\sum_{n=1}^{\infty} M_{A_n} \cap F(X, S))]$. It is easy to see that X is a $Z(R)$-T_0-space but X is $Z(R)$-T_1 at no point; while $\mathbb{N}$ as a subspace of X is $Z(R)$-T_2.
(3) Let X be an infinite discrete space and $R = \mathbb{Z} + (C_K(X) \cap F(X, S))$. Then X with the $Z(R)$-topology is a cofinite space and so it is a T_1-space and anti-Hausdorff (i.e., no two points of X can be separated by disjoint open sets).

The next example reveals the necessity of $F(X, S)$ in constructing the above examples.

Example 2.3.
Let X be a topological space, I be an ideal in $C(X)$, $A = \bigcap Z[I]$ and $R = \mathbb{Z} + I$. Consider X with $Z(R)$-topology, then

(a) Two distinct points $a, b \in X$ are separated by open sets if and only if one of them is not in A.

(b) X is a $Z(R)$-Hausdorff space if and only if A has at most one point.

(c) If A has more than two points, then $x \in X$ is T_1-point at f and only if $x \notin A$.
In addition, X is a T_1-space if and only if A is at most a one-point set.

Definition 2.4. Clearly, $Z(R)$ constitutes a base for a topology on X which we call it Z(R)-topology and denote it by $\tau_{Z(R)}$. The topology induced by $Coz(R)$ is denoted by $\tau_{Coz(R)}$ and is called $Coz(R)$-topology. Also, the weak topology induced by R is denoted by τ_R.

Lemma 2.5. If $\mathbb{R} \subseteq R$, then the following families are both subbases for the weak topology induced by R.

(a) $\mathcal{D}_1 = \{f^{-1}((0, \frac{1}{n})) : \ f \in R,\ n \in \mathbb{N}\}$.

(b) $\mathcal{D}_2 = \{f^{-1}((0, +\infty)) : \ f \in R,\ n \in \mathbb{N}\}$.

Corollary 2.6. If $\mathbb{R} \subseteq R$, then $\mathcal{B}_1 = \{\bigcap_{i=1}^{k} f_i^{-1}(0, \frac{1}{n}) : \ f_i \in R,\ k, n \in \mathbb{N}\}$ and $\mathcal{B}_2 = \{\bigcap_{i=1}^{k} f_i^{-1}(0, +\infty) : \ f_i \in R,\ k \in \mathbb{N}\}$ are both bases for the weak topology induced by R.

Now, we compare the determined topologies. It is evident that $\tau_{Coz(R)} \subseteq \tau_R$ and the inclusion may be strict. For example, if $X = \mathbb{R}$, $f = id_X$ and $R = \{\sum_{i=1}^{n} a_i f^i : a_i \in \mathbb{Z}, n \in \mathbb{N}\}$, then R is a subring of $C(\mathbb{R})$ and for each $g \in R$, $Coz(g)$ is a finite-complement set which implies that $\tau_{Coz(R)} \neq \tau_R$.

Proposition 2.7. $\tau_{Coz(R)} = \tau_R$ if and only if $R \subseteq C(X, \tau_{Coz(R)})$.
Proof. $\Rightarrow$) Let $f \in R$, we are to show that $f \in C(X, \tau_{Coz(R)})$. Let U be open in $\mathbb{R}$, then $f^{-1}(U) \in \tau_R = \tau_{Coz(R)}$ and so $f \in C(X, \tau_{Coz(R)})$.

$\Leftarrow$) It suffices to show that $\tau_R \subseteq \tau_{Coz(R)}$ and this is clear, since τ_R is the smallest topology on X under which the elements of R are continuous. □

By the above proposition, if X is $Coz(R)$-T_2-space and $R \subseteq C(X, \tau_{Coz(R)})$, then $(X, \tau_{Coz(R)})$ is a Tychonoff space.

Definition 2.8. Two subsets $S_1, S_2 \subseteq \mathbb{R}^X$ are called zero-set equivalent, if $Z(S_1) = Z(S_2)$.

Lemma 2.9. Let S and $C(\mathbb{R})$ be two zero-set equivalent subsets of $\mathbb{R}^{\mathbb{R}}$ and R be a subring of $\mathbb{R}^X$, if for each $f \in R$ and each $g \in S$ we have $gof \in R$, then $Z(R) = \{f^{-1}(A) : f \in R \textit{ and } A \subseteq \mathbb{R} \textit{ is closed}\}$.

Proposition 2.10. Let R be a subring of $\mathbb{R}^X$, if S and $C(\mathbb{R})$ are zero-set equivalent subsets of $\mathbb{R}^{\mathbb{R}}$ and $gof \in R$, for each $f \in R$ and each $g \in S$, then

(a) $Coz(R)$ is a base for τ_R, i.e., $\tau_R = \tau_{Coz(R)}$.

(b) $\tau_{Coz(R)} = \tau_R \subseteq \tau_{Z(R)}$ and the equality does not hold, in general.

Proof. (a). By Lemma 2.9, it is clear.

(b). We are to show that $Coz(R) \subseteq \tau_{Z(R)}$. If $x \notin Z(f)$ where $f \in R$, then there is $g \in S$ such that $x \in Z(g)$ and $f^{-1}(Z(g)) \cap Z(f) = \emptyset$. Then $gof \in R$, $x \in Z(gof)$ and $Z(gof) \cap Z(f) = \emptyset$. Now, we show that the inclusion may be proper. Let X be a completely regular space that has at least one non-open zero-set Z and set $R = C(X)$, then $\tau_{Coz(R)} = \tau_R = \tau_X$, whereas $Z \notin \tau_X$, consequently, $\tau_{Coz(R)} = \tau_R \subsetneq \tau_{Z(R)}$. □

Theorem 2.11. The following statements are equivalent.

(a) $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$.

(b) Every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

Proof. (a $\Rightarrow$ b). Let $f \in R$ and $x \notin Z(f)$. Then $x \in Coz(f) \in \tau_{Coz(f)} \subseteq \tau_{Z(R)}$. Therefore, there exists $Z(g) \in Z(R)$ such that $x \in Z(g) \subseteq Coz(f)$.

(b $\Rightarrow$ a). Let $f \in R$ and $x \in Coz(f)$, so $x \notin Z(f)$ and by (2), there exists $g \in R$ such that $x \in Z(g)$ and $Z(f) \cap Z(g) = \emptyset$. Hence, $x \in Z(g) \subseteq Coz(f)$ and therefore $Coz(f)$ is open in $Z(R)$-topology. □

Theorem 2.12. The following statements are equivalent.

(a) $\tau_{Z(R)} \subseteq \tau_{Coz(R)}$.

(b) $Z(f)$ is clopen in the space $(X, \tau_{Coz(R)})$ for every $f \in R$.

(c) For each $f \in R$, $Z(f) = \bigcup_{g \in Ann_R(f)} Coz(g)$.

(d) For each $f \in R$, $(Ann_R(f), f)$ is a free ideal.

Proof. (a $\Rightarrow$ b). By 2.12, it is clear.

(b ⇒ c). It is evident.

(c ⇒ d). If $f \in R$ and I is an ideal in R, then $\bigcap_{h\in(I,f)} Z(h) = \bigcap_{g\in I}(Z(f) \cap Z(g))$. Thus, by (c), $Z(f) = \bigcup_{g\in Ann_R(f)} Coz(g)$. Hence $\bigcap_{g\in Ann_R(f)}(Z(f) \cap Z(g)) = \emptyset$, which implies $\bigcap_{g\in(Ann_R(f),I)} Z(g) = \emptyset$ and it means that the ideal $(Ann_R(f), f)$ is free.

(d ⇒ a). Let $f \in R$ and $x \in Z(f)$. By (d), there exists $g \in Ann_R(f)$ such that $x \notin Z(f) \cap Z(g)$. Hence, $x \notin Z(g)$ and $g \in Ann_R(f)$. Therefore, $x \in Coz(g) \subseteq Z(f)$ and so $Z(f) \in \tau_{Coz(R)}$. □

An immediate consequence of Theorems 2.11 and 2.12 is that $\tau_{Coz(R)} = \tau_{Z(R)}$ if and only if $Z(f)$ is clopen under both $Z(R)$-topology and $Coz(R)$-topology.

In part (b) of the following theorem we assume that "=" is a partial order on X.

Theorem 2.13. For a subring R of $\mathbb{R}^X$ the following statements hold.

(a) The mapping $x \longrightarrow M_x(R)$ is a one-one correspondence if and only if $(X, \tau_{Z(R)})$ is a T_0-space.

(b) The mapping $x \longrightarrow M_x(R)$ is an order-isomorphism between X and the set of all maximal fixed ideals of R if and only if $(X, \tau_{Z(R)})$ is a T_1-space.

Proof. (a ⇒). Let x, y are distinct points of X, so $M_x(R) \neq M_y(R)$, say $M_x(R) \nsubseteq M_y(R)$. Hence, there is an $f \in M_x(R) \setminus M_y(R)$. Thus, $x \in Z(f)$ and $y \notin Z(f)$.

(a ⇐). If x, y are distinct points of X. By our hypothesis, there is a $f \in R$ such that $x \in Z(f)$ and $y \notin Z(f)$ and hence $f \in M_x(R) \setminus M_y(R)$ (i.e., $M_x(R) \neq M_y(R)$).

(b ⇒). Suppose that x and y are two distinct points of X. Since $M_x(R) \subseteq M_y(R)$ and so there exists $f \in M_x(R) \setminus M_y(R)$. Consequently, $x \in Z(f)$ and $y \notin Z(f)$.

(b ⇐). Suppose that $x \in X$ and I is a fixed ideal in R containing $M_x(R)$. Take $y \in \cap_{f\in I} Z(f)$. Clearly, $M_x(R) \subseteq I \subseteq M_y(R)$. It is enough to show $x = y$. On the contrary suppose that $x \neq y$. By our hypothesis, there exists $f \in R$ such that $x \in Z(f)$ and $y \notin Z(f)$. Therefore, $M_x(R) \subseteq M_y(R)$ and this is a contradiction. To complete the proof, it is enough to show that every maximal fixed ideal is of the form $M_x(R)$. On the contrary suppose that $M_x(R)$ is not a maximal fixed ideal in R. Hence, there is $y \in X$ such that $y \neq x$ and $M_x(R) \subseteq M_y(R)$, but then $x = y$.

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Recurrent second fundamental form in submanifolds of Kenmotsu manifolds

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Abstract

In this paper, we study recurrent submanifolds of Kenmotsu manifolds. We show that they are totally geodesic. Moreover, generalized recurrent submanifolds of Kenmotsu manifolds are investigated.

Keywords: Kenmotsu manifold, Second Fundamental form, Submanifold
Mathematics Subject Classification [2010]: 53C50, 53C15

1 Preliminaries

Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g})$ be a $2n+1$ dimensional almost contact manifold, where ϕ, ξ, η and $\tilde{g}$ are $(1,1)$-tensor field, vector field, 1-form and a Riemannian metric respectively, which satisfy the following conditions

$$\phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1,$$

$$\phi^2 X = -X + \eta(X)\xi \ , \ \tilde{g}(\xi, X) = \eta(X),$$

$$(\tilde{\nabla}_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \mathcal{T}\tilde{M}.$$

An almost contact manifold is said to be a Kenmotsu manifold if

$$(\tilde{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{1}$$

where $\tilde{\nabla}$ is the Riemannian connection of $\tilde{g}$ [2]. In a Kenmotsu manifold the following relation holds

$$(\tilde{\nabla}_X \xi) = X - \eta(X)\xi. \tag{2}$$

Let (M, g) be a submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. If ∇ be the Levi-Chivita connections of M, then from Gauss and Weingarten formulas we have [5]

$$\tilde{\nabla}_Y X = \nabla_Y X + B(X,Y) \ , \ \tilde{\nabla}_Y V = D_Y V - A_V Y, \tag{3}$$

for any X and Y in $\mathcal{T}M$ and V in $(\mathcal{T}M)^{\perp}$. In (3), B, A and D are the second fundamental form, associated second fundamental form (shape operator) and normal connection on the $(\mathcal{T}M)^{\perp}$, respectively.

Let M be a submanifold of an almost contact manifold $(\tilde{M}, \phi, \xi, \eta, \tilde{g})$. M is said to be an invariant submanifold if the vector field ξ is tangent to M and $\phi T_p(M) \subset T_p M$ for all $p \in M$. Also, M is said to be an anti-invariant, if $\phi T_p(M) \subset T_p(M)^{\perp}$ for all $p \in M$ [4] .

*Speaker

2 Main results

Definition 2.1. A manifold is called totally geodesic if its second fundamental form vanishes identically (B=0).

Moreover, M is called a parallel submanifold [1] if

$$\bar{\nabla}_Z B(X,Y) = 0, \forall X,Y,Z \in \mathcal{T}M.$$

As a generalization of the previous definitions we have the following definitions.

Definition 2.2. A submanifold M is said to be a recurrent submanifold if there exists a 1-form ω such that B satisfies

$$\bar{\nabla}_Z B(X,Y) = \omega(Z)B(X,Y). \tag{4}$$

Definition 2.3. A submanifold is said to be generalized recurrent submanifold [3] if there exist 1-forms ω and ψ in M and normal vector field V such that B satisfies

$$\bar{\nabla}_Z B(X,Y) = \omega(Z)B(X,Y) + \psi(Z)g(X,Y)V. \tag{5}$$

Theorem 2.4. *A recurrent submanifold of a Kenmotsu manifold is totally geodesic.*

Proof. Let $X \in \mathcal{T}M$, from (3),

$$\tilde{\nabla}_X \xi = \nabla_X \xi + B(X,\xi).$$

On the other hand, from (2) we have $\tilde{\nabla}_X \xi = (X) - \eta(X)\xi \in \mathcal{T}M$. Since $B(X,\xi) \in (\mathcal{T}M)^{\perp}$, thus,

$$B(X,\xi) = 0. \tag{6}$$

On the other hand, since M is a recurrent submanifold, Equation (4) leads to

$$\omega(Z)B(X,Y) = \bar{\nabla}_Z B(X,Y) = D_Z B(X,Y) - B(\nabla_Z X,Y) - B(X,\nabla_Z Y).$$

Now, by substituting Y by ξ and using (6), we have $B(X,\phi Z) = 0$, then by substituting Z by ϕZ implies $B(X,Z) = 0$, thus $B = 0$. □

Theorem 2.5. *Any invariant generalized recurrent submanifolds of Kenmotsu manifolds are totally geodesic.*

Proof. We have

$$\bar{\nabla}_Z B(X,Y) = \omega(Z)B(X,Y) + \psi(Z)g(X,Y)V. \tag{7}$$

In the same way of the previous theorem, we can show that

$$B(X,\xi) = 0. \tag{8}$$

So,

$$B(X,Z) = \psi(\phi Z)\eta(X)V, \tag{9}$$

and

$$0 = B(\xi,Z) = \psi(\phi Z)V \quad \forall Z \in \mathcal{T}M.$$

Therefore, $\psi(\phi Z) = 0$, which emply $B = 0$. □

Now, we suppose that the structure vector field $\xi \in \mathcal{T}M^{\perp}$.

Theorem 2.6. *Let M be a submanifold of Kenmotsu manifold such that $\xi \in \mathcal{T}M^{\perp}$, then $B(X,Y) = -g(X,Y)\xi$.*

Proof. Since $\xi \in \mathcal{T}M^{\perp}$, Equations (3) and (2) imply $-A_{\xi}X = X$. So $B(X,Y) = -g(X,Y)\xi$. □

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Ricci Codazzi homogeneous pseudo-Riemannian manifolds of dimension four*

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Abstract

We study pseudo-Riemannian homogeneous four dimensional manifolds with non-trivial isotropy and completely classify those cases where the Ricci tensor is Codazzi. Specially, proper examples of Codazzi manifolds which are not conformally flat have been presented.

Keywords: Ricci tensor, Codazzi equation, conformally flat
Mathematics Subject Classification [2010]: 53C50, 53C30

1 Introduction

A pseudo-Riemannian manifold (M, g) is called *homogeneous*, if $I(M)$, the group of isometries of M, acts transitively on M. Equivalently, for any given points $p, q \in M$, an isometry ϕ of M exists such that $\phi(p) = q$ [8]. Homogeneous manifolds, for their wide geometrical and physical applications, were studied by several authors in the different dimensions and signatures[4, 5, 7]. Some geometric properties, like Ricci solitons, homogeneous structures and Einstein-like manifolds considered on the homogeneous pseudo-Riemannian manifolds [1, 2, 3].

Let (M, g) be a (pseudo-)Riemannian manifold. We say that, (M, g) admits a *Codazzi Ricci tensor*, or belongs to class $\mathcal{B}$, if

$$(\nabla_X Ric)(Y, Z) = (\nabla_Y Ric)(X, Z), \tag{1}$$

for all X, Y, $Z \in \mathfrak{X}(M)$. This condition which is famous as a kind of *Einstein-like property*, is in fact the generalization of Einstein and Ricci-parallel metrics.

In this paper, refereing to [7], we take four dimensional homogeneous pseudo Riemannian manifolds with non-trivial isotropy under consideration and fully classify examples of class $\mathcal{B}$ which are mentioned above. Finally, we classify proper examples of class $\mathcal{B}$, i.e., those cases which are not conformally flat.

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2 Examples of class $\mathcal{B}$

Referring to case numbering of the paper [7], all examples of class $\mathcal{B}$ is determined in the following theorem.

Theorem 2.1. *Let $(G/H, g)$ be an arbitrary non-Ricci-parallel pseudo-Riemannian four-dimensional homogeneous space with non-trivial isotropy, equipped with an invariant metric g. Then $(G/H, g)$ is in class $\mathcal{B}$ if belongs to one of the cases of the following Table I.*

case	*invariant metric*	*class $\mathcal{B}$*
$1.1^1 : 1$	$2a\theta^1\theta^3 + b\theta^2\theta^2 + 2c\theta^2\theta^4 + d\theta^4\theta^4$, $a(c^2 - bd) \neq 0$	$b = 0, a \neq \pm c$
$1.1^1 : 2$	"	$b = 0, p \neq 0, \frac{1}{2}$
$1.1^2 : 1$	$a\theta^1\theta^1 + b\theta^2\theta^2 + 2c\theta^2\theta^4 + a\theta^3\theta^3 + d\theta^4\theta^4$, $a(c^2 - bd) \neq 0$	$b = 0$
$1.1^2 : 2$	"	$b = 0, p \neq 0, 1$
$1.3^1 : 2$	$-2a\theta^1\theta^4 + 2a\theta^2\theta^3 + b\theta^3\theta^3 + 2c\theta^3\theta^4 + d\theta^4\theta^4$, $a \neq 0$	$\lambda \neq 0$
$1.3^1 : 4$	"	✓
$1.3^1 : 5$	"	$\lambda \neq 0$ *or* $\lambda = 0, \mu \neq 0, 2$
$1.3^1 : 7$	"	✓
$1.3^1 : 12$	"	$\mu \neq \lambda \pm 1$
$1.3^1 : 13$	"	$\lambda \neq -\frac{1}{2}, \frac{3}{2}$
$1.3^1 : 14$	"	$\lambda \neq 0, 1$
$1.3^1 : 15, 16, 19, 22, 26 - 29$	"	✓
$1.3^1 : 21, 24, 25$	"	$\lambda \neq 0, 2$
$1.3^1 : 30$	"	$\lambda \neq 1$ *or* $\lambda = 1, \mu \neq \pm 1$
$1.4^1 : 2$	$-2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + 2c\theta^3\theta^4 + d\theta^4\theta^4$, $ad \neq 0$	$p = 3, b \neq 0$
$1.4^1 : 9$	"	$d \neq -2a(h^2 + h + r)$
$1.4^1 : 10$	"	$r \neq -h - h^2$
$1.4^1 : 11$	"	$d \neq -2ar$
$1.4^1 : 12$	"	$r \neq 0$
$2.2^1 : 2$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^2\theta^2, \ a \neq 0$	$p \neq 0, \pm 2$
$2.2^1 : 3$	"	✓
$2.5^1 : 3 - 4$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3, \ a \neq 0$	$2h - h^2 + 4g \neq 0$
$2.5^2 : 2$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4, \ a \neq 0$	$r^2 + p \neq 0$
$3.3^1 : 1$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3, \ a \neq 0$	$p \neq 0$
$3.3^2 : 1$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4, \ a \neq 0$	$p \neq 0$

Table I: Strict examples of class $\mathcal{B}$.

In the Table I, $\{\theta^1, \ldots, \theta^4\}$ is the dual basis of $\{u_1, \ldots, u_4\}$ and ✓ means that all of the invariant metrics belong to class $\mathcal{B}$.

Proof. The proof is based on case by case study of Komrakow's list. We bring the details of the case $1.1^1 : 1$ and just apply the similar arguments for the other cases. For this homogeneous pseudo-Riemannian four-manifold $M = G/H$, there exists a basis $\{h_1, u_1, \cdots u_4\}$ of $\mathfrak{g}$, where the non-zero brackets are

$$[h_1, u_1] = u_1, \quad [h_1, u_3] = -u_3, \quad [u_1, u_3] = [u_2, u_4] = u_2, \quad [u_3, u_4] = u_3,$$

and the isotropy is $\mathfrak{h} = \mathrm{span}\{h_1\}$ [7]. Then, by taking $\mathfrak{m} = \mathrm{span}\{u_1, \cdots u_4\}$, the invariant metric with respect to $\{\theta^i\}$, the dual basis of $\{u_i\}$, is:

$$g = 2a\theta^1\theta^3 + b\theta^2\theta^2 + 2c\theta^2\theta^4 + d\theta^4\theta^4 \tag{2}$$

for some real constants a, b, c, d. The metric g in this case is non-degenerate if and only if $a^2(c^2 - bd) \neq 0$. It is obvious that if $bd > c^2$ the manifold is Lorentzian, otherwise if $bd < c^2$ then g is of signature $(2,2)$. Levi-Civita connection can be found by using well known Koszul formula and the curvature tensor will be determined dy direct calculations. To keep brevity, we don't present the components of the curvature tensor and just bring the Ricci tensor as following:

$$Ric = 2\left(\frac{b}{2a} + \frac{ab}{c^2 - bd}\right)\theta^1\theta^3 + \frac{b^2}{2}\left(-\frac{1}{a^2} + \frac{4}{c^2 - bd}\right)\theta^2\theta^2 - 2\left(\frac{bc}{2a^2} - \frac{2bc}{c^2 - bd}\right)\theta^2\theta^4 - \left(\frac{3}{2} + \frac{c^2}{2a^2} - \frac{2c^2}{c^2 - bd}\right)\theta^4\theta^4. \quad (3)$$

Moreover, we have the following nonzero components for the covariant derivatives of the Ricci tensor

$$\begin{array}{lll} \Lambda_1 Ric_{23} = \frac{b^2(a^2-c^2+bd)}{2a^2(bd-c^2)}, & \Lambda_1 Ric_{34} = \frac{b(a-2c)(a^2-c^2+bd)}{4a^2(c^2-bd)}, & \Lambda_2 Ric_{24} = \frac{b^2(a^2-c^2+bd)}{2a^2(bd-c^2)}, \\ \Lambda_2 Ric_{44} = \frac{bc(a^2-c^2+bd)}{a^2(bd-c^2)}, & \Lambda_3 Ric_{12} = \frac{b^2(a^2-c^2+bd)}{2a^2(c^2-bd)}, & \Lambda_3 Ric_{14} = \frac{b(a+2c)(a^2-c^2+bd)}{4a^2(c^2-bd)}, \\ \Lambda_4 Ric_{24} = \frac{bc(a^2-c^2+bd)}{2a^2(bd-c^2)}, & \Lambda_4 Ric_{44} = \frac{c^2(a^2-c^2+bd)}{a^2(bd-c^2)}, & \end{array} \quad (4)$$

where by $\Lambda_i Ric_{jk}$ we mean $(\nabla_{u_i} Ric)(u_j, u_k)$. According to the Equation (4), the Equation (1) satisfies if and only if either $b = 0$ or $d = \frac{c^2 - a^2}{b}$. The second solution yields that the Ricci tensor is parallel and we also must exclude the Ricci-parallel solutions from the first solution. Clearly $c \neq \pm a$ since the invariant metric is Ricci parallel if and only if $bd = c^2 - a^2$. □

Now, we study the conformally flat cases. Conformal flatness translates into the following system of algebraic equations:

$$W_{ijkh} = R_{ijkh} - \frac{1}{2}(g_{ik}\varrho_{jh} + g_{jh}\varrho_{ik} - g_{ih}\varrho_{jk} - g_{jk}\varrho_{ih}) + \frac{r}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}) = 0, \forall i, j, k, h = 1, \ldots, 4,$$

where W denotes the *Weyl tensor* and r is the scalar curvature. To belong to class $\mathcal{B}$ is a necessary condition for being conformally flat. A complete classification of four-dimensional conformally flat homogeneous pseudo-Riemannian manifolds were obtained in [6]. As a conclusion of the Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $(G/H, g)$ be a pseudo-Riemannian four-dimensional homogeneous space of the Table I. Then $(G/H, g)$ properly belongs to strict class $\mathcal{B}$ (i.e., it is not conformally flat), if it is one of the cases of the following Table II.*

case	*proper class $\mathcal{B}$*	*case*	*proper class $\mathcal{B}$*
$1.3^1 : 2$	$d \neq 0$	$1.3^1 : 24$	$(b - 2d(\lambda^2 - \lambda))(\lambda - \frac{2}{3}) \neq 0$
$1.3^1 : 4$	$d \neq 0$	$1.3^1 : 25$	$(b + 2d(\lambda^2 - \lambda))(\lambda - \frac{2}{3}) \neq 0$
$1.3^1 : 5$	$b\mu(\mu - 1) \neq 2c\lambda(\mu - 1) - d(\lambda^2 + \mu)$ and $(2c + d\lambda)^2 + \mu^2 \neq 0$	$1.3^1 : 28$	$b \neq 2d$
$1.3^1 : 7$	$d \neq b\lambda - 2c$	$1.3^1 : 29$	$b \neq -2d$
$1.3^1 : 12$	$b(\lambda + \mu - 1)(\mu - \frac{1}{2}) \neq 0$	$1.3^1 : 30$	$b + d - \lambda d - \mu b \neq 2c$
$1.3^1 : 15$	$b \neq -d$	$1.4^1 : 9$	$d^2 + (p^2 + p - r)^2 \neq 0$ and $(p + \frac{1}{2})^2 + (4ar + a + 4d)^2 \neq 0$
$1.3^1 : 16$	$b \neq d$	$1.4^1 : 10$	$r \neq h + h^2$
$1.3^1 : 19$	$b \neq 0$	$2.5^2 : 2$	$s \neq 0$
$1.3^1 : 21$	$b(\lambda - \frac{1}{2}) \neq 0$		

Table II: Proper examples of strict class $\mathcal{B}$.

Proof. We consider cases by case the strict examples of class $\mathcal{B}$, which are presented in the Table I. For the case $1.1^1 : 1$, the non-zero components of the Weyl tensor are:

$$\begin{array}{ll} W_{1223} = -\frac{b^2(a^2-2bd+2c^2)}{12a(bd-c^2)}, & W_{1234} = \frac{b(a^2c-2bcd+2c^3-3abd+3ac^2)}{12a(bd-c^2)}, \\ W_{1313} = \frac{b(a^2-2bd+2c^2)}{6(bd-c^2)}, & W_{1324} = -\frac{b}{2}, \\ W_{1423} = -\frac{b(a^2c-2bcd+2c^3+3abd-3ac^2)}{12a(bd-c^2)}, & W_{1434} = \frac{bd(a^2-2bd+2c^2)}{12a(bd-c^2)}, \\ W_{2424} = -\frac{b(a^2-2bd+2c^2)}{6a^2}. & \end{array}$$

Thus, the Weyl tensor vanishes identically if and only if $b = 0$ and so the strict examples of class $\mathcal{B}$, belonging to the case $1.1^1 : 1$, are conformally flat and so are not contained in the Table II. The other cases were checked by similar arguments. □

Note that, as also showed by the above Corollary 2.2, differently from the Riemannian case, a (locally) homogeneous conformally flat pseudo-Riemannian manifold need not to be (locally) symmetric (see also [6]).

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Semi-symmetric four dimensional homogeneous pseudo-Riemannian manifolds *

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Abstract

In this paper, we study four-dimensional pseudo-Riemannian homogeneous four spaces with non-trivial isotropy and we will determine examples with semi-symmetric curvature operators. We also present non-trivial examples of semi-symmetric homogeneous four-manifolds which are not locally symmetric.

Keywords: Homogeneous manifold, Curvature operators, Semi-symmetric manifolds.
Mathematics Subject Classification [2010]: 53C50, 53C30

1 Introduction

A pseudo-Riemannian manifold (M, g) is said to be *semi-symmetric* if its curvature tensor R satisfies

$$R(X, Y) \circ R = 0, \tag{1.1}$$

for all vector fields X, Y on M. Here, $R(X, Y)$ acts as a derivation on R. Equation (1.1) is the integrability condition of the equation $\nabla R = 0$, which determines locally symmetric spaces.

Riemannian semi-symmetric spaces have been extensively studied in literature. Since they are defined through a condition on the curvature tensor, their definition extends at once to the pseudo-Riemannian manifolds. Locally symmetric spaces are obviously semi-symmetric, but the converse does not hold: in any dimension greater than two, there exist Ricmannian semi-symmetric spaces which are not locally symmetric [2, 9]. However, semi-symmetry implies local symmetry in several classes of Riemannian manifolds. Some examples may be found in [2, 5]. In particular, a locally homogeneous semi-symmetric Riemannian manifold is locally symmetric. In the pseudo-Riemannian case, the following result has been proved by the first author and et al.

Theorem 1.1. [7] A there dimensional Lorentzian manifold is semi-symmetric if and only if it is curvature Ricci commuting.

*Will be presented in English
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Let $M = G/H$ (with H connected) be a homogeneous pseudo-Riemannian manifold, $\mathfrak{g}$ is the Lie algebra of G and the isotropy subalgebra is $\mathfrak{h}$. The factor space is $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ which identifies with a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropy representation

$$\psi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m}), \qquad \psi(x)(y) = [x, y]_{\mathfrak{m}} \quad \text{for all} \;\; x \in \mathfrak{g},\; y \in \mathfrak{m}.$$

A bilinear form on $\mathfrak{m}$ is determined by the matrix g with respect to a basis of $\mathfrak{g}$ by $\{h_1, \cdots, h_r, u_1, \cdots, u_n\}$, where $\{h_j\}$ and $\{u_i\}$ are bases of $\mathfrak{h}$ and $\mathfrak{m}$ for $1 \leq j \leq r = \dim H$ and $1 \leq i \leq n = \dim M$, respectively. Such a bilinear form is invariant if and only if ${}^t\psi(x) \circ g + g \circ \psi(x) = 0$ for all $x \in \mathfrak{h}$. It is well known that invariant pseudo-Riemannian metrics $\hat{g}$ on the homogeneous space $M = G/H$ are in a one-to-one correspondence with nondegenerate invariant symmetric bilinear forms g on $\mathfrak{m}$ [8]. The invariant bilinear form g uniquely defines its invariant linear Levi-Civita connection, described in terms of the corresponding homomorphism of $\mathfrak{h}$-modules $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$, such that $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$. Explicitly, one has

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \qquad \text{for all} \;\; x, y \in \mathfrak{g}, \tag{1.2}$$

where $v : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is the $\mathfrak{h}$-invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}), \qquad \text{for all } x, y, z \in \mathfrak{g}.$$

The *curvature tensor* is then determined by

$$\begin{aligned} R :&\mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}) \\ &(x, y) \to [\Lambda(x), \Lambda(y)] - \Lambda([x, y]), \end{aligned} \tag{1.3}$$

the *Ricci tensor* ϱ of g, will be deduced in terms of its components with respect to $\{u_i\}$, by

$$\varrho(u_i, u_j) = \sum_{r=1}^{4} R_{ri}(u_r, u_j), \qquad i, j = 1, \cdots, 4. \tag{1.4}$$

2 Semi-symmetric examples

Here we focus on Four dimensional homogeneous manifold according to the classification of the reference [8].

Theorem 2.1. *Let $(M = G/H, g)$ be a four-dimensional non-Einstein homogeneous manifold with non-trivial isotropy, then $(M = G/H, g)$ is semi-symmetric if and only if be curvature-Ricci commuting.*

Proof. It is well known that in the case when the curvature operator commutes with the Ricci operator, the manifold is called curvature-Ricci commuting. The proof is based on case by case considering according to the classification in Komrakov's list [8]. First of all we compute the invariant metric tensor of each case, then the connection, curvature operator and Ricci operator of each case will be compute respectively. A straightforward but long computation in each case such that $R(x, y)\varrho(z, w) = \varrho(z, w)R(x, y)$ will show some condition over metric coefficients, on the other hand we use (1.1) formula in order to obtain semi-symmetric condition for each case, it will appear that in all cases both two conditions are the same. ☐

Theorem 2.2. *Let $(M = G/H, g)$ be a homogeneous four-manifold with non-trivial isotropy, semi-symmetric non-locally symmetric cases are specified in the following Table I:*

case	*invariant metric*	*semi-symmetric non-locally symmetric*
$1.1^1 : 1$	$2a\theta^1\theta^3 + b\theta^2\theta^2 + 2c\theta^2\theta^4 + d\theta^4\theta^4, a(c^2 - bd) \neq 0$	$b = 0, bd \neq -a^2 + c^2$
$1.1^1 : 2$	″	$b = 0, p(p - \frac{1}{2}) \neq 0$
$1.1^2 : 1$	$a\theta^1\theta^1 + b\theta^2\theta^2 + 2c\theta^2\theta^4 + a\theta^3\theta^3 + d\theta^4\theta^4, a(c^2 - bd) \neq 0$	$b = 0, bd \neq c^2 + 4a^2$
$1.1^2 : 2$	″	$b = 0, p(p - 1) \neq 0$
$1.3^1 : 2$	$-2a\theta^1\theta^4 + 2a\theta^2\theta^3 + b\theta^3\theta^3 + 2c\theta^3\theta^4 + d\theta^4\theta^4, a \neq 0$	$\lambda \neq 0$
$1.3^1 : 3 - 4$	″	✓
$1.3^1 : 5$	″	$\mu^2 + \lambda^2 \neq 0$, and $(b + d)^2 + \lambda^2 + (\mu - 2)^2 \neq 0$
$1.3^1 : 6 - 7$	″	✓
$1.3^1 : 8$	″	$b \neq 0$
$1.3^1 : 9$	″	$b\lambda(\lambda + 1) \neq 0$
$1.3^1 : 10$	″	✓
$1.3^1 : 12$	″	$b^2 + (\lambda - \mu \pm 1)^2 \neq 0$, and $\mu^2 + (\lambda \pm 1)^2 \neq 0$, and $\lambda^2 + (\mu \pm 1)^2 \neq 0$, and $(\mu - \frac{1}{2})^2 + (\lambda + \frac{1}{2})^2 \neq 0$, and $(\mu - \frac{1}{2})^2 + (\lambda - \frac{3}{2})^2 \neq 0$,
$1.3^1 : 13 - 16$	″	✓
$1.3^1 : 19$	″	✓
$1.3^1 : 20$		$b \neq 0$
$1.3^1 : 21$	″	$\lambda(b^2 + (\lambda - 2)^2) \neq 0$
$1.3^1 : 22$	″	✓
$1.3^1 : 23$	″	✓
$1.3^1 : 24$	″	$\lambda((b + 4d)^2 + (\lambda - 2)^2) = 0$
$1.3^1 : 25$	″	$\lambda((b + 4d)^2 + (\lambda - 2)^2) \neq 0$
$1.3^1 : 26 - 29$	″	✓
$1.3^1 : 30$	″	$c^2 + (\mu - 1)^2 + (\lambda - 1)^2 \neq 0$
$1.4^1 : 2$	$-2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + 2c\theta^3\theta^4 + d\theta^4\theta^4, ad \neq 0$	$p = 1, b \neq 0$
$1.4^1 : 9$	″	$(d + 4a)^2 + (r - 2)^2 + (p + 1)^2 \neq 0$, and $(d + 4a)^2 + r^2 + (p + 2)^2 \neq 0$, and $(d + a)^2 + (r - \frac{3}{4})^2 + (p + \frac{1}{2})^2 \neq 0$
$1.4^1 : 10$	″	$(r^2 + p^2)(r^2 + (p - 1)^2) \neq 0$
$1.4^1 : 11$	″	$(d + 4a)^2 + (r - 2)^2 \neq 0$
$1.4^1 : 12$	″	$r \neq 0$
$1.4^1 : 13$	″	✓
$1.4^1 : 15$	″	$a \neq -d$
$1.4^1 : 16$	″	$a \neq d$
$1.4^1 : 17$	″	✓
$1.4^1 : 18$	″	$a \neq -d$
$1.4^1 : 19$	″	$a \neq d$
$1.4^1 : 20$	″	✓
$2.2^1 : 2$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^2\theta^2,\ a \neq 0$	$\lambda(\lambda \pm 2) \neq 0$
$2.2^1 : 3$	″	✓
$2.5^1 : 3$	$2a\theta^1\theta^3 + a\theta^2\theta^4 + b\theta^3\theta^3$	$(g - 2)^2 + (h + 2)^2 + k^2 \neq 0$
$2.5^1 : 4$	″	$g \neq -\frac{h}{2} + \frac{h^2}{4}$
$2.5^1 : 5$	″	✓
$2.5^2 : 2$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4,\ a \neq 0$	$(p + r^2)^2 + s^2 \neq 0$
$2.5^2 : 3$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4,\ a \neq 0$	$s \neq 0$
$3.3^1 : 1$	$2a\theta^1\theta^3 + 2a\theta^2\theta^4 + b\theta^3\theta^3,\ a \neq 0$	$\lambda \neq 0$
$3.3^2 : 1$	$2a\theta^1\theta^3 + a\theta^2\theta^2 + b\theta^3\theta^3 + a\theta^4\theta^4,\ a \neq 0$	$\lambda \neq 0$

Table I: Non-locally symmetric semi-symmetric examples of homogeneous spaces G/H with non-trivial isotropy.

Here $\{\theta^1, \ldots, \theta^4\}$ is the dual basis of $\{u_1, \ldots, u_4\}$ and ✓ means that all of the invariant metrics are semi-symmetric non-locally symmetric.

Proof. We will consider all the spaces included in Komrakov's classification of $M = G/H$, four-dimensional homogeneous pseudo-Riemannian with nontrivial isotropy witch is appeared in Theorem 2 of [8], and checked the condition (1.1) for each case in Komrakov's clasificasion. So by straightforward but long computation and using the formula (1.2) and (1.3) for each cases, we will have the result which is shown in Table I. □

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Talk Some new subgroupoids of topological fundamental groupoid pp.: 1–4

Some new subgroupoids of topological fundamental groupoid

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Abstract

$\lim_{hh} lk$ In this talk, we introduce some subgroupoids of the fundamental groupoid of locally wild spaces by using the recently emerged subgroups of the fundamental group, $\pi_1^s(X,x)$, $\pi_1^{sg}(X,x)$ and $\pi_1^{sp}(X,x)$, for a given space X which is not semi-locally simply connected. Also, we use the advantages of covering groupoid theory to find categorical universal covering of these spaces.

Keywords: fundamental groupoid, Small loop, Spanier group
Mathematics Subject Classification [2010]: 57M10, 55R05

1 Introduction

A groupoid G is a small category in which each morphism is an isomorphism. In a groupoid G, we call morphisms as elements of G and for $x, y \in O(G) = objec(G)$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. The object group at x is $G(x) = G(x, x)$. For $x \in O(G)$, by $star_G x$ we mean the set of all the elements of G such that initiate at x.

A morphism of groupoids $\widetilde{G}$ and G is a functor, i.e., it consists of a pair of functions $f : \widetilde{G} \longrightarrow G$, $O(f) : O(\widetilde{G}) \longrightarrow O(G)$ preserving all the structure. Let $f : \widetilde{G} \longrightarrow G$ be a morphism of groupoids. Then f is called a covering morphism if for each $\tilde{x} \in \widetilde{G}$, the restriction $star_{\widetilde{G}}\tilde{x} \longrightarrow star_G f(\tilde{x})$ of f is bijective.

Let G be a groupoid. A subgroupoid of G is a subcategory H of G such that $a \in H$ implies that $a^{-1} \in H$; that is, H is a subcategory which is also a groupoid. A subgroupoid N of G is called normal if N is wide in G (as a subcategory) and, for any objects x, y of G and a in $G(x, y)$, $aN(x)a^{-1} \subseteq N(y)$. If N is a normal subgroupoid of G such that $N(x, y) = \emptyset$ for $x \neq y$, the quotient groupoid of G by N is a groupoid G/N by object set as same as G and $G/N(x, y) = \{aN(x) \ : \ a \in G(x, y)\}$, for any $x, y \in Object(G/N)$ with the multiplication that if $a \in G(x, y)$ and $b \in G(y, z)$ then $bN(y)aN(x) = baN(x)$.

For a topological space X, the homotopy classes of the paths in X form a groupoid on X. The composition of paths in X induces a composition of the homotopy classes. This groupoid is called fundamental groupoid and denoted by $\pi_1 X$. (see [1])

When the space X is not semi-locally simply connected, some subgroups of the fundamental group will emerge that have important role in the classification of the categorical universal covering.

*Speaker

Definition 1.1. ([5]) A loop $\alpha : (I, \partial I) \longrightarrow (X, x)$ is *small* if and only if there exists a representative of the homotopy class $[\alpha] \in \pi_1(X, x)$ in every open neighborhood U of x. The *small loop group* $\pi_1^s(X, x)$ of (X, x) is the subgroup of the fundamental group $\pi_1(X, x)$ consisting of all homotopy classes of small loops. The SG subgroup of $\pi_1(X, x)$, denoted by $\pi_1^{sg}(X, x)$, is the subgroup generated by the following set

$$\{[\alpha * \beta * \alpha^{-1}] \mid [\beta] \in \pi_1^s(X, \alpha(1)),\ \alpha \in P(X, x)\},$$

where $P(X, x)$ is the space of all paths in X with initial point x.

Definition 1.2. [3] If $\mathcal{U}$ is an open cover of X, then consider the subgroup of $\pi_1(X, x)$ consisting of the homotopy classes of loops that can be represented by a product of the form

$$\prod_{j=1}^{n} u_j v_j u_j^{-1},$$

where the u_j's are arbitrary paths starting at the base point x and each v_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$. This group is called the *Spanier group with respect to* $\mathcal{U}$, denoted by $\pi(\mathcal{U}, x)$. The Spanier group of the space X, denoted by $\pi_1^{sp}(X, x)$ is as follows:

$$\pi_1^{sp}(X, x) = \bigcap_{open\ covers\ \mathcal{U}} \pi(\mathcal{U}, x),$$

Pakdaman et. al [4, 5, 6, 7] introduced three categorical universal covering related to these new subgroups of the fundamental group. For the existence of them, they use classical relation between covering spaces and fundamental groups. In this article, at first we introduce some normal subgroupoids of the fundamental groupoid which are constructed by $\pi_1^s(X, x), \pi_1^{sg}(X, x), \pi_1^{sp}(X, x)$ and then by using covering groupoid theory, we prove that with some local properties, these groups can be image subgroups by some covering maps.

2 Main results

In this section we assume that all the spaces are locally path connected and by the universal covering we mean the categorical sense, that is, a covering $p : \widetilde{X} \longrightarrow X$ with the property that for every covering $q : \widetilde{Y} \longrightarrow X$ with a path connected space $\widetilde{Y}$ there exists a covering $f : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ f = p$.

Definition 2.1. For a topological space X, small generated fundamental groupoid is a groupoid denoted by $\pi^{sg}X$ with $O(\pi^{sg}X) = X$, $\pi^{sg}X(x) = \pi_1^{sg}(X, x)$ and $\pi^{sg}X(x, y) = \emptyset$, for $x \neq y \in X$.

Proposition 2.2. *For a given space X, $\pi^{sg}X$ is a totally disconnected normal subgroupoid of πX.*

Definition 2.3. For a topological space X, Spanier fundamental groupoid is a groupoid denoted by $\pi^{sp}X$ with $O(\pi^{sp}X) = X$, $\pi^{sp}X(x) = \pi_1^{sp}(X, x)$ and $\pi^{sp}X(x, y) = \emptyset$, for $x \neq y \in X$.

Proposition 2.4. *For a given space X, $\pi^{sp}X$ is a totally disconnected normal subgroupoid of πX.*

R. Brown and G. Danesh-Naruie [2] showed that when N is a totally disconnected normal subgroupoid of the fundamental groupoid of a locally path connected and semi-locally simply connected space X, the the topology of X can be lifted on $\frac{\pi X}{N}$ so that it becomes a topological groupoid over X.

Definition 2.5. [2] Let G be a groupoid and $X = O(G)$. If the set of morphisms of G and X have both topologies such that the source and target maps $s, t : G \longrightarrow X$, the difference map $\delta : G \times G \longrightarrow G$ defined by $\delta(a, b) = a \circ b^{-1}$ and the unit map $1 : X \longrightarrow G$ by $1(x) = 1_x$ are continuous.

Theorem 2.6. *[2] Let X be a semi-locally simply connected space, and let N be a totally disconnected normal subgroupoid of πX. Then the set of elements of the quotient groupoid $\frac{\pi X}{N}$ may be given a topology such that:*
i) $\frac{\pi X}{N}$ becomes a topological groupoid over X with discrete object groups.
ii) For each $x \in X$ the subspace $star_{\frac{\pi X}{N}} x$ is the covering space determined by the subgroup $N(x)$ of $\pi_1(X, x)$.

Definition 2.7. [4, 5, 6] i) A space X is a semi-locally small generated space if and only if for each $x \in X$ there exists an open neighborhood U of x such that $i_*\pi_1(U, x) \subseteq \pi_1^{sg}(X, x)$, where $i : U \longrightarrow X$ is the inclusion map.
ii) A space X is a semi-locally Spanier space if and only if for each $x \in X$ there exists an open neighborhood U of x such that $i_*\pi_1(U, x) \subseteq \pi_1^{sp}(X, x)$, where $i : U \longrightarrow X$ is the inclusion map.

Theorem 2.8. *Let X be a semi-locally small generated space. Then the set of elements of the quotient groupoid $\frac{\pi X}{\pi^{sg}X}$ may be given a topology such that:*
i) $\frac{\pi X}{\pi^{sg}X}$ becomes a topological groupoid over X with discrete object groups $\pi_1^{sg}(X, x)$.
ii) For each $x \in X$ the subspace $star_{\frac{\pi X}{\pi^{sg}X}} x$ is the covering space determined by the subgroup $\pi_1^{sg}(X, x)$ of $\pi_1(X, x)$.
Sketch of the proof: *Let $\mathcal{U}$ be the open cover of X consisting of all open, path connected subsets U of X such that $i_*\pi_1(U, x) \subseteq \pi_1^{sg}(X, x)$, for the inclusion $i : U \longrightarrow X$ and $x \in U$. For each $U \in \mathcal{U}$ and $x \in U$, define $L_x : U \longrightarrow \frac{\pi X}{\pi_1^{sg}(X,x)}$ by $L_x(x') = [\alpha]\pi_1^{sg}(X, x)$, where $\alpha : I \longrightarrow U$ is a path from x to x'. The condition semi-locally small generated makes that L_x be independent of the choice of α. Let $\widetilde{U}_x = L_x(U)$. Then for every $a \in \frac{\pi X}{\pi^{sg}X}(x, y)$ the sets $\widetilde{V}_y a \widetilde{U}_x^{-1}$ for all $U, V \in \mathcal{U}$ such that $x \in U$ and $y \in V$ forms a base for the lifted topology on $\frac{\pi X}{\pi^{sg}X}$.*

We have a similar result for Spanier fundamental groupoid as follow:

Theorem 2.9. *Let X be a semi-locally Spanier space. Then the set of elements of the quotient groupoid $\frac{\pi X}{\pi^{sp}X}$ may be given a topology such that:*
i) $\frac{\pi X}{\pi^{sp}X}$ becomes a topological groupoid over X with discrete object groups $\pi_1^{sp}(X, x)$.
ii) For each $x \in X$ the subspace $star_{\frac{\pi X}{\pi^{sp}X}} x$ is the covering space determined by the subgroup $\pi_1^{sp}(X, x)$ of $\pi_1(X, x)$.

By using the results of [7] we can prove the following corollary.

Corollary 2.10. *For a semi-locally Spanier space X, there exists a covering map $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X}, \tilde{x}) = \pi_1^{sp}(X, x)$. Moreover, this covering map is universal covering of X.*

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Some properties of multi-Fedosove supermanifolds of order 3

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Abstract

In this paper we define multi-Fedosove supermanifolds and show that every multisymplectic supermanifold of order 3 is a multi-Fedosove supermanifolds. Then we study the curvature tensor of a multi-Fedosove supermanifolds.

Keywords: Multisymplectic supermanifold, multi-Fedosove supermanifolds, curvature tensor
Mathematics Subject Classification [2010]: 58A50, 53D05

1 multi-Fedosove supermanifolds

A supermanifold $\mathcal{M}$ of dimension $n|m$ is a pair $(M, \mathcal{O}_{\mathcal{M}})$, where M is a Hausdorff topological space and $\mathcal{O}_{\mathcal{M}}$ is a sheaf of commutative superalgebras with unity over $\mathbb{R}$ locally isomorphic to $\mathbb{R}^{m|n} = (\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n} \otimes \Lambda_{\eta^1,\ldots,\eta^m})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbb{R}^n$ and $\Lambda_{\eta^1,\ldots,\eta^m}$ is the grassmann superalgebra of m generators.

Definition 1.1. Let ξ be a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-supermodules on $\mathcal{M}$, a connection on ξ is a morphism $\nabla : \mathcal{T}_{\mathcal{M}} \otimes_{\mathbb{R}} \xi \to \xi$ of sheaves of supermodules over $\mathbb{R}$ such that
$\nabla_{fX}v = f\nabla_X v$, $\nabla_X fv = (Xf) + (-1)^{\widetilde{X}\widetilde{f}} f\nabla_X v$ and $\widetilde{\nabla_X v} = \widetilde{v} + \widetilde{X}$,
for all homogeneous function f, vector fields X and section v of ξ.

Let us consider a multisymplectic supermanifold of degree k $(\mathcal{M}, \omega)$, i.e. a supermanifold $\mathcal{M}$ with a closed non-degenerate graded differential k-form ω.

Definition 1.2. A multisymplectic connection on $\mathcal{M}$ is a connection for which:
i- The torsion tensor vanishes, i.e.

$$\nabla_X Y - (-1)^{\widetilde{X}\widetilde{Y}} \nabla_Y X = [X, Y].$$

ii- It is compatible to the multisymplectic form, i.e. $\nabla\omega = 0$.

A multi-Fedosov supermanifold $(\mathcal{M}, \omega, \nabla)$ is defined as a multisymplectic supermanifold $(\mathcal{M}, \omega)$ equipped with a a multisymplectic connection ∇.

*Speaker

If ∇ be a connection on multisymplectic supermanifold $\mathcal{M}$ of order 3 then $\nabla\omega = 0$ if and only if

$$X(\omega(Y,Z,V)) = (-1)^{\widetilde{X}\widetilde{\omega}}\omega(\nabla_X^Y, Z, V) + (-1)^{\widetilde{X}(\widetilde{\omega}+\widetilde{Y})}\omega(Y, \nabla_X^Z, V) \tag{1}$$

$$+(-1)^{\widetilde{X}(\widetilde{\omega}+\widetilde{Y}+\widetilde{Z})}\omega(Y, Z, \nabla_X^V),$$

for any vector field X, Y, Z, V.
If (η_i) is a system of coordinates on $\mathcal{U} \subseteq \mathcal{M}$,

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$$

gives well-defined elements $\Gamma_{ij}^k \in \mathcal{O}_\mathcal{M}(U)$ of parity

$$\widetilde{\Gamma_{ij}^k} = \widetilde{\eta_i} + \widetilde{\eta_j} + \widetilde{\eta_k}$$

. The components of ω in these coordinates are $\omega_{ijk} = \omega(\partial_i, \partial_j, \partial_k)$. It is sufficient to write (1) for $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$ and $V = \partial_l$. This gives

$$\partial_l\omega_{ijk} = (-1)^{\epsilon_l\widetilde{\omega}}\omega(\nabla_{\partial_l}\partial_i, \partial_j, \partial_k) + (-1)^{\epsilon_l(\epsilon_i+\widetilde{\omega})}\omega(\partial_i, \nabla_{\partial_l}\partial_j, \partial_k) + (-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\widetilde{\omega})}\omega(\partial_i, \partial_j, \nabla_{\partial_l}\partial_k)$$

$$= (-1)^{\epsilon_l\widetilde{\omega}}\omega_{\lambda jk}\Gamma_{li}^\lambda + (-1)^{\epsilon_l(\epsilon_i+\widetilde{\omega})}\omega_{i\lambda k}\Gamma_{lj}^\lambda + (-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\widetilde{\omega})}\omega_{ij\lambda}\Gamma_{lj}^\lambda$$

$$= (-1)^{\epsilon_l\widetilde{\omega}}\Gamma_{jkli} - (-1)^{\epsilon_l(\epsilon_i+\widetilde{\omega})}\Gamma_{iklj} + (-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\widetilde{\omega})}\Gamma_{ijlk},$$

where $\Gamma_{ijlk} = \omega_{ij\lambda}\Gamma_{lk}^\lambda$ and $\widetilde{\partial_i} = \epsilon_i$.

The equality $d\omega = 0$ means

$$(-1)^{\epsilon_i\widetilde{\omega}}\partial_i\omega_{jkl} - (-1)^{\epsilon_j(\epsilon_i+\widetilde{\omega})}\partial_j\omega_{ikl} + (-1)^{\epsilon_k(\epsilon_i+\epsilon_j+\widetilde{\omega})}\partial_k\omega_{ijl} - (-1)^{\epsilon_l(\epsilon_i+\epsilon_j+\epsilon_k+\widetilde{\omega})}\partial_l\omega_{ijk} = 0.$$

Let Π be a symmetric connection. If we define $\Gamma_{ijkl} = \partial_l\omega_{kij} + \Pi_{ijkl} - \Pi_{jilk} - \Pi_{likj} + \Pi_{ljik}$ then Γ compatible to the ω.

2 Curvature of multi-Fedosove supermanifoldsof order 3

If ∇ be a multisymplectic connection of order 3 on $\mathcal{M}$. The curvature ∇ is defined by usual formula

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Then we have

$$- < R(X,Y)Z, V >= (-1)^{\widetilde{X}\widetilde{Y}} < R(Y,X)Z, V >= (-1)^{\widetilde{Z}\widetilde{V}} < R(X,Y)V, Z > .$$

$$< R(X,Y)Z, V >= (-1)^{(\widetilde{X}+\widetilde{Y})(\widetilde{Z}+\widetilde{V})} < R(Z,W)X, Y > .$$

And

$$R(X,Y)Z + (-1)^{\widetilde{Z}(\widetilde{X}+\widetilde{Y})}R(Z,X)Y + (-1)^{\widetilde{X}(\widetilde{Y}+\widetilde{Z})}R(Y,Z)X = 0.$$

The components of the curvature tensor are introduce by

$$R(\partial_i, \partial_k)\partial_j = R^m_{ijk}\partial_m.$$

The curvature R^m_{klt} satisfies the tensor equations

$$R^i_{mjk} = -(-1)^{\epsilon_j\epsilon_k} R^i_{mkj}.$$

And

$$(-1)^{\epsilon_m\epsilon_k} R^i_{mjk} + (-1)^{\epsilon_j\epsilon_m} R^i_{jkm} + (-1)^{\epsilon_j\epsilon_k} R^i_{kmj} = 0.$$

Denote also

$$R_{ijklt} = \omega_{ijm} R^m_{klt} = \omega(\partial_i, \partial_j, R(\partial_l, \partial_t)\partial_k).$$

The components of the curvature tensor in terms of the Christoffel symbols has the standard form;

$$R^l_{ijk} = (-1)^{\epsilon_j\epsilon_k}\partial_j\Gamma^l_{ki} - \partial_k\Gamma^l_{ij} + (-1)^{\epsilon_j\epsilon_i}\Gamma^m_{ki}\Gamma^l_{mj} - (-1)^{\epsilon_k(\epsilon_i+\epsilon_j)}\Gamma^m_{ij}\Gamma^l_{km}.$$

Instead of R_{ijklt} we can also consider $R(X, Y, Z, V, W)$ which is a multilinear function on any tangent space $T_x\mathcal{M}$:

$$R(X, Y, Z, V, W) = \omega(X, Y, R(V, W)Z).$$

So that

$$R_{ijklt} = R(\partial_i, \partial_j, \partial_k, \partial_l, \partial_t).$$

Then we have

$$R_{ijklt} = -(-1)^{\epsilon_t\epsilon_l} R_{ijktl}.$$

And

$$R_{ij(klt)} = 0.$$

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Some results on Φ-reflexive property

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Abstract

We propose Φ-reflexive property which is generalization of the reflexive concept in [2]. So, we prove some properties of this new concept. Finally, we used Φ-reflexive property to prove some results on manifold theory, Frölicher spaces, differential spaces and diffeology spaces.

Keywords: Φ-reflexive property, Manifold, Frölicher space, Differential space, Diffeology
Mathematics Subject Classification [2010]: 51-06, 51H25.

1 Introduction

The concept of smooth manifold generalized by some mathematicians: Roman Sikroski presented the differential spaces in the 1971s [6]; The diffeological spaces offered by Jean-Marie Souriau in the 1980s and developed by his students Paul Donato and Patrick Iglesias [5]; In the 1982s, the Frölcher spaces introduced by Alfred Frölicher [4]; The (X,Γ)-structure is an another generalization of smooth manifold which consists of all above structures. This structure proposed by the authors in the 2015s [3].

In this paper, we obtain the interesting results about of some above structures by Φ-reflexive property concept.

2 Φ-reflexive property

Batubenge and others offer reflexive concept and used to compare the subcategories of above structures [2]. In this paper, we present Φ-reflexive property which is a generalization reflexive concept in [2]. So, we obtain some interesting results of this concept.

Definition 2.1. [2] Suppose that M is a nonempty set and assume that $\mathcal{D}_0$ is a collection of parametrizations from some open subsets of $\mathbb{R}^n$'s to M. Let $\mathcal{F}_0$ be a family of real functions on M. We define the following sets:

$$\Phi\mathcal{D}_0 := \{f : M \to \mathbb{R} | \forall (p : U \to M) \in \mathcal{D}_0, f \circ p \in C^\infty(U)\},$$
$$\Pi\mathcal{F}_0 := \{parametrizations\ p : U \to M | \forall f \in \mathcal{F}_0, f \circ p \in C^\infty(U)\}$$

We said $\mathcal{D}_0$ or $\mathcal{F}_0$ are reflexive if and only if $\mathcal{D}_0 = \Pi\Phi\mathcal{D}_0$ or $F_0 = \Phi\Pi\mathcal{F}_0$(resp).

*Speaker

Definition 2.2. Suppose that X is a topological space and M is a nonempty set. A map $\phi : U \to M$ from some open subset of X into M is called an X-parametrization on M. A map $f : M \to X$ from M into X is called an X-function on M.

Definition 2.3. Suppose that X_1 and X_2 are topological spaces. Assume that Φ is a collection of continuous X_1-parametrizations on X_2 and let M is a nonempty set. Let $\mathcal{P}_0$ be an X_1-parametrizations collection on M and let $\mathcal{F}_0$ is an X_2-functions family on M. We define two following sets:

$$\Phi^*\mathcal{P}_0 := \{f : M \to X_2 | f \circ p \in \Phi, \ \forall p \in \mathcal{P}\},$$
$$\Phi_*\mathcal{F}_0 := \{p : U \to M | U \text{ is an open subset of } X_1 \textit{ and } f \circ p \in \Phi, \ \forall f \in \mathcal{F}\}.$$

Remark 2.4. The above operators is inclusion-reserving. Also, we always have the following conditions:

$$\Phi_*\Phi^*\mathcal{P}_0 \supseteq \mathcal{P}_0, \qquad \Phi^*\Phi_*\mathcal{F}_0 \supseteq \mathcal{F}_0.$$

Definition 2.5. By the above assuming, we say $\mathcal{F}_0$ has Φ-reflexive property if $\Phi^*\Phi_*\mathcal{F}_0 = \mathcal{F}_0$. Similarly, we say $\mathcal{P}_0$ has Φ-reflexive property if $\Phi_*\Phi^*\mathcal{P}_0 = \mathcal{P}_0$.

The following example show that the Φ-reflexive property is a generalization of definition 2.1.

Example 2.6. Assume that Φ is all smooth maps from the open subsets of $\mathbb{R}^n$'s to $\mathbb{R}$ ($n \in \mathbb{N}$). Let $\mathcal{P}_0$ is a parametrizations collection from some open subsets $\mathbb{R}^n$'s to M and $\mathcal{F}_0$ be a family of real function on M. Then $\mathcal{F}_0$ and $\mathcal{P}_0$ are reflexive in concept of definition 2.1 if and only if have Φ-reflexive property.

Lemma 2.7. *Assume that Φ is a collection of continuous X_1-parametrizations on a topological space X_2 and let M is a nonempty set. Let $\mathcal{P}_0$ is an X_1-parametrizations collection on M and $\mathcal{F}_0$ is an X_2-functions family on M. Then*

- *The X_2-functions family $\mathcal{F} := \Phi^*\mathcal{P}_0$ on M has Φ-reflexive property.*
- *The X_1-parametrizations collection $\mathcal{P} := \Phi_*\mathcal{F}_0$ on M has Φ-reflexive property.*

Lemma 2.8. *Let Φ is a collection of continuous X_1-parametrizations on a topological space X_2. Then Φ has Φ-reflexive property.*

Lemma 2.9. *Consider that Φ is a continuous collection of X_1-parametrizations on a topological space X_2 and M be a nonempty set. Let $\mathcal{F}_0$ be an X_2-function family on M and let $\mathcal{P}_0$ be an X_1-parametrizations collection on M. Denote $\mathcal{T}_{\mathcal{F}_0}$ by the weakest topology on M such that all elements of $\mathcal{F}_0$ are continuous and $\mathcal{T}_{\mathcal{P}_0}$ is the strongest topology on M such that all elements of $\mathcal{P}_0$ are continuous. If we have $f \circ p \in \Phi$, for any $f \in \mathcal{F}_0$ and for all $p \in \mathcal{P}_0$, then*

$$\mathcal{T}_{\mathcal{F}_0} \subseteq \mathcal{T}_{\mathcal{P}_0}.$$

Definition 2.10. Let X be a topological space and M_1, M_2 are nonempty sets and let $\zeta : M_1 \to M_2$ is a map.

- The X-parametrizations collections $\mathcal{P}_1$ on M_1 and $\mathcal{P}_2$ on M_2 are ζ-related, written $\mathcal{P}_1 \sim_\zeta \mathcal{P}_2$ provided $\zeta_\#(\mathcal{P}_1) := \{\zeta \circ p | p \in \mathcal{P}_1\} \subseteq \mathcal{P}_2$.
- The X-functions families $\mathcal{F}_1$ on M_1 and $\mathcal{F}_2$ on M_2 are ζ-related, written $\mathcal{F}_1 \sim_\zeta \mathcal{F}_2$ provided $\zeta^\#(\mathcal{F}_2) := \{f \circ \zeta | f \in \mathcal{F}_2\} \subseteq \mathcal{F}_1$.

Lemma 2.11. *Consider the above assuming, then*

- *If $\mathcal{P}_1 \sim_\zeta \mathcal{P}_2$, then the map $\zeta : (M_1, \mathcal{T}_{\mathcal{P}_1}) \to (M_2, \mathcal{T}_{\mathcal{P}_2})$ is continuous.*
- *If $\mathcal{F}_1 \sim_\zeta \mathcal{F}_2$, then the map $\zeta : (M_1, \mathcal{T}_{\mathcal{F}_1}) \to (M_2, \mathcal{T}_{\mathcal{F}_2})$ is continuous.*

Lemma 2.12. *Suppose that Φ is a collection of continuous X_1-parametrizations on a topological space X_2. Let M_1, M_2 are nonempty sets and $\zeta : M_1 \to M_2$ is a map.*

i) If two X_1-parametrizations collections $\mathcal{P}_1$ on M_1 and $\mathcal{P}_2$ on M_2 are ζ-related. Then $\Phi^*\mathcal{P}_1$ on M_1 and $\Phi^*\mathcal{P}_2$ on M_2 are ζ-related, too.

ii) If two X_2-functions families $\mathcal{F}_1$ on M_1 and $\mathcal{F}_2$ on M_2 are ζ-related. Then $\Phi_*\mathcal{F}_1$ on M_1 and $\Phi_*\mathcal{F}_2$ on M_2 are ζ-related, too.

iii) If $\mathcal{P}_0$ is an X_1-parametrizations collection on M_1, then $\zeta^\#\phi^*\zeta_\#\mathcal{P}_0 \subseteq \phi^*\mathcal{P}_0$.

iv) If $\mathcal{F}_0$ is an X_2-functions family on M_2, then $\zeta_\#\phi_*\zeta^\#\mathcal{F}_0 \subseteq \phi_*\mathcal{F}_0$.

Definition 2.13. Assume that Φ is a collection of continuous X_1-parametrizations on a topological space X_2 and M is a nonempty set. Let V is an open subset of X_1 and let $\mathcal{F}_0$ is an X_2-functions family on M. We define two following sets:

$$\Phi_*|_V\mathcal{F}_0 := \{p \in \Phi_*\mathcal{F}_0 | dom(p) \subseteq V\},$$
$$\Phi_*^V\mathcal{F}_0 := \{p \in \Phi_*\mathcal{F}_0 | dom(p) = V\}.$$

We called an X_2-functions family $\mathcal{F}_0$ (X_1-parametrizations collection $\mathcal{P}_0$) has $\Phi|_V$-reflexive property if $\Phi^*\Phi_*|_V\mathcal{F}_0 = \mathcal{F}_0$ ($\Phi_*|_V\Phi^*\mathcal{P}_0 = \mathcal{P}_0$).

Similarly, we said an X_2-functions family $\mathcal{F}_0$ (X_1-parametrizations collection $\mathcal{P}_0$) has Φ^V-reflexive property if $\Phi^*\Phi_*^V\mathcal{F}_0 = \mathcal{F}_0$ ($\Phi_*^V\Phi^*\mathcal{P}_0 = \mathcal{P}_0$).

Definition 2.14. [3] A pseudomonoid on a topological space X is a collection of continuous maps $\Gamma = \{f : U_f \to X : U_f \subseteq X$ is an open subset$\}$ satisfies the following properties:

- $id_X \in \Gamma$,
- If $f, g \in \Gamma$ and $image(g) \subseteq U_f$, then $f \circ g \in \Gamma$,
- If $f \in \Gamma$, and $V \subseteq U_f$ is an open subset of X, Then $f|_V \in \Gamma$.

Lemma 2.15. *Consider Γ_n is the pseudomonoid on $\mathbb{R}^n$ consists of all locally diffeomorphisms of $\mathbb{R}^n$ and let M be a nonempty set. Let $\mathcal{P}$ be an $\mathbb{R}^n$-parametrizations collection on M such that $M = \cup_{p\in\mathcal{P}} dom(p)$. Then $\mathcal{P}$ is a maximally n-manifold atlas on M if and only if $\mathcal{P}$ has Γ_n-reflexive property.*

Lemma 2.16. *Let Γ be the pseudomonoid consists of all smooth real function on the open subsets of $\mathbb{R}$ and let M be a nonempty set.*

- *Let $\mathcal{C}$ is an $\mathbb{R}$-parametrizations collection on M. Then $(\mathcal{C}, \Gamma^*\mathcal{C})$ is a Frölicher structure on M if and only if $\mathcal{C}$ has $\Gamma^{\mathbb{R}}$-reflexive property.*
- *Let $\mathcal{F}$ is a real functions family on M. Then $(\Gamma_*^{\mathbb{R}}\mathcal{F}, \mathcal{F})$ is a Frölicher structure on M if and only if $\mathcal{F}$ has $\Gamma^{\mathbb{R}}$-reflexive property.*

Proposition 2.17. *Suppose that Φ is a collection of continuous X_1-parametrizations on a topological space X_2. Let V be an open subset of X_1 and M is a nonempty set.*

i) Suppose that $\mathbb{P}_\Phi$ denote all X_1-parametrizations collections on M which have Φ-reflexive property. Let $\mathbb{F}_\Phi$ denote all X_2-functions families on M which have Φ-reflexive property. Then the operator $\Phi^*|_{\mathbb{P}_\Phi} : \mathbb{P}_\Phi \to \mathbb{F}_\Phi$ is bijective and the inverse of its is $\Phi_*|_{\mathbb{F}_\Phi}$.

ii) Let $\mathbb{P}|_V$ and $\mathbb{F}|_V$ denote all X_1-parametrizations collections and all X_2-functions families on M(resp) which have $\Phi|_V$-reflexive property. Then the operators $\Phi^*|_{\mathbb{P}|_V}$ and $\Phi_*|_V$ are inverse of each other.

iii) Let $\mathbb{P}^V$ and $\mathbb{F}^V$ denote all X_1-parametrizations collections and all X_2-functions families on M(resp) which have Φ^V-reflexive property. Then the operators $\Phi^*|_{\mathbb{P}^V}$ and Φ_*^V are inverse of each other.

Lemma 2.18. *Consider the Φ presented in example 2.6. Let $\mathscr{D}_\Phi$ be all diffeologies which have Φ-reflexive on M and $\mathscr{S}_\Phi$ be all differential structures on M which have Φ-reflexive property. Then $\phi^*|_{\mathscr{D}_\Phi}$ and $\Phi_*|_{\mathscr{S}_\Phi}$ are the inverse each other.*

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Topological classification of some orbit spaces arising from isometric actions on flat Riemannian manifolds

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Abstract

We give a topological classification of an orbit space $\frac{M}{G}$, arising from isometric action of a connected Lie group G on a flat Riemannian manifold M, under the conditions that the fixed point set of the action is nonempty and $\dim\frac{M}{G} \leq 3$.

Keywords: Riemannian manifold, orbit space, cohomogeneity

1 Introduction

A G-manifold is a complete differentiable manifold M with a differentiable action of a Lie group G on M. The orbit space which is the collection of all orbits $\{G(x) : x \in M\}$ will be denoted by $\frac{M}{G}$. $\dim\frac{M}{G}$ is called the cohomogeneity of M under the action of G. The most studied families of G-manifolds are cohomogeneity zero G-manifolds (also called homogeneous manifolds), for which the space of orbits consists of a single point. The topology and geometry of these spaces is for the most part well-understood. The next important family of G-manifolds are cohomogeneity one G-manifolds. Mostert proved in [9] that for a compact Lie group G, the orbit space $\frac{M}{G}$ of a cohomogeneity one G-manifold M is either a circle or interval (i.e., it is homeomorphic to S^1, $[0, 1]$, $[0, +\infty)$ or $(-\infty, +\infty)$. Mostert's theorem has been generalized for proper actions with non-compact G. Moreover, If M is endowed with a Riemannian metric, and G is a closed and connected subgroup of the isometries of M, there are more interesting results about the orbit spaces. It is proved that if M is a Riemannian manifold of negative curvature and G is a connected and closed subgroup of isometries of M, acting on M with cohomogeneity one, then the orbit space is not homeomorphic to $[0, 1]$, so by (generalized) Mostert's theorem, it would be homeomorphic to $(0, 1)$ or S^1 or R, and if in addition M is simply connected, then the orbit space is homeomorphic to $(0, 1)$ or R. This result, generalized to flat Riemannian manifolds in [7].

Theorem A. *Let M^n, $n > 2$, be a flat Riemannian manifold which is of cohomogeneity two, under the action of a connected and closed Lie group G of isometries. If $M^G \neq \emptyset$,*

*Speaker

then G is compact and one of the following is true:
(a) *M is isometric to R^n and principal orbits are homogeneous hypersurfaces of spheres. M^G has only one point, or it is isometric to R.*
(b) *M is isometric to $\frac{R^n}{\Gamma}$, where Γ is isomorphic to $(Z,+)$. Each principal orbit is isometric to $S^{n-2}(c)$(c depends onorbits), and M^G is homeomorphic to S^1.*
Theorem B. *Let M be a flat Riemannian manifold, $dimM > 2$, and let G be a closed and connected subgroup of the isometries of M. If M is a cohomogeneity two G-manifold and $M^G \neq \emptyset$, then $\frac{M}{G}$ is homeomorphic to one of the following spaces:*

$$[0,+\infty)\times R,\ S^1\times[0,\infty)$$

Theorem C. *Let M be a flat nonsimply connected Riemannian manifold, $dimM > 3$, and let G be a closed and connected subgroup of the isometries of M such that M is a cohomogeneity three G-manifold and $M^G \neq \emptyset$. Then $\frac{M}{G}$ is homeomorphic to one of the following spaces:*
(1) $[0,+\infty)\times R\times S^1$
(2) $[0,+\infty)\times B$ *such that B is a compact surface with $\pi_1(B) = \pi_1(M)$.*

2 Preliminaries and Proofs

Let M be a connected manifold and G be a connected subgroup of the diffeomorphisms of M, and $\widetilde{M}$ be its universal covering manifold with the covering map $\kappa : \widetilde{M} \to M$. Let G^* be the universal covering group of G with the projection $\pi : G^* \to G$, and let $\Theta : G\times M \to M$ be the diffeomorphic action of G on M. One can show that there is an action $\Theta^* : G^*\times\widetilde{M} \to \widetilde{M}$ that covers Θ and commutes with each deck transformation δ of $\widetilde{M}$ (i.e., $\Theta^*(g^*,\delta\widetilde{x}) = \delta\Theta^*(g^*,\widetilde{x})$ (see [2], pages 62, 63). If the action of G is effective but the action of G^* is not effective, we can replace G^* by its effective factor $\widetilde{G}$.

In the following, M is supposed to be a complete and connected Riemannian manifold and G is supposed to be a closed and connected subgroup of Iso(M), the isometry group of M. So, $\widetilde{M}$ will be a Riemannian manifold and $\widetilde{G}$ will be a closed and connected subgroup of Iso$(\widetilde{M})$. We will denote by Δ the deck transformation group of the covering $\kappa : \widetilde{M} \to M$. For simplicity we will denote $\Theta(g,x)$ by gx (similarly, $\Theta^*(\widetilde{g},\widetilde{x})$ by $(\widetilde{g}\widetilde{x})$). The set of the fixed points of the action of G on M ($\{x\in M : gx = x$ for all $g\in G\}$) is denoted by M^G. The map $M \to \frac{M}{G}$, $x \to G(x)$, is called the canonical projection onto the orbit space. According to the arguments in [2] pages 62-64, one can show that the assertions in the following fact are true.

Fact 2.1.
(1) $dim\frac{M}{G} = dim\frac{\widetilde{M}}{\widetilde{G}}$ and each deck transformation δ maps $\widetilde{G}$-orbits on to $\widetilde{G}$-orbits.
(2) If $x\in M$ and $\widetilde{x}\in\widetilde{M}$ such that $\kappa(\widetilde{x}) = x$ then $\kappa(\widetilde{G}(\widetilde{x})) = G(x)$.
(3) If G has a fixed point in M then $\widetilde{G} = G$ and $(\widetilde{M})^{\widetilde{G}} = \kappa^{-1}(M^G)$.
(4) Following (3), if $\widetilde{G}$ has only one fixed point then $\widetilde{M} = M$.

Fact 2.2. Following Fact 2.1, Put

$$\Delta' = \{\delta \in \Delta : \delta(\widetilde{G}(x)) = \widetilde{G}(x), \forall x \in \widetilde{M}\}$$

and

$$\widetilde{\Delta} = \frac{\Delta}{\Delta'}.$$

If Δ' is a normal subgroup of Δ then $\widetilde{\Delta}$ acts effectively on $\widetilde{\Omega}$ and $\Omega = \frac{\widetilde{\Omega}}{\widetilde{\Delta}}$.

By Fact 2.2 we get the following Fact:

Fact 2.3. If Δ acts effectively on $\widetilde{\Omega}$ then $\Omega = \frac{\widetilde{\Omega}}{\Delta}$.

Lemma 2.4. *If all elements of a non-trivial subgroup $H \subset \Delta$ leave invariant a fixed geodesic L, the H is infinite cyclic.*

Lemma 2.5. let R^n be of cohomogeneity k under the action of G a closed and connected subgroup of the isometries, and let $(R^n)^G \neq \emptyset$. Then
(1) If $k = 1$ then $\widetilde{\Omega}_{n,G,1} = [0, \infty)$.
(2) If $k = 2$ then $\widetilde{\Omega}_{n,G,2} = [0, \infty) \times R$.

Fact 2.6. *Let M be a flat Riemannian manifold and consider $\widetilde{M} = R^n$, its universal covering manifold. Let G be a closed and connected subgroup of the isometris of M which acts by cohomogeneity k on M. Consider the covering group $\widetilde{G}$ of G as mentioned in Fact 2.1, and let $dim\widetilde{M}^{\widetilde{G}} = m > 0$. Then*
(1) $k > m$ and $\widetilde{\Omega}_{n,\widetilde{G},k} = \widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times R^m$.
(2) Δ acts effectively on $\widetilde{\Omega}_{n,\widetilde{G},k}$ and $\frac{\widetilde{\Omega}_{n,\widetilde{G},k}}{\Delta} = \widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times \frac{R^m}{\Delta}$.

Proof: Put $L = \widetilde{M}^{\widetilde{G}}$. It is known that L is a totally geodesic submanifold of R^n, so it is an affine subspace of R^n. Since the elements of $\widetilde{G}$ and Δ are commutative then $\Delta(L) = L$. If $a \in L$ then denote by W_a the affine subspace of R^n which is perpendicular to L at a and $dimL + dimW_a = n$. Without lose of generality we can suppose that $L = \{o\} \times R^m \subset R^{n-m} \times R^m = R^n$. Since $\widetilde{G}$ leaves L invariant point wisely, then $\widetilde{G}$ decomposes as $\widetilde{G} = \hat{G} \times \{I\}$, where $\hat{G} \subset SO(n - m)$ and I is the identity map on R^m. Then for all $(x_1, x_2) \in R^{n-m} \times R^m$, $\widetilde{G}(x_1, x_2) = \hat{G}(x_1) \times \{x_2\}$. So, for all $a \in L$ and all $x \in W_a$, $\widetilde{G}(x) \subset W_a$. Since $\widetilde{G}$ has fixed point, it is compact and $\widetilde{G}(x)$ must be compact. Then $dim\widetilde{G}(x) < dimW_a = n - m$. If $k \leq m$ then $dim\widetilde{G}(x) < dimW_a = n - m \leq n - k$. This means that the cohomogeneity of $\widetilde{G}$ action on R^n must be less than k, and M must be a G-manifold of cohomogeneity less than k, which is a contradiction. Therefore, $k > m$. Now, it is easy to show that the following map is a homeomorphism:

$$\begin{cases} \psi : \widetilde{\Omega}_{n,\widetilde{G},k} = \frac{R^n}{\widetilde{G}} \to \widetilde{\Omega}_{n-m,\hat{G},k-m} \times R^m \\ \psi(\widetilde{G}(x)) = (\hat{G}(x_1), x_2) \quad , \quad x = (x_1, x_2) \in R^{n-m} \times R^m \end{cases}$$

$\hat{G}$ is isomorphic to $\widetilde{G}$, so in the following we will denote it by $\widetilde{G}$. Since by assumption, $\Delta(L) = L$, $L = \{o\} \times R^m \simeq R^m$, we can consider the following action of Δ on $\widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times$

R^m, which is effective.

$$\begin{cases} \Delta \times (\widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times R^m) \to \widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times R^m \\ (\delta, (A, b)) \to (A, \delta(b)) \end{cases}$$

Then we have

$$\frac{\widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times R^m}{\Delta} = \widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times \frac{R^m}{\Delta}$$

Since the elements of Δ are commutative with the elements of $\widetilde{G}$, it is easy to show that the homeomorphism ψ maps Δ-orbits of $\widetilde{\Omega}_{n,\widetilde{G},k}$ on to Δ-orbits of $\widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times R^m$. This means that ψ induces a homeomorphism between $\frac{\widetilde{\Omega}_{n,\widetilde{G},k}}{\Delta}$ and $\frac{\widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times R^m}{\Delta}(= \widetilde{\Omega}_{n-m,\widetilde{G},k-m} \times \frac{R^m}{\Delta})$.

Proof of Theorem B: Consider $\widetilde{M} = R^n$, the universal Riemannian covering manifold of M, and consider the symbols used in Fact 2.1. Put $L = \widetilde{M}^{\widetilde{G}}$. Since $M^G \neq \emptyset$ then by Fact 2.1(3), $\widetilde{M}^{\widetilde{G}} \neq \emptyset$. Put $m = \dim \widetilde{M}^{\widetilde{G}}$. By Fact 2.6(1), we have $2 > m$, so $m = 0$ or $m = 1$.

Proof of Theorem C: Similar to the proof of Theorem B, put $L = \widetilde{M}^{\widetilde{G}}$. L is a totally geodesic submanifold of $\widetilde{M} = R^n$. Since M is supposed to be non-simply connected then by Fact 2.1(4), $dimL \geq 1$. If $dimL \geq 3$ then as like as the proof of previous theorem, we can show that cohomogeneity of the action of $\widetilde{G}$ on R^n must be bigger than three which is contradiction. Thus, $dimL = 1$ or 2.

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Unique Path Lifting from Homotopy Point of View and Fibrations

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Abstract

The aim of this paper is to introduce the concepts of path homotopically lifting and its role in the category of fibrations. At first, we have some various notions, closely related to path lifting and unique path lifting; and their relations are supplemented by examples. Then, we study some results in the category of fibration with these notions instead of unique path lifting.

Keywords: Homotopically lifting, Unique path lifting, Fibration
Mathematics Subject Classification [2010]: 57M10, 57M12, 54D05, 55Q05

1 Introduction

A map $p : E \to B$ is called a fibration if it has homotopy lifting property with respect to an arbitrary space X, namely, given maps $\tilde{f} : X \to E$ and $F : X \times I \to B$ such that $F \circ j = p \circ \tilde{f}$ for $j : X \to X \times I$ by $j(x) = (x, 0)$, there is a map $\tilde{F} : X \times I \to E$ such that $\tilde{F} \circ j = \tilde{f}$ and $p \circ \tilde{F} = F$. Also, a map $p : E \to B$ is said to have unique path lifting property (upl) if, given paths w and w' in E such that $p \circ w = p \circ w'$ and $w(0) = w'(0)$, then $w = w'$.

Fibrations with upl, as a generalization of covering spaces are important. It is well known that every fiber (inverse image of a singleton) of a fibration with unique path lifting has no nonconstant path [4, Theorem 2.2.5].

In fact, unique path lifting causes a lot of results about a fibration $p : E \to B$, like injectivity of p_*, uniqueness of lifting of a given map and being homeomorphic of any two fibers [4]. Unique path lifting has an important role in the various topological concepts such as covering theory and new generalizations of covering theory, for example [1, 2, 3]. At first, we consider path lifting in the homotopy category and also will discuss about the uniqueness of this type of path lifting and classical path lifting. In fact, their relations will be introduced by some examples. Then, in the last section we would supplement the relations between these new notions in the presence of fibrations. For example, we call a map $p : E \to B$ has weakly unique path homotopically lifting property (wuphl) if, given paths w and w' in E such that $w(0) = w'(0)$, $w(1) = w'(1)$, $p \circ w \simeq p \circ w'$ rel $\{0, 1\}$, we have, $w \simeq w'$ $rel\{0, 1\}$. We will show that every loop in each fiber of a fibration with wuphl is nullhomotopic, which is a homotopy analogue of the same result when we have unique path lifting. Throughout this paper, a *map* $f : X \to Y$ means a continuous function and $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ will denote the homomorphism induced by f.

*Speaker

2 Main results

Path lifting is the lifting of paths in the category Top. We can consider path lifting problem in the htop and get a new feature of lifting problem.

Definition 2.1. Let $p : E \to B$ be a map. A path $\tilde{\alpha} : I \to E$ is called a homotopically lifting of a path α if $po\tilde{\alpha} \simeq \alpha \; rel \; \{0, 1\}$.

Definition 2.2. Let $p : E \to B$ be a map and $\tilde{\alpha}$ *and* $\tilde{\beta}$ be paths in E, then we say that
(i) p has **unique path lifting (upl)** if

$$\tilde{\alpha}(0) = \tilde{\beta}(0), \; po\tilde{\alpha} = po\tilde{\beta} \Rightarrow \tilde{\alpha} = \tilde{\beta}.$$

(ii) p has **homotopically unique path lifting (hupl)** if

$$\tilde{\alpha}(0) = \tilde{\beta}(0), \; po\tilde{\alpha} = po\tilde{\beta} \; rel \; \{0, 1\} \Rightarrow \tilde{\alpha} \simeq \tilde{\beta} \; rel \; \{0, 1\}.$$

(iii) p has **weekly homotopically unique path lifting (whupl)** if

$$\tilde{\alpha}(0) = \tilde{\beta}(0), \; \tilde{\alpha}(1) = \tilde{\beta}(1), \; po\tilde{\alpha} = po\tilde{\beta} \Rightarrow \tilde{\alpha} \simeq \tilde{\beta} \; rel \; \{0, 1\}.$$

(iv) p has **unique path homotopically lifting (uphl)** if

$$\tilde{\alpha}(0) = \tilde{\beta}(0), \; po\tilde{\alpha} \simeq po\tilde{\beta} \; rel \; \{0, 1\} \Rightarrow \tilde{\alpha} \simeq \tilde{\beta} \; rel \; \{0, 1\}.$$

(v) p has **weekly unique path homotopically lifting (wuphl)** if

$$\tilde{\alpha}(0) = \tilde{\beta}(0), \; \tilde{\alpha}(1) = \tilde{\beta}(1), \; po\tilde{\alpha} \simeq po\tilde{\beta} \; rel \; \{0, 1\} \Rightarrow \tilde{\alpha} \simeq \tilde{\beta} \; rel \; \{0, 1\}.$$

Example 2.3. Every continuous map from a simply connected space to any space has wuphl and whupl. Note that every injective map has upl and also, for injecive map, wuphl and uphl are equivalent.

By a direct verification we have the following proposition

Proposition 2.4. *Let $p : E \to B$ be a map and $e \in p^{-1}(b)$, for $b \in B$. Then*
i) Injectivity of $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is equivalent to wuphl.
ii) Injectivity of $p_* : \pi_1(E, e) \to \pi_1(B, b)$ implies that p has whupl.

It is notable that that converse of (ii) is not necessarily true, for seeing this, refer to Example 2.8.
In the next proposition, we show that in Top, uniqueness and homotopically uniqueness of path lifting are equivalent.

Proposition 2.5. *upl ⇔ hupl*

Proof. By definitions, $upl \Longrightarrow hupl$. Now let $p : E \to B$ be a map with hupl and $\tilde{\alpha}$ *and* $\tilde{\beta}$ be paths in E such that $\tilde{\alpha}(0) = \tilde{\beta}(0)$, $p \circ \tilde{\alpha} = p \circ \tilde{\beta}$. Define, for every $t \in I$, $\tilde{\alpha}_t$, $\tilde{\beta}_t : I \to E$ such that $\tilde{\alpha}_t(s) = \tilde{\alpha}(st)$ and $\tilde{\beta}_t(s) = \tilde{\beta}(st)$. By definitions, $\tilde{\alpha}_t(0) = \tilde{\beta}_t(0)$ and $p \circ \tilde{\alpha}_t = p \circ \tilde{\beta}_t$. Then hupl imply that $\tilde{\alpha}_t \simeq \tilde{\beta}_t \; rel \; \{0, 1\}$, specially $\tilde{\alpha}_t(1) = \tilde{\beta}_t(1)$ which implies $\tilde{\alpha}(t) = \tilde{\beta}(t)$ and since t is arbitrary, $\tilde{\alpha} = \tilde{\beta}$. □

Proposition 2.6.
(i) upl ⇒ whupl,
(ii) uphl ⇒ whupl,
(iii) uphl ⇒ wuphl,
(iv) uphl ⇒ upl,
(v) wuphl ⇒ whupl.

Proof. Use definitions. Just for (iv), a method like in the proof of the previous proposition is needed. □

Since, uphl imply upl and also, a map with upl has unique lifting property for path connected space, we have

Corollary 2.7. *If a map has uphl, it has the unique lifting property for path connected spaces.*

The following example shows that the converse of all the parts of Proposition 2.6 is not true.

Example 2.8.
For, (i) *wuphl ⇏ uphl*, (ii) *whupl ⇏ uphl* and (iii) *whupl ⇏ upl*, let $E = \{0\} \times [0, 1] \times [0, 1]$ and $B = \{0\} \times [0, 1] \times \{0\}$, and $p : E \to B$ is the vertical projection.
Also, for (iv) *upl ⇏ uphl* and (v) *whupl ⇏ wuphl*, let $E = \{(x, y, 2) \in R^3\} - \{(0, 0, 2)\}$, $B = \{(x, y, 0) \in R^3\}$ and $p : E \to B$ be again the vertical projection.

Remark 2.9. Moreover, there is no relation between upl and wuphl, because the part (i) of the example 2.8 imply that, *wuphl ⇏ upl* and by (ii), we have, *upl ⇏ wuphl*.

3 fibrations and homotopically liftings

In this section, we compare and study the notions introduced in section 2 in presence of fibrations.

Proposition 3.1. *For fibrations we have:*
(i) upl (hupl) ⇒ uphl
(ii) upl (hupl) ⇒ wuphl

Proof. For (i) see [4, Lemma 2.3.3], also, (ii) come from definition and (i). □

Corollary 3.2. *For fibrations, upl (hupl) and uphl are equivalent.*

Remark 3.3. We already saw that even within assumption fibration, the converse of (i) of this proposition is true, moreover, the map in example 2.8 (i), is a fibration with wuphl which has not upl, then, the converse of (ii) is failed.

In the following theorem, we show that considering lifting in the homotopy category makes that paths in fibers are homotopically constant.

Theorem 3.4. *If $p : E \to B$ is a fibration, then p has wuphl if and only if every loop in each fiber is nullhomotopic.*

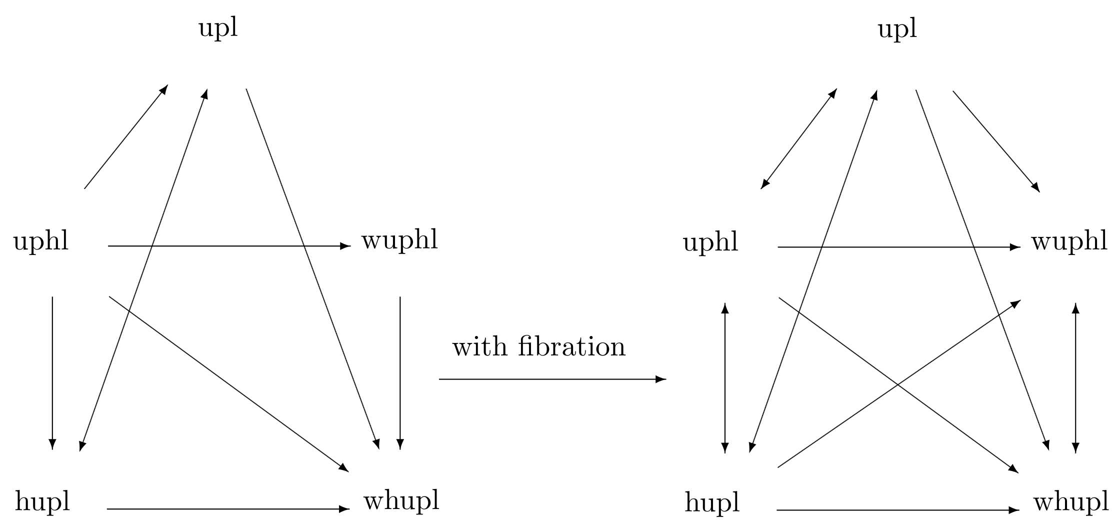
Proof. Refer to preprint. □

Similarly, we can replace, wuphl with whupl, then

Theorem 3.5. *A fibration $p : E \to B$ has whupl if only if every loop in each fiber is nullhomotopic.*

Corollary 3.6. *If $p : E \to B$ is a fibration then whupl and wuphl are equivalent.*

So, the relation between this five kinds of the paths lifting is as the following



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Web geometry of Lorentz dynamical system

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Abstract

The paper is devoted to solve Cartan equivalence problem for a dynamical system that is called Lorenz equations under a web transformation.

Keywords: Cartan equivalence problem, dynamical systems, Web geometry.
Mathematics Subject Classification [2010]: 58A15, 58A20, 58J70

1 Introduction

The method of equivalence of E. Cartan (see [1], [3] and [4]) provides a powerful tool for constructing differential invariants which solve the problem of deciding when two geometric objects are really the same up to some preassigned group of coordinate transformations. In [2] R. B. Gardner gave some examples of solving these problems. For example, he has given the local equivalence problem for $y' = f(x, y)$ under diffeomorphisms of the form $\Phi(x, y) = (\varphi(x), \psi(y))$. We generalize this problem to a system of n first order autonomous ODEs.

We generalize this local equivalence problem to one of the most famous dynamical systems which exhibits chaotic behavior that is the *Lorentz equations*

$$\begin{cases} \dot{x} = -\sigma(x-y) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases} \tag{1}$$

where $\sigma, r, b > 0$ and "." represents derivative with respect to arc length t, under the group of coordinate transformations defined by

$$\Phi(t, x, y, z) = (\xi(t), \varphi_1(x), \varphi_2(y), \varphi_3(z)). \tag{2}$$

that is called the *pseudo-group of web transformations.*

*Speaker

2 The Cartan equivalence method

Let $G \subset \mathrm{GL}(m)$ be a Lie group. Let ω and $\overline{\omega}$ be coframes defined, respectively, on the m-dimensional manifolds M and $\overline{M}$. The G-valued equivalence problem for these coframes is to determine whether or not there exists a local diffeomorphism $\Phi : M \to \overline{M}$ and a G-valued function $g : M \to G$ with the property that

$$\Phi^*(\overline{\omega}) = g(x)\omega. \tag{3}$$

Let $U \subset M$ and the lifted differential forms $\omega = S\omega_U$ on $U \times G$ was defined. We may differentiate the lifted forms to find

$$\begin{aligned} d\omega &= dS \wedge \omega_U + Sd\omega_U \\ &= dSS^{-1} \wedge S\omega_U + Sd\omega_U. \end{aligned}$$

The matrix dSS^{-1} is the Maurer-Cartan matrix of right invariant forms on G, therefore

$$(dSS^{-1})^i_j = \sum_\rho a^i_{j\rho}\pi^\rho, \tag{4}$$

where π^ρ is a basis for the Maurer-Cartan forms and the $a^i_{j\rho}$ are constants, [5].

Recalling that the forms ω_U are basic, that is, both coefficients and differentials can be expressed in terms of coordinates on U alone, we can write the exterior derivatives in the group-fiber representation

$$d\omega^i = \sum a^i_{j\rho}\pi^\rho \wedge \omega^j + \frac{1}{2}\sum \gamma^i_{jk}(u, S)\omega^j \wedge \omega^k, \tag{5}$$

Thus let us write the exterior derivatives in the following form,

$$d\omega^i = \sum \Delta^i_j \wedge \omega^j, \tag{6}$$

where no assumption is made on the Δ^i_j. If we now subtract the group-fiber representation (5) from the above representation we find

$$d\omega^i = \sum(\Delta^i_j - a^i_{j\rho}\pi^\rho) \wedge \omega^j - \frac{1}{2}\sum \gamma^i_{jk}\omega^j \wedge \omega^k, \tag{7}$$

Theorem 2.1. *([2]) (Cartan's lemma) Let $\{\omega^i\}$ be an independent set of 1-forms, and let $\{\pi^i\}$ be an arbitrary set of 1-forms of the same finite cardinality; then*

$$\sum \pi_i \wedge \omega^i = 0, \tag{8}$$

holds if and only if $\pi_i = \sum C_{ij}\omega^j$, where C_{ij} is a symmetric matrix.

3 Main Results

Lorenz arrived at these equations when modeling a two dimensional fluid cell between two parallel plates which are at different temperatures. We try to plot some solutions of

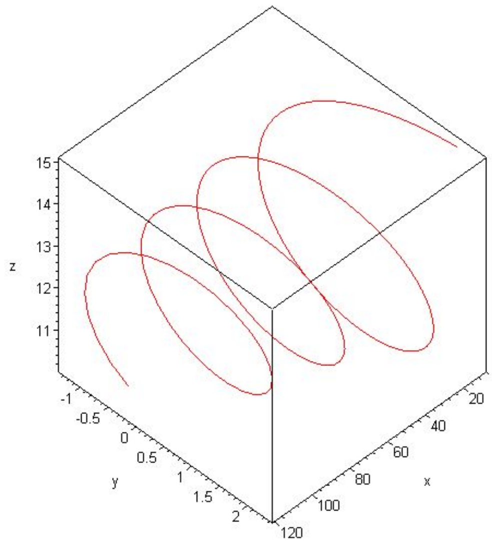

Figure 1: A solution of Lorenz system (1) for $\sigma = 5, r = 12, b = \frac{3}{2}$

Lorenz equation (1) using Maple 14, with initial values $x(0) = 10, y(0) = 2, z(0) = 15$ and $0 < t < 0.5$.

The system (1) is invariant under the transformation

$$(x, y, z) \longrightarrow (-x, -y, z). \tag{9}$$

Moreover, the z axis is an invariant manifold since

$$x(t) = 0, y(t) = 0, z(t) = z_0 e^{-bt} \tag{10}$$

is a solution of our system.

Theorem 3.1. *All web transformations preserving the Lorentz system (1) are*

$$\Phi(t, x, y, z) = (t, ax, ay, az). \tag{11}$$

Proof. Applying the equivalence method of Cartan, the necessary and sufficient conditions for equivalence of two Lorentz system (1) under web transformation (2) is given. The contact 1-forms for Lorentz system are

$$\begin{aligned} \omega_0 &= dt, \\ \omega_1 &= dx + \sigma(x - y)dt, \\ \omega_2 &= dy - (rx - y - xz)dt, \\ \omega_3 &= dz - (xy - bz)dt. \end{aligned} \tag{12}$$

There is a one-to-one correspondence between vector fields

$$\mathbf{X} = \frac{d}{dt} - \sigma(x - y)\frac{\partial}{\partial x} + (rx - y - xz)\frac{\partial}{\partial x} + (xy - bz)\frac{\partial}{\partial x}, \tag{13}$$

and Lorentz systems (1). Using duality of vector field (13), we may choose the following two coframes

$$\Theta_0 = dT, \Theta_1 = -\frac{1}{\sigma(X-Y)}\, dX, \Theta_2 = \frac{1}{rX - Y - XZ}\, dY, \Theta_3 = \frac{1}{XY - bZ}\, dZ, \quad (14)$$

$$\theta_0 = dt, \theta_1 = -\frac{1}{\sigma(x-y)}\, dx, \theta_2 = \frac{1}{rx - y - xz}\, dy, \theta_3 = \frac{1}{xy - bz}\, dz, \quad (15)$$

and smooth function $g : \mathbb{R}^4 \to G$ which satisfy in $\Phi^*\Theta = g.\theta$ relation with structure group

$$G = \{a\,\mathrm{I}_4 \mid a \in \mathbb{R} \setminus \{0\}\}, \quad (16)$$

where I_4 is the 4×4 identity matrix. Note that the coframe (12) is equivalent to coframe (15). Choosing web transformation (2) and equivalence conditions leads to:

$$\begin{aligned}
\Phi^*\Theta_0 &= \Phi^*(dT) = \dot{\xi}(t)dt = dt,\\
\Phi^*\Theta_1 &= \Phi^*\left(-\frac{1}{\sigma(X-Y)}\, dX\right) = -\frac{1}{\sigma(x-y)}\dot{\varphi}_1(x)\, dx = -\frac{a}{\sigma(x-y)}\, dx,\\
\Phi^*\Theta_2 &= \Phi^*\left(\frac{1}{rX - Y - XZ}\, dY\right) = \frac{1}{rx - y - xz}\dot{\varphi}_2(y)\, dy = \frac{a}{rx - y - xz}\, dy,\\
\Phi^*\Theta_3 &= \Phi^*\left(\frac{1}{XY - bZ}\, dZ\right) = \frac{1}{xy - bz}\dot{\varphi}_3(z)\, dz = \frac{a}{xy - bz}\, dz.
\end{aligned}$$

Therefore

$$\dot{\xi}(t) = 1, \qquad \dot{\varphi}_1(x) = \dot{\varphi}_2(y) = \dot{\varphi}_3(z) = a,$$

which concludes that all web transformations preserving the Lorentz system are (11) where $a \in \mathbb{R} \setminus \{0\}$, [4]. □

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A compact finite difference method without using Hopf-Cole transformation for solving 1D Burgers' equation

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Abstract

A new compact finite difference (CFD) method for solving one-dimensional (1D) Burgers' equation without using the Hopf-Cole transformation is analyzed. This method leads to a system of linear equations involving tridiagonal matrices and the rate of convergence of the method is of order $O(k^2+h^4)$ where k and h are the time and space step sizes, respectively. Numerical results obtained by the proposed method are compared with the exact solutions and the results obtained by some other methods.

Keywords: Burgers' equation, compact finite difference method
Mathematics Subject Classification [2010]: 65M06, 65M12

1 Introduction

Burgers' equation was formulated by Bateman in 1915 [2] and later treated by Burgers [3]. This equation is also called the nonlinear advection-diffusion equation, and can be regarded as a qualitative approximation of the Navier-Stocks equations. Recently, Xie et al. [4] applied the Hopf-Cole transformation method to linearize the equation and constructed a CFD method which is unconditionally stable and its accuracy is second- and fourth-order accurate in time and space, respectively. We aim to construct a CFD method for the 1D Burgers' equation without using the Hopf-Cole transformation.

2 Construction of the method

We consider the following one-dimensional nonlinear Burgers' equation

$$u_t + uu_x - \nu u_{xx} = 0, \qquad a < x < b, \quad 0 < t < T, \tag{1}$$

where $\nu = 1/Re$ in which Re is the Reynolds' number. The following boundary and initial conditions are also considered

$$u(a,t) = 0, \quad u(b,t) = 0, \qquad 0 \le t \le T, \qquad u(x,0) = f(x), \qquad a \le x \le b,$$

where f is a given function. In order to construct a CFD method, we select integers $M, N > 0$ and define $h = (b-a)/M,\ k = T/N$. The grid points for this situation are

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(x_i, t_n), where $x_i = a + ih$ for $i = 0, 1, \ldots, M$ and $t_n = nk$ for $n = 0, 1, \ldots, N$. Assuming $u_i^n = u(x_i, t_n)$, we use the following notations for simplicity

$$u_i^{n+1/2} = \frac{u_i^{n+1} + u_i^n}{2}, \quad \partial_t u_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{k}, \quad \delta_x^2 u_i^n = u_{i+1}^n - 2u_i^n + u_{i-1}^n.$$

Setting $V = u_t$ and $F = u_x$, Eq. (1) at the intermediate point $(x_i, t_{n+\frac{1}{2}})$ can be written as

$$V_i^{n+1/2} + (uF)_i^{n+1/2} - \nu(u_{xx})_i^{n+1/2} = 0. \tag{2}$$

To obtain a fourth-order scheme with tridiagonal nature, we use the following relation,

$$(u_{xx})_i^{n+1/2} = \frac{\delta_x^2}{h^2(1 + \frac{1}{12}\delta_x^2)} u_i^{n+1/2} + O(h^4), \quad F_i = \frac{\delta_x}{h(1 + \frac{1}{6}\delta_x^2)} u_i + O(h^4),$$

to change (2) to

$$(1 + \frac{1}{12}\delta_x^2)(V_i^{n+1/2} + (uF)_i^{n+1/2}) = \frac{\nu}{h^2}\delta_x^2 u_i^{n+1/2} + O(h^4), \tag{3}$$

which is nonlinear. For obtaining a simpler implementation, we apply the following linearized approximation [1],

$$(uF)^{n+1} = F^n u^{n+1} + u^n F^{n+1} - (uF)^n + O(k^2),$$

and write Eq. (3) as

$$2(u_{i+1}^{n+1} + 10u_i^{n+1} + u_{i-1}^{n+1}) + k\left(u_{i+1}^n F_{i+1}^{n+1} + 10u_i^n F_i^{n+1} + u_{i-1}^n F_{i-1}^{n+1}\right) +$$

$$k\left(F_{i+1}^n u_{i+1}^{n+1} + 10F_i^n u_i^{n+1} + F_{i-1}^n u_{i-1}^{n+1}\right) - \tfrac{12k\nu}{h^2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

$$= 2(u_{i+1}^n + 10u_i^n + u_{i-1}^n) + \tfrac{12k\nu}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + O(k^2 + h^4),$$

which is linear and leads to the following approximate matrix form

$$(\boldsymbol{I} + \boldsymbol{A})U^{n+1} = (\boldsymbol{I} + \boldsymbol{B})U^n, \tag{4}$$

where

$$U^n = (U_1^n, \ldots, U_{M-1}^n)^T \simeq U_e^n = (u_1^n, \ldots, u_{M-1}^n)^T,$$

$$\boldsymbol{A} = \tfrac{3k}{2h}(\boldsymbol{D}\boldsymbol{T}_1^{-1}\boldsymbol{T}_2 + \boldsymbol{C}) - \tfrac{6k\nu}{h^2}\boldsymbol{T}_3^{-1}\boldsymbol{T}_4, \qquad \boldsymbol{B} = \tfrac{6k\nu}{h^2}\boldsymbol{T}_3^{-1}\boldsymbol{T}_4,$$

$$\boldsymbol{T}_1 = 6\boldsymbol{I} + \boldsymbol{T}_4, \qquad \boldsymbol{T}_3 = 12\boldsymbol{I} + \boldsymbol{T}_4 \qquad \boldsymbol{D} = \mathrm{diag}(U_1^n, \ldots, U_{M-1}^n),$$

$$\boldsymbol{T}_2 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 0 & 1 \\ 0 & \ldots & 0 & -1 & 0 \end{bmatrix}, \qquad \boldsymbol{T}_4 = \begin{bmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \ldots & 0 & 1 & -2 \end{bmatrix},$$

and $\boldsymbol{C} = \text{diag}(\boldsymbol{T}_1^{-1}\boldsymbol{T}_2 U^n)$.
The numerical stability of the scheme (3) is investigated by using the energy method in the following theorem which can be proved without difficulty (see [1, 4]).

Theorem 2.1. *If scheme* (3) *has a unique solution* U^n *and* k *is sufficiently small, then we have*

$$\|U^n\|_{L_2}^2 \leq 2\|U^0\|_{L_2}^2, \qquad 1 \leq n \leq N.$$

By applying Theorem 2.1, we obtain the following discrete $\|.\|_{L_\infty}$-norm inequality

$$\|U^n\|_{L_\infty}^2 \leq M_0^2\|U^n\|_{H^1}^2 \leq 2M_0^2\|U^0\|_{H^1}^2, \qquad 1 \leq n \leq N. \tag{5}$$

where $M_0 = \max\{\sqrt{b-a}, 1/\sqrt{b-a}\}$ and $\|u\|_{H^1}^2 = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2$. Inequality (5) shows that (3) is an unconditional stable scheme.

Theorem 2.2. *Assume that the exact solution* u *of the initial-value problem for the Burgers' equation is sufficiently smooth, and* U *is the numerical solution of* (3)*. Under some mild conditions (see e.g. [1]), if* k *is sufficiently small, then there exists a constant* B *such that*

$$\|u(\cdot, nk) - U^n\|_{L_2} \leq B(k^2 + h^4).$$

Theorem 2.2 can be proved without difficulty (see [1, 4]).

3 Numerical results

The accuracy of the scheme is measured by using the $L_\infty = \| U_{app} - U_{exact} \|_\infty$ error norm.

Example 3.1. We consider the shock-like solution of the Burgers' equation. The exact solution is

$$u(x,t) = \frac{x/t}{1 + \sqrt{t/t_0}\exp(x^2/4\nu t)}, \qquad t \geq 1 \tag{6}$$

where $t_0 = \exp(1/8\nu)$. The initial condition is taken from (6) by setting $t = 1$ and the boundary conditions are considered as $u(a,t) = u(b,t) = 0$. The numerical solution is obtained by the present method at different nodes and times and compared with the exact solution as well as the compact finite difference method presented in [4]. Errors displayed in Table 1 show that the present method has higher accuracy.

Example 3.2. We consider the exact solution of (1) as

$$u(x,t) = \frac{\gamma + \mu + (\mu - \gamma)\exp(\eta)}{1 + \exp(\eta)}, \qquad t \geq 0$$

where $\eta = \gamma(x - \mu t - \varepsilon)/\nu$, and γ, ε, and μ are constants. The initial condition is obtained from the exact solution by setting $t = 0$, and the boundary conditions $u(0,t) = 1$ and $u(1,t) = 0.2$ are used. The smaller value of ν gives the steeper wave. We simulate the movement of the solution by taking parameters $\gamma = 0.4$, $\mu = 0.6$, and $\varepsilon = 0.125$. To show that the method has fourth-order convergence rate with nonhomogeneous boundary conditions, we initially set $h = 0.02$ and $k = 0.02$, then reduce them by a factor of 2 and 4, respectively, in Table 2.

Table 1: Comparison of the numerical and exact solutions, and errors of Example 3.1 for $\nu = 0.001,\ h = 0.005,\ k = 0.01$ and $[a, b] = [0, 1.2]$.

x	T		Xie[4]	Present	Exact
0.6	1.7		0.3507	0.3529	0.3529
		L_∞	0.0143	0.0019	
0.8	2.4		0.0038	0.0033	0.0033
		L_∞	0.0089	0.00069	
0.8	3.1		0.2573	0.2581	0.2581
		L_∞	0.0054	0.00037	

Table 2: Order of convergence for Example 3.2 with $T = 0.02,\ h = 0.02,\ k = 0.02$ and $\nu = 0.005$

	h, k	$\frac{h}{2}, \frac{k}{4}$	$\frac{h}{4}, \frac{k}{16}$	$\frac{h}{8}, \frac{k}{64}$	$\frac{h}{16}, \frac{k}{256}$
$E = L_\infty$	0.0109	$7.4519e-004$	$4.7102e-005$	$2.9337e-006$	$1.8476e-007$
$r = \frac{E(h,k)}{E(\frac{h}{2},\frac{k}{4})}$	-	14.6271	15.8208	16.0555	15.8784
Order $= \log_2 r$	-	3.82454	3.97753	4.00693	3.98478

4 Conclusion

A CFD method for one-dimensional nonlinear Burgers' equation is introduced and analyzed. This method is shown to be second- and fourth-order accurate in time and space, respectively. This method successfully simulates the physical behaviors of the motion of solutions. Our numerical experiments show that the present method offers higher accuracy, and they also confirm very well obtained theoretical results.

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A computational algorithm for the inverse of positive definite tri-diagonal matrices

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Abstract

In this paper, employing the general Cholesky Q.I.F. factorization, an efficient algorithm is developed to find the inverse of a general positive definite tridiagonal matrix.

Keywords: Cholesky Q.I.F. factorization, Positive definite tridiagonal.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

The linear system of equations whose coefficient matrix is of tri-diagonal type of the form

$$T=\begin{bmatrix} a_1 & c_1 & \circ & \cdots & \circ \\ c_1 & a_2 & c_2 & \ddots & \vdots \\ \circ & c_2 & a_3 & \ddots & \circ \\ \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ \circ & \cdots & \circ & c_{n-1} & a_n \end{bmatrix} \tag{1.1}$$

is of special importance in many scientific and engineering applications. For example in parallel computing and in solving differential equations using finite differences.

2 Cholesky Q.I.F. factorization

Consider the linear system $Ax=f$, where A is an $n\times n$ symmetric positive definite matrix. Suppose $n=2m-2$. Assume that there exists a matrix W such that , $A=WW^T$,where

$$W=\begin{bmatrix} w_{1,1} & w_{1,2} & & \cdots & \cdots & & & w_{1,n} \\ \circ & w_{2,2} & & \cdots & \cdots & & w_{2,n} & \circ \\ & \circ & \ddots & & & . & \circ & \\ \vdots & & & w_{m-1,m-1} & w_{m-1,m} & \circ & & \\ \vdots & \vdots & \circ & \circ & w_{m,m} & \circ & \vdots & \vdots \\ & \circ & . & & & . & \circ & \\ \circ & \circ & w_{n-1,3} & \cdots & \cdots & & w_{n-1,n-1} & \circ \\ \circ & w_{n,2} & & \cdots & \cdots & \cdots & \cdots & w_{n,n} \end{bmatrix}$$

*Speaker

Note that W has a "butterfly" or a "bowtie" structure.
Suppose $w_1, w_2, \cdots, w_n$ are columns of W, then we have $W = [w_1, w_2, \cdots, w_n]^T$. Each w_i for $i = 1, 2, \cdots, n$ is of the following form

$$w_i = \begin{cases} [w_{1,i}, \cdots, w_{i,i}, \circ, \cdots, \circ, w_{n-i+2,i}, \cdots, w_{n,i}]^T & for \quad i = 1, ..., m-1 \\ [w_{1,i}, \cdots, w_{n-i+1,i}, \circ, \cdots, \circ, w_{i,i}, \cdots, w_{n,i}]^T & for \quad i = m, ..., n \end{cases}.$$

Algorithm 2.1. To compute elements of W.
for $k = 1, ..., m-1$, $w_{m+k-1,m+k-1} = \sqrt{a^{(k)}_{m+k-1,m+k-1}}$
for $i = 1, ..., m-k$, and, $m+k, ..., n$, $w_{i,m+k-1} = a^{(k)}_{i,m+k-1}/w_{m+k-1,m+k-1}$
$w_{m-k,m-k} = \sqrt{a^{(k)}_{m-k,m-k} - w^2_{m-k,m+k-1}}$
for $i = 1, ..., m-k-1$, and, $m+k, ..., n$, $w_{i,m-k} = (a^{(k)}_{i,m-k} - w_{i,m+k-1}w_{m-k,m+k-1})/w_{m-k,m-k}$
if$(k \neq m-1)$, $A_{k+1} = A_k - w_{m+k-1}w^T_{m+k-1} - w_{m-k}w^T_{m-k}$
Assume the matrix A is the Positive definite tridiagonal matrix, after Cholesky Q.I.F. factorization, we have W in the following form

$$\begin{bmatrix} w_{1,1} & \circ & \cdots & \cdots & \cdots & \cdots & \cdots & \circ & \circ \\ w_{2,1} & w_{2,2} & \circ & \cdots & \cdots & \cdots & \cdots & \circ & \circ \\ \circ & \ddots\ \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \\ \vdots & \cdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots \\ \circ & \cdots & w_{m-1,m-2} & w_{m-1,m-1} & \circ & \circ & \cdots & \circ & \circ \\ \circ & \cdots & \circ & w_{m,m-1} & w_{m,m} & w_{m,m+1} & \circ & \cdots & \circ \\ \circ & \circ & \cdots & \circ & \circ & w_{m+1,m+1} & w_{m+1,m+2} & \cdots & \circ \\ \vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \circ & \circ & \cdots & \cdots & \cdots & \cdots & \circ & w_{n-1,n-1} & w_{n-1,n} \\ \circ & \circ & \circ & \cdots & \cdots & \cdots & \circ & \circ & w_{n,n} \end{bmatrix}$$

where W is a tridiagonal matrix. To find the inverse matrix W^{-1} one can use the Gaussian elimination method:

$$W^{-1} = \begin{bmatrix} R_{1,1} & \circ & \circ & \cdots & \cdots & \cdots & \cdots & \circ & \circ \\ R_{2,1} & R_{2,2} & \circ & \circ & \cdots & \cdots & \cdots & \circ & \circ \\ \circ & \ddots & \ddots & \ddots & & \cdots & \cdots & \vdots & \vdots \\ \circ & \ddots & \ddots & \ddots & & \cdots & \cdots & \vdots & \vdots \\ R_{m,1} & R_{m,2} & \cdots & \cdots & R_{m,m} & R_{m,m+1} & \cdots & \cdots & R_{m,n} \\ \circ & \circ & \cdots & \cdots & \circ & R_{m+1,m+1} & \cdots & \cdots & R_{m+1,n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \circ & \circ & \cdots & \cdots & \cdots & \cdots & \circ & \circ & R_{n,n} \end{bmatrix}_{n\times n}$$

3 computational algorithm

In this section, we present a new computational algorithm for inverting a positive definite tridiagonal matrix using Cholesky Q.I.F. factorization.
Now $A = WW^T$ gives $A^{-1} = (WW^T)^{-1} = (W^T)^{-1}W^{-1} = (W^{-1})^T W^{-1}$.

We see that the inverse matrix A^{-1} of the matrix$A$ may be obtained once the inverse matrix W^{-1} is available.

Algorithm 3.1. INPUT Dimension n; m and elements of A. OUTPUT the entries $T_{i,j}$, $(1 \le i, j \le n)$ of the inverse matrix $T = A^{-1}$ of A.

step 1. Compute W

$w_{1,1} = \sqrt{a_{1,1}}, w_{2,1} = a_{2,1}/w_{1,1}, w_{n,n} = \sqrt{a_{n,n}}, w_{n-1,n} = a_{n-1,n}/w_{n,n}$

for $\quad i = 2, ..., m-2,\ w_{i,i} = \sqrt{a_{i,i} - w_{i,i-1}^2}, w_{i+1,i} = a_{i+1,i}/w_{i,i}$

$w_{n+1-i,n+1-i} = \sqrt{a_{n+1-i,n+1-i} - w_{n+1-i,n+2-i}^2}, w_{n-i,n+1-i} = a_{n-i,n+1-i}/w_{n+1-i,n+1-i}$

$w_{m-1,m-1} = \sqrt{a_{m-1,m-1} - w_{m-1,m-2}^2}, w_{m,m-1} = a_{m,m-1}/w_{m-1,m-1},$

$w_{m,m} = \sqrt{a_{m,m} - w_{m,m-1}^2 - w_{m,m+1}^2}.$

step 2. Compute W^{-1}

for $i = 1, ..., n,\ R_{i,i} = 1/w_{i,i}$

for $i = 2, ..., m,\ R_{i,i-1} = -w_{i,i-1}/w_{i,i}w_{i-1,i-1}$

for $i = 3, ..., m,\ j = i-2, ..., 1,\ R_{i,j} = R_{i-1,j}R_{i,j+1}/R_{i-1,j+1}$

for $i = n-1, ..., m,\ R_{i,i+1} = -w_{i,i+1}/w_{i,i}w_{i+1,i+1}$

for $i = n-2, ..., m,\ j = n, ..., i+2,\ R_{i,j} = R_{i,j-1}R_{i+1,j}/R_{i+1,j-1}$

step 3. Compute A^{-1}

for $i = 1, ..., m,\ T_{i,i} = \sum_{k=i}^{m} R_{k,i}^2$

for $i = m+1, ..., n,\ T_{i,i} = \sum_{k=m}^{i} R_{k,i}^2$

for $i = m, ..., n,\ j = 1, ..., m,\ T_{i,j} = R_{m,j}R_{m,i}$

for $i = 2, ..., m-1,\ j = 1, ..., i-1,\ T_{i,j} = \sum_{k=i}^{m} R_{k,j}R_{k,i}$

for $i = m+2, ..., n,\ j = m+1, ..., n-1,\ T_{i,j} = \sum_{k=m}^{i-1} R_{k,i}R_{k,j}$

for $i = 1, ..., n-1,\ j = i+1, ..., n,\ T(i,j) = T(j,i)$

4 Example

Example 4.1. Consider the 6×6 matrix A given by

$$A = \begin{bmatrix} 2 & 1 & \circ & \circ & \circ & \circ \\ 1 & 3 & 1 & \circ & \circ & \circ \\ \circ & 1 & 4 & 2 & \circ & \circ \\ \circ & \circ & 2 & 5 & 1 & \circ \\ \circ & \circ & \circ & 1 & 6 & 1 \\ \circ & \circ & \circ & \circ & 1 & 6 \end{bmatrix}$$

a) after Cholesky factorization of matrix A, compute A^{-1} by using,
$A^{-1} = (LL^T)^{-1} = (L^T)^{-1}L^{-1} = (L^{-1})^T L^{-1}$.

$$L = \begin{bmatrix} 1.4142 & \circ & \circ & \circ & \circ & \circ \\ \circ.7\circ71 & 1.5811 & \circ & \circ & \circ & \circ \\ \circ & \circ.6325 & 1.8974 & \circ & \circ & \circ \\ \circ & \circ & 1.\circ541 & 1.972\circ & \circ & \circ \\ \circ & \circ & \circ & \circ.5\circ71 & 2.3964 & \circ \\ \circ & \circ & \circ & \circ & \circ.4173 & 2.4137 \end{bmatrix}$$

$$L^{-1}=\begin{bmatrix} \circ.7\circ71 & \circ & \circ & \circ & \circ & \circ \\ -\circ.3162 & \circ.6325 & \circ & \circ & \circ & \circ \\ \circ.1\circ54 & -\circ.21\circ8 & \circ.527\circ & \circ & \circ & \circ \\ -\circ.\circ563 & \circ.1127 & -\circ.2817 & \circ.5\circ71 & \circ & \circ \\ \circ.\circ119 & -\circ.\circ238 & \circ.\circ596 & -\circ.1\circ73 & \circ.4173 & \circ \\ -\circ.\circ\circ21 & \circ.\circ\circ41 & -\circ.\circ1\circ3 & \circ.\circ186 & -\circ.\circ721 & \circ.4143 \end{bmatrix}$$

$$A^{-1}=\begin{bmatrix} \circ.6144 & -\circ.2289 & \circ.\circ722 & -\circ.\circ299 & \circ.\circ\circ51 & -\circ.\circ\circ\circ9 \\ -\circ.2289 & \circ.4577 & -\circ.1443 & \circ.\circ598 & -\circ.\circ1\circ2 & \circ.\circ\circ17 \\ \circ.\circ722 & -\circ.1443 & \circ.36\circ8 & -\circ.1494 & \circ.\circ256 & -\circ.\circ\circ43 \\ -\circ.\circ299 & \circ.\circ598 & -\circ.1494 & \circ.269\circ & -\circ.\circ461 & \circ.\circ\circ77 \\ \circ.\circ\circ51 & -\circ.\circ1\circ2 & \circ.\circ256 & -\circ.\circ461 & \circ.1793 & -\circ.\circ299 \\ -\circ.\circ\circ\circ9 & \circ.\circ\circ17 & -\circ.\circ\circ43 & \circ.\circ\circ77 & -\circ.\circ299 & \circ.1716 \end{bmatrix}$$

b) Now by Algorithm 3.1. we have

$$W=\begin{bmatrix} 1.4142 & \circ & \circ & \circ & \circ & \circ \\ \circ.7\circ71 & 1.5811 & \circ & \circ & \circ & \circ \\ \circ & \circ.6325 & 1.8974 & \circ & \circ & \circ \\ \circ & \circ & 1.\circ541 & 1.9281 & \circ.414\circ & \circ \\ \circ & \circ & \circ & \circ & 2.4152 & \circ.4\circ82 \\ \circ & \circ & \circ & \circ & \circ & 2.4495 \end{bmatrix}$$

$$W^{-1}=\begin{bmatrix} \circ.7\circ71 & \circ & \circ & \circ & \circ & \circ \\ -\circ.3162 & \circ.6325 & \circ & \circ & \circ & \circ \\ \circ.1\circ54 & -\circ.21\circ8 & \circ.527\circ & \circ & \circ & \circ \\ -\circ.\circ576 & \circ.1153 & -\circ.2881 & \circ.5187 & -\circ.\circ889 & \circ.\circ148 \\ \circ & \circ & \circ & \circ & \circ.414\circ & -\circ.\circ69\circ \\ \circ & \circ & \circ & \circ & \circ & \circ.4\circ82 \end{bmatrix}$$

$$A^{-1}=\begin{bmatrix} \circ.6144 & -\circ.2289 & \circ.\circ722 & -\circ.\circ299 & \circ.\circ\circ51 & -\circ.\circ\circ\circ9 \\ -\circ.2289 & \circ.4577 & -\circ.1443 & \circ.\circ598 & -\circ.\circ1\circ2 & \circ.\circ\circ17 \\ \circ.\circ722 & -\circ.1443 & \circ.36\circ8 & -\circ.1494 & \circ.\circ256 & -\circ.\circ\circ43 \\ -\circ.\circ299 & \circ.\circ598 & -\circ.1494 & \circ.269\circ & -\circ.\circ461 & \circ.\circ\circ77 \\ \circ.\circ\circ51 & -\circ.\circ1\circ2 & \circ.\circ256 & -\circ.\circ461 & \circ.1793 & -\circ.\circ299 \\ -\circ.\circ\circ\circ9 & \circ.\circ\circ17 & -\circ.\circ\circ43 & \circ.\circ\circ77 & -\circ.\circ299 & \circ.1716 \end{bmatrix}$$

5 Conclusion

It can be readily verified that in Algorithm 3.1 the arithmetical operations counts in steps 2 and 3 are considerably reduced compared to existing methods.

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A fast iterative method for solving first kind linear integral equations

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Abstract

In this paper, we study the $\mathcal{LS}$-algorithm for solving linear integral equations of the first kind. This method is based on the reducing the solution of first kind linear integral equations to the solution of a least squares problem with bidiagonal matrix. Then applying the QR factorization method leads to a simple recurrence formula for generating the sequence of approximate solutions. Some properties and convergence theorem are proposed. Moreover, regularization property of the new method with a suitable stopping rule is studied. Finally, some numerical examples are presented to show the efficiency of the new method.

Keywords: Linear operators, Compact operators, Ill-posed problems, First kind equations
Mathematics Subject Classification [2010]: 45N05,45Q05,47B34.

1 Introduction

Integral equations of the first kind with a continuous or weakly singular kernel provide a typical example for the following equation

$$\mathcal{L}u = f, \tag{1}$$

where $\mathcal{L} : \mathcal{V} \to \mathcal{W}$ is a compact linear operator from a Hilbert space $\mathcal{V}$ into a Hilbert space $\mathcal{W}$. Due to the compactness, the operator $\mathcal{L}$ is not boundedly invertible. Hence, the equation (1) is ill-posed in the sense of Hadamard [2]. This makes it difficult to solve by straightforward application of numerical methods, developed to solve well-posed problems. A general strategy for solving problem (1) is regularization technique [1, 2]. So far, many regularization schemes have been proposed including Tikhonov's method, Landweber's iteration [2]. Two problems with most regularization methods are first the right choice of the regularization parameter and second they have high computation time. Iterative methods have an inherent regularization property when applied straight to (1). In fact the number of iteration plays the role of the regularization parameter which is controlled by an suitable stopping rule. In this paper, we apply the $\mathcal{LS}$-algorithm [1], to compute the minimum norm solution of (1). Also, we study the regularization properties of the new method by using the discrepancy principle, introduced by Morozov [2], in context of iteration methods.

*Speaker

2 The proposed method

The new method, $\mathcal{LS}$-algorithm, is based on a bidiagonalization process, called $\mathcal{L}$-Bidiad, for the linear operator $\mathcal{L}$. This process generates two orthonormal sets of functions namely $\psi_1, \psi_2, \ldots \in \mathcal{V}$ and $\phi_1, \phi_2, \ldots \in \mathcal{W}$. We use the same symbols $\langle ., . \rangle$ and $\|.\|$ for the inner products and their corresponding norms on the Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$.

$\mathcal{L}$-Bidiag process:

$$\begin{aligned} \beta_1\phi_1 &= f, \qquad \alpha_1\psi_1 = \mathcal{L}^*\phi_1, \\ \beta_{i+1}\phi_{i+1} &= \mathcal{L}\psi_i - \alpha_i\phi_i, \\ \alpha_{i+1}\psi_{i+1} &= \mathcal{L}^*\phi_{i+1} - \beta_{i+1}\psi_i, \qquad i = 1, 2, \cdots, \end{aligned} \tag{2}$$

where $\phi_i \in \mathcal{W}, \psi_i \in \mathcal{V}$ and the scalars $\alpha_i \geq 0$ and $\beta_i \geq 0$ are chosen so that $\|\phi_i\| = \|\psi_i\| = 1$. With the definitions

$$\Psi_k = [\psi_1, \psi_2, \ldots, \psi_k], \qquad \Phi_k = [\phi_1, \phi_2, \ldots, \phi_k],$$

$$G_k = \begin{pmatrix} \alpha_1 & & & \\ \beta_1 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{pmatrix},$$

and by using Definitions 3.4 and 3.5 in [1], the recurrence formula (2) can be rewritten as

$$\begin{aligned} \Phi_{k+1} \star (\beta_1 e_1) &= f, \\ \mathcal{L}\Psi_k &= \Phi_{k+1} \star G_k, \\ \mathcal{L}^*\Phi_{k+1} &= \Psi_k \star G_k^T + \alpha_{k+1}\psi_{k+1}e_{k+1}^T, \end{aligned} \tag{3}$$

where G^T denotes the transpose of G.
$\mathcal{LS}$-algorithm form solution estimates $u_k = \Psi_k \star \Lambda_k$ for some $\Lambda_k \in \mathbb{R}^k$ at kth stage to minimize the corresponding residual $r_k = \mathcal{L}u_k - f$. By using (2) and (3) and by orthonormality of Φ_{k+1}, the subproblem

$$\min_{\Lambda_k} \|\beta_1 e_1 - G_k\Lambda_k\|,$$

is obtained which can be solved by using QR factorization. Finally as [1] the $\mathcal{LS}$-algorithm is summarized as follows.

$\mathcal{LS}$-algorithm

1. Set $u_0 = 0$ as a zero function
2. $\beta_1 = \|f\|, \ \ \phi_1 = \frac{f}{\beta_1}, \ \ \alpha_1 = \|\mathcal{L}^*\phi_1\|, \ \ \psi_1 = \frac{\mathcal{L}^*\phi_1}{\alpha_1}, \ \ \omega_1 = \psi_1, \ \ \overline{\omega}_1 = \beta_1, \ \ \overline{\tau}_1 = \alpha_1$
3. For $i = 1, 2, \ldots$ until convergence, Do
4. $\quad \chi_i = \mathcal{L}\psi_i - \alpha_i\phi_i$
5. $\quad \beta_{i+1} = \|\chi_i\|, \ \ \phi_{i+1} = \frac{\chi_i}{\beta_{i+1}}$
6. $\quad \varpi_i = \mathcal{L}^*\phi_{i+1} - \beta_{i+1}\psi_i$

7. $\alpha_{i+1} = \|\varpi_i\|, \quad \psi_{i+1} = \frac{\varpi_i}{\alpha_{i+1}}$
8. $\tau_i = \sqrt{\overline{\tau}_i^2 + \beta_{i+1}^2}$
9. $c_i = \frac{\overline{\tau}_i}{\tau_i}$
10. $s_i = \frac{\beta_{i+1}}{\tau_i}$
11. $\eta_{i+1} = s_i \alpha_{i+1}$
12. $\overline{\tau}_{i+1} = -c_i \alpha_{i+1}$
13. $\mu_i = c_i \overline{\mu}_i$
14. $\overline{\mu}_{i+1} = s_i \overline{\mu}_i$
15. $u_i = u_{i-1} + \frac{\phi_i}{\tau_i}\omega_i$
16. $\omega_{i+1} = \psi_{i+1} - \frac{\eta_{i+1}}{\tau_i}\omega_i$
17. If $|\overline{\mu}_{i+1}|$ is small enough then stop
18. EndDo

The proof of the following theorem is similar to Theorem 2.23 of [2].

Theorem 2.1. *Let $\mathcal{L}$ and $\mathcal{L}^*$ are injective and assume the $\mathcal{LS}$-algorithm does not stop after finitely many steps. Then*

$$\| \mathcal{L}u_k - f \| \longrightarrow 0 \qquad as \qquad k \to \infty,$$

for every $f \in \mathcal{W}$.

Now we return to the regularization of the operator equation (1). for this end, we consider the perturbed equation $\mathcal{L}u^\delta = f^\delta$ where $||f^\delta - f|| \leq \delta$. We use the following stopping rule which is the discrepancy principle in context of the iteration methods [2].
Stopping rule: Fix $\ell > 1$ and terminate the algorithm at the first time, $k = k(\delta)$, that $|\mu_{k+1}| \leq \ell\delta$.
Now, we let (μ_j, x_j, y_j) be a singular system of $\mathcal{L}$ [2]. The following theorem shows that the $\mathcal{LS}$-algorithm is optimal under the above stopping rule.

Theorem 2.2. *Let $f, f^\delta \notin span\{y_1, y_2, ..., y_N\}$ for all $N \in \mathbb{N}$ and let $f \in (\mathcal{L}^*\mathcal{L})^{\frac{\nu}{2}}(\mathcal{V})$ for some $\nu > 0$ and $\| u \|_\nu \leq R$. If the $\mathcal{LS}$-algorithm is stopped after $k(\delta)$ steps according to mentioned stopping rule with fixed parameter $\ell > 1$, then there exist $c > 0$ such that*

$$||u^\delta_{k(\delta)} - u|| \leq cR^{\frac{\nu}{\nu+1}}\delta^{\frac{\nu}{\nu+1}}.$$

Example 2.3. We consider the following first kind Fredholm integral equation

$$\int_0^1 (t^2 + s^2)^{\frac{1}{2}} u(s) ds = \frac{(1+t^2)^{\frac{3}{2}} - t^3}{3},$$

with the exact solution $u^*(t) = t$.

Table 1: Numerical results for the Example 2.3for 17-point Simpson's rule.

t	$\|u^*(t)-u_3(t)\|$	$\|u^*(t)-u^S(t)\|$
0	$3.99e-04$	$1.06e-2$
0.25	$2.08e-04$	$2.46e+00$
0.5	$2.39e-06$	$6.44e+1$
0.75	$3.82e-04$	$2.00e+1$
1	$7.49e-04$	$9.55e+00$

For this example we compare the 17-point Simpson quadrature rule (Nÿstom like method) with the $\mathcal{LS}$-algorithm when the involved definite integrals in each iteration are approximated by mentioned quadrature rule. The numerical results are given in Table 2. In this table $|u^*(t)-u_k(t)|$ and $|u^*(t)-u^S(t)|$ are the absolute solution errors of the $\mathcal{LS}$-algorithm and Nÿstrom, respectively. We see that the $\mathcal{LS}$-algorithm is clearly superior.

Example 2.4. We consider symm's equation

$$-\frac{1}{2\pi}\int_0^{2\pi}(\ln(4sin^2\frac{t-s}{2})+K(t,s))u(s)=f(t), \qquad K(t,s)=\begin{cases} -\frac{1}{2\pi}\ln\frac{|\sigma(t)-\sigma(s)|^2}{4\sin^2\frac{t-s}{2}}, & t\neq s\\ -\frac{1}{\pi}\ln|\sigma^{\cdot}(t)|, & t=s,\end{cases}$$

where $\sigma(t)=(cost,2sint)$. We use the exact solution $u^*(t)=e^{3sint}$ and define $f(t)$ accordingly.

We approximate the smooth part and weakly singular part by using the trapezoidal rule and trigonometric interpolation, respectively. Here, The node points are $t_j=j\pi/n$, $j=0,1,...,2n-1$ with $n=60$. And we perturbed the right hand side of the discretized form by uniformly distributed random vector depended on δ. The results shown in Table 2, confirm the theorem 2.2. In this table, $U^{\delta}_{k(\delta)}$ and U are approximated solution, obtained from the stopping rule, and exact solution in node points, respectively. Also, $||U||_n^2=\frac{||U||}{2n}$ where $||.||$ is Euclidean norm.

Table 2: Numerical results for the Example 2.3for 17-point Simpson's rule.

δ	0.1	0.01	0.001	0
$\parallel U^{\delta}_{k(\delta)}-U\parallel_n$	1.51e-01	3.17e-02	5.7e-03	4.83e-13

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A greedy meshless method for solving boundary value problems

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Abstract

In this paper we use a meshless method based on a greedy algorithm to solve boundary value problems (BVPs). This method is greedy Kansa's method that use the optimal trial points. In the greedy algorithm , the optimal trial points for interpolation obtained among a huge set of initial points are used for numerical solution of BVPs. This paper shows that selection nodes greedily yields the better conditioning and good approximation in contrast with the usual Kansa method. A well known BVP is solved and compared with the usual Kansa's method.

Keywords: Greedy algorithm, Meshless method, Radial basis function
Mathematics Subject Classification [2010]: 65N35, 65N22

1 Introduction

A greedy algorithm is an algorithm that follows the problem solving heuristic of making the locally optimal choice at each stage with the hope of finding a global optimum. A recent survey of the approximation properties of such algorithms is given in [1]. Schaback and Muller [3] has shown that representations of kernel-based approximants in terms of the standard basis of translated kernels are notoriously unstable. They introduced the Newton bases functions with a recursively computable set of basis functions and vanishing at increasingly many data points turn out to be more stable. In [4] adaptive calculation of Newton bases is used, which turns out to be stable, complete, orthonormal computable. In this work, we will apply the greedy method to meshless method for solving a linear PDE problem is given in the form

$$\begin{aligned} Lu &= f, \quad in\ \Omega, \\ Bu &= g \quad on\ \partial\Omega \end{aligned} \tag{1}$$

with a linear differential operator L and a linear boundary operator B. Consider smooth symmetric positive definite kernel $K : \Omega \times \Omega \to \mathbb{R}$ on spatial domain Ω. This means that for all finite sets $X := \{x_1, \cdots x_N\} \subseteq \Omega$ the kernel matrix $A := (K(x_j, x_k))_{1\leq j,k\leq N}$ is symmetric and positive definite. It is well-known that this kernel is reproducing in a "native" Hilbert space $\mathcal{N}_k = \overline{\text{span}\{K(x,.) : |x \in \Omega\}}$ of functions on Ω in the sense $\langle u, K(x,.)\rangle_{\mathcal{N}_k} = u(x) \ \ \forall x \in \Omega, \ \forall u \in \mathcal{N}_k$.

*Speaker

2 Greedy Algorithm

Here a greedy algorithm will be described in the context of radial basis functions for PDEs. It is based on paper [4]. In this algorithm we let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a continuous positive definite kernel K on it. Also we take a large finite subset X as data points. For a fixed domain, a fixed kernel and a fixed scale, this algorithm gives the first n Newton basis functions on N points, and provides a subset of best trial points among the data site X. Then we shall use these greedy points as trial points for the collocation method on the same domain with the same kernel and scale. The complexity of this algorithm is $O(Nn^2)$ and it requires a total storage of $O(Nn)$.

Algorithm 1: Adaptive calculation of Newton basis on optimal points

Data: $X \in \mathbb{R}^{N\times d}$: data points; n_{max}: maximal number of points to be finally selected; ε: power function tolerance; The symmetric positive definite kernel $K : \Omega \times \Omega \to \mathbb{R}$

Result: I: the indices of greedy selected points from X;

Initialize $I^{n_{max}\times 1} := 0$;

$\mathbf{k} := (K(x_1, x_1), \ldots, K(x_N, x_N))^T$; % $[x_1; \ldots ; x_N] = X$

$i := \text{argmax}_{1\le t\le N}(\mathbf{k}_t)$;

$z := \mathbf{k}_i$;

$\mathbf{v}_1 := \frac{K(X,x_i)}{\sqrt{\mathbf{k}_i}}$; % component-by-component root and division

$\mathbf{w} := \mathbf{v}_1^2$; % component-by-component square

$I_1 := i$;

for $j := 2$ **to** n_{max} **do**

- $i := \text{argmax}_{1\le t\le N}(\mathbf{k}_t - \mathbf{w}_t)$; $z := \mathbf{k}_i - \mathbf{w}_i$;
- **if** $z < \varepsilon$ **then**
  - $j := j - 1$;
  - break;
- $\mathbf{k}^0 := K(X, x_i)$;
- **for** $k := 1$ **to** $j - 1$ **do**
  - $\mathbf{k}^0 := \mathbf{k}^0 - \mathbf{v}_{k,i} * \mathbf{v}_k$;
- $\mathbf{v}_j := \frac{\mathbf{k}^0}{\sqrt{\mathbf{k}_i-\mathbf{w}_i}}$; % component-by-component root and division
- $\mathbf{w} := \mathbf{w} + \mathbf{v}_j^2$; % component-by-component square
- $I_j := i$;

return: I.

3 Meshless methods

The algorithm 1 suggests the best trial points and we use them for Kansa's method. It sufficients to run the algorithm for predetermined kernel, scale and domain Ω. Then selected points are used for PDE solution in the same domain with a possibly different set of test points.

Table 1: RMS-errors and condition number for test problem

	Greedy method		Usual Kansa method	
N	RMS-error	Condition Number	RMS-error	Condition Number
80	0.0302	1.8825e+007	0.0946	4.2819e+010
120	2.5534e-004	1.3433e+009	0.0065	1.3176e+011
160	5.2268e-004	6.9924e+010	0.0012	2.8090e+011
200	8.5355e-005	2.0741e+012	2.0152e-004	2.3162e+013
240	2.4113e-005	5.6810e+013	6.2529e-005	3.7933e+016

4 Numerical Example

In the following numerical result some well known PDE are solved by the greedy Kansa's method and compared with the full Kansa's method. We compare the stability of both methods by examining the condition number of their coefficient matrices. Also, for comparing the accuracy we examin the maximum errors (MAX) and the root mean square (RMS) errors. The maximum errors evaluated by

$$MAX = \max_{1\le j\le N} |u_j - \tilde{u_j}|,$$

and the root mean square errors evaluated by

$$RMS = \left[\frac{1}{N}\sum_{j=1}^{N}(u_j - \tilde{u_j})^2\right]^{1/2},$$

4.1 Test Problem

Consider the following Poisson problem with Dirichlet boundary conditions:

$$\Delta u = 4e^{x^2+y^2} + 4(x^2+y^2)e^{x^2+y^2},$$
$$u = e^{x^2+y^2},$$

and $\partial\Omega$ is an ellipse whose equation is $x^2 + 4y^2 - 1 = 0$. In this case the exact solution is given by $u(x, y) = e^{x^2+y^2}$. By Algorithm 1 we generated different n numbers of trial points also we selected n random points to use in first method. Figure 1 shows the first 80 selected points and the decay of the maximum of the power function for this case. A comparsion between greedy Kansa method and usual Kansa for different n values is implementated and the results are reported in Table 1. It shows that using the Algorithm 1 causes the errors is minimized and the condition number is improved. Decay of power function in Figure 1 and MAX-errors in Figure 2 guarantee the accuracy of greedy kansa method.

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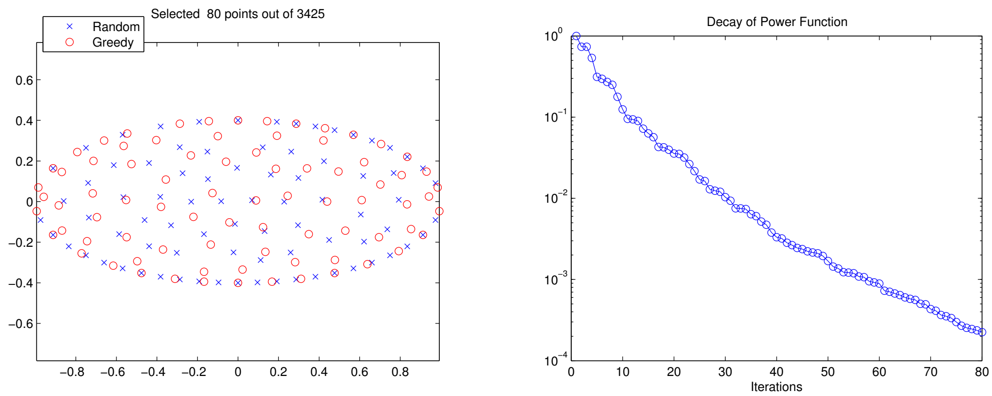

Figure 1: Plot of selected point and power function decay for test problem.

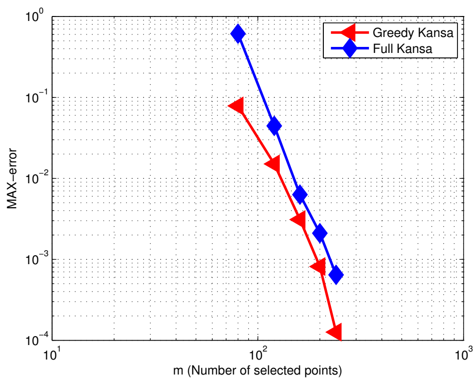

Figure 2: Plot of MAX-errors for greedy and full Kansa method of test problem.

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A method of particular solutions with Chebyshev basis functions for systems of multi-point boundary value problems

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Abstract

This paper presents a new semi-analytic numerical method for solving system of multi-point boundary value problems. The method is based on the use of the particular solutions of the linearized equation. Numerical implementation confirms the validity, efficiency and applicability of the method.

Keywords: Particular solutions, System of Multi-point boundary value problems, Chebyshev basis functions.
Mathematics Subject Classification [2010]: 34B15, 35J57

1 Introduction

We consider the following multi-point boundary value problem (MPBVPs):

$$u^{(s)} = F(u, u', \dots, u^{(s-1)}, x), \quad x \in [0, 1], \tag{1}$$

$$\sum_{j=0}^{s-1} a_{j,k} u^{(j)}(\xi_{j,k}) = d_k, \quad 0 \le \xi_{j,k} \le 1,\ k = 1, \dots, s, \tag{2}$$

where some of the coefficients $a_{j,k}, d_k$ could be equal to zero. Sometimes we write the equation in the form

$$u^{(s)} = F(u, u', \dots, u^{(s-1)}, x) + f(x) \tag{3}$$

highlighting that $f(x)$ does not depend on u. The linear analogs of (3)

$$u^{(s)} = \sum_{k=0}^{s-1} A_k u^{(k)}(\xi) + f(x), \quad x \in [0, 1], \tag{4}$$

is also considered in the paper. We assume that F; A_k and f are smooth enough functions with respect to their arguments.
In this paper we use the semi-analytic method proposed earlier in[1, 3, 4] to solve nonlinear two-point BVPs. This method is described in detail in the next section. Then we apply it to solve the system of nonlinear two-point BVPs. A numerical example illustrating the applicability of the method is placed in Section 3.

*Speaker

2 Main algorithm

Let $\phi_m(x)$ be some system of basis functions on $[0,1]$, here we consider the Chebyshev basis functions

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x,$$
$$\varphi_m(x) = 2x\varphi_{m-1}(x) - \varphi_{m-2}(x), \quad m = 3, \ldots, M. \tag{5}$$

The particular solutions of the equation $\phi_m^{(s)}(x) = \varphi_m(x)$, which correspond to the basis functions φ_m are:

$$\phi_m(x) = \frac{1}{(s-1)!}.\int_0^x (x-t)^{s-1}\varphi_i(t)dt. \tag{6}$$

We denote

$$\Phi_m(x) = \phi_m(x) + c_{m,0} + c_{m,1}x + \ldots + c_{m,s-1}x^{s-1}. \tag{7}$$

So, $\Phi_m^{(s)}$ satisfies $\Phi_m^{(s)}(x) = \phi_m^{(s)}(x) = \varphi_m(x)$. The free coefficients $c_{m,i}$ in (7) are chosen in such a way that Φ_m satisfy the homogeneous boundary conditions (2):

$$\sum_{j=0}^{s-1} a_{j,k}\Phi_m^{(j)}(\xi_{j,k}) = 0, \quad k = 1, \ldots, s. \tag{8}$$

Substituting (7) in (8), one gets a linear system for $c_{m,0}, c_{m,1}, \ldots, c_{m,s-1}$. We assume that the nonlinear term in (3) can be approximated by the linear combinations of the basis functions $\varphi_m(x)$:

$$F(u, u', \ldots, u^{(s-1)}, x) = \sum_{m=0}^{M} q_m\phi_m(x). \tag{9}$$

Substituting this approximation in the initial equation (3), one gets

$$u_M^{(s)}(x) = \sum_{m=0}^{M} q_m\phi_m(x) + f(x). \tag{10}$$

Let $u_f(x)$ satisfy the equation $u_f^{(s)}(x) = f(x)$ and the boundary conditions (2):

$$\sum_{j=0}^{s-1} a_{j,k}u_f^{(j)}(\xi_{j,k}) = d_k. \tag{11}$$

When there exists a particular solution $u_p(x)$ in explicit analytic form, then it can be written in the form:

$$u_f(x) = u_p(x) + c_0 + c_1x + \ldots + c_{s-1}x^{s-1}. \tag{12}$$

When there are no particular solution, $f(x)$ is joined to the nonlinear term and we get $u_f^s(x) = 0$, and $u_f(x) = c_0 + c_1 + \ldots + c_{s-1}x^{s-1}$. Substituting $u_f(x)$ in (11), one gets a linear system for $c_0, c_1, \ldots, c_{s-1}$. So

$$u_M(x, \mathbf{q}) = u_f(x) + \sum_{m=1}^{M} q_m\Phi_m(x), \quad \mathbf{q} = (q_1, \ldots, q_M) \tag{13}$$

satisfies Eq. (10) and the boundary conditions of the initial problem (2). To get unknowns $q_1, \ldots, q_M$ we substitute $u_M(x, \mathbf{q})$ in (9)

$$F\left(u_M(x,\mathbf{q}), u_M^{(1)}(x,\mathbf{q}), \ldots, u_M^{(s-1)}(x,\mathbf{q}), x\right) = \sum_{m=1}^{M} q_m \phi_m(x). \tag{14}$$

Note that we can always get the $u_f(x)$ in the analytic way when $f(x)$ is a simple combination of elementary functions, e.g., quasipolynomial $(b_0 + b_1 x + \ldots + b_p x^p) exp(\mu x)$. Otherwise we can use the formula

$$u_f(x) = \frac{1}{(n-1)!}\int_{x_0}^{x}(x-t)^{s-1} f(t) d(t) + c_0 + c_1 + \ldots + c_{s-1}x^{s-1} \tag{15}$$

and evaluate the integral numerically. Another approach is to join the term $f(x)$ to the nonlinear term F. To solve (14) we use the following algorithm. Let $0 \leqslant x_1 < x_2 < \ldots < x_M \leqslant 1$ be collocation points. In particular, we use the Chebyshev collocation points

$$x_n = \frac{1}{2}\left[1 + \cos\left(\frac{\pi(n-1)}{M-1}\right)\right]. \tag{16}$$

We write the collocation of (14) at these points and get the system of M nonlinear equations

$$F\left(u_M(x_n,\mathbf{q}), u_M^{(1)}(x_n,\mathbf{q}), \ldots, u_M^{(s-1)}(x_n,\mathbf{q}), x_n\right) = \sum_{m=1}^{M} q_m \phi_m(x_n), \quad n = 1, \ldots, M. \tag{17}$$

We solve this system of equations. Dealing with linear problems (4), one gets

$$f(x_n) + \sum_{k=0}^{s-1} A_k(x_n)\left[u_f^{(k)}(x_n) + \sum_{m=1}^{M} q_m \Phi_m^{(k)}(x_n)\right] = \sum_{m=1}^{M} q_m \phi_m(x_n) \tag{18}$$

instead of (17). Rewriting in the form

$$\sum_{m=1}^{M}\left[\sum_{k=0}^{s-1} A_k(x_n)\Phi_m^{(k)}(x_n) - \phi_m(x_n)\right] = -f(x_n) - \sum_{k=0}^{s-1} A_k(x_n) u_f^{(k)}(x_n), \tag{19}$$

we get the linear system for $q_1, \ldots, q_M$ and the linear system is solved by Maple. After determining $q_1, \ldots, q_M$ we get the approximate solution $u_M(x, \mathbf{q})$ (13). We use the maximal absolute errors e_{max} to evaluate the exactness of the solution.

3 Illustration of the method

As a sample we consider the following system of nonlinear BVP with the equations of the second order [2]:

$$\begin{cases} u''(x) + u'(x) + xu(x) + v'(x) + 2xv(x) = f_1(x), \\ v''(x) + v(x) + 2u'(x) + x^2 u(x) = f_2(x), \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{20}$$

Table 1: The maximum absolute errors for different numbers of the basis functions

M	5	10	15
e_{max} for $u(x)$	0.0010063093	0.00000449789	1.5493×10^{-7}
e_{max} for $v(x)$	0.0015402899	0.0000019050	4.92×10^{-8}

where $0 \le x \le 1$, $f_1(x) = -2(1+x)\cos(x) + \pi\cos(\pi x) + 2x\sin(\pi x) + (4x - 2x^2 - 4)\sin(x)$, and $f_2(x) = -4(x-1)\cos(x) + 2(2 - x^2 + x^3)\sin(x) - (\pi^2 - 1)\sin(\pi x)$ with the exact solutions $u_{exact}(x) = 2\sin(x)(1-x)$ and $v_{exact}(x) = \sin(\pi x)$.

For equations (20) we have $s = 2$. We define $\varphi_i(x)$ and $\phi_i(x)$ as said in the previous section and $\Phi_i(x) = c_{i,0} + c_{i,1}x$. The coefficients $c_{i,0}$ and $c_{i,1}$ will be determined by substituting $\Phi_i(x)$ in the homogenous boundary conditions. Also, we set $u_f(x) = c_{1,0} + c_{1,1}x$ and $v_f = c_{2,0} + c_{2,1}x$ and they will be determined by solving the system of equations achived from substituting u_f and v_f in the non-homogenous boundary conditions. Now we set

$$u_M(x) = u_f(x) + \sum_{m=1}^{M} q_m \Phi_m(x), \tag{21}$$

$$v_M(x) = v_f(x) + \sum_{m=1}^{M} q_{M+m} \Phi_m(x) \tag{22}$$

To get unknowns $q_1, \ldots, q_{2M}$ we substitute $u_M(x)$ and $v_M(x)$ in (20) and use collocation method. The maximum absolute errors e_{max} are shown in Table 1 for different numbers of the basis functions M.

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A New Adaptive Element Free Galerkin Algorithm Based on the Background Mesh*

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Abstract

In this work we present an adaptive element free Galerkin procedure based on background mesh for meshless methods using MLS. It comprises a cell energy error estimate and a local domain refinement technique. The error estimate differs from conventional point wise approaches in that it evaluates error based on individual cells instead of points. In this technique, each node is assigned a scaling factor to control local nodal density and achieve high efficiency in domain refinement. Refinement of the neighborhood of a node is accomplished simply by adjusting its scaling factor. Some challenging problems are discussed to show that the proposed adaptive procedure is effective, efficient and convergent.

Keywords: Meshless methods, Adaptive Element Free Galerkin (EFG) method, A posteriori error estimate, Moving Least Squares (MLS) approximation, Crack problem.
Mathematics Subject Classification [2010]: 65M99

1 Introduction

The Element Free Galerkin (EFG) method [2, 3] may be regarded as an alternative to the finite element method especially for problems with discontinuities, e.g. crack propagation problems. The EFG method differs from the FEM by using the Moving Least Squares (MLS) approximation. In practical implementations, EFG formulation requires a background mesh for domain integration.

A posteriori error estimates, initiated in [1], are computable quantities in terms of the discrete solution and known data that measure the actual discrete errors without the knowledge of exact solutions. They are essential in designing algorithms for mesh refinement which optimize the computation. The ability of error control and the asymptotically optimal approximation property make the adaptive finite element methods attractive for complicated physical and industrial processes.

*Will be presented in English
†Speaker

2 Adaptive element free Galerkin method

Let Ω be a bounded polyhedral Lipschitz domain in R^d, $d \geq 2$. In what follows we will study the following second order elliptic equation: Find $u \in V_D$

$$\begin{cases} \mathcal{L}u(\mathbf{x}) := -\mathrm{div}(\mathbf{A}\nabla u) + cu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases} \tag{1}$$

where $V_D := \{u \in H^1(\Omega) : \ u|\Gamma = 0 \ \text{ in the sense of traces}\}$. For any $f \in L^2(\Omega)$, the weak formulation of the problem (1) reads as follows: Find $u \in V_D$ such that

$$\int_\Omega \mathbf{A}\nabla u.\nabla v + cuv = \int_\Omega fv + \int_\Gamma gv, \quad \forall v \in V_D. \tag{2}$$

Let $X = \{x_1, ..., x_N\} \subseteq \Omega$ be a set of meshless points scattered over Ω. The MLS method approximates the function u by its values at points $\mathbf{x}_j$, $j = 1, 2, ..., N$, by

$$\tilde{u}(\mathbf{x}) = \sum_{j=1}^{N} \phi_j(\mathbf{x}) u(\mathbf{x}_j), \qquad \mathbf{x} \in \Omega, \tag{3}$$

where $\phi_j(\mathbf{x})$ are MLS shape functions obtained in such way that $\tilde{u}(\mathbf{x})$ be the best approximation of $u(\mathbf{x})$ in polynomial subspace $\mathbb{P}_m(\mathbb{R}^d) = span\{p_1, ..., p_Q\}$, $Q = \binom{m+d}{d}$, with respect to a weighted, discrete and moving l_2 norm (see [4, 5, 6] for more detailes).

Now suppose that V_h is a subspace built using MLS shape functions, that is, $V_h = \mathrm{span}\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$. Then the EFG solution of this problem is to find $u_h \in V_h$ such that

$$\int_\Omega \mathbf{A}\nabla u_h.\nabla v_h + cu_h v_h = \int_\Omega f v_h + \int_\Gamma g v_h, \quad \forall v_h \in V_h, \tag{4}$$

which leads to the following system

$$\mathbf{K}\mathbf{u} = \mathbf{b}, \tag{5}$$

where $K_{ij} = \int_\Omega \mathbf{A}\nabla\varphi_i.\nabla\varphi_j + c\varphi_i\varphi_j$ and $b_i = \int_\Omega f\varphi_i + \int_\Gamma g\varphi_i$.

2.1 Adaptive strategy

In the following discussion we will consider a sequence of background cells $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$, where $\mathcal{T}_0$ is a given (coarse) mesh and each $\mathcal{T}_{l+1}$ is the standard refinement of $\mathcal{T}_l$. Now we shall give a brief description and some properties of the new adaptive algorithm. Our adaptive algorithm is implemented on Adaptive Element Free Galerkin ($\mathcal{AEFG}$) package, which is developed for solving PDEs by authors of the current work. We define the local a posteriori error estimator over an element $T \in \mathcal{T}_l$ by

$$\eta_l(T)^2 := h_T^2 \|\mathcal{L}|_T u_l - f\|^2_{L^2(T)}, \tag{6}$$

for all $T \in \mathcal{T}_l$ and all $l \in \mathbb{N}$. Here h_T is the radius of Ω_T, the domain of definition of T.

The global a posteriori error estimate over $\mathcal{T}_l$ is defined as the l_2 sum of the element wise contributions

$$\eta_l^2 = \sum_{T \in \mathcal{T}_l} \eta_l(T)^2. \tag{7}$$

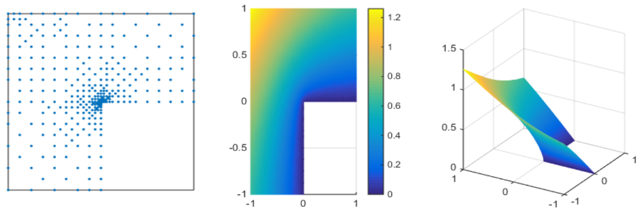

Figure 1: The final nodes distribution, contour curves and elevations of the EFG solution after 20 iterations for the L-shaped domain.

Now we describe the adaptive algorithm used in this paper.

Algorithm 2.1. Input: Initial triangulation $\mathcal{T}_0$ and adaptivity parameter $0 < \theta \leq 1$.
Loop: For $l = 0, 1, 2, \ldots$ do 1-4

1. Compute discrete solution $\mathbf{u}_l$ of (5)
2. Compute refinement indicators $\eta_l(T)$ for all $T \in \mathcal{T}_l$ and η_l.
3. Determine set $\mathcal{M}_l \subseteq \mathcal{T}_l$ of minimal cardinality such that
$$\theta \eta_l^2 \leq \sum_{T \in \mathcal{M}_l} \eta_l(T)^2. \tag{8}$$
4. Refine (at least) the marked elements $T \in \mathcal{M}_l$ to obtain the triangulation $\mathcal{T}_{l+1}$.

Output: Approximate solutions $\mathbf{u}_l$ and error estimators η_l for all $l \in \mathbb{N}$.

We remark that this loop can be controlled by a stopping criterion based on the a posteriori error estimator η_l, avoiding too many iterations on coarse meshes. After reaching to a desirable tolerance, values of $u(\mathbf{x})$ at any point $\mathbf{x}$ can be approximated by MLS approximation.

3 Numerical Examples

This section reports some numerical results regarding the singularities. In this section we demonstrate the performance of the implicit error estimator (6) applied to the second order elliptic equation with singularities on a domain $\Omega \subset \mathbb{R}^2$. Our implementation uses MLS approximation with first order polynomials. In all numerical results, the experimental parameter θ is set 0.5.

Example 3.1. (L-shaped domain) The first experiment is to solve the Laplace equation with Dirichlet boundary condition in the L-shaped domain $\Omega = (-1,1) \times (0,1) \cup (-1,0) \times (-1,0]$, where the exact solution is given by $u(r,\theta) = r^{2/3} \sin\left(\frac{2}{3}\right)$. The elevations and contour plots of the adaptive EFG solution and the final nodes distribution after 20 iterations are shown in Fig. 1.

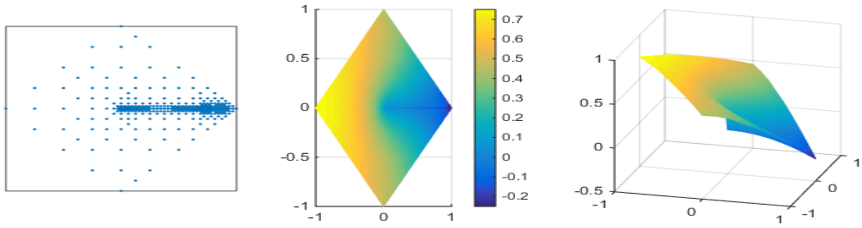

Figure 2: The final nodes distribution, contour curves and elevations of the EFG solution after 20 iterations for crack problem.

Example 3.2. (Crack problem) Let $\Omega = \{|x| + |y| < 1\}\ \{0 \le x \le 1, y = 0\}$, and the solution u satisfies the Poisson equation

$$\begin{aligned} -\Delta u &= 1 \quad in\ \Omega, \\ u &= g \quad on\ \partial\Omega, \end{aligned} \tag{9}$$

and g is chosen so that the exact solution is $u(r,\theta) = r^{1/2}\sin\left(\frac{\theta}{2}\right) - \frac{1}{4}r^2$. The elevations and contour plots of the adaptive EFG solution and the final nodes distribution after 20 iterations are shown in Fig. 2.

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A new iterative method for solving free boundary problems

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Abstract

In this paper, an efficient iterative method is proposed to approximate the solution of free boundary problems (FBP). This method is based on hybrid of the radial basis function (RBF) collocation and finite difference (FD) methods. Finally, a numerical example is given to illustrate the good performance of the new method.

Keywords: Free boundary problem, Multiquadric radial basis functions.
Mathematics Subject Classification [2010]: 35R35, 65N06

1 Introduction

A free-boundary problem is a partial differential equation that in which some part of the boundary is not known, but is to be determined. The segment Γ of the boundary of domain which is not known is called the free boundary. Then, both the free boundary and the solution of the differential equation should be determined.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 1$ with smooth boundary $\partial\Omega$. Assume further that $g \in W^{1,2}(\Omega)$ and takes both positive and negative values over $\partial\Omega$, and $\lambda^{\pm} : \Omega \to \mathbb{R}$ are positive Lipschitz-continuous functions. The study of the following FBP is suggested by Weiss in [3]. Find a weak solution $u \in W^{1,2}(\Omega)$ of $\Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}}$, in Ω such that $u - g \in W_0^{1,2}(\Omega)$ for a given $g \in W^{1,2}(\Omega)$, where χ_A denotes the characteristic function of the set A. This problem can be modeled as follows

$$\begin{cases} \Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}}, & x \in \Omega; \\ u = g, & x \in \partial\Omega. \end{cases} \tag{1}$$

If, in addition, we assume that $\lambda^- = 0$ and g be non-negative on the boundary then we have the one-phase obstacle problem.
The free boundary problems (1) have been studied from different viewpoints, see [1, 2].

In this study, we propose an efficient iterative method to solve two-phase problem, one-phase obstacle problem and FBP of the form

$$\begin{cases} \Delta u = -\begin{cases} \lambda^+ u, & if\ u > 0; \\ 0 & if\ u \leq 0. \end{cases}, & x \in \Omega; \\ u = g, & x \in \partial\Omega. \end{cases}$$

*Speaker

2 Main results

In this section, we present a new iterative method to solve the two-phase boundary problem (1) and the free boundary problem (2). Also, this method is capable for solving one phase obstacle problem. To do so, we consider a uniform mesh on $\Omega \subset \mathbb{R}^2$ and let $\Delta x = \Delta y = h$. For simplicity let $\Omega = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and

$$p_{i,j} = (-1 + (i-1)h, -1 + (j-1)h), \ \ i, j = 1, \dots, m, \ \ h = \frac{2}{m-1}, \ N = m^2.$$

We define $x_l = p_{i,j}, \quad i, j = 1, \dots, m$, and

$$\bar{u}_l = \frac{1}{4}[u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1})], \ \ i, j = 2, \dots, m-1, \tag{2}$$

where $l = j + (i-1)m$ and $\bar{u}_l = \bar{u}(x_l)$. Consider x_l, $l = 1, 2, \dots, N$, as collocation points. Let M of them are located in the domain and $N - M$ of them on the boundary of the problem. The unknown solution u is approximated by a linear combination of the form

$$u(x) \approx \tilde{u}(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x), \tag{3}$$

where $\phi_i(x) = \sqrt{c^2 + ||x - \bar{x}_i||^2}$ is the Multiquadric RBF and $\bar{x}_i$, $i = 1, 2, \dots, N$, are the centers of RBF. Also, α_i, $i = 1, 2, \dots, N$ are the unknown coefficients to be determined. Note that, here we consider the centers of RBF and collocation points the same. Hence, we present an iterative method which is based on combination of RBF collocation and FD methods for solving them. This method has been described as follows. For $N - M$ nodal points are located on the boundaries ($x_l \in \partial\Omega$), the Dirichlet boundary condition is imposed by

$$\tilde{u}_l^{k+1} = g_l, \tag{4}$$

at iteration $k+1$. For nodes which are located in the interior of the domain, we present two methods for two-phase problem (1) and free boundary problem (2).

Method A

Consider the two-phase problem (1). In the interior points of domain, this problem is equivalent with the following problem

$$\begin{cases} \Delta u = \lambda^+, & if\ u > 0; \\ \Delta u = -\lambda^-, & if\ u < 0; \\ u = 0, & \text{otherwise}. \end{cases} \tag{5}$$

By using FD method, the system (5) can be written as

$$\begin{cases} u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1}) - 4u(p_{i,j}) = \lambda_l^+ h^2, & if\ \bar{u}_l - \frac{\lambda_l^+ h^2}{4} > 0; \\ u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1}) - 4u(p_{i,j}) = -\lambda_l^- h^2, & if\ \bar{u}_l + \frac{\lambda_l^- h^2}{4} < 0; \\ u(p_{i,j}) = 0, & \text{otherwise}; \end{cases} \tag{6}$$

where $\lambda_l^+ = \lambda^+(x_l)$, $\lambda_l^- = \lambda^-(x_l)$. Hence, for the two-phase problem, for each node located in the domain ($x_l\ in\ \Omega$), we have the following iterative procedure by combining finite difference and RBF collocation methods

$$\begin{cases} \tilde{u}_l^{k+1} = 0, & if\ \hat{u}_l^k \le 0\ and\ \hat{\hat{u}}_l^k \ge 0; \\ \Delta\tilde{u}^{k+1}(x)\mid_{x=x_l} = \lambda_l^+ \chi_{\hat{u}_l^k>0} - \lambda_l^- \chi_{\hat{\hat{u}}_l^k<0}, & \text{otherwise}; \end{cases} \tag{7}$$

where $\hat{u}_l^k = \bar{u}_l^k - \frac{\lambda_l^+ h^2}{4}$, $\hat{\hat{u}}_l^k = \bar{u}_l^k + \frac{\lambda_l^- h^2}{4}$ and

$$\bar{u}_l^k = \frac{1}{4}[u^k(p_{i-1,j}) + u^k(p_{i+1,j}) + u^k(p_{i,j-1}) + u^k(p_{i,j+1})],\ \ k = 0, 1, \ldots, \tag{8}$$
$$i, j = 2, 3, \ldots, m-1, \quad l = j + (i-1)m.$$

Putting equations (4) and (7) together results in a linear system of equations.

Method B

Consider the FBP (2). In the interior points of domain, this problem is equivalent with

$$\begin{cases} \Delta u = -\lambda^+ u, & if\ u > 0; \\ \Delta u = 0, & if\ u < 0; \\ u = 0, & \text{otherwise}. \end{cases} \tag{9}$$

By using FD method, system (9) can be written as

$$\begin{cases} u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1}) - 4u(p_{i,j}) = -\lambda_l^+ h^2 u(p_{i,j}), & if\ \frac{\bar{u}_l}{1-\frac{\lambda_l^+ h^2}{4}} > 0; \\ u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1}) - 4u(p_{i,j}) = 0, & if\ \bar{u}_l < 0; \\ u(p_{i,j}) = 0, & \text{otherwise}. \end{cases} \tag{10}$$

Let $\bar{M} = \max_{x_l \in \Omega} \sqrt{\lambda_l^+}$. If we choose h such that $h < 2/\bar{M}$, then $1 - \frac{\lambda_l^+ h^2}{4} > 0$, for every $x_l\ in\ \Omega$. Thus system (10), is reduced to

$$\begin{cases} u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1}) - 4u(p_{i,j}) = -\lambda_l^+ h^2 u(p_{i,j}), & if\ \bar{u}_l > 0; \\ u(p_{i-1,j}) + u(p_{i+1,j}) + u(p_{i,j-1}) + u(p_{i,j+1}) - 4u(p_{i,j}) = 0, & if\ \bar{u}_l < 0; \\ u(p_{i,j}) = 0, & if\ \bar{u}_l = 0. \end{cases} \tag{11}$$

Then, similar to method A, For each node located in the domain ($x_l\ in\ \Omega$), we have the following iterative procedure

$$\begin{cases} \tilde{u}_l^{k+1} = 0, & if\ \bar{u}_l^k = 0; \\ \Delta\tilde{u}^{k+1}(x)\mid_{x=x_l} = -\lambda_l^+ \tilde{u}_l^{k+1} \chi_{\bar{u}_l^k>0}, & \text{otherwise}; \end{cases} \tag{12}$$

where $k = 0, 1, 2, \ldots$.

Two above methods are summarized as the following algorithm.

algorithm

step 1: Choose an initial guess as $u_l^0 = \begin{cases} 0, & if\ x_l\ in\ \Omega; \\ g_l, & if\ x_l\ on\ \partial\Omega \end{cases}$

step 2: For $k = 0, 1, 2, \ldots$, until convergence, Do

step 3: Compute $\bar{u}_l^k$ from equation (8) step 4: Solve the linear system obtained from method A or B

step 5: Set the approximate solution $\tilde{u}^k(x) = \sum_{i=1:N}^{N} \alpha_i^k \phi_i(x)$, where $\alpha^k = (\alpha_1^k, \alpha_2^k, \dots, \alpha_N^k)^T$ is the solution obtained from step 4

step 6: Put $u^k = \tilde{u}^k$

step 7: EndDo

3 Numerical results

Example 3.1. consider the following problem $\Delta u = -u\chi_{\{u>0\}}$, $(x, y) \in (-4, 4)^2$, with the analytical solution $u(x, y) = \begin{cases} J_0(r), & if\ r < r_c, \\ A \ln \frac{r_c}{r}, & if r \geq r_c, \end{cases}$ where $r^2 = x^2 + y^2$ and $r_c \approx 2.404826$ is the first zero of $J_0(r)$ and $A \approx 1.248459$. By applying the proposed method for solving this problem, after 6 iterations, we obtain $||e||_\infty = 5.2586e - 04$ (max error) and $e_{RMS} = 2.0659e - 04$ (RMS error) with $m = 31$ and $c = 0.4$. The numerical results are depicted Figure 1.

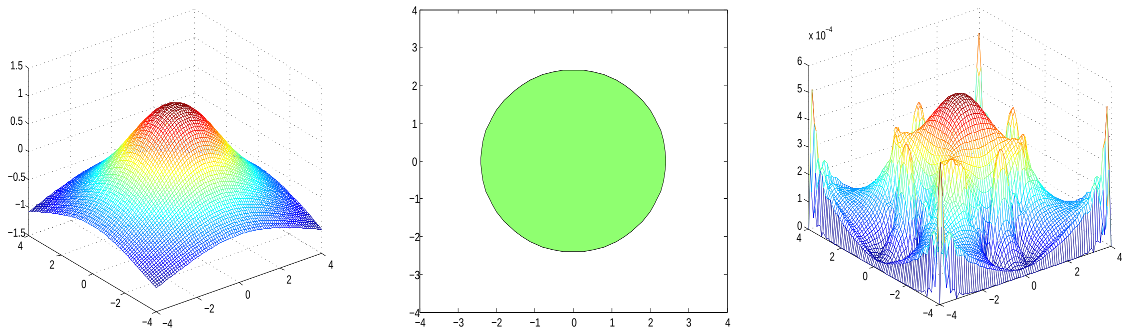

Figure 1: The numerical solution of problem (left), the level set of solution (middle), the error solution (right) for Example 3.1.

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A new method for Lane-Emden type equation in terms of shifted orthonormal Bernestein polynomial

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Abstract

In this paper, we introduce shifted orthonormal Bernstein polynomials (SOBPs) and drive the operational matrix of integration of these functions. Then, we apply Galerkin method with numerical integration to solve linear and nonlinear Lane-Emden type singular initial value problems (IVPs). The idea of obtaining our algorithm is essentially based on converting the differential equation with its initial conditions to a system of linear or nonlinear algebraic equations. Numerical results with comparison are given to confirm the validity, efficiency and applicability of the method.

Keywords: shifted orthonormal Bernstein polynomials , operational matrix, Galerkin method with numerical integration
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Recently, the studies on (IVPs) for second order ordinary differential equations (ODEs) have been the focus of considerable attention. One of the second order equations describing this type of problem is the Lane-Emden singular IVPs, which can be written in the form of

$$y''(x) + \frac{\alpha}{x} y'(x) + f(x, y) = g(x), \alpha \geq 0,\ 0 \leq x \leq L, \tag{1}$$

subject to initial conditions

$$y(0) = A, y'(0) = B, \tag{2}$$

where A and B are constants, $f(x, y)$ is a continuous real valued function, and $g(x) \in C[0, L]$. In this study, a new method based on SOBPs defined on the interval $[0, L]$ is developed for approximate solution of the nonlinear differential equations of Lane-Emden type. Recently, some other approximate solutions of Lane-Emden equations are obtained [1, 2].

*Speaker

2 Shifted Orthonormal Bernstein Polynomials

The explicit representation of the orthonormal Bernstein polynomials of mth degree are defined on the interval $[0,1]$ as

$$\psi_{j,m}(t) = \sqrt{2(m-j)+1}(1-t)^{m-j}\sum_{k=0}^{j}(-1)^k\binom{2m+1-k}{j-k}\binom{j}{k}t^{j-k},$$
$$j = 0, \cdots, m. \quad (3)$$

The shifted orthonormal Bernestein polynomials on $[0, L]$ can easily be obtained by using the transformation $t = \frac{x}{L}$ in (3)

$$\varphi_{i,m}(x) = \frac{1}{\sqrt{L}}\psi_{i,m}(\frac{x}{L}), \quad i = 0, \dots, m,$$

which are shifted orthonormal polynomials on $[0, L]$ respect to weight function $w(x) = 1$.

2.1 Expansion of SOBPs in Terms of Taylor Basis

By using Taylor expansion, $\varphi_{j,m}(x)$, $x \in [0, L]$ can be represented as

$$\varphi_{j,m}(x) = Z_{j+1}T_m(x), \quad j = 0, \dots, m,$$

where Z_{j+1} is a row vector of Taylor coefficients and

$$T_m(x) = [1, x, x^2, \dots, x^m]^T.$$

we denote by Z the matrix whose jth row is Z_j, $(j = 1, \cdots, m+1)$.

2.2 Function Approximation

Theorem 2.1. *For any $u \in L^2_\omega(I)$ and $m \in \mathbf{N}$, there exists a unique $q^*_m \in \mathbf{P}_m$ such that,*

$$\|u - q^*_m\|_{L^2_\omega} = \inf_{q_m \in \mathbf{P}_m}\|u - q_m\|_{L^2_\omega},$$

The SOBPs are orthogonal with respect to the weight function $\omega(x) = 1$ over $I = (0, L)$. Therefore, if f is an arbitrary element in $L^2(0, L)$, by theorem 1, f has the unique best approximation $\pi_m f$, such that

$$\pi_m f = \sum_{k=0}^{m} c_k\varphi_{k,m}, \quad c_k = (f, \varphi_{k,m}), \quad k = 0, \dots, m.$$

2.3 SOBPs Operational Matrix of Integration

Let P be the $(m+1)\times(m+1)$ operational matrix of integration, i.e.

$$\int_0^x \Phi(t)dt \simeq P\Phi(x), \quad 0 \le x \le L, \quad (4)$$

it can be obtained as

$$P = Z\Lambda B,$$

which $\Phi(x) = [\varphi_{0,m}(x), \ldots, \varphi_{m,m}(x)]^T$, and Λ and B can be expressed as follows

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots & \frac{1}{m+1} \end{bmatrix},$$

$$B = [Z_2^{-1}, Z_3^{-1}, \ldots, Z_{m+1}^{-1}, C_{m+1}^T]^T,$$

and

$$C_{m+1} = \int_0^L x^{m+1}\Phi(x)dx.$$

3 Description of method to solve Lane-Emden equations

Now, let us consider Lane-Emden equation (1) subject to the initial conditions (2). If we approximate $y(x)$, $f(x,y)$ and $g(x)$ by the SOBPs as

$$y''(x) \simeq C^T\Phi(x), \quad x \in [0, L], \tag{5}$$

integrating from 0 to x on both sides of (5) and using (4), and initial conditions (2) lead to

$$y'(x) \simeq C^T P\Phi(x) + B, \quad y(x) \simeq C^T P^2\Phi(x) + Bx + A, \quad x \in [0, L].$$

On the other hand, we have

$$f(x,y) \simeq f(x, C^T P^2\Phi(x) + Bx + A),$$

also, we expressed function $g(x)$ as

$$g(x) \simeq G^T\Phi(x),$$

where $C^T = [c_0, c_1, \ldots, c_m]$, and $G^T = [g_0, g_1, \ldots, g_m]$.
Using operational matrix of integration SOBP, the residual $R_m(x)$ for(1) can be written as

$$R_m(x) = C^T\Phi(x) + \frac{\alpha}{x}(C^T P\Phi(x) + B) + f(x, C^T P^2\Phi(x) + Bx + A) - G^T\Phi(x). \tag{6}$$

If we apply the Galerkin method with numerical integration, then (6) is reduce to $(m+1)$ linear or nonlinear equations, namely

$$\langle R_m(x), \varphi_{i,m}(x)\rangle_n = \sum_{j=0}^{n} R_m(x_j)\varphi_{i,m}(x_j)w_j, \quad i = 0, \ldots, m, \tag{7}$$

where $\{x_j, w_j\}_{j=0}^n$ being the set of shifted Legendre-Gauss quadrature nodes and weights. The system (7) can be solved with the aid of Newton's iteration method for the unknown components of vector C, and hence the approximate solution $\pi_m y(x)$ can be obtained.

Table 1: Comparison of $y(x)$, between present method and methods [3, 4], for Example 2

x	present method ($n = 20, m = 15$)	HFC [3] ($N = 30$)	Wazwaz [4]	Padé approximate [24,24]
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	-0.0016658339	-0.0016664188	-0.0016658339	-0.0016658339
0.2	-0.0066533671	-0.0066539713	-0.0066533671	-0.0066533671
0.5	-0.0411539573	-0.0411545150	-0.0411539573	-0.0411539573
1.0	-0.1588276775	-0.1588281737	-.15882767752	-0.1588276775
1.5	-0.3380194248	-0.3380198308	-0.3380194248	-0.3380194248
2.0	-0.5598230043	-0.5598233120	-0.5598230043	-0.5598230043
2.5	-0.8063408706	-0.8063410846	-0.8063408706	-0.8063408706

4 Numerical results

Example 1. We consider the isothermal gas spheres equation as follows

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \quad x \geq 0,$$

subject to the boundary conditions

$$y(0) = 0, \; y'(0) = 0.$$

This equation has been solved by [3]. We solve the equation with $m = 15, n = 20$. In Table 1 we lists a comparison between the values of $y(x)$ obtained by the present method and those obtained by Padé approximate and methods in [3, 4]. The results show that our approach is more accurate.

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Talk A non-standard finite difference method for HIV infection of CD4+T cells . . . pp.: 1–4

A non-standard finite difference method for HIV infection of CD4+T cells model

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Abstract

A dynamical model of HIV infection of CD4+T cells is solved numerically using a non-standard finite difference(NSFD) method. This new discrete system has the same properties as the continuous model. Through discrete Lyapunov function, the global asymptotical stability of the steady-state solution(when the basic reproduction number $R_0 \leq 1$)is determined. The Schur-Cohn criteria is used for local asymptotical stability of the steady-state solution, when $R_0 > 1$ as well. Finally, numerical simulations are provided to illustrate the theoretical results.

Keywords: non-standard finite difference method, asymptotical stability, Lyapunov function, basic reproduction number.
Mathematics Subject Classification [2010]: 65Q10, 65M06

1 Introduction

Consider the dynamic model for HIV infection of CD4+T cells[1]:

$$\begin{cases} \frac{\partial T}{\partial t} = \lambda - \alpha T + rT(1 - \frac{T+I}{T_{max}}) - kVT, \\ \frac{\partial I}{\partial t} = kVT - \beta I, \\ \frac{\partial V}{\partial t} = N\beta I - \gamma V. \end{cases} \quad ,T(0) \geq 0 \ ,I(0) \geq 0 \ ,V(0) \geq 0. \tag{1}$$

where $T(t)$, $I(t)$ and $V(t)$ denote concentration of uninfected, infected and viruse population of CD4+T cells by HIV in the blood. T_{max} is maximum level of CD4+T cells in the body, r is the rate at which T cells multiply through mitosis, λ is the constant rate which the body produces CD4+T cells, α is the death rate of CD4+T cells, β is the death rate of infected cells and γ is the death rate of virus particles. k is the rate of infection of T cells by virus and each infected CD4+ T cell is assumed to produce N virus particles during its life time. All the coefficients of Eq(1) are positive real numbers. Eq(1) has two steady states as follows:

$$E_0 = (T_0, I_0, V_0) = (\frac{T_{max}(r - \alpha + \sqrt{(\alpha - r)^2 + 4r\lambda/T_{max}}}{2r}, 0, 0)$$

$$E_1 = (T_1, I_1, V_1) = (\frac{\gamma}{kN}, \frac{\frac{\lambda kN}{\gamma} + r - \alpha - \frac{r\gamma}{T_{max}kN}}{\frac{r}{T_{max}} + \frac{kN\beta}{\gamma}}, \frac{N\beta}{\gamma} I_1).$$

*Speaker

and the basic reproduction number of infection is given by $R_0 = \frac{kNT_0}{\gamma}$ [2] . This type of models has been considered by several researchers [3, 4]. This paper is organized as follows: In section 2 an NSFD method will be develope for Eq(1) and its steady-states are introduced. The asymptotical stabilities of the steady state solutions are analysed in section 3. In section 4 we will present some numerical simulations.

2 Discretization of the Model

For discretization of Eq(1), consider uniform step size $\Delta t = h$ on t axis. Notationally T_n, I_n, and V_n will be approximate $T(t), I(t)$, and $V(t)$ at nh. With this notation we propose the following NSFD method for Eq(1):

$$\begin{cases} \frac{T_{n+1}-T_n}{\phi(h)} = \lambda - \alpha T_{n+1} + rT_n - r\frac{T_nT_{n+1}}{T_{max}} - r\frac{T_{n+1}I_n}{T_{max}} - kV_nT_{n+1} \\ \frac{I_{n+1}-I_n}{\phi(h)} = kV_nT_{n+1} - \beta I_{n+1} \\ \frac{V_{n+1}-V_n}{\phi(h)} = N\beta I_{n+1} - \gamma V_{n+1} \end{cases} \tag{2}$$

where $\phi(h) = h + O(h^2)$. It is easy to chek that Eq(2) has also the steady state solutions $E_0,\ E_1$. In the next thorem, we want to show positivity of the solutions.

Theorem 2.1. *For arbitary $h > 0$, the solution of Eq(2) satisfies $T_n \geq 0, I_n \geq 0$, and $V_n \geq 0$ for all $n \in N$.*

Proof. The Eq(2) is equivalent to

$$T_{n+1} = \frac{\lambda\phi(h) + (1 + r\phi(h))T_n}{1 + \alpha\phi(h) + r\phi(h)\frac{T_n+I_n}{T_{max}} + k\phi(h)V_n},\ I_{n+1} = \frac{I_n + k\phi(h)V_nT_{n+1}}{1 + \beta\phi(h)}\ ,\ V_{n+1} = \frac{N\beta\phi(h)I_{n+1} + V_n}{1 + \gamma\phi(h)}. \tag{3}$$

For $n = 0$, we have $T_0 \geq 0, I_0 \geq 0$, and $V_0 \geq 0$. Assume that $T_n \geq 0, I_n \geq 0$ and $V_n \geq 0$. Then Eq(3) implies $T_{n+1} \geq 0, I_{n+1} \geq 0$ and $V_{n+1} \geq 0$. □

3 Stability Analysis of the Model

Theorem 3.1. *for arbitary $h > 0$, if $R_0 \leq 1$, then steady state E_0 is globally asymptotically stable.*

Proof. Consider the Lyapunov function $L_n = \frac{1}{\phi(h)}\big[NI_n + (1+\gamma\phi(h))V_n\big]$. It is easy to see that$L_n \geq 0$ and equality is obtained only in E_0. However $L_{n+1} - L_n = NkV_nT_{n+1} - V_n = V_n(NkT_{n+1} - \gamma) = \gamma V_n(R_0\frac{T_{n+1}}{T_0} - 1)$. For all n we have $T_n \leq T_0$[1]. Hence for $R_0 \leq 1$, we will deduce that $L_{n+1} - L_n \leq 0$. It means that $\{L_n\}$ is a monotone decreasing sequance and there exist a constant $\bar{L} \geq 0$ such that $\lim_{n\to\infty} L_n = \bar{L}$. Thus $\lim_{n\to\infty}(L_{n+1} - L_n) = 0$. Therefor we have the following conclusion:
(i)If $R_0 < 1$, then $\lim_{n\to\infty} V_n = 0$ and via the first and second equation in Eq(2), $\lim_{n\to\infty} T_n = T_0$ and $\lim_{n\to\infty} I_n = 0$.
(ii) If $R_0 = 1$ we have $\lim_{n\to\infty} T_n = T_0$ or $\lim_{n\to\infty} V_n = 0$ that both imply previous results. □

To investigate the infection when $R_0 > 1$, we examine the local stability of E_1. The Jacobian matrix of Eq(3) is defined as $M = \frac{\partial(T_{n+1}, I_{n+1}, V_{n+1})}{\partial(T_n, I_n, V_n)}$. Let $P(s)$ be the characteristic polynomial of M.

Lemma 3.2. *(Jury condition, Schur-Cohn criteria, n=3) Suppose the characteristic polynomail* $P(s) = s^3 + p_1s^2 + p_2s + p_3$ *is given. The solutions* $s_i, i = 1, 2, 3$ *of* $P(s)$*satisfy* $|s_i| < 1$ *if the following three conditions are held:(i)* $P(1) > 0$*. (ii)* $(-1)^3P(-1) > 0$*. (iii)* $1 - (p_3)^2 > |p_2 - p_3p_1|$.

Theorem 3.3. *Suppose that* $P(s) = s^3 + p_1s^2 + p_2s + p_3$ *is the characteristic polynomail* M*. If* $R_0 > 1$ *and* $1-(p_3)^2 > |p_2-p_3p_1|$*, then steady state* E_1 *will be locally asymptotically stable.*

Proof. According to the linearized stability theorem, If all the roots of the characteristic polynomial have absolute values less than one, then the equilibrium point $E_1$is locally asymptotically stable. Hence we must investigate if conditions in above lemma are satisfied. The first condition for M is equivalent to $N^2T_{max}k^2\lambda - NT_{max}\alpha k\gamma + NT_{max}k\gamma r - \gamma^2 r > 0$. By definition R_0, it is equivalent to $R_0 > 1$. In the scond condition $\rho(r) = Ar^2 + Br + C$ must be positive, where

$$\begin{aligned}
A =\ & \gamma^2\phi(3NT_{max}\beta k\gamma\phi^2 + 2NT_{max}\beta k\phi + 4NT_{max}k\gamma\phi - \beta\gamma^2\phi^2 + 8NT_{max}k - 2\gamma^2\phi - 4\gamma)\\
B =\ & NT_{max}k\gamma(3NT_{max}\beta^2k\gamma\phi^3 + N\beta k\lambda\gamma\phi^3 + 4NT_{max}\beta^2k\phi^2 + 4NT_{max}\beta k\gamma\phi^2\\
& +\alpha\beta\gamma^2\phi^3 - \beta^2\gamma^2\phi^3 + 2Nk\lambda\gamma\phi^2 + 8NT_{max}\beta k\phi + 2\alpha\beta\gamma\phi^2 - 2\beta^2\gamma\phi^2 + 2\beta\gamma^2\phi^2 + 4Nk\lambda\phi\\
& +4\gamma^2\phi + 8\gamma),\\
C =\ & N^2T_{max}^2\beta k^2(N\beta k\lambda\gamma\phi^3 + \alpha\beta\gamma^2\phi^3 + 2N\beta k\lambda\phi^2 + 2Nk\lambda\gamma\phi^2 + 4\beta\gamma^2\phi^2\\
& +4Nk\lambda\phi + 4\beta\gamma\phi + 4\gamma^2\phi + 8\gamma).
\end{aligned}$$

Above coefficients are positive, becouse T_{max} is much larger than others. However, the scond condition is satisfied and for locally asymptotically stable state, only the third condintion in above lemma must be satisfied and this concludes the proof. □

4 numerical simulations

In this section, the results are checked by doing numerical simulations. For this, we used the *maple*17 and *matlabR*2010*a* software. Conside the initial conditions as follows[1]:

$$T(0) = 0.1, I(0) = 0, V(0) = 0.1, \gamma = 2.4, k = 0.0027,$$
$$\lambda = 0.1, \alpha = 0.02, \beta = 0.3, T_{max} = 1500.$$

If $r = 0.001$, then the basic reproduction number will be $R_0 = 0.0591$. Theorem (3.1), proves that for $R_0 \leq 1$, the steady state E_0 of Eq(2) is globally asymptotically stable. That is , the disease will die out. Figure 1 presents the graph of numerical solution connected to $T(t)$ with $\phi(h) = h = 0.5$. Now if we consider $r > 0.021$, then $R_0 > 1$.Theorem (3.3) shows that if $R_0 > 1$, the steady state E_1 of the NSFD method Eq(2) is locally asymptotically stable if $1 - (p_3)^2 > |p_2 - p_3p_1|$ is true. Now if $\phi(h) = 0.05$, then E_1 will be locally asymptotically stable for $0.021 \leq r < 0.10002$ or $r > 1.7408$. Figure 2 displays the graphs of numerical solutions connected to $T(t),\ V(t)$ when $r = 4,\ 0.05$.

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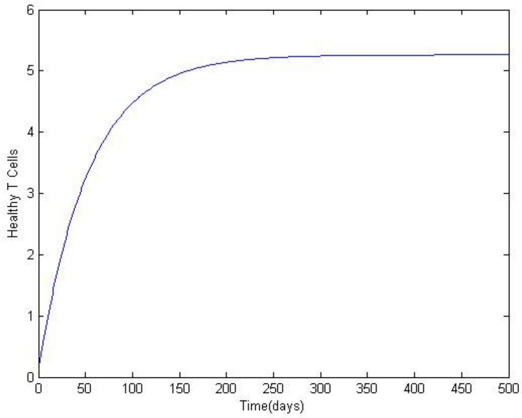

Figure 1: $r = 0.001, R_0 = 0.0591, \phi(h) = h = 0.5.$

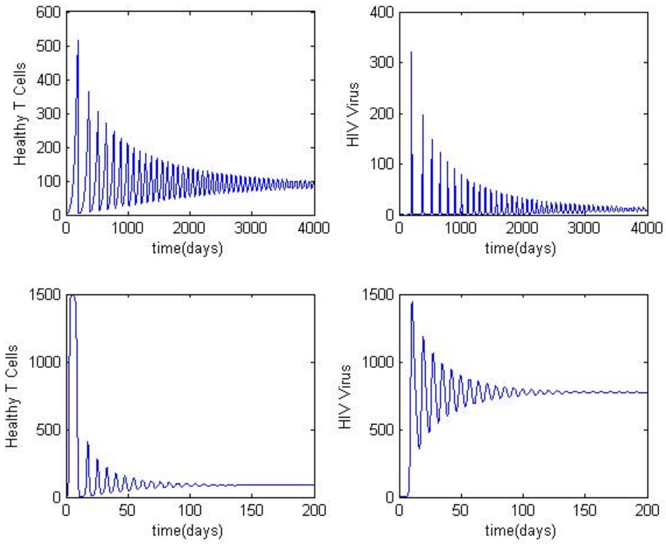

Figure 2: when $r = 0.05, R_0 = 10.1623(top)$, $r = 4, R_0 = 16.790(down)$.

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A numerical study for the MHD Jeffery-Hamel problem based on orthogonal Bernstein polynomials

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Abstract

In this investigation, a collocation method based on orthogonal Bernstein polynomials for solving MHD Jeffery-Hamel problem is introduced. The validity of the proposed method is ascertained by comparing our results with fourth-order Runge-Kutta method (RK4) results.

Keywords: Orthogonal Bernstein polynomials, Jeffery-Hamel flows, Fluid mechanics
Mathematics Subject Classification [2010]: 34B15, 76A10

1 Introduction

The problem of an incompressible, viscous fluid between nonparallel walls, commonly known as the Jeffery-Hamel flow, is an example of one of the most applicable type of flows in fluid mechanics [1]. Consequently, this problem has been well studied in literature, see for example, [2, 3].The classical Jeffery-Hamel problem was extended in [4] to include the effects of an external magnetic field on an electrically conducting fluid. In this study, we are going to introduce and implement a collocation method based on orthogonal Bernstein polynomials [5] to find the approximate solution of the MHD Jeffery-Hamel problem.

2 Mathematical formulation

Consider the steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at the intersection between two rigid plane walls, where the angle between them is 2α as shown in Fig. 1. We assume that the velocity is only along the radial direction and depends on r and θ,$V(u(r,\theta),0)$ [1]. Using continuity and the Navier-Stokes equations in polar coordinates,

$$\frac{\rho\partial}{r\partial r}(ru(r,\theta)) = 0, \tag{1}$$

*Speaker

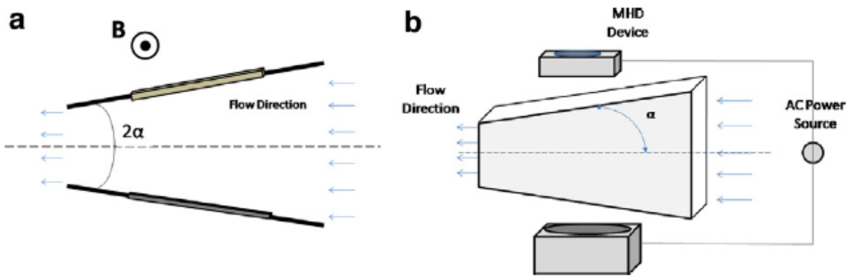

Figure 1: Geometry of the MHD Jeffery-Hamel flow in convergent cannel; (a) 2-D view and (b) Schematic setup of problem.

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + v\left[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}\right], \quad (2)$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{2v}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0. \quad (3)$$

The continuity equation (1) implies that,

$$f(\theta) \equiv ru(r,\theta). \quad (4)$$

Using dimensionless parameters,

$$F(x) \equiv \frac{f(\theta)}{f_{max}}, \quad x \equiv \frac{\theta}{\alpha} \quad (5)$$

and eliminating p between (2) and (3), we obtain an ordinary differential equation for the normalized function profile $F(x)$ as:

$$F'''(x) + 2\alpha ReF(x)F'(x) + (4-H)\alpha^2 F'(x) = 0. \quad (6)$$

Since we have a symmetric geometry, the boundary conditions will be

$$F(0) = 1, \quad F'(0) = 0, \quad F(1) = 0. \quad (7)$$

The Reynolds number is

$$Re \equiv \frac{f_{max}\alpha}{v} = \frac{U_{max} r\alpha}{v}\begin{pmatrix} divergent\ channel : \alpha > 0, U_{max} > 0 \\ convergent\ channel : \alpha < 0, U_{max} < 0 \end{pmatrix}. \quad (8)$$

3 Solution of the problem

In this section, we apply the orthogonal Bernstein collocation method (OBCM) to find solutions for MHD Jeffery-Hamel problem (6) which satisfy the boundary conditions (7). The orthogonal Bernstein polynomials are defined on the interval $[0,1]$ by [5]:

$$\phi_{j,n}(x) = \left(\sqrt{2(n-j)+1}\right)(1-x)^{n-j}\sum_{k=0}^{j}(-1)^k\binom{2n+1-k}{j-k}\binom{j}{k}x^{j-k}. \quad (9)$$

These polynomials can be written in a simpler form in terms of the original non-orthogonal Bernstein basis functions as:

$$\phi_{j,n}(x) = \sqrt{2(n-j)+1}\ \sum_{k=0}^{j}(-1)^k \frac{\binom{2n+1-k}{j-k}\binom{j}{k}}{\binom{n-k}{j-k}} B_{j-k,n-k}(x) \tag{10}$$

where $B_{j,n}(x),\ j = 0, 1, \ldots, n$ are Bernstein polynomials as follows:

$$B_{j,n}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad j = 0, 1, \ldots, n. \tag{11}$$

Let the unknown function $F(x)$ be approximated by a truncated series of orthogonal Bernstein polynomials as:

$$F(x) \simeq F_n(x) = \sum_{j=0}^{n} f_j \phi_{j,n}(x). \tag{12}$$

Then, we construct the residual function by substituting $F(x)$ by $F_n(x)$ in the equation (6):

$$RES(x) = F_n'''(x) + 2\alpha Re F_n(x) F_n'(x) + (4-H)\alpha^2 F_n'(x), \tag{13}$$

The equations for obtaining the coefficients f_is come from equalizing $RES(x)$ to zero at collocation points x_i $i = 0, 1, \ldots n-3$ plus three boundary conditions as follows:

$$RES(x_i) = 0, \quad i = 0, 1, \ldots n-3, \tag{14}$$

$$F_n(0) = 1, \quad F_n'(0) = 0, \quad F_n(1) = 0, \tag{15}$$

where

$$x_i = \frac{1}{2}\left(1 + \cos\left(\frac{(2i+1)\pi}{2n-4}\right)\right), \quad i = 0, 1, \ldots n-3. \tag{16}$$

Equations (14) and (15) generate a set of $n+1$ nonlinear equations that can be solved by Newton's method for the unknown coefficients f_is.

4 Numerical Results

Table 1 shows the numerical data for $F(x)$ using DTM, HPM, HAM [6] and numerical Rung-Kutta method for validity of the presented method (OBCM) with $n = 30$ when $\alpha = 3°, Re = 110$ and $H = 0$. Fig. 2 display the effects of Reynolds number Re and steep angle α of the channel on velocity profile of fluid.

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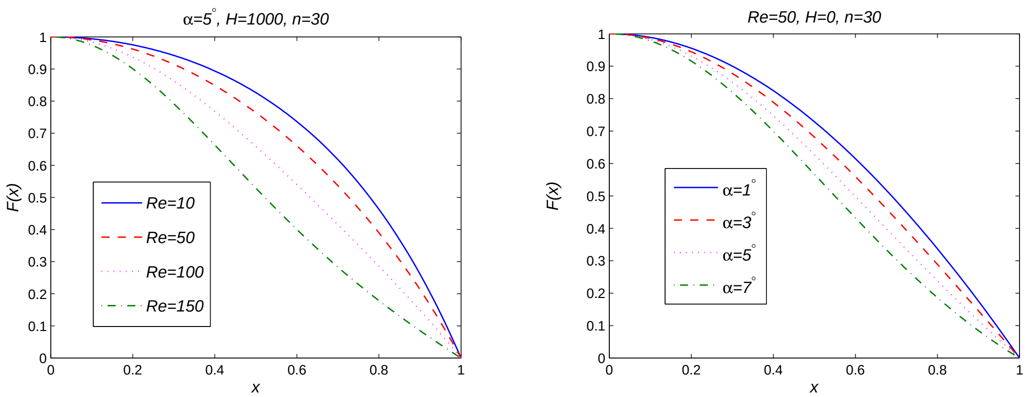
Table 1: Comparison of the numerical results when $\alpha = 3°, Re = 110$ and $H = 0$.

x	DTM[6]	HPM[6]	HAM[6]	OBCM	Numerical
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9789771156	0.9791761778	0.9792357062	0.9792357065	0.9792357065
0.2	0.9182598446	0.9190424983	0.9192658842	0.9192658855	0.9192658855
0.3	0.8243664466	0.8260939720	0.8265336102	0.8265336122	0.8265336122
0.4	0.7065763476	0.7096036928	0.7102211838	0.7102211832	0.7102211832
0.5	0.5751498602	0.5798357741	0.5804994700	0.5804994588	0.5804994588
0.6	0.4397114086	0.4463900333	0.4469350941	0.4469350670	0.4469350670
0.7	0.3081560927	0.3170877938	0.3174084545	0.3174084275	0.3174084275
0.8	0.1862239095	0.1975366451	0.1976410661	0.1976410945	0.1976410945
0.9	0.0784362201	0.0912421454	0.0912302288	0.0912304211	0.0912304211
1.0	0.0000000015	0.0000000007	0.0000004700	0.0000000000	0.0000000000



(a) (b)

Figure 2: Velocity diagram via OBCM for different values of Re (a) and velocity diagram via OBCM for different values of α (b).

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Talk A preconditioned method for approximating the generalized inverse of large . . . pp.: 1–4

A preconditioned method for approximating the generalized inverse of large matrices

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Abstract

In this paper, the preconditioned global least squares algorithm is applied to approximate the generalized inverse of nearly singular or rectangular matrices. This preconditioner is based on the C-orthogonalization process, where C a symmetric positive definite matrix. Finally, some numerical experiments are given to illustrate the efficiency of the new preconditioner.

Keywords: preconditioner, matrix equation, GL-LSQR algorithm, pseudo-inverse.
Mathematics Subject Classification [2010]: 65F08, 65F10

1 Introduction

Throughout this paper, the following notations are used. $||.||_F$ denotes the Frobenius norm. We define the C-inner product by $(x, y)_C =< Cx, y >_2$ where C is symmetric positive definite and $x, y \in \mathbb{R}^n$.

Consider the following matrix equation

$$AXA = A. \tag{1}$$

The solution of (1) is called the generalized inverse of A. The main objective of this paper is computing the generalized inverses of nearly singular matrices and rectangular full rank matrices.

Usually, iterative methods are applied to solve matrix equations with the large and sparse coefcient matrices. Sometimes, these iterative methods may fail or have a low convergence rate. To overcome this problem, one can use an appropriate preconditioner. Recently [3], Toutounian and Karimi proposed the global least squares (GL-LSQR) method for obtaining the approximate solution of matrix equation $AX = B$. Their method is a global version of least squares method for solving linear system of equations with multiple

*Speaker

right hand sides.
In this paper, we present a new preconditioning technique to find the approximate generalized inverses of nearly singular matrices and rectangular matrices by using the GL-LSQR algorithm. This preconditioner is based on the C-orthogonalization, where C is a symmetric positive-denite matrix.

2 The preconditoning technique

In this section, our main goal is to present a right preconditioner for the GL-LSQR algorithm, denoted by R-PGLS, to solve the matrix equation (1). This preconditioner is based on the C-inner product, where C is a symmetric positive matrix. We apply the GL-LSQR algorithm 2 of [1] to the transformed matrix equation

$$ARYA = A, \qquad X = RY, \tag{2}$$

where R is the inverse factor of the upper factorization $(A^TA)^{-1} = RR^T$. We demonstrate that the incomplete inverse factor R can be implemented as a preconditioner.

More recently in [2], Karimi et al. presented a block preconditioner for the block partitioned matrices. They used the incomplete inverse factor $\hat{R}$ of A^TA as a right preconditioner for the GL-LSQR algorithm for solving the partitioned matrix equations.

Now we want to use the approximate inverse factor $\hat{R}$ of A^TA as a right preconditioner for the GL-LSQR algorithm to solve (1). We let $C \in \mathbb{R}^{n\times n}$ be a SPD matrix. In the following we find the inverse factor of C. By using the set of unit basis vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$, where e_j is jth column of the identity matrix of order n, we can construct a C-orthogonal set of vectors $z_1, z_2, \ldots, z_n \in \mathbb{R}^n$ by conjugate Gram-Schmidt with respect to the C-inner product (3). Written as a modified Gram-Schmidt process, the algorithm starts by setting $z_j = e_j$, for $j = 1, 2, \ldots, s$ and then performs the following nested loop:

$$z_i \longleftarrow z_i - \frac{(z_i, z_j)_C}{(z_j, z_j)_C} z_j, \quad j = 1, 2, \ldots, n-1 \quad i = j+1, \ldots, n. \tag{3}$$

Let $Z = [z_1, z_2, \ldots, z_n]$ and $D = diag(d_1, \ldots, d_n)$, where $d_j = (z_j, z_j)_C$, $j = 1, 2, \ldots, n$, we obtain the inverse upper-lower factorization

$$C^{-1} = ZD^{-1}Z^T. \tag{4}$$

Since D is a diagonal matrix with positive diagonal elements, we can define $R = ZD^{-\frac{1}{2}}$. So we have the inverse upper-lower factorization $C^{-1} = RR^T$.
An inverse approximate factorization $C^{-1} \approx \hat{R}\hat{R}^T$, can be obtained by carrying out the updates in the process (3) incompletely. Given a dropping tolerance $0 < \tau < 1$, the entries of z_i are scanned after each update and entries that are smaller than τ in absolute value are discarded. We denote $\hat{z_i}$ the sparse of z_i and by setting

$$\hat{Z} = [\hat{z_1}, \hat{z_2}, , \cdots, \hat{z_n}],$$

we have $\hat{R} = \hat{Z}D^{-\frac{1}{2}}$ as the incomplete inverse factor of C.
Therefore, the C-orthogonalization algorithm can be summarized as follows.

C-orthogonalization process

1. Let $z_j = e_j, j = 1, 2, \cdots, s$
2. For $j = 1, 2, \cdots, s-1$ Do
3. For $i = j+1, j+2, \cdots, s$ Do
4. $z_i = z_i - \frac{(z_i, z_j)_C}{(z_j, z_j)_C} z_j$
5. Use a dropping strategy for the elements of the vector z_i
6. EndDo
7. EndDo

Now we consider matrix equation (1) and suppose that A be full column rank matrix. So $A^T A$ is SPD matrix and by taking $C = A^T A$ in the above C-orthogonalization process, we can obtain the incomplete inverse factor $\hat{R}$ of $A^T A$. We apply this inverse factor as a right preconditioner and present the right preconditioned GL-LSQR algorithm, namely R-PGLS algorithm. The main steps of this algorithm are as follows.

R-PGLS algorithm

1. Compute approximate inverse factor $\hat{R}$ of $A^T A$ by using C-orthogonalization process
2. Set $Y_0 = 0$
3. $\beta_1 = \|A\|_F, \quad U_1 = \frac{A}{\beta_1}$
4. $Q_1 = A^T U_1 A^T, \quad \alpha_1 = \|\hat{R}^T Q_1\|_F$
5. $V_1 = \frac{\hat{R}^T Q_1}{\alpha_1}$
6. Set $W_1 = V_1$, $\bar{\phi}_1 = \beta_1$, $\bar{\rho}_1 = \alpha_1$
7. For $i = 1, 2, \cdots$ until convergence, Do
8. $P_i = \hat{R} V_i \hat{R}_B^T$
9. $\overline{W}_i = A P_i A - \alpha_i U_i$
10. $\beta_{i+1} = \|\overline{W}_i\|_F$
11. $U_{i+1} = \frac{\overline{W}_i}{\beta_{i+1}}$
12. $Q_{i+1} = A^T U_{i+1} A^T$
13. $\overline{S}_i = \hat{R}^T Q_{i+1} - \beta_{i+1} V_i$
14. $\alpha_{i+1} = \|\overline{S}_i\|_F$
15. $V_{i+1} = \frac{\overline{S}_i}{\alpha_{i+1}}$
16. $\rho_i = \sqrt{\overline{\rho}_i^2 + \beta_{i+1}^2}$,
17. $c_i = \frac{\overline{\rho}_i}{\rho_i}$
18. $s_i = \frac{\beta_{i+1}}{\rho_i}$
19. $\theta_{i+1} = s_i \alpha_{i+1}$
20. $\overline{\rho}_{i+1} = -c_i \alpha_{i+1}$
21. $\phi_i = c_i \overline{\phi}_i$
22. $\overline{\phi}_{i+1} = s_i \overline{\phi}_i$
23. $Y_i = Y_{i-1} + \frac{\phi_i}{\rho_i} W_i$
24. $W_{i+1} = V_{i+1} - \frac{\theta_{i+1}}{\rho_i} W_i$
25. If $|\overline{\phi}_{i+1}|$ is small enough then compute $X_i = \hat{R} Y_i$ as a approximate solution
26. EndDo

For more details about R-PGLS algorithm, One can refer to [1].

3 Numerical results

In this section, For the numerical experiment, we use two general matrices $WELL1013$ and $PDE225$ from Harwell-Boeing collection [4]. We apply the GL-LSQR and R-PGLS algorithms for solving the linear matrix equation (1), where the coefficient matrix $A = WELL1033, rand(700, 500)$ and A_1, where 225 by 100 matrix A_1 is the same $PDE225$ in which the last 125 columns are removed. We compare both algorithms in terms of number of iterations. In this examples, the initial iteration matrix is zero and the algorithm stops when the current iterate satisfies

$$RError = \frac{||R_k||_F}{||R_0||_F} \leq \epsilon,$$

where R_k is the residual of the kth iterate and ε is a proper stopping tolerance. We applied the GL-LSQR and R-PGLS algorithms for solving this problem, the results are shown in Figure 1. As we see from this figure , the GL-LSQR algorithm (middle and right figures) does not converge or stagnates while the R-PGLS algorithm converges very fast and this shows that the R-PGLS algorithm is clearly superior. Note that in this example, we have taken the drop tolerance $\tau = 10^{-4}$, for discarding the entries of $\hat{z}_i$ that are smaller than τ in absolute value. However, if one takes $\tau = 0.1, 0.01, 0.001$ then the R-PGLS algorithm will converge in more iterations. For example, if we take $\tau = 0.1, 0.01, 0.001$ for $A = WELL1033$ then the R-PGLS algorithm will stop after 4142, 1372 and 810 iterations, respectively (with almost RError=9.62e-15).

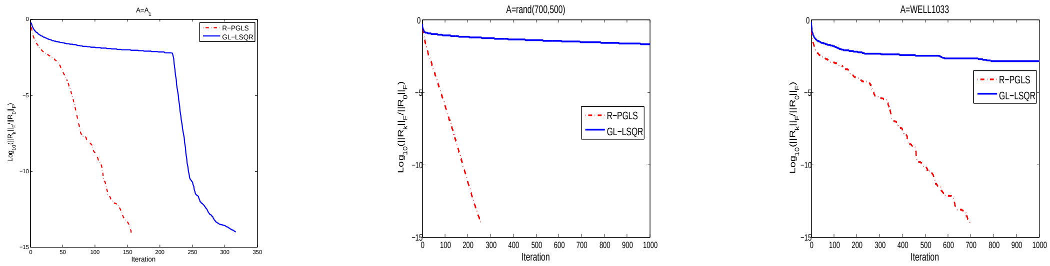

Figure 1: Convergence history of R-PGLS algorithm versus GL-LSQR algorithm.

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Talk A preconditioner based on the shift-splitting method for generalized saddle . . . pp.: 1–4

A preconditioner based on the shift-splitting method for generalized saddle point problems

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Abstract

In this paper, we propose a preconditioner based on the shift-splitting method for generalized saddle point problems with nonsymmetric positive definite (1,1)-block and symmetric positive semidefinite $(2, 2)$-block. The proposed preconditioner is obtained from an basic iterative method which is unconditionally convergent. We also present a relaxed version of the proposed method. Some numerical experiments are presented to show the effectiveness of the method.

Keywords: Generalized saddle point, preconditioner, shift-splitting, Navier-Stokes.
Mathematics Subject Classification [2010]: 65F10, 65F50

1 Introduction

We consider the solution of the following large and sparse generalized saddle point problem

$$\mathcal{A}u = \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsymmetric positive definite ($x^T Ax > 0$ for all $0 \neq x \in \mathbb{R}^n$), $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, the matrix $B \in \mathbb{R}^{m\times n}$ is of full row rank, $x, f \in \mathbb{R}^n$, $y, g \in \mathbb{R}^m$ and $m \leq n$. It can be verified that the system (1) has a unique solution [1, Lemma 1.1]. Saddle point problems of the form (1) arise from finite difference or finite element discretization of the Navier-Stokes problem (see [2] and references therein). Several iterative method have been presented to solve system (1) or some special cases of it in the literature. The main methods have been reviewed in [2]. In [1], Benzi and Golub presented the Hermitian and skew-Hermitian splitting (HSS) method to solve (1). Since, in general, the HSS method is too slow to be used to solve (1), they used the GMRES method in conjunction with the preconditioner extracted from the HSS method to solve (1). Recently, when the matrix A is symmetric positive definite, Salkuyeh et al. in [6] have presented a stationary iterative method based on the shift-splitting method to solve (1). The proposed method naturally serves a preconditioner for the problem (1). More recently, Cao et al. in [3] have considered the same iterative method to solve the system (1) when $C = 0$. In this paper, we consider the problem (1) in its general form and investigate the convergence properties of the proposed iterative method and the corresponding preconditioner.

*Speaker

2 Main results

For the sake of the simplicity we use the nations used in [6]. Assuming $\alpha, \beta > 0$, Salkuyeh et al. in [6] considered the splitting $\mathcal{A} = \mathcal{M}_{\alpha,\beta} - \mathcal{N}_{\alpha,\beta}$ for the saddle point problem (1) with A being symmetric positive definite, where

$$\mathcal{M}_{\alpha,\beta} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B^T \\ -B & \beta I + C \end{pmatrix} \quad \text{and} \quad \mathcal{N}_{\alpha,\beta} = \frac{1}{2}\begin{pmatrix} \alpha I - A & -B^T \\ B & \beta I - C \end{pmatrix}. \tag{2}$$

This splitting gives the following basic iterative method (hereafter is denoted by the MGSS iteration scheme)

$$\mathcal{M}_{\alpha,\beta} u^{(k+1)} = \mathcal{N}_{\alpha,\beta} u^{(k)} + b \tag{3}$$

for solving the linear system (1), where $u^{(0)}$ is an initial guess. In continuation, we show that the symmetry of the matrix A can be omitted. From Eq. (3), we see that the iteration matrix of the proposed method is $\Gamma_{\alpha,\beta} = \mathcal{M}_{\alpha,\beta}^{-1}\mathcal{N}_{\alpha,\beta}$. Hence, the method is convergent if and only if the spectral radius of $\Gamma_{\alpha,\beta}$ is less than 1, i.e., $\rho(\Gamma_{\alpha,\beta}) < 1$.

Lemma 2.1. *([6, Lemma 1]) Assume that α and β are two positive numbers. If λ is an eigenvalue of the matrix $\Gamma_{\alpha,\beta}$, then $\lambda \neq \pm 1$.*

Lemma 2.2. *Let $A \in \mathbb{R}^n$ be a nonsymmetric positive definite matrix. Then, $\Re(x^*Ax) > 0$, for any $0 \neq x \in \mathbb{C}^n$.*

Proof. Let $x = r + is$, where $r, s \in \mathbb{R}^n$. Obviously, both of the vectors r and s can not be zero simultaneously. On the other hand,

$$x^*Ax = (r^T - is^T)A(r + is) = r^TAr + s^TAs + ir^T(A - A^T)s.$$

Hence, $\Re(x^*Ax) = r^TAr + s^TAs > 0$. □

Theorem 2.3. *Let λ be an eigenvalue of the matrix Γ and $\alpha, \beta > 0$. Then $|\lambda| < 1$.*

Proof. Let $u = (x; y)$ be an eigenvector corresponding to the eigenvalue λ of $\Gamma_{\alpha,\beta}$. Then, we have $\mathcal{N}_{\alpha,\beta}u = \lambda\mathcal{M}_{\alpha,\beta}u$ which is equivalent to

$$(\alpha I - A)x - B^Ty = \lambda(\alpha I + A)x + \lambda B^Ty, \tag{4}$$

$$Bx + (\beta I - C)y = -\lambda Bx + \lambda(\beta I + C)y. \tag{5}$$

According to Theorem 1 in [6] we have $x \neq 0$.

Without loss of generality it is assumed that $\|x\|_2 = 1$. Premultiplying both sides of (4) by x^* yields

$$\alpha - x^*Ax - (Bx)^*y = \lambda(\alpha\|x\|_2^2 + x^*Ax) + \lambda(Bx)^*y. \tag{6}$$

Since A is positive definite, according to Lemma 2.2 we have $\Re(x^*Ax) > 0$. If $Bx = 0$, then Eq. (6) implies

$$|\lambda| = \frac{|\alpha - x^*Ax|}{|\alpha + x^*Ax|} = \frac{\sqrt{(\alpha - \Re(x^*Ax))^2 + (\Im(x^*Ax))^2}}{\sqrt{(\alpha + \Re(x^*Ax))^2 + (\Im(x^*Ax))^2}} < 1.$$

We now assume that $Bx \neq 0$. In this case, from Eq. (5) we obtain

$$Bx = \frac{\beta(\lambda - 1)}{\lambda + 1} y + Cy. \tag{7}$$

Substituting Eq. (7) in (6) yields

$$(1 - \lambda)\alpha - (1 + \lambda)x^* Ax = (1 + \lambda)\left(\beta \frac{\overline{\lambda} - 1}{1 + \overline{\lambda}} y^* y + y^* Cy\right).$$

Letting $p = x^* Ax$, $q = y^* y$, and $r = y^* Cy$, it follows from the latter equation that

$$\alpha\omega + \beta q\overline{\omega} = p + r, \quad \text{with} \quad \omega = \frac{1 - \lambda}{1 + \lambda}. \tag{8}$$

Since $\alpha, \beta, \Re(p) > 0$ and $q, r \geq 0$, form (8) we see that

$$\Re(w) = \frac{\Re(p) + r}{\alpha + \beta q} > 0.$$

Hence, we have

$$|\lambda| = \frac{|1 - \omega|}{|1 + \omega|} = \sqrt{\frac{(1 - \Re(\omega))^2 + \Im(\omega)^2}{(1 + \Re(\omega))^2 + \Im(\omega)^2}} < 1,$$

which completes the proof. □

Theorem 2.3 shows that the MGSS method is convergent and therefore it provides the preconditioner $\mathcal{P}_{MGSS} = \mathcal{M}_{\alpha,\beta}$ for a Krylov subspace method such as GMRES, or its restarted version GMRES(m) for solving the saddle point problem (1). Implementation of the method is as described in [6]. We can also use a relaxed version of the MGSS (say RMGSS) preconditioner

$$\mathcal{P}_{RMGSS} = \begin{pmatrix} A & B^T \\ -B & \beta I + C \end{pmatrix}.$$

for the saddle point problem (1). Similar to Theorem 2 in [6] one may discuss about the eigenvalues distribution of the coefficient matrix of the preconditioned system.

3 Numerical experiments

We consider the steady-state Navier-Stokes equation

$$\begin{cases} -\nu\triangle\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla.\mathbf{u} &= 0, \end{cases} \quad \text{in} \quad \Omega = [0,1] \times [0,1],$$

where $\nu > 0$. By the IFISS package [4], this problem is linearized by the Picard iteration and then discretized by using the stabilized Q1-P0 finite elements (see [5]). The stabilization parameter is set to be 0.1. This yields a generalized saddle point problem of the form (1). The right-hand side vectors f and g are taken such that x and y are two vectors of all ones. In Table 1, the generic properties of the coefficient matrix have been given.

We use GMRES(30) in conjunction with the preconditioner $\mathcal{P}_{MGSS}$ to solve the saddle point problem (1). Numerical results are given in Table 1. In this table "Iters" and

Table 1: Numerical results for the test problem with $\nu = 1/50$.

			GMRES(30)		MGSS	
Grid	m	n	Iters	CPU(s)	Iters	CPU(s)
8×8	162	62	59	0.08	6	0.90
16×16	578	256	115	0.28	8	2.02
32×32	2178	1024	608	4.95	12	3.58
64×64	8450	4096	3554	110.1	28	21.48

"CPU" stand, respectively, for the number of iterations and the CPU time (in seconds) for the convergence. To show the effectiveness of the methods we also give the results of GMRES(30) without preconditioning. We use a null vector as an initial guess and the stopping criterion $\|b - \mathcal{A}x^{(k)}\|_2 < 10^{-9}\|b\|_2$. In the implementation of the preconditioner $\mathcal{P}_{MGSS}$ (see Algorithm 1 in [6]), we use the Cholesky factorization of $\beta I + C$ and the GMRES(10) method to solve the inner systems. It is noted that, the inner iteration is terminated when the residual norm is reduced by a factor of 10^2 and the maximum of the inner iterations is set to be 40. In the MGSS method the parameters α and β are set to be 0.01 and 0.001, respectively. As seen, the proposed preconditioner is very effective in reducing the number of iterations and CPU times.

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A Quick Numerical Approach for Solving high order Integro-Differential Equations

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Abstract

A direct method for solving high order integro-differential equations by using Chebyshev wavelet basis is presented. We use operational matrix of integration (OMI) for Chebyshev wavelets to reduce this type of equations to a system of algebraic equations. Some quadrature formula for calculating inner products have been operated by Fast Fourier Transform (FFT).

Keywords: High order integro-differential equations, Chebyshev wavelets, Operational matrix of integration
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In this paper a fast computational method for solving second (or higher) order integro-differential equations is presented. We would like to use Chebyshev Wavelet basis to span the approximating space. The main advantage of this method is that inner products for setting up the matrices can be done at most by $O(N^2 lnN)$ operations as those of the Fast Galerkin scheme [1], which can be compared with at least $O(N^3)$ operation count of early methods.

Definition 1.1. Chebyshev wavelets $\psi_{n,m} = \psi(k, n, m, t)$, have introduced as

$$\psi_{n,m}(t) = \begin{cases} 2^{k/2}\widetilde{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & otherwise \end{cases} \tag{1}$$

where

$$\widetilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}} T_m(t), & m > 0, \end{cases} \tag{2}$$

are orthonormal Chebyshev polynomials of the first kind of degree m ($m = 0, 1, \dots, M - 1$), $n = 1, 2, \dots, 2^{k-1}$ which are orthogonal with respect to the weight function $\omega(t) = 1/\sqrt{1 - t^2}$, on the interval [-1,1].

*Speaker

If we consider truncated series in (3), we obtain

$$x(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = \mathbf{C}^T\Psi(t), \tag{3}$$

(M is specified positive integer which denotes the degree of chebyshev polynomials) where $\mathbf{C}$ and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices.

2 Fast Direct Method

Consider the following second order integro-differential equation,

$$a_2x''(s) + a_1x'(s) + a_0x(s) + \lambda \int_0^1 k(s,t)x(t)dt = y(s), \quad x(0) = x_0, \quad x'(0) = x'_0, \tag{4}$$

where a_2, a_1, a_0 and λ are constants and $y(s) \in L^2_\omega[0,1]$, $k \in L^2_\omega[0,1] \times [0,1]$ and $x(t)$ is an unknown function.

If we approximate functions and initial values by Chebyshev wavelets as $x''(t) \simeq X_2^T\Psi(t)$ also

$$k(s,t) \simeq \Psi^T(s)\mathrm{K}\Psi(t), \quad y(s) \simeq \mathrm{Y}^T\Psi(s), \quad x(0) \simeq X_0^{0^T}\Psi(t), \quad x'(0) \simeq X_1^{0^T}\Psi(t)$$

then we get

$$\begin{aligned} x'(s) = \textstyle\int_0^s x''(t)dt + x'(0) &\simeq \textstyle\int_0^s X_2^T\Psi(t)dt + X_1^{0^T}\Psi(s) \\ &\simeq X_2^TP\Psi(s) + X_1^{0^T}\Psi(s) \\ &= (X_2^TP + X_1^{0^T})\Psi(s) \end{aligned}$$

and with same integration we obtain

$$x(s) = (X_2^TP^2 + X_1^{0^T}P + X_0^{0^T})\Psi(s).$$

Now by substituting into main equation, we have by orthonormality of Chebyshev wavelets

$$\int_0^1 \Psi(t)\Psi^T(t)dt = I,$$

$$\Psi^T(s)[a_2I + a_1P^T + a_0P^{2^T} + \lambda KP^{2^T}]X_2 = \Psi^T(s)[Y - (a_1I + (a_0I + \lambda K)P^T)X_1^0 - (a_0I + \lambda K)X_0^0],$$

where I is identity matrix and this equation holds for each s in interval $[0,1]$, therefore we should solve the following linear system

$$[a_2I + a_1P^T + a_0P^{2^T} + \lambda KP^{2^T}]X_2 = Y - (a_1I + (a_0I + \lambda K)P^T)X_1^0 - (a_0I + \lambda K)X_0^0. \tag{5}$$

Finding vector X_2 leads to an approximation of the unknown function $x(s)$ by

$$x(s) = (P^{2^T} X_2 + P^T X_1^0 + X_0^0)^T \Psi(s).$$

The elements of matrices of our method have calculated by using (p+1)-point closed Gauss-Chebyshev quadrature rule we have, [1]

$$\begin{aligned}
\langle y, \psi_{il} \rangle &= \int_0^1 y(s)\psi_{il}\omega_l(s)ds \\
&= \int_{(l-1)/2^{k-1}}^{l/2^{k-1}} 2^{k/2} y(s)\widetilde{T}_i(2^k s - 2l + 1)\omega(2^k s - 2l + 1)ds \\
&= 2^{-k/2} \int_{-1}^{1} y(2^{-k}(u + 2l - 1))\widetilde{T}_i(u)\omega(u)du \\
&\simeq 2^{-k/2} \frac{\pi}{p} \sum_{m=0}^{p}{}'' y(2^{-k}(\cos(\pi m/p) + 2l - 1)) \cos(\pi i m/p)\delta_i
\end{aligned}$$

for $i = 1, 2, \ldots, 2^{k-1}$ and $l = 0, 1, \ldots, M-1$, where $\langle ., . \rangle$ denotes the inner product, double prime denotes that the first and the last terms are halved, and δ_i is defined as

$$\delta_i = \begin{cases} \sqrt{\frac{1}{\pi}} & i = 0, \\ \sqrt{\frac{2}{\pi}} & i \neq 0. \end{cases}$$

In this and some similar methods we have to calculate the N elements of vector Y and N^2 (where $N = M.2^{k-1}$) elements of matrix K. The number of elements for Hybrid Taylor-Block Pulse, Hybrid Legendre- Block Pulse and Legendre wavelets methods cost at least $O(N^2)$ operations for calculating vector Y and $O(N^3)$ operations for calculating matrix K, but Chebyshev wavelets basis functions and the Fast Fourier Transform (FFT) technique have been used to evaluate Y in $O(N \ln N)$ operations and same as above relations can be done to calculate the elements of matrix K by two dimensional Gauss Chebyshev quadrature formulae in $O(N^2 \ln N)$ operations, [1].

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An algorithm for Jacobi inverse eigenvalue problem

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Abstract

In this paper, for given n eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, we construct a Jacobi matrix $J \in \mathbb{R}^{n\times n}$. Also, the algorithm and numerical examples of this method will be expressed.

Keywords: eigenvalue problem, Inverse eigenvalue problem, Jacobi matrix.
Mathematics Subject Classification [2010]: 65F18

1 Introduction

Jacobi inverse eigenvalue problem, is of the great value for many application, including control theory, vibration theory and structural design. This kind of problem is a problem in which the Jacobi matrix is constructed using the spectral data matrix that consist of spectral data, which can possibly include part of the eigenvalues, eigenvectors or both.

Hochstadt [3] in 1974 constructed a Jacobi matrix using the eigenvalues of matrix and leading principal submatrix; also see [5, 2]. In this paper, we will construct a Jacobi matrix using only it's eigenvalues.

A Jacobi matrix is a tridiagonal symmetric matrix of the form

$$J_n = \begin{pmatrix} \beta_1 & \alpha_1 & & 0 \\ \alpha_1 & \beta_2 & \ddots & \\ & \ddots & \ddots & \alpha_{n-1} \\ 0 & & \alpha_{n-1} & \beta_n \end{pmatrix}, \tag{1}$$

where $\alpha_i > 0$ for $i = 1, \ldots, n-1$ and $\beta_i \in \mathbb{R}$ for $i = 1, \ldots, n$.
We denote J_n Jacobi matrix by $J_n = J(\beta_1, \beta_2, \ldots, \beta_n; \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$, and its leading principle submatrices by J_i; $i = 1, \ldots, n-1$.

Let $\{\lambda_i\}_{i=1}^n$be the set of eigenvalues of matrix J_n and $\{\mu_i\}_{i=1}^n$ be the set of eigenvalues of leading principle submatrix of J_n i.e. J_{n-1}. It is well known [1] that the eigenvalues of J_n are distinct and $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$ satisfy in the following interlacing property

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \lambda_{n-1} < \mu_{n-1} < \lambda_n. \tag{2}$$

*Speaker

Two polynomials $p(x)$ and $q(x)$ are said to be discrete orthogonal relation to the weight function $w(x) > 0$, if

$$\langle p, q\rangle = \sum_{i=1}^{n} w_i p(\xi_i)q(\xi_i) = 0, \tag{3}$$

where $(w_i)_{i=1}^n > 0$ and $(\xi_i)_{i=1}^n$ are n points, satisfying $\xi_1 < \xi_2 < \ldots < \xi_n$.

Theorem 1.1. *Suppose $p_n(\lambda)$ is characteristic polynomial of matrix J_n whose eigenvalues are $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, and $p_{n-1}(\lambda)$ is characteristic polynomial of J_{n-1}, then*

$$p_{n-1}(\lambda_i) = \frac{\gamma}{w_i p_n'(\lambda_i)}, \tag{4}$$

where

$$\gamma = \prod_{j=2}^{n}\prod_{k=1}^{j-1}(\lambda_j - \lambda_k).$$

Theorem 1.2. *Let $J_n = S\Lambda S^*$ be the spectral decomposition of an unreduced J_n. Then the associated inner product of the from (3) is given by*

$$\xi_i = \lambda_i, \qquad w_i = \delta s_{1i}^2, \qquad i = 1, ..., n \tag{5}$$

for any positive δ; $\sum_{i=1}^n w_i = \delta$.

2 Main result

Suppose $\lambda_1 < \lambda_2 < ... < \lambda_n$ are the eigenvalues of the Jacobi matrix J_n. It is clear that J_n is characterized by the $2n-1$ unknown entries $\{\alpha_i\}_{i=1}^{n-1}$ and $\{\beta_i\}_{i=1}^{n}$. Thus it is intuitively true that $2n-1$ pieces of information are needed to solve the inverse problems where $\lambda_1, \lambda_2, ..., \lambda_n$ are n pieces of information and the rest are considered $w_1, w_2, ..., w_{n-1}$.
The following Theorem, states the process of construction Jacobi matrix J_n.

Theorem 2.1. *Let $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ be eigenvalues of*

$$J_n = J(\beta_1, \beta_2, \ldots, \beta_n; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}),$$

and suppose that $p_n(\lambda), p_{n-1}(\lambda), \ldots, p_1(\lambda)$ are characteristic polynomials of $J_n, J_{n-1}, ..., J_1$ such that

$$p_i(\lambda) = k_i\lambda^i + s_i\lambda^{i-1} + \ldots, \tag{6}$$

for $i = 1, \ldots, n$, then

$$\alpha_i = \frac{k_{i-1}}{k_i}, \quad \beta_i = -\left(\frac{s_i}{k_i} - \frac{s_{i-1}}{k_{i-1}}\right), \tag{7}$$

and eigenvector corresponding to λ_i is $X_i = (p_0(\lambda_i), p_1(\lambda_i), \ldots, p_{n-1}(\lambda_i))^T$.

Proof. Since characteristic polynomials of J_n are orthogonal, we can write

$$p_i(\lambda) = (a_i\lambda - b_i)p_{i-1}(\lambda) - c_i p_{i-2}(\lambda); \quad i = 2, \ldots, n \tag{8}$$

Since $p_i(\lambda) = k_i\lambda^i + s_i\lambda^{i-1} + \ldots$, by direct computation we have

$$a_i = \frac{k_i}{k_{i-1}}, \quad b_i = a_i\left(\frac{s_{i-1}}{k_{i-1}} - \frac{s_i}{k_i}\right), \quad c_i = \frac{k_i k_{i-2}}{k_{i-1}^2}. \tag{9}$$

We can rewrite equation (8) as follow

$$\lambda p_{i-1}(\lambda) = \frac{1}{a_i}p_i(\lambda) + \frac{b_i}{a_i}p_{i-1}(\lambda) + \frac{c_i}{a_i}p_{i-2}(\lambda).$$

Since

$$\frac{1}{a_i} = \frac{k_{i-1}}{k_i}, \quad \frac{c_i}{a_i} = \frac{k_{i-2}}{k_{i-1}},$$

thus we have

$$\lambda p_{i-1}(\lambda) = \alpha_i p_i(\lambda) + \beta_i p_{i-1}(\lambda) + \alpha_{i-1}p_{i-2}(\lambda), \tag{10}$$

where

$$\alpha_i = \frac{k_{i-1}}{k_i}, \quad \beta_i = \frac{b_i}{a_i} = -\left(\frac{s_i}{k_i} - \frac{s_{i-1}}{k_{i-1}}\right).$$

Now, if we set

$$p(\lambda) = (p_0(\lambda), \ldots p_{n-1}(\lambda))^T, \; u = (0, \ldots, 0, 1)^T, \; J_n = J(\beta_1, \beta_2, \ldots, \beta_n; \alpha_1, \alpha_2, \ldots, \alpha_{n-1}).$$

then

$$\lambda p(\lambda) = J_n p(\lambda) + \alpha_n p_n(\lambda)u.$$

In other hand, $\{\lambda_i\}_{i=1}^n$ are zeroes of $p_n(\lambda)$, so

$$J_n p(\lambda_j) = \lambda_j p(\lambda_j), \tag{11}$$

and the proof is completed. □

By the previous theorem, we know that the first element for all of the eigenvectors of J_n is equal. Therefore based on Theorem 1.2, all of the weights $w_1, \ldots, w_n$ are equal. At first, we choose $wi = 1$ for $i = 1, \ldots, n$.
Suppose that $p_n(\lambda) = \prod_{i=1}^n(\lambda - \lambda_i)$ is characteristic polynomial of matrix J_n. Thus from Theorem 1.1 we can compute $p_{n-1}(\lambda_i)$. Now if we interpolate $(\lambda_i, p_{n-1}(\lambda_i))$ for $i = 1, \ldots, n$, then we obtain $p_{n-1}(\lambda)$[4]. So, we can compute α_n and β_n by using equation (7) and $p_{n-2}(\lambda)$ by recursive relation (10). By continuing this method, we will obtain all of α_i and β_i.

Note that for computing $p_{i-2}(\lambda)$ in recursive relation (10), we replace α_{i-1} with $\frac{k_{i-2}}{k_{i-1}}$. Also, $p_0(\lambda)$ is a constant polynomial which $k_0 = p_0(\lambda)$ and $s_0 = 0$ for it.

Therefore, we can construct Jacobi matrix J by n given eigenvalues. Also we will enable to compute eigenvector X_i corresponding to eigenvalue λ_i after obtaining $p_i(\lambda); i = 0, 1, \ldots, n$ in recursive relation (10).

Now, consider $w_i = w$ for $i = 1, \ldots, n$, where w is a positive number and unequal to 1. In this case, one verified easily that $p_i(\lambda)$ for $i = 0, \ldots, n-1$ is the product of $\frac{1}{w_i}$ and $p_i(\lambda)$ corresponding to $w = 1$. Therefore, we conclude from formula (7) that the choice of the weight w dose not have any influence on the computation of α_i for $i = 1, \ldots, n-1$ and β_i for $\imath = 1, \ldots, n$.

The above description leads to the following theorem.

Theorem 2.2. *Let $\{\lambda_i\}_{i=1}^n$ be a set of real numbers that $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Then, there exists a unique Jacobi matrix J_n such that $\{\lambda_i\}_{i=1}^n$ are eigenvalues of J_n.*

In sequel, we give an algorithms and numerical examples.

Algorithm(*JIEP*)
1. Input $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ and λ_i; $(\%\lambda_1 < \lambda_2 < ... < \lambda_n)$.
2. Compute $p_n(\lambda) = \prod_{i=1}^n(\lambda - \lambda_i)$.
3. For $i = 1, \ldots, n$
4. $\quad p_{n-1}(\lambda_i) = \frac{\prod_{j=2}^n \prod_{k=1}^{j-1}(\lambda_j - \lambda_k)}{w_i p_n'(\lambda_i)}$; $w_i = 1$.
5. End For
6. Compute $p_{n-1}(\lambda)$ using interpolating $(\lambda_i, p_{n-1}(\lambda_i))$ for $i = 1, ..., n$.
7. Set $\alpha_n := \frac{k_{n-1}}{k_n}$
9. Set $\beta_n := -(\frac{s_n}{k_n} - \frac{s_{n-1}}{k_{n-1}})$
10. For $i = n, \ldots, 2$
11. $\quad$ Compute $p_{i-2}(\lambda)$ by $\lambda p_{i-1}(\lambda) = \alpha_i p_i(\lambda) + \beta_i p_{i-1}(\lambda) + \alpha_{i-1} p_{i-2}(\lambda)$
12. $\quad$ Set $\alpha_{i-1} := \frac{k_{i-2}}{k_{i-1}}$
13. $\quad$ Set $\beta_{i-1} := -(\frac{s_{i-1}}{k_{i-1}} - \frac{s_{i-2}}{k_{i-2}})$
14. End For
15. Set $J_n = J(\beta_1, \beta_2, \ldots, \beta_n; \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$.
16. Set $X_i = (p_0(\lambda_i), p_1(\lambda_i), \ldots, p_{n-1}(\lambda_i))^T$

Example 2.3. Suppose that $\Lambda = (2, 3, 5, 7, 9)$. Then by using the above algorithm, we have the following result:

$$J_5 = J\,(5.2000, 5.7268, 5.7123, 5.0657, 4.2951; 2.5612, 1.9367, 1.7159, 1.5673)$$

Example 2.4. Suppose that $\Lambda = (1, 3, 6, 8, 9, 13)$.Then eigenvector corresponding to the eigenvalue 8 is:

$$X_4 = \left(7.7690 \times 10^6, 2.6264 \times 10^6, -7.3147 \times 10^6, -4.5687 \times 10^6, 3.1331 \times 10^6, 1.4515 \times 10^7\right)^T$$

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Bernoulli operational matrix for solving optimal control problems

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Abstract

In this paper, we use the Bernoulli operational matrix of derivatives and the collocation points, for solving linear and nonlinear optimal control problems (OCPs). By Bernoulli polynomials bases, the Two-Point Boundary Value Problem (TPBVP), derived from the Pontryagins maximum principle, transforms into the matrix equation.

Keywords: Optimal control problems; Bernoulli polynomials; Hamiltonian system.
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1 Introduction

Optimal control problems (OCPs) appear in engineering, science, economics, and many other fields. Since most practical problems are rather too complex to allow analytical solutions, numerical methods are unavoidable for solving these complex practical problems. There are numerous computational methods for solving various practical optimal control problems.

2 Main results

In this paper, we consider following linear optimal control problem (OCP)

$$\begin{aligned} &\dot{x} = Ax(t) + Bu(t),\ x(t_0) = x_0, \\ &J = \frac{1}{2}x(t_f)^T Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (x^T Px + 2x^T Qu + u^T Ru)dt, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathrm{R}^{m\times n}$. The control $u(t)$ is an admissible control if it is piecewise continuous in t for $t \in [t_0, t_f]$. Its values belong to a given closed subset U of $\mathbb{R}^+$. The input $u(t)$ is derived by minimizing the quadratic performance index J, where $S \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times m}$ are positive semi-definite matrices and $R \in \mathbb{R}^{m\times m}$ is positive definite matrix.
we consider Hamiltonian for system (1) as

$$H(x, u, \lambda, t) = \frac{1}{2}(x^T Px + 2x^T Qu + u^T Ru) + \lambda^T (Ax + Bu), \tag{2}$$

*Speaker

where $\lambda \in \mathrm{R}^n$ is co-state vector.
According to the Pontryagin's maximum principle, we have

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -Px - Qu - A^T\lambda, \tag{3}$$

$$\frac{\partial H}{\partial u} = Q^T x + Ru + B^T\lambda = 0. \tag{4}$$

The optimal control is computed by

$$u^* = -R^{-1}Q^T x - R^{-1}B^T\lambda, \tag{5}$$

where λ and x are the solution of the Hamiltonian system:

$$\begin{cases} \dot{x} = [A - BR^{-1}Q^T]x - BR^{-1}B^T\lambda, \\ \dot{\lambda} = [-P + QR^{-1}Q^T]x + [QR^{-1}B^T - A^T]\lambda, \\ x(t_0) = x_0, \quad x(t_f) = x_f, \end{cases} \tag{6}$$

Because the initial value of λ is not known, Thus we rewrite Two-Point Boundary Value Problem (TPBVP) in (6) as following:

$$\begin{cases} \dot{x} = [A - BR^{-1}Q^T]x - BR^{-1}B^T\lambda, \\ \dot{\lambda} = [-P + QR^{-1}Q^T]x + [QR^{-1}B^T - A^T]\lambda, \\ x(t_0) = x_0, \quad \lambda(t_0) = \alpha, \end{cases} \tag{7}$$

where $\alpha \in \mathbb{R}$ is an unknown parameter.

Remark 2.1. For identifing of $\lambda(t_0)$, by considering the final state condition $x(t_f) = x_f$, and since the approximations of Bessel polynomials are functions of both t and α, we have $x_k(t_f, \alpha) = x_f$. That is, α should be a real root of $x_k(t_f, \alpha) - x_f = 0$.

3 Method of solution

Let the solution of (7) is approximated by the first $N + 1$-terms Bernoulli polynomials. Hence if we write

$$x_N(t) = \sum_{n=0}^{N} a_{1,n} B_n(t) = B(t)A, \tag{8}$$

$$\lambda_N(t) = \sum_{n=0}^{N} a_{2,n} B_n(t) = B(t)A, \tag{9}$$

where the Bernoulli coefficient vector A and the Bernoulli vector $B(t)$ are given by

$$A^T = \begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix}, \quad B(t) = \begin{bmatrix} B_0(t) & B_1(t) & \dots & B_N(t) \end{bmatrix} \tag{10}$$

then the kth derivative of $y_N(t)$ can be expressed in the matrix form by

$$y_N^{(k)}(t) = B^{(k)}(t)A \tag{11}$$

Example 3.1. Consider a single-input scalar system as follows:

$$\dot{x} = -x(t) + u(t), \tag{12}$$
$$J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \tag{13}$$

with free terminal condition and the initial condition

$$x(0) = 1. \tag{14}$$

The analytical solution of the problem defined above is

$$x(t) = cosh(\sqrt{2}t) + \beta sinh(\sqrt{2}t),$$
$$u(t) = (1 + \sqrt{2}\beta)cosh(\sqrt{2}t) + (\sqrt{2} + \beta)sinh(\sqrt{2}t),$$

where

$$\beta = -\frac{cosh(\sqrt{2}) + \sqrt{2}sinh(\sqrt{2})}{\sqrt{2}cosh(\sqrt{2}) + sinh(\sqrt{2})}.$$

According to (6) we have

$$\dot{x} = -x(t) - \lambda(t), \tag{15}$$
$$\dot{\lambda} = -x(t) + \lambda(t), \tag{16}$$
$$x(0) = 0, \quad \lambda(0) = \alpha, \tag{17}$$

we can obtain the following optimal control law

$$u^*(t) = -\lambda(t), \tag{18}$$

we also require that

$$\lambda(1) = 0, \tag{19}$$

using Remark 2.1 and considering the final state conditions, we should have $\alpha = 0.38582$, therefore

$$\dot{x} = -x(t) - \lambda(t), \tag{20}$$
$$\dot{\lambda} = -x(t) + \lambda(t), \tag{21}$$
$$x(0) = 0, \quad \lambda(0) = 0.38582. \tag{22}$$

Now, we get the approximate solutions by applying the present method for $N = 6$. In Figs. 1-2, the approximate solutions $x(t)$ and $u(t)$ of the present method applied for $N = 6$ are compared with the exact solution. For the approximate solutions $x(t)$ and $u(t)$ gained by the present method for $N = 6$, we denotes the error functions obtained the accuracy of the solution given by Eqs. (15) and (15) in Figs. 3-4.

Table 2
Comparison of the exact solution whit the present method ($N = 6$).

t	Abs error Bernoulli (control)	Abs error Bernoulli (state)
0	1.4038e-006	0
0.2	1.6091e-006	5.2807e-007
0.4	2.1335e-006	7.2335e-007
0.6	2.8071e-006	1.0185e-006
0.8	3.5868e-006	1.6320e-006
1.0	8.1492e-006	4.3328e-006

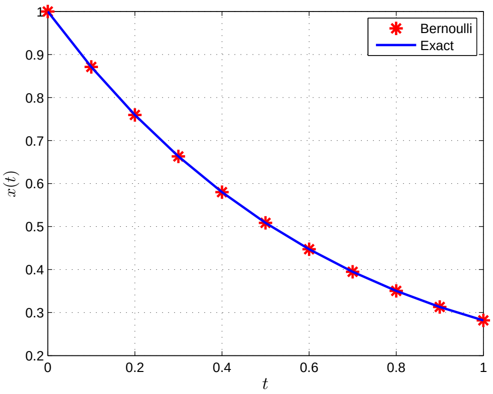

Fig. 1. The optimal state ($N = 6$),

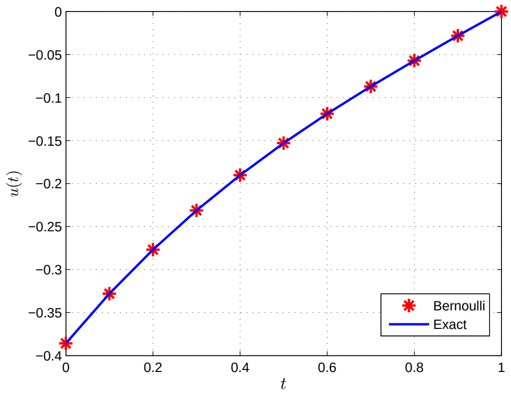

Fig. 2. The optimal control ($N = 6$).

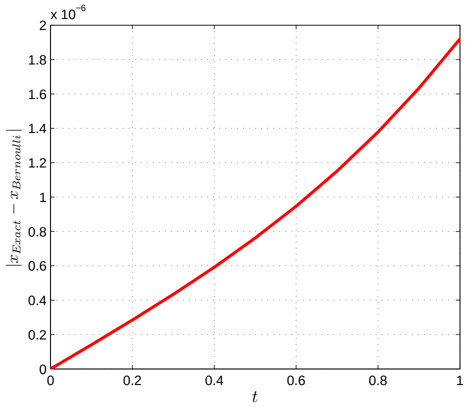

Fig. 3. The absolute error function of state,

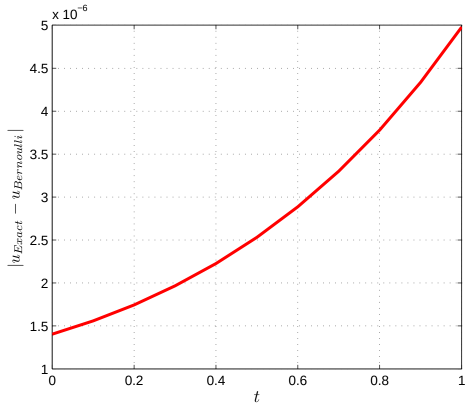

Fig. 4. The absolute error function of control.

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B-spline collocation method to solve the nonlinear fractional Burgers' equation

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Abstract

In this paper, to approximate the solution of nonlinear fractional Burgers' equation, we give a cubic B-spline finite element algorithm. To investigate the stability conditions, we use von-Neumann analysis and finally some numerical results is presented to show the applicability of the new scheme.

Keywords: Fractional Burgers' equation, B-spline functions, Collocation method
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

We denote the following fractional nonlinear Burgers' equation as an initial-boundary value problem:

$$\begin{aligned}
&u_t^\alpha + uu_x - \nu u_{xx} = f(x,t), \qquad x \in [a,b],\ t \in [0,T],\ 0 < \alpha < 1,\\
&u(a,t) = g_1(t), \quad u(b,t) = g_2(t),\\
&u(x,0) = u_0(x)
\end{aligned} \tag{1}$$

where $\nu > 0$ is the coefficient of kinematic viscosity. f, u_0, g_1 and g_2 are smooth enough functions in time and space scales. Let subscripts x and t the space and time differentiations, respectively; and superscript α the order of fractional derivative.

Definition 1.1. The Caputo fractional derivative is defined as

$$u_t^\alpha(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_s(x,s)}{(t-s)^\alpha} ds, \qquad 0 < \alpha < 1, \tag{2}$$

where $\Gamma(\alpha)$ is the Gamma function.

*Speaker

2 Numerical method and main results

For the numerical purpose, we first define a uniform partition $0 = t_0 < t_1 < \cdots < t_n = T$ on $[0, T]$ with $\Delta t = t_{j+1} - t_j$, $j = 0, 1, \cdots, n-1$. To discretization of time fractional derivative, we use the $L1$-formula [4].

$$u_t^{\alpha}(x, t_{k+1}) = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} (u^{k-j+1} - u^{k-j}) \tag{3}$$

where $u^k = u(x, t_k)$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$. Linearizing the nonlinear term uu_x by newton's method, and substituting (3) in (1), for $r = (\Delta t)^{-\alpha}\Gamma(2-\alpha)$, (1) leads to

$$u^{k+1} + ru^{k+1}u_x^k + ru^k u_x^{k+1} - r\nu u_{xx}^{k+1} = b_k u^0 + \sum_{j=0}^{k} (b_j - b_{j+1})u^{k-j} + ru^k u_x^k + rf^{k+1} \tag{4}$$

where $f^{k+1} = f(x, t_{k+1})$. To space discretization, let the solution domain $[a, b]$ is partitioned into uniformly sized finite elements as $a = x_0 < x_1 < \cdots < x_N = b$, with $h = x_{m+1} - x_m$, $m = 0, 1, \cdots, N-1$. In this uniform mesh, the cubic B-spline function $Q_m(x)$ is given by:

$$Q_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3 & x \in [x_{m-2}, x_{m-1}] \\ (x - x_{m-2})^3 - 4(x - x_{m-1})^3 & x \in [x_{m-1}, x_m] \\ (x_{m+2} - x)^3 - 4(x_{m+1} - x)^3 & x \in [x_m, x_{m+1}] \\ (x_{m+2} - x)^3 & x \in [x_{m+1}, x_{m+2}] \\ 0 & otherwise. \end{cases} \tag{5}$$

Since $\{Q_m(x)\}_{m=-1}^{N+1}$ is a basis for the functions over the solution domain, the approximate solution in cubic B-splines collocation method can be considered as:

$$u(x, t) \simeq U(x, t) = \sum_{m=-1}^{N+1} \delta_m(t) Q_m(x), \tag{6}$$

where $\delta_m(t)$ are unknown time dependent parameters that should be computed from the initial and boundary conditions in collocation method process.

From definition (5), the values of U, U_x and U_{xx} at the nodal points are as follows:

$$\begin{cases} U(x_m, t) = \delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t), \\ h\ U_x(x_m, t) = 3(\delta_{m+1}(t) - \delta_{m-1}(t)), \\ h^2\ U_{xx}(x_m, t) = 6(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)). \end{cases} \tag{7}$$

Let $\delta_m^k = \delta_m(t_k)$. Substituting (7) in (4), the completed discretized form of main problem for $m = 0, 1, \cdots, N$ can written as:

$$\begin{aligned} \beta_{m1}\delta_{m-1}^{k+1} + \beta_{m2}\delta_m^{k+1} + \beta_{m3}\delta_{m+1}^{k+1} &= b_k(\delta_{m-1}^0 + 4\delta_m^0 + \delta_{m+1}^0) \\ &+ \sum_{j=1}^{k-1} (b_j - b_{j+1})(\delta_{m-1}^{k-j} + 4\delta_m^{k-j} + \delta_{m+1}^{k-j}) + \beta_{m4}\delta_{m-1}^k + \beta_{m5}\delta_m^k + \beta_{m6}\delta_{m+1}^k \end{aligned} \tag{8}$$

where

$$\beta_{m1} = 1 + r_1 k_m - r_1 z_m - 2r_2, \qquad \beta_{m4} = (b_0 - b_1) + r_1 k_m,$$
$$\beta_{m2} = 4 + 4r_1 k_m + 4r_2, \qquad \beta_{m5} = 4\beta_{m4},$$
$$\beta_{m3} = 1 + r_1 k_m + r_1 z_m - 2r_2, \qquad \beta_{m6} = \beta_{m4}, \tag{9}$$

and

$$r_1 = \frac{3r}{h}, \qquad r_2 = \frac{3r\nu}{h^2}, \qquad k_m = \delta^k_{m+1} - \delta^k_{m-1}, \qquad z_m = \delta^k_{m-1} + 4\delta^k_m + \delta^k_{m+1}. \tag{10}$$

When $k = 0$, the system obtained from (8) can be converted into matrix form as:

$$A_1\delta^1 = (b_0 D_1 + B_1)\delta^0 + C^1 \tag{11}$$

where $\delta^k = (\delta^k_{-1}, \delta^k_0, \delta^k_1, \cdots, \delta^k_{N+1})$. The coefficient matrices, A_1, D_1 and B_1 are tridiagonal and their dimensions are $(N+1) \times (N+3)$. To make the system solvabe, parameters δ^k_{-1} and δ^k_{N+1} may be eliminated from the system by boundary conditions.
With continue this process for various k, we have a recurrence matrix system as:

$$A\delta^{k+1} = D\left[b_k\delta^0 + \sum_{j=1}^{k-1}(b_j - b_{j+1})\delta^{k-j}\right] + B\delta^k + C^{k+1} \tag{12}$$

This system can be solved iteratively. To start the iteration process and obtain the initial vector δ^0, we use the initial condition of the problem. From (7), we have

$$u_0(x_m) = u(x_m, 0) \simeq \delta^0_{m-1} + 4\delta^0_m + \delta^0_{m+1}, \qquad m = 0, 1, \cdots, N. \tag{13}$$

Finally, system (13) with $u_{xx}(x_m, 0) = u_0''(x_m),\ m = 0, N$, gives us the matrix system as:

$$A_0\ \delta^0 = B_0. \tag{14}$$

To investigate the stability conditions , we use von-Neumann analysis. To this, we first linearize the nonlinear term uu_x by taking the solution u as a constant m and then, let $f = 0$. Applying (3) and (7) in linearized form of the equation, we have:

$$\begin{aligned}(1 - r_1 m - 2r_2)e^{k+1}_{m-1} + 4(1 + r_2)e^{k+1}_m + (1 + r_1 m - 2r_2)e^{k+1}_{m+1} \\ = b_k(e^0_{m-1} + 4e^0_m + e^0_{m+1}) + \sum_{j=0}^{k}(b_j - b_{j+1})(e^{k-j}_{m-1} + 4e^{k-j}_m + e^{k-j}_{m+1}).\end{aligned} \tag{15}$$

with e^k_m as the error of scheme at time level k. Then substituting the fourier mode $e^k_m = q^k e^{ip\rho}$ into (15) results

$$q^{k+1} = Q\{b_k q^0 + \sum_{j=0}^{k-1}(b_j - b_{j+1})q^{k-j}\} \tag{16}$$

where $Q = \frac{2+4\cos^2(\frac{\rho}{2})}{2+4\cos^2(\frac{\rho}{2})+8r_2\sin^2(\frac{\rho}{2})+2ir_1 m\sin(\rho)}$ with $i = \sqrt{-1}$. It is easy to verify that $|Q| \le 1$.
Let $|q^{\{k\}}|_{\max} = \max\{|q^0|, |q^1|, |q^2|, \cdots, |q^k|\}$. Therefore, for equation (16) we have,

$$|q^{k+1}| \le |q^{\{k\}}|_{\max}. \tag{17}$$

(17) shows that $|e^k| \le |e^0|$, i.e., the error of this method in time level k, for every k, does not growth and is smaller than or equal to its initial error. So, the method is unconditionally stable.

3 Numerical test

Denote the fractional nonlinear Burger equation defined in (1) with $f(x,t) = \frac{2t^{2-\alpha}e^x}{\Gamma(2-\alpha)} + t^4e^{2x} - \nu t^2 e^x$, where the initial and boundary conditions are

$$\begin{aligned} &u(0,t) = t^2, \quad u(1,t) = et^2, \quad t \geq 0, \\ &u(x,0) = 0, \qquad 0 \leq x \leq 1. \end{aligned} \tag{18}$$

The exact solution of problem is $u(x,t) = t^2 e^x$. The numerical errors between the exact solution and approximate solution have been shown in Tables 1 and 2.

Table 1: Error norms for $\alpha = 0.5$, $\Delta t = 0.00025$, $T = 1$ and $\nu = 1$

	$N = 10$	$N = 20$	$N = 40$	$N = 80$
L_2-norm	$1.9138e-3$	$5.0021e-4$	$6.7823e-5$	$3.5127e-5$
L_∞-norm	$3.2114e-3$	$8.2431e-4$	$2.0010e-4$	$5.8125e-5$

Table 2: Error norms of problem for $\alpha = 0.5$, $h = 0.025$, $T = 1$ and $\nu = 1$

	$\Delta t = 0.002$	$\Delta t = 0.001$	$\Delta t = 0.0005$
L_2-norm	$4.5721e-4$	$1.7811e-4$	$5.9451e-5$
L_∞-norm	$6.5122e-4$	$2.6511e-4$	$2.0025e-4$

4 Conclusion

In the present study, a new scheme based on B-spline basis functions and collocation finite element method is applied to solve the fractional nonlinear Burger's equation with initial and boundary conditions. In the solution process, the discretized Caputo fractional derivative is denoted same as used in [4]. The unconditional stability of the scheme is presented and finally a test example is included to demonstrate the applicability of the new scheme.

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Block pulse operational matrix for solving fractional partial differential equation

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Abstract

In this paper, we first introduce block pulse functions and the block pulse operational matrices of the fractional order integration. Also the block pulse operational matrices of the fractional order differentiation are obtained.Then we present a computational method based on the above results for solving a class of fractional partial differential equations.

Keywords: Block pulse functions, Operational matrix, Fractional partial differential equations.
Mathematics Subject Classification [2010]: 34A08, 35R11

1 Introduction

Fractional differential equations are generalized from integer order ones, which are achieved by replacing integer order of derivatives by fractional ones. Compared with differential equations of integer order, their advantages are more accurate in natural physical process and dynamic systems [2].
In this paper, our study focuses on a class of fractional partial differential equations as the following form:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = -\frac{\partial^{\beta} u}{\partial x^{\beta}} + \lambda u(x,t) + g(x,t), 0 \leq x \leq 1, 0 \leq t \leq T. \tag{1}$$

subject to the initial-boundry conditions:

$$u(0,t) = p(t), u(x,0) = v(x), \tag{2}$$

where $\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}}$ and $\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}}$ are fractional derivative in Caputo sense, $g(x,t)$ is the known continuous function, $u(x,t)$ is the unknown function, $0 < \alpha \leq 1$ and $1 \leq \beta \leq 2$.

*Speaker

2 Fractional calculus

In this section, we give some necessary definition and preliminaries of the fractional calculus theory which will be used in this article. For more details see [3, 4].

Definition 2.1. The Riemann-Liouville fractional integral operator $I^{\alpha}, \alpha \geq 0$ for function $u(t)$ is given by:

$$I^{\alpha}u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s, \alpha > 0, \tag{3}$$

$$I^{0}u(t) := u(t). \tag{4}$$

Definition 2.2. The Caputo fractional derivative operator of order $\alpha \geq 0$ for function $u(t)$ is defined as:

$$D_{\star}^{\alpha}u(t) = \begin{cases} \frac{\mathrm{d}^r u(t)}{\mathrm{d}t^r} & \alpha = r \in \mathbb{N}^+, \\ \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{u^r(s)}{(t-s)^{\alpha-r+1}} \mathrm{d}s, 0 \leq r-1 < \alpha < r. \end{cases} \tag{5}$$

The relation between the Riemann-Liouville operator and Caputo operator is given by the following expressions:

$$D_{\star}^{\alpha} I^{\alpha} u(t) = u(t), \tag{6}$$

$$I^{\alpha} D_{\star}^{\alpha} u(t) = u(t) - \sum_{k=0}^{r-1} u^{(k)}(0^+) \frac{(t)^k}{k!}, t > 0. \tag{7}$$

3 Block pulse functions (BPFs)

Definition 3.1. For a given positive integer m, the BPFs are defined as:

$$b_i(t) = \begin{cases} 1, & (i-1)h \leq t < ih, \\ 0, & \mathit{otherwise}, \end{cases} \tag{8}$$

where $i = 1, 2, \cdots, m$ and $h = \frac{1}{m}$. Some useful properties of BPFs are listed below [1].

Proposition 3.2. *For $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, m$ we have the following:*

1. $supp\{b_i(x)\} = [\frac{i-1}{m}, \frac{i}{m}]$.

2. *Disjointness:*

$$b_i(t) b_j(t) = \begin{cases} b_i(t), & i = j, \\ 0, & i \neq j. \end{cases} \tag{9}$$

3. *Orthogonality:*

$$\int_0^1 b_i(t) b_j(t) = \begin{cases} h, & i = j, \\ 0, & i \neq j. \end{cases} \tag{10}$$

4. *Completenss: For every* $f \in L^2([0,1))$ *when* m *approach to the infinity, Parseval's identity holds:*

$$\int_0^1 f^2(x)\,\mathrm{d}x = \sum_{i=0}^{\infty} f_i^2 \|b_i(x)\|^2, \tag{11}$$

where

$$f_i = \frac{1}{h}\int_0^1 f(x)b_i(x)\,\mathrm{d}x. \tag{12}$$

5. *A function* $f(x) \in L^2([0,1))$*, can be expressed as:*

$$f(x) \cong \sum_{i=1}^{m} f_i b_i(x) = f^T B_m(x), \tag{13}$$

where $f = [f_1, f_2, \cdots, f_m]^T$ *and* $B_m(x) = [b_1(x), b_2(x), \cdots, b_m(x)]^T$*, such that* f_i *for* $i = 1, 2, \cdots, m$ *are defined in* (12).

Remark 3.3. Every two dimensional function $u(x,t) \in L^2([0,1)\times[0,1))$ can be expressed as:

$$u(x,t) \cong B^T(x)UB(t). \tag{14}$$

where

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,m_2} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,m_2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m_1,1} & u_{m_1,2} & \cdots & u_{m_1,m_2} \end{bmatrix}, u_{i,j} = \frac{1}{h_1 h_2}\int_0^1\int_0^1 u(x,t)b_i(x)b_j(t)\,\mathrm{d}x\,\mathrm{d}t,$$

$$h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2} \text{ and } B(x) = [b_1(x), \cdots, b_{m_1}(x)]^T, B(t) = [b_1(t), \cdots, b_{m_2}(t)]^T. \tag{15}$$

3.1 BPFs-operational matrix of fractional integration

In this part, we introduce the operational matrix of fractional integration of block pulse functions.

Definition 3.4. α-Fractional integration order of the BPFs-vector can be expressed by themselve as:

$$I^\alpha B(x) \cong P_\alpha B(x),$$

where

$$P_\alpha = \left(\frac{1}{m}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \epsilon_1 & \epsilon_2 & \cdots & \epsilon_{m-1} \\ 0 & 1 & \epsilon_1 & \cdots & \epsilon_{m-2} \\ 0 & 0 & 1 & \cdots & \epsilon_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

and $\epsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. Here P_α is called the block pulse operatinal matrix of fractional integration.

4 Solution of the fractional partial differential equation

In this section, we suppose $m_1 = m_2 = m$. Consider the fractional partial differential equation given by Eq. (1). We approximate the function $\frac{\partial^\beta u}{\partial x^\beta}$ by the BPFs, it can be written as:

$$\frac{\partial^\beta u}{\partial x^\beta} \cong B^T(x)UB(t). \tag{16}$$

By applying the operator I_x^β on Eq. (16) and using Eq. (7) we have:

$$I_x^\beta(\frac{\partial^\beta u}{\partial x^\beta}) \cong I_x^\beta[B^T(x)UB(t)] = u(x,t) - u(0,t). \tag{17}$$

$$\Longrightarrow u(x,t) = p(t) + B^T(x)P_\beta^T UB(t). \tag{18}$$

Now, we approximate $p(t)$ by $B^T(x)XB(t)$, then we have:

$$u(x,t) = B^T(x)[X + P_\beta^T U]B(t). \tag{19}$$

Hence, by substituting Eqs. (16) and (19) in Eq. (1), we have:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -B^T(x)UB^T(t) + \lambda B^T(x)[X + P_\beta^T U]B(t) + B^T(x)GB(t). \tag{20}$$

By using Eq. (7)

$$u(x,t) = B^T(x)[-U + G + \lambda(X + P_\beta^T U)]P_\alpha B(t). \tag{21}$$

From Eqs. (19) and (21) and using (10) we have:

$$[X + P_\beta^T U] = [-U + G + \lambda(X + P_\beta^T U)]P_\alpha, \tag{22}$$

Finally, we have:

$$(I - \lambda P_\beta^T)^{-1} P_\beta^T U + UP_\alpha + (I - \lambda P_\beta^T)^{-1}[X - (G + \lambda\ X)P_\alpha] = 0, \tag{23}$$

which is a sylvester equation.

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Complete pivoting strategy to compute the IULBF preconditioner

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Abstract

In this paper, a complete pivoting strategy to compute the IULBF preconditioner is presented.

Keywords: pivoting, IULBF preconditioner.
Mathematics Subject Classification [2010]: 65F10, 65F50, 65F08.

1 Introduction

Consider the linear system of equations of the form $Ax = b$, where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular, large, sparse and nonsymmetric and also $x, b \in \mathbb{R}^n$. An IUL preconditioner M for this system is in the form of $M = UDL \approx A$. This preconditioner will change the original system to the left preconditioned system $M^{-1}Ax = M^{-1}b$. For a proper preconditioner, instead of solving the original system, it is better to solve the left preconditioned system by the Krylov subspace methods [4]. In [1, 2], we have proposed an IUL preconditioner for system $Ax = b$. This preconditioner is termed the $IULBF$.

Algorithm 1 (IULBF preconditioner)

Input: $A \in \mathbb{R}^{n\times n}$ and $\tau_z, \tau_w, \tau_l, \tau_u \in (0,1)$ be drop tolerances parameters.
Output: $A \approx UDL$

for $i = n$ to 1 **do**
  $w_i^{(0)} = e_i^T$, $z_i^{(0)} = e_i$.
  for $j = i+1$ to n **do**
    $p_j^{(i-1)} = e_i^T A z_j^{(n-j)}$ $q_j^{(i-1)} = w_j^{(n-j)} A e_i$
    $U_{ij} = \dfrac{p_j^{(i-1)}}{d_{jj}}$, $L_{ji} = \dfrac{q_j^{(i-1)}}{d_{jj}}$
    If $|L_{ji}| < \tau_l$, then set $L_{ji} = 0$. Also if $|U_{ij}| < \tau_u$, then set $U_{ij} = 0$
    $z_i^{(j-i)} = z_i^{(j-i-1)} - \dfrac{q_j^{(i-1)}}{d_{jj}} z_j^{(n-j)}$, $w_i^{(j-i)} = w_i^{(j-i-1)} - \dfrac{p_j^{(i-1)}}{d_{jj}} w_j^{(n-j)}$
    For all $l \geq j$, if $|z_{li}^{(j-i)}| < \tau_z$ and $|w_{il}^{(j-i)}| < \tau_w$, then set $z_{li}^{(j-i)} = 0$ and $w_{il}^{(j-i)} = 0$
  end for
  $d_{ii} = w_i^{(n-i)} A e_i$
end for
Return $U = (U_{ij})_{1\leq i,j\leq n}$, $D = diag(d_{ii})_{1\leq i\leq n}$ and $L = (L_{ji})_{1\leq j,i\leq n}$.

Algorithm 1, computes the $IULBF$ preconditioner. In this algorithm, matrices L and U are computed column-wise and row wise, respectively.

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2 Pivoting strategy for the IULBF preconditioner

Algorithm 2, computes the IULBF preconditioner which is coupled with complete pivoting strategy. The pivoting strategy of this algorithm is based on the complete pivoting strategy of the Backward IJK version of Gaussian Elimination process. In lines 16 and 35 of this algorithm we use the parameter $\alpha \in (0, 1]$ to control the pivoting process.

Algorithm 2 (*IULBF* preconditioner coupled with complete pivoting strategy)

Input: Let $A \in \mathbb{R}^{n\times n}$, $U = L = \Pi = \Sigma = I_n$, $\tau_z, \tau_w, \tau_l, \tau_u \in (0, 1)$ be drop tolerances and prescribe a pivoting tolerace $\alpha \in (0, 1]$.
Output: $\Pi A\Sigma \approx UDL$.

1. **for** $i = n$ to 1 **do**
2. $m_i = n_i = iter = 0$
3. $satisfied\_p = satisfied\_q = false$
4. **while** not $satisfied\_p$ **do**
5. $iter = iter + 1$
6. $z_i^{(0)} = e_i$
7. **for** $j = i + 1$ to n **do**
8. $q_j^{(i-1)} = w_j^{(n-j)}(\Pi A\Sigma)e_i$
9. $z_i^{(j-i)} = z_i^{(j-i-1)} - (\frac{q_j^{(i-1)}}{d_{jj}})z_j^{(n-j)}$
10. For all $l \ge j$, if $|z_{li}^{(j-i)}| < \tau_z$, then set $z_{li}^{(j-i)} = 0$.
11. **end for**
12. If $iter = 1$, then set $p_i^{(i-1)} = e_i^T(\Pi A\Sigma)z_i^{(n-i)}$. Otherwise set $p_i^{(i-1)} = q_i^{(i-1)}$
13. **for** $j = i - 1$ to 1 **do**
14. $p_i^{(j-1)} = e_j^T(\Pi A\Sigma)z_i^{(n-i)}$
15. **end for**
16. **if** $|p_i^{(i-1)}| < \alpha\ max_{m\le i}|p_i^{(m-1)}|$ **then**
17. $m_i = m_i + 1,\quad \pi_{m_i}^{(i)} = I_n$.
18. $satisfied\_q = false$
19. Choose k such that $|p_i^{(k-1)}| = max_{m\le i}|p_i^{(m-1)}|$.
20. interchange the rows i and k of $\pi_{m_i}^{(i)}$ and the elements $p_i^{(i-1)}$ and $p_i^{(k-1)}$
21. $\Pi = \pi_{m_i}^{(i)}\Pi$
22. **end if**
23. $satisfied\_p = true$
24. **if** not $satisfied\_q$ **then**
25. $w_i^{(0)} = e_i^T$
26. **for** $j = i + 1$ to n **do**
27. $p_j^{(i-1)} = e_i^T(\Pi A\Sigma)z_j^{(n-j)}$
28. $w_i^{(j-i)} = w_i^{(j-i-1)} - (\frac{p_j^{(i-1)}}{d_{jj}})w_j^{(n-j)}$
29. For all $l \ge j$, if $|w_{il}^{(j-i)}| < \tau_w$, then set $w_{il}^{(j-i)} = 0$.
30. **end for**
31. $q_i^{(i-1)} = p_i^{(i-1)}$
32. **for** $j = i - 1$ to 1 **do**
33. $q_i^{(j-1)} = w_i^{(n-i)}(\Pi A\Sigma)e_j$
34. **end for**
35. **if** $|q_i^{(i-1)}| < \alpha\ max_{m\le i}|q_i^{(m-1)}|$ **then**
36. $n_i = n_i + 1,\quad \sigma_{n_i}^{(i)} = I_n$
37. $satisfied\_p = false$
38. Choose l such that $|q_i^{(l-1)}| = \max_{m\le i}|q_i^{(m-1)}|$.
39. interchange the columns i and l of $\sigma_{n_i}^{(i)}$ and the elements $q_i^{(i-1)}$ and $q_i^{(l-1)}$
40. $\Sigma = \Sigma\sigma_{n_i}^{(i)}$
41. **end if**
42. $satisfied\_q = true$
43. **end if**
44. **end while**
45. $d_{ii} = p_i^{(i-1)}$
46. **for** $j = i + 1$ to n **do**
47. $L_{ji} = \frac{q_j^{(i-1)}}{d_{jj}},\quad U_{ij} = \frac{p_j^{(i-1)}}{d_{jj}}$
48. If $|L_{ji}| < \tau_l$, then set $L_{ji} = 0$. Also if $|U_{ij}| < \tau_u$, then set $U_{ij} = 0$.
49. **end for**
50. **end for**
51. Return $L = (L_{ji})_{1\le j,i\le n}$, $D = diag(d_{ii})_{1\le i\le n}$, $U = (U_{ij})_{1\le i,j\le n}$, Π and Σ.

3 Numerical results

In this section, we have considered 8 artificial linear systems where the coefficient matrices are downloaded from [3] and the exact solution of these systems is the vector $[1, \cdots, 1]^T$. We have used two parameters 0.75 and 1.0 as α to compute the $IULBF$ preconditioner with complete pivoting strategy. We have used the command $GMRES$ in Matlab software to solve the original and the left precoditioner systems. We have used 10 as the number of restarts for the $GMRES$ method. The stopping criterion for all linear systems is satisfied when the relative residual is less than 10^{-6}. We have considered the zero vector as the initial solution for all linear systems. The density of all preconditioners is defined as:

$$density = \frac{nnz(L) + nnz(U)}{nnz(A)},$$

where $nnz(L)$, $nnz(U)$ and $nnz(A)$ refer to the number of nonzero entrries of matrices L, U and A, respectively. To compute all of the precoditioners we have considered all of the drop tolerance parameters equal to 0.1 .
Table 1, shows the matrix properties and the information of $GMRES$ method to solve the original linear systems. In this table, n and nnz are the dimension and the number of nonzero entries of the matrix.

Table 1: matrix properties and information of the $GMRES(10)$ method

Matrix	n	nnz	without preconditioner			
			$outer$	$inner$	$flag$	$Itime$
$bfwa62$	62	450	161	2	0	0.5252
$tub100$	100	396	724	10	1	12.5472
$bwm200$	200	796	5000	10	1	14.3536
$saylr1$	238	1128	5000	10	1	13.2154
$cage7$	340	4380	2	8	0	0.0134
$tols340$	340	2196	3881	10	1	13.5088
$bfwb398$	398	1654	3	9	0	0.0778
$olm500$	500	1996	4023	10	1	9.0694

In all the tables, the parameters $outer$, $inner$ and $flag$ indicate the $outer$ iterations, the $inne$r iterations and the status of the convergence for $GMRES(10)$ method.

Table 2: properties of the IULBF preconditioner

Method	IULBF			
Matrix	$density$	$outer$	$inner$	$flag$
$bfwa62$	0.9111	2	8	0
$tub100$	1.0051	1	10	0
$bwm200$	1	4	4	0
$saylr1$	0.9592	4	7	0
$cage7$	0.4841	1	8	0
$tols340$	0.9039	2	10	0
$bfwb398$	0.8368	1	6	0
$olm500$	1.1839	4	7	0

In Tables 1 – 3, when $flag$ is equal to 0, it means that the method has been converged to the desired tolerance within the 2500 outer iterations. $flag = 1$ shows that we can not obtain the convergence in 2500 number of iterations.

Table 3: properties of the IULBFP(0.75) and IULBFP(1.0) preconditioners

Method	IULBFP(0.75)						IULBFP(1.0))					
Matrix	*density*	*Rpiv*	*Cpiv*	*outer*	*inner*	*flag*	*density*	*Rpiv*	*Cpiv*	*outer*	*inner*	*flag*
$bfwa62$	0.9022	3	2	2	8	0	0.9000	4	4	2	8	0
$tub100$	1.0657	24	22	1	10	0	1.1086	23	23	1	9	0
$bwm200$	1.1131	51	45	12	9	0	1.1256	51	51	10	2	0
$saylr1$	0.9592	0	0	4	7	0	0.9592	0	0	4	7	0
$cage7$	0.4780	0	0	1	8	0	0.4780	0	0	1	8	0
$tols340$	0.3679	37	76	1	7	0	0.3657	40	77	1	7	0
$bfwb398$	0.8368	0	0	1	6	0	0.8368	0	0	1	6	0
$olm500$	0.9965	499	249	3	6	0	0.9965	499	249	3	6	0

In Tabe 3, notation $IULBFP(\alpha)$ refers to the $IULBF$ preconditioner with complete pivoting strategy which is computed by the parameter α. The columns *Rpiv* and *Cpiv* show the total number of row and column pivoting. In Tables 2 and 3, the information in the columns $flag$, *outer* and *inner* associated to the three preconditioners indicate that for all of the matrices, one of the preconditioners $IULBFP(1.0)$ or $IULBFP(0.75)$ gives better results of the $GMRES(10)$ method than the $IULBF$ preconditioner. This means that the complete pivoting strategy with one of the values $\alpha = 1.0$ or $\alpha = 0.75$ has a good effect on the quality of the $IULBF$ preconditioner.
If we compare the columns $flag$, *outer* and *inner* in Table 2 by the columns $flag$, *outer* and *inner* of Table 1, then it is clear that the two preconditioners $IULBFP(1.0)$ and $IULBFP(0.75)$ are useful tools to decrease the number of iterations of the $GMRES(10)$ method.

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Constructing an $\mathcal{H}$-matrix via Randomized Algorithms

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Abstract

The key point in constructing an $\mathcal{H}$-matrix is to approximate certain subblocks $\boldsymbol{D}_{n'\times m'}$ of a dense matrix $\boldsymbol{A}_{n\times m}$ by data-sparse low-rank matrices that can be represented as $\boldsymbol{R}_{n'\times m'} = \boldsymbol{U}_{n'\times k}\cdot \boldsymbol{V}^T_{k\times m'}$, with $k \ll \min\{n', m'\}$ as the actual rank of $\boldsymbol{R}$. To obtain $\boldsymbol{R}$ from $\boldsymbol{D}$, the most accurate method is based on SVD which is computationally expensive and needs $\mathcal{O}(n'm'\min\{n', m'\})$ operations. In this paper, we consider various randomized algorithms to obtain such approximations with cost $\mathcal{O}(m'n'k)$. We confirm the advantages of these algorithms applied to a BEM model numerically.

Keywords: Hierarchical matrices, low-rank approximation, randomized algorithm
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

$\mathcal{H}$-matrices provide an inexpensive but sufficiently accurate approximation to dense matrices as they appear in boundary element methods (BEM). Solving integral equations by BEM, finally lead to a linear system of equations:

$$\boldsymbol{A}\cdot \boldsymbol{x} = \boldsymbol{b}. \tag{1}$$

The resulting matrix $\boldsymbol{A}_{n\times n}$ is dense and requires complexity $\mathcal{O}(n^2)$ for its storage as well as matrix-vector multiplication. For computing matrix-matrix multiplication and inversion, this cost would be $\mathcal{O}(n^3)$, which for large-scale computations is prohibitively expensive. The hierarchical matrix technique provides a data-sparse structure by which all $\mathcal{H}$-matrix arithmetic can be performed in almost optimal complexity $\mathcal{O}(n\log^q n)$ with moderate constant q.

To build an $\mathcal{H}$-matrix approximation $\boldsymbol{A}_{\mathcal{H}}$ to a given dense matrix $\boldsymbol{A}$, a tree like data-sparse structure is used to store $\boldsymbol{A}$ such that the leaves of the tree are dense or low-rank matrices ($\mathcal{R}(k)$-matrices). A low-rank matrix stored in so-called $\mathcal{R}(k)$-format in the following sense:

Definition 1.1. A matrix block $\boldsymbol{R}_{n'\times m'}$, is called to be stored in an $\mathcal{R}(k)$-matrix representation, if we have $\boldsymbol{R} = \boldsymbol{U}\cdot \boldsymbol{V}^T$, where the two matrices $\boldsymbol{U}_{n'\times k}$ and $\boldsymbol{V}_{m'\times k}$ are dense matrices. We call $\boldsymbol{R}$ a low-rank or $\mathcal{R}(k)$-matrix.

*Speaker

The rank k is assumed to be small compared to the matrix size n', m'. Therefore, we obtain considerable savings in the storage and work complexities of an $\mathcal{R}(k)$-matrix compared to a full matrix, i.e., $(n' + m')k$ versus $n'm'$ memory cells. In the other hand, if a subblock can not be approximated by an $\mathcal{R}(k)$-matrix, it will be represented by a full ran dense matrix.

1.1 Model problem

As an application of $\mathcal{H}$-matrices we consider a realistic example, namely discretization of boundary integral operator associated with Laplace's equation:

$$\alpha u(x) + \int_\Gamma \kappa(x,y)u(y)ds_y = \mathcal{F}(x), \quad x \in \Gamma := \partial([0,1]^d) \subset \mathbb{R}^d, \quad d = 2, 1, \tag{2}$$

with a given right-hand side $\mathcal{F}$. The kernel function $\kappa(x,y)$ is chosen as $\frac{1}{4\pi}\frac{1}{|x-y|}$ and $-\frac{1}{2\pi}\log|x-y|$ for $d = 2$ and $d = 1$ respectively. In order to solve equation (2) numerically, the domain of integration Γ is divided into triangles $\Gamma = \cup_{i\in\mathcal{I}}\pi_i$, $\mathcal{I} = \{0, \ldots, n-1\}$. Applying the standard *Galerkin* method with piecewise constant ansatz functions $\{\varphi_i\}_{i\in\mathcal{I}}$, the equation (2) will be transformed to a linear system with the coefficient matrix $\boldsymbol{A} := (a_{ij})_{i,j\in\mathcal{I}}$, $a_{ij} := \int_\Gamma\int_\Gamma \varphi_i(x)\kappa(x,y)\varphi_j(y)ds_y ds_x$.

Examples of approximated $\mathcal{H}$-matrix $\boldsymbol{A}_\mathcal{H}$ of $\boldsymbol{A}$ with $n = 1024$, rank $k = 7$, and in one and two dimensions are shown in Fig. 1, where the dense blocks are represented in red color while the green blocks are those that approximated by low-rank matrices.

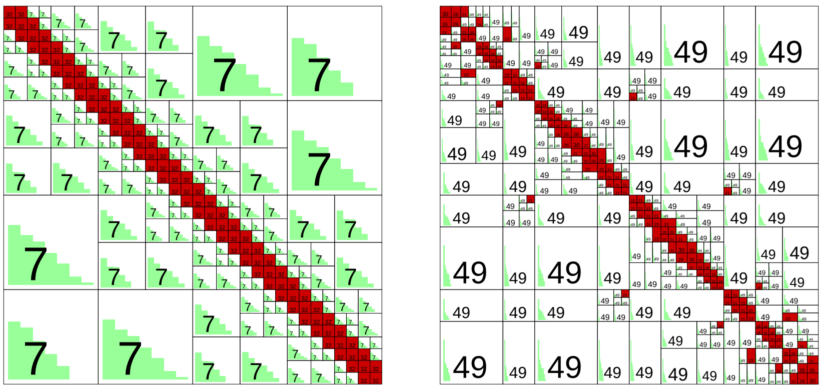

Figure 1: $\mathcal{H}$-matrices corresponding to BEM model for $d = 1$ (left), and $d = 2$ (right).

1.2 Low-rank approximations

Now, the question is how we can obtain a low-rank matrix from a dense matrix? There are several ways to generate a low-rank approximant for each capable block of the original matrix. A class of analytical methods are including but not limited to Taylor series approximation, multipole expansion, and interpolation. On the other hand, algebraic techniques are singular value decomposition (SVD), pseudo-skeletal approximation, cross approximation and its variants, rank-revealing LU and QR factorization.

In this work our focus is on obtaining such approximations based on the SVD that enables us to compute the optimal low-rank approximation of a matrix . An algorithm for constructing a low-rank matrix $\boldsymbol{R} = \boldsymbol{U}\boldsymbol{V}^T$ ($\boldsymbol{U} \in \mathbb{R}^{n' \times k}, \boldsymbol{V} \in \mathbb{R}^{m' \times k}$) from a dense matrix $\boldsymbol{D}_{n' \times m'}$ can be computed by applying a direct SVD [1] as follows:

$$\boldsymbol{D} = \hat{\boldsymbol{U}}_{n' \times p} \hat{\boldsymbol{\Sigma}}_{p \times p} \hat{\boldsymbol{V}}^T_{p \times m'} \approx \tilde{\boldsymbol{U}}_{n' \times k} \tilde{\boldsymbol{\Sigma}}_{k \times k} \tilde{\boldsymbol{V}}^T_{k \times m'} = (\tilde{\boldsymbol{U}}_{n' \times k})(\tilde{\boldsymbol{\Sigma}}_{k \times k} \tilde{\boldsymbol{V}}^T_{k \times m'}) = \boldsymbol{U}\boldsymbol{V}^T,$$

where $p = \min\{n', m'\}$, $\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{V}}$ are the first k columns of the unitary matrices $\hat{\boldsymbol{U}}, \hat{\boldsymbol{V}}$ and the diagonal matrix $\tilde{\boldsymbol{\Sigma}} = \mathbf{diag}(s_0, s_1, \cdots, s_{k'-1}, 0, \cdots, 0)$ is obtained by retaining the first k diagonal elements of $\hat{\boldsymbol{\Sigma}}$ with $s_0 \geq s_1 \geq \cdots \geq s_{k-1} \geq 0$ as singular values. The cost of this algorithm is $\mathcal{O}(n'm' \min\{n', m'\} + m'k)$, which is impractical for large problem sizes.

2 Main results

2.1 Randomized algorithms

Recently, randomized algorithms has been considered as a class of simple but highly efficient tool for computing approximate factorization of matrices that have low numerical rank [3]. Given a matrix $\boldsymbol{D}_{n' \times m'}$, these randomized algorithms operate in two stages. In the first stage, by means of randoms sampling, a low-dimensional subspace is constructed to approximate the range of $\boldsymbol{D}$. The second stage devoted to restricting $\boldsymbol{D}$ to the obtained subspace and performing a standard deterministic factorization (e.g., QR and SVD) of the reduced matrix. To be more precise, the following algorithm will compute an $\mathcal{R}(k)$-matrix factorization of a dense matrix $\boldsymbol{D}_{n' \times m'}$ such that $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{V}^T$.

procedure build_Rk($\boldsymbol{D}, n', m', k, \boldsymbol{R} = \boldsymbol{U}\boldsymbol{V}^T$)
1: Draw an $m' \times k$ Guassian random matrix $\boldsymbol{G}$;
2: Form an $n' \times k$ sample matrix $\boldsymbol{W} = \boldsymbol{D}\boldsymbol{G}$;
3: Form an $n' \times k$ orthogonal matrix $\boldsymbol{Q}$ s.t. $\boldsymbol{W} = \boldsymbol{Q}_W \boldsymbol{R}_W$;
4: Form the $k \times m'$ matrix $\boldsymbol{B} = \boldsymbol{Q}_W^T \boldsymbol{D}$;
5: Compute the SVD of the small matrix $\boldsymbol{B}$: $\boldsymbol{B} = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{V}}^T$;
6: Form the matrix $\boldsymbol{U} = \boldsymbol{Q}_W \hat{\boldsymbol{U}}$;
7: Form the matrix $\boldsymbol{V} = \hat{\boldsymbol{V}}\hat{\boldsymbol{\Sigma}}$;
end;

Algorithm 2.1: Building an $\mathcal{R}(k)$-matrix with fixed rank k from a dense matrix $\boldsymbol{D}$.

In the following we use two modifications of the previous original randomized algorithm. In the first one, to avoid of taking the SVD of the $k \times m'$ matrix $\boldsymbol{B}$, the eigendecomposition of the smaller $k \times k$ matrix $\boldsymbol{B}\boldsymbol{B}^T$ is exploited. We refer to this as `RandSVD1`. Thus, only the lines 5-7 will be changed as follows: Let $\boldsymbol{B} = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{V}}^T$, then

$$\begin{cases} \boldsymbol{B}\boldsymbol{B}^T & = (\boldsymbol{Q}_W^* \boldsymbol{D})(\boldsymbol{Q}_W^* \boldsymbol{D})^T = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Sigma}}^2 \hat{\boldsymbol{U}}^T, \\ \boldsymbol{B}^T \hat{\boldsymbol{U}} & = \hat{\boldsymbol{V}}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{U}}^T \hat{\boldsymbol{U}} = \hat{\boldsymbol{V}}\hat{\boldsymbol{\Sigma}}, \end{cases} \implies \boldsymbol{D} := (\boldsymbol{Q}_W \hat{\boldsymbol{U}})(\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{V}}^T) = \boldsymbol{U}\boldsymbol{V}^T. \quad (3)$$

As the second modification, namely `RandSVD2`, we perform an economic QR factorization of $\boldsymbol{B}^T$ instead of forming $\boldsymbol{B}\boldsymbol{B}^T$. Let $\boldsymbol{B}^T = \hat{\boldsymbol{Q}}\hat{\boldsymbol{R}}$, where $\hat{\boldsymbol{R}}$ is a $k \times k$ matrix. Next performing the SVD gives us $\hat{\boldsymbol{R}} = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{V}}^T$. Therefore we have

$$\boldsymbol{D} = \boldsymbol{Q}_W \boldsymbol{B} = \boldsymbol{Q}_W (\hat{\boldsymbol{Q}}\hat{\boldsymbol{U}}\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{V}}^T)^T = (\boldsymbol{Q}_W \hat{\boldsymbol{V}}\hat{\boldsymbol{\Sigma}})(\hat{\boldsymbol{U}}^T \hat{\boldsymbol{Q}}) = \boldsymbol{U}\boldsymbol{V}^T.$$

Note that, the cost of both algorithms is bounded by $\mathcal{O}(n'm'k)$. We test our randomized algorithm numerically when applied to our BEM model and compare the obtained results with applying a direct SVD to construct an $\mathcal{H}$-matrix in one and two dimensions.

n	SVD t[s]	RandSVD1 t[s]	RandSVD2 t[s]
64	0.001	0.001	0.0001
128	0.007	0.003	0.006
256	0.021	0.011	0.014
512	0.127	0.042	0.041
1024	1.149	0.158	0.160
2048	11.130	0.631	0.631
4096	138.572	2.536	2.554
8192	1483.809	10.174	10.247
16384	13036.270	40.896	41.106
32768	104290.160	164.459	165.362
65536	730031.120	851.235	860.943

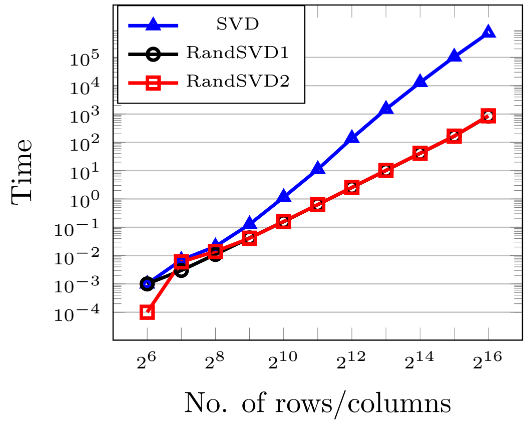

Figure 2: Timing and corresponding plotting for different n in $\mathcal{H}$-matrix construction with rank $k = 10$ for $d = 1$.

n	SVD t[s]	RandSVD1 t[s]	RandSVD2 t[s]
1024	1.186	0.954	0.950
4096	48.103	15.181	14.189
16384	5219.187	238.980	239.299
65536	835069.92	3793.331	3811.280

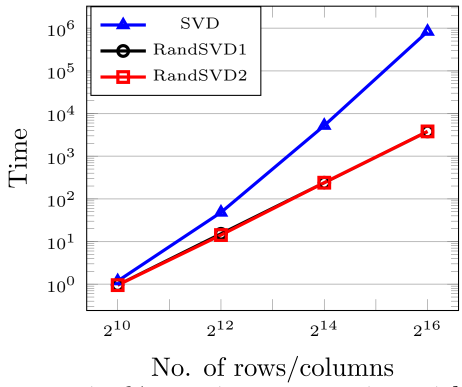

Figure 3: Timing and corresponding plotting for different n in $\mathcal{H}$-matrix construction with rank $k = 10$ for $d = 2$.

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Talk Global CMRH method for solving general coupled matrix equations pp.: 1–4

Global CMRH method for solving general coupled matrix equations

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Abstract

In the present paper, we propose a global CMRH method for solving the large and sparse general coupled matrix equations. We consider the general coupled matrix equations as a linear operator and to give a natural way to derive this new method. A numerical example is given to illustrate the effectiveness of the presented method.

Keywords: linear matrix equations, CMRH method, global Hessenberg
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

We consider the solution of the general coupled matrix equations of the form

$$\sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \quad i = 1, \ldots, p, \tag{1}$$

where $A_{ij} \in R^{m\times m}$, $B_{ij} \in R^{n\times n}$, $C_i \in R^{m\times n}$, $i,j = 1,2,...,p$, are given matrices and $X_i \in R^{m\times n}$, $i = 1,2,...,p$, are the unknown matrices.

For applying the global CMRH method to Eq. (1), as [5], we define the linear operator $\mathcal{M}$ as follows

$$\mathcal{M} : \underbrace{\mathbb{R}^{m\times n} \times \ldots \times \mathbb{R}^{m\times n}}_{p} \longrightarrow \mathbb{R}^{mp\times n}$$

$$X = (X_1^T, X_2^T, \ldots, X_p^T)^T \longrightarrow \mathcal{M}(X) = (\mathcal{A}_1(X)^T, \mathcal{A}_2(X)^T, \ldots, \mathcal{A}_p(X)^T)^T,$$

where

$$\mathcal{A}_i(X) = \sum_{j=1}^{p} A_{ij}X_jB_{ij}, \quad i = 1,2,\ldots,p.$$

*Speaker

Using the linear operator $\mathcal{M}$, we can write Eq. (1) as

$$\mathcal{M}(X) = C, \tag{2}$$

where $C = (C_1^T, C_2^T, \ldots, C_p^T)^T$. In the next section, we use the linear matrix operator $\mathcal{M}$ to present a global CMRH method for solving Eq. (1). As [2], we use the matrix product $*$, for the following product

$$\mathcal{V}_k * \alpha = \sum_{j=1}^{k} \alpha_j V_j,$$

where $\mathcal{V}_k = [V_1, V_2, \ldots, V_k]$, $V_j \in \mathbb{R}^{m\times n}, j = 1, \ldots, k$ and $\alpha \in \mathbb{R}^k$. By the same way, we set

$$\mathcal{V}_k * H = [\mathcal{V}_k * H(:,1), \mathcal{V}_k * H(:,2), \ldots, \mathcal{V}_k * H(:,k)]$$

where H is an $k \times k$ matrix and $H(:,j)$ denotes the jth column of H. It is easy to see that the following relations are satisfied

$$\mathcal{V}_m * (\alpha + \beta) = \mathcal{V}_m * \alpha + \mathcal{V}_m * \beta$$

2 Main results

In this section, we propose a new global CMRH method for solving (1). Let $X^{(0)} = (X_1^{(0)T}, X_2^{(0)T}, \ldots, X_p^{(0)T})^T \in \mathbb{R}^{mp\times n}$ be a given initial approximate solution of the exact solution of Eq. (1) and $R^{(0)} = C - \mathcal{M}(X^{(0)})$ its associated residual. By assuming k is smaller than the grade of $R^{(0)}$, we define the matrix Krylov subspace as follows

$$\mathcal{K}_k(\mathcal{M}, R^{(0)}) = \text{span}\{R^{(0)}, \mathcal{M}(R^{(0)}), \ldots, \mathcal{M}^{(k-1)}(R^{(0)})\}.$$

By using the global Hessenberg process with maximum strategy [1], we can construct a Krylov basis $V_1, V_2, \ldots, V_k$ of $\mathcal{K}_k(\mathcal{M}, R^{(0)})$. As known [1], this process generates $\mathcal{V}_k = [V_1, V_2, ..., V_k]$ and the $(k+1)\times k$ upper Hessenberg matrix $\widetilde{H}_k$ which satisfy the following relation

$$\mathcal{M}(\mathcal{V}_k) \doteq [\mathcal{M}(V_1), \mathcal{M}(V_2), \ldots, \mathcal{M}(V_k)] = \mathcal{V}_{k+1} * \widetilde{H}_k. \tag{3}$$

At the kth iterate, a correction $W^{(k)}$ is determined in the matrix Krylov subspace $\mathcal{K}_k(\mathcal{M}, R^{(0)})$ such that

$$X^{(k)} - X^{(0)} = W^{(k)} \in \mathcal{K}_k(\mathcal{M}, R^{(0)}).$$

By using the basis $\mathcal{V}_k = [V_1, V_2, ..., V_k]$ constructed via the global Hessenberg process, we can write

$$X^{(k)} = X^{(0)} + \mathcal{V}_k * d_k,$$

where $d_k \in \mathbb{R}^k$. The corresponding residual is then expressed by

$$R^{(k)} = R^{(0)} - \mathcal{M}(\mathcal{V}_k) * d_k.$$

From the fact that $R^{(0)} = \beta V_1 = \mathcal{V}_{k+1} * (\beta e_1^{(k+1)})$, where $\beta = |(R^{(0)})_{i_0,j_0}|$ with i_0 and j_0 such that $|(R^{(0)})_{i_0,j_0}| = \max\{(R^{(0)})_{i,j}\}_{1\le i\le mp}^{1\le j\le n}$, the use of Eq. (3) implies that

$$R^{(k)} = \mathcal{V}_{k+1} * (\beta e_1^{(k+1)} - \widetilde{H}_k d_k)$$

The vector d_k can be obtained by imposing the following minimizing norm condition

$$\| R^{(k)} \|_F = \min_{d\in\mathbb{R}^k} \| \mathcal{V}_{k+1} * (\beta e_1^{(k+1)} - \widetilde{H}_k d) \|_F \ . \tag{4}$$

To solve this problem is equivalent to the global GMRES method. As global CMRH method [1] and CMRH method [4], instead of solving Eq. (4), we will solve a smaller problem, namely, minimizing just the Euclidean norm of the coefficient vector in Eq. (4). So, we will obtain d_k from the minimization problem

$$\min_{d\in\mathbb{R}^k} \| (\beta e_1^{(k+1)} - \tilde{H}_k d) \|_2 \tag{5}$$

In practice, the computational and storage requirement grow with iterations. So, we have to use a restarting strategy. The main steps of the restarting global CMRH (denoted by Gl-CMRH) method for solving the general coupled matrix equations can be summarized as shown in Algorithm 1.

Algorithm 1: Gl-CMRH(k) Method

1. Choose $X^{(0)}$, k, and a tolerance ϵ. Compute $R^{(0)} = C - \mathcal{M}(X^{(0)})$.
2. Determine i_0 and j_0 such that $|(R^{(0)})_{i_0,j_0}| = \max\{(R^{(0)})_{i,j}\}_{1\le i\le mp}^{1\le j\le n}$;
   $\beta = |(R^{(0)})_{i_0,j_0}|$; $V_1 = R^{(0)}/\beta$; $p_{1,1} = i_0$; $p_{1,2} = j_0$;
3. Construct the basis $V_1, V_2, \ldots, V_k$ and the matrix $\widetilde{H}_k$ by the global Hessenberg process with maximum strategy [1].
4. Determine d_k as the solution of $\min_{d\in\mathbb{R}^k} \| \beta e_1^{(k+1)} - \widetilde{H}_k d \|_2$.
   Compute the approximate solution $X^{(k)} = X^{(0)} + \mathcal{V}_k * d_k$.
5. Compute $R^{(k)} = C - \mathcal{M}(X^{(k)})$.
   If $\| R^{(k)} \|_F \le \epsilon$, Stop;
   else $X^{(0)} = X^{(k)}$, $R^{(0)} = R^{(k)}$; goto 2.

3 Numerical results

In this section, some numerical results are presented to compare the performance of the Gl-CMRH method with Gl-GMRES method [3]. We consider the general coupled matrix equations

$$\begin{cases} A_{11}X_1B_{11} + A_{12}X_2B_{12} = C_1, \\ A_{21}X_1B_{21} + A_{22}X_2B_{22} = C_2, \end{cases}$$

Table 1

	Gl-GMRES(5)			Gl-CMRH(5)		
m	iters	CPU-Time	Err	iters	CPU-Time	Err
100	48	1.83	5.514703e-006	41	0.81	5.933826e-006
200	46	6.80	7.772339e-006	40	3.98	7.399977e-006
300	44	18.03	1.087575e-005	40	11.61	8.847238e-006
400	44	39.80	1.087349e-005	39	26.30	1.099325e-005

where the coefficient matrices are $m \times m$ matrices and

$$\begin{aligned} A_{11} &= \text{tridiag}(-1,6,-1), & B_{11} &= \text{tridiag}(1,8,-1),\\ A_{12} &= 0.1I_m, & B_{12} &= \text{tridiag}(1,0,1),\\ A_{21} &= 0.1I_m, & B_{21} &= \text{tridiag}(-2,1,-2),\\ A_{22} &= \text{tridiag}(-1,-3,-1), & B_{22} &= \text{tridiag}(1,6,2). \end{aligned}$$

The right-hand side of the corresponding system $\mathcal{M}(X) = C$ was taken such that $X = (X_1, X_2)$ is the exact solution of the system with $X_1 = I_m$ and $X_2 = E_m$, where E_m is $m \times m$ matrix that all of components are equal to one. The initial guess was taken to be zero and the stopping criterion $\|R_j\|_F/\|R_0\|_F < 10^{-8}$ was used for the Gl-CMRH(5) method with Gl-GMRES(5) method. The numerical results are given in Table 1. In this table, "iters" and "CPU-Time" represent the number of iterations and CPU-Time(s) needed for the convergence, respectively, and "Err"stands for

$$Err = \|(X_1, X_2) - (\bar{X}_1, \bar{X}_2)\|_\infty,$$

where $(\bar{X}_1, \bar{X}_2)$ is the approximate solution computed by the numerical methods. As we observe, for this example, the numerical results in terms of iterations and CPU-Time(s) for the Gl-CMRH(5) are better than those of the Gl-GMRES(5) proposed in [3]. From our experiments. we saw that, Gl-CMRH(k) algorithm in general is more suitable than the Gl-GMRES algorithm proposed in [3] for solving the general coupled matrix equations, especially for the large problems.

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How to recognize a fictitious signature?

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Abstract

We present a method to detect an original signature from a fictitious signature with high probability.

Keywords: Signature, Erosion, Multiple knot B-spline

Mathematics Subject Classification [2010]: 68U10, 97P50, 41A10

1 Introduction

It is well-known that signature is used for verification purpose. Based on the application, verification can be performed either Offline or Online. Online systems use dynamic information of a signature captured at the time that the signature is made. Offline systems work on the scanned image of a signature. Khatra in [3], has been used various geometric features to distinguish signatures of different persons. In [1], Chadha et al. introduced a novel method for signature recognition using radial basis function network. In this paper, we want to present a method for offline verification of signatures via isogeometry techniques with smooth multiple knot B-spline functions.

2 Reconstruction

For simplicity, we consider the nodal points in $[0,1]$. We define the simple curve $\mathbf{f}$ and the reconstructed curve $\mathbf{c}$ as follows:

$$\mathbf{f} : [0,1] \longrightarrow \Omega \subset \mathbb{R}^2$$

$$\mathbf{f}(t) = \begin{bmatrix} x(t) \\ f(t) \end{bmatrix},$$

and

$$\mathbf{c} : [0,1] \longrightarrow \Omega \subset \mathbb{R}^2$$
$$\mathbf{c}(t) = \begin{bmatrix} x(t) \\ c(t) \end{bmatrix},$$

where Ω is a polygon domain.

*Speaker

Definition 2.1. Given n control points $d_1, ..., d_n$ and a knot vector $\mathcal{J} = \{t_1, t_2, \ldots, t_{n+m+1}\}$ where $t_1 \leq t_2 \leq \ldots \leq t_{n+m+1}$, the **B-spline curve** defined by the control points and the knot vector is

$$\mathbf{c}(t) = \sum_{i=1}^{n} d_i B_{i,m,\mathcal{J}}(t), \qquad t \in [0,1],$$

where $B_{i,m,\mathcal{J}}$'s are B-spline basis functions of order m.

We should derive the knot vector $\mathcal{J}$ as well as the control points d_i's to calculate

$$\min_{d_i,t_i} \|\mathbf{f} - \mathbf{c}\|^2_{L_2([0,1])} = \min_{d_i,t_i} \|\mathbf{f} - \sum_{i=1}^{n} d_i B_{i,2,\mathcal{J}}\|^2_{L_2([0,1])}.$$

In continue, we want to find the nodal and control points, simultaneously. Since, we consider the knot vector $\mathcal{J} \subset [0,1]$, the first node $t_1 = 0$ and the end node $t_{n+3} = 1$. Also, we use the notation

$$\mathbf{f}^{(k)}(t) := \begin{bmatrix} x^{(k)}(t) \\ f^{(k)}(t) \end{bmatrix}, \qquad k = 0,1,2, \qquad t \in [0,1].$$

and define the following sets:

$\mathcal{I}_1 := \{0, \ 1\}$,

$\mathcal{I}_2 := \{t \in [0,1] \mid f^{'}(t)$ no exists, (the critical points)$\}$,

$\mathcal{I}_3 := \{t \in [0,1] \mid x'(t)$ no exists, (the critical point)$\}$,

$\mathcal{I}_4 := \{t \in [0,1] \mid x'(t) = 0$ (the critical point)$\}$,

$\mathcal{I}_5 := \{t \in [0,1] \mid f^{'}(t) = 0,$ (the critical points)$\}$,

$\mathcal{I}_6 := \{t \in [0,1] \mid f^{''}(t) = 0,$ (the inflection points)$\}$,

$\mathcal{I}_7 := \{t \in [0,1] \mid t \notin \bigcup_{j=1}^{6} \mathcal{I}_j$ & t is the local maximum of the curvature function $\kappa(t)\}$.

Also, we find the nodal and control points such a way that the points in $\bigcup_{j=1}^{7} \mathcal{I}_j$ of $\mathbf{c}$ and $\mathbf{f}$ are coincided. For details, we refer to [4].

3 Algorithm

In this section we present a method how to recognize the fictitious signature. To this end, we remove the noisy effects that are usually happen because of hand motion during the signaturing.

Definition 3.1. The point $\begin{bmatrix} x(t^*) \\ y(t^*) \end{bmatrix}$, is a vertex of the noisy curve

$$\mathbf{f} : [0,1] \longrightarrow \Omega \subset \mathbb{R}^2$$

$$\mathbf{f}(t) = \begin{bmatrix} x(t) \\ f(t) \end{bmatrix},$$

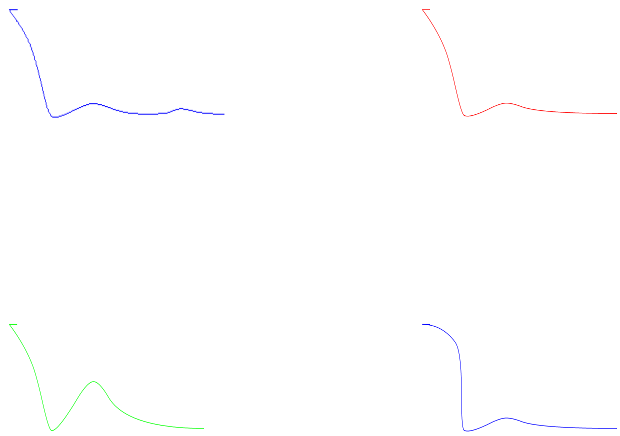

Figure 1: Simulations of a signature (top left signature is original and the other are simulated signatures).

if for a given $\delta > 0$ and $t \in (t^* - \delta, t^* + \delta)$
1. one of the following conditions holds:

$$\begin{array}{c} x(t) \leq x(t^*), \\ x(t) \geq x(t^*), \\ \\ y(t) \leq y(t^*), \\ y(t) \geq y(t^*), \end{array}$$

2. there exists an $\epsilon > 0$ such that:

$$\|x(t) - x(t^*)\| > \epsilon \quad \text{and} \quad \|y(t) - y(t^*)\| > \epsilon.$$

To characterize a fictitious signature, our strategy is finding the curvature on some specified points. Now, we state the main algorithm to detect the fictitious signature from an original one.

Algorithm 1: Recognition of a fictitious signature

1. Scan the signature;
2. Use erosion techniques for thinning the signature curve [2];
3. $S := \emptyset$;
4. **For** $e = 1, 2, \ldots$
5. Find $p \in A_e := \bigcup_{i=1}^{7} \mathcal{I}_j$ where $\mathcal{I}_j$s are located on the simple curve $\mathbf{c}_e$;
6. Find the control and nodal points;
7. Derive the B-spline curve B_e;
8. Put $A_e := A_e \cup H_e$ where H_e is made up of auxiliary points that lie in the middle of

both consecutive points in A_e;
9. Find the curvature κ of the points in A_e on the B-spline curve B_e;
10. Put $E_e := \{i \in \mathbb{N} : x_i \in A_e\}$;
11. Compute $S := S + \sum_{i \in E_e} \|\kappa_{\mathbf{c}_e}(x_i) - \kappa_{B_e}(x_i)\|_{\ell_2}$ where $\kappa_{\mathbf{c}_e}$ and κ_{B_e} are the curvature of the original curve $\mathbf{c}_e$ and B-spline curve B_e on the points in A_e, respectively;
12. **EndFor**
13. If $S > \epsilon$, then the signature would be fictitious.

In Algorithm 1, the identification of a signature depends on the value ϵ. As an example, the first signature in Figure 1 (top left) is original. The three other signatures would be fictitious if $\epsilon > 0.1$ for top right, $\epsilon > 0.7$ for down left and $\epsilon > 0.5$ for down right signatures.

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Inverse eigenvalue problem for a matrix polynomial

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Abstract

Consider an $n \times n$ matrix polynomial $P(\lambda)$ and a set Σ consisting of $k \leq n$ distinct complex numbers. A perturbation of $P(\lambda)$, such that the spectrum of the perturbed matrix polynomial includes the specified set Σ, was recently constructed by Kokabifar, Loghmani, Psarrakos and Karbassi (2015). In this article, we briefly discuss on inverse eigenvalue problem for the case of matrix polynomials as a conceivable application of the topic of the paper.

Keywords: Matrix polynomial, Eigenvalue, Perturbation, Inverse eigenvalue problems

Mathematics Subject Classification [2010]: 15A18, 65F35, 65F18

1 Introduction

Let A be an $n \times n$ complex matrix and let $\mathcal{M}$ be the set of all $n \times n$ complex matrices that have $\mu \in \mathbb{C}$ as a multiple eigenvalue. Malyshev [5] obtained a singular value optimization characterization for the spectral norm distance from A to $\mathcal{M}$. Malyshev's work can be considered as a solution to Wilkinson's problem, that is, the computation of the distance from a matrix $A \in \mathbb{C}^{n \times n}$ with all its eigenvalues simple to the $n \times n$ matrices that have multiple eigenvalues.

In 2008, Papathanasiou and Psarrakos [6] generalized Malyshev's results for the case of matrix polynomials, introducing a (weighted) spectral norm distance from an $n \times n$ matrix polynomial $P(\lambda)$ to the matrix polynomials that have a prescribed $\mu \in \mathbb{C}$ as a multiple eigenvalue, and obtaining an upper and a lower bounds for this distance. A spectral norm distance from $P(\lambda)$ to matrix polynomials that have two distinct eigenvalues, or any $k \leq n$ prescribed eigenvalues, was obtained by Kokabifar, Loghmani, Nazari and Karbassi [3] and Kokabifar, Loghmani, Psarrakos and Karbassi [4], respectively, while constructing a perturbation of $P(\lambda)$ was also considered.

In this article, we are interested to present some conceivable applications of the topic of [4]. Considering the numerous applications of matrices and development of the study and implementation of matrix polynomials, let us concentrate on the subject of *finding a matrix polynomial with some ordered eigenvalues*, extending inverse eigenvalue problem for the case of matrix polynomials and approximating a matrix polynomials with another one that some or all of its eigenvalues are located at desired positions.

*Speaker

For $A_j \in \mathbb{C}^{n\times n}$ $(j = 0,1,\ldots,m)$ and a complex variable λ, we define the *matrix polynomial*

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 = \sum_{j=0}^{m} A_j\lambda^j. \tag{1}$$

If for a scalar $\mu \in \mathbb{C}$ and some nonzero vector $\upsilon \in \mathbb{C}^n$, it holds that $P(\mu)\upsilon = 0$, then the scalar μ is called an *eigenvalue* of $P(\lambda)$ and the vector υ is known as a *(right) eigenvector* of $P(\lambda)$ corresponding to μ. The *spectrum* of $P(\lambda)$, denoted by $\sigma(P)$, is the set of its eigenvalues. The singular values of $P(\lambda)$ are the nonnegative roots of $P(\lambda)^*P(\lambda)$, and they are denoted by $s_1(P(\lambda)) \geq \cdots \geq s_n(P(\lambda))$ (i.e., they are considered in a nondecreasing order) [2].

As it mentioned Kokabifar and etal constructed a perturbation of $P(\lambda)$ such that the perturbed matrix polynomial includes Σ in its spectrum. From now on, for the sake of simplicity and intelligibility some of the results obtained of [4] are reviewed briefly.

Definition 1.1. Let $P(\lambda)$ be a matrix polynomial as in (1) and let $\Delta_j \in \mathbb{C}^{n\times n}$ $(j = 0,1,\ldots,m)$ be arbitrary matrices. Consider perturbations of the matrix polynomial $P(\lambda)$ of the form

$$Q(\lambda) = P(\lambda) + \Delta(\lambda) = \sum_{j=0}^{m}(A_j + \Delta_j)\lambda^j. \tag{2}$$

For $\varepsilon > 0$ and a set of given nonnegative weights $w = \{\omega_0,\ldots,\omega_m\}$, with $\omega_0 > 0$, define the class of admissible perturbed matrix polynomials

$$\mathcal{B}(P,\varepsilon,w) = \{Q(\lambda) \text{ as in (2)} : \|\Delta_j\| \leq \varepsilon\omega_j,\ j = 0,1,\ldots,m\},$$

and the scalar polynomial $w(\lambda) = \omega_m\lambda^m + \omega_{m-1}\lambda^{m-1} + \cdots + \omega_1\lambda + \omega_0$.

Definition 1.2. Consider a complex function f and k distinct scalars $\mu_1,\mu_2,\ldots,\mu_k \in \mathbb{C}$. The *divided difference relative to μ_i and μ_{i+t}* $(1 \leq i \leq k-1,\ 1 \leq t \leq k-i)$ is denoted by $f[\mu_i,\ldots,\mu_{i+t}]$ and is defined by the following recursive formula:

$$f[\mu_i,\ldots,\mu_{i+k}] = \frac{f[\mu_i,\mu_{i+1},\ldots,\mu_{i+k-1}] - f[\mu_{i+1},\mu_{i+2},\ldots,\mu_{i+k}]}{\mu_i - \mu_{i+k}}.$$

Definition 1.3. Suppose that $P(\lambda)$ is a matrix polynomial as in (1) and a set of distinct complex numbers $\Sigma = \{\mu_1,\mu_2,\ldots,\mu_k\}$ $(k \leq n)$ is given. For any scalar $\gamma \in \mathbb{C}$, define the $nk \times nk$ matrix

$$F_\gamma[P,\Sigma] = \begin{bmatrix} P(\mu_1) & 0 & \cdots & 0 \\ \gamma P[\mu_1,\mu_2] & P(\mu_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{k-1}P[\mu_1,\ldots,\mu_k] & \gamma^{k-2}P[\mu_2,\ldots,\mu_k] & \cdots & P(\mu_k) \end{bmatrix}$$

Now, we construct an $n \times n$ matrix polynomial $\Delta_\gamma(\lambda)$ such that the given set $\Sigma = \{\mu_1,\mu_2,\ldots,\mu_k\}$ $(k \leq n)$ is included in the spectrum of the perturbed matrix polynomial $Q_\gamma(\lambda) = P(\lambda) + \Delta(\lambda)$. Without loss of generality, hereafter we can assume that the parameter γ is real nonnegative. Also, for convenience, we set $\rho = nk - k + 1$.

Definition 1.4. Suppose that $u(\gamma) = [u_1(\gamma), \cdots, u_k(\gamma)]^T, v(\gamma) = [v_1(\gamma), \cdots, v_k(\gamma)]^T \in \mathbb{C}^{nk}$, is a pair of left and right singular vectors of $s_\rho(F_\gamma[P,\Sigma])$, respectively, such that $u_j(\gamma), v_j(\gamma) \in \mathbb{C}^n$ for every $j = 1, \ldots, k$. Define the $n\times k$ matrices $U(\gamma) = [\, u_1(\gamma) \ \cdots \ u_k(\gamma)\,]$, $V(\gamma) = [\, v_1(\gamma) \ \cdots \ v_k(\gamma)\,]$.

Suppose that $\gamma > 0$ and $\operatorname{rank}(V(\gamma)) = k$. Consider the quantities $\theta_{ij} = \frac{\gamma}{\mu_i - \mu_j}$, $1 \le i < j \le k$, and for $p = 2, 3, \ldots, k$ define the following vectors

$$\hat{v}_1(\gamma) = v_1(\gamma), \quad \hat{v}_p(\gamma) = v_p(\gamma) + \sum_{i=1}^{p-1}\left[(-1)^i\left(\prod_{j=p-i}^{p-1}\theta_{jp}\right)v_{p-i}(\gamma)\right],$$

the vectors $\hat{u}_p(\gamma), p = 1, \ldots, p$ are defined similarly. Analogously to Definition 1.4, we define the matrices $\hat{U}(\gamma) = [\,\hat{u}_1(\gamma), \cdots, \hat{u}_k(\gamma)]$, $\hat{V}(\gamma) = [\,\hat{v}_1(\gamma), \cdots, \hat{v}_k(\gamma)]$. We also consider the quantities

$$\alpha_{i,s} = \frac{1}{w(|\mu_i|)}\sum_{j=0}^{m}\left(\left(\frac{\bar{\mu}_i}{|\mu_i|}\right)^j \mu_s^j \omega_j\right) \text{ and } \beta_s = \frac{1}{k}\sum_{i=1}^{k}\alpha_{i,s}, \quad i, s = 1, \ldots, k,$$

where we set $\alpha_{i,s} = 1$ whenever $\mu_i = 0$. Then, for nonzero quantities $\beta_i, (i = 1, \ldots, k)$, define

$$\Delta_\gamma = -s_\rho(F_\gamma[P,\Sigma])\hat{U}(\gamma)\operatorname{diag}\left\{\frac{1}{\beta_1}, \frac{1}{\beta_2}, \ldots, \frac{1}{\beta_k}\right\}\hat{V}(\gamma)^\dagger,$$

where $\hat{V}(\gamma)^\dagger$ denotes the *Moore-Penrose pseudoinverse* of $\hat{V}(\gamma)$, and the $n \times n$ matrix polynomial $\Delta_\gamma(\lambda) = \sum_{j=0}^{m}\Delta_{\gamma,j}\lambda^j$, where

$$\Delta_{\gamma,j} = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{1}{w(|\mu_i|)}\left(\frac{\bar{\mu}_i}{|\mu_i|}\right)^j\omega_j\Delta_\gamma\right), \quad j = 1, 2, \ldots, k. \tag{3}$$

Straightforward computations verify that the matrix polynomial $\Delta_\gamma(\lambda)$ satisfies $\Delta_\gamma(\mu_i) = \beta_i\Delta_\gamma$, for $i = 1, \ldots, k$. Notice that $\operatorname{rank}(V(\gamma)) = k$ implies $\hat{v}_i(\gamma) \neq 0$, $(i = 1, \ldots, k)$ and $\hat{V}(\gamma)^\dagger\hat{V}(\gamma) = I_k$. Moreover, since $u(\gamma), v(\gamma)$ is a pair of left and right singular vectors of $s_\rho(F_\gamma[P,\Sigma])$, we have $F_\gamma[P,\Sigma]v(\gamma) = s_\rho(F_\gamma[P,\Sigma])\,u(\gamma)$. Substituting $\hat{u}_1(\gamma), \ldots, \hat{u}_k(\gamma)$ and $\hat{v}_1(\gamma), \ldots, \hat{v}_k(\gamma)$ into these equations yields $s_\rho(F_\gamma[P,\Sigma])\,\hat{u}_i(\gamma) = P(\mu_i)\,\hat{v}_i(\gamma)$, $i = 1, 2, \ldots, k$. Therefore, for the matrix polynomial

$$Q_\gamma(\lambda) = P(\lambda) + \Delta_\gamma(\lambda) = \sum_{j=0}^{m}(A_j + \Delta_{\gamma,j})\,\lambda^j \tag{4}$$

and for every $i = 1, 2, \ldots, k$, it follows $Q_\gamma(\mu_i)\,\hat{v}_i(\gamma) = P(\mu_i)\,\hat{v}_i(\gamma) + \Delta_\gamma(\mu_i)\,\hat{v}_i(\gamma) = 0$. As a consequence, if $\operatorname{rank}(V(\gamma)) = k$, then $\mu_1, \mu_2, \ldots, \mu_k$ are eigenvalues of the matrix polynomial $Q_\gamma(\lambda)$ in (4) with $\hat{v}_1(\gamma), \hat{v}_2(\gamma), \ldots, \hat{v}_k(\gamma)$ as their associated eigenvectors, respectively.

One of straightforward usage of the results is obtaining a matrix polynomial with some prespecified eigenvalues, which can be considered as *inverse eigenvalue problem* for the case of matrix polynomials. In respect of matrices, an inverse eigenvalue problem concerns the reconstruction of a matrix from prescribed spectral data. Inverse eigenvalue problem has a long list of applications in areas such as control theory, mechanics, signal processing and numerical analysis [1].

Returning to the inverse eigenvalue problem for a matrix polynomial, assume that we are asked to find a matrix polynomial having given scalars $\mu_1, \ldots, \mu_l \in \mathbb{C}$ where $l \leq n$. For doing this, one can consider an arbitrary matrix polynomial, namely, $P(\lambda)$ in the craved size. Next, by following procedure briefly described above, the desired matrix polynomial which $\mu_1, \ldots, \mu_l$ are some of its eigenvalues is computable. See the following example.

Example. Suppose that the set $\Sigma = \{1+i, -2, 3\}$ is given and we have find a 3×3 matrix polynomial such that Σ is subset of its spectrum. Consider the matrix polynomial

$$P(\lambda) = \begin{bmatrix} 7 & 9 & -2 \\ 0 & -2 & 0 \\ 6 & -3 & -1 \end{bmatrix} \lambda^2 + \begin{bmatrix} 9 & -3 & 3 \\ -5 & 8 & 10 \\ 4 & -3 & 0 \end{bmatrix} \lambda + \begin{bmatrix} -5 & 0 & 5 \\ -2 & -2 & 10 \\ 1 & 9 & 2 \end{bmatrix},$$

where its coefficients are random matrix generated by MATLAB and assume the set of weights $w = \{12.0731, 14.8523, 11.7991\}$ which are the norms of the coefficient matrices. Then, the matrix polynomial $Q_{1.9457}(\lambda) = P(\lambda) + \Delta_{1.9457}(\lambda)$ is a perturbation of $P(\lambda)$ that includes Σ in its spectrum. Where

$$\begin{aligned} \Delta_{1.9457}(\lambda) &= \begin{bmatrix} -1.5517+0.5809i & -3.6695-3.7570i & 3.2116-2.4259i \\ -1.4161+1.1256i & 0.8042-3.6739i & 1.4734+0.2202i \\ -4.9540+1.3307i & -0.2218-0.1724i & -0.1600-2.5569i \end{bmatrix} \lambda^2 \\ &+ \begin{bmatrix} -1.0060+0.6912i & -3.2915-2.0334i & 1.8646-2.3054i \\ -0.8122+1.0565i & -0.0784-2.7695i & 1.0925-0.1046i \\ -3.3050+1.8322i & -0.1892-0.0838i & -0.5691-1.7995i \end{bmatrix} \lambda \\ &+ \begin{bmatrix} -2.1745-1.0097i & 0.1466-7.5978i & 5.7620+0.8473i \\ -2.5983-0.3167i & 4.6039-2.9017i & 1.2692+1.7425i \\ -6.4023-3.7556i & -0.0475-0.4037i & 2.4733-2.7615i \end{bmatrix}. \end{aligned}$$

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Inverse eigenvalue problem of nonnegative bisymmetric matrices of order ≤ 4

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Abstract

In this paper we solve the inverse eigenvalue problem of nonnegative bisymetric matrices. We try to present some necessary and sufficent conditions to solve this problem for order 3 and 4.

Keywords: Bisymmetric matrices, Bisymmetric nonnegative inverse eigenvalue problem, Spectrum of a matrix, Perron eigenvalue
Mathematics Subject Classification [2010]: 15A29, 15A18

1 Introduction

Bisymmetric matrices have been widely discussed since 1939, and are very useful in communication theory, engineering and statistics [1].

Definition 1.1. A real $n \times n$ matrix $A = (a_{i,j})$ is called a bisymmetric matrix if its elements satisfy the properties

$$a_{i,j} = a_{j.i}, \qquad a_{i,j} = a_{n-j+1,n-i+1}.$$

The set of all $n \times n$ bisymmetric matrices is denoted by $BSR^{n\times n}$.

Clearly, a bisymmetric matrix is a square matrix that is symmetric about both of its main diagonals.

The bisymmetric nonnegative inverse eigenvalue problem is the problem of finding necessary and sufficient conditions for a list of n real numbers to be the spectrum of an $n \times n$ bisymmetric nonnegative matrix. If there exists an $n \times n$ bisymmetric nonnegative matrix A with spectrum σ, we say that is realizable and that A realizes σ. We will denote by N_n the set of all realizable lists of n real numbers.

The nonnegative inverse eigenvalue problem for symmetric matrices is very difficult and it is solved only for $n = 3$ by Loewy and London and for matrices with trace 0 of order $n = 4$ by Reams , respectively.

Through this paper the following notation is used. The spectral radius of nonnegative matrix A denoted by $\rho(A)$. There is a right and a left eigenvector associated with the Perron eigenvalue with nonnegative entries. The Perron eigenvalue is denoted by λ_1.

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Some necessary conditions on the list of real number $\sigma = (\lambda_1, \lambda_2, ..., \lambda_n)$ to be the spectrum of a nonnegative matrix are listed below.

(1) The Perron eigenvalue $\max\{|\lambda_i|, \lambda_i \in \sigma\}$ belongs to σ (Perron- Frobenius theorem).
(2) $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$
(3)$s_k^m \leq n^{m-1} s_{km}, m = 1, 2, ...$

This article is organized as follows. First, we discuss the specified properties and structure of bisymmetric matrices and introduce some lemmas to be used in the subsequent sections in Section 2, then find a solution for BSNIEP by a recursive method.

In recent paper [2] solved this problem in speacial condition of order 2, 3 and 4. In this paper we try to solve BSNIEP problem in more complete condition.

This article is organized as follows. In section 2, we introduce a theorem for 2×2 nonnegative bisymmetric matrix from [2]. In section 3, we find necessary and sufficient conditions for finding a 3×3 nonnegative bisymmetric matrix and in section 4 we discuss the inverse eigenvalue problem of a 4×4 nonnegative bisymmetric matrix.

2 The case $n = 2$

Theorem 2.1. *Let $\sigma = \{\lambda_1, \lambda_2\}$be a set of two real numbers such that $\lambda_1 \geq |\lambda_2|$. Then σ is the set of eigenvalues of a bisymmetric nonnegative matrix such that define as*

$$A = \begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1-\lambda_2}{2} \\ \frac{\lambda_1-\lambda_2}{2} & \frac{\lambda_1+\lambda_2}{2} \end{pmatrix}.$$

3 The case $n = 3$

Theorem 3.1. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of real numbers, then*

$$\begin{aligned} d &= d, \\ a &= \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 - d), \\ c &= \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 + d), \\ b &= \frac{1}{2}\sqrt{2(-d^2 + d(\lambda_1 + \lambda_2) - \lambda_1\lambda_2)}. \end{aligned}$$

are necessary and sufficient conditions for finding a bisymmetric matrix

$$A = \begin{pmatrix} a & b & c \\ b & d & b \\ c & b & a \end{pmatrix},$$

such that σ is spectrum of A.

If we want to solve the inverse eigenvalue problem of nonnegative bisymetric matrices, then we must choose some conditions that all of element of matrix A are nonnegative. For instance it is very impotrtant that $\min \lambda_i \leq d \leq \max \lambda_i$.

Theorem 3.2. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$, the necessary and sufficent condtion that σ be a spectrum of nonnegative bisymmetric matrix of A is*

$$\lambda_1 + \lambda_2 + \lambda_3 > 0, \quad \lambda_1 > |\lambda_i|, \qquad i = 2, 3.$$

Remark 3.3. The solution of problem of Theorem 3.2 is not unique.

Example 3.4. Assume $\sigma = \{6, 5, 3\}$, then two nonnegative bisymmetric matrices are as following

$$A_1 = \begin{pmatrix} 11/2 & 0 & 1/2 \\ 0 & 3 & 0 \\ 1/2 & 0 & 11/2 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

The matrices A_1 and A_2 are nonnegative bisymmetric matrices and their eigenvalues are $\{6, 5, 3\}$.

4 The case $n = 4$

Theorem 4.1. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a set of real numbers, If*

$$\begin{aligned} f &= f, \\ d &= d, \\ a &= \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3) + \frac{3}{2}\lambda_4 - f, \\ h &= \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \frac{3}{2}\lambda_4 - d, \end{aligned}$$

and find b and c from solve system following

$$\begin{aligned} f + a + d + h + \sqrt{(a + d - f - h)^2 + 4(b + c)^2} &= 2\lambda_1, \\ f + a - d - h - \sqrt{(a - d - f + h)^2 + 4(b - c)^2} &= 2\lambda_4, \end{aligned}$$

then there exist a bisymmetric matrix as

$$\begin{pmatrix} a & b & c & d \\ b & f & h & c \\ c & h & f & b \\ d & c & b & a \end{pmatrix},$$

such that σ is spectrum of A.

If we want to solve the inverse eigenvalue problem of nonnegative bisymetric matrices, then we must choose some conditions that all of element of matrix A are nonnegative.

Theorem 4.2. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, with $\lambda_1 + \lambda_2 + \lambda_3 > 0$, and $\lambda_1 > |\lambda_i|$ for $i = 2, 3$, the necessary and sufficient condtions that σ be a spectrum of nonnegative bisymmetric matrix of A is*

$$\lambda_1 + \lambda_2 > \lambda_3 - 3\lambda_4,$$
$$\lambda_1 + \lambda_2 > -(\lambda_3 - 3\lambda_4).$$

Remark 4.3. The solution of problem of Theorem 4.2 is not unique.

Example 4.4. Assume $\sigma = \{10, 6, 4, 1\}$, then we can find two nonnegative bisymmetric matrices as following

$$A_1 = \begin{pmatrix} 5 & 0 & 0 & 1 \\ 0 & \frac{11}{2} & \frac{9}{2} & 0 \\ 0 & \frac{9}{2} & \frac{11}{2} & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} \frac{11}{2} & \frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} & \frac{3}{2} \\ \frac{1}{2}\sqrt{3} & 5 & 4 & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & 4 & 5 & \frac{1}{2}\sqrt{3} \\ \frac{3}{2} & \frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{3} & \frac{11}{2} \end{pmatrix}.$$

It is easy to see that A_1 and A_2 are nonnegative bisymmetric matrices and their eigenvalues are $\{10, 6, 4, 1\}$.

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Nested splitting conjugate gradient method for solving generalized Sylvester matrix equation

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Abstract

In this paper, a matrix version of a nested splitting conjugate gradient (NSCG) iteration method and its convergence conditions are presented for solving generalized Sylvester matrix equation that coefficient matrices are large and nonsymmetric. This method is inner/ outer iterate, which its inner iterations are CG-like method to approximate each outer iterate, while each outer iteration is induced by a convergent and symmetric positive definite splitting of the coefficient matrices.

Keywords: Matrix NSCG, contractive, CG.
Mathematics Subject Classification [2010]: 65F10, 65F50

1 Introduction

In this paper, we consider the generalized Sylvester matrix equation

$$\sum_{j=1}^{p} A_j X B_j = C, \tag{1}$$

where $A_j \in \mathbb{R}^{n\times n}, B_j \in \mathbb{R}^{m\times m}, C, X \in \mathbb{R}^{n\times m}$. The generalized Sylvester equation (1) arises in several areas of applications. They play a cardinal role in the control and communication theory and image restoration; for further details see [2].
Note that the linear matrix equation (1) can be reformulated by the following $nm \times nm$ linear system:

$$\mathcal{A}vec(X) = vec(C), \tag{2}$$

where $\mathcal{A} = \sum_{j=1}^{p}(B_j^T \otimes A_j)$. However, it is quite costly and ill-conditioned to solve this linear equation system.
In this paper, we present an iterative method for solving the matrix equation (1) by using the symmetric and skewsymmetric splitting of the matrices A_j and B_j, $j = 1, 2, ..., p$ in a matrix variant of the nested splitting conjugate gradient (NSCG) method, and give sufficient conditions for convergence. In [1], this method proposed for solving the system of

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linear equations $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a large sparse nonsingular matrix, $x, b \in \mathbb{R}^n$. Thoroughout this paper, we use the following notations. Let $\mathbb{R}^{n\times p}$ be the set of $n \times p$ real matrices. The symbols A^T, $\|A\|_2$ and $trace(A)$ will denote the transpose, 2-Norm and trace, respectively, of a matrix $A \in \mathbb{R}^{n\times p}$. For any matrices A and B in $\mathbb{R}^{n\times p}$, the inner product $< A, B >_F = trace(A^T B)$ denotes the inner product. The associated norm is the Frobenius norm obtained by $\|.\|_F$.
Further, $vec(.)$ will stand for the vec operator, i.e. $vec(C) = (c_1^T, c_2^T, ..., c_m^T)^T$ for the matrix $C = (c_1, c_2, ..., c_m) \in \mathbb{R}^{n\times s}$, where $c_j, j = 1, 2, ..., p$ is the j-th column of C and $A \otimes B = (a_{ij}B)$ denotes the Kronecker product of the matrices A and B. First, we give some definitions and lemmas that we used them.

Definition 1.1. ([3]) Let $\mathcal{H}$ be a symmetric positive definite matrix, we denote $\mathcal{H}-$ norm of a matrix $B \in \mathbb{R}^{n\times n}$ by using the $\|B\|_{\mathcal{H}}$ and define as $\|B\|_{\mathcal{H}} = \|\mathcal{H}^{\frac{1}{2}} B \mathcal{H}^{-\frac{1}{2}}\|_2$.

Lemma 1.2. *([3]) Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Then*

$$\lambda_{max}(A + B) < \lambda_{max}(A) + \lambda_{max}(B), \lambda_{min}(A + B) > \lambda_{min}(A) + \lambda_{min}(B).$$

2 The NSCG method

In this section, we consider the scheme of the NSCG iteration method and its convergence property. $\mathcal{A} = \mathcal{H} - \mathcal{S}$ is called a splitting of the matrix $\mathcal{A}$ if $\mathcal{H}$ is nonsingular. This splitting is convergent if $\rho(\mathcal{H}^{-1}\mathcal{S}) < 1$, a contractive splitting if $\|(\mathcal{H}^{-1}\mathcal{S})\| < 1$ for some matrix norm and symmetric positive definite splitting(spd) if $\mathcal{H}$ is spd matrix.
Let $\mathcal{A} = \mathcal{H} - \mathcal{S}$ is a splitting symmetric positive definite of matrix $\mathcal{A}$. Then the linear systems (2) is equivalent to the fixed point equation: $\mathcal{H}x = \mathcal{S}x + c$. Assume that this splitting is contractive. Given an initial guess $x^{(0)} \in \mathbb{R}^n$. By using CG-like method, we have computed the approximations $x^{(1)}, ..., x^{(l)}$ to the solution x^* of (2). Then the next approximation $x^{(l+1)}$ is a solution of the following linear equation system:

$$\mathcal{H}x = \mathcal{S}x^{(l)} + c.$$

Now, we apply NSCG method for generalized Sylvester equation as follows:
First, we split A_j, B_j, $j = 1, 2, ..., p$ into symmetric and skew-symmetric parts:

$$A_j = H_{A_j} - S_{A_j}, B_j = H_{B_j} - S_{B_j}, j = 1, 2, ..., p,$$

and applying the $vec(.)$ operator, (1) is converted to:

$$\Sigma_{j=1}^{p}((H_{B_j} \otimes H_{A_j}) - (S_{B_j} \otimes S_{A_j}))x = \Sigma_{j=1}^{p}((H_{B_j} \otimes S_{A_j}) - (S_{B_j} \otimes H_{A_j}))x + c,$$

where $x = vec(X)$ and $c = vec(C)$. Define

$$\mathcal{H} = \Sigma_{j=1}^{p}((H_{B_j} \otimes H_{A_j}) - (S_{B_j} \otimes S_{A_j})), \mathcal{S} = \Sigma_{j=1}^{p}((H_{B_j} \otimes S_{A_j}) - (S_{B_j} \otimes H_{A_j})). \quad (3)$$

It is easy to see that $\mathcal{H}$ and $\mathcal{S}$ are symmetric and skew-symmetric parts of the matrix $\mathcal{A}$, respectively. By using lemma 1.2, we have:

$$\lambda_{min}(\mathcal{H}) \geq \sum_{j=1}^{p}(min(\lambda(H_{B_j})\lambda(H_{A_j})) - max(\lambda(S_{B_j})\lambda(S_{A_j}))) := t.$$

If $t > 0$ then $\mathcal{H}$ is a spd matrix. Then, we can apply the NSCG method is said to be in the above with $\mathcal{H}$ and $\mathcal{S}$ in (3) for solving $\mathcal{A}x = c$.
An implementation of the NSCG method is given by the following algorithm.

Algorithm 1 NSCG method for generalized Sylvester matrix equation

1: $X^{(0,0)} = X^{(0)}$; $\quad R^{(0)} = C - \sum_{j=1}^{p} A_j X^{(0)} B_j$.
2: **for** $l = 0 : l_{max}$ **do**
3: $\quad \hat{C} = \sum_{j=1}^{p} H_{A_j} X^{(l,0)} S_{B_j} + \sum_{j=1}^{p} S_{A_j} X^{(l,0)} H_{B_j} + C$.
4: $\quad \hat{R}^{(l,0)} = \hat{C} - \sum_{j=1}^{p} H_{A_j} X^{(l,0)} H_{B_j} - \sum_{j=1}^{p} S_{A_j} X^{(l,0)} S_{B_j}$; $\quad P^{(0)} = \hat{R}^{(l,0)}$.
5: $\quad$ **for** $k = 0 : k_{max}$ **do**
6: $\quad\quad W^{(k)} = \sum_{j=1}^{p} H_{A_j} P^{(k)} H_{B_j} + \sum_{j=1}^{p} S_{A_j} P^{(k)} S_{B_j}$; $\quad \alpha_k = \dfrac{\langle \hat{R}^{(l,k)}, \hat{R}^{(l,k)} \rangle_F}{\langle W^{(k)}, \hat{R}^{(l,k)} \rangle_F}$.
7: $\quad\quad X^{(l,k+1)} = X^{(l,k)} + \alpha_k P^{(k)}$; $\quad \hat{R}^{(l,k+1)} = \hat{R}^{(l,k)} - \alpha_k W^{(k)}$.
8: $\quad\quad$ **if** $\|\hat{R}^{(l,k+1)}\|_F \leq \varepsilon_2 \|\hat{R}^{(l,0)}\|_F$ **then**
9: $\quad\quad\quad$ go to 19.
10: $\quad\quad$ **else**
11: $\quad\quad\quad \beta_k = \dfrac{\langle \hat{R}^{(l,k+1)}, \hat{R}^{(l,k+1)} \rangle_F}{\langle \hat{R}^{(l,k)}, \hat{R}^{(l,k)} \rangle_F}$; $\quad P^{(k+1)} = \hat{R}^{(l,k+1)} + \beta_k P^{(k)}$.
12: $\quad\quad$ **end if**
13: $\quad$ **end for**
14: $\quad X^{(l+1)} = X^{(l,k+1)}$,
15: $\quad$ **if** $\|R^{(l+1)}\|_F \leq \varepsilon_1$ **then**
16: $\quad\quad$ stop.
17: $\quad$ **end if**
18: $\quad X^{(l+1,0)} = X^{(l+1)}$; $\quad l = l + 1$.
19: **end for**

In the following we will give the analysis of the convergence property of the NSCG iteration method:

Lemma 2.1. *Let $\mathcal{H}$ and $\mathcal{S}$ are as in (3). If $t > 0$ and $\theta^3 \tau < t$, then $\mathcal{A} = \mathcal{H} - \mathcal{S}$ is a contractive splitting (with respect to the $\|.\|_{\mathcal{H}} -$ norm), i.e, $\|\mathcal{H}^{-1}\mathcal{S}\|_{\mathcal{H}} < 1$, where:*

$$\tau = \Big(\sum_{j=1}^{p} [-\min(\lambda^2(H_{B_j})\lambda^2(S_{A_j})) - \min(\lambda^2(H_{A_j})\lambda^2(S_{B_j})) + \lambda_{\max}(H_{B_j}S_{B_j} \otimes S_{A_j}H_{A_j}$$
$$+ S_{B_j}H_{B_j} \otimes H_{A_j}S_{A_j})] + \sum_{j>i}^{p} \sum_{i=1}^{p-1} [\lambda_{\max}(H_{B_i}H_{B_j} \otimes S_{A_i}^T S_{A_j} + H_{B_j}H_{B_i} \otimes S_{A_j}^T S_{A_i})$$
$$+ \lambda_{\max}(S_{B_i}S_{B_j}^T \otimes H_{A_i}H_{A_j} + S_{B_j}S_{B_i}^T \otimes H_{A_j}H_{A_i})] + \sum_{\substack{j=1 \\ j\neq i}}^{p} \sum_{i=1}^{p} \lambda_{\max}(H_{B_i}S_{B_j} \otimes S_{A_i}H_{A_j}$$
$$+ S_{B_j}H_{B_i} \otimes H_{A_j}S_{A_i})\Big)^{\frac{1}{2}}, \quad s = \sum_{j=1}^{p} [\max\big(\lambda(H_{A_j})\lambda(H_{B_j})\big) - \min\big(\lambda(S_{B_j})\lambda(S_{A_j})\big)],$$

$$t=\sum_{j=1}^{p}[\min\left(\lambda(H_{B_j})\lambda(H_{A_j}\right)-\max\left(\lambda(S_{B_j})\lambda(S_{A_j})\right)],\quad \theta=(\frac{s}{t})^{\frac{1}{2}}.$$

Theorem 2.2. *Let $\mathcal{H}$ and $\mathcal{S}$ are as in (3). They are symmetric and skew-symmetric parts of the nonsingular and nonsymmetric matrix $\mathcal{A}$, respectively. Let $t>0$, $\eta<1$, the NSCG method is started from an initial guess $X^{(0)}\in\mathbb{R}^{n\times n}$ that produces an iterative sequence $\{X^{(l)}\}_{l=0}^{\infty}$, where $X^{(l)}\in\mathbb{R}^{n\times n}$ is the lth approximation of $X^*\in\mathbb{R}^{n\times n}$ to (1).*
For the error matrix $E^{(l)}=X^{(l)}-X^*$ and the residual matrix $R^{(l)}=C-\sum_{j=1}^{p}A_jX^{(l)}B_j$, we have the following results: For $l=1,2,...,$
a)$\|\sum_{j=1}^{p}H_{A_j}E^{(l)}H_{B_j}+\sum_{j=1}^{p}S_{A_j}E^{(l)}S_{B_j}\|_F\le\omega^{(l)}\|\sum_{j=1}^{p}H_{A_j}E^{(l-1)}H_{B_j}+\sum_{j=1}^{p}S_{A_j}E^{(l-1)}S_{B_j}\|_F$,
b)$\|\sum_{j=1}^{p}H_{A_j}R^{(l)}H_{B_j}+\sum_{j=1}^{p}S_{A_j}R^{(l)}S_{B_j}\|_F\le\tilde{\omega}^{(l)}\|\sum_{j=1}^{p}H_{A_j}R^{(l-1)}H_{B_j}+\sum_{j=1}^{p}S_{A_j}R^{(l-1)}S_{B_j}\|_F$, where:

$$\omega^{(l)}=\left(2\left(\frac{\theta-1}{\theta+1}\right)^{k_l}(1+\eta)+\eta\right)\theta,\qquad \tilde{\omega}^{(l)}=\omega^{(l)}\frac{1+\eta}{1-\eta}.$$

τ,s,t are as in lemma 2.1, $\theta=(\frac{s}{t})^{\frac{1}{2}}$ and $\eta=\frac{\theta^3\tau}{t}$.
Moreover, for $\eta\in(0,\frac{1}{\theta})$ and some $\omega\in(\eta\theta,1)$, if $k_l\ge\dfrac{Ln\left(\dfrac{\omega-\eta\theta}{2\theta(1+\eta)}\right)}{Ln\left(\dfrac{\theta-1}{\theta+1}\right)}$, $l=1,2,3,...,$ then we have $\omega^{(l)}\le\omega$ and the sequence $\{X^{(l)}\}_{l=0}^{\infty}$ converges to the solution X^* of (1).
For $\eta\in\left(0,\dfrac{\sqrt{(\theta+1)^2+4\theta}-(\theta+1)}{2\theta}\right)$, and some $\tilde{\omega}\in\left(\dfrac{(1+\eta)\eta\theta}{(1-\eta)},1\right)$, if
$$k_l\ge\frac{Ln\left(\dfrac{\tilde{\omega}(1-\eta)-\eta\theta(1+\eta)}{2\theta(1+\eta)^2}\right)}{Ln\left(\dfrac{\theta-1}{\theta+1}\right)},$$
$l=1,2,...,$ then we have $\tilde{\omega}^{(l)}\le\tilde{\omega}$ and the residual sequence $\{R^{(l)}\}_{l=0}^{\infty}$ converges to zero.

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Numerical solution for nth order linear Fredholm integro-differential equations by using Chebyshev wavelets integration operational matrix

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Abstract

In this paper, a numerical method for solving nth order linear Fredholm integro-differential equations is proposed. Proposed method is based on using Chebyshev wavelets integration operational matrix (CWIOM). Numerical tests to illustrate applicability of the new approach are presented.

Keywords: Fredholm integro-differential equations, Chebyshev wavelets, Operational matrix.
Mathematics Subject Classification [2010]: 13D45,39B42

1 Introduction

In recent years, numerical solution of integral equations and integro-differential equations by using Haar wavelets, Chebyshev wavelets, Legendre wavelets, CAS wavelets and other hybrid functions based on wavelets via integration operational matrix was discussed by many authors [1, 2, 3, 4]. Here, we consider the following nth order linear Fredholm integro-differential equation

$$\begin{cases} y^{(n)}(x) = f(x) + y(x) + \int\limits_0^1 k(x,t)\left(y^{(n-1)}(t) + y^{(n-2)}(t) + \cdots + y'(t) + y(t)\right) dt \\ y(0) = y_0,\ y'(0) = y_1,\ y''(0) = y_2, \cdots, y^{(n-1)}(0) = y_{n-1}, \end{cases} \tag{1}$$

and proposed a new method based on CWIOM. In [1], the authors a numerical method based on for solving linear Fredholm integro-differential equation as

$$\begin{cases} y^{(n)}(x) = f(x) + y(x) + \int\limits_0^1 k(x,t)y(t)dt \\ y(0) = y_0,\ y'(0) = y_1,\ y''(0) = y_2, \cdots, y^{(n-1)}(0) = y_{n-1}, \end{cases} \tag{2}$$

The main advantage of the proposed method in this paper is that in this method by using CWIOM and without any need to integration, we obtain the approximate solution of equation (1). The paper is organized as follows: In Sections 2 and 3, we recall properties of Chebyshev wavelets, function approximation and the operational matrix, respectively. In Section 4, the proposed method is applied to solve of the nth order linear Fredholm integro-differential equations. Some numerical examples are presented in Section 5.

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2 Wavelets and properties of Chebyshev wavelets

Wavelet $\psi_{a,b}(t)$ is a mother wavelet, where a and b are dilation parameter and translation parameter, respectively. They are defined by [2, 3, 4]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a \in \mathbb{R}, \quad a \neq 0. \tag{3}$$

Chebyshev wavelets $\psi(n,m,k)$ have three arguments; $k = 1,2,\cdots,\ n = 1,2,\cdots,2^{k-1}$ and m is the degree for Chebyshev polynomials. They are defined on the interval $[0,1)$ by

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\widehat{T}_m\left(2^k x - 2n + 1\right), & x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where

$$\widehat{T}_m(x) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\frac{2}{\pi}}T_m(x), & m > 0, \end{cases}$$

and $m = 0,\cdots,M-1$. $T_m(x)$, $m = 0,1,2,\cdots$ are Chebyshev polynomials of the first kind degree m which are with respect to the weight function $w(x) = (1-x^2)^{-\frac{1}{2}}$ on the interval $[-1,1]$, and satisfy the following recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), \qquad m = 1,2,\cdots.$$

Using the Chebyshev wavelets the weight function $w(x)$ is

$$w_n(x) = \left(1 - \left(2^k x - 2n + 1\right)^2\right)^{-\frac{1}{2}}.$$

For function $f(x) \in L^2[0,1)$ using the orthogonal basis functions $T_m(x)$, is defined as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x), \tag{5}$$

where

$$c_{n,m} = \left(f(x), \psi_{n,m}(x)\right)_{w_n}, \tag{6}$$

in which $(.,.)$ denotes the inner product. Now, if the infinite series in equation (5) is truncated, then equation (5) can be written as

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\Psi(x), \tag{7}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices.

3 Integration operational matrix

The integration of the vector $\Psi(x)$ can be written as $\int_0^x \Psi(s)ds = P\Psi(x)$, where P is an $2^{k-1}M \times 2^{k-1}M$ called the integration operational matrix and is given by [2, 3, 4]

$$P = \begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ O & O & L & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & L \end{bmatrix}, \quad F = \frac{1}{2^k}\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{r+1}}{r+1} - \frac{1-(-1)^{r-1}}{r-1}\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{M}}{M} - \frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix},$$

and L, that you can see the matrix in [2]. The integration of the product of two Chebyshev wavelets vector functions is written as $\int_0^1 \Psi(x)\Psi(x)^T dx = I$. Also, $\Psi(x)\Psi(x)^T C = \widehat{C}\Psi(x)$. For more details, see [2, 3].

4 Method solution

Consider the nth order linear Fredholm integro-differential equation (1). To this end, we have:

$$\begin{aligned} y^{(n)}(x) &\simeq y^{(n)^T}\Psi(x), \\ y^{(n-1)}(x) &\simeq y^{(n)^T}P\Psi(x) + y_0^{(n-1)^T}\Psi(x), \\ y^{(n-2)}(x) &\simeq y^{(n)^T}P^2\Psi(x) + y_0^{(n-1)^T}\widehat{C}_1\Psi(x) + y_0^{(n-2)^T}\Psi(x) \\ &\vdots \\ y'(x) &\simeq y^{(n)^T}P^{n-1}\Psi(x) + y_0^{(n-1)^T}\widehat{C}_{n-2}\Psi(x) + \cdots + y_0''^T\widehat{C}_1\Psi(x) + y_0'^T\Psi(x), \\ y(x) &\simeq y^{(n)^T}P^{n}\Psi(x) + y_0^{(n-1)^T}\widehat{C}_{n-1}\Psi(x) + \cdots + y_0''^T\widehat{C}_2\Psi(x) + y_0'^T\widehat{C}_1\Psi(x) + y_0^T\Psi(x). \end{aligned} \tag{8}$$

Finally, substituting equation (8) in equation (1), we conclude that

$$\begin{aligned} &\left[I - K(P^T + \cdots + P^{n^T}) - P^{n^T}\right] y^{(n)} = X + \left[K(I + \widehat{C}_1^T + \cdots + \widehat{C}_{n-1}^T) + \widehat{C}_{n-1}^T\right] y_0^{(n-1)} \\ &+ \left[K\left(\widehat{C}_1^T + \cdots + \widehat{C}_{n-2}^T\right) + \widehat{C}_{n-2}^T\right] y_0^{(n-2)} + \cdots + \left[K(I + \widehat{C}_1^T) + \widehat{C}_1^T\right] y_0' + (I + K)y_0. \end{aligned} \tag{9}$$

5 Numerical examples

In this section, we compute the following integro-differential equations.

Example 5.1. Consider the following third order integro-differential equation with exact solution $y(x) = x^2$,

$$\begin{cases} y'''(x) = -\frac{5}{3}x - x^2 + y(x) + \int_0^1 xt\,(y'(t) + y''(t))\,dt \\ y(0) = y'(0) = 0,\ y''(0) = 2, \end{cases}$$

Example 5.2. Finally consider the linear fourth order integro-differential equation

$$\begin{cases} y^{(4)}(x) = 24x - ex^2 - \frac{x^5}{5} + y(x) + \int_0^1 x^2 e^{t^4} y''(t)dt \\ y(0) = y'(0) = y''(0) = y'''(0) = 0, \end{cases}$$

where the exact solution is $y(x) = \frac{x^5}{5}$.

Table 1: Numerical results of Example 5.1−5.2 for M=3, k=3

x	Absolute error for Example 5.1	x	Absolute error for Example 5.2
0	1.31110×10^{-5}	0	1.90854×10^{-5}
0.1	7.87173×10^{-6}	0.1	2.75047×10^{-5}
0.2	9.69928×10^{-5}	0.2	2.68595×10^{-6}
0.3	3.96844×10^{-4}	0.3	1.52273×10^{-4}
0.4	1.39139×10^{-3}	0.4	1.99042×10^{-5}
0.5	3.42384×10^{-3}	0.5	3.67240×10^{-4}
0.6	6.92173×10^{-3}	0.6	2.97938×10^{-4}
0.7	1.29729×10^{-2}	0.7	3.52965×10^{-4}
0.8	2.19557×10^{-2}	0.8	6.67151×10^{-4}
0.9	3.53960×10^{-2}	0.9	4.78840×10^{-4}

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Numerical solution of an inverse source problem of the time-fractional diffusion equation using a LDG method

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Abstract

This paper is devoted to determine a time-dependent source term in a time-fractional diffusion equation using a fully discrete local discontinous Galerkin (LDG) method. This method is based on a finite difference scheme in time and a local discontinous Galerkin method in space, is numerically stable and has the convergence of order $O((\Delta x)^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}}(\Delta x)^{k+\frac{1}{2}} + (\Delta t)^{\alpha})$.

Keywords: LDG method, time-fractional diffusion equation, inverse source problem.
Mathematics Subject Classification [2010]: 65M32.

1 Introduction

In this paper, we consider the following initial-boundary value problem for the time-fractional diffusion equation

$$\begin{cases} D_t^{\alpha} u = u_{xx} + f(x)p(t), & 0 < x < 1, \quad 0 < t < T, \\ u(0,t) = k_0(t), & \qquad\qquad 0 \leqslant t \leqslant T, \\ u(1,t) = k_1(t), & \qquad\qquad 0 \leqslant t \leqslant T, \\ u(x,0) = \phi(x), & 0 \leqslant x \leqslant 1. \end{cases} \tag{1}$$

Problem (1) is a forward problem when all of the functions f, ϕ, k_0, k_1 and p are given appropriately. The inverse source problem which is considered here is to determine the source term p based on problem (1) and the following additional condition

$$u(x^*, t) = g(t), \qquad 0 \le t \le T,$$

where $x^* \in (0,1)$ is an interior measurement location. D_t^{α} is the Caputo fractional derivatives of order α, i.e.

$$D_t^{\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\cdot, s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \qquad 0 < \alpha < 1, \tag{2}$$

where $\Gamma(.)$ is the Gamma function. The inverse source problem mentioned above has been solved numerically by Wei et al. [1] using a regularized method. We aim to apply the discontinuous Galerkin method to the above mentioned inverse source problem. Of

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course, some DG methods have been applied successfully for the forward fractional diffusion equation. For example, Hesthaven et. al. [2] have been solved some space-fractional diffusion equations using a local discontinuous Galerkin method in a semi-discrete regime while Wei et al. [3] have been applied a fully-discrete LDG for solving a time-fractional diffusion equation.

In the following, we consider a spatial grid $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1$ with cells $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \ldots, N$, the cell lengths $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, 1 \leq j \leq N$, and $h = \Delta x = \max\limits_{1\leq j\leq N} \Delta x_j$. We denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ the values of u at $x_{j+\frac{1}{2}}$, from the right cell I_{j+1} and from the left cell I_j. We define the piecewise-polynomial space V_h^k as the space of polynomials of degree up to k in each cell I_j, i.e., $V_h^k = \{v : v \in P^k(I_j), j = 1, \ldots, N\}$. We point out that the norm $\|\cdot\|$ denotes the usual norm of the $L^2[0,1]$ space.

Let M be a positive integer, $\Delta t = T/M$ be the time meshsize, and $t_n = n\Delta t$, for $n = 0, 1, \ldots, M$ be the mesh points. An approximation to time fractional derivative (2) can be obtained by simple quadrature formule given as [4],

$$D_t^\alpha u(\cdot, t_n) = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} b_i \frac{u(\cdot, t_{n-i}) - u(\cdot, t_{n-i-1})}{\Delta t} + \gamma^n(\cdot), \tag{3}$$

where $b_i = (i+1)^{1-\alpha} - i^{1-\alpha}$, and γ^n is the truncation error with $\|\gamma^n\| \leq C(\Delta t)^{2-\alpha}$ where C is a constant depending on α, u, and T. It is easy to check that $b_n \to 0$ as $n \to \infty$, $b_i > 0,\ i = 0, 1, \ldots$, and $1 = b_0 > b_1 > b_2 > \cdots$.

2 Main results

We rewrite (1) as a first-order system: $q = u_x$, $\ D_t^\alpha u(x,t) - q_x = f(x)p(t)$. Let $u_h^n, q_h^n \in V_h^k$ be the approximation of $u(.,t_n), q(.,t_n)$ respectively, and $p^n = p(t_n), g^n = g(t_n)$. After some manipulations, the following fully discrete local discontinuous Galerkin scheme is obtained: find $u_h^n, q_h^n \in V_h^k$, such that for all test functions $v, w \in V_h^k$,

$$\begin{cases} \displaystyle\int_\Omega u_h^n v dx + \beta \left(\int_\Omega q_h^n v_x dx - \sum_{j=1}^{N} ((\hat{q}_h^n v^-)_{j+\frac{1}{2}} - (\hat{q}_h^n v^+)_{j-\frac{1}{2}}) \right) = \beta p^n \int_\Omega f(x) v dx \\ \qquad\qquad\qquad\qquad \displaystyle + \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega u_h^{n-i} v + b_{n-1} \int_\Omega u_h^0 v dx, \\ \displaystyle\int_\Omega q_h^n w dx + \int_\Omega u_h^n w_x dx - \sum_{j=1}^{N} ((\hat{u}_h^n w^-)_{j+\frac{1}{2}} - (\hat{u}_h^n w^+)_{j-\frac{1}{2}}) = 0, \\ u_h^n(x^*) = g^n, \end{cases} \tag{4}$$

where $\beta = (\Delta t)^\alpha \Gamma(2-\alpha)$ and without lose of generality we assume that x^* is a grid point. The"hat" terms in (4) in the cell boundary terms are the so-called "numerical fluxs", which are single valued functions defined on the cell boundaries and should be designed based on different guiding principles for different equations for ensuring the numerical stability. Among suitable choices, we choose the following numerical fluxs $\hat{u}_h^n = (u_h^n)^-$, $\hat{q}_h^n = (q_h^n)^+$, or $\hat{u}_h^n = (u_h^n)^+$, $\hat{q}_h^n = (q_h^n)^-$. We remark that the choice for the fluxes is not unique. In fact the crucial part is taking $\hat{u}_h^n$ and $\hat{q}_h^n$ from opposite sides [5]. The proof of the following Theorems have been presented in [6].

Theorem 2.1. *Assume that* $u_{xx}(x^*, \cdot)$ *is bounded and* f *is a continuous function on* $[0, 1]$. *For periodic or compactly supported boundary conditions, the fully-discrete LDG scheme (4) is unconditionally stable, and the numerical solution* u_h^n *satisfies*

$$\|u_h^n\| \leq \|u_h^0\| + \kappa, \qquad n = 1, \ldots, M, \tag{5}$$

where κ *is a constant depending on* $\Delta t, f$ *and* u_{xx}.

Theorem 2.2. *If* $u(\cdot, t_n)$ *is the exact solution of the problem (1) which is sufficiently smooth with bounded derivatives and* u_h^n *is the numerical solution of the fully discrete LDG scheme (4), then there holds the following error estimate*

$$\|u(\cdot, t_n) - u_h^n\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}} + c(\Delta t)^{\alpha}), \tag{6}$$

where C *is a constant depending on* u *and* T, *and* c *is a constant depending on* f *and* u_{xx}.

2.1 Numerical results

In this section, we carry out some numerical tests for confirming theoretical results and also investigating the effciency of the proposed method. For simplicity, we set $T = 1$, $\Delta t = 1/M$, and $h = 1/N$. To check the accuracy of the numerical solutions, we compute the relative root mean square error by $\varepsilon(p) = (\sum_{n=1}^{M}(p_h^n - p(t_n))^2 / \sum_{n=1}^{M} p(t_n)^2)^{1/2}$, where p_h^n is an approximation of the exact value of $p(t_n)$ which obtained by the proposed method. For noisy data, we use $g^{\delta}(t_n) = g(t_n)(1 + \delta \cdot \text{rand}(n))$, where $g(t_n)$ is the exact data, $\text{rand}(n)$ is a random number uniformly distributed in $[-1, 1]$ and the magnitude δ indicates a relative noise level.

Example 2.3. In this example all of the functions f, ϕ, k_0, k_1 and g are extracted from the exact solution $u(x, t) = e^{-t}\cos(2\pi x)$ and we set $\alpha = 1$, $M = 1000, \delta = 0$ and $x^* = 0.5$. L^2 and L^{∞} error norms and the numerical orders of accuracy for the function u and the relative root mean square error $\varepsilon(p)$ are reported in Table (a) for piecewise P^1 and P^2 polynomials as the basis functions. In Fig. 1, we show the errors in L^{∞}-norm, L^1-norm and L^2-norm confirming thired-order accuracy for piecewise P^2 polynomials as we expected.

Example 2.4. In this example, we solve a direct problem using the following data: $u(x, 0) = \sin(2\pi x)$, $k_0(t) = k_1(t) = 0$, $f(x) = x^2$, and

$$p(t) = \begin{cases} 1, & t \in [0.25, 0.75], \\ 0, & t \in [0, 0.25) \cup (0.75, 1], \end{cases}$$

and obtain an approximation to g with the aid of the LDG method for $\alpha = 0.5, 0.95$. Then using obtained g, we solve an inverse problem with the help of the LDG method to get an approximation to p. In Table (b), we show the relative root mean square errors $\varepsilon_1(p)$ and $\varepsilon_2(p)$, respectively without and with regularization method using the proposed method in [1], and $\varepsilon_3(p)$ for the proposed LDG method without applying any regularization methods. Our results are considerably better than results reported in [1]. Exact and numerical p with various noise levels $\delta = 5\%, 10\%, 15\%$ are presented in Fig. 2.

	N	$\varepsilon(p)$	L^2-norm	Order	L^∞-norm	Order
$k=1$	10	3.2×10^{-2}	1.0×10^{-2}	-	5.5×10^{-3}	-
	20	5.4×10^{-3}	2.6×10^{-3}	2.0	1.2×10^{-3}	2.2
	30	1.6×10^{-3}	1.1×10^{-3}	2.0	5.2×10^{-4}	2.1
	40	6.7×10^{-4}	6.4×10^{-4}	2.0	2.9×10^{-4}	2.1
$k=2$	10	2.3×10^{-4}	3.8×10^{-4}	-	2.8×10^{-4}	-
	20	1.4×10^{-5}	4.7×10^{-5}	3.0	3.4×10^{-5}	3.1
	30	1.3×10^{-5}	1.4×10^{-5}	3.0	1.0×10^{-5}	3.0
	40	1.2×10^{-5}	6.0×10^{-6}	2.9	4.0×10^{-6}	3.2

(a) Accuracy test for Example 1 with $k=1,2$.

δ		5%	10%	15%
$\alpha=0.5$	$\varepsilon_1(p)$	0.1279	0.1617	0.2056
	$\varepsilon_2(p)$	0.1167	0.1185	0.1185
	$\varepsilon_3(p)$	4.12×10^{-7}	6.23×10^{-7}	8.27×10^{-7}
$\alpha=0.95$	$\varepsilon_1(p)$	0.7368	1.4563	2.1783
	$\varepsilon_2(p)$	0.1235	0.1436	0.1580
	$\varepsilon_3(p)$	8.84×10^{-5}	1.62×10^{-4}	2.59×10^{-4}

(b) The relative mean square error of Example 2.

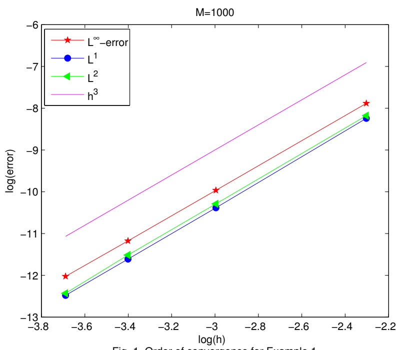

Fig. 1. Order of convergence for Example 1.

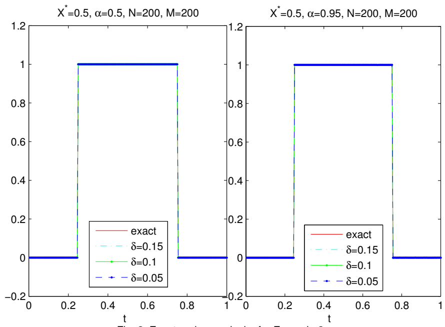

Fig. 2. Exact and numerical p for Example 2.

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Numerical solution of the time fractional Fokker-Planck equation using local discontinuous Galerkin method

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Abstract

In this article, we will offer the numerical solutions of time fractional Fokker-Planck equations (TFFPE). Two methods for discretization in time variable are investigated. The first method is based on a fractional finite difference scheme (FFDS) and in the second method the time fractional derivative is replaced by the Volterra integral equation which could be computed by the trapezoidal quadrature scheme (TQS). Then we have applied the local discontinuous Galerkin method in space for both methods. Some linear and nonlinear test problems have been considered to show the validity and convergence of two proposed methods. The results show that FFDS and TQS are of $2-\alpha$ and second–order accurate in time variable, respectively.

Keywords: Time fractional Fokker-Planck equation; discontinuous Galerkin method.
Mathematics Subject Classification [2010]: 45D05; 45G05; 41A30.

1 Introduction

Fractional calculus have a long history, having been mentioned by Leibniz in a letter to L'Hospital in 1695.
This paper mainly focuses on a numerical algorithm for finding the approximate solution of the nonlinear fractional Fokker–Planck equations with time–fractional derivative of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = [-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)]u(x,t), \quad t>0, \quad \alpha>0, \tag{1}$$

2 Main results

In this section we give some basic definitions and properties of the fractional calculus theory which are needed next.
Definition 2.1. The Caputo derivative is defined as follows:

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^n f(t)}{dx^n}dt, \quad \alpha\in(n-1,n], n\in N,$$

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where $\alpha > 0$ is the order of the derivative and n is the smallest integer greater than α.
Definition 2.2. For n to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$, is defined as,

$$D_*^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & if \quad \alpha \in (n-1,n), \\ \frac{\partial^n u(x,t)}{\partial t^n}, & if \quad \alpha = n \in N. \end{cases}$$

2.1 Local discontinuous Galerkin method

Discontinuous Galerkin (DG) methods are a class of finite element methods using discontinuous piecewise polynomial space for the numerical solution and the test functions. Since the basis functions can be discontinuous, these methods have the flexibility which is not shared by typical finite element methods.
For equations with higher order spatial derivatives, it is not suitable to design DG methods. Local discontinuous Galerkin method is a class of DG methods for solving time dependent partial differential equations (PDEs) with higher derivatives, which are termed local DG (LDG) methods. The idea of LDG methods is to suitably rewrite a higher order PDE into a first order system, then apply the DG method to the system. A key ingredient for the success of such methods is the correct design of interface numerical fluxes.

3 Method of trapezoidal quadrature formula

Now we spot the following fractional ordinary differential equation,

$$D_*^\alpha u(t) = f(u(t),t), \ \ u(0) = u_0, \ \ 0 < \alpha < 1 \tag{2}$$

which is equivalent to the Volterra integral equation,

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(u(\tau),\tau) d\tau \tag{3}$$

in the sense that a continuous function is a solution of the initial value problem (2) if and only if it is a solution of (3). For the numerical computation of (3), the integral is replaced by the trapezoidal quadrature formula at point t_{n+1}

$$\int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} g(\tau) d\tau \approx \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} \widetilde{g}_{n+1}(\tau) d\tau, \tag{4}$$

where $g(\tau) = f(\tau, u(\tau))$ and $\widetilde{g}_{n+1}(\tau)$ is the piecewise linear interpolation of g with nodes and knots chosen at $t_j, j = 0,1,2,...,n+1$. After some elementary calculations, the right hand side of (4) gives [2]

$$\int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} \widetilde{g}_{n+1}(\tau) d\tau = \frac{k^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} g(t_j), \tag{5}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & if \quad j = 0 \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, & if \quad 1 \le j \le n \\ 1, & if \quad j = n+1 \end{cases} \tag{6}$$

and k is the stepsize(the uniform mesh is used). From (4) we immediately get that

$$\left| \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} g(\tau) d\tau - \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} \widetilde{g}_{n+1}(\tau) d\tau \right| \le \max_{0 \le t \le t_{n+1}} \left| g(t) - \widetilde{g}_{n+1}(t) \right| \int_0^{t_{n+1}} |(t_{n+1}-\tau)^{\alpha-1}| d\tau, \tag{7}$$

so that error bounds and orders of convergence for product integration follow from standard results of approximation theory. For a piecewise linear approximation to a smooth function $g(t)$ the product trapezoidal is second order[3].
Combining the above method with the method of lines, the numerical scheme for TFFPE is the following:

$$\begin{aligned} & D_*^{\alpha} u(x,t) = -A_x(x,t,u)u(x,t) - A(x,t,u)u_x(x,t) + B_{xx}(x,t,u)u(x,t) \\ & \qquad +2B_x(x,t,u)u_x(x,t) + B(x,t,u)u_{xx}(x,t), \quad x \in (a,b), \ t \ge 0, \\ & u(x,0) = \psi(x), \quad x \in (a,b) \\ & u(a,t) = \varphi_1(t), \quad u(b,t) = \varphi_2(t), \quad t \ge 0 \end{aligned}$$

where $\psi(x)$, $\varphi_1(t), \varphi_2(t)$ are the initial and boundary conditions, respectively. We can take the numerical fluxes as follows:

$$\widehat{u}_h^n = (u_h^n)^-, \ \widehat{p}_h^n = (p_h^n)^+, \quad or \quad \widehat{u}_h^n = (u_h^n)^+, \ \widehat{p}_h^n = (p_h^n)^-. \tag{8}$$

The above equation is trapezoidal quadrature formula that will used in the numerical solution of example.

4 Numerical results

We consider the problem 1 but, without loss of generality, add a force term $f(x,t)$ on the right–hand side [1]. Now the problem has the analytical solution $p(x,t) = t^2(x-a)^2(x-b)^2$ if taking $A(x,t,u) = -3, \ \ B(x,t,u) = 1$. It can be checked that the corresponding initial condition and force term are, respectively:

$$\begin{aligned} & \psi(x) = 0, \\ & f(x,t) = \tfrac{2\Gamma(2)}{\Gamma(3-\alpha)} t^{2-\alpha}(x-a)^2(x-b)^2 - 6t^2((x-a)(x-b)^2 + (x-a)^2(x-b)) \\ & \qquad - 2t^2((x-a)^2 + (b-x)^2 + 4(x-a)(x-b)). \end{aligned}$$

We compute the errors $\|u_e(T) - u_a(T)\|_{L^2(\omega)}$ and $\|u_e(T) - u_a(T)\|_{L^\infty(\omega)}$ for both FFDS and TQS at time $T = 1$ and with time fractional order $\alpha = 0.8$. In Tables 1,2 appear that the obtaining solutions are of $2-\alpha$ and second–order accurate for FFDS and TQS, respectively.

Table 1: Example 1: The errors with different space step lengths and $\alpha = 0.8$, $dt = 0.0005$

	$L_\infty, FFDS$	$L_2, FFDS$	convergent rate	L_∞, TQS	L_2, TQS	convergent rate
$N = 5$	6.07×10^{-4}	2.56×10^{-4}		6.07×10^{-4}	2.56×10^{-4}	
$N = 10$	8.79×10^{-5}	3.81×10^{-5}	2.74	8.79×10^{-5}	3.81×10^{-5}	2.74
$N = 20$	1.17×10^{-5}	5.22×10^{-6}	2.86	1.17×10^{-5}	5.22×10^{-6}	2.86
$N = 40$	1.51×10^{-6}	6.82×10^{-7}	2.93	1.51×10^{-6}	6.82×10^{-7}	2.93
$N = 80$	1.92×10^{-7}	8.72×10^{-8}	2.96	1.92×10^{-7}	8.72×10^{-8}	2.96
$N = 160$	2.42×10^{-8}	1.10×10^{-8}	2.98	2.42×10^{-8}	1.10×10^{-8}	2.98
$N = 320$	3.04×10^{-9}	1.38×10^{-9}	2.99	3.04×10^{-9}	1.38×10^{-9}	2.99
$N = 640$	3.80×10^{-10}	1.75×10^{-10}	2.98	3.80×10^{-10}	1.75×10^{-10}	2.98
$N = 1280$	5.75×10^{-11}	4.01×10^{-11}	2.12	5.75×10^{-11}	4.01×10^{-11}	2.12

Table 2: Example 1: The errors with different time step lengths and $\alpha = 0.8$, $h = 0.00125$

	$L_\infty, FFDS$	$L_2, FFDS$	convergent rate	L_∞, TQS	L_2, TQS	convergent rate
$dt = 0.1$	2.35×10^{-4}	1.61×10^{-4}		2.01×10^{-6}	1.38×10^{-6}	
$dt = 0.05$	1.03×10^{-4}	7.07×10^{-5}	1.18	5.03×10^{-7}	3.46×10^{-7}	2.00
$dt = 0.025$	4.50×10^{-5}	3.08×10^{-5}	1.19	1.26×10^{-7}	8.66×10^{-8}	2.00
$dt = 0.0125$	1.96×10^{-5}	1.34×10^{-5}	1.20	3.15×10^{-8}	2.16×10^{-8}	2.00
$dt = 0.00625$	8.24×10^{-6}	5.86×10^{-6}	1.19	7.92×10^{-9}	5.42×10^{-9}	1.99
$dt = 0.003125$	3.72×10^{-6}	2.55×10^{-6}	1.20	2.00×10^{-9}	1.35×10^{-9}	2.00
$dt = 0.0015625$	1.61×10^{-6}	1.11×10^{-6}	1.19	5.18×10^{-10}	3.47×10^{-10}	1.95

Conclusion

In this article, we employed two methods for discretization in time variable TFFPE that one of them is based on the fractional finite difference scheme and another is based on the trapezoidal quadrature scheme. Using the second order polynomials as shape functions gave the third–order for linear TFFPE and the maximum second–order of accuracy for nonlinear TFFPE in space variable. It should note that the convergence order of time discretization for FFDS and TQS are $O(\tau^{2-\alpha})$ and $O(\tau^2)$, respectively in time variable.

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Numerical Treatment of Coupling of Two Hyperbolic Conservation Laws By Local Discontinuous Galerkin Methods

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Abstract

In this work, the local discontinuous Galerkin (LDG) method is used to treat a system of differential equations consisting of two hyperbolic conservation laws. The cell entropy inequality is obtained when the upwind flux is utilized. In the linear case, we derive optimal convergence rates of order $\mathcal{O}(h^{k+1})$ in the L_2-norm, in domains where the exact solution is smooth; here h is the mesh width and k is the degree of the (orthogonal Legendre) polynomial functions spanning the finite element subspace. We justify the advantages of the LDG method in a series of numerical examples.

Keywords: Discontinuous Galerkin, coupling equations, error estimates
Mathematics Subject Classification [2010]: 65F05, 65Y05, 5Y20

1 Introduction

The main goal of this paper is to devise, analyze, and implement the local discontinuous Galerkin method (LDG) for the solution of the following coupling of two conservation laws in one space dimension: Find $u : (x, t) \in \mathbb{R} \times \mathbb{R}_+ \longrightarrow u(x, t) \in \mathbb{R}$ such that

$$\begin{cases} u_t + [f_R(u)]_x &= 0, \quad x > 0, \quad t > 0, \\ u_t + [f_L(u)]_x &= 0, \quad x < 0, \quad t > 0, \\ u(x,0) &= u_0, \quad x \in \mathbb{R}, \end{cases} \tag{1}$$

and also a suitable "continuity" condition

$$u(x, t) = u^b(t) \quad t \geq 0,$$

at the interface $x = 0$, to be compatible with initial condition u_0, where $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ is a given function and $f_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$, for $\alpha = L, R$, denote two "smooth" functions ([1, 3]). This type of phenomenon appears for example in an increasing number of problems of fluid mechanics, among others, we emphasize the case of coupled problems involving Euler equation on one side of the interface and Navier-Stokes equation on the other side, as well as modelling certain plasma physical problems cf [1].

For last decades, the technique of discontinuous Galerkin (DG) investigated as an higher-order accurate scheme for treating differential equations specially for those problems with hyperbolic nature and developing discontinuities [2]. The DG methods can

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be viewed as a combination of both finite element methods (FEMs), allowing for discontinuous discrete function, and finite volume methods, with more than one degree of freedom per mesh element. This extended scheme offers great opportunities relative to traditional FEMs when used to discretized hyperbolic problems. The main benefits of the DG methods can be summarized in terms of accuracy, flexibility, and parallelizability.

The purpose of this paper is to investigate the performance of the LDG method when applied to the system (1). The main focus is to implement, derive a priori error estimate of $\mathcal{O}(h^{k+1})$ theoretically, and justify this fact numerically.

1.1 Basic Notations

To start, we begin with the first equation of system (1) and reformulate it as the following initial boundary value problem: Find u such that

$$u_t + [f_R(u)]_x = 0, \qquad (x,t) \in \Omega, \tag{2a}$$

subject to initial and periodic boundary conditions

$$u(x,0) = u_0(x) \qquad x \in \Omega_a, \tag{2b}$$
$$u(0,t) = u(a,t) \qquad t \in \Omega_T, \tag{2c}$$

where our computational domain is $\Omega = \Omega_a \times \Omega_T$ with $\Omega_a = (0,a)$, $\Omega_T = (0,T)$, and $a, T > 0$. For the simplicity we assumed that our boundary conditions are periodic. Let us triangulate the space domain Ω_a with the partition $\mathcal{T}_h = \{K_j\}_{j=1}^N$ where $K_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $1 \le j \le N$, and $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = a$. We set for $1 \le j \le N$

$$x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}), \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}; \quad h = \max_{1 \le j \le N} h_j.$$

We assume that the mesh is quasi-uniform in the sense that there is a constant c independent of h such that $h_j \ge ch$ for all $1 \le j \le N$. To the mesh $\mathcal{T}_h$, we associate the finite element space $\mathcal{V}_h^k$, which is defined as piecewise polynomials space

$$\mathcal{V}_h^k := \{v \in \mathcal{L}_1(\Omega_a) \ : \ v|_K \in \mathbb{P}_k(K)\},$$

where, $\mathbb{P}_q(K)$ denotes the set of polynomials of degree less than or equal to q on the cell $K \in \mathcal{T}_h$. We also write

$$v(x^{\pm}_{j\pm\frac{1}{2}}) = v^{\pm}_{j\pm\frac{1}{2}} = \lim_{s \longrightarrow 0^{\pm}} v(x_{j\pm\frac{1}{2}} + s), \quad u(x^{\pm}_{j\pm\frac{1}{2}}, t) = \lim_{s \longrightarrow 0^{\pm}} u(x_{j\pm\frac{1}{2}} + s, t).$$

The LDG Formulation: We can now formulate the discrete version of the weak forms (2a)-(2c) which are obtained by restricting the trial and test functions to finite dimensional subspace $\mathcal{V}_h^k$ and by exploiting the numerical flux $\mathcal{F}_R(u)$ at the interfaces. Thus the semi-discrete LDG for solving (2) is defined as follows: Find the unique function $u_h = u_h(t) \in \mathcal{V}_h^k$ such that for all test functions $v_h \in \mathcal{V}_h^k$ and for all $1 \le j \le N$ we have

$$\int_{K_j} u_{h,t} v_h dx - \int_{K_j} f_R(u_h) v_{h,x} dx + \mathcal{F}_R^{j+\frac{1}{2}} v_h(x^-_{j+\frac{1}{2}}) - \mathcal{F}_R^{j-\frac{1}{2}} v_h(x^+_{j-\frac{1}{2}}) = 0, \tag{3a}$$

$$\int_{K_j} u_h(x,0) v_h dx = \int_{K_j} u_0(x) v_h dx, \tag{3b}$$

where we have used the notation $\mathcal{F}_R^{j\pm\frac{1}{2}} = \mathcal{F}_R(u_h(x^-_{j\pm\frac{1}{2}},t), u_h(x^+_{j\pm\frac{1}{2}},t))$.

For the periodic boundary conditions (2c), we choose the upwind flux which depends on $f_R(u)$. Depending on whether $\frac{\partial}{\partial u} f_R(u) > 0$ or < 0, we take accordingly

$$\mathcal{F}_R^{j+\frac{1}{2}} = \begin{cases} f_R(u_h(x^-_{N+\frac{1}{2}})), & j=0, \\ f_R(u_h(x^-_{j+\frac{1}{2}})), & j=1,\dots,N, \end{cases} \quad \mathcal{F}_R^{j+\frac{1}{2}} = \begin{cases} f_R(u_h(x^+_{j+\frac{1}{2}})), & j=0,\dots,N-1, \\ f_R(u_h(x^+_{\frac{1}{2}})), & j=N. \end{cases} \tag{4}$$

2 Main results

The next lemma will help us to prove the basic stability estimates for the LDG scheme (3).

Lemma 2.1 (Cell entropy inequality)**.** *The solution u_h to the semi-discrete DG scheme* (3) *satisfies the following cell entropy inequality*

$$\frac{d}{dt}\int_{K_j} U(u_h)dx + \widehat{\mathcal{F}}_R^{j+\frac{1}{2}} - \widehat{\mathcal{F}}_R^{j-\frac{1}{2}} \leq 0, \tag{5}$$

for the square entropy $U(u) = u^2/2$ and for some consistent entropy flux

$$\widehat{\mathcal{F}}_R^{j+\frac{1}{2}} = \widehat{\mathcal{F}}_R(u_h(x^-_{j+\frac{1}{2}},t), u_h(x^+_{j+\frac{1}{2}},t)),$$

satisfying $\widehat{\mathcal{F}}_R(u,u) = F(u)$.

A trivial consequence of the cell entropy inequality is an L_2-stability of the DG scheme:

Corollary 2.2 (L_2-stability)**.** *The solution of u_h to the semi-discrete DG scheme* (3) *satisfies the following L_2-stability*

$$\frac{d}{dt}\int_{\Omega_a} (u_h)^2 dx \leq 0, \quad \textit{or} \quad \|u_h(.,t)\|_{L_2(\Omega_a)} \leq \|u_h(.,0)\|_{L_2(\Omega_a)}. \tag{6}$$

In the next theorem, we show the optimal convergent rate property of the DG solutions toward a particular projection of the exact solution when the upwind fluxes are used:

Theorem 2.3. *Let u be the smooth exact solution of* (2a) *with $f_R(u) = a_R u$, and let u_h be the numerical solution of the LDG scheme* (3) *with the upwind flux* (4)*, then*

$$\|u - u_h\| \leq Ch^{k+1}\|u\|_{k+1,\Omega_a}, \tag{7}$$

where C is a constant independent of h and u.

2.1 Numerical Experiments

By putting $f_\alpha(u) = a_\alpha u$ ($\alpha = L, R$) in (1) and restricting ourselves to the computational domain $\Omega = (-1,1)\times(0,T)$, we get the linear case of our model problem

$$\begin{cases} u_t + a_R u_x &= 0, \quad x \in (0,1), \quad t \in (0,T), \\ u_t + a_L u_x &= 0, \quad x \in (-1,0), \quad t \in (0,T), \\ u(0,t) &= u^b(t), \quad t \in (0,T), \\ u(x,0) &= u_0(x), \quad x \in (-1,1). \end{cases} \tag{8}$$

We solve this problem using the DG method on uniform meshes with the mesh width h obtained by partitioning the domain $[-1, 1]$ into N subintervals with $N = 2^s, s = 4, 5, \cdots, 11$ and using the spaces $\mathbb{P}_k$ with $k = 1, 2, \cdots, 10$. The time interval $[0, T]$ is divided into $nt = T/\Delta t$ small time-step $\Delta t = \frac{h}{2k+1}$, where $h = 2/N$. Here, the final time is taken as $T = 0.5$. We also calculate the L_2-norm error, namely $\|e\| = \|u_h - u_{exact}\|$, and the order of convergence rate of the LDG scheme. The relative error norms of numerical solutions u_h and the convergence ratio are defined by

$$E_h = \frac{\|u_h - u_{exact}\|}{\|u_{exact}\|}, \quad r = \frac{E_h}{E_{h/2}}. \tag{9}$$

Example 2.4. We consider the coupling of two advection equations (8) with wave speeds $a_L = 0.05$ and $a_R = -0.05$ and initial data $u_0(x) = G(x)$, where $G(x)$ is a smooth Gaussian pulse centered at $x = 0$, which is defined by $G(x) = e^{-256x^2}$.

In the following table we use different number of cells and measure the errors in the L_2-norm for various number of polynomials degrees $k = 0, 1, \cdots, 6$. Furthermore, to confirm the obtained theoretical error bounds (7), we calculate the ratios r defined in (9) for a fixed k while the mesh size h is increased. The numerical experiments shown in Table 1 indicate

	$k=0$		$k=1$		$k=2$		$k=3$		$k=4$		$k=5$		$k=6$	
N	E_h	$\log_2 r$	E_h	$\log_2 r$	E_h	$\log_2 r$	E_h	$\log_2 r$	E_h	$\log_2 r$	E_h	$\log_2 r$	E_h	$\log_2 r$
16	4.07E-2	-	1.61E-1	-	1.68E-1	-	7.78E-2	-	1.30E-2	-	5.52E-3	-	1.67E-3	-
32	2.03E-1	-2.31	1.69E-1	-0.08	5.27E-2	1.67	6.20E-3	3.65	1.04E-3	3.64	1.87E-4	4.88	7.38E-6	7.82
64	1.31E-1	0.63	4.73E-2	1.84	5.59E-3	3.24	3.60E-4	4.10	4.02E-5	4.68	2.46E-6	6.25	1.64E-7	5.50
128	7.06E-2	0.89	1.01E-2	2.23	8.06E-4	2.79	3.05E-5	3.56	1.35E-6	4.90	3.73E-8	6.04	1.28E-9	7.00
256	3.73E-2	0.92	2.30E-3	2.13	1.01E-4	3.00	1.71E-6	4.16	3.51E-8	5.26	5.84E-10	6.00	2.31E-11	5.79
512	1.93E-2	0.95	5.49E-4	2.07	1.26E-5	3.00	1.04E-7	4.03	1.19E-9	4.88	8.99E-12	6.02	2.42E-12	3.25
1024	9.87E-3	0.97	1.34E-4	2.03	1.57E-6	3.00	6.40E-9	4.03	3.78E-11	4.98	4.37E-13	4.36	2.63E-13	3.21
2048	4.99E-3	0.98	3.32E-5	2.02	1.97E-7	3.00	3.99E-10	4.00	1.14E-12	5.05	3.02E-13	0.53	3.71E-13	-0.50
4096	2.51E-3	0.99	8.25E-6	2.01	2.46E-8	3.00	2.49E-11	4.00	5.37E-13	1.09	6.59E-13	-1.12	7.68E-13	-1.05

Table 1: Relative L_2 errors and the corresponding convergence rates at time $t = T$ for $\Delta t = h/(2k + 1)$ for different N and k.

that achieving an arbitrary order of accuracy is possible if one use the LDG method. In fact, an accuracy of $(k + 1)$th order of convergence is achieved while the number of cells N is increased.

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On existence, uniqueness and stability of solutions of a nonlinear integral equation

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Abstract

In this paper we investigate the existence, uniqueness and Hyers-Ulam stability for Volterra type integral equations and extension of this type of integral equations. The result is obtained by using the iterative method in the framework of Banach space $X = C([a, b]; \mathbb{R})$. Finally, we give an example to illustrate the applications of our results.

Keywords: Fixed point; Nonlinear functional-integral equation; Iterative method.
Mathematics Subject Classification [2010]: 45D05, 65R20.

1 Introduction

Integral equations play an important role in characterizing many social, physical, biological, and engineering problems. For example, Volterra [l] was investigating the population growth, focusing his study on the hereditary influences, and several authors, (see [2-4]), discussed the integrodifferential modeled integral equations in the field of heat transfer and diffusion process in general neutron diffusion. Generally, several systems are mostly related to uncertainty and un exactness. The problem of un exactness is considered in general exact science, and that of uncertainty is considered as vagueness or fuzzy and accident.
The solutions of integral equations have a major role in the fields of science and engineering. A physical event can be modeled by the differential equation, an integral equation, an integro-differential equation or a system of these. Investigation on existence theorems for diverse nonlinear functional-integral equations has been presented in other references such as [5].

In this paper we intend to prove existence, uniqueness and Hyers-Ulam stability (HUs) of the solutions of the following nonhomogeneous nonlinear Volterra integral equations.

$$u(x) = f(x) + \psi\Big(\int_a^x F(x, t, u(t))dt\Big) \equiv Tu, \qquad u \in X, \tag{1}$$

*Speaker

where $x, t \in [a, b]$, $-\infty < a < b < \infty$, $f : [a, b] \to \mathbb{R}$ is a mapping and F is a continuous function on the domain $D := \{(x, t, u) : x \in [a, b] , t \in [a, x) , u \in X\}$.

In this study, we will use an iterative method to prove that equation (1) has the mentioned cases under some appropriate conditions. On the other hand, in this paper, we prove the HUs theorem of (1) under generalized Lipschitz condition on F. Finally, we offer some examples that verify the application of this kind of nonlinear functional-integral equations.

The stability problem of functional equations originated from a question of Ulam in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?. In the other words, under what condition does there exists a homomorphism near an approximate homomorphism?. The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. Hyers provided a first partial affirmative answer to the question of Ulam for Banach spaces. Let $f : X \longrightarrow Y$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \le \delta$$

for all $x, y \in X$, and for some $\delta > 0$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \le \delta$$

for all $x \in X$. Ever since, the stability problems of functional equations have been extensively investigated by several mathematicians. In below we introduce some preliminaries and use them to obtain our aims in Section 2 and 3. Finally in Section 4 we offer some examples that verify the application of this kind of nonlinear functional-integral equations.

Definition 1.1. Let Ψ denoted the class of those functions $\psi : \mathbb{R} \to \mathbb{R}$ such that there exists $L_\psi > 0$ that for all $s, t \in \mathbb{R}$, $|\psi(s) - \psi(t)| \le L_\psi |s - t|$.

For example every linear function on $\mathbb{R}$ belong to Ψ.

Definition 1.2. Let Φ denoted the class of those functions $\phi : [0, \infty) \to [0, \infty)$ which satisfies the following condition

(i) ϕ is increasing,

(ii) for each $x > 0$, $\phi(x) < x$,

(iii) $\phi : [0, \infty) \to [0, \infty)$ is a upper semi–continuous function such that $\phi(t) = 0$ if and only if $t = 0$ and also for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exists $k \in (0, 1)$ and $n_0 \in \mathbb{N}$, such that $\phi(t_n) \le kt_n$ for each $n \ge n_0$.

For example, $\phi(t) = \mu t$, where $0 \le \mu < 1$, $\phi(t) = \frac{t^2}{t+1}$ and $\phi(t) = t - \ln(1 + \frac{t}{2})$ are in Φ.

2 Existence and uniqueness of the solution of nonlinear integral equations

Now we consider the equation (1) under the following conditions:
(i) $\psi : \mathbb{R} \to \mathbb{R}$ is belong to Ψ.
(ii) $F : D \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous.
(iii) There exists a continuous function $p : [a, b] \times [a, b] \to \mathbb{R}$ and $\phi \in \Phi$ such that

$$|F(x, t, u) - F(x, t, v)| \leq p(x, t)\phi(|u - v|),$$

for each $x, t \in [a, b]$ and $u, v \in \mathbb{R}$.
(iv) $\sup_{x\in[a,b]} \int_a^b p(x, t)dt \leq \frac{1}{L_\psi(b-a)}$.

Theorem 2.1. *Under the assumptions $(i) - (iv)$ the integral equation (1) has a unique solution in X.*

3 Stability of Nonlinear Integral Equations

In this section, we prove that the nonlinear integral equation in (1) has the HUs.

Theorem 3.1. *The equation $Tx = x$, where T is defined by (1), has the Hyers–Ulam stability that is for every $\xi \in X$ and $\epsilon > 0$ with*

$$d(T\xi, \xi) \leq \epsilon,$$

there exists a unique solution $u \in X$ such that

$$Tu = u,$$

and

$$d(\xi, u) \leq K\epsilon,$$

for some $K \geq 0$.

4 Applications

In this section, for efficiency of our theorem, some examples are introduced. Maleknejad et al. presented some examples that the existence of their solutions can be established using their theorem. Generally Examples 4.1 and 4.2 are introduced for first-time in this work.

Example 4.1. Consider the following nonlinear Volterra integral equation

$$u(x) = \sin(\frac{1}{1+x}) + \frac{x}{9}\int_0^x \frac{\cos(x^2 t)}{(1+xt)^2}\arctan(u(t))dt, \qquad (x \in [0, 1]). \tag{2}$$

We write

$$|F(x,t,u)-F(x,t,v)| = |\frac{x\cos(x^2t)}{9(1+xt)^2}(\arctan(u)-\arctan(v))| \leq |\frac{x\cos(x^2t)}{(1+xt)^2}||\frac{u-v}{9}|.$$

Take $p(x,t)=\frac{x\cos(x^2t)}{(1+xt)^2}$ and $\phi(t)=\frac{t}{9}$. Since $\sup_{x\in[0,1]}\int_0^1 p^2(x,t)dt \leq 1$, then the equation (2) has a unique solution in $C([0,1],\mathbb{R})$.

Example 4.2. Consider the following singular Volterra integral equation

$$u(x) = f(x) + \lambda\int_0^x (x-t)^{-\alpha}u(t)dt, \qquad (x\in[0,T]), \tag{3}$$

where $0\leq\lambda<1$ and $0<\alpha<\frac{1}{2}$. Then

$$|F(x,t,u)-F(x,t,v)| = |\lambda(u-v)(x-t)^{-\alpha}| \leq |\lambda||u-v||(x-t)|^{-\alpha}.$$

Put $p(x,t)=(x-t)^{-\alpha}$ and $\phi(t)=\lambda t$. We have

$$\sup_{x\in[0,T]}\int_0^T p^2(x,t)dt = \sup_{x\in[0,T]}\int_0^T |(x-t)|^{-2\alpha}dt = \frac{T^{1-2\alpha}}{1-2\alpha}.$$

It follows that if $T^{1-\alpha}\leq(1-2\alpha)^{1/2}$, then the equation (3) has a unique solution in complete metric space $C([0,T],\mathbb{R})$.

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Parallelization of the adaptive wavelet galerkin method for elliptic BVPs

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Abstract

In this work, an adaptive wavelet galerkin method (AWGM) with optimal computational complexity is parallelized. The method is applied to the solution of the second order elliptic BVPs. With tensor product wavelet basis, the rate of the AWGM is dimension independent. The numerical results indicate the method converge with optimal rate. Our results demonstrate that the AWGM can be implemented in a multiprocessor environment and is scalable.

Keywords: Adaptive method, Tensor product wavelets, Parallel computation
Mathematics Subject Classification [2010]: 35K15, 65F50, 65Y05

1 Introduction

This paper deals with the implementation of the AWGM in shared-memory parallel programming. Recently multiprocessing platforms are available with multi-core processors sharing memory. An efficient way for performing the applications in high performance computing fields is to parallelize them in multiprocessing schemes. In order to achieve the best speed up as possible, *Synchronization* is a natural and essential part of parallel programs. We strongly notice that this main task of parallelization cannot be avoided in the adaptive methods. Shared-memory computing are rendered parallel with threading model extensions such as *OpenMP* and Pthreads. In this context, A *thread* is a sequence of such instructions within a program that can be executed independently of other code. In fact, *OpenMP* and Pthreads programming are two well known and dominant shared-memory programming models.

OpenMP is a portable interface for implementing fork-join parallelism on shared memory multiprocessor machines. It is a library which implemented with *"omp.h"*. *OpenMP* provides suitable level of abstraction to a programmer. It extends and defines a set of directives and library routines for Fortran and C/C++ [1] . Actually it consists of the set of directives, clauses and functions that enables creating, managing, communicating and synchronizing parallel threads.
One of the advantages of the programming in *OpenMP* is that the resulting parallel code is close to its sequential version. It explicitly decalres parallel regions but much of the synchronizations are managed implicitly. The execution performance of the program in *OpenMP* is highly dependent on the quality of the *OpenMP* implementation. An efficient way for designing the data structure of adaptive methods is hash table. In multi core

processing, It turns out often that a number of threads try to create different records simultaneously on the same bucket of hash table. Therefore in this case, a good protection management is necessary. To resolve this problem, we can consider *data lock*. To lock data, we associate a lock variable for each bucket in the hash table. To create records by a group of threads on the same bucket, we set the lock variable of the specific bucket. After inserting, the lock variable should be released. To implement *lock data*, the following directives can be exploited in *OpenMP* C/C++

```
omp_lock_t //Declaration of a lock variable
omp_init_lock //Initialization of a lock variable
omp_set_lock //Blocking thread execution until a lock is available
omp_unset_lock //Releasing ownership of a lock
```

2 Elliptic boundary value problems and the AWGM

The variational formulation of a second order elliptic boundary value problem on a domain $\Omega \subset \mathbb{R}^n$ with homogeneous Dirichlet boundary conditions reads as $Bu = f$, where

$$(Bu)(v) := \int_{\Omega} (A\nabla u \cdot \nabla v + (b \cdot \nabla u)v + cuv)dx.$$

If $A \in L_\infty(\Omega)^{n\times n}, b \in L_\infty(\Omega)^n, c \in L_\infty(\Omega), c \geq 0$ $(a.e.), \nabla \cdot b = 0$ $(a.e.)$ and, for some $\delta > 0, A \geq \delta > 0$ $(a.e.)$, then B is coercive and boundedly invertible. Assume that for any n, the normalized tensor product basis $\Psi := \{\psi_\lambda := \otimes_{m=1}^n \psi_{\lambda_m}^{(m)} / \| \otimes_{m=1}^n \psi_{\lambda_m}^{(m)} \|_B \colon \lambda \in \nabla := \prod_{m=1}^n \nabla^{(m)}\}$, is a Riesz basis for $H_0^1(\square)$ where $\Omega = \square := (0,1)^n$. This space is equipped with energy norm $\|.\|_B$ and $\psi_{\lambda_m}^{(m)}$ for $\lambda_m \in \nabla^{(m)}$ is univariate wavelet function in mth-coordinate.

By writing $u = \mathbf{u}^\top \Psi := \sum_{\lambda \in \nabla} \mathbf{u}_\lambda \psi_\lambda$, we infer that the problem can equivalently be written as the bi-infinite matrix vector problem $\mathbf{Bu} = \mathbf{f}$ where $\mathbf{f}$ is the load vector and $\mathbf{B}$ is the boundedly invertible matrix. We solve this equation with the AWGM that is described here. More details about the AWGM, we refer to [3].

$\texttt{AWGM}[\epsilon] \to \mathbf{w}_\epsilon$:
% Input: $\epsilon > 0$.
% Parameters: $\mu \in (0, \kappa(\mathbf{B})^{-\frac{1}{2}})$ *and* $\gamma \in (0, \mu\kappa(\mathbf{B})^{-1})$.

$i := 0$, $\Lambda_i := \emptyset$, $\mathbf{w}^{(i)} := 0$, $\mathbf{r}^{(i)} := \mathbf{f}$
while$\|\mathbf{r}^{(i)}\| > \epsilon$ do
  $\Lambda_{i+1} := \mathbf{EXPAND}[\Lambda_i, \mathbf{r}^{(i)}, \mu\|\mathbf{r}^{(i)}\|]$
  $\mathbf{w}^{(i+1)} := \mathbf{GALERKIN}[\Lambda_{i+1}, \mathbf{w}^{(i)}, \|\mathbf{r}^{(i)}\|, \gamma\|\mathbf{r}^{(i)}\|]$
  $\mathbf{r}^{(i+1)} := \mathbf{f} - \mathbf{B}\mathbf{w}^{(i+1)}$
  $i := i + 1$
enddo
$\mathbf{w}_\epsilon := \mathbf{w}^{(i)}$

$\mathbf{GALERKIN}[\Lambda, \mathbf{w}, \delta,] \to \bar{\mathbf{w}}$:
% Input: $, \delta > 0$, $\Lambda \subset \nabla$, *a* $\mathbf{w} \in \ell_2(\Lambda)$ *with* $\|\mathbf{f}|_\Lambda - \mathbf{A}|_{\Lambda\times\Lambda}\mathbf{w}\| \leq \delta$.
% Output: $\bar{\mathbf{w}} \in \ell_2(\Lambda)$ *with* $\|\mathbf{f}|_\Lambda - \mathbf{A}|_{\Lambda\times\Lambda}\bar{\mathbf{w}}\| \leq$ *in* $\mathcal{O}(\log(\delta/)\#\Lambda)$ *operations.*

EXPAND$[\Lambda, \mathbf{g}, \sigma] \to \bar{\Lambda}$:
% Input: $\Lambda \subset \nabla$*, a finitely supported* $\mathbf{g} \in \ell_2(\nabla)$*, and a scalar* $\sigma \in [0, \|\mathbf{g}\|_{\ell_2(\nabla)}]$.
% Output: $\Lambda \subset \bar{\Lambda} \subset \nabla$ *with* $\|\mathbf{P}_{\bar{\Lambda}}\mathbf{g}\| \geq \sigma$ *and such that, up to some absolute multiple,*
% $\#(\bar{\Lambda} \setminus \Lambda)$ *is minimal over all such* $\bar{\Lambda}$*, and the cost of the call is* $\mathcal{O}(\#\Lambda \cup \#supp\, \mathbf{g})$,

where $\mathbf{P}_{\bar{\Lambda}}$ is the operator that replaces coefficients outside $\bar{\Lambda}$ by zeros.

3 Hash-Storage Technique

Looking for an efficient way to implement the `AWGM`, one can use *hash* storage strategy. A basic hash table consists of a set of slots. Each entry of the given data has a key or index. The size of a hash table is slightly larger than the size of the input data. Each item of the input data is mapped to one these slots by *hash function.* Each entry of the given data has a key and hash function operates key and associates a unique position in the set of slots. Generally, hash function is not injective function because there are many more possible different entries than different addresses in the hash table. When two or more items try to occupy the same address in hash table, a *Collision* occurs.

In the `AWGM` implementation, the key is multi index $\lambda \in \nabla$ and u_λ is stored and can be retrieved at address in the hash table which produced by hash function. Since the `AWGM` has optimal $\mathcal{O}(N)$ complexity for N unknowns, thus we keep in mind that access to a specific entry in hash table should be performed in constant time. Regarding to this important issue, we should provide a good hash function to minimize collisions as possible. We remark that there is no magic and perfect hash function which produces a unique set of integers within some suitable range.

Hence there exists an isomorphism between $\mathbb{N}_0^n$ and ∇, then w.l.o.g. we can assume that the multi index $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{N}_0^n$. By using the modulo arithmetic %, in one dimension we define the simple hash function $H : \mathbb{N}_0 \to \{0, 1, \cdots, p-1\}$ with $H(\lambda) := \lambda \% p$ such that the prime number p is the size of the storage space, i.e., the length of hash table. This hash function will create a uniform distribution of addresses in hash table. An obvious choice for hash function in multi index is to define $\mathfrak{H} : \mathbb{N}_0^n \to \{0, \cdots, p-1\}$ as $\mathfrak{H}(\lambda) := \sum_{i=1}^n H(\lambda_i)$. This is not good alternative for $\mathfrak{H}$ because $\sum_{i=1}^n H(\lambda_i) = H(\sum_{i=1}^n \lambda_i)$. Therefore in this case all λ with equal ℓ_1-norm get the same bucket in the hash table. In order to have less collisions, we define a profitable isomorphism between $\mathbb{N}_0^n$ and $\mathbb{N}$ denoted by $\mathfrak{K}(n, \cdot) : \mathbb{N}_0^n \to \mathbb{N}_0$. We figure out that the isomorphism $\mathfrak{K}(n, \lambda)$ fulfill the recursive formula

$$\mathfrak{K}(n, \lambda) = \binom{\|\lambda\|_1}{n-1} + \mathfrak{K}(n-1, (\lambda_1, \cdots, \lambda_{n-1})).$$

By setting $s_1 := \lambda_1, s_2 := \lambda_1 + \lambda_2, \cdots, s_n = \lambda_1 + \cdots + \lambda_n$, then

$$\mathfrak{K}(n, \lambda) = \binom{s_n + n - 1}{n} + \binom{s_{n-1} + n - 2}{n-1} + \cdots + \binom{s_1}{1},$$

and so we define $\mathfrak{H}(\lambda) := \mathfrak{K}(n, \lambda) \% p$. In the strategy known as separate chaining, each slot of the bucket array is a pointer to a linked list that contains the pairs (*key,value*) which hashed to the same location. In order to have less and less dynamic memory operations, we will store one record of each chain in the slot array itself. In single thread or serial computations, a satisfactory data structure of the wavelet coefficient **u** consists of a hash table and a linked list of the support of wavelet coefficients, i.e., a linked list containing $\{\lambda : u_\lambda \neq 0, \forall\lambda \in \nabla\}$. This data structure is not designed suitably for parallel programs in shared-memory computation. Because if more than one threads try to insert records on the same slot then the slot and the keys linked list should be protected for one thread against the other threads. To prepare an efficient algorithm in parallel environment, we should use the right locks. We can remove the locks on the keys linked list. This goal can be done by using q^2 linked lists where q is the number of threads. The support of wavelet coefficients **u** is separated equally by q-threads and each thread accesses to q linked lists without any lock mechanism.

4 Numerics

Using quartic (with order $d = 5$) wavelets, with discontinuous piecewise quartic duals as constructed in [2], we solved the Poisson problem of finding $u \in H_0^1(\square)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = f(v) \quad (v \in H_0^1(\square)),$$

by applying the `AWGM` in *OpenMP* where $\square = (0,1)^n$. This method produces a sequence of approximations from the span of the basis that converges in $H^1(\Omega)$-norm with the best possible rate. Assuming a sufficiently smooth right-hand side, this rate is $d - 1 = 4$. In this example, for our convenience we took as right hand side function $f = 1$. We consider the speedups of the `AWGM` in Table 1. We use T_1 as the time for the full simulation on one processor. We calculate the speedup $S_P = T_p/T_1$ where T_p is the observed time on p processors. The results show that the speedup of the `AWGM` in parallel computing is almost better than the sequential implementation.

p	1	2	4	8	16
$n = 2, S_p$	1	1.6	3.05	5.8	11.05
$n = 3, S_p$	1	1.5	2.9	5.4	10.1

Table 1: The `AWGM` on p=1, 2, 4, 8, and 16 processors with speedup S_p for n=2, 3

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Pivoting strategy for an ILU preconditioner

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Abstract

In this paper, a complete pivoting strategy for the right-looking version of $RIF-NS$ preconditioner is presented.

Keywords: preconditioning, pivoting, right-looking version of $RIF-NS$ preconditioner
Mathematics Subject Classification [2010]: 65F10, 65F50, 65F08.

1 Introduction

Consider the linear system of equations of the form $Ax = b$, where the coefficient matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular, large, sparse and nonsymmetric and also $x, b \in \mathbb{R}^n$. An ILU preconditioner M of this system is in the form of $M = LDU \approx A$. This preconditioner will change the original system to the left preconditioned system $M^{-1}Ax = M^{-1}b$. For a proper preconditioner, instead of solving the original system, it is better to solve the left preconditioned system by the Krylov subspace methods [3]. In [1], we have proposed an ILU preconditioner for system $Ax = b$. This preconditioner is termed the $RIF-NS$ and has two left- and right-looking versions.

2 Pivoting strategy for the right-looking $RIF-NS$ preconditioner

Algorithm 1, uses the complete pivoting strategy to compute the right-looking version of $RIF-NS$ preconditioner. Here we explain the step i of this algorithm. At the beginning of this step, $\Pi = \Pi_{i-1}...\Pi_1$ and $\Sigma = \Sigma_1...\Sigma_{i-1}$ are the row and the column permutation matrices, respectively. For $k < i$, the matrices Π_k and Σ_k are the row and the column permutation matrices associated to step k of this algorithm. At the beginning of this step, the parameters m_i, n_i, $iter$, $satisfied\_p$ and $satisfied\_q$ are initialized in line 3. At the end of this step, m_i and n_i will be the total number of row and column pivoting associated to step i. The parameter $iter$ is used to compute the pivot entry in this step. $satisfied\_p$ ($satisfied\_q$) shows whether or not we need to the row (column) pivoting strategy. In line 7 of the algorithm, the vector $(q_i^{(i-1)}, \cdots, q_n^{(i-1)})$ is computed. Suppose that $|q_k^{(i-1)}| = max_{m\geq i+1}|q_m^{(i-1)}|$. If the criterion $|q_i^{(i-1)}| < \alpha\, |q_k^{(i-1)}|$ is satisfied for

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$\alpha \in (0, 1]$, then the row pivoting strategy is applied in lines 8-11 of the algorithm. Suppose that $|p_l^{(i-1)}| = max_{m \geq i+1} |p_m^{(i-1)}|$. If the criterion $|p_i^{(i-1)}| < \alpha\ |p_l^{(i-1)}|$ is satisfied for an $\alpha \in (0, 1]$, then the column pivoting strategy is applied in lines 17-20 of the algorithm. After the column pivoting, $satisfied\_p$ is set to $true$ in line 22 and the algorithm will alternate between the row and the column pivoting. After the internal $while$ loop, the pivot entry d_{ii} is set equal to $q_i^{(i-1)}$. In lines 25-28 of the algorithm, the i-th column of matrices W and L, and the i-th row of matrix U are computed.

3 Numerical results

In this section, we have formed 6 artificial linear systems where the coefficient matrices are downloaded from [2] and the exact solution of these systems is the vector $[1, \cdots, 1]^T$. We have used two parameters 0.1 and 1.0 as α to compute the right-looking version of $RIF - NS$ preconditioner with complete pivoting strategy. We have used The command *bicgstab* in $Matlab$ software to solve the original and the left preconditioned systems by the $BICGSTAB$ method. The stopping criterion for all linear systems is satisfied when the relative residual is less than 10^{-6}. We have considered the zero vector as the initial solution for all systems. The density of the preconditioners is defined as :

$$density = \frac{nnz(L) + nnz(U)}{nnz(A)},$$

where $nnz(L)$, $nnz(U)$ and $nnz(A)$ are the number of nonzero entries of matrices L, U and A. Table 1, shows the matrix properties and the information of $BICGSTAB$ method to solve the original linear systems. In this table, n and nnz are the dimension and the number of nonzero entries of the matrix. In Tables 1 and 2, the parameters it and $flag$ indicate the number of iterations and the status of the convergence. The parameter $iter$ can be an integer+0.5 indicating convergence halfway through an iteration. When $flag$ is equal to 0, it means that the method has been converged to the desired tolerance within the 2500 iterations. $flag = 2$ shows that the preconditioner is ill-conditioned and $flag = 4$ indicates that one of the scalar quantities calculating during the method became too small or too large to continue computing.

Algorithm 1 Right-looking version of $RIF - NS$ with complete pivoting

Input: $A \in \mathbb{R}^{n\times n}$ and $\tau_w, \tau_l, \tau_u \in (0,1)$ be drop tolerances. **Output:** $\Pi A \Sigma \approx M = LDU$

$w_i^{(0)} = e_i,\ 1 \le i \le n$
for $i = 1$ to n **do**
  $m_i = n_i = 0$, $iter = 0$, $satisfied\_p = satisfied\_q = false$
  while not $satisfied\_q$ **do**
    $iter = iter + 1$
    If $iter = 1$, then set $q_i^{(i-1)} = (w_i^{(i-1)})^T(\Pi A \Sigma)e_i$. Otherwise set $q_i^{(i-1)} = p_i^{(i-1)}$.
    $q_j^{(i-1)} = (w_j^{(i-1)})^T(\Pi A\Sigma)e_i,\ i+1 \le j \le n$
    if $|q_i^{(i-1)}| < \alpha\ max_{m \ge i+1}|q_m^{(i-1)}|$ **then**
      $m_i = m_i + 1,\quad \pi_{m_i}^{(i-1)} = I_n$ and $satisfied\_p = false$
      Choose k such that $|q_k^{(i-1)}| = max_{m\ge i+1}|q_m^{(i-1)}|$. Then, interchange columns i and k of $W - I$ and rows i and k of $\pi_{m_i}^{(i-1)}$ and $L - I$. Also interchange elements $q_i^{(i-1)}$ and $q_k^{(i-1)}$ and do the update $\Pi = \pi_{m_i}^{(i-1)}\Pi$
    end if
    $satisfied\_q = true$
    if not $satisfied\_p$ **then**
      $p_i^{(i-1)} = q_i^{(i-1)}$
      $p_j^{(i-1)} = (\Pi A \Sigma)_{ij},\quad i+1 \le j \le n.$
      $p_j^{(i-1)} = p_j^{(i-1)} - L_{ik}d_{kk}U_{kj}$ for $k = 1$ to $i-1$ and $j = i+1$ to n
      if $|p_i^{(i-1)}| < \alpha\ max_{m\ge i+1}|p_m^{(i-1)}|$ **then**
        $n_i = n_i + 1,\quad \sigma_{n_i}^{(i-1)} = I_n$ and $satisfied\_q = false$
        Choose l such that $|p_l^{(i-1)}| = \max_{m\ge i+1}|p_m^{(i-1)}|$. Then, interchange columns i and l of $\sigma_{n_i}^{(i-1)}$ and $U - I$. Also, interchange elements $p_i^{(i-1)}$ and $p_l^{(i-1)}$ and do the update $\Sigma = \Sigma\sigma_{n_i}^{(i-1)}$
      end if
    end if
    $satisfied\_p = true$
  end while
  $d_{ii} = q_i^{(i-1)}$
  for $j = i+1$ to n **do**
    $w_j^{(i)} = w_j^{(i-1)} - (\frac{q_j^{(i-1)}}{d_{ii}})w_i^{(i-1)}$ and for all $l \le i$, if $|w_{lj}^{(i)}| < \tau_w$, then set $w_{lj}^{(i)} = 0$
    $L_{ji} = \frac{q_j^{(i-1)}}{d_{ii}},\quad U_{ij} = \frac{p_j^{(i-1)}}{d_{ii}}$. If $|L_{ji}| < \tau_l$, then set $L_{ji} = 0$. If $|U_{ij}| < \tau_u$, then set $U_{ij} = 0$.
  end for
end for
Return $L = (L_{ij})_{1\le i,j\le n}$, $U = (U_{ij})_{1\le i,j\le n}$, $D = diag(d_{ii})_{1\le i\le n}$, Π and Σ.

Table 1

Matrix	n	nnz	without preconditioner	
			it	$flag$
$bwm200$	200	796	109.5	0
str_400	363	3157	0	4
$tols90$	90	1746	28	4
str_0	363	2454	0	4
$tub100$	100	396	106.5	0
$08blocks$	300	592	1	4

In Table 2, the notation $RLRIF - NSP(\alpha)$ refers to the right-looking version of $RIF - NS$ preconditioner with complete pivoting strategy which is computed by using the parameter α.

Table 2

Method	RLRIF-NSP(0.1)					RLRIF-NSP(1.0)					RLRIF-NS		
Matrix	$density$	$Rpiv$	$Cpiv$	$iter$	$flag$	$density$	$Rpiv$	$Cpiv$	$iter$	$flag$	$density$	$iter$	$flag$
$bwm200$	1.0012	0	0	23.5	0	1.2073	84	81	19.5	0	1.0012	23.5	0
str_400	0.5854	357	5	11	0	0.6097	383	57	0	2	5.5958	0	2
$tols90$	0.1523	18	0	2.5	0	0.4370	20	3	2.5	0	0.3070	12	0
str_0	0.5028	358	0	2	0	0.5676	362	29	2	0	3.9238	0	2
$tub100$	1.0050	0	0	8	0	1.1591	62	59	7.5	0	1.0050	8	0
$08blocks$	1.4797	292	0	1.5	0	118.3513	32	5	0	2	2	0.5	4

The $RLRIF - NS$ is a notation for the right-looking version of $RIF - NS$ preconditioner. The columns *Rpiv* and *Cpiv* show the total number of row and column pivoting. In this table, the information in the columns $flag$ and $iter$ associated to the three preconditioners indicate that for all of the matrices, one of the preconditioners $RLRIF - NSP(1.0)$ or $RLRIF - NSP(0.1)$ gives better results of the $BICGSTAB$ method than the $RLRIF - NS$ preconditioner. This means that the complete pivoting strategy with one of the values $\alpha = 1.0$ or $\alpha = 0.1$ has a good effect on the quality of the right-looking version of $RIF - NS$ preconditioner.

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Talk Reproducing kernel method for solving a class of Fredholm integral equations pp.: 1–4

Reproducing kernel method for solving a class of Fredholm integral equations

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Abstract

This paper presented a numerical method for solving Fredholm integral equations by reproducing kernel method (RKM). On the basis of reproducing kernel Hilbert spaces theory, an iterative algorithm for solving some integral equations is presented. We present two examples which have better results than others.

Keywords: Reproducing kernel, Fredholm integral, Approximate solution.

Mathematics Subject Classification [2010]: 45B05, 74H15, 41A10.

1 Introduction

The opinion integral equations play an important role in both mathematics and other applicable areas. This kind equations have been investigated in many application domains. Here, we study Fredholm integral equations [1].

$$y(x) = g(x) + \int_a^b k(x,t)y(t)\mathrm{d}t, \tag{1}$$

where the function $g(x)$ and $k(x,t)$ are given, and the unknown function $y(t)$ is to be determined. A new method of solving solution for Fredholm integral equations is proposed in a reproducing kernel Hilbert space in this paper. It is called reproducing kernel method. Reproducing kernel theory has important applications in numerical analysis, differential equations, integral equations, probability and statistics, learning theory and so on. Reproducing kernel methods for solving a variety of integral equations were introduced by Jin [2], Du [3], Chen [4], Shen [5].

*Speaker

2 Reproducing kernel Hilbert space

To solve (1), first, we construct reproducing kernel spaces ${}^oW^4[a,b]$.

Definition 2.1. ${}^oW^m[a,b] = \{u^{(m-1)}(x)$ is an absolutely continuous real function, $u^{(m)}(x) \in L^2[a,b], u(a) = 0\}$.

The inner product and norm in ${}^oW^m[a,b]$ are given respectively by

$$\langle u, v\rangle = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)\,\mathrm{d}x, \tag{2}$$

and

$$\|u\|_m = \sqrt{\langle u, u\rangle}_m, \qquad u, v \in {}^oW^m[a,b]. \tag{3}$$

By [6], ${}^oW^4[a,b]$ is a reproducing kernel space and its reproducing kernel $R_y(x)$ can be obtained.
Let $R_x(y)$ be

$$R_y(x) = \begin{cases} R_1(x,y) = \sum_{i=1}^{8} c_i(y)x^{i-1}, & y \le x, \\ R_2(x,y) = \sum_{i=1}^{8} d_i(y)x^{i-1}, & y > x, \end{cases} \tag{4}$$

where coefficients $c_i(y), d_i(y)$, $\{i = 1, 2, \cdots, 8\}$, could be obtained by solving the following equations

$$\frac{\partial^i R_y(x)}{\partial x^i}|_{x=y+0} = \frac{\partial^i R_y(x)}{\partial x^i}|_{x=y-0}, \qquad i = 0, 1, 2, 3, 4, 5, 6, \tag{5}$$

$$\frac{\partial^7 R_y(x)}{\partial x^7}|_{x=y+0} - \frac{\partial^7 R_y(x)}{\partial x^7}|_{x=y-0} = 1, \tag{6}$$

and

$$\begin{cases} \frac{\partial^i R_y(a)}{\partial x^i} - (-1)^{3-i}\frac{\partial^{7-i} R_y(a)}{\partial x^{7-i}} = 0, & i = 1, 2, 3, \\ \frac{\partial^{7-i} R_y(b)}{\partial x^{7-i}} = 0, & i = 0, 1, 2, 3, \\ R_y(a) = 0. \end{cases} \tag{7}$$

3 Solving Eq. (1) in the Reproducing Kernel Space

To solve Eq. (1), we define operator $\mathbb{L} : {}^oW^4[a,b] \to L^2[a,b]$ as follows:

$$(\mathbb{L}y)(x) = y(x) - \int_a^b k(x,t)y(x)\,\mathrm{d}t. \tag{8}$$

Lemma 3.1. $\mathbb{L}$ *is a bounded linear operator.*

Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[a,b]$. Put $\varphi_i(x) = R_x(x_i)$ and $\psi_i(x) = \mathbb{L}^\star\varphi_i(x)$, $\mathbb{L}^\star$ is the adjoint operator of $\mathbb{L}$ and

$$\psi_i(x) = [\mathbb{L}_y R_y(x)](x_i) = R(x, x_i) - \int_a^b k(x,t)R(x,t)\,\mathrm{d}t. \tag{9}$$

The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of ${}^oW^4[a,b]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad (\beta_{ii} > 0, \quad i = 1, 2, \cdots). \tag{10}$$

According to [6], we have the following theorems:

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and solution of (1) is unique, then (i) the exact solution of Eq. (1) can be represented by*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i(x), \tag{11}$$

(ii) the approximate solution $u(x)$ can be obtained by taking finitly many terms in the series representation of $u(x)$ and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i(x). \tag{12}$$

Theorem 3.3. *Suppose $\|u_n(x)\|_{{}^oW^4}$ is bounded in (12), if $\{x_i\}_{i=1}^{\infty}$ is dense in $[a,b]$, then the n-term approximate solution $u_n(x)$ converges to the exact solution $u(x)$ of Eq. (1) and the approximate solution is expressed as*

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\bar{\psi}_i(x). \tag{13}$$

4 Numerical experiments

Our new method has been tested for the following two equations.

Example 4.1. Consider the following Fredholm integral equation:

$$y(x) - \int_0^{\frac{\pi}{2}} k(x,t)y(t)\,dt = g(x), \tag{14}$$

where $g(x) = \sin(x) - x$, $k(x,t) = xt$. The exact solution is $y(x) = \sin(x)$ and $x \in [0, \frac{\pi}{2}]$. Using the method presented in section 3, taking $n = 10$ and $n = 20$, $x_i = \frac{\pi}{2(n+1)} \times i, i = 1, 2, \ldots, n$. The approximate solution, the absolute errors $|u_n(x) - u(x)|$ for $n = 10$ and 20 are graphically shown in figure 1, respectively. However, by increasing n, the behavior improves.

Example 4.2. Consider the following Fredholm integral equation:

$$y(x) - \int_0^{1} k(x,t)y(t)\,dt = g(x), \tag{15}$$

where $g(x) = e^x - \frac{e^{x+1}-1}{x+1}$, $k(x,t) = e^{xt}$. The exact solution is $y(x) = e^x$ and $x \in [0,1]$. Using the method presented in section 3, taking $n = 10$ and $n = 20$, $x_i = \frac{1}{n+1} \times i, i = 1, 2, \ldots, n$. The approximate solution, the absolute errors $|u_n(x) - u(x)|$ for $n = 10$ and $n = 20$ are graphically shown in figure 2, respectively. However, by increasing n, the behavior improves.

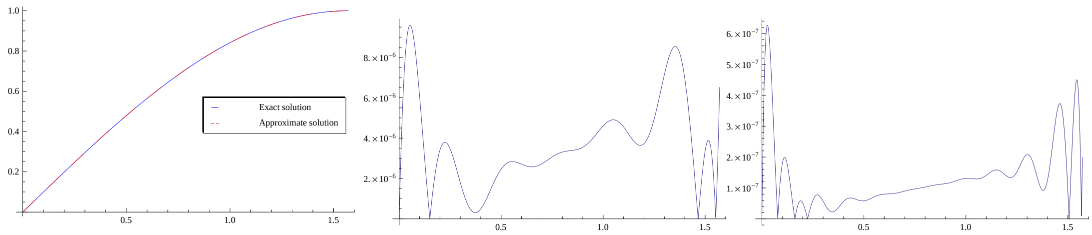

Figure 1: The approximate solution, the absolute errors for $n = 10$ and 20, respectively.

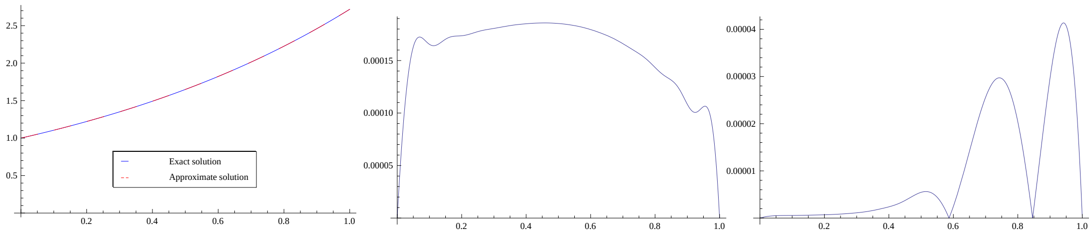

Figure 2: The approximate solution, the absolute errors for $n = 10$ and 20, respectively.

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Semiconvergence of the iterative Monte Carlo method for solving singular linear systems

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Abstract

In this paper, semiconvergence of the iterative Monte Carlo method to solve singular linear systems is discussed. First, sufficient conditions for the semiconvergence of this method are given. Then, Monte Carlo method is employed based on semiconvergence conditions. Finally, the numerical experiment is presented to illustrate the efficiency of the proposed method.

Keywords: Monte Carlo, Singular linear systems, Markov chain, Semiconvergence.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Let us consider the linear system of n equations

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is singular and $b, x \in \mathbb{R}^n$ with b known and x unknown. We assume that the system (1) is solvable, *i.e.*, it has at least one solution. In order to solve the system (1) with plain iterative Monte Carlo (MC) method, the coefficient matrix A is splitted to $A = M - N$, where M is nonsingular. Hence, a stationary iterative method for solving (1) can be presented in the following form

$$x^{(k+1)} = Tx^{(k)} + f, \quad k = 0, 1, 2, \dots, \tag{2}$$

where $f = M^{-1}b$ and the matrix $T = M^{-1}N$ is called the iteration matrix of the iterative method. According to the essential theorem of iterative methods [1], we know that the method (2) is convergent if and only if $\rho(T) < 1$ ($\rho(T)$ is the spectral radius of T). For continuity, we recall some basic concepts.

Definition 1.1. [5] The index of square matrix A is the smallest nonnegative integer k such that the following statement is true,

$$rank(A^k) = rank(A^{k+1}).$$

*Speaker

Let λ is an eigenvalue for A. In this case, the index of the eigenvalue λ is defined to be the index of the matrix $\lambda I - A$. In the other words, $index(\lambda) = index(\lambda I - A)$. Of course the index of the eigenvalue λ can be calculated from Jordan canonical form of matrix A. For singular systems, the method (2) is semiconvergent if it converges to a solution of (1) which depends on the initial guess x^0. From [1], it can be known that the iterative method (2) is semiconvergent if and only if each of the following conditions is established.

(1) $\rho(T) = 1$;
(2) $index(I - T) = 1$, which means that $index(\lambda = 1) = 1$;
(3) If $\mu \in \sigma(T)$ with $|\mu| = 1$, then $\mu = 1$, *i.e.*, $v(T) = \{|\mu|, \mu \in \sigma(T), \mu \neq 1\} < 1$,

where $\sigma(T)$ is spectrum of T. The semiconvergence of the iterative method (2) has been investigated by many authors [6].

2 Main results

The stationary iterative MC method is based on the iterative presentation method (2). The plain MC method has been constructed from the convergence of the iterative method (2) in [3]. In this paper, MC method is produced from the semicovergence of the iterative method (2). So, consider the inner product $< h, x >= \sum_{i=1}^{n} h_i x_i$, where $h \in \mathbb{R}^n$ is a known vector and $x \in \mathbb{R}^n$ is the exact solution of the linear algebraic system (1). In the MC approach, we consider an initial density vector $p \in \mathbb{R}^n$, where its entries satisfy in $p_i \geq 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$ conditions. Also, consider a transition density matrix $P = [p_{ij}] \in \mathbb{R}^{n\times n}$, where its entries satisfy in $p_{ij} \geq 0$, $i, j = 1, \ldots, n$ and $\sum_{j=1}^{n} p_{ij} = 1$, for any $i = 1, \ldots, n$. The initial density vector p and the transition density matrix P have the following properties

$$\begin{cases} p_i > 0, \quad when \quad h_i \neq 0 \quad and \quad p_i = 0, \quad when \quad h_i = 0 \quad for \quad i = 1, \ldots, n, \\ p_{ij} > 0, \quad when \quad t_{ij} \neq 0 \quad and \quad p_{ij} = 0, \quad when \quad t_{ij} = 0 \quad for \quad i, j = 1, \ldots, n. \end{cases}$$

It is obvious that under the above conditions the Markov chains (the random trajectories constructed) never visit zero elements of the matrix T. Suppose the terminated Markov chain $i_0 \to i_1 \to i_2 \to \ldots \to i_k$ of length k with n states, starting from i_0. As [2, 3, 4], we define the weights on the Markov chain in the following form

$$\begin{cases} w_m = \frac{t_{i_0 i_1} t_{i_1 i_2} \cdots t_{i_{m-1} i_m}}{p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{m-1} i_m}}, \quad m = 0, 1, 2, \ldots, k, \\ w_m = w_{m-1} \frac{t_{i_{m-1} i_m}}{p_{i_{m-1} i_m}}, \quad w_0 \equiv 1. \end{cases}$$

The random variable $\eta_k(h)$ is defined as

$$\eta_k(h) = \frac{h_{i_0}}{p_{i_0}} \sum_{m=0}^{k} w_m f_{i_m}.$$

Theorem 2.1. *Let x be a solution of the system* (1) *and the matrix T satisfy in the semiconvergence conditions. Then the mathematical expectation of the random variable $\eta_k(h)$ is equal to the inner product $< h, x^{(k+1)} >$, i.e.,*

$$E[\eta_k(h)] =< h, x^{(k+1)} > . \tag{3}$$

It is noteworthy that if we set $h = (0, 0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0)^T$ then from (3) we obtain the j^{th} element of the solution x *i.e.* x_j. Generally, we simulate n random trajectories $i_0^{(s)} \to i_1^{(s)} \to i_2^{(s)} \to \dots \to i_k^{(s)}$, $s = 1, 2, \dots, n$ and we consider the sample mean (MC estimation) of $\eta_k^{(s)}(h)$, $s = 1, 2, \dots, n$. Based on the Strong Law of Large Numbers (SLLN), we have

$$\begin{cases} \theta_k = \frac{1}{n}\sum_{s=1}^{n} \eta_k^{(s)}(h) \approx< h, x^{(k+1)} >, \\ \\ if \quad h = (0, 0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0)^T = e_j \quad then \quad \theta_k = \dfrac{1}{n}\sum_{s=1}^{n} \eta_k^{(s)}(h) \approx x_j, \end{cases}$$

where θ_k is the MC estimation for the j^{th} element of the solution x. In this way, better estimation of the parameter can be obtained. Similarly, by changing the vector h, we can obtain the other elements of the solution x. In this paper we assume that random walking is realized based on the MAO and UM Monte Carlo methods. The MAO and UM Monte Carlo methods are arisen from constructed transition density matrices; see for example [4] and references therein. The number of Markov chains is given by $n \geq (\frac{0.6745}{\epsilon} . \frac{\|f\|}{1-\|T\|})^2$ and the length of Markov chains can be obtained from

$$k \leq \frac{\log(\delta/\|f\|)}{\log(\|T\|)},$$

where ϵ and δ are given positive real numbers [2, 3].

3 A numerical example

Consider the linear system of equations (1), where

$$A = \frac{1}{4}\begin{pmatrix} 1 & -2 & 1 & -2 \\ -1 & -1 & -1 & -1 \\ 4 & -2 & 4 & -2 \\ 2 & -4 & 2 & -4 \end{pmatrix}, \quad b = \frac{1}{4}\begin{pmatrix} -2 \\ -4 \\ 4 \\ -4 \end{pmatrix}.$$

The matrix A is singular and $b = A(1, 1, 1, 1)^T$ .Hence, the system (1) is solvable and $x = (1, 1, 1, 1)^T$ is a solution of this system. By choosing the matrix M such that M is nonsingular, we have

$$M = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad T = M^{-1}N = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 1 & 2 & 1 \\ 1 & -2 & 1 & 2 \end{pmatrix}.$$

It is easy to compute that $\sigma(T) = \{1, 1, \frac{3}{4}, \frac{1}{4}\}$, $index(I - T) = 1$, which means that index of $\lambda = 1$ is equal to 1 and $\upsilon(T) = \frac{3}{4} < 1$. Therefore the semiconvergence conditions is satisfied. Comparison of the computational complexity of the UM and MAO Monte Carlo methods are given in the Table 1 and Table 2. By assuming $\epsilon = 0.05$, approximate solution converges to a exact solution $(0, 0, 2, 2)^T$.

Table 1: Numerical results by MAO Monte Carlo method

components of approximate solution	absolute error	computational time(s)	iterations
$x_1 = 0.000012$	0.000012	0.025	6
$x_2 = 0.000104$	0.000104	0.021	7
$x_3 = 2.000123$	0.000123	0.017	5
$x_4 = 2.000051$	0.000051	0.020	8

Table 2: Numerical results by UM Monte Carlo method

components of approximate solution	absolute error	computational time(s)	iterations
$x_1 = 0.000033$	0.000033	0.031	8
$x_2 = 0.000169$	0.000169	0.025	8
$x_3 = 2.000301$	0.000301	0.023	7
$x_4 = 2.000093$	0.000093	0.027	10

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Septic B-spline solution of one dimensional Cahn-Hillird equation

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Abstract

The septic B-spline collocation scheme is implemented to find numerical solution of one dimensional Cahn-Hillird equation. The scheme is based on the finite-difference formulation for time integration and septic B-spline functions for space integration. Stability and Convergence of the scheme are discussed. The accuracy of the proposed method is demonstrated a test problem.

Keywords: septic B-spline, Collocation, Cahn-Hillird equation
Mathematics Subject Classification [2010]: 65L10, 65M06, 65M12

1 Introduction

Consider the one-dimensional Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 \varphi(u)}{\partial x^2} = 0, \quad x \in (a, b), \quad t \geq 0, \tag{1}$$

with initial condition

$$u(x, 0) = \phi(x), \quad x \in [a, b], \tag{2}$$

and the boundary conditions

$$\frac{\partial u(a,t)}{\partial x} = \frac{\partial u(b,t)}{\partial x} = 0, \quad \frac{\partial^3 u(a,t)}{\partial x^3} = \frac{\partial^3 u(b,t)}{\partial x^3} = 0, \quad t \geq 0, \tag{3}$$

where $\varphi(u) = \frac{d\psi(u)}{du}$ and $\psi(u) = \frac{1}{4}r_2u^4 + \frac{1}{3}r_1u^3 + \frac{1}{2}r_0u^2$. The constant γ is positive, and r_0, r_1, r_2 are given constants. It is known if the initial data $u_0 \in H^2_E([a, b]) = \{f \in H^2([a, b]) : \frac{\partial f}{\partial x} = 0 \text{ on } a \text{ and } b\}$ then the problem (1)-(3) has a unique solution for all times [1].There are many algorithms for numerical solution of the C-H equations in literature, using different methods (for example see references [2, 3, 4]).

In current work, we will use septic B-spline to solve the Cahn-Hillird partial differential equation (1). The main purpose is to analyze the efficiency of the septic B-spline-difference method for such problems with sufficient accuracy. The time derivative is replaced by horizontal method of line finite-difference representation and the space derivatives by septic B-spline. In comparison with the existing well-known methods, our method is simple with better numerical stability and lower computational cost. Numerical computations show that our results are well accepted.

*Speaker

2 Temporal discretization

We consider a uniform mesh with the grid points $R_{i,j}$ to discretize the region $(a, b) \times (0, T)$. Each $R_{i,j}$ is the vertices of the grid point (x_i, t_j) with $x_i = a + ih, i = 0, 1, 2, ..., N$ and $t_j = jk, j = 0, 1, 2, ..., M$, where h and k are mesh sizes in the space and time directions, respectively, $h = (b - a)/N$ and $k = T/M$.

We discretize the problem (1)-(3) in the temporal direction by means of the θ-finite difference method, $\theta \in [\frac{1}{2}, 1]$. In this case, we get a system of ordinary differential equations with boundary conditions. Discretization by the proposed method yields the following system of differential equations:

$$u^{j+1}(x) + k\gamma\theta\frac{\partial^4 u^{j+1}(x)}{\partial x^4} - k\theta\frac{\partial^4 \varphi(u^{j+1}(x))}{\partial x^2} = F(x, t_j), \quad j = 0, 1, ..., M - 1, \tag{4}$$

$$F(x, t_j) = u^j(x) + k(1 - \theta)(-\gamma\frac{\partial^4 u^j(x)}{\partial x^4} + \frac{\partial^2 \varphi(u^j(x))}{\partial x^2}),$$

where

$$u^0 = \phi(x), \tag{5}$$

$$\frac{\partial u^{j+1}(a)}{\partial x} = \frac{\partial u^{j+1}(b)}{\partial x} = 0, \qquad \frac{\partial^3 u^{j+1}(a)}{\partial x^3} = \frac{\partial^3 u^{j+1}(b)}{\partial x^3} = 0. \tag{6}$$

Here $u(x, t_j)$ approximate the exact solution $U(x, t)$ at the time level $t_j = jk$. For $\theta = \frac{1}{2}$, our method reduces to the Crank-Nicolson method and for $\theta = 1$, our method reduces to the back-ward Euler method. Now in each time level we have a nonlinear ordinary differential equation in the form of (4) with the boundary conditions (6) which can be solved by using septic B-spline collocation method.

3 Numerical scheme in spatial direction

Let B_i be septic B-splines with knots at the points x_i, $i = 0, 1, ..., N$. The set of splines $\{B_{-3}, B_{-2}, ..., B_{N+2}, B_{N+3}\}$ forms a basis for functions defined over $[a, b]$. Thus, an approximation $u^{j+1}(x)$ to the exact solution $U^{j+1}(x)$ can be expressed in terms of the septic B-splines as trial functions:

$$u^{j+1}(x) = \sum_{i=-3}^{N+3} \alpha_i B_i(x), \tag{7}$$

where α_i's are time dependent quantities to be determined from boundary conditions and collocation form of the differential equations.

Septic B-splines B_i with the required properties are defined by

$$B_i(x)=\frac{1}{h^7}\begin{cases}(x-x_{i-4})^7, & x\in[x_{i-4},x_{i-3}],\\ (x-x_{i-4})^7-8(x-x_{i-3})^7, & x\in[x_{i-3},x_{i-2}],\\ (x-x_{i-4})^7-8(x-x_{i-3})^7+28(x-x_{i-2})^7, & x\in[x_{i-2},x_{i-1}],\\ (x-x_{i-4})^7-8(x-x_{i-3})^7+28(x-x_{i-2})^7-56(x-x_{i-1})^7, & x\in[x_{i-1},x_i],\\ (x_{i+4}-x)^7-8(x_{i+3}-x)^7+28(x_{i+2}-x)^7-56(x_{i+1}-x)^7, & x\in[x_i,x_{i+1}],\\ (x_{i+4}-x)^7-8(x_{i+3}-x)^7+28(x_{i+2}-x)^7, & x\in[x_{i+1},x_{i+2}],\\ (x_{i+4}-x)^7-8(x_{i+3}-x)^7, & x\in[x_{i+2},x_{i+3}],\\ (x_{i+4}-x)^7, & x\in[x_{i+3},x_{i+4}],\\ 0, & \text{otherwise},\end{cases} \tag{8}$$

$$i=-3,-2,0,...,N+2,N+3.$$

By using the approximation (7), septic B-splines (8), the nodal value u^{j+1} and its first, second, third, fourth and fifth derivatives with respect to variable x at the nodes x_i are obtained in terms of the element parameters as

$$\begin{aligned}
u^{j+1}(x_i)&=\alpha_{i-3}+120\alpha_{i-2}+1191\alpha_{i-1}+2416\alpha_i+1191\alpha_{i+1}+120\alpha_{i+2}+\alpha_{i+3},\\
u_x^{j+1}(x_i)&=\tfrac{7}{h}(-\alpha_{i-3}-56\alpha_{i-2}-245\alpha_{i-1}+245\alpha_{i+1}+56\alpha_{i+2}+\alpha_{i+3}),\\
u_{xx}^{j+1}(x_i)&=\tfrac{42}{h^2}(\alpha_{i-3}+24\alpha_{i-2}+15\alpha_{i-1}-80\alpha_i+15\alpha_{i+1}+24\alpha_{i+2}+\alpha_{i+3}),\\
u_{xxx}^{j+1}(x_i)&=\tfrac{210}{h^3}(-\alpha_{i-3}-8\alpha_{i-2}+19\alpha_{i-1}-19\alpha_{i+1}+8\alpha_{i+2}+\alpha_{i+3}),\\
u_{xxxx}^{j+1}(x_i)&=\tfrac{840}{h^4}(\alpha_{i-3}-9\alpha_{i-1}+16\alpha_i-9\alpha_{i+1}+\alpha_{i+3}),\\
u_{xxxxx}^{j+1}(x_i)&=\tfrac{2520}{h^5}(-\alpha_{i-3}+4\alpha_{i-2}-5\alpha_{i-1}+5\alpha_{i+1}-4\alpha_{i+2}+\alpha_{i+3}).
\end{aligned} \tag{9}$$

Using Eqs. (7)-(9) and putting the values of $B_i(x)$ and its derivatives in Eqs. (1)-(3) we have

$$\mathcal{A}\mathcal{C}^{j+1}-k\theta\mathcal{P}^{j+1}=\mathcal{F}^j+\mathcal{T}, \tag{10}$$

with $\mathcal{C}^{j+1}=[\alpha_{-1},\alpha_0,...,\alpha_N,\alpha_{N+1}]^T$, $\mathcal{A}=\mathcal{A}_0+\frac{k\gamma\theta}{h^4}\mathcal{A}_1+\frac{k\gamma\theta}{h^5}\mathcal{A}_2$, $\mathcal{T}=\mathcal{O}(h^3)$ and

$$\mathcal{A}_0=\begin{pmatrix}0&0&0&0&0&&&&\\594&2416&1788&240&2&&&&\\114.5&1191&2421.5&1192&120&1&&&\\1&120&1191&2416&1191&120&1&&\\\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&&\\&1&120&1191&2416&1191&120&1&\\&&1&120&1192&2421.5&1191&114.5&\\&&&2&240&1788&2416&594&\\&&&0&0&0&0&0&\end{pmatrix},\quad \mathcal{F}^j=\begin{pmatrix}G_0^j\\F_0^j\\F_1^j\\\vdots\\F_N^j\\G_N^j\end{pmatrix},$$

$$P(u,u_x,u_{xx})=\frac{\partial^2\varphi(u)}{\partial x^2},\quad G(x,t_j)=u^j(x)+k(1-\theta)(-\gamma\frac{\partial^5u_x^j(x)}{\partial x^5}+u^j(x)\frac{\partial^3\varphi(u^j(x))}{\partial x^3}),$$

$$\mathcal{A}_1=\begin{pmatrix}0&0&0&0&0&&&\\45360&13440&-60480&0&1680&&&\\-4620&-7560&18060&-6720&0&840&&\\840&0&-7560&-13440&-7560&0&840&\\\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\ddots&\\&840&0&-7560&-13440&-7560&0&840\\&&840&0&-6720&18060&-7560&-4620\\&&&1680&0&-60480&13440&45360\\&&&0&0&0&0&0\end{pmatrix},$$

$$\mathcal{A}_2=\begin{pmatrix}-226800&0&226800&0&&\\0&0&0&0&&\\&\ddots&\ddots&\ddots&&\\&&0&0&0\\&&-226800&0&226800\end{pmatrix},\quad \mathcal{P}^{j+1}=\begin{pmatrix}0\\P_0^j\\P_1^j\\\vdots\\P_N^j\\0\end{pmatrix}.$$

4 Stability and Convergence

Theorem 4.1. *The time semi-discrete method (4)-(6) is unconditionally stable for all values of* $\theta\in[\frac{1}{2},1]$.

Theorem 4.2. *The septic-spline approximation* u^{j+1} *converges to the exact solution* U^{j+1} *of the boundary value problem defined by Eqs. (4)-(6) with order three by the* $\|.\|_\infty$ *norm, i.e.,* $\|U^{j+1}-u^{j+1}\|_\infty=\mathcal{O}(h^3)$.

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Sinc-Finite difference collocation method for time-dependent convection diffusion equations

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Abstract

In this paper, Sinc-collocation method is used for time-dependent convection-diffusion equations. Sinc-collocation method based on double exponential transformation(DE) is used for space dimension and finite difference method is used for time dimension. The error in the approximation of the solution is shown to converge at an exponential rate, and the numerical results confirm that compared with the results based on single exponential transformation(SE), our method is of high accuracy and of good convergence.

keywords:Sinc-collocation method, Convection diffusion problems, finite difference method

1 Introduction

Sinc methods have been studied extensively and found to be a very effective technique for the solution ODEs and PDEs, particularly for problems with singular solutions and those on unbounded domain.Despite all advantages, it is difficult for the traditional Sinc method to solve some types of two or more dimensional boundary value problem. In these types of problems, it is better to divide a PDE into some ODEs and incorporated the Sinc method with other methods[4]. Now it is known that the Sinc-collocation method based on DE transformation converges more rapidly for some class of equations under proper conditions[3, 5].

2 Notation and background

Definition 2.1. [1]. Let h be a positive constant which represents mesh size of discretization and $k = \circ, \pm 1, \pm 2, \ldots$ .. The Sinc basic functions is defined for all x$\in \mathbb{R}$ by

$$Sinc(\frac{x-kh}{h}) = S(k,h) = \begin{cases} \dfrac{sin\pi(\frac{x-kh}{h})}{\pi(\frac{x-kh}{h})} & x \neq kh \\ 1 & x = kh \end{cases} \tag{1}$$

*Speaker

Definition 2.2. [2].D_d is a restricted trip with $|2d|$ width containing real axis.

$$D_d = \{z \in \mathbb{C} \ , \ |Imgz| < d\}$$

If x belongs to a subinterval of $\mathbb{R}$, at first, this subinterval must be transferrd to D_d by a proper conformal one-to-one map. Let ϕ be this map and

$$x_k = \phi^{-1}(kh) \in D \quad , \quad \phi^{-1} = \psi$$

Now we consider second-order two boundary value equation:

$$Lu(x) \equiv -u^{''}(x) + p(x)u^{'}(x) + q(x)u(x) = f(x) \qquad a < x < b \ , \ u(a) = u(b) = \circ \tag{2}$$

Sinc interpolation formula ,is:

$$u(x) \simeq u_m(x) = \sum_{k=-M}^{M} u_k S(k,h)\phi(x) \qquad k = -M, ..., M \qquad m = 2M+1 \tag{3}$$

Given that there is no guarantee that derivative of $u_m^{'}$ approximates the $u^{'}$ as well as u_m approximates u. in order to get rid of this problem, we can apply the following change of variable

$$\nu(\xi) = (\phi^{'})^l u)o(\psi(\xi)) \quad , \quad \xi = \phi(x) \in D_d \tag{4}$$

Following [1,2], by choosing a proper ϕ_{SE} *and* ϕ_{DE}(SE and DE transformation respectively) , $u_m^{'}$ and$u_m^{''}$ can approximate $u^{'}$ *and* $u^{''}$ inD_d . For $m = 2M+1$ and $n = 1, 2$, we have

$$v_m(\xi) = \sum_{k=-M}^{M} v(kh)S(k,h)(\xi) \quad , \quad \frac{d^n}{d\xi^n} v_m(\xi) = \sum_{k=-M}^{M} v(kh)\frac{d^n}{d\xi^n} S(k,h)(\xi) \tag{5}$$

Choice of l is dependent on conditions of the problem, for example $l = 1/2$ is more convenient for self-adjoint problem [2].Substitute (4),(5)into (2), then we have

$$\left(-\sum_{k=-M}^{M} \left[\frac{d^2}{d\xi^2} S(k,h)(\xi) + \mu_p(\xi)\frac{d}{d\xi} S(k,h)(\xi) + \gamma_q(\xi)S(k,h)(\xi) \right] \right) v(kh) = \left(\psi^{'}(\xi)\right)^{2-l} f\left(\psi(\xi)\right) \tag{6}$$

$$\mu_p(\xi) = \mu_p(\phi(x)) = (2l-1)\frac{\phi^{''}(x)}{(\phi^{'}(x))^2} + \frac{p(x)}{\phi^{'}(x)}$$

$$\gamma_q(t) = \gamma_q(\phi(x)) = -\frac{1}{(\phi^{'}(x))^{2-l}} \left(\frac{1}{(\phi^{'}(x))^l}\right)^{''} - \frac{l\phi^{''}(x)p(x)}{(\phi^{'}(x))^3} + \frac{q(x)}{(\phi^{'}(x))^2}$$

by solving this system of equation find υ, then $(\phi^{'})^{-l}\upsilon$ gives us u.

3 Sinc- Finite difference method

In our approach, the given problem discretized in time direction so that the problem can be converted to an ordinary differential equation in each time level. By using solutions of each level and applying forwarding finite difference method we can approximate solution of next level.Suppose the following convection diffusion equation

$$\frac{\partial}{\partial t}u(x,t) + H(x)\frac{\partial}{\partial x}u(x,t) + R(x)\frac{\partial^2}{\partial x^2}u(x,t) = f(x,t) \tag{7}$$

$$u(x,\circ) = g(x) \qquad a < x < b \text{ ' } \quad u(a,t) = u(b,t) = \circ \qquad \circ < t \leq \tau$$

Rewrite the above equation in the following form

$$-u_{xx} - \frac{H}{R}u_x = \frac{1}{R}(u_t - f) \quad , \qquad u_t = \frac{u^{j+1} - u^j}{\triangle t}$$

$$-u_{xx}^{j+1} - \frac{H}{R}u_x^{j+1} - \frac{1}{R\triangle t}u^{j+1} = \frac{1}{R}(-f^{j+1} - \frac{u^j}{\triangle t})$$

$u^j$is a vector of solution at jth- level of time. For each j we obtain an ODE equation, so it can be writen in the form of(6) . We compare DE and SE transformation which shown by ϕ_{SE}, ϕ_{DE} respectively

$$\phi_{SE} = \ln\left(\frac{x-a}{b-x}\right) \quad , \quad \phi_{DE} = Arcsinh\left(\frac{2}{\pi}Arctgh\left(\frac{-2}{a-b}x + \frac{a+b}{a-b}\right)\right)$$

To investigate the convergence of SE and DE methods, refer to [1,6] . Note that, if u in (7) does not vanish at boundary points and $u(a,t) = p(t), u(b,t) = q(t)$ the following conversion can be considered $w(x,t) = u(x,t) + \frac{x-b}{b-a}p(t) + \frac{a-x}{b-a}q(t)$, we have

4 Numerical examples

Example 4.1. We consider following problem with exact solution $u_{exact}(x,t) = x(1-x)\,t\exp(-t)$

$$\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = [x(1-x)(1-t) + 2t]e^{-t}$$
$$\mathrm{u}(\circ,t) = u(1,t) = \circ \qquad t > \circ$$
$$\mathrm{u(x,\circ)} = \circ \qquad \circ < x < 1$$

let M=32 and $\Delta t = 0.001$ For both methods,E represents maximum error of approximating exact solution by u_m on the Sinc grids. The results are listed in table 1

Example 4.2. Consider following problem with true solution $u = (t^2+1)e^{-(1+\frac{1}{\kappa})t}\,(sin(\pi x) + (1-x))$

$$\frac{\partial}{\partial t}u(x,t) + \kappa\frac{\partial}{\partial x}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = f(x,t) \quad ,$$

$$\mathrm{u}(\circ,t) = (t^2+1)exp(-(1+\tfrac{1}{\kappa})t) \quad , \quad u(1,t) = \circ \qquad t > \circ$$

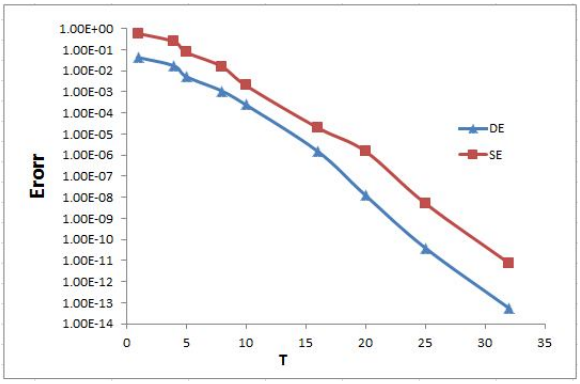
$$\mathrm{u(x,\circ)} = sin\,(\pi x) + (1-x) \quad \circ < x < 1 \ ,$$

Let M=32, $K = 100$ and $\Delta t = 0.1$. The results are shown in figure1
As expected, the DE transformation converges more rapidly.

Table 1

T	E_{DE}		E_{SE}	
	m=5	m=20	m=5	m=20
1	1.34×10^{-4}	5.9×10^{-6}	1.08×10^{-3}	1.73×10^{-5}
5	1.28×10^{-5}	2.81×10^{-7}	1.06×10^{-4}	1.06×10^{-6}
10	1.73×10^{-7}	5.19×10^{-9}	1.45×10^{-6}	1.43×10^{-7}
20	1.57×10^{-11}	5.34×10^{-13}	1.32×10^{-8}	1.30×10^{-11}

Figure 1



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Sinc-Galerkin method for solving parabolic equations

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Abstract

In this paper Sinc-galerkin method is used for a class of time-dependent parabolic equation. The method based on double exponential transformation(DE) and used for both space and time directions and it has been tested the accuracy of method on an example. Finally the obtained results based on DE transformation compared with this method based on single exponantial transformation(SE) . The results confirm that the accurate nature of our method .

Keywords: Sinc-Galerkin, double exponential transformation, parabolic equation, numerical comparision

1 Introduction

We consider the one dimentional time-depndent parabolic equation

$$\frac{\partial}{\partial t}u(x,t) + H(x)\frac{\partial}{\partial x}u(x,t) + R(x)\frac{\partial^2}{\partial x^2}u(x,t) = f(x,t), \tag{1}$$

$$u(x,\circ) = g(x) \qquad a < x < b \ , \ u(a,t) = \gamma(t) \ , \ u(b,t) = \delta(t), \ \ t > \circ$$

convection-diffusion and heat equation are a special model of this model.Many methods have been propsed for this type of equation that mixed Sinc-Galerkin with other methods and also in [2] there are some kind of this problem that solved by Sinc-Galerkin method based on SE transformation.

2 Sinc-Galerkin method

We explain the method on a heat equation with homogenious boundary conditions,but the method can be applied for other parabolic equations

$$\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = f(x,t) \ \ , \ \ u(0,t) = u(t,b) = 0 \quad t > \circ \ \ , \ \ u(0,x) = 0 \quad 0 < x < 1 \tag{2}$$

*Speaker

Definition 2.1. For a mesh size $h > \circ$ and $k = \circ, \pm 1, \pm 2, \ldots$, the basic Sinc functions on the real axis is defined by

$$Sinc(\frac{x-kh}{h}) = S(k,h) = \begin{cases} \frac{sin\pi(\frac{x-kh}{h})}{\pi(\frac{x-kh}{h})} & x \neq kh \\ 1 & x = kh \end{cases} \tag{3}$$

Definition 2.2. D_d is a restricted trip with $|2d|$ width containing real axis. $D_d = \{z \in \mathbb{C} : |Im(z)| < d\}$[2].

If x or t belong to a subinterval of R like $\mathbb{D}$, at first, they must be transferred to D_d by a one-to-one conformal map . Let ϕ be the conformal map for space dimension and Υ for time dimension.

Definition 2.3. [2, 3] δ^i and matrix $I^i = [\delta^i]$ for $i = \circ, 1, 2$ are definde

$$\delta_{jk}^{(\circ)} \equiv [S(j,h)o\phi(x)]|_{x=x_k} = \begin{cases} \circ & j = k \\ 1 & j \neq k \end{cases} \qquad \delta_{jk}^{(1)} \equiv h\frac{d}{d\phi}[S(j,h)o\phi(x)]|_{x=x_k} = \begin{cases} \circ & j = k \\ \frac{(-1)^{k-j}}{k-j} & j \neq k \end{cases} \tag{4}$$

$$\delta_{jk}^{(2)} \equiv h^2\frac{d^2}{d\phi^2}[S(j,k)o\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi}{3} & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & j \neq k \end{cases} \tag{5}$$

Now we proposed some formula and definition in one dimension, x then, use them for both x and t dimension

Definition 2.4. If f and g belong to $L^2((a,b))$, the weighted inner product is defined by

$$< f, g >= \int_a^b f(x)g(x)w(x)dx \tag{6}$$

Definition 2.5. [1] If f be an analytic function on (a,b) and $|\frac{f}{\phi'}| \approx O(exp(-\alpha e^{(-\beta|\phi(x)|)}))$ (it means f/ϕ' decades double exponentially), by choosing a proper h the Sinc Quadrature is defined by

$$\int_a^b f(x)dx \approx \sum_{k=-M}^{M} f(kh)\int_{-\infty}^{\infty} S(k,h)o(x)dx = h\sum_{k=-M}^{M}\frac{f(kh)}{\phi'(kh)} \tag{7}$$

applying (4), (5) and (6), we have

$$\int_a^b u(x)([S(j,h)o\phi]w)'(x)dx = h\sum_{k=-M}^{M}(uw)(x_k)\frac{\delta_{jk}^{(1)}}{h} + h(\frac{u(w)'}{\phi'})(x_j)| \tag{8}$$

$$\int_a^b u([S(j,h)o\phi]''w)(x)dx = h\sum_{k=-M}^{N}u(x_k)[\frac{\delta_{jk}^{(2)}}{h^2}(\phi'w)(x_k)+\frac{\delta_{jk}^{(1)}}{h}(\frac{\phi''}{\phi'}+2w')(x_k)]-h(\frac{w''u}{\phi'})(x_j) \tag{9}$$

The basic functions for two dimension space and time is defined by $S_{kl} = S_k S_l^* = S(k,h)o\phi(x)S(l,h)o\Upsilon(t)$ and the inner product is

$$< f, g >= \int_0^\infty \int_a^b f(x,t)g(x,t)w(x)\nu(t)dx \;, \tag{10}$$

where w and ν are the proper weight functions for space and time dimension respectively.The approximate solution to (2) is defined by Sinc interpolation

$$u_{m_x,m_t}(x,t) = \sum_{j=-M_t}^{M_t} \sum_{i=-M_x}^{M_x} u_{ij} S_{ij}(x,t) \tag{11}$$

Applying the inner product to (2) and integrating by parts, twice in x and once in t , we have

$$\int_0^\infty -\frac{\partial}{\partial t} S_l^* \nu \left(\int_a^b u(x,t)(-\frac{\partial^2}{\partial x^2}[S_k w(x)]dx) \right) dt + P_u = \int_0^\infty \int_a^b f(x,t) S_k S_l^* w(x) dxdt \;, \tag{12}$$

where, P_u is some terms containig $u^{'} and\ u^{''}$.There is no guarantee that the the partial derivative $u^{'}_{m_x,m_t}$ and $u^{''}_{m_x,m_t}$ approximates the $u^{'} andu^{''}$ as well as u_{m_x,m_t} approximates u. To get rid of this problem we apply integrating by parts then, choose a proper weight functions to vanish P_u, such as $w(x) = \frac{1}{\sqrt{\phi'}}$ and $\nu(t) = \sqrt{\Upsilon'}$ for this problem.

First of all, we apply (9) for fixed t for the inner integration , then applying (8) to the first term in left hand side of (12) , then by using I^i in the definition 2.3 we can obtaine the folowing matrix form

$$A_x V + V C_t^T = G \tag{13}$$

$$G = D(w)FD(\frac{\nu}{\sqrt{\Upsilon'}}) \quad , \quad V = D(w)UD(\frac{\nu}{\sqrt{\Upsilon'}})$$

$$A_x = D(\phi') \left[-\frac{1}{h^2} I^{(2)} - \frac{1}{h} I^{(1)} D(\frac{\phi^{''}}{(\phi')^2} + \frac{2w^{'}}{\phi' w}) - D(\frac{w^{''}}{(\phi')^2 w}) \right] D(\phi') \;,$$

$$\mathrm{C}_t = D(\sqrt{\Upsilon'}) \left[-\frac{1}{h} I^{(1)} - D(\frac{\nu^{'}}{\nu' \nu}) \right] D(\sqrt{\Upsilon'}) \quad ,$$

where, D represents diagonal matrix at Sinc grids $x_k = \phi^{-1}(kh)$ and $t_l = \Upsilon^{-1}(lh)$ and C_t^T is transpose of C_t and F is a matrix with elements $f(x_k, t_l)$.

3 Selection of ϕ and Υ

In this paper, we choose

$$\phi_{DE}(x) = Arcsinh\left(\tfrac{2}{\pi} Arctgh\left(\tfrac{-2}{a-b} x + \tfrac{a+b}{a-b} \right) \right) \quad , \quad \Upsilon_{DE}(t) = Arcsinh(\frac{2}{\pi} ln(t))$$

these functions transferred the domain of the given problem into D_d we can convert the equation (2) for fixed time t_l to the second order ODE with respect to x as

$\{-u_{xx}(x,t_l) = f(x,t_l) - u_t(x,t_l) \equiv f_l(x) \ , \ \circ < x < 1 \ , \ -M_t \leq l \leq M_t \ , \ u(a,t_l) = u(b,t_l) = \circ$

similarly, for a fixed x we can convert (2) as the first orderODE with respect to t

$\{u_t = f(x_k,t) + u_{xx}(x_k,t) \equiv g_k(t) \ , \ t > \circ \ ; \ , \ -M_x \leq k \leq M_x \ , \ u(x_k,\circ) = \circ$

If these ODEs are satisfied the proper conditions defined in theorem2.3 of [3] and also by choozing a proper mesh size h we expected the order of this method be $||u - u_{mx,m_t}|| \simeq O(\frac{-k' M_x}{log(M_x)})$ for some $k' > \circ$

4 Numerical result

Example 4.1. We consider equation (2) in the case $f(x,t) = t^{3/2}e^{-t}[(\frac{3}{2t} - 1)xln(x) - \frac{1}{x}]$ with exact solution $u(x,t) = t^{3/2}e^{-t}xln(x)$ and applied our method to this problem and compare our results with the results in [4] which used SE transformation to the problem. we choose the mesh size,$h = \frac{log(\pi^2 M_x/16)}{M_x}$ and $M = M_x = M_t$. e represents maximum error of approximating u by u_m at mesh grids, the results tabulated in Table1 and confirm that our method is more accurate.

Table 1

M	h_{SE}[4]	e_{SE}	h_{DE}	e_{DE}
4	1.57	4.1×10^{-3}	0.225	2.5×10^{-3}
8	1.11	1.1×10^{-3}	0.199	4.92×10^{-4}
16	0.785	2.1×10^{-4}	0.143	8.7×10^{-7}
32	0.555	2.6×10^{-5}	0.093	8.8×10^{-9}

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Solving a multi-order fractional differential equation using the method of particular solutions

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Abstract

This paper presents a new semi-analytic numerical method for solving multi-order fractional differential equations. The method is based on the use of the particular solutions of the linearized equation. Numerical implementation confirms the validity, efficiency and applicability of the method.

Keywords: Particular solution, Fractional differential equation, Multi-point boundary value problem.
Mathematics Subject Classification [2010]: 34A08, 35M12

1 Introduction

Fractional differential equations have been found to be effective to describe some physical phenomenas. In this paper, the method of particular solutions is applied to solve the multi-order fractional differential equation:

$$D^{\alpha}u(t) = f(t, u(t), D^{\beta_1}u(t), \ldots, D^{\beta_n}u(t)) = 0, \quad u^{(k)}(0) = c_k, \quad k = 0, \ldots, m, \tag{1}$$

where $m < \alpha \leq m+1$, $0 < \beta_1 < \beta_2 < \ldots < \beta_n < \alpha$ and D^{α} denotes Caputo fractional derivative of order α. It should be noted that f can be non linear in general. In Daftardar-Gejji and Jafari [1], it was proved that the Eq.(1) can be represented as a system of fractional differential equations (FDEs)

$$\begin{aligned}
&D^{\alpha_i}u_i(t) = u_{i+1}, \quad i = 1, 2, \ldots, n-1,\\
&D^{\alpha_n}u_i(t) = f(t, u_1, u_2, \ldots, u_n);\\
&u_i^k(0) = c_k^i, \quad 0 \leq k \leq m_i, \quad m_i \leq \alpha_i \leq m_i + 1, \quad 1 \leq i \leq n. \qquad (2)
\end{aligned}$$

For more details we refer to [3].

In Section 2, we describe the particular solution method for the solution of multi-point boundary value problems (MPBVPs) and then we present this method to solve multi-order fractional differential equations. A numerical example illustrating the applicability of the method is placed in Section 3.

*Speaker

2 Main algorithm

Consider the following multi-point boundary value problem

$$u^{(s)} = F(u, u', \dots, u^{(s-1)}, x), \quad x \in [0,1], \tag{3}$$

$$\sum_{j=0}^{s-1} a_{j,k} u^{(j)}(\xi_{j,k}) = d_k, \quad 0 \le \xi_{j,k} \le 1,\ k = 1, \dots, s, \tag{4}$$

where some of the coefficients $a_{j,k}, d_k$ could be equal to zero. Sometimes we write the equation in the form

$$u^{(s)} = F(u, u', \dots, u^{(s-1)}, x) + f(x) \tag{5}$$

highlighting that $f(x)$ that does not depend on u.
Let $\phi_m(x)$ be some system of basis functions on $[0,1]$, here we consider the monomials:

$$\varphi_m(x) = x^{m-1}, \quad m = 1, \dots, M. \tag{6}$$

The particular solutions of the equation $\phi_m^{(s)}(x) = \varphi_m(x)$, which correspond to the basis functions φ_m are:

$$\phi_m(x) = \frac{x^{m+s-1}}{m(m+1)\dots(m+s-1)}. \tag{7}$$

We denote

$$\Phi_m(x) = \phi_m(x) + c_{m,0} + c_{m,1}x + \dots + c_{m,s-1}x^{s-1}. \tag{8}$$

So, $\Phi_m^{(s)}$ satisfies $\Phi_m^{(s)}(x) = \phi_m^{(s)}(x) = \varphi_m(x)$. The free coefficients $c_{m,i}$ in (8) are chosen in such a way that Φ_m satisfies the homogeneous boundary conditions (4):

$$\sum_{j=0}^{s-1} a_{j,k} \Phi_m^{(j)}(\xi_{j,k}) = 0, \quad k = 1, \dots, s. \tag{9}$$

Substituting (8) in (9), one gets a linear system of equations for $c_{m,0}, c_{m,1}, \dots, c_{m,s-1}$. We assume that the nonlinear term in (5) can be approximated by the linear combinations of the basis functions $\varphi_m(x)$:

$$F(u, u^{(1)}, \dots, u^{(s-1)}, x) = \sum_{m=0}^{M} q_m \phi_m(x). \tag{10}$$

Substituting this approximation in the initial equation (5), one gets

$$u_M^{(s)}(x) = \sum_{m=0}^{M} q_m \phi_m(x) + f(x). \tag{11}$$

Let $u_f(x)$ satisfies the equation $u_f^{(s)}(x) = f(x)$, and the boundary conditions (4):

$$\sum_{j=0}^{s-1} a_{j,k} u_f^{(j)}(\xi_{j,k}) = d_k. \tag{12}$$

When there exists a particular solution $u_p(x)$ in explicit analytic form, then it can be written in the form:

$$u_f(x) = u_p(x) + c_0 + c_1 x + \ldots + c_{s-1}x^{s-1}. \quad (13)$$

When there are no particular solutions, $f(x)$ is joined to the nonlinear term and we get $u_f^s(x) = 0$, and $u_f(x) = c_0 + c_1 x + \ldots + c_{s-1}x^{s-1}$. Substituting $u_f(x)$ in (12), one gets a linear system for $c_0, c_1, \ldots, c_{n-1}$. So

$$u_M(x, \mathbf{q}) = u_f(x) + \sum_{m=1}^{M} q_m \Phi_m(x), \quad \mathbf{q} = (q_1, \ldots, q_M), \quad (14)$$

satisfies Eq. (11) and the boundary conditions of the initial problem (4). To get unknowns $q_1, \ldots, q_M$ we substitute $u_M(x, \mathbf{q})$ in (10)

$$F\left(u_M(x, \mathbf{q}), u_M^{(1)}(x, \mathbf{q}), \ldots, u_M^{(s-1)}(x, \mathbf{q}), x\right) = \sum_{m=1}^{M} q_m \phi_m(x). \quad (15)$$

Note that we can always get the $u_f(x)$ in the analytic way when $f(x)$ is a simple combination of elementary functions, e.g., quasipolynomial $(b_0 + b_1 x + \ldots + b_p x^p)exp(\mu x)$. Otherwise we can use the well known formula

$$u_f(x) = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{s-1} f(t) d(t) + c_0 + c_1 + \ldots + c_{s-1}x^{s-1} \quad (16)$$

and evaluate the integral numerically. Another approach is to join the term $f(x)$ to the nonlinear term F. To solve (15) we use the following algorithm. Let $0 \leqslant x_1 < x_2 < \ldots < x_M \leqslant 1$ be collocation points. In particular, we use the Chebyshev collocation points

$$x_n = \frac{1}{2}\left[1 + \cos\left(\frac{\pi(n-1)}{M-1}\right)\right]. \quad (17)$$

We write the collocation of (15) at these points and get the system of M nonlinear equations

$$F\left(u_M(x_n, \mathbf{q}), u_M^{(1)}(x_n, \mathbf{q}), \ldots, u_M^{(s-1)}(x_n, \mathbf{q}), x_n\right) = \sum_{m=1}^{M} q_m \phi_m(x_n), \quad n = 1, \ldots, M. \quad (18)$$

We solve this system of equations.

Dealing with linear problems (5), one gets

$$f(x_n) + \sum_{k=0}^{s-1} A_k(x_n)\left[u_f^{(k)}(x_n) + \sum_{m=1}^{M} q_m \Phi_m^{(k)}(x_n)\right] = \sum_{m=1}^{M} q_m \phi_m(x_n) \quad (19)$$

instead of (18). Rewriting in the form

$$\sum_{m=1}^{M}\left[\sum_{k=0}^{s-1} A_k(x_n)\Phi_m^{(k)}(x_n) - \phi_m(x_n)\right] = -f(x_n) - \sum_{k=0}^{s-1} A_k(x_n) u_f^{(k)}(x_n), \quad (20)$$

we get the linear system for $q_1, \ldots, q_M$ and the linear system is solved by maple. After determining $q_1, \ldots, q_M$ we get the approximate solution $u_M(x, \mathbf{q})$ (14). We implement this method to some multi-order FDE in the next section.

3 Illustration of the method

We consider the following initial value problem in case of the inhomogeneous Bagley-Torvik equation [2]:

$$D^2u(t) + +D^{1.5}u(t) + u(t) = 1 + t, \quad u(0) = 0, \quad u'(0) = 1 \tag{21}$$

with the exact solution $u_{exact}(t) = 1 + t$. Similar to (2) it can be viewed as the following system of FDE:

$$D^{1.5}u_1 = u_2, \quad u_1(0) = u_1'(0) = 1, \tag{22}$$

$$D^{0.5}u_2 = -u_2 - u_1 + 1 + t, \quad u_2(0) = 0. \tag{23}$$

We apply the method of particular solutions to the system of equations (22) and (23) with different number of basis functions M. Here we assume $\varphi_m(t)$ as said in (6), but $\phi_m(t)$ will be different because of the type of the derivatives here, i.e. here we have derivatives of type Caputo. We have $s_1 = 1.5$ and $s_2 = 0.5$, so $\phi_{i,m}(t), i = 1, 2$, will be as follows:

$$\phi_{i,m}(t) = \frac{1}{\Gamma(s_i)} \int_0^t (t-\xi)^{s_i-1}\varphi_m(\xi)d\xi, \quad i = 1, 2. \tag{24}$$

We consider $\Phi_{1,m}(t) = \phi_{1,m}(t) + c_{1,m,0} + c_{1,m,1}t$ and $\Phi_{2,m}(t) = \phi_{2,m}(t) + c_2, m, 0$ and $u_{1,f}(t) = c_{1,0} + c_{1,1}t$ and $u_{2,f}(t) = 0$. After determining unknown coefficients as said at the previous section, finally we set

$$u_{i,M}(t) = u_{i,f}(t) + \sum_{m=1}^{M} q_m\Phi_{i,m}(t), \quad i = 1, 2, \tag{25}$$

and substitute (25) in the equations (22) and (23) and determine unknowns $q_1, \ldots, q_M$ by collocation method. We find $u_{1,M}(t) = 1 + t$ and $u_{2,M}(t) = 0$ and so $u_M(t) = u_{1,M}(t) + u_{2,M}(t) = 1 + t$ will be the approximate solution of (21). Notice that for each number of basis functions this method gives us the exact solution.

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Solving Large Sparse Linear Systems by Using QR-Decomposition whit Iterative Refinement

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Abstract

In this article, for solution of a system of linear algebraic equations $Ax = b$ with a large sparse coefficient matrix A, the QR-decomposition with iterative refinement (QRIR) is compared with the QR-decomposition by means of Givens rotations(QRGR), which is without iterative refinement and leads to direct solution. We verify by numerical experiments that the use of sparse matrix techniques with QRIR may result in a reduction of both the computing time and the storage requirements.

Keywords:large sparse linear systems, QR-decomposition with Givens rotations(QRGR), QR-decomposition with iterative refinement(QRIR)

1 Introduction

A system of linear algebraic equations is

$$Ax = b \tag{1}$$

where A is a nonsingular, large, sparse and nonsymmetric matrix of order n and b is a given column vector of order n. To solve the linear system (1) one can try several different algorithms. One method is to find the inverse and multiply it on both sides, which is expensive computationally. Another method is to make a guess of the solution and iteratively refine that guess until the error is suitably small. The method proposed here is an iterative refinement based on the QR-decomposition method. The QR-decomposition of a matrix is a decomposition of a matrix A into a product $A = QR$ of an orthogonal matrix Q and an upper triangular matrix R. There are several methods for actually computing the QR-decomposition, such as the Gram-Schmidt process, Householder transformations, or Givens rotations. Householder transformation has greater numerical stability than the Gram-Schmidt method. Givens rotation procedure is used here, which does the equivalent of the sparse Givens matrix multiplication, without the extra work of handling the sparse elements. The Givens rotation procedure is useful in situations where only a relatively few off diagonal elements need to be zeroed, and is more easily parallelized than Householder transformations. The factorization operation count with Givens rotation is always smaller than other methods. In this paper for computing the QR-decomposition, we use Givens rotations algoritm for sparse matrices[2].

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2 QRIR Algorithm

The QR-decomposition method intermediates Givens rotations are useful for dissolving system of linear algebraic equations where A is a nonsingular, large and sparse matrix. An approximate QR-decomposition of the matrix A is

$$A = \widetilde{Q}\widetilde{R} + E \tag{2}$$

where E is an error matrix. The approximate solution of the system (1) is computed by

$$\widetilde{Q}\widetilde{R}x^{(0)} = b \tag{3}$$

Assume that some technique such as a QR-decomposition is used in the computation of (2) in the decomposition stage and (3) in the solution stage. The decomposition stage (2) is performed by using QR-decomposition with Givens rotation(QRGR). It is well known that the factorization stage is much more expensive than the solution stage. Therefore, it may be advantageous to use Givens rotation to control the sparsity. If this is done, then normally the computing time needed to obtain $x^{(0)}$ and the storage needed for the nonzero entries of $\widetilde{Q}$ and $\widetilde{R}$ are reduced (For more details see[2]). However, the approximation $x^{(0)}$ so computed may be crude and an attempt to regain the accuracy lost by iterative refinement has to be carried out. This means that the computations should be continued after the solution stage (3) by the following formulae:

$$r^{(i)} = b - Ax^{(i)} \tag{4}$$

$$\widetilde{Q}\widetilde{R}d^{(i)} = r^{(i)} \tag{5}$$

$$x^{(i+1)} = x^{(i)} + d^{(i)} \tag{6}$$

$$for \quad i = 0, 1, 2, \ldots$$

Different criteria must be used to stop the iterative process (4)-(6) if the accuracy has not been achieved or if the process does not converge. Normally single precision computations are used in (5) and (6), while the residual vectors $r^{(i)}$, for $i = 0, 1, 2, \ldots$, are accumulated in double precision and then rounded to single precision. If $x^{(i)}$ is accepted as a solution of (1), then it is said that the system is solved directly or that $x^{(i)}$ is a QR-decomposition direct solution with Givens rotation (QRGR). The solution obtained by the use of (4)-(6) is called the QR-decomposition iteratively refined solution (QRIR) by using storage technique[2]. Therefore we can write QRIR algorithm. This algorithm has three steps:

Step 1. QR-decomposition by using Givens rotations by implementing storage technique [2].

Step 2. Solving system $\widetilde{R}x^{(0)} = \widetilde{Q}^t b$ for $x^{(0)}$ by using back substitution.

Step 3. Improvement by using The technique of iterative refinement:

for $i = 0, 1, 2, \ldots$ until the desired accuracy is achieved (say, 10^{-16})

I. Compute $r^{(i)} = b - Ax^{(i)}$;

II. Solving system $\widetilde{R}d^{(i)} = \widetilde{Q}^t r^{(i)}$ for $d^{(i)}$;

III. Compute $x^{(i+1)} = x^{(i)} + d^{(i)}$.

First Step of this algorithm is usful for redusing the computing time and the storage requirements. Step 3 of the Algorithm is optional. If hoping that $\|x^{(0)} - x\| \leq \epsilon$,we can accept $x^{(0)}$ for solving system. If the third Step is carried out and if the process converges, then $\widetilde{x} = x^{(i)}$ and $\|x^{(i)} - x\| \leq \epsilon$. The iteratively refined solution (QRIR) is normally more accurate than $x^{(0)}$ and an estimation of $\|\widetilde{x} - x\|$ is computed by $\|d^{(i)}\|$.

3 Numerical examples

The computational environment used for the tests was an Intel Core i7-3537U, 2.0GHz CPU with 6GB RAM, and the matrices used in the experiments are chosen randomly.
Example 1: Consider the system (1) whose nonzero entries of the coefficient matrix A are given by $a_{ij} = 1/(i+j+1)$. The matrix A is ill-conditioned for even modest size n and it has a large condition number. It is used to illustrate the performance of the algorithms. In this example, the dimensions of the matrices considered are $n = 10, 40, 100$.
In Examples 2-4 , we consider linear systems (1) whose coefficient matrices A are of order n with nz nonzero entries on the diagonal and sparsely distributed throughout, and those are chosen randomly with $n = 200, 400, 1000$ where $k(A)$ is condition number of A:
Example 2: $n = 200$, $nz = 638$, $k(A) = 15.716318$.
Example 3: $n = 400$, $nz = 1276$, $k(A) = 16.10840$.
Example 4: $n = 1000$, $nz = 2190$, $k(A) = 15.04919$.
The matrices given in Examples 2-4 are well-conditioned. The QR-decomposition obtained by (2) is not so accurate. The same is true for the solution $x^{(0)}$ obtained by (3). However, full machine accuracy is often achieved by the iterative process (4)-(6). The computing time may be reduced when QR-decomposition with iterative refinement (QRIR) is used for sparse systems (which may never happen in the case where the matrix is dense). Denote by t_1 , the computing time needed to solve the system by QRIR and by t_2 the computing time for the QR-decomposition with direct solution (QRGR). Our experiments show that $t_1 < t_2$ for the accuracy shown in Table 1.

Table 1: The computing time (in seconds) and the number of iterations obtained by using QRIR for Examples 1-4.

Example	n	nz	$k(A)$	t_2	t_1	Iter.
1	10	26	452.549	Negl.	Negl.	8
1	40	122	1231.3812	Negl.	Negl.	6
1	100	457	831.435	0.0136	Negl.	13
2	200	638	15.716318	0.0165	Negl.	2
3	400	1276	16.10840	0.1524	0.0138	2
4	1000	2190	15.04919	0.2037	0.0564	3

4 Conclusion

In this paper, for solving large sparse linear systems, the QR-decomposition with iterative refinement (QRIR) was compared with the QR-decomposition with Givens rotations(QRGR), which is without iterative refinement. We verify by numerical experiments that the use of sparse matrix techniques with QRIR may result in a reduction of both the computing time and the storage requirements. If the condition number of the coefficient is large (see Example 1), then the Step3 of algorithm may converge slowly. In this way, one may obtain an answer of acceptable (but unknown) accuracy. Assume that A is dense. Consider the solution of (1) by QRGR and by QRIR. Then we have: (i) extra storage is needed when QRIR is used, (ii) the iterative process (4)-(6) requires extra computing time, and (iii) the solution obtained by QRGR is satisfactory, in general. Therefore, QRGR is preferred when the matrix is dense. If the matrix is sparse and QRGR is used, we have found that the solution is satisfactory because the stability requirements in QR-decomposition are satisfied by using Givens rotations. However, it is also possible to use QRIR if the accuracy is more important (but not computing time and computer storage). Moreover, the QRIR procedure converges to the true solution even though the matrix A is ill-conditioned.

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Solving nonlinear fuzzy differential equations by the Adomian-Tau method

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Abstract

In this paper, a numerical method for nonlinear fuzzy differential equations is presented. The method is based on Adomian-Tau method. Numerical examples are presented to verify the efficiency and accuracy of the proposed method.

Keywords: fuzzy differential equation, generalized differentiable, Adomian-Tau method.
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1 preliminary

In this section,we present definitions and concepts that need in throughout papers.
Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis $u : \mathbb{R} \to [0,1]$, such that u is normal, upper semicontinuous and convex fuzzy set with compact support. Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbb{R}; u(x) \geq \alpha\}$ and $[u]^0 = \{x \in \mathbb{R}; u(x) > 0\}$. Then it is well- known that for any $\alpha \in [0,1]$, $[u]^\alpha$ is a bounded closed interval. For $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda.u$ are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda.u]^\alpha = \lambda[u]^\alpha, \forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha = \{x + y : x \in [u]^\alpha, y \in [v]^\alpha\}$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda[u]^\alpha = \{\lambda x : x \in [u]^\alpha\}$ means the usual product between a scalar and a subset of $\mathbb{R}$.
Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$, $D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$, be the Hausdorff distance between fuzzy numbers, where $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$, $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$. The following properties are well-known

- $D(u + w, v + w) = D(u,v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}$,
- $D(k.u, k.v) = |k| D(u,v), \quad \forall k \in \mathbb{R}, u, v \in \mathbb{R}_{\mathcal{F}}$,
- $D(u + v, w + e) \leq D(u,w) + D(v,e), \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$,

and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Definition 1.1. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exist $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the $H-$ difference of x and y and it is denoted by $x \ominus y$.

In this paper the " $\ominus$ " sign stands always for $H-$ difference and let us remark that $x \ominus y \neq x + (-1)y$.

Definition 1.2. [1] Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a,b)$, then f is strongly generalized differential on x_0, if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that

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(i) for all $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), f(x_0) \ominus f(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0), \quad \text{or}$$

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h),\ f(x_0-h) \ominus f(x_0)$ and the limits

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h\to 0} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0), \quad \text{or}$$

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0),\ f(x_0-h) \ominus f(x_0)$ and the limits

$$\lim_{h\to 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0), \quad \text{or}$$

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h),\ f(x_0) \ominus f(x_0-h)$ and the limits

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0).$$

Theorem 1.3. *Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a,b)$.*

- *(i) If f is strongly generalized differentiable on x_0 as in (i) of Definition 1.2 (i-differentaible) then*
$$[f'(x_0)]^\alpha = [(\underline{f}^\alpha)'(x_0), (\overline{f}^\alpha)'(x_0)], \quad \forall \alpha \in [0,1],$$
- *(ii) If f is strongly generalized differentiable on x_0 as in (ii) of Definition 1.2 (ii-differentiable) then*
$$[f'(x_0)]^\alpha = [(\overline{f}^\alpha)'(x_0), (\underline{f}^\alpha)'(x_0)], \quad \forall \alpha \in [0,1].$$

2 Adomian-Tau method

Consider the following nonlinear differential equations system

$$\begin{cases} y_1'(x) = f_2(x, y_1(x), y_2(x)), & y_1(x_0) = \lambda_1 \\ y_2'(x) = f_1(x, y_1(x), y_2(x)), & y_2(x_0) = \lambda_2. \end{cases} \tag{1}$$

Assume that $y_{in}(x), i = 1,2$ is a polynomial approximation of degree n for $y_i(x), i = 1,2$ then, one can write:

$$y_{in} = \sum_{j=0}^{n} a_{ij} x^j = \underline{\underline{a}}_i \, \underline{\underline{X}} \tag{2}$$

where $\underline{\underline{a}}_i = [a_{i0}, a_{i1}, a_{i2}, ..., a_{in}, 0, ...]$ and $\underline{\underline{X}} = [1, x, x^2, ...]^T$. The tau method converts differential equations to algebraic equations. The effect of differentiation or shifting on coefficients $\underline{\underline{P}}_n = [p_0, p_1, p_2, ..., p_n, 0, ...]$ of polynomial $P_n(x) = \underline{\underline{P}}_n \underline{\underline{X}}$ is the same as that of the post-multiplication of P_n by either matrix η or μ, defined by:

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \vdots \\ & & & 0 & \\ & \cdots & & & \ddots \end{bmatrix}, \quad \eta = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \vdots \\ 0 & 0 & 3 & 0 & \\ & & \cdots & & \ddots \end{bmatrix}. \tag{3}$$

Lemma 2.1. *[3] Let $P_n(x)$ be a polynomial of the form $P_n(x) = \sum_{i=0}^{n} P_i x^i = \underline{P}_n \underline{\underline{X}}$, then*

i) $\frac{d^k}{dx^k} P_n(x) = \underline{P}_n \eta^k \underline{\underline{X}}, \qquad i = 0, 1, 2, ...,$

ii) $x^k P_n(x) = \underline{P}_n \mu^k \underline{\underline{X}}$

By using Lemma 2.1, one can write

$$y_i'(x) = a_i \eta \underline{\underline{X}}, \ \ i = 1, 2. \tag{4}$$

We now use Adomian decomposition, to simplify the non-linear term of Equations (1).
By setting $\tilde{f}_i(x) = f_i(x, y_1, y_2)$, and substituting $y_i(x) = \sum_{j=0}^{\infty} a_{ij} x^j$, we get

$$\tilde{f}_i(x) = f_i\left(x, \sum_{j=0}^{\infty} a_{1j} x^j, \sum_{j=0}^{\infty} a_{2j} x^j\right) = \sum_{j=0}^{\infty} A_j^{f_i} x^j = \underline{A}^{f_i} \underline{\underline{X}}, \qquad i = 1, 2,$$

where $\underline{A}^{f_i} = [A_0^{f_i}, A_1^{f_i}, ...]$, with

$$A_k^{f_i} = \frac{1}{k!} \left\{ \frac{d^k}{dx^k} f_i\left(x, \sum_{j=0}^{\infty} a_{1j} x^j, \sum_{j=0}^{\infty} a_{2j} x^j\right) \right\}\Big|_{x=0} = \frac{\tilde{f_i}^{(k)}(0)}{k!}, \quad i = 1, 2, \quad k = 0, 1, ...$$

which depends on $a_{10}, a_{11}, ..., a_{1k}, a_{20}, a_{21}, ..., a_{2k}$, for $k = 0, 1, ...$. From Relations (4) and (5) the matrix form of Equations (1) can be written as

$$\underline{a}_i \eta \underline{\underline{X}} = \underline{A}^{f_i} \underline{\underline{X}}, \qquad i = 1, 2, \tag{5}$$

which yields

$$\underline{a}_i \eta = \underline{A}^{f_i}, \ \ i = 1, 2, \tag{6}$$

since $\underline{\underline{X}}$ is a base vector. Consequently the unknown coefficients in Relation (2) can be determined from Relation (6). In fact, we use initial conditions to write

$$a_{i0} = \lambda_i, \qquad i = 1, 2.$$

and determined other coefficients by forward substituting from the following systems:

$$\begin{cases} a_{1j} = \dfrac{A_{j-1}^{f_1}}{j} \\ a_{2j} = \dfrac{A_{j-1}^{f_2}}{j} \end{cases} forj = 1, 2, ..., n. \tag{7}$$

3 Numerical Example

Example 3.1. Consider the following fuzzy differential equation

$$y'(t) = 2ty(t) + t(r-1), \qquad y(0) = (-1, 0, 1) \quad t \in [0, 1]. \tag{8}$$

In this case (i)-different, the exact solution is

$$[Y(t)]^\alpha = [\underline{Y}^\alpha, \overline{Y}^\alpha(t)] = [\frac{1}{2}(3e^{t^2} - 1)(\alpha - 1), \frac{1}{2}(3e^{t^2} - 1)(1 - \alpha)],$$

and Equation (8) is equivalent with system

$$\begin{cases} \underline{y}^{\alpha} = 2t\underline{y}^{\alpha} + t(\alpha-1), & \underline{y}^{\alpha}(0) = \alpha - 1 \\ \overline{y}^{\alpha} = 2t\overline{y}^{\alpha} + t(1-\alpha), & \overline{y}^{\alpha}(0) = 1-\alpha. \end{cases} \tag{9}$$

Using Adomian-Tau method if $\underline{y}^{\alpha} = \sum_{j=0}^{n} a_j x^j$ and $\overline{y}^{\alpha} = \sum_{j=0}^{n} b_j x^j$ for n=6 we have

$$\begin{cases} \underline{y} = (\alpha-1)(1+\frac{3}{2}t^2+\frac{3}{4}t^4+\frac{t^6}{4}) \\ \overline{y} = (1-\alpha)(1+\frac{3}{2}t^2+\frac{3}{4}t^4+\frac{t^6}{4}) \end{cases} \tag{10}$$

As well as, (ii)-different exact solution is

$$[y(t)]^{\alpha} = [\underline{Y}^{\alpha}, \overline{Y}^{\alpha}(t)] = [\frac{1}{2}(3e^{-t^2}-1)(\alpha-1), \frac{1}{2}(3e^{-t^2}-1)(1-\alpha)]$$

and Equation (8) in this case is equivalent with system

$$\begin{cases} \underline{y}^{\alpha} = 2t\overline{y} + t(1-\alpha), & \underline{y}^{\alpha}(0) = \alpha - 1 \\ \overline{y}^{\alpha} = 2t\underline{y} + t(\alpha-1), & \overline{y}^{\alpha}(0) = 1-\alpha. \end{cases} \tag{11}$$

By apply Adomian-Tau method the same as above approximation for $n = 6$ we have

$$\begin{cases} \underline{y}^{\alpha} = (\alpha-1)(1-\frac{1}{2}t^2+\frac{3}{4}t^4-\frac{1}{4}t^6) \\ \overline{y}^{\alpha} = (1-\alpha)(1-\frac{1}{2}t^2+\frac{3}{4}t^4-\frac{1}{4}t^6) \end{cases} \tag{12}$$

Example 3.2. Consider the following fuzzy nonlinear differential equation from [2]

$$y'(t) = ty^2(t) \quad y(0) = (1.1, 1.2, 1.3), \quad t \in [0,1] \tag{13}$$

where the exact solution in the (i)-differentiable case for $\alpha \in [0,1]$ is

$$[y(t)]^{\alpha} = [\underline{Y}_{\alpha}, \overline{Y}_{\alpha}(t)] = [\frac{-(2\alpha+22)}{(\alpha+11)t^2-20}, \frac{-(2\alpha-26)}{(\alpha-13)t^2+20}]$$

and Equation (13) in this case is equivalent with system

$$\begin{cases} \underline{y} = t\underline{y}^2, & \underline{y}(0) = 1.1 + 0.1\alpha \\ \overline{y} = t\overline{y}^2, & \overline{y}(0) = 1.3 - 0.1\alpha. \end{cases} \tag{14}$$

If $y(t)$ approximate by $\underline{y}_{\alpha}(t) = \sum_{j=0}^{n} a_j t^j$ and $\overline{y}_{\alpha}(t) = \sum_{j=0}^{n} b_j t^j$, hence

$$\begin{cases} \underline{y}_{\alpha}(t) = (1.1+0.1\alpha) + \frac{(1.1+0.1\alpha)^2}{2}t^2 + \frac{(1.1+0.1\alpha)^3}{4}t^4 + \frac{(1.1+0.1\alpha)^4}{24}t^6 \\ \overline{y}_{\alpha}(t) = (1.3-0.1\alpha) + \frac{(1.3-0.1\alpha)^2}{2}t^2 + \frac{(1.3-0.1\alpha)^3}{4}t^4 + \frac{(1.3-0.1\alpha)^4}{24}t^6. \end{cases} \tag{15}$$

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Solving the Black-Scholes equation through a higher order compact finite difference method

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Abstract

In this paper a new compact finite difference (CFD) method for solving Black-Scholes equation is analyzed. Thise method leads to a system of linear equations involving tridiagonal matrices and the rate of convergence of the method is of order $O(k^2 + h^8)$ where k and h are the time and space step-sizes, respectively. Numerical results obtained by the proposed method are compared with the exact solution.

Keywords: Option pricing, Black-Scholes equation, compact finite difference scheme
Mathematics Subject Classification [2010]: 62P05, 65M06

1 Introduction

The Black-Scholes model [4, 5] is a powerful tool for valuation of equity options. This model is used for finding prices of stocks. Analytical approach and Numerical techniques are two ways for solving the European options. In [2] Mellin transformation was used to solve this model. They required neither variable transformation nor solving diffusion equation. R. Company et. al. [3] solved the modified Black-Scholes equation pricing option with discrete dividend. They used a delta-defining sequence of generalized Dirac-Delta function and applied the Mellin transformation to obtain an integral formula. Finally, they approximated the solution by using a numerical quadrature approximation.
Our contribution in this paper is the use of a high-order CFD method [1] for the pricing of options under the standard Black-Scholes model.

2 Construction of the method

Consider following Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \tag{1}$$

where S is the asset value, σ is the volatility and r is the *risk-free interest rate*. If we denote the current price of the underlying by S, then the payoffs at expiry, T, for a given exercise price, K, of European Calls and Puts is

$$C(S,T) = \max(S-K, 0), \qquad P(S,T) = \max(K-S, 0). \tag{2}$$

*Speaker

The closed form solution for the European Put option is

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{\log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t},\ N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}s^2}ds.$$

Consider the transformations of the independent variables $S = Ke^x, t = T - \frac{2\tau}{\sigma^2}$ and the dependent variable

$$v(x,\tau) = \frac{1}{K}V(S,t) = \frac{1}{K}V(Ke^x, T - \frac{2\tau}{\sigma^2}).$$

By the chain rule for functions of several variables, the Black-Scholes equation (1) transforms to a *constant coefficient* one, i. e.

$$v_\tau = v_{xx} + \left(\frac{2r}{\sigma^2} - 1\right)v_x - \frac{2r}{\sigma^2}v,$$

where the subscripts represent the partial derivatives with respect to the corresponding variables. The transformation can be defined by

$$v(x,\tau) = e^{-\alpha x - \beta^2\tau}u(x,\tau) \quad \text{where} \quad \gamma = \frac{2r}{\sigma^2}, \quad \alpha = \frac{1}{2}(\gamma - 1), \quad \beta = \frac{1}{2}(\gamma - 1) = \alpha + 1.$$

Consequently, the equation that to be satisfied by the transformed dependent variable $u = u(x,\tau)$ is the dimensionless form of the heat equation, i. e.

$$u_\tau = u_{xx}. \tag{3}$$

In this paper we will now price a European Put using the compact finite difference method. We first consider the following heat Black-Scholes PDE equation

$$u_\tau = u_{xx}, \qquad X_{\min} = a < x < b = X_{\max}, \quad 0 < \tau < \frac{\sigma^2}{2}T, \tag{4}$$

where

$$X_{\min} = \ln(\frac{S_{\min}}{K}), \qquad X_{\max} = \ln(\frac{S_{\max}}{K})$$

and we cannot, of course, discretely solve for all values of x up to infinity!. The initial and boundary conditions for the European Put are

$$u(x,0) = \max\{e^{\alpha x} - e^{\beta x}, 0\}, \quad u(a,\tau) = e^{\alpha a + (\beta^2 - \gamma)\tau}, \quad u(b,\tau) = 0. \tag{5}$$

To construct a CFD method, we select integers $M, N > 0$ and define $h = (b-a)/M,\ k = \frac{\sigma^2}{2}T/N$. The grid points for this situation are (x_i, τ_n), where $x_i = ih$ for $i = 0, 1, \ldots, M$ and $\tau_n = nk$ for $n = 0, 1, \ldots, N$. Assuming $u_i^n = u(x_i, \tau_n)$, we use the following notations for simplicity

$$u_i^{n+1/2} = \frac{u_i^{n+1} + u_i^n}{2}, \quad \partial_\tau u_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{k}, \quad \delta_x^2 u_i^n = u_{i+1}^n - 2u_i^n + u_{i-1}^n. \tag{6}$$

To obtain a eighth-order scheme with tridiagonal nature, (3) at the intermediate point $(x_i, t_{n+\frac{1}{2}})$ can be written as

$$\partial_\tau u_i^{n+1/2} = \frac{\delta_x^2}{h^2(1+\frac{1}{12}\delta_x^2+\frac{1}{360}\delta_x^4+\frac{1}{20160}\delta_x^6)} u_i^{n+1/2} + O(k^2+h^8)$$
$$= \frac{\delta_x^2}{h^2\,p(\delta_x^2)} u_i^{n+1/2} + O(k^2+h^8), \tag{7}$$

where $p(\delta_x^2) = \left(1+\frac{\frac{1}{12}\delta_x^2}{1-\frac{\frac{1}{30}\delta_x^2}{1-\frac{1}{56}\delta_x^2}}\right)$. With the aid of the approximate matrix $\boldsymbol{B}$ for δ_x^2, (7) can be written as

$$(\boldsymbol{I}+\boldsymbol{A})U^{n+1} = (\boldsymbol{I}-\boldsymbol{A})U^n, \tag{8}$$

where

$$U^n = (U_1^n, \dots, U_{M-1}^n)^T, \qquad \boldsymbol{B} = \begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}$$

and $\boldsymbol{A} = -\frac{k}{h^2}\,p(\boldsymbol{B})$.

The stability of the (7) is investigated by using the matrix method. The error e^n at the nth time level is given by $e^n = u^n_{exact} - u^n_{app}$, where u^n_{exact} and u^n_{app} are the exact and the numerical solutions at the n-th time level, respectively. The error equation for (8) can be written as

$$e^{n+1} = \boldsymbol{H}e^n,$$

where $\boldsymbol{H} = (\boldsymbol{I}+\boldsymbol{A})^{-1}(\boldsymbol{I}-\boldsymbol{A})$. From above argument we have the following theorem that can be proved without difficulty.

Theorem 2.1. *The numerical scheme* (7) *is stable if* $\|\boldsymbol{H}\|_2 \leqslant 1$*, which is equivalent to* $\rho(\boldsymbol{H}) \leqslant 1$*, where* $\rho(\boldsymbol{H})$ *denotes the spectral radius of the matrix* $\boldsymbol{H}$*.*

By using theorem 2.1 it can be seen that the stability is assured if $\rho(\boldsymbol{H})$ satisfy the following condition

$$\left|\frac{1-\rho(\boldsymbol{A})}{1+\rho(\boldsymbol{A})}\right| \leqslant 1.$$

This shows that the scheme (7) is unconditionally stable if $\rho(\boldsymbol{A}) \geqslant 0$.

3 Numerical results

The accuracy of the scheme is measured by using the $L_\infty = \| U_{app} - U_{exact} \|_\infty$ error norm. In Table 1, numerical solution for Put options obtained by the present method at different asset values are displayed and compared with the exact solution and the well-known Crank-Nicolson method. To show that the method has eighth-order convergence rate, we initially set $h = 0.034$ and $k = 0.02$, then reduce them by a factor of 2 and 16, respectively, in Table 2.

Table 1: Comparison of numerical and exact solutions for $S_{\min} = 1$, $S_{\max} = 150$, $\sigma = 0.2$, $T = 1$, $r = 0.05$, $h = 0.034$ and $k = 0.02$.

S_0	K	Crank-Nicolson	Present	Exact	$L_\infty(CN)$	$L_\infty(Present)$
10	30	18.5854	18.5368	18.5369	0.0486	$4.5213e-05$
	60	47.1220	47.0736	47.0738	0.0482	$1.8430e-04$
	100	85.1707	85.1228	85.1229	0.0478	$1.8026e-04$

Table 2: Rate of convergence with $S_{\min} = 1$, $S_{\max} = 100$, $\sigma = 0.2$, $T = 1$, $r = 0.05$, $k = 0.02$ and $h = 0.034$.

	h, k	$\frac{h}{2}, \frac{k}{16}$	$\frac{h}{2^2}, \frac{k}{16^2}$	$\frac{h}{2^3}, \frac{k}{16^3}$	$\frac{h}{2^4}, \frac{k}{16^4}$
$E = L_\infty$	$4.588e-05$	$6.707e-07$	$1.865e-09$	$8.978e-12$	$3.421e-14$
$R = \frac{E(h,k)}{E(\frac{h}{2},\frac{k}{16})}$	–	68.4097	359.5560	207.7704	262.4291
Order $= \log_2 R$	–	6.0961	8.4901	7.6988	8.0358

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Solving two-dimensional FitzHugh-Nagumo model with two-grid compact finite difference (CFD) method

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Abstract

The aim of this paper is to propose a two-grid compact finite difference (CFD) method to obtain the numerical solution of the two-dimensional FitzHugh-Nagumo model. We use the fourth-order CFD and second-order central finite difference methods for discretizing the spatial and time derivatives, respectively. The obtained system has been solved by two-grid (TG) method, where the TG method is used for solving the large sparse linear systems. Also, in the proposed method the spectral radius with local Fourier analysis is calculated for different values of h and Δt.

Keywords: FitzHugh-Nagumo equations (FHN), two-grid method, multigrid technique, compact finite difference method
Mathematics Subject Classification [2010]: 35K57, 35K20, 65N55, 65N06.

1 Introduction

The FHN equations exhibit excitability, a feature in common with Hodgkin-Huxley and other ionic models [3]. The FHN equations [3, 4] (the modelling of propagation of action potentials through excitable tissue for which v represents the non-diffusive gating variable) with diffusion can be written as

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \nabla^2 u + \frac{1}{\epsilon} f(u,v), \\ \frac{\partial v}{\partial t} = g(u,v), \end{cases} \tag{1}$$

with homogeneous Neumann boundary conditions, where elements D_1, known as the diffusion coefficient for u. In the present paper, the second kinetic model studied is the classic cubic FHN local dynamics [3] with the local ion dynamics are defined by $f(u,v) = u(1-u)(u-a) - v,\ g(u,v) = \alpha u - \gamma v$ where a, α and γ are dimensionless constants.

The finite difference approximations for derivatives are one of the simplest and of the oldest methods to solve differential equations. One approach to achieve accurate solutions is to use higher-order or locally exact discretization methods for solving the convection-diffusion equation [5, 8]. One of high-order finite difference methods can be noted compact finite difference method, that was planned by researchers such as Gupta et al. [1, 2].

*Speaker

Table 1: The E_h and E_τ^n errors for u and v obtained by the presented method when $\nu_1 = \nu_2 = 5$ and $T = 1$.

$\Delta t = 0.001$			$h = 200/2^6$		
h	E_h for u	E_h for v	Δt	E_τ^n for u	E_τ^n for v
$200/2^5$	4.33×10^{-2}	1.26×10^{-4}	0.0064	4.88×10^{-3}	1.17×10^{-5}
$200/2^6$	8.09×10^{-3}	1.91×10^{-5}	0.0032	4.53×10^{-3}	1.13×10^{-5}
$200/2^7$	2.44×10^{-3}	6.64×10^{-6}	0.0016	3.87×10^{-3}	9.80×10^{-6}
$200/2^8$	4.24×10^{-4}	1.30×10^{-6}	0.0008	2.57×10^{-3}	6.57×10^{-6}
$200/2^9$	1.58×10^{-5}	5.25×10^{-8}	$---$	$----$	$----$

For problems of large scale and complicated systems, direct solution methods based on Gaussian elimination techniques are expensive and are not efficient in terms of memory usage and CPU time. For this reason, we can consider multigrid method as an effective method that has least computational cost among of iterative methods. Multigrid (MG) schemes in numerical analysis are a group of algorithms for solving differential equations using a hierarchy of discretizations. The studies by J. Zhang [9, 10] show that the fourth-order compact schemes work well with fast iterative solution methods, e.g. the multigrid methods.

2 Main results

Simply substituting the compact and forward finite difference schemes in Eq. (1), we get

$$\begin{cases} \frac{u^n - u^{n-1}}{\Delta t} = D_1\left(\frac{\delta_x^2}{h^2(1+\frac{1}{12}\delta_x^2)}u^n + \frac{\delta_y^2}{h^2(1+\frac{1}{12}\delta_y^2)}u^n\right) + \frac{1}{\epsilon}\left(-au^n - v^n + (1+a)(u^{n-1})^2 - (u^{n-1})^3\right), \\ \frac{v^n - v^{n-1}}{\Delta t} = \alpha u^n - \gamma v^n. \end{cases}$$

Now, v^n is computed using the second relation of Eq. (2) and then by plugging the result in the first relation of Eq. (2) and with some manipulation, we obtain value of u^n. As that from the model is clear the value of v^n must be calculated according to following equation

$$v^n = \frac{1}{1+\gamma\Delta t}(\alpha \Delta t u^n - v^{n-1}).$$

Now, in the following, the standard two-grid algorithm [7] is expressed.

Algorithm 1: Two-grid method $u_h \leftarrow TG(u_h, f_h, \nu_1, \nu_2)$
1) Relax ν_1 times on $A_h v_h = f_h$ on Ω^h with arbitrary initial guess u_h.
2) Compute $r_h = f_h - A_h u_h$.
3) Compute $r_{2h} = I_h^{2h} r_h$.
4) Solve $A_{2h} e_{2h} = r_{2h}$ on Ω^{2h}.
5) Correct fine-grid solution $u_h \leftarrow u_h + I_{2h}^h e_{2h}$.
6) Relax ν_2 times on $A_h v_h = f_h$ on Ω^h with initial guess u_h.

According to the algorithm expressed in above, iterative matrix form of two-grid method is as follows:

$$M_{TG} = S_h^{\nu_2}(I_h - T_{TG})S_h^{\nu_1},$$

which $T_{TG} = I_{2h}^h A_{2h}^{-1} I_h^{2h} A_h$ and I_h is the identity matrix. We identify the coarse-grid operator $A_{2h} = I_{2h}^h A_h I_h^{2h}$.

Table 2: Two-grid convergence factors ρ_{loc} when $\nu_1 = \nu_2 = 5$ and $\omega = 0.9417$.

$\Delta t = 0.0005$		$h = 200/2^7$	
h	ρ_{loc}	Δt	ρ_{loc}
$200/2^6$	9.77×10^{-5}	0.004	9.36×10^{-5}
$200/2^7$	9.74×10^{-5}	0.002	9.58×10^{-5}
$200/2^8$	9.58×10^{-5}	0.001	9.69×10^{-5}
$200/2^9$	8.95×10^{-5}	0.0005	9.74×10^{-5}

Also, we use the local Fourier analysis to show that the spectral radius of the iteration matrix in the two-grid method ($\rho(M_{TG})$) is low for 2D FitzHugh-Nagumo equations. Note that the interested readers can refer to [7] for the asymptotic convergence and error reduction factors with their respective definitions and theorems.

3 Numerical results

In the current paper, we don't have the exact solutions, thus to examine the numerical stability of time difference and the convergence of full discrete schemes, we employ strategy of the reference solution. Thus, we consider W^N and S_h^N as two reference solutions and set W^m and S_h^I as numerical solutions and also apply the following error relations

$$E_\tau^n = \|W^N - W^m\|_\infty, \qquad E_h = \|S_h^N - S_h^I\|_\infty.$$

It should be noted that in this case the iterative method used is the method of ω-Jacobi by $\omega = 0.9417$. We will investigate the FitzHugh-Nagumo monodomain model for kinetic model (II) [6], with homogenous Neumann boundary conditions for both u and v by $D1 = 1$, $a = 0.15$, $\epsilon = 1$, $\alpha = 0.005$ and $\gamma = 0.025$, over the square domain $\Omega = [-100, 100] \times [-100, 100]$. Also we use the following initial condition

$$\begin{cases} u(x, y, 0) = \exp(-((x-30)^2 + (y-30)^2)/16), \\ v(x, y, 0) = 0. \end{cases}$$

Table 1 represents errors obtained corresponding to E_h and E_τ^n. In this table, we considered the present method with $\Delta t = 0.001$, $D1 = 1$, $\epsilon = 1$, $a = 0.15$, $\alpha = 0.005$, $\gamma = 0.025$ and $h = \frac{200}{2^6}$. Table 2 presents the corresponding two-grid convergence factor. As we can see in Table 2, for the case of fixed value Δt and different h, the two-grid convergence factor decreases. In the case of a fixed value h and different Δt, the two-grid convergence factor does not noticeably change.

Graphs of approximation solution for u, v of equations for kinetic model (II) using the present method at $T = 100$ on rectangular domain $\Omega = [-100, 100] \times [-100, 100]$ with $M = 128$, $\Delta t = 0.001$ and $\nu_1 = \nu_2 = 5$ are shown in Fig. 1. The asymptotic convergence factor and error reduction factor with $\Delta t = 0.0005$ and $M = 128$ computed by the presented method and can be observed in Fig. 2.

Conclusion

In the current paper, we employed a numerical algorithm based on the two-grid compact finite difference method for solving two-dimensional FitzHugh-Nagumo model. Numerical simulations show the efficiency of the new technique. It should be said that the present method can be used with some changes for other differential equations.

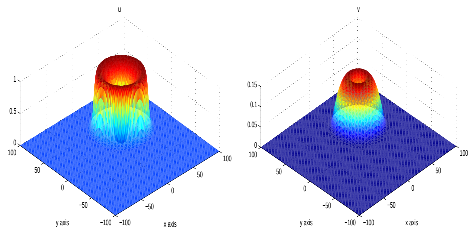

Figure 1: Graphs of approximation solution using the present method with $M = 128$, $\Delta t = 0.001$ and $\nu_1 = \nu_2 = 5$ at $T = 100$ on rectangular domain $\Omega = [-100, 100] \times [-100, 100]$ for kinetic model (II) and $D1 = 1$, $a = 0.15$, $\alpha = 0.005$, $\gamma = 0.025$.

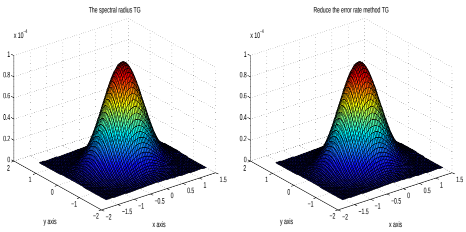

Figure 2: Graphs of the asymptotic convergence factor and error reduction factor by the presented method is computed with $\Delta t = 0.0005$ and $M = 128$.

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The Interval Matrix Equation $\mathbf{A}X\mathbf{B} = \mathbf{C}$

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Abstract

In this paper, we define a solution set for the interval matrix equation $\mathbf{A}X\mathbf{B} = \mathbf{C}$, where $\mathbf{A}$ and $\mathbf{B}$ are the known square interval matrices of dimensions $m \times m$ and $n \times n$, respectively, $\mathbf{C}$ is a rectangular interval matrix of dimension $m \times n$ and the unknown matrix X is also of dimension $m \times n$. Afterwards, some conditions bounding the solution set will be studied. We also present a number of methods for solving the aforementioned interval matrix equation. Finally, we show that whenever $\mathbf{A}$ and $\mathbf{B}$ are inverse positive, hull of solution set can be described explicitly.

Keywords: Interval matrix, Interval linear systems, Linear matrix equations, Solution set.
Mathematics Subject Classification [2010]: 65F30

1 Introduction

Matrix equations have numerous applications in sciences and engineering, including calculation for electromagnetic scattering, structural mechanics and computation of the frequency response matrix in control theory.
An example of these matrix equations is in the form of:

$$AXB = C, \tag{1}$$

where A, B and C, are the known real matrices of dimensions $m \times m$, $n \times n$ and $m \times n$, respectively, while the unknown matrix X is a real matrix with dimension of $m \times n$.

In practical applications, the elements of A, B and C are usually obtained from experiments and thus they may appear with uncertainties. The uncertain elements are shown in interval forms. Therefore with the presence of uncertainties in data, the matrix equations (1) is transformed to the following interval matrix equation

$$\mathbf{A}X\mathbf{B} = \mathbf{C} \tag{2}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are interval matrices. Note that bold-face letters are used to show intervals.

In this paper, we use notations $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ as the field of real numbers and the vector space of $m \times n$ real matrices, respectively. We denote the set of all $m \times n$ interval matrices by $\mathbb{IR}^{m\times n}$.

*Speaker

For the interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$, the center matrix denoted by mid($\mathbf{A}$) or $\check{\mathbf{A}}$ and the radius matrix denoted by rad($\mathbf{A}$) are respectively defined as

$$\check{\mathbf{A}} = \frac{1}{2}(\underline{A} + \overline{A}) \quad , \quad \mathrm{rad}(\mathbf{A}) = \frac{1}{2}(\overline{A} - \underline{A}).$$

It is clear that $\mathbf{A} = [\check{\mathbf{A}} - \mathrm{rad}(\mathbf{A}), \check{\mathbf{A}} + \mathrm{rad}(\mathbf{A})]$.

We assume that the reader is familiar with a basic interval arithmetic and interval operators on the interval matrices; for more detail, refer to [1, 2].

If Σ is a bounded set of $m \times n$ real matrices, then interval hull of Σ denoted by $\mathbf{\Sigma}$ is defined as

$$\mathbf{\Sigma} = [\inf(\Sigma), \sup(\Sigma)].$$

An $n \times n$ interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is said to be regular if each $A \in \mathbf{A}$ is nonsingular. An inverse positive matrix is a regular square matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ with nonnegative inverse.

For two interval matrices $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and $\mathbf{B} \in \mathbb{IR}^{k\times t}$, the Kronecker product denoted by $\otimes$ is defined by the following $mk \times nt$ block interval matrix

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{a}_{ij}\mathbf{B}).$$

Also vec($\mathbf{A}$) is defined as an mn-interval vector and obtained by stacking the columns of $\mathbf{A}$, i.e.,

$$\mathrm{vec}(\mathbf{A}) = (\mathbf{A}_{.1}, \mathbf{A}_{.2}, \cdots, \mathbf{A}_{.n})^T,$$

where $\mathbf{A}_{.j}$ is the j^{th} column of $\mathbf{A}$.

2 Main results

Consider the matrix equation (2). The solution set for this equation is defined as follows:

$$\Sigma(X) = \left\{X \in \mathbb{R}^{m\times n} | \quad AXB = C \quad \text{for some} \quad A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}\right\}. \qquad (3)$$

Much like the solution for interval linear systems presented in other studies, the solution set of an interval matrix equation generally has a complicated structure, [2]. However, we can show that $\Sigma(X)$ is closed and moreover it is connected and compact if $\mathbf{A}$ and $\mathbf{B}$ are regular. If $\Sigma(X)$ is bounded, we look for an enclosure of this set, i.e. for an interval matrices $\mathbf{X}$ satisfying $\Sigma(X) \subseteq \mathbf{X}$. The special case in which, $\mathbf{A}$ and $\mathbf{B}$ are inverse positive, we can present the interval hull of solution set.

2.1 Description and properties of solution set

In this section, we present some properties and descriptions of $\Sigma(X)$ and the conditions that imply boundedness of it. The following theorem shows that the solution set is always a closed set.

Theorem 2.1. *The solution set defined by (3) is closed.*

In the above theorem, we do not suppose $\mathbf{A}$ and $\mathbf{B}$ to be regular. With this assumption, $\Sigma(X)$ will be connected and compact.

Theorem 2.2. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *in the interval matrix equation (2) are regular. Then for each interval matrices* $\mathbf{C} \in \mathbb{IR}^{m\times n}$*, the solution set is compact and connected.*

The following theorem give us a description of a superset of $\Sigma(X)$.

Theorem 2.3. *The solution set* $\Sigma(X)$ *defined by (3) satisfies*

$$\Sigma(X) \subseteq \left\{ X \in \mathbb{R}^{m\times n} : \begin{array}{l} |\check{\mathbf{A}}X\check{\mathbf{B}} - \check{\mathbf{C}}| \leq \\ |\check{\mathbf{A}}||X|\mathrm{rad}(\mathbf{B}) + \mathrm{rad}(\mathbf{A})|X||\mathbf{B}| + \mathrm{rad}(\mathbf{C}) \end{array} \right\}. \tag{4}$$

The following theorems express some conditions for boundedness of $\Sigma(X)$.

Theorem 2.4. *Let* $\mathbf{C} \in \mathbb{IR}^{m\times n}$ *be arbitrary. The solution set of interval matrix equation (2) is bounded if and only if* $\mathbf{A}$ *and* $\mathbf{B}$ *are regular.*

Theorem 2.5. *For all interval* $m \times n$ *matrices* $\mathbf{C}$ *the solution set defined by (3) is bounded if one of the following inequalities has only the trivial solution* $X = 0$.

$$\begin{cases} |\check{\mathbf{A}}X\check{\mathbf{B}}| \leq |\mathbf{A}||X|\mathrm{rad}(\mathbf{B}) & \text{if } A \text{ is thin,} \\ |\check{\mathbf{A}}X\check{\mathbf{B}}| \leq \mathrm{rad}(\mathbf{A})|X||\mathbf{B}| & \text{if } B \text{ is thin.} \end{cases} \tag{5}$$

2.2 Obtaining of enclosure of solution set

In this section, we look for interval matrix $\mathbf{X}$ as an enclosure of the solution set of the interval matrix equation (2) whenever the solution set is bounded. It is clear that for each enclosure $\mathbf{X}$, the inclusion $\mathbf{C} \subseteq \mathbf{AXB}$ is valid.

The matrix equation $\mathbf{A}X\mathbf{B} = \mathbf{C}$ can be transformed to the following form

$$\mathbf{G}z = \mathbf{d}, \tag{6}$$

where $\mathbf{G} = \mathbf{B}^T \otimes \mathbf{A}$, $\mathbf{d} = \mathrm{vec}(\mathbf{C})$ and $z = \mathrm{vec}(X)$. So, by finding interval vector $\mathbf{z}$ as an enclosure of solution set of the interval linear system (6), we can specify the columns of the interval matrices $\mathbf{X}$. To solve the interval linear system (6) see [2].

Example 2.6. Consider the interval matrix equation $\mathbf{A}X\mathbf{B} = \mathbf{C}$, in which

$$\mathbf{A} = \begin{bmatrix} [1,2] & [2,2.5] \\ [-2,-1] & [5,6] \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} [3,4] & [-1,0] \\ [1,1] & [6,8] \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} [6,7] & [1,3] \\ [8,9] & [6,8] \end{bmatrix}.$$

By using Matlab toolbox Intlab [4] and Verintervalhull.m code of Versoft [3] for solving the interval linear system $\mathbf{G}z = \mathbf{d}$, we obtain the enclosure of the solution set as

$$\mathbf{X} = \begin{bmatrix} [-0.2264, 1.7081] & [-0.7964, 0.7098] \\ [0.2838, 0.9075] & [0.0589, 0.4545] \end{bmatrix},$$

that is shaper than the enclosure of the previous example.

However this method may not succeed because due to interval dependencies, it is possible for $\mathbf{G}$ to be singular even if $\mathbf{A}$ and $\mathbf{B}$ are regular. Therefore, we try to find an enclosure $\mathbf{X}$ of $\Sigma(X)$ by an easier and better technique.

We can reduce the interval matrix equation $\mathbf{A}X\mathbf{B} = \mathbf{C}$ to the two interval matrix equations $\mathbf{A}Y = \mathbf{C}$ and $X\mathbf{B} = \mathbf{Y}$, where $\mathbf{Y}$ is an enclosure for the solution set of $\mathbf{A}Y = \mathbf{C}$. Thus we need to solve two interval matrix equation such as $\mathbf{A}X = \mathbf{B}$. To this end, we consider an interval linear system of the form

$$\mathbf{A}X_{.j} = \mathbf{B}_{.j},$$

where $X_{.j}$ and $\mathbf{B}_{.j}$ are j^{th} columns of X and $\mathbf{B}$, respectively.

Example 2.7. Consider the interval matrix equation in previous example . By using the above method and Matlab toolbox Intlab we obtained the following result:

$$\mathbf{Y} = \begin{bmatrix} [0.4999, 3.2501] & [-2.0001, 0.7501] \\ [1.5172, 2.5556] & [0.7272, 1.5834] \end{bmatrix}, \mathbf{X} = \begin{bmatrix} [0.0899, 1.1945] & [-0.3334, 0.2895] \\ [0.3008, 0.8216] & [0.0909, 0.3846] \end{bmatrix}.$$

Theorem 2.8. *In the interval matrix equation (2), suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are inverse positive. Then*

1. $\mathbf{\Sigma}(\mathbf{X}) = [\overline{A}^{-1}\underline{C}\overline{B}^{-1}, \underline{A}^{-1}\overline{C}\underline{B}^{-1}]$ *when* $\underline{C} \geq 0$,
2. $\mathbf{\Sigma}(\mathbf{X}) = [\underline{A}^{-1}\underline{CB}^{-1}, \overline{A}^{-1}\overline{CB}^{-1}]$ *when* $\overline{C} \leq 0$,
3. $\mathbf{\Sigma}(\mathbf{X}) = [\underline{A}^{-1}\underline{CB}^{-1}, \underline{A}^{-1}\overline{C}\underline{B}^{-1}]$ *when* $\underline{C} \leq 0 \leq \overline{C}$.

Example 2.9. Consider the interval matrix equation $\mathbf{A}X\mathbf{B} = \mathbf{C}$, in which

$$\mathbf{A} = \begin{bmatrix} [30,30] & [-12,-1] & [-12,-1] \\ [-12,-1] & [30,30] & [-12,-1] \\ [-12,-1] & [-12,-1] & [30,30] \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} [25,27] & [-4,-2] & [-3,-2] \\ [-2,-1] & [20,23] & [-2,-2] \\ [-4,-1] & [-4,-2] & [30,30] \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} [2,4] & [4,5] & [1,2] \\ [0,2] & [3,4] & [5,6] \\ [1,2] & [8,9] & [4,6] \end{bmatrix}.$$

Since $\mathbf{A}$ and $\mathbf{B}$ are inverse positive and $\mathbf{C} \geq 0$, from the above theorem it follows that

$$\mathbf{\Sigma}(\mathbf{X}) = \begin{bmatrix} [0.0028, 0.0289] & [0.0067, 0.0608] & [0.0021, 0.0308] \\ [0.0005, 0.0274] & [0.0055, 0.0599] & [0.0061, 0.0338] \\ [0.0020, 0.0279] & [0.0126, 0.0660] & [0.0056, 0.0342] \end{bmatrix}.$$

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The use of a tau method based on Bernstein polynomials for solving the viscoelastic squeezing flow between two parallel plates

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Abstract

In this paper, a numerical method based on Bernstein polynomials for solving the viscoelastic squeezing flow between two parallel plates is introduced. This method expands the desired solutions in terms of a set of Bernstein polynomials over a closed interval and then makes use of the tau method to determine the expansion coefficients to construct approximate solutions.

Keywords: Squeezing flow; Bernstein polynomials; Tau method.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Many of the mathematical modeling, which appears in many areas of scientific fields such as fluid dynamics, plasma physics and solid state physics, can be modeled by nonlinear ordinary or partial differential equations. Apart of a limited number of these problems, most of them do not have an exact solution, so these nonlinear equations should be solved using approximate methods. Therefore, several attempts have been made to develop the new techniques for obtaining analytical or numerical solutions which reasonably approximate the exact solutions. These known methods are for example, Runge-Kutta method spectral methods, the Adomian decomposition method, the variational iteration method, the homotopy perturbation method and the homotopy analysis method.
Here, we have considered the viscoelastic squeezing flow between two parallel plates. This problem studied first by Ran et al. [1] in 2009 and solved by using homotopy analysis method (HAM). Zodwa et al. [2] used the successive linearization method (SLM) to solve this problem. In this study, we are going to introduce and implement a new algorithm based on Bernstein polynomials [3] to find the approximate solution of the viscoelastic squeezing flow between two parallel plates. Bernstein polynomials have many useful properties, such as, the positivity, the continuity, and unity partition of the basis set over the interval $[a, b]$[3].

*Speaker

2 Flow analysis and mathematical formulation

The description of the physical problem closely follows that of Zodwa et al. [2]. The problem under consideration is that of a two-dimensional quasi-steady axisymmetric flow of an incompressible viscous fluid between two infinite parallel plates. The velocity is $\mathbf{u} = [u_r(r, z, t), 0, u_z(r, z, t)]$and the governing equations can be expressed as

$$\frac{\partial p}{\partial r} + \frac{\rho}{r}\frac{\partial^2\psi}{\partial t\partial z} - \rho\frac{\partial\psi}{\partial r}\frac{E^2\psi}{r^2} - \frac{\mu}{r}\frac{\partial E^2\psi}{z} = 0, \tag{1}$$

$$\frac{\partial p}{\partial r} - \frac{\rho}{r}\frac{\partial^2\psi}{\partial t\partial z} - \rho\frac{\partial\psi}{\partial z}\frac{E^2\psi}{z^2} + \frac{\mu}{r}\frac{\partial E^2\psi}{z} = 0, \tag{2}$$

where r and z are the radial and axial coordinates respectively, ρ is the fluid density, μ is the coefficient of kinematic viscosity, p is the pressure, $\psi(r, z)$ is the Stokes stream function and $E = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

Eliminating the pressure term from (1) and (2) by the integrability condition $\frac{\partial^2 p}{\partial r\partial z} = \frac{\partial^2 p}{\partial z\partial r}$, we get the compatibility equation

$$\rho\left[\frac{1}{r}\frac{\partial E^2\psi}{\partial t} - \frac{\partial(\psi, \frac{E^2\psi}{r^2})}{\partial(r, z)}\right] = \frac{\mu}{r}E^4\psi. \tag{3}$$

For small values of the approach velocity v of the two plates, the gap $2H$ changes slowly with time and can be assumed to constant, hence from 3 we write

$$-\rho\left[\frac{\partial(\psi, \frac{E^2\psi}{r^2})}{\partial(r, z)}\right] = \frac{\mu}{r}E^4\psi, \tag{4}$$

with the boundary conditions

$$\begin{cases} u_r = 0, & u_z = -V, \qquad at \quad z = H,\\ u_z = 0, & \frac{\partial u_r}{\partial z} = 0, \qquad at \quad z = 0. \end{cases} \tag{5}$$

Using the stream function $\psi\,(r, z) = \mathrm{r}^2 F^*(Z)$ and introducing the non-dimensional parameters $F^* = \frac{2F}{V}$, $Z^* = Z/H$ and $M = \rho HV/\mu$ equation (4) and boundary conditions (5) become

$$F^{(iv)}\,(z) + MF\,(z)\,F'''\,(z) = 0, \tag{6}$$

$$F(0) = 0,\ F''(0) = 0,\ F(1) = 1,\ F'(1) = 0. \tag{7}$$

3 Bernstein polynomials

The Bernstein polynomials of degree n are defined on the interval $[0, 1]$ as [4]

$$B_{i,n}\,(x) = \binom{n}{i} x^i(1-x)^{n-i}, \qquad i = 0, 1, \cdots, n. \tag{8}$$

These Bernstein polynomials form a basis on $[0, 1]$. There are $n+1$, nth-degree polynomials. For convenience, we set $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$. Moreover, the recursive definition for the Bernstein polynomials over the interval $[0, 1]$ is as follows:

$$B_{i,n}(x) = (1 - x) B_{i,n-1}(x) + x B_{i-1,n-1}(x). \tag{9}$$

Suppose that $H = L^2[0, R]$ where $R \in \mathbb{R}$, let $\{B_{0,n}(x), B_{1,n}(x), \cdots, B_{n,n}(x)\} \subset H$ be the set of Bernstein polynomials of nth degree, and suppose that

$$Y = span\{B_{0,n}(x), B_{1,n}(x), \cdots, B_{n,n}(x)\}.$$

Theorem1. For every given x in a Hilbert space H and every given closed subspace Z of H there is a unique best approximation to x from Z.
Proof. See [5].
Since $H = L^2[0, R]$ is Hilbert space and Y is finite-dimensional subspace, so Y is a closed subspace of H, therefore Y is a complete subspace of H. So, if f be an arbitrary element in H, by Theorem 1, f has unique best approximation from Y such as f^*, that is

$$\exists f^* \in Y;; \quad \forall g \in Y \quad ||f - f^*||_2 \leq ||f - g||_2,$$

where $||f||_2 = \sqrt{\langle f, f\rangle}$. Since $f^* \in Y$, there exist unique coefficients $f_0, f_1, \cdots, f_n$ such that

$$f(x) \approx f^*(x) = \sum_{i=0}^{n} f_i B_{i,n}(x),$$

where the coefficients $f_0, f_1, \cdots, f_n$ can be obtained by solving the following linear system

$$\sum_{i=0}^{n} f_i \langle B_{i,n}(x), B_{j,n}(x)\rangle = \langle f(x), B_{j,n}(x)\rangle, \qquad j = 0, 1, \cdots, n.$$

4 Solution of the problem

The tau approach is a modification of the Galerkin method that is applicable to problems with non-periodic boundary conditions. In this section we apply Bernstein-tau method (BTM) for the computation of the viscoelastic squeezing flow between two parallel plates based on the Bernstein polynomials.

For an arbitrary natural number n, we suppose that the approximate solution $F(Z)$ of (6) is as follows:

$$F(\mathrm{z}) \approx \sum_{i=0}^{n} f_i B_{i,n}(z), \tag{10}$$

and the residual function associated to the differential equations (6) is

$$RESF(z) = F^{(iv)}(z) + MF(z) F'''(z). \tag{11}$$

By substituting (10) in the above residual function, we obtain

$$RESF(z) \approx \sum_{i=0}^{n} f_i B_{i,n}^{(iv)}(z), + M\left(\sum_{i=0}^{n} f_i B_{i,n}(z)\right)\left(\sum_{i=0}^{n} f_i B_{i,n}'''(z)\right). \tag{12}$$

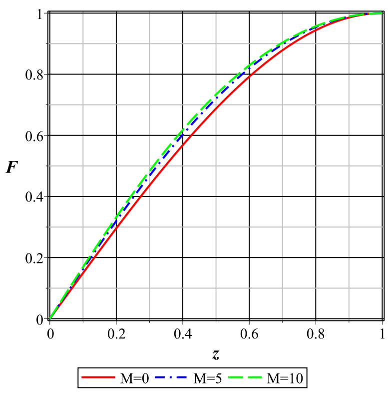

Figure 1: Effect of M on $F(z)$ when $n = 15$.

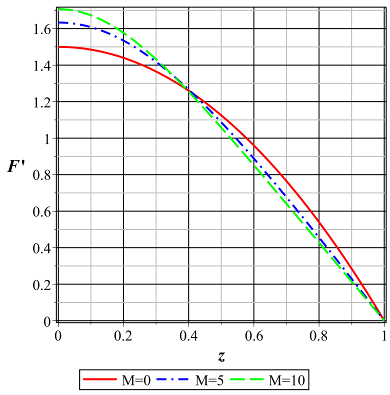

Figure 2: Effect of M on $F'(z)$ when $n = 15$.

In tau method we get the inner product of the above equations with $B_{s,n}(z)$:

$$\langle RESF\,(\eta)\,, B_{s,n}\,(\eta)\rangle = 0, \qquad s = 0, 1, \cdots, n-4, \tag{13}$$

where $\langle f, g\rangle = \int_0^1 f\,(z)\,g\,(z)\,dz$.

Also by imposing the boundary conditions (7), we have

$$\sum_{i=0}^{n} f_i B_{i,n}(0) = 0, \qquad \sum_{i=0}^{n} f_i B''_{i,n}(0) = 0, \sum_{i=0}^{n} f_i B_{i,n}(1) = 1, \qquad \sum_{i=0}^{n} f_i B'_{i,n}(1) = 0. \tag{14}$$

From (13) and (14), a nonlinear system of $n+1$ equations and $n+1$ unknown coefficients is resulted. Solving this system, we can obtain unknown coefficients f_i $i = 0, 1, \cdots$, n and therefore $F(z)$ is identified.

5 Results and discussion

The nonlinear ordinary differential equation (6) subject to boundary conditions (7) has been solved using exponential Bernstein-tau method (BTM) for some values of the parameter. Figs. 1 and 2 represent the effects of the parameter M on $F(z)$ and $F'(z)$, respectively, when $n = 15$. Fig. 1 shows that $F(z)$ increases with increasing the parameter M.

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Two-stage waveform relaxation method for linear system of IVPs with non-constant HPD coefficients

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Abstract

In this paper, a two-stage waveform relaxation method is introduced to solve the system of initial value problems in the form $y'(t) + A(t)y(t) = f(t)$. Convergence of this method is analyzed when $A(t)$ is Hermitian positive definite matrix for every $t \in [t_0, T]$. Finally, a numerical example is presented to illustrate efficiency of the method.

Keywords: Two-stage method, Waveform relaxation, Hermitian positive definite, P-regular splitting.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In [3], the two-stage waveform relaxation (TSWR) method was applied to solve the linear system of ordinary differential equations $y'(t) + Ay(t) = f(t)$, where A is an M-matrix. Indeed this method was obtained by combining the waveform relaxation (WR) method with two-step iterative strategy. Afterwards, in [2, 4] the TSWR was investigated to solve linear systems of ordinary differential equations (ODEs) and differential-algebraic equations, when the coefficient matrices are Hermitian positive definite and Hermitian positive semi-definite. Recently the TSWR method has been applied to solve the linear system of ODEs

$$\begin{cases} y'(t) + A(t)y(t) = f(t), \\ y(t_0) = y_0, \quad t \in [t_0, T], \end{cases} \tag{1}$$

where $A(t) : [t_0, T] \longrightarrow \mathbb{C}^{m\times m}$ is a nonsingular M-matrix for every $t \in [t_0, T]$ with continuous entries and $f(t) : [t_0, T] \longrightarrow \mathbb{C}^m$ is supposed to be continuous (see [1]). In this paper, we study the WR and TSWR methods for (1), when $A(t)$ is Hermitian positive definite for every $t \in [t_0, T]$. We will use the notation $A(t) \succ 0$ ($A(t) \succeq 0$) for a matrix function $A(t)$ to be Hermitian positive (semi-)definite for every $t \in [t_0, T]$.

Definition 1.1. The splitting $A(t) = C(t) - D(t)$ is called P-regular if $C^H(t) + D(t) \succ 0$, and Hermitian P-regular splitting if $C(t) \succ 0$ and $D(t) \succeq 0$.

Definition 1.2. We say that the splitting $A(t) = M(t) - N(t) - D(t)$ is composite P-regular if $C(t) = M(t) - N(t)$ and $A(t) = C(t) - D(t)$ are both P-regular splittings, and a composite Hermitian P-regular splitting if $M(t) \succ 0$, $N(t) \succeq 0$ and $D(t) \succeq 0$.

*Speaker

2 Main results

2.1 Two-stage waveform relaxation method

Similar to [1], we consider the splitting $A(t) = C(t) - D(t)$. Based on this splitting WR iterative method is generated in the form

$$\begin{cases} y_{n+1}^{k+1} = (I + hC_{n+1})^{-1}(y_n^{k+1} + hD_{n+1}y_{n+1}^k + hf(t_{n+1})), \\ y_0^{k+1} = y_0, \quad k = 0, 1, \ldots, \quad n = 0, 1, \ldots, N-1, \end{cases} \tag{2}$$

where y_n^k is an approximation for $y^k(t_n)$ and for brevity of notation, $C(t_n)$ and $D(t_n)$ are denoted by C_n and D_n, respectively. By substituting the composite splitting $A(t) = M(t) - N(t) - D(t)$ in Eq. (1) the TSWR method is defined (see [1]) as

$$\begin{cases} z_{n+1}^{v+1} = H_{n+1}z_n^{v+1} + hb_{n+1}(v, k), \\ z_0^{v+1} = y_0^k = y_0, \quad k = 0, 1, \ldots, \quad v = 0, 1, \ldots, s-1, \end{cases} \tag{3}$$

where

$$\begin{cases} b_n(v, k) = (I + hM_n)^{-1}(N_n z_n^v + D_n y_n^k + f(t_n)), \\ H_n = (I + hM_n)^{-1}. \end{cases}$$

Furthermore, we assume that the number of inner iterations steps is fixed for all outer iterations, for example $v_k \equiv s$, $k = 0, 1, \ldots,$ where s is a positive integer. Similar to [1] the TSWR iterative method (3) can be written in the following matrix form

$$y_n^{k+1} = T_s y_n^k + S_s g_n + p_{s,n}(k). \tag{4}$$

2.2 Convergence analysis

Similar to Theorems 5.4, 5.5 and 5.6 in [4] we state the following theorem and propositions.

Theorem 2.1. *(Convergence theorem of TSWR method). Let $A_n \succ 0$ and $A_n = M_n - N_n - D_n$ is a composite P-regular splitting. If $C_n = M_n - N_n$is a Hermitian matrix, $D_n \succeq 0$, $h > 0$ and $s \geq 1$, then $\rho(T_s) < 1$.*

Proposition 2.2. *Let $A_n \succ 0$ and $A = M_n - N_n - D_n$ is a composite Hermitian P-regular splitting of A_n and $N_n \succ 0$, $h > 0$. Let us indicate with T_{s_1} and T_{s_2} the matrices of convergence of TSWR method with s_1 and s_2 inner iterations, respectively. If $1 \leq s_2 < s_1$, then $\rho(T_{s_1}) < \rho(T_{s_2}) < 1$.*

Proposition 2.3. *Let $A_n \succ 0$ and $A = M_n - N_n - D_n$ a composite Hermitian P-regular splitting of A_n. Let us indicate with $T_s^{(1)}$ and $T_s^{(2)}$ the matrices of convergence of TSWR method with h_1 and h_2, respectively and $0 < h_1 < h_2$. If M_n, N_n and $D_n \succ 0$, for $s \geq 1$ then it is $\rho(T_s^{(1)}) < \rho(T_s^{(2)}) < 1$.*

Table 1: Values of $\rho(T_s)$ for TSWR method at $t = 0.2$.

s	$h = 0.1$	$h = 0.3$	$h = 0.5$	$h = 0.8$	$h = 1$
1	0.1878346	0.3795026	0.4768109	0.5571723	0.5903373
2	0.0608407	0.1744693	0.2596036	0.3423428	0.3797674
3	0.0448056	0.1222679	0.1807858	0.2415781	0.2715328
4	0.0425690	0.1078366	0.1562064	0.2076859	0.2326671
5	0.0422571	0.1037697	0.1475036	0.1936632	0.2161418
6	0.0422136	0.1026236	0.1444222	0.1878613	0.2088976
7	0.0422075	0.1023006	0.1433312	0.1854608	0.2057219
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0.0422065	0.1021738	0.1427331	0.1837667	0.2032431

3 A numerical example

Example 3.1. In initial value problem (1) assume that $A(t) : [t_0, T] \to \mathbb{C}^{m\times m}$ is defined as

$$A(t) = \begin{pmatrix} 4t & t^2 & & & \\ t^2 & 4t & t^2 & & \\ & t^2 & 4t & t^2 & \\ & & \ddots & \ddots & \ddots \\ & & & t^2 & 4t \end{pmatrix}.$$

The function $f(t)$ is computed such that the exact solution is given by

$$y(t) = [t, t^2, t^3, t^4, t^5, \ldots, t, t^2, t^3, t^4, t^5]^T \in \mathbb{C}^m.$$

We set $t_0 = 0.1$, $T = 1$ and $m = 25$ and consider splitting matrices

$$M(t) = \text{diag}(16t), \quad N(t) = \text{tridiag}(t^2, 2t, t^2), \quad D(t) = M(t) - N(t) - A(t).$$

Since $t \in [0.1, 1]$ and from eigenvalue analysis of tridiagonal matrices, we deduce that $A(t) \succ 0$ and $A(t) = M(t) - N(t) - D(t)$ is a composite Hermitian P-regular splitting. According to propositions 2.2 and 2.3, the numerical results given in Table 1 and Table 2 indicate the monotonicity of $\rho(T_s)$ at varying of s and h, at $t = 0.2$ and $t = 0.9$, respectively.

In continuation, we compare the numerical results of the WR and TSWR methods. For the two methods, we set $h = 0.1$, $N = 10$ and all of our computations terminate once the current iterations obey $\|y_n^{k+1} - y_n^k\|_\infty \leq 10^{-3}$, $n = 0, 1, \ldots, N-1$ or $k > 1000$. In the TSWR method the number of the inner iterations is set to be $s = 5$. The number of outer iterations is 15 for the TSWR method but it is 25 for the WR method.

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Table 2: Values of $\rho(T_s)$ for TSWR method at $t = 0.9$.

s	$h = 0.1$	$h = 0.3$	$h = 0.5$	$h = 0.8$	$h = 1$
1	0.5530098	0.7609082	0.8227706	0.8622005	0.8761973
2	0.4012300	0.6492013	0.7332346	0.7892482	0.8095910
3	0.3496918	0.5970103	0.6880012	0.7506266	0.7737566
4	0.3321915	0.5726260	0.6651494	0.7301800	0.7544776
5	0.3262491	0.5612333	0.6536047	0.7193554	0.7441054
6	0.3242313	0.5559105	0.6477724	0.7136248	0.7385252
7	0.3235461	0.5534236	0.6448259	0.7105909	0.7355230
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0.3231939	0.5512428	0.6418174	0.7071779	0.7320269

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Operation Research & Control Theory

A Delayed-Projection Neural Networks to solve Bilevel Programming Problems

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Abstract

Projection-type methods are a class of simple methods for solving mathematical programming problems. In this paper we proposed a new neural network model, delayed-projection neural network, to solve bilevel optimization problems. The properties of the neural network are analyzed and the conditions for Lyapunov stability, global convergently are presented. Simulation experiments on numerical examples demonstrated to show the applicability and validity of the network.

Keywords: Bilevel programming problem, Delayed-projection neural network, Lyapunov stability, global convergently
Mathematics Subject Classification [2010]: 65k05, 90C26

1 Introduction

Bi-level programming (BLP) is a hierarchical optimization problem in which the constraint region is implicitly determined by another optimization problem. In this paper, we will consider BLP as follows:

$$\begin{aligned} (UP) \quad & \min_{x,y} \; F(x,y) \\ s.t \quad & H(x,y) \le 0, \\ (LP) \quad & y \in \left\{ \begin{array}{ll} \min_y & f(x,y) \\ s.t & a \le x \le b \\ & c \le y \le d \end{array} \right. . \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $F : \mathbb{R}^{n\times m} \to \mathbb{R}^1$, $f : \mathbb{R}^{n\times m} \to \mathbb{R}^1$ and $H : \mathbb{R}^{n\times m} \to \mathbb{R}^1$ are continuous differentiable functions. The term (UP) is called the upper-level problem and (LP) is called the lower-level problem. This problem arises in numerous areas of applications such as resource allocation, nance budget, price control, transaction network. In modern science and technology, real time solutions of optimization problems are desired. However, usual numerical methods may not be efficient in such occasions, specially in large scale problems, because of stringent requirements on computing time. The most important advantages of the neural networks are massively parallel processing and fast convergence. According to these points, in past two decades, applications of neural networks have been

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widely investigated [1]-[4]. It is well known that, in the hardware implementation of neural networks, time delays inevitably occur in the signal communication among the neurons. This may lead to the oscillation phenomenon or instability of networks. Therefore, the study on the dynamical behavior of the delayed neural network is attractive both in theory and in practice[5]. In this paper a specific delayed- neural network model based on globally projected dynamical system, is proposed in order to solve problem 1.

2 Neural network for BLP

An appealing way to deal with general BLP is the so called Karush-Kuhn-Tucker (KKT) approach where the lower level constraint, that y is a global minimizer of the program LP, is firstly relaxed to the condition that y is a local minimizer of LP [6]. The latter condition is then replaced by the KKT-conditions.

$$\nabla_y f(x,y) = 0$$

Let $\Omega = \{(x,y) \in \mathbb{R}^{n\times m} | a \le x \le b, c \le y \le d\}$. So the problem 1 will reduce to the following one-level problem:

$$\begin{aligned} &\min_{x,y} F(x,y) \\ s.t.\quad & H(x,y) \le 0, \\ & g(x,y) = \nabla_y f(x,y) = 0, \\ & (x,y) \in \Omega \end{aligned} \tag{2}$$

Definition 2.1. [3] Let C be a closed convex set in $\mathbb{R}^n$. Then for each $\boldsymbol{x} \in \mathbb{R}^n$, there exists a unique point $\boldsymbol{y} \in C$ such that $\|\boldsymbol{x}-\boldsymbol{y}\| \le \|\boldsymbol{x}-\boldsymbol{z}\|$, $\forall \boldsymbol{z} \in C$. The projection of $\boldsymbol{x}$ on the set C with respect to Euclidean norm is $\boldsymbol{y} = P_C(\boldsymbol{x}) = \arg\min_{z\in C} \|\boldsymbol{x}-\boldsymbol{z}\|$.

By the well-known projection theorem [1], it follows that is a solution of 2 if and only if it satises the following projection equation:

$$u^* = p_{s_0}(u^* - \alpha G(u)). \tag{3}$$

where $s_0 = \{u = ((x,y), z_1, z_2)^T | (x,y) \in \Omega, z_1 \ge 0\}$, and

$$G(u) = \begin{pmatrix} \nabla_{x,y} F(x,y) + \nabla_{x,y} H(x,y)^T z_1 + \nabla_{x,y} g(x,y)^T z_2 \\ -H(x,y) \\ -g(x,y) \end{pmatrix}.$$

Let $w = (x,y)^T$. Based on 3 and delayed methods, we proposed the following delayed neural network for solving BLP:

$$\begin{cases} \frac{du}{dt} = \frac{d}{dt}\begin{pmatrix} w \\ z_1 \\ z_2 \end{pmatrix} = \\ \qquad \lambda \begin{pmatrix} P_\Omega(w - (\nabla_w F(w) + \nabla_w H(w)^T z_1 + \nabla_w g(w)^T z_2)) - 2w(t) + w(t-\tau) \\ (z_1 - H(w))^+ - 2z_1(t) + z_1(t-\tau) \\ g(w) - z_2(t) + z_2(t-\tau) \end{pmatrix}, \\ u(t) = \phi(t) \qquad t \in [t_0 - \tau, t_0] \end{cases} \tag{4}$$

Where $\tau \geq 0$ denotes the time delay, $\phi(t) \in C([t_0 - \tau, t_0], \mathbb{R}^{n+m})$. Also, $k^+ = (k_1^+, k_2^+, ..., k_{n+m}^+)$, $(k_i)^+ = \max\{0, k_i\}$ and

$$P_\Omega(x_i) = \begin{cases} a_i & x_i > a_i \\ x_i & a_i \leq x_i \leq b_i \\ b_i & x_i < b_i \end{cases}.$$

3 Stability and convergence analysis

In this section, we state the global convergence and Lyapunov stability of the proposed delayed neural network model 4 for solving problem1.

Definition 3.1. [1] A continuous-time neural network is said to be globally convergent, if for any given initial point, the corresponding trajectory of the related dynamic system converges to an equilibrium point.

Definition 3.2. [1] The equilibrium point $\boldsymbol{u}^*$ of the delayed projection neural network is Lyapunov stable if, for each $\epsilon > 0$, there is $\delta > 0$ such that if $\|\boldsymbol{u}_0 - \boldsymbol{u}^*\| < \delta$, then $\|\boldsymbol{u}(t) - \boldsymbol{u}^*\| < \epsilon$, for $t \geq t_0$.

Lemma 3.3. *For any initial point* $\boldsymbol{u}_0 = (\boldsymbol{w}(t_0)^T, \boldsymbol{z}_1(t_0)^T, \boldsymbol{z}_2(t_0)^T)$ *there exists a unique continuous solution for proposed neural network model. Morover, if* $\boldsymbol{u}_0 \in \boldsymbol{s}_0$ *then* $\boldsymbol{u}(t) \in \boldsymbol{s}_0$.

Theorem 3.4. *If* $\nabla_w^2 \boldsymbol{F}(w) + \nabla_w^2 \boldsymbol{H}(w)^T z_1 + \nabla_w^2 \boldsymbol{g}(w)^T z_2$ *be positive definite on* $\boldsymbol{s}_0$ *then the delayed projection neural network 4 is stable in the Lyapunov sense and globally convergent to a stationary point* $\boldsymbol{u}^* = ((\boldsymbol{w}^*)^T, (\boldsymbol{z}_1^*)^T, (\boldsymbol{z}_2^*)^T)$*, where* $\boldsymbol{w}^*$ *is the solution of BLP.*

4 Illustrative example

Example 4.1. Consider the following bilevel optimization problem:

$$\begin{aligned} \min_{x,y} (x_1 - 30)^2 + (x_2 - 20)^2 - 20y_1 + 20y_2 \\ s.t. \quad x_1 + 2x_2 - 30 \leq 0, \\ -x_1 - x_2 + 20 \leq 0, \\ 0 \leq x_1, x_2 \leq 15, \\ \min_y (x_1 - y_1)^2 + (x_2 - y_2)^2 \\ 0 \leq y_1, y_2 \leq 15. \end{aligned}$$

Solution. This problem has a theoretical optimal solution $\boldsymbol{w}^* = (\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*) = (15, 7.5, 15, 7.5)$. All simulation results show that the delayed-projection neural network 4, is Lyapanov stable at $\boldsymbol{w}^*$. Figure 1 shows the trajectories of proposed model 4, with the five initial function $\phi_k(t) = (\sin(kt), kt, -\cos(kt), kt)^T, \quad k = 1, ..., 5$ and $\tau = 1$. According to the simulation result, all the trajectories are convergent to $\boldsymbol{w}^*$.

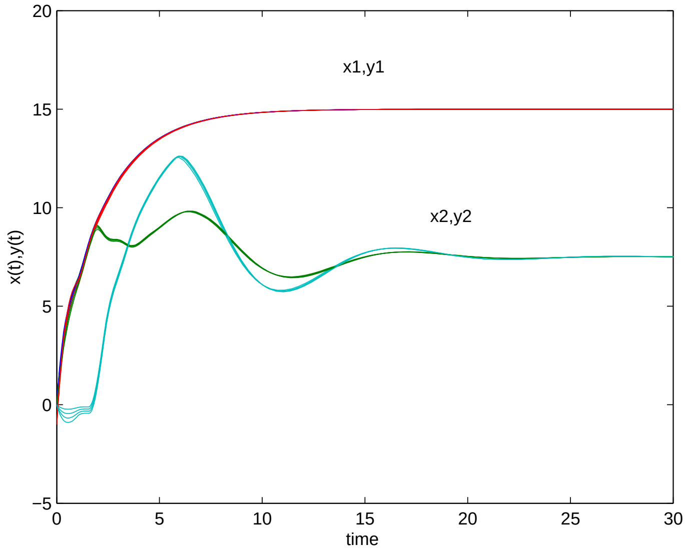

Figure 1: The transient behavior of the neural network mode (4) with five various initial functions.

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A Genetic Algorithm For Finding The Semi-Obnoxious (k,l)-core Of A Network

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Abstract

Let $G = (V, E)$ be a graph, with $\mid V \mid = n$. A (k, l)-core of G is a subtree with at most k leaves and with a diameter of at most l which the sum of the distances from all vertices to this subtree is minimized. In this paper, we present a genetic algorithm for finding the (k, l)-core of a graph with pos/neg weight.

Keywords: Core, Genetic algorithm, Median subtree, Semi-obnoxious
Mathematics Subject Classification [2010]: 90B90, 90B06

1 Introduction

The core of a graph is defined in [6] as a path in the graph minimizing the sum of the distances of all vertices of the graph from the path. This problem is extended to finding a core of specied size l on tree networks in [2, 5, 7]. Peng et al. [8] considered problem with a constraint on numbers of leaves and presented an algorithm for constructing a k-tree core on trees which has time complexity of O(kn). After that, problem is extended to finding a subtree of tree with at most k leaves and with a diameter of at most l so that the sum of the weighted distances from all vertices to the subtree is minimized. This subtree is called a (k, l)-core of tree. Becker et al. [3] presented an efficient algorithm for finding a (k, l)-core of a tree with time complexity of $O(n^2 logn)$.
If some of the vertices have positive weights and some negative weights the problem is referred to as the semi-obnoxious location problem. Burkard and Krarup [4] showed that the positive or negative (for simplicity we write pos/neg) 1-median, problem on a cactus can be solved in linear time.
Many genetic algorithms are applied to solve some location problems such as median problem and hub location problem[1].
In this paper, we consider (k, l)-core of G that is a subtree with at most k leaves and with a diameter of at most l which the sum of the distances from all vertices to this subtree is minimized. Then present a genetic algorithm for finding the (k, l)-core of a graph with pos/neg weight.

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2 Problem formulation

Let $T = (V, E)$ be a tree, that $|V| = n$, $w(v_i)$ be the weight of vertex $v_i \in V$ (for simplicity we write w_i) and $a(i, j)$ be the length of edge (i, j). Then $w(T) = \sum_{i=1}^{n} w_i$ is the weight of the tree T. Also let $d(v_i, v_j)$ be the length of path from v_i to v_j, then the length of shortest path between path p and vertex v is given by

$$d(p, v) = min_{u \in p}\ d(u, v).$$

The diameter d_T of T is the maximum distance between two vertices of T and any path whose length equals d_T is a diameter path.
Suppose $T' = (V', E')$ be a subtree of T. Let $d(v, T')$ be the minimum distance from $v \notin V'$ to a vertex in T'. We show the sum of distances from T' to all the vertices that they are not in V' by $d(T')$, that is called DISTSUM of T'.
A (k, l)-core of a tree is a subtree with at most k leaves and with a diameter of at most l minimizing the sum of the distances of all vertices of the tree to this subtree. In other words, the following function is minimized:

$$F(T') = \sum_{v_i \notin V'} w(v_i) d(v_i, T')$$

3 The genetic algorithm

In this section we present a genetic algorithm for finding the (k, l)-core of a graph with pos/neg weight. In the GAs each chromosome corresponds to a solution for the problem. At first, an initial population of solutions is generated. Then, by using crossover and mutation operators, new chromosomes are produced. If the new member there is not in population and its fitness value is better than the worst fitness value in the population, then the worst member is replaced by the new one.
Genetic algorithm
Input: A graph G with pos/neg weight.
Output: A (k, l)-core S^* of G and its DISTSUM f_{best}.

Initial(T).
For $t := 0$ to $t := 2n$ **do** the following:
Select $T_1 \in S$ with minimum f and $T_2 \in S$ randomly.
Crossover(T_1, T_2).
Replace($T_{crossover}, S$**)**.
Select randomly a subtree $T_m \in S$.
Mutation(T_m**)**.
Replace($T_{mutaion}, S$**)**.
Find a subtree T_f in S with minimum f.
Set $T_{best} := T_f$, $f_{best} := f(T_f)$, t:=t+1.
End for

End
Procedure Initial(T)
Input: A graph G with pos/neg weight.
Output:A set S of subtrees of G.

Set $S := \emptyset$.
For each vertex $v \in V$ with maximum weight that is not already selected and The size of the population **do** the following:
$f_{best} := f(v)$, $S(v) := v$.
P (v) := v, EndP := v.
While $Adj1 := \{u \in V | u \notin P(v), u\ is\ adjacent\ to\ EndP\}$ is not empty and $L(p) \leq L$ do the following
Select a vertex $u \in Adj1$ with maximum weight.
Add u to P (v) and EndP := u.
If $f(p) < f_{best}$ then $f_{best} := f(p)$, $S(v) := P(v)$.
End while
While $Adj2 := \{u \in V | u \notin P(v), u\ is\ adjacent\ to\ internal\ vertices\ of\ P\}$ is not empty and $K \leq k$ **do** the following
Select a vertex $u \in Adj2$ with maximum weight.
Add u to P (v) and K:=K+1.
If $f(p) < f_{best}$ then $f_{best} := f(p)$, $S(v) := P(v)$.
End while
$S := S \cup \{S(v)\}$.
End for
End
Procedure Crossover(T_1, T_2)
Input: Two subtrees T_1, T_2 of G.
Output: A subtree $T_{crossover}$ of G.
Set the common edges in T_1, T_2 to T_{new}.
For each two vertices u and v that cause a disconnection in T_{new}
do the following:
Find the shortest path P_{uv} from u to v.
If by adding P_{uv} to T_{new}, T_{new} does not contain any cycle insert it to T_{new}.
Else if by adding path from u to v in T_1 to T_{new}, T_{new} does not contain any cycle insert this path to T_{new}.
Else delete v and all vertices after it in T_{new} that make a connected subpath.
$f_{crossover} := f(T_{new})$,$T_{crossover} := T_{new}$.
Let x be the end vertex of T_{new}.
While $Adj3 := \{y \in T_1 \cup T_2 \setminus T_{new} y\ is\ adjacent\ to\ x\}$ is not empty and $L(T_{new}) \leq L$ **do** the following:
Select vertex $y \in Adj3$ with maximum weight.
Add y to the end of T_{new}.
If $f(T_{new}) < f_{crossover}$ then $f_{crossover} := f(T_{new})$, $T_{crossover} := T_{new}$.
Set x = y.
End while
While $Adj4 := \{t \in V | u \notin T_{new}, t\ is\ adjacent\ to\ internal\ vertices\ of\ T_{new}\}$ is not empty and $K \leq k$ **do** the following
Select a vertex $t \in Adj4$ with maximum weight.
Add t to T_{new} and K:=K+1.
If $f(T_{new}) < f_{crossover}$ then $f_{crossover} := f(T_{new})$, $T_{crossover} := T_{new}$.
End while

End
Procedure Mutation(T_m)
Input: A subtree T_m of G.
Output: A subtree $T_{mutation}$ of G.
For each leaf v of T_m **do** the following:
Set $Adj5 := \{y \in G \setminus T_m, y\ is\ adjacent\ to\ fother\ of\ v\}$.
Select a vertex $y \in Adj5$ with maximum weight.
Replace v by y.
End For
Set $T_{mutation} := T_m$
End
Procedure Replace(T_{new}, S)
If T_{new} is not in the population and $f(T_{new}) < f_{worst}$ then do the following:
Replace the worst member by T_{new}.
Update the worst member of the population and its tness value, f_{worst}.
If $f(T_{new}) < f_{best}$, set $f_{best} = f(T_{new})$.
End

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A Newton-type method for multiobjective optimization problems

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Abstract

In this paper, we propose a Newton-type algorithm for nonconvex multiobjective optimization problems. The presented terminates, when the termination conditions are satisfied. Convergence of the algorithm is considered.

Keywords: Multiobjective optimization, Newton-type method, Pareto optimality, Critical point.
Mathematics Subject Classification [2010]: 90C29, 49M15

1 Introduction

In multiobjective optimization, several conflicting objectives have to be minimized, simultaneously. Generally, no unique solution exists but a set of mathematically equally good solutions can be identified, by using the concept of Pareto optimality. For solving large scale nonlinear multiobjective optimization problem, iterative methods are very effective. Recently, some iterative approaches for solving multiobjective optimization problems were developed [1, 4]. Newton's method for single objective optimization problems was extended to multiobjective optimization problems by Fliege et al. [1], which uses convexity assumption. Now in this paper, we present a Newton-type algorithm that works for nonconvex functions also under suitable assumptions, denote its global convergence. The necessary assumption is that the objective functions are twice continuously differentiable but no other parameters or ordering of the functions are needed.

2 Basic Definitions

In this paper, we consider the following unconstrained nonconvex multiobjective optimization problem

$$\begin{aligned} \min \quad & F(x) = (F_1(x), \dots, F_m(x)) \\ s.t. \quad & x \in U \subset \mathbb{R}^n \end{aligned}$$

where $F = (F_1, \dots, F_m)^T : U \to R^m$ is continuous differentiable and $U \subset \mathbb{R}^n$ is the domain of F which is assumed to be open. Let $I = \{1, 2, \dots, m\}$, for any $u, v \in \mathbb{R}^m$, we define

$$u \leq v \Longleftrightarrow v - u \in \mathbb{R}^m_+ \Longleftrightarrow v_j - u_j \geq 0, j \in I,$$

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$$u < v \Longleftrightarrow v - u \in \mathbb{R}^m_{++} \Longleftrightarrow v_j - u_j > 0, j \in I,$$

where $\mathbb{R}^m_+ := \{y \in \mathbb{R}^m | y \geq 0\}$ and $\mathbb{R}^m_{++} := \{y \in \mathbb{R}^m | y > 0\}$.

Definition 2.1. A feasible solution $x^* \in U$ is local Pareto optimal of F if and only if there exists a neighborhood $V \subset U$ such that there does not exist $x \in V$ with $F(x) \leq F(x^*)$, and $F(x) \neq F(x^*)$.

Definition 2.2. A point $x^* \in U$ is critical (or stationary) for F , if

$$R(\bigtriangledown F(x^*)) \cap (-\mathbb{R}^m_{++}) = \emptyset, \tag{1}$$

where $R(\bigtriangledown F(x))$ denotes the range or image space of the gradient of the continuously differentiable function F at x.

Note that for $m = 1$, relation (1) reduces to the "gradient-equal-zero" condition. Clearly, if x^* is critical for F, then for all $s \in \mathbb{R}^n$ there exists $j_0 \in I$ such that

$$\bigtriangledown F_{j_0}(x^*)^T s \geq 0.$$

Note that if $x \in U$ is noncritical, then there exists $s \in \mathbb{R}^n$ such that $\bigtriangledown F_j(x)^T s < 0$ for all $j = 1, \ldots, m$. In this case, since F is continuously differentiable, we have:

$$\lim_{\alpha \to 0} \frac{F_j(x + \alpha s) - F_j(x)}{\alpha} = \bigtriangledown F_j(x)^T s < 0, \qquad j = 1, \ldots, m. \tag{2}$$

So s is a descent direction for F at x, i. e., there exists $\alpha_0 > 0$ such that

$$F(x + \alpha s) < F(x) \quad for\, all \quad \alpha \in (0, \alpha_0].$$

3 Main Results

A descent direction s is a direction that reduces every objective function value. Relation (2) implies that $s \in \mathbb{R}^n$ is a descent direction for F at x if and only if

$$\bigtriangledown F_j(x)^T s < 0, \qquad \forall j \in I.$$

Lemma 3.1. *$x^* \in U$ is critical if and only if either one of the following two conditions are satisfied:*
(i) There does not exist $s \in \mathbb{R}^n$ such that for all $j \in I$

$$\bigtriangledown F_j(x^*)^T s < 0.$$

(ii) In the special case, there also exists at least one $j_0 \in I$ such that

$$\bigtriangledown F_{j_0}(x^*) = 0.$$

Proof. The proof can be found in [3]. □

We now proceed by defining a Newton direction for the multiobjective problem under consideration. For $x \in U$, given sufficient small $\epsilon > 0$, we define $s(x)$, the Newton direction at x, as the optimal solution of

$$SP_\epsilon(x) \begin{cases} \min & t \\ \text{s.t.} & \nabla F_j(x)^T s + \frac{1}{2} s^T \nabla^2 F_j(x) s \leq t, \quad j \in I \\ & \|s\| \leq 1, \quad t \leq -\epsilon, \quad (t,s) \in \mathbb{R} \times \mathbb{R}^n \end{cases}$$

The constraint $\|s\| \leq 1$ is used to improve performance as $\|s\| \leq 1$ eliminates the possible case $\|s\| \to \infty$.
Let $\theta(x)$ be the optimal objective function value for subproblem $SP_\epsilon(x)$.

Lemma 3.2. *Given $x \in U$. For a sufficient small positive scalar ϵ, if the feasible set of $SP_\epsilon(x)$ is empty, then x is a critical point for F.*

Proof. Assume that the feasible set is empty. We show that does not exist a descent direction. By contradiction, assume that there exists a direction $\overline{s} \in \mathbb{R}^n$, such that

$$\nabla F_j(x)^T \overline{s} < 0, \quad \forall j \in I.$$

The above inequality implies that there is a positive scalar $\overline{\alpha}$ such that for any $\alpha \in (0, \overline{\alpha}]$,

$$\alpha \nabla F_j(x)^T \overline{s} + \frac{1}{2} \alpha^2 \overline{s}^T \nabla^2 F_j(x) \overline{s} < 0.$$

If for any $\alpha \in (0, \overline{\alpha}]$, we define $-\epsilon = \alpha \nabla F_j(x)^T \overline{s} + \frac{1}{2}\alpha^2 \overline{s}^T \nabla^2 F_j(x) \overline{s}$, then we can see that $\alpha \overline{s}$ is feasible to $SP_\epsilon(x)$. This contradicts that the feasible set of $SP_\epsilon(x)$ is empty. □

Lemma 3.3. *If point x is noncritical then $\theta(x) < 0$.*

Proof. See [1]. □

The following theorem will be a criterion for accepting a step in the multiobjective Newton-type direction.

Theorem 3.4. *If $x \in U$ is a noncritical point for F, then for any $0 < \beta < 1$ there exists $\overline{\alpha} \in (0, 1]$ such that*

$$x + \alpha s(x) \in U \quad \text{and} \quad F_j(x + \alpha s(x)) \leq F_j(x) + \beta \alpha \theta(x)$$

holds for all $\alpha \in [0, \overline{\alpha}]$ and $j \in \{1, \ldots, m\}$.

Proof. Since U is an open set and $x \in U$, there exists $0 < \alpha_1 \leq 1$ such that $x + \alpha s(x) \in U$ for $\alpha \in [0, \alpha_1]$. Therefore, for $\alpha \in [0, \alpha_1]$ and $j = 1, \ldots, m$, we have,

$$F_j(x + \alpha s(x)) = F_j(x) + \alpha \nabla F_j(x)^T s(x) + \frac{\alpha^2}{2} s(x)^T \nabla^2 F_j(x) s(x) + o_j(\alpha),$$

where $\lim_{\alpha \to 0^+} o_j(\alpha)/\alpha = 0$. As $\alpha^2 \leq \alpha$, for $\alpha \in [0, \alpha_1]$ and $j = 1, \ldots, m$, we conclude that:

$$\begin{aligned} F_j(x + \alpha s(x)) &\leq F_j(x) + \alpha \nabla F_j(x)^T s(x) + \frac{\alpha}{2} s(x)^T \nabla^2 F_j(x) s(x) + o_j(\alpha) \\ &\leq F_j(x) + \alpha \beta \theta(x) + \alpha[(1 - \beta)\theta(x) + \frac{o_j(\alpha)}{\alpha}]. \end{aligned}$$

Now observe that, since x is noncritical, $\theta(x) < 0$ (Lemma 3.3) and so, for $\alpha \in [0, \alpha_1]$ small enough, the last term at the right hand side of the above equations is non-positive. □

The method proposed in [1] uses convexity assumption, while the following algorithm works for non-convex functions, too.

Step 1: (Initialization) Choose $x_0 \in U$and $0 < \beta < 1$. Give a sufficient small positive scalar ϵ. Set $k := 0$.

Step 2: (Main loop) Solve the direction search program $SP_\epsilon(x_k)$ to obtain $s(x_k)$ and $\theta(x_k)$. Terminate, if either one of the following two conditions are satisfied:
(i) the problem is infeasible,
(ii) $\bigtriangledown F_j(x_k)^T s(x_k) \geq 0$ for some $j \in \{1, \ldots, m\}$ and problem is feasible.
Else, proceed to the line search, Go to step 3.

Step 3: (Line Search) Choose α_k as the largest α such that

$$x_k + \alpha s(x_k) \in U,$$

$$F_j(x_k + \alpha s(x_k)) \leq F_j(x_k) + \alpha\beta\theta(x_k), \quad j = 1, \ldots, m.$$

Step 4: (Update) Define $x_{k+1} = x_k + \alpha_k s(x_k)$ and set $k := k + 1$. Go to Step 2.

We denote the global convergence of the above algorithm. First we make some basic assumptions.
A1. Assume that the level set $L_0 = \{x \in \mathbb{R}^n : F(x) \leq F(x_0)\}$ is bounded.
A2. Assume that for sufficient large k, the step-length $\alpha_k = 1$ is accepted.

Theorem 3.5. *Denote by $\{x_k\}_k$ a sequence generated by the above algorithm. Suppose that there is a constant c such that $\| \bigtriangledown^2 F_j(x)\| \leq c$, for any $x \in L_0$ and $j \in I$. Under our assumptions A1 and A2, every accumulation point of the sequence $\{x_k\}$ is critical for F.*

Proof. The proof is similar to that of Theorem 5 in [3]. □

4 Conclusion

We proposed a Newton-type method for computing the critical points of smooth multi-objective optimization problems under non-convexity. Under suitable assumptions, global convergence established.

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A three-stage Data Envelopment Analysis model on fuzzy data

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Abstract

Data envelopment analysis (DEA) is a methodology for measuring the relative efficiencies of a set of decision making units (DMUs) that use inputs to produce multiple outputs. The conventional DEA, requires crisp input and output data, but the observed values of the input and output data in real word applications are sometimes imprecise. This paper proposes a methodology for a fuzzy three-stage DEA model, where input-output data are treated as fuzzy numbers. A pair of two-level mathematical programs is formulated to calculate the upper bound and lower bound of the fuzzy efficiency score. Then can be transform this pair of two-level mathematical programs into a pair of conventional mathematical programs to calculate the bounds of the fuzzy efficiency score.

Keywords: Data Envelopment Analysis, Two-stage, Decision Making Unit, Fuzzy Data
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Suppose the operation of a DMU can be divided into three stages. The first process applies input x_{ij} $(i = 1, \ldots, m)$ to produce intermediate products z^1_{tj} $(t = 1, \ldots, G)$ and all of this intermediate products in the second process produce another intermediate products denote by z^2_{kj} $(t = 1, \ldots, F)$, also in the third process this intermediate products applies to produce outputs y_{rj} $(r = 1, \ldots, s)$. The three-stage model to calculating the efficiency

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of DMU_p in set of n DMUs is as follows:

$$\begin{array}{rll} E_k = Max & \sum_{r=1}^{s} u_r y_{rp} / \sum_{r=1}^{m} v_i x_{ip} & \\ \text{s.t.} & \sum_{t=1}^{G} w_t^1 z_{tj}^1 / \sum_{i=1}^{m} v_i x_{ij} \leq 1 & \forall j \\ & \sum_{k=1}^{F} w_k^2 z_{kj}^2 / \sum_{t=1}^{G} w_t^1 z_{tj}^1 \leq 1 & \forall j \\ & \sum_{r=1}^{s} u_r y_{rj} / \sum_{k=1}^{F} w_k^2 z_{kj}^2 \leq 1 & \forall j \\ & u_r, v_i, w_t^1, w_k^2 \geq \varepsilon & \end{array} \tag{1}$$

This model is extension of conventional CCR model[1].

2 Main results

Denote $\tilde{x}_{ij}$, $\tilde{z}_{tj}^1$, $\tilde{z}_{kj}^2$ and $\tilde{y}_{rj}$ as the fuzzy counterparts of x_{ij}, z_{tj}^1, z_{kj}^2 and y_{rj} respectively. Model (1) for fuzzy observations can be formulate as[2]:

$$\begin{array}{rll} \tilde{E}_k = Max & \sum_{r=1}^{s} u_r \tilde{y}_{rp} / \sum_{r=1}^{m} v_i \tilde{x}_{ip} & \\ \text{s.t.} & \sum_{t=1}^{G} w_t^1 \tilde{z}_{tj}^1 / \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 1 & \forall j \\ & \sum_{k=1}^{F} w_k^2 \tilde{z}_{kj}^2 / \sum_{t=1}^{G} w_t^1 \tilde{z}_{tj}^1 \leq 1 & \forall j \\ & \sum_{r=1}^{s} u_r \tilde{y}_{rj} / \sum_{k=1}^{F} w_k^2 \tilde{z}_{kj}^2 \leq 1 & \forall j \\ & u_r, v_i, w_t^1, w_k^2 \geq \varepsilon & \end{array} \tag{2}$$

Assume $(x_{ij})_\alpha = [(x_{ij})_\alpha^L, (x_{ij})_\alpha^U]$, $(z_{tj}^1)_\alpha = [(z_{tj}^1)_\alpha^L, (z_{tj}^1)_\alpha^U]$, $(z_{kj}^2)_\alpha = [(z_{kj}^2)_\alpha^L, (z_{kj}^2)_\alpha^U]$ and $(y_{rj})_\alpha = [(y_{rj})_\alpha^L, (y_{rj})_\alpha^U]$ as the $\alpha-$cuts of $\tilde{x}_{ij}$, $\tilde{z}_{tj}^1$, $\tilde{z}_{kj}^2$ and $\tilde{y}_{rj}$, respectively. To find the membership function $\mu_{\tilde{E}_k(e)}$, it is suffices to find the lower and upper bounds of the $\alpha-$cuts of $\tilde{E}_k(e)$, $(\tilde{E}_k)_\alpha = [(E_k)_\alpha^L, (E_k)_\alpha^U]$, where

$$(E_k)_\alpha^L = \min\{e \mid \mu_{\tilde{E}_k}(e) \geq \alpha\} \tag{3}$$

$$(E_k)_\alpha^U = \max\{e \mid \mu_{\tilde{E}_k}(e) \geq \alpha\} \tag{4}$$

Above expression can be written as follows:

$$\begin{array}{rl} (E_k)_\alpha^U = \text{Max} & E_k(x, z^1, z^2, y) \\ \text{s.t.} & (x_{ij})_\alpha^L \leq x_{ij} \leq (x_{ij})_\alpha^U \\ & (z_{tj}^1)_\alpha^L \leq z_{tj}^1 \leq (z_{tj}^1)_\alpha^U \\ & (z_{kj}^2)_\alpha^L \leq z_{kj}^2 \leq (z_{kj}^2)_\alpha^U \\ & (y_{rj})_\alpha^L \leq y_{rj} \leq (y_{rj})_\alpha^U \\ & \forall i, t, k, r, j \end{array} \tag{5}$$

$$
\begin{aligned}
(E_k)^L_\alpha = \text{Min} \quad & E_k(x, z^1, z^2, y) \\
\text{s.t.} \quad & (x_{ij})^L_\alpha \leq x_{ij} \leq (x_{ij})^U_\alpha \\
& (z^1_{tj})^L_\alpha \leq z^1_{tj} \leq (z^1_{tj})^U_\alpha \\
& (z^2_{kj})^L_\alpha \leq z^2_{kj} \leq (z^2_{kj})^U_\alpha \\
& (y_{rj})^L_\alpha \leq y_{rj} \leq (y_{rj})^U_\alpha \\
& \forall i, t, k, r, j
\end{aligned} \tag{6}
$$

where $E_k(x, z^1, z^2, y)$ was defined in Model (2). Models (5) and (6) are two-level programs. Two-level programs are used for modeling, and they must be converted to a one-level program to can be solved. This model can be converted to one-level mathematical programs as follows:

$$
\begin{aligned}
(E_k)^U_\alpha = \text{Max} \quad & \sum_{r=1}^{s} u_r (y_{rp})^U_\alpha / \sum_{i=1}^{m} v_i (x_{ip})^L_\alpha \\
\text{s.t.} \quad & \sum_{t=1}^{G} \hat{z}^1_{tp} / \sum_{i=1}^{m} v_i (x_{ip})^L_\alpha \leq 1 \\
& \sum_{t=1}^{G} \hat{z}^1_{tj} / \sum_{i=1}^{m} v_i (x_{ij})^U_\alpha \leq 1 \qquad \forall j, j \neq p \\
& \sum_{k=1}^{F} \hat{z}^2_{kj} / \sum_{t=1}^{G} \hat{z}^1_{tj} \leq 1 \qquad \forall j \\
& \sum_{r=1}^{s} u_r (y_{rp})^U_\alpha / \sum_{k=1}^{F} \hat{z}^2_{kp} \leq 1 \\
& \sum_{r=1}^{s} u_r (y_{rj})^L_\alpha / \sum_{k=1}^{F} \hat{z}^2_{kj} \leq 1 \qquad \forall j, j \neq p \\
& w^1_t (z^1_{tj})^L_\alpha \leq \hat{z}^1_{tj} \leq w^1_t (z^1_{tj})^U_\alpha \\
& w^2_k (z^2_{kj})^L_\alpha \leq \hat{z}^2_{kj} \leq w^2_k (z^2_{kj})^U_\alpha \\
& v_i, w^1_t, w^2_k, u_r \geq \varepsilon
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
(E_k)^L_\alpha = \text{Min} \quad & \theta \\
\text{s.t.} \quad & \theta (x_{ip})^U_\alpha - [\lambda^1_p (x_{ip})^U_\alpha + \sum_{j=1, j\neq p}^{n} \lambda^1_j (x_{ij})^L_\alpha] \geq 0 \qquad \forall i \\
& \sum_{j=1}^{n} \lambda^1_j z^1_{tj} - \sum_{j=1}^{n} \lambda^2_j z^1_{tj} \geq 0 \qquad \forall t \\
& \sum_{j=1}^{n} \lambda^2_j z^2_{kj} - \sum_{j=1}^{n} \lambda^3_j z^2_{kj} \geq 0 \qquad \forall k \\
& \lambda^3_p (y_{rp})^L_\alpha + \sum_{j=1, j\neq p}^{n} \lambda^3_j (y_{rj})^U_\alpha \geq (y_{rp})^U_\alpha \qquad \forall r \\
& (z^1_{tj})^L_\alpha \leq z^1_{tj} \leq (z^1_{tj})^U_\alpha \qquad \forall t, j \\
& (z^2_{kj})^L_\alpha \leq z^2_{kj} \leq (z^2_{kj})^U_\alpha \qquad \forall k, j \\
& \lambda^1, \lambda^2, \lambda^3 \geq 0
\end{aligned} \tag{8}
$$

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An Efficient Computational Algebraic Method for Convex Polynomial Optimization

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Abstract

In this paper, we state an algorithm to solve constrained polynomial optimization problems using Computational Algebra methods. The efficiency of our algorithm relies on the intensive properties of Gröbner basis for zero dimensional ideals which carries the problem into Linear Algebra. In order to use Gröbner basis, we assign the KKT ideal to the given optimization problem whose affine variety contains all feasible points. Then, we state an efficient criterion to determine the optimum value.

Keywords: Constrained optimization, Gröbner basis, KKT conditions
Mathematics Subject Classification [2010]: 13P10, 13P25

1 Introduction

Mathematical optimization has a wide broad of applications for instance in mathematics, computer science, economics, management science, model predictive control together with lots of methods an algorithms to solve optimization problems. On the other hand, Computer Algebra contains some novel computational tools such as Gröbner basis to solve lots of problems dealing with algebraic equations [1]. In this paper we use intensive properties of Gröbner basis to solve constrained optimization problems. So we continue to recall the necessary concepts of polynomial rings and some properties of Gröbner basis which are important in this text.

Let $\mathbb{K}$ be a field and $\mathbf{x} = x_1, \ldots, x_n$ be n (algebraically independent) variables. Each power product $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called a monomial where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. We can sort the set of all monomials over $\mathbb{K}$ by special types of total orderings so called monomial orderings, recalled in the following definition.

Definition 1.1. The total ordering $\prec$ on the set of monomials is called a monomial ordering whenever $\prec$ is well-ordering and for each monomials $\mathbf{x}^\alpha, \mathbf{x}^\beta$ and $\mathbf{x}^\gamma$ we have $\mathbf{x}^\alpha \prec \mathbf{x}^\beta \Rightarrow \mathbf{x}^\gamma\mathbf{x}^\alpha \prec \mathbf{x}^\gamma\mathbf{x}^\beta$

Among the monomial orderings, we point to pure and graded reverse lexicographic orderings denoted by $\prec_{lex}$ and $\prec_{grevlex}$ as follows: assuming $x_n \prec \cdots \prec x_1$, we say that $\mathbf{x}^\alpha \prec_{lex} \mathbf{x}^\beta$ whenever $\alpha_1 = \beta_1, \ldots, \alpha_i = \beta_i$ and $\alpha_{i+1} < \beta_{i+1}$ for an integer $1 \leq i < n$, and

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$\mathbf{x}^\alpha \prec_{grevlex} \mathbf{x}^\beta$ if $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$ breaking ties when there exists an integer $1 \leq i < n$ such that $\alpha_n = \beta_n, \ldots, \alpha_{n-i} = \beta_{n-i}$ and $\alpha_{n-i-1} > \beta_{n-i-1}$.

Each $\mathbb{K}-$linear combination of monomials is called a polynomial on $\mathbf{x}$ over $\mathbb{K}$. The set of all polynomials has the ring structure with usual polynomial addition and multiplication, and is called the polynomial ring on $\mathbf{x}$ over $\mathbb{K}$ and denoted by $\mathbb{K}[\mathbf{x}]$. Let f be a polynomial and $\prec$ be a monomial ordering. The greatest monomial w.r.t. $\prec$ contained in f is called the leading monomial of f, denoted by $\mathrm{LM}(f)$ and the coefficient of $\mathrm{LM}(f)$ is called the leading coefficient of f which is pointed by $\mathrm{LC}(f)$. Further, if F is a set of polynomials, $\mathrm{LM}(F)$ is defined to be $\{\mathrm{LM}(f)|f \in F\}$ and if I is an ideal, $\mathrm{in}(I)$ is the ideal generated by $\mathrm{LM}(I)$ and is called the initial ideal of I. We are now going to remind the concept of Gröbner basis of a polynomial ideal.

Definition 1.2. Let I be a polynomial ideal of $K[\mathbf{x}]$ and $\prec$ be a monomial ordering. The finite set $G \subset I$ is called a Gröbner basis of I if $\mathrm{in}(I) = \langle \mathrm{LM}(G) \rangle$.

Let $\{f_1 = 0, \cdots, f_k = 0\}$ be a polynomial system and $I = \langle f_1, \ldots, f_k \rangle$. We define the affine variety associated to the above system or equivalently to the ideal I to be

$$\mathbf{V}(I) = \mathbf{V}(f_1, \ldots, f_k) = \{\alpha \in \overline{\mathbb{K}}^n | f_1(\alpha) = \cdots = f_k(\alpha) = 0\}$$

where $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$.

Definition 1.3. Let $I \subset \mathbb{K}[\mathbf{x}]$ be an ideal. If there exists no variable u for which $I \cap \mathbb{K}[u] = \{0\}$ then we say that I is a zero dimensional ideal.

For a zero dimensional ideal I, the vector space $\mathbb{K}[\mathbf{x}]/I$ is finite dimensional and its basis can be easily found by reading the leading monomials of a Gröbnr basis. As an important fact, the set $B = \mathbb{M} \setminus \mathrm{in}(I)$ constructs a basis for $\mathbb{K}[\mathbf{x}]/I$ where $\mathbb{M}$ is the set of all monomials in $\mathbb{K}[\mathbf{x}]$. The following theorem describes a novel property of zero dimensional ideals which is also one of the main theorems in this paper. But we need to the following definition.

Definition 1.4. Let I be a zero dimensional polynomial ideal and B be a basis for $\mathbb{K}[\mathbf{x}]/I$. For each polynomial $h \in \mathbb{K}[\mathbf{x}]$ we define the linear transformation φ_h as follows:

$$\begin{array}{rcl} \varphi_h : \dfrac{\mathbb{K}[\mathbf{x}]}{I} & \to & \dfrac{\mathbb{K}[\mathbf{x}]}{I} \\ f + I & \mapsto & hf + I \end{array}$$

Let also M_h be the matrix representation of φ_h with respect to B. Then M_h is called the multiplication matrix of h with respect to I.

Theorem 1.5. *The set of eigenvalues of M_h is the set of possible values of h over $\mathbf{V}(I)$.*

2 The new idea

In this section we go back to the optimization problem. Consider the general form of an optimization problem as

$$\begin{array}{llcccccc} \text{Minimum} & f(\mathbf{x}) & & & & & & \\ \text{subject to} & g_1(\mathbf{x}) & \leq & 0, & \cdots, & g_m(\mathbf{x}) & \leq & 0 \\ & h_1(\mathbf{x}) & = & 0, & \cdots, & h_k(\mathbf{x}) & = & 0 \end{array} \tag{1}$$

where $f(\mathbf{x}), g_i(\mathbf{x})$ and $h_j(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable and b_i and c_j are fixed constants for $i = 1, \ldots, m$ and $j = 1, \ldots, k$. The feasible set of the above problem is

$$\Omega = \{\mathbf{x} \in \mathbb{R}^n \mid g_i(\mathbf{x}) \leq 0,\ h_j(\mathbf{x}) = 0,\ i = 1, \ldots, m, j = 1, \ldots, k\}$$

There are some necessary conditions on the minimizer point $\mathbf{x}^* \in \Omega$ which are known as Karush–Kuh–Tucker (KKT) conditions. Posing these condition on the problem causes to a system of equalities and inequalities whose solution set is a superset for critical points. In addition to KKT conditions, some constraint qualifications like (LICQ) must be checked on the minimizer point. Because of the simplicity, we will assume that LICQ holds for the given proble. We can know state the KKT conditions in the following theorem.

Theorem 2.1. *Let $\mathbf{x}^*$ be a minimizer of the Problem (1) for which LICQ holds. Then there exist μ_i^* for $i = 1, \ldots, m$ and λ_j^* for $j = 1, \ldots, k$ such that*

$$\begin{cases} \nabla f(\mathbf{x}^*) + \sum_{i=1}^m \mu_i^* \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^k \lambda_j^* \nabla h_j(\mathbf{x}^*) = 0 \\ \mu_i^* g_i(\mathbf{x}^*) = h_j(\mathbf{x}^*) = 0 \\ g_i(\mathbf{x}^*) \leq 0 \\ \mu_i^* \geq 0 \end{cases}$$

These necessary conditions let us to compute a superset containing all feasible points. For this purpose, we state the concept of KKT ideal. By the KKT ideal associated to the Problem (1), we mean the ideal:

$$\langle \nabla f(\mathbf{x}) + \sum_{i=1}^m \mu_i \nabla g_i(\mathbf{x}) + \sum_{j=1}^k \lambda_j \nabla h_j(\mathbf{x}), \mu_i g_i(\mathbf{x}), h_j(\mathbf{x}) \mid i = 1, \ldots, m, j = 1, \ldots, k\rangle$$

which is defined in $\mathbb{R}[\mathbf{x}]$. Fotiou et al in [2] have suggested to compute the variety of the KKT ideal by computing the eigenvalues of multiplication matrices associated to each variable and multiplier. In doing so it is needed to compute the cartesian product of eigenvalues as a superset of variety which is usually very larger than feasible set. Then they must check the variety elements one by one to find the feasible points which is of course very time consuming. And finally, they must examine the feasible points to figure out the minimizer. However this process makes a large number of points to be checked. In our idea, it is enough just to compute the eigenvalues of $M_{f(\mathbf{x})}$. It is worth noting that, as we don't import inequalities in the generating set of KKT ideal, some of eigenvalues may be incompatible with inequalities. Thus we need to an appropriate criterion to determine compatible eigenvalues. In doing so, we apply the theory of $\mathrm{G}^2\mathrm{V}$ algorithm [3] which is the most efficient known algorithm to compute Gröbner basis.

Proposition 2.2. *($\mathrm{G}^2\mathrm{V}$-based criterion) Let I be the* KKT *ideal of the optimization problem (1) and G be a Gröbner basis for I. Let also M be the reduced row form of $M_{f(\mathbf{x})} - e \cdot Id$ where e is an eigenvalue and Id is the identity matrix. Suppose now that H is the set of polynomials obtained from the non-zero rows of M. Then e is a compatible eigenvalue if and only if H has a real solution satisfying G equations and $g_i(\mathbf{x}) \leq 0$ for each $i = 1, \ldots, m$.*

Theorem 2.3. *The following algorithm solves the Optimization problem 1:*

Algorithm 1 MIN-VALUE

Require: E; the set of eigenvalues of $M_{f(\mathbf{x})}$
Ensure: The minimum value of $f(x)$

```
F := E;
flag := false;
while not flag do
    e := min(F);
    F := F \ {e};
    if e is compatible then
        flag := true;
    end if
end while
Return(e)
```

Example 2.4. Consider the following optimization problem:

$$\begin{array}{rl} Minimum & y^2x - x^2y \\ subject\ to & 0 \le x \le 10, \quad 0 \le y \le 10, \quad y \le x^2 + x + 1 \end{array}$$

Constructing the KKT ideal, we receive to the following set of real eigenvalues:

$$E = \{-145 - 55\sqrt{37}, -250, 0, -145 + 55\sqrt{37}, 250, 112110\}$$

where the minimum member is $-145 - 55\sqrt{37}$. To test whether this eigenvalue is compatible or not, we use G^2V-based criterion which shows that $729 + (5\sqrt{37} - 14)x^3 \in H$ which implies $x < 0$, and so $-145 - 55\sqrt{37}$ is an incompatible eigenvalue. Therefore we continue with -250. After a compatibility test, we see that the system has the real solution $x = 10, y = 5$ which shows that the eigenvalue is compatible and so the minimum value is -250.

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An optimal algorithm for reverse obnoxious center location problems on graphs

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Abstract

This paper is concerned with a reverse obnoxious center location problem on graphs in which the aim is to modify the edge lengths within a given budget such that a predetermined facility location on the underlying graph becomes as far as possible from the existing customer points under the new edge lengths. We develop a combinatorial algorithm which solves this problem in linear time.

Keywords: Obnoxious center location; Reverse optimization; Combinatorial optimization.
Mathematics Subject Classification [2010]: 90C27, 90B80, 90B85, 90C35

1 Introduction

Location problems are basic optimization models in the area of operation research which have significant applications in practice and theory. These problems ask to find the best locations of facilities on graphs or on real spaces in order to serve the existing clients. The facilities on a system could be either desirable or undesirable (obnoxious), where the aim of an obnoxious facility location model is to establish one or more facilities as far as possible from the clients while fulfill their demands.

In practice, some times we are faced with the situations that we should change some input patameters of the graph in order to improve the existing locations of the facilities. Such problems are mainly categorized into *inverse and reverse location problems* in the literature. Whereas, in an inverse location problem the goal is to modify certain parameters of the problem under investigation at minimum total cost such that predetermined facility locations become optimal, the task of a reverse location model is to improve the given locations by changing some parameters within a given budget constraint. In this case, the improved graph works as efficient as possible.

For the reverse 1-center location problem on an unweighted tree, an algorithm with running time $\mathcal{O}(n^2 \log n)$ was proposed by Zhang et al. [5]. In 2009, Alizadeh et al. [2] considered the inverse 1-center location problem with edge length augmentation on tree networks and developed an $\mathcal{O}(n \log n)$ time combinatorial algorithm using a set of suitably extended AVL-search trees. Later, Alizadeh and Burkard [1] showed that the inverse absolute and vertex 1-center model can be solved in $\mathcal{O}(n^2)$ time provided that no

*Speaker

topology change occurs on the tree. For the general case, they proposed an $\mathcal{O}(n^2 r_v)$ time algorithm, where the parameter r_v is bounded by n. The same authors in 2013 investigated the inverse obnoxious center location problem with edge length modification on graphs [3] and proposed a linear time solution algorithm. Recently, Nguyen [4] proposed an $\mathcal{O}(n^2)$ time method for the uniform cost reverse 1-center location problem on weighted trees. In this paper we investigate the reverse obnoxious center location problems on graphs and provide a linear time solution mehod.

2 Problem statement and basic properties

Let a connected graph $N = (V(N), E(N))$ with vertex set $V(N)$, $|V(N)| = n$, and edge set $E(N)$, $|E(N)| = m$, be given such that every edge $e \in E(N)$ has a positive length $l(e)$. The shortest path distance between two vertices u and v on N with respect to edge lengths l is defined by

$$d_l(u,v) = \min \left\{ \sum_{e \in P(u,v)} l(e) : P(u,v) \text{ is a path between } u \text{ and } v \right\},$$

where $l = \{l(e) \ : \ e \in E(N)\}$. We say that point p lies in N, $p \in N$, if p coincides with a vertex or lies on an edge $e \in E(N)$. In the latter case p is fixed by choosing a parameter λ, $0 < \lambda < 1$, such that $d_l(u,p) = \lambda l(e)$. In a classical obnoxious center location problem the aim is to find an optimal solution for the following model

$$\begin{aligned} \max \ & \min_{\substack{v \in V(N) \\ v \neq p}} d_l(v,p) \\ \text{s.t.} \ & \ p \in V(N), \end{aligned}$$

where we assume that the facility location does not coincide with customer points (Dropping the preceeding assumption, the problem is trivial, since any vertex of graph N in this case is an optimal solution). An optimal solution $p^* \in V(N)$ is called an obnoxious center location on graph N.

We are now going to state the reverse obnoxious center location problem: Consider the underlying graph N with edge lengths l. Let s be a prespecified vertex of N as the existing facility location and a known budget $\mathbf{B} > 0$ is given. The task is to use the budget in order to change the length of some edges such that the minimum of distances between s and customers $v \in V(N)$, $v \neq s$ is maximized under the new edge lengths. We are not allowed to modify the edge lengths arbitrarily, so let $u^+(e)$ and $u^-(e)$ be the maximum permissible amounts for increasing and decreasing $l(e)$, $e \in E(N)$, respectively. Suppose that we incur the nonnegative cost $c^+(e)$ if $l(e)$ is increased by one unit and nonnegative cost $c^-(e)$ if $l(e)$ is decreased by one unit.

Therefore, we can state the *reverse obnoxious center location problem* (ROCLP for short) on the given graph N as follows:

Increase the edge lengths $l(e), e \in E(N)$ by an amount $x(e)$ or decrease it by an amount $y(e)$, such that with $\tilde{l}(e) = l(e) + x(e) - y(e)$, the following three statements hold:

i. The budget constraint

$$\sum_{e\in E(N)} (c^+(e)x(e) + c^-(e)y(e)) \le \mathbf{B}$$

is fulfilled.

ii. The objective value $\min_{v\in V(N),v\neq s} d_{\tilde{l}}(s,v)$ is improved under new lengths $\tilde{l}$.

iii. The increase and decrease amounts lie within given modification bounds, namely:

$$x(e) \le u^+(e) \quad \text{for all} \quad e \in E(N),$$
$$y(e) \le u^-(e) \quad \text{for all} \quad e \in E(N).$$

According to the above problem statement, one can formulate ROCLP on the graph N as the following nonlinear optimization model:

$$\begin{aligned}
\max \quad & \min_{\substack{v\in V(N)\\ v\neq s}} d_{\tilde{l}}(s,v) \\
\text{s.t.} \quad & \sum_{e\in E(N)} (c^+(e)x(e) + c^-(e)y(e)) \le \mathbf{B}, \\
& \tilde{l}(e) - x(e) + y(e) = l(e) \quad \forall\, e \in E(N), \\
& 0 \le x(e) \le u^+(e) \qquad \forall\, e \in E(N), \\
& 0 \le y(e) \le u^-(e) \qquad \forall\, e \in E(N).
\end{aligned}$$

From the special structure of the problem, we can observe that any edge length reduction imposes an additional cost. Then we conclude that

Lemma 2.1. *In order to solve ROCLP, it is sufficient to increase the edge lengths of the underlying graph N.*

According to Lemma 2.1, we conclude that any optimal modification on the edge lengths contains $y(e) = 0$ for all $e \in E(N)$. The following lemma describes which edges of N must be considered for modification:

Lemma 2.2. *Let s be a prespesified vertex on the given graph N. The value of* $\min_{v\in V,v\neq s} d_l(s,v)$ *is equal to the length of shortest edge incident to s.*

Let $\deg(s)$ denote the degree of the prespecified vertex s. Now, define the star graph $S = (V(S), E(S))$ by

$$V(S) = \{v_i \ : \ v_i \text{ is adjacent to } s;\ i = 1,\cdots,deg(s)\},$$
$$E(S) = \{e_i = (s,v_i) \ : \ i = 1,\cdots,deg(s)\}.$$

Moreover, for simplicity we consider the following definition.

Definition 2.3. Corresponding to the edge lengths l, the critical-distance of S is defined by

$$\mathrm{CD}(l) = \min\{l(e) \ : \ e \in E(S)\}.$$

3 Main solution idea

The considerations and results mentioned above lead to the following generic solution strategy: increase only the lengths of some edges on star graph S such that the corresponding critical-distance $\mathrm{CD}(\tilde{l})$ of S under the new lengths $\tilde{l}$ is maximized and the budget and bound constraints are satisfied.

Let $z^* = CD(l^*)$ denote the optimal critical distance of the underlying graph under the optimal new edge lengths l^*. Our solution approach is summarized as follows:

i. Obtain the optimal objective value z^*.

ii. An optimal solution of the original problem is determined by

$$x^*(e) = \begin{cases} z^* - l(e) & \text{if } e \in E(S), l(e) < z^*, \\ 0 & \text{otherwise,} \end{cases}$$
$$y^*(e) = 0 \qquad \forall\, e \in E(N).$$

Note that the optimal objective value z^* can be computed in $\mathcal{O}(n)$ time. Then, we conclude that

Theorem 3.1. *The reverse obnoxious center location problem can be solved in $\mathcal{O}(n)$ time on a graph with n vertices.*

Finally, it should be pointed out that the problem under the Chebyshev norm and Hamming distance is also solved in linear time.

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Talk Control of fractional discrete-time linear systems by partial eigenvalue ... pp.: 1–4

Control of fractional discrete-time linear systems by partial eigenvalue assignment

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Abstract

In this article to control of fractional-order discrete-time linear system a special form of state feedback matrix is proposed to assign suitable eigenvalues to closed-loop monodromy matrix of the discrete-time system. It makes large monodromy matrix and changing all of eigenvalues make some problems. By reassigning a part of good spectrums of monodromy matrix, leaving the rest of the spectrums invariant, we have lower oreder matrix to modify the dynamic response of linear system and we lie the poles of this systems at the unit circle by less expenses. The effectiveness of our algorithm is illustrated by an example.

Keywords: fractional, partial pole assignment, discrete-time, linear system
Mathematics Subject Classification [2010]: 93B55,93B52,93D15

1 Introduction

In this article to control of fractional discrete-time system we proposed a special form of state feedback matrix to assign suitable eigenvalues to closed-loop monodromy matrix of system. It makes large monodromy matrix and the conventional numerical methods (e.g. the QR based and Schur methods) for EVA problem do not work well. Furthermore, in most of these applications only a small number of eigenvalues are responsible for instability and others need to be reassigned. Clearly, a complete eigenvalue assignment, in case when only a few eigenvalues are bad, does not make sense.These consideration gives rise to the following partial eigenvalue assignment problem (PEVA) for the linear control system.

2 Preliminaries and definitions

2.1 Fractional-order derivatives

Definition 2.1. The disctere-time fractional derivative defined by Grunwald−Letnikov is

$$ {}_{G}D^{\alpha}x(t_k) = \lim_{\mathrm{h}\to 0} \frac{1}{h^{\alpha}} \sum_{i=0}^{k} (-1)^i \left(\begin{array}{c} \alpha \\ i \end{array} \right) x(t_{k-i}), \left(\begin{array}{c} \alpha \\ i \end{array} \right) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-i)\times\Gamma(i+1)} \quad (1)$$

*Speaker

The generalization of the integer-order difference to a non-integer order (or fractional-order) difference with zero initial time is defined as follows [4].

$$\Delta^\alpha x_k = \Delta^\alpha x(t_k) = \sum_{i=0}^{k} (-1)^i \begin{pmatrix} \alpha \\ i \end{pmatrix} x(t_{k-i}) \tag{2}$$

2.2 Fractional-order discrete-time linear systems

In this section we consider the commensurate fractional discrete-time linear system

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k \tag{3}$$

$$x_{k+1} = (A + \alpha I_n)x_k + \sum_{i=1}^{k} c_i x_{k-i} + Bu_k, \;\; c_i = (-1)^i \begin{pmatrix} \alpha \\ i+1 \end{pmatrix} \tag{4}$$

Stability of this kind of systems is tested by practical stability [4].

3 Stability of fractional discrete-time linear systems

By (4) the sequence c_i converges to zero. Getting $c_i = 0$ for $i > L$ (greater L is better) the system (4) will be a time delay system with L delays .

$$x_{k+1} = (A + \alpha I_n)x_k + \sum_{i=1}^{L} c_i x_{k-i} + Bu_k \tag{5}$$

$$X_{k+1} = \overline{A} X_k + \overline{B} u_k \tag{6}$$

$$X_k = \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ . \\ . \\ . \\ x_{k-L} \end{bmatrix}, \overline{A} = \begin{bmatrix} A+\alpha I_n & c_1 I & c_2 I & \cdots & c_{L-1} I & c_L I \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}, \overline{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

3.1 New special form of state feedback law

With a state feedback law of the form

$$u(k) = \sum_{i=0}^{L} F x_{k-i} \tag{8}$$

where $F_k(i)$ is a feedback gain, applied to the system (5). The closed-loop system is

$$x_{k+1} = (A + \alpha I_n + BF)x_k + \sum_{i=1}^{L} (c_i I_n + BF) x_{k-i} \tag{9}$$

defining

$$\overline{\Gamma} = \begin{bmatrix} A+\alpha I_n+BF & c_1I+BF & \cdots & c_LI+BF \\ I & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \tag{10}$$

the system (9) changes to a standard closed-loop system $X_{k+1} = \overline{\Gamma} X_k$.

3.2 Partial pole assignment of the closed-loop system

Defining

$$\overline{F} = [F \quad F \quad \cdots F], \qquad \overline{\Gamma} = \overline{A} + \overline{B}\,\overline{F} \tag{11}$$

The feedback matricx can be obtained by the algorithm given by Karbassi and Bell [3]. Supposing pair $(\overline{A};\overline{B})$ is controllable. The algorithm given by [2] is
1- Let $\{\lambda_i|\lambda_i \in \mathbb{C}\}$ be the set of the eigenvalues of $\overline{A}$.
2- The bad eigenvalues $\Omega(\overline{A}) = \{\lambda_1, \cdots, \lambda_p\}$ (the set of eigenvalues that $|\lambda_i| \geq 1$) should be changed to $S = \{\mu_1, \cdots, \mu_p\}$ and the remaining eigenvalues be invariant.
3- Find a real feedback matrix F such that

$$\Omega(\overline{\Gamma}) = \Omega(\overline{A} + \overline{B}\,\overline{F}) = \{\mu_1, \cdots, \mu_p; \lambda_{p+1}, \cdots, \lambda_n\} \tag{12}$$

4- Let $Y = \{y_1, y_2, \cdots, y_p\}$ be the left eigenvectors of $\overline{A}$ corresponding to $\{\lambda_1, \cdots, \lambda_p\}$
5- Let $A'_{p\times p} = diag(\lambda_1, \cdots, \lambda_p)$, $B'_{p\times m} = Y^H\overline{B}$
6- Finding feedback matrix $F'_{m\times p}$ such that $eig(A' + B'F') = \{\mu_1, \cdots, \mu_p\}$
7- Let $\overline{F} = F' \times Y^H$
8- Now we have $big(\overline{A} + \overline{B}\,\overline{F}) = \{\mu_1, \cdots, \mu_p; \lambda_{p+1}, \cdots, \lambda_n\}$

4 Numerical examples

In this section, we give two examples to show the success of the proposed method.

Example Check the practical stability of the fractional system

$$\Delta^{0.8} x_{k+1} = Ax_k + Bu_k \tag{13}$$

where

$$A = \begin{bmatrix} -0.625 & 1.8 & 0.9 \\ 0.7 & 0 & 0.2 \\ 1 & 1.2 & -0.8 \end{bmatrix}, B = \begin{bmatrix} 3.2 & 0.8 \\ 4.1 & 1 \\ 0 & 0 \end{bmatrix} \tag{14}$$

$$A_\alpha = \begin{bmatrix} .175 & 1.8 & .9 \\ 0.7 & 0.8 & .2 \\ 1 & 1.2 & 0 \end{bmatrix} \quad \overline{A} = \begin{bmatrix} A_\alpha & 0.08I_3 & 0.032I_3 \\ I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_3 & \mathbf{0} \end{bmatrix}_{9\times 9} \quad \overline{B} = \begin{bmatrix} B \\ \mathbf{0}_{6\times 2} \end{bmatrix} \tag{15}$$

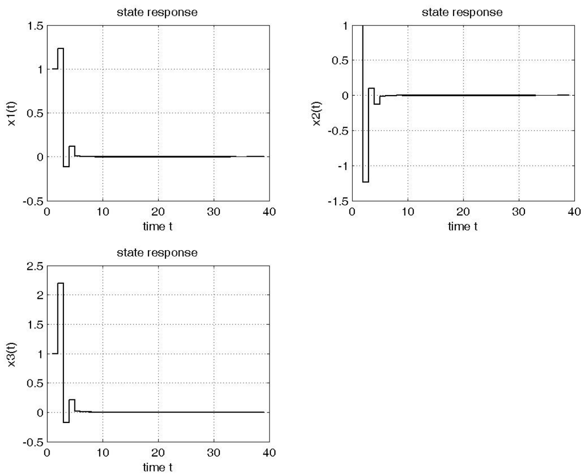

Figure 1: Coverging of $x_i(t)$ to zero

The method adopted for obtaining the feedback matrix by partial pole assignment algorithm is: The eigenvalues of $\overline{A}$ are

$$\{2.1637, -1.0210, 0.3402, 0.2025, -0.2559 \pm 0.1691i, -0.0219 \pm 0.1196i, -0.1548\} \quad (16)$$

we want to change only two first spectums to 0.1 and leave other ones.

$$\overline{F} = [FFF] = \begin{bmatrix} -19.86 & 0.01 & 11.07 & 0.6 & -0.66 & -0.74 & 0.45 & -0.33 & -0.45 \\ 80.09 & -1.5 & -45.53 & -2.47 & 2.64 & 2.98 & -1.83 & 1.34 & 1.84 \end{bmatrix} \quad (17)$$

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Decomposition algorithm for linear programming problem with fuzzy variables

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Abstract

In this paper we focus on large-scale linear programming problems. By considering the vague nature of human judgements, we assume that the decision maker may consider linear programming problem with fuzzy variables. In this paper, using ranking function of fuzzy triangular numbers for the master problem, the corresponding nonfuzzy programming problem is introduced then Dantzig-Wolfe decomposition is applicable.

Keywords: Decomposition method, block angular structure, fuzzy programming, ranking function, master problem
Mathematics Subject Classification [2010]: 90B99

1 Introduction

A lot of actual large-scale optimization problems can be formulated as mathematical programming problems with block angular structure. From such a point of view, since G.B.Dantzig and P.Wolfe [1] proposed the decomposition principle for block angular linear programming problems at the beginning of 1960's, researches for block angular mathematical programming problems have been done actively [4]. In classical optimization model, the objective function and the constraints are represented very precisely under certainty. However, many of the constraints are externally controlled and the variations cannot be predicted to a reliable extent. This may cause difficulties in representing these interacting variables for optimization. To overcome these limitation, Zimmermann [2] introduced fuzzy goal and the fuzzy constraint into the linear programming problem. In this paper we use decomposition algorithm for fuzzy variable linear programming problem.

Definition 1.1. If X is a collection of objected generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_A(x))|x \in X\}$$

$\mu_A(x)$ is a called the membership function. The family of all fuzzy sets in X is denoted by F(X).

Definition 1.2. A fuzzy set $\tilde{A}$ is a convex set if:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)), \;\; x_1, x_2 \in X \;\; and \;\; \lambda \in [0,1]$$

*Speaker

Definition 1.3. Let A_1,A_2,$\cdots$,A_n are fuzzy sets. We define the convex combination of fuzzy sets as follows:

$$\begin{aligned} c = \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n \\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1 \\ \lambda_i \geq 0, \;\; i = 1, 2, \cdots, n \end{aligned}$$

Where membership function of this set is:

$$\mu_c(x) = \lambda_1 \mu_{A_1}(x) + \lambda_2 \mu_{A_2}(x) + \cdots + \lambda_n \mu_{A_n}(x)$$

Definition 1.4. A ray is a collection of the form $\{\tilde{x}_0 + \lambda \tilde{d} : \lambda \geq 0\}$, where vector $\tilde{d}$ is a nonzero fuzzy vector.

Definition 1.5. Given a fuzzy convex set, a nonzero fuzzy vector $\tilde{d}$ is called direction of the set, if for each $\tilde{x}_0$ in the set, the ray $\{\tilde{x}_0 + \lambda \tilde{d} : \lambda \geq 0\}$ also is the set.

1.1 Ranking function

In fact, an efficient approach for ordering the elements is to define a ranking function $\Re : F(R) \longrightarrow R$ which maps for each fuzzy numbers in to the real line, where a natural order exists. we define orders on by:

$$\tilde{A} \geq \tilde{B} \; If \; and \; only \; if \; \Re(\tilde{A}) \geq \Re(\tilde{B})$$

$$\tilde{A} \leq \tilde{B} \; If \; and \; only \; if \; \Re(\tilde{A}) \leq \Re(\tilde{B})$$

$$\tilde{A} = \tilde{B} \; If \; and \; only \; if \; \Re(\tilde{A}) = \Re(\tilde{B})$$

Here we introduce a linear ranking function that is similar to the ranking function [3]. For any arbitrary fuzzy number $\tilde{A} = (\underline{A}(r), \bar{A}(r))$, we use ranking function as follows:

$$\Re(\tilde{A}) = A + \frac{1}{4}(A'' - A')$$

2 The decomposition algorithm for solving linear programming problems with fuzzy variables

Consider the following linear program with fuzzy variables:

$$\begin{aligned} Min \quad & \tilde{z} = c\tilde{\mathrm{x}} \\ s.t. \quad & A\tilde{\mathrm{x}} = \tilde{b} \\ & D\tilde{\mathrm{x}} = \tilde{d} \\ & \tilde{\mathrm{x}} \geq \tilde{0} \end{aligned} \tag{1}$$

where $D\tilde{\mathrm{x}} = \tilde{d}$ is a set of constraints with special structure, the coefficient matrix A is $m \times n$ matrix, $c \in R^n$, $\tilde{b} \in (FT(R))^m$, and $\tilde{\mathrm{x}} \in (FT(R))^n$ where $\tilde{b}_i = (b_i, b'_i, b''_i)$, $i = 1, 2, \cdots, m$. Any point $\tilde{\mathrm{x}}$ with $D\tilde{\mathrm{x}} = \tilde{d}$ and $\tilde{\mathrm{x}} \geq \tilde{0}$ can be represented as a convex combination of the finite number of extreme points of $\tilde{X}$ and nonnegative linear combination of the extreme directions of $\tilde{X}$. If $\tilde{x}_1, \cdots, \tilde{x}_t$ are extreme points and $\tilde{d}_1, \cdots, \tilde{d}_l$ are extreme directions then we have

$$\begin{array}{l} \tilde{x} = \sum_{j=1}^{t} \lambda_j \tilde{x}_j + \sum_{j=1}^{l} \mu_j \tilde{d}_j \\ \sum_{j=1}^{t} \lambda_j = 1 \\ \lambda_j \geq 0 \quad j = 1, 2, \cdots, t \\ \mu_j \geq 0 \quad j = 1, 2, \cdots, l \end{array}$$

The primal problem can be transformed into the problem with variables $\lambda_1, \lambda_2, \cdots, \lambda_t$ and $\mu_1, \mu_2, \cdots, \mu_l$ as follows:

$$\begin{array}{ll} Min & \tilde{z} = \sum_{j=1}^{t} (c\tilde{x}_j)\lambda_j + \sum_{j=1}^{l} (c\tilde{d}_j)\mu_j \\ s.t. & \sum_{j=1}^{t} (A\tilde{x}_j)\lambda_j + \sum_{j=1}^{l} (A\tilde{d}_j)\mu_j = \tilde{b} \\ & \sum_{j=1}^{t} \lambda_j = 1 \\ & \lambda_j \geq 0, \quad j = 1, 2, \cdots, t \\ & \mu_j \geq 0, \quad j = 1, 2, \cdots, l. \end{array} \qquad (2)$$

linear programming (2) is a linear programming with fuzzy variables that it is equivalent with the following linear programming.

$$\begin{array}{lll} Min & \Re(\tilde{z}) = \sum_{j=1}^{t} (c(\Re(\tilde{x}_j))\lambda_j + \sum_{j=1}^{l} (c\Re(\tilde{d}_j))\mu_j & \\ s.t. & \sum_{j=1}^{t} (A\Re(\tilde{x}_j))\lambda_j + \sum_{j=1}^{l} (A\Re(\tilde{d}_j))\mu_j = \Re(\tilde{b}) & (3.1) \\ & \sum_{j=1}^{t} \lambda_j = 1 & (3.2) \\ & \lambda_j \geq 0, \quad j = 1, 2, \cdots, t & \\ & \mu_j \geq 0, \quad j = 1, 2, \cdots, l. & \end{array} \qquad (3)$$

Suppose that we have a basic feasible solution of the foregoing problem with basis B, and let w and α be the dual variables corresponding to Equations (3.1) and (3.2). Further suppose that B^{-1}, $(w, \alpha) = \hat{c}_B B^{-1}$ ($\hat{c}_B$ is the cost of the variables), and $\bar{b} = B^{-1} \begin{pmatrix} \Re(\tilde{b}) \\ 1 \end{pmatrix}$ are known. and displayed.

BASIS INVERSE	RHS
(w, α)	$\hat{c}_B \bar{b}$
B^{-1}	$\bar{b}$

Recall that the current solution is optimal to the overall problem if $z_j - \hat{c}_j \leq 0$ for each variables. In particular, the following conditions must hold at optimality:

For λ_j nonbasic:

$$z_j - \hat{c}_j = (w, \alpha) \begin{pmatrix} A\Re(\tilde{x}_j) \\ 1 \end{pmatrix} - c\Re(\tilde{x}_j) = wA\Re(\tilde{x}_j) - c\Re(\tilde{x}_j) + \alpha \leq 0 \qquad (4)$$

For μ_j nonbasic:

$$z_j - \hat{c}_j = (w, \alpha) \begin{pmatrix} A\Re(\tilde{d}_j) \\ 0 \end{pmatrix} - c\Re(\tilde{d}_j) = wA\Re(\tilde{d}_j) - c\Re(\tilde{d}_j) \leq 0 \qquad (5)$$

Since the number of nonbasic variables is very large, checking conditions (4) and (5) by generating the corresponding extreme points and directions computationally infeasible. However, we may determine whether or not these conditions hold by solving the following subproblem.

$$\begin{aligned} Max \quad & (wA-c)\Re(\tilde{\mathrm{x}})+\alpha \\ s.t. \quad & \Re(D\tilde{\mathrm{x}})=\Re(\tilde{d}) \\ & \Re(\tilde{\mathrm{x}})\geq\Re(\tilde{0}) \end{aligned}$$

First, suppose that the optimal solution value of the subproblem is unbounded. Recall that this is only possible if an extreme direction is found such that $(wA-c)\Re(\tilde{d}_k)>0$.This means that condition (5) is violated. Moreover, $z_k-\hat{c}_k=(wA-c)\Re(\tilde{d}_k)>0$ and μ_k is eligible to enter the basis. In this case $\begin{pmatrix} A\Re(\tilde{d}_k) \\ 0 \end{pmatrix}$ is updated by premultiplying by B^{-1} and the resulting column $\begin{pmatrix} z_k-\hat{c}_k \\ y_k \end{pmatrix}$ is inserted in the foregoing array and the revised simplex method is continued. Now consider the case where the optimal solution value is bounded. A necessary and sufficient condition for boudedness is that $(wA-c)\Re(\tilde{d}_j)\leq 0$ for all extreme directions and so equation (5) holds. Now we check whether (4) holds. Let $\tilde{\mathrm{x}}_k$ be an optimal extreme point and consider the optimal objective, $z_k-\hat{c}_k$, to the subproblem. If $z_k-\hat{c}_k\leq 0$, then by optimality of $\tilde{\mathrm{x}}_k$, for each extreme point $\tilde{\mathrm{x}}_j$, we have

$$(wA-c)\Re(\tilde{\mathrm{x}}_j)+\alpha\leq(wA-c)\Re(\tilde{\mathrm{x}}_k)+\alpha=z_k-\hat{c}_k\leq 0$$

and hence condition (4) holds and we stop with an optimal solution. If, on the other hand, $z_k-\hat{c}_k>0$, then λ_k is introduced in the basis. This is done by inserting the column $\begin{pmatrix} z_k-\hat{c}_k \\ y_k \end{pmatrix}$ in to the foregoing array and pivoting, where $y_k=B^{-1}\begin{pmatrix} A\Re(\tilde{\mathrm{x}}_k) \\ 1 \end{pmatrix}$. Note that, as in the bounded case, if the master problem includes slack or other explicitly present variables, then the $z_j-\hat{c}_j$ values for these variables must be checked before deducing optimality.

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Generalized KKT optimality conditions in an optimization problem with interval-valued objective function and linear-fractional constraints

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Abstract

In this paper, we consider an optimimization problem in which some (all) parameters in the objective function intervals and constraints are linear fractional functions. Indeed, we investigate KKT conditions. A numerical example is carried out to show the efficiency of our method.

Keywords: KKT Condition, Interval Variables, Interval-Valued Objective Function, Linear-Fractional.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In conventional mathematical programming problems, system parameters or model coefficients. are usually determined as crisp values. However, in the real world prob- lems, these parameters are not exactly known. Generally, Interval, stochastic and fuzzy programming approaches are often used to describe imprecise and uncertain components existing in a real decision problem.

Interval programming assumes that the information about the range of variation of some (or all) of the parameters is available, which allows to specify a model with interval coefficients. Some pioneering works about intervals have been done by Moore [1,2]. Since then, a number of interval ordering definitions [3,4] have been developed in different ways. Moreover, there have been many studies about interval optimization problems. For instance, Inuguichi et al. [5] proposed a goal programming approach to solve the interval programming problem.

In this paper, we investigate KKT condition for an optimization programming with interval objective function and linear fractional constraints. Indeed, we investigate this condition for this kind of non-convex programming problems. Finally, using an example we show that the condition which we achieve works succesfully. [5,6]

2 Preliminaries

We consider the following interval-valued mimimization problem:

$$min \quad f(x) = [f^L(x), f^U(x)] \qquad \textbf{(D)}$$

$$s.t \quad x \in S = \{x : \ x \geqslant 0, \ \ g_i(x) = \frac{P_i(x)}{D_i(x)} = \frac{\sum_{j=1}^{n} {p_j}^i x_j + {p_0}^i}{\sum_{j=1}^{n} {d_j}^i x_j + {d_0}^i} \geqslant b_i\}$$

*Speaker

Definition 1. [5] Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in $\mathbb{R}$. We write

$$A \preceq_{LU} B \;\; if \quad a^L \leqslant b^L \; and \; a^U \leqslant b^U$$

For details on interval analysis, we refer to Moore [1].

Definition 2. [5] Let $f(x) = [f^L(x), f^U(x)]$ be an interval-valued function defined on a convex set $X \subseteq \mathbb{R}^n$. We say that f is LU-convex at $x^\star$ if

$$f(\lambda x^\star + (1-\lambda)x) \preceq_{LU} \lambda f(x^\star) + (1-\lambda) f(x)$$

for each $\lambda \in (0,1)$ and each $x \in X$.

Definition 3. [6] Suppose that X is a nonempty, open, convex set in $\mathbb{R}^n$. The function $f(x) : X \to \mathbb{R}$.

a) f is quasiconvex if $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in X, \quad \forall\, \lambda \in [0,1]$.

b) f is quasiconcave if $-f$ is quasiconvex.

Definition 4. [6] Suppose X is a convex set in $\mathbb{R}^n$. The defferentiable function $f(x): X \to \mathbb{R}$ is a pseudoconvex function if

$$\nabla f(x)^t (y - x) \geq 0 \Rightarrow f(y) \geq f(x). \quad \forall x, y \in X$$

Definition 5. [5] Let $x^\star$ be a feasible point. We say that $x^\star$ is a type-**I** solution of problem (D) if there exists no $\bar{x} \in X$ such that $f(\bar{x}) \prec_{LU} f(x^\star)$.

We write $A \prec_{LU} B$ if and only if $A \preceq_{LU} B$ and $A \neq B$. Equivalenty, $A \prec_{LU} B$ if and only if

$$\begin{cases} a^L < b^L \\ a^U \leqslant b^U \end{cases} \; or \; \begin{cases} a^L \leqslant b^L \\ a^U < b^U \end{cases} \; or \; \begin{cases} a^L < b^L \\ a^U < b^U \end{cases} . \tag{1}$$

Definition 6. [5] Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in $\mathbb{R}$ and let $a^C = \frac{1}{2}(a^L + a^U)$ and $b^C = \frac{1}{2}(b^L + b^U)$ and also let $a^W = \frac{1}{2}(a^U - a^L)$ and $b^W = \frac{1}{2}(b^U - b^L)$ so we write $A \preceq_{CW} B$ if and only if $a^C \leqslant b^C$ and $a^W \leqslant b^W$.

Definition 7. [5] Let $x^\star$ be a feasible point. We say that $x^\star$ is a type-**II** solution of problem (D) if there exists no $\bar{x} \in X$ such that $f(\bar{x}) \prec_{LU} f(x^\star)$ or $f(\bar{x}) \prec_{CW} f(x^\star)$.

Theorem 1. Let X be a a convex subset of $\mathbb{R}^n$ and f be an interval-valued function defined on X. Then f is LU-convex at $x^\star$ if and only if f^L and f^U are convex at $x^\star$.

Proof. Proof is straitforward.

Remark 1. Let $x^\star$ be a feasible solution. If $x^\star$ is a type-**I** solution of problem (D) then $x^\star$ is also a type **-II** solution of problem (D).

3 KKT Sufficient Conditions

Theorem 2. Let $x^\star$ be a feasible solution of (D), and suppose $x^\star$ together with multipliers u satisfies

$$\begin{aligned} &\nabla f(x^\star) + \nabla g(x^\star)^t u = 0, \\ &u \leq 0, \qquad\qquad\qquad\qquad \textbf{(KKT)} \\ &u_i g_i(x^\star) = 0, \quad i = 1, ..., m. \end{aligned}$$

If $f(x)$ is a pseudoconvex function, $g_i(x), i = 1, ..., m$ are quasiconcave functions, then $x^\star$ is a global optimal solution of (D).

Proof. Obviosly, the feasibele set is convex. Let $I = \{i \mid g_i(x^\star) = b_i\}$ denote the index of active constraints at $x^\star$. Let x be a feasible point different from $x^\star$. Then $\lambda x + (1-\lambda)x^\star$ is feasible for all $\lambda \in (0,1)$. Thus for $i \in I$ we have

$$g_i(\lambda x + (1-\lambda)x^\star) = g_i(x^\star + \lambda(x - x^\star)) \geq b_i = g_i(x^\star).$$

for any $\lambda \in (0,1)$, and since the value of $g_i(.)$ dose not increase by moving in the direction $x - x^\star$, we must have $\nabla g_i(x^\star)^t(x - x^\star) \geq 0$ for all $i \in I$. Thus, from the **KKT** conditions,

$$\nabla f(x^\star)^t(x - x^\star) = -\big(\nabla g(x^\star)^t u\big)^t (x - x^\star) \geq 0$$

and by pseudoconvexity, $f(x) \geq f(x^\star)$ for any feasible x.

Consider problem (D), Let $x^\star \in S$. Suppose that $g_i, i = 1, ...m$, be quasiconcave on $\mathbb{R}^n$ and continuously differentiable at $x^\star$. Now we are in a position to present the Karush-Kuhn- Tucker optimality conditions for problem (D).

Theorem 3. Suppose that the linear- fractional constraint functions $g_i, i = 1, ..., m$, of problem (D) satisfy the **KKT** assumptions at $x^\star$ and the interval-valude objective function f is LU-convex and weakly continuously differentiable at $x^\star$. If there exist (Lagrange) multipliers $0 < \lambda^L, \lambda^U \in \mathbb{R}$ and $0 \leqslant \mu_i \in \mathbb{R}, i = 1, ..., m$, such that

1. $\lambda^L \nabla f^L(x^\star) + \lambda^U \nabla f^U(x^\star) + \sum_{i=1}^m \mu_i \nabla g_i(x^\star) = 0;$
2. $\mu_i g_i(x^\star) = 0 \quad \forall i = 1, ..., m$

then $x^\star$ is a type-**I** and type-**II** solution of problem (D).

Proof. Since $f(x) = [f^L(x), f^U(x)]$, we can define a real-valued function

$$\bar{f}(x) = \lambda^L f^L(x) + \lambda^U f^U(x). \tag{2}$$

Since f LU-convex and weakly continuously differentiable at $x^\star$, by theorem 1, we see that the real-valued functions f^L and f^U are convex and continuously differentiable at $x^\star$. Therefore, $\bar{f}$ is also convex and continuously differentiable at $x^\star$. Since

$$\nabla \bar{f}(x^\star) = \lambda^L \nabla f^L(x^\star) + \lambda^U \nabla f^U(x^\star).$$

according to conditions 1 and 2 of this theorem, we obtain the following two new conditions

$$(i) \quad \nabla \bar{f}(x^\star) + \sum_{i=1}^m \mu_i \nabla g_i(x^\star) = 0;$$
$$(ii) \quad \mu_i g_i(x^\star) = 0 \quad \forall i = 1, ..., m.$$

using theorem 2 we see that $x^\star$ is an optimal solution of the real-valued objective function $\bar{f}$ subject to the same constraints of problem (D), i.e ,

$$\bar{f}(x^\star) \leqslant \bar{f}(\bar{x}) \tag{3}$$

for any $\bar{x}(\neq x^\star) \in X$. We are going to prove this theorem by contradiction. Suppose that $x^\star$ is not a type-**I** solution of problem (D). Then, according to defintion 5, there exists on $\bar{x} \in X$ such that $f(\bar{x}) \prec_{LU} f(x^\star)$, i.e.,

$$\begin{cases} f^L(\bar{x}) < f^L(x^\star) \\ f^U(\bar{x}) \leqslant f^U(x^\star) \end{cases} \quad or \quad \begin{cases} f^L(\bar{x}) \leqslant f^L(x^\star) \\ f^U(\bar{x}) < f^U(x^\star) \end{cases} \quad or \quad \begin{cases} f^L(\bar{x}) < f^L(x^\star) \\ f^U(\bar{x}) < f^U(x^\star) \end{cases} \quad . \tag{4}$$

Therefore, from expessions (2) and (4), we see that $\bar{f}(\bar{x}) < \bar{f}(x^\star)$ (since $\lambda^L > 0$ and $\lambda^U > 0$) which contradicts (3). From Remark (1), it also shows that $x^\star$ is a type-**II** solution of problem (D). This completes the proof.

Example 1. Let us consider the following interval-valued minimization problem with linear- fractional constraints.

$$\begin{aligned} \min \quad & f(x) = [x^2 + x + 1, x^2 + 3] \\ s.t \quad & \frac{-x+6}{x+2} \geqslant 1 \\ & x \geqslant 0. \end{aligned}$$

we write $g_1(x) = \frac{-x+6}{x+2} - 1 \geq 0$ and $g_2(x) = x$. Then the assumptions presented in theorem 3 are satisfied, and the **KKT** conditions are given below:

1. $\lambda^L(2x^\star + 1) + \lambda^U 2x^\star + \mu_1 \frac{-8}{(x^\star+2)^2} + \mu_2 = 0;$
2. $\mu_1(\frac{-x^\star+6}{x^\star+2} - 1) = 0 = \mu_2 x^\star.$

Let us take $x^\star = 0$. Then condition 2 $\mu_1 = 0$ and conditin 1 $\lambda^L = \mu_2$. Let us take the multipliers $\lambda^L = \lambda^U = \mu_2 = 1$ and $\mu_1 = 0$. Then theorem 3 shows that $x^\star = 0$ is a type-**I** and type-**II** solution.

Theorem 4. Under the same assumptions of Theorem 3, let k be any integer with $1 < k < m$. if there exist (Lagrange) multipliers $0 \geqslant \mu_i \in \mathbb{R}, i = 1, ..., m$, such that

$$\begin{aligned} &(i) \quad \nabla f^L(x^\star) + \Sigma_{i=1}^k \mu_i \nabla g_i(x^\star) = 0; \\ &(ii) \quad \nabla f^U(x^\star) + \Sigma_{i=k+i}^m \mu_i \nabla g_i(x^\star) = 0; \\ &(iii) \quad \mu_i g_i(x^\star) = 0 \quad for\ all\ i = 1, ..., m, \end{aligned}$$

then $x^\star$ is a type-**I** and type-**II** solution of problem (D).

Proof. Proof is a direct result of theorem 3.

Theorem 5. Under the same assumptions of Theorem 3, let $f^C = \frac{1}{2}(f^L + f^U)$. If there exist (Lagrange) multipliers $0 < \lambda^U, \lambda^C \in \mathbb{R}$ and $0 \geqslant \mu_i \in \mathbb{R}, i = 1, .., m$, such that

$$\begin{aligned} &(i) \quad \lambda^U \nabla f^U(x^\star) + \lambda^C \nabla f^C(x^\star) + \Sigma_{i=1}^m \mu_i \nabla g_i(x^\star) = 0; \\ &(ii) \quad \mu_i g_i(x^\star) = 0 \quad \forall i = 1, .., m, \end{aligned}$$

then $x^\star$ is a type-**I** and type-**II** solution of problem (D).

Proof. Proof is a direct result of theorem 3.

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Multiwavelets Galerkin method for solving linear control systems

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Abstract

In this paper a numerical technique is proposed for solving linear control systems. Multiwavelets Galerkin method is applied for solving the extreme conditions obtained from the Pontryagin's maximum principle.

Keywords: Multiwavelets, Galerkin method, Linear control systems
Mathematics Subject Classification [2010]: 42C40, 37L65, 93Cxx

1 Introduction

Optimal control theory has many successful practical applications in areas ranging from economics to various engineering disciplines. The optimal control problem has been studied by many researchers [1]. In this paper, we consider linear optimal problem (OCP)

$$\begin{aligned} &\dot{x} = Ax(t) + Bu(t), x(t_0) = x_0, \\ &J = \tfrac{1}{2}x(t_f)^T Sx(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}(x^T Px + 2x^T Qu + u^T Ru)dt, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. The control $u(t)$ is an admissible control if it is piecewise continuous in t for $t \in [t_0, t_f]$. The input $u(t)$ is derived by minimizing the quadratic performance index J, where $S \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times m}$ are positive semi-definite matrices and $R \in \mathbb{R}^{m\times m}$ is positive definite matrix.

2 Optimality conditions for linear optimal control system

In this section, we try to get the optimal control law $u^*(t) = -k(t)x(t)$ for system (1) by using PMP [2]. For this purpose, one can consider Hamiltonian as

$$H(x, u, \lambda, t) = \frac{1}{2}(x^T Px + 2x^T Qu + u^T Ru) + \lambda^T(Ax + Bu), \tag{2}$$

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where $\lambda \in \mathbb{R}^n$ is co-state vector. According to the PMP, one has $\dot{\lambda} = -\frac{\partial H}{\partial x} = -Px - Qu - A^T\lambda$ and $\frac{\partial H}{\partial u} = Q^T x + Ru + B^T\lambda = 0$. The optimal control is computed by $u^* = -R^{-1}Q^T x - R^{-1}B^T\lambda$, where λ and x are the solution of Hamiltonian system

$$\begin{cases} \dot{x} = [A - BR^{-1}Q^T]x - BR^{-1}B^T\lambda, \\ \dot{\lambda} = [-P + QR^{-1}Q^T]x + [QR^{-1}B^T - A^T]\lambda, \end{cases} \tag{3}$$

with the condition $x(t_0) = x_0$. The terminal condition is assumed as $\lambda(t_f) = Sx(t_f)$. Assuming that the solution of system (3) is

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} F(t,t_f) & G(t,t_f) \\ L(t,t_f) & M(t,t_f) \end{pmatrix} \begin{pmatrix} x(t_f) \\ \lambda(t_f) \end{pmatrix} \tag{4}$$

It can be show that the above-mentioned system can be rewritten in the following form

$$\begin{cases} \dot{V} = [A - BR^{-1}Q^T]V(t) - BR^{-1}B^T W(t), \\ \dot{W} = [-P + QR^{-1}Q^T]V(t) + [QR^{-1}B^T - A^T]W(t), \\ V(t_f) = I, W(t_f) = S, \end{cases} \tag{5}$$

where $V(t) = F(t,t_f) + G(t,t_f)S$ and $W(t) = L(t,t_f) + M(t,t_f)S$.

3 Interpolating scaling functions

Assume that P_r is the Legendre polynomial of order r and r is any fixed nonnegative integer number. Let τ_k denotes the roots of P_r for $k = 0, ..., r-1$. Also suppose ω_k is the Gauss-Legendre quadrature weight $\omega_k = 2(rP'_r(\tau_k)P_{r-1}(\tau_k))^{-1}$. By these assumptions, the interpolating scaling functions (ISF) are given

$$\phi^k(t) = \begin{cases} \sqrt{\frac{2}{\omega_k}} L_k(2t-1), & t \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

where $L_k(t)$ is the Lagrange interpolating polynomial. In this system of wavelets, we assume that $\mathbf{\Phi_0^r} = \{\phi^k\}_{k=0}^{r-1}$ be an orthonormal basis for the Hilbert subspace $V_0^r := \text{span}\{\phi^k : 0 \le k \le r-1\} \subset L^2[0,1)$. Then we can define the projection $P_0 : L^2([0,1)) \to V_0^r$ via

$$P_J(f)(x) := \sum_{k=0}^{r-1} \sum_{l=0}^{2^J-1} \langle f, \phi_{J,l}^k \rangle \phi_{J,l}^k \approx \sum_{k=0}^{r-1} \sum_{l=0}^{2^J-1} 2^{-J/2} \sqrt{\frac{\omega_k}{2}} f\left(2^{-J}(\hat{\tau}_k + l)\right) \phi_{J,l}^k = F^T \Phi_J^r. \tag{6}$$

3.1 The Operational Matrix of Derivative

Suppose that the derivative of $f(x)$ in (6) be given by

$$\frac{d}{dt} f(x) \approx P_J(f')(x) = \sum_{k=0}^{r-1} \sum_{l=0}^{2^J-1} \tilde{f}_{Jl}^k \phi_{Jl}^k(x) = \tilde{F}^T \Phi_J^r(x), \tag{7}$$

One can express a relation between F and $\tilde{F}$ by $\tilde{F} = D_\phi F$ where D_ϕ is the operational matrix of the derivative for the ISFs and express as a block tridiagonal matrix as [3]

$$D_\phi = 2^J \begin{bmatrix} \underline{R} & H & & & & \\ -H^T & R & H & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -H^T & R & H \\ & & & & -H^T & \overline{R} \end{bmatrix}_{N\times N},$$

where, each block is an $r \times r$ matrix and $N = r2^J$. Also for $k, i = 0, ..., r-1$, we have

$$[\underline{R}]_{k+1,i+1} = \frac{1}{2}\phi^i(1)\phi^k(1) - \phi^i(0)\phi^k(0) - q_{k+1,i+1}, [q]_{k+1,i+1} = \sqrt{\frac{\omega_i}{2}}\frac{d}{dt}\phi^k(\hat{\tau}_i).$$

$$[R]_{k+1,i+1} = \frac{1}{2}\phi^i(1)\phi^k(1) - \frac{1}{2}\phi^i(0)\phi^k(0) - q_{k+1,i+1}, [H]_{k+1,i+1} = \frac{1}{2}\phi^i(0)\phi^k(1),$$

$$[\overline{R}]_{k+1,i+1} = \phi^i(1)\phi^k(1) - \frac{1}{2}\phi^i(0)\phi^k(0) - q_{k+1,i+1},$$

4 Description of the Method

Assume that we expand $V(t)$ and $W(t)$ using interpolating scaling functions as

$$V(t) \approx \mathcal{V}^T \otimes \Phi^r_J(t), \quad W(t) \approx \mathcal{W}^T \otimes \Phi^r_J(t), \tag{8}$$

where $\mathcal{V}$ and $\mathcal{W}$ are $(n\times 1)$ unknown vectors and $\otimes$ is the Kronecker product. Using equations (8) and operational matrix of derivative for equation (5), we obtain

$$\begin{cases} \mathcal{V}^T \otimes D\Phi^r_J(t) = [A - BR^{-1}Q^T]\mathcal{V}^T \otimes \Phi^r_J(t) - BR^{-1}B^T\mathcal{W}^T \otimes \Phi^r_J(t), \\ \mathcal{W}^T \otimes D\Phi^r_J(t) = [-P + QR^{-1}Q^T]\mathcal{V}^T \otimes \Phi^r_J(t) + [QR^{-1}B^T - A^T]\mathcal{W}^T \otimes \Phi^r_J(t), \\ \mathcal{V}^T \otimes \Phi^r_J(t_f) = I, \mathcal{W}^T \otimes \Phi^r_J(t_f) = S, \end{cases} \tag{9}$$

The entries of vector $\Phi^r_J(t)$ is independent, so from (9) we get

$$\begin{cases} \mathcal{V}^T \otimes D = [A - BR^{-1}Q^T]\mathcal{V}^T - BR^{-1}B^T\mathcal{W}^T, \\ \mathcal{W}^T \otimes D = [-P + QR^{-1}Q^T]\mathcal{V}^T + [QR^{-1}B^T - A^T]\mathcal{W}^T, \\ \mathcal{V}^T \otimes \Phi^r_J(t_f) = I, \mathcal{W}^T \otimes \Phi^r_J(t_f) = S, \end{cases} \tag{10}$$

From equation (10), one has $2nN$ equations which can be solved for $\mathcal{V}$ and $\mathcal{W}$. Then we be able to obtain the unknown coefficients and approximate $V(t)$ and $W(t)$.

5 Numerical results

In this section to illustrate the effectiveness of the multiwavelets Galerkin method, we consider example of optimal control of linear systems. Consider a single-input scalar system as follows

$$\begin{aligned} \dot{x} &= -x(t) + u(t), \\ J &= \tfrac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \end{aligned} \tag{11}$$

Table 1: Comparison of the presented method with $r = 5$ and BCM.

	$J = 2$		$J = 3$		BCM	
t	$V(t)$	$W(t)$	$V(t)$	$W(t)$	$V(t)$	$W(t)$
0.2	$3.5e-07$	$1.1e-07$	$1.1e-08$	$4.1e-09$	$6.2e-08$	$1.3e-06$
0.6	$1.9e-07$	$5.1e-08$	$5.4e-09$	$9.7e-10$	$5.3e-07$	$1.9e-06$
1.0	$1.3e-07$	$2.2e-49$	$3.8e-09$	$2.2e-49$	$7.5e-06$	$8.3e-11$

According to system (1), we have $A = -1$, $B = 1$, $S = 0$, $Q = 1$, $R = 1$ and $t_f = 1$. By using system (5), we have

$$\dot{V}(t) = -V(t) - W(t), \quad \dot{W}(t) = -V(t) + W(t), \tag{12}$$

The analytical solution of the above-mentioned problem is

$$\begin{cases} V(t) = \cosh(\sqrt{2}t) + \beta\sinh(\sqrt{2}t) \\ W(t) = (1+\sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2}+\beta)\sinh(\sqrt{2}t) \\ \beta = -\frac{\cosh(\sqrt{2})+\sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2})+\sinh(\sqrt{2})} \end{cases}$$

Table 1 consist of absolute error with $r = 5$, $J = 2, 3$ also we compared the approximate solution obtained from the method presented in this paper with the solutions of obtained using Bessel collocation method(BCM) [4].

Acknowledgment

In this paper, multiwavelet Galerkin method has been used successfully for solving optimal control of linear systems. The results reveal the efficiency of the proposed method for solving these systems.

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Solving bi-level integer programming problems with multiple linear objectives at lower level by using particle swarm optimization

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Abstract

Bilevel programming problems are hierarchical optimization problems that consist of the objective of the leader at its first level and that of the follower at the second level. In this paper, we propose a method for solving bi-level integer programming problems with multiple linear objectives at lower level. We begin by finding the convex hull of its original set of constraints using the cutting-plane algorithm. Then, we apply particle swarm optimization (PSO) algorithm to solve this problem. A numerical example illustrates the proposed method.

Keywords: Bi-level optimization, Multiobjective optimization, Particle swarm optimization.
Mathematics Subject Classification [2010]: 90c08,90c10

1 Introduction

Bilevel programming involves two optimization problems where the constraint region of the first level problem is implicitly determined by another optimization problem. Gavete and Gale [1] consider the bilevel problems for which the lower level problem is a linear multiobjective program and constraints at both levels define polyhedra, they proved that the feasible region consists of faces of the polyhedron defined by the constraints. Particle swarm optimization (PSO) is an optimization algorithm proposed by Kennedy and Eberhart in 1995 [2]. The bi-level integer programming with multiple linear objective functions at lower level problem (BIPMLO) can be formulated as:

$$\begin{aligned} \min_{x_1} \quad & f(x_1, x_2) \\ \text{s.t} \quad & A_1^1 x_1 + A_2^1 x_2 \leq b^1 \\ & x_1 \geq 0, \text{ integer} \end{aligned} \tag{1}$$

where x_2 solves

$$\begin{aligned} \min_{x_2} \quad & (d_1 x_2, \ldots, d_k x_2) \\ \text{s.t} \quad & A_1^2 x_1 + A_2^2 x_2 \leq b^2 \\ & x_2 \geq 0, \text{ integer} \end{aligned} \tag{2}$$

*Speaker

$x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are the vectors of variables which controlled by the leader and follower, respectively. $f : R^{n_1} \times R^{n_2} \to R$, $b^1 \in R^{m_1}, b^2 \in R^{m_2}$ and A_1^1, A_1^2, A_2^2 are matrices of suitable dimensions. Also, we introduce the following sets:

$$T = \{(x_1, x_2) \in R^{n_1} \times R^{n_2} : A_1^1 x_1 + A_2^1 x_2 \le b^1, A_1^2 x_1 + A_1^2 x_2 \le b^2, x_1 \ge 0, x_2 \ge 0 \text{ and integers}\}$$
$$T_1 = \{x_1 \in R^{n_1} : \exists x_2 \in R^{n_2} \text{ such that } (x_1, x_2) \in T\}$$
$$V = \{(x_1, x_2) \in R^{n_1} \times R^{n_2} : A_1^1 x_1 + A_2^1 x_2 \le b^1, x_2 \ge 0, \text{ integer}\}$$
$$S = \{(x_1, x_2) \in R^{n_1} \times R^{n_2} : A_1^2 x_1 + A_2^2 x_2 \le b^2, x_2 \ge 0, \text{ integer}\}$$

In what follows, an equivalent problem (BIPMLO) associated with problem (1), (2) can be stated with the help of cutting- plane technique.The equivalent bi-level programming with multiple linear objective functions at lower level problem (BPMLO) can be written in the following form:

$$\begin{aligned} \min_{x_1} \quad & f(x_1, x_2) \\ \text{s.t} \quad & A_1^1 x_1 + A_2^1 x_2 \le b^1 \\ & x_1 \ge 0 \end{aligned} \tag{3}$$

where x_2 solves

$$\begin{aligned} \min_{x_2} \quad & (d_1 x_2, \ldots, d_k x_2) \\ \text{s.t} \quad & A_1^2 x_1 + A_2^2 x_2 \le b^2 \\ & x_2 \ge 0 \end{aligned} \tag{4}$$

2 The Algorithm

In this section, we firstly set up parameters, including Nmaxl the number of iterations of the algorithm PSOL, Nmaxu the number of iterations of the algorighm PSOU, the number of particles ($N\max$), the number of maximum generations (T size) inertial weight (w), two acceleration coefficient (c1 and c2), two random variables, rand 1 and rand 2, are in the interval $0, 1$. Now we are ready to present the algorithm:

Step 1: Convert the problem (BIPMLO) into the equivalent problem (BPMLO), go to step 2.

Step 2: Use Balinski algorithm [3] to find all the vertices of the feasible region T.

Step 3: Set $i = 1$.

Step 4: Select one of the non-integer vertices $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)$ of the solution space. In the tableau of this vertex, choose the row vector where the basic variable has the largest fractional value and construct its corresponding Gomory's fractional cut in the form $h_i x \le r_i$.

Step 5: Add the first cut to the original set of the constraints T. This will yield a new feasible region T^i. If the vertices of the solution space all are integers then go to step 7, otherwise go to step 6.

step 6: Set $i = i + 1$, go to Step 3.

Step 7: Eliminate (drop) all the redundont constraints of the applied cuts.

Step 8: Add all the constraints of applied s-efficent cuts to the original set of constraints T to get $[T]$.

Step 9: Formulate the equivalent problem (BPMLO)

Step 10: Generate upper level's variables, $\tilde{x}_1$, randomly.

Step 11: Solve the lower level problem.

substep 11.1: Generate lower level varibles, $\tilde{x}_2$, randomly.

substep 11.2: set $n := 1$.

substep 11.3: Solve the lower level problem with given $\tilde{x}_1$ from step 10.
In order to check if $(\tilde{x}_1, \tilde{x}_2) \in IR$ or not, we use Benson's approach and check if the optimal objective value of following problem is zero:

$$\begin{aligned} \max \quad & \sum_{i=1}^{k} z_i \\ \text{s.t} \quad & d_i x_2 + z_i = d_i \tilde{x}_2, \quad i = 1, \dots, k \\ & A_2^2 x_2 \leq b^2 - A_1^2 \tilde{x}_1 \\ & x_2 \geq 0 \\ & z_1, \dots, z_k \geq 0 \end{aligned} \tag{5}$$

substep 11.4: Use PSO_L for improving the variable x_2.

substep 11.5: Check if $N < Nmaxl$ go to 11.6, otherwise go to 11.7.

substep 11.6: set $n := n + 1$ and go to 11.3.

substep 11.7: set x_2^* as the optimal solution of problem (5) and go to step 12.

step 12: Solve the following problem:

$$\begin{aligned} \max_{x_1} \quad & f(x_1, x_2) \\ \text{s.t} \quad & A_1^1 x_1 + A_2^1 x_2 \leq b^1 \\ & A_1^2 x_1 + A_2^2 x_2^* \leq b^2 \\ & x_1 \geq 0 \end{aligned} \tag{6}$$

substep 12.1: Generate upper level variable x_{l_i}, randomly.

substep 12.2: Set $n := 1$.

substep 12.3: Solve the upper level problem with given x_2^* from step 11.

substep 12.4: Improve the variables with Psou.

substep 12.5: If $n < Nmaxu$ go to 12.7.

substep 12.6: Set $n = n + 1$ and go to 12.3.

substep 12.7: Set x_1^* as the optimal solution of problem (6).

step 13: (x_1^*, x_2^*) can be considered as an optimal solution for $BIPMLO$.

Example 2.1. Consider the following problem:

$$\begin{aligned} &\min_{x\geq 0,\ \text{integer}} F(x,y) = x - 4y \\ &\min_{y\geq 0,\ \text{integer}} (y, 2y) \\ &\qquad x - y \leq -3 \\ &\qquad -2x + 4y \leq 0 \\ &\qquad 2x + y \leq 12 \\ &\qquad -3x + 2y \leq -4 \end{aligned} \tag{7}$$

This example is taken from [4]. The swarm size are set to 25, the number of maximum generations, T size is set to 50, acceleration cofficient $C1 = chi * phi1, C2 = chi * phi2$, inertia weigth $W = chi$, where $phi1 = 2 : 05$, $phi2 = 2 : 05$, $phi = phi1 + phi2$, $chi = 2/(phi - 2 + sqrt(phi2 - 4 * phi))$.

For this problem, we have $(x^*, y^*) = (2, 1)$ and then $F^*(x, y) = -2$.

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Solving Fuzzy LR Interval Linear Systems Using Ghanbari and Mahdavi-Amiri's Method

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Abstract

Here, we propose a method for solving fuzzy LR interval linear systems with fuzzy coefficients matrix and fuzzy hand-right vector based on the method proposed by Ghanbari and Mahdavi-Amiri for solving fuzzy LR linear systems.

Keywords: Fuzzy LR interval, Fuzzy LR interval linear systems, Least squares model.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Ghanbari and Mahdavi-Amiri in [1] developed the method for solving fuzzy LR triangular linear systems $A\tilde{x} = \tilde{b}$ based on a least squares model. Here, we study the following fuzzy LR interval linear systems:

$$\tilde{A}x = \tilde{b}. \tag{1}$$

To compute an approximate or an exact solution for (1), the proposed method is inspired by Ghanbari and Mahdavi-Amiri's method [1].

2 Basic Concepts and Notations

Definition 2.1. [4] A fuzzy interval $\tilde{a}$ is of LR type, if there exist shape functions L and R (for left and right), and scalars $\alpha \geq 0$, $\beta \geq 0$ and a_l and a_r with the following membership function

$$\mu_{\tilde{a}}(x) = \begin{cases} \mathrm{L}\left(\frac{a_l - x}{\alpha}\right), & a_l - \alpha \leq x \leq a_l, \\ 1, & a_l \leq x \leq a_r, \\ \mathrm{R}\left(\frac{x - a_r}{\beta}\right), & a_r \leq x \leq a_r + \beta, \\ 0, & o.w. \end{cases}$$

The corresponding membership function of a fuzzy LR interval , $\mu_{\tilde{a}}(x)$, denoted by $(a_l, a_r, \alpha, \beta)_{LR}$.

Definition 2.2. [4] Let $\tilde{a} = (a_l, a_r, a_\alpha, a_\beta)_{LR}$, $\tilde{b} = (b_l, b_r, b_\theta, b_\gamma)_{LR}$ and $\delta \in \mathbb{R}$. Then:

*Speaker

1. $\delta \geq 0 \Longrightarrow \delta\tilde{a} = (\delta a_l, \delta a_r, \delta a_\alpha, \delta a_\beta)_{LR}$.
2. $\delta \leq 0 \Longrightarrow \delta\tilde{a} = (\delta a_r, \delta a_l, -\delta a_\beta, -\delta a_\alpha)_{LR}$.
3. $\tilde{a} \bigoplus \tilde{b} = (a_l + b_l, a_r + b_r, a_\alpha + b_\theta, a_\beta + b_\gamma)_{LR}$.

Remark 2.3. we denote the set of LR fuzzy intervals by $\mathbb{I}\left(\Re^1\right)_{LR}$.

Definition 2.4. The system,

$$\tilde{A}x = \tilde{b}$$

where, $\tilde{A} = (A_l, A_r, A_\alpha, A_\beta)_{LR} \in \mathbb{I}(\Re^{m\times n})_{LR}$, and $\tilde{b} = (b_l, b_r, b_\alpha, b_\beta)_{LR} \in \mathbb{I}(\Re^m)_{LR}$ and $x \in \Re^n$ is an unknown vector to be found, is called a fuzzy LR interval linear system (FLRILS).

Corresponding to unknown vector x, we define the two following matrix

$$x^+ = \begin{cases} x_j & x_j \geq 0, \\ 0 & x_j < 0, \end{cases} \qquad x^- = \begin{cases} x_j & x_j < 0, \\ 0 & x_j \geq 0, \end{cases} \tag{2}$$

for $j = 1, ..., n$. and,

$$\tilde{A}x = (A_l x^+ + A_r x^-, A_r x^+ + A_l x^-, A_\alpha x^+ - A_\beta x^-, A_\beta x^+ - A_\alpha x^-).$$

Theorem 2.5. *(Fundamental Theorem of FLRILS) Let $\tilde{A} \in \mathbb{I}(\Re^{m\times n})_{LR}$, and $\tilde{b} \in \mathbb{I}(\Re^m)_{LR}$ and $x \in \mathbb{R}^n$ is a solution of (1) ,if and only if, $({x^+}^T, {x^-}^T)^T$ is solution of the two following systems:*

$$\begin{bmatrix} A_l & A_r \\ A_r & A_l \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} = \begin{bmatrix} b_l \\ b_r \end{bmatrix} \tag{3}$$

and

$$\begin{bmatrix} A_\alpha & -A_\beta \\ A_\beta & -A_\alpha \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} = \begin{bmatrix} b_\alpha \\ b_\beta \end{bmatrix}, \quad x^+ \geq 0, \quad x^- \leq 0. \tag{4}$$

$$\tag{5}$$

Proof. the proof is similar to the proof in [2]. □

3 FLRILS

Here, We define an approximate solution using the concept proposed in [1] and Ming distance function [3]. For two fuzzy LR interval vector $\tilde{x}$ and $\tilde{y}$ defined the following distance function :

$$\begin{aligned} 2D_n^2(\tilde{x}, \tilde{y}) = {} & 2(x_l - y_l)^T(x_l - y_l) + 2(x_r - y_r)^T(x_r - y_r) - 2(x_l - y_l)^T(x_\alpha - y_\alpha) \\ & +2(x_r - y_r)^T(x_\beta - y_\beta) + (x_\alpha - y_\alpha)^T(x_\alpha - y_\alpha) + (x_\beta - y_\beta)^T(x_\beta - y_\beta) \end{aligned} \tag{6}$$

Now, for every x, we define the residual at x as follows:

$$r(x) = 2D_n^2(\tilde{A}x, \tilde{b}) \tag{7}$$

to compute an approximate solution, we must solve the following optimization problem:

$$\begin{cases} \min \ r(x) = 2D_n^2(\tilde{A}x, \tilde{b}) \\ s.t. \\ x \in \mathbb{R}^n. \end{cases} \tag{8}$$

Thus,

$$\begin{aligned} r(x) &= 2(A_l x^+ + A_r x^- - b_l)^T (A_l x^+ + A_r x^- - b_l) \\ &+ 2(A_r x^+ + A_l x^- - b_r)^T ({}_r x^+ + A_l x^- - b_r) \\ &+ (b_\beta - A_\beta x^+ + A_\alpha x^-)^T (b_\beta - A_\beta x^+ + A_\alpha x^-) \\ &+ (b_\alpha - A_\alpha x^+ + A_\beta x^-)^T (b_\alpha - A_\alpha x^+ + A_\beta x^-) \\ &+ 2(A_l x^+ + A_r x^- - b_l)^T (b_\alpha - A_\alpha x^+ + A_\beta x^-) \\ &- 2(A_r x^+ + A_l x^- - b_r)^T (b_\beta - A_\beta x^+ + A_\alpha x^-) \end{aligned} \tag{9}$$

Now, let

$$\begin{aligned} S &= 4{A_l}^T A_l + 4{A_r}^T A_r + 2{A_\beta}^T A_\beta + 2{A_\alpha}^T A_\alpha - 4{A_l}^T A_\alpha + 4{A_r}^T A_\beta \\ R &= 4{A_l}^T A_r + 4{A_r}^T A_l - 2{A_\beta}^T A_\alpha - 2{A_\alpha}^T A_\beta + 4{A_l}^T A_\beta - 4{A_r}^T A_\alpha \\ T &= -4{A_l}^T b_l - 4{A_r}^T b_r - 2{A_\beta}^T b_\beta - 2{A_\alpha}^T b_\alpha + 2{A_l}^T b_\alpha + 2{A_\alpha}^T b_l - 2{A_r}^T b_\beta - 2{A_\beta}^T b_r \\ K &= -4{A_r}^T b_l - 4{A_l}^T b_r + 2{A_\alpha}^T b_\beta + 2{A_\beta}^T b_\alpha + 2{A_r}^T b_\alpha - 2{A_\beta}^T b_l - 2{A_l}^T b_\beta + 2{A_\alpha}^T b_r \end{aligned}$$

Therefore,

$$r(x) = \frac{1}{2}[{x^+}^T {x^-}^T] Q \begin{bmatrix} x^+ \\ x^- \end{bmatrix} + f^T \begin{bmatrix} x^+ \\ x^- \end{bmatrix} + c, \tag{10}$$

where,

$$Q = \begin{bmatrix} S & R \\ R & S \end{bmatrix}, f = \begin{bmatrix} T \\ K \end{bmatrix} \tag{11}$$

and

$$c = 2{b_l}^T b_l + 2{b_r}^T b_r + {b_\alpha}^T b_\alpha + {b_\beta}^T b_\beta - 2{b_l}^T b_\alpha + 2{b_r}^T b_\beta. \tag{12}$$

Now, to compute an approximate solution, we can solve the following quadratic programming problem:

$$\begin{cases} \min \ \frac{1}{2}[{x^+}^T {x^-}^T] Q \begin{bmatrix} x^+ \\ x^- \end{bmatrix} + f^T \begin{bmatrix} x^+ \\ x^- \end{bmatrix} + c \\ s.t. \\ {x^+}^T \geq 0 \\ x^- \leq 0 \\ {x^+}^T x^- = 0. \end{cases} \tag{13}$$

Conclusions

Here, We proposed a method for solving fuzzy LR interval linear systems with fuzzy coefficients matrix and fuzzy hand-right vector using the method proposed by Ghanbari and Mahdavi-Amiri based on a least squares model and obtained the approximate solutions for this systems.

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Talk Using Chebyshev wavelet in state-control parameterization method for ... pp.: 1–4

Using Chebyshev Wavelet in State-control Parameterization Method for Solving Time–varying system

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Abstract

In this paper, a new algorithm based on state-control parameterizatiom method to obtain the solution of time-varying control problem is presented. The state and control variables are expanded by Chebyshev wavelet basis with unknown coefficients and are used to convert optimal control problem into NLP poblem. Applicability of this method is presented by an illustrative example.

Keywords: State-control parameterization, Chebyshew wavelet, Linear time-varying system
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Indirect methods have some drawbacks to obtain the solution of systems that are described by strongly nonlinear differential equations. Thus, many reasearchers proposed direct methods to solve these problems. The direct methods convert optimal control problems into NLP problems and then use existing NLP techniques to solve them.
Direct methods are classified into either discretization [9] or parameterization [8] of the state and/or the control variables. In order to solve various classes of optimal control problems several direct methods that use orthogonal polynomials have been proposed. Wavelets as one of these orthogonal polynomials have good property to approximate functions with discountinous or sharp changes. Many authors have used wavelets for solving optimal control problems such as Haar wavelets [1], harmonic wavelet [4], Shannon wavelet [5], Legendre wavelet [6].
In this paper, the focus is on introducing a state-control parameterization method based on Chebyshev wavelet to find optimal solution for a time-variant system. This work is done as follows: First, a brief description of control problem and Chebyshev wavelet polynomials is given. A mathematical description of proposed state-control parameterization method is presented and finally by presenting an example, we compare our proposed method with other reasearchers to determine the validity of the solution of this example.

*Speaker

2 Problem statement

Find the optimal control that minimizes the quadratic performance index

$$J = \int_0^{t_f} (\{\mathbf{x}'Q\mathbf{x}\} + \{\mathbf{u}'R\mathbf{u}\})dt, \quad (1)$$

Subject to:

$$\dot{x} = A(t)x(t) + B(t)u(t), \quad x(0) = x_0. \quad (2)$$

where $x(.) : I \to \mathbb{R}^s$ is the state variable, $u(.) : I \to \mathbb{R}^r$ is the control variable of system. Also, $A(t)$ and $B(t)$ are time-varying matrices, Q is positive semidefinite matrix and R is a positive definite matrix.

3 The Chebyshev wavelet polynomials

In this section, we briefly describe Chebyshew wavelet polynomials that are used in the next section. By dialation and translation of a single function called the mother wavelet, a family of wavelets can be constructed. An applicable family of wavelets is Chebyshev wavelet $\phi_{nm}(t) = \phi(k, m, n, t)$ that defined on the interval $[0, 1)$ by following:

$$\phi_{nm} = \begin{cases} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \leq t \leq \frac{n}{2^k} \\ 0, & O.W. \end{cases}$$

where $k = 1, 2, ...$,$n = 1, 2, 3, ..., 2^k$,m is the order for Chebyshev polynomials and

$$\alpha_m = \begin{cases} \sqrt{2}, & m = 0 \\ 2, & m = 1, 2, ... \end{cases}$$

$T_m(t)$ are the well-known Chebyshev polynomials that satisfy the following recursive formula:

$$T_0(t) = 1, T_1(t) = t, T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t).$$

4 Main results

In this section, a new state-control parameterization based on Chebyshev wavelet is introduced. Let $Q \subset C^1([0, 1])$ be set of all functions that satisfy initial condition. Also, let $Q_m \subset Q$ be the class of combinations of Chebyshew wavelet polynomials of degree up to m. We can approximate the state and control variables as follows:

$$\hat{x}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} a_{nm}\phi_{nm}(t), \hat{u}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} b_{nm}\phi_{nm}(t). \quad (3)$$

where

$$\phi(t) = [\phi_{10}(t), \phi_{11}(t), ..., \phi_{1M-1}(t), \phi_{20}(t), \phi_{21}(t), ..., \phi_{2M-1}(t), ..., \phi_{2^k0}(t), \phi_{2^k1}(t), ..., \phi_{2^kM-1}(t)],$$
$$a(t) = [a_{10}, a_{11}, ..., a_{1M-1}, a_{20}, a_{21}, ..., a_{2M-1}, ..., a_{2^k0}, a_{2^k1}, ..., a_{2^kM-1}],$$
$$b(t) = [b_{10}, b_{11}, ..., b_{1M-1}, b_{20}, b_{21}, ..., b_{2M-1}, ..., b_{2^k0}, b_{2^k1}, ..., b_{2^kM-1}].$$

Now, we consider the minimization of J on Q_m with a and b as unknowns.
By substituting these approximations of the state and control variables, the performnce index J can be written as:

$$\hat{J}(a_{10}, a_{11}, \ldots, a_{2^kM-1}, b_{10}, b_{11}, \ldots, b_{2^kM-1}) = \int_0^{t_f} (\{\hat{x}'Q\hat{x}\} + \{\hat{u}'R\hat{u}\})dt. \tag{4}$$

We replace equality consraints (2) by (5) to get the initial condition and other constraints as following:

$$\dot{\hat{x}} = A(t)\hat{x}(t) + B(t)\hat{u}(t),$$
$$\hat{x}(t_0) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} a_{nm}\phi_{nm}(t)\Big|_{t=t_0} = x_0. \tag{5}$$

Also, we must add some constraints in order to get the continuity of the state variables between the different sections. $2^k - 1$ points exist that the continuity of the state variables have to be ensured. These points are:

$$t_i = \frac{i}{2^k}, i = 1, 2, \ldots, 2^k - 1$$

So there are $2^k - 1$ equality constraints that must be satisfied.
These process cause to find solution of problem by a new nonlinear programming problem that has $2n + 2$ unknowns as follows:

$$\min_{\mathbf{z}\in\mathbb{R}^{2^k+1M}} \left\{\mathbf{z}'H\mathbf{z}\right\}, \tag{6}$$

Subject to

$$Pz = Q. \tag{7}$$

where $\mathbf{z}' = (\mathbf{a}', \mathbf{b}')$.
Solving this problem is easier than the original problem by well developed optimization algorithms.

Example 4.1. Find the optimal control $u^*(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 (x^2 + u^2)dt.$$

Subject to:

$$\dot{x} = -tx + u, x(0) = 1.$$

The obtained solution for J by our proposed method together with comparison by other researchers for solving this problem is reported in Table 1.

As we see from Table 1, our proposed method has acceptable solution in compare with other methods.

Table 1: Comparison between different reasearches for J value

Research name	Jaddu [7]	Elnagar [3]	Elaydi[2]	Our proposed method
J	0.4842676003768	0.48427022	0.484267810538982	0.4842677532

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Vitality of Nodes in Networks Carrying Flows Over Time

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Abstract

In this paper finding most vital node of networks carrying flows over time is studied, a mathematical model is generalized and a fully combinatorial algorithm is provided adapting an iterative procedure. Given a network and a time horizon T, Most Vital Node Over Time (MVNOT) problem seeks for a node whose removal from network results greatest decrease in the value of maximum flow over time up to time horizon T between two terminal nodes.

Keywords: most vital nodes; maximum flow over time; combinatorial algorithm.
Mathematics Subject Classification [2010]: 90C11

1 Introduction

Vitality problem on networks is firstly introduced by Wolmer [4] at 1963. Wolmer [4] studied looking for a link whose removal from network results greatest decrease in the value of deterministic maximum flow between two predefined nodes. Later, many extensions of the original problem is studied in literature [2]. Recently, a new version of most vital link problem is introduced and studied by Morowati and Mehri [2] which differs from traditional models in the sense that it studies vitality on networks carrying flows over time [3] instead of traditional static flows.

In this paper we study the problem of finding most vital node of a network which aims to transfer maximum flow over time between two terminal nodes up to a predefined time horizon T. The MVNOT problem may simply be reduced to a most vital link problem but this reduction increase problem size significantly. Therefore, providing a direct solution method motivated us to provide an iterative algorithm for MVNOT problem.

2 Preliminaries

Let $G = (N, A, \mathbf{u}, \boldsymbol{\tau}, s, t)$ is given, where N is the set of nodes, A is the set of directed links with a positive capacity $\mathbf{u} = (u_{ij})_{(i,j)\in A}$ and positive transit times $\boldsymbol{\tau} = (\tau_{ij})_{(i,j)\in A}$, s is source node and t is terminal node. A static s-t-flow is a real valued mapping $\mathbf{x}$ on the links of G that satisfies capacity constraints $0 \leq x_{ij} \leq u_{ij}$ for all $(i, j) \in A$ and flow conservation constraints $\sum_{j\in N:(j,i)\in A} x_{ji} - \sum_{j\in N:(i,j)\in A} x_{ij} = 0$, for all $i \in N \setminus \{s, t\}$. The value of a static s-t-flow $\mathbf{x}$ is equal to $|\mathbf{x}| = \sum_{j\in N:(j,t)\in A} x_{jt} - \sum_{j\in N:(t,j)\in A} x_{tj}$.

*Speaker

Given G and a time horizon $T \in \Re^+$, a flow over time on G is defined as an array of nonnegative functions such as $\mathbf{f} = (f_{ij})_{(i,j)\in A}$, where for each link $(i,j) \in A$, $f_{ij} : \Re \to \Re^+$ is a Lebesgue-integrable function which vanishes for all $\theta \in \Re \setminus [0, T - \tau_{ij})$.

Let $\mathbf{f}$, $\theta \in [0,T)$ and $i \in N$ is given then, define the operator $F(\mathbf{f}, \theta, i)$ as follows

$$F_{\boldsymbol{\tau}}(\mathbf{f}, \theta, i) = \int_0^\theta [\sum_{j\in N:(j,i)\in A} f_{ji}(\eta - \tau_{ji}) - \sum_{j\in N:(i,j)\in A} f_{ij}(\eta)]d\eta.$$

Given G and T, maximum flow over time problem seeks a flow over time which has maximum value (i.e. $|\mathbf{f}| = F_{\boldsymbol{\tau}}(\mathbf{f}, T, t))$) among all feasible flows over time. This complicated problem can be formulated as [3]

$$\max_{\mathbf{f}}\{v_{\mathbf{f}}(T) \colon \mathbf{f} \in \Omega(\boldsymbol{\tau})\}, \tag{1}$$

where $\Omega(\boldsymbol{\tau}) = \{\mathbf{f} \mid F_{\boldsymbol{\tau}}(\mathbf{f}, \theta, i) \geq 0, \forall i \in \bar{N}, \theta \in [0,T);\ F_{\boldsymbol{\tau}}(\mathbf{f}, T, i) = 0;\ F_{\boldsymbol{\tau}}(\mathbf{f}, T, s) = -F_{\boldsymbol{\tau}}(\mathbf{f}, T, t) = v_{\mathbf{f}}(T)$ and $0 \leq f_{ij}(\theta) \leq u_{ij}, \forall (i,j) \in A, \theta \in [0,T)\}$ and $\bar{N} = N \setminus \{s,t\}$. Using the concept of temporary repeated flows [3] it is demonstrated that the optimum value of maximum flow over time up to time horizon T is equal to optimum value of following static circulation problem which is defined on the extended network G' assigning an additional artificial link (t,s) with cost $-T$ and infinite capacity:

$$\max_{\mathbf{x}}\{T x_{ts} - \sum_{i\in N}\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij} : \mathbf{x} \in \Lambda\},$$

where $\Lambda = \{\mathbf{x} :\ \sum_{j\in N:(j,i)\in A} x_{ji} - \sum_{j\in N:(i,j)\in A} x_{ij} = 0,\ \forall i \in N;\ x_{ij} \leq u_{ij},\ \forall (i,j) \in A;\ x_{ij} \geq 0,\ \forall (i,j) \in A'\}$ and $A' = A \cup \{(t,s)\}$.

3 Mathematical Formulation and Solution Method

To formulate MVNOT problem we define a set of binary variables ϕ_i assigned to each node $i \in N$. We mean by $\phi_i = 1$ that node i is blocked and otherwise node i is not blocked($\phi_i = 0$). Using these considerations, let Φ be the set of all possible elections for the most vital node in G; that is $\Phi = \{\boldsymbol{\phi} \in \{0,1\}^{|N|} : \sum_{i\in N} \phi_i = 1\}$. As is obvious, Φ is the set of all vectors $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \cdots, \mathbf{e}_{i_{|N|}}$. To block a node i in mathematical model, we can simply increase traverse time of all its outgoing links to a number greater than T, because if the traverse time of these links be grater than T then the traverse time of every i-crossing route will be greater than T, therefore in a maximum flow over time pattern no flow arrives to t from such routes up to time horizon T.

By these notations we can formulate MVNOT problem as following min-max problem:

$$\min_{\boldsymbol{\phi}\in\Phi} H(\boldsymbol{\phi}) = \max_{\mathbf{f}}\{v_{\mathbf{f}}(T) \colon \mathbf{f} \in \Omega((\tau_{ij} + T\phi_i)_{(i,j)\in A,})\} \tag{2}$$

which is a very complicated problem and can not be solved directly therefore, we must do some reformulations on this initial model to provide a solution method. According to the discussion in section 2, for a fixed and constant $\boldsymbol{\phi} \in \Phi$, optimum value of inner maximum flow over time problem in (2) is equal to that's of following circulation problem

$$\mathbf{MP}(\boldsymbol{\phi}, T) : \quad \max_{\mathbf{x}}\{T x_{ts} - \sum_{i\in N} T\phi_i(\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij}) : \mathbf{x} \in \Lambda\}.$$

Notice that for a fixed $\boldsymbol{\phi}$, if $\phi_i = 1$ then the penalized traverse time of all its outgoing links (i, j) is $\tau_{ij} + T$. As a result, the traverse time of every i-crossing route is greater than T. Therefore every positive flow on such routes decreases objective function of MP$(\boldsymbol{\phi}, T)$. This implies that if $\phi_i = 1$ then $f_{ij}(\theta) = 0$ for all $j \in N : (i, j) \in A$ and all $\theta \in [0, T)$, in an optimal flow over time pattern. Since $H(\phi)$ equals optimum value of MP$(\boldsymbol{\phi}, T)$, then (2) may be reformulated as following min-max problem

$$\min_{\boldsymbol{\phi}\in\Phi} H(\boldsymbol{\phi}) = \max_{\mathbf{x}}\{Tx_{ts} - \sum_{i\in N} T\phi_i(\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij}) : \mathbf{x} \in \Lambda\}. \quad (3)$$

Now, for fixed $\boldsymbol{\phi}$ according to strong duality theorem, by taking dual of inner problem in (3) and releasing $\boldsymbol{\phi}$, we can transform (2) to following mixed integer programming problem

$$\min_{\boldsymbol{\phi},\boldsymbol{\alpha},\boldsymbol{\mu}}\{\sum_{(i,j)\in A} u_{ij}\mu_{ij} : (\boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\mu}) \in \Gamma\}, \quad (4)$$

where $\Gamma = \{(\boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\mu}) \in \{0,1\}^{|N|} \times \Re^{|N|} \times \Re^{+|A|} : \mu_{ij} + \alpha_{a_t} - \alpha_{a_h} + T\phi_i \geq -\tau_{ij},\ \forall(i, j) \in A;\ \alpha_t - \alpha_s \geq T;\ \sum_{i\in N}\phi_i = 1\}$. We have transformed the complicated problem (2) into the mixed linear minimization problem (4) which is solvable by all existing methods for solving mixed linear programming problems.

To provide computationally efficient solution method, using special structure of (4) we have provided an improved Benders decomposition based algorithm [1] to MVNOT problem which is a fully combinatorial algorithm.

According to special structure of (4), since all variables of its equivalent problem (4) can be decomposed into two groups (i.e. binary variable $\boldsymbol{\phi}$ and continuous variables $\boldsymbol{\mu}$ and $\boldsymbol{\alpha}$) and the feasible region of its dual (i.e. MP$(\boldsymbol{\phi}, T)$) does not depend on $\boldsymbol{\phi}$, therefore Benders decomposition algorithm [1] is a suitable tool for solving (4). To apply the Benders decomposition algorithm [1] on (4) we must reformulate (4) as min-max programming problem as follows, which makes the Benders decomposition algorithm applicable on it.

$$[\mathrm{Msr}(\hat{X})] \min_{\boldsymbol{\phi}\in\Phi}\{q : Tx_{ts} - \sum_{i\in N} T\phi_i(\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij}) \leq q;\ \forall\mathbf{x} \in \hat{X}\}$$

$$[\mathrm{Sub}(\boldsymbol{\phi})] \qquad \max_{\mathbf{x}}\{Tx_{ts} - \sum_{i\in N} T\phi_i(\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij}) : \mathbf{x} \in \Lambda\},$$

where X is the set of all extreme points of feasible region of inner maximization problem and $\hat{X}$ is a subset of X which is updated in each iteration by adding a new extreme point. Note that $\hat{X}$ starts by $\hat{X} = \{0\}$ initially.

Similar to basic Benders decomposition algorithm, the proposed algorithm solves Sub$(\boldsymbol{\phi})$ in each iteration and updates $\hat{X}$ by adding a new extreme point and then Msr$(\hat{X})$ seeks for the suboptimal $\boldsymbol{\phi}$ to improve previous $\boldsymbol{\phi}$ by examining all $\mathbf{x}$ in updated $\hat{X}$. Note that in each iteration Msr$(\hat{X})$ provides a lower bound and Sub$(\boldsymbol{\phi})$ provides an upper bound on optimal solution of original problem. The algorithm terminates when upper bound and lower bound be equivalent. The proposed algorithm superior to basic Benders algorithm in the sense that it solves no integer programming problem in Msr$(\hat{X})$ directly and solves it using an iterative procedure as follows.

An Iterative and Fully Combinatorial Algorithm

Input:	$G = (N, A, \mathbf{u}, \boldsymbol{\tau}, s, t)$ and T. **Output:** $\boldsymbol{\phi}^*$ defining the most vital node.
Step 0.	UB $\leftarrow +\infty$; LB $\leftarrow -\infty$; $\hat{X} \leftarrow \emptyset$; $\hat{\boldsymbol{\phi}} \leftarrow \mathbf{0}$; $\mathbf{z} \leftarrow -\infty^{\|N\|}$.
Step 1.	Solve MP$(\hat{\boldsymbol{\phi}}, T)$ to obtain optimal solution $\mathbf{x}^*(\hat{\boldsymbol{\phi}})$.
Step 2.	$\hat{X} \leftarrow \hat{X} \cup \mathbf{x}^*(\hat{\boldsymbol{\phi}})$; $s \leftarrow Tx^*_{ts}(\hat{\boldsymbol{\phi}}) - \sum_{(i,j)\in A} \tau_{ij} x^*_{ij}(\hat{\boldsymbol{\phi}})$.
Step 3.	IF $s <$ UB THEN UB $\leftarrow s$ and $\boldsymbol{\phi}^* \leftarrow \hat{\boldsymbol{\phi}}$.
Step 4.	IF UB = LB then STOP$\boldsymbol{\phi}^*$is optimal. ELSE, go to Step 5.
Step 5.	For all $i \in N$, IF $-Tx^*_{ij}(\hat{\boldsymbol{\phi}}) + s > z_i$ THEN $z_i \leftarrow -Tx^*_{ij}(\hat{\boldsymbol{\phi}}) + s$.
Step 6.	Select a node i' such that $z_{i'} = \min_{i\in N}\{z_i\}$; LB $\leftarrow z_{i'}$.
Step 7.	IF UB=LB, STOP; i' is the most vital link. ELSE $\hat{\boldsymbol{\phi}} \leftarrow \mathbf{e}_{i'}$; and go to Step 1.

Theorem 3.1. *The Step 5 and Step 6 of the algorithm is equivalent with solving mater problem Msr($\hat{X}$)in basic Benders algorithm; that is, Step 5 and Step 6 of the iterative algorithm solves Msr($\hat{X}$) correctly.*

Proof. Consider that we are in iteration k and $\hat{X}$ contains $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^k$. After Step 5 of the algorithm for each $\bar{i} \in N$, $z_{\bar{i}}$ is equal to $\max\{Tx_{ts} - \sum_{\substack{i\in N\\ i\neq i'}} T\phi_i(\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij}) - T\phi_{i'}(\sum_{j\in N:(i',j)\in A} \tau_{i'j}x_{i'j}) : \mathbf{x}^i \in \hat{X}\}$. Step 6 selects node i' which has minimum value of $z_{i'}$; Notice that $z_{i'}$ is minimum value which $Tx_{ts} - \sum_{i\in N} T\phi_i(\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij}) \leq z_{i'}$ holds for all $\mathbf{x}^i \in \hat{X}$. Since Msr($\hat{X}$) seeks minimal q^* such that $Tx_{ts} - \sum_{i\in N} T\phi_i(\sum_{j\in N:(i,j)\in A} \tau_{ij}x_{ij}) \leq q^*$ hold for all $\mathbf{x}^i \in \hat{X}$; this implies that $z_{i'} = q^*$. □

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Statistics & Probability Theory

Directionally Uniform Distributions and their applications

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Abstract

One of the main properties of the Gaussian distribution is the existence of a multivariate version which its directional marginals give Gaussian distributions with prescribed covariance. In this article we study the same property for uniform distribution. We formulate the concept of directionally uniform distributions and then prove that in dimensions 2 and 3 such distribution exist but in dimensions greater than 3 it does not exist.

Keywords: Uniform Distribution, Bochner's Theorem, Characteristic Function, Directionally Uniform distributions
Mathematics Subject Classification [2010]: 60E05, 60E10

1 Introduction

Among continuous distributions, the normal distribution is probably the most interesting because of its several useful properties. One of its properties is the existence of the multivariate Gaussian distribution, which all of its linear combinations are Gaussian. This property make the Gaussian distribution computationally efficient and can be used to generate families of normal variables with prescribed mean and covariance.

One could ask if such multivariate version exists for other continuous distributions. In this article we study this problem for uniform distribution.

In section 2 we define the directionally uniform distribution in $\mathbb{R}^n$. In Theorem 2.2 and 2.4 we prove that this distribution exists only in 2 and 3 dimensions.

We also provide an application of the directionally uniform distribution in $\mathbb{R}^3$.

2 Main Results

2.1 Definition

Definition 2.1. By an n dimensional directionally uniform distribution we mean a probability measure on $\mathbb{R}^n$ with the property that its projection on any direction is a uniform distribution on an interval.

The first problem is the existence of such distributions. We will show that in dimensions 2 and 3 it exists but for $n \geq 4$ it does not exist.

2.2 Dimensions 2 and 3

Theorem 2.2. *For $n = 2, 3$, the directionally uniform distribution on $\mathbb{R}^n$ exists.*

Proof. It suffices to prove the statement for $n = 3$, since then the projection of the distribution on $x - y$ plane would satisfy the condition for $n = 2$.

For $n = 3$, let μ be the uniform distribution on the surface of a 3-sphere. We claim that the projection of μ on any direction is a uniform distribution.

Let $x = (x_1, x_2, x_3)$ be the standard coordinate on $\mathbb{R}^3$ and let S^2 be the surface of the unit sphere:

$$S^2 = \{x : x_1^2 + x_2^2 + x_3^2 = 1\}$$

Since μ is rotationally invariant, it suffices to prove the claim for just one direction, say, x_1 direction.

Note that μ, the surface measure of S^2 can be written in coordinates as

$$d\mu = \frac{1}{4\pi x_1} dx_2 dx_3$$

Let $\pi : \mathbb{R}^3 \to \mathbb{R}$ be the projection $\pi(x) = x_1$. We show that $\pi^*\mu$ is a uniform distribution on $[-1, 1]$. For this, we compute

$$\pi^*\mu([1-a, 1]) = \int_{x_1 \in [1-a,1]} d\mu = \frac{1}{4\pi} \int_{x_1 \in [1-a,1]} \frac{1}{x_1} dx_2 dx_3$$

Now, write the last integral in the polar coordinates for (x_2, x_3),

$$= \frac{1}{4\pi} \int_{x_1 \in [1-a,1]} \frac{1}{x_1} r dr d\theta = \frac{1}{2} \int_{x_1 \in [1-a,1]} \frac{r dr}{x_1}$$

By change of variable $r = \sqrt{1 - x_1^2}$,

$$= \frac{1}{2} \int_{1-a}^{1} dx_1 = \frac{a}{2}$$

Which shows that $\pi^*\mu$ is uniform on $[-1, 1]$.

□

2.3 Dimension $n \geq 4$

In order to prove the non-existence of directionally uniform distribution for $n \geq 4$, we study some elementary properties of these distributions.

Let μ be a directionally uniform distribution in $\mathbb{R}^n$.

Lemma 2.3. *μ has bounded support.*

Proof. Let π_i for $i = 1, \ldots, n$ be the projection on the direction of x_i. By definition, $\pi^*\mu$ is a uniform distribution in some finite interval I_i. Hence μ is supported in $I_1 \times \cdots \times I_n$. □

By lemma 2.3, all of the moments of μ are finite. Hence by translation we can assume that its mean is zero, i.e $\int x\mu(dx) = 0$.

Since the projection of μ on any direction is non-degenerate, hence its covariance matrix is non-degenerate. So, by applying a linear transformation we may assume that the covariance matrix of μ is $\frac{1}{3}I$, i.e

$$\int xx^T \mu(dx) = \frac{1}{3}I$$

This implies that for any $u \in \mathbb{R}^n$,

$$\int \|u.x\|^2 \mu(dx) = \frac{1}{3}\|u\|^2 \tag{1}$$

Now let X be a random vector with distribution mu. By assumption, $u.X$ is a uniform distribution with mean zero and by equation (1), it's variance is $\frac{1}{3}\|u\|^2$, hence it should have a uniform distribution on $[-\|u\|, \|u\|]$.

Now, we can compute the characteristic function of μ,

$$\phi_\mu(u) = \int e^{iu.x}\mu(dx) = \mathbb{E}(e^{iu.X})$$

Now, note that the characteristic function of uniform distribution on $[-a, a]$ is $\frac{\sin(ta)}{ta}$, hence

$$\phi_\mu(u) = \frac{\sin \|u\|}{\|u\|}$$

We are ready to prove the theorem.

Theorem 2.4. *For $n \geq 4$, there is no directionally uniform distribution on $\mathbb{R}^n$.*

Proof. It suffices to prove the statement for $n = 4$.

By the above arguments, If a directionally uniform distribution exists, then one can find a directionally uniform distribution with characteristic function

$$\phi_\mu(u) = \frac{\sin \|u\|}{\|u\|}, \qquad u \in \mathbb{R}^4$$

We claim that $\phi_\mu(u)$ is not a positive definite function and hence can not be the characteristic function of a probability distribution.

Recall the definition of positive definiteness:

A function $\phi : \mathbb{R}^n \to \mathbb{C}$ is called positive definite if for any $u_1, \ldots, u_k \in \mathbb{R}^n$, the matrix $[\phi(u_i - u_j)]_{k\times k}$ is positive definite. By Bochner's theorem [1], $\phi : \mathbb{R}^n \to \mathbb{C}$ is the characteristic function of a probability distribution if and only if $\phi(0) = 1$, ϕ is continuous at 0 and ϕ is positive definite. (We actually use the obvious side of Bochner's theorem.)

To show that $\frac{\sin\|u\|}{\|u\|}$ is not positive definite on $\mathbb{R}^4$, we have implemented a simple MATLAB code which for randomly generated $u_1, \ldots, u_{20} \in \mathbb{R}^4$, computes the least eigenvalue of the matrix $\frac{\sin\|u_i-u_j\|}{\|u_i-u_j\|}$ and observed that it is negative. The values of $u_1, \ldots, u_{20}$ and the least eigenvalue are shown in table 1.

□

Table 1

u_1	2.071661	-1.52947	-0.12509	-0.47156
u_2	-0.10042	-0.60975	1.196003	0.872045
u_3	0.27824	-0.09717	2.352896	-0.80128
u_4	0.157121	-0.21488	-0.44896	-0.34094
u_5	-0.70237	0.197078	-0.22346	-0.67959
u_6	1.164527	0.91031	2.075903	0.20048
u_7	-0.32273	-1.90679	-0.99754	0.153502
u_8	-1.02664	2.121854	-0.30072	1.829166
u_9	-0.98766	1.928701	1.167703	-0.45846
u_{10}	0.075282	-0.66725	-0.99968	-1.33551
u_{11}	1.050265	1.385223	-0.63016	-0.91293
u_{12}	1.699063	0.042737	-0.5005	0.661253
u_{13}	-0.45907	1.229852	0.112322	-0.15984
u_{14}	-0.42368	-1.02792	1.080404	1.944249
u_{15}	-0.02356	-0.03711	-1.44475	-2.42517
u_{16}	-0.91353	-0.14023	1.773574	0.384946
u_{17}	-1.76742	-0.28037	0.333058	-0.77013
u_{18}	1.171953	-0.36592	0.784256	1.821137
u_{19}	0.39525	0.436417	-0.49054	-1.01394
u_{20}	-0.80148	0.541668	-0.22287	-1.1707
Least eigenvalue	-0.1548			

3 Application

As an application of the directionally uniform distribution introduced in section 2, we show how it can be used to generate correlated uniform variables with prescribed covariance matrix.

Let X be a uniform point on the surface of S^2. If $X = (X_1, X_2, X_3)$, then by symmetry, X_1, X_2 and X_3 have mean zero and covariance matrix $\frac{1}{3}I$. Hence, for any 3×3 matrix A, AX will be a random vector with covariance matrix $\frac{1}{3}AA^T$.

Therefore if we are given a covariance matrix C, we can put $A = (3C)^{\frac{1}{2}}$ such that $\frac{1}{3}AA^T = C$ and then AX gives us three uniform variables with covariance matrix C.

4 General Distributions

The method used in proof of theorem 2.4 is a general method that can be used to study the existence of distributions with given directional marginals.

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Improved Ridge M-Estimators

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Abstract

The focus of this approach is on parameter estimation in multiple regression model in the presence of multicollinearity and outliers. Some improved ridge M-estimators are define and their performance is evaluated in a real example.

Keywords: M-Estimator; Multicollinearity; Outliers; Ridge regression; Shrinkage M-estimator.
Mathematics Subject Classification [2010]: 62G08 , 62J07

1 Introduction

A traditional linear regression model has form

$$\boldsymbol{y}_n = (y_1, \dots, y_n)^T = X_n\boldsymbol{\beta} + \boldsymbol{\epsilon}_n, \quad \boldsymbol{\epsilon}_n = (\epsilon_1, \dots, \epsilon_n)^T, \tag{1}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_p)^T$ is the vector of unknown (regression) parameters, X_n is an $n \times p$ (design) matrix of known regression constants, $n > p \geq 1$, and the ϵ_is are errors.

There are four assumptions that must be verified before implementing the model: (i) linearity and additivity of the relationship between dependent and independent variables, (ii) statistical independence of the errors, (iii) homoscedasticity, and (iv) normality of the error distribution. When all of the assumptions are true, the best estimator for unknown parameter $\boldsymbol{\beta}$ is the ordinary least squares (OLS) estimator defined as $\hat{\boldsymbol{\beta}}_n^{OLS} = (X_n^T X_n)^{-1} X_n^T \boldsymbol{y}_n$

In real world, we may encounter a data set that doesn't satisfy one or more of the above assumptions, resulting on inappropriateness of the OLS method. Sometimes, there exist highly correlated two or more variables in collection of predictors in a regression setup. This phenomena is called multicollinearity that has been studied by many researchers in different aspects. Horel and Kennard [1] introduced the ridge regression approach to combat multicollinearity, which was already known as Tikhonov regularization. Another common problem in regression analysis is to take normality assumption for the errors, when they are not so in practice, like as fat tailed distributions, that can produce outliers. When outliers exist in the data, the use of robust estimators reduces their effects. When the regressors are fixed, so only allowing for outliers in the dependent variable (the response), it is suggested to use M-estimation, which introduced by Huber [2].

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In practice, it may happen that both the multicollinearity and outliers exist simultaneously. For this case, in 1991, Silvapull [3] suggested a method that was a combination of ridge and M-estimation methods.

In some situations it is possible to have some non-sample information usually subjected to the model as constraints. Our interest is to focus on an estimation problem where both the multicollinearity and outliers exist and some prior information about unknown parameters are also available.

Throughout, we may assume that X_n is of rank p, and consider the partitioning (where $p = p_1 + p_2, p_1 \geq 0, p_2 \geq 0$)

$$\boldsymbol{\beta} = \left(\overset{p_1 \times 1}{\boldsymbol{\beta}_1^T} \quad \overset{p_2 \times 1}{\boldsymbol{\beta}_2^T} \right) \quad \text{and} \quad X_n = \left(\overset{n \times p_1}{X_{n1}} \quad \overset{n \times p_2}{X_{n2}} \right), \tag{2}$$

so that (1) may also be written as

$$\boldsymbol{Y}_n = X_{n1}\boldsymbol{\beta}_1 + X_{n2}\boldsymbol{\beta}_2 + \boldsymbol{\epsilon}_n. \tag{3}$$

We are interested in the estimation of $\boldsymbol{\beta}_1$ when it is plausible that $\boldsymbol{\beta}_2$ is "close to" $\mathbf{0}$.

2 Main Results

For the global (unrestrained) model in (2), we denote an M-estimator of $\boldsymbol{\beta}$ by $\tilde{\boldsymbol{\beta}}_n^{(M)} = (\tilde{\boldsymbol{\beta}}_{1n}^{(M)T}, \tilde{\boldsymbol{\beta}}_{2n}^{(M)T})^T$, so that $\tilde{\boldsymbol{\beta}}_{1n}^{(M)}$ is an *unrestrained M-estimator* (UME) of $\boldsymbol{\beta}_1$. The UME, $\tilde{\boldsymbol{\beta}}_n^{(M)} = (\tilde{\boldsymbol{\beta}}_{1n}^{(M)T}, \tilde{\boldsymbol{\beta}}_{2n}^{(M)T})^T$ of $\boldsymbol{\beta}$ is a solution to $\boldsymbol{M}_n(\boldsymbol{b}) = \mathbf{0}$, where

$$\boldsymbol{M}_n(\boldsymbol{b}) = (M_{n1}(\boldsymbol{b}), \ldots, M_{np}(\boldsymbol{b}))^T = \sum_{i=1}^{n} \boldsymbol{x}_i \psi(y_i - \boldsymbol{x}_i^T \boldsymbol{b}), \qquad X_n^T = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n),\ \boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^p. \tag{4}$$

and $\psi(\cdot)$ is the score function [2]. We also write $\boldsymbol{M}_n(\boldsymbol{b}) = (\boldsymbol{M}_{n1}^T(\boldsymbol{b}_1, \boldsymbol{b}_2), \boldsymbol{M}_{n2}^T(\boldsymbol{b}_1, \boldsymbol{b}_2))^T$, where for $\boldsymbol{M}_n$ and $\boldsymbol{b}$, we use the same partitioning as in (2). Let

$$C_n = X_n^T X_n = \begin{bmatrix} X_{n1}^T X_{n1} & X_{n1}^T X_{n2} \\ X_{n2}^T X_{n1} & X_{n2}^T X_{n2} \end{bmatrix} = \begin{bmatrix} C_{n11} & C_{n12} \\ C_{n21} & C_{n22} \end{bmatrix}, \tag{5}$$

and assume that there exists a positive definite (p.d.) matrix C, such that as $n \to \infty$,

$$n^{-1} C_n \to C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad \text{and} \quad \max_{1 \leq i \leq n} \left\{ \boldsymbol{x}_i^T C_n^{-1} \boldsymbol{x}_i \right\} = O\left(n^{-\frac{1}{2}}\right) = o(1). \tag{6}$$

For the restrained model $X_n = X_{n1}\boldsymbol{\beta}_1 + \boldsymbol{\epsilon}_n$ (i.e. $\boldsymbol{\beta}_2 = 0$), let $\hat{\boldsymbol{\beta}}_{1n}^{(M)}$ be the corresponding M-estimator of $\boldsymbol{\beta}_1$; This estimator is termed a *restrained M-estimator* (RME) of $\boldsymbol{\beta}_1$ and it is a solution to $\boldsymbol{M}_{n(1)}(\boldsymbol{b}_1, \mathbf{0}) = \mathbf{0}$. Following Singer and Sen [4], we can rewrite the restricted estimator as

$$\hat{\boldsymbol{\beta}}_{1n}^{(M)} = \tilde{\boldsymbol{\beta}}_{1n}^{(M)} + C_{n11}^{-1} C_{n12} \tilde{\boldsymbol{\beta}}_{2n}^{(M)}. \tag{7}$$

This RME generally performs better than the UME when $\boldsymbol{\beta}_2$ is $\mathbf{0}$ (or very close to $\mathbf{0}$). Often to incorporate uncertain prior information on $\boldsymbol{\beta}_2$ in the estimation of $\boldsymbol{\beta}_1$, a suitable (M-)

test statistics (for testing $H_0 : \boldsymbol{\beta}_2 = 0$) is taken into consideration. In a *preliminary test M-estimation* (PTME) formulation, the $\hat{\boldsymbol{\beta}}_{1n}^{(M)PT}$ is chosen as the RME or UME, according as the preliminary test leads to the acceptance or rejection of H_0.

For the PTME and SME, we need to introduce a suitable (M-) test statistic for testing the null hypothesis $H_0 : \boldsymbol{\beta}_2 = \mathbf{0}$. Toward this, we proceed as in Singer and Sen [4] let

$$\begin{aligned}
&\hat{\boldsymbol{M}}_{n(2)} = \boldsymbol{M}_{n(2)}(\hat{\boldsymbol{\beta}}_{1n}^{(M)}, \mathbf{0}),\\
&S_n^2 = n^{-1}\sum_{i=1}^{n} \psi^2\left(Y_i - \boldsymbol{x}_{i(1)}^T\hat{\boldsymbol{\beta}}_{1n}\right), \qquad \boldsymbol{x}_i^T = (\boldsymbol{x}_{i(1)}^T, \boldsymbol{x}_{i(2)}^T), \quad i \geq 1,\\
&C_{nrr.s} = C_{nrr} - C_{nrs}C_{nss}^{-1}C_{nsr}, \quad \text{for} \quad r \neq s = 1, 2. \qquad (8)
\end{aligned}$$

Then, an appropriate (aligned M-) test statistic is

$$T_n^{(M)} = S_n^{-2}\left\{\hat{\boldsymbol{M}}_{n(2)}^T C_{n22.1}\hat{\boldsymbol{M}}_{n(2)}\right\}. \qquad (9)$$

Under H_0, $T_n^{(M)}$ has asymptotically the chi-square distribution function with p_2 degrees of freedom (d.f.) where $p_2 \geq 1$. The PTME is then defined by

$$\hat{\boldsymbol{\beta}}_{1n}^{(M)PT} = \tilde{\boldsymbol{\beta}}_{1n}^{(M)} I\left(T_n^{(M)} \geq \chi^2_{p_2,\alpha}\right) + \hat{\boldsymbol{\beta}}_{1n}^{(M)} I\left(T_n^{(M)} < \chi^2_{p_2,\alpha}\right), \qquad (10)$$

where $I(A)$ stands for the indicator function of the set A.

The *Shrinkage M-estimatior* (SME), based on the usual James-Stein [5] rule, incorporates the same test statistic in a smoother manner. It is defined as

$$\hat{\boldsymbol{\beta}}_{1n}^{(M)S} = \tilde{\boldsymbol{\beta}}_{1n}^{(M)} - (p_2 - 2)[T_n^{(M)}]^{-1}(\tilde{\boldsymbol{\beta}}_{1n}^{(M)} - \hat{\boldsymbol{\beta}}_{1n}^{(M)}). \qquad (11)$$

We also consider the following positive-rule SME:

$$\hat{\boldsymbol{\beta}}_{1n}^{(M)S(+)} = \hat{\boldsymbol{\beta}}_{1n}^{(M)S} - (1 - (p_2 - 2)[T_n^{(M)}]^{-1})I(T_n^{(M)} < p_2 - 2)(\tilde{\boldsymbol{\beta}}_{1n}^{(M)} - \hat{\boldsymbol{\beta}}_{1n}^{(M)}), \qquad (12)$$

where a^+ is equal to $a \vee 0$. For more details about these estimators, see Sen and Saleh [6].

2.1 Ridge M-Regression

Following Hoerl and Kennard [1], we define a $R_n(k)$ matrix as analogy to ordinary ridge regression as $(I_{p_1} + kC_{11.2}^{-1})^{-1}$ that it satisfied in the following equation.

$$\lim_{n\to\infty} R_n(k) = [I_p + kC_{n11}]^{-1} = [I_p + kC_{11}]^{-1} = R(k). \qquad (13)$$

We define the unrestricted RR M-estimator (URRME), restricted RR M-estimator (RRRME), preliminary test RR M-estimator (PTRRME), shrinkage RR M-estimator (SRRME) and the positive rule RR M-estimator (PRRRME) are, respectively, as follows:

$$\begin{aligned}
\tilde{\boldsymbol{\beta}}_{1n}^{(M)}(k) &= R_n(k)\tilde{\boldsymbol{\beta}}_{1n}^{(M)}, \quad \hat{\boldsymbol{\beta}}_{1n}^{(M)}(k) = R_n(k)\hat{\boldsymbol{\beta}}_{1n}^{(M)}, \quad \hat{\boldsymbol{\beta}}_{1n}^{(M)PT}(k) = R_n(k)\hat{\boldsymbol{\beta}}_{1n}^{(M)PT},\\
\hat{\boldsymbol{\beta}}_{1n}^{(M)S}(k) &= R_n(k)\hat{\boldsymbol{\beta}}_{1n}^{(M)S}, \quad \hat{\boldsymbol{\beta}}_{1n}^{(M)S+}(k) = R_n(k)\hat{\boldsymbol{\beta}}_{1n}^{(M)S+}. \qquad (14)
\end{aligned}$$

Since for $\boldsymbol{\beta}_2$ the pivot is taken as $\mathbf{0}$, we consider a shrinkage neighborhood of $\mathbf{0}$ and toward this, we consider the sequence $\{K_n\}$ of alternative, where

$$K_{(n)} : \boldsymbol{\beta}_2 = \boldsymbol{\beta}_{2(n)} = n^{-\frac{1}{2}}\boldsymbol{\xi}, \qquad \boldsymbol{\xi} = (\xi_{p_1+1}, \ldots, \xi_p)^T \in \mathbb{R}^{p_2}, \qquad (15)$$

so that the null hypothesis H_0 reduces to $H_0 : \boldsymbol{\xi} = \mathbf{0}$.

Table 1: Average prediction errors and standard deviations

	URRME	RRRMRE	PTRRME	SRRME	PRRRME
mean	328.2775	325.9454	328.2774	328.2768	328.2765
sd	561.3188	577.2734	577.2734	577.2711	577.2711

3 Application

To evaluate the performance of various estimators, a real 10-factor data set of Gorman and Toman [7] is used. This data set is taken from routine operation for a petroleum refining unit. The first column of this data is the response on the log scale, the remaining columns are the predictors. This data contains 36 observations. The variance inflation factor (VIF) values for this data are 56.27, 354.92, 68.55, 20.07, 216.68, 120.04, 899.31, 8.65, 2.051, and 8.14. It reveals severe multicollinearity problem. Also, the Bonfroni test for identifying outliers is done. The result of Bonferonni probabilty (0.50703) shows the existences of outliers.

The performance of the estimators are evaluated using average 10-fold cross validation error. Prediction error, as a squared version of difference between the observed and predicted values of the response variable, is used to evaluate the performance of estimators. Table 1 shows the average and standard deviation of the prediction errors for 1000 repetition of the process. It can be seen that the new estimator PRRRME performs better than others.

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Interval estimation for a general class of exponential distributions under progressive censoring

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Abstract

In this paper, the interval estimation is discussed for a general class of exponential type distributions which includes several well-known lifetime models such as exponential, Burr XII, Weibull, Pareto and Rayleigh. A numerical example is presented to illustrate the proposed interval estimates.

Keywords: Exponential distribution, Interval estimation, Joint confidence region.
Mathematics Subject Classification [2010]: 62F25, 62E15.

1 Introduction

The most common censoring schemes are Type-I and Type-II censoring. In the conventional Type-I and Type-II censoring schemes, we are not allowed to remove units at points other than the terminal point of the experiment. Type-II progressive censoring scheme is a more general censoring which allows for removal of units at points other than the terminal point of the experiment. The progressive Type-II censoring, after starting the life-testing experiment with n units, arises as follows. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then, immediately following the second failure, R_2 surviving units are removed from the test at random. This process continues until, at the time of the m-th failure, all the remaining $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ units are removed from the experiment. Here, the R_i's are fixed prior to study. If $R_1 = R_2 = \cdots = R_m = 0$, then $n = m$ which corresponds to the complete sample situation. If $R_1 = R_2 = \cdots = R_{m-1} = 0$, we have $R_m = n - m$ which corresponds to the conventional Type-II right censoring scheme. For more details, see [1] and [2].

Let us consider the continuous random variable X with cumulative distribution function (cdf) and probability density function (pdf) given by

$$F(x;\beta,\theta) = 1 - \exp\{-\beta Q(x;\theta)\}, 0 < x < \infty, \tag{1}$$

and

$$f(x;\beta,\theta) = \beta q(x;\theta)\exp\{-\beta Q(x;\theta)\}, \tag{2}$$

*Speaker

where β and θ are the model parameters, $Q(x;\theta)$ is increasing in x with $Q(0;\theta) = 0$ and $Q(\infty;\theta) = \infty$, and $q(x,\theta) = \frac{\partial}{\partial x}Q(x,\theta) > 0$. This family is a general class of exponential type distributions and is useful in estimating the survival function for right censored data. It includes several well-known lifetime models such as exponential, Burr type XII, Weibull, Pareto, Rayleigh and so on.

Suppose $X_{1:m:n}, \cdots, X_{m:m:n}$ be a progressive Type-II censored sample from the above family with censored scheme $R_1, \cdots, R_m$. Here a $100(1-\alpha)\%$ confidence interval for θ is constructed. Further, we present an exact joint confidence region for (β, θ).

2 Main Results

For $i = 1, ..., m$, let us define $Y_{i:m:n} = -\ln[1 - F(X_{i:m:n};\beta,\theta)] = \beta\, Q(X_{i:m:n};\theta)$. Then, it can be shown that $Y_{1:m:n}, \cdots, Y_{m:m:n}$ are the progressive Type-II censored sample from a standard exponential distribution. For notation simplicity, let us write X_i for $X_{i:m:n}$. If we define $Z_1 = nY_1$, $Z_2 = (n - R_1 - 1)(Y_2 - Y_1), \cdots, Z_m = (n - R_1 - ... - R_{m-1} - (m-1))(Y_m - Y_{m-1})$, then $Z_1, Z_2, ..., Z_m$ are independent and identically distributed (iid) EXP(1) random variables (see [1]). Hence

$$V = 2Z_1 = 2nY_1 \sim \chi^2_{(2)}, \qquad \text{and} \qquad U = 2\sum_{i=2}^{m} Z_i = 2\left[\sum_{i=1}^{m}(1+R_i)Y_i - nY_1\right] \sim \chi^2_{(2m-2)},$$

and U and V are independent. Let us define

$$T_1 = \frac{U/(2m-2)}{V/2} = \frac{1}{m-1}\left[\frac{\sum_{i=1}^{m}(1+R_i)Y_i - nY_1}{nY_1}\right] \sim F_{(2m-2,2)},$$

and

$$T_2 = U + V = 2\sum_{i=1}^{m}(1+R_i)Y_i \sim \chi^2_{(2m)}.$$

So, we have

$$T_1 = \frac{1}{m-1}\left[\frac{\sum_{i=1}^{m}(1+R_i)Q(x_i,\theta) - nQ(x_1,\theta)}{nQ(x_1,\theta)}\right] \sim F_{(2m-2,2)},$$

and

$$T_2 = 2\beta\sum_{i=1}^{m}(1+R_i)Q(x_i,\theta) \sim \chi^2_{(2m)}.$$

It is clear that T_1 and T_2 are independent. Also, based on Lemma 1 in [4], T_1 is an increasing function of θ. Therefore, we can construct a confidence interval for θ and a joint confidence region for (θ, β). An exact confidence interval for θ is given in the following theorem.

Theorem 2.1. *Suppose that $X_1, \cdots, X_m$ is a progressively Type-II censored sample from the family in (1). Then, for any $0 < \alpha < 1$, the interval*

$$\left(\varphi[x_1, ..., x_m, F_{(2(m-1),2)}(1-\frac{\alpha}{2})],\ \varphi[x_1, ..., x_m, F_{(2(m-1),2)}(\frac{\alpha}{2})]\right),$$

is a $100(1-\alpha)\%$ *confidence interval for* θ*, where* $\varphi(x_1, ..., x_m, t)$ *is the solution of* θ *for the equation*

$$\frac{1}{m-1}\left[\frac{\sum_{i=1}^{m}(1+R_i)Q(x_i,\theta)-nQ(x_1,\theta)}{nQ(x_1,\theta)}\right]=t.$$

An exact joint confidence region for (β, θ) is given in the following theorem.

Theorem 2.2. *Suppose that* $X_1, \cdots, X_m$ *is a progressively Type-II censored sample from the family in (1). Then, the following inequalities determine a* $100(1-\alpha)\%$ *joint confidence region for* (β, θ)*:*

$$\varphi(x_1, ..., x_m, F_{(2(m-1),2)}(\frac{1+\sqrt{1-\alpha}}{2})) < \theta < \varphi(x_1, ..., x_m, F_{(2(m-1),2)}(\frac{1-\sqrt{1-\alpha}}{2})),$$

$$\frac{\chi^2_{(2m)}(\frac{1+\sqrt{1-\alpha}}{2})}{2\sum_{i=1}^{m}(1+R_i)Q(x_i,\theta)} < \beta < \frac{\chi^2_{(2m)}(\frac{1-\sqrt{1-\alpha}}{2})}{2\sum_{i=1}^{m}(1+R_i)Q(x_i,\theta)},$$

where $\varphi(x_1, ..., x_m, t)$ *is the solution of* θ *for following equation*

$$\frac{1}{m-1}\left[\frac{\sum_{i=1}^{m}(1+R_i)Q(x_i,\theta)-nQ(x_1,\theta)}{nQ(x_1,\theta)}\right]=t.$$

3 Example

Here we use a special case of the model (1) with $Q(x, \theta) = \ln(1 + x^\theta)$, which corresponds to the Burr XII model with cdf $F(x, \beta, \theta) = 1 - (1 + x^\theta)^\beta, x > 0,\ \beta > 0,\ \theta > 0$. We apply the proposed estimation methods to the real data set reported in [3]. Data are the time to breakdown of an insulating fluid in an accelerated life test conducted at a voltage of 34 kV. Zimmer et. al. [5] indicated that the Burr type XII distribution is acceptable for these data. A progressively Type II censored sample of size $m = 8$ was randomly generated from these observations. The censoring scheme and corresponding observed sample are presented in Table 1.

Table 1: Progressively Type-II censored sample generated from the times to breakdown data.

i	1	2	3	4	5	6	7	8
x_i	0.19	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	0	0	3	0	3	0	0	5

By Theorem 2.1 and using the S-PLUS package, the 95% confidence interval for θ is $(0.44791, 2.717365)$ with length 2.26946. By Theorem 2.2 and by solving non-linear equation, we obtain the following 95% joint confidence region for β and θ:

$$0.38777 < \theta < 3.06367,$$

$$\frac{8.57797}{2\sum_{i=1}^{m}(1+R_i)\ln(1+x_i^\theta)} < \beta < \frac{36.70271}{2\sum_{i=1}^{m}(1+R_i)\ln(1+x_i^\theta)}.$$

The area of above joint confidence region for β and θ is 1.10082. Figure 1 shows the shape of the 95% joint confidence region for β and θ.

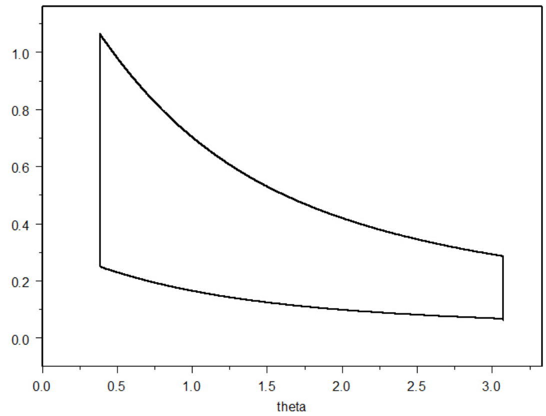

Figure 1: The 95% joint confidence region.

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Matrix Variate Hypergeometric Gamma Distribution

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Abstract

In this paper a generalized matrix gamma distribution including generalized hypergeometric function and zonal polynomials is introduced. Some important statistical characteristics such as the Laplace transformation and expectation of determinant are given.

Keywords: Generalized hypergeometric function, Matrix variate hypergeometric gamma distribution, Multivariate gamma function, Zonal polynomials.
Mathematics Subject Classification [2010]: 62E05; 62E15.

The inverted matrix variate gamma (IMG) distribution, which is the distribution of the inverse of the gamma matrix (GM), is the generalized form of the inverted Wishart (IW) distribution. It can be found in Iranmanesh et al.(2013). It is well known and well documented that the IW and IMG distributions have many applications in inferential problems concerning the covariance matrix. In Bayesian analysis they are used as the conjugate prior for the covariance matrix of a multivariate normal distribution. recently Nagar et al. (2013) defined an extended matrix variate gamma distribution by extending the multivariate gamma function.
In the present article, an attempt has been made to give a generalized definition of MG and IMG distribution including generalized hypergeometric function and zonal polynomials and study some of their properties.

1 Introduction

Definition 1.1. The multivariate gamma function, denoted by $\Gamma_m(a)$ is defined

$$\begin{aligned}\Gamma_m(a) &= \int_{\boldsymbol{X}>0} etr(-\boldsymbol{X})(\det \boldsymbol{X})^{a-\frac{(m+1)}{2}} d\,\boldsymbol{X}, \\ &= \pi^{m(m-1)/4}\prod_{j=1}^{m}\Gamma\left(a-\frac{j-1}{2}\right), \qquad (1)\end{aligned}$$

where $Re(a) > \frac{m-1}{2}$, $etr(.) \equiv exp\,\mathrm{tr}(.)$ and $\boldsymbol{X}(m\times m) > 0$ is a $m\times m$ positive definite matrix. The integral is over the space of positive definite (and hence symmetric) $m\times m$

*Speaker

matrices.See Gupta and Nagar(2000) and Muirhead(1982). A more generalized integral representation of the multivariate gamma function can be obtained as

$$\Gamma_m(a) = \det(\boldsymbol{Y})^a \int_{\boldsymbol{R}>0} etr(-\boldsymbol{Y}\boldsymbol{R}) \det(\boldsymbol{R})^{a-\frac{(m+1)}{2}} d\ \boldsymbol{R}. \tag{2}$$

where $Re(a) > \frac{m-1}{2}$ and $Re(\boldsymbol{Y}) > \frac{m-1}{2}$.

The above result can be stablished for real $\boldsymbol{Y} > 0$ by substituing $\boldsymbol{X} = \boldsymbol{Y}^{1/2}\boldsymbol{R}\boldsymbol{Y}^{1/2}$ with the jacobian $J(\boldsymbol{X} \to \boldsymbol{Y}) = \det(\boldsymbol{Y})^{\frac{(m+1)}{}}$ in (1), see Mathai(1997).

Definition 1.2. The generalized hypergeometric function of one matrix, defined in constantine(1963), is given by

$${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{X}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa ... (a_p)_\kappa}{(b_1)_\kappa ... (b_p)_\kappa} \frac{C_\kappa(\boldsymbol{X})}{k!}, \tag{3}$$

where $a_i, i = 1, ..., p, b_j, j = 1, ..., q$ are arbitrary complex numbers; $\boldsymbol{X}(m \times m)$ is a complex symmetric matrix; $C_\kappa(\boldsymbol{X})$ is the zonal polynomial of complex symmetric matrix $\boldsymbol{X}(m \times m)$ corresponding to the ordered partition $\kappa = (k_1, ..., k_m), k_1 \geq ... \geq k_m \geq 0, k_1 + ... + k_m = k$. The generalized hypergeometric coefficient $(a)_\kappa$ used above is defined by

$$(a)_\kappa = \prod_{i=1}^{m} \left(a - \frac{i-1}{2}\right)_{k_i}, \tag{4}$$

where $(a)_k = a(a+1)...(a+k-1), r = 1, 2, ...$ with $(a)_0 = 1$.

due to Dia'z Garcia (2009) assume $p \leq q$; for $Re(a) > \frac{m-1}{2}$, we have

$$\begin{aligned} \int_{\boldsymbol{X}>0} etr(-\boldsymbol{X}\boldsymbol{Z}) \quad {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{X}\boldsymbol{U}) \det(\boldsymbol{X})^{\alpha-\frac{(m+1)}{2}} d\ \boldsymbol{X} \\ = \det(\boldsymbol{Z})^{-\alpha} \quad {}_{p+1}F_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{U}\boldsymbol{Z}^{-1}). \end{aligned} \tag{5}$$

Definition 1.3. A random matrix $\boldsymbol{X}$ of order m is said to have a matrix hypergeometric gamma (MHG) distribution with parameters α, β, $\boldsymbol{\Sigma}$ and $\boldsymbol{U}$ denoted by $\boldsymbol{X} \sim MHG(\alpha, \beta, \boldsymbol{\Sigma}, \boldsymbol{U})$, if its density function is given by

$$\begin{aligned} f(\boldsymbol{X}) = \frac{\det(\boldsymbol{\Sigma})^{-\alpha}}{\beta^{\alpha p}\Gamma_p(\alpha) \quad {}_{p+1}F_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{U}\beta\boldsymbol{\Sigma})} etr\left(-\frac{1}{\beta}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\right) \\ \times \det(\boldsymbol{X})^{\alpha-(m+1)/2} \quad {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{X}\boldsymbol{U}). \end{aligned} \tag{6}$$

special cases

1. For $\beta = 1$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$,the distribution of (6) reduces to the matrix gamma distribution proposed by Roux (1971).
2. For $\boldsymbol{U} = 0$ the distribution of (6) reduces to the MG distribution introdused by Iranmanesh et al.(2013).

Theorem 1.4. *Let $\boldsymbol{X} \sim MHG(\alpha, \beta, \boldsymbol{\Sigma}, \boldsymbol{U})$. Then $\boldsymbol{Y} = \boldsymbol{X}^{-1}$ has inverse MHG (IMHG) distribution denoted by $\boldsymbol{Y} \sim IMHG_p(\alpha, \beta, \boldsymbol{\Sigma}, \boldsymbol{U})$ with the following density function*

$$f(\boldsymbol{Y}) = \frac{\det(\boldsymbol{\Sigma})^{-\alpha}}{\beta^{\alpha p}\Gamma_p(\alpha) \quad {}_{p+1}F_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{U}\beta\boldsymbol{\Sigma})} etr\left(-\frac{1}{\beta}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}^{-1}\right)$$
$$\times \det(\boldsymbol{Y})^{-\alpha-(m+1)/2} \quad {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{Y}^{-1}\boldsymbol{U}), \boldsymbol{Y} > 0. \tag{7}$$

Proof. The proof follows from the fact that the Jacobian of transformation is given by $J(\boldsymbol{X} \to \boldsymbol{Y}) = \det(\boldsymbol{Y})^{-(m+1)}$. □

2 Main results

In this section, various properties of the MHG and IMHG distributions are derived.

Theorem 2.1. *Let $\boldsymbol{X} \sim MHG(\alpha, \beta, \boldsymbol{\Sigma}, \boldsymbol{U})$. Then the Laplace transformation of $\boldsymbol{X}$ is*

$$\varphi_{\boldsymbol{X}}(\boldsymbol{T}) = \frac{{}_{p+1}F_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{U}(\boldsymbol{T} + \frac{1}{\beta}\boldsymbol{\Sigma}^{-1})^{-1})}{{}_{p+1}F_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{U}\beta\boldsymbol{\Sigma})} \det(\boldsymbol{I}_p + \beta\boldsymbol{\Sigma}\boldsymbol{T})^{-\alpha}, \tag{8}$$

where $\boldsymbol{T}$ is a $m \times m$ matrix.

Theorem 2.2. *Let $\boldsymbol{X} \sim MHG(\alpha, \beta, \boldsymbol{\Sigma}, \boldsymbol{U})$. Then*

$$E(\det(\boldsymbol{X})^h) = \frac{\Gamma_p(\alpha + h)}{\Gamma_p(\alpha)}\beta^{hp} \det(\boldsymbol{\Sigma})^h.$$

Theorem 2.3. *Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be independent, $\boldsymbol{X}_1 \sim MHG(\alpha_1, \beta, \boldsymbol{\Sigma}, \boldsymbol{U})$ and $\boldsymbol{X}_2 \sim MG_p(\alpha_2, \beta, \boldsymbol{\Sigma})$. Then the p.d.f of $\boldsymbol{Z} = \boldsymbol{X}_2^{-1/2}\boldsymbol{X}_1\boldsymbol{X}_2^{-1/2}$ is given by*

$$f(\boldsymbol{Z}) = \frac{\det(\boldsymbol{Z})^{\alpha_1 - \frac{m+1}{2}} \det(\boldsymbol{Z} + \boldsymbol{I}_m)^{-(\alpha_1+\alpha_2)}}{\Gamma_m(\alpha_1)\Gamma_m(\alpha_2)}$$
$$\times \frac{{}_{p+1}F_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{Z}\boldsymbol{U}\beta(\boldsymbol{Z} + \boldsymbol{I}_m)^{-1}\boldsymbol{\Sigma})}{{}_{p+1}F_q(a_1, ..., a_p; b_1, ..., b_q; \boldsymbol{U}\beta\boldsymbol{\Sigma})}. \tag{9}$$

Theorem 2.4. *Let $\boldsymbol{Y} \sim IMHG(\alpha, \kappa, \boldsymbol{\Sigma}, \boldsymbol{U})$. Then*

$$E(\det(\boldsymbol{Y})^h) = \frac{\Gamma_p(\alpha - h)}{\Gamma_p(\alpha)} det(\boldsymbol{\Sigma})^{-h}\beta^{-hp}.$$

Theorem 2.5. *Let $\boldsymbol{Y} \sim IMHG(\alpha, \beta, \boldsymbol{\Sigma}, \boldsymbol{U})$ and $\boldsymbol{A}(m \times m)$ be a constant symmetric matrix. Then*

$$\boldsymbol{A}\boldsymbol{W}\boldsymbol{A}' \sim IMHG_p(\alpha, \beta, \boldsymbol{A}'^{-1}\boldsymbol{\Sigma}\boldsymbol{A}^{-1}, \boldsymbol{A}\boldsymbol{U}\boldsymbol{A}').$$

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Preliminary test shrinkage estimator in the exponential distribution under progressively Type-II censoring

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Abstract

In this paper and based on progressively Type-II censored samples, we propose the preliminary test shrinkage estimation (SPTE) for the unknown parameter of the exponential distribution. It is shown that the proposed estimator dominates the corresponding classical estimators in the neighborhood of null hypothesis.

Keywords: Exponential distribution; MSE; Preliminary test shrinkage estimation; Progressively Type-II censoring, Relative efficiency.
Mathematics Subject Classification [2010]: 62F03, 62F10, 62F30

1 Introduction

The progressive Type-II censoring, after starting the life-testing experiment with n units can be described as follows: n units are put on life test at time 0. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then, immediately following the second failure, R_2 surviving units are removed from the test at random. This process continues until, at the time of the m-th failure, all the remaining $R_m = n - R_1 - R_2 - ... - R_{m-1} - m$ units are removed from the experiment. The R_i's are fixed prior to study. If $R_1 = R_2 = ... = R_m = 0$, we have $n = m$ which corresponds to the complete sample situation. If $R_1 = R_2 = ... = R_{m-1} = 0$, then $R_m = n - m$ which corresponds to the conventional Type-II right censoring scheme. For more details, see Balakrishnan and Aggarwala (2000).

Based on complete, censored and record data, the preliminary test and preliminary test shrinkage estimators have been discussed by some authors in exponential distribution . See for example, Baklizi (2010) and Golam Kibria and Saleh (2010). But these estimators have not been discussed in the literature based on progressively type-II censored data. In this paper, we consider the preliminary test shrinkage estimator for the unknown parameter of the exponential distribution under progressively Type-II censoring.

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2 Main results

Let $X_{1:m:n} = X_1, \cdots, X_{m:m:n} = X_m$ be a progressively Type-II censored sample from the exponential distribution with the probability density function (pdf)

$$f(x) = \frac{1}{\theta}\, e^{-\frac{x}{\theta}}, \quad x > 0, \quad \theta > 0. \tag{1}$$

The maximum likelihood estimator (MLE) and the best linear unbiased estimator (BLUE) of θ is (see Balakrishnan and Aggarwala, 2000)

$$\hat{\theta} = \frac{\sum_{i=1}^{m}(R_i+1)X_i}{m}. \tag{2}$$

Further, using the property of spacings, it can be shown that (see Balakrishnan and Aggarwala, 2000)

$$T = \frac{2m\hat{\theta}}{\theta} = \frac{2\sum_{i=1}^{m}(R_i+1)X_i}{\theta} \sim \chi^2_{2m}.$$

Now, let $1-\alpha = F_{2m}(c_2) - F_{2m}(c_1)$, where $F_{2m}(.)$ stands for the χ^2 cdf with 2m degrees of freedom, $1-\alpha/2 = F_{2m}(c_2)$ and $\alpha/2 = F_{2m}(c_1)$ where c_1 and c_2 are the critical values from the chi-square distribution with 2m degrees of freedom. Our aim is to obtain a preliminary test shrinkage estimator of θ, when a priori suspected $\theta = \theta_0$ is available. Often the information on the value of θ is available from knowledge or previous experiment. This non-sample prior information can be expressed in the form of following test of the hypothesis

$$H_0 : \theta = \theta_0, \quad vs. \quad H_a : \theta \neq \theta_0.$$

Now we will choose $\hat{\theta}$ or θ_0 based on the rejection of H_0 or do not reject of H_0. The preliminary test (PT) estimator of θ denoted by , $\hat{\theta}^{PT}$, is defined as follows: $\hat{\theta}^{PT} = \theta_0$ if we do not reject H_0 and $\hat{\theta}^{PT} = \hat{\theta}$ if we reject H_0. By likelihood ratio test, we reject H_0 when $\chi^2_{(2m)} \in \bar{A}$, where

$$A = \{T : c_1 < T < c_2\} \quad , c_1 = \chi^2_{\alpha/2,(2m)}, \quad c_2 = \chi^2_{1-\alpha/2,(2m)}.$$

For $0 \leq k \leq 1$, the preliminary test shrinkage (PTS) estimator of θ is defined by (see Baklizi, 2010)

$$\hat{\theta}^{PTS} = \hat{\theta}(1 - I(A)) + [k\hat{\theta} + (1-k)\theta_0]I(A). \tag{3}$$

For $k = 0$, this estimator reduces to the preliminary test estimator (PTE)

$$\hat{\theta}^{PTE} = \hat{\theta}(1 - I(A)) + \theta_0 I(A). \tag{4}$$

Notice that

$$E(I(A)) = P(c_1 < T < c_2) = 1 - \alpha.$$

2.1 Comparison of PTS estimator and usual estimator

The MSE of the MLE is

$$MSE(\hat{\theta}) = var(\hat{\theta}) = \frac{\theta^2}{m}. \tag{5}$$

Now, let us define $\lambda = \frac{\theta_0}{\theta}$. The MSE of the PTS estimator can be shown to be

$$\begin{aligned} MSE(\hat{\theta}^{PTS}) = & \frac{\theta^2}{m} + \frac{\theta_0^2(m+1)(k^2-1)}{m\lambda^2}\{F_{2m+4}(c_2) - F_{2m+4}(c_1)\} \\ & - 2\lambda\theta^2(k^2-k)\{F_{2m+2}(c_2) - F_{2m+2}(c_1)\} + \theta^2[(1-k)\lambda^2 - 2\lambda](1-k)(1-\alpha) \\ & + 2(1-k)\theta^2\{F_{2m+2}(c_2) - F_{2m+2}(c_1)\} \end{aligned} \tag{6}$$

Now, the relative efficiency of $\hat{\theta}^{PTS}$ compare to $\hat{\theta}$ is

$$\begin{aligned} RE(\hat{\theta}^{PTS}, \hat{\theta}) = & \frac{MSE(\hat{\theta})}{MSE(\hat{\theta}^{PTS})} \\ & = [1 + (m+1)(k^2-1)\{F_{2m+4}(c_2) - F_{2m+4}(c_1)\} \\ & - 2m\lambda(k^2-k)\{F_{2m+2}(c_2) - F_{2m+2}(c_1)\} \\ & + m(1-k)(1-\alpha)\{(1-k)\lambda^2 - 2\lambda\} \\ & + 2m(1-k)\{F_{2m+2}(c_2) - F_{2m+2}(c_1)\}]^{-1}. \end{aligned} \tag{7}$$

Figure 1 shows several relative efficiency graphs for various values of m. From this Figure, we can see that the $\hat{\theta}^{PTS}$ dominates the usual estimator $\hat{\theta}$ in the neighborhood of the null hypothesis. Table 1 presents the range of λ for which $\hat{\theta}^{PTS}$ dominates $\hat{\theta}$ for $k = 0.5$ and different m and α. From the table, it is also evident that the proposed PTS estimator dominates the usual estimator near the null hypothesis.

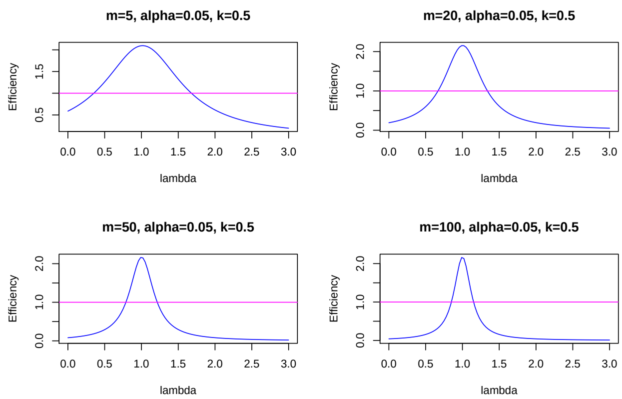

Figure 1: Relative efficiency of $\hat{\theta}^{PTS}$ for different values of m

3 Numerical example

Here we consider the progressively type-II censored data reported in Viveros and Balakrishnan (1994). Data present the results of a life-test experiment in which specimens of a

Table 1: Range of λ for which $\hat{\theta}^{PTS}$ dominates $\hat{\theta}$ for $k = 0.5$ and different m , α

m	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
5	[0.425,1.623]	[0.352,1.679]	[0.265,1.743]
10	[0.584,1.440]	[0.534,1.481]	[0.477,1.527]
20	[0.702,1.310]	[0.668,1.339]	[0.629,1.372]
30	[0.755,1.252]	[0.727,1.277]	[0.697,1.304]

type of electrical insulating fluid were subject to a constant voltage stress(34 KV /minutes). The observations and the censoring scheme applied, are reported in Table 2. For

Table 2: Progressively censored data given in the Example.

i	1	2	3	4	5	6	7	8
X_i	0.19	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	0	0	3	0	3	0	0	5

the progressively censored data reported in Table 2, the MLE of θ is $\hat{\theta} = 14.258$. Let us now consider the estimation of θ, when the prior guess is $\theta_0 = 7$. Therefore, we want to test $H_0 : \theta = 7, \quad vs. \quad H_1 : \theta \neq 7$. The value of the test statistic is

$$T_0 = \frac{2m\widehat{\theta}}{\theta_0} = \frac{2(16)(14.258)}{7} = 32.589$$

Since $T_0 \notin (\chi^2_{2m,\alpha/2}, \chi^2_{2m,1-\alpha/2}) = (6.907, 28.845)$, the preliminary test rejects the null hypothesis that $\theta = 7$, hence the preliminary test shrinkage estimation $\hat{\theta}^{PTS}$ is equal to the MLE. If we consider the prior guess as $\theta_0 = 14$, then since $T_0 = \frac{2(16)(14.258)}{14} = 16.294 \in (6.907, 28.845)$, the null hypothesis is not rejected by the preliminary test. In this case, the PTS estimator for θ is $\hat{\theta}^{PTS} = 14.129$.

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Robust mixture regression model fitting by slash distribution with application to musical tones

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Abstract

The traditional estimation of mixture regression models is based on the normal assumption of component errors and thus is sensitive to outliers or heavy-tailed errors. A robust mixture regression model based on the slash distribution by extending the mixture of slash distributions to the regression setting is proposed. Using the fact that the slash distribution can be written as a scale mixture of a normal and a latent distribution, this procedure is implemented by an EM algorithm. Finally, the proposed method is compared with other procedures, based on a real data set.

Keywords: EM algorithm, Normal mixture regression, Outliers
Mathematics Subject Classification [2010]: 62J05, 62F35

1 Introduction

Mixture regression models (MRM) are well known as switching regression models in the econometrics literature, which were introduced by Goldfeld and Quandt [4]. These models have been widely used to investigate the relationship between variables coming from several unknown latent homogeneous groups and applied in many fields, such as business, marketing, and social sciences.

In general, a normal mixture regression model (N − MRM) is defined as: let Z be a latent class variable such that given $Z = j$, the response y depends on the p-dimensional predictor $\mathbf{x}$ in a linear way

$$Y = \mathbf{x}^\top \boldsymbol{\beta}_j + \epsilon_j, \quad j = 1, \ldots, m, \tag{1}$$

where m is the number of groups (also called components in mixture models) in the population, the $\boldsymbol{\beta}_j$ are unknown p-dimensional vectors of regression coefficients and $\epsilon_j \sim N(0, \sigma_j^2)$ is independent of $\mathbf{x}$. Suppose $P(Z = j) = \pi_j$ and Z is independent of $\mathbf{x}$, then the conditional density of Y given $\mathbf{x}$, without observing Z, is

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$$\psi(y;\mathbf{x},\boldsymbol{\delta}) = \sum_{j=1}^{m} \pi_j \phi(y;\mathbf{x}^\top \boldsymbol{\beta}_j, \sigma_j^2), \tag{2}$$

where $\phi(\cdot;\mu,\sigma^2)$ is the density function of $N(\mu,\sigma^2)$ and $\boldsymbol{\delta} = (\boldsymbol{\delta}_1^\top, \ldots, \boldsymbol{\delta}_m^\top)^\top$ with $\boldsymbol{\delta}_j = (\pi_j, \boldsymbol{\beta}_j^\top, \sigma_j^2)^\top$.

The MLE $\boldsymbol{\delta}$ in (2) works well when the error distribution is normal. However, the normality based MLE is sensitive to outliers or heavy-tailed error distributions. Markatou [2] proposed using a weight factor for each data point to robustify the estimation procedure for mixture regression models. Neykov *et al.* [3] proposed robust fitting of mixtures using the trimmed likelihood estimator.

In this article, we propose a robust mixture regression model based on slash distribution by extending the mixture of slash distribution to the regression setting. In Section 2, we present the slash-MRM, including the EM algorithm for maximum likelihood (ML) estimation. Finally, a real example is given to illustrate the performance of the proposed method.

2 The proposed model

In order to more robustly estimate the mixture regression parameters, we assume that the error density function in (1) is a slash distribution with parameter $q_j > 0$ and scale parameter $\sigma_j > 0$:

$$f(\epsilon_j;\sigma_j,q_j) = \frac{q_j}{\sigma_j}\int_0^1 u^{q_j}\phi\left(\frac{u\epsilon_j}{\sigma_j};0,1\right)du, \quad \epsilon_j \in \mathbb{R}, \quad j=1,\ldots,m.$$

The mixture regression model with slash distribution can be formulated in a similar way to the model defined in (2) as follows:

$$g(y;\mathbf{x},\boldsymbol{\Theta}) = \sum_{j=1}^{m} \pi_j f(y - \mathbf{x}^\top\boldsymbol{\beta}_j;\sigma_j,q_j),$$

where $f(\cdot;\sigma,q)$ is the density function of the slash distribution and $\boldsymbol{\Theta} = (\boldsymbol{\theta}_1^\top,\ldots,\boldsymbol{\theta}_m^\top)^\top$ with $\boldsymbol{\theta}_j = (\pi_j, \boldsymbol{\beta}_j^\top, \sigma_j, q_j)^\top$.

2.1 Maximum likelihood estimation via EM algorithm

In this subsection, we present an EM algorithm for the ML estimation of the mixture regression model with slash distribution. For $j = 1,\ldots,m$, and $i = 1,\ldots,n$, denote Z_{ij} as latent Bernoulli variables such that

$$z_{ij} = \begin{cases} 1, & \text{if the ith observation is from the jth component,} \\ 0, & \text{otherwise.} \end{cases}$$

If the complete data set $\mathbf{T} = \{(\mathbf{x}_i, y_i, z_{ij});\ i=1,\ldots,n, j=1,\ldots,m\}$ is observable, the complete log likelihood function of $\boldsymbol{\Theta}$ can be written as

$$\ell(\boldsymbol{\Theta};\mathbf{T}) = \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}\log\left\{\pi_j f\left(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j;\sigma_j,q_j\right)\right\}.$$

Note that the above maximizer does not have explicit solutions for $\boldsymbol{\beta}_j^\top, \sigma_j$ and q_j. The computation can be further simplified based on the fact that the slash distribution can be considered a scale mixture of normal distributions. Let u be the latent variable such that

$$\epsilon|u \sim N(0, \sigma^2/u^2), \qquad u \sim Beta(q, 1),$$

where $Beta(\alpha, \beta)$ has density $f(u; \alpha, \beta) = b(\alpha, \beta)u^{\alpha-1}(1-u)^{\beta-1}$, $0 < u < 1$, where $b(\alpha, \beta)$ is the Beta function. Then, marginally ϵ has a slash distribution with parameter q and scale parameter σ. Therefore, we can simplify the computation of M step of the proposed EM algorithm by introducing another latent variable u. Therefore, the complete log likelihood function for $(\mathbf{u}, \mathbf{T})$ is

$$\begin{aligned}\ell_c(\boldsymbol{\Theta}; \mathbf{T}, \mathbf{u}) &= \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}\log\pi_j + \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}\log(q_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}(q_j-1)\log(u_i) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}\left\{-\frac{1}{2}\log(2\pi\sigma_j^2) + \log(u_i) - \frac{u_i^2}{2\sigma_j^2}\left(y_i - \mathbf{x}_i^T\beta_j\right)^2\right\},\end{aligned}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ is independent of $\mathbf{z} = (z_{11}, \ldots, z_{nm})$.

Based on the EM algorithm principle, in E-step on the $(k+1)^{th}$ iteration, we need to calculate the conditional expectation of the log-likelihood function of complete data, which is $E\left(\ell_c(\boldsymbol{\Theta}; \mathbf{T}, \mathbf{u})|\mathbf{y}, \mathbf{X}, \boldsymbol{\Theta}^{(k)}\right)$. Based on the above argument, the E-step requires the calculations of $p_{ij}^{(k+1)} = E\left(Z_{ij}|\mathbf{y}, \mathbf{X}, \boldsymbol{\Theta}^{(k)}\right)$, $u_{ij}^{(k+1)} = E\left(U_i^2|\mathbf{y}, \mathbf{X}, z_{ij}=1, \boldsymbol{\Theta}^{(k)}\right)$ and $l_{ij}^{(k+1)} = E\left(\log(U_i)|\mathbf{y}, \mathbf{X}, z_{ij}=1, \boldsymbol{\Theta}^{(k)}\right)$. Thus, the EM algorithm can be written as:

(1) Choose some initial value $\boldsymbol{\Theta}^{(0)} = (\pi_1^{(0)}, \boldsymbol{\beta}_1^{(0)}, \sigma_1^{(0)}, q_1^{(0)}, \ldots, \pi_m^{(0)}, \boldsymbol{\beta}_m^{(0)}, \sigma_m^{(0)}, q_m^{(0)})^\top$.

(2) E-step: On the $(k+1)^{th}$ iteration, according to Bayes theorem, we can compute conditional expectations as follows:

$$p_{ij}^{(k+1)} = \frac{\pi_j^{(k)} f(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j^{(k)}; \sigma_j^{(k)}, q_j^{(k)})}{\sum_{l=1}^{m}\pi_l^{(k)} f(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_l^{(k)}; \sigma_l^{(k)}, q_l^{(k)})}, \quad l_{ij}^{(k+1)} = \int_0^1 \frac{q_j u_i^{q_j}\log(u_i) e^{-\frac{u_i^2(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j)^2}{2\sigma_j^2}}}{\sigma_j\sqrt{2\pi} f(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j; \sigma_j, q_j)} du_i,$$

and

$$u_{ij}^{(k+1)} = \begin{cases} \dfrac{q_j 2^{\frac{q_j}{2}}\left(\frac{(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j)^2}{\sigma_j^2}\right)^{-\frac{q_j+3}{2}}}{\sigma_j\sqrt{\pi} f(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j; \sigma_j, q_j)}\Gamma\left(\frac{q_j+3}{2}\right) G\left(\frac{(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j)^2}{2\sigma_j^2}; \frac{q_j+3}{2}, 1\right), & \text{if } y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j \neq 0, \\ \frac{q_j}{q_j+3}\frac{1}{\sigma_j\sqrt{2\pi} f(0; \sigma_j, q_j)}, & \text{if } y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j = 0, \end{cases}$$

where $\Gamma(\cdot)$ and $G(\cdot; r, 1)$ are the complete gamma function and the cdf of the gamma distribution with parameters shape r and scale 1, respectively.

(3) M-step: On the $(k+1)^{th}$ iteration, compute the estimator of parameters which maximize the expected complete log-likelihood. The estimators can be written as:

$$\pi_j^{(k+1)} = \sum_{i=1}^{n}\frac{p_{ij}^{(k+1)}}{n}, \quad \boldsymbol{\beta}_j^{(k+1)} = \left(\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^\top p_{ij}^{(k+1)} u_{ij}^{(k+1)}\right)^{-1}\left(\sum_{i=1}^{n}\mathbf{x}_i y_i p_{ij}^{(k+1)} u_{ij}^{(k+1)}\right),$$

$$\sigma_j^{(k+1)} = \left\{\frac{\sum_{i=1}^{n} p_{ij}^{(k+1)} u_{ij}^{(k+1)}(y_i - \mathbf{x}_i^\top\boldsymbol{\beta}_j^{(k+1)})^2}{\sum_{i=1}^{n} p_{ij}^{(k+1)}}\right\}^{1/2}, \text{ and } q_j^{(k+1)} = -\frac{\sum_{i=1}^{n} p_{ij}^{(k+1)}}{\sum_{i=1}^{n} p_{ij}^{(k+1)} l_{ij}^{(k+1)}}.$$

(4) Repeat the E-step and M-step until the convergence is obtained. One stopping rule we can choose is to stop the iteration when the change of the likelihood value is smaller than 10^6 or runs are more than 500.

3 A real example

We illustrate our proposed methods with a data set obtained from Cohen [1], representing the perception of musical tones by musicians. In The experiment recorded 150 trials from the same musician. The overtones were determined by a stretching ratio, which is the ratio between adjusted tone and the fundamental tone. The purpose of this experiment was to see how this tuning ratio affects the perception of the tone and to determine whether either of two musical perception theories was reasonable.

These data were analyzed recently by Yao *et al.* [5], leading them to propose a robust mixture regression using the t-distribution. Now we revisit this data set with the aim of expanding the inferential results to the slash distribution. Table 1 present the ML estimates of the parameters from the normal, t and slash models. For comparing purposes of various models, we used Akaike (AIC) and Bayesian (BIC) information criteria.

Table 1: Fitted various models on the tone perception data set.

Model	$\hat{\beta}_{01}$	$\hat{\beta}_{11}$	$\hat{\beta}_{02}$	$\hat{\beta}_{12}$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{q}_1$	$\hat{q}_2$	$\hat{\pi}$	$\hat{\ell}$	AIC	BIC
Normal	-0.039	1.008	1.892	0.056	0.084	0.084	-	-	0.325	107.257	-200.513	-179.439
t	0.006	0.998	1.978	0.017	0.011	0.011	1	1	0.485	202.804	-387.608	-360.513
Slash	0.003	0.999	1.954	0.029	0.002	0.020	0.569	1.455	0.443	229.436	-440.871	-413.776

From Table 1, it appears that the slash model present a better fit than all other models.

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Testing Statistical Hypothesis of exponential populations with multiply sequential order statistics

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Abstract

Sequential order statistics (SOS) coming from non-homogeneous exponential distributions are considered in this paper. The generalized likelihood ratio (GLRT) and the Bayesian tests are derived for testing homogeneity of the exponential populations. It is shown that the GLRT in this case is also scale invariant. The maximum likelihood and the Bayesian estimates of parameters are derived on the basis of observed SOS samples. Explicit expression for SOS-based Bayes factor (BF) are derived.

Keywords: Bayes, GLRT, Sequential order statistics, Estimation
Mathematics Subject Classification [2010]: 62N05, 62G30, 62P30

1 Introduction

Let $X_1, \cdots, X_n$ be independent and identically distributed (i.i.d.) random variables with a common distribution function (DF), say F, and denoted by $X_1, \cdots, X_n \overset{i.i.d.}{\sim} F$. Denote in magnitude order of $X_1, \cdots, X_n$ by $X_{1:n} \leq \cdots \leq X_{n:n}$, which are called *order statistics* (OSs). In engineering system reliability analyses, lifetimes of r-out-of-n systems, say T, coincide to $X_{r:n}$ in which $X_1, \cdots, X_n$ stand for component lifetimes. When the component lifetimes $X_1, \cdots, X_n \overset{i.i.d.}{\sim} F$, the OSs are used for describing the system lifetime. Notice that failing a component does not change here the lifetimes of the surviving components. Motivated by Cramer and Kamps [1], the failure of a component may result in a higher load on the surviving components and hence causes the lifetime distributions change. In these cases, the system lifetimes may be adequate to model by the concept of *sequential order statistics* (SOSs) as an extension of OSs. Cramer and Kamps [1] considered the problem of estimating the parameters on the basis of s independent SOSs samples under a conditional proportional hazard rates (CPHR) model, defined by $\bar{F}_j(t) = \bar{F}_0^{\alpha_j}(t)$ for $j = 1, \cdots, r$, where the underlying CDF $F_0(t)$ is the exponential distribution, i.e.

$$F_0(x;\sigma) = 1 - \exp\left\{-\left(\frac{x}{\sigma}\right)\right\}, \quad x > 0, \quad \sigma > 0. \tag{1}$$

This paper develops testing Statistical Hypothesis for homogeneity of the exponential populations in section 2. In section 3, the Bayesian approach is used and Bayes factor is derived for evaluating support of data for homogeneity of populations.

*Speaker

2 SOS-based likelihood analysis

We here assume that $s \geq 2$ independent SOS samples of equal size r from s heterogeneous populations are available. The data may be represented by $\mathbf{x} = [[x_{ij}]]_{1\leq i\leq s, 1\leq j\leq r}$ where the i-th row of the matrix $\mathbf{x}$ denotes the SOS sample coming from the i-th population. The LF of the available data is then

$$L(\mathcal{F};\mathbf{x}) = (\frac{n!}{(n-r)!})^s \prod_{i=1}^{s} \left(\prod_{j=1}^{r-1} \left[f_j^{[i]}(x_{ij}) \left(\frac{\bar{F}_j^{[i]}(x_{ij})}{\bar{F}_{j+1}^{[i]}(x_{ij})} \right)^{n-j} \right] f_r^{[i]}(x_{ir}) \bar{F}_r^{[i]}(x_{ir})^{n-r} \right), \quad (2)$$

where $\mathcal{F} = \{F_j^{[i]}, i = 1, \cdots, s, j = 1, \cdots, r\}$ and for $i = 1, \cdots, s$, $j = 1, \cdots, r$, $\bar{F_j}^{[i]}(x) = 1 - F_j^{[i]}(x)$. By substituting Equation (1) into Equation(2), under the earlier mentioned CPHR model, the LF of the available data reduces to

$$L(\sigma_1, \cdots, \sigma_s, \boldsymbol{\alpha}; \mathbf{x}) = (\frac{n!}{(n-r)!})^s \left(\prod_{j=1}^{r} \alpha_j \right)^s \left(\prod_{i=1}^{s} \frac{1}{\sigma_i} \right)^r \exp\Big\{ -\sum_{i=1}^{s}\sum_{j=1}^{r} \Big(\frac{x_{ij} m_j}{\sigma_i} \Big) \Big\}. \quad (3)$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_r)$, and for $j = 1, \cdots, r$, $\alpha_j > 0$, and $m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1}$ with convention $\alpha_{r+1} \equiv 0$. We consider the problem of homogeneity testing on the basis of independent SOS samples from different exponential populations, i.e.,

$$H_0 : \sigma_1 = \cdots = \sigma_s \quad v.s \quad H_1 : \sigma_i \neq \sigma_j \quad \exists i \neq j. \quad (4)$$

Following Cramer and Kamps [2] and Esmailian and Doostparast [4], two cases are considered in sequel: (i) $\boldsymbol{\alpha}$ known, and (ii) $\boldsymbol{\alpha}$ unknown. First suppose that the vector parameter $\boldsymbol{\alpha}$ in Equation (3) is known. By Theorem 8.1 in Cramer and Kamps [3] and under the null hypothesis H_0 in (4), the unique ML estimate of the common mean of the s exponential populations, say σ_0, is

$$\hat{\sigma}_0 = \frac{\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij} m_j}{rs} = \frac{\sum_{i=1}^{s}\sum_{j=1}^{r} (n-j+1)\alpha_j D_{ij}}{rs}, \quad (5)$$

where $D_{ij} = x_{ij} - x_{i,j-1}$, for $j = 1, \cdots, r$. When the baseline exponential populations are heterogeneous, from Equation (5), the unique ML estimate of σ_i $(i = 1, \cdots, s)$ is derived as

$$\hat{\sigma}_i = \frac{\sum_{j=1}^{r} x_{ij} m_j}{r} = \frac{\sum_{j=1}^{r} (n-j+1)\alpha_j D_{ij}}{r}. \quad (6)$$

The generalized likelihood ratio test (GLRT) statistic for testing the problem (4) is

$$\Lambda_1 = \frac{\sup_{\Omega_0} L(\sigma_1, \cdots, \sigma_s; \mathbf{x})}{\sup_{\Omega} L(\sigma_1, \cdots, \sigma_s; \mathbf{x})} = \prod_{i=1}^{s} \left(\frac{\hat{\sigma}_i}{\hat{\sigma}_0} \right)^r \exp\Big\{ \sum_{i=1}^{s}\sum_{j=1}^{r} \Big(\frac{1}{\hat{\sigma}_i} - \frac{1}{\hat{\sigma}_0} \Big) m_j x_{ij} \Big\}, \quad (7)$$

The null hypothesis H_0 is rejected if $A(\boldsymbol{T}, \boldsymbol{\alpha}) > c$, where $\boldsymbol{T} = (T_1, \cdots, T_s)$ and $A(\boldsymbol{T}, \boldsymbol{\alpha}) = -\sum_{i=1}^{s} \log(T_i / \sum_{j=1}^{s} T_j)$. Since under the CPHR with the one-parameter exponential baseline CDF, we have $T_i = \sum_{j=1}^{r} (n-j+1)\alpha_j D_{ij} \sim \Gamma(r, \sigma_i)$, for $i = 1, \cdots, s$ (Cramer and Kamps [2]), the rejection region of GLRT reads $A(\boldsymbol{T}, \boldsymbol{\alpha}) > \chi_{2r,1-\gamma}/2$.

Remark 2.1. The family of distribution (3) is invariant with respect to the group of the scale transformations $\mathcal{G} = \{g_a : g_a(\mathbf{x}) = a\mathbf{x} = \{ax^{\star}_{ij}\}_{1\leq i\leq s, 1\leq j\leq r}, a > 0\}$. Also, the problem of hypotheses testing (4) remains invariant under $\mathcal{G}$ since $\bar{G}(\Omega) = \Omega$ and $\bar{G}(\Omega_0) = \Omega_0$ where $\Omega = \{(\sigma_1, \cdots, \sigma_s) : \sigma_i > 0, i = 1, \cdots, s\} = \mathbb{R}^{+s}$, $\Omega_0 = \{(\sigma_1, \ldots, \sigma_s) : \sigma_1 = \cdots = \sigma_s\}$ and $\bar{G} = \bar{g}_a(\sigma_1, \cdots, \sigma_s) = a(\sigma_1, \cdots, \sigma_s)$ is the induced group of transformations on the parameter space Ω by the group of scale transformations $\mathcal{G}$. Fortunately, the GLRT is invariant with respect to the group of the scale transformations.

Remark 2.2. The unique MLEs have asymptotically the multivariate normal distribution with mean vector $(\sigma_1, \cdots, \sigma_s)$ and the variance-covariance matrix $[i(\hat{\sigma}_1, \cdots, \hat{\sigma}_s)]^{-1}$; See, e.g., [7]. An approximate equi-tailed confidence interval for σ_i is $(\hat{\sigma}_i - z_{\gamma/2}\sqrt{\hat{\sigma}_i^2/r}, \hat{\sigma}_i + z_{\gamma/2}\sqrt{\hat{\sigma}_i^2/r})$, where z_γ stands for the γ-percentile of the standard normal distribution.

Now assume that the vector parameter $\boldsymbol{\alpha}$ in Equation (3) is unknown. After some algebraic manipulations, the likelihood equations are

$$\hat{\hat{\sigma}}_i = \frac{\sum_{j=1}^r x_{ij}\hat{m}_j}{r} = \frac{\sum_{j=1}^r (n-j+1)\hat{\alpha}_j D_{ij}}{r}, \quad i = 1, \cdots, s, \tag{8}$$

and

$$\hat{\alpha}_j = \frac{s}{(n-j+1)\sum_{i=1}^s D_{ij}/\hat{\hat{\sigma}}_i}, \quad j = 1, \cdots, r. \tag{9}$$

The ML estimates of the parameters are obtained numerically by solving the likelihood equations given by Equations (8) and (9). Consider again the hypotheses testing problem (4). It is easy to verify that the unique ML estimates of the parameters under the null hypothesis H_0 are

$$\hat{\hat{\sigma}}_0 = \frac{\sum_{i=1}^s \sum_{j=1}^r x_{ij}\hat{m}_{0,j}}{rs} = \frac{\sum_{i=1}^s \sum_{j=1}^r (n-j+1)\hat{\alpha}_{0,j} D_{ij}}{rs}, \tag{10}$$

and

$$\hat{\alpha}_{0,j} = \frac{s\hat{\hat{\sigma}}_0}{(n-j+1)\sum_{i=1}^s D_{ij}}, \quad j = 1, \cdots, r, \tag{11}$$

where $\hat{m}_{0,j} = (n-j+1)\hat{\alpha}_{0,j} - (n-j)\hat{\alpha}_{0,j+1}$, with convention $\hat{\alpha}_{0,r+1} \equiv 0$. Therefore, the GLRT statistic for the hypotheses testing problem (4) is

$$\Lambda_2 = \prod_{j=1}^r \left(\frac{\hat{\alpha}_{0,j}}{\hat{\alpha}_j}\right)^s \prod_{i=1}^s \left(\frac{\hat{\hat{\sigma}}_i}{\hat{\hat{\sigma}}_0}\right)^r \exp\Big\{\sum_{i=1}^s \sum_{j=1}^r \Big(\frac{\hat{m}_j}{\hat{\hat{\sigma}}_i} - \frac{\hat{m}_{0,j}}{\hat{\hat{\sigma}}_0}\Big) x_{ij}\Big\}, \tag{12}$$

where $\hat{m}_j = (n-j+1)\hat{\alpha}_j - (n-j)\hat{\alpha}_{j+1}$. The null hypothesis H_0 rejects if $-2\log\Lambda_2 > c$.

3 SOS-based Bayes analysis

We here consider the problem of estimating unknown parameters via a strict Bayesian approach. To do this, we assume that $\boldsymbol{\alpha}$ is known and suggest the conjugate prior distributions for the scale parameters $\sigma_i, i = 1, \cdots, s$, i.e. $\sigma_i \sim IG(a_i, b_i),\ i = 1, \cdots, s$, be independent random variables. which implies

$\sigma_i \mid \underline{\mathbf{x}} \sim IG\left(a_i + r, \sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} + b_i\right)$, $i = 1, \cdots, s$. As we expected given x, the parameter σ_i are independent.

An equi-tailed credible set at level γ for σ_i $(i = 1, \cdots, s)$ is obtained as

$$CI_i(\gamma) = \left(\frac{\sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} + b_i}{\chi_{2(a_i+r),(1+\gamma)/2}}, \frac{\sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij} + b_i}{\chi_{2(a_i+r),(1-\gamma)/2}} \right). \quad (13)$$

Therefore, a conservative simultaneously credible set at level γ is $CI_1(\gamma^{1/s}) \times \cdots \times CI_s(\gamma^{1/s})$ where "$\times$" stands for the *Cartesian product* in Euclidean space.

3.1 Bayesian Test

Under the null hypothesis $H_0 : \sigma_1 = \cdots = \sigma_s$, we assume that the common value of σ_i $(i = 1, \cdots, s)$, say σ, follows the $IG(a_0, b_0)$-distribution where a_0 and b_0 are known positive hyper parameters. Therefore, the *Bayes factor* is

$$BF = \frac{\Gamma(sr + a_0)}{\Gamma(a_0)} \frac{b_0^{a_0}}{(\sum_{i=1}^{s} T_i + b_0)^{sr+a_0}} \prod_{i=1}^{s} \frac{(T_i + b_i)^{a_i+1}}{a_i b_i^{a_i}}. \quad (14)$$

Under the "$0-K$" loss function, the Bayes test rejects the null hypothesis H_0 if $BF < (K_0\pi_1)/(K_1\pi_0)$, where π_i and K_i, for $i = 1, 2$, are prior rpobability for the hypothesis H_i and the loss of the accepting H_i when $H_j (j \neq i)$ is true, respectively.

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Talk The Exponentiated G Family of Power Series Distributions pp.: 1–4

The Exponentiated G Family of Power Series Distributions

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Abstract

In this paper, we introduce the exponentiated G-power series (EGPS)distributions which is obtained by compounding a new exponentiated family and power series distributions. We obtain several properties of the EGPS distribution such as its probability density function, quantiles, moments, order statistics, mean residual life and reliability function. Sub-models of this family are studied in a real example.

Keywords: Exponentiated family, Maximum likelihood estimation, Power series distributions.
Mathematics Subject Classification [2010]: 60E05, 62E10

1 Introduction

The exponential distribution is commonly used in many applied problems, particularly in lifetime data analysis. A generalization of this distribution is the exponentiated family. It is a lifetime distribution and is often applied to describe the distribution of adult life spans by actuaries and demographers. The exponentiated family is considered for the analysis of survival in some sciences such as biology, gerontology, computer, and marketing science. A random variable X is said to have a Exp-G denoted by $X \sim Exp - G(\alpha)$, if its cumulative distribution function (cdf) and the probability density function (pdf) are given by $H_\alpha(x;\Theta) = [G(x;\Theta)]^\alpha$ and $h_\alpha(x;\Theta) = \alpha g(x;\Theta)[G(x;\Theta)]^{\alpha-1}$ respectively. This family contains many exponentiated distributions such as exponentiated Weibull, exponentiated exponential, exponentiated Pareto and etc.

In this paper, we compound the exponentiated G family and power series distributions, and introduce a new class of distribution. This procedure follows similar way that was previously carried out by some authors: The exponential power series distribution is introduced by [1] ; the Weibull-power series distributions is introduced by [3] and the generalized exponential power series distribution is introduced by [2]

The remainder of our paper is organized as follows: In Section 2, we give the pdf and cdf of EGPS model. Some properties such as quantiles, moments, order statistics, mean residual life, reliability function and maximum likelihood estimator(MLE) are given in Section 3. An application of EGPS model is given in the Section 4.

*Speaker

2 The EGPS model

A discrete random variable, N is a member of power series distributions (truncated at zero) if its probability mass function is given by

$$p_n = P(N = n) = \frac{a_n \lambda^n}{C(\lambda)}, \quad n = 1, 2, \ldots, \tag{1}$$

where $a_n \geq 0$, $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$, and $\lambda \in (0, s)$ is chosen in a way such that $C(\lambda)$ is finite and its first, second and third derivatives are defined and shown by $C'(.)$, $C''(.)$ and $C'''(.)$. This family of distributions includes many of the most common distributions, including the binomial, Poisson, geometric, negative binomial. We define the Exp-G class of distributions as $F(x) = \sum_{n=1}^{\infty} \frac{a_n (\lambda H_\alpha(x))^n}{C(\lambda)} = \frac{C(\lambda (G(x))^\alpha)}{C(\lambda)}$,

and denote by $EGPS(\alpha, \lambda, \Theta)$. The pdf of $EGPS(\alpha, \lambda, \Theta)$ is given by

$$f(x) = \frac{\lambda \alpha g(x) (G(x))^{\alpha-1} C'\left(\lambda (G(x))^\alpha\right)}{C(\lambda)}. \tag{2}$$

This class of distributions can be applied to reliability problems. Some properties of EGPS model are presented in the following propositions.

Proposition 2.1. *The pdf's of EGPS class can be expressed as infinite linear combination of density of order distribution, i.e. it can be written as*

$$f(x) = \alpha \lambda g(x) \frac{(G(x))^{\alpha-1} C'(\lambda (G(x))^\alpha)}{C(\lambda)} = \sum_{n=1}^{\infty} p_n h_{n\alpha}(x), \tag{3}$$

where $h_{n\alpha}(x)$ is the pdf of $Y_{(n)} = \max(Y_1, Y_2, ..., Y_n)$, given by $h_{n\alpha}(x) = n\alpha g(x)[G(x)]^{n\alpha-1}$, i.e. Exp-G distribution with parameter $n\alpha$. Also, we obtained

$F(x) = \sum_{n=1}^{\infty} p_n H_{n\alpha}(x) = \sum_{n=1}^{\infty} p_n (G(x))^{n\alpha}$. So, the EGPS distribution is a mixture of Exp-G family.

Proposition 2.2. $\lim_{\lambda \to 0^+} F(x) = [G(x)]^{c\alpha}$, *which is a Exp-G distribution with parameter $c\alpha$, where $c = min\{n \in N : a_n > 0\}$.*

Proposition 2.3. *If $G(x) = 1 - \exp(-\beta x)$, then $F(x) = \frac{C(\lambda (1-e^{-\beta x})^\alpha)}{C(\lambda)}$. In fact, it is the cdf of the generalized exponential-power series (GEPS) class of distribution and is introduced by [2]. Aslo, if $G(x) = 1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}$, then $F(x) = \frac{C(\lambda [1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}]^\alpha)}{C(\lambda)}$. This is the cdf of the generalized Gompertz-power series (GGPS) class of distribution. The GGPS model contains several lifetime models such as: generalized Gompertz-binomial (GGB), generalized Gompertz-Poisson (GGP), generalized Gompertz-geometric (GGG) and generalized Gompertz-logarithmic (GEL) distributions, generalized Gompertz (GG) as special cases.*

Proposition 2.4. *The hazard rate function of the EGPS class of distributions is*

$$r(x) = \frac{\lambda \alpha g(x) (G(x))^{\alpha-1} C'\left(\lambda (G(x))^\alpha\right)}{C(\lambda) - C(\lambda G(x))}.$$

3 Statistical properties

In this section, quantiles, moments, order statistics, mean residual life and reliability function of EGPS distribution are obtained.

Proposition 3.1. *If U has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation $X = H^{-1}\{\frac{C^{-1}(C(\lambda)U)}{\lambda}\}$ has the $EGPS(\alpha, \lambda, \Theta)$ distribution, where $C^{-1}(.)$ is the inverse function of $C(.)$.*

Proposition 3.2. *The moment generating function of EGPS class can be expressed as $M_X(t) = \sum_{n=1}^{\infty} p_n M_{Y_{(n)}}(t)$. Also, $\mu_r = E[X^r] = \sum_{n=1}^{\infty} p_n E[Y_{(n)}^r]$.*

Proposition 3.3. *The pdf of ith- order statistic is obtained as*

$$f_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{m-i} (-1)^j \binom{m-i}{j} p_n h_{n\alpha}(x) [\frac{C(\lambda H_\alpha(x))}{C(\lambda)}]^{j+i-1}.$$

Proposition 3.4. *An explicit expression of mean residual life function of X are obtained as*

$$m(t) = E[X - t | X > t] = \frac{C(\lambda) \sum_{n=1}^{\infty} p_n E[Z I_{(Z>t)}]}{C(\lambda) - C(\lambda H_\alpha(x))} - t.$$

Proposition 3.5. *In the stress - strength model, $R = P(X > Y)$ is a measure of component reliability . It has many applications especially in engineering concept. The quantity R for EGPS can be expressed as*

$$R = \sum_{n=1}^{\infty} p_n \int_0^{\infty} h_{n\alpha}(x) \frac{C(\lambda G^\alpha(x))}{C(\lambda)} dx.$$

Proposition 3.6. *Let $x_1, ..., x_n$ be observed value from the EGPS distribution wit parameters $\boldsymbol{\xi} = (\alpha, \lambda, \Theta)^T$. The total log-likelihood function for $\boldsymbol{\xi}$ is given by*

$$\begin{aligned} l_n = l_n(\boldsymbol{\xi}; x) &= n[\log(\alpha) + \log(\lambda) - \log(C(\lambda))] + \sum_{i=1}^{n} \log[g(x_i; \Theta)] \\ &+ (\alpha - 1) \sum_{i=1}^{n} \log t_i + \sum_{i=1}^{n} \log(C'(\lambda (t_i)^\alpha)), \end{aligned}$$

where $t_i = G(x_i; \Theta)$. The MLE of $\boldsymbol{\xi}$, say $\hat{\boldsymbol{\xi}}$, is obtained by solving the nonlinear system $U(\boldsymbol{\xi}; \boldsymbol{x}) = (\frac{\partial l_n}{\partial \alpha}, \frac{\partial l_n}{\partial \lambda}, \frac{\partial l_n}{\partial \Theta})^T = \mathbf{0}$. We cannot get an explicit form for this nonlinear system of equations and they can be calculated by using a numerical method, like the Newton method or the bisection method.

4 Real example

In this section, we consider the data consisting of the strengths of 1.5 cm glass fibers given in [4] and fit the Gompertz, GG, GGG, GGP, GGB (with $m = 5$), and GGL distributions. The MLE's of the parameters (with standard deviations) for the distributions are obtained. To test the goodness-of-fit of the distributions, we calculated the maximized log-likelihood, the Kolmogorov-Smirnov (K-S) statistic with its respective p-value, the AIC (Akaike Information Criterion), AICC (AIC with correction) and BIC (Bayesian Information Criterion) for the six distributions. The results are given in Table 1 and show that the GGG distribution yields the best fit among the GGP, GGB, GGL, GG and Gompertz distributions.

Table 1: Parameter estimates (with std.), K-S statistic, p-value, AIC, AICC and BIC.

Distribution	Gompertz	GG	GGG	GGP	GGB	GGL
$\hat{\beta}$	0.0088	0.0356	0.7320	0.1404	0.1032	0.1705
$s.e.(\hat{\beta})$	0.0043	0.0402	0.2484	0.1368	0.1039	0.2571
$\hat{\gamma}$	3.6474	2.8834	1.3499	2.1928	2.3489	2.1502
$s.e.(\hat{\gamma})$	0.2992	0.6346	0.3290	0.5867	0.6010	0.7667
$\hat{\alpha}$	—	1.6059	2.1853	1.6205	1.5999	2.2177
$s.e.(\hat{\alpha})$	—	0.6540	1.2470	0.9998	0.9081	1.3905
$\hat{\theta}$	—	—	0.9546	2.6078	0.6558	0.8890
$s.e.(\hat{\theta})$	—	—	0.0556	1.6313	0.5689	0.2467
$-\log(L)$	14.8081	14.1452	12.0529	13.0486	13.2670	13.6398
K-S	0.1268	0.1318	0.0993	0.1131	0.1167	0.1353
p-value	0.2636	0.2239	0.5629	0.3961	0.3570	0.1992
AIC	33.6162	34.2904	32.1059	34.0971	34.5340	35.2796
AICC	33.8162	34.6972	32.7956	34.78678	35.2236	35.9692
BIC	37.9025	40.7198	40.6784	42.6696	43.1065	43.8521

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Others (Applications of Mathematics in other Sciences)

A generalization of the Mertens' formula and analogue to the Wallis' product over primes

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Abstract

In this paper, we study the asymptotic expansion of the product $\prod_{p\leqslant x}(1+\frac{\alpha}{p})$ for each fixed real $\alpha > -2$ where the p runs over the prime numbers. As an application, we study the Wallis' product and its generalizations, running over primes p, which are analogue to Wallis product for $\frac{\pi}{2}$ running over positive integers.

Keywords: Prime numberse, Wallis' product, analytic computations.
Mathematics Subject Classification [2010]: 11A41, 11Y35 , 11N99

1 Introduction

A generalization of the Mertens formula. Among his interesting three results in number theory related to the density of the primes, Mertens [2] proved a result asserting, in todays notation, that

$$\prod_{p\leqslant x}\Big(1-\frac{1}{p}\Big)=\frac{e^{-\gamma}}{\log x}\Big(1+O\Big(\frac{1}{\log x}\Big)\Big) \tag{1}$$

where the product runs over primes and γ denotes the Eulers constant. Several generalizations, and also improvements on the O-term in the above formula are obtained [3]. In this note we study the following generalization.

Theorem 1.1. *Assume that $\alpha > -2$ and $\alpha \neq 0$ is a fixed real, and define the constant $C(\alpha)$ by*

$$C(\alpha)=e^{\alpha\gamma}\prod_{p}\Big(1-\frac{1}{p}\Big)^{\alpha}\Big(1+\frac{\alpha}{p}\Big). \tag{2}$$

Then for each $x > 1$ we have

$$\prod_{p\leqslant x}\Big(1+\frac{\alpha}{p}\Big)=C(\alpha)(\log x)^{\alpha}\Big(1+O\Big(\frac{1}{\log^2 x}\Big)\Big)$$

Moreover, if we assume that the Riemann Hypothesis is true, then one may reduce the above O-term up to $O(x^{\frac{1}{2}}\log x)$.

*Speaker

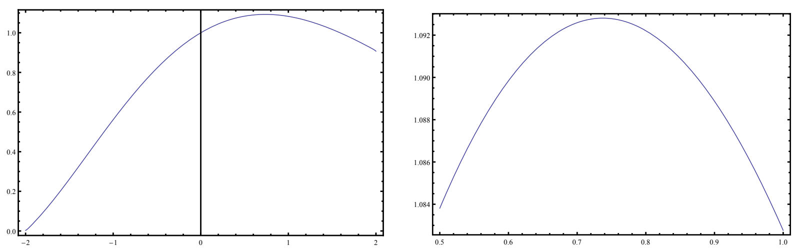

Figure 1: Graphs of $C(\alpha)$ for $\alpha \in (-2, 2)$ (left) and for $\alpha \in [0.5, 1]$ (right), where it attains a local maximum.

Appearance of the constant $C(\alpha)$ is an important part of the above generalization. As classical results we have

$$C(-1) = e^{-\gamma} \qquad \textit{and} \qquad C(1) = \frac{e^{\gamma}}{\zeta(2)} = \frac{6e^{\gamma}}{\pi^2}$$

While it doesnt seem easy to determine other values of $C(\alpha)$ in terms of well-known constants, we establish a method to compute its values for $\alpha \in (-2, 2)$ in terms of rapidly convergent series.
The following result describes this method.

2 Main results

Theorem 2.1. *For each $\alpha \in (-2, 2)$ we have*

$$C(\alpha) = e^{\alpha M + S(\alpha)}, \tag{3}$$

where M is the Meissel-Mertens constant,

$$S(\alpha) = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\alpha^n P(n)}{n}, \tag{4}$$

and P is the prime zeta function, defined for complex s with $\Re(s) > 1$ by

$$P(s) = \sum_{p} \frac{1}{p^s}.$$

related to the prime zeta function.

and consequently, the series of $S(\alpha)$ defined by 4 converges rapidly for $\alpha \in (-2, 2)$. This allows us to compute $S(\alpha)$ for $\alpha \in (-2, 2)$ numerically, and to generate a graph of $C(\alpha)$ for $\alpha \in (-2, 2)$. Moreover, we use the approximate value

$$M \approxeq 0.2614972128476427837554268386 09,$$

for the Meissel-Mertens constant. Figure 1 pictures the graph of $C(\alpha)$ for$\alpha \in (-2, 2)$. As this figure and more precisely numerical computations show, $C(\alpha)$ attains a local maximum at $\alpha_{\max} \approxeq 0.73738444$ with the value

$$\max_{\alpha \in (-2,2)} C(\alpha) \approxeq 1.09280370325023524.$$

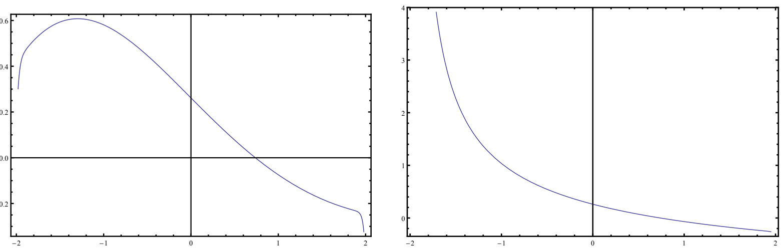

Figure 2: Graphs of $\frac{d}{d\alpha}C(\alpha)$ (left) and $M + \frac{d}{d\alpha}S(\alpha)$ (right) for $\alpha \in (-2, 2)$

More precisely, we have

$$\frac{d}{d\alpha}C(\alpha) = \Big(M + \frac{d}{d\alpha}S(\alpha)\Big)C(\alpha) = \Big(M + \sum_{n=2}^{\infty}(-1)^{n-1}\alpha^{n-1}P(n)\Big)C(\alpha),$$

and α_{max} is the unique solution of the equation$M + \frac{d}{d\alpha}S(\alpha) = 0$ in $(-2, 2)$. The right graph in Figure 2 pictures $M + \frac{d}{d\alpha}S(\alpha)$ for $(-2, 2)$. By numerical solving the above equation, we get more precise value

$$\alpha_{max} \approxeq 0.7373844381548068 61.$$

Wallis product over primes. As a consequence of Theorem 1.2, we obtain the following.

Corollary 2.2. *For each* $\alpha \in (-2, 2)$ *we have* $C(\alpha)C(\alpha) = e^{-T(\alpha)}$, *with*

$$T(\alpha) = \sum_{n=1}^{\infty} \frac{P(2n)\alpha^{2n}}{n}.$$

The function $T(\alpha)$ is useful to formulate an analogue to Wallis product over primes. We recall the Wallis product formula for π, which asserts that

$$\lim_{n\to\infty} \prod_{n=1}^{\infty} \Big(\frac{2n}{2n-1}\frac{2n}{2n+1}\Big) = \frac{\pi}{2}.$$

A formulation of the Wallis product over primes in a more general form is as follows.

Theorem 2.3. *Assume that* a *is a fixed real with* $|a| > \frac{1}{2}$. *Then we have*

$$W_a = \prod_{\pi} \Big(\frac{ap}{ap-1}\frac{ap}{ap+1}\Big) = e^{T(\frac{1}{a})}. \tag{5}$$

Corollary 2.4. *we have*

$$W_2 = \prod_{\pi} \Big(\frac{2p}{2p-1}\frac{2p}{2p+1}\Big) = \exp\Big(\sum_{n=1}^{\infty} \frac{P(2n)}{n2^{2n}}\Big) \approxeq 1.1225029494299445172.$$

Remark 2.5. As numerical computations reported in Table 1 , also know that both $S(\alpha)$ and $T(\alpha)$ are continuous functions for $\in (2, 2)$. It follows immediately that $W_a \to 1$ as $a \to \infty$.

a	$W_a = e^{T(\frac{1}{a})}$
1	1.6449340668482264364724151666460251892189499012068
2	1.1225029494299445171776898915538506025772573240898
3	1.0520419006999488008574935610892686827816227562072
9	1.0056048283030987269017677824391643342543568800620
10	1.0045365888603880522959429628014286647886951159790
50	1.0001809214921512146616681538748770618197138977805
100	1.0000452261496469672340363027559610240466672008116
200	1.0000113062734768999250275966977247497149360978003
500	1.0000018089919323336766173136957541025539760633609

Table 1: Values of $W_a = e^{T(\frac{1}{a})}$ for several values of a (by using Wolfram Mathematica 9.0)

proof of theorem 2.3. For each fixed real a with$|a| > \frac{1}{2}$ we define the partial generalized Wallis product over primes by

$$W_a(x) = \prod_{p \leqslant x} \frac{ap}{ap-1} \frac{ap}{ap+1},$$

and we let

$$W_a = \lim_{x \to \infty} W_a(x) \qquad \textit{and} \quad \mathcal{F}_\alpha(x) = \prod_{p \leqslant x} \left(1 + \frac{\alpha}{p}\right)$$

We have

$$W_a(x) = \left(\mathcal{F}_{-\frac{1}{a}}(x) \mathcal{F}_{\frac{1}{a}}(x)\right)^{-1}$$

and hence Corollary 2.2 implies that

$$\prod_p \frac{ap}{ap-1} \frac{ap}{ap+1} = W_a = \left(C(-\frac{1}{a}) C(-\frac{1}{a})\right)^{-1} = e^{T(\frac{1}{a})}.$$

This completes the proof. □

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A new approach for image compression using normal matrices

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Abstract

In this paper, we present a method for image compression on the basis of eigenvalue decomposition of normal matrices. The proposed method is convenient and self-explanatory, requiring fewer and easier computations as compared to some existing methods. Through the proposed technique, the image is transformed to the space of normal matrices. Then, the properties of spectral decomposition are dealt with to obtain compressed images. Experimental results are provided to illustrate the validity of the method.

Keywords: Image compression, Transform, Normal matrix, Eigenvalue
Mathematics Subject Classification [2010]: 15A18, 94A08, 47B15

1 Introduction

Nowadays, digital images and other multimedia files can become very large in size and, therefore, occupy a lot of storage space. In addition, owing to their size, it takes more time to move them from place to place and a larger bandwidth to download and upload them on the Internet. So, digital images may pose problems if we regard the storage space as well as file sharing. To tackle this problem, *image compression* which deals with reducing the size of an image (or any other multimedia) file can be used. Image compression actually refers to the reduction of the amount of image data (bits) required for representing a digital image without causing any major degradation of the image quality. By eliminating redundant data and efficiently optimizing the contents of a file image, provided that as much basic meaning as possible is preserved, image compression techniques, make image files smaller and more feasible to share and store.

The study of digital image compression has a long history and has received a great deal of attention especially with respect to its many important applications. References for theory and practice of this method are [5, 6], to name but a few.

With respect to the influences of singular values of A in compressing an image, and considering the important point that the singular values of A are the positive square roots of the eigenvalues of matrices A^*A and AA^*, the present study concerns itself with the eigenvalue of the normal matrices $A+A^*$ and $A-A^*$ on the purpose of establishing certain technique for image compression that is efficient, leads to desirable results and needs fewer calculations.

*Speaker

2 Image compression method

In this section, first we review the definition and some properties of normal matrices. See [2, 4] and the references mentioned there as the suggested sources on a series of conditions on normal matrices. Then, we will describe the proposed method on the basis of these presented properties.

A matrix $M \in \mathbb{C}^{n\times n}$ is called *normal* if $M^*M = MM^*$, where * denotes complex conjugate transpose. Assuming M as an n-square normal matrix, there exists an orthonormal basis of $\mathbb{C}^{n\times n}$ that consists of eigenvectors of M, and M is unitarily diagonalizable. That is, let the scalars $\lambda_1,\dots,\lambda_n$, counted according to multiplicity, be eigenvalues of the normal matrix M and let $u_1,\dots,u_n$ be its corresponding orthonormal eigenvectors. Then, the matrix M can be factored as the following:

$$M = U\Lambda U^* = \sum_{i=1}^{n} \lambda_i u_i u_i^*, \qquad \Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n), \qquad U = [u_1,\dots,u_n],$$

where the matrix U satisfies $UU^* = I_n$. Maintaining the generality, assume that eigenvalues are ordered in a non-ascending sequence of magnitude, i.e., $|\lambda_1| \geq |\lambda_2| \dots \geq |\lambda_n|$. If all the elements of the matrix M are real, then $M^* = M^T$, where M^T refers to the transpose of the matrix M. A square matrix M is called *symmetric* if $M = M^T$ and called *skew-symmetric* if $M = -M^T$. That symmetric and skew-symmetric matrices are normal is easy to see. Also, the whole set of the eigenvalues of a real symmetric matrix are real, but all the eigenvalues of a real skew-symmetric matrix are purely imaginary. A general square matrix M satisfies $M = B + C$, for which the symmetric matrix $B = (M + M^T)/2$ is called the *symmetric part* of M and, similarly, the skew-symmetric matrix $C = (M - M^T)/2$ is called the *skew-symmetric* part of M. As a consequence, every square matrix may be written as the sum of two normal matrices: a symmetric matrix and a skew-symmetric one. We specially use this point in the proposed image compression technique.

In what follows, a method for image compression is presented using normal matrices. To this purpose, the matrix representing the image is transformed into the space of normal matrices. Next, the properties of its eigenvalue decomposition are utilized, and some less significant image data are deleted. Finally, by returning to the original space, the compressed image can be constructed.

Let X be an $n \times n$ matrix to represent the image. What is noticeable is that finding the eigenvalues and eigenvectors of a matrix requires fewer calculations than finding its singular values and singular vectors. Moreover, it is possible to calculate the eigenvalues and eigenvectors of a normal (especially symmetric or skew-symmetric) matrix by explicit formulas and, therefore, may yet again need less computation [1, 3].

A new method is presented here about both symmetric and skew-symmetric parts of the matrix X in order to compress the image which is found to be of a remarkably high reliability. Assume B_X and C_X as the symmetric and skew-symmetric parts of the matrix X. The normal matrices B_X and C_X can be factored as in the following:

$$B_X = U_{B_X}\Lambda_{B_X}U_{B_X}^* = \sum_{i=1}^{n} \lambda_{B_{X,i}} u_{B_{X,i}} u_{B_{X,i}}^*, \qquad \Lambda_{B_X} = diag(\lambda_{B_{X,1}},\dots,\lambda_{B_{X,n}}),$$
$$C_X = U_{C_X}\Lambda_{C_X}U_{C_X}^* = \sum_{i=1}^{n} \lambda_{C_{X,i}} u_{C_{X,i}} u_{C_{X,i}}^*, \qquad \Lambda_{C_X} = diag(\lambda_{C_{X,1}},\dots,\lambda_{C_{X,n}}).$$

Now, compress the symmetric and skew-symmetric parts of the image by wiping off the small enough eigenvalues of B_X and C_X. If k of the larger eigenvalues remains, then there is

$$\tilde{B}_X = \sum_{i=1}^{k} \lambda_{B_{X,i}} u_{B_{X,i}} u^*_{B_{X,i}} \qquad \tilde{C}_X = \sum_{i=1}^{k} \lambda_{B_{X,i}} u_{C_{X,i}} u^*_{C_{X,i}}, ; \qquad k \leq n. \tag{1}$$

where reserving the matrix $\tilde{B}_X$, $k(n+1)$ storage spaces are required for saving the matrix $\tilde{C}_X$. As a result, the total storage requirement for $\mathscr{X}$ is $2k(n+1)$. Also, through (1), the compressed image $\mathscr{X}$ will be $\mathscr{X} = \tilde{B}_X + \tilde{C}_X$.

3 Experimental results

In this section, the validity and the influence of the proposed image compression method is examined. The Peak Signal to Noise Ratio (PSNR) is calculated to measure the quality of the compressed image. In the case of gray scale images of size $M \times N$, whose pixels are represented with 8 bits, PSNR is computed as follows:

$$PSNR = 10 \log_{10}^{\frac{255^2}{MSE}}; \qquad MSE = \frac{1}{MN} \sum_{i,j} \left| X_{i,j} - \mathscr{X}_{i,j} \right|^2,$$

where $X_{i,j}$ and $\mathscr{X}_{i,j}$ refer to the elements of the original and the compressed images respectively. In addition, Compression Ratio (CR) may be calculated as an important index to evaluate how much of an image is compressed. Where

$$\text{CR} = \frac{\text{Original Image Size}}{\text{Compressed Image Size}}.$$

In the experiments conducted in this study, a 512×512 gray scale image Lena considered. The PSNR results are shown in Table 1 for some integer values of k. Also, the CR results are given in Table 2 for a 512×512 image. The results obtained by this technique are compared to those achieved by image compression method using Singular Value Decomposition (SVD) [7]. Furthermore, Figure 1 shows the original and compressed image Lena obtained by the proposed technique as well as image compression method using SVD, for $k = 100$.

Table 1: PSNR results for Lena

k	Proposed Method	SVD Method
10	22.0050	22.4065
30	26.9531	27.2243
50	29.8129	30.1761
75	32.6763	33.1093
100	35.1047	35.6641
150	39.2938	39.8988

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Table 2: CR results for an image of size 512×512

k	Proposed Method	SVD Method
10	25.5501	25.5750
20	12.7750	12.7875
50	5.1100	5.1150
75	3.4067	3.4100
100	2.5550	2.5575
150	1.7033	1.7050

Figure 1: Original and compressed image Lena for $k = 100$.

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Adaptive backstepping control of nonlinear systems based on singular perturbation theory

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Abstract

This paper studies adaptive backstepping control of nonlinearly parameterized systems with completely non-affine property. Using parameter separation and time scale separation in back-stepping control procedure, virtual/actual control inputs are defined as solutions of a series of fast dynamic equations. Moreover, the class of systems under consideration is much more general than the previouse work and for deriving the adaptation law of unknown parameters, it is not need to designe state predictor.

Keywords: parameter separation, singular perturbation theory, nonlinear parameterization, non-affine property.

1 INTRODUCTION

Among different nonlinear systems, pure feedback systems can represent more practical process such as biochemical process, aircraft flight control system [1], mechanical systems [2], etc. In the past few years, the control of various pure-feedback systems were considered such as uncertain non-affine pure feedback systems with unknown dead zone [3], with hysteresis input [4], with output constraints [5]. Despite these efforts, control problem of completely non-affine pure-feedback systems with nonlinear parameterization has remained largely open. These systems has been considered in [6]. In this paper, adaptive control of non-linearly parameterized completely non-affine pure-feedback systems is investigated.

2 PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries on singular perturbation theory

Consider the problem of solving the state equation [7]

$$\dot{x}(t) = f(t, x(t), z(t), \varepsilon),$$

$$\varepsilon\dot{z}(t) = g(t, x(t), z(t), \varepsilon), \tag{1}$$

*Speaker

It is assumed that the functions f and g are continuously differentiable in their arguments for $(t, x, z, \varepsilon) \in [0, \infty) \times D_x \times D_z \times [0, \varepsilon_0]$ and$D_z \subset R^m$ and $D_x \subset R^m$ are open connected sets, $\varepsilon_0 \gg 0$. if *g(t,x,z,0)=0* has $l \geq 1$ for each isolated real roots *z=h_a(t,x), a=1,2,. . . ,l,* for each *(t,x)* $\in [0, \infty) \times D_x$ when $\varepsilon = 0$, we say that the model (1) is in standard form. Let *ν=z-h(t,x).* From singular perturbation theory, the reduced system is represented by

$$\dot{x}(t) = f(t, x(t), h(t, x(t)), 0), \tag{2}$$

and the boundary layer system with the new time scale $\tau = t/\varepsilon$ is defined as

$$\frac{d\nu}{d\tau} = g(t, x, \nu + h(t, x(t)), 0), \tag{3}$$

2.2 Problem statement

Consider the following pure feedback system with nonlinear parameterization

$$\dot{x}_i(t) = f_{i1}(\bar{x}_i(t), x_{i+1}(t)) + f_{i2}(\bar{x}_n(t), \theta),\ i = 1, \dots, n-1,$$

$$\dot{x}_n(t) = f_{n1}(\bar{x}_n(t), u(t) + f_{n2}(\bar{x}_n(t), \theta) \tag{4}$$

where $\bar{x}_i = [x_1, x_2, \dots, x_i]^T \in R^i$, and $u \in R$ are the system states and control input, respectively.
The control objective is to design a control law u(t) for system (4) such that the origin of the system is asymptotically stable.

Remark 2.1. The argument$\bar{x}_n(t)$ in the term f_{i2} and f_{n2} leads to larger class of nonlinear systems in comparison to [18].

Definition 2.2. we assume $(\frac{\partial f_{i1}}{\partial x_{i+1}})> 0$ and $(\frac{\partial f_{n1}}{\partial u})> 0$.

Definition 2.3. There exist continues functions $\Gamma_{i2}(\bar{x}_i(t), \theta) \geq 0, i = 1, \dots, n$

Lemma 2.4. *For any real-valued continuous function $f(x, y)$ where $x \in R^m, y \in R^n$,there are smooth scalar functions $a(x) \geq 0, b(y) \geq 0, c(x) \geq 0$ and $d(y) \geq 1$,such that*

$$|f(x, y)| \leq a(x) + b(y), \tag{5}$$

$$|f(x, y)| \leq c(x)d(y), \tag{6}$$

a constructive proof is given in [8].

Remark 2.5. According to Lemma 2.4, there exist two smooth functions $\gamma_i(\bar{x}_i) \geq 1$ and $\Lambda_i(\theta) \geq 1$ satisfying

$$|f_{i2}(\bar{x}_n, \theta)| \leq \Gamma_{i2}(\bar{x}_i(t), \theta) \leq \gamma_i(\bar{x}_i)\Lambda_i(\theta), \qquad i = 1, \dots, n \tag{7}$$

Let $\Theta = \sum_{i=1}^{n} \Lambda_i(\theta)$ be a new unknown constant. Using remark 2.5, it is deduced that

$$|f_{i2}(\bar{x}_n, \theta)| \leq \gamma_i(\bar{x}_i)\Theta, \qquad i = 1, \dots, n. \tag{8}$$

3 CONTROLLER DESIGN

Similar to the backstepping method, This design procedure contains n steps. The design procedure is presented in the following. Introduce the change of coordinates $z_i = x_i - \alpha_{i-1}$ where $i = 1, \ldots, n, \alpha_0 = 0$
The derivative of z_i is expressed as
$\dot{z}_i = f_{i1}(\bar{x}_i(t), x_{i+1}(t)) + f_{i2}(\bar{x}_n(t), \theta)) - \dot{\alpha}_{i-1}$.
we should find α_i such that $f_{i1}(\bar{x}_i, z_{i+1} + \alpha_i) + f_{i2}(\bar{x}_n, \theta) - \dot{\alpha}_{i-1} = -k_i z_i$ where $k_i > 0$ is is the ith positive control gain. To overcome the non-affine property, the ith approximate virtual controller can be designed as the following ith fast dynamics

$$\epsilon_i \dot{\alpha}_i = -sign(\frac{\partial Q_i}{\partial \alpha_i}) Q_i(\bar{z}_{i+1}, \alpha_i, \hat{\Theta}), \tag{9}$$

where $\alpha_i(0) = \alpha_{i,0}, \varepsilon_i \ll 1, \bar{z}_{i+1} = [z_1, z_2, \ldots, z_{i+1}]^T, Q_i(\bar{z}_{i+1}, \alpha_i, \hat{\theta}) = k_i z_i + f_{i1}(\bar{x}_i, z_{i+1} + \alpha_i) + sat(z_i/\mu)\gamma_i(\bar{x}_i)\hat{\Theta} - \dot{\alpha_{i-1}}$
Let $\alpha_i = h_i(\bar{z}_{i+1}, \hat{\Theta})$ be an isolated root of $Q_i(\bar{z}_{i+1}, \alpha_i, \hat{\Theta}) = 0$. Then the reduced system is defined as

$$\dot{z}_i = -k_i z_i + f_{i2}(\bar{x}_n, \theta) - sat(z_i/\mu)\gamma_i(\bar{x}_i)\hat{\Theta}. \tag{10}$$

and the boundary layer system can be represented by

$$\frac{dy_i}{d\tau_i} = -sign(\frac{\partial Q_i}{\partial \alpha_i}) Q_i(\bar{z}_{i+1}, y_i + h_i(\bar{z}_{i+1}, \hat{\Theta}), \hat{\Theta}), \tag{11}$$

where $y_i = \alpha_i - h_i(\bar{z}_{i+1}, \hat{\Theta})$ and $\tau_i = t/\varepsilon_i$. Considering the control Lyapunov function $V_i = V_{i-1} + \frac{1}{2}z_i^2, i = 1, \ldots, n$ and $V = V_n + \frac{1}{2}\tilde{\Theta}^2$. Using the reduced system (10), it is deduced that

$$\dot{V}_n \leq \sum_{j=1}^{n-1} -k_j z_j^2 + |z_j|\gamma_j(\bar{x}_j\tilde{\Theta}) - k_n z_n^2 + |z_n| f_{n2}(\bar{x}_n, \theta) - z_n sat(z_n/\mu)\gamma_n(\bar{x}_n)\hat{\theta} \leq \sum_{j=1}^{n} -k_j z_j^2 + |z_j|\gamma_j(\bar{x}_j)\tilde{\Theta} \tag{12}$$

$$\dot{V} \leq \sum_{j=1}^{n-1} -k_j z_j^2 + |z_j|\gamma_j(\bar{x}_j - \tilde{\Theta})\tilde{\Theta} \tag{13}$$

Finally, we can eliminate the $\tilde{\Theta}$ term from (13) by designing the adaptation law as $\dot{\hat{\Theta}} = \sum_{j=1}^{n} |z_j|\gamma_j(\bar{x}_j)$. Therefore, the derivative of V is $\dot{V} \leq \sum_{j=1}^{n} -k_j z_j^2$. In this design, it is assumed that $\alpha_n = u$ and $\bar{z}_{n+1} = \bar{z}_n$. By using the Lasalles Theorem, this Lyapunov function guarantees the asymptotic stability of the origin of reduced system (10).

Theorem 3.1. *Consider the singular perturbation problem of the pure feedback system (4) and the controller (9). Assume that the following conditions are satisfied for all* $(\bar{z}_{i+1}, \alpha_i - h_i(\bar{z}_i, \hat{\Theta})) \in D_{\bar{z}_{i+1}} \times D_{y_i}$ *for some domains* $D_{\bar{z}_{i+1}} \subset R^{i+1}$ *and* $D_{y_i} \subset R$*,which contain their respective origins, where* $i = 1, \ldots, n, \bar{z}_{n+1} = \bar{z}_n, D_{\bar{z}_{n+1}} = D_{\bar{z}_n}$ *and* $\alpha_n = u$.
B1) $f_{i1}(0,0) = 0, f_{i2}(0,\theta) = 0, Q_i(0,0,\hat{\theta})$
B2) On any compact subset of $D_{\bar{z}_{i+1}} \times D_{y_i}$ *the equation* $0 = Q_i(\bar{z}_{i+1}, \alpha_i, \hat{\Theta})$ *has an isolated root* $\alpha_i = h_i(\bar{z}_{i+1}\hat{\Theta})$ *such that* $h_i(0,\hat{\theta}) = 0$.
B3) The functions Q_i, h_i *and their first partial derivatives respect to their arguments are*

bounded.
B4) $(\bar{z}_{i+1}, y_i) \mapsto (\partial Q_i/\partial \alpha_i)(\bar{z}_{i+1}, y_i + h_i(\bar{z}_{i+1}, \hat{\Theta}))$ *is bounded below by some positive constant for all* $z_{i+1} \in D_{z_{i+1}}$*So the origins of (11) are exponentially stable. Then, there exists a positive constant* ε^* *such that for all* $\varepsilon < \varepsilon^*$ *the origin of (4) is asymptotically stable*

Proof. It should be verified that the conditions in theorem 1 satisfy all assumptions in theorem 11.4 in [7]. First, Assumptions (B1) - (B3) directly imply the first three assumptions in theorem 11.4 hold respectively. Second, we show from Remark 1 that assumption (A4) holds. The exponential stability of the boundary layer system (11) can be easily obtained locally by linearization with respect toy_i[7]. Using Assumption 1 and (B4) yields

$$sign(\frac{\partial Q_i}{\partial \alpha_i}) = sign(\frac{\partial f_{i1}}{\partial \alpha_i}) > 0 \tag{14}$$

This confirms that the boundary layer system has a locally exponentially stable origin. Finally, in previous section we showed that the origin of reduced system (10) is asymptotically stable and the derivative of Lyapunov function of reduced system is $\dot{V} \leq -K||z||^2$ .Therefore theorem 11.4 can be applied. Accordingly, there exists a constant $\varepsilon_i^* > 0$ such that for $0 < \varepsilon < \varepsilon^*$, the origins of the systems (9) and (10) are asymptotically stable. It follows that $z_i \to 0$ and $\alpha_i \to 0$ as $t \to \infty$. Since $x_i = z_i + \alpha_{i-1}$it can be concluded that the origin of the nonlinearly parameterized pure feedback system (4) is asymptotically stable. □

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Algebraic structure of bags and fuzzy bags

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Abstract

Since the notion of bags was introduced, several works have been done using this new concept. However, existing some drawbacks in the first definition of bags, reveal the necessity of a revision of this notion. The proposed definition by Delgado et al. has improved these drawbacks. Considering the vast application of bags, more study on them seems necessary. In this regard, here, algebraic structure of bags and fuzzy bags are studied and it is shown that both sets of bags and fuzzy bags equipped with appropriate operations are complete Boolean algebra.

Keywords: Algebraic structure, Bags, Fuzzy bags, Representation by levels
Mathematics Subject Classification [2010]: 08A72,03E72

1 Introduction

The notion of bag was introduced by Yager [6] as an algebraic set-like structure where an element can appear more than once. Some operations were defined and studied from an algebraic point of view [4, 5]. So far, bags have been used for knowledge representation. For instance, bags have been used in flexible querying, representation of relational information, decision problem analysis, criminal career analysis and even in fields such as biology [1, 3, 6]. In [4], Delgado et al. claimed that although bags are algebraically well defined, they were not well suited with real-world information. Also, they showed that the initial definition for bags has some deficiencies and then, they proposed new definitions for bags and fuzzy bags. They defined fuzzy bags based on the theory of representation by levels (RL) and called it RL-bags. For more details about RL theory see [2].

In this work, we consider proposed definitions in [4] and study the algebraic structure of bags and fuzzy bags.

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2 Preliminaries

Definition 2.1. [4] Let P and O be two universes (sets) called "properties" and "objects", respectively. A bag $\mathcal{B}^f$ is a pair (f, B^f), where $f : P \to \mathcal{P}(O)$ is a function and B^f is the following subset of $P \times \mathcal{N}$

$$B^f = \{(p, card(f(p))), p \in P \text{ and } f(p) \neq \emptyset\},$$

where $\mathcal{N}$ is the set of natural numbers, $\mathcal{P}(O)$ is the power set of O and $card(X)$ is the cardinality of the set X.

Example 2.2. [5] Let the set of objects be O = {John, Mary, Bill, Tom, Sue, Stan, Harry} and $P = \{17, 21, 27, 35\}$. Let $f_1, f_2, f_3, f_4 : P \to \mathcal{P}(O)$ be the functions in Table 1.

Table 1: Several functions: age-people

p	17	21	27	35
$f_1(p)$	{Bill, Sue}	{John, Tom}	$\emptyset$	$\emptyset$
$f_2(p)$	{Bill, Sue}	{John, Tom, Stan}	$\emptyset$	{Harry}
$f_3(p)$	$\emptyset$	{Stan}	{Mary}	{Harry}
$f_4(p)$	{Bill}	{John, Stan}	$\emptyset$	$\emptyset$

So, we can define bags $\mathcal{B}^{f_i} = (f_i, B^{f_i})$, $1 \leq i \leq 4$, where

$$B^{f_1} = \{(17, 2), (21, 2)\}, \qquad B^{f_2} = \{(17, 2), (21, 3), (35, 1)\},$$
$$B^{f_3} = \{(21, 1), (27, 1), (35, 1)\}, \qquad B^{f_4} = \{(17, 1), (21, 2)\}.$$

In the following, we restate some results about bags.

Definition 2.3. [4] Let $* \in \{\cup, \cap, \setminus\}$. Then $\mathcal{B}^f * \mathcal{B}^g = \mathcal{B}^{f*g} = (f * g, B^{f*g})$, where $f * g : P \to \mathcal{P}(O)$ such that $(f * g)(p) = f(p) * g(p)$ for all $p \in P$.

Example 2.4. [5] Table 2 shows some operations between functions in Example 2.2. Where, the corresponding summaries are

$$B^{f_1 \cup f_2} = \{(17, 2), (21, 3), (35, 1)\}, \qquad B^{f_2 \cap f_3} = \{(21, 1), (35, 1)\},$$
$$B^{f_1 \setminus f_3} = \{(17, 2), (21, 2)\}, \qquad B^{f_3 \setminus f_2} = \{(27, 1)\}.$$

Table 2: Operations on functions from Example 2.2

p	17	21	27	35
$(f_1 \cup f_2)(p)$	{Bill, Sue}	{John, Tom, Stan}	$\emptyset$	{Harry}
$(f_2 \cap f_3)(p)$	$\emptyset$	{Stan}	$\emptyset$	{Harry}
$(f_1 \setminus f_3)(p)$	{Bill, Sue}	{John, Tom}	$\emptyset$	$\emptyset$
$(f_3 \setminus f_2)(p)$	$\emptyset$	$\emptyset$	{Mary}	$\emptyset$

Definition 2.5. Set $\mathbf{B}(P, O)$ as the set of all bags $\mathcal{B}^f = (f, B^f)$ defined in Definition 2.1.

Remark 2.6. [4] Operations $\cap$ and $\cup$ in $\mathbf{B}(P, O)$ satisfy in the idempotent, commutative, associative and distributive laws.

Definition 2.7. [5] a) A bag $\mathcal{B}^f$ is a subbag of $\mathcal{B}^g$, denoted by $\mathcal{B}^f \sqsubseteq \mathcal{B}^g$ if $f(p) \subseteq g(p)$ for all $p \in P$.
b) Two bags $\mathcal{B}^f$ and $\mathcal{B}^g$ are equal, denoted by $\mathcal{B}^f = \mathcal{B}^g$ if $\mathcal{B}^f \sqsubseteq \mathcal{B}^g$ and $\mathcal{B}^g \sqsubseteq \mathcal{B}^f$.

Definition 2.8. [4] Let $\mathcal{B}^f = (f, B^f)$. Then, complement of $\mathcal{B}^f$ is $(\mathcal{B}^f)^c = \mathcal{B}^{f^c} = (f^c, B^{f^c})$, where $f^c : P \to \mathcal{P}(O)$ such that $f^c(p) = O \setminus f(p)$ for all $p \in P$.

Now, we quote the definition of RL-bags or fuzzy bags and restate some results about them.

Definition 2.9. [4] A fuzzy bag or a RL-bag $\tilde{\mathcal{B}}^f$ is a pair (Λ_f, ρ_f) where Λ_f is a finite set of levels and $\rho_f : \Lambda_f \to \mathbf{B}(P, O)$ is a function that maps each level into a crisp bag.

It is clear that a crisp bag $\mathcal{B}^g$ is a particular case of fuzzy bag where $\Lambda_g = \{1\}$ and $\mathcal{B}^g = \rho_g(1) = (g, B^g)$ [4].

For each level $\alpha \in \Lambda_f$ we consider the associated bag in that level, (f_α, B^{f_α}) and the corresponding summary is denoted by $\tilde{B}^f(\alpha)$ (or sometimes for the sake of simplicity by $\tilde{B}^{f_\alpha}$) using the same count operation as in the crisp case [4].

Definition 2.10. [4] Let $* \in \{\cup, \cap, \setminus\}$ and $\tilde{\mathcal{B}}^f, \tilde{\mathcal{B}}^g$ be two fuzzy bags. Then, $\tilde{\mathcal{B}}^f * \tilde{\mathcal{B}}^g = \tilde{\mathcal{B}}^{f*g} = (\Lambda_{f*g}, \rho_{f*g})$ is actually a fuzzy bag, where $\Lambda_{f*g} = \Lambda_f \cup \Lambda_g$ and

$$\rho_{f*g}(\alpha) = (f_\alpha * g_\alpha, B^{f*g}(\alpha)) \text{ for all } \alpha \in \Lambda_{f*g},$$

where $f_\alpha * g_\alpha : P \to \mathcal{P}(O)$ such that $(f_\alpha * g_\alpha)(p) = f_\alpha(p) * g_\alpha(p)$ for all $p \in P$, $\alpha \in \Lambda_{f*g}$.

Definition 2.11. Set $\tilde{\mathbf{B}}_\Lambda(P, O)$ as the set of all fuzzy bags $\tilde{\mathcal{B}}^f = (\Lambda, \rho^f)$ defined in Definition 2.9.

Remark 2.12. [4] Operations $\cap$ and $\cup$ in $\tilde{\mathbf{B}}_\Lambda(P, O)$ satisfy in the idempotent, commutative, associative and distributive laws.

Definition 2.13. [4] a) A fuzzy bag $\tilde{\mathcal{B}}^f$ is a subbag of $\tilde{\mathcal{B}}^g$, denoted by $\tilde{\mathcal{B}}^f \tilde{\sqsubseteq} \tilde{\mathcal{B}}^g$, if $f_\alpha(p) \subseteq g_\alpha(p)$ for all $p \in P, \alpha \in \Lambda_f \cup \Lambda_g$.
b) Two fuzzy bags $\tilde{\mathcal{B}}^f$ and $\tilde{\mathcal{B}}^g$ are equal, denoted by $\tilde{\mathcal{B}}^f = \tilde{\mathcal{B}}^g$, if $\tilde{\mathcal{B}}^f \tilde{\sqsubseteq} \tilde{\mathcal{B}}^g$ and $\tilde{\mathcal{B}}^g \tilde{\sqsubseteq} \tilde{\mathcal{B}}^f$.

Note that that Definition 2.13 is direct extension of the crisp case. Actually, it reduces to the crisp bags [4]. The next definition introduces the concept of complement.

Definition 2.14. Let $\tilde{\mathcal{B}}^f = (\Lambda_f, \rho_f)$ be a fuzzy bag. Then, complement of $\tilde{\mathcal{B}}^f$ is $(\tilde{\mathcal{B}}^f)^c = \tilde{\mathcal{B}}^{f^c} = (\Lambda_{f^c}, \rho_{f^c})$, where $\Lambda_{f^c} = \Lambda_f$ and $\rho_{f^c}(\alpha) = (\mathcal{B}^f(\alpha))^c = \mathcal{B}^{f^c}(\alpha)$ for all $\alpha \in \Lambda_{f^c}$.

Remark 2.15. Definition 2.14 is a revised form of Definition 16 in [4] in order to obtain some consistency in the complement of fuzzy bags.

3 Algebraic structure of bags and fuzzy bags

In this section, we characterize some algebraic structure of bags and fuzzy bags or RL-bags. Let $\sqsubseteq$ be the relation defined in Definition 2.7. We have the following results.

Theorem 3.1. *$(\boldsymbol{B}(P, O), \sqsubseteq)$ is a Boolean algebra.*

Proof. Clearly, $\mathbf{B}(P,O)$ is a lattice. Define $\mathcal{B}^0 = (0, B^0)$ and $\mathcal{B}^1 = (1, B^1)$ where, $0(p) = \emptyset$, $1(p) = O$ for all $p \in P$, $B^0 = \{(p, 0), p \in P\}$ and $B^1 = \{(p, card(O)), p \in P\}$. Clearly, $\mathcal{B}^0, \mathcal{B}^1 \in \mathbf{B}(P,O)$, $\sup(\mathcal{B}^0, \mathcal{B}^f) = \mathcal{B}^f$ and $\inf(\mathcal{B}^1, \mathcal{B}^f) = \mathcal{B}^f$. So, $(\mathbf{B}(P,O), \sqsubseteq)$ is bounded. By Definition 2.8, for each $\mathcal{B}^f \in \mathbf{B}(P,O)$, we have $\mathcal{B}^{f^c} \in \mathbf{B}(P,O)$. But, $\sup(\mathcal{B}^f, \mathcal{B}^{f^c}) = \mathcal{B}^1$ and $\inf(\mathcal{B}^f, \mathcal{B}^{f^c}) = \mathcal{B}^0$. Thus, $(\mathbf{B}(P,O), \sqsubseteq)$ is complemented. By Remark 2.6, property of distributivity holds. So, $(\mathbf{B}(P,O), \sqsubseteq)$ or $(\mathbf{B}(P,O), \cup, \cap, ^c, \mathcal{B}^0, \mathcal{B}^1)$ is a Boolean algebra. □

Corollary 3.2. *$(\boldsymbol{B}(P,O), \cup, \cap, ^c, \mathcal{B}^0, \mathcal{B}^1)$ is a De Morgan algebra.*

Theorem 3.3. *$(\boldsymbol{B}(P,O), \cup, \cap, ^c, \mathcal{B}^0, \mathcal{B}^1)$ is a complete Boolean algebra.*

In what follows, we study the algebraic structure of the set of all fuzzy bags. Let $\tilde{\sqsubseteq}$ be the relation defined in Definition 2.13. We have the following results.

Theorem 3.4. *$(\tilde{\boldsymbol{B}}_\Lambda(P,O), \tilde{\sqsubseteq})$ is a Boolean algebra.*

Proof. It is clear that $(\tilde{\mathbf{B}}_\Lambda(P,O), \tilde{\sqsubseteq})$ is a lattice. Define $\tilde{\mathcal{B}}^0 = (\Lambda, \rho_0)$ and $\tilde{\mathcal{B}}^1 = (\Lambda, \rho_1)$, where, ρ_0 and ρ_1 maps all $\alpha \in \Lambda$ to $\mathcal{B}^0$ and $\mathcal{B}^1$, respectively. Clearly, $\tilde{\mathcal{B}}^0, \tilde{\mathcal{B}}^1 \in \tilde{\mathbf{B}}_\Lambda(P,O)$ and also, $\sup(\tilde{\mathcal{B}}^0, \tilde{\mathcal{B}}^f) = \tilde{\mathcal{B}}^f$ and $\inf(\tilde{\mathcal{B}}^1, \tilde{\mathcal{B}}^f) = \tilde{\mathcal{B}}^f$. So, $(\tilde{\mathbf{B}}_\Lambda(P,O), \tilde{\sqsubseteq})$ is bounded. By Definition 2.14, for each $\tilde{\mathcal{B}}^f \in \tilde{\mathbf{B}}_\Lambda(P,O)$, we have $\tilde{\mathcal{B}}^{f^c} \in \tilde{\mathbf{B}}_\Lambda(P,O)$. But, $\sup(\tilde{\mathcal{B}}^f, \tilde{\mathcal{B}}^{f^c}) = \tilde{\mathcal{B}}^1$ and $\inf(\tilde{\mathcal{B}}^f, \tilde{\mathcal{B}}^{f^c}) = \tilde{\mathcal{B}}^0$. Thus, $(\tilde{\mathbf{B}}_\Lambda(P,O), \tilde{\sqsubseteq})$ is complemented. By Remark 2.12, property of distributivity holds. So, $(\tilde{\mathbf{B}}_\Lambda(P,O), \tilde{\sqsubseteq})$ or $(\tilde{\mathbf{B}}_\Lambda(P,O), \cup, \cap, ^c, \tilde{\mathcal{B}}^0, \tilde{\mathcal{B}}^1)$ is a Boolean algebra. □

Corollary 3.5. *$(\tilde{\boldsymbol{B}}_\Lambda(P,O), \cup, \cap, ^c, \tilde{\mathcal{B}}^0, \tilde{\mathcal{B}}^1)$ is a De Morgan algebra.*

Theorem 3.6. *$(\tilde{\boldsymbol{B}}_\Lambda(P,O), \cup, \cap, ^c, \tilde{\mathcal{B}}^0, \tilde{\mathcal{B}}^1)$ is a complete Boolean algebra.*

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An Edge Detection Scheme with Legendre Multiwavelets

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Abstract

Many edge detection methods which based on wavelet transform, use this transform to approximate the gradient of image and detect edges by searching the modulus maximum of gradient vectors. In this paper we present an edge detection scheme based on Legendre multiwavelets. The results of this algorithm are compared with Sobel edge detector.

Keywords: Wavelet transform, Edge detection, Legendre multiwavelets
Mathematics Subject Classification [2010]: 65T60, 68U10, 94A08

1 Introduction

Edge is the important characteristic of image. Edges are among objects, regions, between objects and backgrounds. If all edges in an image identify accurately, all the objects can be located. Edge detection plays an important role in medical imaging [1], computer vision and machine vision [2] and recognition Persian characters [3]. The large class of edge detectors look up points where the gradient of the image has local maximum.

In recent decades, wavelet analysis fostered as a useful research method. Wavelet analysis is a new development in the area of applied mathematics [5]. Parallelly, the theory of wavelets got more demystified and has become an important tool for image processing like edge detection [4]. Some edge detector such as Canny edge detector use wavelet transform. However, multiwavelet system can simultaneously provide perfect reconstruction while preserving length due to orthogonality of filters, good performance at the boundaries, and a high order of approximation (vanishing moments). In this paper we used Legendre multiwavelets to introduce an edge detection scheme.

2 Multiwavelet Transform

Like wavelets, multiwavelets were also based upon multiresolution analyses (MRA). MRA using wavelets comprises of one scaling function $\phi(x)$ and one wavelet function $\psi(x)$, where as multiwavelets possess many number of scaling functions under one vector denoted as

$$\Phi(x) = [\phi_0(x), \phi_1(x), \cdots, \phi_N(x)]^T, \tag{1}$$

*Speaker

and many wavelet functions denoted by

$$\Psi(x) = [\psi_0(x), \psi_1(x), \cdots, \psi_N(x)]^T. \tag{2}$$

Multiwavelets satisfying the followig dilations equations,

$$\Phi(x) = \sum_k H(k)\Phi(2x-k), \tag{3}$$

$$\Psi(x) = \sum_k G(k)\Phi(2x-k), \tag{4}$$

where $H(k)$ and $G(k)$ are $N \times N$ matrices. In other words, the coefficients $H(K)$ and $G(K)$ are $N \times N$ matrices instead of scalar values.

Multiwavelet decomposition produces N low pass subbands and two high pass subbands in each dimension. In image processing, wavelet decomposition yields four subbands after one level of decomposition, whereas in multiwavelets N^4 subbands result after first level of decomposition. When N is two the next figure shows image subband structure for first level of decomposition.

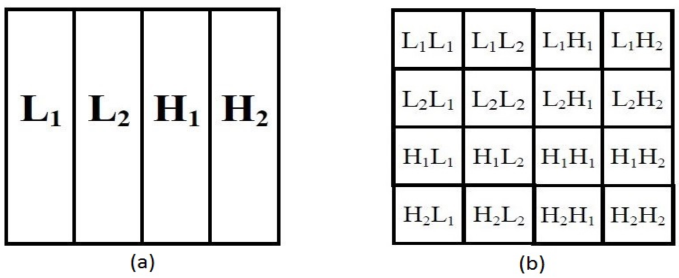

Figure 1: filtering along (a) horizontal direction (b) vertical direction after horizontal direction

Linear Legendre wavelets is an exmaple of multiwavelet with $N = 2$ [6]. One can define a pair of Linear Legendre scaling functions $\phi_0(x)$ and $\phi_1(x)$ as

$$\begin{cases} \phi_0(x) = 1 & 0 \le x < 1 \\ \phi_1(x) = \sqrt{3}(2x-1) & 0 \le x < 1 \end{cases} \tag{5}$$

These scaling function hold in

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \end{bmatrix} = H(0) \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \end{bmatrix} + H(1) \begin{bmatrix} \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{6}$$

where

$$H(0) = \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad and \quad H(1) = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

In this case, $\Psi(x)$ satisfying

$$\begin{bmatrix} \psi_0(x) \\ \psi_1(x) \end{bmatrix} = G(0) \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \end{bmatrix} + G(1) \begin{bmatrix} \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{7}$$

where

$$G(0) = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \quad and \quad G(1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Now, one can use these lowpass filters $H(0)$, $H(1)$ and highpass filters $G(0)$, $G(1)$ to decompose an image. Images can be considered as two variables functions $f(x, y)$, so, the edges of image may be detected by looking up for modulus maximum points according to [7].

$$Mf(x,y) = \sqrt{|W^1 f(x,y)|^2 + |W^2 f(x,y)|^2}, \tag{8}$$

Here we use the Legendre multiwavelet transform which it treats like wavelet transform.

3 Experimental Results

This section consists of experimental results for a set of standard images. In order to verify the efficiency and accuracy of the proposed algoerithm, some images are used as experimental subjects. We compare the proposed method for three standard testing images with Sobel edge detector.

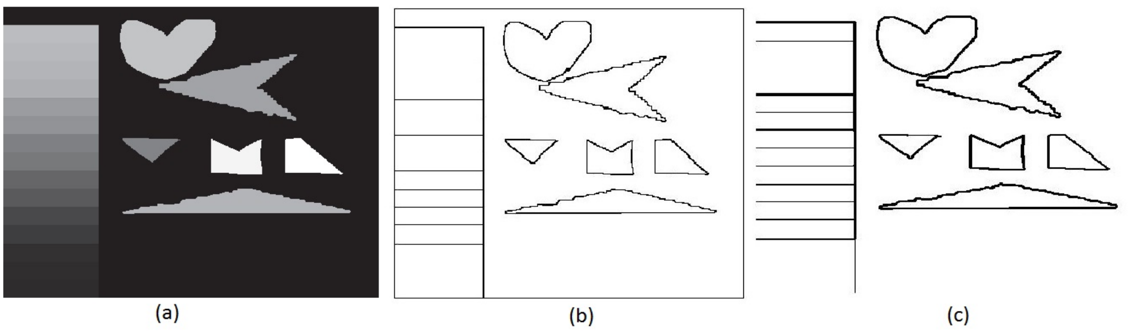

Figure 2: (a) Original Image (b) Edges of Sobel (c) Edges of proposed method

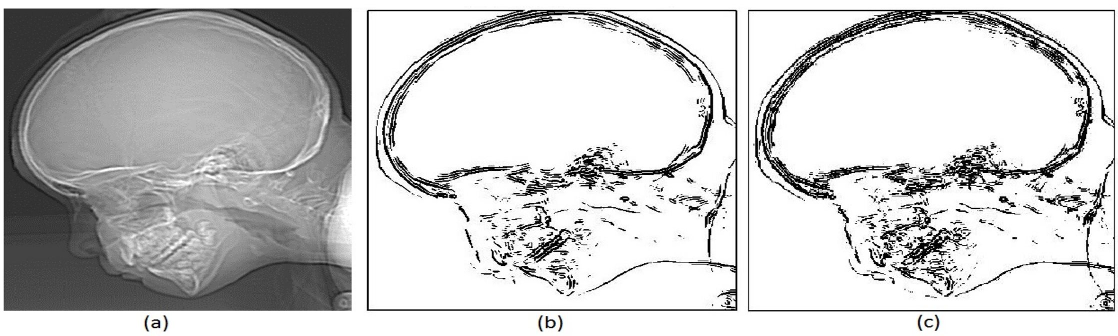

Figure 3: (a) Original Image (b) Edges of Sobel (c) Edges of proposed method

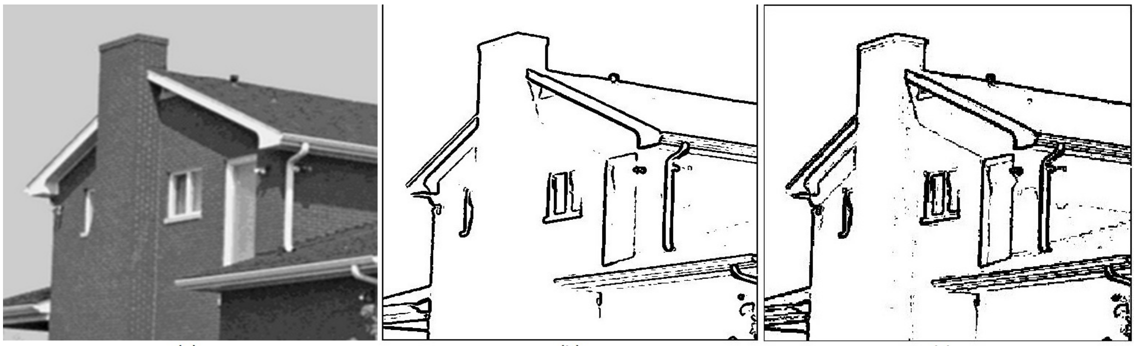

Figure 4: (a) Original Image (b) Edges of Sobel (c) Edges of proposed method

As the experimental results show, the proposed method detected more correct edge pixels in copmaring with Sobel edge detectors. The edges of three images were detected to show the efficiency of our method.

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Talk Coexistence of game theory in social science pp.: 1–4

Coexistence of game theory in social science

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Abstract

The object of this article is to demonstrate the possibilities of games theory as an instrument for study of social science .The approach to be used describe elementary games theoretic models as on integral part of social science with a collection of example to understand subject better. This paper addressed to theoreticians and practitioners of social science not particularly versed in games theory, rather than to those who are fluent in its mathematical language and intricacies.

Keywords: Best strategies, Game theory, Nash equilibrium, Social science
Mathematics Subject Classification [2010]:

1 Introduction

Social science of game theory just as microeconomic theory has sometimes been said to be applied branch of calculus. The following examples present a simplified application of game theory.These provides an opportunity to describe the main steps needed to construct a game theoretical model of real events and also to elaborate on same of the contributions that game theory can make to the study of social science. Reader must know to that target of this article is to avoid from complex mathematical calculation and with a large number of example help reader to be skill to give number to social science events. We hope that we are successful in reaching to this target. We will start with a simple example which all of us have done in childhood.

Example 1.1. "the warfare Game"

This game helps government to solve bad social Phenomenon of begging this game advice which strategy is better to face this phenomenon:

(1)

We can consider that there is not Nash equilibrium. We can understand best strategy for government when the beggar decide to work is supporting, and when they decide to begging is unsporting.

*Speaker

Beggar

government		working	begging
x	support	3,2	-1,3
1-x	unsupport	-1,1	0,0
		y	1-y

Figure 1: EXAM 1

Example 1.2. “Battle of couple ”

Man and women decided to go to garden with trees of apple and peach w game theory can suggest which decision is better for them to be happy which we can show in following table :

(2)

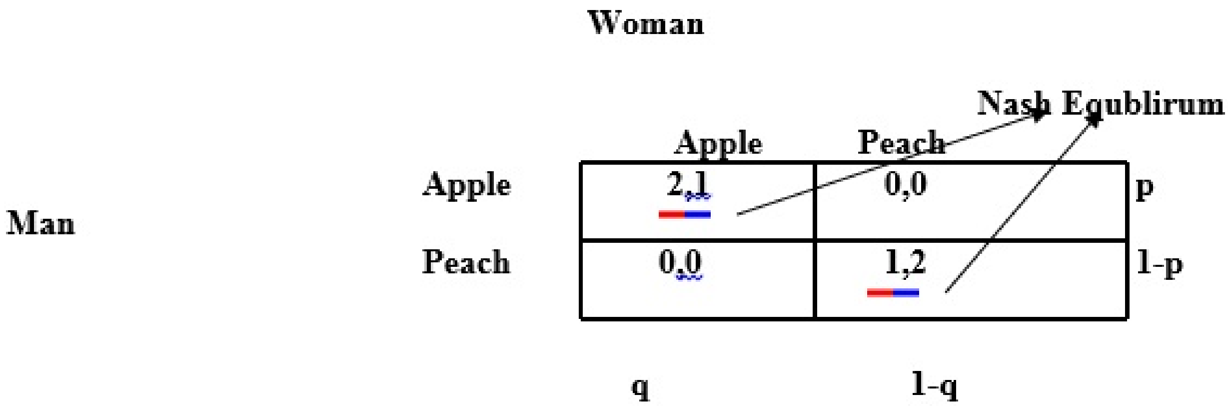

Figure 2: EXAM 2

In this game man interest to apple tree and woman like peach tree. Look to pay matrix result show that in this game we have Nash equilibrium which means it is better man and woman be with together. Question is we have two Nash equilibrium which one is better? By mixed Nash equilibrium we can find best strategy as follow: To find Ne of q using Man payoff

$$MAN \left\{ \begin{array}{l} apple : 2q + 0(1-q) \\ peach : 0q + 1(1-q) \end{array} \right\} \Longrightarrow 2q = 1(1-q) \Longrightarrow q = \frac{1}{3}$$

To find Nash equilibrium of p using woman payoff

$$WOMAN \begin{cases} apple : 1p + 0(1-p) \\ peach : 0p + 2(1-p) \end{cases} \Longrightarrow 1p = 2(1-p) \Longrightarrow p = \frac{2}{3}$$

$P = 2/3$ is BR for man :

$$\left.\begin{array}{l} apple \to 2(\frac{1}{3}) + 0(\frac{2}{3}) \\ peach \to 0(\frac{1}{3}) + 1(\frac{2}{3}) \end{array}\right\} = \frac{2}{3}$$

Man : $p \to \frac{2}{3}[\frac{2}{3}] + \frac{1}{3}[\frac{2}{3}] = \frac{2}{3}$
No strictly profitable pure deviation either
$NE = [(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})] \quad \longrightarrow \quad$ [MAN$\frac{2}{3}$,WOMAN$\frac{2}{3}$]
Payoffs is low because they fail to meet sometimes.
$Prob(meet) = [\frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3}] = \frac{4}{9}$

Example 1.3. we know that in each country steering of car is left or right. in this example we consider which one is better for society?
(3)

		Country 2 L	Country 2 R
Coutary 1	L	2,2	0,0
	R	0,0	1,1

Figure 3: EXAM 3

We can check that NE is on (L, L) and (R, R) but again we can see that (L, L) is better for all countaries.

Next example is about Traffic light game.

Example 1.4. "Traffic light"
If driver is one side and police in another side.
In this game

d:delay

D: congestion

p:Probability to catch by traffic police

F:fine for jumping
(4)

		Driver II	
		Obey	Diobey
DriverI	Obey	d	d+D
	Disobey	0	D

Figure 4: EXAM 4

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Yazd University
Talk Hunter's lemma for forest algebras pp.: 1–4

Hunter's Lemma for Forest Algebras*

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Abstract

Forest algebras are defined for investigating forests [ordered sequences] of unranked trees, where a node may have more than two [ordered] successors [3]. We define a new version of syntactic congruence of a subset of the free forest algebra, not just a forest language, which leads to more general results. We show that for a inverse zero action subset and a forest language which is the restriction of the inverse zero action subset to the horizontal monoid, the two versions of syntactic congruences coincide. We define on the free forest algebra a pseudo-ultrametric associated with a pseudovariety of forest algebras. We show that the basic operations on the free forest algebra are uniformly continuous, this pseudo-ultrametric space is totally bounded, and its completion is a forest algebra. We show that the analog of Hunter's Lemma [5] holds for metric forest algebras, which leads to the result that zero-dimensional compact metric forest algebras are residually finite.

Keywords: Forest Algebra, metric, Hunter's Lemma.
Mathematics Subject Classification [2010]: 68R99

1 Introduction

We recall the forest algebra structures defined in [1]. After that, for a subset K of a forest algebra, we define a binary relation $\backsim_K$ of K and we show that the relation $\backsim_K$ define a congruence relation of elements of S. Then we define a syntactic forest algebra which is the quotient of a forest algebra by $\backsim_K$ for some subset K of the forest algebra the so called syntactic congruence of K. Then we show that for a inverse zero action subset K of a forest algebra the quotient of the forest algebra by $\backsim_K$ is a forest algebra. Then, we define a metric on the free forest algebra A^{Δ} with respect to a pseudovariety of finite forest algebras $\mathbf{V}$ whence the basic operations with respect to this metric are contractive. We establish a lemma similar to Hunter's Lemma[5].

Over a finite alphabet A, finite unranked ordered trees and forests are expressions defined inductively. If s is a forest and $a \in A$, then as is a *tree* where a is the root of the tree and it is the direct ancestor of the root of each tree in the forest s. Suppose that $t_1, \ldots, t_n$ is a finite sequence of trees, if we put each tree t_i on the right side of the tree t_{i-1} for $i = 2, 3, \ldots, n$ denoted by $t_1 + \cdots + t_n$ then the result is a *forest*. This applies as well to the empty sequence of trees, which thus gives rise to the *empty forest*, denoted by

*Will be presented in English
†Speaker

0. The set of all forests is called the *horizontal* set. A set L of forests over A is called a *forest language*. If we take a forest and replace one of the leaves by a special symbol *hole*, which is denoted by $\square$, we obtain a *context*. A forest s can be substituted in place of the hole of a context p; the resulting forest is denoted by ps. There is a natural composition operation on contexts, the context qp is formed by replacing the hole of q with p. The set of all contexts is called the *vertical* set [3, 2].

Definition 1.1. A *forest algebra* S consists of a pair (H, V) of distinct monoids, subject to some additional requirements, which we describe below.

We write the operation in V, the vertical monoid, multiplicatively and the operation in H, the horizontal monoid, additively, although H is not assumed to be commutative. We accordingly denote the identity of V by $\square$ and that of H by 0.

We require that V acts on the left of H. That is, there is a map

$$(v, h) \in V \times H \mapsto vh \in H$$

such that $w(vh) = (wv)h$, for every $h \in H$ and every $v, w \in V$. We also require that this action be *monoidal*, that is, $\square.h = h$, for every $h \in H$.

We further require that for every $h \in H$ and $v \in V$, V contains elements $h + v$ and $v + h$ such that for every $x \in S$,

$$(v + h)x = vx + h \quad \text{and} \quad (h + v)x = h + vx,$$

where vx is given by the action of v on x if x is a forest and by composition (multiplication) if x is a context.

We call the *equational axioms* of forest algebras, the preceding axioms on the elements of the forest algebras.

And also we require that the action be *faithful*, that is, if $vh = wh$, for every $h \in H$, then $v = w$.

Note that, a forest algebra $S = (H, V)$ is finite if and only if H is finite.

A morphism $\alpha : (H_1, V_1) \to (H_2, V_2)$ of algebras has equational axioms of forest algebras is a pair of monoid homomorphisms $\gamma : H_1 \to H_2$ and $\delta : V_1 \to V_2$ such that, for every $h \in H$ and every $v \in V$, $\gamma(vh) = \delta(v)\gamma(h)$ and

$$\delta(h + v) = \gamma(h) + \delta(v) \quad , \qquad \delta(v + h) = \delta(v) + \gamma(h).$$

However, we will abuse notation slightly and denote both component maps by α.

Definition 1.2. A *subalgebra* of a forest algebra is a subset of a forest algebra has the equational axioms of forest algebras.

Definition 1.3. A subset K of a forest algebra $S = (H, V)$ is called a *inverse zero action subset* if, for every context v, $v \in K$ if and only if $v0 \in K$.

Let $S = (H, V)$ be a forest algebra and K a subset of S. We take $H' = K \cap H$ and $V' = K \cap V$. We may define on S a relation $\backsim_K = (\sigma_K, \sigma'_K)$, the so-called *syntactic congruence* of K, as follows:

- for $h_1, h_2 \in H$, $h_1 \; \sigma_K \; h_2$ if for all $t, w, r \in V$:

I. $th_1 \in K \iff th_2 \in K$;
II. $t(rh_1 + w) \in K \iff t(rh_2 + w) \in K$;
III. $t(w + rh_1) \in K \iff t(w + rh_2) \in K$.

- for $u, v \in V$, $u\ \sigma'_K\ v$ if for all $t, w \in V$ and $h \in H$:

I. $tuh\ \sigma_K\ tvh$;
II. $tuw \in K \iff tvw \in K$.

It is easy to check that σ_K and σ'_K are equivalence relations and the following result holds.

Lemma 1.4. *For a forest algebra S and a subset K of S, the equivalence relations σ_K and σ'_K are congruences with respect to the basic operations of S.*

Lemma 1.4, guarantees that the quotient of the forest algebra S with respect to equivalence $\backsim_K$ is well defined. In this quotient, if faithfulness holds then, since the equational axioms of forest algebras are preserved by taking quotients, it is a forest algebra.

Definition 1.5. The *syntactic forest algebra* for K is the quotient of S with respect to the equivalence $\backsim_K$, where the horizontal semigroup H_K consists of equivalence classes σ_K of forests in S, while the vertical semigroup V_K consists of equivalence classes σ'_K of contexts in S. The *syntactic morphism* $\alpha_K = (\gamma_K, \delta_K) : S \longrightarrow S/\backsim_K$ assigns to every element of S its equivalence class in (H_K, V_K).

Proposition 1.6. *The syntactic congruence of K is the largest one that saturates K.*

Definition 1.7. A nonempty class $\mathbf{V}$ of finite forest algebras is called a *pseudovariety* if the following conditions hold:

(i) if $S \in \mathbf{V}$ and B is a forest subalgebra of S, then $B \in \mathbf{V}$;

(ii) if $S \in \mathbf{V}$ and $S \to B$ is an onto morphism, then $B \in \mathbf{V}$;

(iii) $\mathbf{V}$ is closed under finite direct products.

For two elements $u, v \in A^\Delta$ and a forest algebra B if for every morphism $\varphi : A^\Delta \to B$ the equality $\varphi(u) = \varphi(v)$ holds, then we say that B *satisfies the identity* $u = v$ and we write $B \vDash u = v$. For a pseudovariety of finite forest algebras $\mathbf{V}$, define:

$$r(u, v) = \min \{|B| \mid\ B \in \mathbf{V} \ \ \textit{and} \ \ B \nvDash u = v\}$$

and $d(u, v) = 2^{-r(u,v)}$ where we take $\min \emptyset = \infty$ and $2^{-\infty} = 0$.

Proposition 1.8. *Let $\mathbf{V}$ be a pseudovariety of finite forest algebras. The function d is a pseudo-ultrametric on A^Δ, the basic operations are contractive and the pseudo-ultrametric space (A^Δ, d) is totally bounded.*

Note that, by [4, Theorem 1.15], every metric space has a completion. It is natural to consider the completion of A^Δ, denoted $\overline{\mathtt{C}}_A\mathbf{V}$, as the union of the completions of H^A and V^A which denoted respectively $\overline{\mathtt{C}}_{\mathbf{V}}H^A$ and $\overline{\mathtt{C}}_{\mathbf{V}}V^A$. Since operations on A^Δ are uniformly continuous, they do extend to uniformly continuous operations on $\overline{\mathtt{C}}_A\mathbf{V}$. Hence, $\overline{\mathtt{C}}_A\mathbf{V}$ has naturally equational axioms of forest algebras. One can easily show that $\overline{\mathtt{C}}_A\mathbf{V}$ is a forest algebra.

Lemma 1.9. *Let $K = (H', V')$ be a inverse zero action subset of a compact metric forest algebra $S = (H, V)$. Then H' is a clopen subset of H if and only if V' is a clopen subset of V.*

Lemma 1.10. (Similar to Hunter's Lemma) *Let K be a clopen inverse zero action subset of a compact and zero-dimensional metric forest algebra S. Then there is a continuous morphism $\psi : S \to T$ into a finite forest algebra T such that $K = \psi^{-1} \circ \psi(K)$.*

Proof. It suffices to show that the classes of the syntactic congruence of K are open. Then there are only finitely many of them since S is a compact forest algebra. So that $S/\backsim_K = (H/\sigma_K, V/\sigma'_K)$ is a finite forest algebra and the natural mapping $S \to S/\backsim_K$ is a continuous morphism. □

Theorem 1.11. *A zero-dimensional and compact metric forest algebra is residually finite.*

We defined a metric on the free forest algebra with respect to a pseudovariety of finite forest algebras and we showed that the basic operations with respect to this metric are contractive. We showed that the completion of the free forest algebra with respect to the defined metric exists and is a forest algebra. We established in this context an analog of Hunter's Lemma [5]. And we can easily show that the Hausdorff completion of the free forest algebra with respect to a pseudovariety **W** of finite forest algebras is pro-**W**. So, one can easily establish an analog of Reiterman's Theorem, for a pseudovariety **V** of finite algebras a simple basis may be seen as a formalization of a simple algebraic criterion for membership in **V**.

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L2-SVM Problem and a New One-layer Recurrent Neural Network for its Primal Training

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Abstract

This paper presents a brief review on Support Vector Machine (SVM) and L2-SVM problems and a new one-layer Recurrent Neural Network (RNN) for L2-SVM learning. The L2-SVM problem is first converted into a new reformulation, which has some advantage over its original form, then a neural network for its primal is proposed. The proposed neural network is guaranteed to obtain solution of L2-SVM. Moreover, this neural network can converge globally to the optimal solution of L2-SVM and the rate of the convergence is dependent to a scaling parameter, not to the size of data set. Simulation examples based on Iris and Fisher-Iris problems are discussed to show the excellent performance of the proposed neural network.

Keywords: Support vector machine, L2-SVM problem, Primal SVM training, Recurrent neural network, Convex programming, Lyapanov function.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In recent machine learning probelms, Support Vector Machine (SVM) has a great role in binary classification. The main feature of this problem is to classify data in two disjoint classes and its range of application is expanded in manifold fields. As a result, diefferent kind of SVMs such as L2-SVM, Least Square Supprot Vector Machine (LS-SVM) and so forth are introduced. These problems are modeled as convex optimization problems and dealing with them are based on convex programming methods. For instance, SVM and L2-SVM are modeld as a quadratic optimization porblem and different methods for solving them are presented [1, 2].
On the other hand, Recurrent Neural Networks (RRNs) have been received an extreme attention for optimizing problems in recent decades. A great number of RNNs are presented to solve convex, non-convex, smooth and non-smooth problems with different structures [3, 4].
Implementing the structure of RNNs, many engineering problems are solved by RNNs. In [5], Xia and Wang have proposed a one-layer RNN for SVM dual problem. In this paper,

*Speaker

we present a reformulation of SVM and L2-SVM problems and propose our one-layer RNN for L2-SVM in its primal learning. Moreover, the convergence of proposed neural network, its performance and its convergence rate are analyzed on real-world data sets.

2 Problem Statement

2.1 SVM Learning

Let $\{x_i, y_i\}$ be a set of data points where $x_i \in \mathbb{R}^n$ is the ith data in n-dimensional space and y_i shows the label of x_i, in binary classification case $y_i \in \{-1, 1\}$. The SVM problem is to divide these data points into two disjoint groups by a hyperplane such that it has the maximum margin of both classes. In addition, this hyperplane must separate the data of the similar class in the same group. When data are linearly separable, the desired hyperplane $w^T x + b$ can be obtained by solving the following convex optimization problem

$$\begin{array}{lll} \min & 1/2w^T w + c\Sigma_{i=1}^{l}\xi_i & \\ s.t. & y_i(w^T x_i + b) \geq \mathbf{1} - \xi_i, & i = 1, ..., l, \\ & \xi_i \geq 0, & i = 1, ..., l \end{array} \tag{1}$$

where w is a $n \times 1$ vector, $b \in \mathbb{R}$, x_i is a vector of n-dimensional , $c > 0$ is a regularization parameter for the tradeoff between model complexity and training error and ξ_i measures the difference between $w^T x_i + b$ and y_i.

2.2 L2-SVM Learning

The major difference between SVM and L2-SVM appears in dealing with slack variables, and L2-SVM is modeled as follows

$$\begin{array}{lll} \min & 1/2w^T w + c\Sigma_{i=1}^{l}\xi_i^2 & \\ s.t. & y_i(w^T x_i + b) \geq \mathbf{1} - \xi_i, & i = 1, ..., l. \end{array} \tag{2}$$

3 The New Reformulation

In following, we investigate a refomulation of (1) to propose our new recurrent neural network. To do this, let $z = (w^T, b)^T$, problem (1) can be reformulated as

$$\begin{array}{ll} \min & 1/2z^T Qz + 1/2\xi C\xi \\ s.t. & \mathbf{1}_{l\times 1} - Az - \xi \leq 0. \end{array} \tag{3}$$

where C is a matrix such that $cI_{l\times l}$, $\mathbf{1}_{l\times 1}$ denotes a $l \times 1$ vector with elements one, Q is a symmetric and semi-definite positive matrix

$$Q = \begin{bmatrix} I_{n\times n} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} y_1 & 0 & \dots & 0 \\ 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_l \end{bmatrix} \times \begin{bmatrix} x_1^T & 1 \\ x_2^T & 1 \\ \vdots & \vdots \\ x_l^T & 1 \end{bmatrix}$$

and $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T$.
According to Karush-Kuhn-Tucker (KKT) conditions, (z, ξ) is the solution of (3) if and only if there exist $u \in \mathbb{R}^l$ such that (z, ξ, u) satisfies the following conditions

$$\begin{cases} \begin{pmatrix} Qz \\ C\xi \end{pmatrix} + (u^T, u^T) \begin{pmatrix} -A \\ -I_{l\times l} \end{pmatrix} = 0 \\ Qz - A^T u = 0, z \ is \ free \\ C\xi - u = 0, \\ u^T(\mathbf{1}_{l\times 1} - Az - \xi) = 0, \ u \geq 0 \end{cases} \tag{4}$$

The second term implies that $\xi = C^{-1}u$. Hence, using the well-known projection theorem, one can easily obtain the following lemma.

Lemma 3.1. *(z^*, ξ^*) is the solution to (3) if and only if there exist non-negative $u^* \in \mathbb{R}^l$ such that (z^*, ξ^*, u^*) satisfies*

$$\begin{cases} A^T u^* - Qz^* = 0 \\ P_+(u^* + \mathbf{1}_{l\times 1} - Az^* - C^{-1}u^*) = u^* \\ \xi^* = C^{-1}u^* \end{cases} \tag{5}$$

where $P_+(x) = max\{0, x\}$.

Based on the above equivalent formulation, we propose a RNN for solving (1), with dynamical system given by

$$\frac{d}{dt}\begin{pmatrix} z \\ u \end{pmatrix} = \alpha \begin{pmatrix} -Qz + A^T u \\ P_+(u + \mathbf{1}_{l\times 1} - Az - C^{-1}u) - u \end{pmatrix} \tag{6}$$

where $\alpha > 0$ is a scaling parameter.

4 Convergence Analysis

Definition 4.1. A continuous-time nueral network is said to be globally convergent if for any initial point, the trajectory of the corresponding dynamic system converges to an equilibrium point.

Lemma 4.2. *Let $X \in \mathbb{R}^n$ be a closed convex set, and $P_X(.)$ denotes the projection function defined by*

$$P_X(u) = arg \min_{v\in X} \parallel u - v \parallel,$$

then for all $u, v \in \mathbb{R}^n$ and $x \in X$

$$(u - P_X(u))^T(P_X(u) - x) \geq 0 \quad \|P_X(u) - P_X(v)\| \leq \|u - v\|.$$

Lemma 4.3. *For any initial point $s_0 = (z_0^T, u_0^T)^T$ there exist a unique continuous solution $s(t) = (z(t)^T, u(t)^T)^T$ for (6). Moreover the equilibrium point of (6) solves problem (3).*

Theorem 4.4. *The proposed nueral network (6) with the initial point $s_0 \in \mathbb{R}^{n+1} \times \mathbb{R}^l$ is stable in the sense of Lyapunov and globally converges to the solution of (3).*

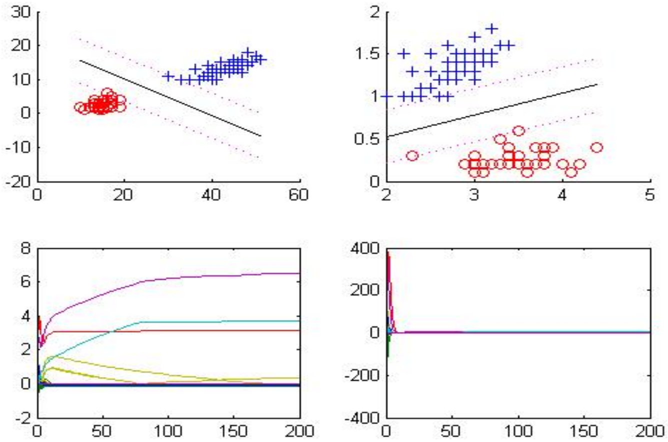

Figure 1

5 Experiment Results

In this section, to illustrate the performance of the proposed recurrent neural networks, we present several simulation results on empirical data sets, Fisher-iris data sets.
To this, let $\alpha = 10$, Fig. 1 show the convergence of the trajectory of (6) with the initial point one and zero. These results confirm the globally convergence of proposed neural network.
In addition, the performance of (6) to classify data sets, Iris and Fisher-Iris, is brought in Fig. 1.

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New Optimization algorithm via Modified Quantum Genetic Computation

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Abstract

This paper proposes a modified method for solving optimization problems by quantum genetic algorithms. This method according to mutation after measurement process, improves the efficiency and accuracy of searching the optimal solution of the optimization problem. To show the advantages of proposed method an example simulation is presented.

Keywords: Quantum Genetic Algorithm, Mutation, Qubit
Mathematics Subject Classification [2010]: 68Q12, 68W20

1 Introduction

Quantum genetic algorithm (QGA) is the product of the combination of quantum computation and genetic algorithms, and it is a new evolutionary algorithm of probability [2]. It was proposed by Narayanan and Moore in 1996. QGA is based on the concept and principles of quantum computing such as qubits and superposition of states. The quantum state vector is introduced in the Genetic Algorithm to express genetic code, and quantum logic gates are used to realize the chromosome evolution [3]. By these means, better results are achieved, but there are still some problems in conventional QGA. In this paper we improve the performance of QGA by mutation of chromosomes before rotating the Genes. This paper is organized as follows. In section 2 a description of the basic concept of quantum computing and QGA principles is presented. Section 3 describes the structure of QGA. An experimental simulation and Concluding remarks follow in Section 4.

2 QGA principles

2.1 Qubit and Its Representation

In quantum information theory, the state $|\psi>$ of a (finite dimensional) quantum system encodes information. In particular, in typical implementations, the information is encoded in a number of two level systems called qubits [1]. The qubit is a two-state quantum system. These two states of a qubit are represented by the computational basis vectors

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$|0>$ and $|1>$ in a two-dimensional Hilbert space. An arbitrary qubit state$|\psi>$ maintains a coherent superposition of the basis states $|0>$ and $|1>$ according to the expression:

$$|\psi>= \alpha|0> + \beta|1>; |\alpha|^2 + |\beta|^2 = 1$$

where α and β are the complex numbers which are called the probability amplitude of corresponding state of qubit.

2.2 Structure of Quantum Chromosomes

A chromosome is a string of m qubits that forms a quantum register. And the jth individual chromosome of the tth generation is defined as follows:

$$q_j^t = \begin{pmatrix} \alpha_{11}^t & \alpha_{12}^t & ... & \alpha_{1k}^t|\alpha_{21}^t & \alpha_{22}^t & ... & \alpha_{2k}^t|...|\alpha_{m1}^t & \alpha_{m2}^t & ... & \alpha_{mk}^t \\ \beta_{11}^t & \beta_{12}^t & ... & \beta_{1k}^t|\beta_{21}^t & \beta_{22}^t & ... & \beta_{2k}^t|...|\beta_{m1}^t & \beta_{m2}^t & ... & \beta_{mk}^t \end{pmatrix}$$

where k represents the number of qubit encoding of each gene; m represents the number of genes in the chromosome. Initialize the quantum encoding (α, β) of each individual in the population with $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, which indicates that when $t = 0$, the probability of collapsing the superposed state into each basic states is equal.

2.3 Quantum Rotating Gates

The construction of Quantum rotating gates is the key issue of QGA, it can be designed according to the practical problems and usually can be defined as

$$U(\theta_i) = \begin{pmatrix} cos(\theta_i) & -sin(\theta_i) \\ sin(\theta_i) & cos(\theta_i) \end{pmatrix}$$

The updated process is

$$\begin{pmatrix} \alpha_i' \\ \beta_i' \end{pmatrix} = U(\theta_i) \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} cos(\theta_i) & -sin(\theta_i) \\ sin(\theta_i) & cos(\theta_i) \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$$

where $(\alpha_i, \beta_i)^T$ and $(\alpha_i', \beta_i')^T$ are the probability amplitudes of the ith qubit in chromosome before and after the quantum rotating gates updating, respectively; θ_i is the rotating angle. Here we use following table to update chromosomes.

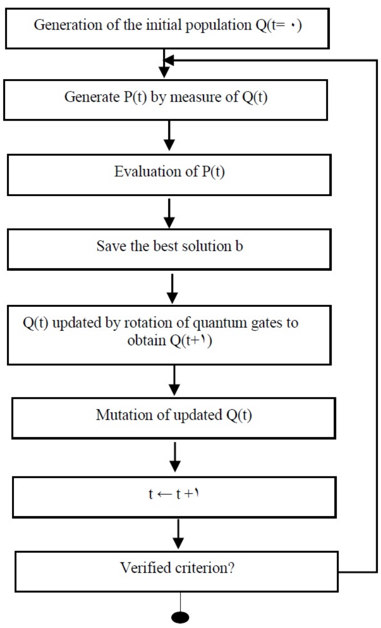
x_i	best_i	$f(x) > f(\text{best})$	$\Delta\theta_i$	$s(\alpha_i, \beta_i)$			
				$\alpha_i\beta_i > 0$	$\alpha_i\beta_i < 0$	$\alpha_i = 0$	$\beta_i = 0$
0	0	FALSE	0	0	0	0	0
0	0	TRUE	0	0	0	0	0
0	1	FALSE	$\Delta\theta_i$	+1	−1	0	±1
0	1	TRUE	$\Delta\theta_i$	−1	+1	±1	0
1	0	FALSE	$\Delta\theta_i$	−1	+1	±1	0
1	0	TRUE	$\Delta\theta_i$	+1	−1	0	±1
1	1	FALSE	0	0	0	0	0
1	1	TRUE	0	0	0	0	0

Figure 1: Adjustment strategy of rotating angle

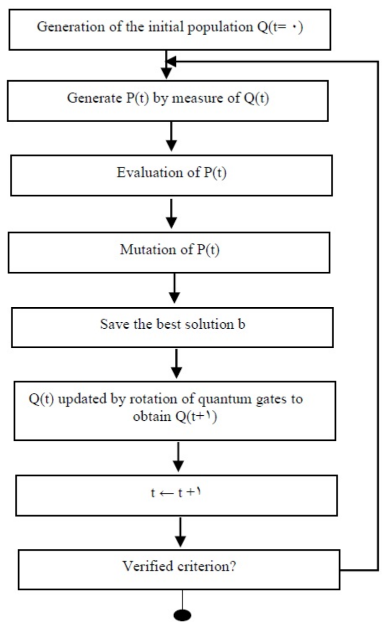
the value and the sign of θ_i are determined by the adjustment strategy. x_i is the ith bit of the current chromosome; $best_i$ is the ith bit of the current optimal chromosome; $f(x)$ is the fitness function; $s(\alpha_i, \beta_i)$ is the direction of the rotating angle; $\delta\theta_i$ is the value of the rotating angle. The value of $\delta\theta_i$ is generally a constant value is around 0.01π.

3 Structure of Genetic Algorithm (GA) and Quantum Genetic Algorithm (QGA)

In [3], an operation named quantum mutation operation that can completely reverse the individuals evolutionary direction by swapping the value of probability amplitude of qubits (α, β), is introduced as an improvement method of quantum genetic algorithms. Quantum NOT gates is adopted to realize chromosomal variation. Quantum mutation operation helps to increase the diversity of the population and reduce the probability of premature convergence. Figure 2.a shows the QGA structure with the Quantum mutation. In this paper we add the mutation operator for measured qubits, that is classic genes mutates with a little mutation rate. The structure of proposed method is shown in figure 2.b.



(a) mutation after updating chromosomes



(b) mutation before updating chromosomes(proposed method)

Figure 2: QGA structure with mutation operation

4 Simulation results

A simple example: find the optimal solution in the binary function:

$$Minf(x) = sin(x)$$

$$0 \leq x \leq 10$$

The conventional quantum genetic algorithm and improved quantum genetic algorithm are encoded by the binary; the evolution generation is 100; the size of population is 20; the length of each binary variable is 8; fitness function is the objective function. The results

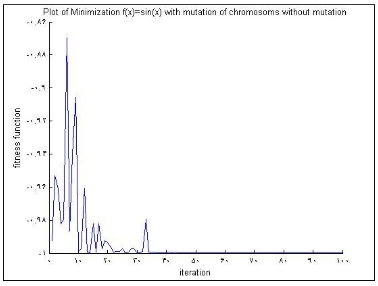
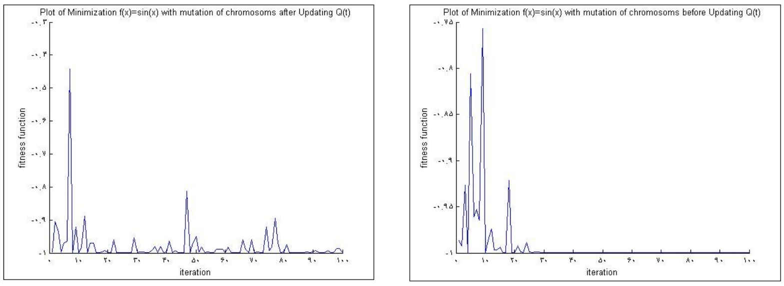


Figure 3: Minimization of $f(x) = sin(x)$ with conventional QGA



(a) mutation after updating chromosomes

(b) mutation of measured chromosomes(before updating Q(t))

Figure 4: Minimization of $f(x) = sin(x)$ with improved QGA

of the conventional quantum genetic algorithm, QGA with quantum mutation operator and QGA with mutation operator on measured qubits are shown in figures 3, 4.a and 4.b, respectively. The x-axis represents the evolutionary generations; y-axis represents the best fitness of every generation. Comparing above figures, we see that the convergence of QGA with mutation of measured qubits (fig 4.b), is more stable than the case of adding quantum mutation (fig 4.a). The modified method presented in this paper causes arising performance ratio of convergence and robustness of algorithm.

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Nonbinary Cycle Codes by Packing Design

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Abstract

In this paper, some $(v, 3, 1)$-packing designs are used to construct a class of nonbinary QC-LDPC codes whose parity-check matrices have uniform column-weight two. The main advantage of this approach is that, the constructed nonbinary QC-LDPC codes can achieve the large girth 36. Simulation results show that the constructed nonbinary QC-LDPC codes perform better than the nonbinary progressive edge growth (PEG) QC-LDPC codes and nonbinary codes from lifting girth-8 cycle codes for moderate block length and low code rate.

Keywords: Nonbinary QC-LDPC codes, Packing Design, Girth.

Mathematics Subject Classification [2010]: 11T71, 68P30

1 Introduction and Preliminaries.

Nonbinary LDPC codes are applicable to match underling modulation in multicarrier underwater acoustic communications. It is feasible via developing a code design procedure to obtain nonbinary LDPC codes with significant performance. It has shown in [1], a code construction method that replaces an appropriate portion of the columns in the parity check matrix of a cycle code by columns having a weight equal or greater than two in order to increasing the codes minimum distance and decreasing the multiplicities of low weight codewords. Due to short cycles in the parity-check matrices create correlation of the extrinsic information during iterative decoding, and cause decoding performance degradation, many approach have been studied to construct nonbinary LDPC codes with large girth. Some girth-8 cycle codes, whose parity-check matrices with girth 24 proposed in [4].

For construct parity-check matrix of column-weight two nonbinary LDPC codes with large girth, We simply employ $(v, 3, 1)$-packing design. A $(v, 3, 1)$-packing design of order v, block size 3, and index one is a collection with k, 3-element subsets, called blocks, of a v-set,$V = \{1, 2, \cdots, v\}$, such that every 2-subset of V occurs in at most one block. We associate with $(v, 3, 1)$- packing design a binary *incidence matrix* $D = (d_{ij})$ of v rows and k columns. Every row of D corresponds to a block and every column corresponds to one object in V, such that $d_{ij} = 1$ if the i-th object belongs to the j-th block and $d_{ij} = 0$, otherwise. In [2] has provided the details of the design construction. For $t \geq 7$, let $\mathcal{X}_t = \{1, 3, \cdots, 2\lceil t/2\rceil - 1\}$ and $\mathcal{Y}_t = \{2, 4, \cdots, 2\lfloor t/2\rfloor\}$ be the odd and even positive

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integers not greater than t, respectively. Set $\mathcal{W} = \mathcal{Y}_t \times \mathcal{X}_t = \{(y, x) : \ y \in \mathcal{Y}_t, \ x \in \mathcal{X}_t\}$. Define the relation $\mathcal{R}$ on $\mathcal{W}$ by $(y_1, x_1) \ \mathcal{R} \ (y_2, x_2) \Leftrightarrow y_1 - x_1 \equiv y_2 - x_2 \pmod{2\lceil \frac{t}{2} \rceil}$. It is easy to see that $\mathcal{R}$ is an equivalent relation on $\mathcal{W}$.

Lemma 1.1. [2] For positive integers $1 \leq i < j \leq \lceil \frac{t}{2} \rceil$, we have $(2, 2i - 1) \not\mathcal{R} (2, 2j - 1)$.

For each $t \geq 7$ and $1 \leq i \leq \lceil \frac{t}{2} \rceil$, let $\mathcal{R}_i = [(2, 2i - 1)]$ be the equivalence class of $(2, 2i - 1)$ under the relation $\mathcal{R}$. By Lemma 1.1, the equivalence classes $\mathcal{R}_i$, $1 \leq i \leq \lceil \frac{t}{2} \rceil$, are distinct. Each $\mathcal{R}_i$ has $\lfloor \frac{t}{2} \rfloor$ elements. Set $K_i = \{\{y, x, t + 1 + u_i\} : (y, x) \in \mathcal{R}_i\}$, where $u_i = 1 - i \pmod{\lceil \frac{t}{2} \rceil}$, and $\mathcal{K}_t = \bigcup_i K_i$. Clearly $\mathcal{K}_t$ is a $(v, k)-$design on $V = \{1, \cdots, v\}$, where $k = \lceil t/2 \rceil \lfloor t/2 \rfloor$ and $v = \lceil 3t/2 \rceil$.

The incidence matrix D_t of $\mathcal{K}_t$ can be used to construct a base matrix of a nonbinary codes with uniform column-weight two. This means that each row and column of the incidence matrix correspond to points and each nonzero entry of the incidence matrix correspond to lines. So that each line is composed of two points and there is one and only one line between two points. Each pair of lines has at most one common point. Now, label each point and line by a pair (i, j), where i and j are the row and column indices of base matrix $\mathbf{B}_t = (b_{ij})$, such $b_{ij} = 1$ if the i-th point lies on the j-th line and $b_{ij} = 0$, otherwise. Figure 1, shows an example of a Tanner graph of $\mathcal{K}_7$ and corresponding incidence matrix $\mathbf{B}_7$. Hereinafter, in order to properly apply the notations, we show the base matrix $\mathbf{B}_t$ by

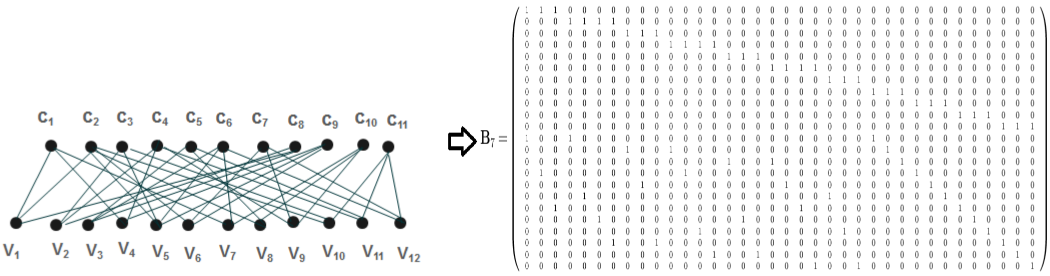

Figure 1: Tanner graph of $\mathcal{K}_7$ and corresponding incidence matrix $\mathbf{B}_7$.

a (m, n) block-design $\mathcal{B} = [B_1, B_2, \ldots, B_n]$, i.e. a list of blocks $B_i \subseteq M = \{1, 2, \ldots, m\}$, where B_i, $1 \leq i \leq n$, is the row-indices of non-zero elements in the ith column of $\mathbf{B}_t$. The number of blocks of $\mathbf{B}_t$ depend on choice of t, is $(t/2)\lceil 3t/2 \rceil$ for even t and $3\lceil t/2 \rceil \lfloor t/2 \rfloor$, otherwise. Also, for each $t \geq 7$, the girth of base matrix $\mathbf{B}_t$ is 12. Let $\mathcal{C}_t$ be the nonbinary QC-LDPC codes with the parity-check matrices H_t. The following theorem states that for each $t \geq 7$, $\mathbf{B}_t$ is irregular (the proof is clear).

Theorem 1.2. If t is even, then $\mathbf{B}_t$ has $\lceil 3t/2 \rceil$ rows of weight $\frac{t}{2}$ and $\lceil t/2 \rceil \lfloor t/2 \rfloor$ rows of weight 3. Moreover, if t is odd, then $\mathbf{B}_t$ has $\lfloor t/2 \rfloor$ rows of weight $\lceil t/2 \rceil$, $2\lceil t/2 \rceil$ rows of weight $\lfloor t/2 \rfloor$ and $\lceil t/2 \rceil \lfloor t/2 \rfloor$ rows of weight 3.

It has been shown in [2], while the girth of base matrix of QC-LDPC codes is $2g$, the maximum achievable girth of QC-LDPC codes is at least $6g$. So, according to this fact, for each $t \geq 7$ the maximum girth of parity-check matrix of nonbinary QC-LDPC code $\mathcal{C}_t$ is at least 36. Let g, m and n be some positive integers such that $g \geq 6$ and $m < n$.

For given positive integer P and base matrix $\mathbf{B}_t$ with corresponding block-design $\mathcal{B}$, let $A = (a_1, a_2, \cdots, a_n)$ be a slope-vector, such that each a_i belongs to $\{0, 1, 2, \ldots, P-1\}$. By $\mathcal{H}$, we mean the $mP \times nP$ parity-check matrix of a binary code with CPM size P which is obtained by replacing each zero and (i, j) non-zero element of $\mathbf{B}_t$ by the $P \times P$ zero matrix and I^{a_i}, respectively. I^{a_i} is $P \times P$ identity matrix by cyclic shifting of each column a_i position. For each j, $1 \leq j \leq n$, if $B_j = \{j_1, j_2\}$, $j_1 < j_2$, then we define $a_{j_1,j_2} := a_j$ and $a_{j_2,j_1} := -a_j \bmod P$. Let for $z \leq n$, $g(a_1, \cdots, a_z)$ be the minimum $2l$ cycle in $\mathcal{H}$, such that $a_{j_0,j_1} + a_{j_1,j_2} + \cdots + a_{j_{l-2},j_{l-1}} + a_{j_{l-1},j_0} \equiv 0 \bmod P$, where for each $0 \leq k \leq l-1$, we have $j_k \neq j_{k+1 \bmod l}$, and if $b_k = \{j_k, j_{k+1 \bmod l}\}$, then $b_k \neq b_{k+1 \bmod l}$ and $\{b_0, \cdots, b_{l-1}\} \subseteq \{B_1, B_2, \cdots, B_z\}$. For enough large P, the following algorithm finds A, such that $g(\mathcal{H}) \geq 2g$.

Algorithm.

1. Select an arbitrary positive integer $t \geq 7$.
2. Set $P = 1$.
3. Set $W = \{0, 1, ..., P-1\}$, $z = 1$, $W_1 = W$.
4. If $z = 0$ then $z \to z + 1$ and go to 3.
5. If $W_z = \emptyset$, then $W_{z-1} \to W_{z-1} - \{a_{z-1}\}$,$z \to z - 1$, and go to step 4.
6. Select arbitrary element $a_z \in W_z$.
7. If $z < n$ then $z \to z + 1$.
8. If $z = n$ then go to step 10.
9. Let $W_z := \{a \in W | g(a_1, \cdots, a_{z-1}, a_z := a) \geq 2g\}$ and go to step 5.
10. Print $a = (a_1, \cdots, a_n)$ as a solution.
11. END

Then the nonzero elements of the parity-check matrix $\mathcal{H}$ is replaced by some nonzero elements of $GF(q)$, randomly, to generate an ensemble of nonbinary QC-LDPC codes with different lengths and girths.

2 Simulation Results

In this section, we have provided some bit-error-rate (BER) performance comparisons between the constructed nonbinary column-weight two QC-LDPC codes, on one hand, and some other nonbinary QC-LDPC codes. In Figure 2, $NB(gb)$ and $NC(gb)$ are used to denote constructed nonbinary codes and cycle codes in [4], respectively, with girth b. This figure shows that the constructed nonbinary code with girth 24 performs remarkably better than nonbinary codes [4] and PEG [3] codes. Moreover, Figure 3, has provided some performance comparisons between the constructed nonbinary codes with different girth and it shows that the nonbinary constructed code having base matrices with girth 36 have greatly improved error correction performance than other codes.

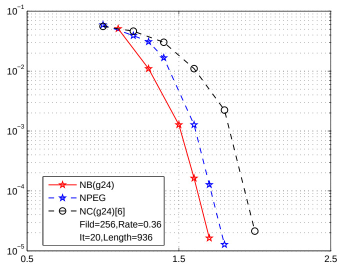

Figure 2: Nonbinary constructed code versus nonbinary PEG and cycle code in[4].

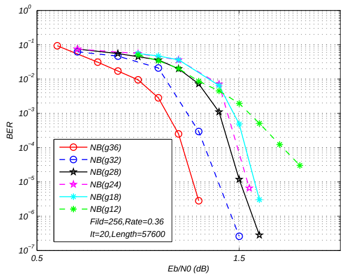

Figure 3: Performance comparison nonbinary constructed code with different girth.

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On uniqueness of a spacewise-dependent heat source in a time-fractional heat diffusion process

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Abstract

In this paper, a multi-dimensional inverse source problem for the time-fractional diffusion equation is investigated. Uniqueness results have been proved under some conditions on the problem. The fractional differentiation is considered to be of Riesz-Caputo type.

Keywords: time-fractional equation, uniqueness result, heat source, inverse problem, parabolic heat equation.
Mathematics Subject Classification [2010]: 35R30, 58J35, 58J90

1 Introduction

In recent years, fractional differential equation have attracted wide attentions. Various models using fractional partial differential equations have been successfully applied to describe problems in biology, physics, chemistry and biochemistry, and finance. These new fractional-order models are more adequate than the integer-order models, because the fractional order derivatives and integrals enable the description of the memory and hereditary properties of different substance. Time-fractional diffusion equation is deduced by replacing the standard time derivative with a time fractional derivative and can be used to describe the superdiffusion and subdiffusion phenomena. The direct problems, i.e., initial value problem and initial boundary value problems for time-fractional diffusion equation have been studied extensively in recent years, for instances, on maximum principle, on some uniqueness and existence results, on numerical solutions by finite element methods and finite difference methods, on exact solutions [7]. The early papers on inverse problems were provided by Murio in [1, 2] for solving sideways fractional heat equations by mollification methods. After that, some works have been published. In [3], Cheng et al. considered an inverse problem for determining the order of fractional derivative and diffusion coefficient in fractional diffusion equation and gave a uniqueness result. In [4], Liu and Yamamoto solved a backward problem for the time-fractional diffusion equation by a quasi- reversibility regularization method. Zheng and Wei in [5, 6] solved the Cauchy problems for time fractional diffusion equation on a strip domain by a Fourier regularization and a modified equation method. In [7] the one dimentional initial-boundary value problem for time fractional diffusion equation has been dealt with in terms of left-sided

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Caputo fractional derivative. Following the ideas in [8], in this paper we are going to prove a uniqueness result for the inverse multi-dimentional problem

$$\begin{cases} {}_0^{RC}D_T^{\alpha}u + Lu = f(x) & \text{in} \quad \Omega \times (0,T), \quad 0<\alpha<1, \\ u = 0 & \text{on} \quad \Gamma \times (0,T), \\ u(x,0) = u_0(x) & \text{for} \quad x \in \Omega, \end{cases} \tag{1}$$

with additional information

$$u(x,T) = \psi_T(x), \tag{2}$$

where ${}_0^{RC}D_T^{\alpha}u$ is the Riesz-Caputo fractional derivative of u taken in terms of the time variable.

2 Main resulte

Definition 2.1.

1) The left and right Riemann-Liouville fractional integrals of order α are defined respectively by

$${}_aI_x^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}y(t)dt \quad and \quad {}_xI_b^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}y(t)dt.$$

2) The Riesz fractional integral ${}_a^RI_b^{\alpha}y$ is given by

$${}_a^RI_b^{\alpha} = \frac{1}{2}({}_aI_x^{\alpha}y(x) + {}_xI_b^{\alpha}y(x)).$$

3) The left and right Riamann-Liouville fractional derivatives of order α are defined respectively by

$${}_aD_x^{\alpha}y(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}y(t)dt \quad and \quad {}_xD_b^{\alpha}y(x) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b (t-x)^{-\alpha}y(t)dt.$$

4) The Riesz fractional derivative ${}_a^RD_b^{\alpha}y$ is given by

$${}_a^RD_b^{\alpha}y(x) = \frac{1}{2}({}_aD_x^{\alpha}y(x) - {}_xD_b^{\alpha}y(x)).$$

5) The left and right Caputo fractional derivatives of order α are defined respectively by

$${}_a^CD_x^{\alpha}y(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}\frac{d}{dx}y(t)dt \quad \text{and} \quad {}_x^CD_b^{\alpha}y(x) = \frac{-1}{\Gamma(1-\alpha)}\int_x^b (t-x)^{-\alpha}\frac{d}{dx}y(t)dt.$$

6) The Riesz-Caputo fractional derivative ${}_a^{RC}D_b^{\alpha}y$ is given by

$${}_a^{RC}D_b^{\alpha}y(x) = \frac{1}{2}({}_a^CD_x^{\alpha}y(x) - {}_x^CD_b^{\alpha}y(x)).$$

Lemma 2.2. *Let* ${}_aI_x^{\alpha}y(x)$, ${}_a^RI_b^{\alpha}y$, ${}_aD_x^{\alpha}y(x)$, ${}_a^RD_b^{\alpha}y(x)$, ${}_a^CD_x^{\alpha}y(x)$, ${}_a^{RC}D_b^{\alpha}y(x)$ *be as above. Then we have*

$$\int_0^T {}_0^{RC}D_T^{\alpha}u(s) \cdot {}_0^{RC}D_T^{2\alpha}u(s)ds =$$

$$\frac{1}{2}{}_0^RI_T^{1-\alpha}\left({}_0^{RC}D_T^{\alpha}u(s)\right)^2 |_{s=0}^T + \frac{1}{4\Gamma(1-\alpha)}\int_0^T \left(\frac{{}_0^{RC}D_T^{\alpha}u(T)}{(T-s)^{\alpha}} - \frac{{}_0^{RC}D_T^{\alpha}u(0)}{s^{\alpha}}\right){}_0^{RC}D_T^{\alpha}u(s)ds.$$

Theorem 2.3. *Consider a linear differential operator*

$$Lu(x,t) = \nabla \cdot (-A(x)\nabla u(x,t)) + b^t(x)\nabla u(x,t) + c(x)u(x,t),$$

with bounded (dis-continuous) coefficients obeying

$$\forall u: \quad (Lu, u) \geq 0,$$

and Lu does not change sign. Let u_0, $\psi_T \in L^2(\Omega)$. Then there exists at most one spacewise-dependent heat source $f \in L^2(\Omega)$ such that (1) together with condition (2) hold.

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Talk Open questions concerning Hindman's theorem pp.: 1–4

Open questions concerning Hindman's theorem

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Abstract

Hindman's theorem states that for every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite set A such that the set of numbers which can be written as a sum of distinct elements of A is monochromatic. In this paper, we survey some interesting questions concerning this theorem.

Keywords: Reverse mathematics, Hindman's theorem, Ultrafilters
Mathematics Subject Classification [2010]: 03B30, 05C55

1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. Given $X \subseteq \mathbb{N}$ let $FS(X)$ be the set of all sums of finite nonempty subsetes of X. Hindma's theorem is the following statement.

Theorem 1.1. *(Hindman) If $\mathbb{N} = C_0 \cup \ldots \cup C_l$, then there exists an infinite set $X \subseteq \mathbb{N}$ such that $FS(X) \subseteq C_i$ for some $i \leq l$.*

There are four proofs of Hindman's theorem:
(1) The original combinatorial proof due to Hindman [6];
(2) The simplified combinatorial proof due to Baumgartner [1];
(3) The dynamical proof due to Furstenberg and weiss [2];
(4) The ultrafilter proof due to Glazer [4].

The notion of an ultrafilter is a powerful tool in set theory, combinatorics and topology. We here give a short proof of Hindman's theorem using ultrafilters. For more details see [3]. An ultrafilter on a set X is a set of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying

1. $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
3. For all $A \subseteq X$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Theorem 1.2. *Let $\mathcal{F}$ be an ultrafilter on a set X.*
1. *If B is such that $A \cap B \neq \emptyset$ for all $A \in \mathcal{F}$ then $B \in \mathcal{F}$.*
2. *If A and B are such that $A \cup B \in \mathcal{F}$ then at least one of $A, B \in \mathcal{F}$.*

*Speaker

Proof. We prove the second and left the first as an exercise. If we have both $A, B \notin \mathcal{F}$ then by the first statement there are $C, D \in \mathcal{F}$ with $A \cap C = \emptyset$ and $B \cap D = \emptyset$ so $(A \cup B) \cap (C \cup D) = \emptyset$, so $A \cap B \notin \mathcal{F}$ since $C \cap D \in \mathcal{F}$. □

By induction, one can extend the second statement above: If $\mathcal{F}$ is an ultrafilter and $A \in \mathcal{F}$ then whenever we write $A = A_1 \cup \cdots \cup A_n$ as a disjoint union of finitely many sets, exactly one of the A_i is in $\mathcal{F}$. An ultrafilter on X is called nonprincipal if it is not of the form $\mathcal{F}(x) = \{A : x \in A\}$ for some $x \in X$. It is known that for infinite X, there is a nonprincipal ultrafilter on X.

Let βX be the set of all ultrafilters on X. If X is finite then $|X| = |\beta X|$ with a natural bijection: $x \mapsto \mathcal{F}(x)$. If X is infinite, βX again contains a copy of X, the collection of principle ultrafilters $\{\mathcal{F}(x) : x \in X\}$. We'll just consider here the simple case $X = \mathbb{N}$. We define a binary operation $+ : \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$ as follows. For $\mathcal{F}, \mathcal{G} \in \beta\mathbb{N}$,

$$\mathcal{F} + \mathcal{G} = \{A \subseteq \mathbb{N} : \{n \in \mathbb{N} : A - n \in \mathcal{G}\} \in \mathcal{F}\},$$

where $A - n = \{a - n : a \in A\}$. It it shown that for all $\mathcal{F}, \mathcal{G} \in \beta\mathbb{N}$, $\mathcal{F} + \mathcal{G} \in \beta\mathbb{N}$. Another important property is the following.

Theorem 1.3. *(Idempotent lemma) There is $\mathcal{F} \in \beta\mathbb{N}$ with $\mathcal{F} + \mathcal{F} = \mathcal{F}$.*

For $A \subseteq \mathbb{N}$, let $A' = \{n \in \mathbb{N} : A - n \in \mathcal{F}\}$, then by Idempotent lemma,

$$\mathcal{F} + \mathcal{F} = \{A \subseteq \mathbb{N} : A' \in \mathcal{F}\} = \mathcal{F}.$$

So for all $A \in \mathcal{F}$, $A' \in \mathcal{F}$ and so $A \cap A' \in \mathcal{F}$. We are now ready to prove the Hindman's theorem.

Proof. (Proof of theorem 1.1) Fix a colouring χ. We'll inductively construct sequences $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ and distinct $a_1, a_2, \ldots$ with the properties that $a_i \in A_{i-1}$, $A_i \in \mathcal{F}$, and $a_{i+1} + A_{i+1} \subseteq A_i$, and with χ constant on A_0. This will give the result, for consider any finite sum from among the a_i's, say

$$a_7 + a_4 + a_3.$$

We have $a_7 \in A_6 \subseteq A_5 \subseteq A_4$, so $a_7 + a_4 \in A_3$, so $a_7 + a_4 + a_3 \in A_2 \subseteq A_1 \subseteq A_0$. To complete the proof, fix an idempotent ultrafilter $\mathcal{F} \in \beta\mathbb{N}$. There is a unique colour i with $A_0 := \{n \in \mathbb{N} : \chi(n) = i\} \in \mathcal{F}$. Since $\mathcal{F}$ is idempotent, $A'_0 \in \mathcal{F}$ and so also $A_0 \cap A'_0 \in \mathcal{F}$. Select $a_1 \in A_0 \cap A'_0$ and set $A_1 = A_0 \cap (A_0 - a_1) - \{a_1\}$ So $A_1 \subseteq A_0$, $a_1 + A_1 \subseteq A_0$, and $A_1 \in \mathcal{F}$ (removing one element from a set in $\mathcal{F}$ does not take it out of $\mathcal{F}$, since $\mathcal{F}$ is nonprincipal). Having defined A_n, select $a_{n+1} \in A_n \cap A'_n$ and set $A_{n+1} = A_n \cap (A_n - a_{n+1}) - \{a_{n+1}\}$. □

As a corrolary to Hindman's theorem one can prove that for every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite set A such that finite products of elements of A lie entirely inside one partition class. That raises a natural question:

Conjecture. whenever the natural numbers are partitioned into finitely many classes, it is possible to find an infinite set A such that both finite sums and products of A lie entirely inside one partition class.

The following simple case is only known when the number of classes is two.

Conjecture. Whenever the natural numbers are partitioned into finitely many classes, it is possible to find two numbers a and b such that a, b, $a+b$, and ab all lie in one partition class.

Conjecture. Whenever the natural numbers are partitioned into finitely many classes, it is possible to find two numbers a and b such that $a+b$ and ab all lie in one partition class.

1.1 $SP(a,r)$

For $a, r \in \mathbb{N}$, let $SP(a,r)$ be the first $n \in \mathbb{N}$, if such exists, such that whenever $\{1, 2, \ldots, n\}$ is r-colored, there exist x and y with $a \leq x < y$ such that $\{x+y, xy\}$ is monochromatic. If no such n exists, the number is defined to be infinite. It is an old result of R. Graham that $SP(a,2)$ is finite for all a. The exact value of $SP(a,2)$ is known for $a \leq 105$ [7]. In all computed cases, $SP(a,2)$ is divisible by a^2. This seems less likely to be a random occurrence.

Conjecture. For all $a \in \mathbb{N}$, $SP(a,2)$ is divisible by a^2.

Even the following seemingly simple case is open.

Conjecture. For all $a \in \mathbb{N}$, $SP(a,2)$ is divisible by a.

2 Hindman's theorem and reverse mathematics

An important question in mathematical logic is that which set existence axioms are needed to prove the known theorems of ordinary mathematics. This is the theme of a research program in foundations of mathematics called reverse mathematics. This question is studied in the language of second order arithmetic, the weakest language rich enough to express and develop the bulk of mathematics. Note that the formalization of mathematics within second order arithmetic goes back to Dedekind and was developed by Hilbert and Bernays [5]. In many cases, if a mathematical theorem is proved from appropriately weak set existence axioms, then the axioms will be logically equivalent to the theorem. Furthermore, only a few specific subsystems of second order arithmetic, called the big five, arise repeatedly in this context: RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. For details we refere the reader to [8]. Let HT denotes the statement of Hindman's theorem. Within RCA_0 one can prove that

1. HT implies ACA_0
2. HT can be proved in ${\mathsf{ACA}_0}^+$.

An interesting open question is the strength of Hindman's theorem.

Question. Is HT equivalent to ${\mathsf{ACA}_0}^+$, or to ACA_0, or does it lie strictly between them?

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Talk Private quantum channels and higher rank numerical range pp.: 1–4

Private quantum channels and higher rank numerical range*

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Abstract

In this note, by using the notion of dual map and cojugate quantum channel, we show that the higher rank numerical ranges and (p, k) numerical range can be used to describe private quantum codes and private quantum subsystems, respectively. We also show how this description provide a bridge between quantum error correction and cryptography.

Keywords: private quantum code, quantum cryptography, completely positive map, quantum error correction, higher rank numerical range
Mathematics Subject Classification [2010]: 15A60, 47A12, 81P68, 81P94

1 Introduction

First introduced in [1], private quantum channels are at the heart of quantum cryptography. They were introduced as the quantum analogue of the classical one-time pad.

In this paper we will restrict our attention to finite-dimensional Hilbert spaces $\mathbb{C}^n$. The set of $n \times m$ complex matrices is denoted by $M_{n\times m}$. $X^\dagger$ will be a notation to denotes the complex conjugate transpose of $X \in M_n \equiv M_{n\times n}$. We will use Dirac (bra-ket) notation: a unit column vector in $\mathbb{C}^n$ will be denoted $|\psi\rangle$, its dual (row) vector $|\psi\rangle^\dagger$ will be denoted $\langle\psi|$, and the rank-one projection associated to $|\psi\rangle$ is its outer product $|\psi\rangle\langle\psi|$. A mixed state is a convex combination of rank one projections. We call mixed states and outer products of pure states density operators, which are precisely the trace-one positive operators.

Given a linear map $\Phi : M_n \to M_m$, its dual map $\Phi^\dagger : M_m \to M_n$ is defined via the Hilbert–Schmidt inner product: it is the unique map $\Phi^\dagger$ satisfying $\mathrm{Tr}(\rho\Phi^\dagger(A)) = \mathrm{Tr}(\Phi(\rho)A)$ for all $A \in M_m$ and all density matrices $\rho \in M_n$. Quantum channels are described by completely positive trace preserving linear (CPTP) maps. The dual of a CPTP map is a unital (i.e. $\Phi^\dagger(I_n) = I_m$) completely positive linear (UCP) map. The Kraus operators of a channel Φ are the operators $\{E_i\}_{i=1}^r \subset M_{m\times n}$ given by operator sum representation $\Phi(\rho) = \sum_{i=1}^r E_i\rho E_i^\dagger$ for all $\rho \in M_n$. This representation of Φ is not unique, however, in general, results do not depend on the choice of Kraus operators.

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2 Private subspace codes and higher rank numerical ranges

A mathematical definition of private quantum channel can be given as follows.

Definition 2.1. Let $\mathcal{S} \subseteq \mathbb{C}^n$ be a subspace, $\Phi : M_n \to M_m$ be a quantum channel and let $\rho_0 \in M_m$ be a density matrix. Then $\mathcal{S}$ is *private subspace code* for Φ with output ρ_0 if

$$\Phi(|\psi\rangle\langle\psi|) = \rho_0, \qquad \forall\, |\psi\rangle \in \mathcal{S}.$$

Often both the channel Φ itself, as well as the triple $[\mathcal{S}, \Phi, \rho_0]$ from the above definition are called a private quantum channel.

Motivated by the theory of quantum error correction, researchers study the (joint) higher rank numerical range defined as follows, see for example [4].

Definition 2.2. Given $X_1, \ldots, X_m \in M_n$. The (joint) rank-k numerical range $\Lambda_k(\mathbf{X})$ of the m-tuple $\mathbf{X} = (X_1, \ldots, X_m)$ is defined as the collection of vectors $(a_1, \ldots, a_m)$ such that $PX_jP = a_jP$, $j = 1, \ldots, m$, for some rank-k orthogonal projection $P \in M_n$.

In the following result, a characterization of private quantum codes in terms of the dual map of a channel is derived.

Theorem 2.3 ([2]). *Let $\Phi : M_n \to M_m$ be a quantum channel. Then a subspace $\mathcal{S}$ of $\mathbb{C}^n$ is a private subspace code for Φ with output state $\rho_0 \in M_m$, if and only if for any $X \in M_m$, there exists a $\lambda_X \in \mathbb{C}$ such that*

$$P_{\mathcal{S}}\Phi^\dagger(X)P_{\mathcal{S}} = \lambda_X P_{\mathcal{S}},$$

where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$. Moreover, in this case $\lambda_X = \mathrm{Tr}(\rho_0 X)$.

Remark 2.4. Let $\Phi : M_n \to M_m$ be a quantum channel. By using Theorem 2.3, there exists a private subspace code of dimension k for Φ if and only if the joint rank-k numerical range of $\Phi^\dagger(X)$ for all $X \in M_m$ is nonempty.

Let $\Phi : M_n \to M_m$ be a quantum channel with operator sum representation $\Phi(\rho) = \sum_{j=1}^r V_j\rho V_j^\dagger$, where $V_j \in M_{m\times n}$ and $1 \le r \le mn$ is the smallest. Define $F : M_n \to M_{mr}$ with $F(\rho) = \left[\, V_i\rho V_j^\dagger \,\right]_{(i,j)}$. Then the *conjugate channel* of Φ is a quantum channel $\Phi^\# : M_n \to M_r$ defined by $\Phi^\#(\rho) = \left[\, \mathrm{Tr}(V_i\rho V_j^\dagger) \,\right]_{(i,j)}$. Fix an orthonormal basis $\{|e_i\rangle\}_{i=1}^m$ for $\mathbb{C}^m$. Then $\Phi^\#$ has operator sum representation $\Phi^\#(\rho) = \sum_{j=1}^k R_j\rho R_j^\dagger$, where $R_j^\dagger = \left[\, V_1^\dagger|e_j\rangle \; V_2^\dagger|e_j\rangle \; \cdots \; V_r^\dagger|e_j\rangle \,\right]$.

Theorem 2.5. *Let $\Phi : M_n \to M_m$ be a quantum channel with operator sum representation $\Phi(\rho) = \sum_{j=1}^r V_j\rho V_j^\dagger$, where $V_j \in M_{m\times n}$. Then a subspace $\mathcal{S}$ of $\mathbb{C}^n$ is a private subspace code for $\Phi^\#$ if and only if there exist $\Lambda = [\lambda_{ij}] \in M_r$ such that*

$$P_{\mathcal{S}}V_j^\dagger V_i P_{\mathcal{S}} = \lambda_{ij}P_{\mathcal{S}},$$

where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$. Moreover, $\rho_0 = [\lambda_{ij}]_{(i,j)}$.

By using Knill-Laflamme theorem in quantum error correction theory [4] we have the following corollary.

Corollary 2.6. *Given a conjugate pair of quantum channels Φ and $\Phi^{\#}$, a code is an error correction code for one if and only if it is a private subspace code for the other.*

Example 2.7. Consider the quantum channel $\Phi : M_4 \to M_4$ with Kraus operators $V_1 = \frac{1}{\sqrt{2}} I_4$ and $V_2 = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \otimes I_2$. Let $\mathcal{S}$ be a subspace of $\mathbb{C}^4$ spanned by $\{|00\rangle, |01\rangle\}$. Since $P_{\mathcal{S}} V_j^{\dagger} V_i P_{\mathcal{S}} = \lambda_{ij} P_{\mathcal{S}}$, where $\Lambda = [\lambda_{ij}]_{(i,j)} = \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, by using Theorem 2.5, $\mathcal{S}$ is a private subspace code for the conjugate channel $\Phi^{\#}$, where $\Phi^{\#}(\rho) = \sum_{j=1}^{4} R_j^{\dagger} \rho R_j$ and

$$R_1 = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right), \quad R_2 = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right), \quad R_1 = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{smallmatrix}\right), \quad R_1 = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right).$$

Every state $|\psi\rangle \in \mathcal{S}$ is of the form $|\psi\rangle = \left[\begin{smallmatrix} a & b & 0 & 0 \end{smallmatrix}\right]^T$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. By direct calculation, we find $\Phi^{\#}(|\psi\rangle\langle\psi|) = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Thus the code $\mathcal{S}$ is indeed private for the channel $\Phi^{\#}$ and $\rho_0 = \Lambda$.

3 Private subsystems and (p, k) numerical range

Let $\mathcal{A}$ and $\mathcal{B}$ be subspaces of $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$, respectively, such that $\dim \mathcal{A} = p \le n_1$, $\dim \mathcal{B} = k \le n_2$ and $\mathbb{C}^n = (\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes \mathcal{B})^{\perp}$. We call $\mathcal{A}$ and $\mathcal{B}$ subsystems of $\mathbb{C}^n$. The subspaces of $\mathbb{C}^n$ can be viewed as subsystems $\mathcal{B}$ for which $p = n_1 = 1$. A subscript on a state will indicate to which subsystem the state belongs, e.g. $|\psi_{\mathcal{A}}\rangle$ means the state belongs to $\mathcal{A}$.

Definition 3.1. Let $\Phi : M_n \to M_m$ be a quantum channel and let $\mathcal{B}$ be a subsystem of $\mathbb{C}^n$. Then $\mathcal{B}$ is called a *private subsystem* for Φ if for any $|\psi_{\mathcal{A}}\rangle \in \mathcal{A}$ there exists $\rho_{|\psi_{\mathcal{A}}\rangle} \in M_m$ such that

$$\Phi(|\psi_{\mathcal{A}}\rangle\langle\psi_{\mathcal{A}}| \otimes |\psi_{\mathcal{B}}\rangle\langle\psi_{\mathcal{B}}|) = \rho_{|\psi_{\mathcal{A}}\rangle}, \qquad \text{for all } |\psi_{\mathcal{B}}\rangle \in \mathcal{B}.$$

Motivated by the theory of *correctable quantum subsystems*, an extension of rank-k numerical range, known as (p, k) *numerical range* of $X \in M_n$ is defined as follows; see [3].

$$\Lambda_{(p,k)}(X) = \left\{ Y \in M_p \ : \ W^{\dagger} X W = Y \otimes I_k \text{ for some } W \in M_{n \times pk} \text{ with } W^{\dagger} W = I_{pk} \right\}.$$

Simillarly, we can define the joint (p, k) numerical range of an m-tuple of matrices in M_n. Note that $\Lambda_{(1,k)}(X) = \Lambda_k(X)$.

In the following result, we derive a characterization of private quantum subsystems in terms of the dual map.

Theorem 3.2. *Let $\Phi : M_n \to M_m$ be a quantum channel. Then a subsystem $\mathcal{B}$ of $\mathbb{C}^n$ is a private subsystem for Φ, if and only if for any $X \in M_m$, there exist $W_{\mathcal{A}} \in M_{n_1 \times p}$, $W_{\mathcal{B}} \in M_{n_2 \times k}$ and $Y_{X, W_{\mathcal{A}}} \in M_p$ such that $W_{\mathcal{A}}^{\dagger} W_{\mathcal{A}} = I_p$, $W_{\mathcal{B}}^{\dagger} W_{\mathcal{B}} = I_k$ and*

$$(W_{\mathcal{A}} \otimes W_{\mathcal{B}})^{\dagger} \Phi^{\dagger}(X) (W_{\mathcal{A}} \otimes W_{\mathcal{B}}) = Y_{X, W_{\mathcal{A}}} \otimes I_k.$$

Moreover, if $W_{\mathcal{A}} = \left[\, |\phi_1\rangle \cdots |\phi_p\rangle \,\right]$, then $Y_{X, W_{\mathcal{A}}} = \operatorname{diag}\left(\operatorname{Tr}(X \rho_{|\phi_1\rangle}), \ldots, \operatorname{Tr}(X \rho_{|\phi_p\rangle})\right)$.

Remark 3.3. Let $\Phi : M_n \to M_m$ be a quantum channel. By using Theorem 3.2, there exists a private subsystem of dimension k for Φ if and only if the joint (p, k) numerical range of $\Phi^{\dagger}(X)$ for all $X \in M_m$ is nonempty.

Theorem 3.4. *Let $\Phi : M_n \to M_m$ be a quantum channel. Then a subsystem $\mathcal{B}$ of $\mathbb{C}^n$ is a private subsystem for $\Phi^\#$, if and only if there exist $W_\mathcal{A} \in M_{n_1 \times p}$, $W_\mathcal{B} \in M_{n_2 \times k}$ and $Y_{ij,W_\mathcal{A}} \in M_p$ such that $W_\mathcal{A}^\dagger W_\mathcal{A} = I_p$, $W_\mathcal{B}^\dagger W_\mathcal{B} = I_k$ and*

$$(W_\mathcal{A} \otimes W_\mathcal{B})^\dagger V_j^\dagger V_i (W_\mathcal{A} \otimes W_\mathcal{B}) = Y_{ij,W_\mathcal{A}} \otimes I_k.$$

Moreover, if $W_\mathcal{A} = [\,|\phi_1\rangle \cdots |\phi_p\rangle\,]$, then $\rho_{|\phi_t\rangle} = \left[(Y_{ij,W_\mathcal{A}})_{tt} \right]_{(i,j)}$ for all $t = 1, \ldots, p$ and $Y_{ij,W_\mathcal{A}} = \operatorname{diag}\left((\rho_{|\phi_1\rangle})_{ij}, \ldots, (\rho_{|\phi_p\rangle})_{ij} \right)$.

By using theory of operator quantum error correction [3] we have the following corollary. Another approach to the following corollary is given in [2].

Corollary 3.5. *Given a conjugate pair of quantum channels Φ and $\Phi^\#$, a subsystem is an operator error correction subsystem for one if it is a private subsystem for the other.*

Example 3.6. Consider quantum channel $\Phi : M_4 \to M_4$ with Kraus operators

$$V_1 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{1-\alpha} \end{bmatrix} \otimes I_2 \quad \text{and} \quad V_2 = \begin{bmatrix} 0 & \sqrt{\alpha} \\ \sqrt{1-\alpha} & 0 \end{bmatrix} \otimes I_2,$$

for some $0 \le \alpha \le 1$. Decompose $\mathbb{C}^4 = \mathcal{A} \otimes \mathcal{B}$ with respect to the standard basis so that $\mathcal{A} = \mathcal{B} = \mathbb{C}^2$. Note that $V_i^\dagger V_j = Y_{ij} \otimes I_2$, for some $Y_{ij} \in M_2$. So by using the Theorem 3.4, the subsystem $\mathcal{B}$ is private for the conjugate channel $\Phi^\#$, where $\Phi^\# : M_4 \to M_2$ has Kraus operators

$$R_1 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix} \otimes [\,1\;0\,], \qquad R_2 = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha} \end{bmatrix} \otimes [\,0\;1\,],$$
$$R_3 = \begin{bmatrix} 0 & \sqrt{1-\alpha} \\ \sqrt{1-\alpha} & 0 \end{bmatrix} \otimes [\,1\;0\,], \qquad R_4 = \begin{bmatrix} 0 & \sqrt{1-\alpha} \\ \sqrt{1-\alpha} & 0 \end{bmatrix} \otimes [\,0\;1\,].$$

Let $|\psi_\mathcal{A}\rangle \in \mathcal{A}$ and $|\psi_\mathcal{B}\rangle \in \mathcal{B}$. By direct calculation, we find $\Phi^\#(|\psi_\mathcal{A}\rangle\langle\psi_\mathcal{A}| \otimes |\psi_\mathcal{B}\rangle\langle\psi_\mathcal{B}|) = \alpha |\psi_\mathcal{A}\rangle\langle\psi_\mathcal{A}| + (1-\alpha)\sigma_x |\psi_\mathcal{A}\rangle\langle\psi_\mathcal{A}| \sigma_x$, where is not related to $|\psi_\mathcal{B}\rangle$. Thus the subsystem $\mathcal{B}$ is indeed private for the channel $\Phi^\#$.

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Range of charged particle in matter: the Mellin transform

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Abstract

There are some integrals which cannot be evaluated in terms of elementary functions or even the standard special functions for general values of parameters. In these cases it may be used a general method that builds on Mellin transform method. In a physical quantity case, particles generally lose energy when travelling in a medium. They will eventually have lost all their kinetic energy and come to rest. The distance travelled by the particles is referred to as the *"range"*. In this paper, we calculate the physical quantity *"range"* R, by integrating of the energy loss per unit path, $\frac{dE}{dx}$, while it first turns into Mellin convolution of two functions and, finally it is expressed in terms of Meijer G-function.

Keywords: Mellin convolution, Inverse Mellin transform, Meijer G-function, Energy loss, Range
Mathematics Subject Classification [2010]: 44A35, 33C60, 44A10

1 Introduction

Using of energetic ion beams to synthesize and modify materials has evolved over the past several decades. Ion beam modification of materials applies energetic ions over a broad range of energies controllably to change electrical, optical, structural, mechanical and chemical properties of materials for a wide range of research and applications [1]. The energy loss per unit path, $-\frac{dE}{dx}$, depends on the velocity of particle. The calculation of this value by the quantum mechanics methods gives the following expression for a heavy particle wth charge ze, moving at velocity $v \ll c$

$$-\frac{dE}{dx} = \frac{e^4}{4\pi\epsilon_0^2 m}.\frac{z^2}{v^2}.NZ\ln\frac{2mv^2}{I}, \tag{1}$$

where the first term in the right hand side includes the universal constants; the second term, the characteristics of the particle; and the third, the parameters of the medium. NZ is the concentration of electrons in the substance, equal to the product of the number of atoms per unit volume per nuclear charge, and I is the mean excitation energy of the atoms of the medium. The paths of the motion of slowing down heavy charged particles

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are linear. In the overwhelming majority of collisions, the energy is transferred to very light electrons and because of this no significant deflection from the direction of the initial motion of the particle is observed [2].

Since particles lose energy when travelling in a medium, they will eventually have lost all their kinetic energy and come to rest. The distance travelled by the particles is referred to as the *range*. The energy loss increases towards the end of the range. Close to the end it reaches a maximum and then abruptly drops to zero. However, all the particles with a given kinetic energy do not have exactly the same range. This is due to the statistical nature of the energy loss process. There are fluctuations on the range called range straggling. The range is computed on the basis of the relationship between the energy lost and the distance traversed [2].

$$R = \int_0^R dx = \frac{4\pi\epsilon_0^2 m}{e^4}.\frac{M}{z^2}\int_0^{v_0}\frac{v^3 dv}{NZ\ln\frac{2mv^2}{I}} \tag{2}$$

As the Meijer G-function is nowadays available both in symbolic computer algebra packages and as high-performance computing codes, this opens up the possibility to compute the range of particles because of energy loss.

There are some integrals which cannot be evaluated in terms of elementary functions or even the standard special functions for general values of parameters. We use a general method that builds on Mellin transform method [3,4].

Definition 1.1. The Mellin transform, of a function f(x) defined on the interval $[0,\infty)$ is given by

$$\mathfrak{m}_f = \int_0^\infty f(x)x^{s-1}dx, \tag{3}$$

and its inverse integral is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\mathfrak{m}_f ds \tag{4}$$

The driving force behind the Mellin transform method is the Mellin convolution theorem. The Mellin convolution of two functions $f_1(z)$ and $f_2(z)$ is defined as

$$(f_1 \star f_2)(z) = \int_0^\infty f_1(t)f_2(\frac{z}{t})\frac{dt}{t} \tag{5}$$

The Mellin convolution theorem states that the Mellin transform of a Mellin convolution is equal to the products of the Mellin transforms of the original functions,

$$\mathfrak{m}_{f_1\star f_2}(u) = \mathfrak{m}_{f_1(u)}\mathfrak{m}_{f_2}(u) \tag{6}$$

Now it can be shown that any definite integral

$$f(z) = \int_0^\infty g(t,z)dt \tag{7}$$

can be written as the Mellin convolution of two functions f_1 and f_2 are of the hypergeometric type, which is true for many elementary functions and majority of special functions, the integral turns out to be a Mellin-Barnes integral. Depending on the involved coefficients, this integral can be evaluated as a Fox H function, or in simpler cases, a Meijer G-function.

2 Main results

Now we compute the "range" through using Mellin convolution and inverse Mellin transform. Changing variable $\ln \frac{2mv^2}{I} = t$ gives

$$R = \int_0^R dx = \frac{4\pi\epsilon_0^2 m}{e^4} \cdot \frac{M}{z^2} \int_0^{v_0} \frac{v^3 dv}{NZ \ln \frac{2mv^2}{I}} = \frac{4\pi\epsilon_0^2 m}{e^4} \cdot \frac{M}{z^2} \cdot \frac{I^2}{8m^2 NZ} \int_{t_0}^{\infty} e^{-2t} t^{-1} dt \qquad (8)$$

The form of equation (8) allows us to apply the Mellin transform method (equation (5)), with $z = 1$ and

$$f_1(t) = H(t - t_0); \quad f_2(t) = e^{-\frac{2}{t}}$$

Here $H(x)$ is Heaviside step function.

The Mellin transform of these functions are readily computed

$$\mathfrak{m}_{f_1} = -\frac{t_0^2}{s} \qquad (9)$$

$$\mathfrak{m}_{f_2} = 2^{-s}\Gamma(s) \qquad (10)$$

As a result, the definite integral (5) can be transformed to an inverse Mellin transform as follows

$$f(z) \;\; = \;\; -t_0^2 G_{1,0}^{0,1}\left({}^{2}_{-}|2z\right) \qquad (11)$$

where $z = 1, t_0 = \ln \frac{2mv_0^2}{I}$. Finally we compute the *range*

$$R \;\; = \;\; \frac{4\pi\epsilon_0^2 m}{e^4} \cdot \frac{M}{z^2} \cdot \frac{I^2}{8m^2 NZ} (\ln \frac{2mv_0^2}{I})^2 G_{1,0}^{0,1}\left({}^{2}_{-}|2\right) \qquad (12)$$

3 Acknowledgment

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Schmidt rank-k numerical range and numerical radius

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Abstract

Numerical range of a Hermitian matrix X is defined as the set of all possible expectation values of this observable among a normalized quantum state. In this paper, we study a modification of this definition in which the expectation value is taken among a certain subset of the set of all quantum states, known as k-entangled pure states. We also analyze basic properties of the related numerical radius and its applications in quantum information theory.

Keywords: numerical range, tensor product, quantum information, entanglement
Mathematics Subject Classification [2010]: 15A60, 15A69, 47A12, 81P68

1 Introduction

In the Schrödinger picture of quantum mechanics, quantum information is contained in quantum states, which come in two varieties: *pure* and *mixed*. Mathematically, pure quantum states are described by unit column vectors $|v\rangle \in \mathbb{C}^n$.

Within quantum information theory, the theory of entanglement is one of the most important and active areas of research. A pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called *separable*, if it can be written as an elementary tensor: $|v\rangle = |a\rangle \otimes |b\rangle$, for some pure states $|a\rangle \in \mathbb{C}^m$ and $|b\rangle \in \mathbb{C}^n$. Otherwise, $|v\rangle$ is said to be *entangled*. The notion of *Schmidt rank* extends the notion separability. The Schmidt rank of a pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$, written $\mathrm{SR}(|v\rangle)$, is defined as the least k such that we can write $|v\rangle$ as a linear combination of k separable pure states. Although this definition perhaps seems difficult to use at first glance, the Schmidt decomposition theorem [2, Theorem 2.7] provides a simple method of computing Schmidt rank.

Theorem 1.1 (Schmidt decomposition)**.** *For any pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ there exists $1 \le k \le \min\{m, n\}$, non-negative real scalars $\{\alpha_i\}_{i=1}^k$ with $\sum_{i=1}^k \alpha_i^2 = 1$, and orthonormal sets of pure states $\{|a_i\rangle\}_{i=1}^k \subset \mathbb{C}^m$ and $\{|b_i\rangle\}_{i=1}^k \subset \mathbb{C}^n$ such that*

$$|v\rangle = \sum_{i=1}^{k} \alpha_i |a_i\rangle \otimes |b_i\rangle .$$

*Speaker

The least possible k in Theorem 1.1 is equal to the Schmidt rank of $|v\rangle$. Also, the constants $\{\alpha_i\}_{i=1}^{k}$ are known as the Schmidt coefficients of $|v\rangle$.

The Schmidt rank can be interpreted as the amount of entanglement contained within a pure state. A pure state is separable if and only if its Schmidt rank equals 1, and $1 \leq SR(|v\rangle) \leq \min\{m, n\}$, for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$. The set of k-*entangled* pure states is defined as the collection of all pure states with Schmidt rank at most k.

In this paper, the set of $n \times n$ complex matrices is denoted by M_n. $X^\dagger$ denotes the complex conjugate transpose of $X \in M_n$. We will use Dirac (bra-ket) notation: the dual (row) vector $|v\rangle^\dagger$ will be denoted $\langle v|$, and the rank-one projection associated to $|v\rangle$ is its outer product $|v\rangle\langle v|$.

The classical numerical range of $X \in M_n$, denoted by $W(X)$, is defined as $W(X) = \{\langle v|X|v\rangle \ : \ |v\rangle \in \mathbb{C}^n, \langle v|v\rangle = 1\}$. Also, the related concept of numerical radius, is defined as $w(X) = \max\{|\lambda| \ : \ \lambda \in W(X)\}$. Note that a Hermitian matrix $X \in M_n \otimes M_m$ is positive semidefinite, if and only if $W(X) \subseteq [0, +\infty)$. The notion of k-*block positivity* of a Hermitian matrix $X \in M_n \otimes M_m$ is a useful tool in studying of entanglement [3] and is the starting point of our investigations in this paper. A Hermitian matrix $X \in M_n \otimes M_m$ is said k-*block positive* if $\langle v|X|v\rangle \geq 0$ for all pure states $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $SR(|v\rangle) \leq k$. Observe that if $k = \min\{m, n\}$, then this definition reduces to simply the usual notion of positive semideffiniteness.

The main goal of this paper is to study restricted numerical range and radius of a Hermitian matrix $X \in M_n \otimes M_m$ with respect to the set of k-entangled pure states, where $1 \leq k \leq \min\{m, n\}$. In fact, we define Schmidt rank-k numerical range and radius of X as follows.

Definition 1.2. Let $X \in M_m \otimes M_n$ and let $1 \leq k \leq \min\{m, n\}$. Then the *Schmidt rank-k numerical range* of X, denoted by $W_{S(k)}(X)$, is defined as

$$W_{S(k)}(X) = \{\langle v|X|v\rangle \ : \ |v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n, SR(|v\rangle) \leq k\}.$$

Also, the Schmidt rank-k numerical radius of X is defined as

$$w_{S(k)}(X) = \sup\left\{|\lambda| \ : \ \lambda \in W_{S(k)}(X)\right\}.$$

2 Schmidt rank-k numerical range

It is not difficult to establish the basic properties of the Schmidt rank-k numerical range. We list them below.

Proposition 2.1. *Let $X, Y \in M_m \otimes M_n$ and let $1 \leq k \leq \min\{m, n\}$. Then*

(1) *If $m = 1$ or $n = 1$ or $k = \min\{m, n\}$, then $W_{S(k)}(X) = W(X)$;*

(2) $\emptyset \neq \Lambda^{\otimes}(X) = W_{S(1)}(X) \subseteq W_{S(2)}(X) \subseteq \cdots \subseteq W_{S(\min\{m,n\})}(X) = W(X)$, *where $\Lambda^{\otimes}(X)$, is the product numerical range of X, that is defined in [1].*

(3) (Subadditivity) $W_{S(k)}(X + Y) \subseteq W_{S(k)}(X) + W_{S(k)}(Y)$.

(4) (Translation) $W_{S(k)}(X + \lambda I_{mn}) = W_{S(k)}(X) + \{\lambda\}$, *for all* $\lambda \in \mathbb{C}$.

(5) (Scalar multiplication) $W_{S(k)}(\lambda X) = \lambda W_{S(k)}(X)$, *for all* $\lambda \in \mathbb{C}$.

(6) $W_{S(k)}(X) = \{\lambda\}$, *for some* $\lambda \in \mathbb{C}$, *if and only if* $X = \lambda I_m \otimes I_n$.

(7) *The Schmidt rank-k numerical range of X forms a connected and compact set in the complex plane, but does not need to be convex.*

(8) (Product unitary invariance) *If* $U \in M_m$ *and* $V \in M_n$ *are unitary matrices, then* $W_{S(k)}\left((U \otimes V)^* X (U \otimes V)\right) = W_{S(k)}(X)$.

(9) (Projection) *Let* $\mathrm{Re}(X) = \frac{1}{2}(X + X^\dagger)$ *and* $\mathrm{Im}(x) = \frac{1}{2\imath}(X - X^\dagger)$, *then*

$$W_{S(k)}(\mathrm{Re}(X)) = \mathrm{Re}(W_{S(k)}(X)), \quad \textit{and} \quad W_{S(k)}(\mathrm{Im}(X)) = \mathrm{Im}(W_{S(k)}(X)).$$

(10) *The Schmidt rank-k numerical range of A includes the barycenter of the spectrum; i.e.* $\frac{1}{mn}\mathrm{tr}(X) \in W_{S(k)}(X)$.

(11) $W_{S(k)}(A \otimes I_n + I_m \otimes B) = W(A) + W(B)$, *for all* $A \in M_m$ *and* $B \in M_n$.

Proposition 2.2. *Let* $A \in M_m$, $B \in M_n$ *and let* $1 \le k \le \min\{m, n\}$. *Then*

$$W_{S(k)}(A \otimes B) = \bigcup_{\substack{U_1 \in \mathcal{X}_{m,k} \\ U_2 \in \mathcal{X}_{n,k}}} W(U_1^\dagger A U_1 \circ U_2^\dagger B U_2),$$

where $\mathcal{U}_{n,k} := \left\{U \in M_{n\times k} \;:\; U^\dagger U = I_k\right\}$ *and* $A \circ B$ *denotes the Hadamard Product of A and B.*

Proposition 2.3. *Let* $A \in M_m$, $B \in M_n$ *and let* $1 \le k \le \min\{m, n\}$. *Then*

1. *If one of A and B is normal then* $\mathrm{conv}\left(W_{S(k)}(A \otimes B)\right) = W(A \otimes B)$.
2. *If* $e^{\imath\theta}A$ *is positive semidefinite for some* $\theta \in [0, 2\pi)$, *then* $W_{S(k)}(A \otimes B) = W(A \otimes B)$.

Proposition 2.4. *Let* $X \in M_m \otimes M_n$ *and let* $1 \le k \le \min\{m, n\}$. *Then*

$$W_{S(k)}(X) = \bigcup_{U \in \mathcal{U}_k^{\mathrm{sep}}} W(U^\dagger X U),$$

where $\mathcal{U}_k^{\mathrm{sep}} := \left\{U = \begin{bmatrix}|v_1\rangle & \cdots & |v_k\rangle\end{bmatrix} \;:\; |v_i\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n, SR(|v_i\rangle) = 1, U^\dagger U = I_k\right\}$.

Proposition 2.5. *For any Hermitian* $X \in M_m \otimes M_n$, *its Schmidt rank-k numerical range* $W_{S(k)}(X)$ *is convex and forms an interval of the real line.*

Consider a Hermitian $X \in M_m \otimes M_n$ with ordered spectrum $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{nm}$. By using Proposition 2.5, $W_{S(k)}(A) = [\lambda_{S(k)}^{\min}(X), \lambda_{S(k)}^{\max}(X)]$. The bounds $\lambda_{S(k)}^{\min}(X)$ and $\lambda_{S(k)}^{\max}(X)$ determine the minimal and maximal expectation values of an observable X among all k-entangled pure states.

Lemma 2.6 ([4])**.** *The maximum dimension of a subspace* $\mathcal{V} \subseteq \mathbb{C}^m \otimes \mathbb{C}^n$ *such that* $SR(|v\rangle) > k$ *for all* $|v\rangle \in \mathcal{V}$ *is given by* $(m-k)(n-k)$.

Theorem 2.7. *For any Hermitian $X \in M_m \otimes M_n$ with ordered spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{nm}$, we have*

$$\lambda_{S(k)}^{\max}(X) \geq \lambda_{nm-(m-k)(n-k)}, \qquad \lambda_{S(k)}^{\min}(X) \leq \lambda_{(m-k)(n-k)+1}.$$

Proposition 2.8. *Suppose $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ has Schmidt coefficients $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ and let $1 \leq k \leq \min\{m, n\}$. Then $W_{S(k)}(|v\rangle\langle v|) = \left[0, \sum_{i=1}^{k} \alpha_i^2\right]$.*

3 Schmidt rank-k numerical radius

Indeed, Schmidt rank-k numerical radius of $X \in M_m \otimes M_n$ is a very powerful tool in quantum information and specially detecting k-block positivity of X.

Proposition 3.1. *Let $X, Y \in M_m \otimes M_n$ and let $1 \leq k \leq \min\{m, n\}$. Then*

(1) *Schmidt rank-k numerical radius is a vector norm on $M_m \otimes M_n$.*

(2) *If $m = 1$ or $n = 1$ or $k = \min\{m, n\}$, then $w_{S(k)}(X) = w(X)$.*

(3) *$r^{\otimes}(X) = w_{S(1)}(X) \leq w_{S(2)}(X) \leq \cdots \leq w_{S(\min\{m,n\})}(X) = w(X)$, where $r^{\otimes}(X)$, is the product numerical radius of X, that is defined in [1]..*

(4) (Product unitary invariance) *If $U \in M_m$ and $V \in M_n$ are unitary matrices, then $w_{S(k)}\left((U \otimes V)^* X (U \otimes V)\right) = w_{S(k)}(X)$.*

Corollary 3.2. *For any Hermitian $X \in M_m \otimes M_n$, we have*

$$w_{S(k)}(X) = \max\left\{|\lambda_{S(k)}^{\max}(X)|, |\lambda_{S(k)}^{\min}(X)|\right\} \geq \lambda_{nm-(m-k)(n-k)}.$$

Theorem 3.3. *Let $X \in M_m \otimes M_n$ be positive semidefinite. Then*

$$w_{S(k)}(X) = \sup\left\{|\langle w|X|v\rangle| \ : \ SR(|v\rangle) \leq k, SR(|w\rangle) \leq k\right\}.$$

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Shifted Legendre pseudospectral approach for solving population projection models

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Abstract

In this investigation, a numerical technique based on shifted Legendre polynomials for solving population projection models is proposed. The approach reduces the solution of the main problem to the solution of a system of nonlinear algebraic equations. The comparison of the results with the analytical and numerical solution show the efficiency and accuracy of presented method.

Keywords: Population projection models, Logistic growth model, Pseudospectral method, Shifted Legendre polynomials
Mathematics Subject Classification [2010]: 34B15, 76A10, 34B16

1 Introduction

Population dynamics has traditionally been the dominant branch of mathematical biology, whose history spans more than 200 years [1, 5]. A projection may be defined as the numerical outcome of a particular set of assumptions regarding the future population [1]. Most mathematical models that describe the dynamics of a population over time $u(t)$ are based on first order differential equation of the form:

$$u'(t) = Au(t) - u(t)F(u(t)) + B, \quad u(0) = \beta, \quad t \geq 0. \tag{1}$$

In population models the solution $u(t)$ of (1) corresponds to the population density at time t, the linear term $Au(t)$ corresponds to intrinsic growth, loss, or transition processes in the population independent of population density. The nonlinear logistic term $-u(t)F(u(t))$ in (1) corresponds to loss processes due to crowding at a rate proportional to a functional of the population density. Lastly, the constant term B corresponds to an external source of population growth, independent of the population density.

*Speaker

2 The shifted Legendre pseudospectral approach

2.1 The shifted Legendre polynomials

Assuming that the Legendre polynomial of degree n is denoted by $P_n(x)$. Then $P_n(x)$ can be generated by the recurrence formulae [2]:

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \quad , n = 1, 2, \ldots, \tag{2}$$

$$P_0(x) = 1, \quad P_1(x) = x. \tag{3}$$

The shifted Legendre polynomial of order n on interval $[0, L]$, which we denote it by $P_n^*(t)$, is constructed by Legendre polynomials $P_n(x)$ by replacing the independent variable, x, by $x = \frac{2}{L}t - 1$:

$$P_n^*(t) = P_n\left(\frac{2}{L}t - 1\right), \quad , n = 0, 1, 2, \ldots. \tag{4}$$

2.2 The Pseudospectral Method

In this section, we apply the shifted Legendre pseudospectral method to solve the population projection model (1) on interval $[0, L]$. By choosing the $P_n^*(t)$ as basis, we expand the function $u(t)$ in terms of these polynomials as:

$$u(t) \simeq u_N(t) = \sum_{n=0}^{N} u_n P_n^*(t), \tag{5}$$

Substituting $u_N(t)$ in (1), we have the residual function, as follows:

$$RES(t) = u_N'(t) - Au_N(t) + u_N(t)F(u_N(t)) - B, \tag{6}$$

Since the residual function is identically equal to zero for the exact solution, the challenge is to choose the series coefficients u_n so that the residual function is minimized. The different spectral methods differ mainly in their minimization strategies. As it was explained in the previous section, the pseudospectral technique associates a grid of points with each basis set [3]. Here, we have chosen the shifted Legendre-Gauss-Radau nodes as nodal points. These points are represented by t_i. The nodes are the roots of the function $P_N^*(t) + P_{N-1}^*(t)$, which contain the zero point. Now, by equalizing the residual to zero at these nodes, we form the system of nonlinear equations

$$RES(t_i) = 0, \quad i = 0, 1, \ldots, N-1, \tag{7}$$

$$u_N(0) = \beta. \tag{8}$$

Solution of this system gives the unknown coefficients u_ns. We used the `fsolve` function of the Maple software to solve this system.

3 Applications

In this section, two types of the known population growth models are investigated.

3.1 Ordinary logistic growth model

Consider the Verhulst-Pearl [4] logistic population growth model

$$u'(t) = [a - bu(t)]u(t), \quad u(0) = \beta, \quad t \geq 0, \quad a, b > 0. \tag{9}$$

The model (9) describes a population growth rate with a linear term $au(t)$, where the parameter a may be considered to be the "per capita" birth rate per aphid, also called the "intrinsic rate of natural increase". The growth of the population in (9) is constrained by the nonlinear term, $bu^2(t)$. This may be interpreted as having per capita death rate $bu(t)$, where the parameter b describes the strength of the "density dependent" mortality. The exact solution of this problem is $u(t) = \frac{k}{1+[(k-\beta)/\beta]\exp(-at)}$ where $k = a/b$. This problem is solved by proposed method for $\beta = 0.5, a = b = 1$ and $N = 10$ on interval $[0, 8]$. In Figures 1 and 2, the analytical and approximate solution and the error function $|u(t) - u_N(t)|$ are plotted, respectively.

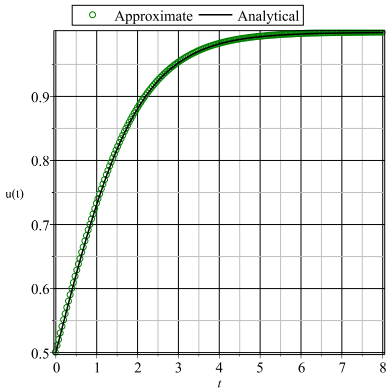

Figure 1: Analytical and estimated function.

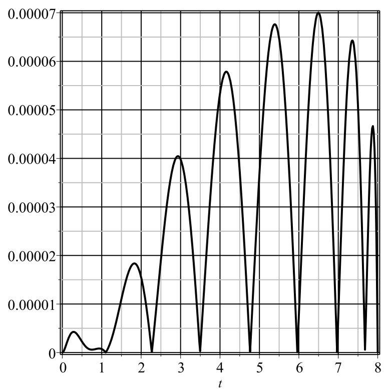

Figure 2: Absolute error function $|u(t) - u_N(t)|$.

3.2 Growth model based on cumulative size dynamics

Consider the Prajneshu [5] logistic population growth model

$$u'(t) = \lambda u(t) - \delta u(t) \int_0^t u(\tau)d\tau, \quad u(0) = \beta, \quad t \geq 0, \quad \lambda, \delta > 0. \tag{10}$$

Model (10) is depending on the principle: "Aphid population growth is constrained by the 'cumulative size' of the past population" [6]. In this equation, the rate of change of the aphid population may be considered to be the net difference of a 'birth' and a 'death' rate. Note that, as in (10), the population birth rate is assumed to have form $\lambda u(t)$, where the intrinsic birth rate is denoted as λ. However, population size control is assumed now to come through a unique density-dependent death function. The per capita death rate is assumed to be proportional to the cumulative density, $\int_0^t u(\tau)d\tau$, as opposed to the current size, $u(t)$, in (10). The new death rate parameter is denoted δ.

This problem is solved by proposed method for $\beta = 0.0082, \lambda = 2.453, \delta = 0.02307$ and $N = 80$ on interval $[0, 10]$. Figure 3, shows the proposed solution, of (10) and the non-linear regression model (NRM) which suggested by Prajneshu [5] and may be written as:

$$u_{NRM}(t) = a\exp(-bt)(1 + d\exp(-bt))^{-2}, \tag{11}$$

where the regression model parameters a, b and d are functions of the mechanistic parameters λ, δ, and β, the initial value [6] and can be obtained by solving the system $\delta = 2b^2 d/a,\ \lambda = b\left(d^2 - 1\right),\ \beta = a/(1+d)^2$. We have reported, in Figure 4 the absolute error function $|u_{NRM}(t) - u_N(t)|$.

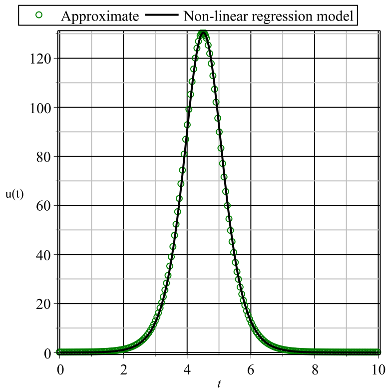

Figure 3: Analytical and estimated function.

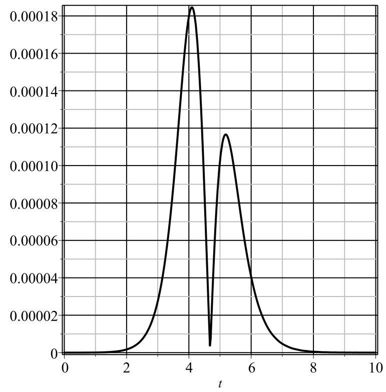

Figure 4: Absolute error function $|u_{NRM}(t) - u_N(t)|$.

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Uniqueness of Solutions to Fuzzy Differential Equations Driven by Liu's Process with Weak Lipschitz Coefficients

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Abstract

Fuzzy differential equations (FDEs) is a type of differential equations driven by Liu process. These equations are frequently used in financial. This paper is devoted to build the existence and uniqueness theorem of solution to fuzzy differential equations which a fuzzy process in the sense of Liu. Under the Lipshitz condition, the linear growth condition is weak. Furthermore, the estimate for the error between approximate solution and accurate solution is given.

Keywords: Fuzzy differential equation, liu process, credibility space condition
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In this paper, the following is considered fuzzy differential equation

$$dx(t) = f(t, x(t)) + g(t, x(t))d\mathbf{C}_t \tag{1}$$

where $\mathbf{C}_t$ is Liu process, f, g are functions, and $x(t)$ is the solution to the Eq. (1.1) which is a parameter of a fuzzy process. Existence and uniqueness of solution to the Eq. (1.1) by employing Lipshitz and linear growth conditions were studied by (A New Existence and Uniqueness Theorem for Fuzzy Differential Equations, [3]; Existence and Uniqueness Theorems for Fuzzy Differential Equations, [25]) and non-Lipschitz condition was explained by (Uniqueness of solutions to fuzzy differential equations driven by Liu's process with non-Lipschitz coefficients, [9]). However a little attention has been paid to weaker conditions, because we these weaker conditions, it opens a door to finding solutions for wider range of equations.
Furthermore, instead of Linear growth condition, a weaker condition was introduced, in order to solve of function such as $-|x|^2x$.
In this paper, a weak condition will be expressed, using this condition, some problems that are not solvable in linear growth condition can be solved. A new existence and uniqueness theorem will be prove in Section 2 and theorem will be prove for estimate of solution of equation (1.1).

*Speaker

2 Main results

Throughout this paper, we consider the fuzzy differential equations

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{C}_t \tag{2}$$

where $\mathbf{C}_t$ is a standard Liu process and f, g are some given functions. $x(t)$ is the solution to the Eq. (3.3) which is a fuzzy process in the sense of liu.
By the definition of fuzzy differential, this equation is equivalent to the following fuzzy integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)d\mathbf{C}_s. \tag{3}$$

Furthermore, let us state the following conditions.

(**D**) **The Lipshitz condition**: For all $x(t), y(t) \in \mathbf{R}^d$ and $t \in [t_0, T]$, there exists a positive constant $\mathbf{L}$ such that

$$|f(x(t), t) - f(y(t), t)|^2 \vee |g(x(t), t) - g(y(t), t)|^2 \leq \mathbf{L}|x(t) - y(t)|^2.$$

(**H**) **Weak condition**: For $t \in [t_0, T]$, there is

$$f(0, t), g(0, t) \in \mathbf{L}^2[t_0, T]$$

Remark 3.1. Assume coefficient $f(x(t), t)$ and $g(x(t), t)$ of E.q (2.3) satisfied the conditions (**D**) and (**H**). Let $\mathbf{I} = |f(0, t)|^2_{\mathbf{L}^2[0,T]}$, $\mathbf{J} = |g(0, t)|^2_{\mathbf{L}^2[0,T]}$. If $x(t)$ is the solution of equation (2.3), then

$$\mathbf{E}(\sup_{t_0 \leq t \leq T} |x(t)|^2) \leq \mathbf{K}\ e^{6\ \mathbf{L}(T - t_0 + 1)(T - t_0)}. \tag{4}$$

Particularly $x(t) \in \mathbf{M}^2([t_0, T], \mathbf{R}^d)$, where $\mathbf{K} = (3|x_0|^2 + 6((T - t_0)\mathbf{I} + \mathbf{J}))$.

Theorem 3.4. Let coefficients $f(x(t), t)$ and $g(x(t), t)$ of Eq. (2.3) satisfy the conditions (**D**) and (**H**). Then there is a unique solution $x(t)$ to equation (2.3) and $x(t) \in \mathbf{M}^2([t_0, T], \mathbf{R}^n)$.
Proof: The uniqueness follows from the conditions (**D**) and (**H**). Let $x(t)$ and $\overline{x}(t)$ are solutions of equation (2.3),
put $\mathbf{a}(w, s) = f(x(s), s) - f(\overline{x}(s), s)$ and $\mathbf{b}(w, s) = g(x(s), s) - g(\overline{x}(s), s)$ where $w \in \theta$. Then

$$x(t) - \overline{x}(t) = \int_{t_0}^{t} \mathbf{a}ds + \int_{t_0}^{t} \mathbf{b}\mathrm{d}\mathbf{C}(s).$$

Using Holder inequality and Lipschitz condition, we obtain

$$|x(t) - \overline{x}(t)|^2 \leq 2|\int_{t_0}^{t} \mathbf{a}ds|^2 + 2|\int_{t_0}^{t} \mathbf{b}\mathrm{d}\mathbf{C}_s|^2 \leq 2(t - t_0)\int_{t_0}^{t} \mathbf{L}|x_s - \overline{x}(s)|^2ds + 2|\int_{t_0}^{t} \mathbf{b}\mathrm{d}\mathbf{C}_s|^2.$$

Thus, we get

$$\sup_{t_0 \leq s \leq t} |x(s) - \overline{x}(s)|^2 \leq 2\mathbf{L}(T - t_0)\int_{t_0}^{t} |x(s) - \overline{x}(s)|^2ds + 2\sup_{t_0 \leq s \leq t} |\int_{t_0}^{t} \mathbf{b}\mathrm{d}\mathbf{C}(s)|^2.$$

Taking the expectation and noting Doob inequality, we may deduce that

$$\mathbf{E}(\sup_{t_0\le s\le t}|x(s)-\overline{x}(s)|^2)\le 2\mathbf{L}(T+4)\int_{t_0}^t \mathbf{E}(\sup_{t_0\le r\le s}|x(r)-\overline{x}(r)|^2)ds.$$

According to Gronwall inequality, we have

$$\mathbf{E}(\sup_{t_0\le t\le T}|x(t)-\overline{x}(t)|^2)=0. \tag{5}$$

Hence $x(t)=\overline{x}(t)$ for all $t_0\le t\le T$ a.s. The uniqueness has been proved.
The proof of the existence of the solution. Let $x^0(t)=x(0), t\in[t_0,T]$, and for $n=1,2,\ldots,$ define Picard iterations sequence

$$x^n(t)=x(0)+\int_{t_0}^t f(x^{n-1}(s),s)ds+\int_{t_0}^t g(x^{n-1}(s),s)\mathrm{d}\mathbf{C}_s.$$

Clearly $x^0(0)\in\mathbf{M}^2([0,T],\mathbf{R}^n)$. It is easy to see the induction of $x^n(0)\in\mathbf{M}^2([0,T],\mathbf{R}^n)$. Using inequality $(a+b)^2\le 2(a^2+b^2)$ and Holder inequality, we have

$$(x^n(t))^2=3|x(0)|^2+3(t-t_0)\int_{t_0}^t f^2(x^{n-1}(s),s)ds+3(\int_{t_0}^t g(x^{n-1}(s),s)d\mathbf{C}_s)^2. \tag{6}$$

Taking the expectation

$$\le \mathbf{A}+6\mathbf{L}[T-t_0+1]\int_{t_0}^t \mathbf{E}|x^{n-1}(s)|^2ds, \tag{7}$$

where

$$\mathbf{A}=3\mathbf{E}|x(0)|^2+6[(T-t_0)\mathbf{I}+\mathbf{J}].$$

By virtue of Eq. (2.8), for any $k\le 1$, we have

$$\mathbf{B}=\mathbf{A}+6\ \mathbf{L}\ (T-t_0)(T-t_0+1)\mathbf{E}|x(0)|^2.$$

By using Gronwall inequality, for $t_0\le t\le T,\ n\ge 1$ we obtain

$$\max_{1\le n\le k}\mathbf{E}|x^n(t)|^2\le \mathbf{B}e^{6\ \mathbf{L}\ (T+1)(T-t_0)}, \tag{8}$$

noting that

$$\le 2(T-t_0)\int_{t_0}^t f^2(x(0),s)ds+2\mid\int_{t_0}^t g(x(0),s)ds\mid^2.$$

Taking the expectation

$$\le 4\mathbf{L}(T-t_0)^2\mathbf{E}(|x(0)|^2)+4(T-t_0)\mathbf{I}+4\mathbf{L}(T-t_0)^2\mathbf{E}|x(0)|^2+4\mathbf{J}\le\mathbf{Q}, \tag{9}$$

where

$$\mathbf{Q}=4\mathbf{L}(T-t_0+1)(T-t_0)\mathbf{E}(|x(0)|^2)+4(T-t_0)\mathbf{I}+4\mathbf{J}.$$

Now we prove that for any $n \geq 0$, we have

$$\mathbf{E}|x^{n+1}(t) - x^n(t)|^2 \leq \frac{\mathbf{Q}[\mathbf{R}(T-t_0)]^n}{n!}, \quad t_0 \leq t \leq T, \tag{10}$$

where $\mathbf{R} = 2\mathbf{L}(T - t_0 + 1)$. From Eq. (2.10), we see that under $n = 0$, Eq. (2.11) holds. Noting that

$$|x^{n+1}(t) - x^n(t)|^2$$
$$\leq 2\mathbf{L}(T-t_0)\int_{t_0}^{t} |x^n(s) - x^{n-1}(s)|^2 ds + 2 \mid \int_{t_0}^{t} [g(x^n(s), s) - g(x^{n-1}(s), s)] ds \mid^2 . \tag{11}$$

Taking the expectation and using $\mathbf{D}$ condition, we have

$$\sum_{n=0}^{\infty} \frac{4\mathbf{Q}[4\mathbf{R}(T-t_0)]^n}{n!} < \infty,$$

by Borel-Cantell lemma, for almost all for $\omega \in \theta$. There exists a positive integer $n_0 = n_0(\omega)$, such that $n \geq n_0$, we have

$$\sup_{t_0 \leq t \leq T} |x^{n+1}(t) - x^n(t)| \leq \frac{1}{2^n}.$$

From the partial sums

$$x^0(t) + \sum_{i=0}^{n-1} [x^{i+1}(t) - x^i(t)] = x^n(t)$$

are uniformly in $t \in [0, T]$. Clearly, $x(t)$ is continuous and $\mathcal{P}_t$ is adapted. On the other hand, from Eq. (2.11), $\{x^n(t)\}_{n \leq 1}$ is a Cauchy in $\mathbf{L}^2$ for every t. Hence $x(t) \in \mathbf{L}^2[0, T]$ in Eq. (2.9). Let $n \to \infty$ in Eq. (2.8) gives sequence, we have

$$\mathbf{E}|x(t)|^2 \leq \mathbf{B}e^{6\mathbf{L}(T+1)(T-t_0)}, t_0 \leq t \leq T.$$

Therefore, $x(t) \in \mathbf{M}^2([t_0, T], \mathbf{R}^d)$. We deduce that $x(t)$ satisfies equation (2.3).
Note that $(n \to \infty)$, we Hence in Eq. (2.7), letting $n \to \infty, t_0 \leq t \leq T$.
We have

$$x(t) = x(0) + \int_{t_0}^{t} f(x(s), s) ds + \int_{t_0}^{t} g(x(s), s) d\mathbf{C}_s.$$

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Weighted Hermite-Hadamard's inequality without symmetry condition for fractional integral

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Abstract

Weighted Hermite-Hadamard's inequality without symmetry condition for fractional integral is discussed. The main results of this paper improve and generalize some previous results obtained by many researchers.

Keywords: Fractional integral, Hermite–Hadamard's inequality.
Mathematics Subject Classification [2010]: 26D15

1 Preliminaries and some results

One of the most well-known inequalities for the class of convex functions is the Hermite-Hadamard inequality given in [4]

$$\frac{f(a+b)}{2} \leqslant \frac{1}{b-a} \int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1}$$

which plays an important role in nonlinear analysis. Weighted generalization of 1 based on the symmetry condition was proved by Fejér [2].

Theorem 1.1. *[2] If $f : [a, b] \to \mathbb{R}$ is a convex function, then the following inequality holds*

$$\frac{f(a+b)}{2} \int_a^b \omega(x)\,dx \leqslant \int_a^b \omega(x) f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \int_a^b \omega(x)\,dx$$

where $\omega : [a, b] \to (0, \infty)$ is a non-negative function which is integrable and symmetric about $\frac{a+b}{2}$.

However, the lack of symmetry condition in many problems in statistics, probability and engineering is reasonable. Therefore, finding a weighted generalization of Hermite-Hadamard's inequality without the symmetry condition is interesting for researchers [1].

Theorem 1.2. *[1] Let $f : [a, b] \subset (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function and $\omega : [a, b] \longrightarrow (0, \infty)$ be an integrable function.*
(i) If the function $\frac{f'}{\omega}$ is increasing, then the following inequality is hold,

$$\frac{\int_a^b \omega(x) f(x)\,dx}{\int_a^b \omega(x)\,dx} \leqslant \frac{f(a)+f(b)}{2} \tag{2}$$

*Speaker

(ii) If the function $\frac{f'}{\omega}$ *is decreasing, then the following inequality is hold,*

$$\frac{\int_a^b \omega(x) f(x)\, dx}{\int_a^b \omega(x)\, dx} \geqslant \frac{f(a)+f(b)}{2} \tag{3}$$

Now we can obtain a general weighted Hermite-Hadamard's inequality without the symmetry condition by using the fractional integral.

Definition 1.3. [3] The Riemann-Liouville fractional integral of a function $y \in L^1([a,\ b],\ \mathbb{R})$ of order $\alpha > 0$ is defined as $I^{\alpha}_{a^+} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(s)}{(t-s)^{1-\alpha}} ds \quad t > a$.

Theorem 1.4. *Let* $\alpha \geqslant 1$. *Let* $f : [a,\ b] \subset (0,\infty) \longrightarrow \mathbb{R}$ *be a differentiable function,* $\omega : [a,\ b] \longrightarrow (0,\ \infty)$ *be an integrable function.*
(i) If $\left(\frac{b-a}{b-x}\right)^{\alpha-1} \frac{f'(x)}{\omega(x)}$ *for any* $x \in [a,\ b)$ *is increasing, then*

$$\frac{\mathbb{I}^{2\alpha-1}_{a+} f\omega(b)}{\mathbb{I}^{\alpha}_{a+}\omega(b)} \leqslant \frac{1}{2}(b-a)^{\alpha-1}[f(a)+f(b)]. \tag{4}$$

(ii) If $\left(\frac{b-a}{b-x}\right)^{\alpha-1} \frac{f'(x)}{\omega(x)}$ *is decreasing, then*

$$\frac{\mathbb{I}^{2\alpha-1}_{a+} f\omega(b)}{\mathbb{I}^{\alpha}_{a+}\omega(b)} \geqslant \frac{1}{2}(b-a)^{\alpha-1}[f(a)+f(b)]. \tag{5}$$

Proof. We will prove (i) and the other case is similar. Let

$$H(x) = \int_a^x \frac{1}{(\Gamma(\alpha))^2}(b-t)^{2\alpha-2} f(t)\,\omega(t)\, dt - \frac{1}{2(\Gamma(\alpha))^2}(b-a)^{\alpha-1}[f(a)+f(x)] \int_a^x (b-t)^{\alpha-1}\omega(t)\, dt.$$

Then $H'(x) =$

$$\frac{1}{2(\Gamma(\alpha))^2}\left([f(x)-f(a)](b-a)^{\alpha-1}\omega(x)(b-x)^{\alpha-1} - f'(x)(b-a)^{\alpha-1}\int_a^x (b-t)^{\alpha-1}\omega(t)\, dt\right).$$

By our assumption and the extended mean value theorem, we have

$$\frac{(b-a)^{\alpha-1}(f(x)-f(a))}{\int_a^x (b-t)^{\alpha-1}\omega(t)\, dt} = \frac{(b-a)^{\alpha-1} f'(\xi)}{(b-\xi)^{\alpha-1}\omega(\xi)} \leqslant \frac{(b-a)^{\alpha-1} f'(x)}{(b-x)^{\alpha-1}\omega(x)},\ (a < \xi < x).$$

Thus, $H'(x) = \frac{1}{2(\Gamma(\alpha))^2}\left(\begin{array}{c}(b-a)^{\alpha-1}[f(x)-f(a)]\,\omega(x)(b-x)^{\alpha-1} - \\ f'(x)(b-a)^{\alpha-1}\int_a^x (b-t)^{\alpha-1}\omega(t)\, dt\end{array}\right) \leqslant 0$. So, for $b \geqslant a$, we have $H(b) \leqslant H(a) = 0$, and the proof is completed. □

Remark 1.5. If $\alpha = 1$, in Theorem 1.4, we get Theorem 1.2 obtained by Jaksic *et al.* [1].

In the next theorem, we provide a more general case of the Theorem 1.2.

Theorem 1.6. *Let $f : [a, b] \subset (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function, $\omega : [a, b] \longrightarrow (0, \infty)$ be an integrable function and $\mathbf{k}_\alpha(x, y) : [a, b] \times (0, \infty) \to (0, \infty)$ be a positive differentiable kernel which may depend on a parameter $\alpha > 0$.*
(i) If $\frac{\mathbf{k}_\alpha((b,a))f'(x)}{\mathbf{k}_\alpha(b,x)\omega(x)}$ is increasing, then

$$\frac{\int_a^b (\mathbf{k}_\alpha((b,t)))^2 f(t)\,\omega(t)\,dt}{\int_a^b \mathbf{k}_\alpha((b,t))\,\omega(t)\,dt} \leqslant \frac{1}{2}\mathbf{k}_\alpha((b,a))\,[f(a) + f(b)] \tag{6}$$

(ii) If $\frac{\mathbf{k}_\alpha((b,a))f'(x)}{\mathbf{k}_\alpha(b,x)\omega(x)}$ is decreasing, then

$$\frac{\int_a^b (k_\alpha((b,t)))^2 f(t)\,\omega(t)\,dt}{\int_a^b \mathbf{k}_\alpha((b,t))\,\omega(t)\,dt} \geqslant \frac{1}{2}\mathbf{k}_\alpha((b,a))\,[f(a) + f(b)] \tag{7}$$

Remark 1.7. Clearly, for $\mathbf{k}_\alpha(b, x) = \frac{1}{\Gamma(\alpha)}(b - x)^{\alpha-1}$, the Riemann Liouville fractional integral $\mathbb{I}_{a+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b - t)^{\alpha-1} f(t)dt$ is obtained, thus generalizing Theorem 1.4.

Finaly, we prove mean value theorems of Lagrange and Cauchy type. The following lemma will be needed.

Lemma 1.8. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be twice differentiable function and let $\omega : [a, b] \longrightarrow \mathbb{R}^+$ be a differentiable integrable function. Denote*

$$G_f(x) = \frac{f''(x) - f'(x)\omega'(x)}{\omega^2(x)}. \tag{8}$$

Let $\mathbf{k}_\alpha(y, x) : \mathbb{R}^+ \times [a, b] \to \mathbb{R}^+$ be a positive differentiable kernel which may depend on a parameter $\alpha > 0$ and $\varphi_1,\ \varphi_1 : [a, b] \longrightarrow \mathbb{R}$ be the functions defined by

$$\varphi_1(x) = M \int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} t\omega(t)dt - \int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} f'(t)dt - f(a), \tag{9}$$

$$\varphi_2(x) = f(a) + \int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} f'(t)dt - m \int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} t\omega(t)dt, \tag{10}$$

where $M = \max\{\Lambda_f(x) : x \in [a, b]\}$ and $m = \min\{G_f(x) : x \in [a, b]\}$. Then $\frac{\mathbf{k}_\alpha((b,a))\varphi_1'(x)}{\mathbf{k}_\alpha((b,x))\omega(x)}$ and $\frac{\mathbf{k}_\alpha(b,a)\varphi_2'(x)}{\mathbf{k}_\alpha(b,x)\omega(x)}$ are increasing functions.

Proof. It is sufficient to show that the $\left(\frac{\mathbf{k}_\alpha((b,a))\varphi_1'(x)}{\mathbf{k}_\alpha(b,x)\omega(x)}\right)'$ and $\left(\frac{\mathbf{k}_\alpha((b,a))\varphi_2'(x)}{\mathbf{k}_\alpha(b,x)\omega(x)}\right)'$ are positive. □

Theorem 1.9. *Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a twice differentiable function, $\omega : [a, b] \longrightarrow \mathbb{R}^+$ be a differentiable integrable function, and $\mathbf{k}_\alpha(y, x) : \mathbb{R}^+ \times [a, b] \to \mathbb{R}^+$ be a positive differentiable kernel which may depend on a parameter $\alpha > 0$ such that $\mathbf{k}_\alpha(b, x) \leqslant \mathbf{k}_\alpha(b, a), x \geq a$ and let $G_f \in C[a, b])$ be as defined in Lemma 1.8. Then there exists $\eta \in [a, b]$ such that*

$$\frac{\int_a^b \frac{\mathbf{k}_\alpha(b,x)}{\mathbf{k}_\alpha(b,a)} f'(x)dx + 2f(a)}{2} - \frac{\int_a^b (\mathbf{k}_\alpha(b,x))^2 [\int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} f'(t)dt + f(a)]\omega(x)dx}{\mathbf{k}_\alpha(b,a) \int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx} = \lambda G_f(\eta) \tag{11}$$

where

$$\lambda = \frac{\int_a^b \frac{\mathbf{k}_\alpha(b,x)}{\mathbf{k}_\alpha(b,a)} x\omega(x)dx}{2} - \frac{\int_a^b [(\mathbf{k}_\alpha(b,x))^2 \omega(x) \int_a^x \mathbf{k}_\alpha(b,t) t\omega(t)dt]dx}{(\mathbf{k}_\alpha(b,a))^2 \int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx}.$$

Proof. Since G_f is continuous on a compact set, it attains its maximum and minimum value on it. Let us consider. Let us consider $M = \max\{G_f(x)\}$ and $m = \min\{G_f(x)\}$. Since $\frac{\mathbf{k}_\alpha((b,a))\varphi_1'(x)}{\mathbf{k}_\alpha((b,x))\omega(x)}$ and $\frac{\mathbf{k}_\alpha((b,a))\varphi_2'(x)}{\mathbf{k}_\alpha((b,x))\omega(x)}$, are increasing functions, Theorem 1.5 yields

$$\frac{\int_a^b \frac{\mathbf{k}_\alpha(b,x)}{\mathbf{k}_\alpha(b,a)} f'(x)dx + 2f(a)}{2}\mathbf{k}_\alpha(b,a) + \frac{\int_a^b (\mathbf{k}_\alpha(b,x))^2 \varphi_1(x)\omega(x)dx}{\int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx} \leqslant \frac{1}{2}\mathbf{k}_\alpha(b,a)M\lambda_1, \quad (12)$$

$$\frac{\int_a^b \frac{\mathbf{k}_\alpha(b,x)}{\mathbf{k}_\alpha(b,a)} f'(x)dx + 2f(a)}{2}\mathbf{k}_\alpha(b,a) - \frac{\int_a^b (\mathbf{k}_\alpha(b,x))^2 \varphi_2(x)\omega(x)dx}{\int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx} \geqslant \frac{1}{2}\mathbf{k}_\alpha(b,a)m\lambda_1 \quad (13)$$

Substituting $\varphi_1(x)$ and $\varphi_2(x)$ in (12) and (13) respectively we have

$$\frac{\int_a^b \frac{\mathbf{k}_\alpha(b,x)}{\mathbf{k}_\alpha(b,a)} f'(x)dx + 2f(a)}{2} - \frac{\int_a^b (\mathbf{k}_\alpha(b,x))^2 [\int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} f'(t)dt + f(a)]\omega(x)dx}{\mathbf{k}_\alpha(b,a)\int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx}$$

$$\leqslant M\left[\frac{\lambda_1}{2} - \frac{\int_a^b [(\mathbf{k}_\alpha(b,x))^2\omega(x)\int_a^x \mathbf{k}_\alpha(b,t)t\omega(t)dt]dx}{(\mathbf{k}_\alpha(b,a))^2\int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx}\right] = M\lambda,$$

$$\frac{\int_a^b \frac{\mathbf{k}_\alpha(b,x)}{\mathbf{k}_\alpha(b,a)} f'(x)dx + 2f(a)}{2} - \frac{\int_a^b (\mathbf{k}_\alpha(b,x))^2 [\int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} f'(t)dt + f(a)]\omega(x)dx}{\mathbf{k}_\alpha(b,a)\int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx}$$

$$\geqslant m\left[\frac{\lambda_1}{2} - \frac{\int_a^b (\mathbf{k}_\alpha(b,x))^2\omega(x)\int_a^x \mathbf{k}_\alpha(b,t)t\omega(t)dtdx}{(\mathbf{k}_\alpha(b,a))^2\int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx}\right] = m\lambda.$$

Therefore

$$m\lambda \leqslant \frac{\int_a^b \frac{\mathbf{k}_\alpha(b,x)}{\mathbf{k}_\alpha(b,a)} f'(x)dx + 2f(a)}{2} - \frac{\int_a^b (\mathbf{k}_\alpha(b,x))^2 [\int_a^x \frac{\mathbf{k}_\alpha(b,t)}{\mathbf{k}_\alpha(b,a)} f'(t)dt + f(a)]\omega(x)dx}{\mathbf{k}_\alpha(b,a)\int_a^b \mathbf{k}_\alpha(b,x)\omega(x)dx} \leqslant M\lambda.$$

Since G_f is continuous on $[a,b]$, there exist $\eta \in [a,b]$ such that (11) is holds. □

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