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# A NEW PROOF OF GROMOV'S THEOREM ON GROUPS OF POLYNOMIAL GROWTH

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## 1. Introduction

1.1. **Statement of results.** Let $G$ be a group with a finite symmetric generating set $S$, and let $B_G(r) \subset G$ denote the ball centered at $e \in G$ with respect to the word norm on $G$ given by $S$:

$$B_G(r) = \{g \in G \mid g = g_1 \cdots g_k \text{ for some } g_1, \ldots, g_k \in S,\ k \leq r\}\,.$$

**Definition 1.1.** The group $G$ has **polynomial growth** if for some $d \in (0, \infty)$

$$\limsup_{r \to \infty} \frac{|B_G(r)|}{r^d} < \infty, \tag{1.1}$$

and it has **weakly polynomial growth** if for some $d \in (0, \infty)$

$$\liminf_{r \to \infty} \frac{|B_G(r)|}{r^d} < \infty. \tag{1.2}$$

Our main result is a new proof of the following theorem of Gromov and Wilkie-van den Dries [Gro81, vdDW84]:

## Gromov's Theorem

$\exists H \leq G$ f.i $H$: Nilpotent.

* Heisenberg group.

* Abelian group.

$G$ : finitely generated group

Then; $G$ is virtually nilpotent $\Longleftrightarrow$ $G$ is a group with polynomial growth

## Theorem.

$\Gamma$: Cayley graph of a group $G$ of weakly polynomial growth and $d \in [0, \infty)$

Then;

The space of Harmonic functions on $\Gamma$ with polynomial growth at most $d$ is finite dimensional.

Theorem $\Longrightarrow$ Corollaries I, II

**Cor I**

G : Infinite group of weakly polynomial growth

Then; G admits a finite dimensional representation

$$G \longrightarrow GL(n, \mathbb{R})$$ with infinite image.

**Cor II**

G : group with weakly polynomial growth Then

G is virtually nilpotent

Preliminaries

poincare inequality

Calculus on Cayley graph

Poincare ineq on cayley graph

$\Downarrow$

Estimate dimension of Harmonic functions on cayley graph

(part I)

## * Growth of finitely generated groups

Growth rate [J. Milnor]

$G = \langle \Gamma \rangle,\ \Gamma = \{\gamma_1, \dots, \gamma_n\}$

$\ell(g) = \ell_\Gamma(g) = \min\{\ell(\omega) : \omega \text{ represents } g\}$

$d(g_1, g_2) = \ell(g_1 g_2^{-1})$ (d: Geodesic distance on Ca gley graph)

$B_{\Gamma, G}(n) = \{ g \in G \mid \|g\| \leq n \}$

$V(n) = \# B_{G,\Gamma}(n)$

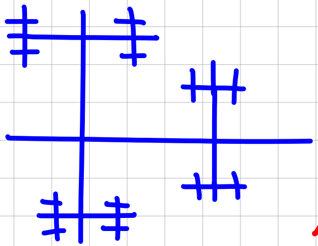
# The Cayley graph

$G = \langle \Gamma \rangle, \Gamma = \{\gamma_1, \dots, \gamma_n\}$

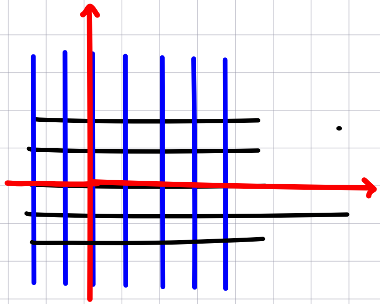
$\downarrow$

Graph

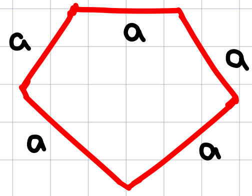
$\mathbb{F}_2 = \langle a, b \rangle \Longrightarrow$



$\mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle \Longrightarrow$



$\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle \Longrightarrow$




$n = 5$

# Calculus on cayley graph

$(G, \cdot)$ f.g gp
$G = \langle S \rangle$

$\Longrightarrow \Gamma : \Gamma(G, S)$ cayley graph

$(\Gamma, d_s)$ : metric space

$(\Gamma, \mathbb{B})$ : Borel $\sigma$-algebra

$(\Gamma, \mathbb{B}, \mu)$ : measurable space

$\mu$ : counting measure.

Consider locally uniformly finite assumption

$$d = \sup_{g \in \Gamma} \deg(g) < \infty$$

Gradiant

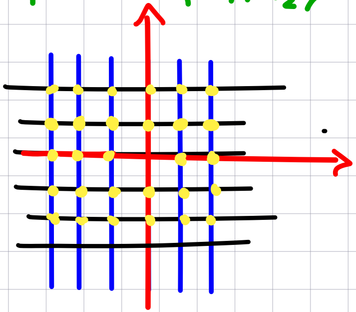
$u : \Gamma \longrightarrow \mathbb{R}$ $\quad |\nabla_G u|(x) = \sum_{y \sim x} |u(y) - u(x)|$

# Harmonic

$u : \Gamma \longrightarrow \mathbb{R}$ is called Harmonic if

$$u(g) = \sum_{h \sim g} u(h) \Big/ \deg(g)$$

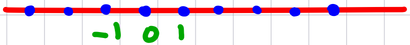
$G = \mathbb{Z}^2 = \langle e_1, e_2 \rangle$

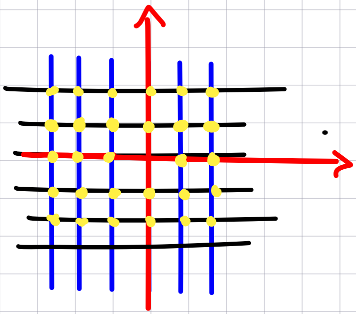


$u(a) = c \;;\; c \in \mathbb{R}$

$(\mathbb{Z},+)$ , $\mathbb{Z}=\langle 1\rangle$

$B_\Gamma(n)=2n+1$



$G=\mathbb{Z}^2=\langle e_1,e_2\rangle$

$\# B_\Gamma(n)=2n^2-2n+1$

$(\mathbb{Z}^k,+)$ , $\Gamma=\{e_1,e_2,\dots,e_k\}$

$\# B_\Gamma(n)=2n^k+O(n^{k-1})$

$\mathbb{F}_2$

$\# B_{\{a,b\}}(n)=1+4(1+3+3^2+\dots+3^{n-1})=1+2(3^n-1)$

# Asymptotic growth of $\#B_{G,\Gamma}(n) = V(n)$

- $V(m+n) \leq V(n)\cdot V(m) \implies \log V(n)$ is subadditive sequence.

  $\implies \lim \frac{1}{n}\log V(n) = \inf_n \log V(n)/n$

- Growth rate of finitely generated groups

  * Exponential growth

    $\exists\, \alpha>0\,;\quad \alpha = \lim_n \frac{1}{n}\log V(n)$

  * polynomial growth

    $\exists\, \alpha \geq 1, C>0 \quad V(n) \leq C n^{\alpha} \iff \exists\, d \in (0,\infty)\,;\ \limsup V(n)/n^d < \infty$

* Weakly polynomial growth

$\exists d \in (0, \infty)$, $\liminf V(n)/n^d < \infty$

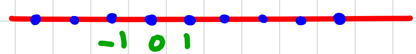
These classification are independent of generating set

$V_\Gamma(n)$ grows polynomially / exponentially $\Leftrightarrow$ $V_{\Gamma'}(n)$ grows polynomially / exponentially

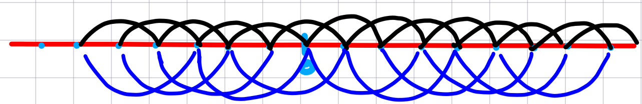
$\Gamma . \Gamma'$ : generating sets

$(\mathbb{Z}, +)$, $\Gamma = \langle 1 \rangle$ , $\Gamma' = \langle 2, 3 \rangle$

$V_\Gamma(n) = 2n+1$




$V_{\Gamma'}(n) = O(n)$



* Poly. growth

$(\mathbb{Z}^k, +)$ , $U^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

* Exp. growth

$\mathbb{F}_2$ ; $\mathbb{F}_k$ ; $SL(n, \mathbb{Z})$

## Theorem (Tits' alternative Theorem)

$G$ : f.g linear group

either $\begin{cases} G \text{ virtually solvable} \\ \text{It's contains a non-abelian free group} \longrightarrow \text{Exp growth} \end{cases}$

## Theorem (Milnor-Wolf)

$G$ : virtually solvable, $G$ : f.g

either $\begin{cases} G \text{ has exponential growth} \\ G \text{ is virtually nilpotent.} \end{cases}$

## Reduction of Gromov's Theorem for linear group.

$G$ : f.g linear group

$G$ polynomial growth $\Longleftrightarrow$ $G$ virtually nilpotent

(Idea of proof)

(M.W Thm + Tit's thm)

# Poincare inequality on cayley graph

For $R \in [0,\infty) \cap \mathbb{Z}$ , $\forall$ smooth function $f : B(3R) \to \mathbb{R}$.

$$\int_{B(R)} |f - f_R|^2 \leq 8 |S|^2 R^2 \frac{V(2R)}{V(R)} \int_{B(3R)} |\nabla f|^2$$

$f_R :=$ average of $f$ over $B(R)$

# Poincare inequality

## Sobolev space.

** Weak derivatives**

$u, v \in L^1(U)$ ; $U \subseteq \mathbb{R}^n$. $U$: open

$v$ is $\alpha^{th}$- weak derivative of $u$ $(D^\alpha u = v)$

when ; $\displaystyle\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx$

$$\forall \phi \in C_c^\infty(U)$$

$$\begin{cases} \phi : \text{Test function} \\ D^\alpha \phi = \dfrac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \dfrac{\partial^{\alpha_2}}{\partial x^{\alpha_2}} \cdots \dfrac{\partial^{\alpha_k}}{\partial x^{\alpha_k}} \qquad |\alpha| = \alpha_1 + \cdots + \alpha_k \end{cases}$$

- $D^\alpha u$ is unique

Example

$$u(x) = \begin{cases} x & 0 < x \le 1 \\ 1 & 1 \le x < 2 \end{cases} \qquad V(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & ; 1 < x < 2 \end{cases}$$

$u' = V$ in Weak Sense.

$$\int u \phi' dx = -\int_0^2 v \phi \, dx \quad , \phi \in C_c^\infty(U).$$

## ** Sobolev space

$$W^{k,p}(U) = \{u : U \to \mathbb{R} : D^\alpha u \in L^p(U),\ |\alpha| \le k\}$$

$u \in W^{k,p}(U)$

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \sum_{|\alpha| \le k} \|D^\alpha u\|_p & 1 \le p < \infty \\ \sum_{|\alpha| \le k} \|D^\alpha u\|_\infty & p = \infty \end{cases}$$

- $(W^{k,p}(U), \|\cdot\|_W)$ is Banach space.

- $W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_W}$

- $(W^{2,p}(U), \|\cdot\|_W)$ Hilbert space.

$$(u)_U = \fint_U u\,dx = \frac{1}{\mathrm{vol}(U)} \int_U u\,dx$$

## Theorem (Poincare inequality)

$U$ : bounded, connected open subset of $\mathbb{R}^n$

$\partial U$ : $C^1$

Assume that $1 \le p \le \infty$ Then; $\exists\, C = C(n, p, U)$ s.t

$$\|u - (u)_U\|_p \le C \|Du\|_p$$

$$u \in W^{1,p}(U)$$

$$Du = (u_{x_1}, \ldots, u_{x_n})$$

## Sketch of proof

Contradiction :

$\exists\, u_k \in W^{1,p}(U)$ , $k=1,2,\dots$ s.t $\|u_k-(u_k)_U\|_p > k\|Du_k\|_p$

$v_k := (u_k-(u_k)_U) / \|u_k-(u_k)_U\|_p$ $k=1,2,\dots$

$\Rightarrow (v_k)_U = 0$ , $\|v_k\|_p = 1$

$\Rightarrow \|Dv_k\|_p < 1/k$ ; $k=1,2,\dots$ (*)

$\{v_k\}_{k=1}^{\infty}$ is $L_p$- bounded ← Sobolev embedding Thm

## Theorem ( Rellich- kondrachov compactness)

$U$: bounded open subset of $\mathbb{R}^n$

$\partial U : C^1$

$1 \leq p < n$

Then $W^{1,p}(U) \subset\subset L^q(U)$ for $1 \leq q \leq p^*$

compactly embeded

* $\|x\|_W \leq c\|x\|_q$
* each bounded seq in $W^{1,p}$ is precmp in $L^q$.

(R-k cmp) $\Longrightarrow \exists \{v_{k_j}\}_{j=1}^{\infty} \quad v_{k_j} \xrightarrow{q} v$

$\Longrightarrow (v)_U = 0 \ , \ \|v\|_q = 1$

$* \Rightarrow \forall i = 1, 2, \dots, n \ , \ \phi \in C_c^{\infty}(U)$

$$\int_U v \phi_{x_i} \, dx = \lim_{k_j \to \infty} \int_U v_{k_j} \phi_{x_i} \, dx = - \lim_{k_j \to \infty} \int_U v_{k_j, x_i} \phi \, dx = 0$$

$\Longrightarrow v \in W^{1,p}(U) \ ; \ Dv = 0$

$\Longrightarrow v = cte$

$\Longrightarrow v = 0$

$\Longrightarrow \|v\|_q = 0 \ \because$ X

# Proof of Corollaries I.II.

## Some definitions and theorems

### Amenable

$G$ is called Amenable group if $G$ admits Følner sequence.

$$F = \{F_n\} \quad \forall s \in G \; ; \; \lim \frac{|sF_n \Delta F_n|}{|F_n|} = 0$$

Equivalent definition

For every continuous Action $G \curvearrowright X$, $X$ cmp Hausdorff

has regular invariant measure.

$$\mu_n = \frac{1}{|F_n|} \sum_{s \in F_n} \delta_{sx_0}$$

$$\mu_n \in \mathbb{P}(X)$$

$$\mu_n \to \mu.$$

Ex//

1) $(\mathbb{Z}, +)$ $F_n = \{1, 2, \cdots, n\}$

Følner sequence.

2) Abelian groups

3) group with sub-Exponential growth is Amenable

$\lim_n |B_n|^{1/n} = 1$

$F = \{B_\Gamma(n)\}$

Shalom's Theorem.

Amenable linear group are virtually solvable.

## Property T.

$G$ has propeity T $\Leftrightarrow$ If every isometric action $G$ on Hilbert space has a fixed pt.

Ex //

$G$ with polynomial growth doesn't have property (T).

## Theorem (Korevaar-Schoen)

The following are equivalent:

1) $G$ doesn't have property (T)

2) There is an isometric Action $G \curvearrowright H$ on Hilbert space $\mathcal{H}$ and a non constant $G$-$\underbrace{\text{equivaraint}}_{f(gx) = g f(x) \ \forall g \in G}$ harmonic map $f: \Gamma \to \mathcal{H}$

**Cor I**

$G$: Infinite group of weakly polynomial growth

Then; $G$ admits a finite dimensional representation

$$G \longrightarrow GL(n,\mathbb{R}) \text{ with infinite image.}$$

**proof:**

$G$ doesnot have property (T)

$G$ Amenable

$\Downarrow$ k.s Thm.

$\exists\ G \curvearrowright H$ isometric which has no fixed pt. and a nonconstant $G$-equivariant

$H$: Hilbert space

$$f : \Gamma \longrightarrow H$$

$$\begin{cases} \phi : H^* \longrightarrow \mathcal{V} := \{\text{lipschitz harmonic functions}\} \\ \phi(\varphi) = \varphi \circ f \end{cases}$$

Thm $\Longrightarrow$ $\dim(V) < \infty$

$\Longrightarrow$ $\operatorname{codim}(\ker(\phi)) < \infty$

$\Longrightarrow$ $\ker(\phi)^{\perp} \subset H$ is finite dimensionl s.t

$\ker(\phi)^{\perp} \supset \operatorname{Im}(f)$

$A = \operatorname{Aff\,hull}(\operatorname{Im}(f))$

$\dim(A) < \infty$

$\Longrightarrow$ $G \longrightarrow \operatorname{Isom}(A)$

finite dimensional representation.

**Cor II**

$G$ : group with weakly polynomial growth Then

$G$ is virtually nilpotent

**Proof [Induction on $d$]**

$G$ : f.g $\rightsquigarrow$ $\deg(G) = \min\{ d \geq 0 : \liminf_n V(n)/n^d < \infty \}$

- $d = 0 \Longrightarrow G$ finite group $\Longrightarrow G$ : Virtually Nilpotent.

- $d-1 \overset{?}{\Longrightarrow} d$

Cor I $\Longrightarrow G \xrightarrow{\beta} GL_n(\mathbb{R})$

$H = \beta(G)$ , $H$ has polynomial growth.

$\left.\begin{array}{c}\text{Tits' Alternative thm} \\ + \\ \text{Shalom's thm}\end{array}\right\} \Longrightarrow$ Virtually solvable $\xRightarrow{\text{W.M Thm}}$ H: Virtually Nilpotent.

W.L.o.G

* H: Nilpotent (up to finite index subgroups)
* H: It's abelianization is torsion-free

A short exact sequence

$$1 \to k \to G \xrightarrow{\alpha} \mathbb{Z} \to 1$$

$k = \ker(\alpha)$

$\dim(k) \leq \deg(G) - 1 \Longrightarrow k$: Virtually nilpotent.

$k' \trianglelefteq G$, $k'$: finite index nilpotent subgroup.

$L = \langle a \rangle \leq G$

$\alpha(L) = \mathbb{Z}$

$\Longrightarrow K'L \leq G$ : finite index solvable subgroup of $G$

$+$

$K'L$ : polynomial growth

$$\Big\Downarrow \text{W.M Thm}$$

$G$ : virtually nilpotent.

# THANK YOU