Logic, Algebra and Computation

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CPL: Classical Proposition Logic

- Syntax:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi$$

- Semantics:

  True- False

- CLP- formulas denote propositions

CPL: Classical Proposition Logic

- Proof system for CLP:
  - Axioms- Deductive rules- Theorem- Deduction $\Gamma \vdash \varphi$
- Model Theory for CLP:
  - Valuations

$$V\colon \text{Atoms} \longrightarrow \{T, F\}$$

$$\bar{V}\colon \text{Formulas} \longrightarrow \{T, F\}$$

$$\bar{V}(\neg\varphi) \neq \bar{V}(\varphi)$$

$$\bar{V}(\bot) = F$$

$$\bar{V}(\varphi \vee \psi) = F \text{ iff } \bar{V}(\varphi) = \bar{V}(\psi) = F.$$

- Semantical consequence relation $\Gamma \vDash \varphi$.

Elementary Algebra as Logic

- Algebraic equations
- Syntax:
  - Terms or formulas in the language

$$x, y, z, \ldots., +, . , -.$$

- Semantics:
  - Terms as matrices or real numbers together with operations between them.

Programming Languages

- Syntax describes the form of a valid program.

- Semantics describes the meaning of the program or the result of executing that program.

Boolean Algebra

- $\mathcal{A} = (A, +, -, 0)$ where

$$a.\,b = -(-a + -b)$$

$$1 = -0$$

- Examples of BA
  - $\mathbf{2} = (\{0, 1\}, +, -, 0)$

$$-a = 1 - a$$

$$a + b = \max(a, b)$$

  - Power Set Algebra

A a set

$$\mathbf{B(A)} = (P(A), \cup, -, \emptyset)$$

- A Set Algebra is a subalgebra of a Power Set Algebra.

- The set of propositional formulas (Form) can be considered as an Algebra:

$$\mathcal{F}\!orm = (\text{Form}, +, -, \bot)$$

$$-\varphi = \neg\varphi$$

$$\varphi + \psi = \varphi \vee \psi$$

- A valuation can be considered as a homomorphism between algebras:

$$\theta\colon \mathcal{F}\!orm \longrightarrow \mathbf{2}$$

$$\theta(\bot) = 0$$

$$\theta(\neg\varphi) = 1 - \theta(\varphi)$$

$$\theta(\varphi \vee \psi) = \max\,(\theta(\varphi)\,,\theta(\psi))$$

- *Theorem*:
  1. $\vDash \varphi$ iff $\mathbf{2} \vDash \varphi \approx \top$
  2. $\mathbf{2} \vDash \varphi \approx \top$ iff $\vDash \varphi \leftrightarrow \psi$
  3. $\vDash \varphi \leftrightarrow (\varphi \leftrightarrow \top)$

- An equation $s \approx t$ is valid in an algebra if for any assignment to the variables occurring in s and t they have the same value (meaning) in the algebra.

Lindenbaum-Tarski Algebra

- $\mathcal{L} = (\text{Form}/{\equiv}, +, -, 0)$

$$\varphi \equiv \psi \quad \text{iff} \quad \vdash \varphi \leftrightarrow \psi$$

$$[\varphi] + [\psi] = [\varphi \vee \psi]$$

$$-[\varphi] = [\neg \varphi]$$

$$0 = [\bot]$$

- *Proposition*:
  1. $\mathcal{L}$ is a Boolean Algebra
  2. $\vdash \varphi$ iff $\mathcal{L} \vdash \varphi \approx \top$

Proof of 2: (⇒) Easy.

(⇐) Define $e(p)=[p] \Rightarrow e(\varphi)=[\varphi] \Rightarrow [\varphi]=1 \Rightarrow \vdash \varphi \leftrightarrow \top \Rightarrow \vdash \varphi$.

- *Corollary(Algebraic Completeness Theorem):*

$$\vdash \varphi \text{ iff } \mathcal{BA} \vDash \varphi \approx \top$$

Proof: (⇒) Induction on the complexity of proofs.

(⇐) $\mathcal{BA} \vDash \varphi \approx \top \Rightarrow$

$\mathcal{L} \vDash \varphi \approx \top \Rightarrow$

$\vdash \varphi$.

- *Stone Representation Theorem:*

  Any Boolean Algebra is isomorphic to a set algebra.

- *Note:* This theorem shows that we have completeness w.r.t. concrete Boolean Algebras.

Modal Logic

- Syntax:

$$\varphi \ ::= \ p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi$$

$$\Box\varphi :\models \neg\Diamond\neg\,\varphi$$

- Kripke semantics for modal logic:

  $F = (W, R)$ frame

  W a set

  R a binary relation on W.

- $M = (W, R, V)$ a Kripke model
  - $M, w \Vdash p$ iff $w \in V(p)$
  - $M, w \nVdash \bot$
  - $M, w \Vdash \neg\varphi$ iff $M, w \nVdash \varphi$
  - $M, w \Vdash \varphi \vee \psi$ iff $M, w \Vdash \varphi$ or $M, w \Vdash \psi$
  - $M, w \Vdash \Diamond\varphi$ iff $\exists v$ (Rwv and $M, v \Vdash \varphi$).

- Example of valid formulas:

(K) $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

(Dual) $\Box p \rightarrow \neg\Diamond\neg\, p$

Normal modal Logic

- A set *Λ* of modal formulas containing (K) and (Dual) and closed under the following rules

(MP),

(Generalization or Nessesitation) and

(Uniform substitution).

- *Definition*: If $\varphi \in \Lambda$ we say φ is a theorem of Λ and write $\vdash_{\Lambda} \varphi$.
- K= The smallest normal modal logic.

- *Definition:* For a class F of formulas,

$$\Lambda_{\mathrm{F}} = \{\varphi : \varphi \text{ is valid in F}\}.$$

- *Definition:* A normal modal logic is sound w.r.t. Λ if $\Lambda \subseteq \Lambda_{\mathrm{F}}$, and complete w.r.t. Λ if $\Lambda_{\mathrm{F}} \subseteq \Lambda$.

- *Definition:* A normal modal logic is complete if it is the logic of some frame F, i.e. $\Lambda = \Lambda_F$.

- *Theorem:*
  1. K is the logic of all frames
  2. K4 = K + ◊◊p→ p is the logic of transitive frames
  3. T = K + p → ◊p is the logic of reflexive frames.

- *Fact:* There are modal logics that are not the logic of any Frame, a weak point for Frame-semantics.

Boolean Algebras with operator (BAO)

- $\mathbb{A} = (A, +, -, 0, f_\Diamond)$ where

$$f_\Diamond : A \longrightarrow A$$

$$f_\Diamond(0) = 0$$

$$f_\Diamond(x + y) = f_\Diamond(x) + f_\Diamond(y)$$

- In the definition of assignment

$$\theta : \mathcal{Form} \longrightarrow \mathbb{A}$$

$$\theta(\Diamond\varphi) = f_\Diamond(\theta(\varphi))$$

Lindenbaum- Tarski Algebra for Modal Logic

- If Λ is normal modal logic, then

$$\mathcal{L}_{\Lambda} = (\,\mathrm{Form}/{\equiv_{\Lambda}}, +, -, 0, f_{\Diamond}\,)$$

$$f_{\Diamond}([\varphi]) = [\Diamond\varphi]$$

- *Proposition*: $\mathcal{L}_{\Lambda}$ is a Boolean Algebra with operator.

- *Proposition*: $\vdash_{\Lambda} \varphi$ iff $\mathcal{L}_{\Lambda} \vDash_{\Lambda} \varphi \approx \top$.

- *Corollary(Algebraic Completeness of Λ)*

$$\vdash_{\Lambda} \varphi \text{ iff } \mathcal{BAO} \vDash \varphi \approx \top.$$

- *Definition(Complex Algebra):* $F = (W,R)$ a frame and $X \subseteq W$
  - $m_R(X) \overset{def}{=} \{w \in W: Rwx \text{ for some } x \in X\}$
- *Note:* $V(\Diamond\varphi) = m_R(v(\varphi))$ for any valuation.

- Full Complex Algebra of $F = F^+ = (\mathbb{P}(W), \cup, -, \emptyset, m_R)$.

- A Complex Algebra is a subalgebra of a full complex Algebra.
- Complex Algebras are Concrete Boolean Algebra with Operator.

- *Theorem (Jonsson- Tarski):* Every $\mathcal{BAO}$ is isomorphic to a Complex Algebra.

- This theorem is a generalization of Ston's representation theorem to the new context of Modal Logic.

Applications of Modal Logic in TCS

- Temporal logic
  - □ ⇝ Always true
  - ◊ ⇝ Sometime true

- Temporal logic can be used to express properties of a transition system.

- Indeed, a transition system can be considered as certain Kripke model. In each state, some propositions are true.

- Extra temporal operators can be used.
  - Linear- time operators:
    - Xp : p holds true next time
    - Fp : p holds true sometime in the future
    - Gp : p holds true globally in the future
    - PUq : p holds true until q holds true
  - Path quantifiers:
    - A : for every path
    - E : there exists a path
- Model checking: To check whether a real system interpreted as a transition system satisfies certain good condition.

Epistemic Logic

- $\Box_a \varphi$: Agent a knows φ
- $E\varphi$: Everyone knows φ
- $C\varphi$: Everyone knows φ and everyone knows that everyone knows φ and …

- Epistemic logic is used in AI and Game theory.

Dynamic Logic

- DL is a formal system for reasoning about programs.
- A correctness specification is a formal description (in the language of DL) of how the program is supposed to behave.
- A given program is correct if its behavior fulfills the specification.

- Some expressions of DL:
  - $[\alpha](\varphi \wedge \psi) \leftrightarrow [\alpha]\varphi \wedge [\alpha]\psi$

    After each execution of program α, $\varphi \wedge \psi$ is true iff after each execution of α, φ and ψ is true.
  - $[\alpha; \beta]\, \varphi \leftrightarrow [\alpha]\, [\beta]\varphi$

    Where $[\alpha; \beta]$ is sequentional composition
  - $\alpha \cup \beta$ nondeterministic choice
  - α^* iteration
  - $\varphi?$ test

IPL: Intuitionistic Propositional Logic

- IPL is obtained from CPL by omitting PEM: $\varphi \vee \neg \varphi$

- Semantics: Kripke Model- Heyting Algebra

- **BHK**-interpretation of IPL is a basis for λ-Calculus.
  - *Example:* a proof of $\varphi \rightarrow \psi$ is a construction(function) which maps each proof of φ to a proof of ψ.

- Term

  $\lambda x^{A}. x$

  $\lambda x^{A}. \lambda y^{B}. x$

- Type

  $A \rightarrow A$

  $A \rightarrow (B \rightarrow A)$

- Some Typing rules

$$\frac{\Gamma \vdash M: A \rightarrow B \qquad \Gamma \vdash N: A}{\Gamma \vdash MN:B}$$

- Corresponding Natural Deduction rule in IPL

$$\frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Correspondence

Logic	Programming
Intuitionistic Logic	λ-Calculus
Formulas	Types
Proofs	Terms
Simplifications	Reduction

Fuzzy Logic

- PC($*$)

  $*$ a t-norm

  $\Rightarrow_*$: the implication related to $*$

- Language: $p_1, p_2, \ldots$

  $\&, \rightarrow, \neg, \wedge, \vee, \bar{0}$

- Valuations:

$$V: \text{Atoms} \longrightarrow [0, 1]$$

$$\bar{V}: \text{Formulas} \longrightarrow [0, 1]$$

$$\bar{V}(\varphi \& \psi) = \bar{V}(\varphi) * \bar{V}(\psi)$$

$$\bar{V}(\varphi \rightarrow \psi) = \bar{V}(\varphi) \Rightarrow_* \bar{V}(\psi)$$

$$\bar{V}(\neg\varphi) = (-)_* (\varphi) = \bar{V}(\varphi) \Rightarrow_* \mathbf{0}$$

$$\bar{V}(\varphi \wedge \psi) = \min(\bar{V}(\varphi), \bar{V}(\psi))$$

$$\bar{V}(\varphi \vee \psi) = \max(\bar{V}(\varphi), \bar{V}(\psi)).$$

- φ is a tautology of PC($*$) if for every valuation V in PC($*$), V(φ)=1.

- BL: Basic Logic introduced by Hajek
- BL-Algebra: the algebraic semantics for BL

- *Theorem:* BL is sound and complete w.r.t. BL-algebras.

- *Theorem:* BL is the logic of t-norms (providing concrete BL-algebras).

Thank you for your attention