Deflection of Beams and Shafts

12

CHAPTER OBJECTIVES

Often limits must be placed on the amount of deflection a beam or shaft may undergo when it is subjected to a load, and so in this chapter we will discuss various methods for determining the deflection and slope at specific points on beams and shafts. The analytical methods include the integration method, the use of discontinuity functions, and the method of superposition. Also, a semigraphical technique, called the moment-area method, will be presented. At the end of the chapter, we will use these methods to solve for the support reactions on a beam or shaft that is statically indeterminate.

12.1 The Elastic Curve

The deflection of a beam or shaft must often be limited in order to provide integrity and stability of a structure or machine, and prevent the cracking of any attached brittle materials such as concrete or glass. Furthermore, code restrictions often require these members not vibrate or deflect severely in order to safely support their intended loading. Most important, though, deflections at specific points on a beam or shaft must be determined if one is to analyze those that are statically indeterminate.

Before the slope or the displacement at a point on a beam (or shaft) is determined, it is often helpful to sketch the deflected shape of the beam when it is loaded, in order to "visualize" any computed results and thereby partially check these results. The deflection curve of the longitudinal axis that passes through the centroid of each cross-sectional area of a beam is called the *elastic curve*. For most beams the elastic curve can be sketched without much difficulty. When doing so, however, it is necessary to know how the slope or displacement is restricted at various types of supports. In general, supports that resist a *force*, such as a pin, restrict *displacement*, and those that resist a *moment*, such as a fixed wall, restrict *rotation* or *slope* as well as displacement. With this in mind, two typical examples of the elastic curves for loaded beams (or shafts), sketched to an exaggerated scale, are shown in Fig. 12–1.

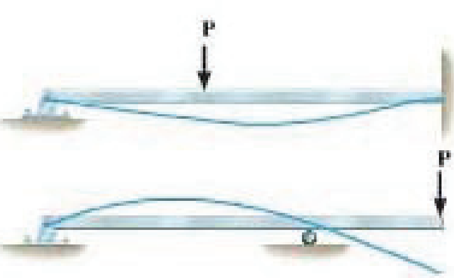


Fig. 12–1

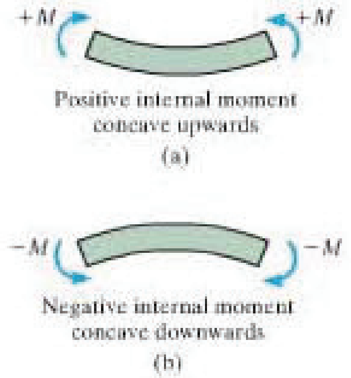


Fig. 12–2

If the elastic curve for a beam seems difficult to establish, it is suggested that the moment diagram for the beam be drawn first. Using the beam sign convention established in Sec. 6.1, a positive internal moment tends to bend the beam concave upward, Fig. 12–2*a*. Likewise, a negative moment tends to bend the beam concave downward, Fig. 12–2*b*. Therefore, if the moment diagram is *known*, it will be easy to construct the elastic curve. For example, consider the beam in Fig. 12–3*a* with its associated moment diagram shown in Fig. 12–3*b*. Due to the roller and pin supports, the displacement at B and D must be zero. Within the region of negative moment, AC, Fig. 12–3*b*, the elastic curve must be concave downward, and within the region of positive moment, CD, the elastic curve must be concave upward. Hence, there must be an *inflection point* at point C, where the curve changes from concave up to concave down, since this is a point of zero moment. Using these facts, the beam's elastic curve is sketched in Fig. 12–3*c*. It should also be noted that the displacements Δ_A and Δ_E are especially critical. At point E the *slope* of the elastic curve is *zero*, and there the beam's *deflection* may be a *maximum*. Whether Δ_E is actually greater than Δ_A depends on the relative magnitudes of $\mathbf{P}_1$ and $\mathbf{P}_2$ and the location of the roller at B.

Following these same principles, note how the elastic curve in Fig. 12–4 was constructed. Here the beam is cantilevered from a fixed support at A and therefore the elastic curve must have both zero displacement and zero slope at this point. Also, the largest displacement will occur either at D, where the slope is zero, or at C.

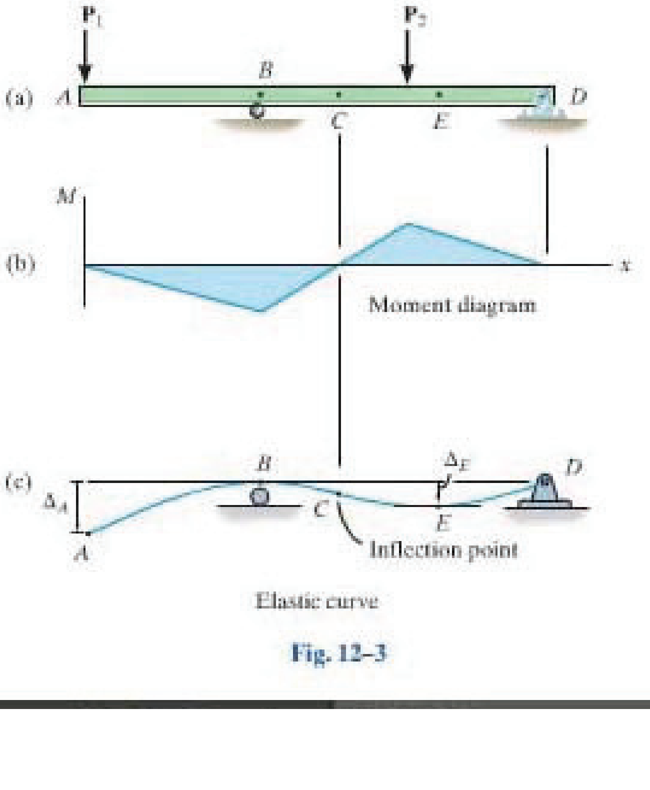


Fig. 12–3

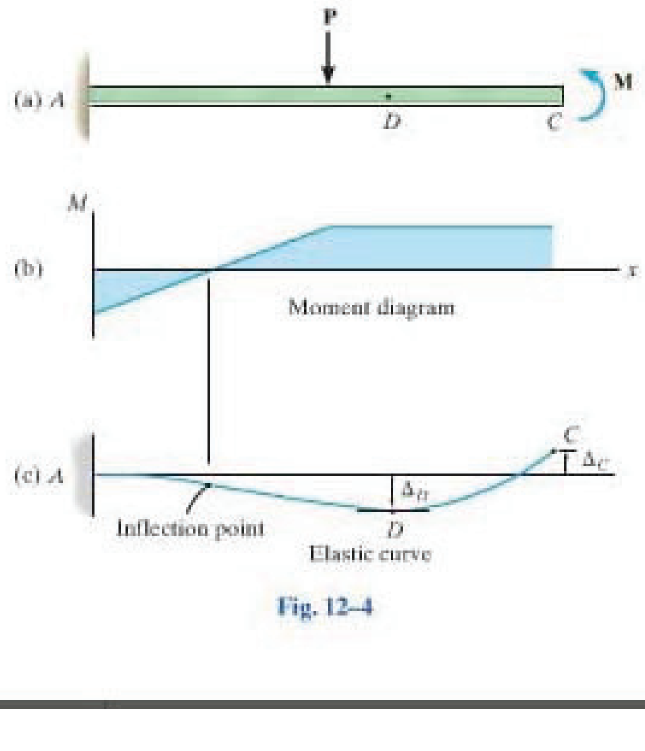


Fig. 12–4

Moment-Curvature Relationship. We will now develop an important relationship between the internal moment and the radius of curvature ρ (rho) of the elastic curve at a point. The resulting equation will be used for establishing each of the methods presented in the chapter for finding the slope and displacement at points on the elastic curve.

The following analysis, here and in the next section, will require the use of three coordinates. As shown in Fig. 12–5*a*, the x axis extends positive to the right, along the initially straight longitudinal axis of the beam. It is used to locate the differential element, having an undeformed width dx. The v axis extends *positive upward* from the x axis. It measures the *displacement* of the elastic curve. Lastly, a "localized" y coordinate is used to specify the position of a fiber in the beam element. It is measured *positive upward* from the neutral axis (or elastic curve) as shown in Fig. 12–5*b*. Recall that this same sign convention for x and y was used in the derivation of the flexure formula.

To derive the relationship between the internal moment and ρ, we will limit the analysis to the most common case of an initially straight beam that is elastically deformed by loads applied perpendicular to the beam's x axis and lying in the x–v plane of symmetry for the beam's cross-sectional area. Due to the loading, the deformation of the beam is caused by both the internal shear force and bending moment. If the beam has a length that is much greater than its depth, the greatest deformation will be caused by bending, and therefore we will direct our attention to its effects. Deflections caused by shear will be discussed in Chapter 14.

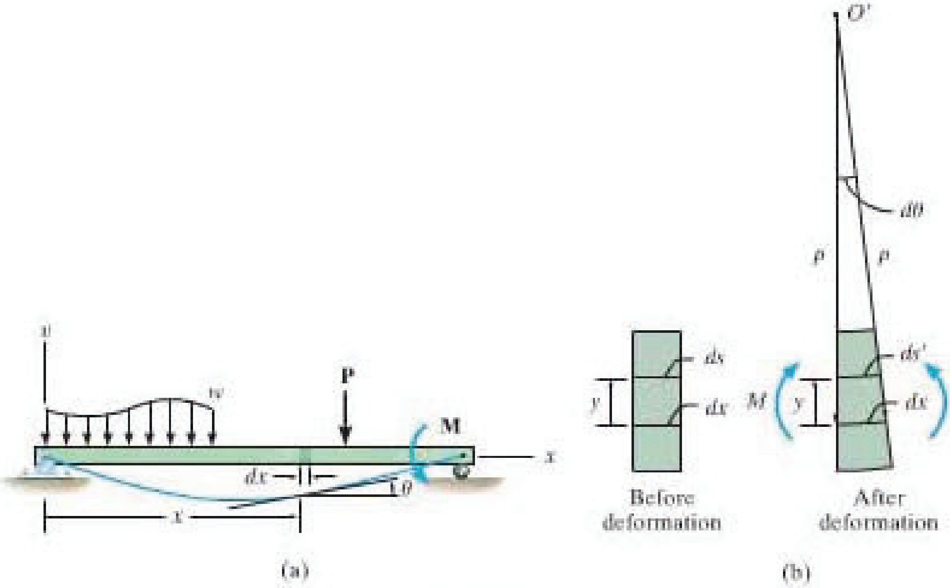


Fig. 12–5

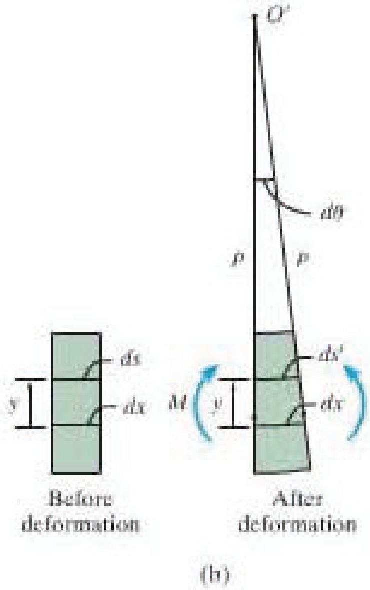


Fig. 12-5 (cont.)

When the internal moment M deforms the element of the beam, the angle between the cross sections becomes $d\theta$, Fig. 12–5*b*. The arc dx represents a portion of the elastic curve that intersects the neutral axis for each cross section. The *radius of curvature* for this arc is defined as the distance ρ, which is measured from the *center of curvature* O' to dx. Any arc on the element other than dx is subjected to a normal strain. For example, the strain in arc ds, located at a position y from the neutral axis, is $\epsilon = (ds' - ds)/ds$. However, $ds = dx = \rho\, d\theta$ and $ds' = (\rho - y)\, d\theta$, and so $\epsilon = [(\rho - y)\, d\theta - \rho\, d\theta]/\rho\, d\theta$ or

$$\frac{1}{\rho} = -\frac{\epsilon}{y} \tag{12–1}$$

If the material is homogeneous and behaves in a linear-elastic manner, then Hooke's law applies, $\epsilon = \sigma/E$. Also, since the flexure formula applies, $\sigma = -My/I$. Combining these two equations and substituting into the above equation, we have

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}} \tag{12–2}$$

where

ρ = the radius of curvature at the point on the elastic curve ($1/\rho$ is referred to as the *curvature*)

M = the internal moment in the beam at the point

E = the material's modulus of elasticity

I = the beam's moment of inertia about the neutral axis

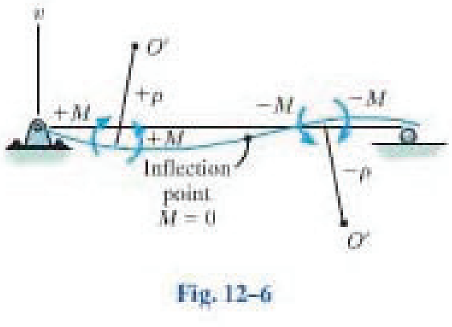


Fig. 12–6

The product EI in this equation is referred to as the *flexural rigidity*, and it is always a positive quantity. The sign for ρ therefore depends on the direction of the moment. As shown in Fig. 12–6, when M is *positive*, ρ extends *above* the beam, i.e., in the positive v direction; when M is *negative*, ρ extends *below* the beam, or in the negative v direction.

Using the flexure formula, $\sigma = -My/I$, we can also express the curvature in terms of the stress in the beam, namely,

$$\frac{1}{\rho} = -\frac{\sigma}{Ey} \tag{12–3}$$

Both Eqs. 12–2 and 12–3 are valid for either small or large radii of curvature. However, the value of ρ is almost always calculated as a *very large quantity*. For example, consider an A-36 steel beam made from a W14 × 53 (Appendix B), where $E_{st} = 29(10^3)$ ksi and $\sigma_Y = 36$ ksi. When the material at the outer fibers, $y = \pm 7$ in., is about to *yield*, then, from Eq. 12–3, $\rho = \pm 5639$ in. Values of ρ calculated at other points along the beam's elastic curve may be even *larger*, since σ cannot exceed σ_Y at the outer fibers.

12.2 Slope and Displacement by Integration

The equation of the elastic curve for a beam can be expressed mathematically as $v = f(x)$. To obtain this equation, we must first represent the curvature $(1/\rho)$ in terms of v and x. In most calculus books it is shown that this relationship is

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

Substituting into Eq. 12–2, we have

$$\frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}} = \frac{M}{EI} \qquad (12\text{–}4)$$

This equation represents a nonlinear second-order differential equation. Its solution, which is called the *elastica*, gives the exact shape of the elastic curve, assuming, of course, that beam deflections occur only due to bending. Through the use of higher mathematics, elastica solutions have been obtained only for simple cases of beam geometry and loading.

In order to facilitate the solution of a greater number of deflection problems, Eq. 12–4 can be modified. Most engineering design codes specify *limitations* on deflections for tolerance or esthetic purposes, and as a result the elastic deflections for the majority of beams and shafts form a shallow curve. Consequently, the *slope* of the elastic curve, which is determined from dv/dx, will be *very small*, and its square will be negligible compared with unity.* Therefore the curvature, as defined above, can be *approximated* by $1/\rho = d^2v/dx^2$. Using this simplification, Eq. 12–4 can now be written as

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \qquad (12\text{–}5)$$

It is also possible to write this equation in two alternative forms. If we differentiate each side with respect to x and substitute $V = dM/dx$ (Eq. 6–2), we get

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x) \qquad (12\text{–}6)$$

Differentiating again, using $w = dV/dx$ (Eq. 6–1), yields

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = w(x) \qquad (12\text{–}7)$$

*See Example 12.1.

For most problems the flexural rigidity (EI) will be constant along the length of the beam. Assuming this to be the case, the above results may be reordered into the following set of three equations:

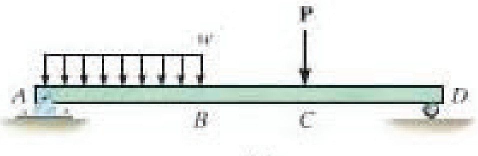
$$EI\frac{d^4v}{dx^4} = w(x) \tag{12-8}$$

$$EI\frac{d^3v}{dx^3} = V(x) \tag{12-9}$$

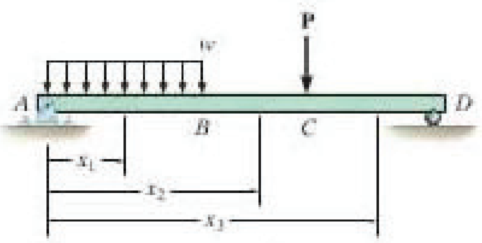
$$EI\frac{d^2v}{dx^2} = M(x) \tag{12-10}$$

Solution of any of these equations requires successive integrations to obtain the deflection v of the elastic curve. For each integration it is necessary to introduce a "constant of integration" and then solve for all the constants to obtain a unique solution for a particular problem. For example, if the distributed load w is expressed as a function of x and Eq. 12–8 is used, then four constants of integration must be evaluated; however, if the internal moment M is determined and Eq. 12–10 is used, only two constants of integration must be found. The choice of which equation to start with depends on the problem. Generally, however, it is easier to determine the internal moment M as a function of x, integrate twice, and evaluate only two integration constants.

Recall from Sec. 6.1 that if the loading on a beam is discontinuous, that is, consists of a series of several distributed and concentrated loads, then several functions must be written for the internal moment, each valid within the region between the discontinuities. Also, for convenience in writing each moment expression, *the origin* for each x coordinate can be *selected arbitrarily*. For example, consider the beam shown in Fig. 12–7*a*. The internal moment in regions AB, BC, and CD can be written in terms of the x_1, x_2, and x_3 coordinates selected, as shown in either Fig. 12–7*b* or Fig. 12–7*c*, or in fact in any manner that will yield $M = f(x)$ in as simple a form as possible. Once these functions are integrated twice through the use of Eq. 12–10 and the constants of integration determined, the functions will give the slope and deflection (elastic curve) for each region of the beam for which they are valid.



(a)



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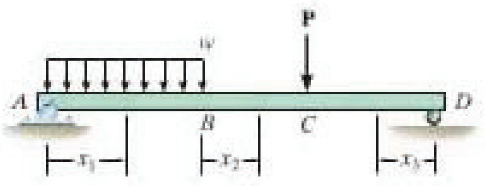

(c)

Fig. 12–7

Sign Convention and Coordinates. When applying Eqs. 12–8 through 12–10, it is important to use the proper signs for M, V, or w as established by the sign convention that was used in the derivation of these equations. For review, these terms are shown in their *positive directions* in Fig. 12–8*a*. Furthermore, recall that *positive deflection*, v, is *upward*, and as a result, the *positive slope angle* θ will be measured *counterclockwise* from the x axis when x is *positive to the right*. The reason for this is shown in Fig. 12–8*b*. Here positive increases dx and dv in x and v create an increased θ that is counterclockwise. If, however, *positive* x is directed to the *left*, then θ will be *positive clockwise*, Fig. 12–8*c*.

Realize that by assuming dv/dx to be very small, the original horizontal length of the beam's axis and the arc of its elastic curve will be about the same. In other words, ds in Fig. 12–8*b* and 12–8*c* is approximately equal to dx, since $ds = \sqrt{(dx)^2 + (dv)^2} = \sqrt{1 + (dv/dx)^2}\, dx \approx dx$. As a result, points on the elastic curve are assumed to be *displaced vertically*, and not horizontally. Also, since the *slope angle* θ will be *very small*, its value in radians can be determined *directly* from $\theta \approx \tan\theta = dv/dx$.

The design of a roof system requires a careful consideration of deflection. For example, rain can accumulate on areas of the roof, which then causes ponding, leading to further deflection, then further ponding, and finally possible failure of the roof.

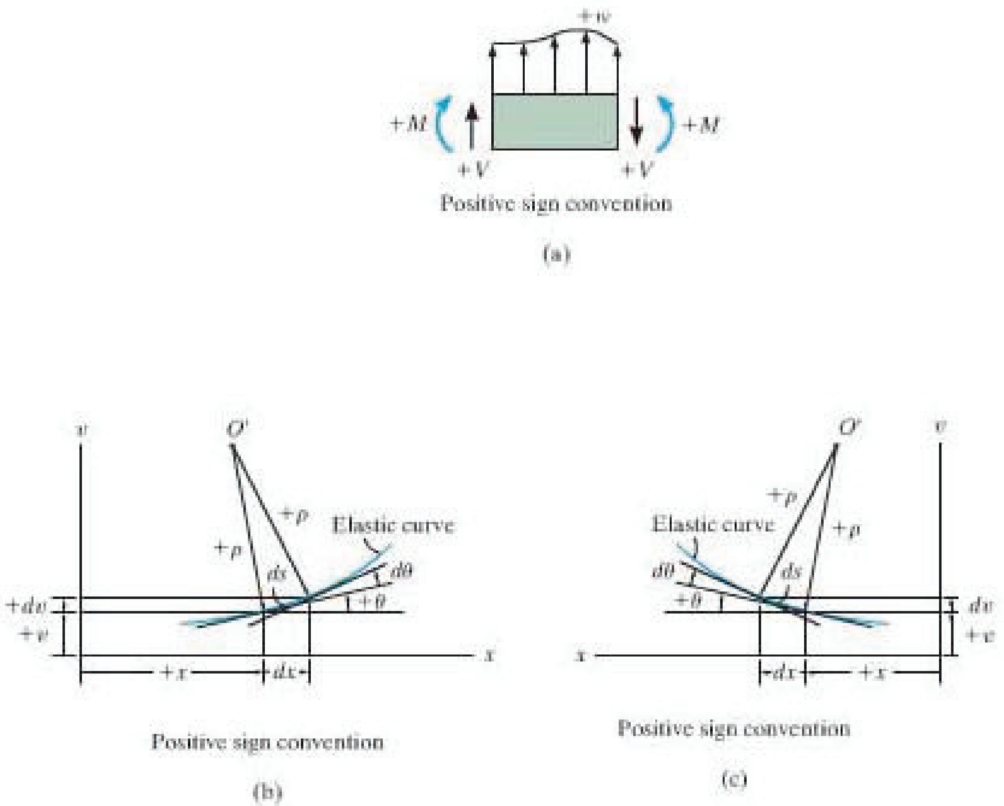

Fig. 12–8

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Boundary and Continuity Conditions.

When solving Eqs. 12–8, 12–9, or 12–10, the constants of integration are determined by evaluating the functions for shear, moment, slope, or displacement at a particular point on the beam where the value of the function is known. These values are called ***boundary conditions***. Several possible boundary conditions that are often used to solve beam (or shaft) deflection problems are listed in Table 12–1. For example, if the beam is supported by a roller or pin (1, 2, 3, 4), then it is required that the displacement be *zero* at these points. Furthermore, if these supports are located at the *ends of the beam* (1, 2), the internal moment in the beam must also be zero. At the fixed support (5), the slope and displacement are both zero, whereas the free-ended beam (6) has both zero moment and zero shear. Lastly, if two segments of a beam are connected by an "internal" pin or hinge (7), the moment must be zero at this connection.

If the elastic curve cannot be expressed using a single coordinate, then ***continuity conditions*** must be used to evaluate some of the integration constants. For example, consider the beam in Fig. 12–9*a*. Here two x coordinates are chosen with origins at A. Each is valid only within the regions $0 \le x_1 \le a$ and $a \le x_2 \le (a + b)$. Once the functions for the slope and deflection are obtained, they must give the *same values* for the slope and deflection at point B so the elastic curve is physically *continuous*. Expressed mathematically, this requires that $\theta_1(a) = \theta_2(a)$ and $v_1(a) = v_2(a)$. These conditions can be used to evaluate two constants of integration. If instead the elastic curve is expressed in terms of the coordinates $0 \le x_1 \le a$ and $0 \le x_2 \le b$, shown in Fig. 12–9*b*, then the continuity of slope and deflection at B requires $\theta_1(a) = -\theta_2(b)$ and $v_1(a) = v_2(b)$. In this particular case, a *negative* sign is necessary to match the slopes at B since x_1 extends positive to the right, whereas x_2 extends positive to the left. Consequently, θ_1 is positive counterclockwise, and θ_2 is positive clockwise. See Figs. 12–8*b* and 12–8*c*.

TABLE 12–1

1

$\Delta = 0$
$M = 0$
Roller

2

$\Delta = 0$
$M = 0$
Pin

3

$\Delta = 0$
Roller

4

$\Delta = 0$
Pin

5

$\theta = 0$
$\Delta = 0$
Fixed end

6

$V = 0$
$M = 0$
Free end

7

$M = 0$
Internal pin or hinge

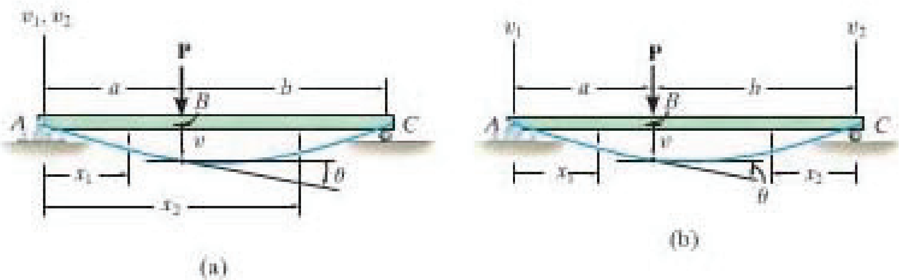

Fig. 12–9

Procedure for Analysis

The following procedure provides a method for determining the slope and deflection of a beam (or shaft) using the method of integration.

Elastic Curve.

- Draw an exaggerated view of the beam's elastic curve. Recall that zero slope and zero displacement occur at all fixed supports, and zero displacement occurs at all pin and roller supports.
- Establish the x and v coordinate axes. The x axis must be parallel to the undeflected beam and can have an origin at any point along the beam, with a positive direction either to the right or to the left.
- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuities. Choose these coordinates so that they will simplify subsequent algebraic work.
- In all cases, the associated positive v axis should be directed upward.

Load or Moment Function.

- For each region in which there is an x coordinate, express the loading w or the internal moment M as a function of x. In particular, *always* assume that M acts in the *positive direction* when applying the equation of moment equilibrium to determine $M = f(x)$.

Slope and Elastic Curve.

- Provided EI is constant, apply either the load equation $EI\, d^4v/dx^4 = w(x)$, which requires four integrations to get $v = v(x)$, or the moment equation $EI\, d^2v/dx^2 = M(x)$, which requires only two integrations. For each integration it is important to include a constant of integration.
- The constants are evaluated using the boundary conditions for the supports (Table 12–1) and the continuity conditions that apply to slope and displacement at points where two functions meet. Once the constants are evaluated and substituted back into the slope and deflection equations, the slope and displacement at *specific points* on the elastic curve can then be determined.
- The numerical values obtained can be checked graphically by comparing them with the sketch of the elastic curve. Realize that *positive* values for *slope* are *counterclockwise* if the x axis extends *positive* to the *right*, and *clockwise* if the x axis extends *positive* to the *left*. In either of these cases, *positive displacement* is *upward*.

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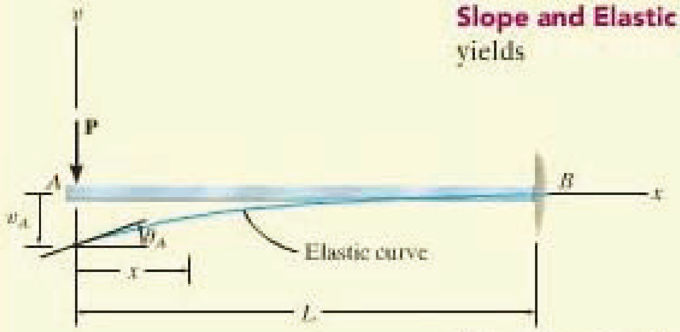
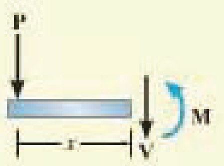
EXAMPLE 12.1

The cantilevered beam shown in Fig. 12–10*a* is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.



(a)



(b)

Fig. 12–10

SOLUTION I

Elastic Curve. The load tends to deflect the beam as shown in Fig. 12–10*a*. By inspection, the internal moment can be represented throughout the beam using a single *x* coordinate.

Moment Function. From the free-body diagram, with **M** acting in the *positive direction*, Fig. 12–10*b*, we have

$$M = -Px$$

Slope and Elastic Curve. Applying Eq. 12–10 and integrating twice yields

$$EI\frac{d^2v}{dx^2} = -Px \quad (1)$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \quad (2)$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \quad (3)$$

Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, Eqs. 2 and 3 become

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1L + C_2$$

Thus, $C_1 = PL^2/2$ and $C_2 = -PL^3/3$. Substituting these results into Eqs. 2 and 3 with $\theta = dv/dx$, we get

$$\theta = \frac{P}{2EI}(L^2 - x^2)$$

$$v = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \qquad \textit{Ans.}$$

Maximum slope and displacement occur at $A(x = 0)$, for which

$$\theta_A = \frac{PL^2}{2EI} \quad (4)$$

$$v_A = -\frac{PL^3}{3EI} \quad (5)$$

The *positive* result for θ_A indicates *counterclockwise* rotation and the *negative* result for v_A indicates that v_A is *downward*. This agrees with the results sketched in Fig. 12–10*a*.

In order to obtain some idea as to the actual *magnitude* of the slope and displacement at the end A, consider the beam in Fig. 12–10*a* to have a length of 15 ft, support a load of $P = 6$ kip, and be made of A-36 steel having $E_{st} = 29(10^3)$ ksi. Using the methods of Sec. 11.2, if this beam was designed without a factor of safety by assuming the allowable normal stress is equal to the yield stress $\sigma_{allow} = 36$ ksi; then a W12 × 26 would be found to be adequate ($I = 204\ \text{in}^4$). From Eqs. 4 and 5 we get

$$\theta_A = \frac{6\ \text{kip}(15\ \text{ft})^2(12\ \text{in./ft})^2}{2[29(10^3)\ \text{kip/in}^2](204\ \text{in}^4)} = 0.0164\ \text{rad}$$

$$v_A = -\frac{6\ \text{kip}(15\ \text{ft})^3(12\ \text{in./ft})^3}{3[29(10^3)\ \text{kip/in}^2](204\ \text{in}^4)} = -1.97\ \text{in.}$$

Since $\theta_A^2 = (dv/dx)^2 = 0.000270\ \text{rad}^2 \ll 1$, this justifies the use of Eq. 12–10, rather than applying the more exact Eq. 12–4, for computing the deflection of beams. Also, since this numerical application is for a *cantilevered beam*, we have obtained *larger values* for θ and v than would have been obtained if the beam were supported using pins, rollers, or other fixed supports.

SOLUTION II

This problem can also be solved using Eq. 12–8, $EI\, d^4v/dx^4 = w(x)$. Here $w(x) = 0$ for $0 \le x \le L$, Fig. 12–10*a*, so that upon integrating once we get the form of Eq. 12–9, i.e.,

$$EI\frac{d^4v}{dx^4} = 0$$

$$EI\frac{d^3v}{dx^3} = C_1' = V$$

The shear constant C_1' can be evaluated at $x = 0$, since $V_A = -P$ (negative according to the beam sign convention, Fig. 12–8*a*). Thus, $C_1' = -P$. Integrating again yields the form of Eq. 12–10, i.e.,

$$EI\frac{d^3v}{dx^3} = -P$$

$$EI\frac{d^2v}{dx^2} = -Px + C_2' = M$$

Here $M = 0$ at $x = 0$, so $C_2' = 0$, and as a result one obtains Eq. 1 and the solution proceeds as before.

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EXAMPLE 12.2

The simply supported beam shown in Fig. 12–11*a* supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.

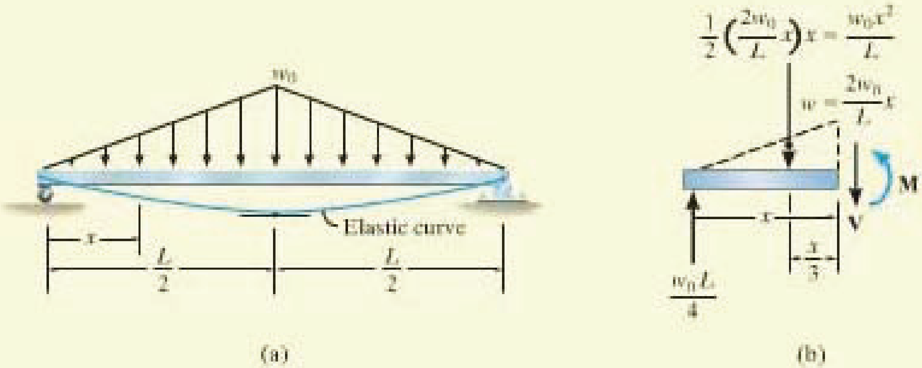


Fig. 12–11

SOLUTION I

Elastic Curve. Due to symmetry, only one x coordinate is needed for the solution, in this case $0 \leq x \leq L/2$. The beam deflects as shown in Fig. 12–11*a*. The maximum deflection occurs at the center since the slope is zero at this point.

Moment Function. A free-body diagram of the segment on the left is shown in Fig. 12–11*b*. The equation for the distributed loading is

$$w = \frac{2w_0}{L}x \tag{1}$$

Hence,

$$\circlearrowleft + \Sigma M_{NA} = 0; \qquad M + \frac{w_0x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0L}{4}(x) = 0$$

$$M = -\frac{w_0x^3}{3L} + \frac{w_0L}{4}x$$

Slope and Elastic Curve. Using Eq. 12–10 and integrating twice, we have

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x \qquad (2)$$

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

The constants of integration are obtained by applying the boundary condition $v = 0$ at $x = 0$ and the symmetry condition that $dv/dx = 0$ at $x = L/2$. This leads to

$$C_1 = -\frac{5w_0L^3}{192} \qquad C_2 = 0$$

Hence,

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 - \frac{5w_0L^3}{192}$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 - \frac{5w_0L^3}{192}x$$

Determining the maximum deflection at $x = L/2$, we have

$$v_{max} = -\frac{w_0L^4}{120EI} \qquad \textit{Ans.}$$

SOLUTION II

Since the distributed loading acts downward, it is negative according to our sign convention. Using Eq. 1 and applying Eq. 12–8, we have

$$EI\frac{d^4v}{dx^4} = -\frac{2w_0}{L}x$$

$$EI\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + C_1'$$

Since $V = +w_0L/4$ at $x = 0$, then $C_1' = w_0L/4$. Integrating again yields

$$EI\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + \frac{w_0L}{4}$$

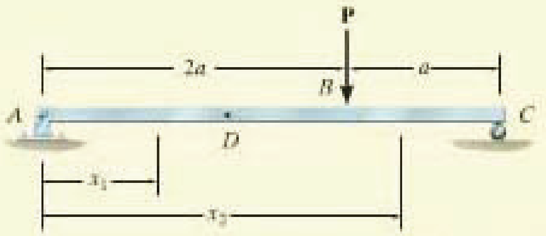
$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x + C_2'$$

Here $M = 0$ at $x = 0$, so $C_2' = 0$. This yields Eq. 2. The solution now proceeds as before.

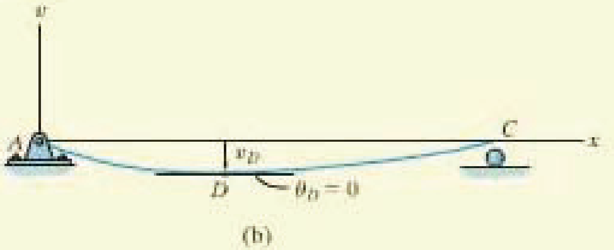
12

EXAMPLE 12.3

The simply supported beam shown in Fig. 12–12*a* is subjected to the concentrated force **P**. Determine the maximum deflection of the beam. *EI* is constant.



(a)



(b)

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 12–12*b*. Two coordinates must be used, since the moment function will change at *P*. Here we will take x_1 and x_2, having the *same origin* at *A*.

Moment Function. From the free-body diagrams shown in Fig. 12–12*c*,

$$M_1 = \frac{P}{3}x_1$$

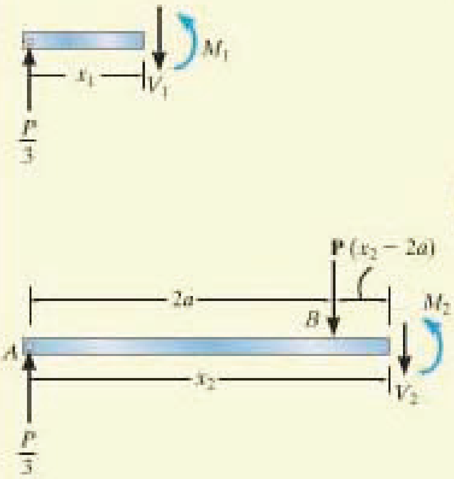
$$M_2 = \frac{P}{3}x_2 - P(x_2 - 2a) = \frac{2P}{3}(3a - x_2)$$

Slope and Elastic Curve. Applying Eq. 12–10 for M_1, for $0 \le x_1 < 2a$, and integrating twice yields

$$EI\frac{d^2v_1}{dx_1^2} = \frac{P}{3}x_1$$

$$EI\frac{dv_1}{dx_1} = \frac{P}{6}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = \frac{P}{18}x_1^3 + C_1x_1 + C_2 \qquad (2)$$



(c)

Fig. 12–12

Likewise for M_2, for $2a < x_2 \le 3a$,

$$EI\frac{d^2v_2}{dx_2^2} = \frac{2P}{3}(3a - x_2)$$

$$EI\frac{dv_2}{dx_2} = \frac{2P}{3}\left(3ax_2 - \frac{x_2^2}{2}\right) + C_3 \qquad (3)$$

$$EIv_2 = \frac{2P}{3}\left(\frac{3}{2}ax_2^2 - \frac{x_2^3}{6}\right) + C_3x_2 + C_4 \qquad (4)$$

The four constants are evaluated using *two* boundary conditions, namely, $x_1 = 0$, $v_1 = 0$ and $x_2 = 3a$, $v_2 = 0$. Also, *two* continuity conditions must be applied at B, that is, $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$ and $v_1 = v_2$ at $x_1 = x_2 = 2a$. Substitution as specified results in the following four equations:

$v_1 = 0$ at $x_1 = 0$; $\quad 0 = 0 + 0 + C_2$

$v_2 = 0$ at $x_2 = 3a$; $\quad 0 = \frac{2P}{3}\left(\frac{3}{2}a(3a)^2 - \frac{(3a)^3}{6}\right) + C_3(3a) + C_4$

$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}$; $\quad \frac{P}{6}(2a)^2 + C_1 = \frac{2P}{3}\left(3a(2a) - \frac{(2a)^2}{2}\right) + C_3$

$v_1(2a) = v_2(2a)$; $\quad \frac{P}{18}(2a)^3 + C_1(2a) + C_2 = \frac{2P}{3}\left(\frac{3}{2}a(2a)^2 - \frac{(2a)^3}{6}\right) + C_3(2a) + C_4$

Solving, we get

$$C_1 = -\frac{4}{9}Pa^2 \qquad C_2 = 0$$

$$C_3 = -\frac{22}{9}Pa^2 \qquad C_4 = \frac{4}{3}Pa^3$$

Thus Eqs. 1–4 become

$$\frac{dv_1}{dx_1} = \frac{P}{6EI}x_1^2 - \frac{4Pa^2}{9EI} \qquad (5)$$

$$v_1 = \frac{P}{18EI}x_1^3 - \frac{4Pa^2}{9EI}x_1 \qquad (6)$$

$$\frac{dv_2}{dx_2} = \frac{2Pa}{EI}x_2 - \frac{P}{3EI}x_2^2 - \frac{22Pa^2}{9EI} \qquad (7)$$

$$v_2 = \frac{Pa}{EI}x_2^2 - \frac{P}{9EI}x_2^3 - \frac{22Pa^2}{9EI}x_2 + \frac{4Pa^3}{3EI} \qquad (8)$$

By inspection of the elastic curve, Fig. 12–12*b*, the maximum deflection occurs at D, somewhere within region AB. Here the slope must be zero. From Eq. 5,

$$\frac{1}{6}x_1^2 - \frac{4}{9}a^2 = 0$$

$$x_1 = 1.633a$$

Substituting into Eq. 6,

$$v_{max} = -0.484\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.

12

EXAMPLE 12.4

The beam in Fig. 12–13*a* is subjected to a load **P** at its end. Determine the displacement at *C*. *EI* is constant.

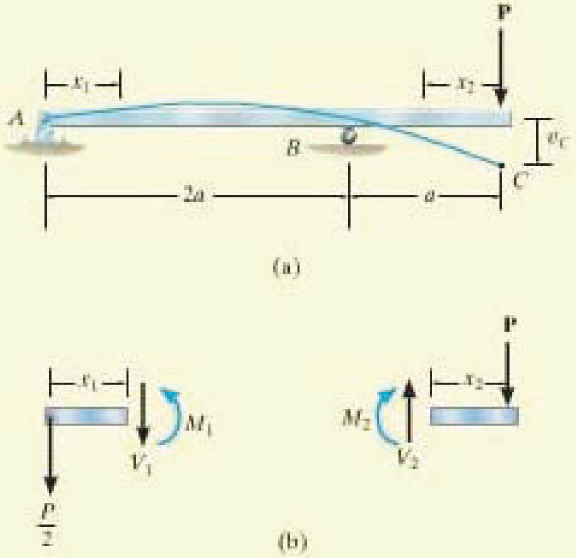


Fig. 12–13

SOLUTION

Elastic Curve. The beam deflects into the shape shown in Fig. 12–13*a*. Due to the loading, two *x* coordinates will be considered, namely, $0 \leq x_1 < 2a$ and $0 \leq x_2 < a$, where x_2 is directed to the left from *C*, since the internal moment is easy to formulate.

Moment Functions. Using the free-body diagrams shown in Fig. 12–13*b*, we have

$$M_1 = -\frac{P}{2}x_1 \qquad M_2 = -Px_2$$

Slope and Elastic Curve. Applying Eq. 12–10,

For $0 \leq x_1 \leq 2a$: $$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

For $0 \le x_2 \le a$:
$$EI\frac{d^2v_2}{dx_2^2} = -Px_2$$

$$EI\frac{dv_2}{dx_2} = -\frac{P}{2}x_2^2 + C_3 \quad (3)$$

$$EIv_2 = -\frac{P}{6}x_2^3 + C_3x_2 + C_4 \quad (4)$$

The *four* constants of integration are determined using *three* boundary conditions, namely, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$, and $v_2 = 0$ at $x_2 = a$, and *one* continuity equation. Here the continuity of slope at the roller requires $dv_1/dx_1 = -dv_2/dx_2$ at $x_1 = 2a$ and $x_2 = a$. Why is there a negative sign in this equation? (Note that continuity of displacement at B has been indirectly considered in the boundary conditions, since $v_1 = v_2 = 0$ at $x_1 = 2a$ and $x_2 = a$.) Applying these four conditions yields

$v_1 = 0$ at $x_1 = 0$; $\quad 0 = 0 + 0 + C_2$

$v_1 = 0$ at $x_1 = 2a$; $\quad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$

$v_2 = 0$ at $x_2 = a$; $\quad 0 = -\frac{P}{6}a^3 + C_3a + C_4$

$$\frac{dv_1(2a)}{dx_1} = -\frac{dv_2(a)}{dx_2}; \quad -\frac{P}{4}(2a)^2 + C_1 = -\left(-\frac{P}{2}(a)^2 + C_3\right)$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{7}{6}Pa^2 \qquad C_4 = -Pa^3$$

Substituting C_3 and C_4 into Eq. 4 gives

$$v_2 = -\frac{P}{6EI}x_2^3 + \frac{7Pa^2}{6EI}x_2 - \frac{Pa^3}{EI}$$

The displacement at C is determined by setting $x_2 = 0$. We get

$$v_C = -\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

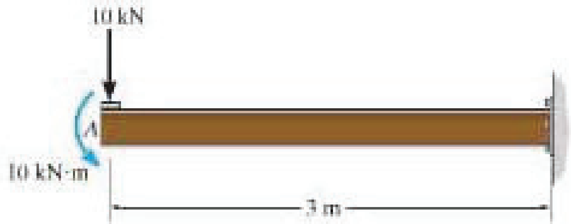
12

FUNDAMENTAL PROBLEMS

F12–1. Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.

F12–1

F12–2. Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.

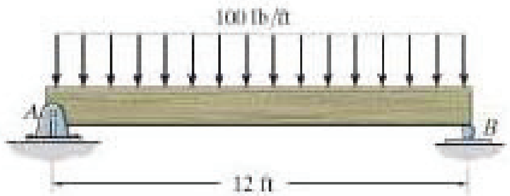


F12–2

F12–3. Determine the slope of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.

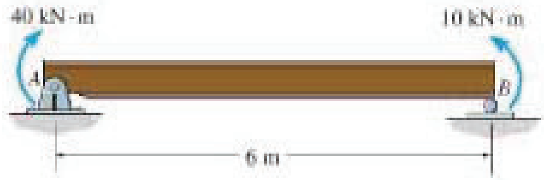
F12–3

F12–4. Determine the maximum deflection of the simply supported beam. The beam is made of wood having a modulus of elasticity of $E_w = 1.5(10^3)$ ksi and a rectangular cross section of $b = 3$ in. and $h = 6$ in.



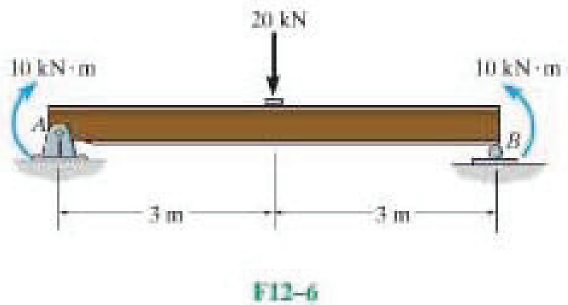
F12–4

F12–5. Determine the maximum deflection of the simply supported beam. $E = 200$ GPa and $I = 39.9(10^{-6})$ m^4.



F12–5

F12–6. Determine the slope of the simply supported beam at A. $E = 200$ GPa and $I = 39.9(10^{-6})$ m^4.

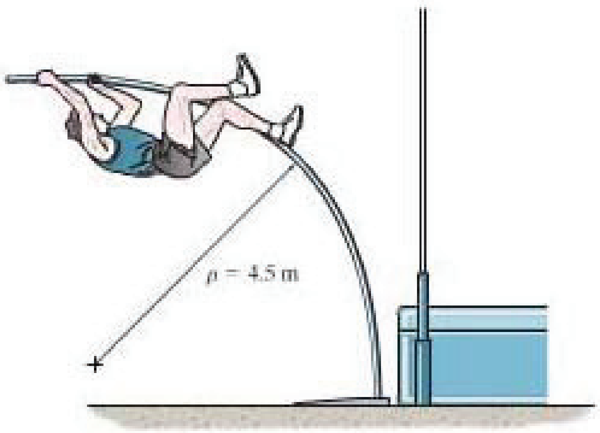


F12–6

PROBLEMS

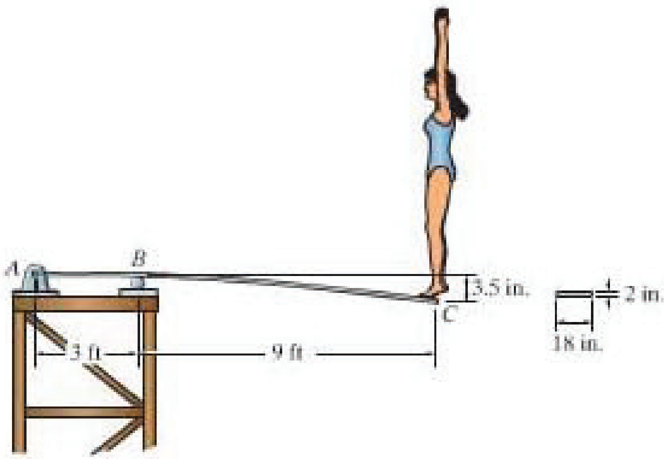
•**12–1.** An A-36 steel strap having a thickness of 10 mm and a width of 20 mm is bent into a circular arc of radius $\rho = 10$ m. Determine the maximum bending stress in the strap.

12–2. A picture is taken of a man performing a pole vault, and the minimum radius of curvature of the pole is estimated by measurement to be 4.5 m. If the pole is 40 mm in diameter and it is made of a glass-reinforced plastic for which $E_g = 131$ GPa, determine the maximum bending stress in the pole.



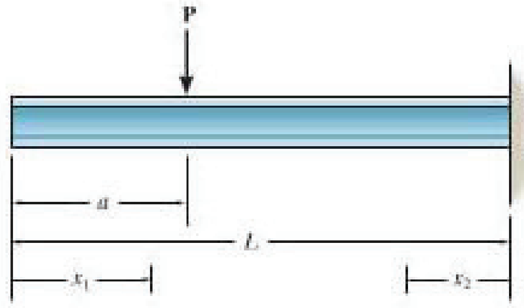
Prob. 12–2

12–3. When the diver stands at end C of the diving board, it deflects downward 3.5 in. Determine the weight of the diver. The board is made of material having a modulus of elasticity of $E = 1.5(10^3)$ ksi.



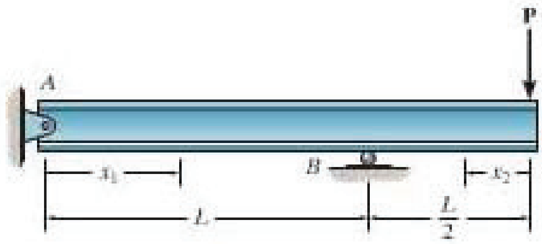
Prob. 12–3

***12–4.** Determine the equations of the elastic curve using the x_1 and x_2 coordinates. EI is constant.



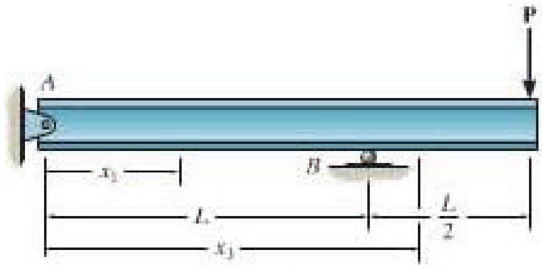
Prob. 12–4

•**12–5.** Determine the equations of the elastic curve for the beam using the x_1 and x_2 coordinates. EI is constant.



Prob. 12–5

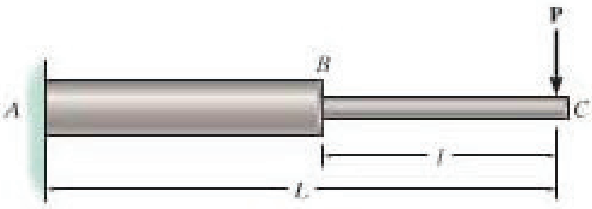
12–6. Determine the equations of the elastic curve for the beam using the x_1 and x_3 coordinates. Specify the beam's maximum deflection. EI is constant.



Prob. 12–6

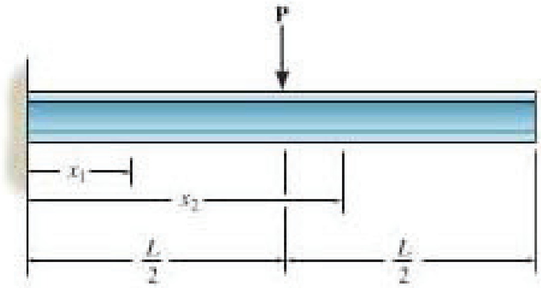
12

12–7. The beam is made of two rods and is subjected to the concentrated load **P**. Determine the maximum deflection of the beam if the moments of inertia of the rods are I_{AB} and I_{BC}, and the modulus of elasticity is E.



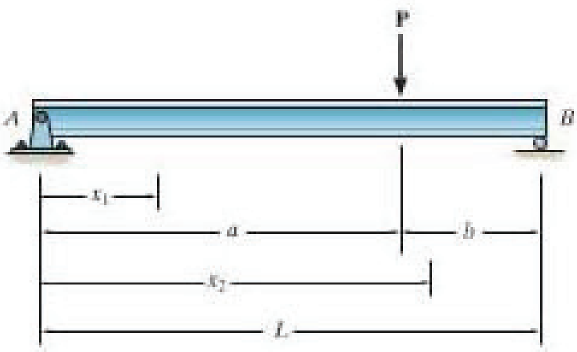
Prob. 12–7

***12–8.** Determine the equations of the elastic curve for the beam using the x_1 and x_2 coordinates. EI is constant.



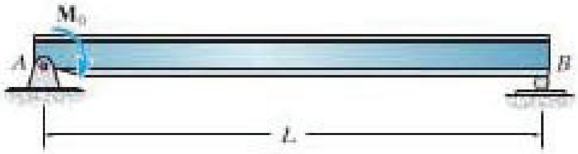
Prob. 12–8

•12–9. Determine the equations of the elastic curve using the x_1 and x_2 coordinates. EI is constant.



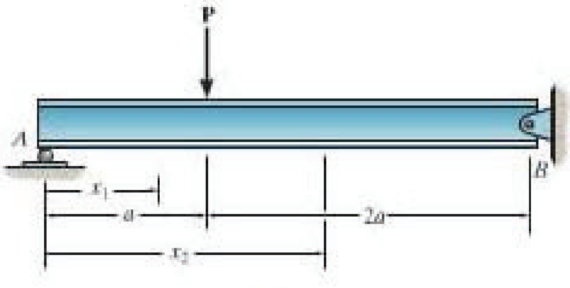
Prob. 12–9

12–10. Determine the maximum slope and maximum deflection of the simply supported beam which is subjected to the couple moment $\mathbf{M}_0$. EI is constant.



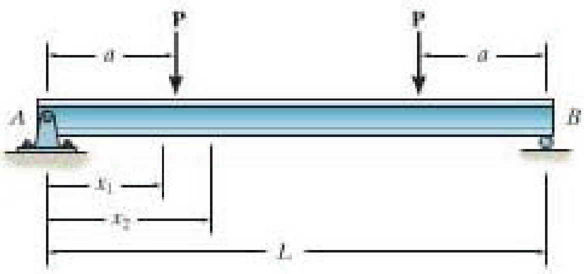
Prob. 12–10

12–11. Determine the equations of the elastic curve for the beam using the x_1 and x_2 coordinates. Specify the beam's maximum deflection. EI is constant.



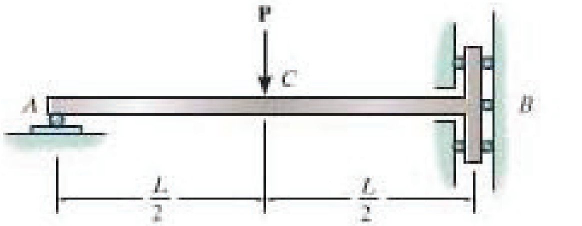
Prob. 12–11

***12–12.** Determine the equations of the elastic curve for the beam using the x_1 and x_2 coordinates. Specify the slope at A and the maximum displacement of the shaft. EI is constant.



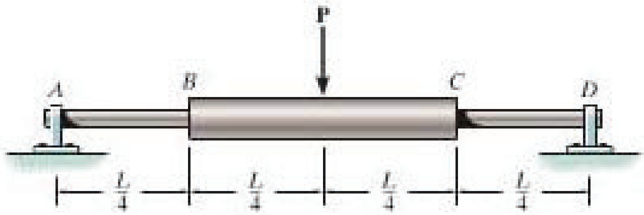
Prob. 12–12

12–13. The bar is supported by a roller constraint at *B*, which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope at *A* and the deflection at *C*. *EI* is constant.



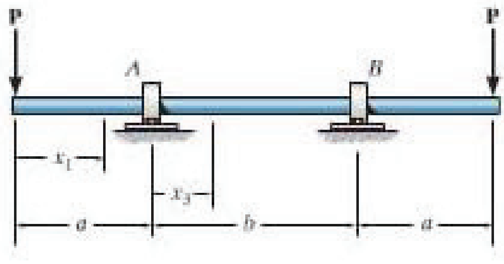
Prob. 12–13

12–14. The simply supported shaft has a moment of inertia of $2I$ for region *BC* and a moment of inertia *I* for regions *AB* and *CD*. Determine the maximum deflection of the beam due to the load **P**.



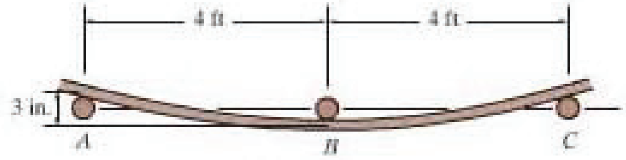
Prob. 12–14

12–15. Determine the equations of the elastic curve for the shaft using the x_1 and x_3 coordinates. Specify the slope at *A* and the deflection at the center of the shaft. *EI* is constant.



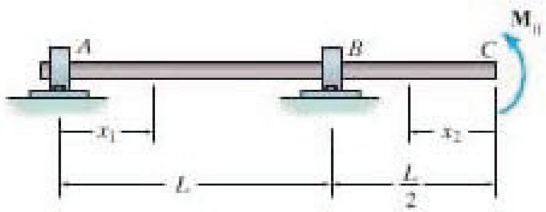
Prob. 12–15

***12–16.** The fence board weaves between the three smooth fixed posts. If the posts remain along the same line, determine the maximum bending stress in the board. The board has a width of 6 in. and a thickness of 0.5 in. $E = 1.60(10^3)$ ksi. Assume the displacement of each end of the board relative to its center is 3 in.



Prob. 12–16

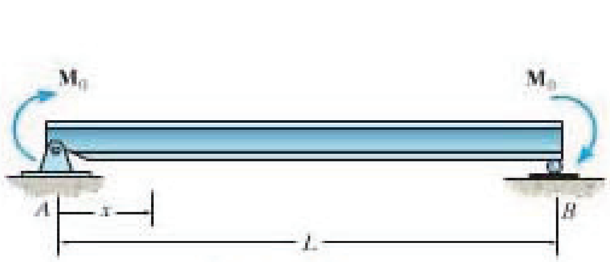
•12–17. Determine the equations of the elastic curve for the shaft using the x_1 and x_2 coordinates. Specify the slope at *A* and the deflection at *C*. *EI* is constant.



Prob. 12–17

12–18. Determine the equation of the elastic curve for the beam using the *x* coordinate. Specify the slope at *A* and the maximum deflection. *EI* is constant.

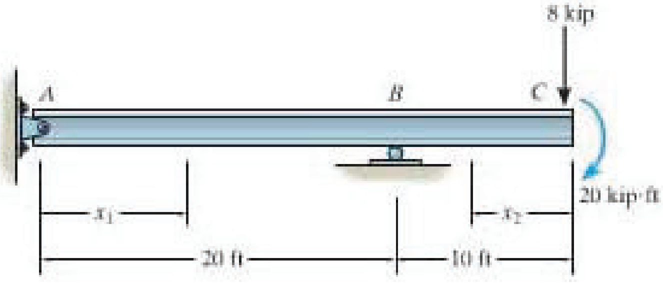
12–19. Determine the deflection at the center of the beam and the slope at *B*. *EI* is constant.



Probs. 12–18/19

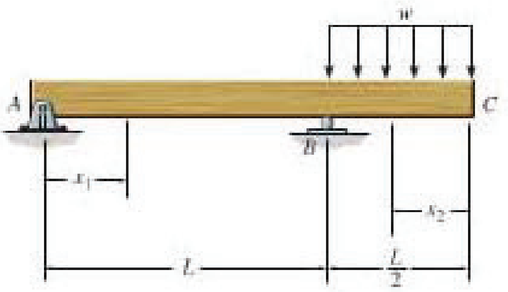
12

*12–20. Determine the equations of the elastic curve using the x_1 and x_2 coordinates, and specify the slope at A and the deflection at C. EI is constant.



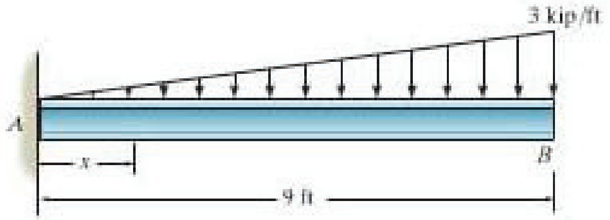
Prob. 12–20

•12–21. Determine the elastic curve in terms of the x_1 and x_2 coordinates and the deflection of end C of the overhang beam. EI is constant.



Prob. 12–21

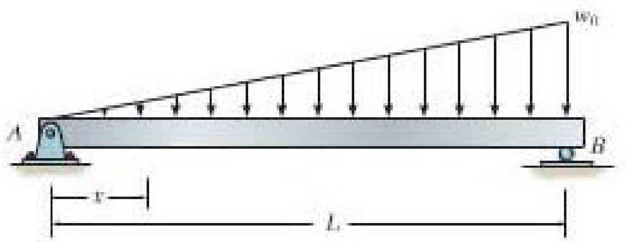
12–22. Determine the elastic curve for the cantilevered W14 × 30 beam using the x coordinate. Specify the maximum slope and maximum deflection. $E = 29(10^3)$ ksi.



Prob. 12–22

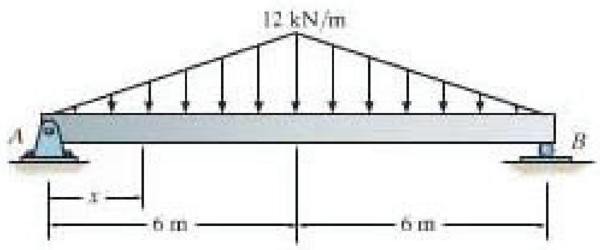
12–23. The beam is subjected to the linearly varying distributed load. Determine the maximum slope of the beam. EI is constant.

*12–24. The beam is subjected to the linearly varying distributed load. Determine the maximum deflection of the beam. EI is constant.



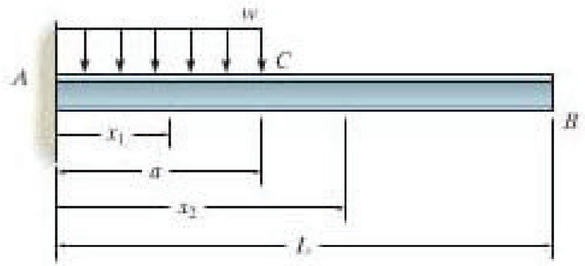
Probs. 12–23/24

•12–25. Determine the equation of the elastic curve for the simply supported beam using the x coordinate. Determine the slope at A and the maximum deflection. EI is constant.



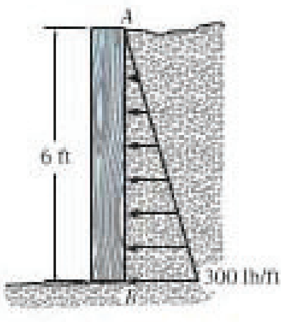
Prob. 12–25

12–26. Determine the equations of the elastic curve using the coordinates x_1 and x_2, and specify the slope and deflection at B. EI is constant.



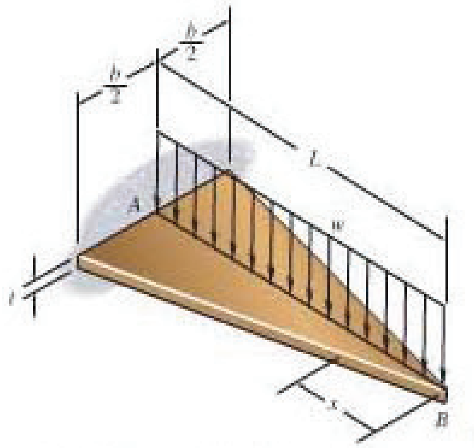
Prob. 12–26

12–27. Wooden posts used for a retaining wall have a diameter of 3 in. If the soil pressure along a post varies uniformly from zero at the top A to a maximum of 300 lb/ft at the bottom B, determine the slope and displacement at the top of the post. $E_w = 1.6(10^3)$ ksi.



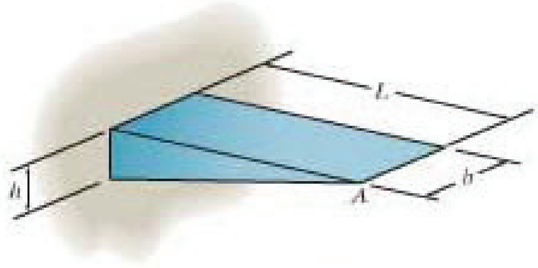
Prob. 12–27

*12–28. Determine the slope at end B and the maximum deflection of the cantilevered triangular plate of constant thickness t. The plate is made of material having a modulus of elasticity E.



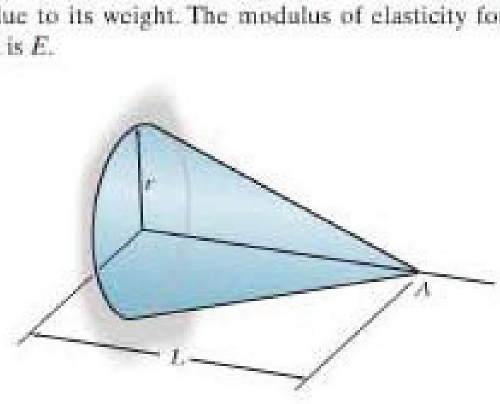
Prob. 12–28

•**12–29.** The beam is made of a material having a specific weight γ. Determine the displacement and slope at its end A due to its weight. The modulus of elasticity for the material is E.



Prob. 12–29

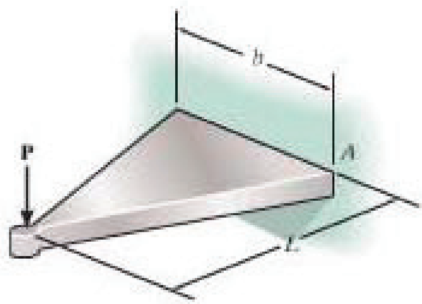
12–30. The beam is made of a material having a specific weight of γ. Determine the displacement and slope at its end A due to its weight. The modulus of elasticity for the material is E.



Prob. 12–30

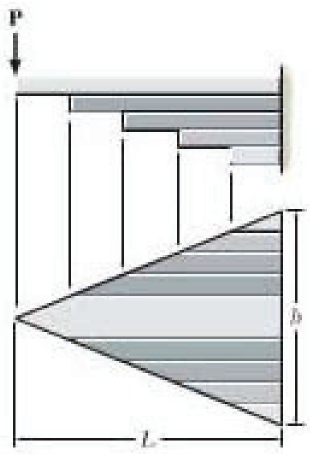
12

12–31. The tapered beam has a rectangular cross section. Determine the deflection of its free end in terms of the load P, length L, modulus of elasticity E, and the moment of inertia I_0 of its fixed end.



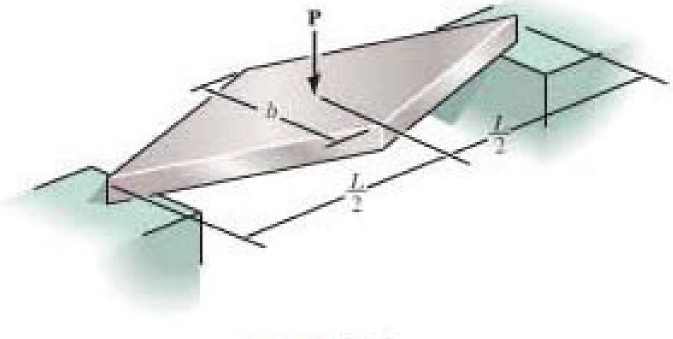
Prob. 12–31

***12–32.** The beam is made from a plate that has a constant thickness t and a width that varies linearly. The plate is cut into strips to form a series of leaves that are stacked to make a leaf spring consisting of n leaves. Determine the deflection at its end when loaded. Neglect friction between the leaves.



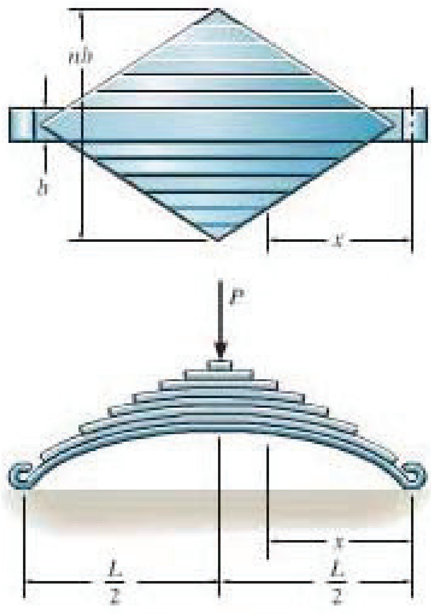
Prob. 12–32

•**12–33.** The tapered beam has a rectangular cross section. Determine the deflection of its center in terms of the load P, length L, modulus of elasticity E, and the moment of inertia I_c of its center.



Prob. 12–33

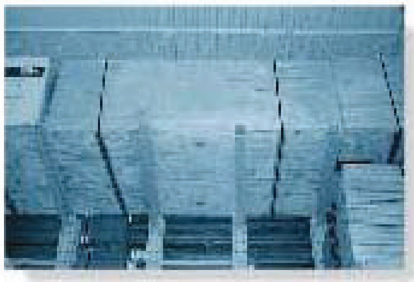
12–34. The leaf spring assembly is designed so that it is subjected to the same maximum stress throughout its length. If the plates of each leaf have a thickness t and can slide freely between each other, show that the spring must be in the form of a circular arc in order that the entire spring becomes flat when a large enough load **P** is applied. What is the maximum normal stress in the spring? Consider the spring to be made by cutting the n strips from the diamond-shaped plate of thickness t and width b. The modulus of elasticity for the material is E. *Hint*: Show that the radius of curvature of the spring is constant.



Prob. 12–34

*12.3 Discontinuity Functions

The method of integration, used to find the equation of the elastic curve for a beam or shaft, is convenient if the load or internal moment can be expressed as a continuous function throughout the beam's entire length. If several different loadings act on the beam, however, the method becomes more tedious to apply, because separate loading or moment functions must be written for each region of the beam. Furthermore, integration of these functions requires the evaluation of integration constants using both boundary and continuity conditions. For example, the beam shown in Fig. 12–14 requires four moment functions to be written. They describe the moment in regions *AB*, *BC*, *CD*, and *DE*. When applying the moment-curvature relationship, $EI\,d^2v/dx^2 = M$, and integrating each moment equation twice, we must evaluate eight constants of integration. These involve two boundary conditions that require zero displacement at points *A* and *E*, and six continuity conditions for both slope and displacement at points *B*, *C*, and *D*.



For safety purposes these cantilevered beams that support sheets of plywood must be designed for both strength and a restricted amount of deflection.

In this section, we will discuss a method for finding the equation of the elastic curve for a *multiply loaded beam* using a *single expression*, either formulated from the loading on the beam, $w = w(x)$, or from the beam's internal moment, $M = M(x)$. If the expression for w is substituted into $EI\,d^4v/dx^4 = w(x)$ and integrated four times, or if the expression for M is substituted into $EI\,d^2v/dx^2 = M(x)$ and integrated twice, the constants of integration will be determined only from the boundary conditions. Since the continuity equations will not be involved, the analysis will be greatly simplified.

Discontinuity Functions. In order to express the load on the beam or the internal moment within it using a single expression, we will use two types of mathematical operators known as *discontinuity functions*.

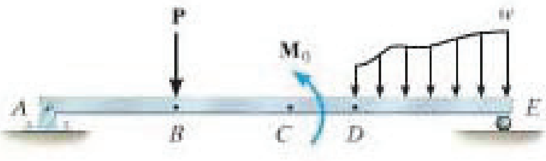


Fig. 12–14

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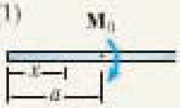
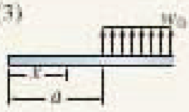
TABLE 12–2

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
(1) 	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^{0}$
(2) 	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^{0}$	$M = P\langle x-a\rangle^{1}$
(3) 	$w = w_0\langle x-a\rangle^{0}$	$V = w_0\langle x-a\rangle^{1}$	$M = \frac{w_0}{2}\langle x-a\rangle^{2}$
(4) 	$w = m\langle x-a\rangle^{1}$	$V = \frac{m}{2}\langle x-a\rangle^{2}$	$M = \frac{m}{6}\langle x-a\rangle^{3}$

Macaulay Functions. For purposes of beam or shaft deflection, Macaulay functions, named after the mathematician W. H. Macaulay, can be used to describe ***distributed loadings***. These functions can be written in general form as

$$\langle x - a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases} \qquad n \geq 0 \tag{12–11}$$

Here x represents the coordinate position of a point along the beam, and a is the location on the beam where a "discontinuity" occurs, namely the point where a distributed loading *begins*. Note that the Macaulay function $\langle x - a\rangle^n$ is written with angle brackets to distinguish it from the ordinary function $(x - a)^n$, written with parentheses. As stated by the equation, only when $x \geq a$ is $\langle x - a\rangle^n = (x - a)^n$, otherwise it is zero. Furthermore, these functions are valid only for exponential values $n \geq 0$. Integration of Macaulay functions follows the same rules as for ordinary functions, i.e.,

$$\int \langle x - a\rangle^n \, dx = \frac{\langle x - a\rangle^{n+1}}{n+1} + C \tag{12–12}$$

Note how the Macaulay functions describe both the *uniform load* w_0 $(n = 0)$ and *triangular load* $(n = 1)$, shown in Table 12–2, items 3 and 4. This type of description can, of course, be extended to distributed loadings having other forms. Also, it is possible to use superposition with the uniform and triangular loadings to create the Macaulay function for a trapezoidal loading. Using integration, the Macaulay functions for shear, $V = \int w(x)\,dx$, and moment, $M = \int V\,dx$, are also shown in the table.

Singularity Functions. These functions are only used to describe the point location of concentrated forces or couple moments acting on a beam or shaft. Specifically, a concentrated force **P** can be considered as a special case of a distributed loading, where the intensity of the loading is $w = P/\epsilon$ such that its length is ϵ, where $\epsilon \to 0$, Fig. 12–15. The area under this loading diagram is equivalent to P, *positive upward*, and so we will use the singularity function

$$w = P\langle x - a\rangle^{-1} = \begin{cases} 0 & \text{for } x \neq a \\ P & \text{for } x = a \end{cases} \qquad (12\text{–}13)$$

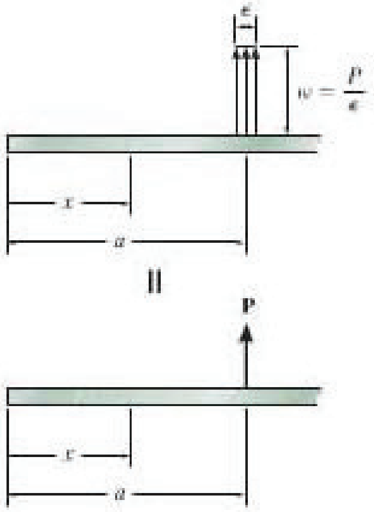


Fig. 12–15

to describe the force P. Here $n = -1$ so that the units for w are force per length, as it should be. Furthermore, the function takes on the value of **P** only at the point $x = a$ where the load occurs, otherwise it is zero.

In a similar manner, a couple moment $\mathbf{M}_0$, considered *positive clockwise*, is a limit as $\epsilon \to 0$ of two distributed loadings as shown in Fig. 12–16. Here the following function describes its value.

$$w = M_0\langle x - a\rangle^{-2} = \begin{cases} 0 & \text{for } x \neq a \\ M_0 & \text{for } x = a \end{cases} \qquad (12\text{–}14)$$

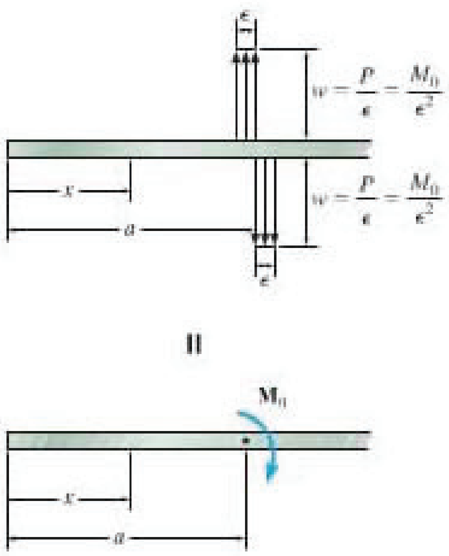


Fig. 12–16

The exponent $n = -2$, in order to ensure that the units of w, force per length, are maintained.

Integration of the above two singularity functions follow the rules of operational calculus and yields results that are *different* from those of Macaulay functions. Specifically,

$$\int \langle x - a\rangle^n dx = \langle x - a\rangle^{n+1}, n = -1, -2 \qquad (12\text{–}15)$$

Using this formula, notice how M_0 and P, described in Table 12–2, items 1 and 2, are integrated once, then twice, to obtain the internal shear and moment in the beam.

Application of Eqs. 12–11 through 12–15 provides a rather direct means for expressing the loading or the internal moment in a beam as a function of x. When doing so, close attention must be paid to the signs of the external loadings. As stated above, and as shown in Table 12–2, *concentrated forces and distributed loads are positive upward, and couple moments are positive clockwise.* If this sign convention is followed, then the internal shear and moment are in accordance with the beam sign convention established in Sec. 6.1.

12

As an example of how to apply discontinuity functions to describe the loading or internal moment consider the beam loaded as shown in Fig. 12–17*a*. Here the reactive 2.75-kN force created by the roller, Fig. 12–17*b*, is positive since it acts upward, and the 1.5-kN·m couple moment is also positive since it acts clockwise. Finally, the trapezoidal loading is negative and has been separated into triangular and uniform loadings. From Table 12–2, the loading at any point x on the beam is therefore

$$w = 2.75\text{ kN}\langle x - 0\rangle^{-1} + 1.5\text{ kN}\cdot\text{m}\langle x - 3\text{ m}\rangle^{-2} - 3\text{ kN/m}\langle x - 3\text{ m}\rangle^{0} - 1\text{ kN/m}^2\langle x - 3\text{ m}\rangle^{1}$$

The reactive force at B is not included here since x is never greater than 6 m, and furthermore, this value is of no consequence in calculating the slope or deflection. We can determine the moment expression directly from Table 12–2, rather than integrating this expression twice. In either case,

$$M = 2.75\text{ kN}\langle x - 0\rangle^{1} + 1.5\text{ kN}\cdot\text{m}\langle x - 3\text{ m}\rangle^{0} - \frac{3\text{ kN/m}}{2}\langle x - 3\text{ m}\rangle^{2} - \frac{1\text{ kN/m}^2}{6}\langle x - 3\text{ m}\rangle^{3}$$

$$= 2.75x + 1.5\langle x - 3\rangle^{0} - 1.5\langle x - 3\rangle^{2} - \frac{1}{6}\langle x - 3\rangle^{3}$$

The deflection of the beam can now be determined after this equation is integrated two successive times and the constants of integration are evaluated using the boundary conditions of zero displacement at A and B.

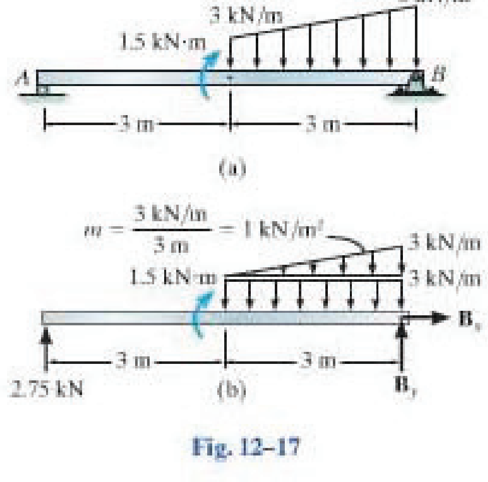


Fig. 12–17

Procedure for Analysis

The following procedure provides a method for using discontinuity functions to determine a beam's elastic curve. This method is particularly advantageous for solving problems involving beams or shafts subjected to *several loadings*, since the constants of integration can be evaluated by using *only* the boundary conditions, while the compatibility conditions are automatically satisfied.

Elastic Curve.

- Sketch the beam's elastic curve and identify the boundary conditions at the supports.
- Zero displacement occurs at all pin and roller supports, and zero slope and zero displacement occur at fixed supports.
- Establish the x axis so that it extends to the right and has its origin at the beam's left end.

Load or Moment Function.

- Calculate the support reactions at $x = 0$ and then use the discontinuity functions in Table 12–2 to express either the loading w or the internal moment M as a function of x. Make sure to follow the sign convention for each loading as it applies for this equation.
- Note that the distributed loadings must extend all the way to the beam's right end to be valid. If this does not occur, use the method of superposition, which is illustrated in Example 12.6.

Slope and Elastic Curve.

- Substitute w into $EI\, d^4v/dx^4 = w(x)$, or M into the moment curvature relation $EI\, d^2v/dx^2 = M$, and integrate to obtain the equations for the beam's slope and deflection.
- Evaluate the constants of integration using the boundary conditions, and substitute these constants into the slope and deflection equations to obtain the final results.
- When the slope and deflection equations are evaluated at any point on the beam, a *positive slope* is *counterclockwise*, and a *positive displacement* is *upward*.

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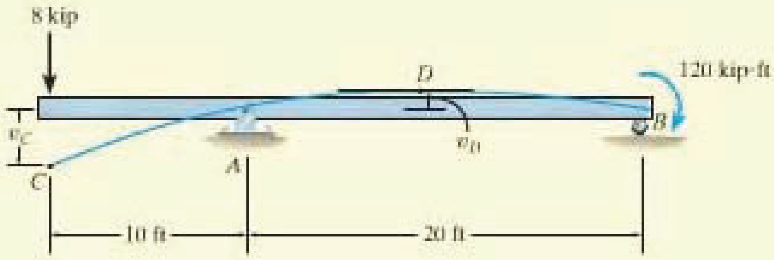
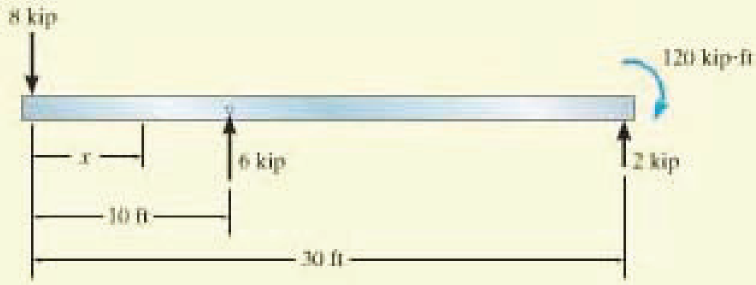
EXAMPLE 12.5

Determine the maximum deflection of the beam shown in Fig. 12–18*a*. *EI* is constant.



(a)



(b)

Fig. 12–18

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 12–18*a*. The boundary conditions require zero displacement at *A* and *B*.

Loading Function. The reactions have been calculated and are shown on the free-body diagram in Fig. 12–18*b*. The loading function for the beam can be written as

$$w = -8 \text{ kip} \langle x - 0\rangle^{-1} + 6 \text{ kip} \langle x - 10 \text{ ft}\rangle^{-1}$$

The couple moment and force at *B* are not included here, since they are located at the right end of the beam, and *x* cannot be greater than 30 ft. Integrating $dV/dx = w(x)$, we get

$$V = -8\langle x - 0\rangle^{0} + 6\langle x - 10\rangle^{0}$$

In a similar manner, $dM/dx = V$ yields

$$M = -8\langle x - 0\rangle^{1} + 6\langle x - 10\rangle^{1}$$
$$= (-8x + 6\langle x - 10\rangle^{1}) \text{ kip}\cdot\text{ft}$$

Notice how his equation can also be established *directly* using the results of Table 12–2 for moment.

Slope and Elastic Curve. Integrating twice yields

$$EI\frac{d^2v}{dx^2} = -8x + 6\langle x - 10\rangle^1$$

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x - 10\rangle^2 + C_1$$

$$EIv = -\frac{4}{3}x^3 + \langle x - 10\rangle^3 + C_1x + C_2 \qquad (1)$$

From Eq. 1, the boundary condition $v = 0$ at $x = 10$ ft and $v = 0$ at $x = 30$ ft gives

$$0 = -1333 + (10 - 10)^3 + C_1(10) + C_2$$

$$0 = -36\,000 + (30 - 10)^3 + C_1(30) + C_2$$

Solving these equations simultaneously for C_1 and C_2, we get $C_1 = 1333$ and $C_2 = -12\,000$. Thus,

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x - 10\rangle^2 + 1333 \qquad (2)$$

$$EIv = -\frac{4}{3}x^3 + \langle x - 10\rangle^3 + 1333x - 12\,000 \qquad (3)$$

From Fig. 12–18*a*, maximum displacement may occur either at *C*, or at *D*, where the slope $dv/dx = 0$. To obtain the displacement of *C*, set $x = 0$ in Eq. 3. We get

$$v_C = -\frac{12\,000 \text{ kip}\cdot\text{ft}^3}{EI} \qquad \textit{Ans.}$$

The *negative* sign indicates that the displacement is *downward* as shown in Fig. 12–18*a*. To locate point *D*, use Eq. 2 with $x > 10$ ft and $dv/dx = 0$. This gives

$$0 = -4x_D^2 + 3(x_D - 10)^2 + 1333$$

$$x_D^2 + 60x_D - 1633 = 0$$

Solving for the positive root,

$$x_D = 20.3 \text{ ft}$$

Hence, from Eq. 3,

$$EIv_D = -\frac{4}{3}(20.3)^3 + (20.3 - 10)^3 + 1333(20.3) - 12\,000$$

$$v_D = \frac{5006 \text{ kip}\cdot\text{ft}^3}{EI}$$

Comparing this value with v_C, we see that $v_{max} = v_C$.

12

EXAMPLE 12.6

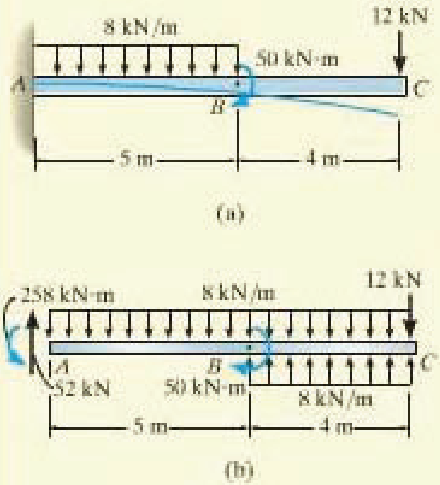


Fig. 12–19

Determine the equation of the elastic curve for the cantilevered beam shown in Fig. 12–19*a*. EI is constant.

SOLUTION

Elastic Curve. The loads cause the beam to deflect as shown in Fig. 12–19*a*. The boundary conditions require zero slope and displacement at A.

Loading Function. The support reactions at A have been calculated and are shown on the free-body diagram in Fig. 12–19*b*. Since the distributed loading in Fig. 12–19*a* does not extend to C as required, we can use the superposition of loadings shown in Fig. 12–19*b* to represent the same effect. By our sign convention, the beam's loading is therefore

$$w = 52\text{ kN}\langle x-0\rangle^{-1} - 258\text{ kN}\cdot\text{m}\langle x-0\rangle^{-2} - 8\text{ kN/m}\langle x-0\rangle^{0} + 50\text{ kN}\cdot\text{m}\langle x-5\text{ m}\rangle^{-2} + 8\text{ kN/m}\langle x-5\text{ m}\rangle^{0}$$

The 12-kN load is *not included* here, since x cannot be greater than 9 m. Because $dV/dx = w(x)$, then by integrating, neglecting the constant of integration since the reactions are included in the load function, we have

$$V = 52\langle x-0\rangle^{0} - 258\langle x-0\rangle^{-1} - 8\langle x-0\rangle^{1} + 50\langle x-5\rangle^{-1} + 8\langle x-5\rangle^{1}$$

Furthermore, $dM/dx = V$, so that integrating again yields

$$M = -258\langle x-0\rangle^{0} + 52\langle x-0\rangle^{1} - \frac{1}{2}(8)\langle x-0\rangle^{2} + 50\langle x-5\rangle^{0} + \frac{1}{2}(8)\langle x-5\rangle^{2}$$
$$= (-258 + 52x - 4x^2 + 50\langle x-5\rangle^{0} + 4\langle x-5\rangle^{2})\text{ kN}\cdot\text{m}$$

This same result can be obtained *directly* from Table 12–2.

Slope and Elastic Curve. Applying Eq. 12–10 and integrating twice, we have

$$EI\frac{d^2v}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^{0} + 4\langle x-5\rangle^{2}$$

$$EI\frac{dv}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5\rangle^{1} + \frac{4}{3}\langle x-5\rangle^{3} + C_1$$

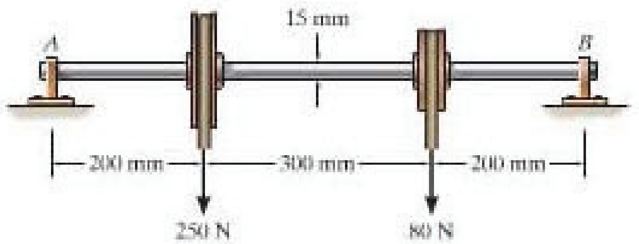
$$EIv = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^{2} + \frac{1}{3}\langle x-5\rangle^{4} + C_1x + C_2$$

Since $dv/dx = 0$ at $x = 0$, $C_1 = 0$; and $v = 0$ at $x = 0$, so $C_2 = 0$. Thus,

$$v = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^{2} + \frac{1}{3}\langle x-5\rangle^{4}\right)\text{ m} \qquad \textit{Ans.}$$

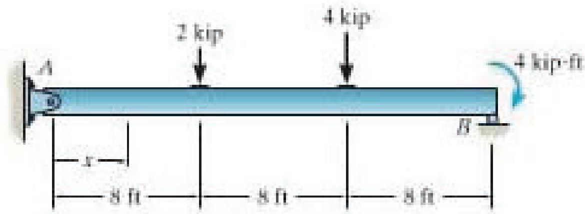
PROBLEMS

12–35. The shaft is made of steel and has a diameter of 15 mm. Determine its maximum deflection. The bearings at A and B exert only vertical reactions on the shaft. $E_{st} = 200$ GPa.



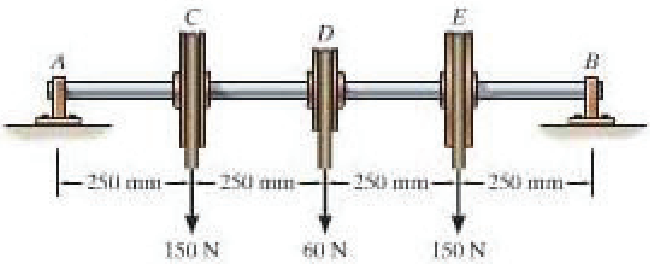
Prob. 12–35

*12–36. The beam is subjected to the loads shown. Determine the equation of the elastic curve. EI is constant.



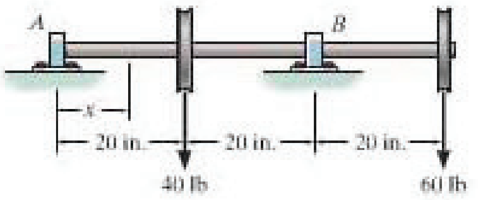
Prob. 12–36

•**12–37.** Determine the deflection at each of the pulleys C, D, and E. The shaft is made of steel and has a diameter of 30 mm. The bearings at A and B exert only vertical reactions on the shaft. $E_{st} = 200$ GPa.



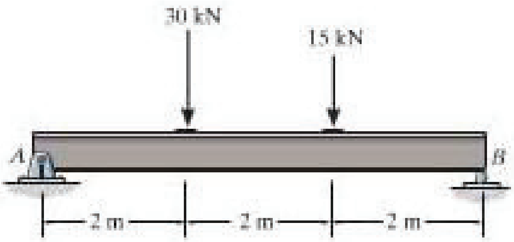
Prob. 12–37

12–38. The shaft supports the two pulley loads shown. Determine the equation of the elastic curve. The bearings at A and B exert only vertical reactions on the shaft. EI is constant.



Prob. 12–38

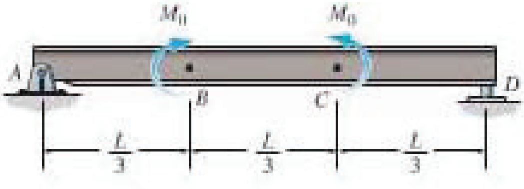
12–39. Determine the maximum deflection of the simply supported beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.



Prob. 12–39

*12–40. Determine the eqution of the elastic curve, the slope at A, and the deflection at B of the simply supported beam. EI is constant.

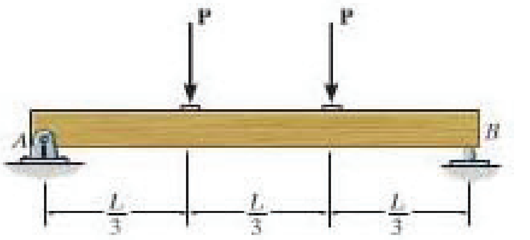
•**12–41.** Determine the equation of the elastic curve and the maximum deflection of the simply supported beam. EI is constant.



Probs. 12–40/41

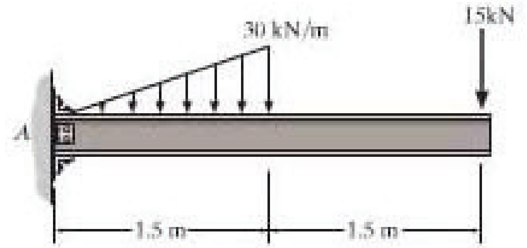
12

12–42. Determine the equation of the elastic curve, the slope at A, and the maximum deflection of the simply supported beam. EI is constant.



Prob. 12–42

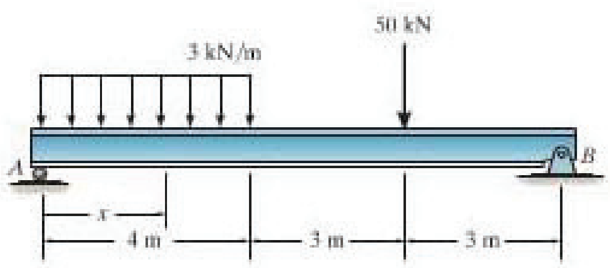
12–43. Determine the maximum deflection of the cantilevered beam. The beam is made of material having an $E = 200$ GPa and $I = 65.0(10^6)$ mm^6.



Prob. 12–43

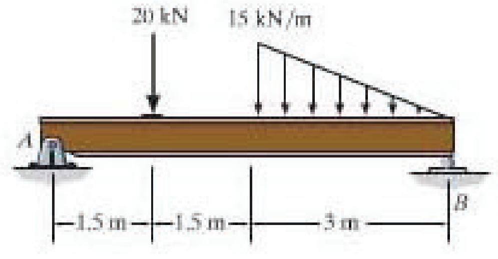
***12–44.** The beam is subjected to the load shown. Determine the equation of the elastic curve. EI is constant.

•**12–45.** The beam is subjected to the load shown. Determine the displacement at $x = 7$ m and the slope at A. EI is constant.



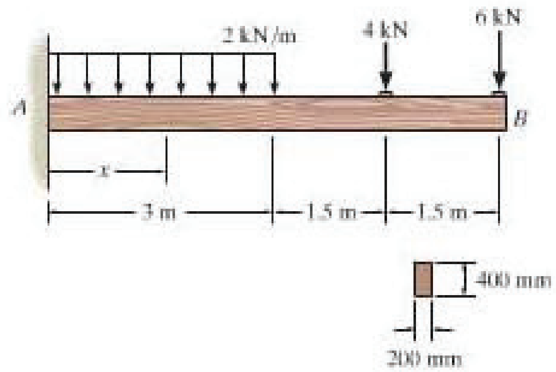
Probs. 12–44/45

12–46. Determine the maximum deflection of the simply supported beam. $E = 200$ GPa and $I = 65.0(10^6)$ mm^4.



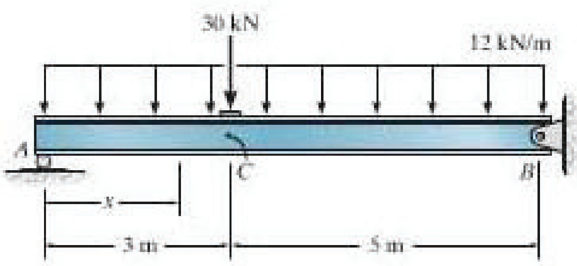
Prob. 12–46

12–47. The wooden beam is subjected to the load shown. Determine the equation of the elastic curve. If $E_w = 12$ GPa, determine the deflection and the slope at end B.



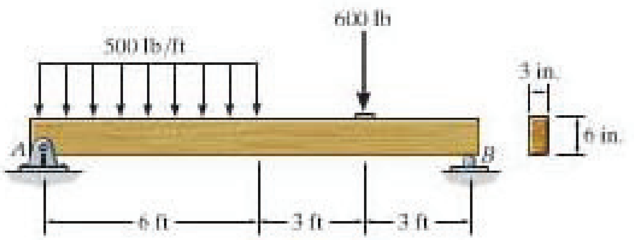
Prob. 12–47

***12–48.** The beam is subjected to the load shown. Determine the slopes at A and B and the displacement at C. EI is constant.



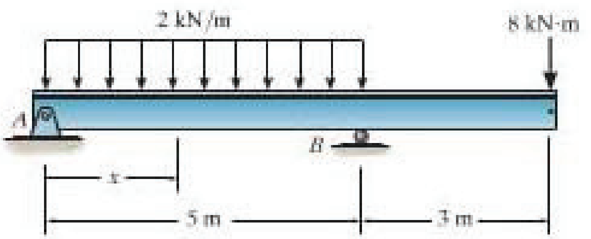
Prob. 12–48

•**12–49.** Determine the equation of the elastic curve of the simply supported beam and then find the maximum deflection. The beam is made of wood having a modulus of elasticity $E = 1.5(10^3)$ ksi.



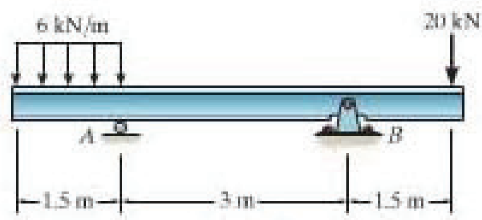
Prob. 12–49

12–50. The beam is subjected to the load shown. Determine the equations of the slope and elastic curve. EI is constant.



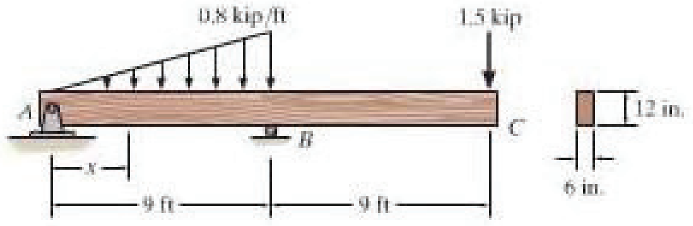
Prob. 12–50

12–51. The beam is subjected to the load shown. Determine the equation of the elastic curve. EI is constant.



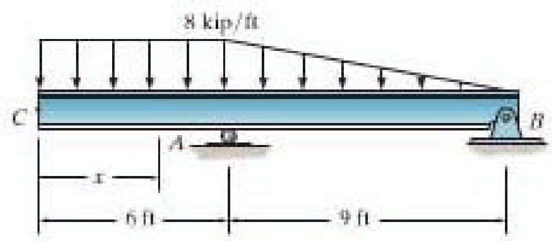
Prob. 12–51

***12–52.** The wooden beam is subjected to the load shown. Determine the equation of the elastic curve. Specify the deflection at the end C. $E_w = 1.6(10^3)$ ksi.



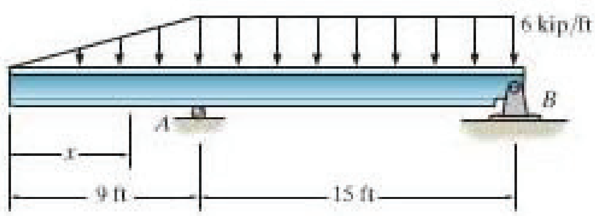
Prob. 12–52

12–53. Determine the displacement at C and the slope at A of the beam.



Prob. 12–53

12–54. The beam is subjected to the load shown. Determine the equation of the elastic curve. EI is constant.



Prob. 12–54

*12.4 Slope and Displacement by the Moment-Area Method

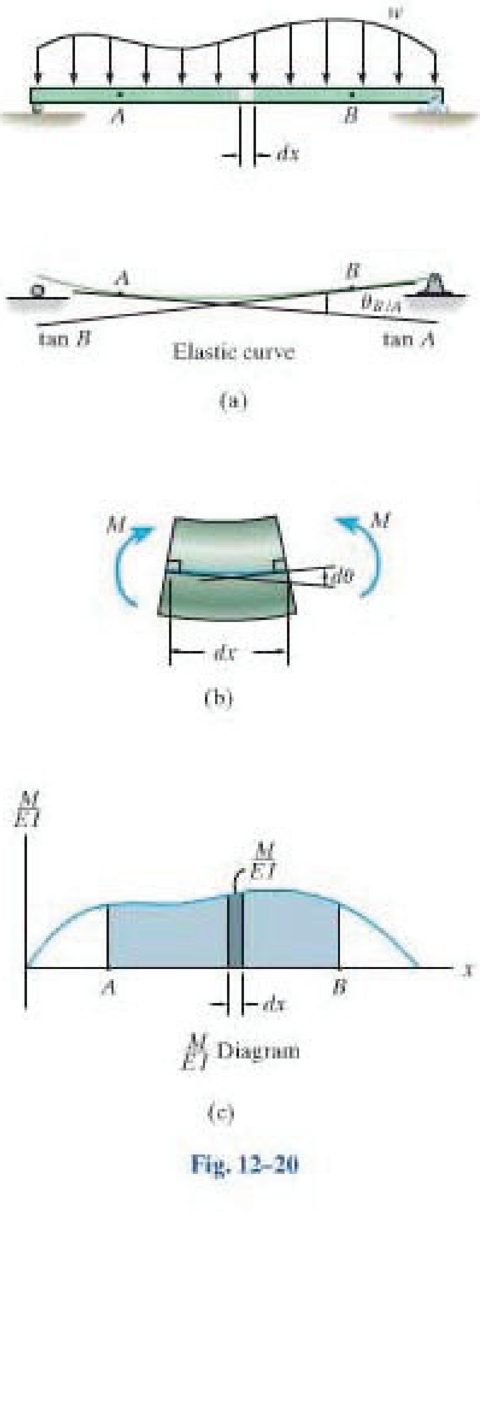


Fig. 12–20

The moment-area method provides a semigraphical technique for finding the slope and displacement at specific points on the elastic curve of a beam or shaft. Application of the method requires calculating areas associated with the beam's moment diagram; and so if this diagram consists of simple shapes, the method is very convenient to use. Normally this is the case when the beam is loaded with concentrated forces and couple moments.

To develop the moment-area method we will make the same assumptions we used for the method of integration: The beam is initially straight, it is elastically deformed by the loads, such that the slope and deflection of the elastic curve are very small, and the deformations are only caused by bending. The moment-area method is based on two theorems, one used to determine the slope and the other to determine the displacement at a point on the elastic curve.

Theorem 1. Consider the simply supported beam with its associated elastic curve, shown in Fig. 12–20*a*. A differential segment dx of the beam is isolated in Fig. 12–20*b*. Here the beam's internal moment M deforms the element such that the *tangents* to the elastic curve at each side of the element intersect at an angle $d\theta$. This angle can be determined from Eq. 12–10, written as

$$EI\frac{d^2v}{dx^2} = EI\frac{d}{dx}\left(\frac{dv}{dx}\right) = M$$

Since the *slope* is *small*, $\theta = dv/dx$, and therefore

$$d\theta = \frac{M}{EI}dx \qquad (12\text{–}16)$$

If the moment diagram for the beam is constructed and divided by the flexural rigidity, EI, Fig. 12–20*c*, then this equation indicates that $d\theta$ is equal to the *area* under the "M/EI diagram" for the beam segment dx. Integrating from a selected point A on the elastic curve to another point B, we have

$$\theta_{B/A} = \int_A^B \frac{M}{EI}dx \qquad (12\text{–}17)$$

This equation forms the basis for the first moment-area theorem.

Theorem 1: *The angle between the tangents at any two points on the elastic curve equals the area under the M/EI diagram between these two points.*

The notation $\theta_{B/A}$ is referred to as the angle of the tangent at B measured *with respect to* the tangent at A. From the proof it should be evident that this angle is measured *counterclockwise*, from tangent A to tangent B, if the area under the M/EI diagram is *positive*. Conversely, if the area is *negative*, or lies below the x axis, the angle $\theta_{B/A}$ is measured clockwise from tangent A to tangent B. Furthermore, from the dimensions of Eq. 12–17, $\theta_{B/A}$ will be in *radians*.

Theorem 2. The second moment-area theorem is based on the relative deviation of tangents to the elastic curve. Shown in Fig. 12–21*a* is a greatly exaggerated view of the vertical deviation dt of the tangents on each side of the differential element dx. This deviation is caused by the curvature of the element and has been measured along a vertical line passing through point A on the elastic curve. Since the slope of the elastic curve and its deflection are assumed to be very small, it is satisfactory to approximate the length of each tangent line by x and the arc ds' by dt. Using the circular-arc formula $s = \theta r$, where r is the length x and s is dt, we can write $dt = x\,d\theta$. Substituting Eq. 12–16 into this equation and integrating from A to B, the vertical deviation of the tangent at A *with respect to* the tangent at B can then be determined; that is,

$$t_{A/B} = \int_A^B x\frac{M}{EI}dx \qquad (12\text{–}18)$$

Since the centroid of an area is found from $\bar{x}\int dA = \int x\,dA$, and $\int(M/EI)\,dx$ represents the area under the M/EI diagram, we can also write

$$t_{A/B} = \bar{x}\int_A^B \frac{M}{EI}dx \qquad (12\text{–}19)$$

Here $\bar{x}$ is the distance from A to the *centroid* of the area under the M/EI diagram between A and B, Fig. 12–21*b*.

The second moment-area theorem can now be stated in reference to Fig. 12–21*a* as follows:

Theorem 2: *The vertical distance between the tangent at a point (A) on the elastic curve and the tangent extended from another point (B) equals the moment of the area under the M/EI diagram between these two points (A and B). This moment is calculated about the point (A) where the vertical distance ($t_{A/B}$) is to be determined.*

Note that $t_{A/B}$ is *not* equal to $t_{B/A}$, which is shown in Fig. 12–21*c*. Specifically, the moment of the area under the M/EI diagram between A and B is calculated about point A to determine $t_{A/B}$, Fig. 12–21*b*, and it is calculated about point B to determine $t_{B/A}$, Fig. 12–21*c*.

If the moment of a *positive* M/EI area between A and B is found for $t_{A/B}$, it indicates that point A is *above* the tangent extended from point B, Fig. 12–21*a*. Similarly, *negative* M/EI areas indicate that point A is *below* the tangent extended from point B. This same rule applies for $t_{B/A}$.

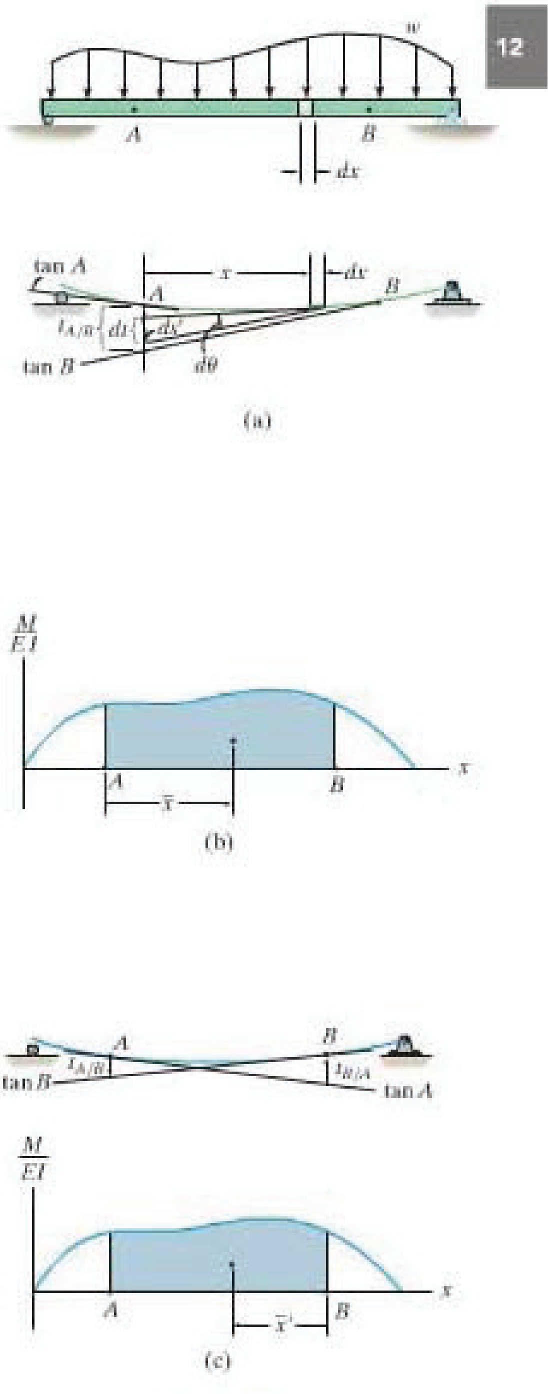


Fig. 12–21

Procedure for Analysis

The following procedure provides a method that may be used to apply the two moment-area theorems.

M/EI Diagram.

- Determine the support reactions and draw the beam's M/EI diagram. If the beam is loaded with concentrated forces, the M/EI diagram will consist of a series of straight line segments, and the areas and their moments required for the moment-area theorems will be relatively easy to calculate. If the loading consists of a series of distributed loads, the M/EI diagram will consist of parabolic or perhaps higher-order curves, and it is suggested that the table on the inside front cover be used to locate the area and centroid under each curve.

Elastic Curve.

- Draw an exaggerated view of the beam's elastic curve. Recall that points of zero slope and zero displacement always occur at a fixed support, and zero displacement occurs at all pin and roller supports.
- If it becomes difficult to draw the general shape of the elastic curve, use the moment (or M/EI) diagram. Realize that when the beam is subjected to a *positive moment*, the beam bends *concave up*, whereas *negative moment* bends the beam *concave down*. Furthermore, an inflection point or change in curvature occurs where the moment in the beam (or M/EI) is zero.
- The unknown displacement and slope to be determined should be indicated on the curve.
- Since the moment-area theorems apply *only between two tangents*, attention should be given as to which tangents should be constructed so that the angles or vertical distance between them will lead to the solution of the problem. In this regard, *the tangents at the supports should be considered*, since the beam has zero displacement and/or zero slope at the supports.

Moment-Area Theorems.

- Apply Theorem 1 to determine the *angle* between any two tangents on the elastic curve and Theorem 2 to determine the vertical distance between the tangents.
- The algebraic sign of the answer can be checked from the angle or vertical distance indicated on the elastic curve.
- A *positive* $\theta_{B/A}$ represents a *counterclockwise* rotation of the tangent at B with respect to the tangent at A, and a *positive* $t_{B/A}$ indicates that point B on the elastic curve lies *above* the extended tangent from point A.

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EXAMPLE 12.7

Determine the slope of the beam shown in Fig. 12–22a at point B. EI is constant.

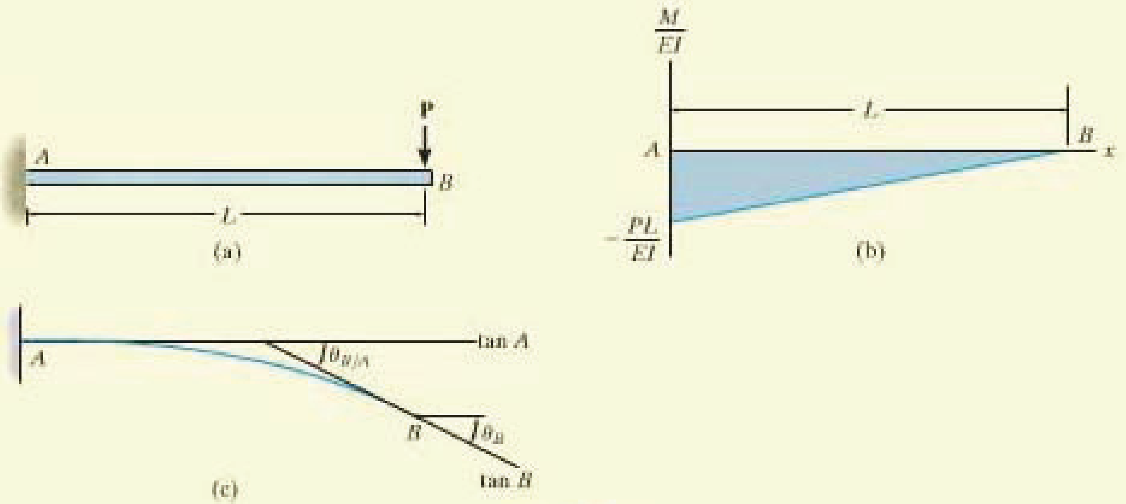


Fig. 12–22

SOLUTION

M/EI Diagram. See Fig. 12–22b.

Elastic Curve. The force **P** causes the beam to deflect as shown in Fig. 12–22c. (The elastic curve is concave downward, since M/EI is negative.) The tangent at B is indicated since we are required to find θ_B. Also, the tangent at the support (A) is shown. This tangent has a *known* zero slope. By the construction, the angle between tan A and tan B, that is, $\theta_{B/A}$, is equivalent to θ_B, or

$$\theta_B = \theta_{B/A}$$

Moment-Area Theorem. Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between points A and B; that is,

$$\theta_B = \theta_{B/A} = \frac{1}{2}\left(-\frac{PL}{EI}\right)L$$

$$= -\frac{PL^2}{2EI} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle measured from the tangent at A to the tangent at B is *clockwise*. This checks, since the beam slopes downward at B.

12

EXAMPLE 12.8

Determine the displacement of points B and C of the beam shown in Fig. 12–23a. EI is constant.

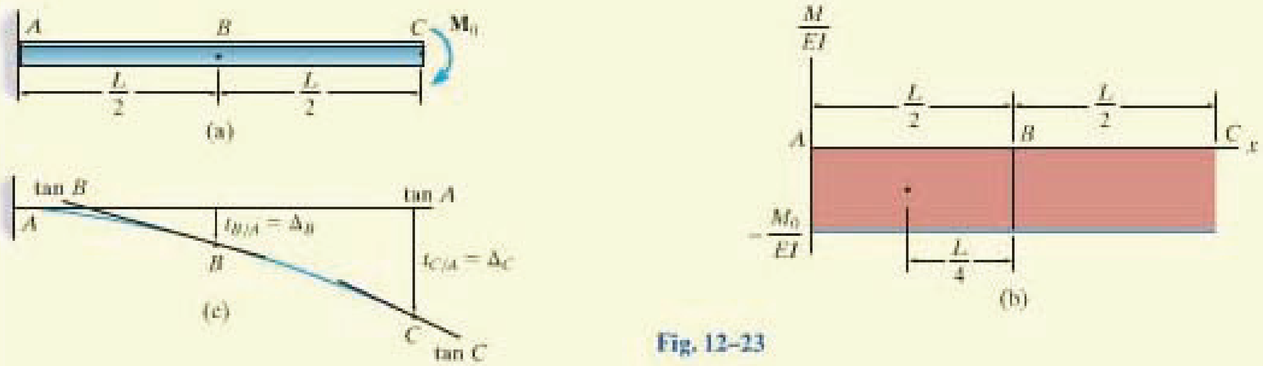


Fig. 12–23

SOLUTION

***M/EI* Diagram.** See Fig. 12–23b.

Elastic Curve. The couple moment at C causes the beam to deflect as shown in Fig. 12–23c. The tangents at B and C are indicated since we are required to find Δ_B and Δ_C. Also, the tangent at the support (A) is shown since it is horizontal. The required displacements can now be related directly to the vertical distance between the tangents at B and A and C and A. Specifically,

$$\Delta_B = t_{B/A}$$

$$\Delta_C = t_{C/A}$$

Moment-Area Theorem. Applying Theorem 2, $t_{B/A}$ is equal to the moment of the shaded area under the M/EI diagram between A and B calculated about point B (the point on the elastic curve), since this is the point where the vertical distance is to be determined. Hence, from Fig. 12–23b,

$$\Delta_B = t_{B/A} = \left(\frac{L}{4}\right)\left[\left(-\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\right] = -\frac{M_0L^2}{8EI} \qquad \textit{Ans.}$$

Likewise, for $t_{C/A}$ we must determine the moment of the area under the *entire* M/EI diagram from A to C about point C (the point on the elastic curve). We have

$$\Delta_C = t_{C/A} = \left(\frac{L}{2}\right)\left[\left(-\frac{M_0}{EI}\right)(L)\right] = -\frac{M_0L^2}{2EI} \qquad \textit{Ans.}$$

NOTE: Since both answers are *negative*, they indicate that points B and C lie *below* the tangent at A. This checks with Fig. 12–23c.

EXAMPLE 12.9

Determine the slope at point C of the shaft in Fig. 12–24a. EI is constant.

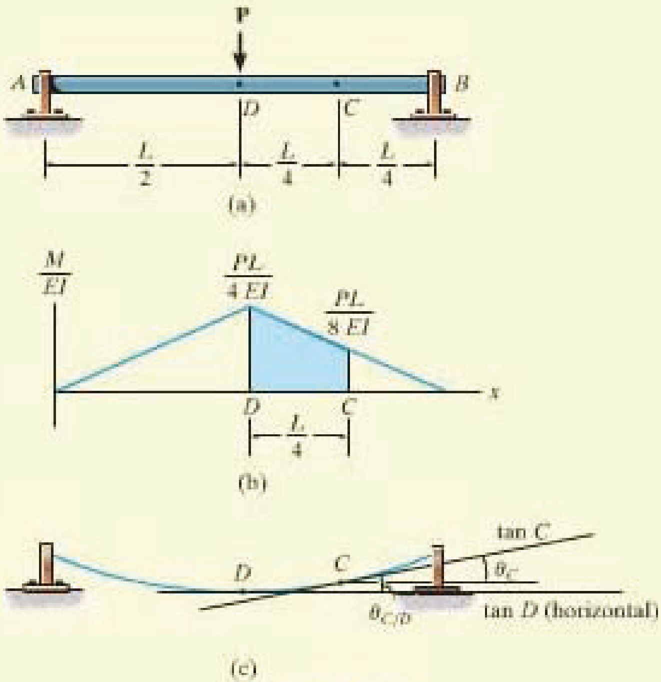


Fig. 12–24

SOLUTION

M/EI Diagram. See Fig. 12–24b.

Elastic Curve. Since the loading is applied symmetrically to the beam, the elastic curve is symmetric, and the tangent at D is horizontal, Fig. 12–24c. Also the tangent at C is drawn, since we must find the slope θ_C. By the construction, the angle $\theta_{C/D}$ between the tangents at tan D and C is equal to θ_C; that is,

$$\theta_C = \theta_{C/D}$$

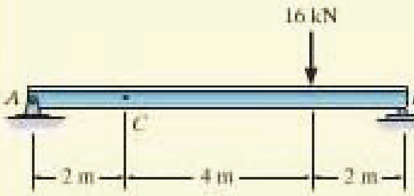
Moment-Area Theorem. Using Theorem 1, $\theta_{C/D}$ is equal to the shaded area under the M/EI diagram between points D and C. We have

$$\theta_C = \theta_{C/D} = \left(\frac{PL}{8EI}\right)\left(\frac{L}{4}\right) + \frac{1}{2}\left(\frac{PL}{4EI} - \frac{PL}{8EI}\right)\left(\frac{L}{4}\right) = \frac{3PL^2}{64EI} \qquad \textit{Ans.}$$

What does the positive result indicate?

12

EXAMPLE 12.10



Determine the slope at point C for the steel beam in Fig. 12–25*a*. Take $E_{st} = 200$ GPa, $I = 17(10^6)\ \text{mm}^4$.

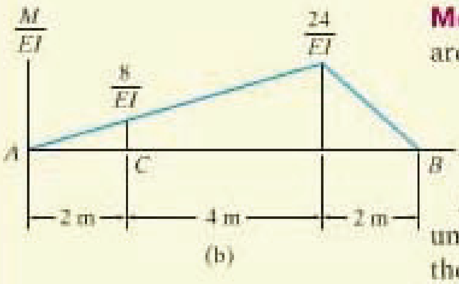
SOLUTION

***M/EI* Diagram.** See Fig. 12–25*b*.

Elastic Curve. The elastic curve is shown in Fig. 12–25*c*. The tangent at C is shown since we are required to find θ_C. Tangents at the *supports*, A and B, are also constructed as shown. Angle $\theta_{C/A}$ is the angle between the tangents at A and C. The slope at A, θ_A, in Fig. 12–25*c* can be found using $|\theta_A| = |t_{B/A}|/L_{AB}$. This equation is valid since $t_{B/A}$ is actually very small, so that $t_{B/A}$ in meters can be approximated by the length of a circular arc defined by a radius of $L_{AB} = 8$ m and a sweep of θ_A in radians. (Recall that $s = \theta r$.) From the geometry of Fig. 12–25*c*, we have

$$|\theta_C| = |\theta_A| - |\theta_{C/A}| = \left|\frac{t_{B/A}}{8}\right| - |\theta_{C/A}| \qquad (1)$$

Note that Example 12.9 could also be solved using this method.



Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(2\text{ m})\left(\frac{8\text{ kN}\cdot\text{m}}{EI}\right) = \frac{8\text{ kN}\cdot\text{m}^2}{EI}$$

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B (the point on the elastic curve), since this is the point where the vertical distance is to be determined. We have

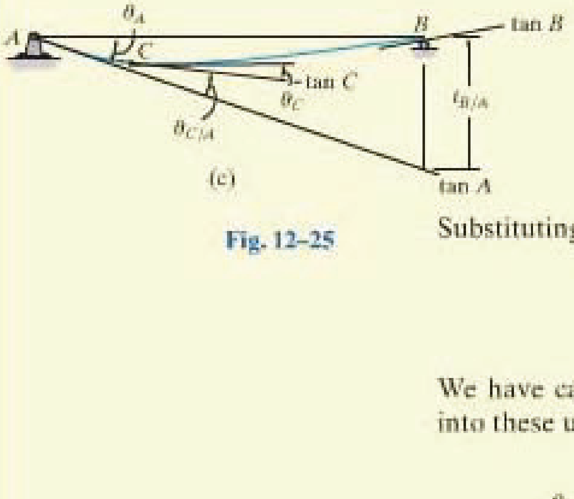


Fig. 12–25

$$t_{B/A} = \left(2\text{ m} + \frac{1}{3}(6\text{ m})\right)\left[\frac{1}{2}(6\text{ m})\left(\frac{24\text{ kN}\cdot\text{m}}{EI}\right)\right] + \left(\frac{2}{3}(2\text{ m})\right)\left[\frac{1}{2}(2\text{ m})\left(\frac{24\text{ kN}\cdot\text{m}}{EI}\right)\right] = \frac{320\text{ kN}\cdot\text{m}^3}{EI}$$

Substituting these results into Eq. 1, we get

$$\theta_C = \frac{320\text{ kN}\cdot\text{m}^2}{(8\text{ m})EI} - \frac{8\text{ kN}\cdot\text{m}^2}{EI} = \frac{32\text{ kN}\cdot\text{m}^2}{EI} \curvearrowright$$

We have calculated this result in units of kN and m, so converting EI into these units, we have

$$\theta_C = \frac{32\text{ kN}\cdot\text{m}^2}{[200(10^6)\text{ kN/m}^2][17(10^{-6})\text{ m}^4]} = 0.00941\text{ rad} \curvearrowright \qquad \textit{Ans.}$$

EXAMPLE 12.11

Determine the displacement at C for the beam shown in Fig. 12–26a. EI is constant.

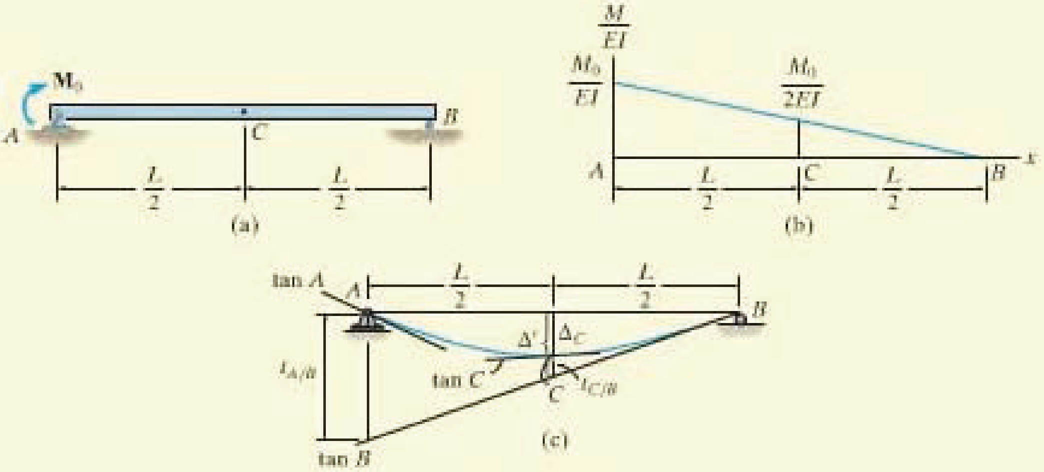


Fig. 12–26

SOLUTION

***M/EI* Diagram.** See Fig. 12–26b.

Elastic Curve. The tangent at C is drawn on the elastic curve since we are required to find Δ_C, Fig. 12–26c. (Note that C is *not* the location of the maximum deflection of the beam, because the loading and hence the elastic curve are *not symmetric*.) Also indicated in Fig. 12–26c are the tangents at the supports A and B. It is seen that $\Delta_C = \Delta' - t_{C/B}$. If $t_{A/B}$ is determined, then Δ' can be found from proportional triangles, that is, $\Delta'/(L/2) = t_{A/B}/L$ or $\Delta' = t_{A/B}/2$. Hence,

$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B} \qquad (1)$$

Moment-Area Theorem. Applying Theorem 2 to determine $t_{A/B}$ and $t_{C/B}$, we have

$$t_{A/B} = \left(\frac{1}{3}(L)\right)\left[\frac{1}{2}(L)\left(\frac{M_0}{EI}\right)\right] = \frac{M_0L^2}{6EI}$$

$$t_{C/B} = \left(\frac{1}{3}\left(\frac{L}{2}\right)\right)\left[\frac{1}{2}\left(\frac{L}{2}\right)\left(\frac{M_0}{2EI}\right)\right] = \frac{M_0L^2}{48EI}$$

Substituting these results into Eq. 1 gives

$$\Delta_C = \frac{1}{2}\left(\frac{M_0L^2}{6EI}\right) - \left(\frac{M_0L^2}{48EI}\right)$$

$$= \frac{M_0L^2}{16EI} \downarrow \qquad \textit{Ans.}$$

12

EXAMPLE 12.12

Determine the displacement at point C for the steel overhanging beam shown in Fig. 12–27a. Take $E_{st} = 29(10^3)$ ksi, $I = 125$ in^4.

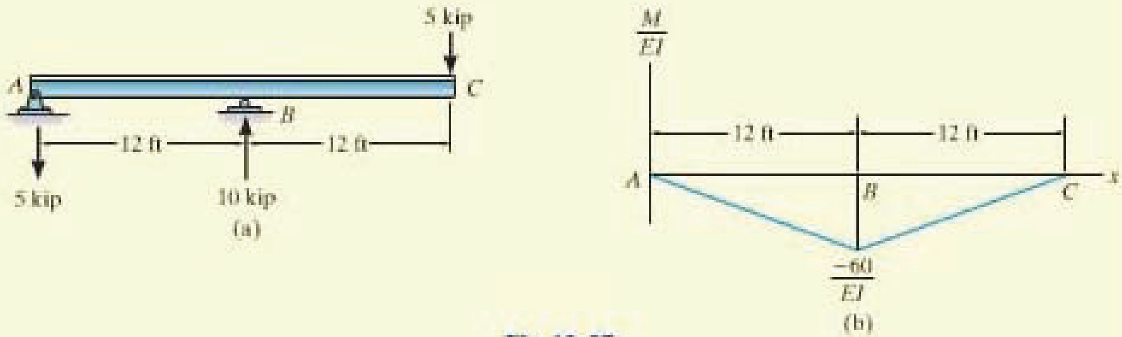


Fig. 12–27

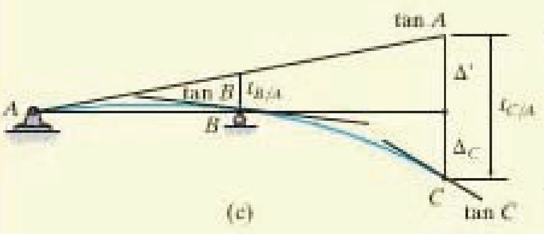
SOLUTION

***M/EI* Diagram.** See Fig. 12–27b.

Elastic Curve. The loading causes the beam to deflect as shown in Fig. 12–27c. We are required to find Δ_C. By constructing tangents at C and at the supports A and B, it is seen that $\Delta_C = |t_{C/A}| - \Delta'$. However, Δ' can be related to $t_{B/A}$ by proportional triangles; that is, $\Delta'/24 = |t_{B/A}|/12$ or $\Delta' = 2|t_{B/A}|$. Hence

$$\Delta_C = |t_{C/A}| - 2|t_{B/A}| \qquad (1)$$

Moment-Area Theorem. Applying Theorem 2 to determine $t_{C/A}$ and $t_{B/A}$, we have



$$t_{C/A} = (12\text{ ft})\left(\frac{1}{2}(24\text{ ft})\left(-\frac{60\text{ kip}\cdot\text{ft}}{EI}\right)\right)$$

$$= -\frac{8640\text{ kip}\cdot\text{ft}^3}{EI}$$

$$t_{B/A} = \left(\frac{1}{3}(12\text{ ft})\right)\left[\frac{1}{2}(12\text{ ft})\left(-\frac{60\text{ kip}\cdot\text{ft}}{EI}\right)\right] = -\frac{1440\text{ kip}\cdot\text{ft}^3}{EI}$$

Why are these terms negative? Substituting the results into Eq. 1 yields

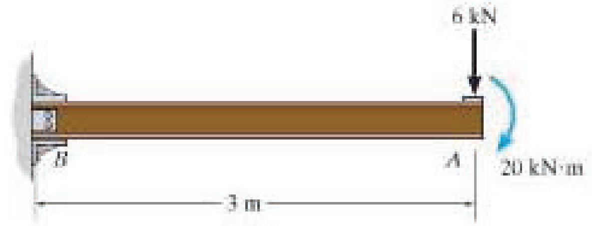
$$\Delta_C = \frac{8640\text{ kip}\cdot\text{ft}^3}{EI} - 2\left(\frac{1440\text{ kip}\cdot\text{ft}^3}{EI}\right) = \frac{5760\text{ kip}\cdot\text{ft}^3}{EI}\downarrow$$

Realizing that the calculations were made in units of kip and ft, we have

$$\Delta_C = \frac{5760\text{ kip}\cdot\text{ft}^3(1728\text{ in}^3/\text{ft}^3)}{[29(10^3)\text{ kip/in}^2](125\text{ in}^4)} = 2.75\text{ in.}\downarrow \qquad \textit{Ans.}$$

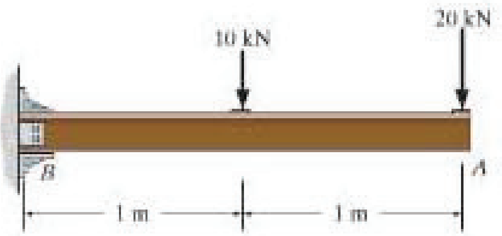
FUNDAMENTAL PROBLEMS

F12–7. Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 65.0(10^{-6})$ m^4.



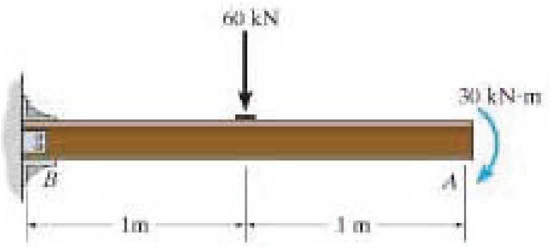
F12–7

F12–8. Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 126(10^{-6})$ m^4.



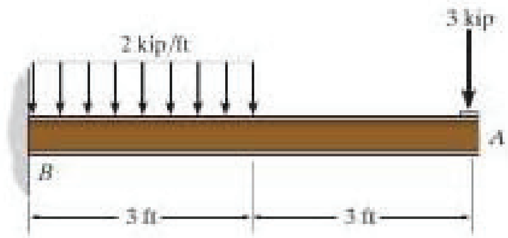
F12–8

F12–9. Determine the slope and deflection of end A of the cantilevered beam. $E = 200$ GPa and $I = 121(10^{-6})$ m^4.



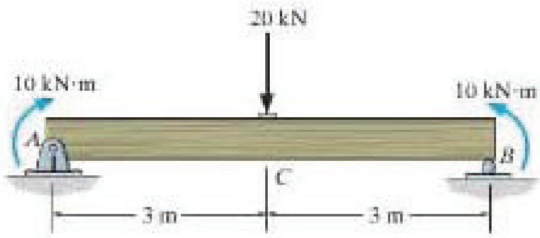
F12–9

F12–10. Determine the slope and deflection at A of the cantilevered beam. $E = 29(10^3)$ ksi, $I = 24.5$ in^4.



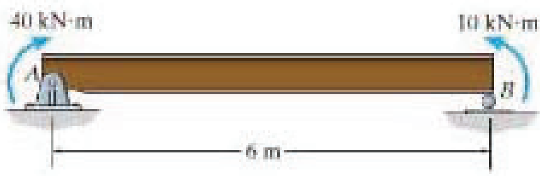
F12–10

F12–11. Determine the maximum deflection of the simply supported beam. $E = 200$ GPa and $I = 42.8(10^{-6})$ m^4.



F12–11

F12–12. Determine the maximum deflection of the simply supported beam. $E = 200$ GPa and $I = 39.9(10^{-6})$ m^4.

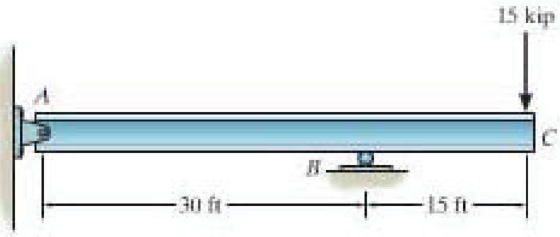


F12–12

12

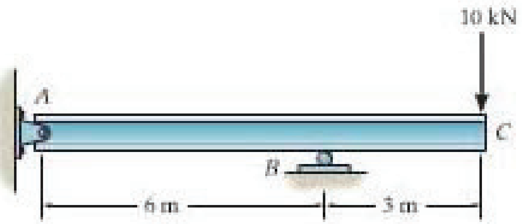
PROBLEMS

12–55. Determine the slope and deflection at C. EI is constant.



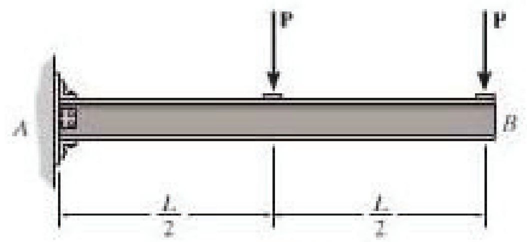
Prob. 12–55

*__12–56.__ Determine the slope and deflection at C. EI is constant.



Prob. 12–56

•**12–57.** Determine the deflection of end B of the cantilever beam. E is constant.

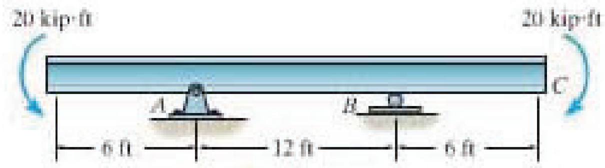


Prob. 12–57

12–58. Determine the slope at A and the maximum deflection. EI is constant.

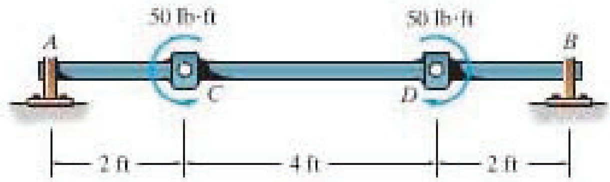
Prob. 12–58

12–59. Determine the slope and deflection at C. EI is constant.



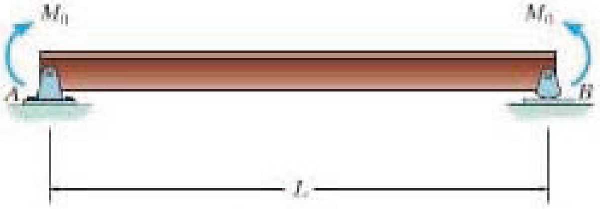
Prob. 12–59

*__12–60.__ If the bearings at A and B exert only vertical reactions on the shaft, determine the slope at A and the maximum deflection of the shaft. EI is constant.



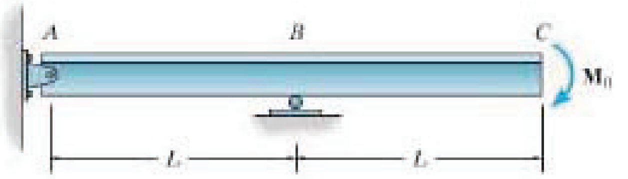
Prob. 12–60

•**12–61.** Determine the maximum slope and the maximum deflection of the beam. EI is constant.



Prob. 12–61

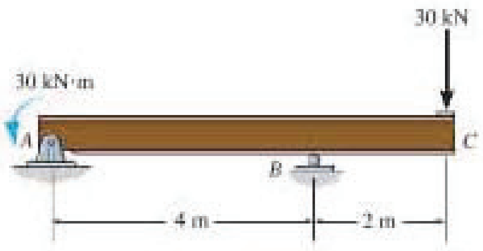
12–62. Determine the deflection and slope at C. EI is constant.



Prob. 12–62

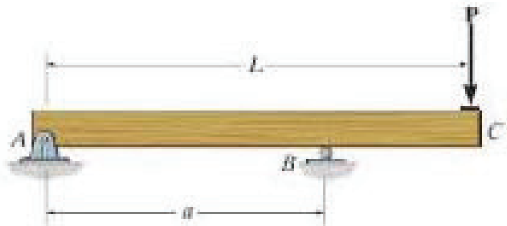
12–63. Determine the slope at A of the overhang beam. $E = 200$ GPa and $I = 45.5(10^6)$ mm^4.

*__12–64.__ Determine the deflection at C of the overhang beam. $E = 200$ GPa and $I = 45.5(10^6)$ mm^4.



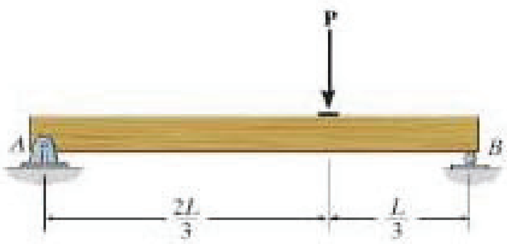
Probs. 12–63/64

•**12–65.** Determine the position a of roller support B in terms of L so that the deflection at end C is the same as the maximum deflection of region AB of the overhang beam. EI is constant.



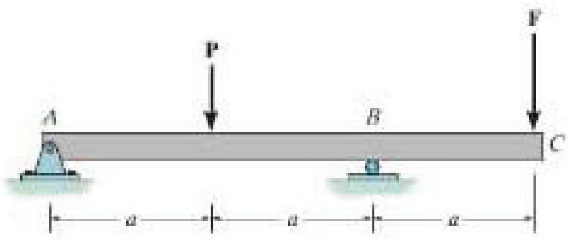
Prob. 12–65

12–66. Determine the slope at A of the simply supported beam. EI is constant.



Prob. 12–66

12–67. The beam is subjected to the load **P** as shown. Determine the magnitude of force **F** that must be applied at the end of the overhang C so that the deflection at C is zero. EI is constant.

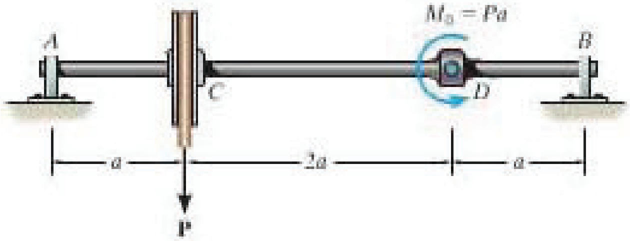


Prob. 12–67

12

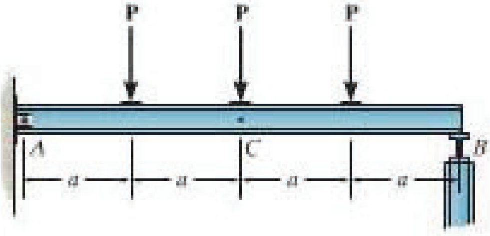
12

*12–68. If the bearings at A and B exert only vertical reactions on the shaft, determine the slope at A and the maximum deflection.



Prob. 12–68

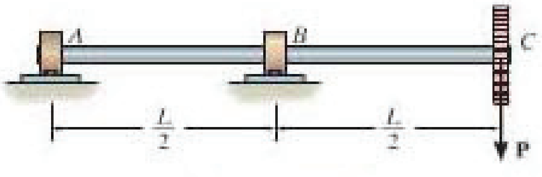
•12–69. The beam is subjected to the loading shown. Determine the slope at A and the displacement at C. Assume the support at A is a pin and B is a roller. EI is constant.



Prob. 12–69

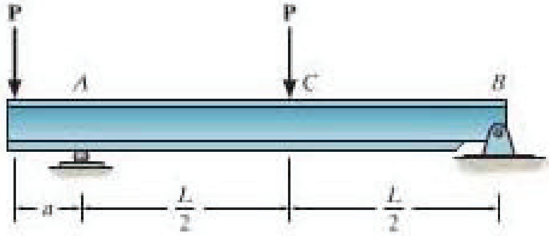
12–70. The shaft supports the gear at its end C. Determine the deflection at C and the slopes at the bearings A and B. EI is constant.

12–71. The shaft supports the gear at its end C. Determine its maximum deflection within region AB. EI is constant. The bearings exert only vertical reactions on the shaft.



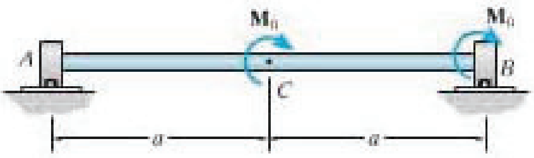
Probs. 12–70/71

*12–72. Determine the value of a so that the displacement at C is equal to zero. EI is constant.



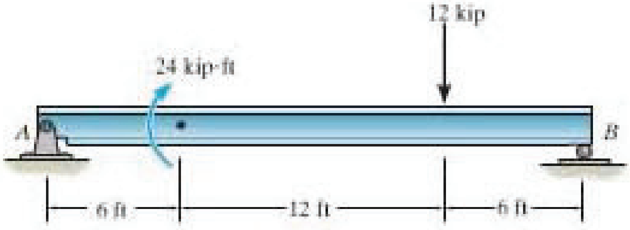
Prob. 12–72

•12–73. The shaft is subjected to the loading shown. If the bearings at A and B only exert vertical reactions on the shaft, determine the slope at A and the displacement at C. EI is constant.



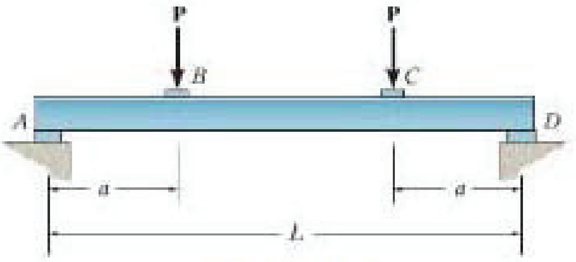
Prob. 12–73

12–74. Determine the slope at A and the maximum deflection in the beam. EI is constant.



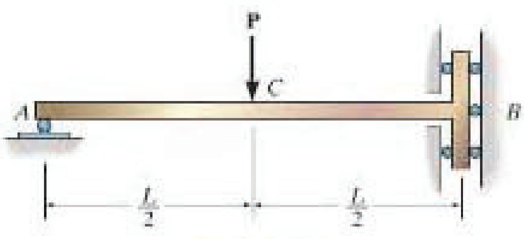
Prob. 12–74

12–75. The beam is made of a ceramic material. In order to obtain its modulus of elasticity, it is subjected to the elastic loading shown. If the moment of inertia is I and the beam has a measured maximum deflection Δ, determine E. The supports at A and D exert only vertical reactions on the beam.



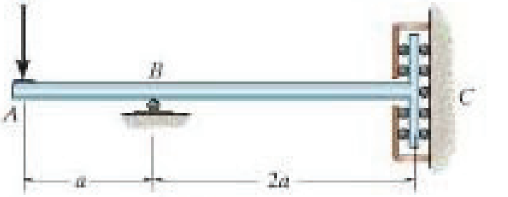
Prob. 12–75

***12–76.** The bar is supported by a roller constraint at B, which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope at A and the deflection at C. EI is constant.



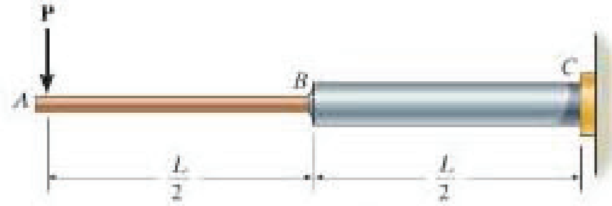
Prob. 12–76

•12–77. The bar is supported by the roller constraint at C, which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope and displacement at A. EI is constant.



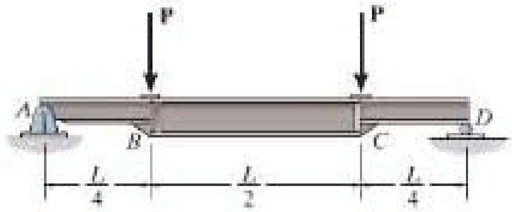
Prob. 12–77

12–78. The rod is constructed from two shafts for which the moment of inertia of AB is I and of BC is $2I$. Determine the maximum slope and deflection of the rod due to the loading. The modulus of elasticity is E.



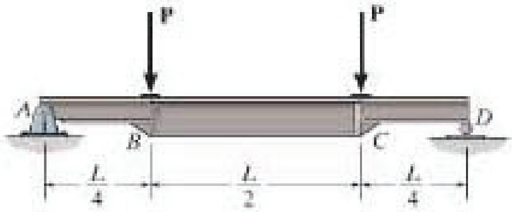
Prob. 12–78

12–79. Determine the slope at point D and the deflection at point C of the simply supported beam. The beam is made of material having a modulus of elasticity E. The moment of inertia of segments AB and CD of the beam is I, while the moment of inertia of segment BC of the beam is $2I$.



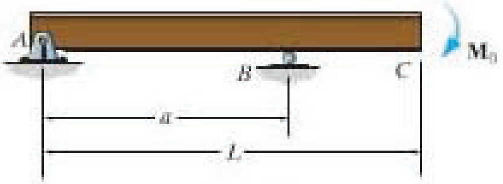
Prob. 12–79

***12–80.** Determine the slope at point A and the maximum deflection of the simply supported beam. The beam is made of material having a modulus of elasticity E. The moment of inertia of segments AB and CD of the beam is I, while the moment of inertia of segment BC is $2I$.



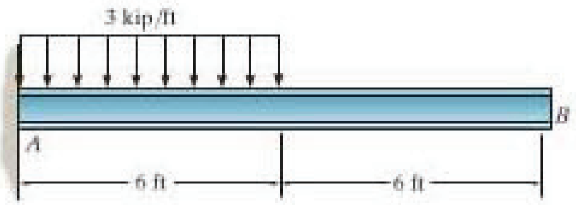
Prob. 12–80

•**12–81.** Determine the position a of roller support B in terms of L so that deflection at end C is the same as the maximum deflection of region AB of the simply supported overhang beam. EI is constant.



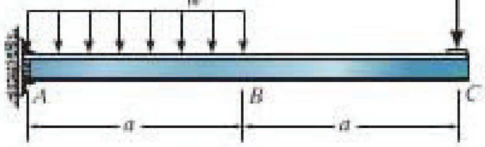
Prob. 12–81

12–82. The W10 × 15 cantilevered beam is made of A-36 steel and is subjected to the loading shown. Determine the slope and displacement at its end B.



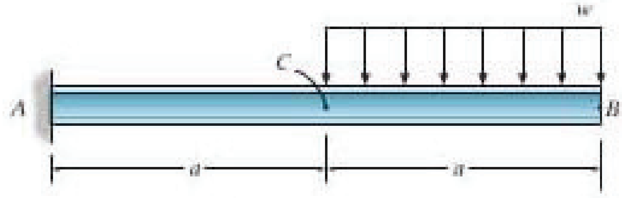
Prob. 12–82

12–83. The cantilevered beam is subjected to the loading shown. Determine the slope and displacement at C. Assume the support at A is fixed. EI is constant.



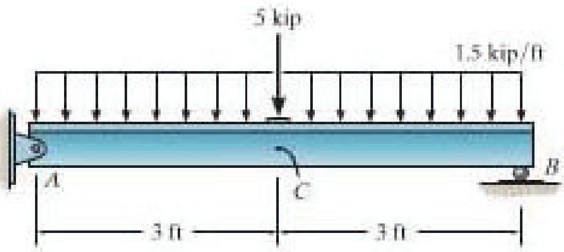
Prob. 12–83

*__12–84.__ Determine the slope at C and deflection at B. EI is constant.



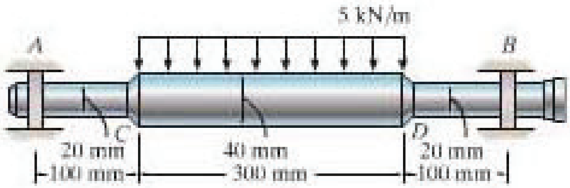
Prob. 12–84

•**12–85.** Determine the slope at B and the displacement at C. The member is an A-36 steel structural tee for which $I = 76.8\ \text{in}^4$.



Prob. 12–85

12–86. The A-36 steel shaft is used to support a rotor that exerts a uniform load of 5 kN/m within the region CD of the shaft. Determine the slope of the shaft at the bearings A and B. The bearings exert only vertical reactions on the shaft.



Prob. 12–86

12.5 Method of Superposition

The differential equation $EI\, d^4v/dx^4 = w(x)$ satisfies the two necessary requirements for applying the principle of superposition; i.e., the load $w(x)$ is linearly related to the deflection $v(x)$, and the load is assumed not to change significantly the original geometry of the beam or shaft. As a result, the deflections for a series of separate loadings acting on a beam may be superimposed. For example, if v_1 is the deflection for one load and v_2 is the deflection for another load, the total deflection for both loads acting together is the algebraic sum $v_1 + v_2$. Using tabulated results for various beam loadings, such as the ones listed in Appendix C, or those found in various engineering handbooks, it is therefore possible to find the slope and displacement at a point on a beam subjected to several different loadings by algebraically adding the effects of its various component parts.

The following examples illustrate how to use the method of superposition to solve deflection problems, where the deflection is caused not only by beam deformations, but also by rigid-body displacements, such as those that occur when the beam is supported by springs.

The resultant deflection at any point on this beam can be determined from the superposition of the deflections caused by each of the separate loadings acting on the beam.

12

EXAMPLE 12.13

Determine the displacement at point C and the slope at the support A of the beam shown in Fig. 12–28a. EI is constant.

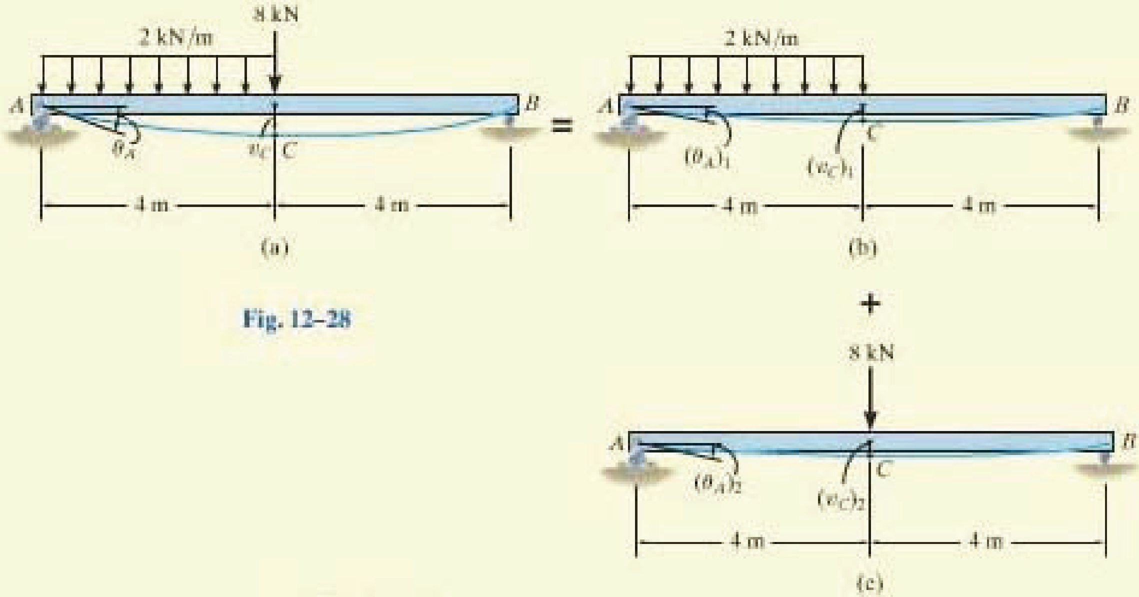


Fig. 12–28

SOLUTION

The loading can be separated into two component parts as shown in Figs. 12–28b and 12–28c. The displacement at C and slope at A are found using the table in Appendix C for each part.

For the distributed loading,

$$(\theta_A)_1 = \frac{3wL^3}{128EI} = \frac{3(2\text{ kN/m})(8\text{ m})^3}{128EI} = \frac{24\text{ kN}\cdot\text{m}^2}{EI} \curvearrowright$$

$$(v_C)_1 = \frac{5wL^4}{768EI} = \frac{5(2\text{ kN/m})(8\text{ m})^4}{768EI} = \frac{53.33\text{ kN}\cdot\text{m}^3}{EI} \downarrow$$

For the 8-kN concentrated force,

$$(\theta_A)_2 = \frac{PL^2}{16EI} = \frac{8\text{ kN}(8\text{ m})^2}{16EI} = \frac{32\text{ kN}\cdot\text{m}^2}{EI} \curvearrowright$$

$$(v_C)_2 = \frac{PL^3}{48EI} = \frac{8\text{ kN}(8\text{ m})^3}{48EI} = \frac{85.33\text{ kN}\cdot\text{m}^3}{EI} \downarrow$$

The displacement at C and the slope at A are the algebraic sums of these components. Hence,

$$(+\curvearrowright) \qquad \theta_A = (\theta_A)_1 + (\theta_A)_2 = \frac{56\text{ kN}\cdot\text{m}^2}{EI} \curvearrowright \qquad \textit{Ans.}$$

$$(+\downarrow) \qquad v_C = (v_C)_1 + (v_C)_2 = \frac{139\text{ kN}\cdot\text{m}^3}{EI} \downarrow \qquad \textit{Ans.}$$

12

EXAMPLE 12.14

Determine the displacement at the end C of the overhanging beam shown in Fig. 12–29a. EI is constant.

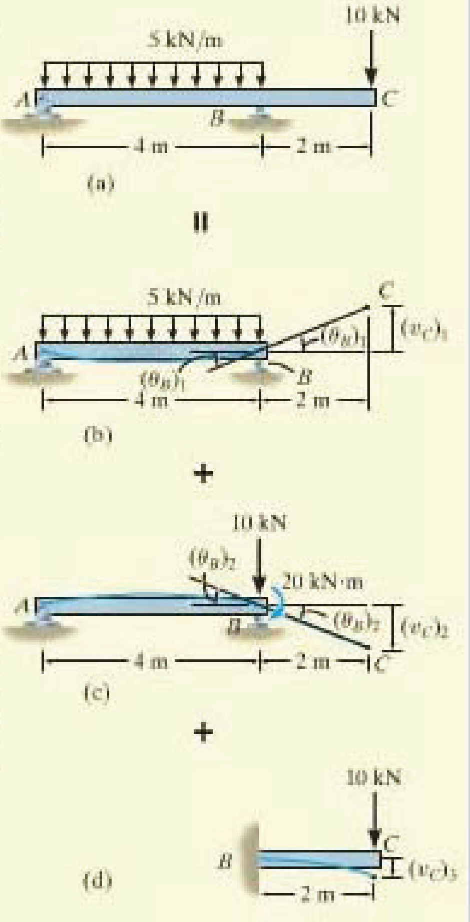


Fig. 12–29

SOLUTION

Since the table in Appendix C *does not* include beams with overhangs, the beam will be separated into a simply supported and a cantilevered portion. First we will calculate the slope at B, as caused by the distributed load acting on the simply supported span, Fig. 12–29b.

$$(\theta_B)_1 = \frac{wL^3}{24EI} = \frac{5\ \text{kN/m}(4\ \text{m})^3}{24EI} = \frac{13.33\ \text{kN}\cdot\text{m}^2}{EI} \circlearrowleft$$

Since this angle is *small*, $(\theta_B)_1 \approx \tan(\theta_B)_1$, and the vertical displacement at point C is

$$(v_C)_1 = (2\ \text{m})\left(\frac{13.33\ \text{kN}\cdot\text{m}^2}{EI}\right) = \frac{26.67\ \text{kN}\cdot\text{m}^3}{EI}\uparrow$$

Next, the 10-kN load on the overhang causes a statically equivalent force of 10 kN and couple moment of 20 kN·m at the support B of the simply supported span, Fig. 12–29c. The 10-kN force does not cause a displacement or slope at B; however, the 20-kN·m couple moment does cause a slope. The slope at B due to this moment is

$$(\theta_B)_2 = \frac{M_0L}{3EI} = \frac{20\ \text{kN}\cdot\text{m}(4\ \text{m})}{3EI} = \frac{26.67\ \text{kN}\cdot\text{m}^2}{EI} \circlearrowright$$

so that the extended point C is displaced

$$(v_C)_2 = (2\ \text{m})\left(\frac{26.7\ \text{kN}\cdot\text{m}^2}{EI}\right) = \frac{53.33\ \text{kN}\cdot\text{m}^3}{EI}\downarrow$$

Finally, the cantilevered portion BC is displaced by the 10-kN force, Fig. 12–29d. We have

$$(v_C)_3 = \frac{PL^3}{3EI} = \frac{10\ \text{kN}(2\ \text{m})^3}{3EI} = \frac{26.67\ \text{kN}\cdot\text{m}^3}{EI}\downarrow$$

Summing these results algebraically, we obtain the displacement of point C,

$$(+\downarrow) \qquad v_C = -\frac{26.7}{EI} + \frac{53.3}{EI} + \frac{26.7}{EI} = \frac{53.3\ \text{kN}\cdot\text{m}^3}{EI}\downarrow \qquad \textit{Ans.}$$

12

EXAMPLE 12.15

Determine the displacement at the end C of the cantilever beam shown in Fig. 12–30. EI is constant.

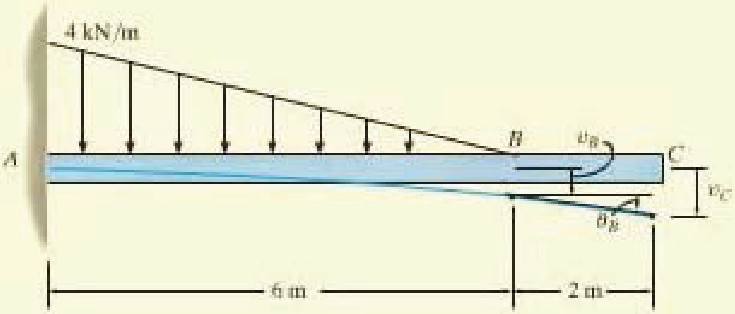


Fig. 12–30

SOLUTION

Using the table in Appendix C for the triangular loading, the slope and displacement at point B are

$$\theta_B = \frac{w_0 L^3}{24EI} = \frac{4\text{ kN/m}(6\text{ m})^3}{24EI} = \frac{36\text{ kN}\cdot\text{m}^2}{EI}$$

$$v_B = \frac{w_0 L^4}{30EI} = \frac{4\text{ kN/m}(6\text{ m})^4}{30EI} = \frac{172.8\text{ kN}\cdot\text{m}^3}{EI}$$

The unloaded region BC of the beam remains straight, as shown in Fig. 12–30. Since θ_B is small, the displacement at C becomes

$$(+\downarrow) \qquad v_C = v_B + \theta_B(L_{BC})$$

$$= \frac{172.8\text{ kN}\cdot\text{m}^3}{EI} + \frac{36\text{ kN}\cdot\text{m}^2}{EI}(2\text{ m})$$

$$= \frac{244.8\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

Ans.

EXAMPLE 12.16

The steel bar shown in Fig. 12–31*a* is supported by two springs at its ends *A* and *B*. Each spring has a stiffness of $k = 15$ kip/ft and is originally unstretched. If the bar is loaded with a force of 3 kip at point *C*, determine the vertical displacement of the force. Neglect the weight of the bar and take $E_{st} = 29(10^3)$ ksi, $I = 12$ in^4.

SOLUTION

The end reactions at *A* and *B* are calculated and shown in Fig. 12–31*b*. Each spring deflects by an amount

$$(v_A)_1 = \frac{2\text{ kip}}{15\text{ kip/ft}} = 0.1333\text{ ft}$$

$$(v_B)_1 = \frac{1\text{ kip}}{15\text{ kip/ft}} = 0.0667\text{ ft}$$

If the bar is considered to be *rigid*, these displacements cause it to move into the position shown in Fig. 12–31*b*. For this case, the vertical displacement at *C* is

$$(v_C)_1 = (v_B)_1 + \frac{6\text{ ft}}{9\text{ ft}}[(v_A)_1 - (v_B)_1]$$

$$= 0.0667\text{ ft} + \frac{2}{3}[0.1333\text{ ft} - 0.0667\text{ ft}] = 0.1111\text{ ft}\downarrow$$

We can find the displacement at *C* caused by the *deformation* of the bar, Fig. 12–31*c*, by using the table in Appendix C. We have

$$(v_C)_2 = \frac{Pab}{6EIL}(L^2 - b^2 - a^2)$$

$$= \frac{3\text{ kip}(3\text{ ft})(6\text{ ft})[(9\text{ ft})^2 - (6\text{ ft})^2 - (3\text{ ft})^2]}{6[29(10^3)\text{kip/in}^2](144\text{ in}^2/1\text{ ft}^2)(12\text{ in}^4)(1\text{ ft}^4/20\,736\text{ in}^4)(9\text{ ft})}$$

$$= 0.0149\text{ ft}\downarrow$$

Adding the two displacement components, we get

$(+\downarrow)\quad v_C = 0.1111\text{ ft} + 0.0149\text{ ft} = 0.126\text{ ft} = 1.51\text{ in.}\downarrow$ *Ans.*

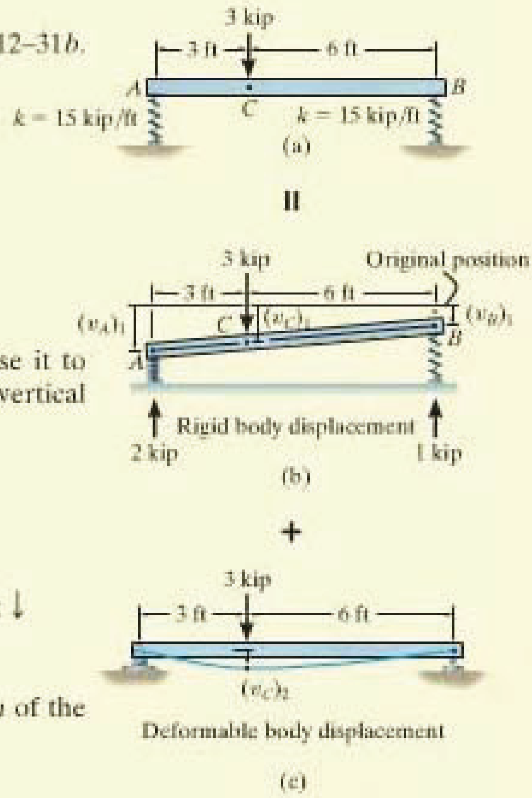
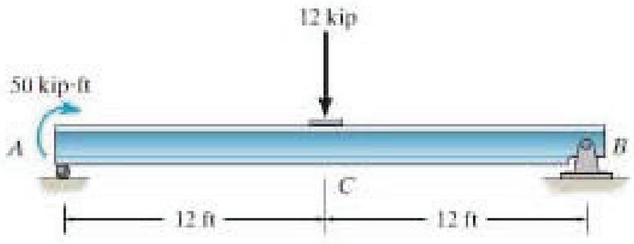


Fig. 12–31

12

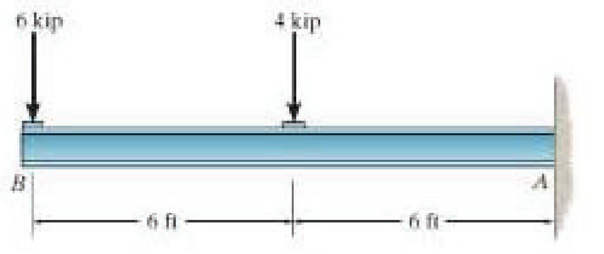
PROBLEMS

12–87. The W12 × 45 simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center C.



Prob. 12–87

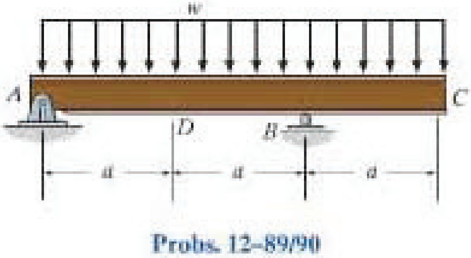
***12–88.** The W10 × 15 cantilevered beam is made of A-36 steel and is subjected to the loading shown. Determine the displacement at B and the slope at A.



Prob. 12–88

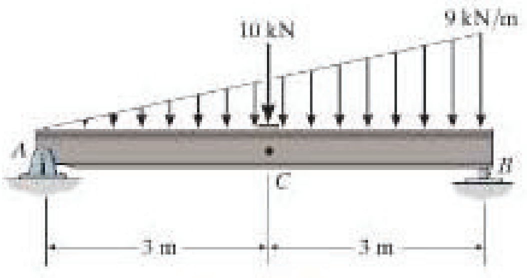
•**12–89.** Determine the slope and deflection at end C of the overhang beam. EI is constant.

12–90. Determine the slope at A and the deflection at point D of the overhang beam. EI is constant.



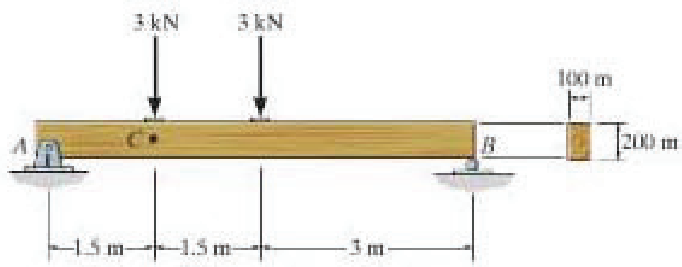
Probs. 12–89/90

12–91. Determine the slope at B and the deflection at point C of the simply supported beam. $E = 200$ GPa and $I = 45.5(10^6)\ \text{mm}^4$.



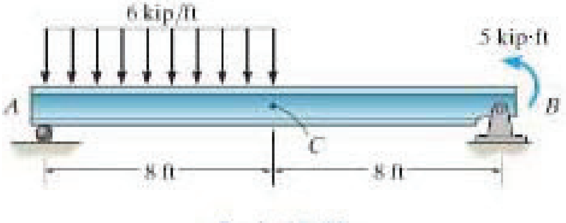
Prob. 12–91

***12–92.** Determine the slope at A and the deflection at point C of the simply supported beam. The modulus of elasticity of the wood is $E = 10$ GPa.



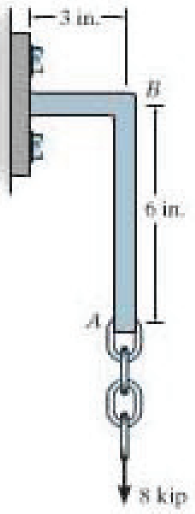
Prob. 12–92

•**12–93.** The W8 × 24 simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center C.



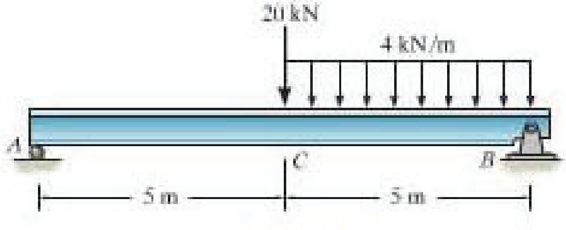
Prob. 12–93

12–94. Determine the vertical deflection and slope at the end A of the bracket. Assume that the bracket is fixed supported at its base, and neglect the axial deformation of segment AB. EI is constant.



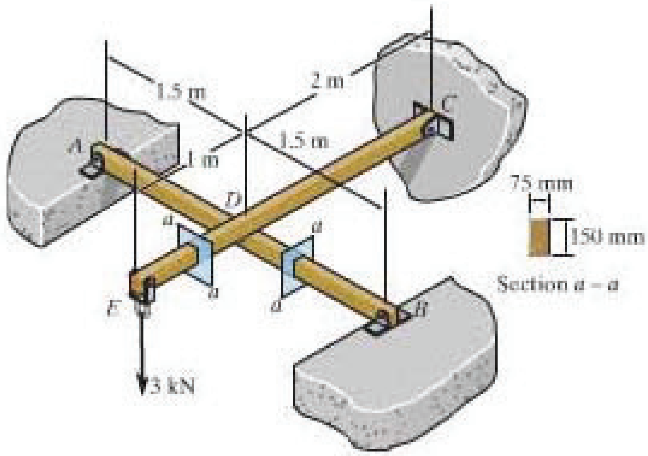
Prob. 12–94

12–95. The simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center C. $I = 0.1457(10^{-3})\ \text{m}^4$.



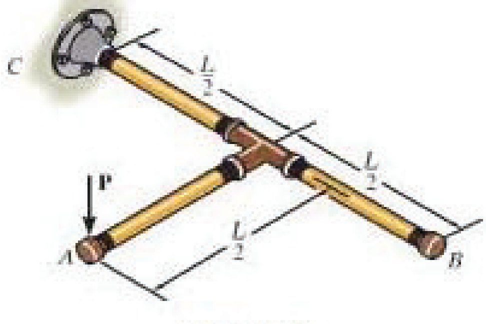
Prob. 12–95

***12–96.** Determine the deflection at end E of beam CDE. The beams are made of wood having a modulus of elasticity of $E = 10$ GPa.



Prob. 12–96

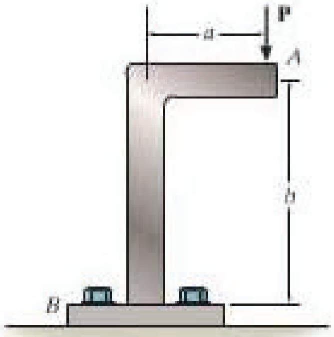
•**12–97.** The pipe assembly consists of three equal-sized pipes with flexibility stiffness EI and torsional stiffness GJ. Determine the vertical deflection at point A.



Prob. 12–97

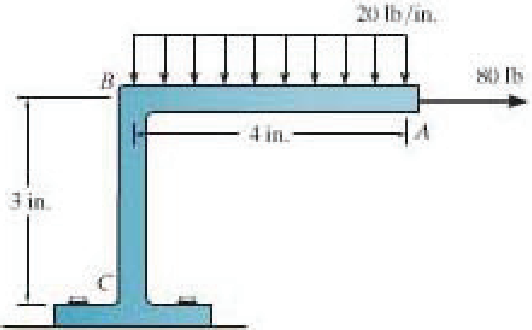
12

12–98. Determine the vertical deflection at the end *A* of the bracket. Assume that the bracket is fixed supported at its base *B* and neglect axial deflection. *EI* is constant.



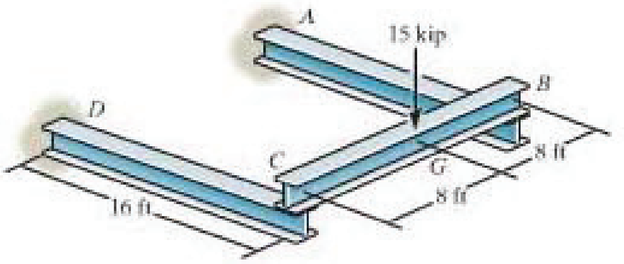
Prob. 12–98

12–99. Determine the vertical deflection and slope at the end *A* of the bracket. Assume that the bracket is fixed supported at its base, and neglect the axial deformation of segment *AB*. *EI* is constant.



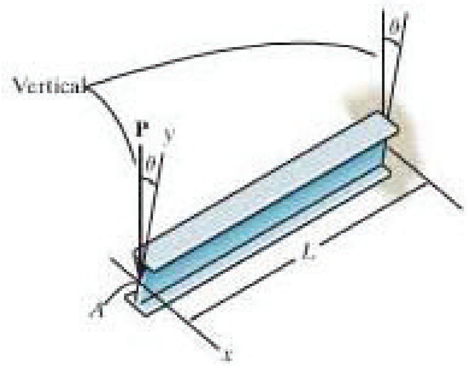
Prob. 12–99

***12–100.** The framework consists of two A-36 steel cantilevered beams *CD* and *BA* and a simply supported beam *CB*. If each beam is made of steel and has a moment of inertia about its principal axis of $I_x = 118\ \text{in}^4$, determine the deflection at the center *G* of beam *CB*.



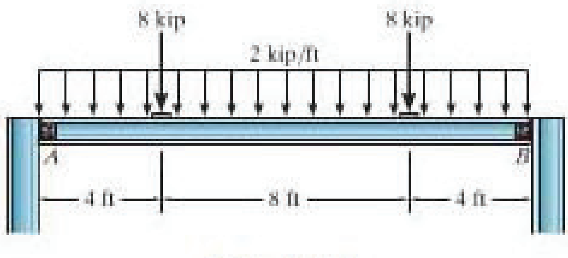
Prob. 12–100

•12–101. The wide-flange beam acts as a cantilever. Due to an error it is installed at an angle θ with the vertical. Determine the ratio of its deflection in the *x* direction to its deflection in the *y* direction at *A* when a load **P** is applied at this point. The moments of inertia are I_x and I_y. For the solution, resolve **P** into components and use the method of superposition. *Note:* The result indicates that large lateral deflections (*x* direction) can occur in narrow beams, $I_y \ll I_x$, when they are improperly installed in this manner. To show this numerically, compute the deflections in the *x* and *y* directions for an A-36 steel W10 × 15, with $P = 1.5$ kip, $\theta = 10°$, and $L = 12$ ft.



Prob. 12–101

12–102. The simply supported beam carries a uniform load of 2 kip/ft. Code restrictions, due to a plaster ceiling, require the maximum deflection not to exceed 1/360 of the span length. Select the lightest-weight A-36 steel wide-flange beam from Appendix B that will satisfy this requirement and safely support the load. The allowable bending stress is $\sigma_{\text{allow}} = 24$ ksi and the allowable shear stress is $\tau_{\text{allow}} = 14$ ksi. Assume *A* is a pin and *B* a roller support.



Prob. 12–102

12.6 Statically Indeterminate Beams and Shafts

The analysis of statically indeterminate axially loaded bars and torsionally loaded shafts has been discussed in Secs. 4.4 and 5.5, respectively. In this section we will illustrate a general method for determining the reactions on statically indeterminate beams and shafts. Specifically, a member of any type is classified as *statically indeterminate* if the number of unknown reactions *exceeds* the available number of equilibrium equations.

The additional support reactions on the beam or shaft that are *not needed* to keep it in stable equilibrium are called *redundants*. The number of these redundants is referred to as the *degree of indeterminacy*. For example, consider the beam shown in Fig. 12–32*a*. If the free-body diagram is drawn, Fig. 12–32*b*, there will be four unknown support reactions, and since three equilibrium equations are available for solution, the beam is classified as being indeterminate to the first degree. Either $\mathbf{A}_y$, $\mathbf{B}_y$, or $\mathbf{M}_A$ can be classified as the redundant, for if any one of these reactions is removed, the beam remains stable and in equilibrium ($\mathbf{A}_x$ cannot be classified as the redundant, for if it were removed, $\Sigma F_x = 0$ would not be satisfied.) In a similar manner, the *continuous beam* in Fig. 12–33*a* is indeterminate to the second degree, since there are five unknown reactions and only three available equilibrium equations, Fig. 12–33*b*. Here the two redundant support reactions can be chosen among $\mathbf{A}_y$, $\mathbf{B}_y$, $\mathbf{C}_y$, and $\mathbf{D}_y$.

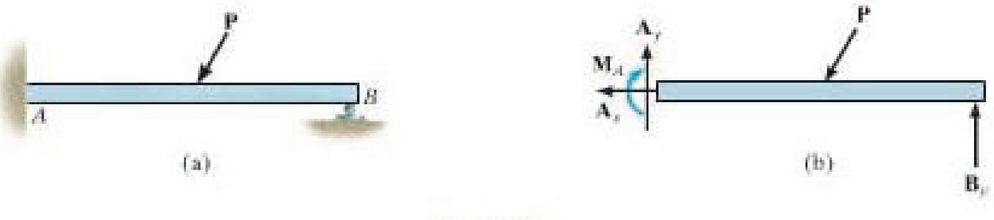


Fig. 12–32

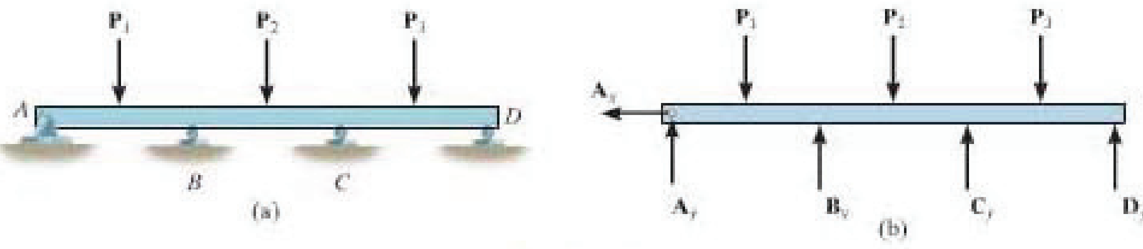


Fig. 12–33

To determine the reactions on a beam (or shaft) that is statically indeterminate, it is first necessary to specify the redundant reactions. We can determine these redundants from conditions of geometry known as *compatibility conditions*. Once determined, the redundants are then applied to the beam, and the remaining reactions are determined from the equations of equilibrium.

In the following sections we will illustrate this procedure for solution using the method of integration, Sec. 12.7; the moment-area method, Sec. 12.8; and the method of superposition, Sec. 12.9.

12.7 Statically Indeterminate Beams and Shafts—Method of Integration

The method of integration, discussed in Sec. 12.2, requires two integrations of the differential equation $d^2v/dx^2 = M/EI$ once the internal moment M in the beam is expressed as a function of position x. If the beam is statically indeterminate, however, M can also be expressed in terms of the *unknown* redundants. After integrating this equation twice, there will be two constants of integration along with the redundants to be determined. Although this is the case, these unknowns can always be found from the boundary and/or continuity conditions for the problem.

The following example problems illustrate specific applications of this method using the procedure for analysis outlined in Sec. 12.2.

An example of a statically indeterminate beam used to support a bridge deck.

EXAMPLE 12.17

The beam is subjected to the distributed loading shown in Fig. 12–34*a*. Determine the reaction at *A*. *EI* is constant.

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 12–34*a*. Only one coordinate *x* is needed. For convenience we will take it directed to the right, since the internal moment is easy to formulate.

Moment Function. The beam is indeterminate to the first degree as indicated from the free-body diagram, Fig. 12–34*b*. We can express the internal moment *M* in terms of the redundant force at *A* using the segment shown in Fig. 12–34*c*. Here

$$M = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

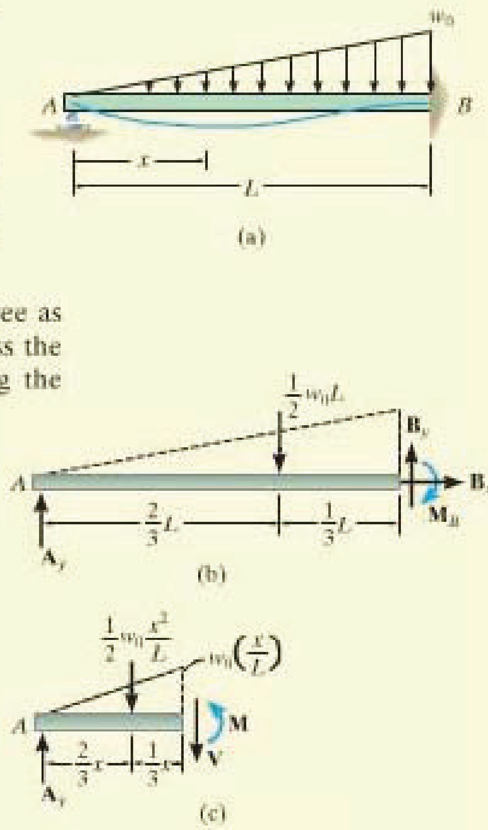


Fig. 12–34

Slope and Elastic Curve. Applying Eq. 12–10, we have

$$EI\frac{d^2v}{dx^2} = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

$$EI\frac{dv}{dx} = \frac{1}{2}A_y x^2 - \frac{1}{24} w_0 \frac{x^4}{L} + C_1$$

$$EIv = \frac{1}{6}A_y x^3 - \frac{1}{120} w_0 \frac{x^5}{L} + C_1 x + C_2$$

The three unknowns A_y, C_1, and C_2 are determined from the boundary conditions $x = 0$, $v = 0$; $x = L$, $dv/dx = 0$; and $x = L$, $v = 0$. Applying these conditions yields

$x = 0, v = 0;$ $\qquad 0 = 0 - 0 + 0 + C_2$

$x = L, \frac{dv}{dx} = 0;$ $\qquad 0 = \frac{1}{2}A_y L^2 - \frac{1}{24} w_0 L^3 + C_1$

$x = L, v = 0;$ $\qquad 0 = \frac{1}{6}A_y L^3 - \frac{1}{120} w_0 L^4 + C_1 L + C_2$

Solving,

$$A_y = \frac{1}{10} w_0 L \qquad \textit{Ans.}$$

$$C_1 = -\frac{1}{120} w_0 L^3 \quad C_2 = 0$$

NOTE: Using the result for A_y, the reactions at *B* can be determined from the equations of equilibrium, Fig. 12–34*b*. Show that $B_x = 0$, $B_y = 2w_0L/5$, and $M_B = w_0L^2/15$.

12

EXAMPLE 12.18

The beam in Fig. 12–35*a* is fixed supported at both ends and is subjected to the uniform loading shown. Determine the reactions at the supports. Neglect the effect of axial load.

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 12–35*a*. As in the previous problem, only one x coordinate is necessary for the solution since the loading is continuous across the span.

Moment Function. From the free-body diagram, Fig. 12–35*b*, the respective shear and moment reactions at A and B must be equal, since there is symmetry of both loading and geometry. Because of this, the equation of equilibrium, $\Sigma F_y = 0$, requires

$$V_A = V_B = \frac{wL}{2} \qquad \textit{Ans.}$$

The beam is indeterminate to the first degree, where M' is redundant. Using the beam segment shown in Fig. 12–35c, the internal moment **M** can be expressed in terms of M' as follows:

$$M = \frac{wL}{2}x - \frac{w}{2}x^2 - M'$$

Slope and Elastic Curve. Applying Eq. 12–10, we have

$$EI\frac{d^2v}{dx} = \frac{wL}{2}x - \frac{w}{2}x^2 - M'$$

$$EI\frac{dv}{dx} = \frac{wL}{4}x^2 - \frac{w}{6}x^3 - M'x + C_1$$

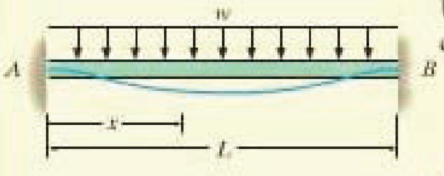
$$EIv = \frac{wL}{12}x^3 - \frac{w}{24}x^4 - \frac{M'}{2}x^2 + C_1x + C_2$$

The three unknowns, M', C_1, and C_2, can be determined from the *three* boundary conditions $v = 0$ at $x = 0$, which yields $C_2 = 0$; $dv/dx = 0$ at $x = 0$, which yields $C_1 = 0$; and $v = 0$ at $x = L$, which yields

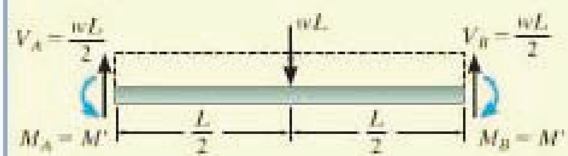
$$M' = \frac{wL^2}{12} \qquad \textit{Ans.}$$

Using these results, notice that because of symmetry the remaining boundary condition $dv/dx = 0$ at $x = L$ is automatically satisfied.

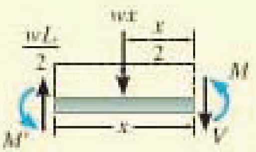
NOTE: It should be realized that this method of solution is generally suitable when only one x coordinate is needed to describe the elastic curve. If several x coordinates are needed, equations of continuity must be written, thus complicating the solution process.



(a)



(b)



(c)

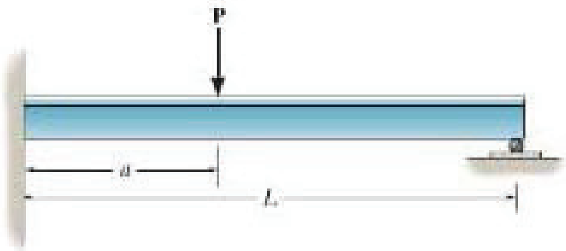
Fig. 12–35

PROBLEMS

12–103. Determine the reactions at the supports A and B, then draw the moment diagram. EI is constant.

Prob. 12–103

***12–104.** Determine the value of a for which the maximum positive moment has the same magnitude as the maximum negative moment. EI is constant.



Prob. 12–104

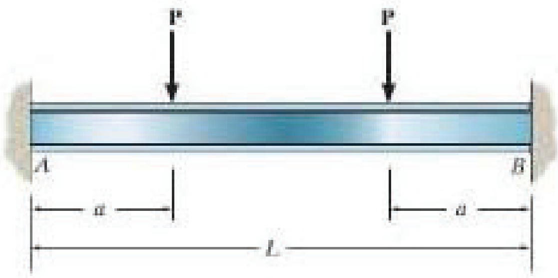
•12–105. Determine the reactions at the supports A, B, and C; then draw the shear and moment diagrams. EI is constant.

Prob. 12–105

12–106. Determine the reactions at the supports, then draw the shear and moment diagram. EI is constant.

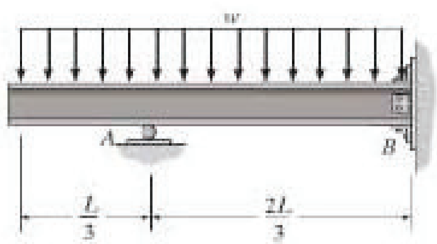
Prob. 12–106

12–107. Determine the moment reactions at the supports A and B. EI is constant.



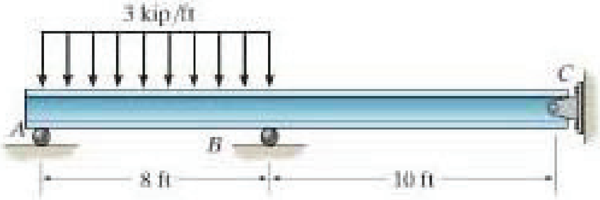
Prob. 12–107

***12–108.** Determine the reactions at roller support A and fixed support B.



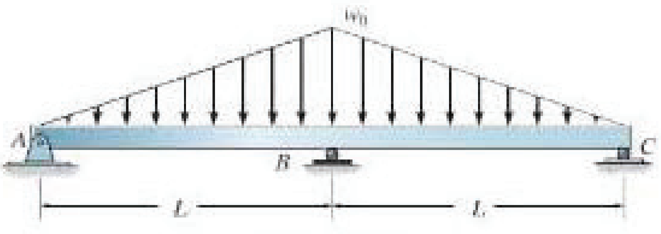
Prob. 12–108

•12–109. Use discontinuity functions and determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant.



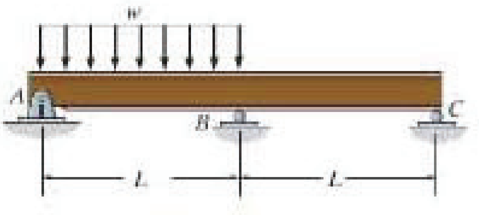
Prob. 12–109

12–110. Determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant.



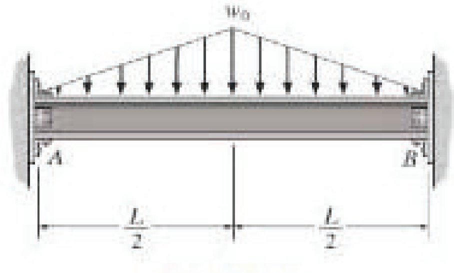
Prob. 12–110

12–111. Determine the reactions at pin support A and roller supports B and C. EI is constant.



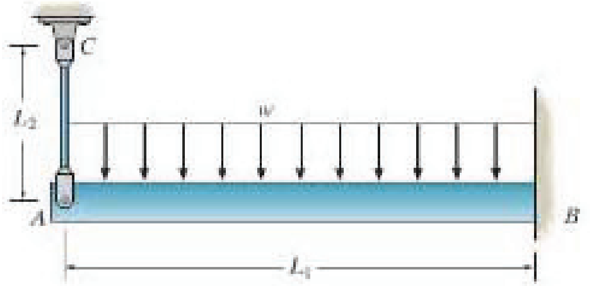
Prob. 12–111

*12–112. Determine the moment reactions at fixed supports A and B. EI is constant.



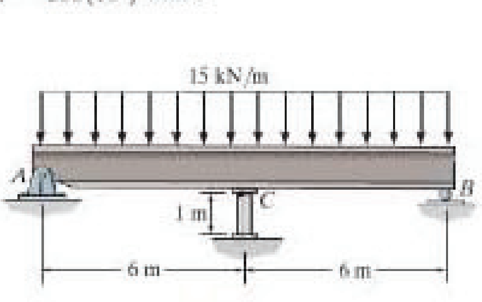
Prob. 12–112

•12–113. The beam has a constant E_1I_1 and is supported by the fixed wall at B and the rod AC. If the rod has a cross-sectional area A_2 and the material has a modulus of elasticity E_2, determine the force in the rod.



Prob. 12–113

12–114. The beam is supported by a pin at A, a roller at B, and a post having a diameter of 50 mm at C. Determine the support reactions at A, B, and C. The post and the beam are made of the same material having a modulus of elasticity $E = 200$ GPa, and the beam has a constant moment of inertia $I = 255(10^6)$ mm^4.



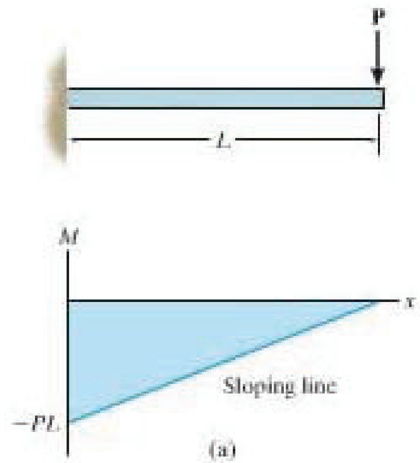
Prob. 12–114

*12.8 Statically Indeterminate Beams and Shafts—Moment-Area Method

If the moment-area method is used to determine the unknown redundants of a statically indeterminate beam or shaft, then the M/EI diagram must be drawn such that the redundants are represented as unknowns on this diagram. Once the M/EI diagram is established, the two moment-area theorems can then be applied to obtain the proper relationships between the tangents on the elastic curve in order to meet the conditions of displacement and/or slope at the supports of the beam. In all cases the number of these compatibility conditions will be equivalent to the number of redundants, and so a solution for the redundants can be obtained.

Moment Diagrams Constructed by the Method of Superposition. Since application of the moment-area theorems requires calculation of both the area under the M/EI diagram and the centroidal location of this area, it is often convenient to use *separate* M/EI diagrams for *each* of the known loads and redundants rather than using the *resultant diagram* to calculate these geometric quantities. This is especially true if the resultant moment diagram has a complicated shape. The method for drawing the moment diagram in parts is based on the principle of superposition.

Most loadings on cantilevered *beams or shafts* will be a combination of the four loadings shown in Fig. 12–36. Construction of the associated moment diagrams, also shown in this figure, has been discussed in the examples of Chapter 6. Based on these results, we will now show how to use the method of superposition to represent the resultant moment diagram by a series of separate moment diagrams for the cantilevered beam shown in Fig. 12–37*a*. To do this, we will first replace the loads by a system of statically equivalent loads. For example, the three cantilevered beams shown in Fig. 12–37*a* are statically equivalent to the



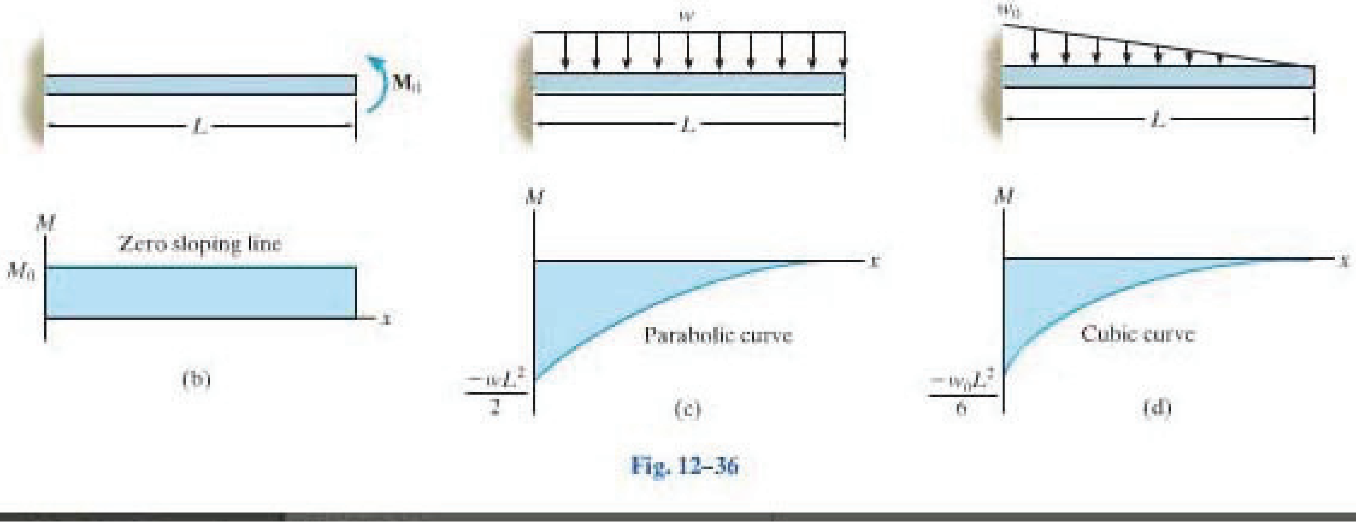


Fig. 12–36

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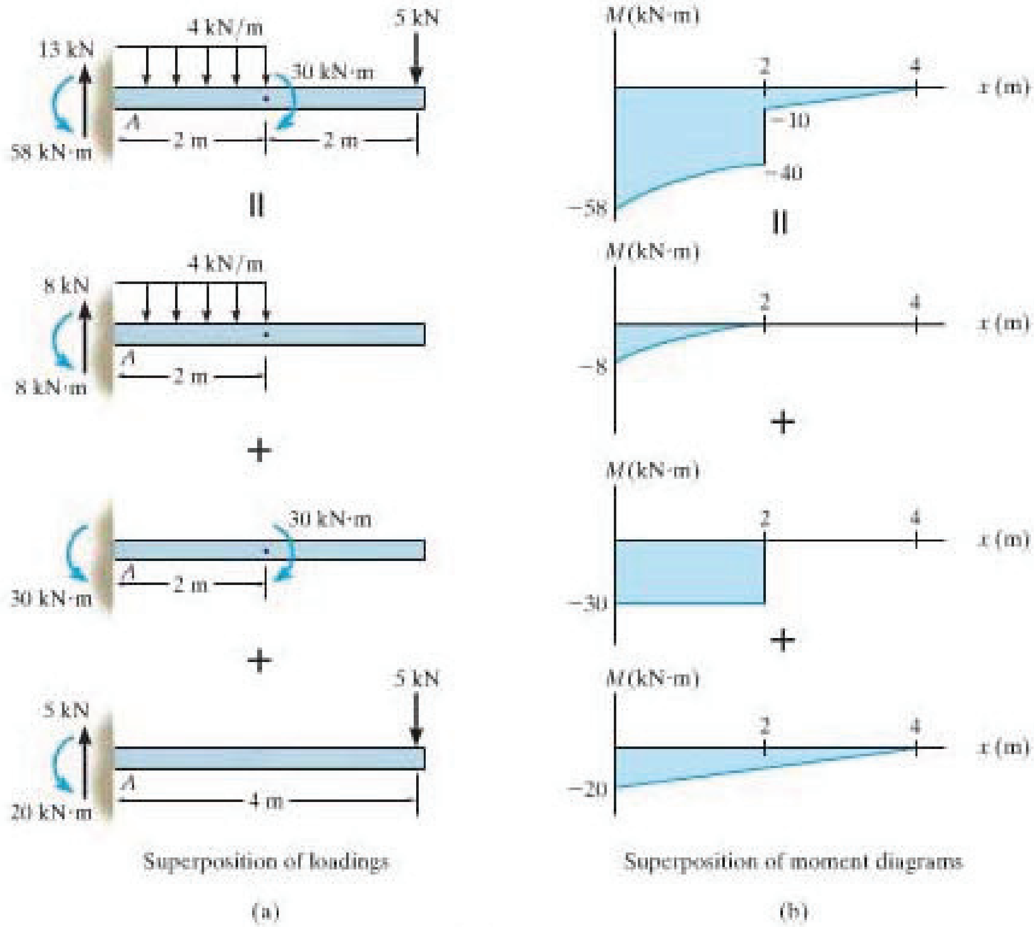


Fig. 12–37

resultant beam, since the load at each point on the resultant beam is equal to the superposition or addition of the loadings on the three separate beams. Thus, if the moment diagrams for each separate beam are drawn, Fig. 12–37*b*, the superposition of these diagrams will yield the moment diagram for the resultant beam, shown at the top. For example, from each of the separate moment diagrams, the moment at end A is $M_A = -8\text{ kN}\cdot\text{m} - 30\text{ kN}\cdot\text{m} - 20\text{ kN}\cdot\text{m} = -58\text{ kN}\cdot\text{m}$, as verified by the top moment diagram. This example demonstrates that it is sometimes easier to construct a series of separate statically equivalent moment diagrams for the beam, *rather* than constructing its more complicated resultant moment diagram. Obviously, the area and location of the centroid for each part are easier to establish than those of the centroid for the resultant diagram.

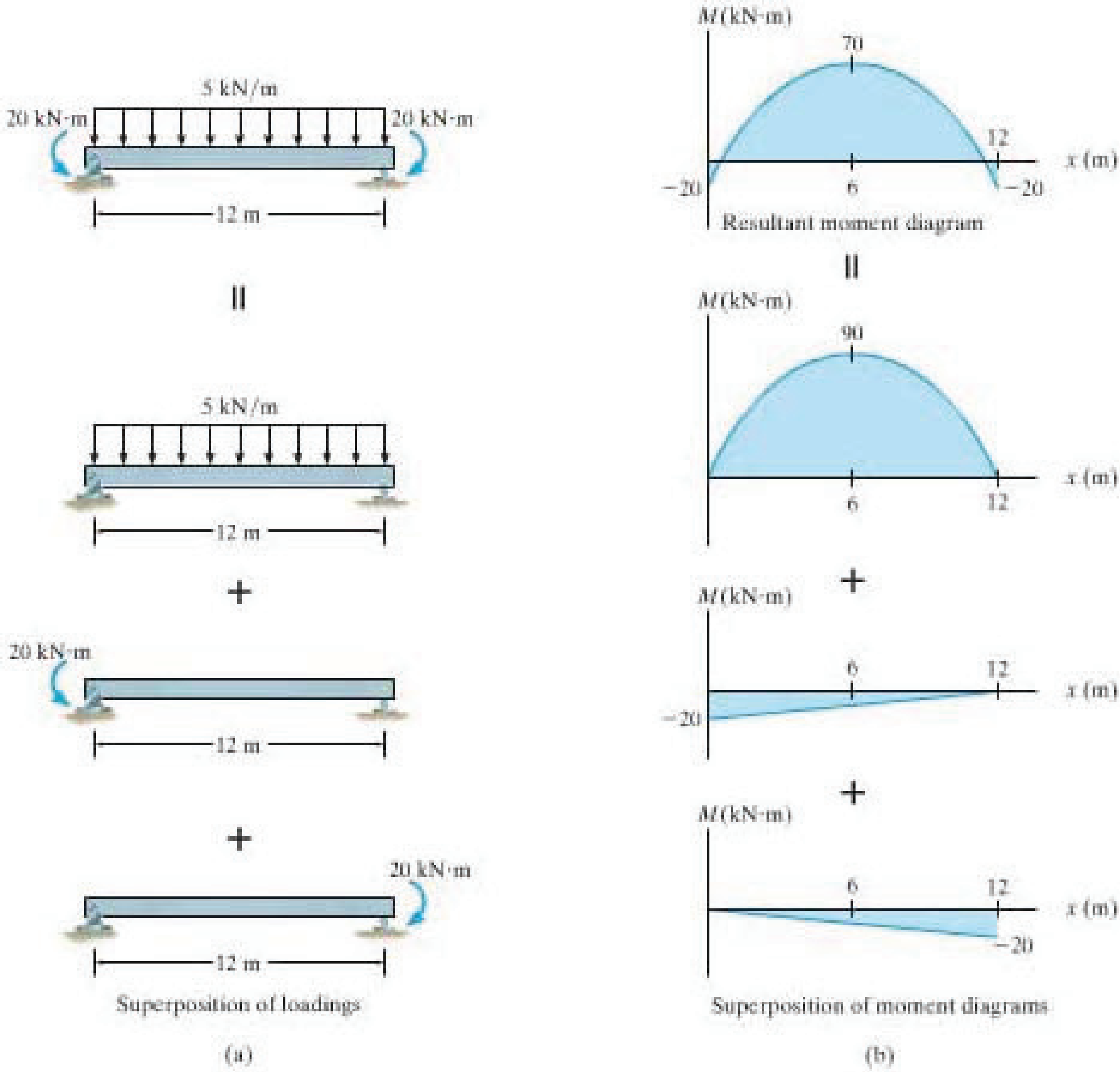


Fig. 12–38

In a similar manner, we can also represent the resultant moment diagram for a *simply supported beam* by using a superposition of moment diagrams for each loading acting on a series of simply supported beams. For example, the beam loading shown at the top of Fig. 12–38*a* is equivalent to the sum of the beam loadings shown below it. Consequently, the sum of the moment diagrams for each of these three loadings can be used rather than the resultant moment diagram shown at the top of Fig. 12–38*b*.

The examples that follow should also clarify some of these points and illustrate how to use the moment-area theorems to obtain the redundant reactions on statically indeterminate beams and shafts. The solutions follow the procedure for analysis outlined in Sec. 12.4.

EXAMPLE 12.19

The beam is subjected to the concentrated force shown in Fig. 12–39*a*. Determine the reactions at the supports. *EI* is constant.

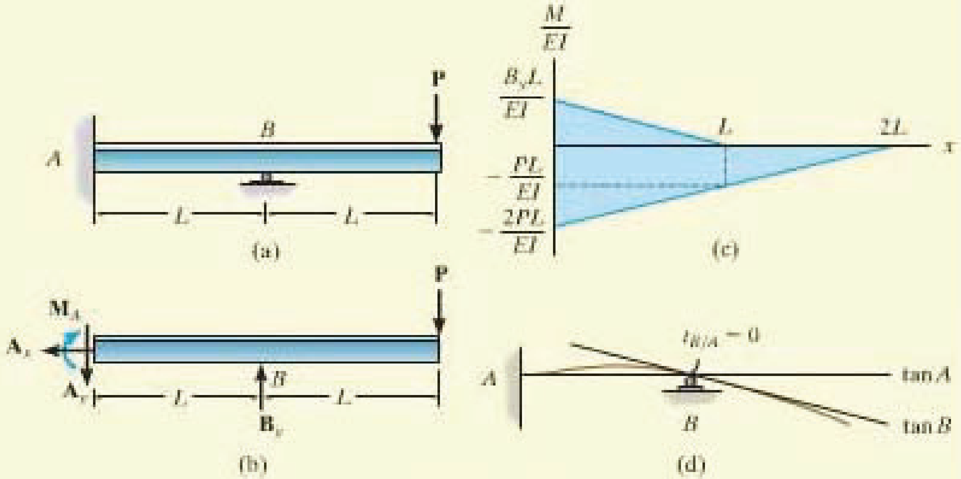


Fig. 12–39

SOLUTION

***M/EI* Diagram.** The free-body diagram is shown in Fig. 12–39*b*. Using the method of superposition, the separate *M/EI* diagrams for the redundant reaction $\mathbf{B}_y$ and the load **P** are shown in Fig. 12–39c.

Elastic Curve. The elastic curve for the beam is shown in Fig. 12–39*d*. The tangents at the supports *A* and *B* have been constructed. Since $\Delta_B = 0$, then

$$t_{B/A} = 0$$

Moment-Area Theorem. Applying Theorem 2, we have

$$t_{B/A} = \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{B_yL}{EI}\right)L\right] + \left(\frac{L}{2}\right)\left[\frac{-PL}{EI}(L)\right]$$

$$+ \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{-PL}{EI}\right)(L)\right] = 0$$

$$B_y = 2.5P \qquad \textit{Ans.}$$

Equations of Equilibrium. Using this result, the reactions at *A* on the free-body diagram, Fig. 12–39*b*, are

$\xrightarrow{+} \Sigma F_x = 0;$ $\qquad A_x = 0$ *Ans.*

$+\uparrow \Sigma F_y = 0;$ $\qquad -A_y + 2.5P - P = 0$

$$A_y = 1.5P \qquad \textit{Ans.}$$

$\circlearrowleft + \Sigma M_A = 0;$ $\qquad -M_A + 2.5P(L) - P(2L) = 0$

$$M_A = 0.5PL \qquad \textit{Ans.}$$

EXAMPLE 12.20

The beam is subjected to the couple moment at its end C as shown in Fig. 12–40a. Determine the reaction at B. EI is constant.

SOLUTION

M/EI Diagram. The free-body diagram is shown in Fig. 12–40b. By inspection, the beam is indeterminate to the first degree. In order to obtain a direct solution, we will choose $\mathbf{B}_y$ as the redundant. Using superposition, the M/EI diagrams for $\mathbf{B}_y$ and $\mathbf{M}_0$, each applied to a simply supported beam, are shown in Fig. 12–40c. (Note that for such a beam A_x, A_y, and C_y do not contribute to an M/EI diagram.)

Elastic Curve. The elastic curve for the beam is shown in Fig. 12–40d. The tangents at A, B, and C have been established. Since $\Delta_A = \Delta_B = \Delta_C = 0$, then the vertical distances shown must be proportional; i.e.,

$$t_{B/C} = \frac{1}{2}t_{A/C} \qquad (1)$$

From Fig. 12–40c, we have

$$t_{B/C} = \left(\frac{1}{3}L\right)\left[\frac{1}{2}\left(\frac{B_yL}{2EI}\right)(L)\right] + \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{-M_0}{2EI}\right)(L)\right] + \left(\frac{L}{2}\right)\left[\left(\frac{-M_0}{2EI}\right)(L)\right]$$

$$t_{A/C} = (L)\left[\frac{1}{2}\left(\frac{B_yL}{2EI}\right)(2L)\right] + \left(\frac{2}{3}(2L)\right)\left[\frac{1}{2}\left(\frac{-M_0}{EI}\right)(2L)\right]$$

Substituting into Eq. 1 and simplifying yields

$$B_y = \frac{3M_0}{2L} \qquad \textit{Ans.}$$

Equations of Equilibrium. The reactions at A and C can now be determined from the equations of equilibrium, Fig. 12–40b. Show that $A_x = 0$, $C_y = 5M_0/4L$, and $A_y = M_0/4L$.

Note from Fig. 12–40e that this problem can also be worked in terms of the vertical distances,

$$t_{B/A} = \frac{1}{2}t_{C/A}$$

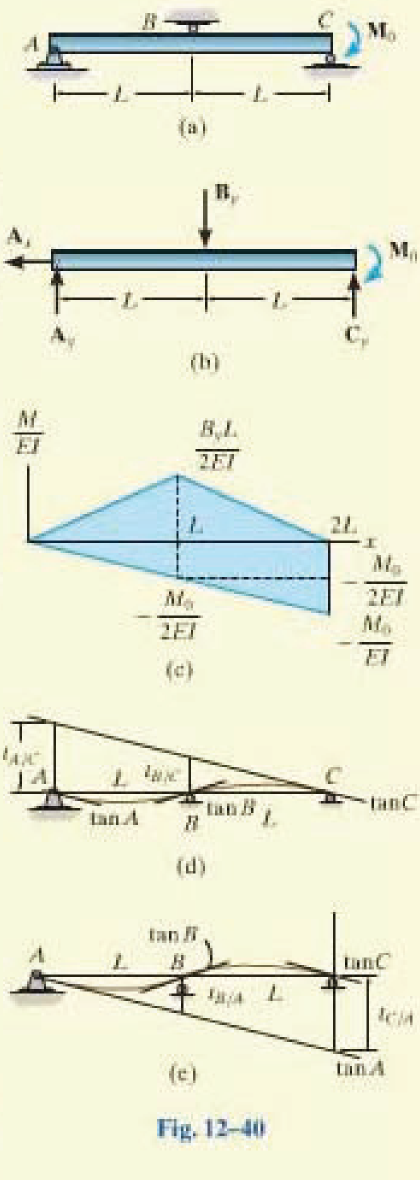


Fig. 12–40

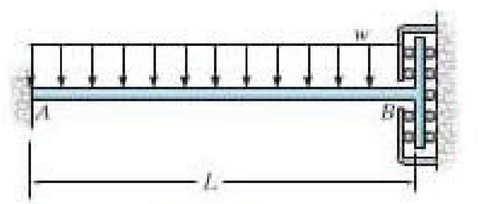
12

PROBLEMS

12–115. Determine the moment reactions at the supports A and B, then draw the shear and moment diagrams. EI is constant.

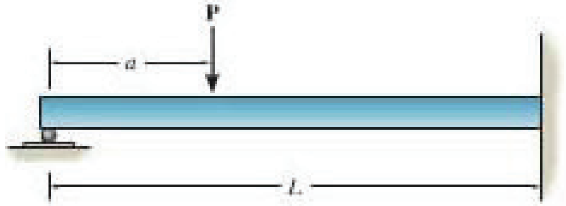
Prob. 12–115

***12–116.** The rod is fixed at A, and the connection at B consists of a roller constraint which allows vertical displacement but resists axial load and moment. Determine the moment reactions at these supports. EI is constant.



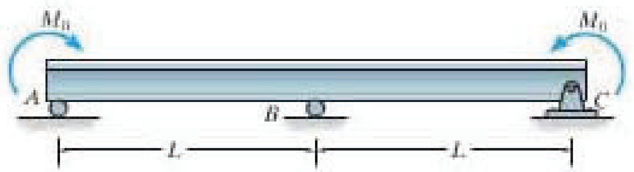
Prob. 12–116

•12–117. Determine the value of a for which the maximum positive moment has the same magnitude as the maximum negative moment. EI is constant.



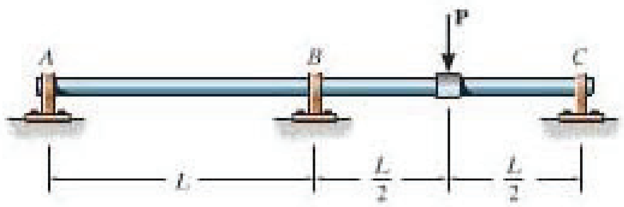
Prob. 12–117

12–118. Determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant.



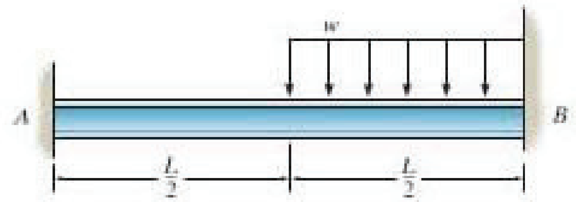
Prob. 12–118

12–119. Determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant. Support B is a thrust bearing.



Prob. 12–119

***12–120.** Determine the moment reactions at the supports A and B. EI is constant.



Prob. 12–120

12.9 Statically Indeterminate Beams and Shafts—Method of Superposition

The method of superposition has been used previously to solve for the redundant loading on axially loaded bars and torsionally loaded shafts. In order to apply this method to the solution of statically indeterminate beams (or shafts), it is first necessary to identify the redundant support reactions as explained in Sec. 12.6. By *removing* them from the beam we obtain the so-called *primary beam*, which is statically determinate and stable, and is subjected *only* to the external load. If we add to this beam a succession of similarly supported beams, each loaded with a *separate* redundant, then by the principle of superposition, we obtain the actual loaded beam. Finally, in order to solve for the redundants, we must write the *conditions of compatibility* that exist at the supports where each of the redundants acts. Since the redundant forces are determined directly in this manner, this method of analysis is sometimes called the ***force method***. Once the redundants are obtained, the other reactions on the beam can then be determined from the three equations of equilibrium.

To clarify these concepts, consider the beam shown in Fig. 12–41*a*. If we choose the reaction $\mathbf{B}_y$ at the roller as the redundant, then the primary beam is shown in Fig. 12–41*b*, and the beam with the redundant $\mathbf{B}_y$ acting on it is shown in Fig. 12–41*c*. The displacement at the roller is to be zero, and since the displacement of point *B* on the primary beam is v_B, and $\mathbf{B}_y$ causes point *B* to be displaced upward v'_B, we can write the compatibility equation at *B* as

$$(+\uparrow) \qquad 0 = -v_B + v'_B$$

The displacements v_B and v'_B can be obtained using any one of the methods discussed in Secs. 12.2 through 12.5. Here we will obtain them directly from the table in Appendix C. We have

$$v_B = \frac{5PL^3}{48EI} \qquad \text{and} \qquad v'_B = \frac{B_yL^3}{3EI}$$

Substituting into the compatibility equation, we get

$$0 = -\frac{5PL^3}{48EI} + \frac{B_yL^3}{3EI}$$

$$B_y = \frac{5}{16}P$$

Now that $\mathbf{B}_y$ is known, the reactions at the wall are determined from the three equations of equilibrium applied to the free-body diagram of the beam, Fig. 12–41*d*. The results are

$$A_x = 0 \qquad A_y = \frac{11}{16}P$$

$$M_A = \frac{3}{16}PL$$

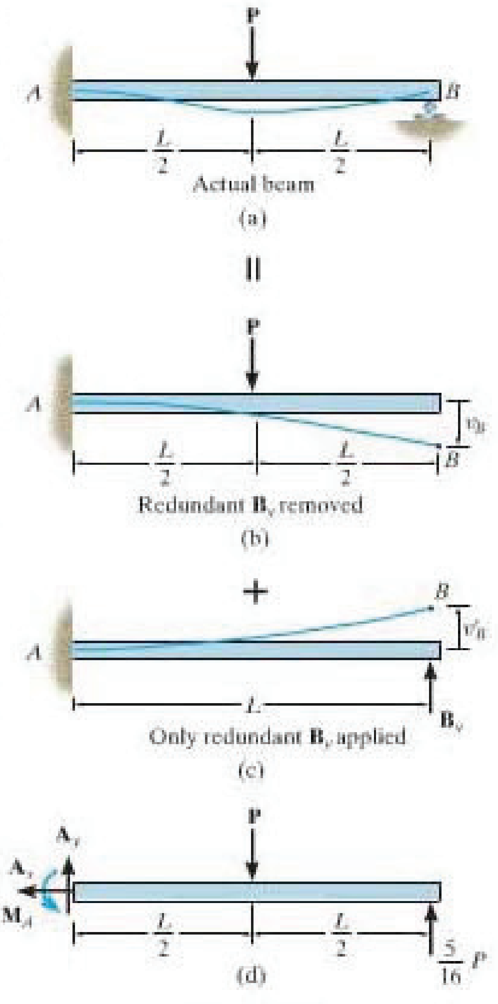


Fig. 12–41

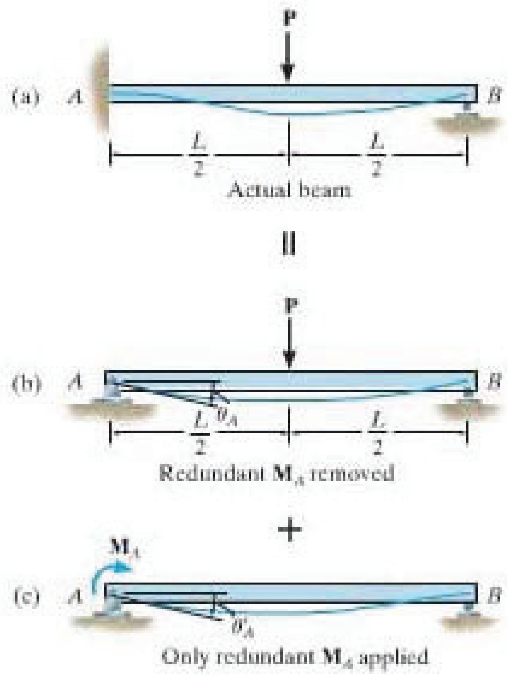


Fig. 12–42

As stated in Sec. 12.6, choice of the redundant is *arbitrary*, provided the primary beam remains stable. For example, the moment at A for the beam in Fig. 12–42*a* can also be chosen as the redundant. In this case the capacity of the beam to resist $\mathbf{M}_A$ is removed, and so the primary beam is then pin supported at A, Fig. 12–42*b*. To it we add the beam with the redundant at A acting on it, Fig. 12–42*c*. Referring to the slope at A caused by the load $\mathbf{P}$ as θ_A, and the slope at A caused by the redundant $\mathbf{M}_A$ as θ'_A, the compatibility equation for the slope at A requires

$$(\uparrow+) \qquad 0 = \theta_A + \theta'_A$$

Again using the table in Appendix C, we have

$$\theta_A = \frac{PL^2}{16EI} \qquad \text{and} \qquad \theta'_A = \frac{M_A L}{3EI}$$

Thus,

$$0 = \frac{PL^2}{16EI} + \frac{M_A L}{3EI}$$

$$M_A = -\frac{3}{16}PL$$

This is the same result determined previously. Here the negative sign for M_A simply means that $\mathbf{M}_A$ acts in the opposite sense of direction of that shown in Fig. 12–42*c*.

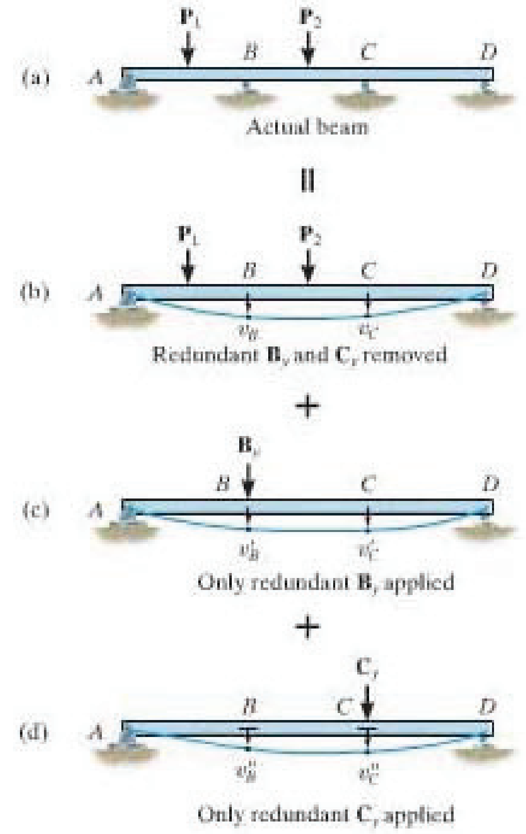


Fig. 12–43

Another example that illustrates this method is given in Fig. 12–43*a*. In this case the beam is indeterminate to the second degree and therefore *two* compatibility equations will be necessary for the solution. We will choose the forces at the roller supports *B* and *C* as redundants. The primary (statically determinate) beam deforms as shown in Fig. 12–43*b* when the redundants are removed. Each redundant force deforms this beam as shown in Figs. 12–43*c* and 12–43*d*, respectively. By superposition, the compatibility equations for the displacements at *B* and *C* are

$$(+\downarrow) \qquad 0 = v_B + v'_B + v''_B$$
$$(+\downarrow) \qquad 0 = v_C + v'_C + v''_C \qquad (12\text{–}20)$$

Here the displacement components v'_B and v'_C will be expressed in terms of the unknown $\mathbf{B}_y$, and the components v''_B and v''_C will be expressed in terms of the unknown $\mathbf{C}_y$. When these displacements have been determined and substituted into Eq. 12–20, these equations may then be solved simultaneously for the two unknowns $\mathbf{B}_y$ and $\mathbf{C}_y$.

Procedure for Analysis

The following procedure provides a means for applying the method of superposition (or the force method) to determine the reactions on statically indeterminate beams or shafts.

Elastic Curve.

- Specify the unknown redundant forces or moments that must be removed from the beam in order to make it statically determinate and stable.
- Using the principle of superposition, draw the statically indeterminate beam and show it equal to a sequence of corresponding *statically determinate beams*.
- The first of these beams, the primary beam, supports the same external loads as the statically indeterminate beam, and each of the other beams "added" to the primary beam shows the beam loaded with a separate redundant force or moment.
- Sketch the deflection curve for each beam and indicate symbolically the displacement (slope) at the point of each redundant force (moment).

Compatibility Equations.

- Write a compatibility equation for the displacement (slope) at each point where there is a redundant force (moment).
- Determine all the displacements or slopes using an appropriate method as explained in Secs. 12.2 through 12.5.
- Substitute the results into the compatibility equations and solve for the unknown redundants.
- If a numerical value for a redundant is *positive*, it has the *same sense of direction* as originally assumed. Similarly, a *negative* numerical value indicates the redundant acts *opposite* to its assumed *sense of direction*.

Equilibrium Equations.

- Once the redundant forces and/or moments have been determined, the remaining unknown reactions can be found from the equations of equilibrium applied to the loadings shown on the beam's free-body diagram.

The following examples illustrate application of this procedure. For brevity, all displacements and slopes have been found using the table in Appendix C.

EXAMPLE 12.21

Determine the reactions at the roller support B of the beam shown in Fig. 12–44a, then draw the shear and moment diagrams. EI is constant.

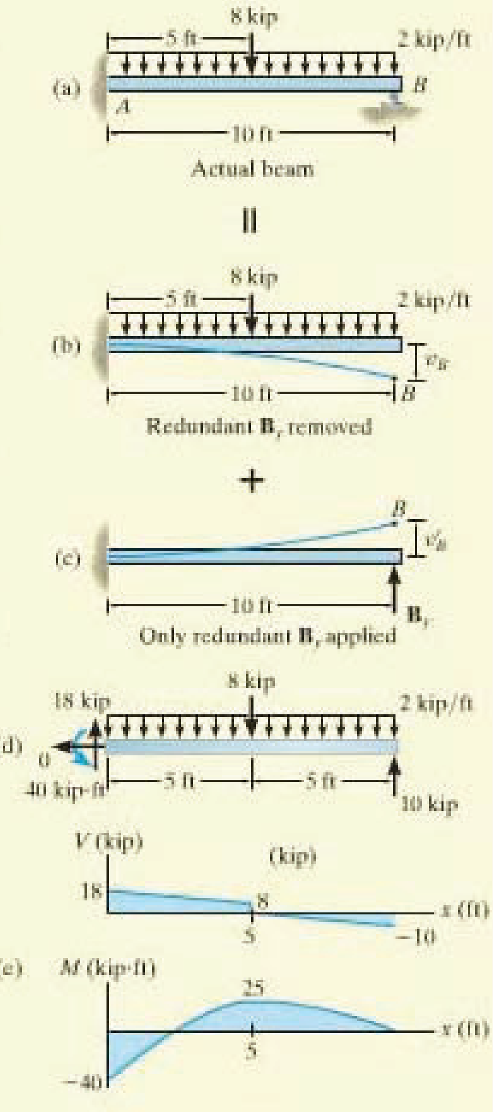


Fig. 12–44

SOLUTION

Principle of Superposition. By inspection, the beam is statically indeterminate to the first degree. The roller support at B will be chosen as the redundant so that $\mathbf{B}_y$ will be determined *directly*. Figures 12–44b and 12–44c show application of the principle of superposition. Here we have assumed that $\mathbf{B}_y$ acts upward on the beam.

Compatibility Equation. Taking positive displacement as downward, the compatibility equation at B is

$$(+\downarrow) \qquad 0 = v_B - v'_B \qquad (1)$$

These displacements can be obtained directly from the table in Appendix C.

$$v_B = \frac{wL^4}{8EI} + \frac{5PL^3}{48EI}$$

$$= \frac{2\text{ kip/ft}(10\text{ ft})^4}{8EI} + \frac{5(8\text{ kip})(10\text{ ft})^3}{48EI} = \frac{3333\text{ kip}\cdot\text{ft}^3}{EI}\downarrow$$

$$v'_B = \frac{PL^3}{3EI} = \frac{B_y(10\text{ ft})^3}{3EI} = \frac{333.3\text{ ft}^3\,B_y}{EI}\uparrow$$

Substituting into Eq. 1 and solving yields

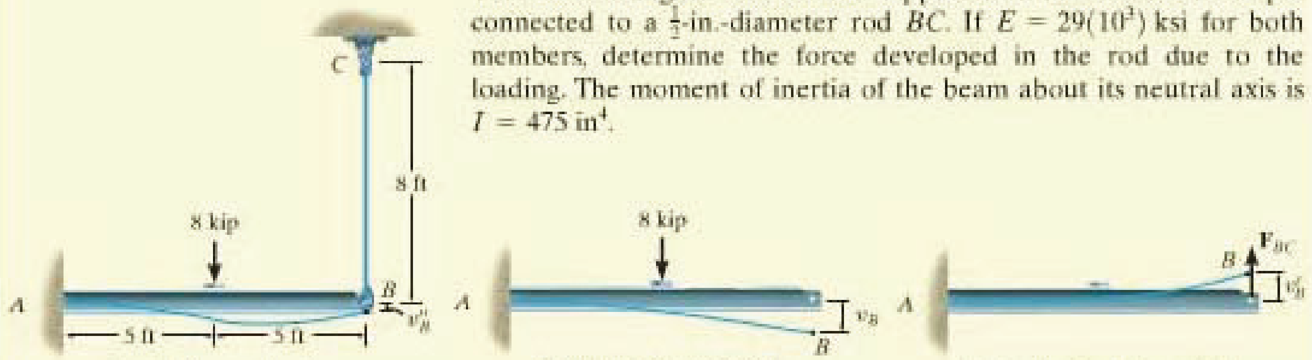
$$0 = \frac{3333}{EI} - \frac{333.3B_y}{EI}$$

$$B_y = 10\text{ kip} \qquad \textit{Ans.}$$

Equilibrium Equations. Using this result and applying the three equations of equilibrium, we obtain the results shown on the beam's free-body diagram in Fig. 12–44d. The shear and moment diagrams are shown in Fig. 12–44e.

EXAMPLE 12.22

The beam in Fig. 12–45*a* is fixed supported to the wall at *A* and pin connected to a $\frac{1}{2}$-in.-diameter rod *BC*. If $E = 29(10^3)$ ksi for both members, determine the force developed in the rod due to the loading. The moment of inertia of the beam about its neutral axis is $I = 475 \text{ in}^4$.



Actual beam and rod (a) | Redundant $\mathbf{F}_{BC}$ removed (b) | Only redundant $\mathbf{F}_{BC}$ applied (c)

Fig. 12–45

SOLUTION I

Principle of Superposition. By inspection, this problem is indeterminate to the first degree. Here *B* will undergo an unknown displacement v''_B, since the rod will stretch. The rod will be treated as the redundant and hence the force of the rod is removed from the beam at *B*, Fig. 12–45*b*, and then reapplied, Fig. 12–45*c*.

Compatibility Equation. At point *B* we require

$$(+\downarrow) \qquad v''_B = v_B - v'_B \qquad (1)$$

The displacements v_B and v'_B are determined from the table in Appendix C. v''_B is calculated from Eq. 4–2. Working in kilopounds and inches, we have

$$v''_B = \frac{PL}{AE} = \frac{F_{BC}(8 \text{ ft})(12 \text{ in./ft})}{(\pi/4)(\frac{1}{2}\text{in.})^2[29(10^3) \text{ kip/in}^2]} = 0.01686F_{BC}\downarrow$$

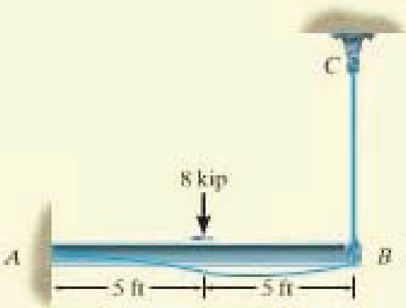
$$v_B = \frac{5PL^3}{48EI} = \frac{5(8 \text{ kip})(10 \text{ ft})^3(12 \text{ in./ft})^3}{48[29(10^3) \text{ kip/in}^2](475 \text{ in}^4)} = 0.1045 \text{ in.}\downarrow$$

$$v'_B = \frac{PL^3}{3EI} = \frac{F_{BC}(10 \text{ ft})^3(12 \text{ in./ft})^3}{3[29(10)^3 \text{ kip/in}^2](475 \text{ in}^4)} = 0.04181F_{BC}\uparrow$$

Thus, Eq. 1 becomes

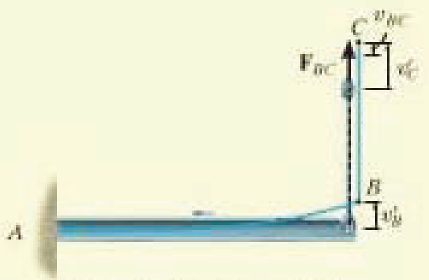
$$(+\downarrow) \qquad 0.01686F_{BC} = 0.1045 - 0.04181F_{BC}$$

$$F_{BC} = 1.78 \text{ kip}$$ *Ans.*



Actual beam and rod
(d)

Redundant $\mathbf{F}_{BC}$ removed
(e)



Only redundant $\mathbf{F}_{BC}$ applied
(f)

Fig. 12–45 (cont.)

SOLUTION II

Principle of Superposition. We can also solve this problem by removing the pin support at C and keeping the rod attached to the beam. In this case the 8-kip load will cause points B and C to be displaced downward the *same amount* v_C, Fig. 12–45*e*, since no force exists in rod BC. When the redundant force $\mathbf{F}_{BC}$ is applied at point C, it causes the end C of the rod to be displaced upward v'_C and the end B of the beam to be displaced upward v'_B, Fig. 12–45*f*. The difference in these two displacements, v_{BC}, represents the stretch of the rod due to $\mathbf{F}_{BC}$, so that $v'_C = v_{BC} + v'_B$. Hence, from Figs. 12–45*d*, 12–45*e*, and 12–45*f*, the compatibility of displacement at point C is

$$(+\downarrow) \qquad 0 = v_C - (v_{BC} + v'_B) \qquad (2)$$

From Solution I, we have

$$v_C = v_B = 0.1045 \text{ in.} \downarrow$$
$$v_{BC} = v''_B = 0.01686 F_{BC} \uparrow$$
$$v'_B = 0.04181 F_{BC} \uparrow$$

Therefore, Eq. 2 becomes

$$(+\downarrow) \qquad 0 = 0.1045 - (0.01686 F_{BC} + 0.04181 F_{BC})$$
$$F_{BC} = 1.78 \text{ kip} \qquad \textit{Ans.}$$

12

EXAMPLE 12.23

Determine the moment at B for the beam shown in Fig. 12–46*a*. EI is constant. Neglect the effects of axial load.

SOLUTION

Principle of Superposition. Since the axial load on the beam is neglected, there will be a vertical force and moment at A and B. Here there are only two available equations of equilibrium ($\Sigma M = 0, \Sigma F_y = 0$) and so the problem is indeterminate to the second degree. We will assume that $\mathbf{B}_y$ and $\mathbf{M}_B$ are redundant, so that by the principle of superposition, the beam is represented as a cantilever, loaded *separately* by the distributed load and reactions $\mathbf{B}_y$ and $\mathbf{M}_B$, Figs. 12–46*b*, 12–46*c*, and 12–46*d*.

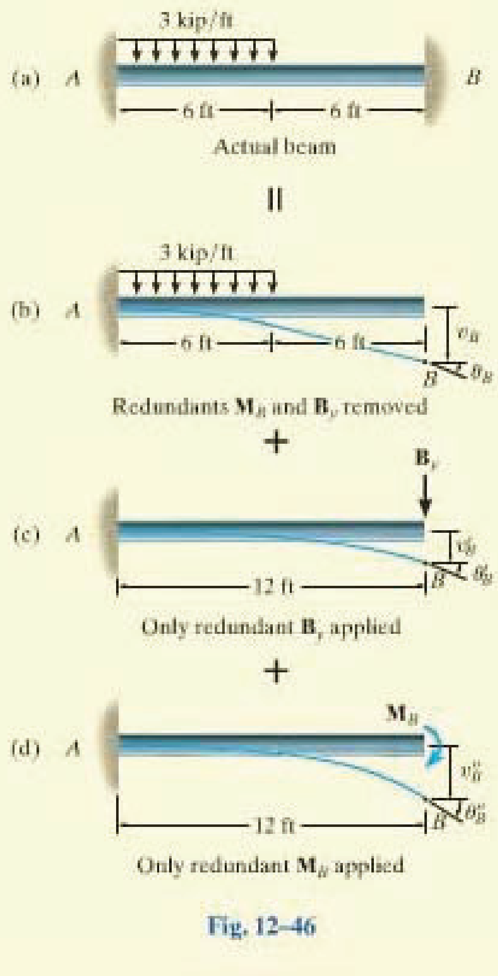


Fig. 12–46

Compatibility Equations. Referring to the displacement and slope at B, we require

$$(\curvearrowleft +) \qquad 0 = \theta_B + \theta'_B + \theta''_B \qquad (1)$$

$$(+\downarrow) \qquad 0 = v_B + v'_B + v''_B \qquad (2)$$

Using the table in Appendix C to calculate the slopes and displacements, we have

$$\theta_B = \frac{wL^3}{48EI} = \frac{3\text{ kip/ft }(12\text{ ft})^3}{48EI} = \frac{108\text{ kip}\cdot\text{ft}^2}{EI} \curvearrowright$$

$$v_B = \frac{7wL^4}{384EI} = \frac{7(3\text{ kip/ft})(12\text{ ft})^4}{384EI} = \frac{1134\text{ kip}\cdot\text{ft}^3}{EI} \downarrow$$

$$\theta'_B = \frac{PL^2}{2EI} = \frac{B_y(12\text{ ft})^2}{2EI} = \frac{72B_y}{EI} \curvearrowright$$

$$v'_B = \frac{PL^3}{3EI} = \frac{B_y(12\text{ ft})^3}{3EI} = \frac{576B_y}{EI} \downarrow$$

$$\theta''_B = \frac{ML}{EI} = \frac{M_B(12\text{ ft})}{EI} = \frac{12M_B}{EI} \curvearrowright$$

$$v''_B = \frac{ML^2}{2EI} = \frac{M_B(12\text{ ft})^2}{2EI} = \frac{72M_B}{EI} \downarrow$$

Substituting these values into Eqs. 1 and 2 and canceling out the common factor EI, we get

$$(\curvearrowleft +) \qquad 0 = 108 + 72B_y + 12M_B$$

$$(+\downarrow) \qquad 0 = 1134 + 576B_y + 72M_B$$

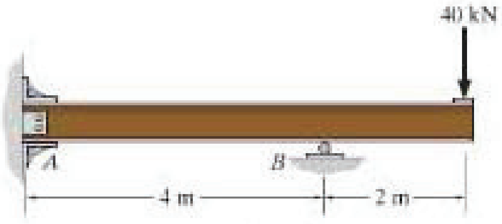
Solving these equations simultaneously gives

$$B_y = -3.375\text{ kip}$$

$$M_B = 11.25\text{ kip}\cdot\text{ft} \qquad \textit{Ans.}$$

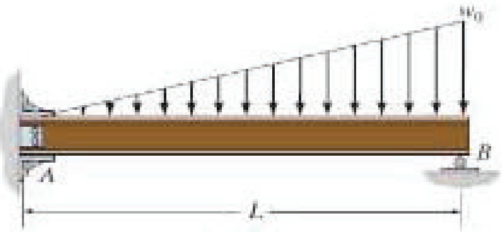
FUNDAMENTAL PROBLEMS

F12–13. Determine the reactions at the fixed support A and the roller B. EI is constant.



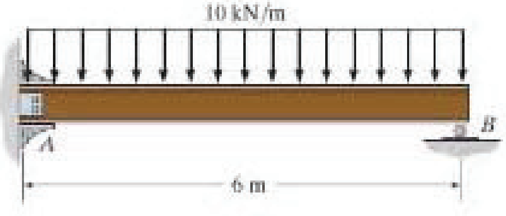
F12–13

F12–14. Determine the reactions at the fixed support A and the roller B. EI is constant.



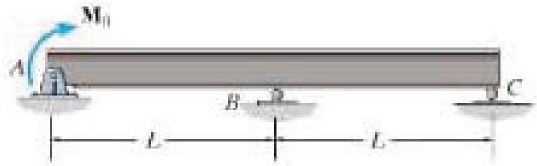
F12–14

F12–15. Determine the reactions at the fixed support A and the roller B. Support B settles 2 mm. E = 200 GPa, $I = 65.0(10^{-6})$ m^4.



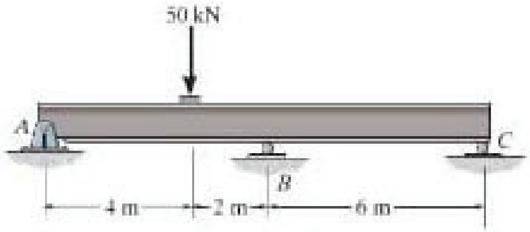
F12–15

F12–16. Determine the reaction at the roller B. EI is constant.



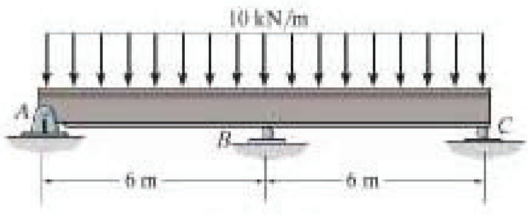
F12–16

F12–17. Determine the reaction at the roller B. EI is constant.



F12–17

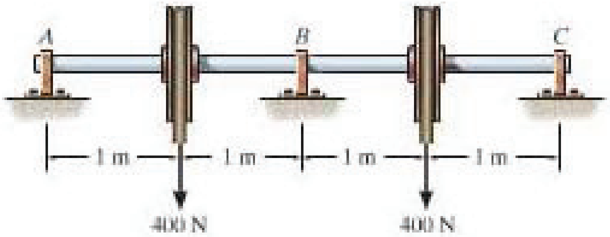
F12–18. Determine the reaction at the roller support B if it settles 5 mm. E = 200 GPa and $I = 65.0(10^{-6})$ m^4.



F12–18

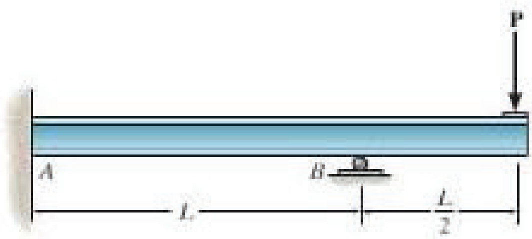
PROBLEMS

•**12–121.** Determine the reactions at the bearing supports *A*, *B*, and *C* of the shaft, then draw the shear and moment diagrams. *EI* is constant. Each bearing exerts only vertical reactions on the shaft.



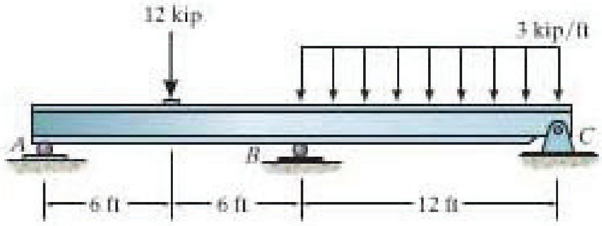
Prob. 12–121

12–122. Determine the reactions at the supports *A* and *B*. *EI* is constant.



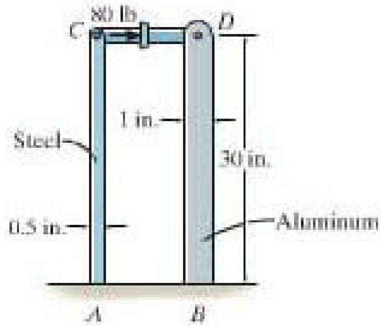
Prob. 12–122

12–123. Determine the reactions at the supports *A*, *B*, and *C*, then draw the shear and moment diagrams. *EI* is constant.



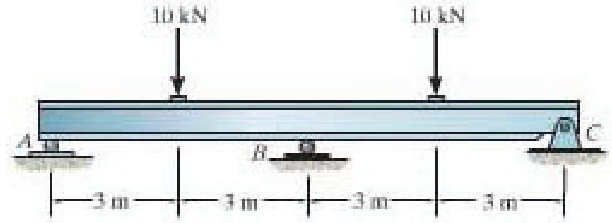
Prob. 12–123

*__12–124.__ The assembly consists of a steel and an aluminum bar, each of which is 1 in. thick, fixed at its ends *A* and *B*, and pin connected to the *rigid* short link *CD*. If a horizontal force of 80 lb is applied to the link as shown, determine the moments created at *A* and *B*. $E_{st} = 29(10^3)$ ksi, $E_{al} = 10(10^3)$ ksi.



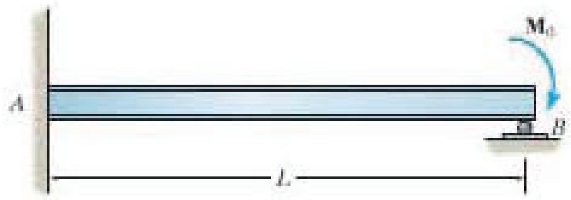
Prob. 12–124

•**12–125.** Determine the reactions at the supports *A*, *B*, and *C*, then draw the shear and moment diagrams. *EI* is constant.



Prob. 12–125

12–126. Determine the reactions at the supports *A* and *B*. *EI* is constant.



Prob. 12–126

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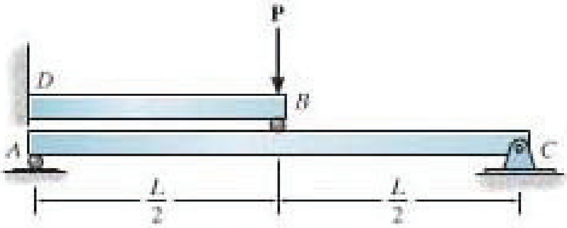
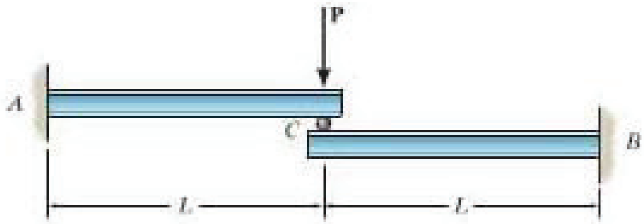
12–127. Determine the reactions at support C. EI is constant for both beams.



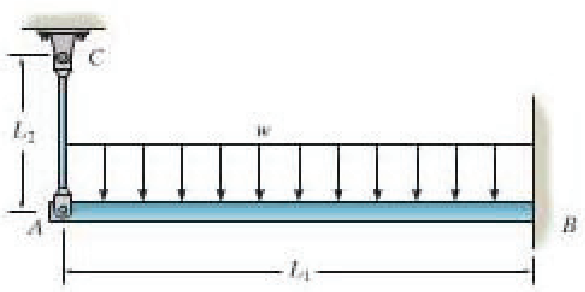
Prob. 12–127

*__12–128.__ The compound beam segments meet in the center using a smooth contact (roller). Determine the reactions at the fixed supports A and B when the load **P** is applied. EI is constant.



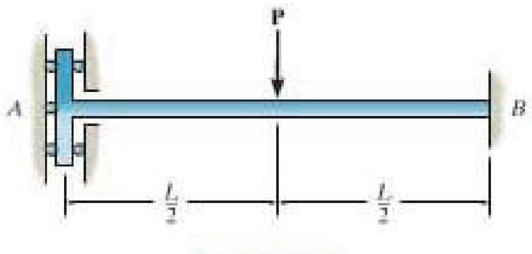
Prob. 12–128

•**12–129.** The beam has a constant E_1I_1 and is supported by the fixed wall at B and the rod AC. If the rod has a cross-sectional area A_2 and the material has a modulus of elasticity E_2, determine the force in the rod.



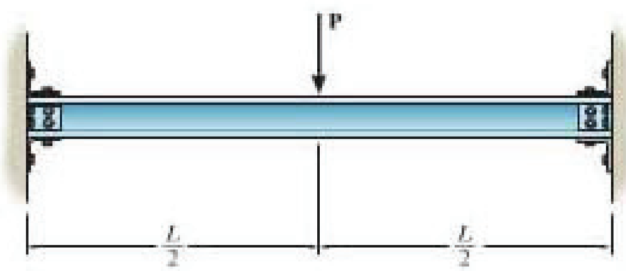
Prob. 12–129

12–130. Determine the reactions at A and B. Assume the support at A only exerts a moment on the beam. EI is constant.



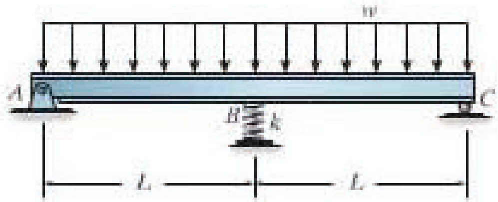
Prob. 12–130

12–131. The beam is supported by the bolted supports at its ends. When loaded these supports do not provide an actual fixed connection, but instead allow a slight rotation α before becoming fixed. Determine the moment at the connections and the maximum deflection of the beam.



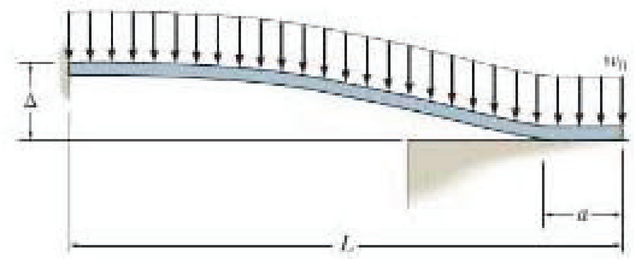
Prob. 12–131

*__12–132.__ The beam is supported by a pin at A, a spring having a stiffness k at B, and a roller at C. Determine the force the spring exerts on the beam. EI is constant.



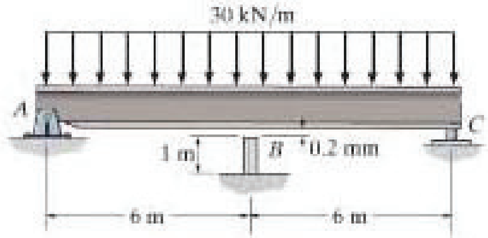
Prob. 12–132

•12–133. The beam is made from a soft linear elastic material having a constant EI. If it is originally a distance Δ from the surface of its end support, determine the distance a at which it rests on this support when it is subjected to the uniform load w_0, which is great enough to cause this to happen.



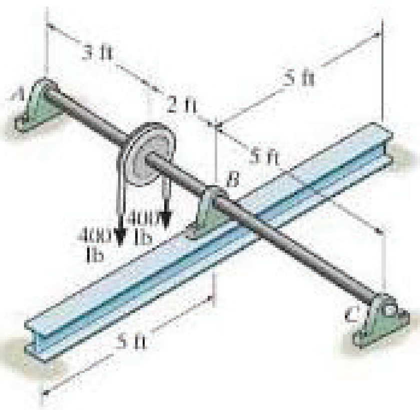
Prob. 12–133

12–134. Before the uniform distributed load is applied on the beam, there is a small gap of 0.2 mm between the beam and the post at B. Determine the support reactions at A, B, and C. The post at B has a diameter of 40 mm, and the moment of inertia of the beam is $I = 875(10^6)\ \text{mm}^4$. The post and the beam are made of material having a modulus of elasticity of $E = 200$ GPa.



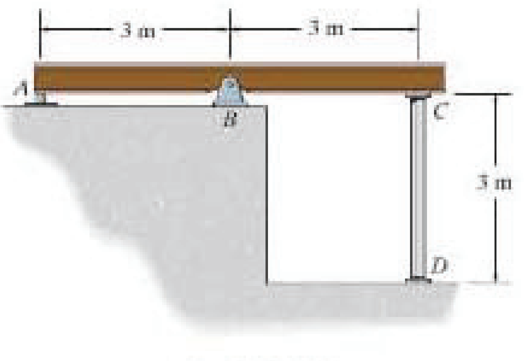
Prob. 12–134

12–135. The 1-in.-diameter A-36 steel shaft is supported by unyielding bearings at A and C. The bearing at B rests on a simply supported steel wide-flange beam having a moment of inertia of $I = 500\ \text{in}^4$. If the belt loads on the pulley are 400 lb each, determine the vertical reactions at A, B, and C.



Prob. 12–135

***12–136.** If the temperature of the 75-mm-diameter post CD is increased by 60°C, determine the force developed in the post. The post and the beam are made of A-36 steel, and the moment of inertia of the beam is $I = 255(10^6)\ \text{mm}^4$.

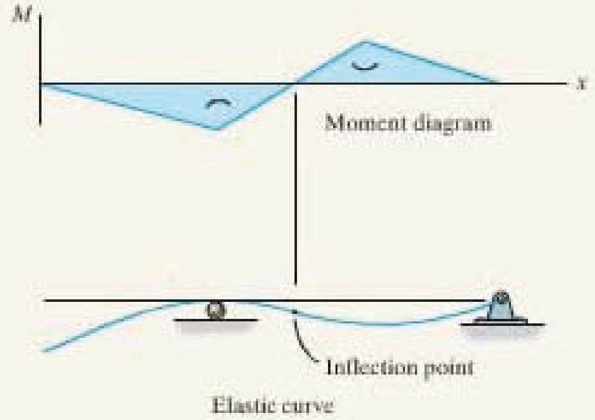


Prob. 12–136

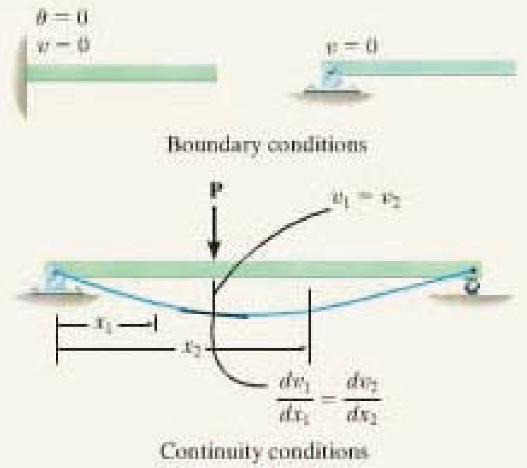
CHAPTER REVIEW

The elastic curve represents the centerline deflection of a beam or shaft. Its shape can be determined using the moment diagram. Positive moments cause the elastic curve to be concave upwards and negative moments cause it to be concave downwards. The radius of curvature at any point is determined from

$$\frac{1}{\rho} = \frac{M}{EI}$$

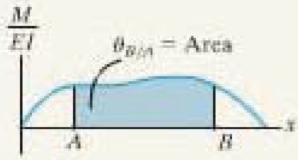


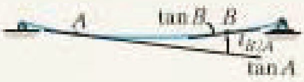
The equation of the elastic curve and its slope can be obtained by first finding the internal moment in the member as a function of x. If several loadings act on the member, then separate moment functions must be determined between each of the loadings. Integrating these functions once using $EI(d^2v/dx^2) = M(x)$ gives the equation for the slope of the elastic curve, and integrating again gives the equation for the deflection. The constants of integration are determined from the boundary conditions at the supports, or in cases where several moment functions are involved, continuity of slope and deflection at points where these functions join must be satisfied.

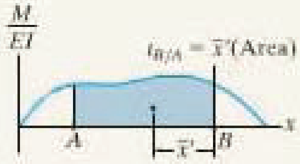


Discontinuity functions allow one to express the equation of the elastic curve as a continuous function, regardless of the number of loadings on the member. This method eliminates the need to use continuity conditions, since the two constants of integration can be determined solely from the two boundary conditions.

The moment-area method is a semi-graphical technique for finding the slope of tangents or the vertical distance between tangents at specific points on the elastic curve. It requires finding area segments under the M/EI diagram, or the moment of these segments about points on the elastic curve. The method works well for M/EI diagrams composed of simple shapes, such as those produced by concentrated forces and couple moments.







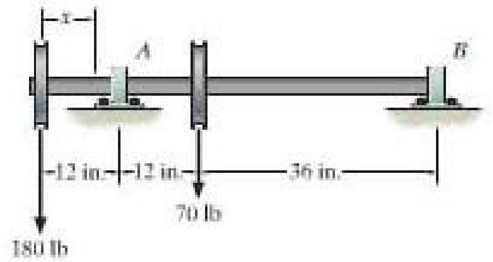
The deflection or slope at a point on a member subjected to combinations of loadings can be determined using the method of superposition. The table in Appendix C is available for this purpose.

Statically indeterminate beams and shafts have more unknown support reactions than available equations of equilibrium. To solve, one first identifies the redundant reactions. The method of integration or the moment-area theorems can then be used to solve for the unknown redundants. It is also possible to determine the redundants by using the method of superposition, where one considers the conditions of continuity at the redundant. Here the displacement due to the external loading is determined with the redundant removed, and again with the redundant applied and the external loading removed. The tables in Appendix C can be used to determine these necessary displacements.

12

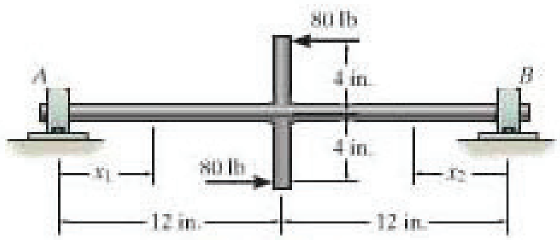
REVIEW PROBLEMS

•**12–137.** The shaft supports the two pulley loads shown. Using discontinuity functions, determine the equation of the elastic curve. The bearings at A and B exert only vertical reactions on the shaft. EI is constant.



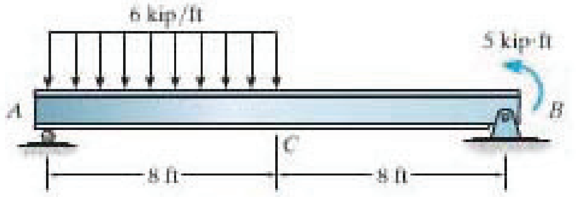
Prob. 12–137

12–138. The shaft is supported by a journal bearing at A, which exerts only vertical reactions on the shaft, and by a thrust bearing at B, which exerts both horizontal and vertical reactions on the shaft. Draw the bending-moment diagram for the shaft and then, from this diagram, sketch the deflection or elastic curve for the shaft's centerline. Determine the equations of the elastic curve using the coordinates x_1 and x_2. EI is constant.



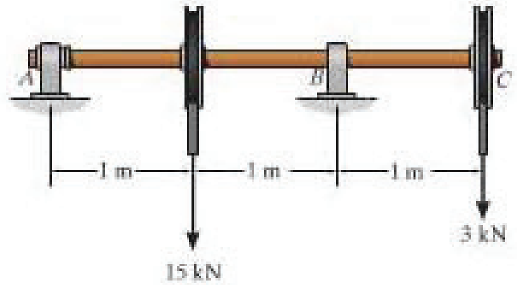
Prob. 12–138

12–139. The W8 × 24 simply supported beam is subjected to the loading shown. Using the method of superposition, determine the deflection at its center C. The beam is made of A-36 steel.



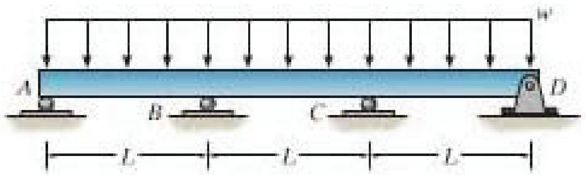
Prob. 12–139

*__12–140.__ Using the moment-area method, determine the slope and deflection at end C of the shaft. The 75-mm-diameter shaft is made of material having $E = 200$ GPa.



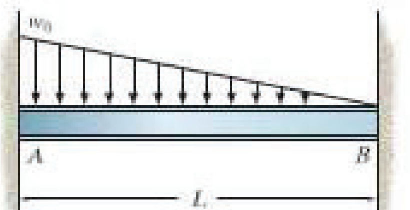
Prob. 12–140

•**12–141.** Determine the reactions at the supports. EI is constant. Use the method of superposition.



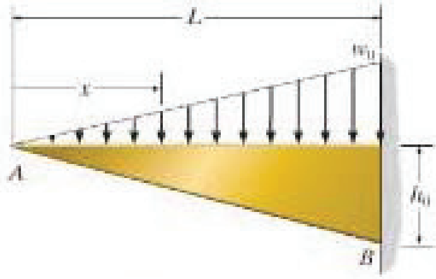
Prob. 12–141

12–142. Determine the moment reactions at the supports A and B. Use the method of integration. EI is constant.



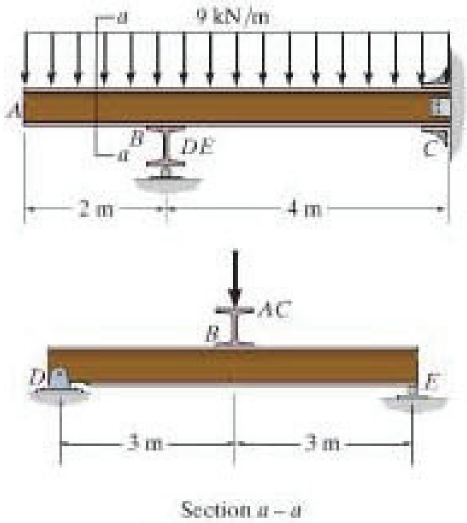
Prob. 12–142

12–143. If the cantilever beam has a constant thickness t, determine the deflection at end A. The beam is made of material having a modulus of elasticity E.



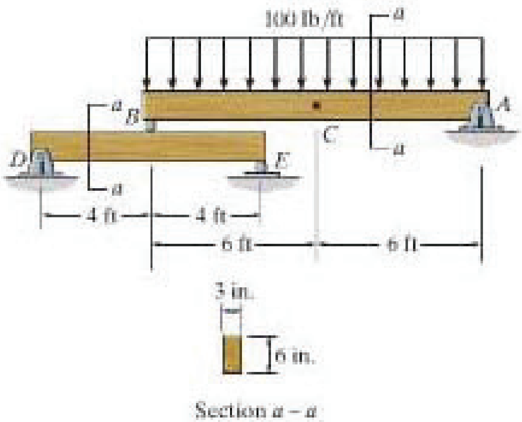
Prob. 12–143

***12–144.** Beam ABC is supported by beam DBE and fixed at C. Determine the reactions at B and C. The beams are made of the same material having a modulus of elasticity $E = 200$ GPa, and the moment of inertia of both beams is $I = 25.0(10^6)$ mm^4.



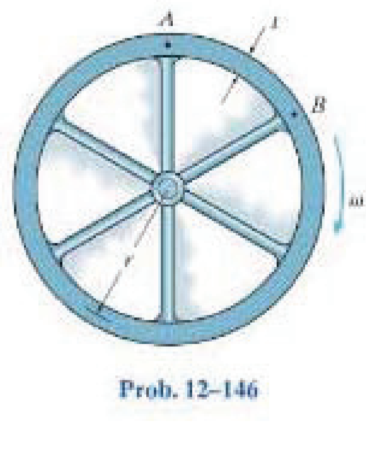
Prob. 12–144

•**12–145.** Using the method of superposition, determine the deflection at C of beam AB. The beams are made of wood having a modulus of elasticity of $E = 1.5(10^3)$ ksi.



Prob. 12–145

12–146. The rim on the flywheel has a thickness t, width b, and specific weight γ. If the flywheel is rotating at a constant rate of ω, determine the maximum moment developed in the rim. Assume that the spokes do not deform. *Hint:* Due to symmetry of the loading, the slope of the rim at each spoke is zero. Consider the radius to be sufficiently large so that the segment AB can be considered as a straight beam fixed at both ends and loaded with a uniform centrifugal force per unit length. Show that this force is $w = bt\gamma\omega^2 r/g$.



Prob. 12–146