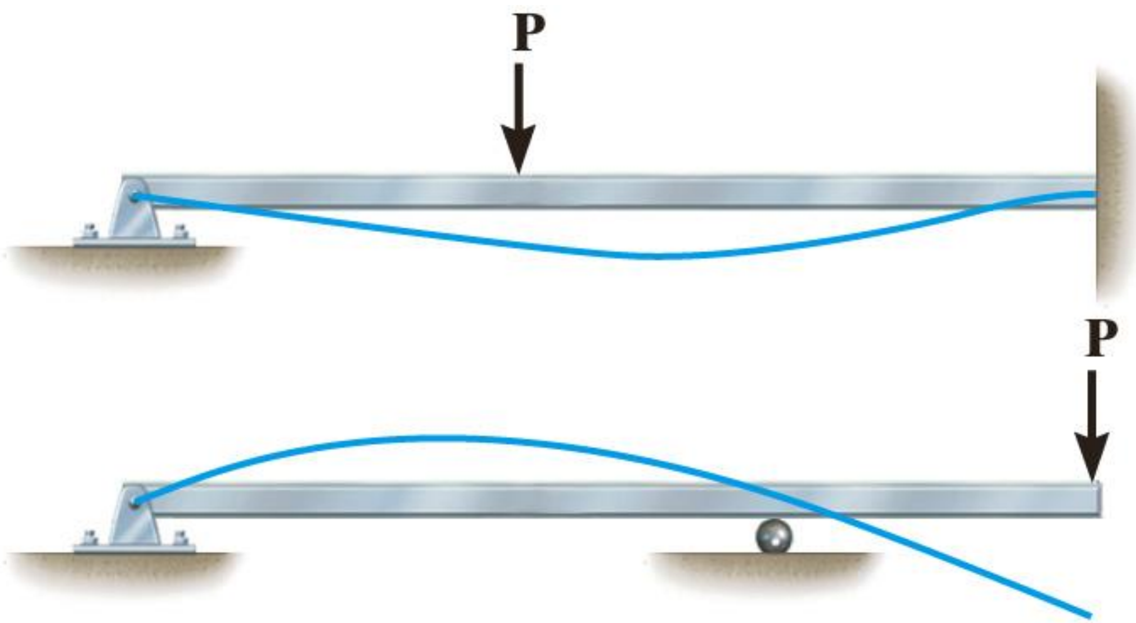
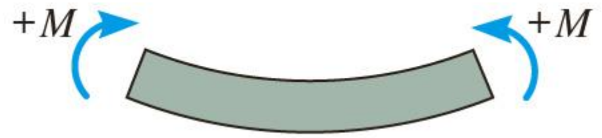

PORTLAND CEMENT

۱۲-۱: منحنی الاستیک



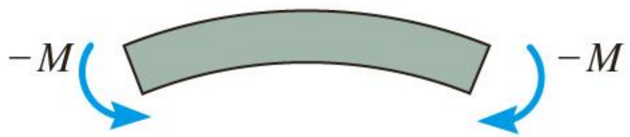
توجه کنید که دیاگرام تغییر شکل (منحنی الاستیک) برای هر تیر متفاوت است. تکیه گاه ها چه تاثیری بر شیب و تغییر شکل دارند؟

دیاگرام گشتاور و علامت آن می تواند یک ابزار کمکی عالی در ترسیم منحنی الاستیک باشد.



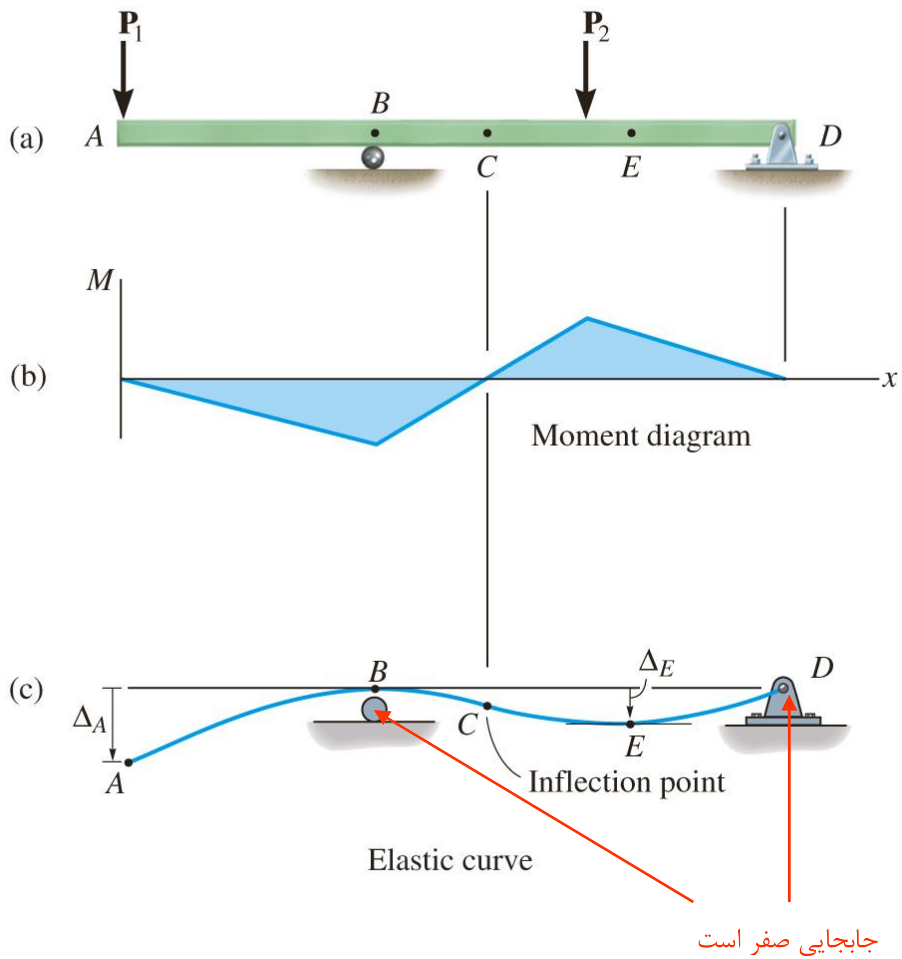
Positive internal moment
concave upwards

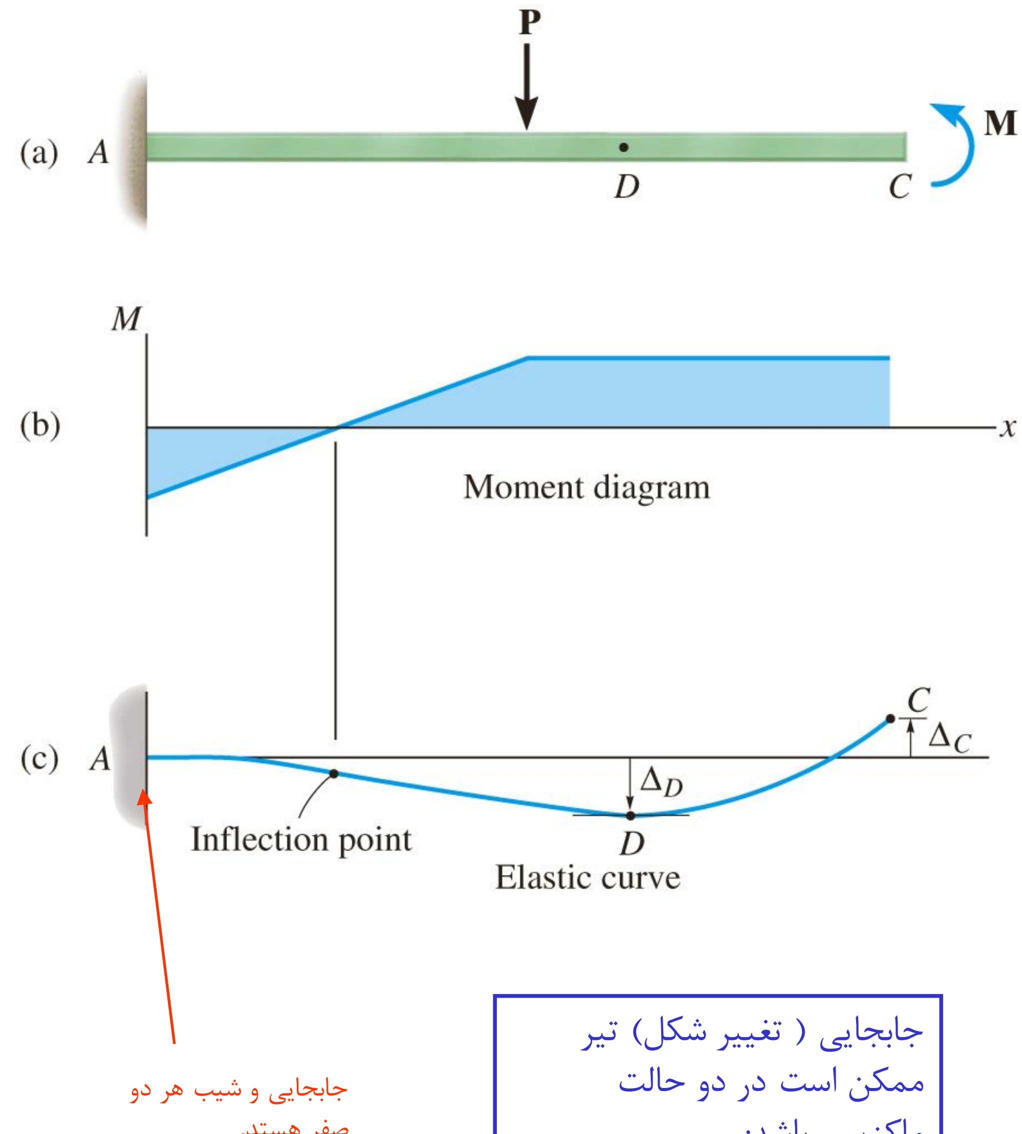
(a)



Negative internal moment
concave downwards

(b)



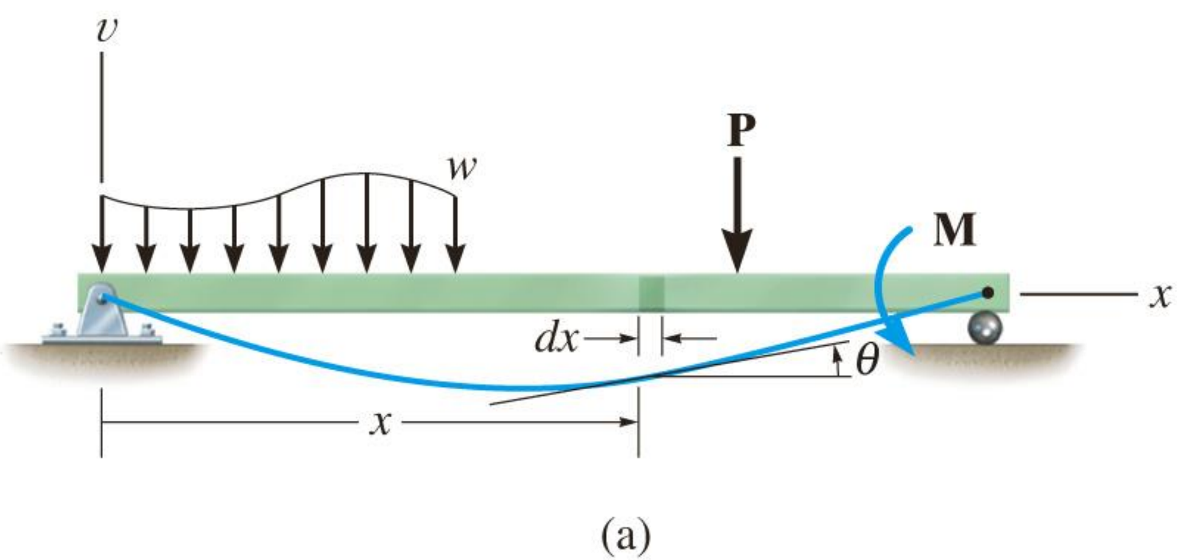


جابجایی (تغییر شکل) تیر
ممکن است در دو حالت
ماکزیمم باشد:

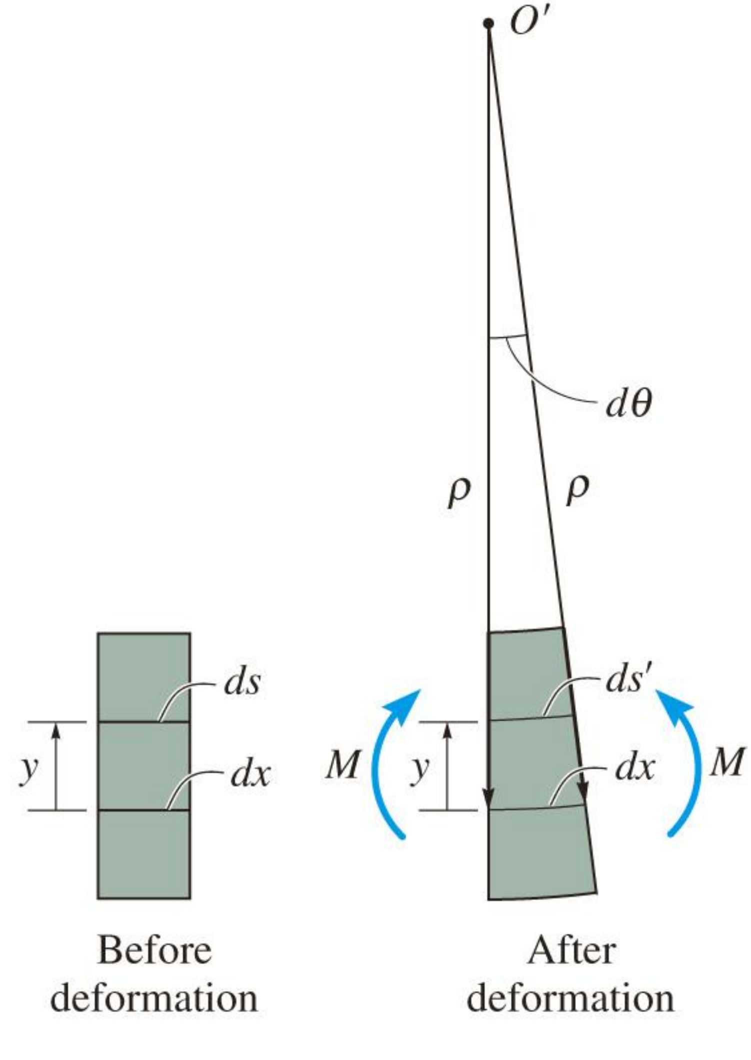
۱) جایی که شیب صفر است.

۲) در منتهی الیه تیر

ارتباط بین گشتاور و انحنا

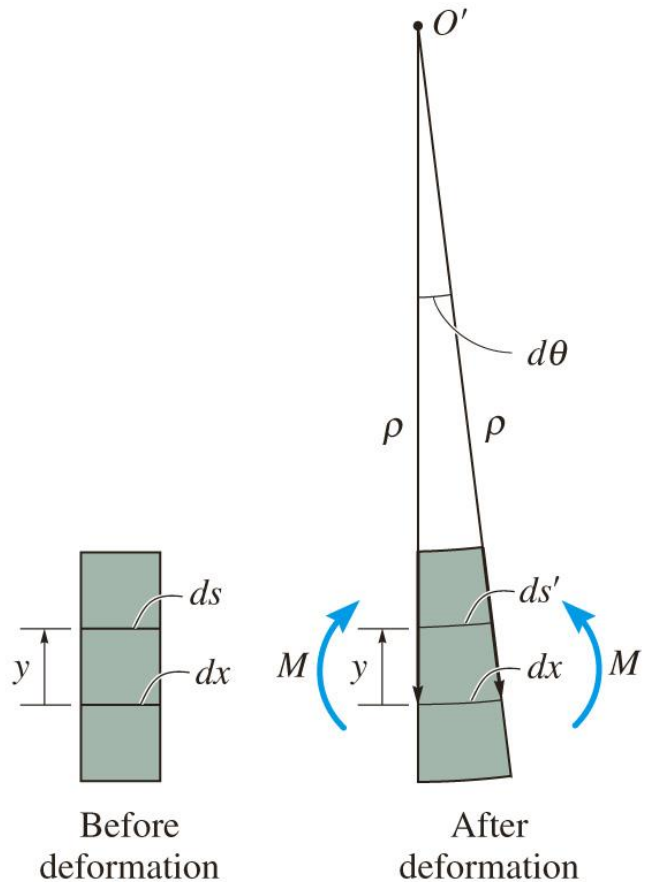


(a)



(b)

تعریف می کنیم : X, υ



ممان داخلی

$$\frac{1}{\rho} = \frac{M}{EI}$$

شاع انحنا

مدول الاستیسیته

ممان اینرسی

۲-۱۲: محاسبه شیب و جابجایی با روش انتگرال گیری

$$\frac{1}{\rho} = \frac{M}{E\,I}$$

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}} \quad , \left(\frac{dv}{dx}\right)^2 \ll 1 \longrightarrow \frac{1}{\rho} = \frac{d^2v}{dx^2}$$

بنابراین:

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

صلبیت خمشی = EI

فرض میکنیم EI در طول تیر ثابت است.

$$EI\frac{d^4v}{dx^4}=w(x)$$

یا

$$EI\frac{d^3v}{dx^3}=V(x)$$

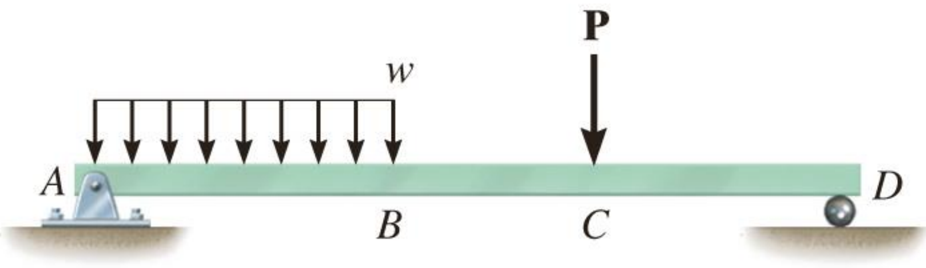
یا

$$EI\frac{d^2v}{dx^2}=M(x)$$

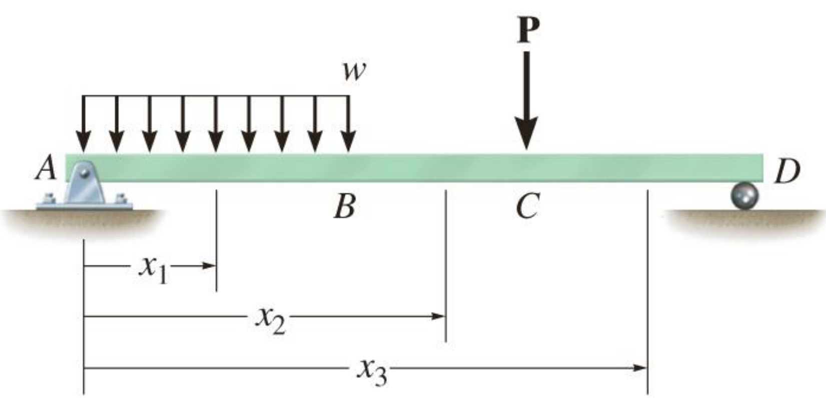
هر یک از این معادلات دیفرانسیلی می توانند برای یافتن $\upsilon(x)$ استفاده شوند. کافی است فقط انتگرال بگیریم و با استفاده از شرایط مرزی ثوابت انتگرال را بدست آوریم.

این روش را انتگرال گیری مستقیم برای محاسبه جابجایی تیر می نامند.

توجه کنید که برای هر بخش (هر زمانی که معادله گشتاور تغییر کند) به M(x) نیاز داریم.

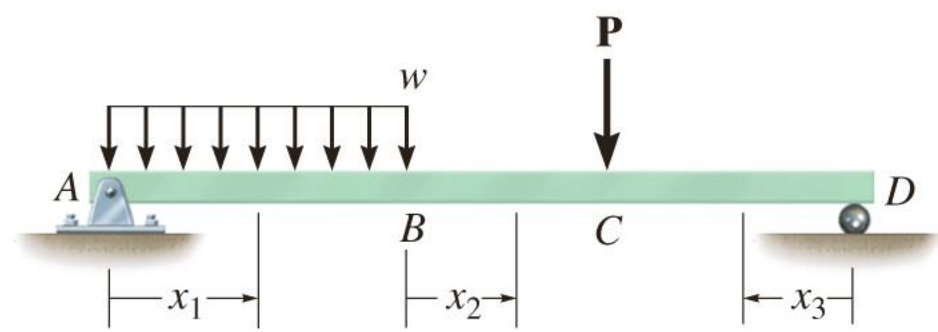


(a)



(b)

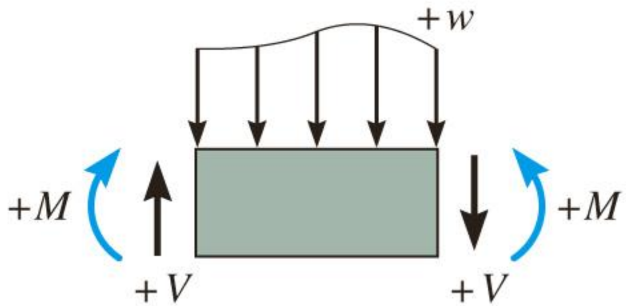
یا



(c)

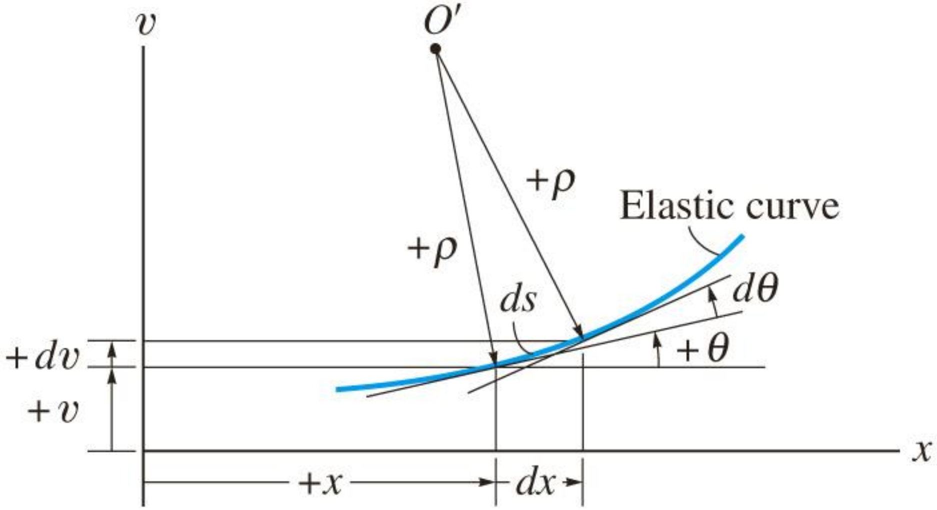
قرارداد علامتها:

همه مقادیر مثبت نشان داده شده اند.



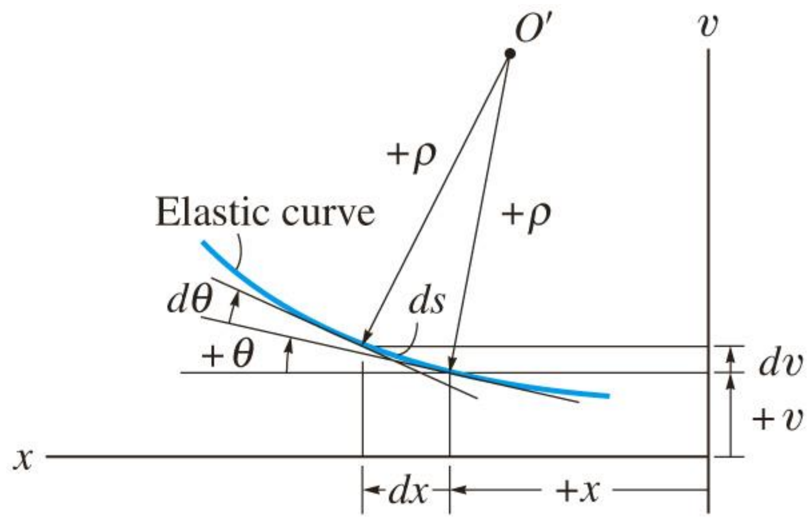
Positive sign convention

(a)



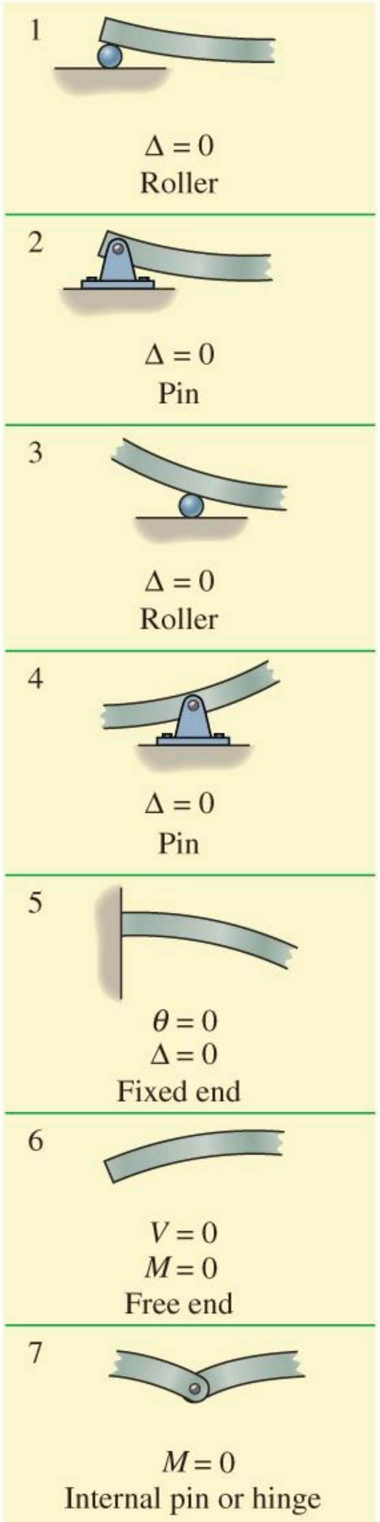
Positive sign convention

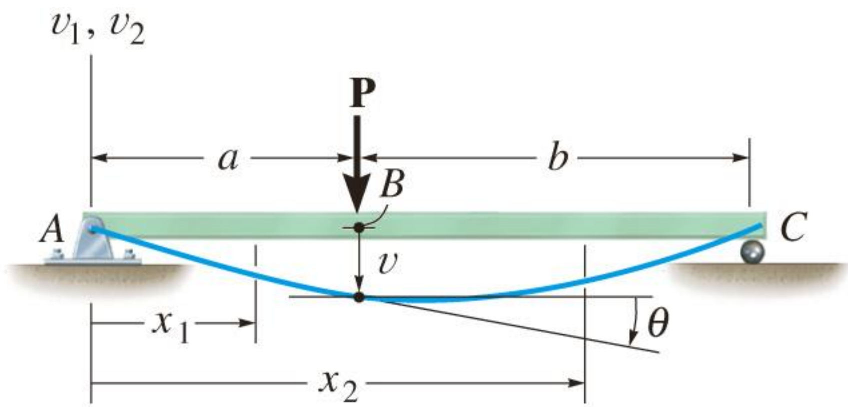
(b)



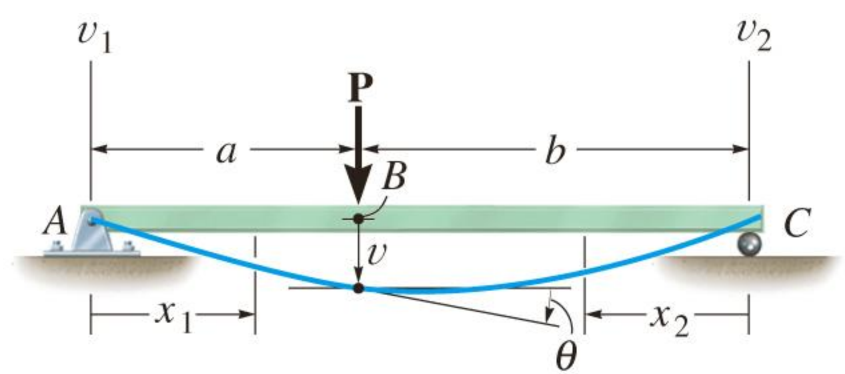
Positive sign convention

(c)


1
Δ = 0
Roller
2
Δ = 0
Pin
3
Δ = 0
Roller
4
Δ = 0
Pin
5
θ = 0
Δ = 0
Fixed end
6
V = 0
M = 0
Free end
7
M = 0
Internal pin or hinge



(a)



(b)

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EXAMPLE 12.1

The cantilevered beam shown in Fig. 12–10*a* is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.

Solution I

Elastic Curve. The load tends to deflect the beam as shown in Fig. 12–10*a*. By inspection, the internal moment can be represented throughout the beam using a single *x* coordinate.

Moment Function. From the free-body diagram, with **M** acting in the *positive direction*, Fig. 12–10*b*, we have

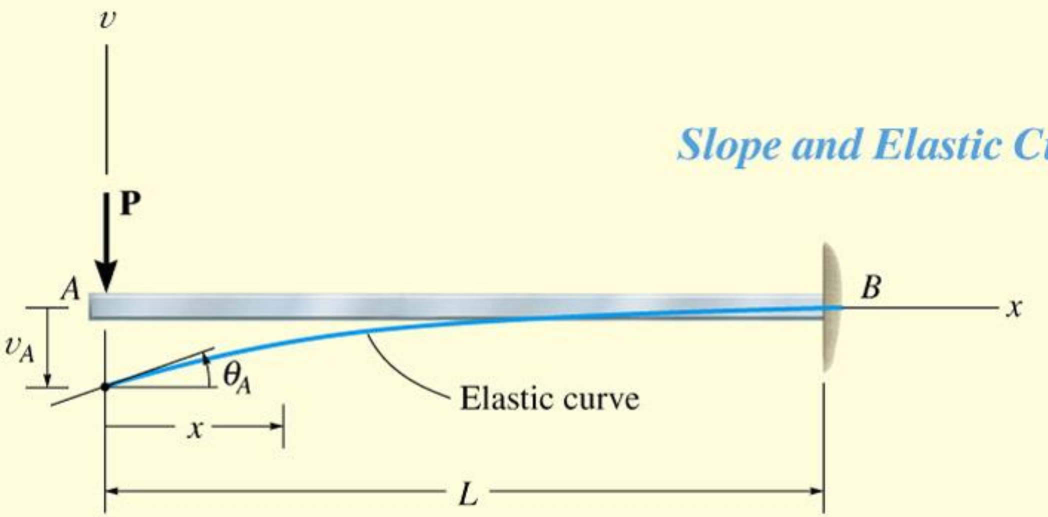
$$M = -Px$$

Slope and Elastic Curve. Applying Eq. 12–10 and integrating twice yields

$$EI\frac{d^2v}{dx^2} = -Px \qquad (1)$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \qquad (2)$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \qquad (3)$$



(a)

Using the boundary conditions $dv/dx = 0$ at $x = L$ and $v = 0$ at $x = L$, Eqs. 2 and 3 become

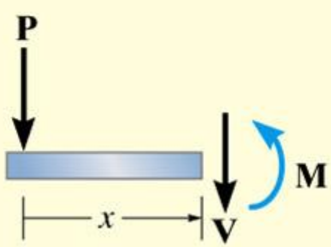
$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1L + C_2$$

Thus, $C_1 = PL^2/2$ and $C_2 = -PL^3/3$. Substituting these results into Eqs. 2 and 3 with $\theta = dv/dx$, we get

$$\theta = \frac{P}{2EI}(L^2 - x^2)$$

$$v = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \qquad \textit{Ans.}$$



(b)

Fig. 12–10

Maximum slope and displacement occur at $A(x = 0)$, for which

$$\theta_A = \frac{PL^2}{2EI} \tag{4}$$

$$v_A = -\frac{PL^3}{3EI} \tag{5}$$

The *positive* result for θ_A indicates *counterclockwise* rotation and the *negative* result for v_A indicates that v_A is *downward*. This agrees with the results sketched in Fig. 12–10*a*.

In order to obtain some idea as to the actual *magnitude* of the slope and displacement at the end A, consider the beam in Fig. 12–10*a* to have a length of 15 ft, support a load of $P = 6$ kip, and be made of A-36 steel having $E_{st} = 29(10^3)$ ksi. Using the methods of Sec. 11.3, if this beam was designed without a factor of safety by assuming the allowable normal stress is equal to the yield stress $\sigma_{allow} = 36$ ksi, then a W12 $\times$ 26 would be found to be adequate ($I = 204$ in.^{4}). From Eqs. 4 and 5 we get

$$\theta_A = \frac{6 \text{ kip}(15 \text{ ft})^2(12 \text{ in./ft})^2}{2[29(10^3) \text{ kip/in.}^2](204 \text{ in.}^4)} = 0.0164 \text{ rad}$$

$$v_A = -\frac{6 \text{ kip}(15 \text{ ft})^3(12 \text{ in./ft})^3}{3[29(10^3) \text{ kip/in.}^2](204 \text{ in.}^4)} = -1.97 \text{ in.}$$

Since $\theta_A^2 = (dv/dx)^2 = 0.000270 \text{ rad}^2 \ll 1$, this justifies the use of Eq. 12–10, rather than applying the more exact Eq. 12–4, for computing the deflection of beams. Also, since this numerical application is for a *cantilevered beam*, we have obtained *larger values* for θ and v than would have been obtained if the beam was supported using pins, rollers, or other fixed supports.

Solution II

This problem can also be solved using Eq. 12–8, $EI\, d^4v/dx^4 = -w(x)$. Here $w(x) = 0$ for $0 \le x \le L$, Fig. 12–10*a*, so that upon integrating once we get the form of Eq. 12–9, i.e.,

$$EI \frac{d^4v}{dx^4} = 0$$

$$EI \frac{d^3v}{dx^3} = C_1' = V$$

The shear constant C_1' can be evaluated at $x = 0$, since $V_A = -P$ (negative according to the beam sign convention, Fig. 12–8*a*.) Thus, $C_1' = -P$. Integrating again yields the form of Eq. 12–10, i.e.,

$$EI \frac{d^3v}{dx^3} = -P$$

$$EI \frac{d^2v}{dx^2} = -Px + C_2' = M$$

Here $M = 0$ at $x = 0$, so $C_2' = 0$, and as a result one obtains Eq. 1 and the solution proceeds as before.

EXAMPLE 12.2

The simply supported beam shown in Fig. 12–11*a* supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.

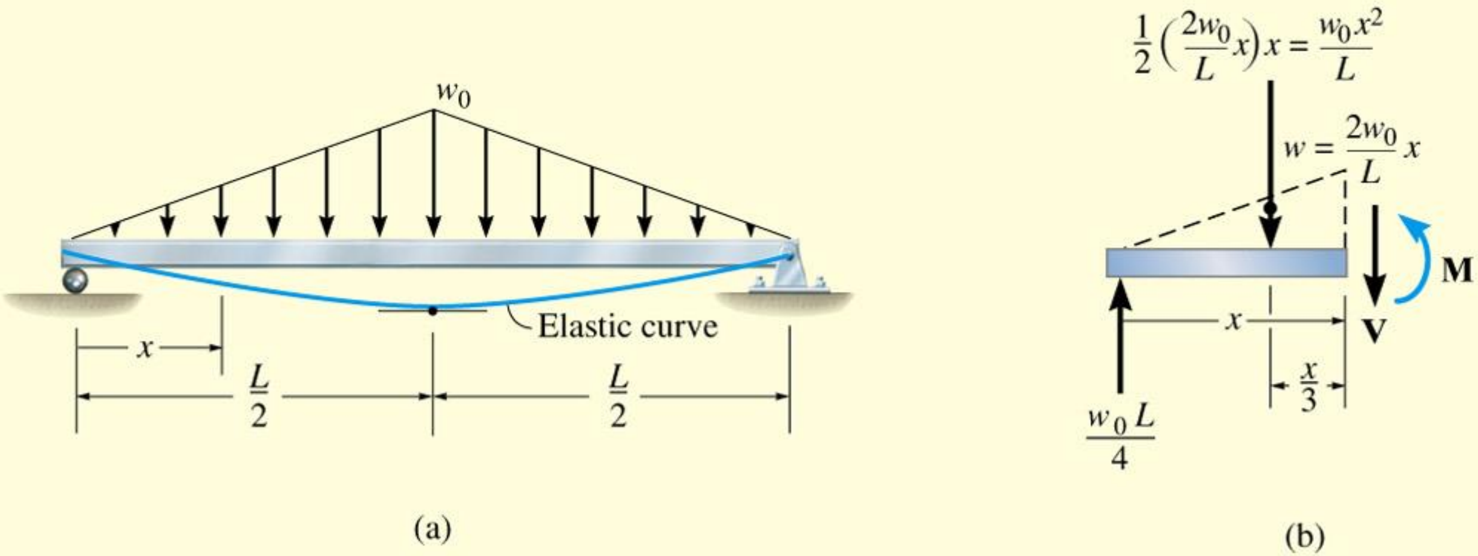


Fig. 12–11

Solution I

Elastic Curve. Due to symmetry, only one x coordinate is needed for the solution, in this case $0 \le x \le L/2$. The beam deflects as shown in Fig. 12–11*a*. Notice that maximum deflection occurs at the center since the slope is zero at this point.

Moment Function. The distributed load acts downward, and therefore it is positive according to our sign convention. A free-body diagram of the segment on the left is shown in Fig. 12–11*b*. The equation for the distributed loading is

$$w = \frac{2w_0}{L}x \tag{1}$$

Hence,

$$\circlearrowleft + \Sigma M_{NA} = 0; \quad M + \frac{w_0 x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0 L}{4}(x) = 0$$

$$M = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4}x$$

Slope and Elastic Curve.

Using Eq. 12–10 and integrating twice, we have

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x \qquad (2)$$

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

The constants of integration are obtained by applying the boundary condition $v = 0$ at $x = 0$ and the symmetry condition that $dv/dx = 0$ at $x = L/2$. This leads to

$$C_1 = -\frac{5w_0L^3}{192} \qquad C_2 = 0$$

Hence,

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 - \frac{5w_0L^3}{192}$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 - \frac{5w_0L^3}{192}x$$

Determining the maximum deflection at $x = L/2$, we have

$$v_{\max} = -\frac{w_0 L^4}{120EI} \qquad \textit{Ans.}$$

Solution II

Starting with the distributed loading, Eq. 1, and applying Eq. 12–8, we have

$$EI\,\frac{d^4v}{dx^4} = -\frac{2w_0}{L}x$$

$$EI\,\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + C_1'$$

Since $V = +w_0L/4$ at $x = 0$, then $C_1' = w_0L/4$. Integrating again yields

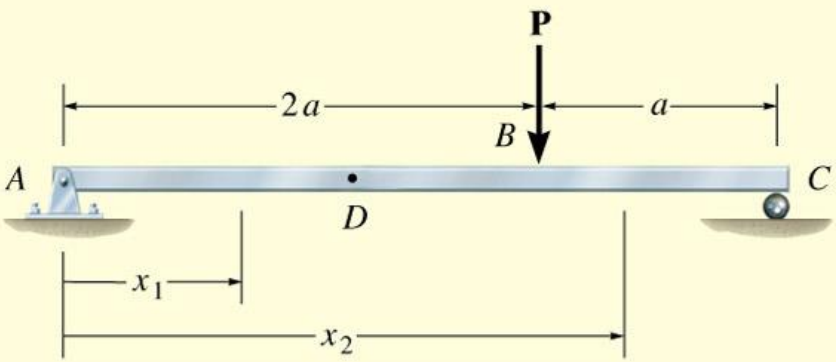
$$EI\,\frac{d^3v}{dx^3} = V = -\frac{w_0}{L}x^2 + \frac{w_0L}{4}$$

$$EI\,\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x + C_2'$$

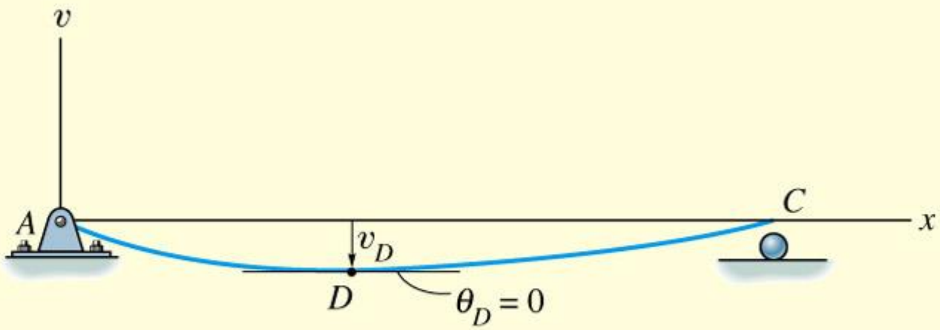
Here $M = 0$ at $x = 0$, so $C_2' = 0$. This yields Eq. 2. The solution now proceeds as before.

EXAMPLE 12.3

The simply supported beam shown in Fig. 12–12*a* is subjected to the concentrated force **P**. Determine the maximum deflection of the beam. *EI* is constant.



(a)



(b)

Solution

Elastic Curve. The beam deflects as shown in Fig. 12–12*b*. Two coordinates must be used, since the moment becomes discontinuous at *P*. Here we will take x_1 and x_2, having the *same origin* at *A*, so that $0 \leq x_1 < 2a$ and $2a < x_2 \leq 3a$.

Moment Function. From the free-body diagrams shown in Fig. 12–12*c*,

$$M_1 = \frac{P}{3}x_1$$

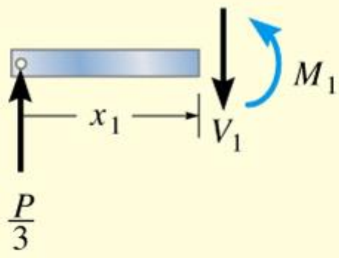
$$M_2 = \frac{P}{3}x_2 - P(x_2 - 2a) = \frac{2P}{3}(3a - x_2)$$

Slope and Elastic Curve. Applying Eq. 12–10 for M_1 and integrating twice yields

$$EI\,\frac{d^2v_1}{dx_1^{\,2}} = \frac{P}{3}x_1$$

$$EI\,\frac{dv_1}{dx^1} = \frac{P}{6}x_1^{\,2} + C_1 \qquad (1)$$

$$EIv_1 = \frac{P}{18}x_1^{\,3} + C_1x_1 + C_2 \qquad (2)$$

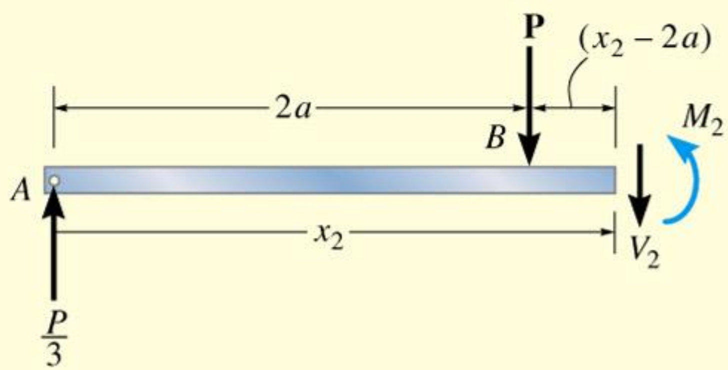


Likewise for M_2,

$$EI\,\frac{d^2v_2}{dx_2^{\,2}} = \frac{2P}{3}(3a - x_2)$$

$$EI\,\frac{dv_2}{dx_2} = \frac{2P}{3}\left(3ax_2 - \frac{x_2^{\,2}}{2}\right) + C_3 \qquad (3)$$

$$EIv_2 = \frac{2P}{3}\left(\frac{3}{2}ax_2^{\,2} - \frac{x_2^{\,3}}{6}\right) + C_3x_2 + C_4 \qquad (4)$$



(c)

Fig. 12–12

The four constants are evaluated using *two* boundary conditions, namely, $x_1 = 0, v_1 = 0$ and $x_2 = 3a, v_2 = 0$. Also, *two* continuity conditions must be applied at B, that is, $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$ and $v_1 = v_2$ at $x_1 = x_2 = 2a$. Substitution as specified results in the following four equations:

$v_1 = 0$ at $x_1 = 0$; $\qquad 0 = 0 + 0 + C_2$

$v_2 = 0$ at $x_2 = 3a$; $\qquad 0 = \dfrac{2P}{3}\left(\dfrac{3}{2}a(3a)^2 - \dfrac{(3a)^3}{6}\right) + C_3(3a) + C_4$

$\dfrac{dv_1(2a)}{dx_1} = \dfrac{dv_2(2a)}{dx_2}$; $\qquad \dfrac{P}{6}(2a)^2 + C_1 = \dfrac{2P}{3}\left(3a(2a) - \dfrac{(2a)^2}{2}\right) + C_3$

$v_1(2a) = v_2(2a)$; $\qquad \dfrac{P}{18}(2a)^3 + C_1(2a) + C_2 = \dfrac{2P}{3}\left(\dfrac{3}{2}a(2a)^2 - \dfrac{(2a)^3}{6}\right) + C_3(2a) + C_4$

Solving these equations we get

$$C_1 = -\frac{4}{9}Pa^2 \qquad C_2 = 0$$

$$C_3 = -\frac{22}{9}Pa^2 \qquad C_4 = \frac{4}{3}Pa^3$$

Thus Eqs. 1–4 become

$$\frac{dv_1}{dx_1} = \frac{P}{6EI}x_1^{\,2} - \frac{4Pa^2}{9EI} \quad (5)$$

$$v_1 = \frac{P}{18EI}x_1^{\,3} - \frac{4Pa^2}{9EI}x_1 \quad (6)$$

$$\frac{dv_2}{dx_2} = \frac{2Pa}{EI}x_2 - \frac{P}{3EI}x_2^{\,2} - \frac{22Pa^2}{9EI} \quad (7)$$

$$v_2 = \frac{Pa}{EI}x_2^{\,2} - \frac{P}{9EI}x_2^{\,3} - \frac{22Pa^2}{9EI}x_2 + \frac{4Pa^3}{3EI} \quad (8)$$

By inspection of the elastic curve, Fig. 12–12*b*, the maximum deflection occurs at *D*, somewhere within region *AB*. Here the slope must be zero. From Eq. 5,

$$\frac{1}{6}x_1^{\,2} - \frac{4}{9}a^2 = 0$$

$$x_1 = 1.633a$$

Substituting into Eq. 6,

$$v_{\max} = -0.484\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.

EXAMPLE 12.4

The beam in Fig. 12–13*a* is subjected to a load **P** at its end. Determine the displacement at C. EI is constant.

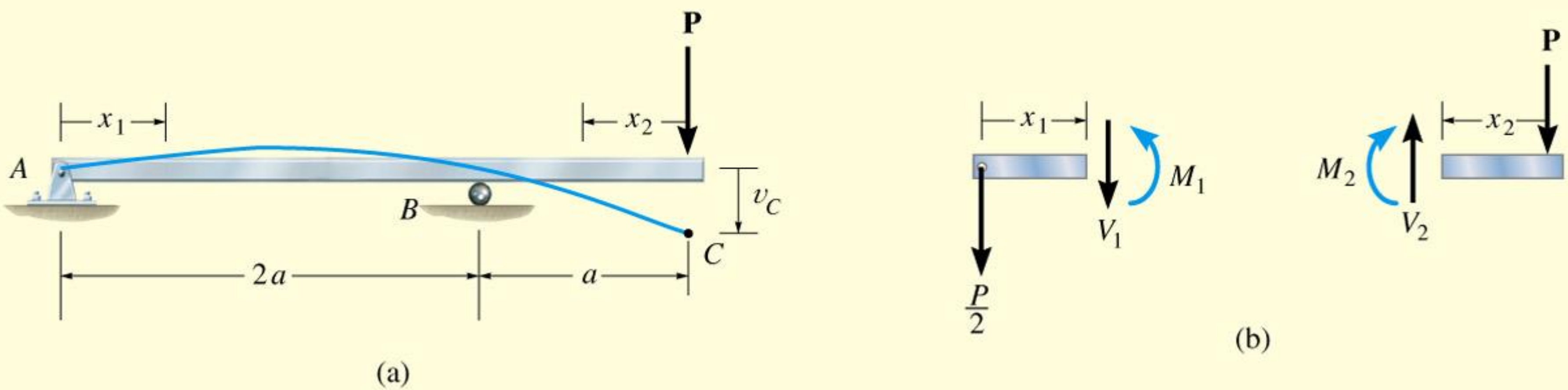


Fig. 12–13

Solution

Elastic Curve. The beam deflects into the shape shown in Fig. 12–13*a*. Due to the loading, two x coordinates will be considered, namely, $0 \leq x_1 < 2a$ and $0 \leq x_2 < a$, where x_2 is directed to the left from C, since the internal moment is easy to formulate.

Moment Functions. Using the free-body diagrams shown in Fig. 12–13*b*, we have

$$M_1 = -\frac{P}{2}x_1 \qquad M_2 = -Px_2$$

Slope and Elastic Curve. Applying Eq. 12–10,

for $0 \le x_1 < 2a$, $\quad EI = \dfrac{d^2v_1}{dx_1{}^2} = -\dfrac{P}{2}x_1$

$$EI\,\frac{dv_1}{dx_1} = -\frac{P}{4}x_1{}^2 + C_1 \tag{1}$$

$$EIv_1 = -\frac{P}{12}x_1{}^3 + C_1x_1 + C_2 \tag{2}$$

For $0 \le x_2 < a$, $\quad EI = \dfrac{d^2v_2}{dx_2{}^2} = -Px_2$

$$EI\,\frac{dv_2}{dx_2} = -\frac{P}{2}x_2{}^2 + C_3 \tag{3}$$

$$EIv_2 = -\frac{P}{6}x_2{}^3 + C_3x_2 + C_4 \tag{4}$$

The *four* constants of integration are determined using *three* boundary conditions, namely, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$, and $v_2 = 0$ at $x_2 = a$ and *one* continuity equation. Here the continuity of slope at the roller requires $dv_1/dx_1 = -dv_2/dx_2$ at $x_1 = 2a$ and $x_2 = a$. Why is there a negative sign in this equation? (Note that continuity of displacement at B has been indirectly considered in the boundary conditions, since $v_1 = v_2 = 0$ at $x_1 = 2a$ and $x_2 = a$.) Applying these four conditions yields

$$v_1 = 0 \text{ at } x_1 = 0; \qquad 0 = 0 + 0 + C_2$$

$$v_1 = 0 \text{ at } x_1 = 2a; \qquad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$$

$$v_2 = 0 \text{ at } x_2 = a; \qquad 0 = -\frac{P}{6}a^3 + C_3 a + C_4$$

$$\frac{dv_1(2a)}{dx_1} = -\frac{dv_2(a)}{dx_2}; \qquad -\frac{P}{4}(2a)^2 + C_1 = -\left(-\frac{P}{2}(a)^2 + C_3\right)$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{7}{6}Pa^2 \qquad C_4 = -Pa^3$$

Substituting C_3 and C_4 into Eq. 4 gives

$$v_2 = -\frac{P}{6EI}{x_2}^3 + \frac{7Pa^2}{6EI}x_2 - \frac{Pa^3}{EI}$$

The displacement at C is determined by setting $x_2 = 0$. We get

$$v_C = -\frac{Pa^3}{EI}$$ *Ans.*