Seventh Edition

CHAPTER

18

VECTOR MECHANICS FOR ENGINEERS:

DYNAMICS

Ferdinand P. Beer

E. Russell Johnston, Jr.

Lecture Notes:

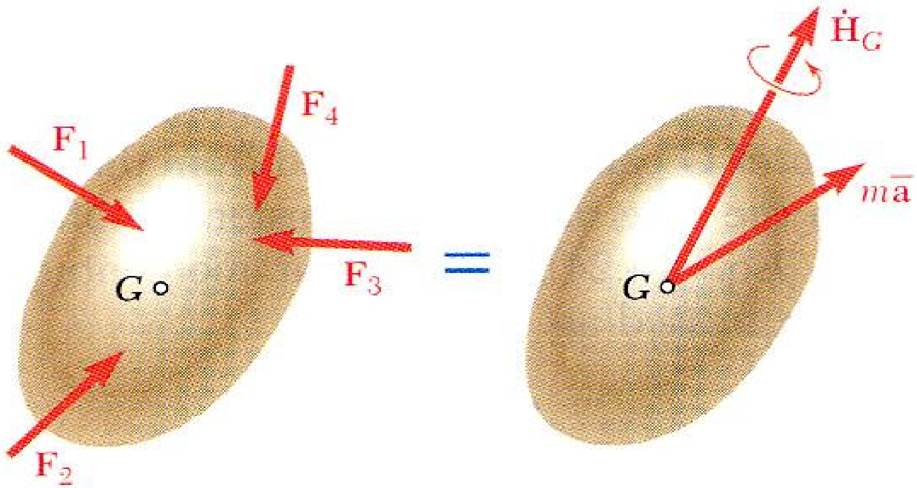
J. Walt Oler

Texas Tech University

Kinematics of Rigid Bodies in Three Dimensions

Contents

Introduction

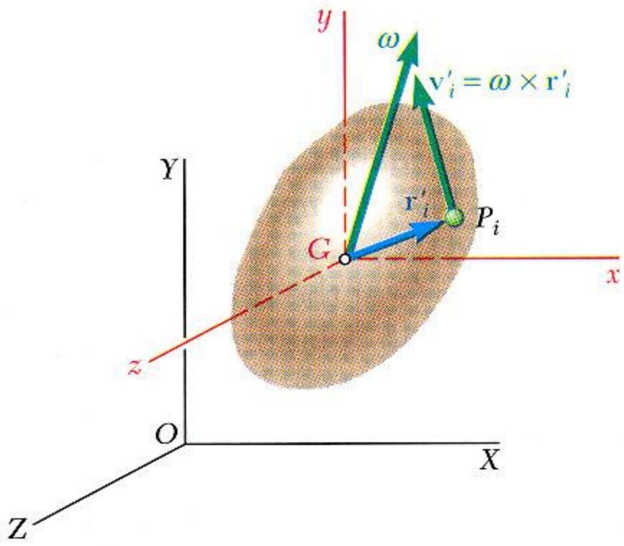


$$\sum \vec{F} = m\vec{\bar{a}}$$

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

- The fundamental relations developed for the plane motion of rigid bodies may also be applied to the general motion of three dimensional bodies.
- The relation $\vec{H}_G = \bar{I}\vec{\omega}$ which was used to determine the angular momentum of a rigid slab is not valid for general three dimensional bodies and motion.
- The current chapter is concerned with evaluation of the angular momentum and its rate of change for three dimensional motion and application to effective forces, the impulse-momentum and the work-energy principles.

Rigid Body Angular Momentum in Three Dimensions



- Angular momentum of a body about its mass center,

$$\vec{H}_G = \sum_{i=1}^{n} \left(\vec{r}_i' \times \vec{v}_i \Delta m_i\right) = \sum_{i=1}^{n} \left[\vec{r}_i' \times \left(\vec{\omega} \times \vec{r}_i'\right) \Delta m_i\right]$$

- The x component of the angular momentum,

$$H_x = \sum_{i=1}^{n} \left[y_i \left(\vec{\omega} \times \vec{r}_i'\right)_z - z_i \left(\vec{\omega} \times \vec{r}_i'\right)_y\right] \Delta m_i$$

$$= \sum_{i=1}^{n} \left[y_i \left(\omega_x y_i - \omega_y x_i\right) - z_i \left(\omega_z x_i - \omega_x z_i\right)\right] \Delta m_i$$

$$= \omega_x \sum_{i=1}^{n} \left(y_i^2 + z_i^2\right) \Delta m_i - \omega_y \sum_{i=1}^{n} x_i y_i \Delta m_i - \omega_z \sum_{i=1}^{n} z_i x_i \Delta m_i$$

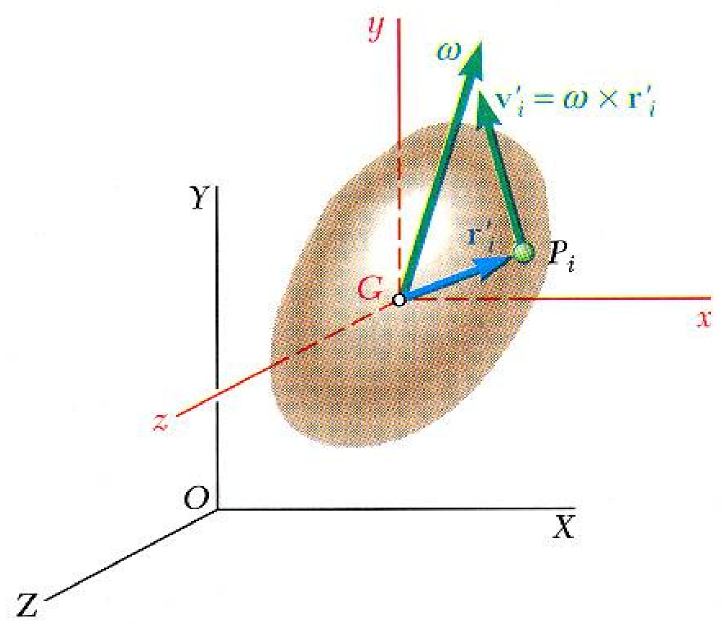
$$H_x = \omega_x \int \left(y^2 + z^2\right) dm - \omega_y \int xy\, dm - \omega_z \int zx\, dm$$

$$= +\bar{I}_x \omega_x - \bar{I}_{xy} \omega_y - \bar{I}_{xz} \omega_z$$

$$H_y = -\bar{I}_{yx} \omega_x + \bar{I}_y \omega_y - \bar{I}_{yz} \omega_z$$

$$H_z = -\bar{I}_{zx} \omega_x - \bar{I}_{zy} \omega_y + \bar{I}_z \omega_z$$

Rigid Body Angular Momentum in Three Dimensions



$$H_x = +\bar{I}_x\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$

$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{I}_{yz}\omega_z$$

$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\omega_z$$

- Transformation of $\vec{\omega}$ into $\vec{H}_G$ is characterized by the inertia tensor for the body,

$$\begin{pmatrix} +\bar{I}_x & -\bar{I}_{xy} & -\bar{I}_{xz} \\ -\bar{I}_{yx} & +\bar{I}_y & -\bar{I}_{yz} \\ -\bar{I}_{zx} & -\bar{I}_{zy} & +\bar{I}_z \end{pmatrix}$$

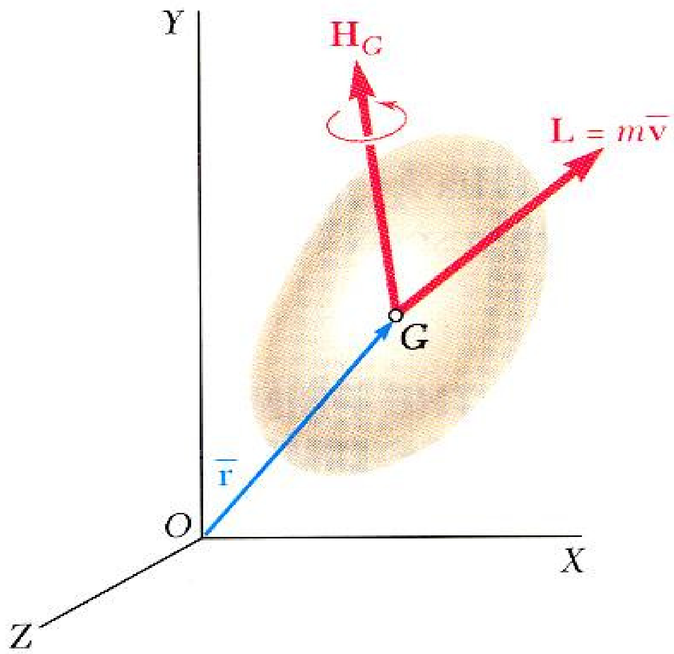
- With respect to the principal axes of inertia,

$$\begin{pmatrix} \bar{I}_{x'} & 0 & 0 \\ 0 & \bar{I}_{y'} & 0 \\ 0 & 0 & \bar{I}_{z'} \end{pmatrix}$$

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \quad H_{y'} = \bar{I}_{y'}\omega_{y'} \quad H_{z'} = \bar{I}_{z'}\omega_{z'}$$

- The angular momentum $\vec{H}_G$ of a rigid body and its angular velocity $\vec{\omega}$ have the same direction if, and only if, $\vec{\omega}$ is directed along a principal axis of inertia.

Rigid Body Angular Momentum in Three Dimensions



- The momenta of the particles of a rigid body can be reduced to:

$$\vec{L} = \text{linear momentum}$$
$$= m\vec{\bar{v}}$$

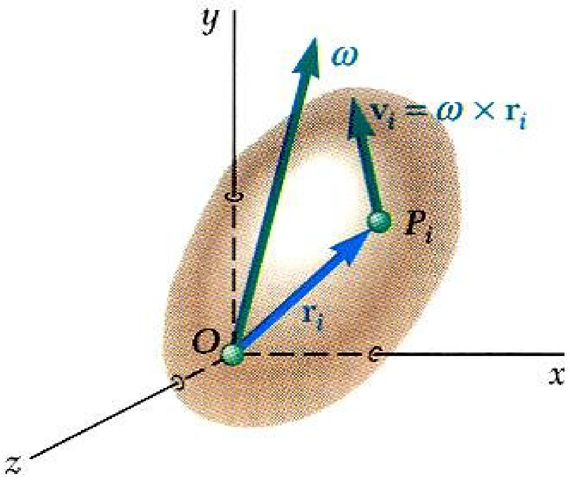
$$\vec{H}_G = \text{angular momentum about } G$$

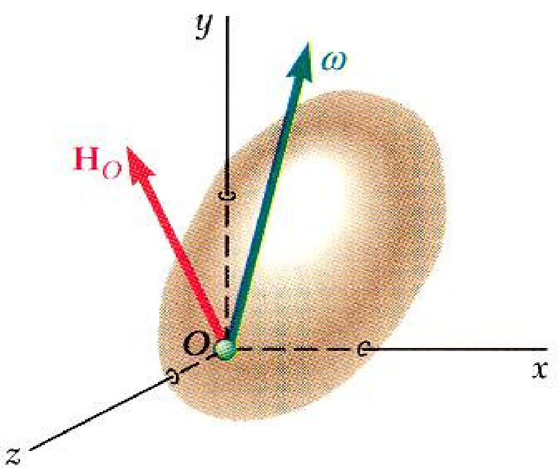
$$H_x = +\bar{I}_x\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$
$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{I}_{yz}\omega_z$$
$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\omega_z$$

- The angular momentum about any other given point O is

$$\vec{H}_O = \vec{\bar{r}} \times m\vec{\bar{v}} + \vec{H}_G$$

Rigid Body Angular Momentum in Three Dimensions





- The angular momentum of a body constrained to rotate about a fixed point may be calculated from

$$\vec{H}_O = \bar{\vec{r}} \times m\bar{\vec{v}} + \vec{H}_G$$

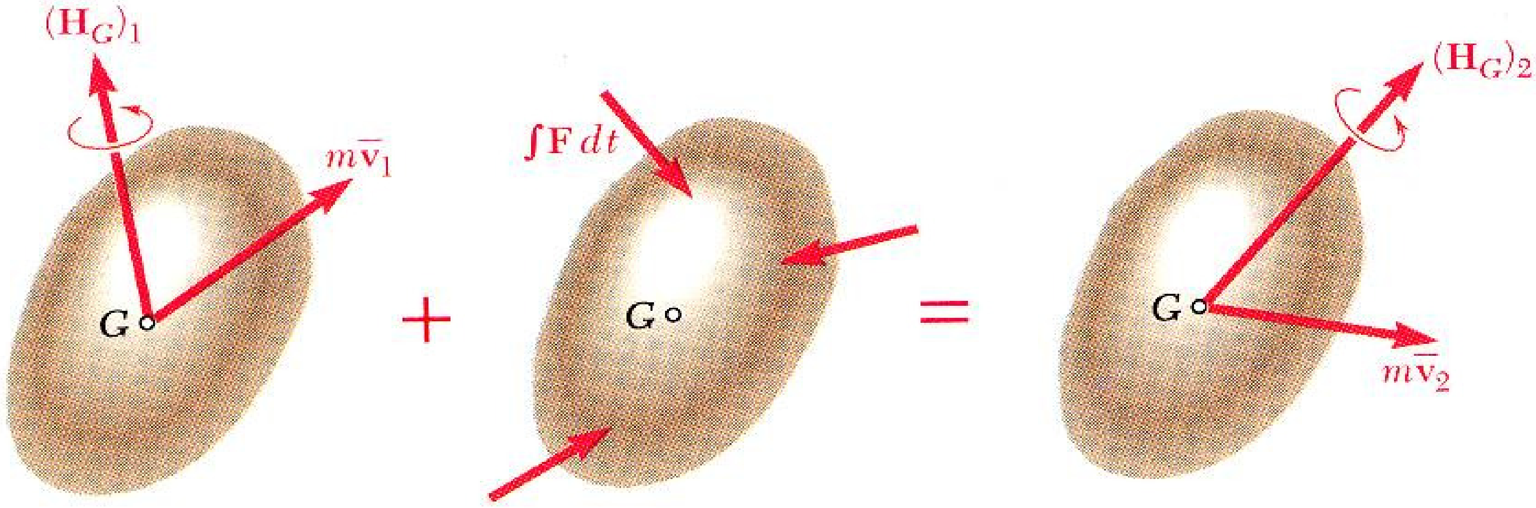
- Or, the angular momentum may be computed directly from the moments and products of inertia with respect to the *Oxyz* frame.

$$\vec{H}_O = \sum_{i=1}^{n} \left(\vec{r}_i \times \vec{v}_i \Delta m\right)$$

$$= \sum_{i=1}^{n} \left[\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\Delta m_i\right]$$

$$H_x = +I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$
$$H_y = -I_{yx}\omega_x + I_y\omega_y - I_{yz}\omega_z$$
$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_z\omega_z$$

Principle of Impulse and Momentum



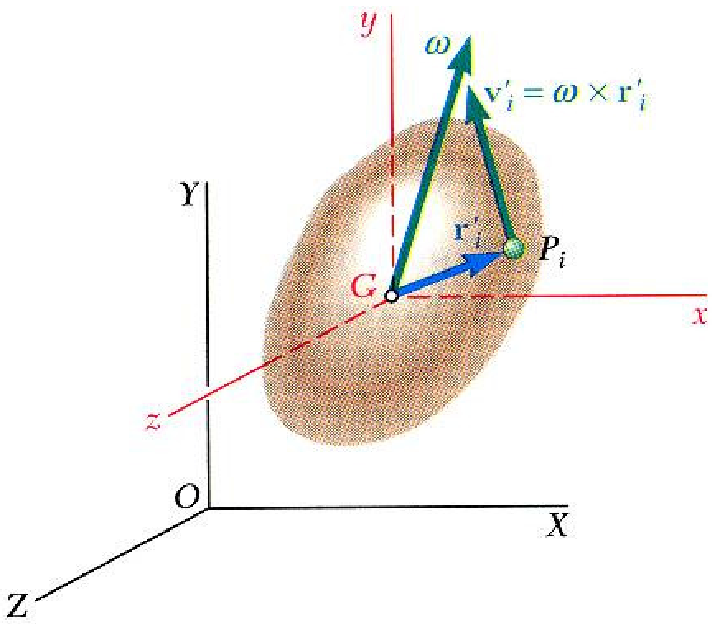
- The principle of impulse and momentum can be applied directly to the three-dimensional motion of a rigid body,

$$Syst\ Momenta_1 + Syst\ Ext\ Imp_{1\text{-}2} = Syst\ Momenta_2$$

- The free-body diagram equation is used to develop component and moment equations.

- For bodies rotating about a fixed point, eliminate the impulse of the reactions at *O* by writing equation for moments of momenta and impulses about *O*.

Kinetic Energy



- Kinetic energy of particles forming rigid body,

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2} \sum_{i=1}^{n} \Delta m_i \bar{v}_i'^2$$

$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2} \sum_{i=1}^{n} \left|\vec{\omega} \times \vec{r}_i'\right|^2 \Delta m_i$$

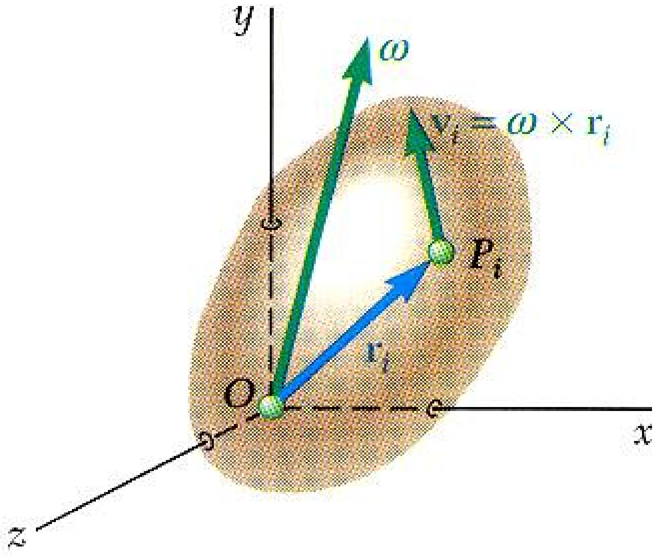
$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_x \omega_x^2 + \bar{I}_y \omega_y^2 + \bar{I}_z \omega_z^2 - 2\bar{I}_{xy}\omega_x\omega_y - 2\bar{I}_{yz}\omega_y\omega_z - 2\bar{I}_{zx}\omega_z\omega_x)$$

- If the axes correspond instantaneously with the principle axes,

$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2)$$

- With these results, the principles of work and energy and conservation of energy may be applied to the three-dimensional motion of a rigid body.

Kinetic Energy



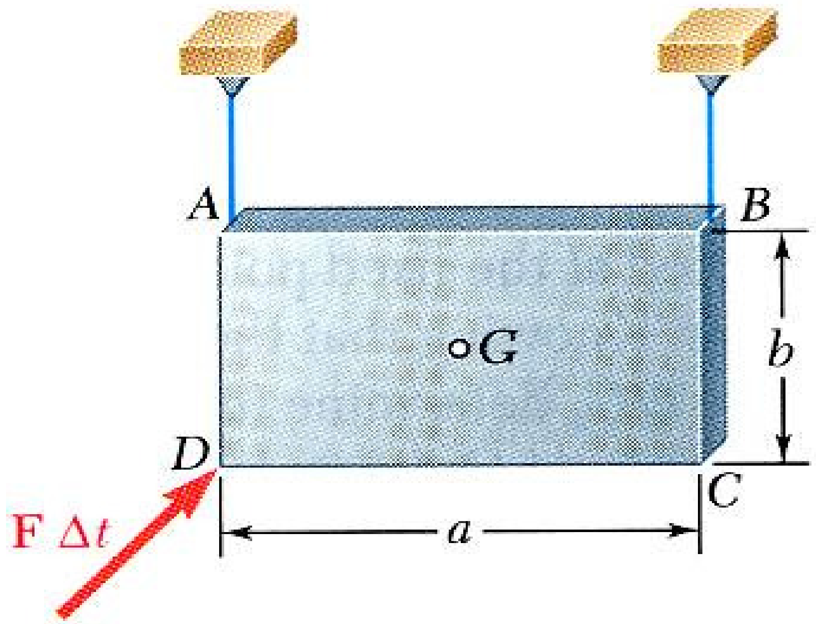
- Kinetic energy of a rigid body with a fixed point,

$$T = \tfrac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{zx}\omega_z\omega_x\right)$$

- If the axes *Oxyz* correspond instantaneously with the principle axes *Ox'y'z'*,

$$T = \tfrac{1}{2}\left(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2\right)$$

Sample Problem 18.1



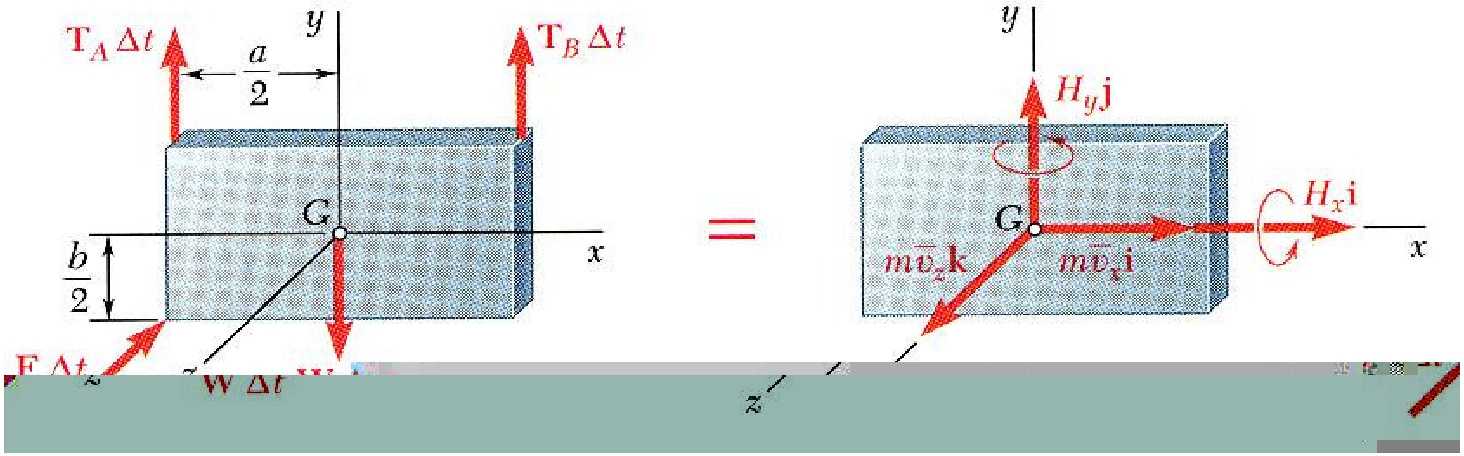
Rectangular plate of mass m that is suspended from two wires is hit at D in a direction perpendicular to the plate.

Immediately after the impact, determine *a*) the velocity of the mass center G, and *b*) the angular velocity of the plate.

SOLUTION:

- Apply the principle of impulse and momentum. Since the initial momenta is zero, the system of impulses must be equivalent to the final system of momenta.
- Assume that the supporting cables remain taut such that the vertical velocity and the rotation about an axis normal to the plate is zero.
- Principle of impulse and momentum yields to two equations for linear momentum and two equations for angular momentum.
- Solve for the two horizontal components of the linear and angular velocity vectors.

Sample Problem 18.1



SOLUTION:

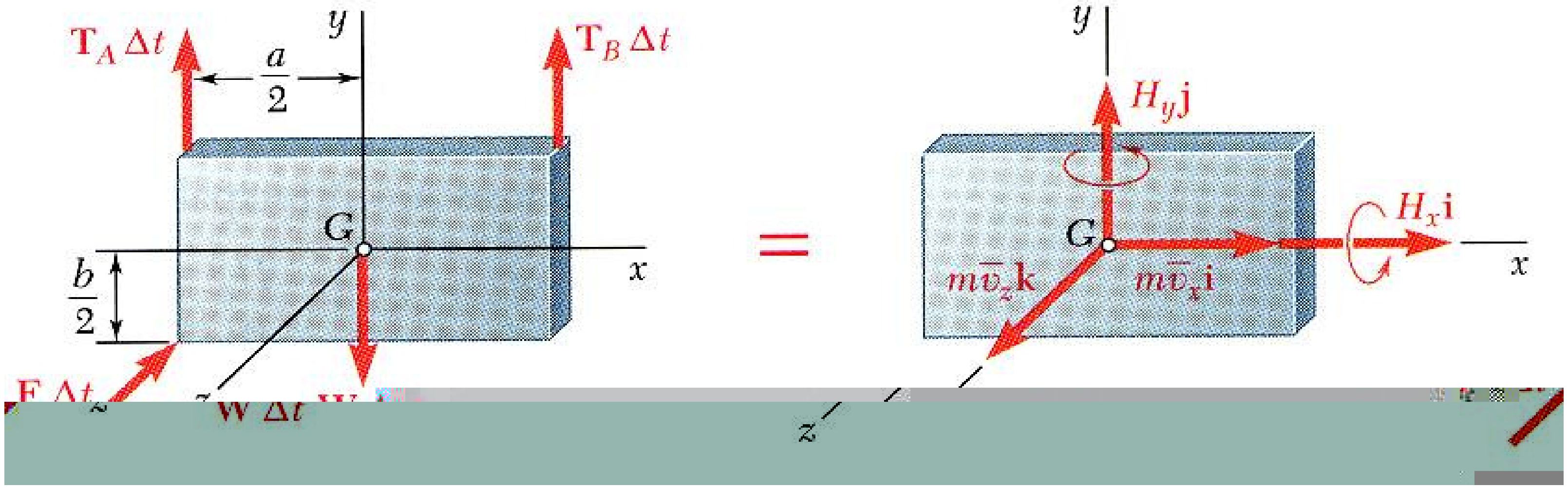
- Apply the principle of impulse and momentum. Since the initial momenta is zero, the system of impulses must be equivalent to the final system of momenta.
- Assume that the supporting cables remain taut such that the vertical velocity and the rotation about an axis normal to the plate is zero.

$$\vec{\bar{v}} = \bar{v}_x \vec{i} + v_z \vec{k} \qquad \vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j}$$

Since the *x*, *y*, and *z* axes are principal axes of inertia,

$$\vec{H}_G = \bar{I}_x \omega_x \vec{i} + \bar{I}_y \omega_y \vec{j} = \tfrac{1}{12} m b^2 \omega_x \vec{i} + \tfrac{1}{12} m a^2 \omega_y \vec{j}$$

Sample Problem 18.1



- Principle of impulse and momentum yields two equations for linear momentum and two equations for angular momentum.

- Solve for the two horizontal components of the linear and angular velocity vectors.

$$0 = mv_x \qquad v_x = 0$$

$$-F\Delta t = m\bar{v}_z \qquad \bar{v}_z = -F\Delta t/m$$

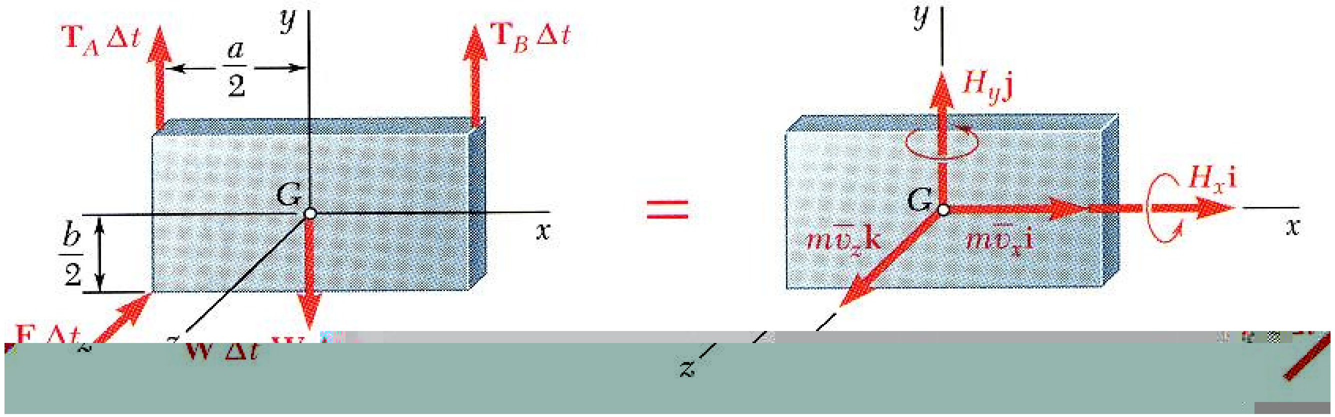
$$\vec{v} = -(F\Delta t/m)\vec{k}$$

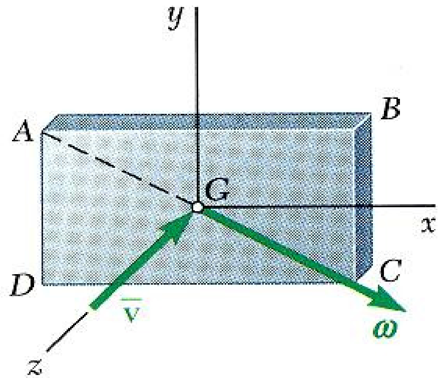
$$\tfrac{1}{2}bF\Delta t = H_x = \tfrac{1}{12}mb^2\omega_x \qquad \omega_x = 6F\Delta t/mb$$

$$-\tfrac{1}{2}aF\Delta t = H_y = \tfrac{1}{12}ma^2\omega_y \qquad \omega_y = -(6F\Delta t/ma)$$

$$\vec{\omega} = \frac{6F\Delta t}{mab}\left(a\vec{i} + b\vec{j}\right)$$

Sample Problem 18.1



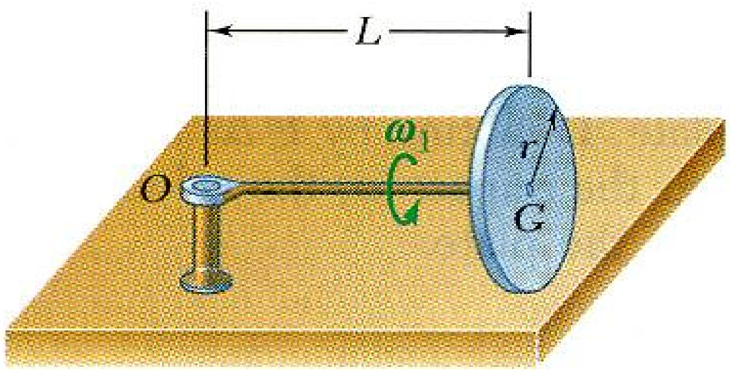


$$\vec{v} - (F\Delta t/m)\vec{k}$$

$$\vec{\omega} = \frac{6F\Delta t}{mab}\left(a\vec{i} + b\vec{j}\right)$$

$$\vec{H}_G = \tfrac{1}{12}mb^2\omega_x\vec{i} + \tfrac{1}{12}ma^2\omega_y\vec{j}$$

Sample Problem 18.2



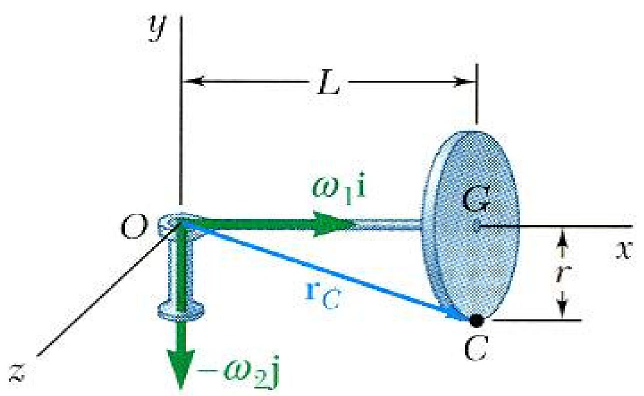
A homogeneous disk of mass *m* is mounted on an axle *OG* of negligible mass. The disk rotates counter-clockwise at the rate ω_1 about *OG*.

Determine: *a*) the angular velocity of the disk, *b*) its angular momentum about *O*, *c*) its kinetic energy, and d) the vector and couple at *G* equivalent to the momenta of the particles of the disk.

SOLUTION:

- The disk rotates about the vertical axis through *O* as well as about *OG*. Combine the rotation components for the angular velocity of the disk.
- Compute the angular momentum of the disk using principle axes of inertia and noting that *O* is a fixed point.
- The kinetic energy is computed from the angular velocity and moments of inertia.
- The vector and couple at *G* are also computed from the angular velocity and moments of inertia.

Sample Problem 18.2



SOLUTION:

- The disk rotates about the vertical axis through *O* as well as about *OG*. Combine the rotation components for the angular velocity of the disk.

$$\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j}$$

Noting that the velocity at *C* is zero,

$$\vec{v}_C = \vec{\omega} \times \vec{r}_C = 0$$

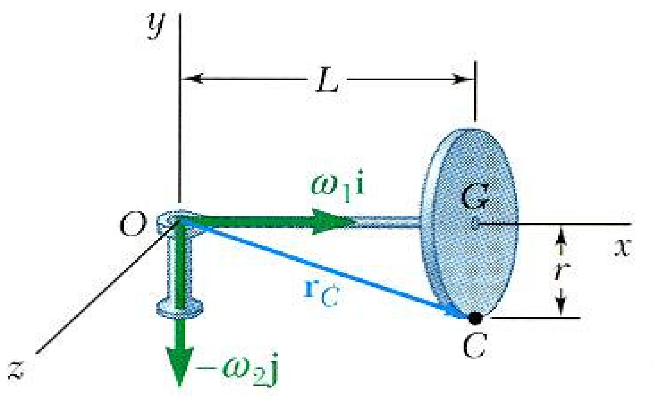
$$0 = \left(\omega_1 \vec{i} + \omega_2 \vec{j}\right) \times \left(L\vec{i} - r\vec{j}\right)$$

$$= \left(L\omega_2 - r\omega_1\right)\vec{k}$$

$$\omega_2 = r\omega_1 / L$$

$$\boxed{\vec{\omega} = \omega_1 \vec{i} - \left(r\omega_1 / L\right)\vec{j}}$$

Sample Problem 18.2



$$\vec{\omega} = \omega_1 \vec{i} - (r\omega_1/L)\vec{j}$$

- Compute the angular momentum of the disk using principle axes of inertia and noting that O is a fixed point.

$$\vec{H}_O = I_x\omega_x\vec{i} + I_y\omega_y\vec{j} + I_z\omega_z\vec{k}$$

$$H_x = I_x\omega_x = \left(\tfrac{1}{2}mr^2\right)\omega_1$$

$$H_y = I_y\omega_y = \left(mL^2 + \tfrac{1}{4}mr^2\right)\left(-r\omega_1/L\right)$$

$$H_z = I_z\omega_z = \left(mL^2 + \tfrac{1}{4}mr^2\right)0 = 0$$

$$\boxed{\vec{H}_O = \tfrac{1}{2}mr^2\omega_1\vec{i} - m\left(L^2 + \tfrac{1}{4}r^2\right)\left(r\omega_1/L\right)\vec{j}}$$

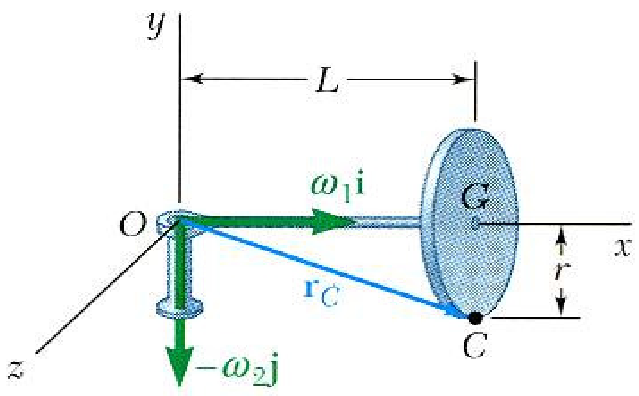
- The kinetic energy is computed from the angular velocity and moments of inertia.

$$T = \tfrac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right)$$

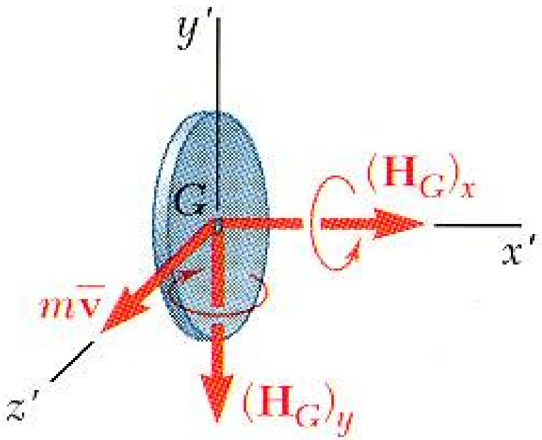
$$= \tfrac{1}{2}\left[mr^2\omega_1^2 + m\left(L^2 + \tfrac{1}{4}r^2\right)\left(-r\omega_1/L\right)^2\right]$$

$$\boxed{T = \tfrac{1}{8}mr^2\left(6 + \frac{r^2}{L^2}\right)\omega_1^2}$$

Sample Problem 18.2



$$\vec{\omega} = \omega_1 \vec{i} - (r\omega_1/L)\vec{j}$$



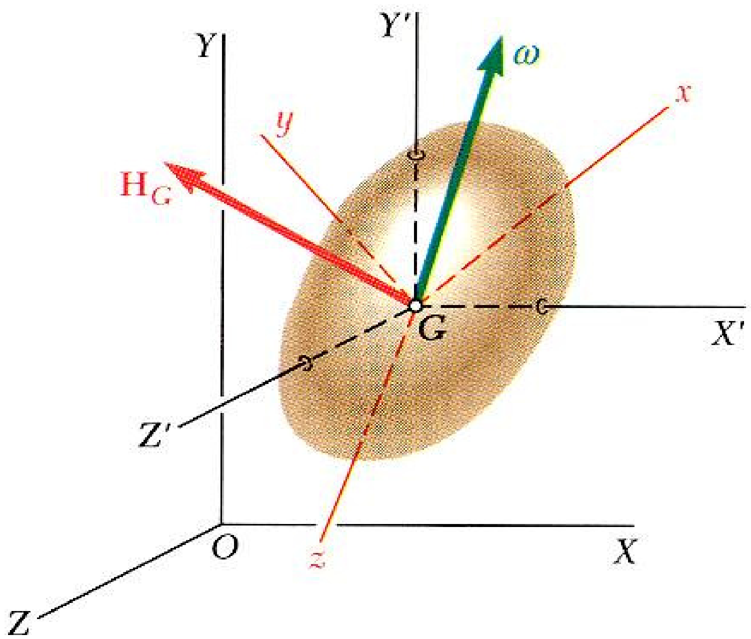
- The vector and couple at *G* are also computed from the angular velocity and moments of inertia.

$$m\vec{\bar{v}} = mr\omega_1\vec{k}$$

$$\vec{H}_G = \bar{I}_{x'}\omega_x\vec{i} + \bar{I}_{y'}\omega_y\vec{j} + \bar{I}_{z'}\omega_z\vec{k}$$
$$= \tfrac{1}{2}mr^2\omega_1\vec{i} + \tfrac{1}{4}mr^2(-r\omega/L)\vec{j}$$

$$\vec{H}_G = \tfrac{1}{2}mr^2\omega_1\left(\vec{i} - \frac{r}{2L}\vec{j}\right)$$

Motion of a Rigid Body in Three Dimensions



$$\sum \vec{F} = m\vec{a}$$

$$\sum \vec{M} = \dot{\vec{H}}_G$$

- Angular momentum and its rate of change are taken with respect to centroidal axes *GX'Y'Z'* of fixed orientation.
- Transformation of $\vec{\omega}$ into $\vec{H}_G$ is independent of the system of coordinate axes.
- Convenient to use body fixed axes *Gxyz* where moments and products of inertia are not time dependent.
- Define rate of change of change of $\vec{H}_G$ with respect to the rotating frame,

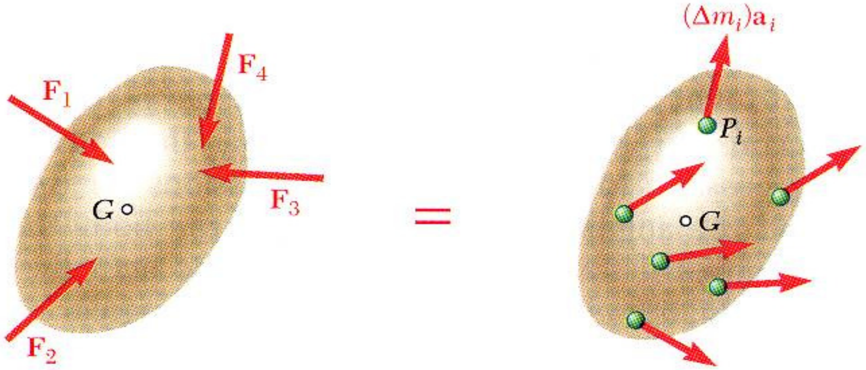
$$\left(\dot{\vec{H}}_G\right)_{Gxyz} = \dot{H}_x\vec{i} + \dot{H}_y\vec{j} + \dot{H}_z\vec{k}$$

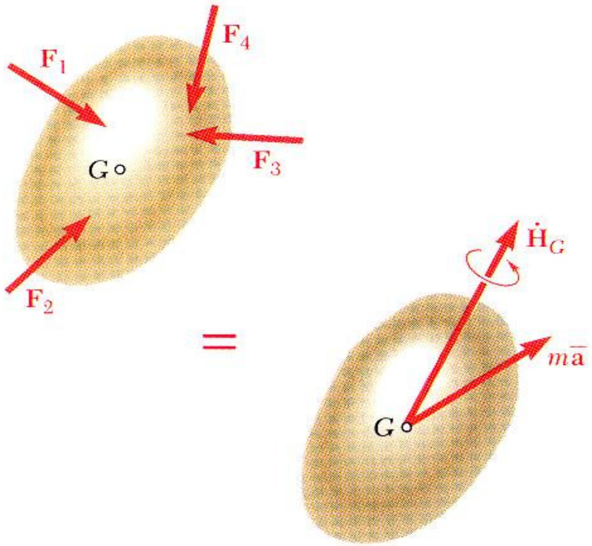
Then,

$$\dot{\vec{H}}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\Omega}\times\vec{H}_G \qquad \vec{\Omega} = \vec{\omega}$$

Vector Mechanics for Engineers: Dynamics

Euler's Eqs of Motion & D'Alembert's Principle





- With $\vec{\Omega} = \vec{\omega}$ and $Gxyz$ chosen to correspond to the principal axes of inertia,

$$\sum \vec{M}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\Omega} \times \vec{H}_G$$

Euler's Equations:

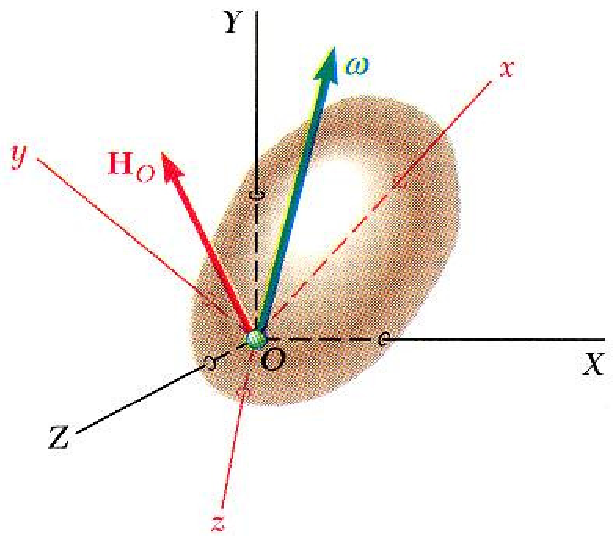
$$\sum M_x = \bar{I}_x \dot{\omega}_x - \left(\bar{I}_y - \bar{I}_z\right)\omega_y \omega_z$$
$$\sum M_y = \bar{I}_y \dot{\omega}_y - \left(\bar{I}_z - \bar{I}_x\right)\omega_z \omega_x$$
$$\sum M_z = \bar{I}_z \dot{\omega}_z - \left(\bar{I}_x - \bar{I}_y\right)\omega_x \omega_y$$

- System of external forces and effective forces are equivalent for general three dimensional motion.

- System of external forces are equivalent to the vector and couple, $m\vec{a}$ and $\dot{\vec{H}}_G$.

Vector Mechanics for Engineers: Dynamics

Motion About a Fixed Point or a Fixed Axis

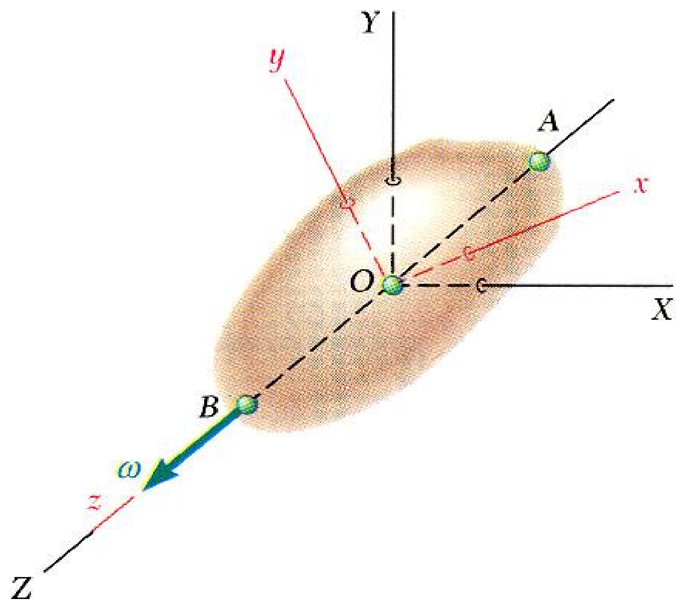


- For a rigid body rotation around a fixed point,

$$\sum \vec{M}_O = \dot{\vec{H}}_O = \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\Omega} \times \vec{H}_O$$

- For a rigid body rotation around a fixed axis,

$$H_x = -I_{xz}\omega \qquad H_y = -I_{yz}\omega \qquad H_z = -I_z\omega$$



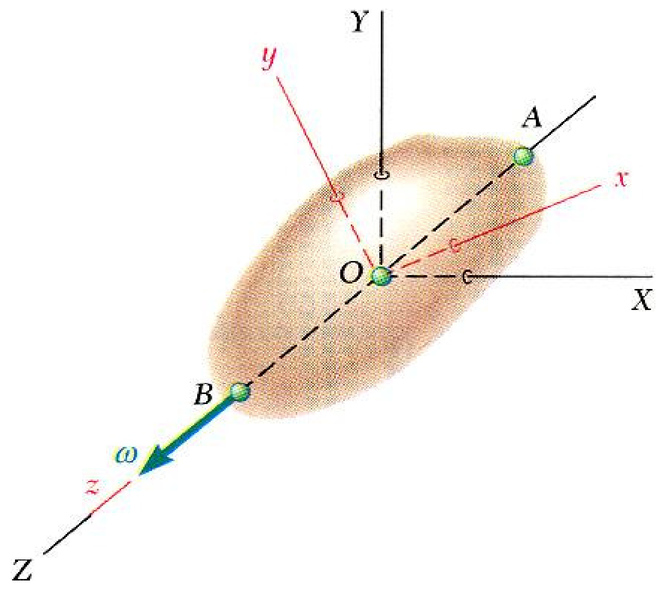
$$\begin{aligned}\sum \vec{M}_O &= \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\omega} \times \vec{H}_O \\ &= \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\dot{\omega} \\ &\quad + \omega\vec{k} \times \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\omega \\ &= \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\alpha + \left(-I_{xz}\vec{j} + I_{yz}\vec{i}\right)\omega^2\end{aligned}$$

$$\sum M_x = -I_{xz}\alpha + I_{yz}\omega^2$$

$$\sum M_y = -I_{yz}\alpha + I_{xz}\omega^2$$

$$\sum M_z = I_z\alpha$$

Rotation About a Fixed Axis



- For a rigid body rotation around a fixed axis,

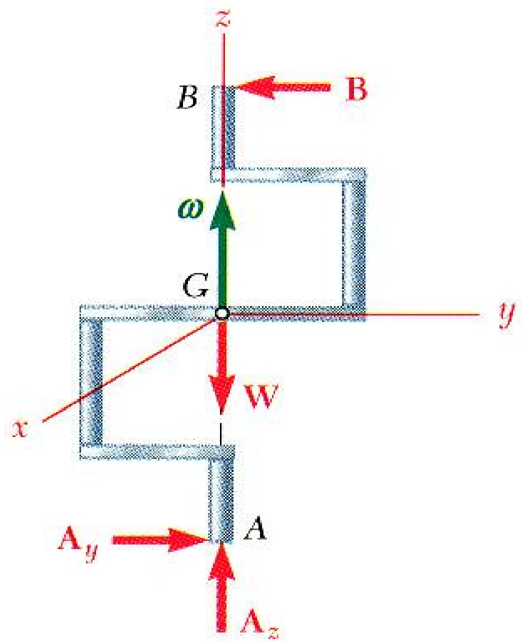
$$\sum M_x = -I_{xz}\alpha + I_{yz}\omega^2$$

$$\sum M_y = -I_{yz}\alpha + I_{xz}\omega^2$$

$$\sum M_z = I_z\alpha$$

- If symmetrical with respect to the xy plane,

$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = I_z\alpha$$

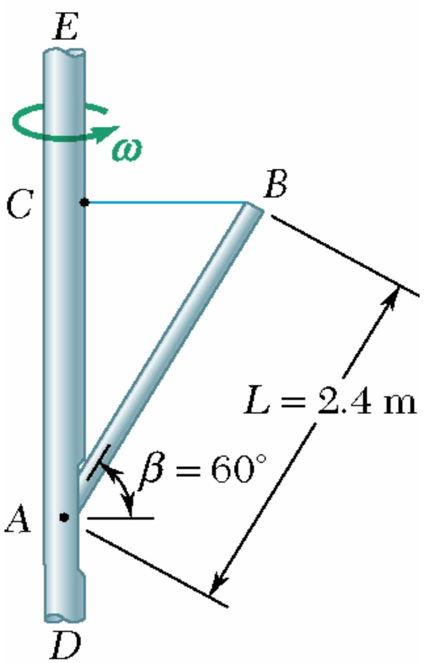


- If not symmetrical, the sum of external moments will not be zero, even if $\alpha = 0$,

$$\sum M_x = I_{yz}\omega^2 \quad \sum M_y = I_{xz}\omega^2 \quad \sum M_z = 0$$

- A rotating shaft requires both static $(\omega = 0)$ and dynamic $(\omega \neq 0)$ balancing to avoid excessive vibration and bearing reactions.

Sample Problem 18.3



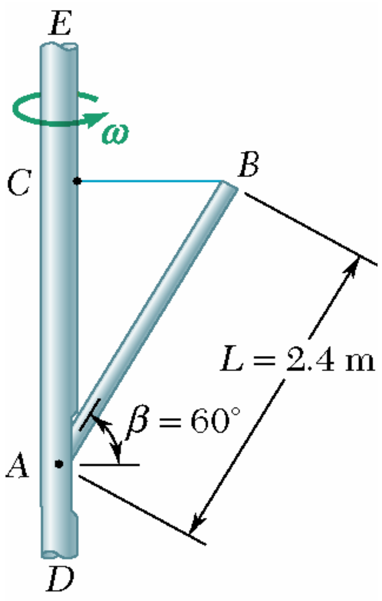
Rod *AB* with weight *W* = 20 kg is pinned at *A* to a vertical axle which rotates with constant angular velocity ω = 15 rad/s. The rod position is maintained by a horizontal wire *BC*.

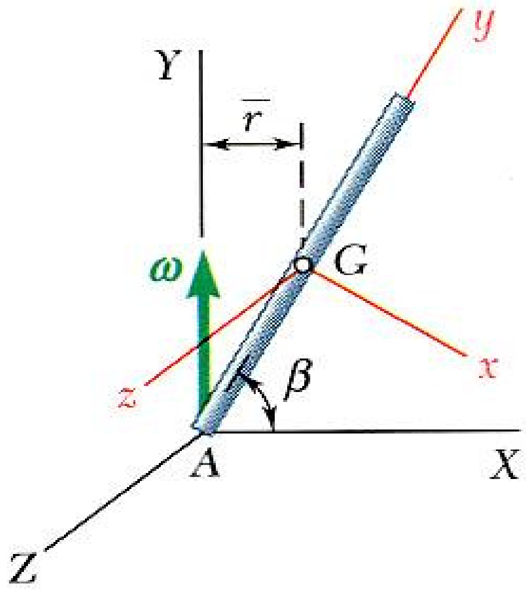
Determine the tension in the wire and the reaction at *A*.

SOLUTION:

- Evaluate the system of effective forces by reducing them to a vector $m\vec{a}$ attached at *G* and couple $\vec{H}_G$.
- Expressing that the system of external forces is equivalent to the system of effective forces, write vector expressions for the sum of moments about *A* and the summation of forces.
- Solve for the wire tension and the reactions at *A*.

Sample Problem 18.3





SOLUTION:

- Evaluate the system of effective forces by reducing them to a vector $m\bar{\vec a}$ attached at G and couple $\vec{H}_G$.

$$\bar{\vec a} = \vec a_n = -r\omega^2\vec I = -\left(\tfrac{1}{2}L\cos\beta\right)\omega^2\vec I$$

$$= -\left(135\,\mathrm{m/s^2}\right)\vec I$$

$$m\bar{\vec a} = (20\,\mathrm{kg})(-135\,\mathrm{m/s^2}) = -(2700\,\mathrm{N})\vec I$$

$$\vec H_G = \bar I_x\omega_x\vec i + \bar I_y\omega_y\vec j + \bar I_z\omega_z\vec k$$

$$\bar I_x = \tfrac{1}{2}mL^2 \qquad \bar I_y = 0 \qquad \bar I_z = \tfrac{1}{2}mL^2$$

$$\omega_x = -\omega\cos\beta \qquad \omega_y = \omega\sin\beta \qquad \omega_z = 0$$

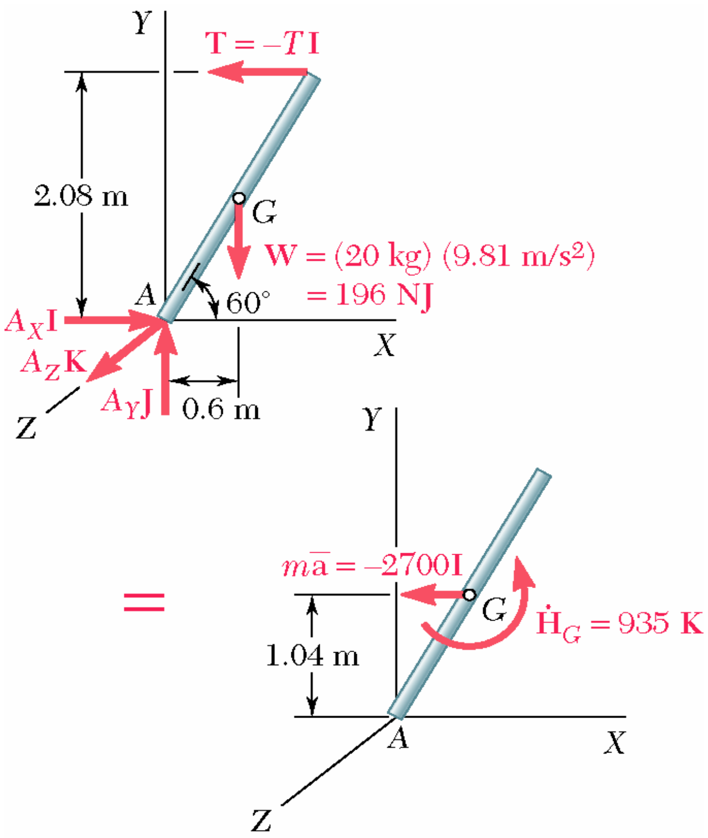
$$\vec H_G = -\tfrac{1}{12}mL^2\omega\cos\beta\,\vec i$$

$$\dot{\vec H}_G = (\dot{\vec H}_G)_{Gxyz} + \vec\omega\times\vec H_G$$

$$= 0 + \left(-\omega\cos\beta\vec i + \omega\sin\beta\vec j\right)\times\left(\tfrac{1}{12}mL^2\omega\cos\beta\vec i\right)$$

$$= \tfrac{1}{12}mL^2\omega^2\sin\beta\cos\beta\vec k = (935\,\mathrm{N\cdot m})\vec k$$

Sample Problem 18.3



- Expressing that the system of external forces is equivalent to the system of effective forces, write vector expressions for the sum of moments about A and the summation of forces.

$$\sum \vec{M}_A = \sum \left(\vec{M}_A\right)_{eff}$$

$$2.08\vec{J} \times \left(-T\vec{I}\right) + 0.6\vec{I} \times \left(-196\vec{J}\right) = 1.04\vec{J} \times \left(-2700\vec{I}\right) + 935\vec{K}$$

$$(2.08T - 118)\vec{K} = (2808 + 935)\vec{K}$$

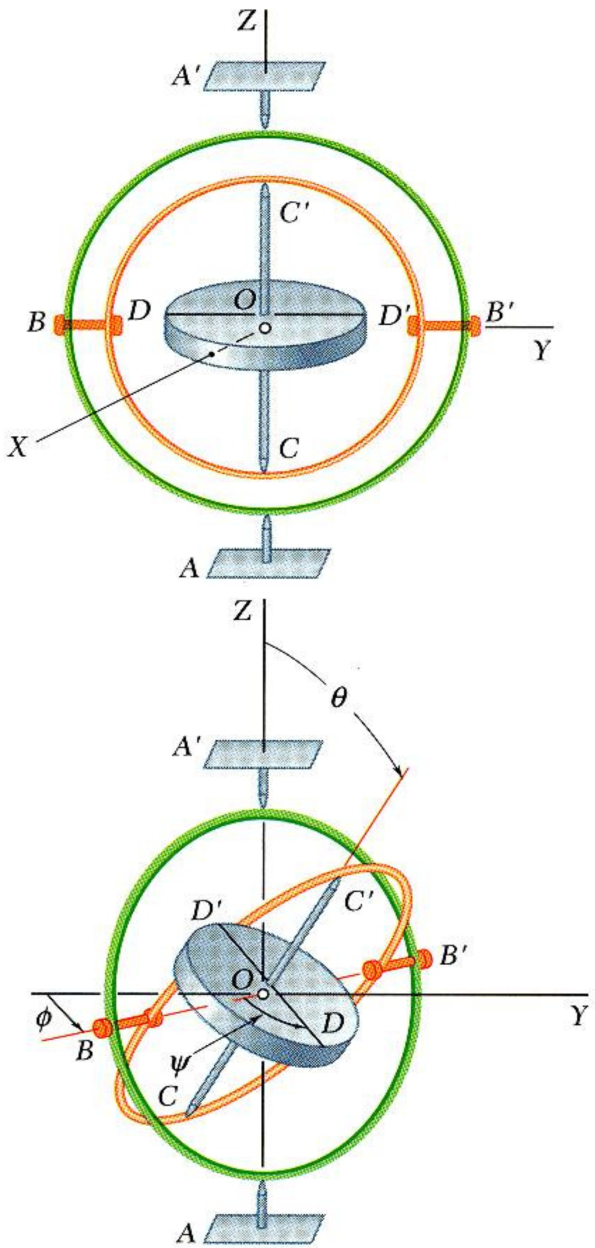
$$\boxed{T = 1856\,\text{N}}$$

$$\sum \vec{F} = \sum \left(\vec{F}\right)_{eff}$$

$$A_X\vec{I} + A_Y\vec{J} + A_Z\vec{K} - 1856\vec{I} - 196\vec{J} = -2700\vec{I}$$

$$\boxed{\vec{A} = -(844\ \text{N})\vec{I} + (196\ \text{N})\vec{J}}$$

Motion of a Gyroscope. Eulerian Angles



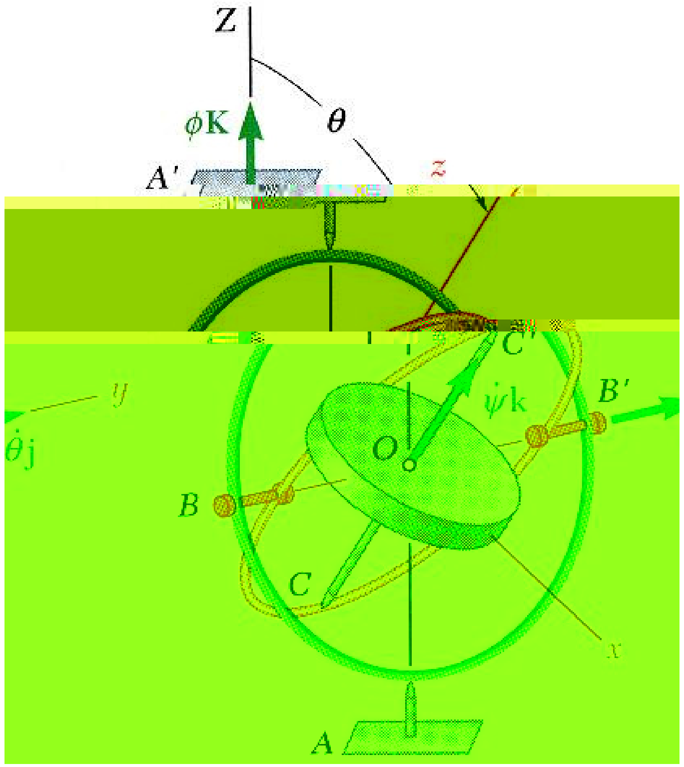
- A gyroscope consists of a rotor with its mass center fixed in space but which can spin freely about its geometric axis and assume any orientation.
- From a reference position with gimbals and a reference diameter of the rotor aligned, the gyroscope may be brought to any orientation through a succession of three steps:
  1) rotation of outer gimbal through φ about *AA'*,
  2) rotation of inner gimbal through θ about *BB'*,
  3) rotation of the rotor through ψ about *CC'*.
- φ, θ, and ψ are called the *Eulerian Angles* and

$$\dot{\phi} = \text{rate of precession}$$

$$\dot{\theta} = \text{rate of nutation}$$

$$\dot{\Psi} = \text{rate of spin}$$

Motion of a Gyroscope. Eulerian Angles



- The angular velocity of the gyroscope,

$$\vec\omega = \dot\phi\vec K + \dot\theta\vec j + \dot\Psi\vec k$$

with $\vec K = -\sin\theta\vec i + \cos\theta\vec j$

$$\vec\omega = -\dot\phi\sin\theta\vec i + \dot\theta\vec j + \left(\dot\Psi + \dot\phi\cos\theta\right)\vec k$$

- Equation of motion,

$$\sum\vec M_O = \left(\dot{\vec H}_O\right)_{Oxyz} + \vec\Omega\times\vec H_O$$

$$\vec H_O = -I'\dot\phi\sin\theta\vec i + I'\dot\theta\vec j + I\left(\dot\Psi + \dot\phi\cos\theta\right)\vec k$$

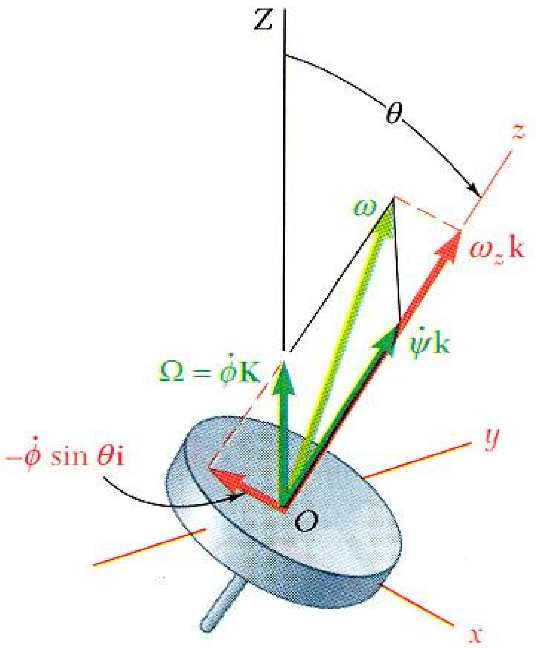
$$\vec\Omega = \dot\phi\vec K + \dot\theta\vec j$$

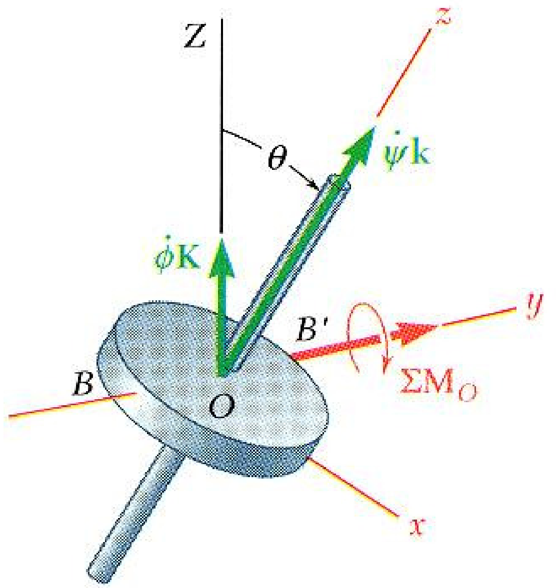
$$\sum M_x = -I'\left(\ddot\phi\sin\theta + 2\dot\theta\dot\phi\cos\theta\right) + I\dot\theta\left(\dot\Psi + \dot\phi\cos\theta\right)$$

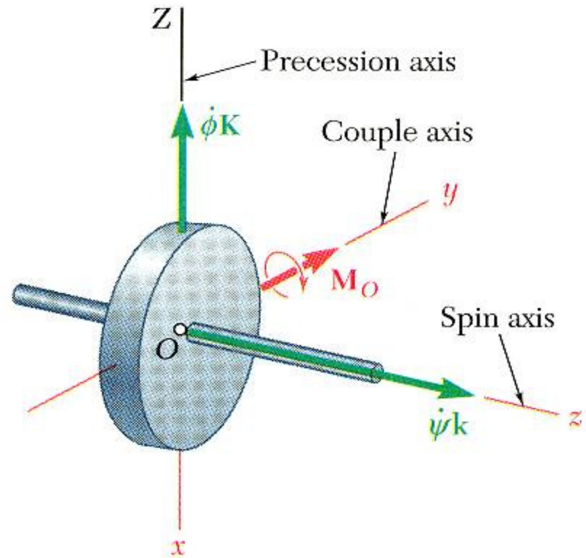
$$\sum M_y = I'\left(\ddot\theta - \dot\phi^2\sin\theta\cos\theta\right) + I\dot\phi\sin\theta\left(\dot\Psi + \dot\phi\cos\theta\right)$$

$$\sum M_z = I\frac{d}{dt}\left(\dot\Psi + \dot\phi\cos\theta\right)$$

Steady Precession of a Gyroscope







Steady precession,

$\theta, \dot{\phi}, \dot{\psi}$ are constant

$$\vec{\omega} = -\dot{\phi}\sin\theta\,\vec{i} + \omega_z\vec{k}$$

$$\vec{H}_O = -I'\dot{\phi}\sin\theta\,\vec{i} + I\omega_z\vec{k}$$

$$\vec{\Omega} = -\dot{\phi}\sin\theta\,\vec{i} + \dot{\phi}\cos\theta\,\vec{k}$$

$$\sum\vec{M}_O = \vec{\Omega}\times\vec{H}_O = \left(I\omega_z - I'\dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta\,\vec{j}$$

Couple is applied about an axis perpendicular to the precession and spin axes

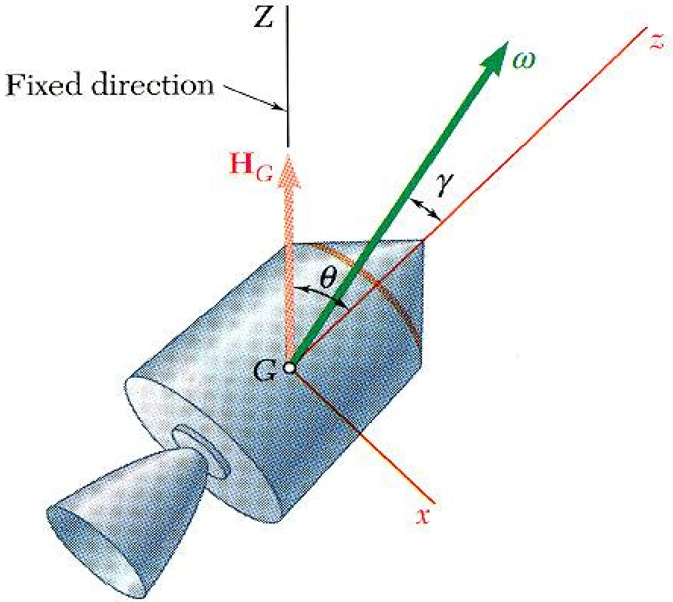
When the precession and spin axis are at a right angle,

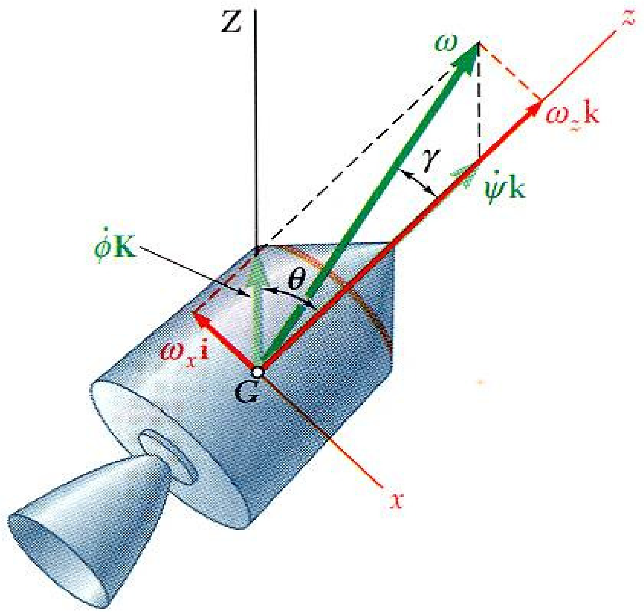
$$\theta = 90°$$

$$\sum\vec{M}_O = I\dot{\Psi}\dot{\phi}\,\vec{j}$$

Gyroscope will precess about an axis perpendicular to both the spin axis and couple axis.

Motion of an Axisymmetrical Body Under No Force





- Consider motion about its mass center of an axisymmetrical body under no force but its own weight, e.g., projectiles, satellites, and space craft.

$$\dot{\vec{H}}_G = 0 \qquad \vec{H}_G = \text{constant}$$

- Define the Z axis to be aligned with $\vec{H}_G$ and z in a rotating axes system along the axis of symmetry. The x axis is chosen to lie in the Zz plane.

$$H_x = -H_G \sin\theta = I'\omega_x \qquad \omega_x = -\frac{H_G \sin\theta}{I'}$$

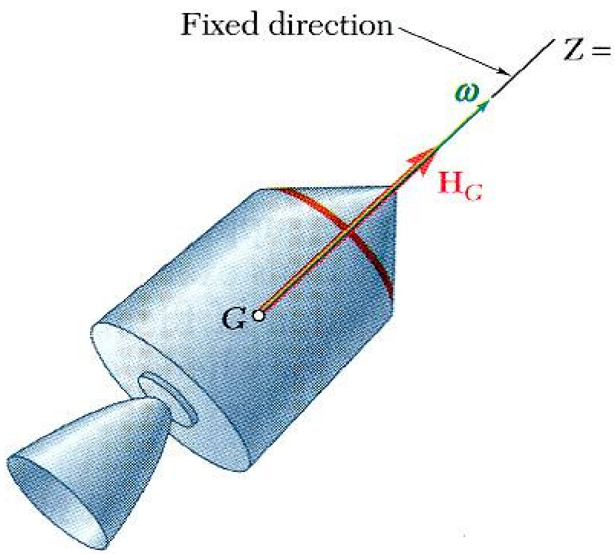
$$H_y = 0 = I'\omega_y \qquad \omega_y = 0$$

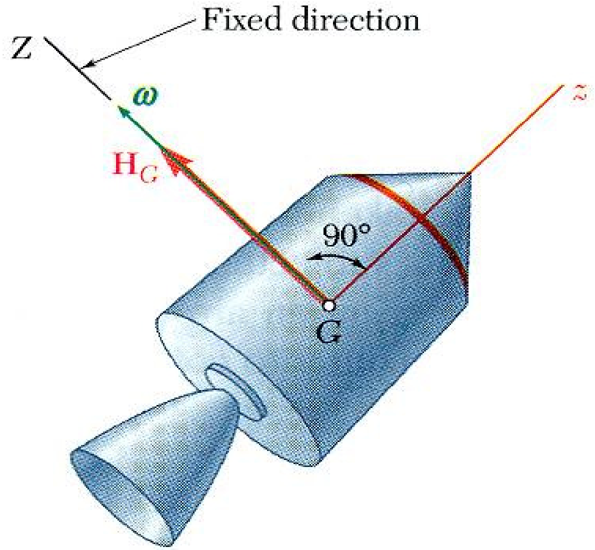
$$H_z = H_G \cos\theta = I\omega_z \qquad \omega_z = \frac{H_G \cos\theta}{I}$$

- θ = constant and body is in steady precession.

- Note: $-\dfrac{\omega_x}{\omega_z} = \tan\gamma = \dfrac{I}{I'}\tan\theta$

Motion of an Axisymmetrical Body Under No Force





Two cases of motion of an axisymmetrical body which under no force which involve no precession:

- Body set to spin about its axis of symmetry,

  $$\omega_x = H_x = 0$$

  $$\vec{\omega} \text{ and } \vec{H}_G \text{ are aligned}$$

  and body keeps spinning about its axis of symmetry.

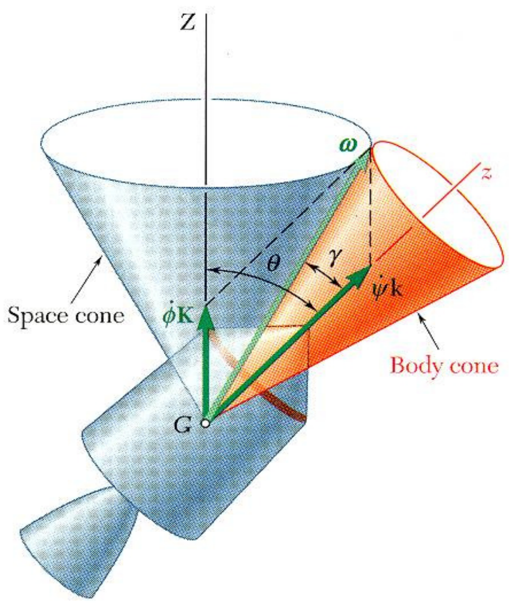
- Body is set to spin about its transverse axis,

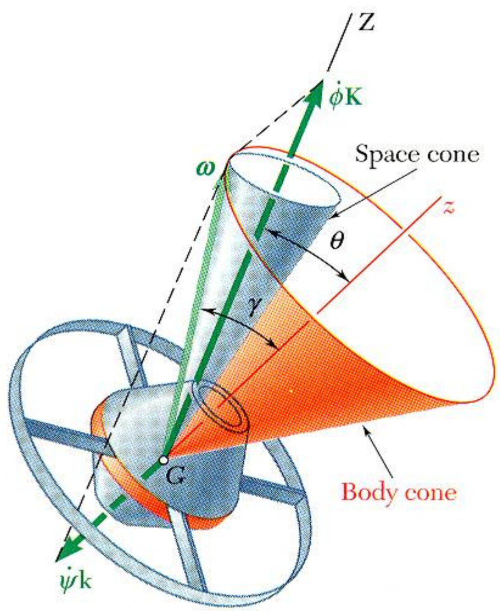
  $$\omega_z = H_z = 0$$

  $$\vec{\omega} \text{ and } \vec{H}_G \text{ are aligned}$$

  and body keeps spinning about the given transverse axis.

Motion of an Axisymmetrical Body Under No Force





The motion of a body about a fixed point (or its mass center) can be represented by the motion of a body cone rolling on a space cone. In the case of steady precession the two cones are circular.

- $I < I'$. Case of an elongated body. $\gamma < \theta$ and the vector ω lies inside the angle ZGz. The space cone and body cone are tangent externally; the spin and precession are both counterclockwise from the positive z axis. The precession is said to be *direct*.

- $I > I'$. Case of a flattened body. $\gamma > \theta$ and the vector ω lies outside the angle ZGz. The space cone is inside the body cone; the spin and precession have opposite senses. The precession is said to be *retrograde*.