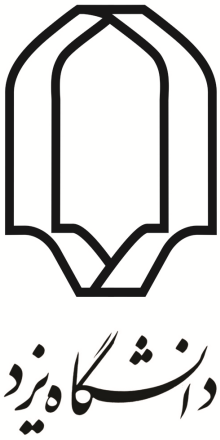

The 46th Annual Iranian Mathematics Conference

25-28 August 2015, Yazd University, Yazd, Iran

Proceedings of the Conference

Poster

Preface

The Annual Iranian Mathematics Conference (AIMC) has been held since 1970. It is the oldest Iranian scientific gathering which takes place regularly each year at one of Iranian universities. The 36^{th} annual Iranian mathematics conference was held at Yazd University and now we are pleased to organize the 46^{th} conference. The 46^{th} AIMC will be held at Yazd University in Yazd (the most beautiful and historical city of Iran) from August 25 until August 28, 2015. The Iranian Mathematical Society and Yazd University have jointly sponsored the 46^{th} AIMC. This conference is an international conference and includes Keynote speakers, Invited speakers, Presentations of contributed research papers, and Poster presentations.

It is our pleasure to publish the Proceedings of the 46^{th} AIMC. More than 700 mathematicians from our country and abroad have taken part in the conference. By kind cooperation of contributors, more than 1100 papers were received. The scientific committee put a considerable effort on referral process in order to arrange a conference of excellent scientific quality. We have 15 plenary speakers from universities of Iran, as well as from Australia, South Korea, Canada, China, Czech Republic, India, Serbia and Spain. Moreover, our invited speakers are about 12.

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46^{th} Annual Iranian Mathematics Conference

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Algebra

A prefilter generated by a set in EQ-algebras

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Abstract

In this paper we introduce the notion of a prefilter generated by a nonempty subset of an EQ-algebra E and we investigate some properties of it. After that by some theorems we characterize a generated prefilter.Then by constituting the set of all prefilters of an EQ-algebra E denoted by $PF(E)$, we show that it s an algebric lattice. Finally, we prove that, the set of all principle prefilters of an ℓEQ-algebra E is a sublattice of $PF(E)$.

Keywords: (Good, separated) EQ-algebra, ℓEQ-algebra.
Mathematics Subject Classification [2010]: 03G1, 03G05

1 Introduction

V. Novák and B. De Baets introduced a spacial algebra called EQ-algebra in [5]. An EQ-algebras have three binary (meet, multiplication and a fuzzy equality) and a top element and also a binary operation implicatin is drived from fuzzy equality. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. These algebras intended to develop an algebric structure of truth values for fuzzy type theory. EQ-algebras are interesting and important algebra for studing and researching and also residuated lattices [3] and BL-algebras [1, 4, 7] are particular casses of EQ-algebras.

Definition 1.1. [2] An algebra $(E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is called an EQ-algebra where for all $a, b, c, d \in E$:

$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $a \leq b$ iff $a \wedge b = a$,
$(E2)$ $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $a \leq b$,
$(E3)$ $a \sim a = 1$, (reflexivity axiom)
$(E4)$ $(a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$, (substitution axiom)
$(E5)$ $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$, (congruence axiom)
$(E6)$ $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$, (monotonicity axiom)
$(E7)$ $a \otimes b \leq a \sim b$,
for all $a, b, c \in E$.

*Speaker

The binary operations $\wedge$, $\otimes$ and $\sim$ are called meet, multiplication and fuzzy equality, respectively.

Clear, $(E, \leq)$ is a partial order. We will also put, for $a, b \in E$

$$\widetilde{a} = a \sim 1 \text{ and } a \to b = (a \wedge b) \sim a$$

The binary operation $\to$ will be called implication.

If E is a nonempty set with three binary operations $\wedge, \otimes, \sim$ such that $(E, \wedge, 1)$ is a $\wedge$-semilattice, $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone with respect to $\leq$, then $(E, \otimes, \wedge, \sim, 1)$ is an EQ-algebra, where $a \sim b = 1$, for all $a, b \in E$.

Lemma 1.2. [2] *Let $(E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then the following properties hold for all $a, b, c, d \in E$:*

(e_1) $a \sim b = b \sim a$,
(e_2) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$,
(e_3) $(a \to b) \otimes (b \to c) \leq (a \to c)$ *and* $(b \to c) \otimes (a \to b) \leq (a \to c)$,
(e_4) $a \sim d \leq (a \wedge b) \sim (d \wedge b)$,
(e_5) $(a \sim d) \otimes ((a \wedge b) \sim c) \leq (d \wedge b) \sim c$,
(e_6) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$,
(e_7) $a \otimes b \leq a \wedge b \leq a, b$,
(e_8) $b \leq \widetilde{b} \leq a \to b$,
(e_9) *If $a \leq b$, then $a \to b = 1$, $b \to a = a \sim b$, $\widetilde{a} \leq \widetilde{b}$, $c \to a \leq c \to b$ and $b \to c \leq a \to c$,*
(e_{10}) *If $a \leq b \leq c$, then $a \sim c \leq a \sim b$ and $a \sim c \leq b \sim c$,*
(e_{11}) $a \otimes (a \sim b) \leq \widetilde{b}$,
(e_{12}) $(a \wedge b) \to c) \otimes (d \to a) \leq (d \wedge b) \to c$.

Throughout this paper, E will be denoted an EQ-algebra unless otherwise stated.

Definition 1.3. [6] Let E be an EQ-algebra. We say that it is
(i) good, if for all $a \in E$, $\widetilde{a} = a$,
(ii) separated, if for all $a, b \in E$, $a \sim b = 1$ implies $a = b$,
(iii) semi-separated, if for all $a \in E$, $a \sim 1 = 1$ implies $a = 1$,
$iv)$ an ℓEQ-algebra, if it has a lattice reduct and for all $a, b, c, d \in E$, $((a \vee b) \sim c) \otimes (d \sim a) \leq c \sim (d \vee b)$.

Definition 1.4. [5] A nonempty subset $F \subseteq E$ is called

A prefilter of E, if for all $a, b \in E$, the following conditions hold
(PF_1) $1 \in F$,
(PF_2) $a, a \to b \in F$, then $b \in F$.

A filter of E, if F is a prefilter of E and for all $a, b, c \in E$,
(F_3) $a \to b \in F$ implies $(a \otimes c) \to (b \otimes c) \in F$.

A positive implication prefilter of E, if F is a prefilter of E and for all $a, b, c \in E$,
(IPF_4) $a \to (b \to c) \in F$ and $a \to b \in F$ imply $a \to c \in F$.

The set of all (filters)prefilters of E is denoted by $(F(E))$ $PF(E)$.

2 A generated prefilter in EQ-algebras

For a nonempty subset $X \subseteq E$, the smallest prefilter of E which contains X, i.e. $\bigcap\{F \in PF(E) : X \subseteq F\}$, is said to be a prefilter of E generated by X and will be denoted by $< X >$.

If $a \in E$ and $X = \{a\}$, we denote by $< a >$ the prefilter generated by $\{a\}$ ($< a >$ is called principal).

For $F \in PF(E)$ and $a \in E$, we denote by $F(a) =< F \cup \{a\} >$. It is clear that $a \in F$ implies $F(a) = F$.

Theorem 2.1. *Let $\emptyset \neq X \subseteq E$. Then*

$$< X >= \{a \in E : x_1 \to (x_2 \to (x_3 \to ...(x_n \to a)...)) = 1, \text{ for some } x_i \in X \text{ and } n \geq 1\}.$$

ω is the set of nonegative integers. For $a, z \in E$ and $n \in \omega$ we define $a \to^0 z = z$, $a \to^{n+1} z = a \to (a \to^n z)$. If $a = 1$, $a \to^{n+1} z$ denoted by $\widetilde{z}^{n+1}$.

Theorem 2.2. *In every EQ-algebra E, for $\emptyset \neq X \subseteq E$ we have*

$$< X >\subseteq \{a \in E : (x_1 \otimes ... \otimes x_n) \to \widetilde{a^k} = 1, \text{ for some } x_i \in X,\ n \geq 1 \text{ and } k \in \omega\}.$$

Moreover in any good EQ-algebra

$$< X >\subseteq \{a \in E : (x_1 \otimes ..., \otimes x_n) \to a = 1, \text{ for some } x_i \in X \text{ and } n \geq 1\}.$$

Theorem 2.3. *Let E be an EQ-alebra and $a, b \in E$. Then for all a, b in E the following are satisfay:*

(i) $a \leq b$ implies $< b >\subseteq< a >$,

(ii) $a^2 = a$ implies $< a >= \{z \in E : a \to \widetilde{z}^k = 1, \text{ for some } k \in \omega\}$,

(iii) If E is a good EQ-algebra and $a^2 = a$, for $a \in E$, then $< a >= \{z \in E : a \leq z\}$,

(iv) Let F be a prefilter of an ℓEQ-algebra E. Then $a \vee b \in F$ implies $F(a) \cap F(b) = F$,

(v) In an ℓEQ-algebra E, we have $< a \vee b >=< a > \cap < b >$,

(vi) $< a \wedge b >=< a > \vee < b >$,

Theorem 2.4. *(i) Let F be a prefilter of an EQ-algebra E. Then*

$$F(a) = \{z \in E : f \to (a \to^n z) = 1, \text{ for some } f \in F \text{ and } n \in \omega\}.$$

(ii) Let F be a positive implication prefilter of E. Then

$$F(a) = \{z \in E : a \to z \in F\}.$$

Let F and G be two prefilters of E. We denote $F \vee G :=< F \cup G >$.

Theorem 2.5. *Let F and $\{F_i\}_{i \in I}$ be prefilters of an ℓEQ- algebra E. Then $F \wedge (\vee_{i \in I} F_i) = \vee_{i \in I}(F \wedge F_i)$.*

A lattice L is called Brouwerian if $a \wedge (\vee_{i \in I} b_i) = \vee_{i \in I}(a \wedge b_i)$, whenever the arbitrary unions exists. Let E be a complete lattice and let a be an element of E. Then a is called compact if $a \leq \vee X$ for some $X \subseteq E$ implies that $a \leq \vee X_1$ for some $X_1 \subseteq X$. A complete lattice is called algebric if every element is the join of compact elements.
By Theorems 2.3 and 2.5 we have the following theorem.

Theorem 2.6. *Let E be an ℓEQ-algebra. Then*

(1) The lattice $(PF(E), \subseteq)$ is a complete Brouwerian lattice.

(2) If we denote by $PF_p(E)$ the family of all principal prefilter of E, then $PF_p(E)$ is a bounded sublattice of $PF(E)$.

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A Simple Method For Hashing To Elliptic Curves

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Abstract

In cryptography, it has been an important problem to transform a random value in $\mathbb{F}_q$ into a random point on an elliptic curve in a deterministic and efficient method. In this paper we propose a simpler form of Shallue-Woestijne-Ulas algorithm in order to hash an element of finite field to a point of an elliptic curves. This subject can be used in cryptosystems based on elliptic curves.

Keywords: Elliptic curves, Quadratic residue, Hash
Mathematics Subject Classification [2010]: 14H52, 11T71

1 Introduction

For a number of elliptic curve cryptosystems it is necessary to hash into an elliptic curve. For instance Boneh-franklin identity based scheme [1]. Before 2006 the usual method was to take $x \in \mathbb{F}_q$ and check whether this value corresponds to a valid abscissa of a point on the elliptic curve. If not, try another abscissa until one of them works. One defect of this algorithm is that the number of operation is not constant. namely the number of steps depends on the input x.
The first algorithm for generating elliptic curve points in deterministic polynomial time was published in ANTS 2006 by Shallue and Woestijne [5] .
The algorithm is based on the skalba equality which says that there exist four maps $X_1(t), X_2(t), X_3(t), X_4(t)$ such that

$$g(X_1(t))g(X_2(t))g(X_3(t)) = (X_4(t))^2$$

where $g(x) = x^3+ax+b$. Then in a finite field for a fixed parameter t, there exists $1 \leq j \leq 3$ such that $g(X_j(t))$ is a quadratic residue, which implies that this $(X_j(t), \sqrt{X_j(t)})$ is a point on the elliptic curve $y^2 = g(x)$.
The maps were simplified and generalized to hyperelliptic curves by Ulas in 2007 [4]. We recall these maps in the following result.

Lemma 1.1. *Let*

$$X_1(t,u) = u \qquad X_2(t,u) = \frac{-b}{a}(1 + \frac{1}{t^4g(u)^2 + t^2g(u)})$$
$$X_3(t,u) = t^2g(u)X_2(t,u) \qquad U(t,u) = t^3g(u)^2g(X_2(t,u))$$

*Speaker

Then

$$U(t,u)^2 = g(X_1(t,u)).g(X_2(t,u)).g(X_3(t,u))$$

Brier et al [6] propose a further simplification of the Shallue-Woestijne-Ulas algorithm for elliptic curves over finite field $\mathbb{F}_q$ with $q \equiv 3(mod\ 4)$.
In this paper we propose a simpler form of the shallue-woestijne-ulas (SWU) algorithm over $\mathbb{F}_q$ for $q \equiv 2(mod\ 3)$.

2 Main result

In this section we suppose $\mathbb{F}_q$ is a finite field where $q \equiv 2(mod\ 3)$. In this case, the function

$$x \longrightarrow x^3$$

is a bijection with the inverse function

$$x \longrightarrow x^{1/3} = x^{2q-1/3}$$

.

Lemma 2.1. *Let $g(x) = x^3 + ax + b$. If u is a quadratic non-residue such that for some $x \in \mathbb{F}_q$ we have*

$$g(u.x) = ug(x) \tag{1}$$

then either x or $u.x$ is the abscissa of a point on the $y^2 = g(x)$. Moreover for each u the value

$$x = \sqrt[3]{\frac{b}{u(u+1)}} \tag{2}$$

satisfies (1).

Proof. Since u is not a quadratic residue, if x satisfies (1) then either $g(u.x)$ or $g(x)$ must be a square in $\mathbb{F}_q$. Therefor either x or $u.x$ must be abscissa of a point on the curve $y^2 = g(x)$. Moreover we have:

$$\begin{aligned} g(ux) = ug(x) &\Longleftrightarrow (ux)^3 + a(ux) + b = u(x^3 + ax + b) \\ &\Longleftrightarrow u^3x^3 + b = ux^3 + ub \\ &\Longleftrightarrow x = \sqrt[3]{\frac{b(u-1)}{u^3-u}} = \sqrt[3]{\frac{b}{u(u+1)}} \end{aligned}$$

□

Theorem 2.2. *Let*

$$X_2(t) = \sqrt[3]{\frac{b}{-3t^2(-3t^2+1)}} \quad X_3(t) = -3t^2X_2(t) \quad U(t) = tg(X_2(t)) \tag{3}$$

Then

$$(U(t))^2 = \frac{-1}{3}g(X_2(t))g(X_3(t))$$

Proof. since $q \equiv 2(mod\ 3)$, -3 is a quadratic non-residue and we can take $u = -3t^2$ in previous lemma. Therefore $X_2(t) = x$ and $X_3(t) = ux$ and we have

$$g(X_2(t))g(X_3(t)) = g(x)g(u.x) = ug(x)^2 = -3t^2g(x)^2 = -3(tg(x))^2 = -3(U(t))^2$$

□

2.0.1 Simplified SWU Algorithm

Input: $\mathbb{F}_q$ such that $q \equiv 2(mod\ 3)$, parameters a, b and $t \in \mathbb{F}_q$
Output: $(x, y) \in E_{a,b}(\mathbb{F}_q)$ where $E_{a,b} : y^2 = x^3 + ax + b$
1. $u \longleftarrow -3t^2$
2. $X_2 \longleftarrow \sqrt[3]{\frac{b}{u^2+u}}$
3. $X_3 \longleftarrow u.X_2$
4. $g_2 \longleftarrow X_2^3 + aX + b; g_3 = X_3^3 + aX_3 + b$
5. If g_2 is a square, return $(X_2, \sqrt{g_2})$, otherwise return $(X_3, \sqrt{g_3})$

Remark 2.3. In the final step of the previous algorithm we need to compute a square root. Although no deterministic algorithm is known for computing square roots, since in our case $q \equiv 2(mod\ 3)$ and hence -3 is a non-quadratic residue, the Tonelli-Shanks algorithm can compute the square root deterministicly in polynomial time. (see [2])

Remark 2.4. In order to compute the pre-images of a point $P = (X_P, Y_P)$ we should solve the equations $X_2(t) = X_P$ and $X_3(t) = X_P$. Since $degX_2(t) = 4$ and $degX_3(t) = 4$ each equation has at most 4 solutions. Hence a point has at most 8 pre-images.

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Poster A Zariski topology on the primary-like spectrum of ν-multiplication modules pp.: 1–4

A Zariski topology on the primary-like spectrum of ν-multiplication modules

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Abstract

Let R be a commutative ring with identity and M be a unitary R-module. Primary-like spectrum of M is the set of all primary-like submodules Q of M where M/Q is a primeful module. In this paper, we introduce a base for the Zariski topology on the primary-like spectrum of a ν-multiplication R-module M.

Keywords: Primary-like submodule, ν-Multiplication module, Zariski topology.
Mathematics Subject Classification [2010]: 13C13, 13C99

1 Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N : M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and annihilator of M, denoted by $Ann(M)$, is the ideal $(0 : M)$. A submodule P of an R-module M is said to be prime (resp. p-prime) if $P \neq M$ and whenever $rm \in P$ (where $r \in R$ and $m \in M$), then $m \in P$ or $r \in p = (P : M)$ [6]. The prime spectrum of M, denoted by $Spec(M)$ (resp. $Spec_p(M)$), is the set of all prime (resp. p-prime) submodules of M. If N is a submodule of M, then $V(N) = \{P \in Spec(M); N \subseteq P\}$ and the radical of N, denoted by $\mathrm{rad}(N)$, is the intersection of all elements of $V(N)$ [5]. An R-module M is called a top module if $\{V(N); N \leq M\}$ induces a topology over $Spec(M)$ [4].

A submodule Q of M is said to be primary-like if $Q \neq M$ and whenever $rm \in Q$ (where $r \in R$ and $m \in M$) implies that $r \in (Q : M)$ or $m \in \mathrm{rad}(Q)$ [2]. An R-module M is said to be primeful if either $M = (0)$ or $M \neq (0)$ and the map

$$\psi : Spec(M) \to Spec(R/Ann(M))$$

defined by $\psi(P) = (P : M)/Ann(M)$ is surjective [3]. If M/N is a primeful module over R, then $\sqrt{(N : M)} = (\mathrm{rad}(N) : M)$ [3, Proposition 5.3]. The primary-like spectrum of M denoted by $\mathcal{X}$ is defined to be the set of all primary-like submodules Q of M where M/Q is a primeful module [2].

Recently, modules whose spectrums having various types of Zariski topologies have been received a good deal of attention (see for example [4]). In [2], we study the algebraic properties of a new class of modules which are equipped with a new Zariski topology defined as follows. Let N be a submodule of an R-module M. We set

*Speaker

$$\nu(N) = \{Q \in \mathcal{X}; N \subseteq \mathrm{rad}(Q)\}.$$

Some elementary facts about ν have been collected in the following lemma.

Lemma 1.1. *Let M be an R-module. Let N, N' and $\{N_i : i \in I\}$ be submodules of M. Then the following statements hold.*

(1) $\nu(M) = \emptyset$.

(2) $\nu(0) = \mathcal{X}$.

(3) $\cap_{i\in I}\nu(N_i) = \nu(\sum_{i\in I} N_i)$.

(4) $\nu(N) \cup \nu(N') \subseteq \nu(N \cap N')$.

(5) $\nu(N) = \nu(\mathrm{rad}(N))$.

(6) $\nu(N) = \nu((N : M)M)$.

Put $\eta(M) = \{\nu(N) \mid N \text{ is a submodule of } M\}$. From (1), (2), (3) and (4) in Lmma 1.1, we can see easily that there exists a topology, $\mathcal{T}$ say, on $\mathcal{X}$ having $\eta(M)$ as the collection of all closed sets if and only if $\eta(M)$ is closed under finite union. An R-module M is called a top-like module if $\eta(M)$ induces the topology $\mathcal{T}$ over $\mathcal{X}$ [2]. Let M be an R-module. Then M is called ν-multiplication if for every submodule N of M there exists an ideal I of R such that $\nu(N) = \nu(IM)$. It is easy to see that every ν-multiplication R-module is a top-like module. In this paper we show that if M is a ν-multiplication R-module, then $(\mathcal{X}, \mathcal{T})$ has a base (Theorem 2.8).

2 Main results

Definition 2.1. An R-module M is called a multiplication module if for every submodule N of M, there exits an ideal I of R such that $N = IM$ [1].

Note that, if N is a submodule of a multiplication R-module M, then $I \subseteq (N : M) = \{r \in R \mid rM \subseteq N\}$ and hence $N = IM \subseteq (N : M)M \subseteq N$ so that $N = (N : M)M$.

Lemma 2.2. *Any multiplication R-module is a ν-multiplication R-module.*

By an easy verication we have the following lemma.

Lemma 2.3. *Let M be a ν-multiplication R-module. Then any homomorphic image of M is a ν-multiplication R-module.*

Lemma 2.4. *An R-module M is a ν-multiplication module if and only if for each $m \in M$ there exists an ideal I of R such that $\nu(Rm) = \nu(IM)$.*

By Lemma 2.2, [4, Proposition 3.2] and [6, Theorem 3.5] and the fact that, if M is a finitely generated R-module, then the topological space over $Spec(M)$ is a topological subspace of $(\mathcal{X}, \mathcal{T})$ we have the following.

Theorem 2.5. *Let M be a finitely generated R-module. Then the following statements are equivalent.*

(1) M is multiplication;

(2) M is ν-multiplication;

(3) M is top-like;

(4) M is top;

(5) $|Spec_p(M)| \leq 1$ for every $p \in Spec(R)$;

(6) If $V(P) = V(P')$ for $P, P' \in Spec(M)$, then $P = P'$;

(7) The natural map ψ is injective;

(8) For every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$;

(9) M_p is a top R_p-module for every prime ideal p of R;

(10) M/mM is cyclic for every maximal ideal m of R.

In [4, Proposition 4.3], it has been proved that $\{X_r; r \in R\}$ forms a base for the topology τ on $Spec(M)$, where $X_r = Spec(M)\backslash V(rM)$. In this paper, we introduce a base for the topology $\mathcal{T}$ on $\mathcal{X}$ when M is a ν-multiplication module. For each $r \in R$, we define $\mathcal{X}_r = \mathcal{X}\backslash\nu(rM)$. Now, we have the following.

Proposition 2.6. *Let M be an R-module. If r is a unit element of R, then $\mathcal{X}_r = \mathcal{X}$. The converse is true, if M is a faithful primeful module.*

By [1, Corollary 2.11] and [3, Proposition 4.5] we have the following.

Proposition 2.7. *Let M be a multiplication R-module and $r \in R$. Consider the following statements.*

(1) $rM \subseteq \mathrm{rad}(0)$.

(2) $X_r = \emptyset$.

(3) $\mathcal{X}_r = \emptyset$.

(4) $rM \subseteq \bigcap_{Q\in\mathcal{X}} \mathrm{rad}(Q)$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4). Moreover, if M is primeful, then the above conditions are all equivalent.

Theorem 2.8. *Let M be a ν-multiplication R-module. Then the set $\mathcal{B} = \{\mathcal{X}_r; r \in R\}$ forms a base for the Zariski topology on $\mathcal{X}$.*

Proof. If $\mathcal{X} = \emptyset$, then $\mathcal{B} = \emptyset$ and the proposition is trivially true. Hence we assume that $\mathcal{X} \neq \emptyset$ and let $\mathcal{U}$ be any open set in $\mathcal{X}$. Hence $\mathcal{U} = \mathcal{X}\backslash\nu(IM)$ for some ideal I of R. Note that $\nu(IM) = \nu(\sum_{a_i\in I} a_iM) = \nu(\sum_{a_i\in I}(a_iM : M)M) = \cap_{a_i\in I}\nu(a_iM)$. Hence $\mathcal{U} = \mathcal{X}\backslash \cap_{a_i\in I} \nu(a_iM) = \cup_{a_i\in I}\mathcal{X}_{a_i}$. This proves that $\mathcal{B}$ is a base for the Zariski topology on $\mathcal{X}$. □

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(Amply) *Rad*-Supplemented Lattices

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Abstract

In this paper, we introduce and investigate (amply) *Rad*-supplemented lattices. If L is a *Rad*-supplemented lattice and $a \in L$, then $1/a$ is *Rad*-supplemented. It is shown that an algebraic lattice L is amply *Rad*-supplemented iff L is a *Rad*-supplemented. If $a/0$ and $1/a$ are *Rad*-supplemented and a has a *Rad*-supplement b in $d/0$ for every sublattice $d/0$ with $a \leq d$, then L is *Rad*-supplemented.

Keywords: *Rad*-Supplement, ample *Rad*-Supplement, *Rad*-Supplemented Lattice, amply *Rad*-Supplemented Lattice
Mathematics Subject Classification [2010]: 06CXX, 16D10

1 Introduction

Throughout this paper, we assume that L is a complete modular lattice with smallest element 0 and greatest element 1. An element a of a lattice L is called *small* in L (notation $a \ll L$), if $a \vee b \neq 1$ for every $b \neq 1$.

Let a and b be elements of a lattice L. a is called a *supplement* of b in L if a is minimal with respect to $1 = a \vee b$. a is a supplement of b in L iff $1 = a \vee b$ and $a \wedge b \ll a/0$ (see [3]). A lattice L is called *supplemented* if every element of L has a supplement in L. L is called *amply supplemented* if for any two elements a and b of L with $1 = a \vee b$, $b/0$ contains a supplement of a. A subset D of L is called *upper directed* if each finite subset of D has an upper bound in D. A lattice L is called *upper continuous* if $a \wedge (\bigvee D) = \bigvee_{d \in D}(a \wedge d))$ holds for every $a \in L$ and upper directed subset $D \subseteq L$. An element $a \in L$ is called *compact* if for every subset X of L and $a \leq \bigvee X$ there is a finite subset $F \subseteq X$ such that $a \leq \bigvee F$ and L is said to be compact if 1 is compact. A lattice L is called *algebraic* if each of its elements is a join of compact elements. An element $e \in L$ is called *essential* in L if $e \wedge a = 0$ holds for each element $a \in L$, $a \neq 0$. A lattice L is called *coatomic* if every proper element of L is contained in a maximal element of L. $Rad(L)$ will indicate *radical* of L (the intersection of all the maximal elements $\neq 1$ in L). We have the following properties of $Rad(L)$ in a lattice L.

Lemma 1.1. *[3, Lemma 7.8 and Proposition 12.2] Let a be an element in a lattice L.*
(1) $a \vee R(L) \leq R(1/a)$;
(2) If $a \leq R(L)$ then $R(1/a) = R(L)$;
(3) If L is algebraic, then $R(a/0) = a \wedge R(L)$.

*Speaker

All definitions and related properties on lattices not given here, can be found in [2] and [3].

2 Main results

Let L be a lattice. If $a, b \in L$ and $1 = a \vee b$ then b is a *Rad-supplement* of a in case $a \wedge b \leq Rad(b/0)$. L is called *Rad-supplemented* lattice in case every element of L has a *Rad*-supplement in L. An element a of a lattice L has *ample Rad-supplements* in L if for every element b of L with $a \vee b = 1$, $b/0$ contains a *Rad*-supplement of a in L. L is called *amply Rad-supplemented* lattice in case every element of L has an ample *Rad*-supplement in L. Clearly each supplemented and amply *Rad*-supplemented lattice is *Rad*-supplemented. Recall that a lattice L is called radical if has no maximal element, that is, $1 = Rad(L)$. $a \in L$ is called radical if $a = Rad(a/0)$.

We start with the following.

Lemma 2.1. *Let L be a radical lattice. Then L is Rad-supplemented.*

By $P(L)$, which is the largest radical element of a lattice L, we will indicate the join of all radical elements of L.

Since $P(L)$ is a radical element of a lattice L, we have the following corollary.

Corollary 2.2. *For every lattice L, $P(L)/0$ is Rad-supplemented.*

Proposition 2.3. *Let L be a Rad-supplemented lattice and b an element of L with $b \wedge Rad(L) = 0$. Then $b/0$ is complemented. In particular, a Rad-supplemented lattice L with $Rad(L) = 0$ is complemented.*

Proposition 2.4. *Let L be an amply Rad-supplemented lattice and l a direct summand of L. Then $l/0$ is amply Rad-supplemented.*

Proposition 2.5. *Let L be an upper continuous lattice. If L is Rad-supplemented, then $1 = x \vee y$ for some complemented $x/0$ and some $y/0$ with essential radical.*

Proposition 2.6. *Let $a_1, b \in L$ and $a_1/0$ be Rad-supplemented. If $a_1 \vee b$ has a Rad-supplement in L, then so does b.*

Proposition 2.7. *If L is a Rad-supplemented lattice, Then*
(1) For every $a \in L$, $1/a$ is Rad-supplemented;
(2) $1/Rad(L)$ is complemented.

Lemma 2.8. *Let x, y and z be elements of a lattice L such that $1 = x \vee y \vee z$. If x is a Rad-supplement of $y \vee z$ in L and y is a Rad-supplement of $x \vee z$ in L, then $x \vee y$ is a Rad-supplement of z in L.*

Proposition 2.9. *Let L be a lattice and $1 = x_1 \vee x_2$. If x_1, x_2 have ample Rad-supplements in L, then $x_1 \wedge x_2$ also has ample Rad-supplements in L.*

Lemma 2.10. *Let a, b be elements of L and b a Rad-supplement of a in L. If a is a maximal element of L, then $a \wedge b = Rad(b/0)$ is a unique maximal element of $b/0$.*

Proposition 2.11. *Let L be a lattice such that $Rad(L) \ll L$. If every element of L has a Rad-supplement, then L is coatomic.*

Proposition 2.12. *Let L be a lattice. If every element of L is a Rad-supplemented sublattice, then L is an amply Rad-supplemented lattice.*

Theorem 2.13. *Let L be an algebraic lattice. Then the following assertions are equivalent.*
(1) L is an amply Rad-supplemented lattice;
(2) L is a Rad-supplemented lattice.

Theorem 2.14. *Let L be a lattice and $a \in L$. If $a/0$ and $1/a$ are Rad-supplemented and a has a Rad-supplement b in $d/0$ for every sublattice $d/0$ with $a \leq d$, then L is Rad-supplemented.*

Lemma 2.15. *Let L be a lattice which contains a radical element a. Then a has a Rad-supplement b in $d/0$ for every sublattice $d/0$ of L with $a \leq d$.*

Since $P(L)/0$ is *Rad*-supplemented, then by combining last lemma and Theorem 2.14, we get the following result.

Corollary 2.16. *If $1/P(L)$ is Rad-supplemented, then L is a Rad-supplemented lattice.*

Proposition 2.17. *The following three statements are equivalent for a lattice L which contains a radical element a.*
(1) L is Rad-supplemented;
(2) $1/a$ is Rad-supplemented;
(3) Every element of L containing a has a Rad-supplement in L.

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Embedding locally inverse semigroups into Rees matrix semigroups

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Abstract

We give a necessary and sufficient condition for a locally inverse semigroup to be embeddable into a Rees matrix semigroup over a generalized inverse semigroup.

Keywords: Inverse semigroup, Rees matrix semigroup
Mathematics Subject Classification [2010]: 20M10, 20M17

1 Introduction

For the standard notions and notation in semigroup theory the reader is referred to [1]. In particular the set of idempotents of a semigroup S will be denoted by $E(S)$, the set of inverses of an element $s \in S$ by $V(s)$. For nonempty subsets $H, K \subseteq S$, HK denotes the usual product of subsets.

Definition 1.1. A regular semigroup is called locally inverse if the submonoid eSe is an inverse subsemigroup of S, for all $e \in E(S)$. If in addition $E(S)$ is a subsemigroup of S, then S is called a generalized inverse semigroup. In this case $E(S)$ is a normal band, i.e. an idempotent semigroup satisfying the equation $xyzx = xzyx$.

Definition 1.2. On a regular semigroup S, a partial order relation $\leq$ is defined by $s \leq t$, if $s = et = tf$, for some $e, f \in E(S)$. It is called the natural partial order. A regular semigroup is locally inverse, if and only if the natural partial order is compatible with the multiplication.

Definition 1.3. An order ideal H of a regular semigroup S, is a nonempty subset H of S, such that $x \leq h$ implies $x \in H$, for all $h \in H$. For $s \in S$ we shall use the notation $[s] = \{x \in S : \ x \leq s\}$. In any locally inverse semigroup S the equality $[s][t] = [st]$ holds, for all $s, t \in S$.

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Structure: Let T be a regular semigroup, and let I, Λ be sets. Let P be a $\Lambda \times I$ matrix over T. Then the set of all triples $(i, t, \lambda) \in I \times T \times \Lambda$ is a semigroup under the multiplication $(i, t, \lambda)(j, u, \mu) = (i, tp_{\lambda j}u, \mu)$, denoted by $\mathcal{M}(T; I, \Lambda; P)$. $\mathcal{M}(T; I, \Lambda; P)$ is called *a Rees matrix semigroup* over T. In general $\mathcal{M}(T; I, \Lambda; P)$ is not regular, however the set of regular elements forms a subsemigroup (see [2]). Moreover if T is a regular monoid with subgroup of units U, and if all entries of P belong to U, then obviously $\mathcal{M}(T; I, \Lambda; P)$ is regular.

Let S be a semigroup, and let $I \times \Lambda$ be a rectangular band. S is termed a rectangular band $I \times \Lambda$ of semigroups $S_{i\lambda}$, if S is the disjoint union of the $S_{i\lambda}$, $(i, \lambda) \in I \times \Lambda$, such that $S_{i\lambda}S_{j\mu} \subseteq S_{i\mu}$, for all $(i, \lambda), (j, \mu) \in I \times \Lambda$.

2 Main results

In what follows let S be a locally inverse semigroup satisfying the following condition (E):

(E1) S is a rectangular band $I \times \Lambda$ of semigroups $S_{i\lambda}$;

(E2) for all $(i, \lambda) \in I \times \Lambda$, $e, f \in S_{i\lambda} \cap E(S)$, $s, t \in S$, the equality $seft = sfet$ holds.

Let S be a locally inverse semigroup satisfying condition (E). For $(i, \lambda) \in I \times \Lambda$, let $E_{i\lambda}$ denote the subset of idempotents of $S_{i\lambda}$. Let further denote $S_i = \bigcup_{\lambda \in \Lambda} S_{i\lambda}$, where $i \in I$, and $S_\lambda = \bigcup_{i \in I} S_{i\lambda}$, where $\lambda \in \Lambda$.

Lemma 2.1. *The sets $E_i = \bigcup_{\lambda \in \Lambda} E_{i\lambda}$, where $i \in I$, and $E_\lambda = \bigcup_{i \in I} E_{i\lambda}$, where $\lambda \in \Lambda$, are normal subbands of S.*

The following concept was developed in [4].

Definition 2.2. A subset H of S is called permissible, if

(i) H is an order ideal of S;

(ii) there exists an order ideal H' of S and $(i, \lambda), (j, \mu) \in I \times \Lambda$, such that $HH'H = H$, $H'HH' = H'$, $HH' \subseteq E_{i\lambda}$, and $H'H \subseteq E_{j\mu}$.

Note that if H is permissible, then H', mentioned in the above definition, is permissible too.

Lemma 2.3. *Let H and H' be permissible subsets of S, and let $a \in H$. Then there exists $a' \in V(a) \cap H'$.*

Lemma 2.4. *Let $H \subseteq S_{i\mu}$ be a permissible subset of S, such that $H^2 = H$. Then $H \subseteq E_{i\mu}$.*

Lemma 2.5. *Let $H \subseteq S_{i\mu}$ be a permissible subset of S. Then for all $j \in I$, $\lambda \in \Lambda$, $c \in S$, we have $\{c\}E_{i\lambda}H \subseteq \{c\}H$, and $HE_{j\mu}\{c\} \subseteq H\{c\}$.*

Lemma 2.6. *The set $\mathcal{A}(S)$ of permissible subsets of S forms a regular semigroup under the usual multiplication of subsets.*

Lemma 2.7. *Let $A \subseteq S_{i\lambda}$, $B \subseteq S_{j\mu}$ be permissible subsets of S. Then $AE_{j\lambda}B = AB$.*

The following construction appeared already in [3] in disguised form.
Let $C(I \times \Lambda)$ be the set of all finite sequences of elements of the rectangular band $I \times \Lambda$, satisfying the property that no two adjacent pairs coincide in their first or second component. For $g_{i_1\lambda_n}$, $g_{j_1\mu_m} \in C(I \times \Lambda)$, where $g_{i_1\lambda_n} = ((i_1, \lambda_1), (i_2, \lambda_2), ..., (i_n, \lambda_n))$, and $g_{j_1\mu_m} = ((j_1, \mu_1), (j_2, \mu_2), ..., (j_m, \mu_m))$, we define the product $g_{i_1\lambda_n} g_{j_1\mu_m} \in C(I \times \lambda)$ to be the sequence which arises from $((i_1, \lambda_1), (i_2, \lambda_2), ..., (i_n, \lambda_n), (j_1, \mu_1), (j_2, \mu_2), ..., (j_m, \mu_m))$ by a successive application of the following cancellation rule (C): Each two adjacent pairs (k_1, κ_1), (k_2, κ_2) (in that order) shall be replaced by (k_2, κ_2) if $k_1 = k_2$, and by (k_1, κ_1) if $\kappa_1 = \kappa_2$, respectively. Obviously $g_{i_1\lambda_n} g_{j_1\lambda_m}$ is a well defined element of $C(I \times \Lambda)$. Moreover we obtain:

Lemma 2.8. *$C(I \times \Lambda)$ is a completely simple semigroup under the multiplication defined above. In particular $C(I \times \Lambda)$ is a rectangular band $I \times \Lambda$ of groups $G_{i\lambda} = \{g_{i\lambda} \in C(I \times \Lambda) : \ g_{i\lambda} = ((i_1, \lambda_1,), (i_2, \lambda_2), ..., (i_n, \lambda_n)), \ i = i_1, \ \lambda_n = \lambda\}$. The group identity $1_{i\lambda}$ of $G_{i\lambda}$ is $((i, \lambda))$, and the group inverse $g_{i\lambda}^{-1}$ of $g_{i\lambda} = ((i_1, \lambda_1,), (i_2, \lambda_2), ..., (i_n, \lambda_n)), \ i = i_1, \ \lambda_n = \lambda$, in $G_{i\lambda}$ is the element of $C(I \times \Lambda)$ arising from $((i, \lambda), (i_n, \lambda_{n-1}), ..., (i_2, \lambda_1), (i, \lambda))$ by applying (C), if necessary.*

For $G_{i\lambda} \in C(I \times \Lambda)$ with $G_{i\lambda} = \{g_{i\lambda} \in C(I \times \Lambda) : \ g_{i\lambda} = ((i_1, \lambda_1,), (i_2, \lambda_2), ..., (i_n, \lambda_n)), \ i = i_1, \ \lambda_n = \lambda\}$, let $< g_{i\lambda} >= E_{i\lambda_1} E_{i_2\lambda_2} ... E_{i_n\lambda}$. By Lemma 2.6, $< g_{i\lambda} >$ belongs to $\mathcal{A}(S)$.

Lemma 2.9. *Let $g_{i\lambda}, \ g_{j\mu} \in C(I \times \Lambda)$. Then $< g_{i\lambda} >< g_{j\mu} > \ \subseteq \ < g_{i\lambda} g_{j\mu} >$.*

Lemma 2.10. *Let $g_{i\lambda} = ((i_1, \lambda_1,), (i_2, \lambda_2), ..., (i_n, \lambda_n)) \in C(I \times \Lambda), \ i = i_1, \ \lambda_n = \lambda$, and let*

$$g'_{i\lambda} = \begin{cases} ((i_n, \lambda_{n-1}), (i_{n-1}, \lambda_{n-2}), ..., (i_2, \lambda_1)), & if \quad n \geq 2 \\ ((i, \lambda)), & if \quad n = 1. \end{cases}$$

Then $< g_{i\lambda} >=< g_{i\lambda} >< g'_{i\lambda} >< g_{i\lambda} >$.

Lemma 2.11. *Let $C = \bigcup \{G_{i\lambda} : \ i \in I, \lambda \in \Lambda\}$ be a completely simple semigroup. For $(i, \lambda) \in I \times \Lambda$ let $1_{i\lambda}$ denote the identity of the maximal subgroup $G_{i\lambda}$. We may assume that $0 \in I \cap \Lambda$. Let $g_{k\lambda} \in G_{k\lambda}$, $g_{k\mu} \in G_{k\mu}$, $g_{l\lambda} \in G_{l\lambda}$, and let $g_{l\mu} \in G_{l\mu}$. Let further $g'_{k\lambda}$, $g'_{k\mu}$, $g'_{l\lambda}$, and $g'_{l\mu}$ be inverses of $g_{k\lambda}$, $g_{k\mu}$, $g_{l\lambda}$ and $g_{l\mu}$, such that one of the following equalities hold: $g_{k\lambda} = g_{k\mu} g'_{l\mu} g_{l\lambda}$, $g_{k\mu} = g_{k\lambda} g'_{l\lambda} g_{l\mu}$, $g_{l\lambda} = g_{l\mu} g'_{k\mu} g_{k\lambda}$, $g_{l\mu} = g_{l\lambda} g'_{k\lambda} g_{k\mu}$. Then there are $g_{k0} \in G_{k0}$, $g_{l0} \in G_{l0}$, $g_{0\lambda} \in G_{0\lambda}$, and $g_{0\mu} \in G_{0\mu}$, such that $g_{k\lambda} = g_{k0} g_{0\lambda}$, $g_{k\mu} = g_{k0} g_{0\mu}$, $g_{l\lambda} = g_{l0} g_{0\lambda}$, and $g_{l\mu} = g_{l0} g_{0\mu}$.*

On the other hand, let $g_{k0} \in G_{k0}$, $g_{l0} \in G_{l0}$, $g_{0\lambda} \in G_{0\lambda}$, and let $g_{0\mu} \in G_{0\mu}$. Then the equalities of the first part of the assumption hold for $g_{k\lambda} = g_{k0} g_{0\lambda}$, $g_{k\mu} = g_{k0} g_{0\mu}$, $g_{l\lambda} = g_{l0} g_{0\lambda}$, and $g_{l\mu} = g_{l0} g_{0\mu}$, and for arbitrary inverses $g'_{k\lambda}$, $g'_{k\mu}$, $g'_{l\lambda}$, $g'_{l\mu}$ of $g_{k\lambda}$, $g_{k\mu}$, $g_{l\lambda}$, and $g_{l\mu}$.

Lemma 2.12. *Let $g_{k\lambda}$, $g_{k\mu}$, $g_{l\lambda}$, $g_{l\mu} \in C(I \times \Lambda)$ be as in Lemma 2.11. Let further $B \in \mathcal{A}(S)$, where $B =< Bg_{l\lambda}B >$. Then $< g_{k\lambda} B g_{l\mu} > \subseteq < g_{k\mu} >$.*

Lemma 2.13. *Let $g_{k\lambda}$, $g_{k\mu}$, $g_{l\lambda}$, $g_{l\mu} \in C(I \times \Lambda)$ be as in Lemma 2.11. Let further $A, B, C \in \mathcal{A}(S)$, where $< Bg_{l\lambda}B >= B$. Then $< Ag_{k\lambda} B g_{l\mu} C > \subseteq < Ag_{k\mu} C >$.*

Lemma 2.14. *Let T be a regular semigroup which satisfies the equality $abca = acba$, for all $a, b, c \in E(T)$. Then T is a generalized inverse semigroup.*

Theorem 2.15. *Let $0 \in I \cap \Lambda$ and let $T = \bigcup_{\lambda\in\Lambda} G_{0\lambda} \times \mathcal{A}(S) \times \bigcup_{i\in I} G_{i0}$, where G_{i0} are maximal subgroups of $C(I \times \Lambda)$. On T let a multiplication be defined by $(g_{0\lambda}, H, g_{i0})(g_{0\mu}, K, g_{j0}) = (g_{0\lambda}, < Hg_{i\mu}K >, g_{j0})$, where $g_{i\mu} = g_{i0}g_{0\mu}$. Then T is a generalized inverse semigroup.*

Theorem 2.16. *Let S be a locally inverse semigroup. Then the following statements are equivalent:*

(i) S satisfies condition (E);

(ii) S is embeddable into a Rees matrix semigroup $\mathcal{M}(T; I, \Lambda; P)$ over a generalized inverse semigroup T;

(iii) S is embeddable into a Rees matrix semigroup $\mathcal{M}(T'; I, \Lambda; Q)$ over an orthodox monoid T' with group of units U, where $T = T' \setminus U$ is a generalized inverse subsemigroup of T', all entries of Q belong to U, and the image of S under the embedding is contained in $I \times T \times \Lambda$.

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Poster Generalizations of primary submodules pp.: 1–4

GENERALIZATIONS OF PRIMARY SUBMODULES

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Abstract

Let R be a commutative ring with $1 \neq 0$ and M be a unitary R-module. Let $S(M)$ be the set of all submodules of M. In this paper, we extend the concept of 2-absorbing primary submodules to the context of ϕ-2-absorbing primary submodules. Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function. A proper submodule N of M is said to be a ϕ-2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $abx \in N \setminus \phi(N)$ implies $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$. Anumber of results concerning ϕ-2-absorbing primary submodules are given.

Keywords: primary submodule, 2-absorbing submodule, 2-absorbing primary submodule, ϕ-primary submodule, ϕ-2-absorbing primeary submodule.
Mathematics Subject Classification [2010]: 13A15,13F05, 13G05

1 Introduction

Throughout this paper R denotes a commutative ring with $1 \neq 0$ and M denotes a unitry R-module and the set of all submodules of M is denoted by $S(M)$. A submodule N of M is said to be proper if $N \neq M$. Let N be a submodule of M. Then $(N : M) = \{r \in R | rM \subseteq N\}$ is an ideal of R.

One of the natural generalisations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodules, (see for example [1],[3-6]).Generalizations of prime submodules to the context of ϕ-prime submodules are studied extensively in [2], [7], [8]. Recall that a proper submodule N of M is called a 2-absorbing submodule of M as in [2] if whenever $abx \in N$ for some $a, b \in R$ and $x \in M$, then $ab \in (N : M)$ or $ax \in N$ or $bx \in N$. A proper submodule N of M is called a weakly prime submodule of M as in [7] if whenever $0 \neq ax \in N$ for some $a \in R$ and $x \in M$, then $a \in (N : M)$ or $x \in N$. We say that a proper submodule N of M is a weakly primary submodule of M if whenever $0 \neq ax \in N$ for some $a \in R$ and $x \in M$, then $a \in (N : M)$ or $x \in rad(N)$.

Also, we say that a proper submodule N of M is a 2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $abx \in N$, then $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$. A proper submodule N of M is a weakly 2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $0 \neq abx \in N$ implies $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$. Recall that a proper submodule N of M is called a ϕ-2-absorbing submodule of M as in [2] if whenever $a, b \in R$ and $x \in M$ with $abx \in N \setminus \phi(N)$

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implies $ab \in (N : M)$ or $ax \in N$ or $bx \in N$. We say that a proper submodule N of M is a ϕ-primary submodule of M if whenever $a \in R$ and $x \in M$ with $ax \in N \setminus \phi(N)$ implies $a \in (N : M)$ or $x \in rad(N)$. We show that ϕ-2-absorbing primary submodules enjoy analogs of many of the properties of (weakly) 2-absorbing primary submodules.

Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function. A proper submodule N of M is said to be a ϕ-2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $abx \in N \setminus \phi(N)$ implies $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$.

Let N be a proper submodule of M and suppose that N is a ϕ-2-absorbing primary submodule of M. Then

(i) If $\phi(P) = \emptyset$ for every $P \in S(M)$, then we say that $\phi = \phi_\emptyset$ and N is called a $\phi_\emptyset$-2-absorbing primary submodule of M, and hence N is a 2-absorbing primary submodule of M.

(ii) If $\phi(P) = 0$ for every $P \in S(M)$, then we say that $\phi = \phi_0$ and N is called a ϕ_0-2-absorbing primary submodule of M, and thus N is a weakly 2-absorbing primary submodule of M.

(iii) If $\phi(P) = P$ for every $P \in S(M)$, then we say that $\phi = \phi_1$ and N is called a ϕ_1-2-absorbing primary submodule of M.

(iv) If $n \geq 2$ and $\phi(P) = (P : M)^{n-1}P$ for every $P \in S(M)$, then we say that $\phi = \phi_n$ and N is called a ϕ_n-2-absorbing primary submodule of M. In particular, if $n = 2$ and $\phi(P) = (P : M)P$ for every $P \in S(M)$, then we say that N is an almost-2-absorbing primary submodule of M.

(v) If $\phi(N) = \cap_{n=1}^{\infty}(P : M)^{n-1}P$ for every $P \in S(M)$, then we say that $\phi = \phi_\omega$ and N is called a ϕ_ω-2-absorbing primary submodule of M.

Since $N \setminus \phi(N) = N \setminus (N \cap \phi(N))$, without loss of generality, we may assume that $\phi(N) \subseteq N$. Given two functions $\psi_1, \psi_2 : S(M) \longrightarrow S(M) \cup \emptyset$. we say $\psi_1 \leq \psi_2$ if $\psi_1(P) \subseteq \psi_2(P)$ for each $P \in S(M)$. Hence it can be easily seen that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$.

2 Main results

Lemma 2.1. *Let N be a proper Submodule of M and $\psi_1, \psi_2 : S(M) \longrightarrow S(M) \cup \emptyset$ are functions with $\psi_1 \leq \psi_2$. If N is a ψ_1-2-absorbing primary submodule of M, then N is a ψ_2-2-absorbing primary submodule of M.*

Theorem 2.2. *Let N be a proper submodule of M. Then*

(i) N is a 2-absorbing primary submodule of $M \Longrightarrow N$ is a weakly 2-absorbing primary submodule of $M \Longrightarrow N$ is a ϕ_ω-2-absorbing primary submodule of $M \Longrightarrow N$ is a ϕ_{n+1}-2-absorbing primary submodule of M for every $n \geq 2 \Longrightarrow N$ is a ϕ_n-2-absorbing primary submodule of M for every $n \geq 2 \Longrightarrow N$ is an almost 2-absorbing primary submodule of M.

(ii) N is an idempotent submodule of M (i.e. $N = (N : M)N$) $\Longrightarrow N$ is a ϕ_ω-2-absorbing primary submodule of M and N is a ϕ_n-2-absorbing submodule of M for every $n \geq 1$.

(iii) If N is a radical submodule of M (i.e. $rad(N) = N$), then N is a ϕ_n- 2-absorbing primary submodule of M if and only if N is a ϕ_n-2-absorbing submodule of M.

(iv) N is a ϕ_n-2-absorbing primary submodule of M, for all $n \geq 2$ if and only if N is a ϕ_ω-2-absorbing primary submodule of M.

Theorem 2.3. *Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function. Set $\frac{M}{\emptyset} = M$, and let N be a proper submodule of M. Then*

(i) N is a ϕ-2-absorbing primary submodule of M if and only if $\frac{N}{\phi(N)}$ is a weakly 2-absorbing primary submodule of $\frac{M}{\phi(N)}$.

(ii) N is a ϕ-prime submodule of M if and only if $\frac{N}{\phi(N)}$ is a weakly prime submodule of $\frac{M}{\phi(N)}$.

(iii) N is a ϕ-primary submodule of M if and only if $\frac{N}{\phi(N)}$ is a weakly primary submodule of $\frac{M}{\phi(N)}$.

Definition 2.4. Let N be a ϕ-2-absorbing primary submodule of M and suppose that $abx \in \phi(N)$ for some $a, b \in R$ and $x \in M$ such that $ab \notin (N : M)$, $ax \notin rad(N)$ and $bx \notin rad(N)$, then we say (a, b, x) is a ϕ-triple-zero of N.

Remark 2.5. Note that a proper submodule N of an R-module M is a ϕ-2-absorbing primary submodule of M that is not a 2-absorbing primary submodule of M if and only if N has a ϕ- triple-zero (a, b, x) for some $a, b \in R$ and $x \in M$.

Theorem 2.6. *Let N be a ϕ-2-absorbing primary submodule of M for some function ϕ and suppose that (a, b, x) is a ϕ-triple-zero of N for some $a, b \in R$ and $x \in M$ (hence N is not a 2-absorbing primary submodule of M). Then*

(i) $abN, (N : M)ax, (N : M)bx \subseteq \phi(N)$.

(ii) $a(N : M)N, b(N : M)N, (N : M)^2x \subseteq \phi(N)$.

(iii) $(N : M)^2N \subseteq \phi(N)$.

Corollary 2.7. *Let N be a ϕ-2-absorbing primary submodule of M such that $(N : M)^2N \nsubseteq \phi(N)$. Then N is a 2-absorbing primary submodule of M.*

Corollary 2.8. *If N is a ϕ-2-absorbing primary submodule of M that is not a 2-absorbing primary submodule of M, then $rad(N) = rad(\phi(N))$*

Corollary 2.9. *Let ϕ be a function and let N be a proper submodule of M such that $\phi(N)$ is a prime submodule of M. Then N is a ϕ-2-absorbing primary submodule of M if and only if N is a 2-absorbing primary submodule of M.*

Corollary 2.10. *Let N be a proper ϕ-2-absorbing primary submodule of M such that $\phi \leq \phi_4$. Then*

(i) N is a ϕ_n-2-absorbing primary submodule of M for every $n \geq 3$.

(ii) N is a ϕ_ω-2-absorbing primary submodule of M.

Theorem 2.11. *Let M be an R-module and a be an element of R such that $aM \neq M$. Suppose that $(0 :_M a) \subseteq rad(aM)$. Then aM is a ϕ-2-absorbing primary submodule of M, for some ϕ with $\phi \leq \phi_3$ if and only if it is a 2-absorbing primary submodule of M.*

Definition 2.12. Let N be a ϕ-2-absorbing primary submodule of M for some function ϕ. Suppose that $I_1I_2L \subseteq N$ but $I_1I_2L \nsubseteq \phi(N)$, for some ideals I_1, I_2 of R, and submodule L of M. We say that N is a free-ϕ-triple-zero with respect to I_1I_2L if (a, b, x) is not a ϕ-triple-zero of N for every $a \in I_1, b \in I_2$, and $x \in L$.

Theorem 2.13. *Let N be a ϕ-2-absorbing primary submodule of M for some function ϕ. Suppose that $I_1I_2L \subseteq N$ but $I_1I_2L \nsubseteq \phi(N)$, for some ideals I_1, I_2 of R, and submodule L of M such that N is a free-ϕ-triple-zero with respect to I_1I_2L. Then $I_1I_2 \subseteq (N : M)$ or $I_1L \subseteq rad(N)$ or $I_2L \subseteq rad(N)$.*

Theorem 2.14. *Let R_1, R_2 be quasi-local commutative rings that are not fields with maximal ideals $\sqrt{0_{R_1}}, \sqrt{0_{R_2}}$, respectively and M_i be a finitely generated R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ as an R-module. Then every proper submodule of M is a 2-absorbing primary submodule of M. In particular, if $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ is a function, then every proper submodule of M is a ϕ-2-absorbing primary submodule of M.*

Theorem 2.15. *Let $R = R_1 \times R_2$, where R_1, R_2 are commutative rings and $M = M_1 \times M_2$ as an R-module, where M_i is a finitely generated R_i-module, for $i = 1, 2$. The following statements are equivalent.*

(i) Every proper submodule of M is a 2-absorbing primary submodule of M.

(ii) Every proper submodule of M_1 is a primary submodule of M_1 and every proper submodule of M_2 is a primary submodule of M_2.

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Improving the results of I. M. Isaacs on derived subgroups and centers of capable groups*

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Abstract

Some results on derived subgroups and centers of capable groups are given by I. M. Isaacs. The goal of this talk is to improve these results under the weaker hypothesises. Moreover, we show that there exists an upper bound for the index of the center of 2-generated finite p-groups with cyclic derived subgroup in terms of the order of its derived subgroup.

Keywords: Capable group, Derived subgroup, Center factor.
Mathematics Subject Classification [2010]: 20D99.

1 Introduction

A group G is called capable if there exists a group H such that $G \cong H/Z(H)$. Capability plays an important role in P. Hall's classification scheme for p-groups up to isoclinism [2]. It is really very interesting to find the relation between the concept of capability and " the Schur's theorem". Schur [4] proved that the finiteness of $G/Z(G)$ implies the finiteness of $\gamma_2(G)$. A natural question which arises here is when the converse of the theorem of Schur does hold? Infinite extra special groups show that the converse of the Schur's theorem does not hold in general. I. M. Isaacs finded the relationship between the capable groups and the converse the Schur's theorem. Isaacs in [3] proved :

Theorem 1.1. *Let G be a capable group and $|\gamma_2(G)| = n$ then $|G : Z(G)|$ is bounded above by some function f of n.*

In fact he showed that the converse of the Schur's theorem holds for the capable groups. We focus our attention on results of Isaacs in [3]. We state the same results under new hypothesises and other results as follows.

2 Main results

First, we recall the following lemmas from [3].

*Will be presented in English
†Speaker

Lemma 2.1 (See [3]). *Let $A \subseteq G$, where A is abelian, and suppose that $|G : A| = m < \infty$ and $|\gamma_2(G)| = n < \infty$. Then*

$$|G : Z(G)| \leqslant m^{1+log_2(n)}.$$

Every group with trivial center is clearly capable and next result gives the bound of the order of such groups.

Lemma 2.2 (See [3]). *There is a function $F(n)$ defined on the natural numbers such that if $Z(G) = 1$ and $|\gamma_2(G)| = n < \infty$, then $|G| < F(n)$.*

The following corollary is an immediate consequence from Lemmas 2.1 and 2.2.

Corollary 2.3. *Let G be a group and suppose that $|G : C_G(\gamma_2(G))| = m < \infty$ and $|\gamma_2(G)| = n < \infty$. If $Z(G) = 1$, then*

$$|G| \leqslant m^{1+log_2(n)}.$$

Now, a generalization of Lemma 2.2 and Corollary 2.3 are as follow.

Lemma 2.4. *There is a function $F(n)$ defined on the natural numbers such that if $Z(G) \cap \gamma_2(G) = 1$ and $|\gamma_2(G)| = n < \infty$, then $|G/Z(G)| < F(n)$.*

Corollary 2.5. *Let G be a group and suppose that $|G : C_G(\gamma_2(G))| = m < \infty$ and $|\gamma_2(G)| = n < \infty$. If $Z(G) \cap \gamma_2(G) = 1$, then*

$$|G/Z(G)| \leqslant m^{1+log_2(n)}.$$

By using the following lemma [3], we deal with the capability of finite groups with cyclic dervied subgroup of the order of power prime.

Lemma 2.6 (See [3]). *Let G be a finite and assume that $\gamma_2(G)$ is a cyclic p-group for some prime p. If $Z(G) \cap \gamma_2(G) \neq 1$, then G has a normal $p-$complement.*

Proposition 2.7. *Let G be a finite and assume that $\gamma_2(G)$ is a cyclic p-group for some prime p. If $Z(G) \cap \gamma_2(G) \neq 1$, then $G = P \times H$ such that $(|H|, p) = 1$ and $P \in Syl_p(G)$. In particular, G is capable if and only if P and H are so.*

Now, we are going to find the normal $\Pi'-$complement M of finite group with cyclic derived subgroup in the following theorem.

Theorem 2.8 (See [3]). *Let G be finite and assume that $\gamma_2(G)$ is cyclic. Let Π be the set of prime divisors of $|Z(G) \cap \gamma_2(G)|$ and let b be the $\Pi'-$part of $|\gamma_2(G)|$. Then*

(a) G has a normal $\Pi'-$complement M and G/M is nilpotent.

(b) $|M : M \cap Z(G)|$ divides $b\varphi(b)$, where φ is Euler's function.

(c) $|G/Z(G)|$ divides $b\varphi(b)|G : V|$, where $V/M = Z(G/M)$.

Next we introduce the normal $\Pi'-$complement M of finite group with cyclic derived subgroup Theorem 2.9.

Theorem 2.9 (See [3]). *Let G be a non-nilpotent finite group and assume that $\gamma_2(G)$ is cyclic. Let Π be the set of prime divisors of $|Z(G) \cap \gamma_2(G)|$ and let b be the $\Pi'-$part of $|\gamma_2(G)|$ and denote by M the smallest term of the lower central series of G. Then*

(a) M is a normal $\Pi'-$complement M and G/M is nilpotent.

(b) $M \cap Z(G) = 1$ and $b = |M| = q$, where q is prime.

(c) $|G/Z(G)|$ divides $b^2|G : V|$, where $V/M = Z(G/M)$.

Finally, we obtain the following equality is always holding for the 2-generated finite p-groups with cyclic derived.

Theorem 2.10. *Let G be finite and capable, and suppose that $\gamma_2(G)$ is cyclic and that all elements of order 4 in $\gamma_2(G)$ are central in G. Then*

$$|G/Z(G)| \leqslant |\gamma_2(G)|^2$$

and the equality holds if G is nilpotent.

Lemma 2.11. *Let G be a 2-generated finite p-groups with cyclic derived and $p > 2$. Then*

$$|G/Z(G)| = |\gamma_2(G)|^2.$$

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J-Armendariz Rings Relative to a Monoid

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Abstract

For a monoid M, we introduce J-M-Armendariz rings, which is a common generalization of J-Armendariz and weak M-Armendariz rings, and investigate their properties. We show that every NI-ring is J-M-Armendariz, for any unique product monoid M. Also, we provide various examples and classify how the J-M-Armendariz rings behave under various ring extensions. It is shown that if R is semicommutative ring and M-Armendariz then R is J-(M×N)-Armendariz, where N is a unique product monoid.

Keywords: J-M-Armendariz ring; Semicommutative ring; Jacobson radical; u.p.-monoid.
Mathematics Subject Classification [2010]: 16N20, 16N40.

1 Introduction

Throughout this paper every ring is an associative ring with identity unless otherwise stated. For a ring R, we denote by $nil(R)$ the set of all nilpotent elements of R and by $J(R)$ the Jacobson radical of R. The n-by-n full (resp. upper triagular) matrix ring over R is denoted by $Mat_n(R)$ (resp. $T_n(R)$), and E_{ij}'s denote the matrix units. $\mathbb{Z}$ and $\mathbb{C}$ denote the ring of integers and the field of complex numbers. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. A ring R is said to be Armendariz if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. This definition was coined by Rege and Chhawchharia in [1] in recognition of Armendariz' s proof in [2] that reduced rings (i.e., rings without nonzero nilpotent elements) satisfy this condition. Recently, several types of generalizations of Armendariz rings have been introduced (see, e.g., [3, 4, 5]). Liu and Zhao [4], studied the structure of the set of nilpotent elements in Armendariz rings and introduced weak Armendariz rings as a generalization. A ring R is said to be weak Armendariz ring if the product of two polynomials in $R[x]$ is zero, then the product of their coefficients is nilpotent. C. Zhang and J. Chen [5], studied a generalization of weak Armendariz rings, which is called weak M-Armendariz rings. A ring is called weak M-Armendariz (weak Armendariz relative to M) if whenever $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$, with $g_i, h_j \in M$

*Speaker

satisfy $\alpha\beta = 0$, then $a_ib_j \in nil(R)$ for each i, j. The Jacobson radical is an important tool for studying the structure of noncommutative rings. Motivated by the above results, Sanaei et al. in [6], studied the structure of the Jacobson radical in Armendariz rings and introduced J-Armendariz rings as a generalization of weak Armendariz rings. A ring R is said to be J-Armendariz ring if the product of two polynomials in $R[x]$ is zero, then the product of their coefficients in Jacobson radical. In this paper we continue to study J-Armendariz rings. We generalize and unify the above concepts by introducing the notion of J-M-Armendariz rings. For a monoid M we introduce J-M-Armendariz rings (J-Armendariz rings relative to a monoid M) which are a common generalization of weak M-Armendariz rings and J-Armendariz rings.

2 Main results

Definition 2.1. Let R be a ring and M a monoid. A ring R is said to be J-M-Armendariz ring (J-Armendariz ring relative to a monoid M), if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n,\ \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j \in J(R)$ for each i, j.

Clearly, (weak) M-Armendariz rings are J-M-Armendariz, but the following example shows that J-M-Armendariz rings are not necessary (weak) M-Armendariz rings.

Example 2.2. Let A be the 3 by 3 full matrix ring over the power series ring $F[[t]]$ over a field F. Let

$$B = \{M = (m_{ij}) \in A | m_{ij} \in tF[[t]] \text{ for } 1 \leq i, j \leq 2 \text{ and } m_{ij} = 0 \text{ for } i = 3 \text{ or } j = 3\}$$

$$C = \{M = (m_{ij}) \in A | m_{ij} \in F \text{ and } m_{ij} = 0 \text{ for } i \neq j\}.$$

Let R be the subring of A generated by B and C. Let $F = \mathbb{Z}_2$. Note that element of R is of the form

$$\begin{pmatrix} a + f_1 & f_2 & 0 \\ f_3 & a + f_4 & 0 \\ 0 & 0 & a \end{pmatrix} \text{ for some } a \in F \text{ and } f_i \in tF[[t]] \quad (i = 1, 2, 3, 4)$$

Let N be a monoid with $|N| \geq 2$. We can show that R is J-N-Armendariz, but R is not (weak) N-Armendariz.

Recall that a monoid M is called a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$, there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$.

Theorem 2.3. *For any u.p.-monoid M, every NI-ring is weak M-Armendariz.*

Corollary 2.4. *For any u.p.-monoid M, every NI-ring is J-M-Armendariz.*

The following example shows that the condition "M is a u.p.-monoid" in Corollary 2.4 is not superfluous.

Example 2.5. Let $M = \{I, E_{11}, E_{12}, E_{21}, E_{22}, 0\}$, where E_{ij} is a units matrix of $Mat_2(\mathbb{Z})$, for each $1 \leq i, j \leq 2$. Then M is a monoid, but not u.p.-monoid. Let $\alpha = 1.E_{11}, \beta = 1.E_{21} - 1.E_{22} \in \mathbb{Z}[M]$. Then $\alpha\beta = 1.0 - 1.0 = 0.0$, but $1.1 = 1 \notin J(\mathbb{Z})$. Hence $\mathbb{Z}$ is not J-M-Armendariz.

Proposition 2.6. *Let M be a strictly totally ordered monoid and I an ideal of R such that R/I is J-M-Armendariz. If $I \subseteq J(R)$, then R is J-M-Armendariz.*

Corollary 2.7. *Let M be a strictly totally ordered monoid and R be a local ring. Then R is J-M-Armendariz.*

Theorem 2.8. *A ring R is J-M-Armendariz, if and only if $R[[x]]$ is J-M-Armendariz.*

The following example shows that subrings of J-M-Armendariz rings need not inherit the property.

Example 2.9. Let M be a monoid with $|M| \geq 2$, F be a field, $R = Mat_2(F)$ and $R_1 = R[[t]]$. Consider the ring

$$S = \{\textstyle\sum_{i=0}^{\infty} a_i t^i \in R_1 \mid a_0 \in kI \text{ for } k \in F\},$$

where I is the identity matrix. Since S is local and so is J-M-Armendariz by Corollary 2.7, So $S[[x]]$ is J-M-Armendariz by Theorem 2.8, Take $e \neq g \in M$. Now for $\alpha = E_{11}txe - E_{12}txg$, $\beta = E_{21}txe + E_{11}txg \in (S[x])[M]$, we have $\alpha\beta = 0$, but $(E_{11}tx)^2 \notin J(R)$, and so $S[x]$ is not J-M-Armendariz.

Theorem 2.10. *Let R_k be a ring, for each $1 \leq k \leq n$. Then any direct product of rings $R = \prod_{k=1}^{n} R_k$, is J-M-Armendariz if and only if any R_k is J-M-Armendariz.*

Theorem 2.11. *Let M be a monoid with $|M| \geq 2$. Then the following conditions are equivalent:*

1. *R is J-M-Armendariz.*
2. *$T_n(R)$ is J-M-Armendariz.*

Proposition 2.12. *Let R_1 and R_2 be rings and M is an (R_1, R_2)-bimodule. Then $R = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$ is J-M-Armendariz if and only if R_1, R_2 are J-M-Armendariz.*

The Morita invariance of a property of R can be checked by testing if it passes to matrix rings $Mat_n(R)$ and corner rings eRe, with $e^2 = e$ a full idempotent ($ReR = R$). It turns out that the J-M-Armendariz property is badly behaved with regards to Morita invariance.

Example 2.13. Let M be a monoid with $|M| \geq 2$, R_1 be any ring and $R = Mat_2(R_1)$ for any $n \geq 2$. Let

$$\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g, \quad \beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} g$$

be two elements in $R[M]$, where $e \neq g \in M$. Then $\alpha\beta = 0$. But

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin J(R).$$

Thus R is not J-M-Armendariz.

Let $T(G)$ be the set of elements of finite order in an Abelian group G. G is said to be torsion-free if $T(G) = \{e\}$.

Theorem 2.14. *Let G be a finitely generated Abelian group. Then the following conditions on G are equivalent:*

1. *G is torsion-free.*
2. *There exists a ring R with $|R| \geq 2$ such that R is J-G-Armendariz.*

Proposition 2.15. *If M is a finite monoid, then $\mathbb{C}$ is not J-M-Armendariz.*

A ring R is called right Ore if given $a, b \in R$ with b regular (elements that are neither left nor right zero-divisors), there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$.

Theorem 2.16. *Let M be monoid and let R be a right Ore ring with classical right quotient ring Q. Then R is J-M-Armendariz if and only if Q is J-M-Armendariz.*

Proposition 2.17. *Let M be a monoid and N a u.p.-monoid. If R is a semicommutative and M-Armendariz ring, then $R[M]$ is J-N-Armendariz.*

Theorem 2.18. *Let M be a monoid and N a u.p.-monoid. If R is a semicommutative and M-Armendariz ring, then $R[N]$ is J-M-Armendariz.*

Theorem 2.19. *Let M be a monoid and N a u.p.-monoid. If R is a semicommutative and M-Armendariz ring, then R is J-($M\times N$)-Armendariz.*

Let $M_i, i \in I$, be u.p.-monoids. Denote $\bigsqcup_{i\in I} M_i = \{(g_i)_{i\in I} \mid$ there exist only finite i's such that $g_i \neq e_i$, the identity of $M_i\}$. Then $\bigsqcup_{i\in I} M_i$ is a monoid with the operation $(g_i)_{i\in I}(g_i')_{i\in I} = (g_i g_i')_{i\in I}$.

Corollary 2.20. *Let $M_i, i \in I$, be u.p.-monoids and R a semicommutative ring. If R is M_{i_0}-Armendariz for some $i_0 \in I$, then R is J-$\bigsqcup_{i\in I} M_i$-Armendariz.*

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Number of proper nilpotent subgroups for direct products

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Abstract

For a group G we denote by $\eta(G)$ the number of proper nilpotent subgroups of G. In this paper we give $\eta(G)$ for $G = H \times K$ such that H and K are nilpotent or nonnilpotent.

Keywords: nilpotent subgroup; soluble group; simple group
Mathematics Subject Classification [2010]: 20D99, 20E07

1 Introduction and results

In group theory, it is well-known that the quantitative properties of some special subgroups (such as, maximal subgroups, normalizer subgroups, centralizer subgroup, solvable subgroups, and derived subgroups) play an important part in characterizing the solubility of groups. For example, Pazderski in[2], showed that a group with less than 21 maximal subgroups is solvable and also showed that a group is solvable if it has at most 2 conjugacy classes of maximal subgroups. Also Zarrin in [6] showed that every finite group with at most 20 normalisers is solvable (see also [7, 8, 9, 10]).

Further problems of a similar nature, with slightly different aspects, have been studied by many people ([1, 3, 4, 5]).

In this paper, we investigate groups with finite number of proper nilpotent subgroups. It is easy to see that if a group has a finite number of nilpotent subgroups if and only if it is a finite group. Therefore in considering such groups we need only consider finite cases. For a group H we denote by $\eta(H)$ the number of proper nilpotent subgroups of H. In theis paper we gives $\eta(G)$ for $G = H \times K$ such that H and K are nilpotent or nonnilpotent.

2 Main results

Theorem 2.1. *Let G be any group and $H \leq G$. Then*

1. *$\eta(H) \leq \eta(G)$;*
2. *$\eta(G/K) \leq \eta(G)$, where K is a normal subgroup of G with $K \leq Z^*(G)$ and $Z^*(G)$ is the hypercenter of G.*

*Speaker

Proof. The proof is straightforward. □

Theorem 2.2. *Let $G = H \times K$ and $(|H|, |K|) = 1$.*

1. *If H, K are nonnilpotent, then $\eta(G) = \eta(H)\eta(K)$;*
2. *If H is nilpotent and K is nonnilpotent, then $\eta(G) = (\eta(H) + 1)\eta(K)$;*
3. *If H, K are nilpotent, then $\eta(G) = (\eta(H) + 1)(\eta(K) + 1) - 1$.*

Proof. (1). It is easy to see that if T and L are two nilpotent subgroups of H and K, respectively, then $T \times L$ is a nilpotent subgroup of G. It follows that $\eta(H)\eta(K) \leq \eta(G)$. Let π be the set of primes dividing the order of H. Then H is a normal Hall π-subgroup of G and K is a normal Hall π'-subgroup of G. Now let A be any subgroup of G. Then $A \cap H$ is a normal Hall π-subgroup of A and $B \cap H$ is a normal Hall π'-subgroup of A. Therefore we have $A = (A \cap H) \times (A \cap K)$. From which it follows that $\eta(H)\eta(K) \geq \eta(G)$. Hence $\eta(H)\eta(K) = \eta(G)$, as wanted.
(2). It is enough to note that $H \times \langle e \rangle$ is a proper nilpotent subgroup of G, where e is the trivial element of K.
(3). In this case it is enough to note that $H \times \langle e \rangle$ and $\langle e \rangle \times K$ are proper nilpotent subgroups of G. □

Corollary 2.3. *If $G = H \times K$, then $\eta(H)\eta(K) \leq \eta(G)$.*

Proof. By the proof of Lemma 2.2, is clear. □

Now we introduce two questions for researchers, because answer to bellow questions is very important for classification of nilpotent and solvable groups by the number of proper nilpotent subgroups .

Question 2.4. *Let n be a positive integer number and G is a group such that $\eta(G) = n$, for which of n the group G is nilpotent?*

Question 2.5. *Let n be a positive integer number and G is a group such that $\eta(G) = n$, for which of n the group G is soluble?*

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Obstinate prefilters in EQ-algebras

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In this paper, we introduce the notions of obstinate prefilters (filters) in an EQ-algebra. We establish properties of obstinate prefilters in an EQ-algebra. We prove some relationships between obstinate prefilters and the other types of prefilters an EQ-algebra .

Keywords: EQ-algebra, (obstinate, prime, implicative and positive implicative) prefilter.
Mathematics Subject Classification [2010]: 03G1, 03G05.

1 Introduction and Preliminaries

A special algebra called EQ-algebra has been recently introduced by Vilém Novák and B. De Baets [2]. Its original motivation comes from fuzzy type theory, in which the main connective is fuzzy equality. An EQ-algebras have three binary (meet, multiplication and a fuzzy equality) and a top element and also a binary operation implicatin is drived from fuzzy equality. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. These algebras intended to develop an algebric structure of truth values for fuzzy type theory. EQ-algebras are interesting and important algebra for studing and researching and also residuated lattices and BL-algebras are particular cases of EQ-algebras.
In this section, we present some definitions and results about EQ-algebras that will be used in the sequel.

Definition 1.1. [1] An EQ-algebra is an algebra $(E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ satisfies the following :
(E_1) $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $a \leq b$ if and only if $a \wedge b = a$,
(E_2) $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone in arguments w.r.t $a \leq b$,
(E_3) $a \sim a = 1$,

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(E_4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq (c \sim (d \wedge b))$,
(E_5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$,
(E_6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$,
(E_7) $a \otimes b \leq a \sim b$,
for all $a, b, c \in E$.
We denote $\widetilde{a} := a \sim 1$ and $a \to b := (a \wedge b) \sim a$, for all $a, b \in E$.

Theorem 1.2. [1, 2] *Let E be an EQ-algebra. For all $a, b, c \in E$ we have*
(e_1) $a \sim b = b \sim a$, *(e_2)* $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$,
(e_3) $a \sim d \leq (a \wedge b) \sim (d \wedge b)$, *$(e_4)$* $a \otimes b \leq a \wedge b \leq a, b$,
(e_5) $b \leq \widetilde{b} \leq a \to b$, *$(e_6)$* $a \sim b \leq (a \to b) \wedge (b \to a)$.

Definition 1.3. [2] Let E be an EQ-algebra. We say that it is
(i) *spanned*, if it contains a bottom element 0 and $\widetilde{0} = 0$,
(ii) *separated*, if for all $a, b \in E$, $a \sim b = 1$ implies $a = b$,
(iii) *semi-separated*, if for all $a \in E$, $a \sim 1 = 1$ implies $a = 1$.

Definition 1.4. [1] A nonempty subset F of an EQ-algebra E is called a prefilter of E, whenever for all $a, b, c \in E$:
(F_1) $1 \in F$,
(F_2) $a,\ a \to b \in F$ implies $b \in F$.
A prefilter F of E is called a filter, if it satisfies the following :
(F_3) $a,\ a \to b \in F$ implies $a \otimes c \to b \otimes c \in F$, for any $a, b, c \in E$.
A prefilter (filter) F of an EQ-algebra E is called proper, whenever $F \neq E$.

Theorem 1.5. [1] *Let F be a prefilter of an EQ-algebra E. The following hold, for all $x, y, z, s, t \in E$:*
(i) If $x \in F$ and $x \leq y$, then $y \in F$,
(ii) If $x, x \sim y \in F$, then $y \in F$,
(iii) If $x \sim y \in F$ and $y \sim z \in F$, then $x \sim z \in F$,
(iv) If $x \to y \in F$ and $y \to z \in F$, then $x \to z \in F$,
(v) If $x \sim y \in F$,$s \sim t \in F$, then $(x \wedge s) \sim (y \wedge t) \in F$, $(x \sim s) \sim (y \sim t) \in F$ and $(x \to s) \sim (y \to t) \in F$.

We denote $a \Leftrightarrow b := (a \to b) \wedge (b \to a)$ and $a \Leftrightarrow^{\circ} b := (a \to b) \otimes (b \to a)$, for all $a, b, c \in E$.

Definition 1.6. [1] A prefilter F of an EQ-algebra E is said to be a prime prefilter if for all $a, b \in E$, $a \to b \in F$ or $b \to a \in F$.

Definition 1.7. [3] A prefilter F of an EQ-algebra E is called a positive implicative prefilter if it satisfies for any $x, y, z \in E$:
(F_4) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

Definition 1.8. [3]A nonempty subset F of E is called an implicative prefilter if it satisfies (F_1) and
(F_5) $z \to ((x \to y) \to x) \in F$ and $z \in F$ imply $x \in F$, for any $x, y, z \in E$.

2 Obstinate prefilter (filters) in EQ-algebras

Definition 2.1. A prefilter F of E is called an obstinate prefilter of an EQ-algebra E if for all $x, y \in E$,
(F_6) $x, y \notin F$ implies $x \to y \in F$ and $y \to x \in F$.
If F is a filter and satisfies (F_6), then F is called an obstinate filter.

Example 2.2. Let $E = \{0, a, b, c, d, 1\}$ such that $0 < a < b < d < 1,\ 0 < a < c < d < 1$. The following binary operations " $\otimes$ " and " $\sim$ " define an EQ-algebra[2]. Also the implication of E is given as follow:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	d	1	d	1	1
c	0	d	d	1	1	1
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Then $\{a, b, c, d, 1\}$ is an obstinate prefilter of E while $\{1, d\}$ is not an obstinate prefilter, because $0, b \notin \{1, d\}$ and $b \to 0 = \{0\} \notin \{1, d\}$.

Theorem 2.3. *$\{1\}$ is a prefilter of an EQ-algebra E if and only if E is a semi-separated EQ-algebra.*

Lemma 2.4. *Let E be a separated EQ-algebra. Then $\{1\}$ is an obstinate prefilter of E if and only if E has at most 2 elements.*

Lemma 2.5. *Let F be a prefilter of an EQ-algebra E. Then F is an obstinate prefilter of E if and only if $x, y \notin F$ implies $x \sim y \in F$.*

Theorem 2.6. *Let F be a filter of an EQ-algebra E. Then F is an obstinate filter of E if and only if $a \Leftrightarrow^{\circ} b \in F$, for all $a, b \in E - F$.*

Theorem 2.7. *Let bottom element $0 \in E$ and F be a proper prefilter of an EQ-algebra E . Then F is an obstinate prefilter of E if and only if $x \notin F$ implies $\neg x \in F$, for all $x \in E$.*

Theorem 2.8. *If $a \to 0 = 0$, for all $a \in E - \{0\}$, then $F = E - \{0\}$ is the only obstinate proper prefilter of an EQ-algebra E.*

Theorem 2.9. *(Extension property) Let F be an obstinate prefilter of an EQ-algebra E and $F \subseteq G$. Then G is also an obstiante prefilter of E.*

Theorem 2.10. *Let F be an obstinate filter of an EQ-algebra E. Then E/F is a chain.*

Proposition 2.11. *Let F be a prefilter of an EQ-algebra E. If $a, b \in F$ then $a \to b, b \to a, a \sim b, a \wedge b \in F$.*

Lemma 2.12. *Every an obstinate prefilter of an EQ-algebra E is an implicative prefilter.*

Theorem 2.13. *Every an obstinate prefilter F of an EQ-algebra E is a prime prefilter.*

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On $RP(S)$

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Abstract

For a regular semigroup S, the set $RP(S)$ of regularity-preserving elements of S is, if non-empty, a completely simple subsemigroup of S. In this paper, we develop some of the properties of Rees matrix semigroups $S = M(T; I, \Lambda; P)$, where T is a regular monoid, and the entries of the matrix P lie in the group of units of T. We show that, if the matrix P is uniform, then S is isomorphic to the direct product of a rectangular band and a regular monoid. Also, we generalize two results that are known for completely simple semigroups.

Keywords: Regular semigroup, Completely simple semigroup, Rees matrix semigroup
Mathematics Subject Classification [2010]: 20M17, 20M10

1 Introduction

We refer the reader to [3] for basic definitions and terminology relating to semigroups and monoids. Let S be a semigroup and $a \in S$. By a pre-inverse of a, we mean an element $b \in S$ such that $aba = a$. We denote the set of pre-inverses of a by $Pre(a)$.

Lemma 1.1. *([2]) Let T be a regular monoid, with group of units G. Then, for $x \in T$ and $g \in G$, we have $x \leq g \Leftrightarrow g^1 \in Pre(x)$.*

We recall that, a mididentity, in a semigroup S, is an element $u \in S$, such that $xuy = xy$ for all $x, y \in S$. The binary operation $\circ$ defined on the set S by $x \circ y = xay$ is associative, the resulting semigroup is denoted by (S, a), and is called a *variant* of S. For an arbitrary semigroup S, it is clear that every element that is regular in a variant (S, a), must also be regular in S, but the converse is not true. If a, x are elements of S, we say that a preserves the regularity of x, if x is regular in the variant (S, a). If a preserves the regularity of every regular element of S, then we say that a is a regularity-preserving element, and the set of such elements in S is denoted by $RP(S)$.

Definition 1.2. Let S be a regular semigroup with $RP(S) \neq \phi$. We shall say that the (Hartwig-Nambooripad) order on S is *RP-compliant*, if the condition

$$(\forall x, y \in S)(\forall g, h \in RP(S))\ x \leq g, y \leq h \Rightarrow xy \leq gh$$

holds in S.

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We note that the order on a completely simple semigroup is RP-compliant, since here $RP(S) = S$, and the order on S is equality.

Theorem 1.3. *Let S be a regular monoid. Then the order on S is RP-compliant if and only if S is orthodox.*

Let S be a monoid with group of units G. We recall that S, is said to be unit regular (uniquely unit regular), if every element of S, has at least one (precisely one) pre-inverse lying in G. For a regular semigroup S with $RP(S) \neq \phi$, we say that S is RP-dominated (uniquely RP-dominated), if for every element $x \in S$, there exists $g_x \in RP(S)$ (a unique $g_x \in RP(S)$), with $x \leq g_x$. It is immediate that a completely simple semigroup S is uniquely RP-dominated, since here $S = RP(S)$ and the order relation reduces to equality. We note also that a regular monoid is (uniquely) RP-dominated, if and only if it is (uniquely) unit regular.

2 Main Result

Throughout this section, we assume that I and Λ are sets, and T is a regular monoid with group of units G, and identity element 1. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix, and $S = M(T; I, \Lambda; P)$ be a Rees matrix semigroup over T, where the entries of P lie in G.

Lemma 2.1. *Let $(j, b, \mu), (k, c, \nu) \in S$. Then:*
(i) $(k, c, \nu) \in Pre((j, b, \mu))$ in $S \Leftrightarrow c \in p_{\mu K}^{-1} Pre(b) p_{\nu j}^{-1}$ in T;
(ii) $(j, b, \mu) \in E(S) \Leftrightarrow b \leq p_{\mu j}$ in T .

Proof. (i): It follows from the fact that $(j, b, \mu)(k, c, \nu)(j, b, \mu) = (j, b, \mu)$ in $S \Leftrightarrow bp_{\mu k} cp_{\nu j} b = b$ in T.

(ii): $(j, b, \mu) \in E(S) \Leftrightarrow bp_{\mu j} b = b \Leftrightarrow p_{\mu j} \in Pre(b)$ in T. Hence $b \leq p_{\mu j}^{-1}$ in T, by Lemma1.1, as required. □

The following result is well-known [3], and is also a consequence of Lemma 2.1(i).

Theorem 2.2. *If S is as above, then S is regular.*

The subset $\{(i, g, \lambda) \in S : g \in G, i \in I, \lambda \in \Lambda\}$ is a subsemigroup of S, and may in an obvious way, be identified with the Rees matrix semigroup $M(G; I, \Lambda; P)$. We then have that $M(G; I, \Lambda; P)$ is a completely simple subsemigroup of S.

Definition 2.3. For $g \in G, k \in I$ and $\nu \in \Lambda$, we write $P(k, g, \nu)$ for the $\Lambda \times I$, matrix $\widehat{P} = (\widehat{p}_{\lambda j})$, where $(\widehat{p}_{\lambda j}) = p_{\lambda k} g p_{\nu j}$.

We have the following description of those variants of S, that are determined by elements of $M(G; I, \Lambda; P)$.

Lemma 2.4. *Let $(k, g, \nu) \in M(G; I, \Lambda; P)$. Then $(S, (k, g, \nu)) = M(T; I, \Lambda; P_{(k,g,\nu)})$.*

Theorem 2.5. $RP(S) = M(G; I, \Lambda; P)$.

Proof. First we show that $RP(S) \subseteq M(G; I, \Lambda; P)$. Let $x = (i, a, \lambda)$ be an element of $RP(S)$. Then x preserves the regularity of $e_{i\lambda} = (i, p_{\lambda i}^{-1}, \lambda)$. Thus $e_{i\lambda} = e_{i\lambda}xwxe_{i\lambda}$ for some $w \in S$. Since $e_{i\lambda}x = x = xe_{i\lambda}$, this gives $e_{i\lambda} = xwx$. Let $w = (j, b, \mu)$. Then $(i, p_{\lambda i}^{-1}, \lambda) = (i, a, \lambda)(j, b, \mu)(i, a, \lambda)$, so $p_{\lambda i}^{-1} = ap_{\lambda j}bp_{\mu i}a$. Thus $a(p_{\lambda j}bp_{\mu i}ap_{\lambda i}) = 1 = (p_{\lambda i}ap_{\lambda j}bp_{\mu i})a$, and hence $a \in G$. It follows that $x \in M(G; I, \Lambda; P)$ and so $RP(S) \subseteq M(G; I, \Lambda; P)$.

Conversely, let $(k, g, \nu) \in M(G; I, \Lambda; P)$. Then $g \in G$. We consider the variant $(S, (k, g, \nu))$ of S. By Lemma 2.4, we have that $(S, (k, g, \nu)) = M(T; I, \Lambda; P_{(k,g,\nu)})$. Since all the entries of $P_{(k,g,\nu)}$, lie in G, $(S, (k, g, \nu))$ is regular by Lemma 2.2, and it follows that $(k, g, \nu) \in RP(S)$. We thus have that $M(G; I, \Lambda; P) \subseteq RP(S)$, as required. □

Theorem 2.6. *For $(i, a, \lambda), (j, b, \mu) \in S$, $(i, a, \lambda) \leq (j, b, \mu)$ in $S \Leftrightarrow i = j$, $\lambda = \mu$ and $a \leq b$ in T.*

Theorem 2.7. *The order on S is RP-compliant if, and only if, the order on T is RP-compliant.*

Proof. Suppose that the order on S is RP-compliant, and let the elements $a, b \in T$, $g, h \in G$ be such that $a \leq g$ and $b \leq h$. Choose $i \in I$ and $\lambda \in \Lambda$ and consider $(i, a, \lambda), (i, b, \lambda)$, and $(i, p_{\lambda i}^{-2}, \lambda)$ in S. Using Theorem 2.6, we have that $(i, a, \lambda) \leq (i, g, \lambda), (i, p_{\lambda i}^{-2}, \lambda) \leq (i, p_{\lambda i}^{-2}, \lambda), (i, b, \lambda) \leq (i, h, \lambda)$, and so, $(i, a, \lambda)(i, p_{\lambda i}^{-2}, \lambda)(i, b, \lambda) \leq (i, g, \lambda)(i, p_{\lambda i}^{-2}, \lambda)(i, h, \lambda)$, that is, $(i, ab, \lambda) \leq (i, gh, \lambda)$. Thus, by Theorem 2.6, $ab \leq gh$. It follows that the order on T is RP-compliant.

Conversely, suppose that the order on T is RP-compliant, and let $(i, a, \lambda), (j, b, \mu) \in S$, and $x, y \in RP(S)$, be such that $(i, a, \lambda) \leq x, (j, b, \mu) \leq y$. Then, by Theorems 2.5 and 2.6, $x = (i, g, \lambda), y = (j, h, \mu)$ for some $g, h \in G$ with $a \leq g$, $b \leq h$. We now have $a \leq g$, $p_{\lambda j} \leq p_{\lambda j}, b \leq h$ in T, and $p_{\lambda j} \in G$, so $ap_{\lambda j}b \leq gp_{\lambda j}h$, since the order on T is RP-compliant. This gives that $(i, a, \lambda)(j, b, \mu) \leq (i, g, \lambda)(j, h, \mu)$, and so the order on S is RP-compliant, as required. □

Corollary 2.8. *The order on S is RP-compliant if, and only if, T is orthodox.*

Theorem 2.9. *S is (uniquely) RP-dominated if, and only if, T is (uniquely) unit regular.*

A standard refinement of the Rees Theorem says that, for a completely simple semigroup $M(G; I, \Lambda; P)$, we may, up to isomorphism, take the matrix P to be normal. The argument used to prove this, with minor adjustments, will show that this result may be generalised to our semigroup $S = M(T; I, \Lambda; P)$. So let $i \mapsto u_i$, $\lambda \mapsto v_\lambda$ be mappings from I into G and Λ into G, respectively, and with P as before, let the $\Lambda \times I$ matrix $\overline{P} = (\overline{p}_{\lambda i})$ be defined by $\overline{p}_{\lambda i} = v_\lambda p_{\lambda i} u_i$. We note that the entries of $\overline{P}$ all lie in G.

We recall that, P is row-uniform (column-uniform), if the rows (columns), of P are pairwise left-proportional (right-proportional).

Lemma 2.10. *The following are equivalent for a $\Lambda \times I$ matrix P :*

(i) P is row-uniform;

(ii) P is column-uniform;

(iii) $(\forall \lambda, \mu \in \Lambda)(\forall i, j \in I) q_{ij} = 1$.

Lemma 2.11. *Let P be a $\Lambda \times I$ matrix over the group G. Then P is uniform if, and only if, $P = P_{(k,g,\nu)}$ for some $g \in G, k \in I$ and $\nu \in \Lambda$.*

Theorem 2.12. *Let the $\Lambda \times I$ matrix $\overline{P}$ be defined as above. Then:*

(i) $S = M(T; I, \Lambda; P)$ is isomorphic to $\overline{S} = M(T; I, \Lambda; \overline{P})$.

(ii) If P is uniform, then $\overline{P}$ is uniform.

(iii) The mappings $i \mapsto u_i, \lambda \mapsto v_\lambda$, may be chosen so that $\overline{P}$ is normal.

Lemma 2.13. *If P is uniform, then $S = M(T; I, \Lambda; P)$ is isomorphic to the direct product of the rectangular band $I \times \Lambda$ and the regular monoid T.*

Theorem 2.14. *If $S = M(T; I, \Lambda; P)$ is orthodox, then T is orthodox and S is isomorphic to the direct product of the rectangular band $I \times \Lambda$ and T.*

Theorem 2.15. *$S = M(T; I, \Lambda; P)$ is orthodox if, and only if, T is orthodox and P is uniform.*

Theorem 2.16. *Let $S = M(T; I, \Lambda; P)$. Then every variant (S, a), where $a \in RP(S)$, is isomorphic to the direct product of the rectangular band $I \times \Lambda$ and the regular monoid T.*

Proof. Let $a \in RP(S)$. By Theorem 2.5 we have that $a = (k, g,)$ for some $g \in G, k \in I$ and $\nu \in \Lambda$. Then $(S, a) = M(T; I, \Lambda; P_{(k,g,\nu)})$, by Lemma 2.4. But the matrix $P_{(k,g,\nu)}$ is uniform by Lemma 2.11, so the result follows by Lemma 2.13. □

We note that a mididentity in a semigroup is necessarily regularity-preserving.

Corollary 2.17. *If $S = M(T; I, \Lambda; P)$ has a mididentity then it is isomorphic to the direct product of the rectangular band $I \times \Lambda$ and the regular monoid T.*

The last two results are generalisations of the corresponding ones for completely simple semigroups [1, Lemma 3.4 and Corollary 3.5].

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On V-regular semigroups

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Abstract

A regular semigroup S is V-regular if $V(ab) \subseteq V(b)V(a)$, for all $a, b \in S$. A characterization of a V-regular semigroup is given. Congruences on V-regular semigroups are described in terms of certain congruence pairs.

Keywords: regular semigroup, V-regular semigroup, congruence, congruence pair
Mathematics Subject Classification [2010]: 20M17

1 Introduction

A regular semigroup S is called V-regular, if $V(ab) \subseteq V(b)V(a)$ for all $a, b \in S$. This concept was introduced by Onstad [4]. This class of semigroups is dual to orthodox semigroups, namely, regular semigroups satisfy that $V(b)V(a) \subseteq V(ab)$ for all elements $a, b \in S$. Properties of V-regular semigroups were given by Nambooripad and Pastijn in [3]. Congruences on regular semigroups have been explored extensively. The kernel-trace approach is an effective tool for handling congruences on regular semigroups, which had been investigated by many authors. The purpose of this paper is to give a characterization of a V-regular semigroup, and to describe congruences on V-regular semigroups in terms of certain congruence pairs. We refer the reader to [2] for basic definitions and terminology relating to semigroups and monoids. If S is a regular semigroup, $a \in S$, then $V(a)$ denotes the set of inverses of a in S. The set of idempotents of S is denoted by $E(S)$. On $E(S)$, we define the natural partial order ω given by $e\omega f \Leftrightarrow ef = fe = e$. For $e, f \in E(S)$, $S(e, f) = fV(ef)e$, is the sandwich set of e and f. The following simple statements will be applied without further mention: for $e, f \in E(S)$,

$$eLf \Rightarrow S(e, f) = f,$$

$$eRf \Rightarrow S(e, f) = e.$$

If ρ is a congruence on S and $h \in S(e, f)$, then $h\rho \in S(e\rho, f\rho)$. Let τ be a relation on S. The congruence generated by τ is denoted by τ^*. If γ is an equivalence on S, then γ^0 is the greatest congruence on S contained in γ. $C(S)$ is the lattice of congruences on S.

Lemma 1.1 (3). *A regular semigroup S is V-regular if and only if the partial band $(E(S), \circ)$ determined by S satisfies the following:*

*Speaker

(1) $\omega L = L\omega$;

(2) $\omega R = R\omega$;

(3) for all $e, f \in E(S), h \in S(e, f)$, *there exist* $e_1, f_2 \in E(S)$ *such that* $e_1 L e, f_2 R f$, *and* $h = f_2 e_1$.

Lemma 1.2 (2)**.** *Let* S *is a regular semigroup,* $\rho \in C(S)$. *If* $a\rho \in E(S/\rho)$, *then there exists* $e \in E(S)$ *such that* $a\rho = e\rho$.

Lemma 1.3. *Let* S *be a* V*-regular semigroup,* $\rho \in C(S), a\rho \in E(S/\rho), x\rho \in S/\rho$. *If* $(a\rho)R(x\rho)$ *in* S/ρ, *then there exists* $e \in E(S)$ *such that* $a\rho = e\rho$, *and* eRx.

Proof. By Lemma 1.2, there exists $f \in E(S)$, such that $a\rho = f\rho$. Let $g \in E(S)$ be such that gRx. Then $(g\rho)R(x\rho)$. Since $a\rho = f\rho$, and $(a\rho)R(x\rho)$, we have $(f\rho)R(g\rho)$. Let $h \in S(f, g)$. Then $h\rho \in S(f\rho, g\rho)$, and so $h\rho = f\rho$. Notice that $h\rho \in E(S), hR(hg)\omega g$, it follows from Lemma 1.1 that there exists $e \in E(S)$, such that $h\omega eRg$. Since gRx, eRx. Now $(h\rho)\omega(e\rho)R(g\rho)$, implies that $(f\rho)\omega(e\rho)R(f\rho)$. Hence $a\rho = f\rho = f\rho e\rho = e\rho$. □

Corollary 1.4. *Let* S *be a* V*-regular semigroup,* $\rho \in C(S), e, f \in E(S)$. *If* $(e\rho)R(f\rho)$, *then there exist* $g, h \in E(S)$, *such that* $gRf, hRe, g\rho = e\rho$ *and* $h\rho = f\rho$.

Remark The dual results of Lemma 1.3 and Corollary 1.4 hold.

2 Main Results

The theorem below give a characterization of a V-regular semigroup.

Theorem 2.1. *A regular semigroup* S *is* V*-regular if, and only if, for all* $a, b \in S, (ab) \in V(ab)$, *there exist* $e_1, e_2, f_1, f_2 \in E(S)$, *such that* $b(ab)'a = f_2 e_1, e_1 LaR\ e_2$, $f_1 LbR f_2$, $ab(ab)'\omega e_2$, *and* $(ab)'ab\omega f_1$.

Proof. Since S is V-regular, for all $a, b \in S, (ab)' \in V(ab)$ there exist $a' \in V(a), b' \in V(b)$ such that $(ab)' = b'a'$. Let $e_1 = a'a, f_1 = b'b, e_2 = aa', f_2 = bb'$. Then $e_1, e_2, f_1, f_2 \in E(S)$, and

$$b(ab)'a = bb'a'a = f_2 e_1, e_1 = a'aLaRaa' = e_2, f_1 = b'bLbRbb' = f_2.$$

Now

$$(ab)(ab)'e_2 = (ab)(ab)'aa' = (ab)(b'a'aa') = (ab)b'a' = (ab)(ab)',$$

and

$$e_2(ab)(ab)' = (aa')(ab)(ab)' = (aa'a)b(ab)' = (ab)(ab)'.$$

It follows that $(ab)(ab)'\omega e_2$. Similarly, $(ab)'ab\omega f_1$.

Conversely, let a, b satisfy the above condition. Now $e_1 LaRe_2, f_1 LbRf_2$, imply that there exist $a' \in V(a) \cap (Le_2 \cap Re_1), b' \in V(b) \cap (Lf_2 \cap Rf_1)$, such that $a'a = e_1, aa' = e_2, b'b = f_1, bb' = f_2$. Since $b(ab)'a = f_2 e_1$, we have that $b'a' = (b'f_2)(e_1 a') = b'(f_2 e_1)a' = b'b(ab)'aa'$. Thus

$$(b'a')(ab)(b'a') = (b'b(ab)'aa')(ab)(b'b(ab)'aa') =$$

$$b'b(ab)'ab(ab)'aa' = b'b(ab)'aa' = b'a',$$

and

$$(ab)(b'a')(ab) = ab(b'b(ab)'aa')ab = ab(ab)'ab = ab.$$

That is, $b'a' \in V(ab)$. Also,

$$(b'a')(ab) = (b'b(ab)'aa')ab = bb(ab)'ab = f1(ab)'ab = (ab)'ab,$$

since $(ab)'ab\omega f_1$, and $(ab)(b'a') = (ab)(b'b(ab)'aa') = ab(ab)'aa' = (ab)(ab)'e_2 = (ab)(ab)'$, since $(ab)(ab)'\omega e_2$. It follows that $(ab)' = (ab)'(ab)(ab)' = (b'a')(ab)\ (ab)' = (b'a')(ab)(b'a') = b'a'$. Therefore, S is V-regular, as required. □

Theorem 2.2. *Let S be a V-regular semigroup, $\rho \in C(S), a, b \in S$. If $a\rho b$, then for any $a \in V(a)$, there exists $b \in V(b)$ such that $a\rho b$.*

Proof. Let $a' \in V(a)$. Then $a'\rho \in V(a\rho)$. Since $a\rho b$, we have that $a'\rho \in V(a\rho) = V(b\rho)$. Let $f\rho = b\rho a'\rho, f'\rho = a'\rho b\rho$. Then $(f\rho)R(b\rho), (f'\rho)L(b\rho), f\rho,\ f'\rho \in E(S/\rho)$. By Lemma 1.3 and its dual, there exist $e, e' \in E(S)$ such that $eRbLe', f\rho = e\rho$, and $f'\rho = e'\rho$. Take $b' \in V(b) \cap Le \cap Re'$. Then $b'\rho \in Le\rho \cap Re'\rho$. Hence $b'\rho = e'\rho b'\rho e\rho = f'\rho b'\rho f\rho = a'\rho b\rho b'\rho b\rho a\rho = a'\rho b\rho b'\rho b\rho a'\rho = a'\rho b\rho a'\rho = a'\rho a\rho a'\rho = a'\rho$, that is, $a'\rho b'$. □

To provide a characterization of congruences on V-regular semigroups in terms of certain congruence pairs, we need the following results.

Lemma 2.3. *Let S be a V-regular semigroup, $\rho \in C(S)$ with $\tau = tr\rho$.*

(1) $(e\rho)R(f\rho)$ in $S/\rho \Leftrightarrow e(\tau R)f$ in $S \Leftrightarrow e(R\tau)f$ in $S(e, f), e, f \in E(S)$;

(2) $R\tau R\tau R = R\tau R$.

An equivalence τ on the set $E(S)$ of idempotents of a regular semigroup S, is normal if $\tau = tr\tau^*$ [5]. It follows from [5, Lemma 2.3], that an equivalence τ on $E(S)$ is normal if, and only if, τ is the trace of a congruence on S. Let K be a subset of a regular semigroup S. A congruence ρ on S saturates K, if $a \in K$ implies $a\rho \subseteq K$. The greatest congruence on S which saturates K is denoted by π_K. Recall from [5, 1.5], that for $a, b \in S, a\pi_K b$, if and only if, $xay \in K \Leftrightarrow xby \in K (x, y \in S^1)$, and $\pi_K = \theta_k^0$, , where the equivalence relation θ_K on S is defined by $a\theta_K b \Leftrightarrow a, b \in K$ or $a, b \in S \setminus K$. We recall from [5] that a subset K of a regular semigroup S is normal, if $K = ker\pi_K$, and a subset K of S is normal, if and only if K is the kernel of a congruence on S. The pair (K, τ), is a congruence pair for a regular semigroup S if,

- (i) K is a normal subset of S;
- (ii) τ is a normal equivalence on $E(S)$;
- (iii) $K \subseteq ker(L\tau L\tau L \cap R\tau R\tau R)^0$,
- (vi) $\tau \subseteq tr\pi_K$.

In such a case $\rho(K, \tau)$ is defined by $\rho(K, \tau) = \pi_K \cap (L\tau L\tau L \cap R\tau R\tau R)^0$. Note that $\rho(K, \tau) = (L\tau L\tau L \cap \theta_K \cap R\tau R\tau R)^0$. When S is a V-regular semigroup it follows from [1, Lemma 2.3], and its dual result that $\rho(K, \tau) = (L\tau L \cap \theta_K \cap R\tau R)^0$.

Theorem 2.4. *If (K, τ) is a congruence pair for a V-regular semigroup S, then $\rho(K, \tau)$ is the unique congruence on S such that $ker\rho(K, \tau) = K$, and $tr\rho(K, \tau) = \tau$. Conversely, if ρ is a congruence on S, then $(ker\rho, tr\rho)$ is a congruence pair for S and $\rho = \rho(ker\rho, tr\rho)$.*

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On generalization of derivation on MV-algebras

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Abstract

In this paper, we extend the notion of derivation on MV-algebras. Moreover, we introduce four types of f-derivation on MV-algebras as a generalization of derivation and obtain some related results. Also, some connections among different types of f-derivation is studied.

Keywords: MV-Algebra, Boolean algebra, Derivation, f-Derivation, Isotone.
Mathematics Subject Classification [2010]: 06D35, 47B47

1 Introduction

The notion of derivation, introduced from the analytic theory is helpful to the research of structure and property in algebraic systems. Jun and Xin [4] applied the notion of derivation to BCI-algebras, which is defined in a way similar to the notion in ring theory, and investigated some properties related to this concept. In [8] and [5], authors introduced the notion of f-derivation and (f, g)-derivation in BCI-algebras, respectively. In [7], Szász introduced the concept of derivation on lattices and investigated some of its properties. Then, f-derivation on lattices were defined and studied in [2].

In [1], Alshehri applied the notion of derivation to MV-algebras and investigated some of its properties. After in [3, 6], the notion of derivation and generalization of it on MV-algebras is studied.

In this paper, we review some notions related to MV-algebras. Also, we recall and introduce some of types of f-derivation on MV-algebras as a generalization of derivation. Then, we obtain some related results.

Definition 1.1. An *MV-algebra* is a structure $(M, \oplus, *, 0)$ where M is a non-empty set, "$\oplus$" is a binary operation, "$*$" is a unary operation, and "0" is a constant such that the following axioms are satisfied for any $a, b \in M$,

($MV1$) $(M, \oplus, 0)$ is a commutative monoid;

($MV2$) $(a^*)^* = a$;

($MV3$) $0^* \oplus a = 0^*$;

($MV4$) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

*Speaker

We define the constant $1 = 0^*$ and the auxiliary operations $\odot, \vee$ and $\wedge$ by

$$a \odot b = (a^* \oplus b^*)^*,\ a \ominus b = a \odot b^*,\ a \vee b = a \oplus (b \odot a^*),\ a \wedge b = a \odot (b \oplus a^*).$$

Let $(M, \oplus, *, 0)$ be an MV-algebra. The partial ordering " $\leq$ " on M is defined by

$$a \leq b \Longleftrightarrow a \wedge b = a, \text{for all } a, b \in M.$$

$a \wedge b = a$ is equivalent to $a \vee b = b$. The structure $(M, \vee, \wedge, , 0, 1)$ is a bounded distributive lattice. If the order relation $\leq$, defined over M, is total, then we say that M is *linearly ordered*. Also, the structure $(M, \ominus, 0)$ is a bounded $BCI \setminus BCK$-algebra.

Let M and N be two MV-algebra. The function $f : M \longrightarrow N$ is called a *homomorphism* if it satisfies the following conditions:

(1) $f(0_M) = 0_N$;

(2) $f(x \oplus_M y) = f(x) \oplus_N f(y)$;

(3) $f(x^*) = f(x)^*$.

If f is a homomorphism, then $f(1_M) = 1_N$ and $f(x \odot_M y) = f(x) \odot_N f(y)$.

The function $f : M \longrightarrow M$ is called *isotone*, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in M$.

Let $B(M) = \{x \in M \mid x \oplus x = x\} = \{x \in M \mid x \odot x = x\}$. Then, $(B(M), \oplus, *, 0)$ is both a largest subalgebra of M and a Boolean algebra.

2 Main results

Let M be an MV-algebra and $f : M \longrightarrow M$ be a homomorphism. A function $D : M \longrightarrow M$ is called

(1) an *f-derivation of type* 1, if $D(x \odot y) = (D(x) \odot f(y)) \oplus (f(x) \odot D(y))$, for all $x, y \in M$ [6];

(2) an *f-derivation of type* 2, if $D(x \wedge y) = (D(x) \wedge f(y)) \vee (f(x) \wedge D(y))$, for all $x, y \in M$ [6];

(3) an *f-derivation of type* 3, if $D(x \ominus y) = (D(x) \ominus f(y)) \wedge (f(x) \ominus D(y))$, for all $x, y \in M$ [6];

(4) an *f-derivation of type* 4, if $D(x \ominus y) = (D(x) \ominus f(y)) \odot (f(x) \ominus D(y))$, for all $x, y \in M$.

In the above definition, if we choose the function f as the identity function, then the f-derivation of type 1 (2, 3 and 4, respectively) is ordinary derivation of type 1 (2, 3 and 4, respectively).

Let M be an MV-algebra and f be arbitrary homomorphism on M . The function $D : M \longrightarrow M$, defined by $D(x) = 0$, for all $x \in M$, is an f-derivation of type 1, 2, 3 and 4 on M.

If MV-algebra M is a Boolean algebra, then for all $x, y \in M$, $x \oplus y = x \vee y$ and $x \odot y = x \wedge y$. So, in this case, every derivation of type 1 on M is coincide with derivation of type 2 on M.

Let $(M, \oplus, *, 0)$ be an MV-algebra. Then, the definition of derivation of type 2 on $(M, \oplus, *, 0)$ is coincide with the definition of derivation on lattice $(M, \wedge, \vee, 0, 1)$. Also, the definition of derivation of type 3 on $(M, \oplus, *, 0)$ is coincide with the definition of derivation on bounded $BCI \setminus BCK$-algebra $(M, \ominus, 0)$.

The following theorem shows the relation between the f-derivation of type 1 and the f-derivation of type 2.

Theorem 2.1. *Let $D : M \longrightarrow M$ be a function on a linearly ordered MV-algebra. Then, D is an isotone f-derivation of type 1 if and only if D is an isotone f-derivation of type 2 and $D(M) \subseteq B(M)$.*

The properties of f-derivation of type 2 (3, respectively) on MV-algebras is similar to the properties of f-derivation on lattices ($BCI \setminus BCK$-algebras, respectively). Also, the properties of f-derivation of type 1 on MV-algebras is studied in [6].

In the following, we consider some properties of f-derivation of type 4.

Lemma 2.2. *Let D be a f-derivation of type 4 on an MV-algebra M. Then, for all $x \in M$, the following conditions hold:*

(1) $D(0) = 0$;

(2) $D(x) = D(x) \odot f(x)$;

(3) $D(x) \leq f(x)$;

(4) $D(x^*) \leq (D(x))^*$;

(5) if $x \leq y$, then $D(x) \leq D(y)$;

(6) $D(x) \leq D(1)$.

Lemma 2.3. *Let D be a f-derivation on MV-algebra M. Then, $D(x) = D(1) \odot f(x)$, for all $x \in B(M)$.*

Proposition 2.4. *Let D be a map on an MV-algebra M such that $D(x) = a \odot f(x)$, for all $x \in M$, where $a \in M$ is a fixed element. If $D(M) \subseteq B(M)$, then D is an f-derivation of type 4.*

Now, we give our main result in the following theorem which shows the relation between the f-derivation of type 1 and the f-derivation of type 4.

Theorem 2.5. *(1) If D is an isotone f-derivation of type 1 on MV-algebra M, then D is an f-derivation of type 4.*

(2) If D is an f-derivation of type 4 on a boolean algebra M, then D is an f-derivation of type 1.

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ON GENERALIZATIONS OF PRIMARY SUBMODULES (II)

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Abstract

Assume that R be is a commutative ring, M is a unitary R-module and $S(M)$ is the set of all submodules of M. Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function. We say that a proper submodule N of M is a ϕ-2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $abx \in N \setminus \phi(N)$ implies $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$. In this paper we show that ϕ-2-absorbing primary submodules enjoy analogs of many of the properties of 2-absorbing primary submodules.

Keywords: 2-absorbing primary submodule, ϕ-primary submodule, ϕ-2-absorbing primeary submodule.
Mathematics Subject Classification [2010]: 13A15, 13F05, 13G05

1 Introduction

We assume throughout that R is a commutative ring with $1 \neq 0$ and M is a unitry R-module. We denote the set of all submodules of M by $S(M)$. Let N be a submodule of M. It is clear that $(N : M) = \{r \in R | rM \subseteq N\}$ is an ideal of R.

A proper submodule P of M is called prime if $r \in R$ and $x \in M$, with $rx \in P$ implies that $r \in (P :_R M)$ or $x \in P$ (see [1-9]). These have led to more information on the structure of the R-module M.

Recall that a proper submodule N of M is called a 2-absorbing submodule of M as in [2] if whenever $abx \in N$ for some $a, b \in R$ and $x \in M$, then $ab \in (N : M)$ or $ax \in N$ or $bx \in N$. We say that a proper submodule N of M is a weakly primary submodule of M if whenever $0 \neq ax \in N$ for some $a \in R$ and $x \in M$, then $a \in (N : M)$ or $x \in rad(N)$. Also, we say that a proper submodule N of M is a 2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $abx \in N$, then $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$. A proper submodule N of M is a weakly 2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $0 \neq abx \in N$ implies $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$. Recall that a proper submodule N of M is called a ϕ-2-absorbing submodule of M as in [2] if whenever $a, b \in R$ and $x \in M$ with $abx \in N \setminus \phi(N)$ implies $ab \in (N : M)$ or $ax \in N$ or $bx \in N$. We say that a proper submodule N of M is a ϕ-primary submodule of M if whenever $a \in R$ and $x \in M$ with $ax \in N \setminus \phi(N)$ implies $a \in (N : M)$ or $x \in rad(N)$.

*Speaker

Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function. A proper submodule N of M is said to be a ϕ-2-absorbing primary submodule of M if whenever $a, b \in R$ and $x \in M$ with $abx \in N \setminus \phi(N)$ implies $ab \in (N : M)$ or $ax \in rad(N)$ or $bx \in rad(N)$.

Let P be a submodule of M. Since $P\backslash\phi(P) = P\backslash(P \cap \phi(P))$, so without loss of generality, throughout this paper we will consider $\phi(P)P$. We hence forth make the assumption that given two functions $\psi_1, \psi_2 : S(M)S(M) \cup \{\emptyset\}$, then $\psi_1 \leq \psi_2$ if $\psi_1(N)\psi_2(N)$, for each $N \in S(M)$. Suppose that $\phi_\alpha : S(M)S(M) \cup \{\emptyset\}$ be one of the following functions we have the corresponding ϕ_α-2-absorbing primary submodules.

$\phi_\emptyset$	$\phi(N) = \emptyset$	2-absorbing primary submodule
ϕ_0	$\phi(N) = 0$	weakly 2-absorbing primary submodule
ϕ_1	$\phi(N) = N$	any module
ϕ_2	$\phi(N) = (N : M)N$	ϕ_2-2-absorbing primary submodule
$\phi_n (n \geq 2)$	$\phi(N) = (N : M)^{n-1}N$	ϕ_n-2 absorbing primary submodule
ϕ_ω	$\phi(N) = \bigcap_{i=1}^{\infty}(N : M)^i N$	ω-2-absorbing primary submodule

Observe that $\phi_\phi \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$. Then it is clear that $\phi_\emptyset$-2 absorbing primary and ϕ_0-2-absorbing primary submodules are 2-absorbing primary and weakly 2-absorbing primary submodules respectively.

In this paper we show that ϕ-2-absorbing primary submodules enjoy analogs of many of the properties of 2-absorbing primary submodules and a number of results concerning ϕ-2-absorbing primary submodules are given.

2 Main results

Theorem 2.1. *Let N_1, N_2 be proper submodules of M such that $N_1 \subseteq N_2$, and let $n \geq 2$. If N_2 is a ϕ_n-2-absorbing primary submodule of M, then $\frac{N_2}{N_1}$ is a ϕ_n-2-absorbing primary submodule of $\frac{M}{N_1}$.*

Theorem 2.2. *Let N_1, N_2 be proper submodules of M such that $N_1 \subseteq N_2$. If N_2 is a ϕ_ω-2-absorbing primary submodule of M, then $\frac{N_2}{N_1}$ is a ϕ_ω-2-absorbing primary submodule of $\frac{M}{N_1}$.*

Definition 2.3. Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function. We remind the reader that we always assume that $\phi(N) \subseteq N$. Let N be a proper submodule of M and S be a multiplicatively closed subset of R. Then

(i) A proper submodule $\frac{L}{N}$ of $\frac{M}{N}$ (as $\frac{R}{(N:M)}$-module), where L is a proper submodule of M such that $N \subseteq L$, is called a ϕ_N-2-absorbing primary submodule of $\frac{M}{N}$ if whenever $a, b \in \frac{R}{(N:M)}$, $x \in \frac{M}{N}$ with $abx \in \frac{L}{N} \setminus \frac{\phi(L)+N}{N}$ implies $ab \in (\frac{L}{N} : \frac{M}{N})$ or $ax \in rad(\frac{L}{N})$ or $bx \in rad(\frac{L}{N})$.

(ii) A proper submodule L_S of M_S, where L is a proper submodule of M such that $(L : M) \cap S = \emptyset$, is called a ϕ_S-2-absorbing primary submodule of M_S if whenever $a, b \in R_S$ and $x \in M_S$ with $abx \in L_S \setminus \phi(L)_S$ implies $ab \in (L_S : M_S)$ or $ax \in rad(L_S)$ or $bx \in rad(L_S)$.

Theorem 2.4. *Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function, P be a proper submodule of M and let N be a submodule of M such that $N \subseteq P$. If P is a ϕ-2-absorbing primary submodule of M, then $\frac{P}{N}$ is a ϕ_N-2-absorbing primary submodule of $\frac{M}{N}$.*

Theorem 2.5. *Let $\phi : S(M) \longrightarrow S(M)\cup\emptyset$ be a function and let P, I be proper submodules of M such that $N \subseteq \phi(P)$. The following statements are equivalent.*

(i) P is a ϕ-2-absorbing primary submodule of M.

(ii) $\frac{P}{N}$ is a ϕ_N-2-absorbing primary submodule of $\frac{M}{N}$.

(iii) $\frac{P}{(N:M)^{n-1}N}$ is a $\phi_{(N:M)^{n-1}N}$-2-absorbing primary submoduleof $\frac{M}{(N:M)^{n-1}N}$ for every $n \geq 1$.

Corollary 2.6. *Let $\phi : S(M) \longrightarrow S(M)\cup\emptyset$ be a function and let P be a proper submodule of M that is not a weakly 2-absorbing primary submodule of M. The following statements are equivalent.*

(i) P is a ϕ-2-absorbing primary submodule of M.

(ii) $\frac{P}{(P:M)^2P}$ is a $\phi_{(P:M)^2P}$-2-absorbing primary submodule of $\frac{M}{(P:M)^2P}$.

(iii) $\frac{P}{(P:M)^{n-1}P}$ is a $\phi_{(P:M)^{n-1}P}$-2-absorbing primary submodule of $\frac{M}{(P:M)^{n-1}P}$ for every $n \geq 3$.

For a commutative ring R with $1 \neq 0$, Let $Z(R)$ be the set of all zero-divisors of R.

Theorem 2.7. *Let $\phi : S(M) \longrightarrow S(M)\cup\emptyset$ be a function. Let P be a proper submodule of M and S be a multiplicatively closed subset of R such that $S \cap Z(R) = S \cap (P : M) = \emptyset$. The following statements are equivalent.*

(i) P is a ϕ-2-absorbing primary submodule of M.

(ii) P_S is a ϕ_S-2-absorbing primary submodule of M_S.

Lemma 2.8. *Let $\phi : S(M) \longrightarrow S(M) \cup \emptyset$ be a function. Set $\frac{M}{\emptyset} = M$, and let N be a proper submodule of M. Then*

(i) N is a 2-absorbing primary submodule of M if and only if $\frac{N}{\phi(N)}$ is a 2-absorbing primary submodule of $\frac{M}{\phi(M)}$.

(ii) N is a prime submodule of M if and only if $\frac{N}{\phi(N)}$ is a prime submodule of $\frac{M}{\phi(M)}$.

(iii) N is a primary submodule of M if and only if $\frac{N}{\phi(N)}$ is a primary submodule of $\frac{M}{\phi(M)}$.

Theorem 2.9. *Let R_1 and R_2 be commutative rings with $1 \neq 0$ and M_i be an R_i-module, for $n = 1, 2$ and consider $M = M_1 \times M_2$ as $R = R_1 \times R_2$-module. Let N_1 be a proper submodule of M_1 and $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \emptyset (i = 1, 2)$ be functions such that $\psi_2(M_2) \neq M_2$, and let $\phi = \psi_1 \times \psi_2$. Then the following statements are equivalent.*

(i) $N_1 \times M_2$ is a ϕ-2-absorbing primary submodule of M.

(ii) $N_1 \times M_2$ is a 2-absorbing primary submodule of M.

(iii) N_1 is a 2-absorbing primary submodule of M_1.

Theorem 2.10. *Let R_1, R_2 be commutative rings with $1 \neq 0$ and M_i be an R_i-module, for $n = 1, 2$ and consider $M = M_1 \times M_2$ as $R = R_1 \times R_2$-module. Let N_1 be a proper submodule of M_1 and $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \emptyset (i = 1, 2)$ be functions and let $\phi = \psi_1 \times \psi_2$. Then the following statements are equivalent.*

(i) $N_1 \times M_2$ is a ϕ-2-absorbing primary submodule of M that is not a 2-absorbing primary submodule of M.

(ii) $\phi(N_1 \times M_2) \neq \emptyset$, $\psi_2(M_2) = M_2$, and N_1 is a ψ_1-2-absorbing primary submodule of M_1 that is not a 2-absorbing primary submodule of M_1.

Theorem 2.11. *Let R_i be a commutative ring, M_i be an R_i-module, for $i = 1, 2$ and consider $M = M_1 \times M_2$ as $R = R_1 \times R_2$-module. Let N_1, N_2 be non-zero submodules of M_1 and M_2, respectively. Let $\psi_i : S(M_i) \to S(M_i) \cup \emptyset$ $(i = 1, 2)$ be functions such that $\psi_1(N_1) \neq N_1$ and $\psi_2(N_2) \neq N_2$. Let $\phi = \psi_1 \times \psi_2$. If $N_1 \times N_2$ is a proper submodule of M, then the following statements are equivalent.*

(i) $N_1 \times N_2$ is a ϕ-2-absorbing primary submodule of M.

(ii) $N_1 = M_1$ and N_2 is a 2-absorbing primary submodule of M_2 or $N_2 = M_2$ and N_1 is a 2-absorbing primary submodule of M_1 or N_1, N_2 are primary submodules of M_1, M_2, respectively.

(iii) $N_1 \times N_2$ is a 2-absorbing primary submodule of M.

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On Generalized Fibonacci Length Of Polygroups

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Abstract

In this paper, we introduce generalized Fibonacci length of polygroups as a generalization of Fibonacci length of groups. We mention a connection between the fundemental group obtain from polygroup an the class of Fibonacci group.
Keywords: Fibonacci lenght, Fibonacci orbit, Polygroups.
MSC(2010): 20N20.

1 Introduction

The concept of a hypergroup which is based on the notion of hyperoperation was introduced by marty in [7] and studied extensively by many mathematicians. Hypergroup theory extends some well-known results in group theory and introduces new topics leading to a wide variety of applications, as well as to broadening of the fields of investigation. Surveys of the theory can be found in the books of Corsini [3], Davvaz and Leoreanu-Fotea [5] Corsini and Leoreanu [4]. Polygroups which form an important subclass of hypergroups were studied by Comer [2]. Quasicanonical hypergroups were introduced for the first time in [1], as a generalization of canonical hypergroups, introduced in [8]. In 1960 D. D. Wall investigated the length of the period of the Fibonacci numbers modulo a given positive integer n, see [9]. In this paper we generalized the notion of Fibonacci length for the class of polygroups and investigate some properties of this notion.

Definition 1.1. A set H endowed with a mapping $* : H \times H \longrightarrow P^*(H)$, named *hyperoperation*, is called a *hypergroupoid* with a carrier H or briefly a *hypergroupoid*.

Remark 1.2. A hyperoperation $* : H \times H \longrightarrow P^*(H)$ yields an operation $\otimes : P^*(H) \times P^*(H) \longrightarrow P^*(H)$, defined by $A \otimes B = \bigcup_{a \in A, b \in B} a * b$. Conversely an operation on $P^*(H)$ yields a hyperoperation on H, defined by $x * y = \{x\} \otimes \{y\}$.

Definition 1.3. (i) A *semihypergroup* is a hypergroupoid $(H, *)$ such that for all a, b and c in H we have $(a * b) * c = a * (b * c)$,
(ii) A *quasihypergroup* is a hypergroupoid $(H, *)$ which satisfies the reproductive law, i.e., for all $a \in H$, $H * a = a * H = H$,
(iii) A *hypergroup* is a semihypergroup which is also a quasihypergroup.

Definition 1.4. Let $(H, \circ)$ be a hypergroup and R be an equivalence on H. If A and B are non-empty subsets of H, then

*Speaker

(1) $A \bar{R} B$ means that $\forall a \in A, \exists b \in B$ such that aRb and $\forall b' \in B, \exists a' \in A$ such that aRb,

(2) $A \bar{\bar{R}} B$ means that $\forall a \in A, \forall b \in B$ we have aRb.

Definition 1.5. The equivalence relation R is called

(1) regular on the right (on the left) if for all x of H, from aRb, it follows that $(a \circ x) \bar{R} (b \circ x)$ $((x \circ a) \bar{R} (x \circ b) respectively)$,

(2) strongly regular on the right (on the left) if for all x of H, from aRb, it follows that $(a \circ x) \bar{\bar{R}} (b \circ x)$ $((x \circ a) \bar{\bar{R}} (x \circ b) respectively)$,

(3) R is called regular (strongly legular) if it is regular (strongly legular) on the right and on the left.

Theorem 1.6. *Let R be a strongly regular equivalence on a semi-hypergroup H, then*

(1) $\frac{H}{R}$ *is a semigroup,*

(2) if H is a hypergroup, $\frac{H}{R}$ *is a group.*

Definition 1.7. For all $n > 1$, we define the relation β_n on a semihypergroup H, as follows:

$$a\beta_n b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

and $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Theorem 1.8. *β^* is the smallest strongly regular relation on H.*

Theorem 1.9. *If H is a hypergroup, then $\beta^* = \beta$.*

Definition 1.10. A *polygroup* is a system $\wp = \langle P, \cdot, e, -1 \rangle$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\cdot$ maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all x, y, z in P:

(P_1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

(P_2) $e \cdot x = x \cdot e = x$,

(P_3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

Let X be a non-empty subset of a polygroup $\langle P, \cdot, e, -1 \rangle$. Let $\{A_i \mid i \in J\}$ be the family of all subpolygroups of P in which contain X. Then, $\bigcap_{i \in J} A_i$ is called the *subpolygroup generated* by X. This subpolygroup is denoted by $< X >$ and we have $< X >= \cup \{x_1^{\varepsilon_1} \cdot \ldots \cdot x_k^{\varepsilon_k} \mid x_i \in X, k \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}\}$. If $X = \{x_1, x_2, \ldots, x_n\}$. Then, the subpolygroup $< X >$ is denoted $< x_1, x_2, \ldots, x_n >$.

Definition 1.11. Let $\langle P, \cdot, e, -1\rangle$ be a polygroup that generated by $A = \{a_1, a_2, \ldots, a_n\}$. Then the Fibonacci orbit of P with respect to the (generating) set A, where A is written as the ordered n-tuple $(a_1, a_2, \ldots, a_n)$, denoted by $GF_A(P)$, is the sequence $F_0^A = a_1$, $F_1^A = a_2, \ldots, F_{n-1}^A = a_n$ and $F_{n+i}^A = \prod_{j=1}^{n} F_{i+j-1}$, $i \geq 0$. Moreover we define the generalized Fibonacci length of P, as follows:

$$m = min\{k \mid a_i \in F_{k+i}^A, \quad 1 \leq i \leq n\},$$

and denoted by $GLEN_A(P)$.

Remark 1.12. If P be is a group, Fibonacci length of P denoted by $LEN(P)$.

Example 1.13. Let $P = \{e, a, b, c, d, f, g\}$. We consider the proper non-commutative polygroup $\langle P, \cdot, e, -1\rangle$, where $\cdot$ is defined on P as follows:

$\cdot$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	$\{e,a\}$	g	f	d	c
c	c	c	f	$\{e,a\}$	g	b	d
d	d	d	g	f	$\{e,a\}$	c	b
f	f	f	c	d	b	g	$\{e,a\}$
g	g	g	d	b	c	$\{e,a\}$	f

It is easy to see that $GLEN_{\langle b,c\rangle}(P) = 6$.

Proposition 1.14. *Let $\langle P, \cdot, e, -1\rangle$ be a polygroup generated by $A = \{a_1, \ldots, a_n\}$. Then $GLEN_A(P) \geq LEN_{\tilde{A}}(\frac{P}{\beta^*})$, where $\tilde{A} = \{\beta^*(a_1), \ldots, \beta^*(a_n)\}$.*

Theorem 1.15. *Let G be group with generating set $A = \{a_1, \ldots, a_n\}$ and let $LEN_A(G) = m$ for finite m. Then G is a epimorphic image of $F(n, m)$.*

Proposition 1.16. *Let $\langle P, \cdot, e, -1\rangle$ be a polygroup generating set $A = \{a_1, \ldots, a_n\}$ and let $GLEN_A(P) = \ell$. Then $\frac{P}{\beta^*}$ is a epimorphic image of $F(n, \ell')$, where $LEN(\frac{P}{\beta^*}) = \ell'$.*

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Poster On induced closed subobjects by certain morphism classes pp.: 1–4

On Induced Closed Subobjects by certain morphism classes*

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Abstract

In this article, we start with a collection $\mathcal{M}$ of morphisms of a small category $\mathcal{X}$, that satisfies certain conditions and we construct an universal closure operation by general method. Next describe closed subobjects relative to this universal closure by a different view point.

Keywords: closed subobject, Lawvere-Tierney topology, morphism class, sieve.
Mathematics Subject Classification [2010]: 18F10, 18F20

1 Introduction

Throughout this article we let $\mathcal{X}$ be a small category and $\mathcal{M}$ be a set of morphisms of $\mathcal{X}$. The collection, $\mathcal{X}_1/x$, of all the $\mathcal{X}$-morphisms with codomain x is a preordered class by the relation $f \leq g$ if there exists a morphism h such that $f = g \circ h$. The equivalence relation generated by this preorder is $f \sim g$ if $f \leq g$ and $g \leq f$. For a class $\mathcal{M}$ of $\mathcal{X}$-morphisms, we write $f \sim \mathcal{M}$ whenever $f \sim m$ for some $m \in \mathcal{M}$. We say $\mathcal{M}$ is saturated provided that $f \in \mathcal{M}$ whenever $f \sim \mathcal{M}$.

Denoting the domain and codomain of a morphism f by $d_0 f$ and $d_1 f$ respectively, recall that a sieve in $\mathcal{X}$, [6], generated by one morphism f is called a principal sieve and is denoted by $\langle f \rangle$ and for a sieve S on x and a morphism f with $d_1 f = x$, $S \cdot f = \{g : f \circ g \in S\}$. We say sieve S is an k-sieve provided that minimum number of morphisms generates S is equal to k.

For a class $S \subseteq \mathcal{X}_1/x$, and a morphism f with codomain x, the class of all the maximal elements w in $\mathcal{X}_1/d_0 f$ satisfying $f \circ w \leq s$ for some $s \in S$ is denoted by $(f \Rightarrow S)$. Obviously for a sieve S on x, see [6], $(f \Rightarrow S)$ is just the class of maximal elements of $S \cdot f$.

We say $\mathcal{M}$ has $\mathcal{X}$-pullbacks if the pullback, $f^{-1}(m)$, of each $m \in \mathcal{M}$ along each $f \in \mathcal{X}$ exists and belongs to $\mathcal{M}$.

Definition 1.1. A class $\mathcal{M}$ of $\mathcal{X}-$morphisms is said to satisfy the principality property, if for each x, $f \in \mathcal{X}_1/x$ and $m \in \mathcal{M}/x$, $(f \Rightarrow \langle m \rangle) \subseteq \mathcal{M}/d_0 f, card((f \Rightarrow \langle m \rangle)) = 1$.

If $\mathcal{X}_1$ satisfies the principality property, the mapping $P^n : \mathcal{X}^{op} \to Set$ with $P^n(x) = \{\langle f_1, f_2, ..., f_n \rangle | \text{for all } 1 \leq i \leq n, f_i \in \mathcal{X}_1/x\}$ and for $f : y \to x$, $P^n(f) : P^n(x) \to P^n(y)$ the function taking $\langle g_1, g_2, ..., g_n \rangle$ to $\langle g_1, g_2, ..., g_n \rangle \cdot f$, is a functor. Also the mapping $P : \mathcal{X}^{op} \to Set$ with $P(x) = \{S \in \Omega(x) \mid S \text{ is finitely generated sieve on } x\}$ and for $f : y \to x$, $P(f) : P(x) \to P(y)$ the function taking S to $S \cdot f$, is a functor.

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2 Main results

According to the construction stated in [3], each class $\mathcal{M}$ with principality property which is satisfy in certain conditions, see [3], introduce a Lawvere-Tierney topology, [6]. We use this method and get the Lawvere-Tierney topologies and universal closure operations which are corresponding to each other, see [6]. Know we find a description of closed subobjects with respect to these closures.

Suppose that $\mathcal{X}_1$ satisfies the principality property. Every set $\mathcal{M}$ of morphisms of $\mathcal{X}$ which satisfies the principality property yields a subobject $M^n : \mathcal{X}^{op} \to Set$ of P^n where $M^n(x) = \{\langle m_1, m_2, \cdots, m_n\rangle \mid \text{for all } 1 \leq i \leq n, m_i \in \mathcal{M}/x\}$.

know subobjects $i : M^n \rightarrowtail \Omega$ in $Set^{\mathcal{X}^{op}}$ are in one to one correspondence with the morphisms $j^n : \Omega \to \Omega$ via the following pullback square, see [6],

$$\begin{array}{ccc} M^n & \xrightarrow{!_{M^n}} & 1 \\ {\scriptstyle i}\downarrow & p.b. & \downarrow{\scriptstyle t} \\ \Omega & \xrightarrow[j^n]{} & \Omega \end{array} \qquad (I)$$

Note that for a given M^n, j^n is defined by the maps j_x that take each sieve S on x to $\{f \in \mathcal{M}/x \mid S \cdot f \in M^n(d_0(f))\}$; and for a given j^n, M^n is defined by $M^n(x) = \{S : {j^n}_x(S) = T_x\}$. With M^n and j^n corresponding to each other, we obviously have, ${j^n}_x(S) = T_x$ if and only if $S \in M^n(x)$.

It is known that, in any topos, the Lawvere-Tierney topologies correspond to universal closure operations, see [6].

This correspondence also holds between arrows $j^n : \Omega \xrightarrow{\cdot} \Omega$ and universal operations "$-^n$". Here is how this correspondence works. For a given $j^n : \Omega \longrightarrow \Omega$, for each X, $-^n : \mathcal{S}ub(X) \longrightarrow \mathcal{S}ub(X)$ is defined by the following pullbacks:

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ {\scriptstyle \alpha}\downarrow & p.b. & \downarrow{\scriptstyle t} \\ X & \xrightarrow[\hat{\alpha}]{} & \Omega \end{array} \qquad (II) \qquad\qquad \begin{array}{ccc} \bar{A} & \xrightarrow{!_{\bar{A}}} & 1 \\ {\scriptstyle \bar{\alpha}}\downarrow & p.b. & \downarrow{\scriptstyle t} \\ X & \xrightarrow[j^n \circ \hat{\alpha}]{} & \Omega \end{array} \qquad (III)$$

where $\hat{\alpha}_c(x) = \{f : a \to c \in \mathcal{X}_1 \mid X(f)(x) \in \alpha_a(A(a))\}$ and we have $\hat{\bar{\alpha}} = j^n \circ \hat{\alpha}$.

Conversely given a universal operation "$-^n$", the morphism $j^n : \Omega \xrightarrow{\cdot} \Omega$ is obtained by the following pullback.

$$\begin{array}{ccc} \bar{1} & \xrightarrow{!_{\bar{1}}} & 1 \\ {\scriptstyle \bar{t}}\downarrow & p.b. & \downarrow{\scriptstyle t} \\ \Omega & \xrightarrow[j^n]{} & \Omega \end{array} \qquad (IV)$$

We call $\bar{A}$ the closure of $A \rightarrowtail X$ and we say that it is closed when $\bar{A} = A$.
If we start with collection $\mathcal{M}$ of morphisms of X and next induces subobject $M^n : \mathcal{X}^{op} \to Sets$ and so induces universal closure operation $-^n$ as the above construction stated before. This means subobject $\alpha : A \rightarrowtail X$ is closed iff A and $\bar{A}$ have the same classifying maps, i.e., $\hat{\alpha} = \hat{\bar{\alpha}}$ and this equivalent to $j^n \circ \hat{\alpha} = \hat{\alpha}$.

Theorem 2.1. *Let $\alpha : A \rightarrowtail X$ be subobject of X and $\hat{\alpha}$ be its classifying map.*
α is a closed subobject w.r.t. $-^n$ if and only if $\hat{\alpha}_x(S) \in M^n(x) \Longleftrightarrow \hat{\alpha}_x(S) = T_x$ for all x

in $\mathcal{X}$ and S in $X(x)$.

Proof. We apply definition of map $j_x^n(S)$ and the fact that $j^n \circ \hat{\alpha} = \hat{\alpha}$ as stated above and so by extending the definitions of classifying maps with some calculations result follows. □

Definition 2.2. [8] Let $\mathcal{M}$ be a set of $\mathcal{X}$-morphisms. $\mathcal{M}$ is said to have:

(a) enough retractions, if for all objects x in $\mathcal{X}$, $\mathcal{M}/x$ has a retraction.

(b) almost enough retractions, if for all objects x in $\mathcal{X}$, $\mathcal{M}/x = \emptyset$ or $\mathcal{M}/x$ has a retraction.

(c) the n-identity property if for all objects x in $\mathcal{X}$ and for all sieves S on x whenever $\mathcal{M}_S = \{f \in \mathcal{X}_1/x | (f \Rightarrow S) \subseteq \mathcal{M}/d_0 f, card((f \Rightarrow S)) \leq n\}$ has at the most n maximal elements which are in $\mathcal{M}/x$, then $1_x \in \mathcal{M}_S$.

(d) the n-maximal property if for all objects x in $\mathcal{X}$ and for all sieves S on x, whenever $S \cap \mathcal{M}/x \neq \emptyset$, then S has at the most n maximal elements which are in $\mathcal{M}/x$.

(e) the n-quasi meet property if for all objects x in $\mathcal{X}$ and $m_1, ..., m_k \in \mathcal{M}/x$ and $n_1, ..., n_l \in \mathcal{M}/x$ such that $k, l \leq n$, there exists maximum element $h_i \in (m_i \Rightarrow \langle n_1, ..., n_l \rangle)$ such that $m_i \circ h_i \sim \mathcal{M}/x$ for $i = 1, ..., k$.

Proposition 2.3. *[8] Let $\mathcal{M}$ be a class of $\mathcal{X}$-morphisms that satisfies the principality property. The induced map $j^n : \Omega \to \Omega$ ($j : \Omega \to \Omega$) is a Lawvere-Tierney topology if and only if $\mathcal{M}$ satisfies (a), (c), (d) and (e) of Definition 2.2.*

suppose that Lawvere-Tierny topology j^n induced from the collection $\mathcal{M}$ which is having the conditions of proposition 2.3. Therefore we may have the following result for each n:

Corollary 2.4. *If $-^n$ universal closure operation induced from a Lawvere-Tierney topology j^n, then subobject $\alpha : A \rightarrowtail X$ is closed if and only if A_S has at the most n maximal elements which are in $\mathcal{M}/x$, then $1_x \in A_S$.*

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On the planar, outer planar and end-regular zero divisor graph of the ring $C(X)$

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Abstract

In this paper, we study the planarity, outerplanarity, and end-regularity of the zero-divisor graph of the ring of all real valued continuous functions $C(X)$, which is denoted by $\Gamma(C(X))$. Also, by using the ring properties of $C(X)$, the graph properties of $\Gamma(C(X))$, and the topological properties of X, we investigate the end-regularity of the graph $\Gamma(C(X))$.

Keywords: Zero divisor graph, The ring of continuous functions, Planar graph, Outerplanar graph, End-regular graph
Mathematics Subject Classification [2010]: 05C10, 46E25

1 Introduction

The idea of a zero-divisor graph of a commutative ring was first introduced by I. Beck [2] in 1988, where he was mainly interested in coloring. This investigation of colorings of a commutative ring was then continued by D. D. Anderson and M. Naseer. Their definition was slightly different than ours; they let all elements of ring be vertices and distinct vertices x and y are adjacent if and only if $xy = 0$. Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors.

Let $C(X)$ be the ring of all real valued continuous functions on a completely regular Hausdorff space X. By the zero divisor graph $\Gamma(C(X))$ of $C(X)$ we mean the graph with vertices consists of all nonzero zero-divisors of $C(X)$ such that there is an edge between distinct vertices f and g if and only if $fg = 0$.

In this paper, we determine the planarity, outerplanarity, and end-regularity of $\Gamma(C(X))$ by using the ring properties of $C(X)$, the graph properties of $\Gamma(C(X))$, and the topological properties of X. Also, we show that, in some cases, the graph $\Gamma(C(X))$ is not end-regular.

2 Main results

In this section, we first state some preliminaries from the ring $C(X)$, topology and graph theory which are expected to be useful in this paper. We use the standard terminology from [3] and [4].

*Speaker

In this paper X is a completely regular Hausdorff space. Let $C(X)$ denote the set of all continuous, real valued functions defined on the space X.

$C(X)$ is provided with a ring structure through addition and multiplication defined by:

$$(f+g)(x) = f(x) + g(x), (fg)(x) = f(x)g(x).$$

It is obvious that both of the above operations are associative and commutative, and the distributive law holds. Thus $C(X)$ is a commutative ring. Let $Z(C(X))$ denote the set of the zero-divisors of $C(X)$.

We associate a simple graph $\Gamma(C(X))$ to $C(X)$ with vertex-set $Z(C(X))^* = Z(C(X))\setminus\{0\}$, and for distinct $f, g \in Z(C(X))^*$, the vertices f and g are adjacent if and only if $fg = 0$.

If $|X| = 1$, then $C(X)$ is a field. In fact, in this case, $C(X)$ is isomorphic to the field of real numbers and $\Gamma(C(X))$ is an empty graph. Hence in the rest of the paper, we assume that $|X| \geq 2$.

Recall that a subset S of X is said to be a zeroset if there exists $f \in C(X)$ such that $S = f^{-1}(0)$. The zeroset $f^{-1}(0)$ is denoted by $Z(f)$.

The following lemma which is from [1], determine the vertices of the zero-divisor graph $\Gamma(C(X))$, and it is very useful in the rest of the paper.

Lemma 2.1. *f is a vertex in $\Gamma(C(X))$ if and only if $intZ(f) \neq \emptyset$ and $Z(f) \neq X$.*

A graph is said to be planar if it can be drawn in the plane, so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski 's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

In the following theorem, we study the planarity of $\Gamma(C(X))$.

Theorem 2.2. *If X is finite, then $\Gamma(C(X))$ is not planar.*

An undirected graph is outerplanar if it can be drawn in the plane without crossing in such away that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or$K_{2,3}$.

Let G be a graph with n vertices and q edges. We recall that a chord is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say C is a primitive cycle if it has no chords. Also, a graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The number frank(G) is called the free rank of G and it is the number of primitive cycles of G. Also, the number rank$(G) = q - n + r$ is called the cycle rank of G, where r is the number of connected components of G. A graph G is called a ring graph if it satisfies one of the following equivalent conditions:

(i) rank(G) =frank(G),

(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Theorem 2.3. *Let X be a finite space. Then $\Gamma(C(X))$ is niether ring graph nor outerplanar.*

Let G and H be graphs. A homomorphism f from G to H is a map from $V(G)$ to $V(H)$ such that for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an isomorphism from G to H, and in this case we say G is isomorphic to H, denoted by $G \cong H$. A homomorphism (resp, an isomorphism) from G to itself is called an endomorphism (resp, automorphism) of G. An endomorphism f is said to be half-strong if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By $End(G)$, we denote the set af all the endomorphisms of G. It is well-known that $End(G)$ is a monoid with respect to the composition of mappings. Let S be a semigroup. An element a in S is called regular if $a = aba$ for some $b \in B$ and S is called regular if every element in S is regular. Also, a graph G is called end-regular if $End(G)$ is regular.

Now, we recall the following Lemma from [5].

Lemma 2.4. *Let G be a graph. If there are pairwise distinct vertices a, b, c in G satisfying $N(c) \subsetneqq N(a) \subseteq N(b)$, then G is not end-regular.*

We end this section with the following theorem which is about the end-regularity of the graph $\Gamma(C(X))$.

Theorem 2.5. *Let X be a finite space. Then $\Gamma(C(X))$ is not end-regular.*

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Prime Ideals In The Kumjian-Pask Algebras

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Abstract

Let Λ be a row-finite k-graph with no sources and R be a unital commutative ring. In this note, we investigate conditions for Λ and R, which under these, $KP_R(\Lambda)$ is a prime ring. Then by applying this result, we characterize basic graded prime ideals of $KP_R(\Lambda)$.

Keywords: Kumjian-Pask algebra, prime ideal, basic graded ideal
Mathematics Subject Classification [2010]: 16W50, 46L05

1 Introduction

Higher rank graphs and their associated C^*-algebras were introduced by Kumjian and Pask in 2000. These C^*-algebras are natural generalizations of directed graph C^*-algebras. The motivation of these definitions was to provide graphical forms for higher rank Cuntz-Krieger algebras.

For a unital commutative ring R and k-graph Λ, Kumjian-Pask algebra $KP_R(\Lambda)$ is the algebraic analogue of the C^*-algebra $C^*(\Lambda)$, that introduced in [5] as higher rank analogues of the Leavitt path algebras. Some important results such as Graded and Cuntz-Krieger uniqueness theorems were proved for these algebras and also analysed their ideal structure. Studing of k-graph algebras has been interested for many authors. One of the reasons, is that they give examples for complicated mathematical concepts (See [1],[2]).

In this note, first, we explain some definitions and preliminaries which we need to prove the main result. Then, in Section 2, we characterize prime Kumjian-Pask algebras $KP_R(\Lambda)$ by giving some equivalent conditions for R and the underlting higher rank graphs Λ. Also, in Corollary 2.2, we determine basic graded prime ideals in $KP_R(\Lambda)$.

Definition 1.1. For a positive integer k, we view the additive semi group $\mathbb{N}^k$ ($\mathbb{N}$ is the set of natural numbers including zero) as a category with one object. A k-graph (or higher rank graph), is a countable category $\Lambda = (\Lambda^\circ, \Lambda, r, s)$ eqquiped with a functor $d : \Lambda \to \mathbb{N}^k$,called the degree map, satisfying the factorization property: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$, if $d(\lambda) = m + n$, there exist unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$

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and $d(\mu) = m$, $d(\nu) = n$.

In k-gragh Λ , for each $\lambda \in \Lambda^{\neq\circ}$, we introduce ghost pathλ^*. for $v \in \Lambda^\circ$, we define $v^* := v$. We define r, s and d on ghost paths by: $d(\lambda^*) = -d(\lambda)$, $r(\lambda^*) = s(\lambda)$, $s(\lambda^*) = r(\lambda)$; Let $v, w \in \Lambda^\circ$, we define $v\Lambda w = \{\lambda \in \Lambda : s(\lambda) = w, r(\lambda) = v\}$ and also for $n \in \mathbb{N}^k$ we define $\Lambda^n = \{\lambda \in \Lambda : d(\lambda) = n\}$. A k-graph Λ is called row-finite if for each $v \in \Lambda^\circ$ and $n \in \mathbb{N}^k$, the set $v\Lambda^n = \{\mu \in \Lambda^n : r(\mu) = v\}$ is finite. Also we say Λ has no sources if $v\Lambda^n \neq \varnothing$ for every $v \in \Lambda^\circ$ and $n \in \mathbb{N}^k$.

Example 1.2. Let $E = (E^\circ, E^1, r_E, s_E)$ be a directed graph. Then, the path category is a 1-graph. In this 1-graph, Λ° is the set of vertices E°. The set of morphisms in this 1-graph are finite paths $E^* = \cup_{n=1}^{\infty} E^n$. Also, the degree map $d : E^* \to \mathbb{N}$ is defined by $d(\mu) =\mid \mu \mid$.

Definition 1.3. Let Λ be a row-finite k-graph with no sources and R be a unital commutative ring. A Kumjian-Pask Λ-family in an R-algebra A, is a set $\{p_v, s_\lambda, s_{\lambda^\star} : v \in \Lambda^\circ, \lambda \in \Lambda\}$ such that the following conditions are satisfied:

1) $\{p_v : v \in \Lambda^\circ\}$ is a family of mutually orthogonal idempotents.

2) $s_\lambda s_\mu = s_{\lambda\mu}$, $s_{\mu^\star} s_{\lambda^\star} = s_{(\lambda\mu)^\star}$, $p_{r(\lambda)} s_\lambda = s_\lambda p_{s(\lambda)} = s_\lambda$, $p_{s(\lambda)} s_{\lambda^\star} = s_{\lambda^\star} p_{r(\lambda)} = s_{\lambda^\star}$ for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu) = s(\lambda)$.

3) $s_{\lambda^\star} s_\mu = \delta_{\lambda,\mu} p_{s(\lambda)}$ for all $\mu, \lambda \in \Lambda^{\neq 0}$ with $d(\lambda) = d(\mu)$.

4) $p_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_{\lambda^\star}$ for all $v \in \Lambda^\circ$ and $n \in \mathbb{N}^k \setminus \{0\}$.

The Kumjian-Pask algebra of Λ with coefficients in R, denoted by $KP_R(\Lambda)$, is the universal R-algebra generated by Kumjian-Pask Λ-family. We write $\{p_v, s_\lambda, s_{\lambda^*}\}$ for the universal Kumjian-Pask Λ-family in $KP_R(\Lambda)$. The universal property of $KP_R(\Lambda)$ means that if A is a R-algebra and $\{q_v, t_\lambda, t_{\lambda^\star} : v \in \Lambda^\circ, \lambda \in \Lambda\}$ is a Kumjian-Pask Λ-family in A, then there exists an R-algebra homomorphism $\pi_{q,t} : KP_R(\Lambda) \to A$ such that $\pi_{q,t}(p_v) = q_v$, $\pi_{q,t}(s_\lambda) = t_\lambda$, $\pi_{q,t}(s_{\lambda^\star}) = t_{\lambda^\star}$. The existence of such universal $KP_R(\Lambda)$ was shown in [2, Theorem 3.4].

Definition 1.4. A ring R is called $\mathbb{Z}^k$-*graded* (or, more concisely, *graded*) if there is a collection of additive subgroups $\{R_n\}_{n\in\mathbb{Z}^k}$ of R such that

1) $R = \bigoplus_{n\in\mathbb{Z}^k} R_n$,

2) $R_j R_n \subseteq R_{j+n}$ for all $j, n \in \mathbb{Z}^k$.

The subgroup R_n is said the *homogeneous component of R of degree n*. If R is graded, an ideal I of R is called *graded* if $I = \bigoplus_{n\in\mathbb{Z}^k}(I \cap R_n)$.

By [2, Theorem 3.4], there is a $\mathbb{Z}^k$-grading on $KP_R(\Lambda)$ satisfying

$$KP_R(\Lambda)_n = span_R\{s_\alpha s_{\beta^*} : \alpha, \beta \in \Lambda, d(\alpha) - d(\beta) = n\}.$$

The hereditary and saturated subsets of Λ° play important roles in the ideal structure of Kumjian-Pask algebras. A subset H of Λ° is hereditary if $\lambda \in \Lambda$ and $r(\lambda) \in H$ imply

$s(\lambda) \in H$. A subset H is saturated if $v \in \Lambda^\circ$, $n \in \mathbb{N}^k$ and $s(v\Lambda^n) \subseteq H$ imply $v \in H$. For a saturated hereditary subset H, we write I_H for the ideal of $KP_R(\Lambda)$ generated by $\{p_v : v \in H\}$. Recall from [2, Lemma 5.1], I_H is a basic and graded ideal in $KP_R(\Lambda)$. Also,for an ideal I of $KP_R(\Lambda)$, we define $H_I = \{v \in \Lambda^\circ : p_v \in I\}$. By [2, Lemma 5.2], H_I is a saturated hereditary subset of Λ°.

Lemma 1.5. *Let Λ be a row-finite k-graph without sources and R be a unital commutative ring. Let I be a nonzero graded ideal of $KP_R(\Lambda)$. Then there exists $rp_v \in I$ for some $v \in \Lambda^\circ$ and $r \in R \setminus \{\circ\}$. Therefore every graded basic ideal I of $KP_R(\Lambda)$ contains a vertex idempotent p_v.*

Proof. Suppose that I is a graded ideal in $KP_R(\Lambda)$. So, $I = \oplus_{n\in\mathbb{Z}^k} I_n$. Take an element $x \in I_n = I \cap KP_R(\Lambda)_n$. Since $x \in KP_R(\Lambda)_n$, we can write $x = \sum_{i=1}^m r_i s_{\alpha_i} s_{\beta_i{}^*}$ such that $d(\alpha_i) - d(\beta_i) = n$. Hence $x = \sum_{i=1}^m r_i p_{v_i} s_{\gamma_i}$ which $\gamma_i = \alpha_i \beta_i{}^*$ and $v_i = r(\alpha_i)$, also $\gamma_i \neq \gamma_j$ for every $i \neq j$. So we have, $s_{\gamma_j{}^*} x = \sum_{i=1}^m r_i s_{\gamma_j{}^*} p_{v_i} s_{\gamma_i} = r_j p_{s(\gamma_j)} \in I$. □

2 The Main results

Recall that an ideal I is called basic, if $rp_v \in I$ implies $p_v \in I$ for $v \in \Lambda^\circ$ and $r \in R\setminus\{\circ\}$. An ideal I of a ring R is prime if for every pair of ideals I_1 and I_2 of R with $I_1 I_2 \subseteq I$ at least one of I_1 and I_2 is contained in I. A ring R is prime if the zero ideal of R is prime. In the graded algebras, a graded ideal I is prime if and only if for any two graded ideals I_1, I_2 with $I_1 I_2 \subseteq I$, we have $I_1 \subseteq I$ or $I_2 \subseteq I$. We will use this fact in the proof of proposition 2.1. A nonempty subset γ of Λ° is said to be satisfy Condition $MT(3)$ if for every $v_1, v_2 \in \gamma$ there exists $w \in \gamma$ such that $v_1\Lambda w \neq \varnothing$ and $v_2\Lambda w \neq \varnothing$.

First, we give some equivalent conditions for the primness of Kumjian-Pask algebras.

Proposition 2.1. *Let Λ be a row-finite k-graph without sources and R be a unital commutative ring. Then the following are equivalent:*

1) *$KP_R(\Lambda)$ is a prime ring.*
2) *R is an ID (Integral Domain) and Λ satisfies Condition $MT(3)$.*
3) *R is an ID and $H_I \cap H_J \neq \emptyset$ for every nonzero graded basic ideals I and J of $KP_R(\Lambda)$.*

Proof. $1 \Rightarrow 2$: Suppose that $KP_R(\Lambda)$ is a prime ring. Let $v, w \in \Lambda^\circ$. Then $KP_R(\Lambda)p_v KP_R(\Lambda)$ and $KP_R(\Lambda)p_w KP_R(\Lambda)$ are two nonzero ideals. So $KP_R(\Lambda)p_w KP_R(\Lambda)p_v KP_R(\Lambda)$ and $p_w KP_R(\Lambda)p_v$ are nonzero. This implies, there are $\alpha, \beta \in \Lambda$ such that $p_w s_\alpha s_{\beta^*} p_v \neq \circ$ and $s(\alpha) = s(\beta) = z$. So $\alpha \in w\Lambda z$ and $\beta \in v\Lambda z$ and this means that Λ satisfies Condition $MT(3)$. Now we show that R is an ID. By contradiction, if there exist nonzero elements $r, s \in R$ such that $r.s = 0$, then $rKP_R(\Lambda)$ and $sKP_R(\Lambda)$ are nonzero ideals in $KP_R(\Lambda)$. But $rKP_R(\Lambda).sKP_R(\Lambda) = rsKP_R(\Lambda) = \{\circ\}$. This is contradiction, because $KP_R(\Lambda)$ is a prime ring.

$2 \Rightarrow 3$: Recall from 1.5 that every nonzero graded basic ideals of $KP_R(\Lambda)$ contains a vertex idempotent p_v. Let I, J be two nonzero graded basic ideals of $KP_R(\Lambda)$. So, there are $v, w \in \Lambda^\circ$ such that $p_v \in I$ and $p_w \in J$. Now by Condition $MT(3)$, there is $z \in \Lambda^\circ$

such that $v\Lambda z \neq \emptyset$ and $w\Lambda z \neq \emptyset$. So $p_z \in I \cap J$ and $H_I \cap H_J \neq \varnothing$.
$3 \Rightarrow 1$: Let R be an ID and $H_I \cap H_J \neq \emptyset$ for every nonzero basic graded ideals I, J of $KP_R(\Lambda)$. We show that the zero ideal of $KP_R(\Lambda)$ is prime. Since the zero ideal is graded, it is sufficient to show $IJ \neq \{\circ\}$ for every nonzero graded ideals I, J of $KP_R(\Lambda)$. If I, J are two nonzero graded ideals, by 1.5 there are $v_1, v_2 \in \Lambda^\circ$ and $r_1, r_2 \in R \setminus \{\circ\}$ such that $r_1 p_{v_1} \in I$ and $r_2 p_{v_2} \in J$. Put $H_i = \{z \in \Lambda^\circ \ : \ v_i \Lambda z \neq \varnothing\}$ for $i \in \{1, 2\}$. It is clear that H_i are two hereditary subsets of Λ°. Let $I_1 = I_{\overline{H_1}}$ and $J_1 = J_{\overline{H_2}}$. Then $p_{v_1} \in I_1$ and $p_{v_2} \in J_1$. Also we have, $r_1 I_1 \subseteq I$ and $r_2 J_1 \subseteq J$. Since I_1 and J_1 are basic, graded ideals of $KP_R(\Lambda)$, by the hypothesis (3), we get $\overline{H_1} \cap \overline{H_2} \neq \varnothing$. Thus, there is $z \in \overline{H_1} \cap \overline{H_2}$ such that $r_1 r_2 p_z \in IJ$. Furtheremore, since R is an ID, we have $r_1 r_2 \neq \circ$. So $\circ \neq r_1 r_2 p_z \in IJ$ and $IJ \neq \{\circ\}$. □

For a saturated hereditary subset H of Λ°, we define a k-graph $\Lambda \setminus H$ as bellow: $(\Lambda \setminus H)^\circ := \Lambda^\circ \setminus H$, $\Lambda \setminus H := \{\mu \in \Lambda : s(\mu) \in \Lambda^\circ \setminus H\}$ such that r, s and d are defined as in k-graph Λ. By [2, Proposition 5.5], there is an isomorphism between $\dfrac{KP_R(\Lambda)}{I_H}$ and $KP(\Lambda \setminus H)$.

Now, we may characterize prime graded basic ideals in the Kumjian-Pask algebras.

Corollary 2.2. *Let Λ be a row-finite k-graph without sources and R be a unital commutative ring. For a saturated, hereditary subset H of Λ°, I_H is a prime ideal of $KP_R(\Lambda)$, if and only if R is an ID and $\Lambda^\circ \setminus H$ satisfies Condition $MT(3)$.*

Proof. Let I_H be a prime ideal. Since $\dfrac{KP_R(\Lambda)}{I_H} \cong KP_R(\Lambda \setminus H)$, $KP_R(\Lambda \setminus H)$ is a prime ring. Therefore, Proposition 2.1 implies that R is an ID and $\Lambda^\circ \setminus H$ satisfies Condition $MT(3)$. Conversely, let $\Lambda^\circ \setminus H$ satisfy Condition $MT(3)$ and R be an ID. So by Proposition 2.1, $KP_R(\Lambda \setminus H)$ is a prime ring. Since $\dfrac{KP_R(\Lambda)}{I_H} \cong KP_R(\Lambda \setminus H)$, we conclude that I_H is a prime ideal of $KP_R(\Lambda)$. □

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Semi Implication Homomorphisms on Lattice Implication Algebras

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Abstract

In this paper, the concept of semi homomorphism of lattice implication algebras is introduced and some properties investigated. The relation between prime filters and semi homomorphisms of lattice implication algebras is obtained. Finally, a condition is obtained which a semi homomorphism is a homomorphism.

Keywords: Lattice implication algebras, Semi implication homomorphism, Prime filter.
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1 Introduction

Lattice implication algebras were first investigate by Y.Xu [3] in 1992. He proposed the concept of lattice and implication algebra to research the logical system whose propositional value is given in a lattice from the semantic view point. Many researchers have investigated this important logic. The properties of lattice implication algebras were studied in [4], [5]. The notion of homomorphism play an important role in study algebraic structure, such as lattices, Hilbert algebras and BCK-algebras [1], [2]. We introduce the notion of a semi implication homomorphism as a generalization of a homomorphism on lattice implication algebras and obtain some results about it.

Definition 1.1. A lattice implication algebra is a structure $L = (L, \vee, \wedge, \rightarrow, ', 0, 1)$ of type $(2, 2, 2, 1, 0, 0)$ such that:[5]
$(L1)$ $L = (L, \vee, \wedge, \rightarrow, ', 0, 1)$ is a bounded lattice with an order reversing involution $'$, 1

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and 0 the greatest and the smallest element of L respectively,
$(L2)$ $x \to (y \to z) = y \to (x \to z)$,
$(L3)$ $x \to x = 1$,
$(L4)$ $x \to y = y' \to x'$,
$(L5)$ $x \to y = y \to x = 1$ implies $x = y$,
$(L6)$ $(x \to y) \to y = (y \to x) \to x$,
$(L7)$ $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
$(L8)$ $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,
for all $x, y, z \in L$.

In the sequel, we will also refer to lattice implication algebra $L = (L, \vee, \wedge, \to, ', 0, 1)$ by its universe L. A lattice implication algebra L is called a lattice H implication algebra, if $(x \vee y \vee ((x \wedge y) \to z) = 1$, for all $x, y, z \in L$ [3].

Definition 1.2. Let J be a subset of a lattice implication algebra L. Then J is called a filter of L, if it satisfies the following conditions:
(1) $1 \in L$,
(2) if $x \in J$ and $x \to y \in J$, then $y \in J$,
for all $x, y \in L$.
The set of all filters of a lattice implication algebra L denoted by $\mathcal{F}(L)$ [5].

Definition 1.3. Let F be a non-empty subset of a lattice implication algebra L. Then F is called a dual ideal of L, if it satisfies the following conditions:[4]
(1) if $x \in F, y \in L, x \leq y$ then $y \in F$,
(2) if $x, y \in F$ then $x \wedge y \in F$.

Definition 1.4. A proper filter P of a lattice implication algebra is called a prime filter if $a \vee b \in P$ implies $a \in P$ or $b \in P$ for all $a, b \in L$.
The set of all prime filters of a lattice implication algebra L denoted by $\mathcal{PF}(L)$ [5].

Definition 1.5. Let L_1 and L_2 be two lattice implication algebras. Then a mapping f from L_1 to L_2 is called an implication homomorphism, if $f(x \to y) = f(x) \to f(y)$, hold for all $x, y \in L_1$ [5].

Proposition 1.6. *Let L_1 and L_2 be two lattice implication algebras and f be an implication homomorphism from L_1 to L_2. Then* [3]
(1) $f(x \vee y) = f(x) \vee f(y)$, *for all* $x, y \in L_1$,
(2) $f(1) = 1$.

2 Semi Implication Homomorphism

Definition 2.1. Let L_1 and L_2 be two lattice implication algebras. A mapping f from L_1 to L_2 is called a semi implication homomorphism if it satisfies:
(1) $f(1) = 1$,
(2) $f(x \to y) \leq f(x) \to f(y)$, for all $x, y \in L_1$.
It is clear that every implication homomorphism of lattice implication algebras is a semi implication homomorphism but the converse may not be true in general. See the following example.

Example 2.2. Let $L_1 = \{0, a, b, 1\}$ with $0 \leq a, b \leq 1$ where a, b are incomparable and let $L_2 = \{0, c, d, e, 1\}$ with $0 \leq c \leq d \leq e \leq 1$. In L_1, we define $'$ as $0' = 1, a' = b, b' = a$ and $1' = 0$ and in L_2, we define $'$ as $0' = 1, c' = e, d' = d, e' = c$ and $1' = 0$. Define implication operator $\to$ for L_1 as Table $1(a)$ and implication operator $\to$ for L_2 as Table $1(b)$. Then L_1 and L_2 are lattice implication algebras. Define $f : L_1 \to L_2$ by $f(0) = 0, f(a) = f(b) = c$ and $f(1) = 1$. Then f is a semi implication homomorphism but it is not an implication homomorphism, because $c = f(a \to b) = f(b) \leq f(a) \to f(b) = 1$.

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

(a) L_1

$\to$	0	c	d	e	1
0	1	1	1	1	1
c	e	1	1	1	1
d	d	d	1	1	1
e	c	d	d	1	1
1	0	c	d	e	1

(b) L_2

Table 1: Table for implication operator.

Definition 2.3. Let L be a lattice implication algebra and $x_0 \in L$. Define $\theta_l, \theta_r : L \to L$ as follows: for any $x \in L$:

$$\theta_l(x) = x_0 \to x,\ \theta_r(x) = x \to x_0,$$

θ_l is called left-mapping and θ_r is called right-mapping.

Proposition 2.4. *Let L be a lattice implication algebra and $x_0 \in L$. Then θ_r is a semi implication homomorphism.*

Remark 2.5. In general θ_l is not a semi implication homomorphism. Suppose $L = L_2$ and $x_0 = c$ in Example 2.2. Then $\theta_l(d \to 0) = c \to (d \to 0) = c \to d = 1$, but $\theta_l(d) \to \theta_l(0) = (c \to d) \to (c \to 0) = 1 \to e = e$. Hence θ_l is not a semi implication homomorphism.

In the following theorem, we will obtain the conditions that θ_l is a semi implication homomorphism.

Theorem 2.6. *Let L be a lattice implication algebra and $x_0 \in L$. Then the following conditions are equivalent:*
(1) *L is a lattice H implication algebra,*
(2) *θ_l is a semi implication homomorphism,*
(3) *θ_l is an implication homomorphism.*

Proposition 2.7. *Let $f : L_1 \to L_2$ be a semi implication homomorphism of lattice implication algebras. Then for all $x, y \in L_1$:*
(1) *if $x \leq y$, then $f(x) \leq f(y)$,*
(2) *if $f(0) = 0$, then $f(x') \leq (f(x))'$,*
(3) *$f(x \wedge y) \leq f(x) \wedge f(y)$,*
(4) *$f(x) \vee f(y) \leq f(x \vee y)$.*

Remark 2.8. In the Proposition 2.7 may not establish equality for parts $2, 3$ and 4 in general. Consider Example 2.2. We have
$c = f(a') \lneqq (f(a))' = d$,
$0 = f(a \wedge b) \lneqq f(a) \wedge f(b) = c$,
$c = f(a) \vee f(b) \lneqq f(a \vee b) = 1$.

Proposition 2.9. *Let L_1 and L_2 be two lattice implication algebras. If $f : L_1 \to L_2$ is a semi implication homomorphism, then* $\ker(f) = \{x \in L_1 | f(x) = 1\}$ *is a filter of L_1.*

Proposition 2.10. *Let L_1 and L_2 be two lattice implication algebras and $f : L_1 \to L_2$ be a semi implication homomorphism. If f is one to one, then* $\ker(f) = \{1\}$.

Remark 2.11. The converse of Proposition 2.10 may not be true in general. In Example 2.2, we have $\ker(f) = \{1\}$ but f is not one to one, because $a \neq b$ but $f(a) = f(b) = c$.

Theorem 2.12. *Let L_1 and L_2 be two lattice implication algebras and $f : L_1 \to L_2$ be a map. Then the following conditions are equivalent:*
(1) *f is a semi implication homomorphism,*
(2) *$f^{-1}(Y) \in \mathcal{F}(L_1)$, for all $Y \in \mathcal{F}(L_2)$.*

Proposition 2.13. *Let L_1 and L_2 be two lattice implication algebras and $f : L_1 \to L_2$ be a implication homomorphism. Define $\{f(L_1/X)\} \downarrow = \{y \in L_2 | f(x) \geq y$, for some $x \in L_1/X\}$, for any $X \in \mathcal{PF}(L_1)$. Then $\{f(L_1/X)\} \downarrow$ is a lattice ideal of L_2.*

Theorem 2.14. *Let L_1 and L_2 be two lattice implication algebras and $f : L_1 \to L_2$ be a semi implication homomorphism. Then the following conditions are equivalent:*
(1) *f is an implication homomorphism,*
(2) *$If X \in \mathcal{PF}(L_1)$ and $Y \in \mathcal{PF}(L_2)$ such that $f^{-1}(Y) \subseteq X$, then there exists $Z \in \mathcal{PF}(L_2)$ such that $Y \subseteq Z$ and $f^{-1}(Z) = X$.*

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Semi-maximal ·-ideals in Product MV-algebras

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Abstract

In this paper, we introduce the notion of the radical of ·-ideal of PMV-algebras. We have also presented several different characterizations and many important properties of the radical of a ·-ideal in a PMV-algebra. This leads us to introduce the notion of semi-maximal ·-ideal. Finally, we show that I is a semi-maximal ·-ideal of A if and only if A/I has no nilpotent elements of A.

Keywords: PMV-algebra, ·-ideal, radical
Mathematics Subject Classification [2010]: $06D35$, $06B10$

1 Introduction

A. Dvurecenskij and A. Di Nola in [3] introduced the notion of PMV-algebras, that is MV-algebras whose product operation $(\cdot)$ is defined on the whole MV-algebra. This operation is associative and left/right distributive with respect to partially defined addition. They showed that the category of product MV-algebras is categorically equivalent to the category of associative unital l-rings. In addition, they introduced and studied MVF-algebras [3]. They also introduced ·-ideals in PMV-algebras. Then they showed that: Any MVF-algebra is a subdirect product of subdirectly irreducible MVF-algebras [3, Corollary 5.6]. Thus they concluded that a product MV-algebra is an MVF-ring if and only if it is a subdirect product of linearly ordered product MV-algebras [3, Theorem 5.8].

In this paper, we introduce the notion of the radical of a ·-ideal in PMV-algebras. Several characterizations of this radical is given. We define the notion of a semimaximal ·-ideal in a PMV-algebra. Finally we show that A/I has no nilpotent elements if and only if I is a semi-maximal ·-ideal of A.

2 Preliminaries

In this section, we summarize properties of the basic notions in MV-algebras and PMV-algebras. For more details about these concepts, we refer the readers to [1, 3, 2].

Definition 2.1. [1] An MV-algebra is a structure $(M, \oplus, *, 0)$, where $\oplus$ is a binary operation, * is a unary operation, and 0 is a constant satisfying the following conditions, for any $a, b \in M$:
$(MV1)$ $(M, \oplus, 0)$ is an abelian monoid, $(MV2)$ $(a^*)^* = a$, $(MV3)$ $0^* \oplus a = 0^*$, $(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Definition 2.2. [1] An ideal of an MV-algebra A is a nonempty subset I of A if it satisfies the following conditions:
$(I1)$ If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$, $(I2)$ If $x, y \in I$, then $x \oplus y \in I$. We denote the set of all ideals of an MV-algebra A by $Id(A)$.

Definition 2.3. [5] Let I be a proper ideal of A. The intersection of all maximal ideals of A which contain I is called the radical of I and it is denoted by $Rad(I)$.

Theorem 2.4. *[5] Let I be a proper ideal of A. Then $Rad(I) = \{a \in A : na \odot a \in I, \quad for\ all \quad n \in \mathbb{N}\}$.*

The categorical equivalence between MV-algebras and lu-groups leads also to the problem of defining a product operation on MV-algebras, in order to obtain structures corresponding to l-rings. We recall that an l-ring [**?**] is a structure $(R, +, \cdot, 0, \leq)$, where $(R, +, 0, \leq)$ is an l-group such that, for any $x, y \in R$ $x \geq 0$ and $y \geq 0$ implies $x \cdot y \geq 0$.

Definition 2.5. [3] A product MV-algebra (or PMV-algebra, for short) is a structure $(A, \oplus, *, \cdot, 0)$, where $(A, \oplus, *, 0)$ is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied:if $x + y$ is defined, then $x \cdot z + y \cdot z$ and $z \cdot x + z \cdot y$ are defined and $(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y$, where $+$ is a partial addition on A, as follows: for any $x, y \in A, \quad x + y$ is defined if and only if $x \leq y^*$ and in this case, $x + y := x \oplus y$. If A is a PMV-algebra, then a unity for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$ for any $x \in A$. A PMV-algebra that has unity for the product is called unital.A ·-ideal of a PMV-algebra A is an ideal I of MV-algebra A such that if $a \in I$ and $b \in A$ entail $a \cdot b \in I$ and $b \cdot a \in I$. We denote by $Id_p(A)$ the set of ·-ideals of a PMV-algebra A.

In the sequel, an lu-ring will be a pair (R, u) where $(R, \oplus, \cdot, \leq)$ is an l-ring and u is a strong unit of R such that $u \cdot u \leq u$. The last conditions imply that the interval $[0, u]$ of an lu-ring (R, u) is closed under the product of R. Thus, if we consider the restriction of $\cdot$ to $[0, u] \times [0, u]$, then the interval $[0, u]$ has a canonical PMV-algebra structure: $x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \quad x \cdot y := x \cdot y$, for any $0 \leq x, y \leq u$. We shall denote this structure $[0, u]_R$.

If $\mathcal{UR}$ is the category of lu-rings, whose objects are pairs (R, u) as above and whose morphisms are l-rings homomorphisms which preserve the strong unit, then we get a functor $\Gamma : \mathcal{UR} \to \mathcal{PMV}$, $\Gamma(R, u) := [0, u]_R$, for any lu-ring (R, u), $\Gamma(h) := h\ |_{[0,u]}$ for any lu-rings homomorphism h. In [3] is proved that Γ establishes a categorical equivalence between $\mathcal{UR}$ and $\mathcal{PMV}$.

Definition 2.6. [4] Let P be a ·-ideal of A. P is called a ·-prime if (i) $P \neq A$, (ii) for every $a, b \in A$, if $a \cdot b \in P$, then $a \in P$ or $b \in P$.

Definition 2.7. [2] An element a in MV-algebra A is said to be infinitesimal if and only if $a \neq 0$ and $na \leq a^*$ for each integer $n \geq 0$. The set of all infinitesimals in A will be denoted by $Inf(A)$.

3 Main results

From now on $(A, \oplus, *, 0)$ (or simply A) is a PMV-algebra.

Definition 3.1. Let I be a proper ·-ideal of A. The intersection of all ·-prime ideals of A which contain I is called the radical of I and it is denoted by $Rad(I)$. If there are not ·-prime ideals of A containing I, then $Rad(I) = A$.

Example 3.2. Let $\Omega = \{1, 2\}$ and $\mathcal{A} = \mathcal{P}(\Omega)$. Which is a PMV-algebra with $\oplus = \cup$ and $\odot = \cdot = \cap$. Obviously, $P_1 = \{\emptyset, \{1\}\}$ and $P_2 = \{\emptyset, \{2\}\}$ are ·-prime ideals of A. Hence $Rad(P_1) = P_1$ and $Rad\{\emptyset, \{2\}\} = P_2$ and $Rad\{\emptyset\} = \{\emptyset, \{1\}\} \cap \{\emptyset, \{2\}\} = \{\emptyset\}$.

Example 3.3. Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and 0 be the matrix with all element 0. If we define the order relation on components $A = (a_{ij})_{i,j=1,2} \geq 0$ iff $a_{ij} \geq 0$ for any i, j, such that $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, then $A = \Gamma(M_2(\mathbb{R}), v) = [0, v]$ is a PMV-algebra. Obviously, $Id(A) = \{\{0\}, A\}$. Hence we show that $P = \{0\}$ is not a ·-prime ideal of A [4]. Thus $Rad\{0\} = A$.

Lemma 3.4. *Let I be ·-ideal of A. Then $Rad(I) = \{a \in A : a^n = a \cdot a \cdot \ldots \cdot a \in I$, for some $n \in \mathbb{N}\}$.*

We recall that $x \in I \to J$ if and only if $(x] \cap I \subseteq J$, for ideals I and J of A [2].

Theorem 3.5. *Let I and J be proper ·-ideals of A and $a, b \in A$. Then the following condition hold:*
(1) *If $x \in B(A)$, for any $x \in A$, then $a \oplus b \in I$,* (2) *If $I \subseteq J$, then $Rad(I) \subseteq Rad(J)$,* (3) *If A is a unital PMV-algebra, then $Rad(I) = A$ iff $I = A$,* (4) *$Rad(Rad(I)) = Rad(I)$,* (5) *$Rad(I) \cup Rad(J) \subseteq Rad(I \cup J]$,* (6) *$Rad(I) \to Rad(J) \subseteq I \to Rad(J)$,* (7) *$Rad(I \to J) \subseteq Rad(I \to Rad(J))$,* (8) *If for every $a \in I$ there exists $k \in \mathbb{N}$ such that $ka \in J$, then $Rad(I) \subseteq Rad(J)$.*

Theorem 3.6. *Let $\{I_i\}_{i \in I}$ be a family of proper ·-ideals of A. Then $Rad(\cap_{i \in I} I_i) = \cap_{i \in I} Rad(I_i)$.*

Proposition 3.7. *Let $f : A \to B$ be a PMV-homomorphism. Then $Rad(Ker(f)) = f^{-1}(Rad(\{0\}))$.*

Definition 3.8. The set of nilpotent elements of a PMV-algebra A is $Nil(A) = \{x \in A : x^n = x \cdot \cdots \cdot x = 0,$ for some $n \geq 1\}$.

Theorem 3.9. *Let I be a ·-ideal of a PMV-algebra A. Then $Nil(A) \subseteq Rad(I)$.*

Remark 3.10. If I is a ·-ideal of A, then $a \in Rad(I)$ if and only if $a/I \in Nil(A/I)$.

Definition 3.11. Let I be a proper ideal of A. If $Rad(I) = I$, then I is called a semi-maximal ·-ideal of A. By Theorem 3.4, a ·-ideal I of A is a semi-maximal if and only if $I = \{a \in A : a^n \in I \quad \text{for some} \quad n \in \mathbb{N}\}$.

Example 3.12. In Example 3.2, we have $Rad(P_1) = P_1$, hence P_1 is a semi-maximal ·-ideal.

Proposition 3.13. *Let A, B be PMV-algebras and $f : A \to B$ be a PMV-homomorphism. Then the following statements hold: (a) If I is a semi-maximal ·-ideal of B, then $f^{-1}(I)$ is a semi-maximal ·-ideal of A, (b) If f is onto, I is a semi-maximal ·-ideal of A and $Ker(f) \subseteq I$, then $f(I)$ is a semi-maximal ·-ideal of B.*

Theorem 3.14. *Let I be a ·-ideal of A. Then the following statements hold:* (1) *$Rad(I)$ is the smallest semi-maximal ·-ideal of A such that $I \subseteq Rad(I)$,* (2) *$Rad(I)/I$ is a semi-maximal ·-ideal of A/I.*

Corollary 3.15. *Let $\{I_i\}_{i\in I}$ be a family of semi-maximal ·-ideals of A. Then $\bigcap_{i\in I} I_i$ a semi-maximal ·-ideal of A.*

Theorem 3.16. *If A is a PMV-algebra and I is a ·-ideal of A, then A/I has no nilpotent elements if and only if I is a semi-maximal ·-ideal of A.*

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Sequential Noetherian Topological Space of Rings

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Abstract

In this paper R is a commutative ring with identity. We prove that if the lattice of ideals R is a principal lattice, then the prime spectrum of R is a sequential Noetherian topological space.

Keywords: Sequential Noetherian topological space, s-compact, Principal lattice.
Mathematics Subject Classification [2010]: 54D55, 06F05, 13A15.

1 Introduction

Throughout this paper $\mathcal{L}$ will denote a K-lattice whose maximal element is compact and $\mathcal{M}$ will be an $\mathcal{L}$-module which is also a K-lattice. A subset A of a topological space X is called sequentially closed if it has the following property: if a sequence in A converges in X to a point x, then $x \in A$. A subset E of a topological space X is called sequentially open if $X \backslash E$ is sequentially closed. A topological space (X, τ) is called s-compact if (X, τ_s) is compact, or equivalently, every sequentially open cover of X has a finite subcover. By a multiplicative lattice, we mean a complete lattice $\mathcal{L}$, with least element $0 = 0_{\mathcal{L}}$ and compact greatest element $\mathbf{R}$, on which there is defined a commutative, associative, completely join distributive product for which $\mathbf{R}$ is a multiplicative identity. Multiplicative lattices have been studied extensively by E. W. Johnson and C. Jayaram (see [3] and [4]). In this article, relationships among the multiplicative lattices with ACC on radical elements, s-compact, and sequential Noetherian topological space are considered. Also, we suppose that R is a commutative ring with identity and $L(R)$ is the lattice of ideals R. It is proved that if $L(R)$ is a principal lattice, then the prime spectrum $\mathrm{Spec}(R)$ is a sequential Noetherian topological space.

2 Sequential Noetherian topological space and Radical elements

A left lattice module over $\mathcal{L}$, or simply an $\mathcal{L}$-module, is a complete lattice $\mathcal{M}$, together with a multiplication $\mathcal{L} \times \mathcal{M} \longrightarrow \mathcal{M}$ satisfying the following for $a, b \in \mathcal{L}$, $\{a_\lambda | \lambda \in \Lambda\} \subseteq \mathcal{M}$, $A \in \mathcal{M}$, $\{B\gamma | \gamma \in \Gamma\} \subseteq \mathcal{M}$:
(i) $(ab)A = a(bA)$;

*Speaker

(ii) $(\bigvee_\lambda a_\lambda)(\bigvee_\gamma B_\gamma) = \bigwedge_{\lambda,\gamma} a_\lambda B_\gamma$;
(iii) $\mathbf{R}A = A$;
(iv) $0_\mathcal{L} A = 0_\mathcal{M}$, where $0_\mathcal{M}$ is the smallest of $\mathcal{M}$.

An element $A \in \mathcal{M}$ is said to be compact if whenever $A \leq \bigvee\{B_\lambda | \lambda \in \Lambda\}$ for some family $\{B_\lambda | \lambda \in \Lambda\}$ of members of $\mathcal{M}$, then there is a finite subset $\Gamma \subseteq \Lambda$ such that $A \leq \bigvee\{B_\lambda | \lambda \in \Gamma\}$. If each element of $\mathcal{M}$ the join of a family of compact elements of $\mathcal{M}$, then $\mathcal{M}$ is said to be compactly generated or a CG-module. An $\mathcal{L}$-module $\mathcal{M}$ is called a K-lattice if it is a CG-module and AH is compact for each compact element $A \in \mathcal{L}$ and each compact element $H \in \mathcal{M}$.

We follow the terminology of [8], [1]. We say that $\mathcal{L}$ is a CG-lattice or K-lattice respectively if this holds when $\mathcal{L}$ is considered as an $\mathcal{L}$-module. In this paper, will deal with various applications of lattice concepts to general topology-i.e., to the general theory of topological spaces. The ideas of general topology can be most simply introduced through the concept of a metric space. For results on rings with ACC on radical ideals see ([9], [10], [6]). Let Spec($\mathcal{L}$) denote the set of prime elements of $\mathcal{L}$, which we give the Zariski topology. That is, the closed sets are the sets of the from $V(A) = \{P \in \text{Spec}(\mathcal{L}) \,|\, A \leq P\}$ with $A \in \mathcal{L}$. If $\mathcal{P} \subseteq \text{Spec}(\mathcal{L})$, we always give $\mathcal{P}$ the relative topology induced from the Zariski topology on Spec($\mathcal{L}$). Recall that a topological space $\mathcal{P}$ is said to be Noetherian if $\mathcal{P}$ satisfies the descending chain condition on closed sets. If $A \in \mathcal{L}$, we define the $\mathcal{P}$-radical of A to be $\mathcal{P}\text{-rad}(A) = \wedge\{P \in \mathcal{P} \,|\, A \leq P\}$, and call A a $\mathcal{P}$-radical element if $A = \mathcal{P}\text{-rad}(A)$. If $\mathcal{P} =$Spec($\mathcal{L}$), we will omit the $\mathcal{P}$. It follows that a subset $\mathcal{P}$ of Spec ($\mathcal{L}$) is Noetherian if and only if $\mathcal{L}$ has ACC on the $\mathcal{P}$-radical elements of $\mathcal{L}$. If $\mathcal{P} \subseteq \text{Spec}(\mathcal{L})$, we say that an element A of $\mathcal{L}$ is $\mathcal{P}$-radically finite if there exists a compact element $F \leq A$ such that $\mathcal{P}\text{-rad}(F) = \mathcal{P}\text{-rad}(A)$. If $\mathcal{P} =$Spec($\mathcal{L}$) we say radically finite for $\mathcal{P}$-radically finite.

Theorem 2.1. *If $\mathcal{P} \subseteq$ Spec($\mathcal{L}$), then each $A \in \mathcal{L}$ is $\mathcal{P}$-radically finite if and only if $\mathcal{P}$ is a Noetherian topological space.*

Proof. ($\Longleftarrow$) Suppose there exists an $H \in \mathcal{L}$ that is not $\mathcal{P}$-radically finite. Let $h_1 \leq H$ be compact. Then $H \nleq \mathcal{P}\text{-rad}(h_1)$. Let $h_2 \leq H$ be compact with $h_2 \nleq \text{rad}(h_1)$. We have $\mathcal{P}-\text{rad}(h_1) < \mathcal{P}\text{-rad}(h_1 \vee h_2)$ and $H \nleq \mathcal{P}-\text{rad}(h_1 \vee h_2)$, and so on, a contradiction to $\mathcal{P}$ being Noetherian.

($\Longrightarrow$) To show that $\mathcal{P}$ is Noetherian, let $H_1 \leq H_2 \leq \ldots$ be a chain of $\mathcal{P}$-radical elements. Let $H = \vee_{i=1}^{\infty} H_i$. Since H is $\mathcal{P}$-radically finite, there exists a compact element h$\leq H$ such that $\mathcal{P}\text{-rad}(H) = \mathcal{P}\text{-rad}(h)$. Then h$\leq H_j$ for some j and $\mathcal{P}\text{-rad}(h) \leq H_j \leq \mathcal{P}\text{-rad}(H) = \mathcal{P}\text{-rad}(h)$. Therefore $H_j = H_k$ for all $k \geq j$. □

Definition 2.2. The maximal irreducible subsets of X are called the irreducible components of X.

Example. The irreducible components of the topological space with the trivial topology is X itself. The irreducible components of the topological space X with the discrete

topology are the points of X. The topological space X with the finite complement topology is irreducible exactly when X consists of infinitely many points, or consists of one point.

The following theorem is one of the main results on principal element of a lattice. Dilworth overcame this in [2], with a new notion of a principal element. Basically, an element E of a multiplicative lattice L, is said to be meet-(join-)principal if $(A \wedge (B : E))E = (AE) \wedge B$ (if $(BE \vee A) : E = B \vee (A : E)$) for all A and B in L. A principal element is an element that is both meet-principal and join-principal or $A \wedge E = (A : E)E$ and $AE : E = A \vee (0 : E)$, for all $A \in L$. A lattice L, is called a principal lattice, when each of its elements is principal. Here, the residual quotient of two elements A and B is denoted by $A : B$, so $A : B = \vee\{X \in L \,|\, XB \leq A\}$. Now, we can prove the following theorem.

Theorem 2.3. *Let R be a commutative ring with identity. If $L(R)$ is a principal lattice, then the prime spectrum Spec(R) is a sequential Noetherian topological space.*

Proof. We know, if $L(R)$ is a principal lattice, then R is Noetherian multiplication ring and Spec(R) is Noetherian topological space (see [7] and [5]) . Also, let X be a Noetherian topological space in which every irreducible closed subset F has a generic point. Then the space(X) is sequential if and only if $h(X) \leq \omega_1$. By using Theorem 2.2, Spec(R) is a sequential Noetherian topological space. □

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Poster Some properties on non-abelian tensor product of groups pp.: 1–4

Some properties on non-abelian tensor product of groups

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Abstract

we prove that the non-abelian tensor product $G \otimes H$ is locally nilpotent or locally solvable if such information is given in terms of $D_H(G)$, the derivative subgroup of G afforded by the action of H on G. This derivative subgroup reduce to the commutator subgroup G' of G if $G = H$ and the actions are conjugation. Also we present a survey of results into the tensor analogues of 3-Engel groups. Finally, we present some results about subgroup which is generalization of the tensor analogue of right 2-Engel elements of a group.

Keywords: Nonabelian tensor product, Engel elements of a group, locally nilpotent.
Mathematics Subject Classification [2010]: 20F19, 20J99, 20F45.

1 Introduction

For any group G, the *the nonabelian tensor square* is a group generated by the symbols $g \otimes h$, subject to the relations,

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h) \text{ and } g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'})$$

where $g, g', h, h' \in G$ and $g^h = h^{-1}gh$.
The nonabelian tensor square is a special case of the nonabelian tensor product which has its origins in homotopy theory. It was introduced by Brown and Loday in [3] and [4], extending ideas of Whitehead in [10]. In [2], Brown, Johnson, and Robertson start the investigation of nonabelian tensor squares as group theoretical objects. If $G = H$ and all actions are given by conjugation, then $G \otimes G$ is called the non-abelian tensor square. One notes that the non-abelian tensor square of a given group always exists.

Definition 1.1. Let G and H be groups with H acting on G. Then the subgroup

$$D_H(G) = \langle g^{-1}g^h \mid g \in G, h \in H \rangle$$

of G is the *derivative of G by H*. The derivative H by G, $D_G(H)$, is defined as similar. Taking $G = H$, and all actions to be conjugation, the derivative subgroup reduce to the commutator subgroup G' of G.

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Visscher [9] showed that if $D_H(G)$ is nilpotent(or solvable), then $G \otimes H$ is nilpotent (or solvable). We prove this result for locally nilpotent (or solvable) groups. Recall *set of right n-Engel elements* of a group G is defined by $R_n(G) = \{a \in G : [a,_n x] = 1, \forall x \in G\}$. (Here $[a, x]$ denotes $a^{-1}x^{-1}ax$, and $[a,_n x]$ denotes the left-normed commutator $[[...[a, x], ...], x]$.) Tensor analogues of $R_n(G)$ can be easily defined as

$$R_n^{\otimes}(G) = \{a \in G : [a,_{n-1} x] \otimes x = 1_{\otimes} \ \forall x \in G\},$$

is called *the set of right $n_{\otimes}$-Engel elements* of a group G. Biddle and Kappe in [1] studied the tensor analogues of the centralizers and introduced the tensor analogues of right 2-Engel elements in groups. They proved that $R_2^{\otimes}(G)$ is a characteristic subgroup of G containing $Z(G)$ and being contained in $R_2(G)$. The properties of the set of all these elements $R_2^{\otimes}(G) = \{a \in G : [a, x] \otimes x = 1_{\otimes}, \forall x \in G\}$ have been studied by Moravec [7]. Moreover he described the structure of $2_{\otimes}$-Engel groups, i.e. groups satisfying the identity $[x, y] \otimes y = 1_{\otimes}$. Kappe and Kappe [6] proved that a group G is a 3-Engel group if and only if the normal closure of every element in G is 2-Engel group. In [8] Nasrabadi and the second auther proved tensor analogue of this result.
Let β be group-theoretic property. A group G is said to have a finite covering by β-subgroups if G be union of finite family of β-subgroups. Moravec [7] proved that a group G has a finite covering by $2_{\otimes}$-Engel groups if and only if $|G : R_2^{\otimes}(G)|$ is finite. Another result of [5] in this direction is that G has a finite covering by 2-Engel normal subgroups if and only if G is a 3-Engel group and $|G : R_2(G)| < \infty$. In [8] Nasrabadi and the second auther proved the situation similar in the context of $3_{\otimes}$-Engel groups.

2 Main results

Lemma 2.1. *([4]) (i) There exist homomorphisms $\lambda : G \otimes H \to G, \mu : G \otimes H \to H$ such that $\lambda(g \otimes h) = g^{-1}g^h, \mu(g \otimes h) = h^{-g}h$.*
(ii) $ker\lambda$ and $ker\mu$ are central subgroups of $G \otimes H$.

Lemma 2.2. *([9]) Let G and H be groups which act on each other in a compatible way. If $t_1, t_2, ..., t_n \in G \otimes H$, then*

$$[t_1, t_2, ..., t_n] = [\lambda(t_1), \lambda(t_2), ..., \lambda(t_{n-1})] \otimes \mu(t_n).$$

Visscher [9] showed that if $D_H(G)$ is nilpotent(or solvable), then $G \otimes H$ is nilpotent (or solvable). We prove this result for locally nilpotent (or solvable) groups.

Theorem 2.3. *Let G and H be groups which act on each other in a compatible way.*
(i) If $D_H(G)$ is locally nilpotent, then so is $G \otimes H$.
(ii) If $D_H(G)$ is locally solvable, then so is $G \otimes H$.

The following corollary follows from the above theorem.

Corollary 2.4. *If G' is locally nilpotent (or solvable), then so is $G \otimes G$.*

Kappe and Kappe [6] proved that a group G is a 3-Engel group if and only if the normal closure of every element in G is a 2-Engel group. The following Theorem is a tensor analogue of this result.

Theorem 2.5. *A group G is a $3_\otimes$-Engel group if and only if the normal closure of every element in G is a $2_\otimes$-Engel group.*

Kappe [5] proved a group G is a 3-Engel group if and only if $\langle x^G\rangle R_2(G)$ is a 2-Engel group for all $x \in G$. Now we have

Theorem 2.6. *A group G is a $3_\otimes$-Engel group if and only if $\langle x^G\rangle R_2{}^\otimes(G)$ is a $2_\otimes$-Engel group for all $x \in G$.*

Also Kappe [5] proved that a group has a finite covering by 2-Engel normal subgroups if and only if $|G : R_2(G)| < \infty$ and G is a 3-Engel group. Now we have

Theorem 2.7. *A group G has a finite covering by $2_\otimes$-Engel normal subgroups if and only if G is a $3_\otimes$-Engel group and $|G : R_2^\otimes(G)| < \infty$.*

Now we introduce a subgroup of G that is generalization of $R_2{}^\otimes(G)$.

Definition 2.8. Let G be a group. We define

$$B^\otimes(G) = \{a \in G | [a, g, x] \otimes g = 1_\otimes \ \ \forall g, x \in G\}.$$

Theorem 2.9. *Let G be a group. Then we have*
i) $B^\otimes(G)$ is a subgroup of G.
ii) $R_2^\otimes(G) \subseteq B^\otimes(G)$.
iii) $[a, g, x, h] \otimes g = 1_\otimes$, for $a \in B^\otimes(G)$ and $g, x, h \in G$.
iv) $([a, g, x, b, c] \otimes h) = 1_\otimes$, for $a \in B^\otimes(G)$ and $g, x, h \in G$.

Theorem 2.10. *Every group G of the variety with the law*

$$[x, g, y] \otimes g = 1_\otimes,$$

for all $x, g, y \in G$ is nilpotent of class at most 3.

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β_3 near - ring

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Abstract

In this paper, we introduce β_3 near - rings and give some examples. By some examples and theorems, we find relations between β_3 near - rings, β_1 near - rings and strong B_1 near - rings. Finally, we show that every β_3 near - ring N is isomorphic to a subdirect product of subdirectly irreducible β_3 near - rings.

Keywords: β_1 near - ring, β_3 near - ring, strong B_1 near - ring, mate function
Mathematics Subject Classification [2010]: 16Y30

1 Introduction and preliminaries

near - rings were introduce by Dickson in 1905 and we proved some theorems. Then B_1 near - rings and strong B_1 near - ring were defined by S.Silviya, and etal [4]. After that U.Sugantha and R.Balakrishnan defined β_1 near - rings and investigated the relation between these notions and (strong) B_1 near - rings [5].
In this paper, we defined β_3 near - rings and find some relations between β_3 near - rings, β_1 near - rings and strong B_1 near - rings.
At first we recall the definition a near - ring.

Definition 1.1. [3] A near - ring is a non - empty set N together with two binary operations "+" and "." such that
(a) (N , +) is a group (not necessarily ablian),
(b) (N , .) is a semigroup,
(c) $\forall n_1$,n_2 $n_3 \in N$: $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ (" right distributive law ")
Obviously $0n = 0$ for all $n \in N$. If, in addition, $n0 = 0$ for all $n \in N$, we say that N is zero symmetric.
In a near - ring N, and $\phi \neq S \subseteq N$, we denote:

$$L = \{a \in N \mid \exists n \in \mathbb{N} \; s.t \; a^n = 0\}$$

$$E = \{a \in N \mid a^2 = a\}$$

$$C(S) = \{n \in N \mid nx = xn; \forall x \in N\}$$

*Speaker

Example 1.2. [4] Consider $(Z_4\ ,+\ ,.\ )$ where $(Z_4\ ,+\ )$ is the group of integers modulo "4" and "." defined as follows

.	0	1	2	3
0	0	0	0	0
1	0	0	1	0
2	0	0	3	0
3	0	0	2	0

Then $(Z_4\ ,+\ ,.)$ is a near - ring.
In the sequal, a near - ring is denoted by N.

Definition 1.3. [3] (i) A subgroup $(M\ ,+\ )$ of $(\ N\ ,+\ )$ is called a subnear - ring of N if $MM \subseteq M$,
(ii) A subgroup $(M\ ,+\ )$ of $(\ N\ ,+\ )$ is called a N - subgroup of N if $NM \subseteq M$ and an invariant N-subgroup of N if $MN \subseteq M$.

Definition 1.4. [3] A normal subgroup I of $(\ N\ ,+\ )$ is called an ideal of N $(I \trianglelefteq N)$ if
(a) $IN \subseteq I$,
(b) $\forall\ n,\ n^{'} \in N, \forall\ i \in I$: $n(n^{'}+i)-nn^{'} \in I$.

Definition 1.5. [1, 4, 5] A near - ring N is called:
(i) a β_1 near - ring if $aNb = Nab$, for all $a, b \in N$,
(ii) a β_2 near - ring, if $aNb = abN$, for all $a, b \in N$,
(iii) a strong B_1 near - ring, if $Nab = Nba$ for all $a, b \in N$,
(iv) an integral, if N has no non - zero zerodivisors.

Definition 1.6. [1]A map f:$N \to N$ is called a mate function for N, if $x = xf(x)x$ for all $x \in N$.

Definition 1.7. [3] An element $a \in N$ is said to be nilpotent if for some positive integer k, $a^k = 0$. N is called nil if every element of N is nilpotent.

Theorem 1.8. [3] Let N be zero symmetric. Then the following statement are equivalent:
(i) N has no non - zero nilpotent elements,
(ii) N is a subdirect product of integral near - rings.

Definition 1.9. [3] A near - ring N is said to fulfill the insertion of factors - property (brively: IFP) provided that, $\forall$ a,b, $n \in N$: $ab = 0 \Rightarrow anb = 0$.

Definition 1.10. [3] If N has IFP and for all $x, y \in N$, $xy = 0$ implyes $yx = 0$, then we say that N has (* , IFP).

Lemma 1.11. [1] Let N be a zero symmetric near - ring with a mate function " m ". Then N has (* , IFP) if and only if L $=\{0\}$

Definition 1.12. [3] An ideal I of N is called a completely semiprime ideal if for $a \in N$, $a^2 \in I \Rightarrow a \in I$

Theorem 1.13. [3]A near - ring N hasno non-zero nilpotent element if and only if for all $x \in N$, $x^2 = 0$ implies $x = 0$.

Lemma 1.14. [5] Let " f " be a mate function for N. Then for every $x \in N$, *$xf(x)$, $f(x)x \in E$ and $Nx = N\,f(x)x$, $xN = xf(x)N$.*

Definition 1.15. [3] A near - ring N is said to be regular if for every a ∈ N there exists b ∈ N such that a = aba.

2 β_3 near - ring

Definition 2.1. We say that N is a β_3 near - ring, if $aNb = Nba$, for all $a, b \in N$.

Example 2.2. [3] Consider the near - ring (N , + , .), where (N , +) is the *kleins'* four group $\{0, a, b, c\}$ and " . " defined as follows:

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	c	b
c	0	a	b	c

Then (N, + , .) is a β_3 near - ring.
(b)In the example (a), we define "." as follows:

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Then *(*N, + , .) is a β_1 near - ring, while it is not a β_3 near - ring.

Proposition 2.3. Let N be a strong B_1 near - ring. Then N is a β_1 near - ring if and only if N is a β_3 near - ring.

Proposition 2.4. Let N be a β_3 near - ring. If N has identity 1, then N is a zero symmetric.

Theorem 2.5. Any homomorphic image of a β_3 near - ring is a β_3 near - ring.

Theorem 2.6. Let N be a strong B_1 near - ring. If N is a β_3 near - ring, then $Nax \subseteq Na \cap Nx$, for all $x, a \in N$.

Lemma 2.7. Let N be a β_3 near - ring with identity 1 . Then every N- subgroup of N is invariant .

Proposition 2.8. Let N be a β_3 near - ring. Then N admits a mate function if and only if $x \in Nx^2$ for all $x \in N$

Lemma 2.9. Let N be a β_3 near - ring. Then for all $a, b, c \in N$, there exists $n \in N$, such that $abc = nca$.

Example 2.10. Consider the near - ring (N , + , .) where (N , +) is the *kleins'* four group $\{0, a, b, c\}$ and " . " defined as following:

.	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	b	0
c	0	0	0	c

Then for all $a, b, c \in N$, there exists $n \in N$, such that abc = nca. While N is not a β_3 near - ring.

Theorem 2.11. Let N be a zero - symmetric β_3 near - ring with a mate function "f". Then:
(i) $L = \{0\}$,
(ii) N has (* , *IFP*) ,
(iii) $E \subseteq C(N)$,
(iv) N is a subdirect product of integral near - rings .

Theorem 2.12. Let N be a zero symmetric strong B_1 near - ring with a mate function "f ".Then $L = \{0\}$ and $E \subseteq C(N)$ *iff* N *is* β_3 *near - ring.*

Theorem 2.13. Every β_3 near - ring N is isomorphic to a subdirect product of subdirectly irreducible β_3 near - ring.

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Weakly spatial frames

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Abstract

The aim of this paper is to determine weakly spatial frames. The concept of weakly spatiality is actually weaker than spatiality and they are equivalent in the case of regular frames. For compact conjunctive frames, the notion of spatiality, weak spatiality and dual atomicity coincide.

Keywords: Frame, Weakly spatial frame, Regular frame, spatial frame, Ring of real-valued continuous functions on a frame
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1 Introduction

Throughout, L denotes a frame, ΣL denotes the set of prime elemets of L, and $\mathcal{R}L$ denotes the ring of real-valued continuous functions on L.

In the theroy of frames (or "pointfree topology"), several authors have tried to find a suitable form of separation axioms. In [6], T_2- frames are describe also authors investigate almost compact frames and H-closed extensions of T_2- frames. All unexplained facts concerning separation axioms can be found in [6] or in [7].

The concept of a weakly spatial frame is introduced and the main results of the note are given in Section 2. The weakly spatial frames play an important role in this note. For conjunvtive frames, they are equivalent with spatial frames. There are many examples of frames which are weakly spatial but they are not spatial (Remark 2.3). Using the Axiom of Choice, compact frames are weakly spatial (Proposition 2.4).

Let L be a weakly spatial frame. In Proposition 2.8, it is proved that if $\alpha \in \mathcal{R}L$ and $\Sigma_{coz(\alpha)} = \emptyset$, then $coz(\alpha) = \bot$, i.e., $\alpha = \mathbf{0}$. Also for every $\alpha \in \mathcal{R}L$, $Z(\alpha) = \emptyset$ if and only if $coz(\alpha) = \top$, i.e., α is a unit of $\mathcal{R}L$(Proposition 2.10). Finally, in the last proposition, it is shown that ΣL is a compact space if and only if L is a compact frame.

Here, we recall some definitions and results from the literature on frames and the pointfree version of the ring of continuous real valued functions. For more details see the appropriate references given in [1, 5, 7].

A *frame* is a complete lattice M in which the distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$ holds for all $x \in L$ and $S \subseteq M$. We denote the top element and the bottom element of M

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by $\top$ and $\bot$ respectively. The frame of open subsets of a topological spase X is denoted by $\mathfrak{O}X$.

A *frame homomorphism (or frame map)* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

An element a of a frame L is said to be *rather below* an element b, written $a \prec b$, in case there is an element s, called a *separating element*, such that $a \wedge s = \bot$ and $s \vee b = \top$. On the other hand, a is *completely below* b, written $a \prec\prec b$, if there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame L is said to be *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee\{x \in L \mid x \prec\prec a\}$ for each $a \in L$.

An element $a \in L$ is said to be *compact* if $a = \bigvee S$, $S \subseteq L$, implies $a = \bigvee T$ for some finite subset $T \subseteq S$. A frame L is said to be *compact* whenever its top element $\top$ is compact. An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. An element $m \in L$ is said to be *maximal (or dual atom)* if $m < \top$ and $m \leq x \leq \top$ implies $m = x$ or $x = \top$. As it is well known, every maximal element is prime.

Recall the contravariant *functor* Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$ $(a \in L)$ as its open sets. Also, for a frame map $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$, where $h_* : M \to L$ is the *right adjoint* of h characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$. Note that h_* preserves primes and arbitrary meets. For more details about *functor* Σ and its properties which are used in this note see [7].

Recall [1] that the frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators. The set $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f-ring in [1]. Corresponding to every continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, ., \wedge, \vee$) we have an operation on $\mathcal{R}L$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p, q) = \bigvee\{\alpha(r, s) \wedge \beta(u, w) : (r, s) \diamond (u, w) \leq (p, q)\},$$

where $(r, s) \diamond (u, w) \leq (p, q)$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$.

The *cozero map* is the map $coz : \mathcal{R}L \to L$, defined by

$$coz(\alpha) = \bigvee\{\alpha(p, 0) \vee \alpha(0, q) : p, q \in \mathbb{Q}\} = \alpha((-, 0) \vee (0, -))$$

where

$$(0, -) = \bigvee\{(0, q)) : q \in \mathbb{Q}, q > 0\}$$

and

$$(-, 0) = \bigvee\{(p, 0)) : p \in \mathbb{Q}, p < 0\}.$$

For $A \subseteq \mathcal{R}L$, let $Coz(A) = \{coz(\alpha) : \alpha \in A\}$ with the cozero part of a frame L, $Coz(\mathcal{R}L)$, called $CozL$ by previous authors. For more details about *cozero map* and its properties which are used in this note see [1].

2 Main results

Weakly spatial frames play a key role the present argument. The weakly spatiality is indeed weaker than spatiality. A frame L is said to be *spatial* if there is a topological space X such that $L \cong OX$, as frames. Also, it is known that a frame L is spatial if and only if for each $a, b \in L$ with $a \not\leq b$ there exists a prime element p of L such that $a \not\leq p$, $b \leq p$.

Definition 2.1. A frame L is said to be *weakly spatial* if $a < \top$ implies $\Sigma_a \neq \Sigma_\top$.

A condition equivalent to the weakly spatiality based on the prime elements of L is provided as follows:

Lemma 2.2. *A frame L is weakly spatial if and only if there is a prime element $p \in L$ such that $a \leq p < \top$, for every $a < \top$.*

Remark 2.3. It is clear that if L is spatial, then L is weakly spatial. The inverse is clearly not true. In fact the spatiality and the weakly spatiality are very much different. As an example, let L be a nonspatial frame and $M = L \cup \{\top_M\}$, where the order of M is the same as in L for the elements of L and for every $x \in L$, $x < \top_M$. The top element $\top_L$ of L is a prime element of M, so M is weakly spatial for all L. Now since $\Sigma M = \Sigma L \cup \{\top_L\}$, M is nonspatial.

The following proposition explains that compact frames are weakly spatial.

Proposition 2.4. *Every compact frame is weakly spatial.*

Recall that a frame L is *conjunctive* if for any $a, b \in L$ with $a \not\leq b$ there is an element $c \in L$ such that $a \vee c = \top$, $b \vee c \neq \top$. For more details about *conjunctive frames* and separation Axioms, see [6, 7].

Proposition 2.5. *Let L be a conjunctive. Then the following statements are equivalent:*

1. *L is a spatial frame.*
2. *L is a weakly spatial frame.*

It is clear that any regular frame is a conjunctive frame [6]. So, by the previous proposition we have:

Corollary 2.6. *For regular frames, the notion of spatiality and weak spatiality coincide.*

Recall that a frame L is dually atomic if for any $\top \neq a \in L$, there is a maximal element $m \in L$ such that $a \leq m$ [6]. This show that $m \notin \Sigma_a$. So any dually atomic frame is a weakly spatial frame. Also, a compact frame L is dually atomic. Because if $\top \neq a \in L$, then there exists a maximal element $m \in L$ such that $a \leq m$. Therefore we have:

Remark 2.7. For compact frames, the notion of dual atomicity and weak spatiality coincide.

Notice that by Proposition 2.5 and Remark 2.7 we can conclude that for compact conjunctive frames, the notion of spatiality, weak spatiality and dual atomicity coincide.

We now turn our attention to describing open set $\Sigma_{coz(\alpha)}$ for $\alpha \in \mathcal{R}L$. It is clear that if $\alpha = \mathbf{0}$, i.e., $coz(\alpha) = \bot$, then $\Sigma_{coz(\alpha)} = \emptyset$. It's reverse holds when L is weakly spatial.

Proposition 2.8. *Let L be weakly spatial and $\alpha \in \mathcal{R}L$. If $\Sigma_{coz(\alpha)} = \emptyset$, then $coz(\alpha) = \bot$.*

Using this proposition, we deduce from Proposition 2.4 the following corollary.

Corollary 2.9. *Let L be a compact frame and $\alpha \in \mathcal{R}L$. If $\Sigma_{coz(\alpha)} = \emptyset$, then $coz(\alpha) = \bot$.*

Regarding the linear map $\widetilde{p} : \mathcal{R}L \longrightarrow \mathbb{R}$, we use the notation of [2]. In [3, 4] we associated with each $\alpha \in \mathcal{R}L$ the zero set $Z(\alpha)$ in L defined by

$$Z(\alpha) = \{p \in \Sigma L : \alpha[p] = \widetilde{p}(\alpha) = 0\}.$$

It is shown in [4] that:

1. $Z(\alpha) = \{p \in \Sigma L : coz(\alpha) \leq p\}$, i.e., $Z(\alpha) = \Sigma L - \Sigma_{coz(\alpha)}$.
2. If α is a unit of $\mathcal{R}L$, then $Z(\alpha) = \emptyset$.

We show below that for every weakly spatial frame L, the reverse (2) always holds.

Proposition 2.10. *Let L be a weakly spatial frame and $\alpha \in \mathcal{R}L$. If $Z(\alpha) = \emptyset$, then α is a unit of $\mathcal{R}L$.*

Proposition 2.11. *Let L be a weakly spatial frame. Then L is a compact frame if and only if ΣL is a compact space.*

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A note on Feng Qi type inequality for pseudo-integral

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Abstract

We present a Feng Qi type inequality for the generalized Sugeno integral and a much wider class of functions than the comonotone functions. There are considered two cases of the real semiring with pseudo-operations: one, when pseudo-operations are dened by monotone and continuous function g, the second semiring $([a, b], \sup, \odot)$, where $\odot$ is generated and the third semiring where both pseudo-operations are idempotent, i.e., $\oplus = sup$ and$\odot = inf$.

Keywords: Hlders inequality Feng Qi inequality, Semiring, Pseudo-addition, Pseudo-multiplication, Pseudo-integral
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the eld of real numbers a semiring is dened on a real interval $[a, b] \subset [-1, 1]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$, see [7, 9]. Based on this structure there were developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. The advantages of the pseudo-analysis are that there are covered with one theory, and so with unied methods, problems (usually nonlinear and under uncertainty) from many different elds (system theory, optimization, decision making, control theory, differential equations, difference equations, etc.). Pseudo-analysis uses many mathematical tools from different elds as functional equations, variational calculus, measure theory, functional analysis, optimization theory, semiring theory, etc.
The integral inequalities are good mathematical tools both in theory and application. Different integral inequalities including Chebyshev, Jensen, Holder and Minkowski inequalities are widely used in various fields of mathematics, such as probability theory, differential equations, decision-making under risk and information sciences.
The In this paper, we use Pseudo-analysis for the generalization of the classical analysis, where instead of the field of the numbers a semiring is defined on a real interval $[a, b] \to [1, 1]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$. Thus it would be an interesting topic to generalize an inequality from the classical analysis as special

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cases. We prove generalizations of the Feng Qi type inequality for pseudo-integrals. The classical Feng Qi [8] is a very interesting integral inequality. More precisely,

Theorem 1.1. *Let n be a positive integer. Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) \geqslant 0$, for $0 \leqslant i \leqslant n-1$, and $f^{(n)}(x) \geqslant n!$, then*

$$\int_a^b [f(x)]^{(n+2)}\, dx \geqslant \left(\int_a^b f(x)dx\right)^{(n+1)}.$$

In [1] the Feng Qi type inequality for Sugeno integral is presented with several examples given to illustrate the validity of this inequalities.

Theorem 1.2. *Let μ be the Lebesgue measure on R and let $f : [0,1] \to [0,\infty)$ be a real valued function such that $(S)\int_0^1 f d\mu = p$. If f is a continuous and strictly decreasing function, such that $f\left(p^{n+1}\right) \geqslant p^{\left(\frac{n+1}{n+2}\right)}$, then the inequality:*

$$(S)\int_0^1 f^{n+2} d\mu \geqslant \left((S)\int_0^1 f d\mu\right)^{n+1}$$

holds for all $n \geqslant 0$.

Definition 1.3. Let Σ be a σ-algebra of subsets of X and let $\mu : \Sigma \to [0,\infty)$ be a non-negative, extended real-valued set function, we say that μ is a fuzzy measure iff:

(FM1) $\mu(\emptyset) = 0$;

(FM2) $E, F \in \sum$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);

(FM3) $E_n \subseteq \sum$, $E_1 \subseteq E_2 \subseteq \ldots$ imply $\lim \mu(E_n) = \mu(\bigcup_{i=1}^{\infty} E_n)$ (continuity from below);

(FM4) $E_n \subseteq \sum$, $E_1 \supseteq E_2 \supseteq \ldots, \mu(E_1) < \infty$ imply $\lim \mu(E_n) = \mu(\bigcap_{i=1}^{\infty} E_n)$ (continuity from above).

If f is a non-negative real-valued function on X, we will denote $F_\alpha = \{x \in X \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$, the $\alpha - level$ of f, for $\alpha > 0$. $F_0 = \{x \in X \mid f(x) > 0\} = supp(f)$ is the support of f. We know that: $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

If μ is a fuzzy measure on X , we define the following:

$$\mathfrak{F}^{\mu}(X) = \{f : X \to [0,\infty) | \quad \text{f is } \mu - \text{measurable}\}.$$

Definition 1.4. Let μ be a fuzzy measure on (X,Σ). If $f \in \mathfrak{F}^{\mu}(X)$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined [10] as

$$\oint_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

Where $\vee$, $\wedge$ denotes the operation sup and inf on $[0,\infty)$ respectively. In particular, if $A = X$ then:

$$\oint_X f d\mu = \oint f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

Definition 1.5. Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b]\times[a,b]\to[a,b]$ which is for $x,y,z,\mathbf{0}$ (zero element) $\in[a,b]$ it satisfies the following requirements:
(i) $x\oplus y = y\oplus x$;
(ii) $(x\oplus y)\oplus z = x\oplus(y\oplus z)$;
(iii) $x\preceq y \Rightarrow x\oplus z\preceq y\oplus z$;
(iv) $\mathbf{0}\oplus x = x$;
Let $[a,b]_+ = \{x | x\in[a,b], \mathbf{0}\preceq x\}$.

Definition 1.6. A binary operation function $\odot : [a,b]\times[a,b]\to[a,b]$ is called a pseudo-multipication, for $x,y,z,\mathbf{1}$ (unit element) $\in[a,b]$ it satisfies the following requirements:
(i) $x\odot y = y\odot x$;
(ii) $(x\odot y)\oplus z = x\odot(y\odot z)$;
(iii) $x\preceq y\Rightarrow x\odot z\preceq y\odot z$ for all $z\in[a,b]_+$;
(iv) $(x\oplus y)\odot z = (x\odot z)\oplus(x\odot y)$;
(v) $\mathbf{1}\oplus x = x$;
(vi) $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist and finit then $\lim_{n\to\infty}(x_n\odot y_n) = \lim_{n\to\infty} x_n\odot \lim_{n\to\infty} y_n$.

Let X be a non-empty set. Let $\mathbb{A}$ be a σ-algebra of subsets of a set X.

We shall consider the semiring $([a,b],\oplus,\odot)$, when pseudo-operations are generated by a monotone and continuous function $g:[a,b]\to[0,\infty]$, i.e., pseudo-operations are given with

$$x\oplus y = g^{-1}(g(x)+g(y)), \qquad x\odot y = g^{-1}(g(x)g(y)).$$

For $x\in[a,b]_+$ and $p\in]0,\infty[$, we will introduce the pseudo-power $x^{(p)}_{\odot}$ as follows: if $p=n$ is a natural number then

$$x^n_{\odot} = \underbrace{x\odot x\odot \ldots\odot x}_{n}$$

Let m be a $\oplus$- measure, where $\oplus$ has a monotone and cotinuous generator g, then gom is a σ-addetive measure in the following two important case of integral based on semiring $([a,b],\oplus,\otimes)$ are discussed. thus, the psudo-integral of function $f:X\to[a,b]$ is defined by

$$\int_X^{\oplus} f\odot dm = g^{-1}\left(\int_X (gof)d(gom)\right).$$

where the integral applied on the right side is the standard Lebesgue integral. In fact, let $m = g^{-1}o\mu$, μ is the standard Lebesgue measure on X, then we obtain

$$\int_X^{\oplus} f(x)dx = g^{-1}\left(\int_X g\left(f(x)\right)dx\right).$$

More on this structure as well as corresponding measures and integrals can be found in [2]. The second class is when $x\oplus y = \max(x,y)$ and $x\odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f:\mathbb{R}\to[a,b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f\odot dm = \sup\Big(f(x)\odot\psi(x)\Big),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuouse denisty can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additive [6]. For any continuouse function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [6].

2 Main results

The aim of this section is to show Feng Qi type inequality derived from [1] for the Pseudo-integral.

Now we peresent generalation of two above theorem for pseudo-integral

Theorem 2.1. *For a given measurable space (X, A) let $f : [0, 1] \to [0, 1]$ be a real valued function such that $(S) \int_0^1 f d\mu = p$. If f is a continuous and strictly decreasing function, such that $f\left(p^{n+1}\right) \geqslant p^{\left(\frac{n+1}{n+2}\right)}$ and let a generator $g : [0, 1] \to [0, \infty)$ of Pseudo-addition $\oplus$ and Psudo-multipication $\odot$ be decreasing function. then the inequality:*

$$\int_{[0,1]}^{\oplus} f_{\odot}^{n+2} \odot dm \geqslant \left(\int_{[0,1]}^{\oplus} f_{\odot} \odot dm \right)_{\odot}^{n+1}$$

holds for all $n \geqslant 0$ and $\sigma - \oplus$-measure m.

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An additive-quadratic equation, its solutions and generalize Hyers-Ulam-Rassias stability

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Abstract

The aim of this paper is to establish general solutions and Hyeres-Ulam-Rassias stability of the following function equation

$$f(x+4y)+4f(x-y)=f(x-4y)+4f(x+y). \tag{1}$$

on Banach spaces. It will be shown that this equation is equivalent to the so-called quadratic functional equation.

Keywords: Hyers–Ulam–Rassias stability; Quadratic equation.
Mathematics Subject Classification [2010]: Primary 39B05; Secondary 39B82, 39B62

1 Introduction

In 1940, S.M. Ulam raised the following question concerning the stability of homomorphisms:
Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$, for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.
A more general problem which imposed by Gruber [4] is:
Suppose a mathematical object satisfies a certain property approximately. Is it the possible to approximate the objects by objects satisfying the property exactly?
This problem is of particular interest in probability theory and in the case of functional equations of different types. Ulam s question for approximately additive mappings was first solved by D.H. Hyers [5] in a special case and then it generalized by Th.M. Rassias [9].

The following functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2}$$

is called the quadratic functional equation. Solutions and the stability of varus kinds of quadratic functionals also studied by many other authors in different cases (see for example [1, 2], [3], [6]- [8], [10]-[11]).

*Speaker

In this paper, we investigate the general solutions and the stability of the following quadratic functional equation

$$f(x+4y)+4f(x-y)=f(x-4y)+4f(x+y) \tag{3}$$

in the class of functions between real vector spaces. Solutions and the Hyers-Ulam-Rassias stability of the following functional equation

$$f(x+2y)+2f(x-y)=f(x-2y)+2f(x+y)$$

was also studied in [7].

2 Solutions of (3)

In this section we present the general solution of (3). Throughout this section let both X and Y be real vector spaces.

Theorem 2.1. *A function $f : X \longrightarrow Y$ satisfies the functional equation (3) if and only if there exist functions $B : X \times X \longrightarrow Y$, $A : X \longrightarrow Y$ and a constant c in Y such that $f(x) = B(x,x) + A(x) + c$ for all $x \in X$, where B is symmetric biadditive and A is additive.*

3 HYERS-ULAM-RASSIAS STABILITY OF THE EQUATION (3)

We now investigate the Hyers-Ulam stability problem for the equation (3). Thus we find the condition that there exists a true solution near a approximate solution for (3). From now on, let X be a real vector space and Y be a real Banach space.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow \mathbb{R}^+$ be a function such that the series*

$$\Phi(x,y) := \sum_{i=0}^{\infty} \frac{\phi(2^{2i+1}x, 2^{2i+1}y)}{4^{2i+1}} \tag{4}$$

converges, for all $x, y \in X$. Suppose that an even function $f : X \longrightarrow Y$ satisfies

$$\|f(x+4y)+4f(x-y)-f(x-4y)-4f(x+y)\| \leq \phi(x,y), \tag{5}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow Y$ which satisfies the equation (3) and the inequality

$$\|f(x)-f(0)-Q(x)\| \leq \frac{1}{4}\Phi(x,\frac{x}{4}) + \Phi(\frac{x}{4},\frac{x}{4}) \tag{6}$$

for all $x \in X$. The function Q is given by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \tag{7}$$

for all $x \in X$. If furthermore either f is measurable or for each fixed $x \in X$ the mapping $t \longmapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $Q(rx) = r^2 Q(x)$, for all $r \in \mathbb{R}$.

Proof. If we replace y by x in (5), we have

$$\|f(5x) + 4f(0) - f(3x) - 4f(2x)\| \leq \phi(x, x) \tag{8}$$

Substituting y by $\frac{y}{4}$ in (5), and then replacing y by x in the resulting inequality, one obtains that

$$\|f(2x) + 4f(\frac{3x}{4}) - f(0) - 4f(\frac{5x}{4})\| \leq \phi(x, \frac{x}{4}) \tag{9}$$

Substituting x by $\frac{x}{4}$ in (8),and then multiplying in 4 in the resulting inequality, one obtains that

$$\|4f(\frac{5x}{4}) + 16f(0) - 16f(\frac{x}{2}) - 4f(\frac{3x}{4})\| \leq 4\phi(\frac{x}{4}, \frac{x}{4}). \tag{10}$$

(9) and (10), imply that

$$\|16f(0) - 16f(\frac{x}{2}) + f(2x) - f(0)\| \leq \phi(x, \frac{x}{4}) + 4\phi(\frac{x}{4}, \frac{x}{4}) \tag{11}$$

Thus with $2x$ instead of x in the last inequality, we get

$$\|\frac{f(4x) - f(0)}{16} - [f(x) - f(0)]\| \leq \frac{\phi(2x, \frac{x}{2}) + 4\phi(\frac{x}{2}, \frac{x}{2})}{16}, \tag{12}$$

for all $x \in X$. Using induction on m, we may show that

$$\|\frac{f(4^m x) - f(0)}{4^{2m}} - [f(x) - f(0)]\| \leq \sum_{i=0}^{m-1} \frac{\phi(2^{2i+1}x, 2^{2i-1}x) + 4\phi(2^{2i-1}x, 2^{2i-1}x)}{4^{2i+2}}. \tag{13}$$

Hence

$$\|\frac{f(2^n x) - f(0)}{4^n} - [f(x) - f(0)]\| \leq \sum_{i=0}^{\frac{n}{2}-1} \frac{\phi(2^{2i+1}x, 2^{2i-1}x) + 4\phi(2^{2i-1}x, 2^{2i-1}x)}{4^{2i+2}}, \tag{14}$$

for all $x \in X$. Note that the right hand side of (14) is a convergent series by assumption. The rest of the proof is similar to Theorem 3.1 in [7]. □

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An Invers probleme of Lyapunov type inequality for Sugeno integral

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Abstract

Integral inequalities play important roles in classical probability and measure theory. Sugenos integral is a useful tool in several theoretical and applied statistics which has been built on non-additive measure. Lyapunov type inequalities for the Sugeno integral on abstract spaces are studied in a rather general form, thus closing the series of papers on the topic dealing with special cases restricted to the(pseudo-)additive operation. Moreover, a strengthened version of Lyapunov type inequality for Sugeno integrals on a real interval based on a binary operation $\star$ is presented.

Keywords: Lyapunov inequality, Semiring, Fuzzy integral inequality; Nonadditive.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Some integral inequalities, such as Lyapunov inequality, Jensen type inequality, Holders inequality and Minkowski inequality, play important roles in classic measure space. A natural thought is whether these integral inequalities still hold in fuzzy measure space under the condition of non-additive measure. The study of inequalities for Sugeno integral was developed by Mesiar, Pap [7, 6] and so on. All of them enrich the fuzzy measure theory. We focus on the inequalities for Sugeno integral on abstract space. There are hardly any papers concern about inequalities for Sugenio ntegral. Hun Hong [5] has done this work, but the Lyapunov type inequality for Sugeno integral on abstract space are obviously uncorrect. Its easy to nd errors in the procedure of the proof and to give counterexamples. Thus the conditions under what the Lyapunov integral are discussed. In [4], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang et al. [8]. Furthermore, Chebyshev type inequalities for fuzzy integral were proved in a rather general form by Mesiar and Ouyang [7]. They obtained the following result:

Theorem 1.1. *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such tath both $(S)\int_A f d\mu$ and $(S)\int_A g d\mu$ are finite. and Let $\star : [o, \infty)^2 \to [0, \infty)$ be continuous and non-decreasing in both arguments and bounded from above by minimum. if f, g are comonotone, then the inequality*

$$(S)\int_A f \star g d\mu \geqslant \left((S)\int_A f d\mu\right) \star \left((S)\int_A g d\mu\right)$$

*Speaker

holds.
It is known that

$$(S)\int_A f\star g d\mu \leqslant \left((S)\int_A f d\mu\right)\star\left((S)\int_A g d\mu\right)$$

where f, g are comonotone functions whenever $\star \geq \max$ (for a similar result, see [16]), it is of great interest to determine the operator $\star$ such that

$$(S)\int_A f\star g d\mu = \left((S)\int_A f d\mu\right)\star\left((S)\int_A g d\mu\right) \tag{1}$$

holds for any comonotone functions f, g, and for any fuzzy measure $\star$ and any measurable set A.

Ouyang, Mesiar and Li [9, 10] proved that there are only 18 operators such that (1.1) holds, including the four well-known operators: minimum, maximum, PF(called the first projection, PF for short, if $x\star y = x$ for each pair (x, y)) and PL(called the last projection, PL for short, if $x \star y = y$ for each pair (x, y)).

The following inequality is a classical Lyapunov type inequality [3]:

$$\left(\int_A (f)^s d\mu\right)^{r-t} \leq \left(\int_A (f)^t d\mu\right)^{r-s}\left(\int_A (f)^r d\mu\right)^{s-t} \tag{2}$$

where $0 < t < s < r, f : [0, 1] \to [0, \infty)$ is an integrable function.

Dug Hun Hong have proved Lyapunov type inequality for fuzzy integrals in [5]. In this context, the purpose of this work is to study the Lyapunov inequality and its connections with some fundamental properties of the Sugeno integrals on abstract spaces. Now, we recall some basic definitions and properties of the fuzzy measure and integral . For details we refer the readers to Refs [2, 7, 11].

Let $\overline{R_+}$ denote $[0, +\infty]$. Throughout this paper, we fix the measurable space $(X, \mathcal{F})$, and all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 1.2. Let Σ be a σ-algebra of subsets of X and let $\mu : \Sigma \to [0, \infty]$ be a non-negative, extended real-valued set function, we say that μ is a fuzzy measure iff:

(FM1) $\mu(\emptyset) = 0$;

(FM2) $E, F \in \sum$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);

(FM3) $(E_n) \subseteq \sum$, $E_1 \subseteq E_2 \subseteq \ldots$ imply $\lim \mu(E_n) = \mu(\bigcup_{i=1}^{\infty} E_n)$ (continuity from below);

(FM4) $(E_n) \subseteq \sum$, $E_1 \supseteq E_2 \supseteq \ldots, \mu(E_1) < \infty$ imply $\lim \mu(E_n) = \mu(\bigcap_{i=1}^{\infty} E_n)$ (continuity from above).

If f is a non-negative real-valued function on X, we will denote $F_\alpha = \{x \in X \mid f(x) \geq \alpha\} = \{f \geq \alpha\}$, the $\alpha - level$ of f, for $\alpha > 0$. $F_0 = \{x \in X \mid f(x) > 0\} = supp(f)$ is the support of f. We know that: $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

If μ is a fuzzy measure on X , we define the following:

$$\mathfrak{F}^\mu(X) = \{f : X \to [0, \infty)| \quad f \text{ is non} - \text{negative and } \mu - \text{measurable}\}.$$

Definition 1.3. Let μ be a fuzzy measure on (X, Σ). If $f \in \mathfrak{F}^{\mu}(X)$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined [**?**] as

$$\oint_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

Where $\vee$, $\wedge$ denotes the operation *sup* and inf on $[0, \infty)$ respectively. In particular, if $A = X$ then:

$$\oint_X f d\mu = \oint f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

The following proposition gives most elementary properties of the fuzzy integral and can be found in [**?**].

Remark 1.4. Let $F(\alpha) = \mu(A \cap F_\alpha)$, from parts (5) and (6) of the above Proposition, it very important to note that

$$F(\alpha) = \alpha \Rightarrow \oint_A f d\mu = \alpha.$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

Notice that in our results the transformation theorem for Sugeno integrals(see [**?**]), plays a fundamental role.

Lemma 1.5. *Let* $(S) \int_A f d\mu = p$. *Then*

$$\forall r \geq p, (S) \int_A f d\mu = (S) \int_0^r \mu(A \cap F_\alpha) dm.$$

where m *is the Lebesgue measure.*

In this contribution, we will prove general Liapunov type inequality for the Sugeno integral of comonotone functions. Recall that two functions
$f, g : X \to R$ are said to be comonotone if for all

$$(x, y) \in X^2, (f(x) - f(y))(g(x) - g(y)) \geq 0$$

. Clearly, if f and g are comonotone, then for all non-negative real numbers p, q either $F_p \subset G_q$ or $G_q \subset F_p$. Indeed, if this assertion does not hold, then there are $x \in F_p \setminus G_q$ and $y \in G_q \setminus F_p$. That is,

$$f(x) \geq p, g(x) \leq q \quad \text{and} \quad f(y) < p, g(y) \geq q,$$

and hence $(f(x) - f(y))(g(x) - g(y)) < 0$, contradicts with the comonotonicity of f and g. Notice that comonotone functions can be defined on any abstract space.

2 Main results

The aim of this section is to show Feng Qi type inequality derived from [**?**] for the Pseudo-integral.

Theorem 2.1. *Let $f \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $(S)\int_A(f)d\mu$ and is finit. Let $\star : [0.\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by manimum. If f^t, f^r are comonotone, then the inequality*

$$\left((S)\int_A(f)^s d\mu\right)^{r-t} \geq \left((S)\int_A(f)^t d\mu\right)^{r-s} \star \left((S)\int_A(f)^r d\mu\right)^{s-t} \tag{3}$$

hold where $0 < t < s < r < \infty$.

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Aproximate Fixed Point Theorem For Asymptotically Nonexpansive Mappings

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Abstract

In this paper, we obtain the existence of approximate fixed points theorems for asymptotically nonexpansive mapping in a Banach space. Also we prove the set $AF(T)$ of approximate fixed points of T is closed and convex.

Keywords: Approximate fixed points, Asymptotically nonexpansive mappings, Uniformly convex Banach space.

Mathematics Subject Classification [2010]: 46A32, 46M05, 41A17

1 Introduction

Nowadays, fixed point and aproximate fixed point play an important role in different areas of mathematics, and its applications, particularly in mathematics, differential equation and dynamic programming. Also,There are plenty of problems in applied mathematics which can be solved by means of fixed point theory. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of fixed points is not strictly required, but that of nearly fixed points. Another type of practical situations that lead to this approximation is when the conditions that have to be imposed in order to guarantee the existence of fixed points are far too strong for the real problem one has to solve.

In 1965, F. E. Browder [1] proved that every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point. Also, This result was also obtained by W. A. Kirk [3].

In 1972 Goebel et al [2] extended Browder's result to a more general class of transformations which it shall call "asymptotically nonexpansive" mappings, and we give some approximate fixed points of such mappings.

Definition 1.1. [5] *Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map. Then $x_0 \in X$ is $\epsilon-$fixed point for T if $\|Tx_0 - x_0\| < \epsilon$.*

*Speaker

Remark 1.2. [5] *In this paper we will denote the set of all $\epsilon-$ fixed points of T, for a given ϵ, by :*

$$AF(T) = \{x \in X \mid x \text{ is an } \epsilon - \text{fixed point of } T\}.$$

Theorem 1.3. *[5] Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$ be a map also for all $x, y \in X$,*

$$\|Tx - Ty\| \leq c \ \|x - y\| \quad : 0 < c < 1$$

then T has an $\epsilon-$ fixed point in completely norm space. Moreover, if $x, y \in X$ are $\epsilon-$ fixed points of T, then $\|x - y\| \leq \frac{2\epsilon}{1-c}$.

2 Main results

We begin by recalling some needed definitions and results. In 1936, Clarkson (see [4]) defined uniformly convex for Banach spaces.

Definition 2.1. [4] A Banach space X is called uniformly convex if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $\|x\| = \|y\| = 1$ then $\|(x + y)/2\| \leq 1 - \delta(\epsilon)$.

Remark 2.2. [4] *In such a space, it is easily seen that the inequalities $\|x\| \leq d$, $\|y\| \leq d$, $\|x - y\| \geq \epsilon$ imply $\|(x+y)/2\| \leq (1-\delta(\epsilon/d)d)$. Furthermore, the function $\delta : (0, 2] \to (0, 1]$ may be assumed to be increasing.*

Definition 2.3. [2] Let U be a subset of a Banach space X. A mapping $T : U \to U$ is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that for any $x, y \in U$,

$$d(T^i x, T^i y) \leq \ Ld(x, y), \qquad \forall i \geq 1.$$

Definition 2.4. [2] Let U be a subset of a Banach space X. A mapping $T : U \to U$ is said to be nonexpansive if for any $x, y \in U$,

$$d(Tx, Ty) \leq \ d(x, y).$$

Definition 2.5. [2] Let U be a bounded, closed and convex subset Banach space X. A mapping $T : U \to U$ is said to be asymptotically nonexpensive if for any $x, y \in U$,

$$d(T^i x, T^i y) \leq \ u_i d(x, y), \qquad \forall i \geq 1.$$

where $\{u_i\}$ is a sequence of real numbers such that $lim_{i\to\infty} u_i = 1$.

Theorem 2.6. *Let U be a bounded, closed and convex subset of a uniformly convex Banach space X, and let $T : U \to U$ be asymptotically nonexpansive, $x_0 \in X$ and $\epsilon > 0$. Then T has an $\epsilon-$ fixed point.*

Theorem 2.7. *Let U be a bounded, closed and convex subset of a uniformly convex Banach space X, and let $T : U \to U$ be asymptotically nonexpansive, $x_0 \in X$ and $\epsilon > 0$. the set $AF(T)$ of $\epsilon-$ fixed points of T is closed and convex.*

Theorem 2.8. *Let $(X, \|.\|)$ be a uniformly convex Banach space, and $T : X \to X$ be a map also if there exists $0 < c < 1$ such that for any $x, y \in X$,*

$$\|Tx - Ty\| \leq c \ \|x - y\|.$$

The set AFT of $\epsilon-$ fixed points of T is closed and convex.

Theorem 2.9. *Let U be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space X, and let $T : U \to U$ be an arbitrary (even noncontinuous) transformation such that for some integer n,*

$$d(T^i x, T^i y) \leq \ u_i d(x, y), \qquad \forall i \geq n,$$

where $lim_{i\to\infty} u_i = 1$. Then T has an $\epsilon-$ fixed point.

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Poster Combinatorically view to integral majorization pp.: 1–3

Combinatorically view to integral majorization

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Abstract

In this paper we look a little combinatorically to integral majorization. Integral majorization is a majorization relation on vectors with integer entries. We define for each path in a grid a vector, called vector grid, and then relate gird vectors to integral vectors and majorization. Then we propose some properties of these concepts and their relations.

Keywords: Integral vector, Grid, Majorization
Mathematics Subject Classification [2010]: 05A17, 15A39

1 Introduction

Relations between combinatorics and linear algebra is very interesting and there are a lot of research in this area. Recently there are some researches on majorizations and combinatorics. For vectors $x, a \in \mathbb{R}^n$, we say that $x$is majoroized by a and denoted by $x \prec a$, provided that

$$\sum_{j=1}^{k} x_{[j]} \leq \sum_{j=1}^{k} a_{[j]}, \qquad \text{for} \quad k = 1, 2, \ldots, n-1,$$

and

$$\sum_{j=1}^{n} x_{[j]} = \sum_{j=1}^{n} a_{[j]},$$

where by $x_{[j]}$ we mean the j^{th} largest element of a vector x[3].

Let $M(a)$ be a polytope of all vectors majorized by a given vector $a \in \mathbb{R}^n$. A vector x is said integral vector if all of its elements are integer. For an integral vector a let $M_I(a)$ be the set of all integral vectors that are majorized by a. In [1] Dahl proposed an algorithm for computing combinatorically the cardinality of $M_I(a)$. Also Dahl propose a relation between $p(n)$, the number of different partitions of a natural number n, and majorization. $p(n)$ has been related to majorization [1]. Consider $(n, 0, 0, \ldots, 0)$ in $\mathbb{R}^n$. There are $p(n)$ nonincreasing nonnegative integral vectors in $\mathbb{R}^n$ that are majorized by $(n, 0, 0, \ldots, 0)$. In this paper we look more precisely to $M_I((n, 0, ..., 0))$ and $M_I((n, m, 0, ..., 0))$ and find some relations between $M_I(a)$ and grid paths.

*Speaker

2 Vector grid paths

Consider the following $n \times m$ grid, $\mathcal{A}$

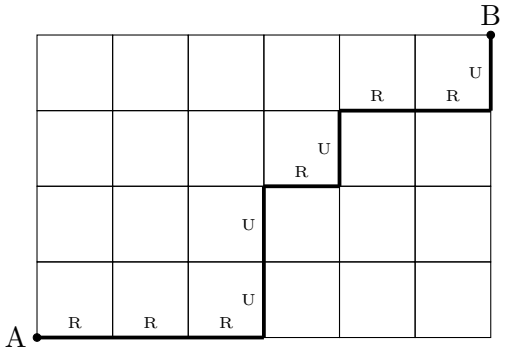

Figure 1

A shortest way on edges from A to B is called a path from A to B. We know each path has m right moves and n up moves. It is easy to show that the number of all paths is $\rho(\mathcal{A}_{n\times m}) = \begin{pmatrix} n+m \\ n \end{pmatrix}$[2]. We show each path by a sequence of its moves. For example the path shown in figure 1 is denoted by $RRRUURURRU$. We will assign to each sequence an integral vector in the following way. Let our sequence be like above. The assigned vector would be $(3, 2, 1, 1, 2, 1, 0, 0, 0, 0)$. It means we have 3 R's first, then 2 of the other letter, and etc. We add zeros to the end of the vector so that its dimension is the length of the sequence. We add zeros to the end of the vector so that the number of its components is the length of the sequence. We use the notation "vector grid of $\mathcal{A}$" for an integer vector related to a path in $\mathcal{A}$.

Lemma 2.1. *If x is a vector grid of $\mathcal{A}_{l\times n-l}$, $k \neq l$ and $n-k \neq l$, then x is not a vector grid of $\mathcal{A}_{k\times n-k}$.*

Proof. If x is a vector grid of $\mathcal{A}_{k\times n-k}$, then $\sum x_{2i+1} = k$ or $\sum x_{2i} = k$, but here it is equal to l or $n-l$ (because it is a vector grid of $\mathcal{A}_{l\times n-l}$). □

Definition 2.2. A nonnegative integral vector is said to be nonvanish at the first if $x_i \neq 0$ there is no $j \leq i$ with $x_j = 0$.

Each grid vector is nonvanish at the first.

3 Majorization and grid paths

Lemma 3.1. *a) Let $\mathcal{A}$ be an $n \times m$ grid and the vector x be a nonnegative integral vector in $\mathbb{R}^{n+m}$. If x is a grid vector of $\mathcal{A}$, then x is majorized by $(n, m, 0, \ldots, 0)$.*
b)If x x is a nonnegative integral vector in $\mathbb{R}^{n+m}$ that is nonvanish at the first, $\sum x_{2i+1} = n$ and $\sum x_{2i} = m$ or $\sum x_{2i+1} = m$ and $\sum x_{2i} = n$, then x is a grid vector of $\mathcal{A}_{n\times m}$ or $\mathcal{A}_{m\times n}$.

Proof. a) If $x = (x_1, \ldots, x_{n+m})$ is a grid vector of $\mathcal{A}$, then $x_{[1]}$ means the number of right steps at the first of the path, hence $x_{[1]} \leq max\{m, n\}$. Also $\sum_{i=1}^{d} x_i \leq n + m$ and $\sum_{i=1}^{n+m} x_i = n + m$. Consequently $x \prec (m, n, 0, \ldots, 0)$.

b) We construct the path by x_1 steps to the right then x_2 steps to the up and etc. Because of the mentioned condition path never go out the grid and the end point of the path is in the top-right point of the grid.

□

Theorem 3.2. *Let $x \in \mathbb{R}^n$ be a vector that is nonvanish at first and n be a natural number. x is majorized by $(n, 0, ..., 0)$ if and only if there is k such that x is an integral vector of the grid $\mathcal{A}_{k\times n-k}$.*

Proof. If $x \prec (n, 0, ..., 0)$, then $\sum x_i = n$. Considering $k = \sum x_{2i+1}$, by the part (a) of the above lemma we have x is an integral vector of the grid $\mathcal{A}_{k\times n-k}$. Conversely since x is an integral vector of the grid $\mathcal{A}_{k\times n-k}$, hence $\sum x_i = n$ that means $x \prec (n, 0, ..., 0)$. □

Corollary 3.3. *Let n be a natural number and $(n, 0, ..., 0), (k, n-k, 0..., 0) \in \mathbb{R}^n$. Then $M_I((n, 0, ..., 0)) = \bigcup_{k=1}^{n} M_I((k, n-k, 0..., 0)) \bigcup \{(n, 0, ..., 0)\}$*

Proof. If $x \prec (n, 0, ..., 0)$, and $x \neq (n, 0, ..., 0)$, then $x \prec (x_1, n - x_1, 0, ..., 0)$ and consequently $x \in M_I((x_1, n - x_1, 0, ...0))$. Obviously if $x \in \bigcup_{k=1}^{n} M_I((k, n-k, 0..., 0)) \bigcup \{a\}$, then $x \in M_I((n, 0, ..., 0))$.

□

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Poster Continuous and exact-continuous frame for Hilbert spaces pp.: 1–3

Continuous and Exact-Continuous Frame for Hilbert Spaces

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Abstract

In this paper we remaind some property for the continuous frame for the Hilbert space and show that the continuous frame and exact-continuous frame are equal under some conditions.

Keywords: Hilbert space, Continuous frame, Exact-Continuous frame, Bases, Measure space.
Mathematics Subject Classification [2010]: 57R25, 42C20

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{M}$ be a measure space with a positive measure μ. A continuous frame is a family $\{\psi_k\}_{k\in\mathcal{M}}$ for which the following hold:
(c1) For all $h \in \mathcal{H}$, the mapping

$$\Phi : \mathcal{M} \to \mathcal{C}, \Phi(k) =< h, \psi_k >$$

is a measurable function on $\mathcal{M}$.
(c2) There exist constants $A, B > 0$ such that

$$A\|h\|^2 \leq \int_{\mathcal{M}} |< h, \psi_k >|^2 \, d\mu(k) \leq B\|h\|^2, \forall h \in \mathcal{H}$$

If $A = B$ then the continuous frame is called continuous tight frame and if $A = B = 1$ then the continuous frame is called normalized continuous tight frame.

For the sake of simplicity we assume that the mapping $x \longmapsto < f, \psi_k >$ is weakly continuous for all $k \in \mathcal{M}$. Note that if $\mathcal{M}$ be a countable set and μ the counting measure then we obtain the usual definition of a (discrete) frame. By Cauchy-Schwartz inequality

$$\int_{\mathcal{M}} < f, \psi_k >< \psi_k, g > d\mu(k)$$

is well defined for all $f, g \in \mathcal{H}$.

*Speaker

2 Main Results

For a fixed $f \in \mathcal{H}$, the mapping $\Phi : \mathcal{H} \to \mathcal{C}$ defined by

$$\Phi(g) = \int_{\mathcal{M}} < f, \psi_k >< \psi_k, g > d\mu(k)$$

is conjugated linear, and bounded because

$$| \int_{\mathcal{M}} < f, \psi_k >< \psi_k, g > d\mu(k) |^2 \leq B^2. \parallel f \parallel^2 . \parallel g \parallel^2 .$$

So for all $g \in \mathcal{H}$

$$< \int_{\mathcal{M}} < f, \psi_k > .\psi_k d\mu(k), g >= \int_{\mathcal{M}} < f, \psi_k >< \psi_k, g > d\mu(k)$$

By this $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{S}f = \int_{\mathcal{M}} < f, \psi_k > .\psi_k d\mu(k).$$

is a linear mapping and

$$\parallel \mathcal{S} \parallel = Sup_{\parallel g \parallel = 1} |< \mathcal{S}f, g >|$$

The operator $\mathcal{S}$ is called frame operator and $\mathcal{S}$ is invertible and has the norm at most B, Thus, if $f \in \mathcal{H}$ then f can represents by

$$f = \mathcal{S}^{-1}\mathcal{S}f = \int_{\mathcal{M}} < f, \psi_k > \mathcal{S}^{-1}\psi_k d\mu(k)$$

$$f = \mathcal{S}\mathcal{S}^{-1}f = \int_{\mathcal{M}} < f, \mathcal{S}^{-1}\psi_k > \psi_k d\mu(k).$$

Representations have to be interpreted in the weak sense. The discrete frames $\{\psi_k\}_{k=1}$ are actually a special case of the continuous frames, corresponding to the case where $\mathcal{M} = \mathcal{N}$, equipped with the counting measure.

A continuous frame $\{\psi_k\}_{k\in\mathcal{M}}$ for complex Hilbert space $\mathcal{H}$ that ceases to be a frame when an arbitrary element is removed, is called an Exact-Continuous frame. Because $\{\psi_k\}_{k\in\mathcal{M}}$ being a continuous frame or not depends on the measure space, it would be more exact to speak about a continuous frame for $\mathcal{H}$ with respect to the measure space $(\mathcal{M}, \mu)$.

Theorem 2.1. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{M}$ be a measure space with a positive measure μ, also let $\{\psi_k\}_{k\in\mathcal{M}}$ be an continuous frame for $\mathcal{H}$. The removal of a vector ψ_j from $\{\psi_k\}_{k\in\mathcal{M}}$ leaves either a continuous frame or an incomplete set. Also the fo1lowing holds:*

(i) If $< \psi_j, \mathcal{S}^{-1}\psi_j > \neq 1$, then $\{\psi_k\}_{k\in\mathcal{M},k\neq j}$ is an exact continuous frame for $\mathcal{H}$.

(ii) If $< \psi_j, \mathcal{S}^{-1}\psi_j >= 1$, then $\{\psi_k\}_{k\in\mathcal{M},k\neq j}$ is incomplete.

Proof. For arbitrary j as a natural number, take

$$\psi_j = \int_{\mathcal{M}} < \psi_j, \mathcal{S}^{-1}\psi_k > \psi_k d\mu(k)$$

So $\psi_j = \int_{\mathcal{M}} \lambda_k \psi_k d\mu(k)$ when $\lambda_j =< \psi_j, \mathcal{S}^{-1}\psi_k >$. Also

$$\psi_j = \int_{\mathcal{M}} \sigma_{j,k}\psi_k d\mu(k)$$

That is

$$\begin{aligned} 1 &= \int_{\mathcal{M}} \mid \sigma_{j,k} \mid^2 d\mu(K) \\ &= \int_{\mathcal{M}} \mid \lambda_k \mid^2 d\mu(K) + \int_{\mathcal{M}} \mid \lambda_k - \sigma_{j,k} \mid^2 d\mu(K) \\ &= \mid \lambda_j \mid^2 + \int_{\mathcal{M}} |\lambda_k|^2 d\mu(k) + \mid \lambda_j - 1 \mid^2 + \int_{\mathcal{X}\setminus\mathcal{M}} |\lambda_k|^2 d\mu(k) \end{aligned}$$

If $\lambda_j \neq 1$ then we can find

$$\psi_j = (1-\lambda_j)^{-1} \int_{\mathcal{M}} < \psi_j, \mathcal{S}^{-1}\psi_k > \psi_k d\mu(k)$$

That is $\{\psi_k\}_{k\in\mathcal{M}, j\neq k}$ is an exact continuous frame for $\mathcal{H}$ with respect to the measure space $(\mathcal{M}, \mu)$.

Now if $\lambda_j = 1$ then $\int_{\mathcal{X}\setminus\mathcal{M}} |\lambda_k|^2 d\mu(k) = 0$ in this case

$$\lambda_j =< \psi_j, \mathcal{S}^{-1}\psi_k >= 0 \quad \forall k \neq j$$

But we have

$$\lambda_j =< \psi_j, \mathcal{S}^{-1}\psi_j >= 1, \quad \mathcal{S}^{-1}\psi_j \neq 0, \quad \forall j$$

Indeed we find a non-zero element $\mathcal{S}^{-1}\psi_k$ that is orthogonal to sequence $\{\psi_j\}_{j\neq k}$ and this means that $\{\psi_j\}_{j\neq k}$ is incomplete sequence, that is $\{\psi_k\}_{k\in\mathcal{M}, j\neq k}$ is an exact continuous frame for $\mathcal{H}$ with respect to the measure space $(\mathcal{M}, \mu)$.

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Poster Continuous fusion frame and continuous resolution of the identity pp.: 1–4

Continuous fusion frame and continuous resolution of the identity

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Abstract

Continuous resolution of the identity (CRI) was introduced, a new family of CRI was constructed, and Moreover, a new operator was then defined for two Bessel continuous fusion sequences and accordingly, a number of reconstruction formulas and a family of CRI were obtained.

Keywords: continuous fusion frame, Continuous g-frame, Continuous resolution of the identity.
Mathematics Subject Classification [2010]: 42C15, 42C99

1 Introduction

Frames for Hilbert space were formally defined by Duffin and Schaeffer [4] in 1952 for studying some problems in non-harmonic Fourier series. Continuous frames were proposed by G. Kaiser [8] and independently by Ali, Antoine, and Gazeau [2] to a family indexed by some locally compact space endowed by a Radon measure. Abdollahpour and Faroughi [1] introduced the concept of continuous g-frames as a generalization of discrete g-frames. Throughout this paper, (Ω, μ) is a measure space, H and K are two Hilbert spaces, and $\{K_\omega\}_{\omega\in\Omega}$ is a sequence of closed Hilbert subspaces of K. For each $\omega \in \Omega$, $\mathcal{B}(H, K_\omega)$ is the collection of all bounded linear operators from H to K_ω. We also denote

$$\bigoplus_{\omega\in\Omega} K_\omega = \left\{ \{g_\omega\}_{\omega\in\Omega} : g_\omega \in K_\omega \quad \text{and} \quad \int_\Omega \|g_\omega\|^2 d\mu(\omega) < \infty \right\}$$

Definition 1.1. A sequence $\mathbf{\Lambda} := \{\Lambda_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ is called a continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$, if

1. for each $f \in H$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,
2. there are two constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \le B\|f\|^2; \qquad (f \in H). \tag{1}$$

*Speaker

Definition 1.2. Let $\{K_\omega\}_{\omega\in\Omega}$ be a family of closed subspaces of a Hilbert space H and (Ω,μ) be a measure space with positive measure μ and $\mathrm{m}:\Omega\longrightarrow\mathbb{R}^+$. Then $\mathcal{K}=\{(K_\omega\ ,\mathrm{m}(\omega))\}_{\omega\in\Omega}$ is called a continuous fusion frame with respect to $(\Omega\ ,\mu)$ and m, if

1. for each $f\in H$, $\{\Pi_{K_\omega}f\}_{\omega\in\Omega}$ is strongly measurable and m is measurable function from Ω to $\mathbb{R}^+$;
2. there are two constants $0<C\leq D<\infty$ such that

$$C\|f\|^2\leq\int_\Omega \mathrm{m}^2(\omega)\|\Pi_{K_\omega}f\|^2d\mu(\omega)\leq D\|f\|^2 \qquad (f\in H), \tag{2}$$

where Π_{K_ω} is the orthogonal projection onto the subspace K_ω.

Definition 1.3. Let $\mathcal{K}=\{(K_\omega\ ,\mathrm{m}(\omega))\}_{\omega\in\Omega}$ be a continuous fusion frame and $S_\mathcal{K}$ be the continuous fusion frame operator. Also, $\mathcal{G}=\{(G_\omega\ ,\mathrm{n}(\omega))\}_{\omega\in\Omega}$ is a Bessel continuous fusion sequence. $\mathcal{G}$ is an alternate dual of $\mathcal{K}$ if:

$$\langle f,g\rangle=\int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f,\Pi_{G_\omega}S_\mathcal{K}^{-1}\Pi_{K_\omega}g\rangle d\mu(\omega) \tag{3}$$

for all $f,g\in H$.

In this paper, continuous resolution of identity (simply CRI) was introduced. Moreover, a new operator was then defined for two Bessel continuous fusion sequences and accordingly, a number of reconstruction formulas and a family of CRI were obtained.

2 Main results

Definition 2.1. A sequence $\{T_j\in\mathcal{B}(H):j\in J\}$ is a resolution of identity on H if, for each $f\in H$:

$$f=\sum_{j\in J}T_jf,$$

where the series converges unconditionally for all $f\in H$.

This definition leads us to introduce the following definition:

Definition 2.2. A sequence $\{T_\omega\in\mathcal{B}(H):\omega\in\Omega\}$ is a continuous resolution of identity (simply CRI) on H if, for each $f\ ,g\in H$:

1. $\omega\longmapsto\langle f,T_\omega^*g\rangle$ is a measurable functional on Ω.
2. $\langle f,g\rangle=\int_\Omega\langle f,T_\omega^*g\rangle\ \mathrm{d}\mu(\omega)$.

Below, two Bessel continuous fusion sequences are considered:
$\mathcal{K}=\{(K_\omega\ ,\mathrm{m}(\omega))\}_{\omega\in\Omega}$ with Bessel bound $D_\mathcal{K}$ and $\mathcal{G}=\{(G_\omega\ ,\mathrm{n}(\omega))\}_{\omega\in\Omega}$ with Bessel bound $D_\mathcal{G}$. The operator

$$\langle S_{KG}f,g\rangle=\int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f,\Pi_{K_\omega}\Pi_{G_\omega}g\rangle d\mu(\omega), \qquad (f,g\in H), \tag{4}$$

is introduced. By Cauchy-Schwartz inequality S_{KG} is a bounded operator and

$$\|S_{KG}\| \leq \sqrt{D_{\mathcal{K}}}\sqrt{D_{\mathcal{G}}}.$$

So

$$\|S_{KG}f\| \leq \sqrt{D_{\mathcal{G}}}\left(\int_{\Omega} \mathrm{m}^2(\omega)\|\Pi_{K_\omega}f\|^2 d\mu(\omega)\right)^{\frac{1}{2}} \tag{5}$$

and also $S_{KG}^* = S_{GK}$.

For this operator, the following result can be obtained.

Proposition 2.3. *Let $\mathcal{K} = \{(K_\omega\ , \mathrm{m}(\omega))\}_{\omega\in\Omega}$ be a continuous fusion frame with continuous fusion frame bounds C and D and continuous fusion frame operator $S_{\mathcal{K}}$ for a Hilbert space H. Let $\mathcal{G} = \{(G_\omega\ , \mathrm{n}(\omega))\}_{\omega\in\Omega}$ be an alternate dual continuous fusion frame for $\mathcal{K}$ with required positivity. Then we have*

$$CI_H \leq S_{GK} \leq DI_H$$

and also S_{GK} is invertible.

Proof. Let f be an arbitrary element of H. Then we have

$$\begin{aligned}\|f\|^2 = \langle f, f\rangle &= \int_{\Omega} \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{G_\omega}S_{\mathcal{K}}^{-1}\Pi_{K_\omega}f\rangle d\mu(\omega)\\ &\leq \frac{1}{C}\int_{\Omega} \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{G_\omega}\Pi_{K_\omega}f\rangle d\mu(\omega)\\ &= \frac{1}{C}\langle S_{GK}f, f\rangle.\end{aligned}$$

Similarly, $\langle S_{GK}f, f\rangle \leq D\|f\|^2$, hence $CI_H \leq S_{GK} \leq DI_H$. By the same argument as in the proof of Proposition 2.9 in [7], S_{GK} is invertible and $\frac{1}{D} \leq \|S_{GK}^{-1}\| \leq \frac{1}{C}$. □

Remark 2.4. By this Proposition we have the following reconstruction formulas:

1. $\langle f, g\rangle = \langle S_{KG}f, S_{GK}^{-1}g\rangle = \int_{\Omega} \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{K_\omega}\Pi_{G_\omega}S_{GK}^{-1}g\rangle d\mu(\omega)$
2. $\langle f, g\rangle = \langle S_{KG}S_{KG}^{-1}f, g\rangle = \int_{\Omega} \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, S_{KG}^{-1}\Pi_{K_\omega}\Pi_{G_\omega}g\rangle d\mu(\omega)$
3. $\langle f, g\rangle = \langle S_{GK}f, S_{KG}^{-1}g\rangle = \int_{\Omega} \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{G_\omega}\Pi_{K_\omega}S_{KG}^{-1}g\rangle d\mu(\omega)$
4. $\langle f, g\rangle = \langle S_{GK}S_{KG}^{-1}f, g\rangle = \int_{\Omega} \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, S_{KG}^{-1}\Pi_{G_\omega}\Pi_{K_\omega}g\rangle d\mu(\omega)$

Theorem 2.5. *The following are equivalent:*

1. *S_{KG} is bounded below;*
2. *There exist $U \in \mathcal{L}(H)$ such that $\{T_\omega\}_{\omega\in\Omega}$ is a CRI, where*

$$T_\omega = \mathrm{m}(\omega)\mathrm{n}(\omega)U\Pi_{G_\omega}\Pi_{K_\omega}, \qquad (\omega \in \Omega).$$

If one of conditions holds, then $\mathcal{G}$ is a continuous fusion frame.

Proof. (1) $\Rightarrow$ (2) If S_{KG} is bounded below, then there exists $U \in \mathcal{L}(H)$ such that $US_{KG} = I_H$. It follows

$$\begin{aligned}\langle f, g\rangle &= \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{K_\omega}\Pi_{G_\omega}U^*g\rangle d\mu(\omega)\\ &= \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, (U\Pi_{G_\omega}\Pi_{K_\omega})^*g\rangle d\mu(\omega).\end{aligned}$$

(2) $\Rightarrow$ (1) If (2) holds, then for $f, g \in H$ we have

$$\langle US_{KG}f, g\rangle = \langle S_{KG}f, U^*g\rangle = \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, (U\Pi_{G_\omega}\Pi_{K_\omega})^*g\rangle d\mu(\omega) = \langle f, g\rangle,$$

hence $US_{KG} = I_H$. It follows that S_{KG} is bounded below.
If S_{KG} is bounded below, from (5) it follows that $\mathcal{G}$ is a continuous fusion frame. □

Corollary 2.6. *The following are equivalent:*

1. *S_{KG} is invertible;*
2. *There exist invertible operator $U \in \mathcal{L}(H)$ such that $\{T_\omega\}_{\omega\in\Omega}$ is a CRI, where*

$$T_\omega = \mathrm{m}(\omega)\mathrm{n}(\omega)U\Pi_{G_\omega}\Pi_{K_\omega}, \qquad (\omega \in \Omega).$$

If one of conditions holds, then $\mathcal{K}$ and $\mathcal{G}$ are continuous fusion frames.

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Coupled fixed point results for (ψ, φ)-weakly contractive mappings in ordered GP-metric spaces

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Abstract

In this paper, we present some coupled fixed point results for (ψ, φ)-weakly contractive mappings in the setup of partially ordered GP-metric spaces. Our results extend, generalize and modify several comparable results in the literature.

Keywords: GP-metric space, Partially ordered set, Coupled fixed point, Mixed monotone property.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

The concepts of mixed monotone mapping and coupled fixed point have been introduced in [1] by Bhaskar and Lakshmikantham. Also, they established some coupled fixed point theorems for a mixed monotone mapping in partially ordered metric spaces.

Definition 1.1. [1] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. F has the mixed monotone property if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for any $y \in X$, and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$ for any $x \in X$.

Definition 1.2. [1] An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.3. [2] The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied:

1. ψ is continuous and nondecreasing.
2. $\psi(t) = 0$ if and only if $t = 0$.

The concept of a partial metric space has been given by Matthews (see [3, 4]) as follows:

*Speaker

Definition 1.4. Let X be a nonempty set and let $p : X \times X \to \mathbb{R}^+$ satisfies:
(P1) $x = y \iff p(x, x) = p(y, y) = p(x, y)$, for all $x, y \in X$;
(P2) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;
(P3) $p(x, y) = p(y, x)$, for all $x, y \in X$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$, for all $x, y, z \in X$.
Then, the pair (X, p) is called a partial metric space and p is called a partial metric on X.

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims [5].

Definition 1.5. [5] Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:
(G1) $G(x, y, z) = 0$ iff $x = y = z$;
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Recently, Zand and Nezhad [6] introduced a new generalized metric space, ($GP-$metric space), as a generalization of both partial metric spaces and G-metric spaces.

Definition 1.6. [6] Let X be a nonempty set. Suppose that a mapping $G_p : X \times X \times X \to \mathbb{R}^+$ satisfies:
$(GP1)$ $x = y = z$ if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;
$(GP2)$ $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$;
$(GP3)$ $G_p(x, y, z) = G_p(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry);
$(GP4)$ $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then G_p is called a $GP-$metric and the pair (X, G_p) is called a $GP-$metric space.

Definition 1.7. Let (X, G_p) be a $GP-$metric space. Let $\{x_n\}$ be a sequence of points of X.
1. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ if $\lim_{n,m\to\infty} G_p(x, x_n, x_m) = G_p(x, x, x)$.
2. $\{x_n\}$ is said to be a $GP-$Cauchy sequence, if $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$ exists (and is finite).
3. (X, G_p) is said to be GP-complete if and only if every GP-Cauchy sequence in X is GP-convergent to an $x \in X$ such that $G_p(x, x, x) = \lim_{n,m\to\infty} Gp(x_n, x_m, x_m)$.

Lemma 1.8. *If G_p be a GP-metric on X, then mappings $d_{G_p}, d'_{G_p} : X \times X \to R^+$ given by*

$$d_{G_p}(x, y) = G_p(x, y, y) + G_p(y, x, x) - G_p(x, x, x) - G_p(y, y, y)$$

and

$$d'_{G_p}(x, y) = max\{G_p(x, y, y) - G_p(x, x, x), G_p(y, x, x) - G_p(y, y, y)\}$$

define equivalent metrics on X.

Lemma 1.9. *(1) A sequence $\{x_n\}$ is a GP-Cauchy sequence in a GP-metric space (X, G_p) if and only if it is a Cauchy sequence in the metric space (X, d_{G_p}).*

(2) A partial metric space (X, G_p) is GP-complete if and only if the metric space (X, d_{G_p}) is complete. Moreover, $\lim_{n\to\infty} d_{G_p}(x, x_n) = 0$ if and only if

$$\lim_{n\to\infty} G_p(x, x_n, x_n) = \lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_n, x_m) = \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x, x, x).$$

2 Main results

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and G_p be a GP−metric on X such that (X, G_p) is a complete GP−metric space. Let $F : X \times X \to X$ be a mapping such that*

$$\begin{aligned}&\psi\big(\max\{G_p(F(x,y), F(u,v), F(s,t)), G_p(F(y,x), F(v,u), F(t,s))\}\big)\\ &\leq \psi\big(\max\{G_p(x,u,s), G_p(y,v,t)\}\big) - \varphi\big(\max\{G_p(x,u,s), G_p(y,v,t)\}\big),\end{aligned} \tag{1}$$

for every pairs $(x, y), (u, v), (s, t) \in X \times X$ such that $x \preceq u \preceq s$ and $y \succeq v \succeq t$, or $s \preceq u \preceq x$ and $t \succeq v \succeq y$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Also, suppose that F has the mixed monotone property and F is continuous. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

In the following theorem we omit the continuity assumption of F.

Theorem 2.2. *Let $(X, \preceq, G_p)$ be a partially ordered complete GP-metric space and let $F : X^2 \to X$ be such that F has the mixed monotone property and satisfies 1 for every $x, y, u, v, s, t \in X$ with $x \preceq u \preceq s$ and $y \succeq v \succeq t$. Also, suppose that:*

(i) If $\{x_n\}$ is a nondecreasing sequence and $x \in X$ with $\lim_{n,m\to\infty} G_p(x_n, x_m, x) = G_p(x, x, x) = 0$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.

(ii) If $\{x_n\}$ is a nonincreasing sequence and $x \in X$ with $\lim_{n,m\to\infty} G_p(x_n, x_m, x) = G_p(x, x, x) = 0$, then $x_n \succeq x$, for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

Theorem 2.3. *Under the hypotheses of Theorem 2.2, suppose that x_0 and y_0 are comparable. Then, it follows that $x = F(x, y) = F(y, x) = y$, that is, F has a coupled fixed point of the form (x, x).*

The following corollary can be deduced from our previous obtained results.

Corollary 2.4. *Let $(X, \preceq, G_p)$ be a partially ordered complete GP-metric space and let $F : X^2 \to X$ be a mapping having the mixed monotone property. Assume that*

$$\begin{aligned}&\psi(\max\{G_p(F(x,y), F(u,v), F(s,t)), G_p(F(y,x), F(v,u), F(t,s))\})\\ &\leq \psi(\tfrac{G_p(x,u,s)+G_p(y,v,t)}{2}) - \varphi(\max\{G_p(x,u,s), G_p(y,v,t)\}),\end{aligned} \tag{2}$$

for every $x, y, u, v, s, t \in X$ with $x \preceq u \preceq s$ and $y \succeq v \succeq t$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions.

Also, suppose that F is continuous, or, X has the following properties:

(i) If $\{x_n\}$ is a nondecreasing sequence and $x \in X$ with $\lim_{n,m\to\infty} G_p(x_n, x_m, x) = G_p(x, x, x) = 0$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.

(ii) If $\{x_n\}$ is a nonincreasing sequence and $x \in X$ with $\lim_{n,m\to\infty} G_p(x_n, x_m, x) = G_p(x, x, x) = 0$, then $x_n \succeq x$, for all $n \in \mathbb{N}$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

In the following theorem, we give a sufficient condition for the uniqueness of the coupled fixed point.

Theorem 2.5. *In addition to the hypotheses of Theorems 2.1 suppose that for every (x, y) and $(x^*, y^*) \in X^2$, there exists $(u, v) \in X^2$ which is comparable with (x, y) and (x^*, y^*). Then, F has a unique coupled fixed point.*

Theorem 2.6. *Under the hypotheses of Theorem 2.2, suppose in addition that for every (x, y) and (x^*, y^*) in X^2, there exists $(u, v) \in X^2$ which is comparable with (x, y) and (x^*, y^*). Then F has a unique coupled fixed point of the form (x, x).*

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Coupled Fixed Points via Measurer of Noncompactness

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Abstract

In this paper, using the technique of measure of nono compactness and Darbo fixed point theorem we prove some theorems on coupled fixed point theorems for a class of functions

Keywords: Keywords: Measure of noncompactness,Banach space, Coupled fixed point
Mathematics Subject Classification [2010]: Subject[2000]: 47H09, 34A12

1 Introduction

Bhaskar and Lakshmikantham [5] introduced the concept of a coupled fixed point for a operator and obtained some coupled fixed point existence theorems for a class of operators. In this paper, using the technique of measure of noncompactness, we prove some the existence theorems of coupled fixed point for a class of operators. Measure of noncompactness have been successfully applied in theories of differential and integral equations(see [7]) . This concept was first introduced by Kuratowski. In some Banach spaces, there are known formulas of measure of noncompactness (see [2]).

Throughout this paper we assume that E is a Banach space. For a subset X of E, the closure and closed convex hull of X in E are denoted by $\bar{X}$, $co(X)$, respectively. Also let $\bar{B}_r$is the closed ball in E centered at zero and with radius r and we write $B(x_0, r)$ to denote the closed ball centered at x_0 with radius r. Moreover, we symbolize by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ subfamily consisting of all relatively compact subsets of E. In addition to, The norm $\|.\|$ in $E \times E$ is defined by $\|(x, y)\| = \|x\| + \|y\|$ for any $x, y \in E \times E$.

The following definitions will be needed in the sequel.

Definition 1.1. ([3]) A mapping $\mu : \mathfrak{M}_E \longrightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions;

($\mathbf{B}_1$) The family $Ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and Ker$\mu \subseteq \mathfrak{N}_E$.

($\mathbf{B}_2$) If $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.

(B_3) $\mu(\overline{X}) = \mu(X)$.

(B_4) $\mu(CoX) = \mu(X)$.

(B_5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1)$. ($\mathbf{B}_6$) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subseteq X_n$, $(n \geq 1)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

*Speaker

The family $Ker\mu$ described in (B_1) said to be the kernel of the measure of noncompactness μ. Observe that the intersection set X_∞ from (B_6) is a member of the family $Ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in Ker\mu$.
The following theorems and examples are basic to all the results of this work.

Definition 1.2. ([6]) An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $G : X \times X \to X$ if $G(x, y) = x$ and $G(y, x) = y$.

The following theorems and examples are basic to all the results of this work.

Theorem 1.1. ([4]) Suppose $\mu_1, \mu_2, ..., \mu_n$ be the measures in $E_1, E_2, ..., E_n$ respectively. Moreover assume that the function $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, ..., x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 3, ..., n$. Then

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), ..., \mu_n(X_n))$$

defines the measure of noncompactness in $E_1 \times E_2 \times ... \times E_n$ where X_i denotes the natural projection of X into E_i for $i = 1, 2, ..., n$.

Now, as results from Theorem 1.1, we present the following examples.

Example 1.1 Let μ be a measure of noncompactness, considering $F(x, y) = \{x, y\}$ for any $(x, y) \in [0, \infty)^2$, then the conditions of Theorem 1.1 are satisfied. Therefore, $\widetilde{\mu}(X) = \max\{\mu_1(X_1), \mu_2(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where X_i denotes the natural projection of X into E. for $i = 1, 2$.

Example 1.2 Let μ be a measure of noncompactness. We define $F(x, y) = x+y$ for any $(x, y) \in [0, \infty)^2$. Then F has the properties mentioned in Theorem 1.1.Hence $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$ where X_i denotes the natural projection of X into E for $i = 1, 2$.

Theorem 1.2. (Darbo [3]) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $G : \Omega \longrightarrow \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that

$$\mu(G(X)) \leq k\mu(X) \tag{1}$$

for any $X \subset \Omega$.Then G has a fixed point.

Theorem 1.3. (Schauder [1]) Let Ω be a closed, convex subset of a normed linear space E. Then every compact, continuous map $G : \Omega \to \Omega$ has at least one fixed point.

2 Main results

Before starting the main results, we always suppose that Ω is a nonempty, bounded, closed, and convex subset of E, moreover

$$\Lambda = \left\{\delta : [0, \infty) \to [0, \infty) : \delta \text{ is increasing map and } \lim_{n\to\infty} \delta^n(t) = 0\right\}.$$

Theorem 2.1. Let $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that

$$\mu(G(X_1 \times X_2) \leq k \max\{\mu(X_1), \mu(X_2)\} \tag{2}$$

for any $X_1, X_2 \subset \Omega$, where μ is an arbitrary measure of noncompactness and k is a constant with $0 \leq k < 1$. Then G has at least a coupled fixed point.

Corollary 2.1. Let $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that

$$\|G(x, y) - G(u, v)\| \leq k \max\{\|x - y\|, \|u - v\|\}$$

for any $(x,y),(u,v) \in \Omega \times \Omega$. Moreover, $0 \leq k < 1$ be a constant. Then G has a coupled fixed point.

Lemma 2.1. Let $\delta \in \Lambda$ and $G : \Omega \longrightarrow \Omega$ be a continuous function satisfying

$$\mu(G(X)) \leq \delta(\mu(X)) \tag{3}$$

for each $X \subset \Omega$, where μ is an arbitrary measure of noncompactness. Then G has at least one fixed point.

Theorem 2.2. Let μ be an arbitrary measure of noncompactness and $\delta \in \Lambda$. Suppose that the mapping $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function satisfying

$$\mu(G(X_1 \times X_2)) \leq \delta(\frac{\mu(X_1)+\mu(X_2)}{2}) \tag{4}$$

for all $X_1, X_2 \subset \Omega$. Then G has at least a coupled fixed point.

Proof: We define a mapping $\widetilde{G} : \Omega \times \Omega \longrightarrow \Omega \times \Omega$ by

$$\widetilde{G}(x,y) = (G(x,y), G(y,x)).$$

It is obvious that $\widetilde{G}$ is continuous. On other hand, from Example 1.2, we have

$$\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$$

is a measure of noncompactness in $E \times E$. Now let $X \subset \Omega \times \Omega$ be any nonempty subset. Then by $(\mathbf{B}_2)$ and (7) we obtain

$$\begin{aligned}
\widetilde{\mu}(\widetilde{G}(X)) &\leq \widetilde{\mu}(G(X_1 \times X_2), G(X_2 \times X_1)) \\
&= \mu(G(X_1 \times X_2)) + \mu(G(X_2 \times X_1))) \\
&\leq \delta(\frac{\mu(X_1)+\mu(X_2)}{2}) + \delta(\frac{\mu(X_2)+\mu(X_1)}{2}) \\
&= 2\delta(\frac{\mu(X_1)+\mu(X_2)}{2}) \\
&= 2\delta(\frac{\widetilde{\mu}(X)}{2}).
\end{aligned}$$

Hence

$$\frac{1}{2}\widetilde{\mu}(\widetilde{G}(X)) \leq \delta(\frac{1}{2}\widetilde{\mu}(X)).$$

Taking $\widetilde{\mu}' = \frac{1}{2}\widetilde{\mu}$, we get

$$\widetilde{\mu}'(\widetilde{G}(X)) \leq \delta(\widetilde{\mu}'(X)).$$

Since, $\widetilde{\mu}'$ is measure of noncompactness, therefore, all the conditions of Lemma 2.1 are satisfied. Hence G has a coupled fixed point.

Now, we will show that many results can be deduced from previous obtained results.

Let, $0 \leq k < 1$ be a constant and $\delta(t) = kt$ for each $t \in [0,\infty)$. Then, Theorem 2.2 reduces to the following corollary.

Corollary 2.2. Assume that $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function such that

$$\mu(G(X_1 \times X_2)) \leq \frac{k}{2}(\mu(X_1) + \mu(X_2)) \tag{5}$$

for each $X_1, X_2 \subset \Omega$ where $0 \leq k < 1$ is a constant. Then G has a coupled fixed point.

Corollary 2.3. Let $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function. In addition, suppose that

$$\|G(x,y)-G(u,v)\| \leq \delta(\frac{\|x-y\|+\|u-v)\|}{2})$$

for any $(x,y),(u,v) \in \Omega \times \Omega$ where $\delta \in \Lambda$. Then G has a coupled fixed point.

Proof: It is clear that the function $\mu : \mathfrak{M}_E \longrightarrow [0,\infty)$ defined by $\mu(X) = diam(X)$ is a measure of noncompactness. Now, let $X_1, X_2 \subset \Omega$ and $(x,y),(u,v) \in X_1 \times X_2$. Then

$$\begin{aligned} \|G(x,y)-G(u,v)\| &\leq \delta(\frac{\|x-y\|+\|u-v)\|}{2}) \\ &\leq \delta(\frac{diam(X_1)+diam(X_2)}{2}). \end{aligned}$$

This yields

$$diam(G(X)) \leq \delta(\frac{diam(X_1)+diam(X_2)}{2})$$

Therefore, Theorem 2.2 show that G has a coupled fixed point.

Corollary 2.4. Let $G : \Omega \times \Omega \longrightarrow \Omega$ be a continuous function. Assume that, there exists a $k \in [0,1)$ with

$$\|G(x,y)-G(u,v)\| \leq \frac{k}{2}(\|x-y\|+\|u-v)\|)$$

$f$or any $(x,y),(u,v|) \in \Omega \times \Omega$. Then G has a coupled fixed point.

Proof: Taking $\delta(t) = \frac{k}{2}$ in Corollary 2.3, we obtain Corollary 2.4.

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Exist and uniqueness of p-best approximation in fuzzy normed spaces

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Abstract

In this paper, we define a fuzzy normed space and study the concept of p-best approximation in fuzzy normed spaces. We also define a p-proximal set and p-Chebyshev set and prove some interesting results in this newsetup.

Keywords: Fuzzy Normed Spaces; p-Best Approximation; p-Proximal Set; p-Chebyshev Set.
Mathematics Subject Classification [2010]: 03Bxx, 90C59

1 Introduction

In this section we recall some notations and basic definitions used in this paper. A function $f : \mathbb{R} \to \mathbb{R}_0^+ = [0,1]$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $sup_{t\in\mathbb{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$. If $a \in \mathbb{R}_0^+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1 & t > a, \\ 0 & t \leq a. \end{cases}$$

A t-norm is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is an abelian monoid with unit one and $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c \in [0,1]$.

Definition 1.1. Let X be a linear space of a dimension greater than one, $*$ a t-norm continuous, and let N be a mapping from $X \times \mathbb{R}$ into D^+. The following conditions are satisfied for all $x, y \in X$ and $t, s > 0$,
(i) $N(x;t) = H_0(t)$ if and only if $x = \theta$ (θ is the null vector in X),
(ii) $N(\alpha x;t) = N(x; \frac{t}{|\alpha|})$ for all t in $\mathbb{R}^+$,
(iii) $N(x + y; t + s) \geq N(x;t) * N(y;s)$.
Triple $(X, N, *)$ is called a fuzzy normed space. If in addition, $t > 0$, $(x) \to N(x;t)$ is a continuous map on X, then $(X, N, *)$ is called a strong fuzzy normed space.

*Speaker

Example 1.2. Let $(X, \|.\|)$ be a normed space and $a * b = ab$ for $a, b \in [0, 1]$. For all $x \in X$ and $t > 0$, consider

$$N_s(x;t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0, \\ 0 & t \leq 0\,. \end{cases}$$

and

$$N(x;t) = H_0(t - \|x\|).$$

Then $(X, N_s, *)$ and $(X, N, *)$ are fuzzy normed spaces.

Definition 1.3. Let $(X, N, *)$ be a fuzzy normed space. Then we define the open ball $B_x(r,t)$ and the closed ball $B_x[r,t]$ with center $x \in X$, radius $t > 0$, $0 < r < 1$ as follows

$$B_x(r,t) = \{y \in X : N(x-y;t) > 1-r\}.$$
$$B_x[r,t] = \{y \in X : N(x-y;t) \geq 1-r\}.$$

2 p-best approximation

Definition 2.1. Let A be a nonempty subset of a fuzzy normed space $(X, N, *)$. For $x \in X$and $t > 0$, let

$$N(x-A;t) = sup\{N(x-y;t) : \ \ y \in A\}.$$

An element $y_0 \in A$ is said to be a p-best approximation to x from A if

$$N(x-y_0;t) = N(x-A;t).$$

By $P_A^t(x)$, we denote the set of elements of p-best approximation of x by elements of the set A, i.e.,

$$P_A^t(x) = \{y \in A : \quad N(x-A;t) = N(x-y;t)\}.$$

Definition 2.2. Let A be a nonempty subset of a fuzzy normed space $(X, N, *)$. Then A is said to be a p-proximal set if $P_A^t(x)$ is a nonvoid for every $x \in X$. A is called a p-Chebyshev set if $P_A^t(x)$ contains exactly one element for every $x \in X$.

Theorem 2.3. *Let A be a subspace of a fuzzy normed space $(X, N, *)$ and $t > 0$.*
(i) If $x \in A$ then $P_A^t(x) = \{x\}$.
(ii) If $x \in cl(A) \setminus A$ then $P_A^t(x) = \{0\}$ ($cl(A)$ is the clouser of A).

Example 2.4. Let $X = \mathbb{R}^2$. For $a, b \in [0, 1]$, let $a * b = ab$. Define $N : \mathbb{R}^2 \to D^+$ by

$$N(x;t) = \begin{cases} (\exp \frac{\sqrt{x_1^2+x_2^2}}{t})^{-1} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Then $(X, N, *)$ is a fuzzy normed space. Let

$$A = \{(x_1, x_2) \in \mathbb{R}^2 | \quad -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\}$$

and $x = (0, 3)$. Then for every $t > 0$,

$$N((-1,1)-(0,3);t) = N((1,1)-(0,3);t) = (exp\frac{\sqrt{5}}{t})^{-1}.$$

On the other hand,

$$\begin{aligned} N(A-(0,3);t) &= sup\{N((x_1,x_2)-(0,3);t)| \quad -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\} \\ &= (exp\frac{\sqrt{5}}{t})^{-1}. \end{aligned}$$

So, for every $t > 0$, $y_0 = (-1, 1)$ and $y_1 = (1, 1)$ are p-best approximations of (0, 3) from A. Therefore $y_0 = (-1, 1)$ and $y_1 = (1, 1)$ are p-best approximations of $x = (0, 3)$ from A. Therefore A is a p-proximal set and is not a p-Chebyshev set.

Theorem 2.5. *(p-proximal sets are closed.) Let A be a p-proximal subset of a fuzzy normed linear space X. Then A is closed.*

Lemma 2.6. *Let $(X, \|.\|)$ be a normed space and*

$$N_s(x;t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0 \\ \\ 0 & t \leq 0 \end{cases}$$

Then $y_0 \in A$ is a best approximation to $x \in X$ in the normed space if and only if y_0 is a p-best approximation to x in the fuzzy normed space $(X, N, *)$ for each $t > 0$.

3 Exist and Uniqueness of p-Best Approximation

Definition 3.1. For a subset A of X, $t > 0$ and $x \in X$. Put

$$\widehat{A}^t = \{x \in X : \ N(x;t) = N(x-A;t)\} = \left(P_A^t\right)^{-1}(\{0\}).$$

Lemma 3.2. *Let X be a fuzzy normed linear space and A a subspace of X, $t > 0$ and $x \in X$. Then*
(i) $y_0 \in P_A^t(x)$ if and only if $x - y_0 \in \widehat{A}^t$.
(ii) $0 \in P_A^t(x)$ if and only if $x \in \widehat{A}^t$.

Theorem 3.3. *Let A be a subspace of fuzzy normed linear space X, $t > 0$ and $x \in X$. Then A is p-proximal set if and only if $X = A + \widehat{A}^t$.*

Proof. If A is p-proximal set, let $x \in X$ and $y_0 \in P_A^t(x)$ Then by Lemma (3.2), $X = A + \widehat{A}^t$. Conversely, Let $X = A + \widehat{A}^t$, and let $x \in X$ Then $x = y_0 + \widehat{y}$ for some $y_0 \in A$ and some $\widehat{y} \in \widehat{A}^t$. Then by Lemma (3.2), $P_A^t(\widehat{y}) = P_A^t(x - y_0)$, this implies that $0 \in P_A^t(x - y_0)$, then for all $y \in A$,

$$N(x - y_0 - 0;t) \geq N(x - y_0 - y;t).$$

This means that $y_0 \in P_A^t(x)$. Therefore A is p-proximal.

Theorem 3.4. *Let A be linear subspace of fuzzy normed linear space X, $t > 0$ and $x \in X$. Then A is p-Chebyshev subspace of X if and only if $X = A \oplus \widehat{A}^t$.*

Proof. Suppose that A is p-Chebyshev subspaces of a fuzzy normed linear space X. Let $x \in X$, Then by Theorem(3.3), $X = A + \widehat{A}^t$. Now, let $x = y_1 + \widehat{y}_1 = y_2 + \widehat{y}_2$, where $y_1, y_2 \in A$ and $\widehat{y}_1, \widehat{y}_2 \in \widehat{A}^t$ show $y_1 = y_2$ and $\widehat{y}_1 = \widehat{y}_2$. Therefore $X = A \oplus \widehat{A}^t$.
Conversely, let$X = A \oplus \widehat{A}^t$, and $x \in X$, suppose that $y_1, y_2 \in P_A^t(x)$ show $y_1 = y_2$, Therefore A is p-Chebyshev.

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Existence results of three weak solutions for a two-point boundary value problem

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Abstract

we prove the existence of at least three weak solutions for one-dimensional fourth-order equations via two three critical points theorems.

Keywords: Dirichlet boundary condition, Variational methods, Critical points.
Mathematics Subject Classification [2010]: 34B15, 34B18, 58E05.

1 Introduction

In this note, we consider the following fourth-order boundary value problem

$$\begin{cases} u''''h(x,u') - u'' = [\lambda f(x,u) + g(u)]h(x,u'), & \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases} \tag{1}$$

where λ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, with $g(0) = 0$, and $h : [0,1]\times\mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$. Due to the importance of fourth-order two-point boundary value problems in describing a large class of elastic deflection, many researchers have studied the existence and multiplicity of solutions for such a problem, we refer the reader to [1, 4, 5] and references therein. In the present paper, employing two three critical points theorems, we establish the existence three weak solutions for the problem (1). We say that a function $u \in H^2([0,1]) \cap H_0^1([0,1])$ is a *weak solution* of problem (1) if

$$\int_0^1 u''(x)v''(x)\,dx + \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}d\tau\right) v'(x)\,dx - \lambda\int_0^1 f(x,u(x))v(x)\,dx - \int_0^1 g(u(x))v(x)\,dx = 0$$

holds for all $v \in H^2([0,1]) \cap H_0^1([0,1])$.

*Speaker

2 Main results

Put

$$A := \frac{\pi^4 - L}{2\pi^4}, \qquad B := \frac{\pi^2 + m(\pi^4 + L)}{2m\pi^4},$$

and suppose that $B \leq 4A\pi^2$. We formulate our main results as follows.

Theorem 2.1. *Assume that there exist two positive constants c, d, with $c < \frac{32}{3\sqrt{3}\pi} d$, such that*

(A1) $F(x,t) \geq 0$ for all $(x,t) \in \left([0, \frac{3}{8}] \cup [\frac{5}{8}, 1]\right) \times [0,d]$;

(A2)

$$\frac{\int_0^1 \max_{|t|\leq c} F(x,t)\, dx}{c^2} < \frac{27}{4096} \frac{\int_{3/8}^{5/8} F(x,d)\, dx}{d^2};$$

(A3)

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} \leq \frac{\pi^4 A}{B} \frac{\int_0^1 \max_{|t|\leq c} F(x,t)\, dx}{c^2}.$$

Then, for every λ in

$$\Lambda := \left] \frac{4096Bd^2}{27\int_{3/8}^{5/8} F(x,d)\, dx}, \frac{Bc^2}{\int_0^1 \max_{|t|\leq c} F(x,t)\, dx} \right[,$$

problem (1) *has at least three distinct weak solutions.*

Proof. For every $u \in X := H^2([0,1]) \cap H_0^1([0,1])$, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ by setting

$$\Phi(u) := \frac{1}{2}\|u\|^2 + \int_0^1 H(x, u'(x))\, dx + \int_0^1 G(u(x))\, dx,$$

$$\Psi(u) := \int_0^1 F(x, u(x))\, dx,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \qquad \forall\ u \in X.$$

Note that the weak solutions of (1) are exactly the critical points of I_λ. The functionals Φ, Ψ satisfy the regularity assumptions of Theorem 3.6 in [3]. We prove, for each

$$\lambda \in \Lambda \subseteq \left] \frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)} \right[,$$

the functional I_λ has at least three distinct critical points in X, which are the weak solutions of the problem (1). This completes the proof. □

Lemma 2.2. *Assume that $f(x,t) \geq 0$ for all $(x,t) \in [0,1] \times \mathbb{R}$. If u is a weak solution of (1), then $u(x) \geq 0$ for all $x \in [0,1]$.*

Proof. Arguing by contradiction, if we assume that u is negative at a point of $[0,1]$, the set

$$\Omega := \{x \in [0,1] : u(x) < 0\},$$

is non-empty and open. Moreover, let us consider $\bar{v} := \min\{u, 0\}$, one has, $\bar{v} \in X$. So, taking into account that u is a weak solution and by choosing $v = \bar{v}$, from our assumptions, one has

$$\begin{aligned} 0 &\geq \lambda \int_\Omega f(x, u(x))u(x)\,dx \\ &= \int_\Omega |u''(x)|^2\,dx + \int_\Omega \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)} d\tau \right) u'(x)\,dx - \int_\Omega g(u(x))u(x)\,dx \\ &\geq \frac{\pi^4 - L}{\pi^4} \|u\|^2_{H^2(\Omega)\cap H_0^1(\Omega)}. \end{aligned}$$

Therefore, $\|u\|_{H^2(\Omega)\cap H_0^1(\Omega)} = 0$ which is absurd. Hence, the conclusion is achieved. □

Our other main result is as follows.

Theorem 2.3. *Assume that there exist three positive constants c_1, c_2, d, with $\frac{3\sqrt{3}\pi}{16\sqrt{2}} c_1 < d < \frac{3\sqrt{3}}{64\sqrt{2}} c_2$, such that*

(B1) $f(x,t) \geq 0$ for all $(x,t) \in [0,\ 1] \times [0, c_2]$;

(B2)

$$\frac{\int_0^1 F(x, c_1)\,dx}{c_1^2} < \frac{9}{2048} \frac{\int_{3/8}^{5/8} F(x,d)\,dx}{d^2};$$

(B3)

$$\frac{\int_0^1 F(x, c_2)\,dx}{c_2^2} < \frac{9}{4096} \frac{\int_{3/8}^{5/8} F(x,d)\,dx}{d^2}.$$

Let

$$\Lambda' := \left] \frac{2048}{9} \frac{Bd^2}{\int_{3/8}^{5/8} F(x,d)\,dx}, B \min\left\{ \frac{c_1^2}{\int_0^1 F(x,c_1)\,dx}, \frac{c_2^2}{2\int_0^1 F(x,c_2)\,dx} \right\} \right[.$$

Then, for every $\lambda \in \Lambda'$ the problem (1) has at least three weak solutions u_i, $i = 1,2,3$, such that $0 < \|u_i\|_\infty \leq c_2$.

Proof. Without loss of generality, we can assume $f(x,t) \geq 0$ for all $(x,t) \in [0,1] \times \mathbb{R}$. Fix λ as in the conclusion and take X, Φ and Ψ as in the proof of Theorem 2.1. Put

$$w(x) = \begin{cases} -\frac{64d}{9}(x^2 - \frac{3}{4}x), & x \in [0, \frac{3}{8}[, \\ d, & x \in [\frac{3}{8}, \frac{5}{8}], \\ -\frac{64d}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}), & x \in]\frac{5}{8}, 1], \end{cases}$$

$r_1 = Bc_1^2$ and $r_2 = Bc_2^2$. Therefore, one has $2r_1 < \Phi(w) < \frac{r_2}{2}$ and we have

$$\begin{aligned}\frac{1}{r_1}\sup_{\Phi(u)<r_1}\Psi(u) &\leq \frac{1}{Bc_1^2}\int_0^1 F(x,c_1)\,dx < \frac{1}{\lambda}\\ &< \frac{9}{2048}\frac{\int_{3/8}^{5/8}F(x,d)\,dx}{Bd^2} \leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},\end{aligned}$$

and

$$\begin{aligned}\frac{2}{r_2}\sup_{\Phi(u)<r_2}\Psi(u) &\leq \frac{2}{Bc_2^2}\int_0^1 F(x,c_2)\,dx < \frac{1}{\lambda}\\ &< \frac{9}{2048}\frac{\int_{3/8}^{5/8}F(x,d)\,dx}{Bd^2} \leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

So, conditions (j) and (jj) of Corollary 3.1 in [2] are satisfied. Finally, let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then, u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1). Hence, owing to Lemma 2.2, we obtain $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for all $x \in [0,1]$. So, one has $\Psi(su_1 + (1-s)u_2) \geq 0$ for all $s \in [0,1]$. From [2] the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are weak solutions of (1). This complete the proof. □

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F-Contraction Type Mappings in 0-Complete Partial Metric Spaces

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Abstract

In this paper we define the F-contraction type mapping and prove the existence of fixed point theorem for F-contractive mappings defined on 0-complete partial metric spaces.

Keywords: Partial metric space, F- Contraction, Fixed point
Mathematics Subject Classification [2010]: 47H10, 54H25

1 Introduction

The notion of partial metric space has been introduced by Matthews [2] in 1994 as a part of the study of denotational semantics of dataflow network. In partial metric space, the usual distance was replaced by partial metric, with an interesting porperty nonzero self-distance of points.

Recently, Wardowski [4] introduced a new concept of F-contraction and proved a fixed point theorem which generalizes the Banach contraction principle in a different way than the known results of the literature on complete metric space.

Definition 1.1. [1] A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow [0, \infty)$ such that for all $x, y, z \in X$,

$(P1)$ $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$,

$(P2)$ $p(x, x) \leq p(x, y)$,

$(P3)$ $p(x, y) = p(y, x)$,

$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Suppose that p is a partial metric on X, then it can be shown that the function $p^s : X \times X \longrightarrow [0, \infty)$ is given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

*Speaker

Definition 1.2. [2]
(a) A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if

$$lim_{n\to\infty} p(x_n,x) = p(x,x).$$

(b) A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if $\lim_{n\to\infty} p(x_n,x_m)$ exists and finite.
(c) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Lemma 1.3. [3] *Let (X,p) be a partial metric space. Then*
(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*
(b) *A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Definition 1.4. Let $F : [0,\infty] \longrightarrow R$ be a mapping satisfying:
(i) F is strictly increasing, i.e. for all $\alpha,\beta \in [0,\infty]$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,
(ii) For each sequence $\{\alpha_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
(iii) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Suppose that (X,p) is a partial metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x,y \in X$

$$\text{if } p(Tx,Ty) > 0 \text{ then } \tau + F(p(Tx,Ty)) \le F(p(x,y)). \tag{1}$$

When we consider in (1) the different types of the mapping F then we obtain the variety of contractions.

Following is an example of an F-cotraction.

Example 1.5. Let $F : [0,\infty] \to R$ defined by $F(\alpha) = \ln\alpha$. It is clear that F satisfyies in (i),(ii) and (iii), for any $k \in (0,1)$. Each mapping $T : X \to X$ satisfying (1) is an F-contraction such that

$$p(Tx,Ty) \le e^{-\tau} p(x,y),$$

for all $x,y \in X, Tx \neq Ty$.

It is clear that for $x,y \in X$ with $Tx = Ty$, the inequality $p(Tx,Ty) \le e^{-\tau} P(x,y)$ also hold, i.e., T is a Banach contraction.

By using Definition 1.4, one can see that:

Proposition 1.6. *Every F-contraction T is a contractive mapping, i.e.*

$$p(Tx,Ty) < P(x,y),$$

for all $x,y \in X$ with $Tx \neq Ty$.

Let F_1 and F_2 be two mappings satisfying (i)-(iii). If $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and a mapping $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is F_2-contraction. Indeed, we have $G(p(Tx, Ty)) \leq G(p(x, y))$ for all $x, y \in XwithTx \neq Ty$. Thus, for all $x, y \in X, Tx \neq Ty$ we obtain that

$$\tau + F_2(p(Tx, Ty)) \leq F_1(p(x, y)) + G(p(x, y)) = F_2(p(x, y)).$$

2 Main results

Definition 2.1. A sequence $\{x_n\}_{n\in N}$ in partial metric space (X, p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Note that every 0-Cauchy sequence in (X, p) is Cauchy in (X, p^s), and that every complete partial metric space is 0-complete.

Theorem 2.2. *Let (X, p) be a 0- complete partial metric space and $T : X \to X$ be F-cotraction. Then T has a unique fixed point $u \in X$ and for every $u_0 \in X$ a sequence $\{T^n u_o\}_{n\in N}$ is convergent to u.*

To prove theorem suppose that T has two fixed point, say $u_1 \neq u_2 \in X$. So $Tu_1 = u_1, Tu_2 = u_2$. Then we get

$$\tau \leq F(p(u_1, u_2)) - F(p(Tu_1, Tu_2)) = 0$$

and so $\tau \leq 0$, which is a contradiction.

Now let u_0 be an arbitrary point in X and $\{u_n\}$ be a sequence in X such that $Tu_n = u_{n+1}$, $n = 0, 1, 2, ...$ Denote $\beta_n = p(u_{n+1}, u_n)$, $n = 0, 1, ...$

If there exists $n_0 \in N$ for which $u_{n_0+1} = u_{n_0}$ then $Tu_{n_0} = u_{n_0}$ and the proof is complete. So suppose that $u_{n+1} \neq u_n$ for every $n \in N$. Thus by using of definition 1.4 it can be shown that $\{u_n\}_{n\in N}$ is a Cauchy sequence in (X, p) and also it is a 0-Cauchy. Then by continuity of T, we have $p(u, Tu) = p(Tu, Tu) = 0$. This means that $T_u = u$.

By the following theorems we can find the unique common fixed point of two mappings.

Theorem 2.3. *Let (X, p) be a partial metric space and $f, g : X \to X$ be mappings. Let $F : [0, \infty] \longrightarrow R$ be such that satisfies condition (i)-(iii) of definition 1.4 and suppose that $\tau > 0$ and for all $x, y \in X$ and $p(fx, gy) > 0$, be such that*

$$\tau + F(p(fx, gy)) \leq F(p(x, y)).$$

If f has a fixed point $u \in X$, then u is a unique common fixed point of f and g, and $p(u, u) = 0$.

Theorem 2.4. *Let (X, p) be a 0-complete partial metric space and $f, g : X \to X$ be mappings. Suppose that there exists $F : [0, \infty] \longrightarrow R$ such that satisfies condition (i)-(iii) of definition 1.4 and suppose that for $\tau > 0$ and $x, y \in X$, with $p(fx, gy) > 0$, we have*

$$\tau + F(p(fx, gy)) \leq F(p(x, y)).$$

If (i)f or g is continuous
or
(ii) F is continuous,
then f and g have a unique common fixed point $u \in X$, and $p(u, u) = 0$.

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Poster Fixed point and the stability of $*$-derivations on C^*-ternary algebras pp.: 1–4

Fixed point and the stability of $*$-derivations on C^*-ternary algebras

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Abstract

By using the fixed point method, we prove the Hyers-Ulam stability of $*$-derivations associated with the following additive mapping on C^*-ternary algebras:

$$\sum_{k=2}^{n}\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f\left(\sum_{i=1,i\neq i_1,\ldots,i_{n-k+1}}^{n} x_i-\sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right)=2^{n-1}f(x_1) \tag{1}$$

for a fixed positive integer n with $n\geq 2$.

Keywords: C^*-ternary algebra, $*$-Derivation, Fixed point method, Hyers-Ulam stability
Mathematics Subject Classification [2010]: 39A30, 58J20

1 Introduction

The first stability problem waz raised in 1940 and it is solved in 1941. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

Suppose that $\mathcal{A}$ be a complex banach space and $(x,y,z)\to[x,y,z]$ on $\mathcal{A}^3$ to $\mathcal{A}$ be a ternary multiplier. $\mathcal{A}$ with this ternary multiplier is called a C^*-ternary algebra

Definition 1.1. Let $\mathcal{A}$ be a C^*-ternary algebra. A $\mathbb{C}$-linear mapping $\delta:\mathcal{A}\to\mathcal{A}$ is called a derivation on $\mathcal{A}$, if for all $a,b,c\in\mathcal{A}$ we have: $\delta([a,b,c])=[\delta(a),b,c]+[a,\delta(b),c]+[a,b,\delta(c)]$.

If $\mathcal{A}$ is a C^*-ternary algebra, then for given mapping $f:\mathcal{A}\to\mathcal{A}$ we define the difference operator $D_\mu f:\mathcal{A}^n\to\mathcal{A}$ by

$$D_\mu f(x_1,\ldots,x_n):=\sum_{k=2}^{n}\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f\left(\sum_{i=1,i\neq i_1,\ldots,i_{n-k+1}}^{n}\mu x_i-\sum_{r=1}^{n-k+1}\mu x_{i_r}\right)+f\left(\sum_{i=1}^{n}\mu x_i\right)-2^{n-1}\mu f(x_1) \tag{2}$$

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for all $\mu \in T^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x_i \in \mathcal{A}$ $(i = 1, 2, ..., n)$.

We will use the following lemmas in this paper.

Lemma 1.2. *(see [2]) Let V and W be linear spaces and let $f : V \to W$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in V$ and all $\mu \in T^1$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 1.3. *(see [3]) A function $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1) if and only if $f : X \to Y$ is additive.*

2 Main results

Assume that $\mathcal{A}$ be a C^*-ternary algebra, $f(0) = 0$ for given mapping $f : \mathcal{A} \to \mathcal{A}$ we define for all $a, b, c \in \mathcal{A}$

$$C_f(a, b, c) := f([a, b, c]) - [f(a), b, c] - [a, f(b), c] - [a, b, f(c)]\,. \tag{3}$$

Theorem 2.1. *Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which exist functions $\varphi\colon \mathcal{A}^n \to [0, \infty)$ and $\psi\colon \mathcal{A}^3 \to [0, \infty)$ such that*

$$\lim_{m\to\infty} 2^m \varphi\Big(\frac{x_1}{2^m}, \frac{x_2}{2^m}, ..., \frac{x_n}{2^m}\Big) = 0\ , \tag{4}$$

$$\lim_{m\to\infty} 8^m \psi\Big(\frac{a}{2^m}, \frac{b}{2^m}, \frac{c}{2^m}\Big) = 0\ , \tag{5}$$

$$\|D_\mu f(x_1, ..., x_n)\| \le \varphi(x_1, x_2, ..., x_n)\ , \tag{6}$$

$$\|C_f(a, b, c)\| \le \psi(a, b, c), \tag{7}$$

$$f(\frac{a^*}{2^m}) - f(\frac{a}{2^m})^* \le \varphi(\frac{a}{2^m}, ..., \frac{a}{2^m}) \tag{8}$$

for all $a, b, c, x_1, x_2, ..., x_n \in \mathcal{A}$, $\mu \in T^1$. If there exists a constant $0 < L < 1$ such that

$$\varphi(x_1, x_2, ..., x_n) \le \frac{\alpha}{2} L\varphi(2x_1, 2x_2, ..., 2x_n)\ , \tag{9}$$

for all $x_1, x_2, ..., x_n \in \mathcal{A}$, where $\alpha = 2^{n-1}$, then there exists a unique $\mathbb{C}$-linear mapping $d\colon \mathcal{A} \to \mathcal{A}$ which is a $*$-derivation and for all $x \in \mathcal{A}$ we have:

$$\|f(x) - d(x)\| \le \frac{L}{1 - L}\varphi(x, x, 0, .., 0), \tag{10}$$

$$\|f(x) - d(x)\| \le \frac{L}{1 - L}\varphi(\frac{x}{2}, \frac{x}{2}, 0, .., 0). \tag{11}$$

Proof. Let $\mu = 1$. We use the following relation

$$1 + \sum_{k=1}^{n-1} \binom{n-1}{k} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1} \tag{12}$$

for all $n > k$. Put $\mu = 1$, $x_1 = x_2 = x$ and $x_3 = ... = x_n = 0$ in (6). Then we obtain

$$\|2f(\frac{1}{2}x) - f(x)\| \leq \frac{2}{\alpha}\varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0) \leq L\varphi(x, x, 0, ..., 0) \tag{13}$$

for all $x \in \mathcal{A}$. Let $F := \{u\colon A \to A\}$. We introduce the generalized complete metric $d_\varphi(u, v) := \inf\{c \in [0, \infty] : \|u(x) - v(x)\| \leq c\varphi(x, x, 0, ..., 0)\}$ on F for all $x \in \mathcal{A}$.

Now we consider the mapping $(\Lambda u)(x) := 2u(\frac{x}{2})$ for all $u \in F$ and all $x \in \mathcal{A}$. Let $u, v \in F$ and let $c \in [0, \infty]$ be an arbitrary constant with $d_\varphi(u, v) \leq c$. We have

$$\|(\Lambda u)(x) - (\Lambda v)(x)\| = 2\|u(\frac{x}{2}) - v(\frac{x}{2})\| \leq 2c\ \varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0) \leq \alpha cL\ \varphi(x, x, 0, ..., 0)$$

for all $x \in \mathcal{A}$. So $d_\varphi(\Lambda f, f) \leq L$. Therefore according to the fixed point alternative theorem (see [1]), the sequence $\{\Lambda^m f\}$ converges to a fixed point d of Λ, i.e.,

$$d\colon A \to A, \qquad d(x) = \lim_{m\to\infty}(\Lambda^m f) = \lim_{m\to\infty} 2^m f(\frac{x}{2^m}).$$

Also d is the unique fixed point of Λ in the set $F_\varphi = \{u \in F : d_\varphi(u, v) < \infty\}$ which that

$$d_\varphi(d, f) \leq \frac{1}{1-L} d_\varphi(\Lambda f, f) \leq \frac{L}{1-L} \quad .$$

So d satisfies in (1) and is additive by Lemma 2.2 . Since $\alpha = |\alpha| = 2^{n-1}$ we have

$$\|\alpha f(\mu x) - \mu\alpha f(x)\| = \|2^{n-1} f(\mu x) - 2^{n-1}\mu f(x)\|$$

for all $x \in \mathcal{A}, \mu \in T^1$. By the relations (12) and (2) and setting $x_1 = x$ and $x_2 = x_3 = ... = x_n = 0$ in (6) we get

$$d(\mu x) = \lim_{m\to\infty} 2^m f(\frac{\mu x}{2^m}) = \lim_{m\to\infty} 2^m \mu f(\frac{x}{2^m}) = \mu d(x)$$

for all $x \in \mathcal{A}$ and all $\mu \in T^1$. Hence by Lemma 2.1, the mapping $d\colon A \to A$ is $\mathbb{C}$-linear. Moreover we have

$$d(x^*) = d(x)^*$$

for all $x \in \mathcal{A}$. By the relations (5) and (7) for all $a, b, c \in \mathcal{A}$ we get

$$8^m \left\| C_f(\frac{a}{2^m}, \frac{b}{2^m}, \frac{c}{2^m}) \right\| \leq 8^m \psi(\frac{a}{2^m}, \frac{b}{2^m}, \frac{c}{2^m})$$

Since the right hand side tends to zero as $m \to \infty$ and d is $\mathbb{C}$-linear, we have:

$$d([a, b, c]) = [d(a), b, c] + [a, d(b), c] + [a, b, d(c)]$$

□

Corollary 2.2. *Let θ_2, θ_1, q, p be non-negative real numbers with $p, q > 1$. Suppose that $f\colon \mathcal{A} \to \mathcal{A}$ be a mapping such that*

$$\|D_\mu f(x_1, x_2, ..., x_n)\| \leq \theta_1 \sum_{i=1}^{n} \|x_i\|^p \quad , \quad \|C_f(a, b, c)\| \leq \theta_2 \ (\|a\|^q + \|b\|^q + \|c\|^q),$$

$$f(\frac{a^*}{2^m}) - f(\frac{a}{2^m})^* \leq \frac{\theta_1 + \theta_2}{2^{mp}} \qquad (m = 0, 1, 2, ...)$$

for all $a, b, c, x_1, x_2, ..., x_n \in \mathcal{A}$, $\mu \in T^1$. Then exists a unique $\mathbb{C}$-linear $*$-derivation $d\colon \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - d(x)\| \leq \frac{2\theta_1}{2^{p-1} - 1} \|x\|^p$$

Proof. The proof follows from Theorem 2.3 for all $a, b, c, x_1, x_2, ..., x_n \in \mathcal{A}$ by choosing $L = 2^{1-p}$, $\varphi(x_1, x_2, ..., x_n) := \theta_1 \sum_{i=1}^{n} \|x_i\|^p$ and taking $\psi(a, b, c) := \theta_2 \Big(\|a\|^q + \|b\|^q + \|c\|^q \Big)$ □

Corollary 2.3. *Let θ_2, θ_1, q, p be non-negative real numbers with $p, q \in (0, 1)$. Suppose that $f\colon \mathcal{A} \to \mathcal{A}$ be a mapping such that*

$$\|D_\mu f(x_1, x_2, ..., x_n)\| \leq \theta_1 + \theta_2 \sum_{i=1}^{n} \|x_i\|^p \quad , \quad \|C_f(a, b, c)\| \leq \theta_1 + \theta_2 \ (\|a\|^q + \|b\|^q + \|c\|^q),$$

$$f(2^m a^*) - f(2^m a)^* \leq (\theta_1 + \theta_2) 2^{mp} \qquad (m = 0, 1, 2, ...),$$

Then exists a unique $\mathbb{C}$-linear $*$-derivation $d\colon \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - d(x)\| \leq \frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}} \|x\|^p$$

Proof. The proof follows from Theorem 2.3 for all $b, c, x_1, x_2, ..., x_n \in \mathcal{A}$ a by choosing $L = 2^{p-1}$ and taking $\varphi(x_1, x_2, ..., x_n) := \theta_1 + \theta_2 \sum_{i=1}^{n} \|x_i\|^p$ and $\psi(a, b, c) := \theta_1 + \theta_2 \Big(\|a\|^q + \|b\|^q + \|c\|^q \Big)$

□

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FIXED POINT FOR COMPATIBLE MAPPINGS OF TYPE (γ) IN COMPLETE FUZZY METRIC SPACES

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Abstract

In this work, we prove common fixed point theorems satisfying some conditions in fuzzy metric spaces in the sense of Sedghi, Turkoglu and Shobe [16]. Our main theorems extend, generalize and improvement some known results in fuzzy metric spaces, in particular produce a general style for prove common fixed point theorems.

Keywords: Compatible map of type (γ) , Complete fuzzy metric space, .
Mathematics Subject Classification [2010]: 48H10

1 Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [10] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [3] and Kramosil and Michalek [5] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and ε^{∞} theory which were given and studied by El Naschie [2]. Many authors have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.2. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X $(non - empty)$ set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

1. $M(x, y, t) > 0$,

*Speaker

2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.
6. $lim_{t\to\infty} M(x, y, t) = 1$.

2 Compatible Maps of Type (γ)

In this section, we give the concept of compatible maps of type (γ) in fuzzy metric spaces and some properties of these maps.

Definition 2.1. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible maps of type (γ) if satisfying:

1. A and S are compatible, that is
$$lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \quad \forall t > 0$$
whenever $\{x_n\}$ is a sequence in X such that
$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x \in X,$$
2. They are continuous at x.

On the other hand we have,
$$\begin{aligned} A(x) &= A(lim_{n\to\infty} Ax_n) = A(lim_{n\to\infty} Sx_n) = lim_{n\to\infty} ASx_n \\ &= lim_{n\to\infty} SAx_n = S(lim_{n\to\infty} Ax_n) = S(x) \end{aligned}$$

Lemma 2.2. *Let $(X, M, *)$ be a fuzzy metric space.*
(i) If we define $E_{\lambda,M} : X^2 \to \mathbb{R}^+ \bigcup \{0\}$ by
$$E_{\lambda,M}(x, y) = inf\{t > 0 : M(x, y, t) > 1 - \lambda\}$$
for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ and $x, y \in X$ such that
$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + ... + E_{\lambda,M}(x_{n-1}, x_n)$$
for any $x_1, x_2, ..., x_n \in X$.
(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if it is a Cauchy sequence with $E_{\lambda,M}$.

Lemma 2.3. *Let $(X, M, *)$ be fuzzy metric space. If a sequence $\{x_n\}$ in X is such that, for any $n \in \mathbb{N}$,*
$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$$
for all $k > 1$, then sequence $\{x_n\}$ is a cauchy sequence.

Lemma 2.4. *If for all $x, y \in X$, $t > 0$ and for a number $k \in (0, 1)$*
$$M(x, y, kt) \geq M(x, y, t)$$
then $x = y$.

3 Main Results

In this section, we prove some common fixed point theorems for compatible mappings of type (γ) under satisfying some conditions in fuzzy metric spaces.

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t = t$ for all $t \in [0, 1]$. Let $P_1, P_2, ..., P_{2m}$ and Q_0, Q_1 be self-mappings continuous of a complete fuzzy metric space, such that:*

(i) $Q_0(X) \subseteq P_1P_3...P_{2m-1}(X)$, $Q_1(X) \subseteq P_2P_4...P_{2m}(X)$,

(ii) there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} M(Q_0x, Q_1y, kt) \geq & M(P_1P_3...P_{2m-1}x, Q_0x, t) \\ & * M(P_2P_4...P_{2m}y, Q_1y, t) \\ & * M(P_2P_4...P_{2m}y, Q_0x, \alpha t) \\ & * M(P_1P_3...P_{2m-1}x, Q_1y, (2-\alpha)t) \\ & * M(P_1P_3...P_{2m-1}x, P_2P_4...P_{2m}y, t) \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$,

(iii) the pairs $(Q_0, P_1P_3...P_{2m-1})$ and $(Q_1, P_2P_4...P_{2m})$ are weak compatible of type (γ),

(iv) for all $1 \leq i = 2n - 1 \leq 2m$ and $2 \leq j = 2n \leq 2m$ such that

$P_iQ_0 = Q_0P_i$,

$P_iP_1P_3...P_{2m-1} = P_1P_3...P_{2m-1}P_i$,

$P_jQ_1 = Q_1P_j$,

$P_jP_2P_4...P_{2m} = P_2P_4...P_{2m}P_j$.

Then $P_1, P_2, ..., P_{2m}$ and Q_0, Q_1 have a unique common fixed point in X.

Corollary 3.2. *Let $\{Q_\mu\}_{\mu\in A}$, $\{Q_\nu\}_{\nu\in B}$ and $\{P_k\}_{k=1}^{2m}$ be the set of all self-mappings a complete fuzzy metric spaces $(X, M, *)$ with $t * t = t$ for all $t \in [0, 1]$, such that:*

(i) $Q_\mu(X) \subseteq P_1, P_2, ..., P_{2m}(X)$ and $Q_\nu(X) \subseteq P_1, P_3, ..., P_{2m-1}(X)$ for all $\mu \in A, \nu \in B$,

(ii) there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} M(Q_\mu x, Q_\nu y, kt) \geq & M(P_1P_3...P_{2m-1}x, Q_\mu x, t) \\ & * M(P_2P_4...P_{2m}y, Q_\nu y, t) \\ & * M(P_2P_4...P_{2m}y, Q_\mu x, \alpha t) \\ & * M(P_1P_3...P_{2m-1}x, Q_\nu y, (2-\alpha)t) \\ & * M(P_1P_3...P_{2m-1}x, P_2P_4...P_{2m}y, t) \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0, 2)$, $\mu \in A, \nu \in B$ and $t > 0$,

(iii) there exists $\mu_0 \in A$, such that pairs $(Q_{\mu_0}, P_1P_3...P_{2m-1})$ and $(Q_\nu, P_2P_4...P_{2m})$ are weak compatible of type (γ),

(iv) for all $\mu \in A, \nu \in B$, $1 \leq i = 2n - 1 \leq 2m$ and $2 \leq j = 2n \leq 2m$ such that

$P_iQ_\mu = Q_\mu P_i$,

$P_iP_1P_3...P_{2m-1} = P_1P_3...P_{2m-1}P_i$,

$P_jQ_\nu = Q_\nu P_j$,

$P_jP_2P_4...P_{2m} = P_2P_4...P_{2m}P_j$.

Then all P_k and $\{Q_\mu\}_{\mu\in A}$, $\{Q_\nu\}_{\nu\in B}$ have a unique common fixed point in X.

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Fixed Point Theorems for G-Nonexpansive Mappings in Ultrametric Spaces and non-Archimedean Normed Spaces Endowed with a Graph

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Abstract

The purpose of this article is to present some new fixed point results for G-nonexpansive mappings defined on an ultrametric space and non-Archimedean normed space which are endowed with a graph. In particular, we investigate the relationship between weak connectivity and the existence of fixed points for these mappings.

Keywords: Fixed point; ultrametric space; spherically complete ultrametric spaces; non-Archimedean space: nonexpansive mapping.
Mathematics Subject Classification [2010]: 47H10, 32P05

1 Introduction

Let (X, d) be a metric space. (X, d) is called an ultrametric space if the metric d satisfies the strong triangle inequality, i.e., for all $x, y, z \in X$:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\},$$

in this case d is said to be ultrametric [4].

We denote by $B(x, r)$, the closed ball $B(x, r) = \{y \in X : d(x, y) \leq r\}$, where $x \in X$ and $r \geq 0$ ($B(x, 0) = \{x\}$). A known characteristic property of ultrametric spaces is the following:

If $x, y \in X$, $0 \leq r \leq s$ and $B(x, r) \cap B(y, s) \neq \emptyset$, then $B(x, r) \subset B(y, s)$.

An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection [4]. [4] Let $\mathbb{K}$ be a non-Archimedean valued field. A norm on a vector space X over $\mathbb{K}$ is a map $\|\cdot\|$ from X into $[0, \infty)$ with the following properties:

1) $\|x\| \neq 0$ if $x \in E \setminus \{0\}$;

2) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ $\quad (x, y \in X)$;

3) $\|\alpha x\| = |\alpha|\|x\|$ $\quad (\alpha \in \mathbb{K}, x \in X)$.

In 1993, Petalas and Vidalis [3] proved the following theorem:

Theorem 1.1 ([3]). *Let X be a non-Archimedean spherically complete normed space. If $T : X \longrightarrow X$ is a nonexpansive mapping, i.e., $d(Tx, Ty) \leq d(x, y)$ $\quad x, y \in X$, Then either T has at least one fixed point or there exists a T-invariant closed ball B with radius $r > 0$ such that $\|b - Tb\| = r$ for all $b \in B$.*

*Speaker

In 2012, Kirk and Shahzad [1] proved the following theorem in a constructive way.

Theorem 1.2 ([1]). *Suppose that (X, d) is a spherically complete ultrametric space and $T : X \longrightarrow X$ is a nonexpansive mapping. Then every closed ball of the form*

$$B(x, d(x, Tx)) \qquad (x \in X),$$

contains either a fixed point of T or a minimal T-invariant closed ball.
Where a ball $B(x, r)$ is called T-invariant if $T(B(x, r)) \subset B(x, r)$ and is called minimal T-invariant if $B(x, r)$ is T-invariant and $d(u, Tu) = r$ for all $u \in B(x, r)$.

In this paper, motivated by the works of Petalas and Vidalis [3], Kirk and Shazad [1] and Jachymski [2], we introduce two new conditions for nonexpansive mappings on complete ultrametric spaces (non-Archimedean spaces) and, using these conditions, obtain some fixed point theorems.

2 The Main Theorem

Let $G = (V(G), E(G))$ be a directed graph. By $\tilde{G}$ we denote the undirected graph obtained from G by ignoring the direction of edges. If x and y are two vertices in a graph G, then a path in G from x to y of length n is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, n$, we always suppose that paths are of the shortest length. A graph G is called connected if there is a path between any two vertices and is called weakly connected if $\tilde{G}$ is connected.
Subsequently, in this paper X is a complete ultrametric space or non-Archimedean normed space with ultrametric d, Δ is the diagonal of the cartesian product $X \times X$ and G is a directed graph such that the set $V(G)$ of its vertices coincides with X, the set $E(G)$ of its edges contains Δ and G has no parallel edges. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. We give our first two results with constructive proofs. In fact, we generalize Kirk and Shahzad's result on nonexpansive mappings on ultrametric spaces and non-Archimedean normed spaces endowed with a graph.

Definition 2.1. Let (X, d) be a metric space endowed with a graph G. We say that a mapping $T : X \longrightarrow X$ is is G-nonexpansive if

1) T preserves the edges of G, i.e., $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

2) $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$ with $(x, y) \in E(G)$.

Definition 2.2. Suppose that (X, d) is an ultrametric space endowed with a graph G and $T : X \longrightarrow X$ a mapping. We would say that a ball $B(x, r)$ is graphically T-invariant if for any $u \in B(x, r)$ that there exists a path between u and x in $\tilde{G}$ with vertices in $B(x, r)$, we have

$$Tu \in B(x, r).$$

Also, a ball $B(x, r)$ is graphically minimal T-invariant if $Tu \in B(x, r)$ and $d(u, Tu) = r$ for any $u \in B(x, r)$ that there exists a path between u and x in $\tilde{G}$ with vertices in $B(x, r)$.

Theorem 2.3. *Let (X, d) be an ultrametric space endowed with a graph G. If G-nonexpansive mapping $T : X \longrightarrow X$ satisfies the following conditions*

(A) *There exists an $x_0 \in X$ such that $d(x_0, Tx_0) < 1$;*

(B) *If $d(x, Tx) < 1$, then there exists a path in $\tilde{G}$ between x and Tx with vertices in $B(x, d(x, Tx))$;*

(C) *If $\{B(x_n, d(x_n, Tx_n))\}$ is a nonincreasing sequence of closed balls in X and for each $n \geq 1$, there exists a path in $\tilde{G}$ between x_n and x_{n+1} with vertices in $B(x_n, d(x_n, Tx_n)$, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a $z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that for each $k \geq 1$, there exists a path in $\tilde{G}$ between x_{n_k} and z with vertices in $B(x_{n_k}, d(x_{n_k}, Tx_{n_k}))$,*

Then for each $x \in X_{)}$ with $d(z, Tz) < 1$, the closed ball $B(x, d(x, Tx))$ contains a fixed point of T or a graphically minimal T-invariant ball.

Corollary 2.4. *Suppose that $(X, d, \preceq)$ is a partially ordered ultrametric space, $G = (V(G), E(G))$ is a directed graph with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$ and $T : X \longrightarrow X$ is a G-nonexpansive mapping such that* (A), (B) *and* (C) *in Theorem 2.3 hold. Then for every $x \in X$ with $d(x, Tx) < 1$, the closed ball $B\big(x, d(x, Tx)\big)$ contains a fixed point of T or a graphically minimal T-invariant ball.*

Corollary 2.5. *Suppose that $(X, \|\cdot\|)$ is a non-Archimedean vector space over a non-Archimedean valued field $\mathbb{K}$ that X endowed with a partial ordering $\preceq$, $G = (V(G), E(G))$ is a graph with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$ and $T : X \longrightarrow X$ is a G-nonexpansive mapping such that* (A), (B) *and* (C) *in Theorem 2.3 hold. Then for every $x \in X$ with $\|x - Tx\| < 1$, the closed ball $B\big(x, \|x - Tx\|\big)$ contains a fixed point of T or a graphically minimal T-invariant ball.*

In the previous theorem, we obtained some results on the closed balls $B(x, d(x, Tx))$ with $d(x, Tx) < 1$. In the following Theorem we obtain these results on every weakly connected ball of the form $B(x, d(x, Tx))$ by adding weak connectivity.

Theorem 2.6. *Let (X, d) be a spherically complete ultrametric space endowed with graph G and $T : X \longrightarrow X$ be a G-nonexpansive mapping. Let for each $x \in X$ the ball $B(x, d(x, Tx))$ is weakly connected. Then for each $z \in X$ the closed ball $B(z, d(z, Tz))$ contains either a fixed point of T or a minimal T-invariant ball.*

Remark 2.7. In the same way, in this case the corollaries (1), (2) and (3) hold, too.

3 Examples

In this section, we will give some examples to support our Theorems. We also compare the hypotheses of Theorems 2.3 and 2.6 in Examples 2 and 3. In the first example, we present a spherically complete ultrametric space endowed with a weakly connected graph to support Theorem 2.6.

Example 3.1. Let X be the space c_0 over a non-Archimedean valued field $\mathbb{K}$ with the valuation of $\mathbb{K}$ discrete and pick a $\pi \in \mathbb{K}$ with $0 < |\pi| < 1$. Define graph $G := (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : \text{ either } x = y \text{ or there exists just one } i \in \mathbb{N} \text{ such that } x_i = y_i\}.$$

Let $B(x, r)$ be an arbitrary closed ball in X. $B(x, r)$ is weakly connected. It is well known when $\mathbb{K}$ is discrete (X, d) is spherically complete. So, all conditions of Theorem 2.6 hold. On the other hand, if there exists $x_0 \in X$ such that $d(x_0, Tx_0) < 1$ for G-nonexpansive mapping $T : c_0 \longrightarrow c_0$ then, since (X, d) is weakly connected, for each $x \in X$ there exists a path between x and Tx. Thus, in this case, hypothesis (B) of the Theorem 2.3 also hold.

In the following example we showe conditions of Theorem 2.3 are independent of conditions of Theorems 2.6.

Example 3.2. Let X be the space c_0 over a non-Archimedean valued field $\mathbb{K}$ with the valuation of field $\mathbb{K}$ discrete and $\pi \in \mathbb{K}$ with $1 < |\pi|$. Suppose $w \in B(0,1)$ has just one zero coordinate. Define graph G', with $V(G') = X$, and

$$E(G') = \{(x,y) \in X \times X : x = y \text{ or } (x,y) \in E(G), (x,w) \in E(G) \text{ and } (y,w) \in E(G)\}.$$

G' isn't weakly connected because if $x \in X$ is such that (x,w) isn't an edg of G, then there is no paths between x and w in G'. Indeed, if $(x = x_0, x_1, x_2, ..., x_N = w)$ is a path between x and w in G', then $(x, x_1) \in E(G')$, therefore $(x,w) \in E(G)$, which is a contradiction. So, G' isn't weakly connected. Therefore conditions of Theorems 2.6 do not hold. Now, define $T : X \longrightarrow X$ by

$$T(x) = \begin{cases} (x_1, x_2, x_3, ...), & (x,w) \in E(G) \\ (1 + x_1, 2x_2, 2x_3, ...), & o.w \end{cases}$$

T is a G'-nonexpansive mapping. Clearly, hypothesis (A) of Theorem 2.3 holds. If $d(x,Tx) < 1$, then $(x,w) \in E(G)$. Therefore, (x,Tx) is a path between x and Tx in $\tilde{G'}$. This means that hypothesis (B) of Theorem 2.3 holds. If $\{B(x_n, d(x_n, Tx_n))\}$ is a family of decreasing balls, such that for each $n \in \mathbb{N}$, there exists a path between x_n and x_{n+1} in $\tilde{G'}$, then for each $n \in \mathbb{N}$, $(x_n, w) \in E(G)$, so $d(x_n, Tx_n) = 0$. Hence there exists $z \in X$ such that $B(x_n, d(x_n, Tx_n)) = \{z\}$ for each $n \in \mathbb{N}$. Therefore, conditions of Theorem 2.3 hold, althought, conditions of Theorem 2.6 don't hold.

The last example shows that the hypotheses of Theorem 2.6 are independent of the hypotheses of Theorem 2.3.

Example 3.3. Let X be the space c_0 over a non-Archimedean valued field $\mathbb{K}$ with the valuation of $\mathbb{K}$ discrete. We endow X with the graph G defined in Example 3.1. Let $e \in \mathbb{K}$ with $|e| > 1$. As we have shown in Example 3.1, for every G-nonexpansive mapping T the hypotheses of Theorem 2.6 hold. Define $T : X \longrightarrow X$ by

$$T(x_1, x_2, \ldots) = (e, x_1, x_2, \ldots)$$

For each $x \in X$. We have

$$d(x,Tx) = \sup\{|x_1 - e|, |x_2 - x_1|, |x_3 - x_2|, ...\},$$

so $|x_1 - e| \leq d(x,Tx)$. Since $|x_1 - e| = \max\{|x_1|, |e|\}$ and $|e| \geq 1$, we infere $d(x,Tx) \geq 1$ for all $x \in X$. Hence for each $x \in X$, $d(x,Tx) \geq 1$ and the hypotheses of Theorem 2.3 do not hold.

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Hyers-Ulam-Rassias stability of functional equations on quasi-normed liner spaces

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Abstract

In this paper, we use the definition of quasi-normed spaces and the behaviors of solutions of the additive functional equation are described. The Hyers-Ulam stability problem of this equation is discussed and theorems concerning the Hyers-Ulam-Rassias stability of the equation are proved on quasi-normed linear space.

Keywords: Complte quasi-normed liner space, Functional equation, Quasi-morm.
Mathematics Subject Classification [2010]: 47J05.

1 Introduction

Defining, in some way, the class of approximate solutions of the given functional equation one can ask whether each mapping from this class can be somehow approximated by an exact solution of the considered equation. Such a problem was formulated by Ulam in 1940 (cf. [4]) and solved in the next year for the Cauchy functional equation by Hyers [2]. In 1950, Aoki [1] and in 1978, Rassias [3] proved a generalization of Hyers theorem for additive and linear mappings, respectively:
The result of Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations.

2 Main results

Definition 2.1. Let X is a vector space over F and $k \geq 1$. Furthermore, let $\|.\|_k : X \longrightarrow [0, \infty)$ be a function such that for all $x, y \in X$ and $c \in F$:
(i) $\|cx\|_k = |c|\|x\|_k$,
(ii) $\|x + y\|_k \leq k(\|x\|_k + \|y\|_k)$,
(iii) $\|x\|_k = 0$ if and only if $x = 0$.

Theorem 2.2. *Let $f : X \longrightarrow Y$ be a function between complete quasi-normed linear spaces such that*

$$\|f(x + y) - f(x) - f(y)\|_k \leq \delta, \text{ for all } x, y \in X,$$

*Speaker

for some $0 < \delta$. *Then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$ *and* $A : X \longrightarrow Y$ *is the unique additive function such that*

$$\|A(x) - f(x)\|_k \leq \delta, \text{ for all } x \in X. \tag{1}$$

Moreover, if $f(tx)$ *is continuous in* t *for each fixed* $x \in X$ *then* A *is linear.*

Proof. Let $x \in X$. We have

$$\|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\|_k \leq (2^{-m} - 2^{-n})\delta, \text{ for all } m < n.$$

Thus $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence for each $x \in X$. Hence the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$.
Let $x, y \in X$. We have

$$\|2^{-n} f(2^n x + 2^n y) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y)\|_k \leq 2^{-n}\delta, \text{ for all } n \in \mathbb{N}.$$

As $n \longrightarrow \infty$, we obtain that A is additive function. Now we have

$$\|2^{-n} f(2^n x) - f(x)\|_k \leq (1 - 2^{-n})\delta, \text{ for all } n \in \mathbb{N}.$$

As $n \longrightarrow \infty$, we get $\|A(x) - f(x)\|_k \leq \delta$, for all $x \in X$.
Let $A' : X \longrightarrow Y$ be another additive function satisfying in (1) and $A(y) \neq A'(y)$ for some $y \in X$. Then there exists $n \in \mathbb{N}$ such that $n > 2k\delta/\|A(y) - A'(y)\|_k$. Hence $2k\delta < \|A(ny) - A'(ny)\|$. On the other hand we have

$$\|A(ny) - A'(ny)\|_k \leq k(\|A(ny) - f(ny)\|_k + \|f(ny) - A'(ny)\|_k) \leq 2k\delta,$$

this is a contradiction. Hence, A is the unique additive function satisfying the inequality (1).
Assume that f is continuous at $y \in X$. If A is not continuous at y, then there exist an integer $\epsilon > 0$ and a sequence $\{x_n\}$ in X converging to zero such that $\|A(x_n)\|_k > \epsilon$, for all $n \in \mathbb{N}$. Let m be an integer greater than $3k^2\delta/\epsilon$. Then

$$\|A(mx_n + y) - A(y)\|_k = \|A(mx_n)\|_k \geq m\epsilon > 3k^2\delta.$$

We have

$$\begin{aligned}\|A(mx_n + y) - A(y)\|_k \leq k(&\|A(mx_n + y) - f(mx_n + y)\|_k + \\ &k(\|f(mx_n + y) - f(y)\|_k + \|A(y) - f(y)\|_k)) \leq 3k^2\delta,\end{aligned}$$

for sufficiently large n. This is a contradiction. So A is continuous at y. Let $f(tx)$ be continuous in t. Then $A(tx)$ is continuous in t, hence A is linear. □

Theorem 2.3. *Let $(X, \|.\|_k)$ and $(Y, \|.\|_k)$ be complete quasi-normed linear spaces and $N \in \mathbb{N}$. Also let $f : X \longrightarrow Y$ be a function such that*

$$\|f(\sum_{i=1}^{N} x_i) - \sum_{i=1}^{N} f(x_i)\|_k \leq \theta \sum_{i=1}^{N} \|x_i\|_k^p, \text{ for all } x_i \in X, \text{ and } 1 \leq i \leq N,$$

for some $0 < \theta$, $p \in [0, 1)$ and $1 \leq k < N^{1-p}$. Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$\|A(x) - f(x)\|_k \leq Nk\theta\|x\|_k^p/(N - kN^p), \text{ for all } x \in X. \tag{2}$$

Moreover, if f is continuous for some $x \in X$ then A is linear.

Proof. We have $\|f(Nx) - Nf(x)\|_k \leq N\theta\|x\|_k^p$, for all $x \in X$. Hence

$$\|N^{-1}f(Nx) - f(x)\|_k \leq \theta\|x\|_k^p, \text{ for all } x \in X.$$

By induction on n, we prove that

$$\|N^{-n}f(N^n x) - f(x)\|_k \leq \theta\|x\|_k^p \sum_{i=0}^{n-1} k^{(i+1)} N^{i(p-1)}, \text{ for all } x \in X.$$

Now we have

$$\begin{aligned}
\|N^{-n-1}f(N^{n+1}x) - f(x)\|_k &\leq k(\|N^{-1}f(Nx) - f(x)\|_k + \\
&\quad \|N^{-n-1}f(N^{n+1}x) - N^{-1}f(Nx)\|_k) \\
&\leq k(N^{-1}\theta\|Nx\|_k^p \sum_{i=0}^{n-1} k^{(i+1)} N^{i(p-1)} + \\
&\quad \theta\|x\|_k^p) \\
&= \theta\|x\|_k^p \sum_{i=0}^{n-1} k^{(i+2)} N^{(i+1)(p-1)} + k\theta\|x\|_k^p \\
&= \theta\|x\|_k^p \sum_{i=0}^{n} k^{(i+1)} N^{i(p-1)},
\end{aligned}$$

for all $x \in X$. Since $1 \leq k < N^{1-p}$, $\sum_{i=0}^{n-1} k^{(i+1)} N^{i(p-1)}$ converges. Thus $\{N^{-n}f(N^n x)\}$ is a Cauchy sequence for each $x \in X$. Hence the limit $A(x) = \lim_{n\to\infty} N^{-n}f(N^n x)$ exists for each $x \in X$. Now we have

$$\|N^{-n}f(N^n x) - f(x)\|_k \leq \theta\|x\|_k^p \sum_{i=0}^{n-1} k^{(i+1)} N^{i(p-1)},$$

for all $x \in X$ and all $n \in \mathbb{N}$. As $n \longrightarrow \infty$, we obtain that

$$\|A(x) - f(x)\|_k \leq k\theta\|x\|_k^p/(1 - kN^{p-1}) = Nk\theta\|x\|_k^p/(N - kN^p),$$

for all $x \in X$. We have

$$\|N^{-n}f(N^n x + N^n y) - N^{-n}f(N^n x) - N^{-n}f(N^n y)\|_k \leq$$

$$N^{n(p-1)}\theta(\|x\|_k^p + \|y\|_k^p),$$

for all $x, y \in X$. As $n \longrightarrow \infty$, we see that A is additive function.
Let $A' : X \longrightarrow Y$ be another additive function satisfying in (2). We have

$$\begin{aligned} \|A(x) - A'(x)\|_k &= (1/n)\|A(nx) - A'(nx)\|_k \\ &\leq k/n(\|A(nx) - f(nx)\|_k + \|f(nx) - A'(nx)\|_k) \\ &\leq k/n(2Nk\theta\|nx\|_k^p/(N - kN^p)) \\ &= 2Nk^2n^{p-1}\theta\|x\|_k^p/(N - kN^p), \end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. As $n \longrightarrow \infty$, we obtain that $A(x) = A'(x)$, for all $x \in X$.
Assume that f is continuous at $x_0 \in X$. Since

$$\|A(x) - f(x)\|_k \leq Nk\theta\|x\|_k^p/(N - kN^p), \text{ for all } x \in X,$$

A is continuous at $x_0 \in X$. We have A is additive, Hence A is continuous on X and $A(qx) = qA(x)$ for all $q \in \mathbb{Q}$. Thus $A(ax) = aA(x)$, for any $a \in \mathbb{R}$. Therefore A is a linear function. □

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Infinitely many solutions for a boundary value problem
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INFINITELY MANY SOLUTIONS FOR A BOUNDARY VALUE PROBLEM

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Abstract

The purpose of this paper is the study of hemivariational inequalities with Neumann boundary condition. Our approach is based on nonsmooth critical point Theorem.

Keywords: hemivariational inequality, Nonsmooth critical point theory, p-Laplacian
Mathematics Subject Classification [2010]: 35J87, 49J40, 49J52

1 Introduction

The applications to nonsmooth variational problems have been seen in (cf. [2]), Bonanno and Candito studied a class of variational-hemivariational inequalities; In (cf. [1]), Alimohammady studied variational-hemivariational inequality on bounded domains.

The aim of this paper is to study the following boundary value problem, depending on the parameters λ, μ with non-smooth Neumann boundary condition:

$$\begin{cases} -\Delta_p u + a|u|^{p-2}u = 0 & in\ \Omega \\ -|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} \in -\lambda \partial F(x,u) - \mu \partial G(x,u) & on\ \partial\Omega \end{cases} \tag{1}$$

We assume that it is given a functional $\chi : X \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicontinuous, proper whose effective domain $dom(\chi) = \{x \in X : \chi(x) < +\infty\}$ is a (nonempty, closed, convex) cone in X.
Our aim is to study the following hemivariational inequalities problem:
Find $u \in dom(\chi)$ which is called a weak solution of problem (1), i.e; if for all $v \in dom(\chi)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v-u)dx + \int_\Omega a|u|^{p-2}u(v-u)dx$$

$$-\lambda \int_{\partial\Omega} F^0(x,u,v-u)d\sigma - \mu \int_{\partial\Omega} G^0(x,u,v-u)d\sigma \geq 0. \tag{2}$$

*Speaker

Denote $X = W^{1,p}(\Omega)$ be endowed with the norm

$$\|u\| = \{\int_\Omega [|\nabla u|^p + a|u|^p]dx\}.$$

Let

$$\mathcal{K} = \sup_{u\in X\setminus\{0\}} \frac{\max_{x\in\bar{\Omega}} |u(x)|^p}{\|u\|^p}, \tag{3}$$

since $p > N$, X are compactly embedded in $C^0(\bar{\Omega})$, and hence $\mathcal{K} < \infty$.
Let X be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we will denote the norm in X and by $<\cdot,\cdot>$ the duality brackets for the pair $(X, X^\star)$. A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighbourhood U of x and a constant $K > 0$ depending on U such that $|h(y) - h(z)| \le K\|y - z\|$ for all $y, z \in U$.
For a locally Lipschitz function $h : X \to \mathbb{R}$ we define the generalized directional derivative of h at $u \in X$ in the direction $\gamma \in X$ by

$$h^0(u;\gamma) = \limsup_{w\to u, t\to 0^+} \frac{h(w + t\gamma) - h(w)}{t}.$$

The generalized gradient of h at $u \in X$ is defined by

$$\partial h(u) = \{x^\star \in X^\star \ : < x^\star, \gamma >_X \le h^0(u;\gamma),\ \forall\gamma \in X\},$$

which is a non-empty, convex and $w^\star-$compact subset of $X^\star$, where $<\cdot,\cdot>_X$ is the duality pairing between $X^\star$ and X.

The generalized Lebesgue-Sobolev space $W^{L,p(x)}(\Omega)$ for $L = 1, 2, ...$ is defined as

$$W^{L,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le L\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}$ with $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a multi-index and $|\alpha| = \Sigma_{i=1}^N \alpha_i$.
In this paper, we denote by $X = W^{1,p(x)}(\Omega)$ and $X^\star$ the dual space.
For a locally Lipschitz function $h : X \to \mathbb{R}$ we define the generalized directional derivative of h at $u \in X$ in the direction $\gamma \in X$ by

$$h^0(u;\gamma) = \limsup_{w\to u, t\to 0^+} \frac{h(w + t\gamma) - h(w)}{t}.$$

The generalized gradient of h at $u \in X$ is defined by

$$\partial h(u) = \{x^\star \in X^\star \ : < x^\star, \gamma >_X \le h^0(u;\gamma),\ \forall\gamma \in X\},$$

which is a nonempty, convex and $w^\star-$compact subset of $X^\star$, where $<\cdot,\cdot>_X$ is the duality pairing between $X^\star$ and X.

We have the following Definitions from (cf. [**?**]).

Definition 1.1. Let X be a Banach space, $\mathcal{I} : X \to (-\infty, +\infty]$ is called a Motreanu-Panagiotopoulos-type functional, if $\mathcal{I} = h + \chi$, where $h : X \to \mathbb{R}$ is locally Lipschitz and $\chi : X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous.

Definition 1.2. An element $u \in X$ is said to be a critical point of $\mathcal{I} = h + \chi$ if

$$h^0(u; v - u) + \chi(v) - \chi(u) \geq 0, \quad \forall v \in X.$$

We introduce the functionals $\phi, \mathcal{F}, \mathcal{G} : X \to \mathbb{R}$, defined as follows

$$\phi(u) = \frac{1}{p}\|u\|, \qquad \forall u \in W^{1,p}(\Omega),$$

$$\mathcal{F}(u) = \int_{\partial\Omega} F(x,u)d\sigma, \qquad \forall u \in W^{1,p}(\Omega),$$

$$\mathcal{G}(u) = \int_{\partial\Omega} G(x,u)d\sigma, \qquad \forall u \in W^{1,p}(\Omega).$$

Theorem 1.3. *(cf. [?]) Let X be a separable and reflexive Banach space, Λ be a real interval, $\mathcal{B}$ a nonempty, closed, convex subset of X. $\phi \in C^1(X, \mathbb{R})$ a sequentially weakly l.s.c. functional, bounded on any bounded subset of X, such that ϕ' is of type $(S)_+$, $\mathcal{F} : X \to \mathbb{R}$ a locally Lipschitz functional with compact gradient. Assume that:*
(i) $lim_{\|u\|\to+\infty}[\phi - \lambda\mathcal{F}] = +\infty, \quad \forall \lambda \in \Lambda,$
(ii) There exists $\rho_0 \in \mathbb{R}$ such that

$$\sup_{\lambda\in\Lambda} \inf_{u\in X} [\phi + \lambda(\rho_0 - \mathcal{F}(u))] < \inf_{u\in X} \sup_{\lambda\in\Lambda} [\phi + \lambda(\rho_0 - \mathcal{F}(u))].$$

Then, there exist $\lambda_1, \lambda_2 \in \Lambda$ $(\lambda_1 < \lambda_2)$ and $\sigma > 0$ such that, for every $\lambda \in [\lambda_1, \lambda_2]$ and every locally Lipschitz functional $\mathcal{G} : X \to \mathbb{R}$ with with compact gradient, there exists $\mu_1 > 0$ such that for every $\mu \in]0, \mu_1[$ the functional $\phi - \lambda\mathcal{F} + \mu\mathcal{G}$ has at least three critical points whose norms are less than σ.

2 Main results

Theorem 2.1. *Assume that*
(i_1) $\alpha < \frac{\mathcal{K}\beta}{c^p}$,
(i_2) $F(x,t) \geq 0$ for every $(x,t) \in \partial\Omega \times \mathbb{R}$.
(i_3) $\phi^{-1}(]-\infty, r[) \cap D(\chi) \neq \emptyset, \quad \forall r > \inf_X \phi$.
Then, for each $\lambda \in (\lambda_1, \lambda_2)$, where

$$\lambda_1 = \frac{1}{p\mathcal{K}\beta},$$

and

$$\lambda_2 = \frac{1}{pc^p\alpha},$$

for every arbitrary Carathéodory function $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfying G is a non-negative function satisfying the condition

$$G_\infty = \lim_{\omega\to+\infty} \frac{\int_{\partial\Omega} \max_{|t|\leq\omega} G(x,t)d\sigma}{|\omega|^p} < +\infty, \tag{4}$$

and for every $\mu \in [0, \mu_{G,\lambda})$, *where*

$$\mu_{G,\lambda} = \frac{1}{p\mathcal{K}G_\infty}(1 - p\mathcal{K}\lambda\alpha),$$

the problem (1) *has a sequence of weak solutions for every* $\mu \in [0, \mu_{G,\lambda})$ *in* X *such that*

$$\frac{1}{p}\|u\| \to +\infty.$$

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Poster Lie ternary (σ, τ, ξ)-derivations on Banach ternary algebras pp.: 1–4

Lie Ternary (σ, τ, ξ)-Derivations on Banach Ternary Algebras

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Abstract

Let A be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and X be a Banach ternary A-module. Let σ, τ and ξ be linear mappings on A. We define a Lie ternary (σ, τ, ξ)-derivation. Moreover, we prove the generalized Hyers-Ulam-Rassias stability of lie ternary (σ, τ, ξ)-derivations on Banach ternary algebras.

Keywords: Banach ternary A-module, Lie ternary (σ, τ, ξ)-derivation, Hyers–Ulam–Rassias stability.

1 Introduction

Let A be a Banach ternary algebra and X be a Banach space. Then X is called a ternary Banach A-module, if module operations $A \times A \times X \to X$, $A \times X \times A \to X$, and $X \times A \times A \to X$ are $\mathbb{C}$-linear in every variable. Moreover satisfy:

$$\max\{\|[xab]_X\|, \|[axb]_X\|, \|[abx]_X\|\} \leq \|a\|\|b\|\|x\|$$

for all $x \in X$ and all $a, b \in A$.

The stability of functional equations was started in 1940 with a problem raised by S. M. Ulam [6]. In 1941 Hyers affirmatively solved the problem of S. M. Ulam in the context of Banach spaces. In 1950 T.Aoki [2] extended the Hyers' theorem. in 1978, Th. M. Rassias [5] formulated and proved the following Theorem:
Assume that E_1 and E_2 are real normed spaces with E_2 complete, $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$ the mapping $t \to f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\epsilon \geq 0$ and $p \in [0, 1)$ such that $\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$. Then there exists a unique linear mapping $T : E_1 \in E_2$ such that $\|f(x) - T(x)\| \leq \epsilon \frac{\|x\|^p}{(1-2^p)}$ for all $x \in E_1$.
The equality $\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ has provided extensive influence in the development of of what we now call Hyers-Ulam-Rassias stability of functional equations [3]. In 1994, a generalization of Rassias' theorem was obtained by Gavruta [4], in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function.

*Speaker

2 Lie ternary (σ, τ, ξ)-derivations on Banach ternary algebras

In this section, We define a Lie ternary (σ, τ, ξ)-derivation on an A-bimodule X.

If A is a normed algebra, σ and τ two mappings on A and X is an A-bimodule. A linear mapping $L : A \to X$ is called a Lie (σ, τ)-derivation, if

$$L([a, b]) = [L(a), b]_{\sigma,\tau} - [L(b), a]_{\sigma,\tau}$$

for all $a, b \in A$, where $[a, b]_{\sigma,\tau}$ is $a\tau(b) - \sigma(b)a$ and $[a, b]$ is the commutator $ab - ba$ of elements a, b.
Throughout this section, Let $(A, [\,]_A)$ be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and $(X, [\,]_X)$ be a ternary Banach A-module. Let σ, τ and ξ be linear mappings on A.

Definition 2.1. A linear mapping $D : (A, [\,]_A) \to (X, [\,]_X)$ is called a Lie ternary (σ, τ, ξ)-derivation, if

$$D([a, b, c]) = [[D(a)bc]_X]_{(\sigma,\tau,\xi)} - [[D(c)ba]_X]_{(\sigma,\tau,\xi)} \tag{1.1}$$

for all $a, b, c \in A$, where $[abc]_{(\sigma,\tau,\xi)} = a\tau(b)\xi(c) - \sigma(c)\tau(b)a$ and $[a, b, c]$ is the commutator $[abc]_A - [cba]_A$ of elements a, b, c.

Let A be a unital Banach ternary algebra and X be a ternary Banach A-module. If $D : A \to X$ is a Lie ternary (σ, τ, ξ)-derivation such that σ, τ and ξ are linear mappings on A, additionally, $\tau(e) = e$, then it is easy to prove that D is a Lie (σ, ξ)-derivation.

Theorem 2.2. *Suppose $f : A \to X$ is a mapping with $f(0) = 0$ for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ and a function $\varphi : A \times A \times A \times A \times A \to [0, \infty]$ such that*

$$\widetilde{\varphi}(x, y, u, v, w) = \frac{1}{2}\sum_{n=0}^{\infty} \varphi(2^n x, 2^n y, 2^n u, 2^n v, 2^n w) < \infty \tag{2.2}$$

$$\begin{aligned} &\|f(\lambda x + \lambda y + [u, v, w]) - \lambda f(x) - \lambda f(y) - [[f(u)vw]_X]_{(g,h,k)} + [[f(w)vu]_X]_{(g,h,k)}\| \\ &\leq \varphi(x, y, u, v, w) \end{aligned} \tag{2.3}$$

$$\begin{aligned} \|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| &\leq \varphi(x, y, 0, 0, 0) \\ \|h(\lambda x + \lambda y) - \lambda h(x) - \lambda h(y)\| &\leq \varphi(x, y, 0, 0, 0) \\ \|k(\lambda x + \lambda y) - \lambda k(x) - \lambda k(y)\| &\leq \varphi(x, y, 0, 0, 0) \end{aligned}$$

for all $\lambda \in \mathbb{T}^1 (:= \{\lambda \in \mathbb{C} \,;\, |\lambda| = 1\})$ and for all $x, y, u, v, w \in A$.
Then there exist unique linear mappings σ, τ and ξ from A to A satisfying

$$\|g(x) - \sigma(x)\| \leq \widetilde{\varphi}(x, x, 0, 0, 0) \tag{2.4}$$

$$\|h(x)-\tau(x)\| \leq \widetilde{\varphi}(x,x,0,0,0) \tag{2.5}$$

and

$$\|k(x)-\xi(x)\| \leq \widetilde{\varphi}(x,x,0,0,0) \tag{2.6}$$

and there exist a unique Lie ternary (σ,τ,ξ)-derivation on $D:A\to X$ such that

$$\|f(x)-D(x)\| \leq \widetilde{\varphi}(x,x,0,0,0) \tag{2.7}$$

for all $x\in A$.

3 stability of C^*-Lie ternary (σ,τ,ξ)-derivations in C^*-ternary algebras

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\to[xyz]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables,conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[xy[zwv]]=[x[wzy]v]=[[xyz]wv]$, and satisfies $\|[xyz]\|\leq\|x\|.\|y\|.\|z\|$ and $\|[xxx]\|=\|x\|^3$ (see [1]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[xyz]:=\langle x,y\rangle z$.

A Lie (σ,τ,ξ)-ternary derivation $L:A\to A$ on a C^*-ternary algebra A is called a C^*-Lie ternary (σ,τ,ξ)-derivation.

Throughout this section, assume that A is a C^*-ternary with norm $\|.\|_A$. Let q be a positive rational number. For a given mapping $f:A\to A$ and a given $\mu\in\mathbb{C}$, we define $D_\mu f:A^n\to A$ by

$$D_\mu f(x_1,...,x_n):=\sum_{i=1}^{n} f(\sum_{j=1}^{n} q\mu(x_i-x_j))+nf(\sum_{i=1}^{n} q\mu x_i)-nq\mu\sum_{i=1}^{n} f(x_i)$$

for all $x_1,...,x_n\in A$.
In this section our aim is to establish the Hyers-Ulam stability of C^*-Lie ternary (σ,τ,ξ)-derivations in C^*-ternary algebras for the Euler-Lagrange type additive mapping.

Theorem 3.1. *Assume that $r>3$ if $nq>1$ and that $0<r<1$ if $nq<1$. Let θ be a positive real number, and let $f:A\to A$ be an odd mapping for which there exist mappings $g,h,k:A\to A$ with $g(0)=h(0)=k(0)=0$ satisfying*

$$\|D_\mu f(x_1,...,x_n)\| \leq \theta\sum_{j=1}^{n}\|x_j\|^r \tag{3.1}$$

$$\|g(q\mu x_1+...+q\mu x_n)-q\mu g(x_1)-...-q\mu g(x_n)\| \leq \theta(\|x_1\|^r+...+\|x_n\|^r) \tag{3.2}$$

$$\|h(q\mu x_1+...+q\mu x_n)-q\mu h(x_1)-...-q\mu h(x_n)\| \leq \theta(\|x_1\|^r+...+\|x_n\|^r) \tag{3.3}$$

$$\|k(q\mu x_1+...+q\mu x_n)-q\mu k(x_1)-...-q\mu k(x_n)\| \leq \theta(\|x_1\|^r+...+\|x_n\|^r) \tag{3.4}$$

such that

$$\|f([x,y,z]) - [f(x)yz]_{(g,h,k)} + [f(z)yx]_{(g,h,k)}\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{3.5}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-Lie ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying

$$\|g(x) - \sigma(x)\| \leq \frac{n\theta}{(nq)^r - nq}\|x\|^r \tag{3.6}$$

$$\|h(x) - \tau(x)\| \leq \frac{n\theta}{(nq)^r - nq}\|x\|^r \tag{3.7}$$

$$\|k(x) - \xi(x)\| \leq \frac{n\theta}{(nq)^r - nq}\|x\|^r \tag{3.8}$$

such that

$$\|f(x) - L(x)\| \leq \frac{\theta}{(nq)^r - nq}\|x\|^r \tag{3.9}$$

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Linear preservers rlt-majorization on $\mathbb{R}_2$

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Abstract

A $1 \times n$ matrix x is said to be rlt-majorized by a $1 \times n$ matrix y, and write $x \prec_{rlt} y$, if there exists an lower triangular row stochastic matrix R such that $x = yR$. We characterize the structure of all linear functions $T : \mathbb{R}_2 \to \mathbb{R}_2$ preserving (resp. strongly preserving) rlt-majorization.

Keywords: Row stochastic matrix, Rlt-majorization, (Strongly) linear preserver.
Mathematics Subject Classification [2010]: 34B15, 76A10

1 Introduction

Majorization is a topic of much interest in various areas of mathematics and statistics. In recent years, this concept has been attended specially and many papers in this topic have been published. For example, one can see [1]-[12].
The following notation will be fixed throughout the paper: $\mathbf{M}_n$ the set of all $n \times n$ real matrices; $\mathbb{R}_n$ for the set of all $1 \times n$ (row) real vectors; $\mathcal{R}_n^{lt}$ for the collection of all $n \times n$ row stochastic lower triangular matrices; $\{\epsilon_1, \ldots, \epsilon_n\}$ for the standard basis of $\mathbb{R}_n$; $tr(x) = \sum_{i=1}^n x_i$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$; G the $n \times n$ matrix with all of the entries of the first column equal to one and the other entries equal to zero; H the $n \times n$ matrix with $1i^{th}$ entries equal to $(-1)^{i+1}$, for all $i(1 \leqslant i \leqslant n)$, and the other entries equal to zero; A^t for the transpose of a given matrix A; $\mathbb{N}_k$ for the set $\{1, \ldots, k\} \subset \mathbb{N}$; $Co(A) := \{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N},\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1,\ a_i \in A,\ i \in \mathbb{N}_m\}$,
for a subset $A \subset \mathbb{R}_n$; $[T]$ for the matrix representation of a linear function $T : \mathbb{R}_n \to \mathbb{R}_n$ with respect to the standard basis.
Let $\sim$ be a relation on $\mathbb{R}_n$. A linear function $T : \mathbb{R}_n \to \mathbb{R}_n$ is said to be a linear preserver (or strong linear preserver) of $\sim$, if $Tx \sim Ty$ whenever $x \sim y$ (or $Tx \sim Ty$ if and only if $x \sim y$).

1.1 Row stochastic and rlt-majorization

Here we introduce the relation $\prec_{rlt}$ on $\mathbb{R}_n$ and we study some properties of rlt-majorization on $\mathbb{R}_2$.

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Definition 1.1. A matrix $R \in \mathbf{M}_n$ with nonnegative entries is called row stochastic if the sum of entries of each row of R is equal to one.

The collection of all $n \times n$ row stochastic lower triangular matrices is denoted by $\mathcal{R}_n^{lt}$. Now we pay attention to the row stochastic lower triangular matrices and introduce a new type of majorization on $\mathbb{R}_n$.

Definition 1.2. For $x, y \in \mathbb{R}_n$, it is said that x is rlt-majorized by y, and write as $x \prec_{rlt} y$, if there exists $R \in \mathcal{R}_n^{lt}$ such that $x = yR$.

In this paper, we investigate this relation on $\mathbb{R}_2$. The following proposition gives an equivalent condition for rlt-majorization on $\mathbb{R}_2$.

Proposition 1.3. *Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}_2$. Then $x \prec_{rlt} y$ if and only if $x_2 \in Co\{y_2, 0\}$ and $tr(x) = tr(y)$.*

Proof. First let $x \prec_{rlt} y$. Then there exists $R \in \mathcal{R}_n^{lt}$ such that $x = yR$. Thus $R = \begin{bmatrix} 1 & 0 \\ r_{21} & r_{22} \end{bmatrix}$, $r_{21} + r_{22} = 1$, $r_{21}, r_{22} \geqslant 0$, $x_2 = r_{22}y_2$, and $x_1 = r_{21}y_2 + y_1$. Therefore, $x_2 \in Co\{y_2, 0\}$ and $tr(x) = tr(y)$.
Now suppose that $x_2 \in Co\{y_2, 0\}$ and $tr(x) = tr(y)$. Hence there exists $0 \leqslant r_{22} \leqslant 1$ such that $x_2 = r_{22}y_2$. So $x_1 = r_{21}y_2 + y_1$ in which $r_{21} = 1 - r_{22}$. Set $r_{12} = 0$, $r_{11} = 1$, and put $R = (r_{ij})$. It is clear that $R \in \mathcal{R}_2^{lt}$ and $x = yR$. Therefore, $x \prec_{rlt} y$. □

Some properties of $\prec_{rlt}$ on $\mathbb{R}_2$ are stated in the following proposition.

Proposition 1.4. *Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}_2$. Then*

(a) $x \prec_{rlt} y \nRightarrow y \prec_{rlt} x$.

(b) $x \prec_{rlt} y$ *and* $y \prec_{rlt} x \Rightarrow x = y$.

(c) $x \prec_{rlt} y$ *and* $y \prec_{rlt} z \Rightarrow x \prec_{rlt} z$.

Proof. Proof, which is easy, is omitted for the sake of brevity. □

2 Main results

In this section the structure of all (strong) linear preservers of $\prec_{rlt}$ on $\mathbb{R}_2$ will be characterized. Before that, some results are needed. The proof of the Theorem 2.4 is divided into a sequence of lemmas.

Lemma 2.1. *Let $T : \mathbb{R}_2 \to \mathbb{R}_2$ be a linear preserver of $\prec_{rlt}$. Then there exist α, β, $\gamma \in \mathbb{R}$ such that $[T] = \alpha(I + \beta H) + \gamma G$.*

Proof. Since $\epsilon_1 \prec_{rlt} \epsilon_2$, this follows that $T\epsilon_1 \prec_{rlt} T\epsilon_2$. Thus there exists $0 \leqslant \beta \leqslant 1$ such that $a_{12} = \beta a_{22}$ and $a_{11} = (1 - \beta)a_{22} + a_{21}$. So $[T] = a_{22}(I + \beta H) + a_{21}G$. Put $\alpha = a_{22}$ and $\gamma = a_{21}$. □

Lemma 2.2. *Let $T : \mathbb{R}_2 \to \mathbb{R}_2$ be a linear function such that $[T] = \alpha(I + \beta H) + \gamma G$, for some α, β, $\gamma \in \mathbb{R}$. Then if $\beta = 0$ or 1 or $\alpha = 0$, then T preserves $\prec_{rlt}$.*

Proof. We see $[T] = \begin{bmatrix} \gamma + \alpha(1-\beta) & \alpha\beta \\ \gamma & \alpha \end{bmatrix}$. Let $x = (x_1, x_2) \prec_{rlt} y = (y_1, y_2) \in \mathbb{R}_2$. Then there exists $0 \leqslant \omega \leqslant 1$ such that $x_2 = \omega y_2$ and $tr(x) = tr(y)$. First assume that $\beta = 0$. It follows that $Tx = ((\alpha+\gamma)x_1+\gamma x_2, \alpha x_2)$ and $Ty = ((\alpha+\gamma)y_1+\gamma y_2, \alpha y_2)$. It is seen that $Tx \prec_{rlt} Ty$. Now consider the case $\beta = 1$. So $Tx = (\gamma tr(x), \alpha tr(x))$ and $Tx = (\gamma tr(y), \alpha tr(y))$. Since $tr(x) = tr(y)$, this shows that $Tx \prec_{rlt} Ty$. Therefore, T preserves $\prec_{rlt}$. If $\alpha = 0$; Then $Tx = (\gamma tr(x), 0)$ and $Ty = (\gamma tr(y), 0)$. We see that $Tx \prec_{rlt} Ty$. □

Lemma 2.3. *Let $T : \mathbb{R}_2 \to \mathbb{R}_2$ be a linear function such that $[T] = \alpha(I + \beta H) + \gamma G$, for some α, β, $\gamma \in \mathbb{R}$. Then if $\beta \notin \{0, 1\}$ and $\alpha \neq 0$, then T does not preserve $\prec_{rlt}$.*

Proof. The proof is divided into five steps. Fix $y = (\frac{-1}{\beta}, 1)$.
Step (I)- $0 < \beta < 1$: Consider $x = (1 - \frac{1}{\beta} - \beta, \beta)$.

Step (II)- $\beta > 1$: Choose $x = (1 - \frac{2}{\beta}, \frac{1}{\beta})$.

Step (III)- $-1 < \beta < 0$: Consider $x = (1 - \frac{1}{\beta} + \beta, -\beta)$.

Step (IV)- $\beta < -1$: Put $x = (1, \frac{-1}{\beta})$.

Step (IV)- $\beta = -1$: Set $x = (\frac{-3\beta}{2}, -\beta)$.
In each case $x \prec_{rlt} y$ and $Tx \not\prec_{rlt} Ty$. It means that T does not preserve $\prec_{rlt}$. □

The following theorem characterizes all the linear preservers of $\prec_{rlt}$ on $\mathbb{R}_2$.

Theorem 2.4. *Let $T : \mathbb{R}_2 \to \mathbb{R}_2$ be a linear function. Then T preserves $\prec_{rlt}$ if and only if $[T] = \alpha(I + H) + \gamma G$ or $[T] = \alpha I + \gamma G$, or $[T] = \gamma G$ for some α, $\gamma \in \mathbb{R}$.*

We need the following lemma to prove the last result of this section.

Lemma 2.5. *Let $T : \mathbb{R}_2 \to \mathbb{R}_2$ be a linear function which strongly preserves $\prec_{rlt}$. Then T is invertible.*

Proof. Let $x \in \mathbb{R}_2$ and let $Tx = 0$. Since $Tx = T0$ and T strongly preserves $\prec_{rlt}$, it shows that $x \prec_{rlt} 0$. So $x = 0$ and hence T is invertible. □

The following theorem characterizes all the linear functions $T : \mathbb{R}_2 \to \mathbb{R}_2$ which strongly preserves rlt-majorization.

Theorem 2.6. *A linear function $T : \mathbb{R}_2 \to \mathbb{R}_2$ strongly preserves $\prec_{rlt}$ if and only if $[T] = \alpha I + \gamma G$, for some α, $\gamma \in \mathbb{R}$ such that $\alpha(\alpha + \gamma) \neq 0$.*

Proof. First assume that T strongly preserves $\prec_{rlt}$. By Theorem 2.4 and Lemma 2.5, there exist α, $\gamma \in \mathbb{R}$ such that $[T] = \alpha I + \gamma G$ and $\alpha(\alpha + \gamma) \neq 0$.
Next we prove the sufficiency of the condition. By Theorem 2.4, T preserves $\prec_{rlt}$. So it is enough to show that $Tx \prec_{rlt} Ty$ implies that $x \prec_{rlt} y$. Since $[T] = \alpha I + \gamma G$, and α, $\alpha + \gamma \neq 0$, we have T is invertible. Let $x, y \in \mathbb{R}_2$ such that $Tx \prec_{rlt} Ty$. So there exists $R \in \mathcal{R}_2^{lt}$ such that $Tx = TyR$, and hence $x = T^{-1}TyR$. Thus $x = yR$. Therefore, T strongly preserves $\prec_{rlt}$. □

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Numerical Range of Self-Inverse Matrices

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Abstract

For an $n \times n$ matrix A , let $M(A)$ be the smallest possible constant in the inquality $D_p(A) \leq M(A)R_p(A)$. Here P is a point on the smooth portion of the boundary $\partial W(A)$ of the numerical range A. $R_p(A)$ is the radius of curvature of $\partial W(A)$ at this point, and $D_p(A)$ is the distance from P to the spectrum of$A$. In this paper we compute the $M(A)$ for matrix A is self-invers. 200 Mathematics Subject Classification. Primary 47A12; Secondary 15A42, 14H50.

Keywords: Numerical range, Matrices, Self-inverse
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Let$A$ be an $n \times n$ matrix with complex entries: $\mathbb{C}^{n\times n}$. The numerical range of A is defined as

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^{n\times n}, \|x = 1\}.$$

where $\langle ., .\rangle$ and $|.|$ are the standard scalar product and norm on $\mathbb{C}^{n\times n}$, respectively. Thus the numerical range of an operator, like spectrum, is a subset of the complex plane whose geometrical properties should say something about the operator.

2. Preliminaries and Auxiliary Results

One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toepliz-Hausdorff Theorem. Other important property of $W(A)$ is that its closure contains the spectrum of the operator. $W(A)$ is a connected set with a piecewise analytic boundary $\partial W(A)$. for details see [2] Hence, for all but finitely many point $P \in \partial W(A)$ the radius of curvature $R_p(A)$ of$\partial W(A)$ at P is well defined. By convention, $R_p(A) = 0$ if P is a corner point of $W(A)$ and $R_p(A) = \infty$ if P lies inside a flat portion of $\partial W(A)$. Let $D_p(A)$ denote the distance from P to we define $M(A)$ the smallest constant such that

$$D_p(A) \leq M(A)R_p(A). \tag{1}$$

for all $P \in \partial W(A)$, where $R_p(A)$ is defined. By Donoghes theorem $D_p(A) = 0$ whenever $R_p(A) = 0$. Therefor, $M(A) = 0$ for all convexoid element A. Recall that convexoid element

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is an element such that its numerical range coincides with the convex hull of its spectrum. For non-convexoid A,

$$M(A) = \sup \frac{D_p(A)}{R_p(A)} \tag{2}$$

where the supremum in the right-hand side is taken along all points $P \in \partial W(A)$ with finite non-zero curvature. Computation of $M(A)$ for arbitrary A is an interesting problem. Computation of $M(A)$ for arbitrary $n \times n$ matrix A is also an interesting open problem. For we $n > 3$, do not have an exact value of $M_n = \sup\{M(A) : A \in \mathbb{C}^{n\times n}\}$ the question whether there exists a universal constant $M = \sup_n M_n$, posed by Mathias [4]. In [1] the authors have proved that

$$\frac{n}{2}\sin(\frac{\Pi}{n}) \leq M_n \leq \frac{n}{2} \tag{3}$$

In [5] the author find a sequence of $n \times n$ Toeplitz nilpotent matrices A_n with $M(A_n)$ algorowing asymptotically as $\log n$. Hence, the ansewer to Mathias question is negative However, the lower bound in (3) is still of some interest, at least for small values of The question of the exact rate of growth of M_n (is it $\log n$ or n or something in between) remains open.

2 Main results

Let the operator A be self-inverse, i.e., $A^2 = I$ but $A \neq \pm I$,so $\sigma(A) = \{\pm 1\}$. Also $\partial W(A)$ is an ellipse with foci at ± 1 and major/minor axis $\|A \pm \frac{1}{\|A\|}\|$ [3]. If $\partial W(A) = a\cos\theta + b\sin\theta$ with $a^2 = b^2 + 1$ then $M(A) = \max\{\frac{\sqrt{a^2-1}}{a}, \frac{a}{a+1}\}$. Then we have following main Theorem: Let the operator be non trivial self-inverse, then $M(A) = \max\{\frac{\|A\|^2-1}{\|A\|^2+1}, \frac{\|A\|^2+1}{(\|A\|+1)^2}\}$.

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On Fuglede–Putnam Theorem

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Abstract

For operators A and B, let $\mathrm{Com}(A,B)$ stand for the set of operators X such that $AX = XB$. A pair (A,B) is said to have the (FP)-property if $\mathrm{Com}(A,B) \subseteq \mathrm{Com}(A^*,B^*)$. Let $\tilde{C}$ denote the Aluthge transform of a bounded linear operator C, (i) if A and B are invertible operators and (A,B) has the (FP)-property, then so is $(\tilde{A},\tilde{B})$; (ii) if A and B are invertible and $U \pm \mathrm{i}I$ and $V \pm \mathrm{i}I$ are all invertible and $(\tilde{A},\tilde{B})$ has the (FP)-property, then so is (A,B); (iii) if $U^2|A|U$ has the (FP)-property and $A^3 = I$, then A is a unitary operator.

Keywords: Fuglede–Putnam theorem; Aluthge transform; polar decomposition.
Mathematics Subject Classification [2010]: 47B20; 47B15

1 Introduction

Let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on (separable) complex Hilbert spaces $\mathcal{H}$, and let $I \in \mathbb{B}(\mathcal{H})$ be the identity operator. A subspace $\mathcal{K} \subseteq \mathcal{H}$ is said to reduce $A \in \mathbb{B}(\mathcal{H})$ if $A\mathcal{K} \subseteq \mathcal{K}$ and $A^*\mathcal{K} \subseteq \mathcal{K}$. Let $\mathbb{K}(\mathcal{H})$ denote the two-sided ideal of all compact operators on $\mathcal{H}$. For $p > 0$, an operator A is called p-hyponormal if $(A^*A)^p \geq (AA^*)^p$. If A is an invertible operator satisfying $\log(A^*A) \geq \log(AA^*)$, then it is called log-hyponormal. If $p = 1$, then A is said to be hyponormal. If A is invertible and p-hyponormal then A is called log-hyponormal.

Let $A = U|A|$ be the polar decomposition of A. It is known that if A is invertible then U is unitary and $|A|$ is also invertible. The Aluthge transform $\tilde{A}$ of A is defined by $\tilde{A} := |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. This notion was first introduced by Aluthge [1] and is a powerful tool in the operator theory. An interesting application of Aluthge transform deals with an generalizing the Fuglede–Putnam theorem [3]. Let $A, B \in \mathbb{B}(\mathcal{H})$ and . For such pair (A,B), denote by $\mathrm{Com}(A,B)$ the set of operators $X \in \mathbb{B}(\mathcal{H})$ such that $AX = XB$. A pair (A,B) is said to have the (FP)-property if $\mathrm{Com}(A,B) \subseteq \mathrm{Com}(A^*,B^*)$. The Fuglede–Putnam theorem is well-known in the operator theory. It asserts that for any normal operators A and B, the pair (A,B) has the (FP)-property. First Fuglede proved it in the case when $A = B$ and then Putnam proved it in a general case; see [4]. There exist many generalizations of this theorem which most of them go into relaxing the normality of A and B; see [4] and references therein. The two next Theorems are concerned the Fuglede–Putnam theorem.

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Theorem 1.1. *[5] Let $A, B \in \mathbb{B}(\mathcal{H})$. Then the following assertions are equivalent*
(i) *The pair (A, B) has the (FP)-property.*
(ii) *If $X \in Com(A, B)$, then $\overline{R(X)}$ reduces A, $(\ker X)^{\perp}$ reduces B, and $A|_{\overline{R(X)}}$, $B|_{(\ker X)^{\perp}}$ are unitarily equivalent normal operators.*

Theorem 1.2. *[3] Let $A, B^* \in \mathbb{B}(\mathcal{H})$ be either log-hyponormal or p-hyponormal operators. Then the pair (A, B) has the (FP)-property.*

2 Main results

In this section we assume that $A, B \in \mathbb{B}(\mathcal{H})$ are invertible operators with the polar decompositions $A = U|A|$ and $B = V|B|$, where U and V are unitary operators.

Lemma 2.1.
(i) $X \in Com(A, B) \iff |A|X|B|^{-1} = U^*XV$
(ii) $X \in Com(A, B) \bigcap Com(A^*, B^*) \iff |A|X|B|^{-1} = U^*XV = X$

Lemma 2.2.
(i) $X \in Com(A, B) \iff |A|^{\frac{1}{2}}X|B|^{\frac{-1}{2}} \in Com(\tilde{A}, \tilde{B})$.
(ii)$X \in Com(A^*, B^*) \iff |A|^{\frac{-1}{2}}X|B|^{\frac{1}{2}} \in Com((\tilde{A})^*, (\tilde{B})^*)$

Lemma 2.3. *The pair $(\tilde{A}, \tilde{B})$ has the (FP)-property, that is $Com(\tilde{A}, \tilde{B}) \subseteq Com((\tilde{A})^*, (\tilde{B})^*)$ if and only if $U^2X = XV^2$ for any $X \in Com(A, B)$.*

Theorem 2.4. *If (A, B) has the (FP)-property, then so is $(\tilde{A}, \tilde{B})$.*

The iterated Aluthge transforms of A are the operators $\Delta_n(A)$ defined inductively by $\Delta_1(A) := \tilde{A}$ and $\Delta_n(A) := \Delta_1(\Delta_{n-1}(A))$ for $n > 1$.

Corollary 2.5. *If (A, B) has the (FP)-property, then so is $(\Delta_n(A), \Delta_n(B))$ for any positive integer n.*

Theorem 2.6. *If both $U \pm \mathrm{i}I$ and $V \pm \mathrm{i}I$ are all invertible, then the (FP)-property for (A, B) is equivalent to the (FP)-property for $(\tilde{A}, \tilde{B})$.*

Proposition 2.7. *[4] Let $A = U|A|$ be the polar decomposition of A and $A^2 = I$ then $U^2 = I$.*

In [4] the authors show that the Proposition 2.7 is not valid when the power 2 is replaced by 3. For example if $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, then $A^3 = I$, but $U^3 \neq I$.

An interesting problem is that under what condition or conditions on operator A the Proposition is still valid for the power 3. The following proposition is about this problem.

Proposition 2.8. *Let $A \in \mathbb{B}(\mathcal{H})$ and $A = U|A|$ be the polar decomposition and $U^2|A|U$ has the (FP)-property and $A^3 = I$. Then A is a unitary operator.*

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On re Generalized Weighted Bergman Spaces

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Abstract

In this paper is generalized the weighted Bergman space $B_{w,\omega}$ and define

$$B_{w^p,\omega}(U) = \{f \in H(U) \mid ||f||^P_{B_{w^p,\omega}} = \int_U w^p(|f(z)|)\omega(z)dm(z) < +\infty\},$$

on the unit disk U, and we study the composition operator C_φ on the $B_{w^p,\omega}$. A counterexample for Lemma 1 in [3]. On Generalized Weighted Bergman Spaces, Complex Variables, 49 (2), 109-124] is provided and a corrected version of the Lemma and corrections on some other results are presented.

Keywords: Modulus function, Composition operator, Compact operator, Generalized weighted Bergman space
Mathematics Subject Classification [2010]: 47B33;46E10

1 Introduction and Preliminaries

In [3], the weighted Bergman space is extended by Stevic and the continuity and compression of the composition operator C_φ is studied on the extended weighted Bergman space. All the theorems proved in the above-mentioned paper are based on Lemma 1 in that paper. This is while the inequality claimed in the lemma, as well as its proof, is incorrect, thus leading to the incorrectness of some other theorems in the paper, such as Theorem 8. The following result, which is Lemma 1 in [3], was proved by Stevic.

Lemma S. *Let w be a modulus function such that $w(|f|)$ is subharmonic for all $f \in B_{w,\omega}$, then*

$$w(|f(z)|) \leq \frac{1}{2G(1-|z|)} \int_U w(|f(\zeta)|)\omega(\zeta)dm(\zeta) \tag{1.1}$$

for all $z \in U$, where $G(r) = \int_0^r \omega(\rho)\rho d\rho$. Unfortunately, Lemma S is not true in general. In this paper, we first re-extend the extended weighted Bergman space $B_{w,\omega}$ to $B_{w^p,\omega}$, and then present and prove Lemma S in a manner similar to [3]. Next, we show that the theorems presented throughout [3] also hold in the extended space $B_{w^p,\omega}$. In other words, it can be said that we study the composition operator C_φ on $B_{w^p,\omega}$.
Let U be the unit disc in the complex plane $\mathbb{C}$, $dm(z) = rdr(d\theta/\pi)$ the normalized Lebesgue area measure on U, and $H(U)$ the space of all analytic functions in U. Further, suppose

*Speaker

$\omega : [0,1) \to (0,\infty)$ is a continuous function and $\omega(r), 0 \le r < 1$, is a weight function which is positive and integrable on $(0,1)$. We extend ω on U by setting $\omega(z) = \omega(|z|)$. Our weights are assumed to be normalized so that $\int_U \omega(z)dm(z) = 1$.

A weight ω is said to be almost classical if it satisfies the following condition:
For each continuous function $\delta : [0,1] \to (0,1)$ there is a positive constant $C_\omega = C(\delta,\omega)$ such that $\sup \frac{\omega(r)}{\omega(r+\delta(r)(1-r))} \le \mathcal{C}_\omega, for, \ \ 0 \le r < 1$.
We study the composition operator C_φ on the space $B_{w^p,\omega}$.

We end this section with an example to which Lemma S fails.

Example 1.2. Let $U = \{z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2}\}$, $z = r$, $\zeta = \rho e^{i\theta}$, $w(z) = 2z^2$ and $w(|f|) = 1$.

2 The Space $B_{w^p,\omega}(U)$

Lemma 2.1. *Let w be a modulus function such that $w(|f|)$ is subharmonic for all $f \in B_{w^p,\omega}$ then*

$$w^p(|f(z)|) \le \frac{C_R}{2G(R)} \int_U w^p(|f(\xi)|)\omega(\xi)dm(\xi) \tag{1}$$

for all $z \in U$, where $G(R) = \int_0^R \omega(\rho)\rho d(\rho)$, and $R < 1 - |z|$

Theorem 2.2. *Let w be a modulus function such that $w(|f|)$ is subharmonic for all $f \in B_{w^p,\omega}$. Then $B_{w^p,\omega}$ is a complete metrizable topological vector space.*

Theorem 2.3. *Let w be a modulus function such that $w(|f|)$ is subharmonic for all $f \in B_{w^p,\omega}$. Then the polynomials are dense in $B_{w^p,\omega}$.*

Theorem 2.4. *Let w be a modulus function such that $w(xy) \le w(x) + w(y)$ and $w(|f|)$ is subharmonic for all $f \in B_{w^p,\omega}$. Then every bounded subset A of $B_{w^p,\omega}$ is relatively compact.*

3 Composition Operator on $B_{w^p,\omega}$

3.1 Continuity

Theorem 3.1. *Let $\phi : U \to U$ be analytic and nonconstant, $\omega(z)$ be an almost classical weight and w be a modulus function. Then the composition operator $C_\phi(f) = fo\phi$ on $B_{w^p,\omega}(U)$ satisfies:*

$$||C_\phi(f)||_{B_{w^p,\omega}} \le C||f||_{B_{w^p,\omega}}$$

for some C independent of f.

Theorem 3.2. *Let $\varphi : U \to U$ be a nonconstant analytic function of bounded valence such that $l = \inf_{z\in U} |\varphi'(z)| > 0$ and ω be a weight function such that $M = \sup_{z\in U} \omega(z)/\omega(|\varphi(z)|) < \infty$. Then the composition operator $C_\varphi(f) = fo\varphi$ is continuous on $B_{w^p,\omega}(U)$.*

Theorem 3.3. *Let $\varphi : U \to U$ be a nonconstant analytic function and ω be a weight function such that*

$$M = \sup_{z\in U} \omega(z)/\omega(|\varphi(z)|) < \infty.$$

Then the composition operator $C_\varphi(f) = fo\varphi$ is continuous on $B_{w^p,\omega}(U)$.

3.2 Compact operators

Theorem 3.4. *If $\varphi : U \to U$ is a nonconstant analytic function and $w(|f|)$ is subharmonic for all $f \in B_{w^p,\omega}$ and there is $0 < r_0 < 1$ such that*

$$\int_U \frac{\omega(z)C_{r_0(1-|\varphi(z)|)}}{G(r_0(1-|\varphi(z)|))} dm(z) < \infty. \tag{2}$$

then C_φ is a metrically compact operator on $B_{w^p,\omega}$, where $G(r)$ is as in Lemma 2.1 and $C_{r_0(1-|\varphi(z)|)}$ is calculated as follows: Since ω is continuous and nonzero, it attains its extremum on closed ball $\overline{B(\varphi(z), r_0(1-|\varphi(z)|))}$. Thus, there exist constants d_1 and C_1 such that $\omega(\rho) < d_1$, $C_1 < \omega(\varphi(z) + \rho e^{i\theta})$. Then it is sufficient to set $C_{r_0(1-|\varphi(z)|)} = \frac{d_1}{C_1}$.

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Poster On some nonlocal elliptic systems with multiple parameters pp.: 1–4

ON SOME NONLOCAL ELLIPTIC SYSTEMS WITH MULTIPLE PARAMETERS

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Abstract

Using variational methods, we study the existence of positive solution for a class of Nonlocal eliptic systems with multiple parameters. The proofs rely essentially on sub and supersoloutions method.

Keywords: Nonlocal ellliptic systems, positive solutions, sub and supersolutions method, Variational methods

2010 mathematics subject classificition: 35 D 05, 35 J 60, 35 P 15.

1 Introduction

In this paper we study the existence of positive solutions to the following nonlocal elliptic systems

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\ div\left(h_1(|\nabla u|^p)\,|\nabla u|^{p-2}\,\nabla u\right) = \alpha_1 a(x) f_1(v) + \beta_1 b(x) g_1(u) & x \in \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\ div\left(h_2(|\nabla v|^q)\,|\nabla v|^{q-2}\,\nabla v\right) = \alpha_2 c(x) f_2(u) + \beta_2 d(x) g_2(v) & x \in \Omega, \\ u = v = 0. \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $1 < p, q < N$,
$M_i : \mathbb{R}_0^+ \to \mathbb{R}$. $i = 1, 2$, are continuous and nondecreasing functions,where $\mathbb{R}_0^+ = [0, +\infty)$, $a, b, c, d \in C(\overline{\Omega})$, and $\alpha_i, \beta_i, i = 1, 2$, are positive parameters
We assume throughout this paper the following hypotheses

(H1) $a, b, c, d \in C(\overline{\Omega})$ and $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$, $c(x) \geq c_0 > 0$, $d(x) \geq d_0 > 0$ for all $x \in \Omega$.,

(H2) $M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$, $i = 1, 2$, are two continnous and increasing functions and $0 < m_i \leq M_i(t) \leq m_{i,\infty}$ for all $t \in \mathbb{R}_0^+$.m

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(H3) $f_i, g_i \in C^1(0,\infty)\cap[0,\infty)$, $i=1,2$, are monotone functions such that $\lim_{t\to\infty} f_i(t) = \lim_{t\to\infty} g_i(t) = \infty$.,

(H4) $\lim_{t\to\infty} f_1\left(L[f_2(t)]^{\frac{1}{q-1}}\right)\diagup t^{p-1} = 0$ for every $L>0$,

(H5) $\lim_{t\to\infty} g_1(t)\diagup t^{p-1} = \lim_{t\to\infty} g_2(t)\diagup t^{q-1} = 0$.,

(H6) $h_i : [0,+\infty)\to\mathbb{R}$, $i=1,2$ are continuous and there exist $\acute{\alpha_i}, \acute{\beta_i} > 0$, such that

$$\acute{\alpha_i} \le h_i(t) \le \acute{\beta_i} \quad for\, all\, t>0.$$

Let $W_0^{1,r}(\Omega)$ $(1\le r<\infty)$ be the completion of $C_0^\infty(\Omega)$, with respect to the norm

$$||u||_r = \left(\int_\Omega |\nabla u|^r\, dx\right)^{\frac{1}{r}}.$$

Definition 1.1. A pair of functions(ψ_1,ψ_2) is said to be subsolution of problem (1) if it is in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ such that

$$M_1\left(\int_\Omega |\nabla\psi_1|^p\, dx\right)\int_\Omega h_1(|\nabla\psi_1|^p)\,|\nabla\psi_1|^{p-2}\,\nabla\psi_1\,\nabla w\, dx$$
$$\le \alpha_1\int_\Omega a(x)f_1(\psi_2)w\, dx + \beta_1\int_\Omega b(x)g_1(\psi_1)w\, dx \quad \forall w\in W$$

and

$$M_2\left(\int_\Omega |\nabla\psi_2|^q\, dx\right)\int_\Omega h_2(|\nabla\psi_2|^q)\,|\nabla\psi_2|^{q-2}\,\nabla\psi_2\,\nabla w\, dx$$
$$\le \alpha_1\int_\Omega c(x)f_2(\psi_1)w\, dx + \beta_1\int_\Omega d(x)g_2(\psi_2)w\, dx \quad \forall w\in W$$

where $W := \{w\in C_0^\infty(\Omega) : w\ge 0\, in\,\Omega\}$. A pair of functions $(z_1,z_2)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ is said to be a supersolution if

$$M_1\left(\int_\Omega |\nabla z_1|^p\, dx\right)\int_\Omega h_1(|\nabla z_1|^p)\,|\nabla z_1|^{p-2}\,\nabla z_1\,\nabla w\, dx$$
$$\ge \alpha_1\int_\Omega a(x)f_1(z_2)w\, dx + \beta_1\int_\Omega b(x)g_1(z_1)w\, dx \quad \forall w\in W$$

and

$$M_2\left(\int_\Omega |\nabla z_2|^q\, dx\right)\int_\Omega h_2(|\nabla z_2|^q)\,|\nabla z_2|^{q-2}\,\nabla z_2\,\nabla w\, dx$$
$$\ge \alpha_1\int_\Omega c(x)f_2(z_1)w\, dx + \beta_1\int_\Omega d(x)g_2(z_2)w\, dx \quad \forall w\in W$$

Our paper extends or complements the previous results eventually in the case $h_i \equiv 1$, $i = 1, 2$ (see [5]).
Our main result in this paper is given by the following theorem.

Theorem 1.2. *Assume that the conditions (H1)-(H6) are satisfied. Then system (1) has a positive weak solution provided $a_0\alpha_1 + b_0\beta_1$ and $c_0\alpha_2 + d_0\beta_2$ are large.*

2 Main results

Let $K_0 > 0$ such that $f_i(t) \geq -K_0$ and $g_i(t) \geq -K_0$ for all $t \geq 0$, $i = 1, 2$. We now construct our positive subsolution. We shall verify that (ψ_1, ψ_2) is a subsolution of (1) for $a_0\alpha_1 + b_0\beta_1$ and $c_0\alpha_2 + d_0\beta_2$ large, where

$$\psi_1 = \left[\frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1\acute{\beta_1}}\right]^{\frac{1}{p-1}} \left(\frac{p-1}{p}\right) \phi_{1,p}^{\frac{p}{p-1}}.$$

$$\psi_2 = \left[\frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_2\acute{\beta_2}}\right]^{\frac{1}{q-1}} \left(\frac{q-1}{q}\right) \phi_{1,q}^{\frac{q}{q-1}}.$$

Let the test function $w \in W := \{w \in C_0^\infty(\Omega) : \ w \geq 0 \ in \, \Omega\}$. We have

$$\int_\Omega h_1(|\nabla\psi_1|^p)|\nabla\psi_1|^{p-2}\nabla\psi_1 \, \nabla w \, dx \leq \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} \int_\Omega [\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p]w \, dx$$

Similary, we have

$$\int_\Omega h_2(|\nabla\psi_2|^q)|\nabla\psi_2|^{q-2}\nabla\psi_2 \, \nabla w \, dx \leq \frac{k_0(c_0\alpha_2 + d_0\beta_2)}{mm_2} \int_\Omega [\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q]w \, dx$$

Now by (3), we have in $\overline{\Omega_\eta}$, $\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p \leq -m$ and $\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q \leq -m$. it follows that in $\overline{\Omega_\eta}$,

$$M_1\left(\int_\Omega |\nabla\psi_1|^p dx\right) \int_\Omega h_1(|\nabla\psi_1|^p)|\nabla\psi_1|^{p-2}\nabla\psi_1 \, \nabla w \, dx \leq \alpha_1 \int_\Omega a(x)f_1(\psi_2)w dx + \beta_1 \int_\Omega b(x)g_1(\psi_1)dx \tag{2}$$

and

$$M_2\left(\int_\Omega |\nabla\psi_2|^q dx\right) \int_\Omega h_2(|\nabla\psi_2|^q)|\nabla\psi_2|^{q-2}\nabla\psi_2 \, \nabla w \, dx \leq \alpha_1 \int_\Omega c(x)f_2(\psi_1)w dx + \beta_1 \int_\Omega d(x)g_2(\psi_2)dx \tag{3}$$

for all $x \in \Omega$. From (5)-(6), it follows that (ψ_1, ψ_2) is a subsolution of system (1).

Next, we construct a supersolution (z_1, z_2) of system (1). Let

$$z_1 = \frac{C}{\acute{\alpha_1}^{\frac{1}{p-1}}} e_p, \quad z_2 = \left(\frac{||c||_\infty\alpha_2 + ||d||_\infty\beta_2}{m_2\acute{\alpha_2}}\right)^{\frac{1}{q-1}} \left(f_2(\frac{C}{\acute{\alpha_1}^{\frac{1}{p-1}}})\right)^{\frac{1}{q-1}} e_q$$

where e_p, e_q are given by (4) and $C > 0$ is large and to be chosen later. we shall verify that (z_1, z_2) is a supersolutions of system(1).

$$M_1\left(\int_\Omega |\nabla z_1|^p dx\right)\int_\Omega h_1(|\nabla z_1|^p)|\nabla z_1|^{p-2}\nabla z_1\,\nabla w\,dx \geq M_1\left(\int_\Omega |\nabla z_1|^p dx\right)\acute{\alpha_1}\frac{C^{p-1}}{\acute{\alpha_1}}\int_\Omega |\nabla e_p|^{p-2}\nabla e_p\,\nabla w\,dx$$
$$\geq m_1 C^{p-1}\int_\Omega w\,dx$$

By (H4) and (H5), we can chose C large enough so that

$$\begin{aligned} m_1 C^{p-1} \geq & \alpha_1||a||_\infty f_1\left[\left(\frac{||c||_\infty\alpha_2 + ||d||_\infty\beta_2}{m_2}\right)^{\frac{1}{q-1}}(f_2(C||e_p||_\infty))^{\frac{1}{q-1}}||e_q||_\infty\right] \\ & + \beta_1||b||_\infty g_1(C||e_p||_\infty) \\ & \geq \alpha_1 a(x)f_1(z_2) + \beta_1 b(x)g_1(z_1). \end{aligned}$$

for all $x \in \Omega$. Hence

$$M_1\left(\int_\Omega |\nabla z_1|^p dx\right)\int_\Omega h_1(|\nabla z_1|^p)|\nabla z_1|^{p-2}\nabla z_1\,\nabla w\,dx \geq \alpha_1\int_\Omega a(x)f_1(z_2)w\,dx + \beta_1\int_\Omega b(x)g_1(z_1)w\,dx$$

Also

$$M_2\left(\int_\Omega |\nabla z_2|^q dx\right)\int_\Omega h_2(|\nabla z_2|^q)|\nabla z_2|^{q-2}\nabla z_2.\nabla w\,dx \geq \alpha_2\int_\Omega c(x)f_2(z_1)w\,dx + \beta_2\int_\Omega d(x)g_2(z_2)w\,dx$$

and thus$(z_1, z_2)$is a supersolution of system(1).

Obviously, we have $\psi_i(x) \leq z_i(x)$ in Ω with large C for $i = 1, 2$. Thus, by the comparison principle, there exist a solution (u, v) of (1) with $\psi_1 \leq u \leq z_1$ and $\psi_2 \leq v \leq z_2$. This completes the proof of theorem 1.2

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Poster On subspace-hypercyclic vectors pp.: 1–3

On Subspace-hypercyclic Vectors

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Abstract

In this paper we state some properties of subspace-hypercyclic vectors. We show that if X be an F-space and M be a closed subspace of X, then for an operator T, the set $HC(T, M) \cap M$ is empty or dense in M.

Keywords: Hypercyclic vectors, Hypercyclic operators, Subspace-hypercyclic operators
Mathematics Subject Classification [2010]: 47A16, 47B37

1 Introduction

Let X be a Banach space. An operator T on X is hypercyclic, if there exists a vector $x \in X$ whose orbit under T, $orb(T, x) = \{x, Tx, T^2x, ...\}$, is dense in X. Such a vector x is called a hypercyclic vector for T.

Hypercyclic operators have been actively studied for more than twenty years. One can refer to [1] and [2] for more information about the subject.

Recently, B. F. Madore and R. A. Martinez-Avendano in [4] introduced the concept of subspace-hypercyclicity. One can see [3],[5] and [6] to find more results about subspace-hypercyclic operators.
Let us recall some preliminaries from [4].

Definition 1.1. Let $T \in B(X)$ and let M be a closed nonzero subspace of X. We say T is M-hypercyclic, if there exists $x \in X$ such that $orb(T, x) \cap M$ is dense in M. Such a vector x is called an M-hypercyclic vector for T.

We show the set of M-hypercyclic vectors of T by $HC(T, M)$.

Definition 1.2. Let $T \in B(X)$. We say T is M-transitive, if for any non-empty open sets $U \subseteq M$ and $V \subseteq M$, both relatively open, there exists $n \in N_0$ such that $T^{-n}(U) \cap V$ contains a relatively open non-empty subset of M.

The following lemma states two equivalent conditions for subspace-transitivity.

Lemma 1.3. *Let $T \in B(X)$. The following conditions are equivalent:*

*Speaker

(i) T *is subspace-transitive with respect to* M.

(ii) for any non-empty open sets $U \subseteq M$ *and* $V \subseteq M$, *there exists* $n \in \mathbb{N}_0$ *such that* $T^{-n}(U) \cap V$ *is a relatively open non-empty subset of* M.

(iii) for any non-empty sets $U \subseteq M$ *and* $V \subseteq M$ *both relatively open, there exists* $n \in \mathbb{N}_0$ *such that* $T^{-n}(U) \cap V$ *is non-empty and* $T^n(M) \subseteq M$.

Theorem 1.4. *Let* $T \in B(X)$. *If* T *is subspace-transitive with respect to* M, *then* T *is subspace-hypercyclic with respect to* M.

Madore and Martinez-Avendano showed in [4] that the converse of the above theorem is not true. They also showed in [4] that subspace-hypercyclicity, like hypercyclicity, is a purely infinite-dimensional concept.

Theorem 1.5. *([4])Let* X *be finite-dimensional. If* $T \in B(X)$, *then* T *is not subspace-hypercyclic for any nonzero closed subspace* M.

Theorem 1.6. *([4])Let* $T \in B(X)$. *If* T *is subspace-hypercyclic for a nonzero closed subspace* M, *then* M *is not finite dimensional.*

2 Main results

In what follows X always is an F-space, a complex and complete metrizable topological vector space. $B(X)$ is the space of bounded linear operators on X. M always is a closed nonempty subspace of X. We also assume that M is separable, since subspace-hypercyclicity can only occur with respect to separable and infinite dimensional subspaces.

Lemma 2.1. *([4]) Let* $T \in B(X)$ *be* M*-transitive. Then*

$$HC(T, M) \cap M = (\bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} T^{-n}(B_j)) \cap M$$

is a dense subset of M, *where* $HC(T, M)$ *is the set of* M*-hypercyclic vectors for* T *and* $\{B_j\}$ *is a countable open basis for the relative topology of* M *as a subspace of* X.

Theorem 2.2. *([7])Let* $T \in B(X)$ *be* M*-hypercyclic. If* x *is an* M*-hypercyclic vector for* T, *and* M *has no isolated point, then for every* $n \in N$, $T^n(x)$ *is a* M*-hypercyclic vector for* T.

Remark 2.3. We know that if X is an F-space, it has no isolated points. If M is a closed subspace of X, it is also an F-space and hence has no isolated points.

Theorem 2.4. *Let* X *be an* F*-space and* $T \in B(X)$. *Then* $HC(T, M) \cap M$ *is empty or dense in* M.

Proof. Let $x \in HC(T, M) \cap M$. Then $orb(T, x) \cap M$ is dense in M. But each member of $orb(T, x) \cap M$ is an M-hypercyclic vector for T, by Theorem2.2. So if $HC(T, M) \cap M$ is nonempty, it is dense in M. □

So once there is one subspace-hypercyclic vector for an operator T, there is a dense set of them.

Corollary 2.5. *Let $T \in B(X)$ be M-hypercyclic. Then $HC(T, M) \cap M$ is dense in M.*

Lemma 2.6. *Let $T \in B(X)$ be an invertible and M-hypercyclic operator. Then for any $n \in N$, $T^{-n}(x)$ is an M-hypercyclic vector for T.*

Proof. First note that

$$orb(T, T^{-n}(x)) = \{T^{-n}(x), T^{-n+1}(x), ..., x, Tx, ..., T^n(x), ...\}.$$

So $orb(T, T^{-n}(x)) \cap M$ contains $orb(T, x) \cap M$ and it is a subset of M. Therefore the closure of $orb(T, T^{-n}(x)) \cap M$ is equal to M. Hence $T^{-n}(x)$ is a M-hypercyclic vector for T .

□

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On the fixed point theorem for $\mathbf{C}^*$-algebra-valued 2-metric spaces

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Abstract

In this paper we first establish the structure of C^*-algebra-valued 2-metric space and then we give some fixed point theorems for self-maps with contractive or expansive conditions on such spaces.

Keywords: C^*-algebra, contractive mapping, expansive mapping, fixed point
Mathematics Subject Classification [2010]: 46L07, 47H10, 54H25

1 Introduction

The notion of C^*-algebra-valued metric spaces has been investigated by Z. Ma, L. Jiang and H. Sun [4]. They presented some fixed point theorems for self-maps with contractive or expansive conditions on such spaces. Very recently, the authors [1, 3] proved some fixed point theorems by introducing the notion of 2-metric spaces. Using the concepts of 2-metric spaces and C^*-algebra-valued metric spaces, we define a new type of extended metric spaces. Then, we prove some fixed point theorems in this structure.

We provide some notations, definitions and auxiliary facts which will be used later in this paper.
Let $\mathbb{A}$ be a unital algebra with unit I. An involution on $\mathbb{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathbb{A}$, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. An assign to each $*$-algebra is $(\mathbb{A}, *)$. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ $(a \in \mathbb{A})$. For more details we refer the reader to [2].
Throughout this manuscript, $\mathbb{A}$ stands for a unital C^*-algebra with unit I. We say an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x = x^*$ and $\sigma(x) \subseteq \mathbb{R}_+ = [0, \infty)$, where θ means the zero element in $\mathbb{A}$ and $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ as follows: $x \preceq y$ if and only if $y - x \succeq \theta$ $(x, y \in \mathbb{A})$. From now on, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

Definition 1.1. ([4]) Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:
1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

*Speaker

3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.
Then d is called a C^*-algebra-valued metric on X and $(X,\mathbb{A},d)$ is called a C^*-algebra-valued metric space.

2 Main results

Definition 2.1. Let X be a nonempty set, $\mathbb{A}$ be a C^*-algebra and $d : X \times X \times X \to \mathbb{A}$ be a map satisfying the following conditions:
(M1) for every pair of distinct elements $x,y \in X$, there exists $z \in X$ such that $d(x,y,z) \neq \theta$;
(M2) if at least two of three elements x,y,z are the same, then $d(x,y,z) = \theta$;
(M3) the symmetry: $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ for all $x,y,z \in X$;
(M4) the rectangle inequality: $d(x,y,z) \preceq d(t,x,y) + d(t,y,z) + d(t,x,z)$ for all $x,y,z,t \in X$.
Then d is called a *C^*-algebra-valued 2-metric* on X and $(X,\mathbb{A},d)$ is called a *C^*-algebra-valued 2-metric* space.

Remark 2.2. Using condition (M1) it readily verified that if for all $a \in X$, $d(x,y,a) = \theta$, then $x = y$.

Example 2.3. Let X be a set with the cardinal $card(X) \geq 4$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 3$ and $\mathbb{A}$ is a unital C^*-algebra. Let $\theta \preceq A \preceq \frac{3}{2}I$. It is easy to verify that $d : X \times X \times X \to \mathbb{A}$ defined by

$$\mathrm{d}(x,y,z) = \begin{cases} \theta, & \text{if at least two of three elements } x,y,z \text{ are the same,} \\ A, & x,y,z \in X_1, \\ I, & \text{otherewise,} \end{cases}$$

is a C^*-algebra-valued 2-metric on X.

Definition 2.4. Let $\{x_n\}$ be a sequence in a C^*-algebra-valued 2-metric space $(X,\mathbb{A},d)$.
1. $\{x_n\}$ is said to be a 2-convergent to $x \in X$ with respect to $\mathbb{A}$, written as $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} \|d(x_n,x,a)\| = 0$.
2. $\{x_n\}$ is said to be a 2-Cauchy sequence with respect to $\mathbb{A}$ in X, if for all $a \in X$, $\lim_{n,m\to\infty} \|d(x_n,x_m,a)\| = 0$.
3. $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued 2-metric space if every 2-Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Definition 2.5. Suppose that $(X,\mathbb{A},d)$ is a *C^*-algebra-valued 2-metric* space. We call a mapping $T : X \to X$ is a C^*-algebra-valued 2-contractive on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that fulfills the following condition:

$$d(Tx,Ty,a) \preceq A^* d(x,y,a)A \quad \text{for all } x,y,a \in X.$$

Theorem 2.6. *Assume that $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued 2-metric space and $T : X \to X$ is a C^*-algebra-valued 2-contractive mapping, then T has a unique fixed point in X.*

Proof. First suppose that $A = \theta$, then T maps X into a single point and so it has a unique fixed point. Thus without loss of generality, one can suppose that $A \neq \theta$. Choose $x_0 \in X$ and set $x_n = Tx_{n-1} = T^n x_0,\ n = 1, 2, \ldots$. Let a be an arbitrary and fixed element of X. For convenience, by B_a we denote the element $d(x_1, x_0, a)$ in $\mathbb{A}$. We are going to show that $\{x_n\}$ is a 2-Cauchy sequence with respect to $\mathbb{A}$. For, by the contraction of T we get

$$\begin{aligned} d(x_{n+1}, x_n, a) = d(Tx_n, Tx_{n-1}, a) &\preceq A^* d(x_n, x_{n-1}, a)A \\ &= A^* d(Tx_{n-1}, Tx_{n-2}, a)A \\ &\preceq (A^*)^2 d(x_{n-1}, x_{n-2}, a)A^2 \\ &\vdots \\ &\preceq (A^*)^n d(x_1, x_0, a)A^n \\ &= (A^*)^n B_a A^n. \end{aligned}$$

Using this fact one observes that $\{x_n\}$ is a 2-Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since
$\theta \preceq d(Tx, x, a) \preceq d(Tx_n, Tx, x) + d(Tx_n, x, a) + d(Tx_n, Tx, a)$

$$\preceq d(x_{n+1}, Tx, x) + d(x_{n+1}, x, a) + A^* d(x_n, x, a)A \to \theta \ (as\ n \to \infty),$$

hence $Tx = x$,i.e., x is a fixed point of T.
We will show that T has a unique fixed point, suppose that $y(\neq x)$ is another fixed point of T, it yields that

$$\theta \preceq d(x, y, a) = d(Tx, Ty, a) \preceq A^* d(x, y, a)A.$$

Consequently, one observes that

$$\begin{aligned} 0 &\leq \|d(x, y, a)\| = \|d(Tx, Ty, a)\| \\ &\leq \|A^*\|\|d(x, y, a)\|\|A\| \\ &= \|A\|^2 \|d(x, y, a)\| \\ &< \|d(x, y, a)\|, \end{aligned}$$

it is impossible, so $x = y$. □

Corollary 2.7. *Suppose $(X, \mathbb{A}, d)$ is a C^*-algebra-valued 2-metric space and $T : X \to X$ is a mapping which T^N is a C^*-algebra-valued 2-contractive on X for some positive integer N. Then T has a unique fixed point.*

Definition 2.8. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued 2-metric space. We call a mapping $T : X \to X$ is a C^*-algebra-valued 2-expansion mapping on X, if it satisfies the following conditions:
(E1) $T(X) = X$;
(E2) $d(Tx, Ty, a) \succeq A^* d(x, y, a)A$, for each $x, y, a \in X$,
where A is an invertible element in $\mathbb{A}$ such that $\|A^{-1}\| < 1$.

Theorem 2.9. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued 2-metric space, $T : X \to X$ be a C^*-algebra-valued 2-expansion mapping on X. Then T has a unique fixed point in X.*

Proof. We first show that T is injective. If there exist $x, y \in X$ such that $Tx = Ty$, then for each $a \in X$ we have

$$\theta = d(Tx, Ty, a) \succeq A^* d(x, y, a)A.$$

Since $A^*d(x, y, a)A \in \mathbb{A}_+$, thus $A^*d(x, y, a)A = \theta$. Now invertibility of A implies that $d(x, y, a) = \theta$ for each $a \in X$. Applying Remark 2.2, we have $x = y$. Therefore T is injective and so by (E1) it is invertible.
Next, we will show T has a unique fixed point in X. Substitute x, y with $T^{-1}x, T^{-1}y$, respectively in (E2), then

$$d(x, y, a) \succeq A^* d(T^{-1}x, T^{-1}y, a)A.$$

This means

$$(A^*)^{-1} d(x, y, a)A^{-1} \succeq d(T^{-1}x, T^{-1}y, a),$$

and thus

$$(A^{-1})^* d(x, y, a)A^{-1} \succeq d(T^{-1}x, T^{-1}y, a).$$

Then T^{-1} is a C^*-algebra-valued 2-contractive mapping and according to Theorem 2.7, it has a unique fixed point x. On the other hand, the equality $T^{-1}x = x$ gives us $Tx = x$, and so x is a unique fixed point of T. □

Theorem 2.10. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued 2-metric space. Suppose the mapping $T : X \to X$ satisfies the following conditions for all $x, y, a \in \mathbb{A}$*

$$d(Tx, Ty, a) \preceq A[d(Tx, x, a) + d(Ty, y, a)],)$$

$$d(Tx, Ty, a) \preceq A[d(Tx, y, a) + d(Ty, x, a)],$$

where $A \in \mathbb{A}'_+$ and $\|A\| < \frac{1}{2}$. Then T has a unique fixed point in X.

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Properties Hypergeometric Functions by Ruscheweyh Derivative*

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Abstract

We study properties of starlike and convexity for the hypergeometric function F(a, b; c; z) defined by Ruscheweyh derivative through putting conditions on a, b, c, to ensure that zF(a, b; c; z) will be in various subclasses of starlike and convex functions.

Keywords: Starlike, Convex , Ruscheweyh Derivative, Hypergeometric functions.
Mathematics Subject Classification [2010]: 30C45, 30C55

1 Introduction

let S denote the class of all functions f of the form

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the open unit disk $\Delta = \{z \in C :| z |< 1\}$.

Definition 1.1. A function $f \in S$ is said to be starlike of order $\beta(0 \leq \beta < 1)$ if and only if $Re\Big(\frac{zf'(z)}{f(z)}\Big) > \beta$.
Denote the class of all starlike functions of order β in Δ by $S^{\star}(\beta)$.

Definition 1.2. A function $f \in S$ is said to be convex of order $\beta(0 \leq \beta < 1)$ if and only if $Re\Big(\frac{1+zf''(z)}{f'(z)}\Big) > \beta$.
Denote the class of all convex functions of order β in Δ by $C(\beta)$.

Let (a,n) denote symbol for the generalized factorial ,

$(a, 0) = 1$ for $a \neq 0$, $(a, n) = a(a + 1)(a + 2)...(a + n - 1)$ for $n \in N$.

and the Gaussian hypergeometric function given by the analyti function ,

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${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}z^n,\ z\in\Delta.$

let T be the subclass of functions f in S of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,(a_n\geq 0). \tag{2}$$

that are analytic and univalent in the open unit disk Δ.
This paper deals with the generalization of starlike and convexity properties for hypergeometric functions defined by Ruscheweyh derivative.H. Silverman [3] has studied starlike and cinvexity properties for hypergeometric function.Also E. S. Aqlan [1] has studied the generalization of starlike and convexity properties for hypergeometric functions.

Definition 1.3. [2],[4] The Ruscheweyh derivative of order λ is denoted by $D^\lambda f$ and is defined as following:
If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ then

$$D^\lambda f(z)=\frac{z}{(1-z)^{\lambda+1}}\star f(z)=z+\sum_{n=2}^{\infty}B_n(\lambda)a_nz^n,(\lambda>-1,z\in\Delta) \tag{3}$$

where

$$B_n(\lambda)=\frac{(\lambda+1)(\lambda+2)...(\lambda+n-1)}{(n-1)!} \tag{4}$$

Definition 1.4. Let $S^\star(\alpha,\beta,\varepsilon,\lambda)$ be a class of starlike functions of order α and type β defined by Ruscheweyh derivative that satisfies

$$\left|\frac{\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)}-1}{2\varepsilon\left[\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)}-\alpha\right]-\left[\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)}-1\right]}\right|<\beta \tag{5}$$

where $0\leq\beta<1,\frac{1}{2}\leq\varepsilon\leq 1,0\leq\alpha<\frac{1}{2\varepsilon},\lambda>-1,z\in\Delta$.

Definition 1.5. Let $C(\alpha,\beta,\varepsilon,\lambda)$ be a class of convex functions of order α and type β defined by Ruscheweyh derivative that satisfies

$$\left|\frac{\frac{z\left(D^\lambda f(z)\right)''}{(D^\lambda f(z))'}}{2\varepsilon\left[1+\frac{z\left(D^\lambda f(z)\right)''}{(D^\lambda f(z))'}-\alpha\right]-\frac{z\left(D^\lambda f(z)\right)''}{(D^\lambda f(z))'}}\right|<\beta \tag{6}$$

where $0\leq\beta<1,\frac{1}{2}\leq\varepsilon\leq 1,0\leq\alpha<\frac{1}{2\varepsilon},\lambda>-1,z\in\Delta$.

2 Main results

Theorem 2.1. *Let f(z) be defined by (2), then*

(i) f(z) is in the class $S^\star(\alpha,\beta,\varepsilon,\lambda)$ if and if

$\sum_{n=2}^{\infty}\Big[(n-1)(1-\beta)+2\varepsilon\beta(n-\alpha)\Big]B_n(\lambda)a_n \le 2\varepsilon\beta(1-\alpha),$

where $0\le\beta<1, \frac{1}{2}\le\varepsilon\le 1, 0\le\alpha<\frac{1}{2\varepsilon}, \lambda>-1, n\in N, n\ge 2.$

(ii) f(z) is in the class $C(\alpha,\beta,\varepsilon,\lambda)$ *if and if*

$\sum_{n=2}^{\infty} n\Big[(n-1)(1-\beta+2\varepsilon\beta)+2\varepsilon\beta(1-\alpha)\Big]B_n(\lambda)a_n \le 2\varepsilon\beta(1-\alpha),$

where $0\le\beta<1, \frac{1}{2}\le\varepsilon\le 1, 0\le\alpha<\frac{1}{2\varepsilon}, \lambda>-1, n\in N, n\ge 2.$

Theorem 2.2. *Let f(z) be defined by (1),*

(i) then a sufficient condition for f(z) to be in the class $S^{\star}(\alpha,\beta,\varepsilon,\lambda)$ *is that*

$\sum_{n=2}^{\infty}\Big[(n-1)-\beta(n-1+2\alpha\varepsilon-2n\varepsilon)\Big]B_n(\lambda)a_n \le 2\varepsilon\beta(1-\alpha),$

where $0\le\beta<1, \frac{1}{2}\le\varepsilon\le 1, 0\le\alpha<\frac{1}{2\varepsilon}, \lambda>-1, n\in N, n\ge 2.$

(ii) then a sufficient condition for f(z) to be in the class $C(\alpha,\beta,\varepsilon,\lambda)$ *is that*

$\sum_{n=2}^{\infty}\Big[n(n-1)(1-\beta+2\varepsilon\beta)+2\varepsilon\beta(1-\alpha)\Big]B_n(\lambda)a_n \le 2\varepsilon\beta(1-\alpha),$

where $0\le\beta<1, \frac{1}{2}\le\varepsilon\le 1, 0\le\alpha<\frac{1}{2\varepsilon}, \lambda>-1, n\in N, n\ge 2.$

Theorem 2.3. *(i) Let a,b,c and* $\alpha,\beta,\varepsilon,\lambda$ *satisfy the following condition such that*

$T_1(a,b,c,\alpha,\beta,\varepsilon,\lambda)\le 2\varepsilon\beta(1-\alpha).\ a,b>0,\ c>a+b+1,\ \beta\in(0,1],\ 0\le\alpha<\frac{1}{2\varepsilon},\ \frac{1}{2}\le\varepsilon\le 1,\ \lambda>-1$ *and*

$T_1(a,b,c,\alpha,\beta,\varepsilon,\lambda)=\sum_{n=2}^{\infty}\Big[(n-1)(1-\beta+2\varepsilon\beta)+2\varepsilon\beta(1-\alpha)\Big]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$

where

$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\left[\frac{(n-1)(1-\beta+2\varepsilon\beta)}{2\varepsilon\beta(1-\alpha)}+1\right]\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\le 1.$

(ii) Let a,b,c and $\alpha,\beta,\varepsilon,\lambda$ *satisfy the following condition such that*

$T_2(a,b,c,\alpha,\beta,\varepsilon,\lambda)\le 2|\frac{c}{ab}|\varepsilon\beta(1-\alpha).\ a,b>-1,\ c>0,\ ab<0,\ \beta\in(0,1],\ 0\le\alpha<\frac{1}{2\varepsilon},\ \frac{1}{2}\le\varepsilon\le 1,\ \lambda>-1$ *and*

$T_2(a,b,c,\alpha,\beta,\varepsilon,\lambda)=\sum_{n=2}^{\infty}\Big[(n-1)(1-\beta+2\varepsilon\beta)+2\varepsilon\beta(1-\alpha)\Big]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$

where

$|\frac{ab}{c}|\frac{\Gamma(c+1)}{\Gamma(\lambda)\Gamma(a+1)\Gamma(b+1)}\sum_{n=2}^{\infty}\Big[\frac{(n-1)(1-\beta+2\varepsilon\beta)}{2\varepsilon\beta(1-\alpha)}+1\Big]\ \frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\leq 1.$

then $zF(a,b;c;z)\in S^{\star}(\alpha,\beta,\varepsilon,\lambda)$.

Theorem 2.4. *(i) Let a,b,c and* $\alpha,\beta,\varepsilon,\lambda$ *satisfy the following condition such that*

$T_3(a,b,c,\alpha,\beta,\varepsilon,\lambda)\leq 2\varepsilon\beta(1-\alpha).\ a,b>0,\ c>a+b+1,\ \beta\in(0,1],\ 0\leq\alpha<\frac{1}{2\varepsilon},\ \frac{1}{2}\leq\varepsilon\leq 1,\ \lambda>-1$ *and*

$T_3(a,b,c,\alpha,\beta,\varepsilon,\lambda)=\sum_{n=2}^{\infty}\Big[(n-1)^2(1-\beta+2\varepsilon\beta)+(n-1)\Big(1-\beta+2\varepsilon\beta(2-\alpha)\Big)+2\varepsilon\beta(1-\alpha)\Big]B_n(\lambda)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$

where

$\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=2}^{\infty}\Big[\frac{(n-1)^2(1-\beta+2\varepsilon\beta)+(n-1)(1-\beta+2\varepsilon\beta(2-\alpha))}{2\varepsilon\beta(1-\alpha)}+1\Big]\ \frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\leq 1,$

(ii) Let a,b,c and $\alpha,\beta,\varepsilon,\lambda$ *satisfy the following condition such that*

$T_4(a,b,c,\alpha,\beta,\varepsilon,\lambda)\leq 2\frac{c}{ab}\varepsilon\beta(1-\alpha).\ a,b>-1,\ c>0,\ ab<0,\ \beta\in(0,1],\ 0\leq\alpha<\frac{1}{2\varepsilon},\ \frac{1}{2}\leq\varepsilon\leq 1,\ \lambda>-1$ *and*

$T_4(a,b,c,\alpha,\beta,\varepsilon,\lambda)=\sum_{n=2}^{\infty}\big[(n-1)^2(1-\beta+2\varepsilon\beta)+(n-1)(1-\beta+2\varepsilon\beta(2-\alpha))\times 2\varepsilon(1-\alpha)\big]B_n(\lambda)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$

where

$\frac{|ab|}{c}\frac{\Gamma(c+1)}{\Gamma(\lambda)\Gamma(a+1)\Gamma(b+1)}\sum_{n=2}^{\infty}\Big[\frac{(n-1)^2(1-\beta+2\varepsilon\beta)+(n-1)(1-\beta+2\varepsilon\beta(2-\alpha))}{2\varepsilon\beta(2-\alpha)}+1\Big]$
$\times\frac{\Gamma(\lambda+n)\Gamma(a+n-1)\Gamma(b+n-1)}{\Gamma(c+n-1)[(n-1)!]^2}\leq 1,$

then $zF(a,b;c;z)\in C(\alpha,\beta,\varepsilon,\lambda)$.

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Redundancy of Dual Frames

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Abstract

In this manuscript we investigate the ralationship between the redundancies of frames and their duals.

Keywords: Frame, Dual frame, Redundancy
Mathematics Subject Classification [2010]: Primary 42C40; Secondary 41A58, 47A58.

1 Introduction

Frames for Hilbert spaces have been introduced in 1952 by Duffin and Schaeffer in their fundamental paper [6] and have been studied in the last two decades as a powerful framework for robust and stable representation of signals by introducing redundancy. The customary definition of redundancy was improved by Bodmann, Casazza and Kutyniok in [2] by providing a quantitative measure, which coined upper and lower redundancies. Redundancy is applied in areas such as: filter bank theory [3] , sigma-delta quantization [1], and signal and image processing [4]. Dual frames play an important role in studying frames and their applications, specially in the reconstruction formula. Therefore it is natural to study and consider their redundancy and its relationship with redundancy of original frame. In this paper, we will show that the ratio between redundancies of frames and dual frames is bounded from below and above by some significant numbers. First, we will review the definitions of frames and redundancy function for finite frames .

Definition 1.1. [5] Let $\mathcal{H}$ be a Hilbert space and I be a countable index set. The family $\phi = \{\varphi_i\}_{i\in I}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H}.$$

The frame ϕ is called a *tight* frame, if $A = B$ and it is a *Parseval* frame if, $A = B = 1$.

Reader can see [5] for the definitions of synthesis, analysis and frame operators.

For a frame the dual frame (canonical dual frame) defined as follows;

*Speaker

Definition 1.2. [5] Let $\phi = \{\varphi_i\}_{i\in I}$ be a frame for $\mathcal{H}$. Then a frame $\psi = \{\psi_i\}_{i\in I}$ is called a dual frame for $\varphi = \{\varphi_i\}_{i\in I}$, if

$$x = \sum_{i\in I} \langle x, \varphi_i\rangle \psi_i \quad for \quad all \quad x \in \mathcal{H}.$$

The frame $\{S_\phi^{-1}\varphi_i\}_{i\in I}$ is the canonical dual frame, where S_ϕ is the frame operator for $\phi = \{\varphi_i\}_{i\in I}$. Dual frames which do not coincide with the canonical dual frame, are often coined *alternate dual frame*.

Now, we present a lemma that we need in the sequel.

Lemma 1.3. *[5] Let $\phi = \{\varphi_i\}_{i\in I}$ be a frame for $\mathcal{H}$. Then the following are equivalent.*

1. *$\phi = \{\varphi_i\}_{i\in I}$ is tight;*
2. *$\phi = \{\varphi_i\}_{i\in I}$ has a dual of the form $\psi = \{\psi_i\}_{i\in I} = \{C\varphi_i\}_{i\in I}$ for some constant $C > 0$.*

The redundancy function is defined from the unit sphere $\mathbb{S} = \{x \in \mathcal{H} : \|x\| = 1\}$ to the set of positive real numbers $\mathbb{R}^+$, [2].

Definition 1.4. [2] Let $\phi = \{\varphi_i\}_{i=1}^N$ be a frame for a finite dimensional Hilbert space $\mathcal{H}^n$. For each $x \in \mathbb{S}$, the redundancy function

$$\mathcal{R}_\phi : \mathbb{S} \to \mathbb{R}^+$$

is defined by

$$\mathcal{R}_\phi(x) = \sum_{i=1}^{N} \|P_{\langle\varphi_i\rangle}(x)\|^2,$$

where $\langle\varphi_i\rangle$ denotes the span of $\varphi_i \in \mathcal{H}$ and $P_{\langle\varphi_i\rangle}$ denotes the orthogonal projection onto $\langle\varphi_i\rangle$. The upper redundancy of ϕ is defined by

$$\mathcal{R}_\phi^+ = max_{x\in\mathbb{S}}\mathcal{R}_\phi(x),$$

and the lower redundancy of ϕ by

$$\mathcal{R}_\phi^- = min_{x\in\mathbb{S}}\mathcal{R}_\phi(x).$$

2 Main results

In this section, we present main results.

Theorem 2.1. *Let $\phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}^n$ and $\psi = {S_\phi}^{-1}\phi$ be the canonical dual of ϕ. Then*

$$\mathcal{R}_\phi^{\pm}(k(S_\phi))^{-2} \leq \mathcal{R}^{\pm}_{{S_\phi}^{-1}\phi} \leq \mathcal{R}_\phi^{\pm}(k(S_\phi))^2,$$

where $k(S_\phi) = \|S_\phi\|\|S_\phi^{-1}\|$ denotes the condition number of the frame operator S_ϕ.

It is known that the redundancy is invariant under scaling. So, using the lemma 1.3, we have:

Proposition 2.2. *Let $\phi = \{\varphi_i\}_{i=1}^N$ be a tight frame for $\mathcal{H}^n$, and $\psi = \{\psi_i\}_{i=1}^N$ be its canonical dual frame. Then*

$$\mathcal{R}_\phi(x) = \mathcal{R}_\psi(x).$$

For general frames (not necessary tight frames) and their alternate duals, we have a relationship between their redundancies in a particular case. First, we will state a lemma from [5].

Lemma 2.3. *[5] Let $\phi = \{\varphi_i\}_{i=1}^N$ be a frame for $\mathcal{H}^n$ with canonical dual ${S_\phi}^{-1}\phi$. If $\psi = \{\psi_i\}_{i=1}^N$ is the alternate dual of ϕ, then*

$$\|(\langle x, {S_\phi}^{-1}\varphi_i\rangle)_{i=1}^N\|_2 \leq \|(\langle x, \psi_i\rangle)_{i=1}^N\|_2.$$

In particular, suppose that ${S_\phi}^{-1}\phi$ and $\psi = \{\psi_i\}_{i=1}^N$ in the previous lemma are equal norm frames, i.e., $\|{S_\phi}^{-1}\varphi_i\| = c$ and $\|\psi_i\| = d$ for some $c, d > 0$ and $i = 1, ..., N$. Then

$$\mathcal{R}_{{S_\phi}^{-1}}(x) = \sum_{i=1}^N \|{S_\phi}^{-1}\varphi_i\|^{-2} |\langle x, {S_\phi}^{-1}\varphi_i\rangle|^2 = c^{-2} \sum_{i=1}^N |\langle x, {S_\phi}^{-1}\varphi_i\rangle|^2$$

and

$$\mathcal{R}_\psi(x) = \sum_{i=1}^N \|\psi_i\|^{-2} |\langle x, \psi_i\rangle|^2 = d^{-2} \sum_{i=1}^N |\langle x, \psi_i\rangle|^2$$

so

$$\mathcal{R}_{{S_\phi}^{-1}}(x) \leq (\frac{d}{c})^2 \mathcal{R}_\psi(x).$$

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Results for The Daugavet Property and Examples

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Abstract

In present paper, results of the Daugavet property for Banach spaces. Also express several examples that show in general the Daugavet property is not transmitted from space into subspace and vice-versa.

Keywords: The Daugavet property ,Banach space, Almost narrow and narrow operators
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In this section we state several results and examples for Banach spaces with the Daugavet property.
Definition 1.1 Banach space X has the Daugavet property if every rank-1 operator $T \in L(X)$ satisfies (1).
Definition 2.1. Let $T : X \to E$ be an operator between Banach spaces.
(a) T is called almost narrow (or strong Daugavet operator) if for every two elements $x, y \in S_X$ and every $\varepsilon > 0$ there is some $z \in B_X$ such that $\|T(y - z)\| \leq \varepsilon$ and $\|x + z\| \geq 2 - \varepsilon$.
(b) T is called narrow if for every functional $x^* \in X^*$ the operator $T \oplus x^* : X \to E \oplus_1 \mathbb{R}$ defined by

$$(T \oplus x^*)(x) = (T(x), x^*(x))$$

is almost narrow.
definition 3.1. A subspace Y of a Banach space X is called rich (respect. almost rich) if the quotient map from X onto X/Y is narrow (respect. almost narrow).
We say that a subspace Y of a Banach space X with the DP is wealthy if Y and every subspace of X containing Y have the DP.

2 Main results

Clearly, every narrow operator is almost narrow. By [1] if X has the DP, then the narrow and weakly compact operators on X are equivalent. The following example shows that for

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every Banach space this condition is not held.

Example 1.2. Put $X = C([0,1])^*$. The following illustrates operator T on X,

$$T : X \to X$$
$$T(\mu) = -\mu(\{0\})\delta_0.$$

According to [1] X does not have the DP. We now claim that T is weakly compact, but it is not narrow. Clearly, rank$(T) = 1$. So T is compact, then by [2, Theorem 4.18] T is weakly compact. Suppose T be narrow, thus it is almost narrow. By [1] T satisfies (1). It is well known that $\|Id + T\| < 2$ and $\|T\| = 1$, therefore T does not satisfy (1). It is a contradiction. Let K be the Cantor set on $[0,1]$ and $G : C([0,1]) \to C(K)$ is defined by $G(f) = f|K$. By [9. Example 1] G is a narrow operator. Since $C([0,1])$ has the DP, according to [1] G is a weakly compact operator.

Corollary 1.2. If every weakly compact operator on the Banach space X is narrow, then X has the DP.

Proof. Let $T : X \to X$ be with rank$(T) = 1$. By [2, Theorem 4.18] T is compact and, then, it is weakly compact. According to our assumption T is narrow. By [1, Lemma 4.3] T satisfies (1). Therefore X has the DP.

Also if every weakly compact operator on the Banach space X satisfies (1), then X has the DP.

In [19] proved for l_1-sum and $l_\infty-$sum (in finite state) maintine the Daugavet property. In this paper with restrict metre show that l_1-sum and $l_\infty-$sum in infinity state preserv the Daugavet property.

Theorem 1.2. If $\{X_i\}_{i\in I}$ has the Daugavet property and d be disceret metre on R , then $X = \bigoplus_1\{X_i\}_{i\in I}$ and $X = \bigoplus_\infty\{X_i\}_{i\in I}$ have the Daugavet property.

Proof. Let $\varepsilon > 0$, $x^* \in S_{X^*}$ and $S = S(x_0, \varepsilon)$ be a slice of B_{X^*}. Assume $x^* = (x_1^*, x_2^*, ...)$ so that $1 = ||x^*||_\infty = sup\{||x_1^*||, ||x_2^*||, ...\} \subset R$, Since d is disceret metre so there is a $i \in I$ so that $\|x_i^*\| = 1$, therefore $x_i^* \in S_{X_i^*}$. Set $S_i = S(\frac{x_0^i}{\|x_0^i\|}, \varepsilon)$, it is clear that $S_i \subseteq B_{X_i^*}$. Since X_i has the Daugavet property for $i \in I$, so there is $y_i^* \in S_i$ so that

$$\|x_i^* + y_i^*\| \geq 2 - \varepsilon.$$

Put $y^* = (y_1^*, y_2^*, ...)$. Clearly $y^* \in B_{X^*}$ and

$$x_0(y^*) = (x_0^1, x_0^2, ...)(y_1^*, y_2^*, ...) = x_0^1(y_1^*) + x_0^2(y_2^*) + ... \geq 1 - \varepsilon.$$

In result $y^* \in S$ and $\|x^* + y^*\| \geq 2 - \varepsilon$.

A similar way, shows that $X = \bigoplus_\infty\{X_i\}_{i\in I}$ has the DP.

Theorem 2.2. If $X_1 \bigoplus_\infty X_2$ has the Daugavet property, then X_1 and X_2 have the Daugavet property.

Proof Let $T : X_1 \to X_1$ be with rank$(T) = 1$ and $||T|| = 1$. Define

$$S : X_1 \bigoplus_\infty X_2 \to X_1 \bigoplus_\infty X_2$$
$$S(x_1, x_2) = (T(x_1), 0).$$

It is obvious that , $||S|| = 1$ and $rank(S) = 1$. Now since $X_1 \bigoplus_\infty X_2$ has the Daugavet property therefore, $||Id + S|| = 2$. So for $\varepsilon > 0$,

$$||Id + S|| = sup_{||(x_1,x_2)||_\infty \leq 1}||Id(x_1, x_2) + S(x_1, x_2)||_\infty \geq 2 - \varepsilon.$$

In result there is $x_1 \in S_{X_1}$ and $x_2 \in S_{X_2}$ so that $||(x_1 + T(x_1), x_2)||_\infty \geq 2 - \varepsilon$. Since $\|x_2\| \leq 2 - \varepsilon$, therefore $||Id + T|| \geq 2 - \varepsilon$. In similar way, we can show X_2 has the Daugavet property.

Corollary 2.2. If X_1 and X_2 have the Daugavet property, then $X_1 \bigoplus_{c_0} X_2$ has the Daugavet property.

Proof. Let $\varepsilon > 0, (x_1, x_2) \in S_{X_1 \bigoplus_{c_0} X_2}$ and $S = S((x_1^*, x_2^*), \varepsilon)$ be a slice of $B_{X_1 \bigoplus_{c_0} X_2}$. Clearly, $x_1 \in S_{X_1}$ and $S_1 = S(\frac{x_1^*}{\|x_1^*\|}, \varepsilon) \subset B_{X_1}$. Since X_1 has the Daugavet property there is $y_1 \in S_1$ so that $||x_1 + y_1|| \geq 2 - \varepsilon$. In similar way there is $y_2 \in S_2$ so that $||x_2 + y_2|| \geq 2 - \varepsilon$. In result there is $(y_1, y_2) \in X_1 \bigoplus_{c_0} X_2$ so that $(x_1^*, x_2^*)(y_1, y_2) = x_1^*(y_1) + x_2^*(y_2) \geq 1 - \varepsilon$ and

$$||(x_1, x_2)+(y_1, y_2)||_\infty = ||(x_1+y_1, x_2+y_2)||_\infty = max\{||(x_1+y_1)||, ||(x_2+y_2)||\} \geq ||x_1+y_1|| \geq 2-\varepsilon.$$

With attention to [1, Lemmas 2.2 and 2.4] if X^{**} has the DP, then X has the DP. An example shows that its reverse does not hold essentially.

Example 2.2. Put $X = C([0, 1])$. According to [1] Banach space X has the DP. We claim that X^{**} does not has the DP. Suppose X^{**} has the DP, then X^* has the DP. But, by [1] X^* has no the DP. It is a contradiction. If X has the DP, then X^* contains an isometric copy of $l_1(\mathbb{N})$ (see [5, Corollary 2.13]). In following example we shows that the reversed result are not true in general.

Example 3.2. Set $Y = L_2(\mu)$ (μ is the Counting measure on $\mathbb{N}$). It is clear that $L_2(\mu)^* = L_2(\mu)$ and $L_2(\mu) = l_2(\mathbb{N})$, therefore, $l_1(\mathbb{N}) \subseteq Y^*$. Since Y is reflexive, by [1, Corollary 3.2.] it does not has the DP.

Let (Ω, Σ, μ) be a positive measure space. A E set is called a atom for μ whenever $\mu(E) > 0$ and for every measurable set $A \subset E$, $\mu(A) = 0$ or $\mu(A^c) = 0$ and μ measure is called nonatomic if has no atomic. We now present two examples that shows generally the DP from space into subspace is not transmitted and vice-versa.

Example 4.2. Put $X = \mathbb{N}$. Let μ be the Counting measure on X. Apparently, $L_1(\mu) \subseteq L_2(\mu)$ and $L_1(\mu)$ has the DP. Since $L_2(\mu)$ is reflexive, by [1, Corollary 2.5] it does not has the DP.

Example 5.2. Let $X = [0, 1]$ and let μ be the Lebesgue measure on X. Obviously, $L_2(\mu) \subseteq L_1(\mu)$ and $L_1(\mu)$ has the DP. Since $L_2(\mu)$ is reflexive, according to [1, Corollary 2.5] it has no the DP.

A slice of the unit ball of X is a set given by

$$S(x^*, \alpha) = \{x \in B_X : \quad x^*(x) \geq 1 - \alpha\}$$

for some functional $x^* \in X^*$ of norm 1 and some $\alpha > 0$. If Banach space X has the DP, then every slice of its unit ball has the diameter 2 (see[1, Corollary 2.5]). An example shows that its reverse is not true in general.

Example 6.2. Put $X = C([0, 1])$ ($[0, 1]$ is considered with discrete metre). By [1] Banach space X does not has the DP. Consider a slice $S = S(x^*, \varepsilon)$ of B_X. Clearly,

$$W = \{x \in B_X : \quad x^*(x) > 1 - \varepsilon\} \subseteq S \subseteq B_X.$$

It is well known that W is a weakly open relative subset of B_X. By [8, Corollary 2.4] diameter $(W) = 2$. Moreover, diameter $(B_X) \leq 2$ and hence diameter$(S) = 2$.

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Some $\mu-$Fixed Point Theorems for $\mu-$continuous Maps on $\sigma-$algebras

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Abstract

Let μ be a positive finite measure on a $\sigma-$algebra $\mathcal{A}$. In this paper, we introduce the concept of μ-fixed point for mappings $f : \mathcal{A} \to \mathcal{A}$ and obtain conditions for the existence of common μ-fixed points of such mappings. We show that for any $\mu-$complete measure space if there exist $m \in \mathbb{N}$ and $0 \leq k < 1$ such that for every $A \in \mathcal{A}$, $\mu(f^m(A)) \leq k\, \mu(A)$, then all $\{f^n\}_{n\in\mathbb{N}}$ have a unique common $\mu-$fixed point.

Keywords: fixed point theorem, contraction mapping, measure
Mathematics Subject Classification [2010]: 47H10, 47H09

1 Introduction

Schauder fixed point theorem states that any compact map from a nonempty, closed, convex, bounded subset E of a Banach space into itself has a fixed point in E [3]. Darbo [4] extended Schauder's fixed point theorem to the setting of noncompact operators by using the concept of $\alpha-k-$set contraction, where $0 \leq k < 1$ and α denotes the Kuratowski measure of noncompactness [6]. In fact, he proved the following theorem.

Theorem 1.1. *Let X be a nonempty, closed, bounded and convex subset of a Banach space and $f : X \to X$ be a bounded continuous map with*

$$\alpha(f(B)) \leq k\, \alpha(B)$$

for all bounded subsetes B of X, where $0 \leq k < 1$. Then f has a fixed point.

Sadovski [7] proved that above theorem is true for a bounded continuous map f such that

$$\alpha(f(B)) \leq \alpha(B)$$

for all bounded subsets B of X with $\alpha(B) > 0$. Banas [2] proved a fixed point result using the concept of $\beta - k-$set contraction, where $0 \leq k < 1$ and β denotes the De Blasi measure of weak noncompactness [5]. Amini-Harandi, Fakhar and Zafarani [1] have introduced a type of generalized set contraction in topological spaces with respect to a measure of noncompactness and proved a fixed point theorem which are either generalized set contraction or condensing ones.

*Speaker

These findings motivate us to define the concept of μ-fixed point for mapping $f : \mathcal{A} \to \mathcal{A}$, where μ is a positive finite measure on a $\sigma-$algebra $\mathcal{A}$, and obtain conditions for the existence of common μ-fixed points of such mappings. We show that if $(X, \mathcal{A}, \mu)$ is a μ-complete measure space and $f : \mathcal{A} \to \mathcal{A}$ is a $\mu-$continuous mapping such that for every $A \neq \varnothing$, $f(A) \neq \varnothing$ and $\mu(f^m(A)) \leq k\ \mu(A)$ for some $m \in \mathbb{N}$ and $0 \leq k < 1$, then all $\{f^n\}_{n\in\mathbb{N}}$ have a unique common $\mu-$fixed point.

2 Main results

Throughout the paper, $\mathcal{A}$ denotes a σ-algebra in a non-empty set X; that is, a collection of subsets of X with the following properties.

(i) $X \in \mathcal{A}$.

(ii) if $A \in \mathcal{A}$, then the complement of A relative to X is an element of $\mathcal{A}$.

(iii) if $A = \bigcup_{n=1}^{\infty} A_n$ and $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$, then $A \in \mathcal{A}$.

Let also μ denote a positive finite measure on σ-algebra $\mathcal{A}$; that is, a countably additive function from $\mathcal{A}$ into $[0, \infty)$ and which is . Now, let us give some new definitions which are needed in the following.

Definition 2.1. Let $\{A_n\}$ be a sequence in a measure space $(X, \mathcal{A}, \mu)$. Then

(i) the sequence $\{A_n\}$ is called μ-*converges* to $A \in \mathcal{A}$, if the sequence $\{\mu(A_n)\}$ converges to $\mu(A)$.

(ii) the sequence $\{A_n\}$ is called μ-*Cauchy* if the sequence $\{\mu(A_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

(iii) the measure space $(X, \mathcal{A}, \mu)$ is called μ-*complete* if every μ-Cauchy sequence in $\mathcal{A} \setminus \{\emptyset\}$, μ-converges to a non-empty element of $\mathcal{A}$.

Definition 2.2. Let $(X, \mathcal{A}, \mu)$ be a measure space. A mapping f from $\mathcal{A}$ into $\mathcal{A}$ is called μ-*continuous* at $A \in \mathcal{A}$ if for every sequence $\{A_n\}$ of elements of $\mathcal{A}$, μ-convergent to A, the sequence $\{f(A_n)\}$, μ-converges to $f(A)$.

Definition 2.3. A non-empty element A in a measure space $(X, \mathcal{A}, \mu)$ is said to be a μ-*fixed point* of a mapping $f : \mathcal{A} \to \mathcal{A}$ if $f(A) = A$ almost every where with respect to μ. Furthermore, f has unique μ-fixed point if A and B are μ-fixed points of f, then $A = B$ almost every where with respect to μ.

The following lemma is needed to prove our results.

Lemma 2.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : \mathcal{A} \to \mathcal{A}$ be a mapping. Then the following statements hold.*

(i) *$A \in \mathcal{A}$ is a $\mu-$fixed point of f if and only if $\mu(f(A)) = \mu(A)$.*

(ii) *f has a unique $\mu-$fixed point if and only if $\mu(A) = \mu(B)$ whenever A and B are $\mu-$fixed points of f.*

The main result of the paper is the following.

Theorem 2.5. *Let $(X, \mathcal{A}, \mu)$ be a μ-complete measure space and let $f : \mathcal{A} \to \mathcal{A}$ be a μ-continuous mapping with $f(A) \neq \emptyset$ for all $A \neq \emptyset$. If there exist $m \in \mathbb{N}$ and $0 \leq k < 1$ such that for every $A \in \mathcal{A}$*

$$\mu(f^m(A)) \leq k\ \mu(A).$$

Then all $\{f^n\}_{n\in\mathbb{N}}$ have a unique common μ-fixed point.

In the next result, we give a corollary for case $k = 1$ in Theorem 2.5.

Proposition 2.6. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space and $f : \mathcal{A} \to \mathcal{A}$ be a $\mu-$continuous mapping with $f(A) \neq \emptyset$ for all $A \neq \emptyset$. If there exists $m \in \mathbb{N}$ such that*

$$\mu(f^m(A)) \leq \mu(A)$$

for all $A \in \mathcal{A}$, then f^m has a μ-fixed point.

Corollary 2.7. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space and $f : \mathcal{A} \to \mathcal{A}$ be a $\mu-$continuous mapping with $f(A) \neq \emptyset$ for all $A \neq \emptyset$. If there exists $m \in \mathbb{N}$ such that*

$$f^m(A) \subseteq A$$

for all $A \in \mathcal{A}$, then f^m has a μ-fixed point.

Theorem 2.8. *Let $(X, \mathcal{A}, \mu)$ be a μ-complete measure space, $\psi : \mathcal{A} \to \mathcal{A}$ be a mapping and $f : \mathcal{A} \to \mathcal{A}$ be a μ-continuous mapping with $f(A) \neq \emptyset$ for all $A \neq \emptyset$. If there exist $m, j_0 \in \mathbb{N}$ such that for every $A \in \mathcal{A}$*

$$\mu(f^m(A)) \leq \mu(\psi(A)) - \mu(\psi(f^{j_0}(A))).$$

Then all $\{f^n\}_{n\in\mathbb{N}}$ have a common μ-fixed point.

As a consequence of this theorem, we have the following result.

Corollary 2.9. *Let $(X, \mathcal{A}, \mu)$ be a μ-complete measure space and $f : \mathcal{A} \to \mathcal{A}$ be a μ-continuous mapping with $f(A) \neq \emptyset$ for all $A \neq \emptyset$. Let also there exist $m, j_2 \in \mathbb{N}$ and $j_1 \in \mathbb{Z}$ such that $j_2 > j_1$ and for every $A \in \mathcal{A}$*

$$\mu(f^m(A)) \leq \mu(f^{j_1}(A)) - \mu(f^{j_2}(A)).$$

Then all $\{f^n\}_{n\in\mathbb{N}}$ have a unique common μ-fixed point.

As an immediate consequence of this corollary, we present the following result.

Corollary 2.10. *Let $(X, \mathcal{A}, \mu)$ be a μ-complete measure space and $f : \mathcal{A} \to \mathcal{A}$ be a μ-continuous mapping with $f(A) \neq \emptyset$ for all $A \neq \emptyset$. Let also there exist $m, j_0 \in \mathbb{N}$ such that for every $A \in \mathcal{A}$*

$$\mu(f^m(A)) \leq \mu(A) - \mu(f^{j_0}(A)).$$

Then all $\{f^n\}_{n\in\mathbb{N}}$ have a unique common μ-fixed point.

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Some Fixed Point Theorems For Mappings on a G-Metric Space Endowed with a Graph

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Abstract

In this paper, we discuss the existence of fixed points for Banach and Kannan contractions defined on G-metric spaces, which were introduced by Mustafa and Sims, endowed with a graph. Our results generalize and unify some recent results by Jachymski, Bojor and Mustafa and those contained therein. Moreover, we provide some examples to show that our results are substantial improvement of some known results in literature.

Keywords: Fixed point, G-metric spaces, Banach contraction, Kannan contraction.
Mathematics Subject Classification [2010]: 47H10, 47H09.

1 Introduction

Investigation of the existence and uniqueness of fixed points of certain mappings in the framework of metric spaces is one of the centers of interests in nonlinear functional analysis. Fixed point theory has a wide application in almost all fields of quantitative sciences such as economics, biology, physics, chemistry, computer science and many branches of engineering. It is quite natural to consider various generalizations of metric spaces in order to address the needs of these quantitative sciences. Different mathematicians tried to generalize the usual notion of metric space (X, d). In the 1960s, Gähler [4] tried to generalize the notion of metric and introduced the concept of 2-metric spaces inspired by the mapping that associated the area of a triangle to its three vertices. But different authors proved that there is no relation between these two functions [5]. Then, Dhage [3] in his Ph. D. thesis introduce a new class of generalized metric space called D-metric spaces. Unfortunately, both kinds of metrics appear not to have as good properties as their authors announced ([5] ,[8]). To overcome these drawbacks, in 2003 Mustafa and Sims [7] showed that most of the results claimed concerning of such spaces are invalid. Then they introduced a generalization of metric spaces (X, d), which are called G-metric spaces ([8], [9]). The G-metric space is defined as follows:

Definition 1.1 ([9])**.** Let X be a nonempty set, and $G : X \times X \times X \to [0, +\infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$; for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric spaces.

A G-metric space (X, G) is called symmetric G-metric space if $G(x, x, y) = G(x, y, y)$; for all $x, y \in X$.

*Speaker

Example 1.2 ([9]). Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined as

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}, \quad (1)$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x), \quad (2)$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 1.3 ([9]). Let (X, G) be a G-metric space, then a sequence $\{x_n\}$ is said to be G-Cauchy sequence if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.

Proposition 1.4 ([9]). *Let (X, G) be a G-metric space. the following are equivalent:*
1) *The sequence $\{x_n\}$ is G-Cauchy,*
2) *for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 1.5 ([9]). Let (X, G) be a G-metric space, and $\{x_n\}$ be a sequence of points of X. we say that $\{x_n\}$ is G-convergent to $x \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Proposition 1.6 ([9]). *Let (X, G) be a G-metric space. the following statements are equivalent:*
1) *$\{x_n\}$ is G-convergent to x,*
2) *$G(x_n, x, x) \to 0$ as $n \to +\infty$,*
3) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$,*
4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.7 ([9]). Let (X, G) be a G-metric space. A mapping $f : X \to X$ is said to be G-continuous if $\{f(x_n)\}$ is G-convergent to $f(x)$ where $\{x_n\}$ is any G-convergent sequence converging to x.

Definition 1.8 ([9]). A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Theorem 1.9 ([8]). *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \leq kG(x, y, z),$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 1.10 ([10]). *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \leq k\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\},$$

where $k \in [0, 1/3)$. Then T has a unique fixed point.

We next review some notions in graph theory. All of them can be found in, e.g., [1].

Let X be a G-metric space. Consider a directed graph H with $V(H) = X$ and $E(H) \supseteq \{(x, x) : x \in X\}$, i.e., $E(H)$ contains all loops. Suppose further that H has no parallel edges. With these assumptions, we may denote H by the pair $(V(H), E(H))$. In this way, the G-metric space X is endowed with the graph H. The notation $\widetilde{H}$ is used to denote the undirected graph obtained from H by deleting the directions of the edges of H. Thus,

$$V(\widetilde{H}) = X \qquad E(\widetilde{H}) = \big\{(x, y) \in X \times X : (x, y) \in E(H) \ \vee \ (y, x) \in E(H)\big\}.$$

By a path in H from a vertex x to a vertex y, it is meant a finite sequence $(x_s)_{s=0}^{N}$ of vertices of H such that $x_0 = x$, $x_N = y$, and $(x_{s-1}, x_s) \in E(H)$ for $s = 1, \ldots, N$. A graph H is called connected if there is a path between any two vertices and is called weakly

connected if $\widetilde{H}$ is connected, i.e., there exists an undirected path in H between its each two vertices.
If H is such that $E(H)$ is symmetric, then for $x \in V(H)$, the symbol $[x]_H$ denotes the equivalence class of the relation $\mathcal{R}$ defined on $V(H)$ by the rule:

$$y\mathcal{R}z \text{ if there is a path in } H \text{ from } y \text{ to } z.$$

Recall that if $T : X \to X$ is an operator, then by $FixT = \{x \in X : T(x) = x\}$ we denote the set of all fixed points of T. Denote also $X_T = \{x \in X : (x, Tx) \in E(H)\}$. Moreover, we may treat H as a G-weighted graph by assigning to each three vertices x, y and z in X the G-distance $G(x, y, z)$.

2 Main Results

The aim of this paper is to study the existence of fixed points for Banach and Kannan H-contractions in G-metric spaces endowed with a graph H by introducing the concept of Banach and Kannan H-contractions according to the articles of Jakhymski and Bojor [6, 2].

Definition 2.1. Let (X, G) be a G-metric space with a graph H and $T : X \to X$ be a mapping. We call T a Banach H-contraction if T preserves edges of H,(i.e., $\forall x, y \in X$, $(x, y) \in E(H) \Rightarrow (Tx, Ty) \in E(H)$), and there exists $k \in [0, 1)$ such that $G(Tx, Ty, Tz) \leq kG(x, y, z)$ for all x, y and z that are on a path to length at most 2 in H.

Proposition 2.2. *Let X be a G-metric space with a graph H. If a mapping T from X into itself is a Banach H-contraction, then T is also a Banach $\widetilde{H}$-contraction.*

Example 2.3. Let $\preceq$ be a partial order on a G-metric space X and consider a graph H_1 by $V(H_1) = X$ and $E(H_1) = \{(x, y) \in X \times X : x \preceq y\}$. Then Banach H_1-contractions are precisely the nondecreasing ordered H-contractions.

Theorem 2.4. *Let X be a complete G-metric space endowed with a graph H and the triple (X, G, H) have the following property:*

(∗) *if $\{x_n\} \to x$ is a sequence in X whose consecutive terms are adjacent, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}$such that whose consecutive terms are adjacent and every term's is adjacent to x.*

Then a Banach $\widetilde{H}$-contraction $T : X \to X$ has a fixed point if and only if $X_T \neq \emptyset$.

Example 2.5. Take the complete G-metric space $X = [0, +\infty)$ with the G-distance $G(x, y, z) = d(x, y) + d(x, z) + d(y, z)$, where d is Euclidean metric on X, and consider the graph H with $V(H) = X$ and

$$(x, y) \in E(H) \Leftrightarrow \begin{cases} x, y \in [0, 1], x \leq y, \\ or \\ x, y \in (n, n + 1], \text{ for some } n = 1, 2, \ldots, x \leq y. \end{cases}$$

Let T be defined as

$$T(x) = \begin{cases} \frac{1}{2}x, & x \in [0, 1], \\ (n - 1) + \frac{1}{2}(x - n), & x \in (n, n + 1], n \text{ even}, \\ n - \frac{1}{2}(x - n), & x \in (n, n + 1], n \text{ odd}, \end{cases}$$

T is not a Banach G-contraction because T is not continuous, but T is a Banach H-contraction with a constant $k = \frac{1}{2}$.

Definition 2.6. Let (X, G) be a G-metric space with a graph H and $T : X \to X$ be a mapping. We call T a Kannan H-contraction if

K1) T preserves edges of H;

K2) there exists $k \in [0, \frac{1}{3})$ such that

$$G(Tx, Ty, Tz) \leq k\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\}$$

for all x, y and z that are on a path to length at most 2 in H.

Theorem 2.7. *Let X be a complete G-metric space endowed with a graph H and the triple (X, G, H) have Property $(*)$. Then a Kannan $\widetilde{H}$-contraction $T : X \to X$ has a fixed point if and only if $X_T \neq \emptyset$.*

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Some iterative methods for solving an operator equation by using g-frames

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Abstract

This paper proposes some iterative methods for solving an operator equation on a separable Hilbert space H equipped with a g-frame. We design some algorithms based on the Richardson and Chebyshev methods and investigate the convergence and optimality of them.

Keywords: Hilbert space, g-frame, operator equation, iterative method, Chebyshev polynomials.
MSC(2010): Primary: 47J25; Secondary: 46C05.

1 Introduction and preliminaries

G-frames are natural generalization of frames and provide more choices on analyzing functions from frame expansion coefficients. Let J be a countable index set and $\{\Lambda_j\}_{j\in J}$ be a set of operators from a separable Hilbert space H to another separable Hilbert space V_j for $j \in J$. The sequence $\{\Lambda_j\}_{j\in J}$ is called a $g-frame$ for H with respect to $\{V_j\}_{j\in J}$ if there are two positive A and B such that

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \qquad \forall f \in H.$$

A and B is called the lower and upper frame bound, respectively. If $A = B$ then $\{\Lambda_j\}_{j\in J}$ is called a tight g-frame. The g-frame operator S for a g-frame $\{\Lambda_j\}_{j\in J}$, for H with respect to $\{V_j\}_{j\in J}$, is defined by

$$Sf = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \qquad \forall f \in H,$$

where Λ_j^* is the adjoint operator of Λ_j.
It is easy to check that S is a bounded, invertible and self-adjoint operator and

$$AI \leq S \leq BI \ , \ \frac{1}{B}I \leq S^{-1} \leq \frac{1}{A}I.$$

Writing $\tilde{\Lambda_j} = \Lambda_j S^{-1}$, then for any $f \in H$ we have

$$f = \sum_{j\in J} \Lambda_j^* \tilde{\Lambda_j} f = \sum_{j\in J} \tilde{\Lambda_j^*} \Lambda_j f.$$

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It is prove that the sequence $\Lambda_j S^{-1}$ is also a g-frame (called canonical dual g-frame) for H with respect to $\{V_j\}_{j\in J}$. For more details we refer to [5].

In this work we present two iterative methods in order to approximate the solution of the operator equation

$$Lu = f, \tag{1}$$

where $L : H \to H$ is bounded invertible and symmetric operator on a separable Hilbert space H. In [1, 3, 4] you can see some developments of numerical methods for solving this problem by using frames.

2 Using g-frames in Richardson iterative method

In this chapter by using Richardson iterative method and g-frames, we wish to solve the operator equation (1). First of all we give and exact solution by using a g-frame.

Theorem 2.1. *Let $L : H \to H$ be a bounded and invertible operator and $\{\Lambda_j\}_{j\in J}$ be a g-frame for H. Then $\{\Lambda_j L\}_{j\in J}$ is also a g-frame for H.*

The most straight forward approach to an iterative solution of a linear system is to rewrite the equation (1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:
write $Lu = f$ as

$$u = (I - L)u + f.$$

For given $u_0 \in H$, define for $k \geq 0$,

$$u_{k+1} = (I - L)u_k + f. \tag{2}$$

Since $Lu - f = 0$,

$$\begin{aligned} u_{k+1} - u &= (I - L)u_k + f - u - (f - Lu) = (I - L)u_k - u + Lu \\ &= (I - L)(u_k - u). \end{aligned}$$

Hence

$$\|u_{k+1} - u\|_H \leq \|I - L\|_{H\to H}\|u_k - u\|_H,$$

so that (2) converges if

$$\|I - L\|_{H\to H} < 1.$$

It is sometimes possible to precondition (1) by multiplying both sides by a matrix B,

$$BLu = Bf,$$

so that convergence of iterative methods is improved. This is very effective technique for solving differential equations, integral equations, and related problems. The following theorem designs an iterative method based on Richardson iterative method and knowledge of g-frames.

Theorem 2.2. *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame with g-frame operator S, and A and B be the bounds of the g-frame $\{\Lambda_j L\}_{j\in J}$.*
Put $u_0 = 0$ and for $k \geq 1$, $u_k = u_{k-1} + \frac{2}{A+B} LS(f - Lu_{k-1})$, Then

$$\|u - u_k\| \leq (\frac{B - A}{B + A})^k \|u\|.$$

3 Chebyshev method by using g-frames

Before introducing our next method, we wish to state without proof some basic facts about Chebyshev polynomials. These polynomial are defined by

$$c_n(x) = \begin{cases} \cos(ncos^{-1}(x)), & |x| \leq 1 \\ \cosh(n\cosh^{-1}(x)) = \frac{1}{2}\left((x+\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^{-n}\right), & |x| \geq 1 \end{cases}$$

and satisfy the recurrence relation

$$c_0(x) = 1, \quad c_1(x) = x, \quad c_n(x) = 2xc_{n-1}(x) - c_{n-2}(x), \quad \forall n \geq 2.$$

The following lemma holds.[2].

Lemma 3.1. *Given $a \leq b \leq 1$, set $P_n(x) = \frac{c_n(\frac{2x-a-b}{b-a})}{c_n(\frac{2-a-b}{b-a})}$, then*

$$\max_{a\leq x\leq b} \mid P_n(x) \mid \leq \max_{a\leq x\leq b} \mid Q_n(x) \mid$$

for all polynomial Q_n of degree n satisfying $Q_n(1) = 1$. Furthermore

$$\max_{a\leq x\leq b} \mid P_n(x) \mid = \frac{1}{c_n(\frac{2-a-b}{b-a})}.$$

Now let $h_n = \sum_{k=1}^{n} a_{n_k} u_k$ such that $\sum_{k=1}^{n} a_{n_k} = 1$, where u_k is the approximated solutions induced by the iterative method represented in the Theorem 2.2.
The condition $\sum_{k=1}^{n} a_{n_k} = 1$ guaranteed if $u_1 = u_2 = ... = u_n = u$, then $h_n = \sum_{k=1}^{n} a_{n_k} u_k = u\sum_{k=1}^{n} a_{n_k} = u$.

In this case, by Theorem 2.2

$$u - h_n = \sum_{k=1}^{n} a_{n_k} u - \sum_{k=1}^{n} a_{n_k} u_k = \sum_{k=1}^{n} a_{n_k}(u - u_k) = \sum_{k=1}^{n} a_{n_k}(I - \frac{2}{A+B}LSL)^k(u - u_0).$$

Writing $R = I - \frac{2}{A+B}LSL$ and $Q_n(x) = \sum_{k=1}^{n} a_{n_k} x^k$, we obtain

$$u - h_n = \sum_{k=1}^{n} a_{n_k} R^k(u - u_0) = Q_n(R)(u - u_0),$$

that means the error is a polynomial in R applied to the initial error $u - u_0$.
Also we note that the spectrum of R is obtained in $[-\rho, \rho]$ where $\rho = \frac{B-A}{B+A}$, and since LSL is a positive definite operator, the spectral theorem yields

$$\|u - h_n\| \leq \|Q_n(R)\|\|u - u_0\| \leq \max_{|x|\leq\rho} \mid Q_n(x) \mid \|u - u_0\|.$$

In order to minimize this error we try to find

$$\min_{Q_n(1)=1} \max_{|x|\leq\rho} \mid Q_n(x) \mid, \tag{3}$$

where the min is taken over all polynomials of degree less than or equal to n, with $Q_n(1) = \sum_{k=1}^{1} a_{n_k} = 1$. By Lemma 3.1 the answer can be given in terms of the Chebyshev

polynomials.
First we note that, replacing $a = -\frac{B-A}{B+A}$ and $b = \frac{B-A}{B+A}$ in Lemma 3.1 gives

$$P_n(x) = \frac{c_n(\frac{2x+\frac{B-A}{B+A}-\frac{B-A}{B+A}}{\frac{B-A}{B+A}+\frac{B-A}{B+A}})}{c_n(\frac{2+\frac{B-A}{B+A}-\frac{B-A}{B+A}}{\frac{B-A}{B+A}+\frac{B-A}{B+A}})} = \frac{c_n(\frac{x}{\rho})}{c_n(\frac{1}{\rho})}.$$

This polynomials solve (3). Now, based on the above argument we can organize the following algorithm in order to induce an approximated solution to the equation (1). Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H with frame operator S and let A and B be the bounds of the g-frame $\{\Lambda_j L\}_{j\in J}$.

Algorithm $[A, B, \epsilon] \to u_\epsilon$

(i) put $\rho = \frac{B-A}{B+A}$, $\sigma = \frac{\sqrt{B}+\sqrt{A}}{\sqrt{B}-\sqrt{A}}$ set $h_0 = 0$, $h_1 = \frac{2}{A+B}LSf$, $\beta_1 = 2$, $n = 1$

(ii) while $\frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{m} > \epsilon$

(1) $n = n + 1$

(2) $\beta_n = (1 - \frac{\rho^2}{4}\beta_{n-1})^{-1}$

(3) $h_n = \frac{2}{\rho}\beta_n(h_{n-1} + \frac{2}{A+B}LS(f - Lh_{n-1})) + (1 - \beta_n)h_{n-2}$

(iii) $u_\epsilon := h_n$.

The following theorem verifies the convergence of this algorithm.

Theorem 3.2. *If u is the exact solution of the equation (1) then, the approximated solution h_n satisfies $\|u - h_n\| \le \frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{m}$. Consequently the output u_ϵ in the* ***Algorithm*** *$[A, B, \epsilon]$ satisfies*

$$\|u - u_\epsilon\| < \epsilon.$$

Remark 3.3. It is obvious that for every $n > 1$, $\frac{2\sigma^n}{1+\sigma^2 n} \le \rho^n$. Therefore this algorithm present an iterative method that is convergence is faster than the Richardson iterative method that is presented in Theorem 2.2.

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Poster Some properties of fundamentally nonexpansive mappings in CAT(0) spaces pp.: 1–4

Some properties of fundamentally nonexpansive mappings in CAT(0) spaces

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Abstract

The purpose of this paper is to give a fixed point theorem and a convergence theorem for fundamentally affine nonexpansive self-mappings in a complete CAT(0) space. Specially, we show that the fixed points set of such mappings defined on a nonempty bounded closed convex subset of a a complete CAT(0) space is always nonempty and closed.

Keywords: Affine mapping, Fixed point, Fundamentally nonexpansive mapping, CAT(0) space
Mathematics Subject Classification [2010]: 47H09, 47H10

1 Introduction and preliminares

Let K be a nonempty subset of a metric space (X, d), and let $T : K \to K$ be a mapping. We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$. The mapping T is said to be
(i) nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in K;$$

(ii) fundamentally nonexpansive if

$$d(T^2x, Ty) \leq d(Tx, y) \text{ for all } x, y \in K;$$

(iii) quasi-nonexpansive if $F(T)$ is nonempty and

$$d(Tx, u) \leq d(x, u) \text{ for all } x \in K \text{ and } u \in F(T).$$

It is evident that fundamental nonexpansiveness is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [6] and [7]). He showed that every nonexpansive mapping defined on a nonempty bounded closed convex subset of a a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and many papers have been appeared. The aim of this paper is to present

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a fixed point theorem and a convergence theorem for fundamentally affine nonexpansive mappings defined on a nonempty bounded closed convex subset of a complete CAT(0) space.

We now review the needed definitions and lemmas. Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ $(d(x, y) = l)$ is a map c from closed interval $[0, l]$ to X such that $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s-t|$ for all $s, t \in [0, l]$. In particular, c is an isometry. The image of c is called a geodesic segment joining x and y. When it is unique this geodesic denoted by $[x, y]$. The metric space (X, d) is said to be geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for all $x, y \in X$. A subset K of X is said to be convex if it includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X, and $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x_1}, \bar{x_2} \in \overline{\triangle}$, $d(x, y) \leq d_{\mathbb{R}^2}(\bar{x_1}, \bar{x_2})$.

Every CAT(0) space is uniquely geodesic (see [1]).

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

Definition 1.1. (see [8, 9]) A sequence $\{x_n\}$ in a CAT(0) space X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim_n x_n = x$ and call x the $\triangle - \lim$ of $\{x_n\}$.

We now collect some lemmas which will be used to prove our main results.

Lemma 1.2. *(see [3]) Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X, then $A(\{x_n\})$ consists of exactly one point.*

Lemma 1.3. *(see [8]) Every bounded sequence in a complete CAT(0) space always has a $\triangle$-convergent subsequence.*

Lemma 1.4. *(see [2]) If K be a nonempty closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 1.5. *(see [4]) Let (X, d) be a CAT(0) space. Then the following statements hold: (i) For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y). \tag{1}$$

It is used the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (1).
(ii) For all $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

Lemma 1.6. *(see [5]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space X and $t \in (0, 1)$. Suppose that $x_{n+1} = ty_n \oplus (1 - t)x_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Definition 1.7. Let K a nonempty convex subset of a CAT(0) space X. A mapping $T : K \to K$ is called affine if

$$T((1 - t)x \oplus ty) = (1 - t)Tx \oplus tTy$$

for all $x, y \in K$ and $t \in [0, 1]$.

2 Main results

The following lemmas play a basic role to prove our main results.

Lemma 2.1. *Let K be a nonempty subset of a metric space (X, d), and let $T : K \to K$ be a fundamentally nonexpansive mapping. Then for all $x, y \in K$, we have*

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y) \ .$$

In addition, if K is closed, then $F(T)$ is also closed.

Lemma 2.2. *Let K be a nonempty bounded convex subset of a CAT(0) space X, and $T : K \to K$ be a fundamentally affine nonexpansive mapping. Then the sequence $\{x_n\}$ defined by $x_1 \in K$ and*

$$x_{n+1} = (1 - t)x_n \oplus tTx_n \tag{2}$$

for all $n \in \mathbb{N}$, where $t \in (0, 1)$, is an approximating fixed point sequence(in short AFPS) for T, i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Using Lemma 1.6, the proof is finished. □

Lemma 2.3. *Let K be a nonempty convex subset of a CAT(0) space X, and $T : K \to K$ be a fundamentally nonexpansive mapping with $F(T) \neq \emptyset$. Consider sequence (2), then the sequence $\{d(x_n, u)\}$ is convergent for all $u \in F(T)$.*

Proposition 2.4. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, and $T : K \to K$ be a fundamentally nonexpansive mapping. If $\{x_n\}$ is an AFPS for T and $\triangle$-$\lim_n x_n = x$ for some $x \in X$. Then x is a fixed point of T.*

Proof. By Lemma 1.4, we have $x \in K$. Applying Lemmas 1.2 and 2.1, the proof is completed. □

Theorem 2.5. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, and $T : K \to K$ be a fundamentally nonexpansive mapping. If T is affine, then $F(T)$ is nonempty and closed.*

Proof. Consider sequence (2) in Lemma 2.2. Now by using Lemmas 1.3, 2.1 and 2.2 along with Proposition 2.4, the proof will be completed. □

Theorem 2.6. *Let K be a nonempty bounded compact convex subset of a complete CAT(0) space X, and $T : K \to K$ be a fundamentally nonexpansive mapping. If T is affine, then sequence* (2) *in Lemma 2.2 converges strongly to a fixed point of T.*

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Some properties of nonnegative integral majorization

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Abstract

A majorization permutahedron $M(a)$ is polytope defined by $M(a) = \{x \in \mathbb{R}^n : x \preceq a\}$. In this paper we look more precisely to $M_I^+(a)$, all positive integer vectors that are majorized by a, and we discuss about its cardinality.

Keywords: integral vector, majorization, permutahedron
Mathematics Subject Classification [2010]: 15A39, 15B36

1 Introduction

Inequalities in matrix theory and specially majorization is one of the interesting areas that has been researched on it in several ways.

For a vector $x \in \mathbb{R}^n$ we say $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is nonincreasing if

$$x_1 \geq x_2 \geq ... \geq x_n.$$

For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we use the notation $x^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_n^{\downarrow})$ for the nonincreasing vector consisting elements of x.

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We say x is majorized by y, $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \quad 1 \leq k \leq n$$

with the equality when $k = n$[3].

A majorization permutahedron is defined by $M(a) = \{x \in \mathbb{R}^n : x \preceq a\}$. Actually this is a special polytope associated with a majorization in $\mathbb{R}^n$, the set of all vectors majorized by a. . In [1] there are some works on the properties of majorization permutahedrons and their cardinality. In this paper we are interested in the set of nonnegative integer vectors majorized by a and discussing about its cardinality. We use the notation $\mathbb{R}_+^n$ for the set of all nonnegative vectors in $\mathbb{R}^n$.

*Speaker

2 Some properties of $M_I^+(a)$

In this section our focus is on integral vectors. An integral vector is a vector that all of its entries are integer. We first state some theorems that are proved on $M_I(a)$, the set of all integer vectors in $M(a)$. For subsets S and T of $\mathbb{R}^n$, we define $S+T = \{s+t : s \in S, t \in T\}$. The following theorems are proved in [2, 1] about integral majorization permutahedrons and their cardinalities.

Theorem 2.1. *Let $a, b \in \mathbb{R}^n$ are monotone vectors. Then:*
i) $M(a+b) = M(a) + M(b)$
ii) $M_I(a+b) = M_I(a) + M_I(b)$

Using the notation $\upsilon(a)$ for the cardinality of $M_I(a)$, we have the following theorem:

Theorem 2.2. *Let $a, b \in \mathbb{R}^n$ be integral vectors. Then the following hold:*
i) If $a \preceq b$, then $\upsilon(a) \leq \upsilon(b)$.
ii) If a is a constant vector, then $\upsilon(a+b) = \upsilon(b)$.
iii) $\upsilon(a+b) \leq \upsilon(a)\upsilon(b)$ and equality holds if and only if a or b is a constant vector.
iv) $\upsilon(ka) \leq \upsilon^k(a)$ for every positive integer k.

Now let a be a nonnegative integarl vector. We will discuss how the above theorems are true about nonnegative integral majorization permutahedrons. We use the notation $M_I^+(a)$ for the set of all nonnegative integar vectors majorized by a and the notation $\upsilon_+(a)$ for the cardinality of $M_I^+(a)$.

Lemma 2.3. *Let $a = (a_1, \cdots, a_n) \in \mathbb{R}_+^n$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n \setminus \mathbb{R}_+^n$ and k be the first indice that x_k is negative. If $a \neq 0$ and $x \preceq a$, then there is $k_0 \in \mathbb{N}$ such that:*

$$\sum_{j=k_0+1}^{k-1} x_j < \sum_{j=k}^{n} |x_j| \leq \sum_{j=k_0}^{k-1} x_j \tag{1}$$

Proof. We have $x \preceq a$, hence $\sum_{i=1}^n a_i = \sum_{i=1}^n x_i$. Since $a \neq 0$ is a nonnegative vector, we have $\sum_{i=1}^n x_i > 0$. Consequently $\sum_{i=k}^n |x_i| < \sum_{i=1}^{k-1} x_i$. Hence we can choose k_0 that the implies the relation1. □

Theorem 2.4. *Let $a, b \in \mathbb{R}_+^n$ are monotone vectors. Then:*
i) $M^+(a+b) = M^+(a) + M^+(b)$
ii) $M_I^+(a+b) = M_I^+(a) + M_I^+(b)$

Proof. i) It is obvious that $M^+(a) + M^+(b) \subseteq M^+(a+b)$. To show the converse let $z \in M^+(a+b)$. Since $M^+(a+b) \subseteq M(a+b)$, by theorem ?? we know there are $x \in M(a)$ and $y \in M(b)$ such that $z = x + y$. If x and y are nonnegative vectors, then $z + y \in M^+(a) + M^+(b)$, else we construct nonnegative vectors x' and y' such that $z = x' + y'$, $x' \preceq a$ and $y' \preceq b$ in the following way. With out loss of generality we may consider $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ are nonincreasing vectors, that means

$$x_1 \geq x_2 \geq \cdots \geq x_n$$

$$y_1 \geq y_2 \geq \cdots \geq y_n$$

Let k and l be respectively the first indices that x_k and y_l are negative and suppose $k \leq l$. Put $x' = (x'_1, \cdots, x'_n)$ in which

$$x'_i = \begin{cases} x_i & \text{if } \ 1 \leq i \leq k_0 - 1 \\ \sum_{j=k_0}^{k-1} x_j - \sum_{j=k}^{n} |x_j| & \text{if } \ i = k_0 \\ 0 & \text{if } \ i \geq k_0 + 1 \end{cases},$$

where k_0 is introduced in lemma2.3 that $\sum_{j=k_0+1}^{k-1} x_j < \sum_{j=k}^{n} |x_j| \leq \sum_{j=k_0}^{k-1} x_j$. Obviously x' is a nonnegative vector. To show $x' \preceq a$, if $d \leq k_0 - 1$, then:

$$\sum_{j=1}^{d} x'_j = \sum_{j=1}^{d} x_j \leq \sum_{j=1}^{d} a_j$$

if $d \geq k_0$, considering the positivity of vector a we have:

$$\sum_{j=1}^{d} x'_j = \sum_{j=1}^{k_0-1} x_j + \sum_{j=k_0}^{k} x_j - \sum_{j=k+1}^{n} |x_j| = \sum_{j=1}^{k_0} x_j - (\sum_{j=k+1}^{n} |x_j| - \sum_{j=k_0+1}^{k} x_j) \leq \sum_{j=1}^{k_0} x_j \leq \sum_{j=1}^{k_0} a_j \leq \sum_{j=1}^{d} a_j$$

and finally for $d = n$ we have:

$$\sum_{j=1}^{n} x'_j = \sum_{j=1}^{k_0-1} x_j + \sum_{j=k_0}^{k-1} x_j - \sum_{j=k}^{n} |x_j| = \sum_{j=1}^{k-1} x_j - \sum_{j=k}^{n} |x_j| = \sum_{j=1}^{k-1} x_j + \sum_{j=k}^{n} x_j = \sum_{j=1}^{n} a_j$$

Hence $x' \preceq a$. Now we must show that $y' = z - x'$ is nonnegative and is majorized by b. Precisely enteries of y are as below:

$$y'_i = \begin{cases} y_i & \text{if } \ 1 \leq i \leq k_0 - 1 \\ y_{k_0} - (\sum_{j=k_0+1}^{k-1} x_j - \sum_{j=k}^{n} |x_j|) & \text{if } \ i = k_0 \\ y_i + x_i & \text{if } \ i \geq k_0 + 1 \end{cases}$$

We know $k_0 < k \leq l$, hence for $i \leq k_0 - 1$, $y'_i \geq 0$. For $i = k_0$, we have $y'_{k_0} = y_{k_0} - (\sum_{j=k_0+1}^{k-1} x_j - \sum_{j=k}^{n} |x_j|)$ and by the above lemma we know $(\sum_{j=k_0+1}^{k-1} x_j - \sum_{j=k}^{n} |x_j|) \leq 0$, hence considering positivity of y_{k_0}, we conclude y'_{k_0} is positive. For $i \geq k_0 + 1$, we have $y'_i = z_i$ that is positive. Hence we have y' is positive. Now we show that $y' \preceq b$. If $d \leq k_0 - 1$, then $\sum_{j=1}^{d} y'_j = \sum_{j=1}^{d} y_j \leq \sum_{j=1}^{d} b_j$. If $d = k_0$, then $\sum_{j=1}^{d} y'_i = \sum_{j=1}^{k_0} y_j - (\sum_{j=k_0+1}^{k-1} x_j - \sum_{j=k}^{n} |x_j|) \leq \sum_{j=1}^{k_0} b_j$, where the last inequality results from the fact that $(\sum_{j=k_0+1}^{k-1} x_j - \sum_{j=k}^{n} |x_j|) \leq 0$. If $d > k_0$, then:

$$\sum_{j=1}^{d} y'_j = \sum_{j=1}^{k_0} y_j - (\sum_{j=k_0+1}^{k-1} x_j - \sum_{j=k}^{n} |x_j|) + \sum_{j=k_0+1}^{d} y_j + x_j = \sum_{j=1}^{d} y_j - \sum_{j=k_0+1}^{k-1} x_j + \sum_{j=k}^{n} |x_j| + \sum_{j=k_0+1}^{d} x_j$$

$$= \sum_{j=1}^{d} y_j + x_j - \sum_{j=1}^{k-1} x_j + \sum_{j=k}^{n} |x_j| = \sum_{j=1}^{d} y_j + x_j - \sum_{j=1}^{n} a_j \leq \sum_{j=1}^{d} b_j$$

Finally for $d = n$, from the above relation we have:

$$\sum_{j=1}^{n} y'_j = \sum_{j=1}^{n} y_j + x_j - \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j$$

Hence $y' \preceq b$ and the proof is complete. □

Theorem 2.5. *Let $a, b \in \mathbb{R}^n_+$ be integral vectors. Then the following hold:*
i) If $a \preceq b$, then $\upsilon_+(a) \leq \upsilon_+(b)$.
ii) If a is a constant vector, then $\upsilon_+(a+b) = \upsilon_+(b)$.
iii) $\upsilon_+(a+b) \leq \upsilon_+(a)\upsilon_+(b)$ and equality holds if and only if a or b is a constant vector.
iv) $\upsilon_+(ka) \leq \upsilon^k_+(a)$ for every positive integer k.

Proof. Parts (i) and (ii) follows from theorem 2.2. The proof of part (iii) is like the proof of part (iii) of theorem 2.2 mentioned in [2], with some changes as follows. We know $a + b \preceq a^{\downarrow} + b^{\downarrow}$, hence $\upsilon_+(a+b) = |M^+_I(a+b)| \leq |M^+_I(a^{\downarrow} + b^{\downarrow})| = |M^+_I(a^{\downarrow}) + M^+_I(b^{\downarrow})| \leq \upsilon_+(a)\upsilon_+(b)$, where the second inequality is by theorem 2.4. We show that if a and b are not constant then the strict inequality holds. Since they are not constants, there are permutations a' and b' of a and b such that $a' = (\alpha, \beta, \cdots)$ with $\alpha > \beta$ and $b' = (p, q, \cdots)$ with $p < q$. Consider vectors $\tilde{a} = (\alpha - 1, \beta + 1, \cdots)$ and $\tilde{b} = (p + 1, q - 1, \cdots)$ with other components equal to a' and b' respectively. Since $\alpha > \beta \geq 0$, we have $\alpha - 1 \geq 0$ and hence $\tilde{a}$ is a nonnegative vector. The same is also true for $\tilde{b}$. Hence we have $\tilde{a} \preceq a$ and $\tilde{b} \preceq b$, also $\tilde{a} \neq a'$ and $\tilde{b} \neq b$, but $a' + b' = \tilde{a} + \tilde{b}$. Part (iv) follows from part (iii). □

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Some results on 2-modular spaces

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Abstract

In this talk, first we review and discuss the concepts of 2-norm and 2-modular. Then, we prove that every 2-modular induces a 2-F-norm. In particular, we show that a β-homogeneous 2-modular induces a 2-F-norm with a special form.

Keywords: 2-modular, 2-modular spaces, 2-norm
Mathematics Subject Classification [2010]: 46A80

1 Introduction

A real valued function $\rho(\cdot,\cdot)$ on X^2 which X is a linear space, is said to be a 2-modular on X if it satisfies the following properties:

1. $\rho(x,y)=0$ if and only if x,y are linearly dependent,
2. $\rho(x,y)=\rho(y,x)$,
3. $\rho(-x,y)=\rho(x,y)$,
4. $\rho(x,\alpha y+\beta z)\leq\rho(x,y)+\rho(x,z)$, for any nonnegative real numbers α,β with $\alpha+\beta=1$.

The spaces equipped with two-modulars introduced by J. Musielak and A. Waszak [5] and generalized by K. Nourouzi and S. Shabanian [4]. In [1] and [2], Gahler developed the notion of a normed space to 2-normed spaces.

This work is devoted to study the relation between two-modular spaces and two-norm spaces.

Example 1.1. Let $X=\mathbb{R}^2$. Then

$$\rho(x_1,x_2)=\begin{cases}1 & x_1,x_2 \text{ are linearly independent,}\\ 0 & x_1,x_2 \text{ are linearly dependent,}\end{cases}$$

is a 2-modular on X.

Definition 1.2. The set defined by

$$X_\rho=\{x\in X : for\ each\ y\in X,\ \rho(\lambda x,y)\to 0\ as\ \lambda\to 0\}$$

is called a 2-modular space.

*Speaker

Evidently X_ρ is a linear subspace of X. Also, we obtain for $x, y \in X_\rho$, $\rho(\lambda x, \lambda y) \to 0$ as $\lambda \to 0$.

Definition 1.3. [3] Let X be a real linear space and $\|\cdot,\cdot\| : X^2 \to \mathbb{R}$ a function. A linear 2-F-norm space, denoted $(X, \|\cdot,\cdot\|)$, is defined by:

1. $\|x, y\| = 0 \Leftrightarrow x$ and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|-x, y\| = \|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,
5. $\|\alpha_k x_k - \alpha x, y\| \to 0$, if $\alpha_k \to \alpha$ and $\|x_k - x, y\| \to 0$ for any $\alpha_k \in \mathbb{R}$ and $x, y, z \in X$. The function $\|\cdot,\cdot\|$ is called a 2-F-norm on X.

For a positive number $\beta \leq 1$, a 2-F-norm is said to be β-homogeneous if it satisfies the condition

6. $\|tx, y\| = |t|^\beta \|x, y\|$ for every $x, y \in X$ and every $t \in \mathbb{R}$.

If $\beta = 1$, we obtain a 2-norm.

It can easily checked that every 2-norm is 2-F-norm. Every 2-F-norm $\|\cdot,\cdot\|$ such that $\|\alpha x, y\|$ is a nondecreasing function of $\alpha > 0$ for every $x, y \in X$ is a 2-modular on X.

Note that every 2-modular on X is a non-negative real functional and the function $\alpha \to \rho(\alpha x, y)$ is nondecreasing for every $x, y \in X$.

Definition 1.4. A 2-modular ρ is said to satisfy the β-homogeneous if there exists $\beta > 0$ such that $\rho(tx, y) = t^\beta \rho(x, y)$ for every $0 \leq t$ and each $x, y \in X_\rho$.

Example 1.5. Suppose that $(X, \|\cdot,\cdot\|)$ is a 2-normed space. Then X is 2-modular with the following 2-modular:

$$\rho(x, y) = \|x, y\|^k,$$

where $k \in \mathbb{N}$. Then ρ is a 2-modular on X and satisfies the k-homogeneous.

2 Main results

We show in next theorem that every 2-modular induces a 2-F-norm.

Theorem 2.1. *If ρ is a 2-modular on X, then*

$$\|x, y\|_\rho = \inf\{u > 0 : \rho(\frac{x}{u}, \frac{y}{u}) \leq u\}$$

is a 2-F-norm on X_ρ.

Corollary 2.2. *If ρ is a 2-modular with β-homogeneity on a linear space X, then*

$$\|x, y\|_\rho = \rho^{\frac{1}{2\beta+1}}(x, y)$$

for any $x, y \in X_\rho$.

Proof. The result is trivial of $\rho(x,y)=0$. Suppose that $\rho(x,y)>0$. Since $\rho(tx,ty)=t^{2\beta}\rho(x,y)$, $\rho(tx,ty)$ is continuous with respect to t in $(0,+\infty)$. We can easily show that $\|x,y\|$ is a unique solution of the equation

$$t=\rho(\frac{x}{t},\frac{y}{t}).$$

Therefore, $\|x,y\|_\rho$ satisfies the equation $t=\frac{\rho(x,y)}{t^{2\beta}}$, that is,

$$\|x,y\|_\rho=\rho^{\frac{1}{1+2\beta}}(x,y)$$

for every $x,y\in X_\rho$. □

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some starlike and convex properties for Hypergeometric functions *

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Abstract

In this paper, we finding conditions on the triplet (a, b, c) so that the function $zF(a, b; c; z)$ is starlike in Δ, where $F(a, b; c; z)$ denotes the hypergeometric function. Also the geometric problem of starlikeness and close to convexity of $zF(a, b; c; z)$ is studied.

Keywords: Starlike, Convex , univalent, Hypergeometric functions.
Mathematics Subject Classification [2010]: 30C45, 30C55

1 Introduction

let S denote the class of all functions f of the form

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the open unit disk $\Delta = \{z \in C :| z |< 1\}$.

Definition 1.1. A function $f \in S$ is said to be starlike of order $\beta(0 \leq \beta < 1)$ if and only if $Re\left(\frac{zf'(z)}{f(z)}\right) > \beta$.
Denote the class of all starlike functions of order β in Δ by $S^{\star}(\beta)$.

Definition 1.2. A function $f \in S$ is said to be convex of order $\beta(0 \leq \beta < 1)$ if and only if $Re\left(\frac{1+zf''(z)}{f'(z)}\right) > \beta$.
Denote the class of all convex functions of order β in Δ by $C(\beta)$.

Definition 1.3. A function $f \in S$ is said to be close-to-convex if there is a convex function $g(z)$ such that $Re\left(\frac{f'(z)}{g'(z)}\right) > 0$.

We note that $f(z)$ is not required to be univalent, and $g(z)$ need not be a function belonging to the class S. It is readily observed that every close-to-convex function is univalent [4]. Merkes and Scott [3] proved an interesting result characterizing starlike hypergeometric functions, and Carlson and Shaffer [5] studied various interesting classes of starlike and convex hypergeometric functions.

*Will be presented in English
†Speaker

let a, b, c be complex numbers with $c \neq 0, -1, -2, \ldots$ .Then the Gaussian hypergeometric function given by the analytic function ,

$F(a,b;c;z) =_2 F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n \quad (z \in \Delta)$,

Where (a,n) denote symbol for the generalized factorial ,

$(a,0) = 1$ for $a \neq 0$, $(a,n) = a(a+1)(a+2)...(a+n-1)$ for $n \in N$.

If $Re(c-a-b) > 0$ then $F(a,b;c;1) = \sum_{n=0}^{\infty} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. For more details we refer to [1, 6 ,9].

The convolution or Hadamard product of f given by (1) and $g \in S$ given by

$g(z) = z + \sum_{n=0}^{\infty} b_n z^n$ is defined as:

$$(f * g)(z) = z + \sum_{n=0}^{\infty} a_n b_n z^n \tag{2}$$

that are analytic and unevalent in the open unit disk Δ.

Integral operater of the hypergeometric function $F(a,b;c;z)$ we have the operator [3]

$$V_{a,c;z}(f)(z) := zF(a,b;c;z) * f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \tag{3}$$

where $Rea > 0, Reb > 0, Re(c+1) > Re(a+b)$ and

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1}(1-t)^{c-a-b} F\Big(c-a, 1-a; c-a-b+1; 1-t\Big). \tag{4}$$

In particular, the CarlsonSchaffer operator $G_{b,c}(f)(z)$ defined as $G_{b,c}(f)(z) := V_{1,b;c}(f)(z)$ is given by

$$G_{b,c}(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad \lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} t^{b-1}(1-t)^{c-b-1}, \tag{5}$$

for $Rec > Reb > 0, f \in S$. Several basic geometric properties of this operator are known in the literature, for example see [2, 7, 8]. Thus, the convolution $zF(a,b;c;z) * f(z)$ can be regarded as an extension of the study of integral operators of functions f in suitable subclasses S, a classical topic in geometric function theory. An interesting observation is that the Bernardi transform is related to zero-balanced hypergeometric function with functions in S.

2 Main results

Theorem 2.1. *Let the hypergeometric function $F(a,b;c;z)$ satisfy the condition*

$$\Big|F'(a,b;c;z) - \frac{ab}{c}\Big|^{1-\alpha} \Big|\frac{zF''(a,b;c;z)}{F'(a,b;c;z)}\Big|^{\alpha} < (\frac{ab}{c})^{1-\alpha}(\frac{1}{2})^{\alpha}, \quad \alpha \geq 0, \ abc > 0 \tag{6}$$

Then $F(a,b;c;z)$ is univalent in Δ.

Theorem 2.2. *Let the hypergeometric function $F(a,b;c;z)$ satisfy the condition*

$$\Big|\frac{zF'(a,b;c;z)}{F(a,b;c;z)}\Big| < 1 \tag{7}$$

Then $zF(a,b;c;z) \in S^*$.

Theorem 2.3. *Let a, b and c be nonzero real numbers such that $F(a,b;c;z)$ has no zeros in Δ. Then $zF(a,b;c;z)$ is a starlike of order $\beta \in [0,1)$ if*

(1) $c \geq 1 + a + b - \frac{ab}{1-\beta}$,

(2) $(1-\beta)\big(c-(a+b)\big) + ab \geq 2(1-\beta)\Big((1-\beta)-(a+b)\Big)$,

(3) $(1-\beta)(3-2\beta)\Big((1-\beta)\big(c-a-b-1\big)+ab\Big) + (1-\beta)^2(c-a-b-1)(a+b+c)$.

Theorem 2.4. *Let b and c be nonzero real numbers such that $F(2,b;c;z)$ has no zeros in Δ. Also, suppose that $c \geq max\{3-b, 3b\}$. then*

(1) $V_{1,b,c}(S^*) \subset S^*$
and

(2) $V_{1,b,c}(C) \subset (C)$.

Theorem 2.5. *Let a, b and c be real numbers and suppose that*

(1) $c \geq |ab + a + b + 2| - 1$,

(2) $c \geq |a + b - ab| + 1$.

If $F(a,b;c;z) \neq 0$ in Δ, Then $zF(a,b;c;z)$ is a starlike of order 1.

Theorem 2.6. *Let the hypergeometric function $F(a,b;c;z)$ satisfy the condition (6). then $F(a+1,b+1;c+1;1) \leq 1$, provided further that $c > a+b$.*

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The existence of best proximity points for set-valued p-cyclic contractions

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Abstract

In this paper the concept of set-valued p-cyclic contraction map is introduced. The existence of best proximity point for such mappings on a metric space with the WUC property is presented.

Keywords: Best proximity point; Property WUC; Set-valued p-cyclic contraction map.
Mathematics Subject Classification [2010]: 47H10, 54H25, 54C60

1 Introduction

In 2003, Kirk et al. [6] established the following fixed point theorem.

Theorem 1.1. *[6] Let A and B be nonempty closed subsets of a complete metric space (X, d), and suppose $f : A \cup B \to A \cup B$ satisfies in the following condition:*

(i) $f(A) \subset B$ and $f(B) \subset A$.

(ii) $d(f(x), f(y)) \leq kd(x, y),\ \forall x \in A\ ,\ y \in B$, where $k \in (0, 1)$.

Then f has a unique fixed point in $A \cap B$.

Each map which satisfying in the assumption (i) of the above theorem is called cyclic map. Later on, Eldred and Veeramani [2] extended the contraction condition (ii) of the above theorem for cyclic maps as follows:

$$d(f(x), f(y)) \leq kd(x, y) + (1 - k)d(A, B),\ \forall x \in A\ \ y \in B,\ \ k \in (0, 1). \tag{1}$$

Every map which satisfying in (1) is said to be a cyclic contraction map. If f is a cyclic map on $A \cup B$, then a point $x \in A \cup B$ is called a best proximity point if $d(x, f(x)) = d(A, B)$, where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Eldred and Veeramani [2] studied cyclic contraction maps and obtained the existence of a best proximity point for cyclic contraction maps in metric spaces and uniformly convex Banach spaces.Then, in [7] the property UC occurs in a large collection of pairs of subsets

*Speaker

of uniformly convex Banach spaces. Next, in [3] the property WUC is introduced and then an existence, uniqueness and convergence theorem is proved for cyclic mappings with property WUC. In the recent years many authors studied the existence of a best proximity point for single valued maps under some suitable contraction conditions, for more details; see [1, 3, 4, 7, 9, 8] and references therein.

Our goal in this paper is to extend the concept of p-cyclic contraction for single valued maps to set-valued maps and obtain the existence of a best proximity point for such maps in metric spaces with the property WUC.

2 Main results

In this section, we prove the existence of a best proximity point for set-valued p-cyclic contraction maps.
Let us first review definitions and propositions that we need in following.

Let (X, d) be a metric space, $\mathcal{CB}(X)$ and $\mathcal{K}(X)$ denote the family of all nonempty closed and bounded subsets of X and the family of all nonempty compact subsets of X, respectively. Then, the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$ is given by

$$H(C, D) = \max\{e(C, D), e(D, C)\},$$

where $e(C, D) = \sup_{a\in C} d(a, D)$ and $d(a, D) = \inf_{b\in D} d(a, b)$. It is well known that if (X, d) is a complete metric space, then $(\mathcal{K}(X), H)$ is a complete metric space.

Definition 2.1. ([3]) Let A and B be nonempty subsets of a metric space (X, d). Then pair (A, B) is said to satisfy the WUC property if for any $\{x_n\} \subset A$ such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \leq d(A, B) + \varepsilon$ for $n \geq n_0$, then it is the case that $\{x_n\}$is convergent.

Proposition 2.2. *Let (X, d) be a metric space and A and B be nonempty subsets of X such that the pair (A, B) satisfies the property WUC. Then, the pair $(\mathcal{K}(A), \mathcal{K}(B))$ also satisfies the property WUC in $(\mathcal{CB}(X), H)$.*

Now, we introduce the notion of set-valued cyclic contraction mappings.

Definition 2.3. Let (X, d) be a metric space, let A and B be nonempty subsets of X. Then a set-valued map $T : A \cup B \multimap A \cup B$ is called a set-valued cyclic contraction if the following are satisfied:

(i) $T(A) \subset B$ and $T(B) \subset A$.

(ii) There exists $k \in (0, 1)$ such that

$$H(T(x), T(y)) \leq kd(x, y) + (1 - k)d(A, B)$$

for all $x \in A$ and $y \in B$.

Theorem 2.4. *Let (X, d) be a metric space and A and B be nonempty subsets of X such that (A, B) satisfies the property WUC. Assume that A is complete and $T : A\cup B \multimap A\cup B$ is a set-valued cyclic contraction such that $T(D)$ is compact for any $D \in \mathcal{K}(A) \cup \mathcal{K}(B)$. Then T has a best proximity point x in A, i.e., $d(x, T(x)) = d(A, B)$. Furthermore, if $y \in T(x)$ and $d(x, y) = d(A, B)$, then y is a best proximity point in B and x is a fixed point of T^2.*

Now we define p-cyclic contraction set-valued map and we give sufficient conditions for the existence best proximity point for such map. Also, we show that, the obtained best proximity point is a fixed point of T^p. Moreover, if $x_i \in A_i$ is a best proximity point in A_i, then each point $y_{i+j} \in T^j(x_i) \subset A_{i+j}$ such that $d(y_{i+j}, x_i) = d(A_1, A_p)$ is a best proximity point in A_{i+j} for $j = 1, \ldots, p-1$.

Definition 2.5. Let $A_1, \ldots, A_p$ be nonempty subsets of a metric space X Then a mapping $f : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ is said p-cyclic contraction if the following are satisfied:

(i) $f(A_i) \subset A_{i+1}$ for $i = 1, \ldots, p,$ where $A_{p+1} = A_1$.

(ii) There exists $k \in (0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y) + (1 - k)d(A_i, A_{i+1})$$

for all $x \in A_i, y \in A_{i+1}$, for $i = 1, \ldots, p$

Theorem 2.6. *Let (X, d) be a complete metric space and $A_1, \ldots, A_p$ be nonempty subsets of X such that (A_i, A_{i+1}) satisfies the property WUC for $i = 1, \ldots, p$. Suppose $f : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ is p-cyclic contraction, then there exists a unique point $z_i \in A_i$ such that $d(z_i, f(z_i)) = d(A_i, A_{i+1})$ and z_i is a fixed point of f^p in A_i. Also $f^{pn}(x)$ converges to z_i for every $x \in A$ and $T^j(z_i)$ is a best proximity point in A_{i+j} for $j = 1, 2, \ldots, p-1$.*

Definition 2.7. Let $A_1, \ldots, A_p$ be nonempty subsets of a metric space X. Then, $T : \cup_{i=1}^p A_i \multimap \cup_{i=1}^p A_i$ is called a p-cyclic set-valued mapping if

$$T(A_i) \subset A_{i+1} \text{ for } i = 1, \ldots, p, \text{ where } A_{p+1} = A_1. \tag{2}$$

A point $x \in A_i$ is said to be a best proximity point in A_i if $d(x, T(x)) = d(A_i, A_{i+1})$.

Definition 2.8. Let $A_1, \ldots, A_p$ be nonempty subsets of a metric space X, and $T : \cup_{i=1}^p A_i \multimap \cup_{i=1}^p A_i$ be a p-cyclic set-valued mapping. T is called p-cyclic nonexpansive set-valued mapping if

$$H(T(x), T(y)) \leq d(x, y) \ \forall x \in A_i, y \in A_{i+1}, \ 1 \leq i \leq p. \tag{3}$$

It is an interesting fact to note that the distances between the adjacent sets are equal under the p-cyclic nonexpansive mapping.

Lemma 2.9. *Let $X, A_1, \ldots, A_p, T$ be as in Definition 2.8. Then $d(A_i, A_{i+1}) = d(A_{i+1}, A_{i+2}) = d(A_1, A_2)$ for all $i = 1, \ldots, p$.*

Now, we introduce the following new class of set-valued p-cyclic contraction maps.

Definition 2.10. Let $A_1, \ldots, A_p$ be nonempty subsets of a metric space X and $T : \cup_{i=1}^p A_i \multimap \cup_{i=1}^p A_i$ be a p-cyclic set-valued mapping. T is called p-cyclic contraction if There exists $k \in (0, 1)$ such that

$$H(T(x), T(y)) \leq kd(x, y) + (1 - k)d(A_i, A_{i+1})$$

for all $x \in A_i, y \in A_{i+1}$, for $i = 1, \ldots, p$.

Theorem 2.11. *Let (X, d) be a complete metric space and $A_1, \ldots, A_p$ be nonempty subsets of a metric space X such that (A_i, A_{i+1}) satisfies the property WUC for $i = 1, \ldots, p$. Suppose that a compact valued map $T : \cup_{i=1}^{p} A_i \multimap \cup_{i=1}^{p} A_i$ is cyclic contraction. Then T has a best proximity point x_i in A_i. i.e. $d(x, T(x)) = d(A_i, A_{i+1})$ and x_i is a fixed point of T^p. Also, there is point $y_{i+j} \in T^j(x_i) \subset A_{i+j}$ that is a best proximity point in A_{i+j} for $j = 1, \ldots, p-1$.*

Corollary 2.12. *Let (X, d) be a complete metric space and $A_1, \ldots, A_p$ be nonempty subsets of a metric space X such that (A_i, A_{i+1}) satisfies the property WUC for $i = 1, \ldots, p$ and $\cap_{i=1}^{p} A_i$ is nonempty. Suppose that a compact valued map $T : \cup_{i=1}^{p} A_i \multimap \cup_{i=1}^{p} A_i$ is contraction. Then T has a fixed point $x \in \cap_{i=1}^{p} A_i$.*

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Poster The metric space of H-varieties and it's convex structure pp.: 1–4

The metric space of H-varieties and it's convex structure

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Abstract

It has shown that, the set of all varieties of banach algebras L is a metric space and the set of all H-varieties of Banach algebras L_H is also a metric space as a subspace of L. In this article convexity of L_H has investigated and the convex structure on this space has introduced. In addition, some convex subsets of L_H has been mentioned.

Keywords: H-variety, Covex space and convex structure

1 Introduction

Definition 1.1. For each Banach algebra $\mathcal{A}$. and $\delta \geq 0$, we define

$$\|p\|_{\mathcal{A},\delta} = \sup\{\|p(x_1, ..., x_n)\| : x_i \in \mathcal{A}, \|x_i\| \leq \delta, 1 \leq i \leq n\}$$

We shall denote $\|p\|_{\mathcal{A},1} = \|p\|_{\mathcal{A}}$, where $p = p(X_1, ..., X_n)$ is a polynomial.
Throughout this paper a polynomial is a non-commuting polynomial without constant term.

Definition 1.2. A non-empty class V of Banach algebras is said to be a variety if there exists a non-negative real-valued function, $p \to f(p)$ on the set of all polynomials, such that V is precisely a class of Banach algebras $\mathcal{A}$ for which, $\|p\|_{\mathcal{A}} \leq f(p)$ for each $p = p(X_1, ..., X_n)$.

Theorem 1.3. *(Dixon [2]) For each non-empty class v of Banach algebras, the followings are equivalent,*
(i) V is closed under taking closed subalgebras, quotient algebras, products(direct sums) and images under isometric isomorphisms.
(ii) V is a variety.

As a law, we mean a formal expression $\|p\| \leq K$, where $K \in \mathbb{R}$ and p is a polynomial. We say that $\mathcal{A}$ satisfies the above law, if $\|p\|_{\mathcal{A}} \leq K$ and $\|p\| \leq K$ is homogeneous if p is a homogeneous polynomial.

Definition 1.4. For each $n \in \mathbb{N}$ such that $n \neq 1$, the variety determined by the law $\|X_1...X_n\| = 0$, is called N_n. .

*Speaker

We denote the variety of all Banach algebras by 1, this variety is the largest variety of Banach algebras. Also, the smallest variety of Banach algebras is shown by N_2.

Definition 1.5. Let V to be a variety of Banach algebras and p a polynomial then, we define

$$|p|_V = \sup\{\|p\|_{\mathcal{A}} : \mathcal{A} \in V\}.$$

Definition 1.6. Let C be a class of Banach algebras and $V(C)$ be the intersection of all varieties containing C. Then, $V(C)$ is a variety called the variety generated by C. If C consists of a single Banach algebra $\mathcal{A}$, then $V(C)$ is written as $V(\mathcal{A})$ and is said to be singly generated.

Theorem 1.7. *[3] For each variety V , there exists a $\mathcal{A} \in V$ such that for all polynomial p, we have*

$$|p|_V = \|p\|_{\mathcal{A}}.$$

that this supremum is always obtaind.

Corollary 1.8. *[3] Each variety of Banach algebras is singly generated.*

Corollary 1.9. *[3] Let V_1 and V_2 be two varietirs. Then, $V_1 \subseteq V_2$ if and only if for all polynomials p, we have*

$$|p|_{V_1} \leq |p|_{V_2}.$$

By Theorem1.7, the supremum in Definition1.5 is always obtain and Corollary1.9 concluded that, there exists a worthy property for varieties saying that they can be compared by their laws.

Definition 1.10. Let P be the set of all polynomials, and P_H be the set of all homogeneous polynomials. We define

$$P_1 = \{p \in P : |p|_1 < 1\}$$
$$P_{H1} = \{p \in P_H : |p|_1 < 1\}$$

Definition 1.11. If $V \in L$, then we define $\phi_V : P_1 \to \mathbb{C}$ as follows

$$\phi_V(p) = |p|_V.$$

It is easy to show that the mapping $\Phi : L \to L^\infty(P_1, \mathbb{C})$ with $\Phi(V) = \phi_V$ is one to one. So, $L^\infty(P_1, \mathbb{C})$ induces a metric on L as below

$$\begin{aligned} d_L(V, W) = d(\phi_V, \phi_W) &= \|\phi_V - \phi_W\|_\infty \\ &= \sup_{p \in P_1} |\phi_V(p) - \phi_W(p)| \\ &= \sup_{p \in P_1} \big||p|_V - |p|_W\big|. \end{aligned}$$

Therefore, (L, d_L) is a metric space. Similarly, (L_H, d_H) is also a metric space with the following norm

$$d_H(V, W) = \sup_{p \in P_{H1}} \big||p|_V - |p|_W\big|.$$

Definition 1.12. If $x \in \mathbb{R}$ and $V \neq N_2$ be a variety, then V_x is the variety that determined by following lows

$$\|p\| \leq |x||p|_V$$

where p is a homogeneous polynomial with $\deg(p) > 1$.

2 Main results

In this section we try to show that, L_H is a convex mrtric space and we will discuss about fixed points of this space.

Definition 2.1. Define

$$L_{\mathbb{R}} = \{V_x : x \in \mathbb{R}, V \in L\}$$

and

$$L_{H_{\mathbb{R}}} = \{V_x : x \in \mathbb{R}, V \in L_H\}$$

Lemma 2.2. *If $L_{\mathbb{R}}$ and $L_{H_{\mathbb{R}}}$ are defined as above then we have $L = L_{\mathbb{R}}$ and $L_H = L_{H_{\mathbb{R}}}$.*

Proof. It is obvious. □

Definition 2.3. Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \longrightarrow X$ is said to be convex structure on X, if for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$ the following condition is satisfied:

$$d\big(u, W(x, y; \lambda)\big) \leq \lambda d(x, y) + (1 - \lambda)d(x, y).$$

A metric space X with convex structure is called a convex metric space.

Also, a subset C of a convexmetric space X with covex structure W on it, is said to be convex if $W(x, y; \lambda) \in C$, for all $x, y \in C$ and $\lambda \in [0, 1]$.

Theorem 2.4. *The metric space $X = (L_H, d_H)$ is a convex metric space.*

Proof. Define $W : X \times X \times [0, 1] \longrightarrow X$ such that for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, $W(V_x, V_y; \lambda) = V_{\lambda x+(1-\lambda)y}$. Obviously W is continuous. For each $V_u \in X$ and $V_x, V_y \in X, \lambda \in [0, 1]$ we will have,

$$\begin{aligned} d_H(V_u, W(V_x, V_y; \lambda)) &= d_H(V_u, V_{\lambda x+(1-\lambda)y}) \\ &= \sup_{p \in P_{H1}} |u - (\lambda x + (1 - \lambda)y)||p|_V \\ &\leq \lambda \sup_{p \in P_{H1}} |u - x||p|_V + (1 - \lambda) \sup_{p \in P_{H1}} |u - y||p|_V \\ &= \lambda d_H(V_u, V_x) + (1 - \lambda)d_H(V_u, V_y) \end{aligned}$$

□

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Poster The relation between 2-norm spaces and no inverting 2-modular spaces pp.: 1–3

The relation between 2-norm spaces and no inverting 2-modular spaces

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Abstract

In this talk, we study 2-norm spaces, and 2-modular spaces. In particular, no inverting 2-modular spaces are studied, and then given a corollary about the relation between two 2-modulars which one of them has β-homogeneity and the other one has some special properties.

Keywords: 2-modular spaces, 2-norm spaces, inverting 2-modular spaces, β-homogeneity
Mathematics Subject Classification [2010]: 46A80

1 Introduction

A linear 2-norm space, denoted $(X, \|\cdot,\cdot\|)$, which $\|\cdot,\cdot\| : X^2 \to \mathbb{R}$ is a function, is defined by:

1. $\|x, y\| = 0 \Leftrightarrow x$ and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\| - x, y\| = \|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,
5. $\|tx, y\| = |t|\|x, y\|$ for every $x, y \in X$ and every $t \in \mathbb{R}$.

   The function $\|\cdot,\cdot\|$ is called a 2–norm on X.

The theory of 2-norm on a linear space was investigated by Gahler in [1].

Definition 1.1. Let X be a real vector space of dimension more than two. A real valued function $\rho(\cdot,\cdot)$ on X^2 satisfying the following properties is called a 2-modular on X:

1. $\rho(x, y) = 0$ if and only if x, y are linearly dependent,
2. $\rho(x, y) = \rho(y, x)$,
3. $\rho(-x, y) = \rho(x, y)$,

*Speaker

4. $\rho(x, \alpha y+\beta z) \leq \rho(x, y)+\rho(x, z)$, for any nonnegative real numbers α, β with $\alpha+\beta = 1$.

Example 1.2. A 2-modular ρ(the absolute value of determinant) on $X = \mathbb{R}^2$ can be defined by

$$\rho(x_1, x_2) = \text{abs} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix},$$

where $x_i = (x_{i1}, x_{i2})$, for each $i = 1, 2$.

Note that every 2-modular on X is a non-negative real functional and the function $\alpha \to \rho(\alpha x, y)$ is nondecreasing for every $x, y \in X$ and every 2-norm $\|\cdot, \cdot\|$ such that $\|\alpha x, y\|$ is a nondecreasing function of $\alpha > 0$ for every $x, y \in X$ is a 2-modular on X.

Definition 1.3. The set defined by

$$X_\rho = \{x \in X : for\ each\ y \in X,\ \rho(\lambda x, y) \to 0\ as\ \lambda \to 0\},$$

is called a 2-modular space.

Definition 1.4. A 2-modular ρ is said to satisfy the β-homogeneous if there exists $\beta > 0$ such that $\rho(tx, y) = t^\beta \rho(x, y)$ for every $0 \leq t$ and each $x, y \in X_\rho$.

Example 1.5. [2] Suppose that $(X, \|\cdot, \cdot\|)$ is a 2-normed space. Then X is 2-modular with the following 2-modular:

$$\rho(x, y) = \|x, y\|^2.$$

Then ρ is a 2-modular on X and satisfies the 2-homogeneous.

2 Main Results

In the following, after defining inverting 2-modular, we prove our main results.

Let $\|\cdot, \cdot\|$ be a 2-norm of a linear space X. If there exists a 2-modular ρ on X such that $X_\rho = X$ and that $\|\cdot, \cdot\|$ is equal to

$$\|x, y\|_\rho = \inf\{u > 0 : \rho(\frac{x}{u}, \frac{y}{u}) \leq u\},$$

we say that the 2-norm $\|\cdot, \cdot\|$ can be inverted, and ρ is called an inverting 2-modular of $\|\cdot, \cdot\|$.

Theorem 2.1. *For any 2-normed space $(X, \|\cdot, \cdot\|)$, there is no inverting 2-modular with the property that $\rho(tx, ty)$ is continuous with respect to t in $(0, \infty)$ for any linearly independent vectors $x, y \in X$.*

Corollary 2.2. Let $\sigma(x, y)$ be a 2-modular with β-homogeneity on a linear space X and $\rho(x, y)$ a 2-modular defined on X that satisfies

$$(1) \qquad (1-\varepsilon)\sigma(x, y) \leq \rho(x, y) \leq (1+\varepsilon)\sigma(x, y)$$

for some ε with $0 < \varepsilon < 1$. Then $\|x, y\|_\rho$ and the $(\beta/(1+2\beta))$-norm $\sigma^{1/(1+2\beta)}(x, y)$ satisfy the relation

$$(2) \qquad (1-\varepsilon)^{\frac{1}{1+2\beta}} \sigma^{\frac{1}{1+2\beta}}(x, y) \leq \|x, y\|_\rho \leq (1+\varepsilon)^{\frac{1}{1+2\beta}} \sigma^{\frac{1}{1+2\beta}}(x, y)$$

A 2-modular $\rho(\cdot,\cdot)$ on a linear space X is said to be equivalent to a 2-modular $\sigma(\cdot,\cdot)$ on X if there are positive numbers a and b such that

$$a\sigma(x,y) \leq \rho(x,y) \leq b\sigma(x,y),$$

for all $x, y \in X$. Note that the above inequality implies

$$b^{-1}\rho(x,y) \leq \sigma(x,y) \leq a^{-1}\rho(x,y),$$

for every $x, y \in X$, so that equivalence of 2-modulars is a symmetric relation on the set of all 2-modulars on X. In fact, it easy to check that equivalence of 2-modulars is an equivalence relation on the set of all 2-modulars on X. If σ and ρ are 2-modulars defined as in Corollary 2.2 and they satisfy the relation (1), that is, ρ is equivalent to σ.

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Poster Vector ultrametric spaces and modular locally constant mappings pp.: 1–4

Vector ultrametric spaces and modular locally constant mappings

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Abstract

In this talk, we give some sufficient conditions for mappings defined on vector ultrametric spaces to be modular locally constant.

Keywords: Modular Locally Constant; Vector Ultrametric Spaces; Spherically complete.
Mathematics Subject Classification [2010]: 26E30

1 Introduction

A metric space (X, d) in which the triangle inequality is replaced by

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \qquad (x, y, z \in X),$$

is called an ultrametric space.

In [6], the notion of a metric locally constant function on an ultrametric space was given in order to investigate certain groups of isometries and describe various Galois groups over local fields. Locally constant functions also appear in contexts such as higher ramification groups of finite extensions of $\mathbf{Q}_p$, and the Fontaine ring B^+_{dR}. Also, metric locally constant functions were studied in [2]. On the other hand, vector ultrametric spaces are given in [1] as vectorial generalizations of ultra metrics. Hence, locally constant functions, in modular sense, can play the same role in vector ultrametric spaces as they do in usual ultrametric spaces. In this note, we introduce modular locally constant mappings in vector ultrametric spaces. Some sufficient conditions are given for mappings defined on vector ultrametric spaces to be modular locally constant. A *modular* on a real linear space $\mathcal{A}$ is a real valued functional ρ on $\mathcal{A}$ satisfying the conditions:

1. $\rho(x) = 0$ if and only if $x = 0$,
2. $\rho(x) = \rho(-x)$,
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in \mathcal{A}$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

*Speaker

Then, the linear subspace

$$\mathcal{A}_\rho = \{x \in \mathcal{A} : \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}$$

of $\mathcal{A}$ is called a *modular space.*

A sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{A}_\rho$ is called *ρ-convergent* (briefly, convergent) to $x \in \mathcal{A}_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, and is called *Cauchy sequence* if $\rho(x_m - x_n) \to 0$ as $m, n \to \infty$. By a *ρ-closed* (briefly, closed) set in $\mathcal{A}_\rho$ we mean a set which contains the limit of each of its convergent sequences. Then, $\mathcal{A}_\rho$ is a *complete modular* space if every Cauchy sequence in $\mathcal{A}_\rho$ is convergent to a point of $\mathcal{A}_\rho$. We refer to [3, 4, 5, 6] for more details.

A *cone* $\mathcal{P}$ in a complete modular space $\mathcal{A}_\rho$ is a nonempty set such that:

1. $\mathcal{P}$ is ρ-closed, and $\mathcal{P} \neq \{0\}$;
2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in \mathcal{P} \Rightarrow ax + by \in \mathcal{P}$;
3. $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$, where $-\mathcal{P} = \{-x : x \in \mathcal{P}\}$.

Let $\preceq$ be the partial order on $\mathcal{A}_\rho$ induced by the cone $\mathcal{P}$, i.e., $x \preceq y$ whenever $y - x \in \mathcal{P}$. The cone $\mathcal{P}$ is called *normal* if

$$0 \preceq x \preceq y \Rightarrow \rho(x) \leq \rho(y), \qquad (x, y \in \mathcal{A}_\rho).$$

The cone $\mathcal{P}$ is said to be *unital* if there exists a vector $e \in \mathcal{P}$ with modular 1 such that

$$x \preceq \rho(x)e \qquad (x \in \mathcal{P}).$$

Throughout this note, we suppose that $\mathcal{P}$ is a cone in complete modular space $\mathcal{A}_\rho$, and $\preceq$ is the partial order induced by $\mathcal{P}$.

Definition 1.1. A *vector ultrametric* on a nonempty set $\mathcal{X}$ is a mapping $d : \mathcal{X} \times \mathcal{X} \to \mathcal{A}_\rho$ satisfying the conditions:

(CUM1) $d(x, y) \succeq 0$ for all $x, y \in \mathcal{X}$ and $d(x, y) = 0$ if and only if $x = y$;

(CUM2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(CUM3) If $d(x, z) \preceq p$ and $d(y, z) \preceq p$, then $d(x, y) \preceq p$, for any $x, y, z \in \mathcal{X}$, and $p \in \mathcal{P}$.

Then the triple $(\mathcal{X}, d, \mathcal{P})$ is called a vector ultrametric space. If $\mathcal{P}$ is unital and normal, then $(\mathcal{X}, d, \mathcal{P})$ is called a *unital-normal* vector ultrametric space.

For unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$, since

$$d(x, y) \preceq \rho(d(x, y))e \qquad \text{and} \qquad d(y, z) \preceq \rho(d(y, z))e,$$

from (CUM3) we get

$$d(x, z) \preceq \max\{\rho(d(x, y)), \rho(d(y, z))\}e,$$

and therefore

$$\rho(d(x, z)) \leq \max\{\rho(d(x, y)), \rho(d(y, z))\}.$$

Let $(\mathcal{X}, d, \mathcal{P})$ be a unital-normal vector ultrametric space. If $x \in \mathcal{X}$ and $p \in \mathcal{P} \setminus \{0\}$, *the ball* $B(x,p)$ *centered at* x *with radius* p is defined as

$$B(x,p) := \{y \in \mathcal{X} : \rho(d(x,y)) \leq \rho(p)\}.$$

The unital-normal vector ultrametric space $(\mathcal{X}, d, \mathcal{P})$ is called *spherically complete* if every chain of balls (with respect to inclusion) has a nonempty intersection.

The following lemma may be easily obtained.

Lemma 1.2. *Let* $(\mathcal{X}, d, \mathcal{P})$ *be a unital-normal vector ultrametric space.*

1. *If* $a, b \in \mathcal{X}$, $0 \preceq p$ *and* $b \in B(a,p)$, *then* $B(a,p) = B(b,p)$.
2. *If* $a, b \in \mathcal{X}$, $0 \prec p \preceq q$, *then either* $B(a,p) \bigcap B(b,q) = \emptyset$ *or* $B(a,p) \subseteq B(b,q)$.

2 Main results

Definition 2.1. Let $(\mathcal{X}, d, \mathcal{P})$ be a unital-normal vector ultrametric space. A mapping $f : \mathcal{X} \to \mathcal{P} \setminus \{0\}$ is said to be *modular locally constant* provided that for any $x \in \mathcal{X}$ and any $y \in B(x, f(x))$ one has $\rho(f(x)) = \rho(f(y))$.

Theorem 2.2. *[1] Let* $(\mathcal{X}, d, \mathcal{P})$ *be a spherically complete unital-normal vector ultrametric space and* $T : \mathcal{X} \to \mathcal{X}$ *be a mapping such that for every* $x, y \in \mathcal{X}, x \neq y$, *either*

$$\rho(d(Tx,Ty)) < \max\{\rho(d(x,Tx)), \rho(d(y,Ty))\} \tag{1}$$

or

$$\rho(d(Tx,Ty)) \leq \rho(d(x,y)). \tag{2}$$

Then there exists a subset B *of* $\mathcal{X}$ *such that* $T : B \to B$ *and the mapping*

$$f(x) = d(x,Tx), \qquad (x \in B) \tag{3}$$

is modular locally constant.

In the following, we assume that $(\mathcal{X}, d, \mathcal{P})$ *is a spherically complete unital-normal vector ultrametric space.*

Corollary 2.3. *[1] Let* $T : \mathcal{X} \to \mathcal{X}$ *be a mapping such that for all* $x, y \in \mathcal{X}, x \neq y$,

$$\rho(d(Tx,Ty)) < \max\{\rho(d(y,Tx)), \rho(d(x,Ty))\}. \tag{4}$$

Then there exists a subset B *of* $\mathcal{X}$ *such that* $T : B \to B$ *and the mapping* f *defined in (3) is modular locally constant.*

Corollary 2.4. *[1] Let* $T : \mathcal{X} \to \mathcal{X}$ *be a mapping such that for all* $x, y \in \mathcal{X}, x \neq y$,

$$\rho(d(Tx,Ty)) < \rho(d(x,y)). \tag{5}$$

Then there exists a subset B *of* $\mathcal{X}$ *such that* $T : B \to B$ *and the mapping* f *defined in (3) is modular locally constant.*

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A classification of Ramanujan complements of unitary Cayley graphs

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Abstract

The unitary Cayley graph on n vertices, X_n, has vertex set $\mathbb{Z}_n$, and two vertices a and b are connected by an edge if and only if they differ by a multiplicative unit modulo n, i.e. $\gcd(ab, n) = 1$. A k-regular graph X is Ramanujan if and only if $\lambda(X) \leq 2\sqrt{k-1}$ where $\lambda(X)$ is the second largest absolute value of the eigenvalues of the adjacency matrix of X. We obtain a complete characterization of the cases in which the complements of unitary Cayley graph $\bar{X}_n$ is a Ramanujan graph.

Keywords: Graph, Cayley Graph, Ramanujan Graph

1 Introduction

We define the Cayley graph $X = Cay(G, S)$ to be the graph whose vertex set is G, and in which two vertices v and u in G are connected by an edge if and only if vu^{-1} is in S.

The unitary Cayley graph on n vertices, X_n, is defined to be the undirected graph whose vertex set is $\mathbb{Z}_n$, and in which two vertices a and b are connected by an edge if and only if $\gcd(a - b, n) = 1$. This can also be stated as $X_n = Cay(\mathbb{Z}_n, \mathbb{U}_n)$, where $\mathbb{Z}_n$ is the additive group of integers modulo n and $\mathbb{U}_n = \mathbb{Z}_n^*$ is the set of multiplicative units modulo n. X_n is a simple, $\varphi(n)$-regular graph, where φ is the Euler totient function. Here $\varphi(n)$ is defined by $\varphi(1) = 1$, and for an integer $n > 1$ with distinct prime power factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for distinct primes $p_1, \ldots, p_k$ and nonnegative integers $\alpha_1, \ldots, \alpha_k$, with $k > 0$, $\varphi(n) = p_1^{\alpha_1 - 1} p_2^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$. When discussing X_n, we always assume $n > 3$

Lemma 1.1. *The eigenvalues of any adjacency matrix of X_n are*

$$\lambda_m(n) = \mu\left(\frac{n}{(n,m)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)} \tag{1}$$

Proof. see [3, Klotz, W. and Sander, T. (2007)] □

When $\frac{n}{(n,m)}$ is square-free,

$$|\lambda_m(n)| = \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)} \tag{2}$$

*Speaker

Corollary 1.2.

$$\mathrm{Spec}(X_{p^\alpha}) = \begin{pmatrix} p^\alpha - p^{\alpha-1} & -p^{\alpha-1} & 0 \\ 1 & p-1 & p^\alpha - 1 \end{pmatrix} \tag{3}$$

The complement $\bar{X}$ of a graph X is the graph with the same vertex set as X such that two vertices are adjacent in $\bar{X}$ if and only if they are not adjacent in X.

2 Ramanujan unitary Cayley graphs

Recall that the adjacency matrix of any r-regular graph X has eigenvalues between k and k, and k is an eigenvalue with multiplicity precisely equal to the number of connected components of X. Furthermore, if $\lambda(X)$ denotes the largest absolute value of the eigenvalues of the adjacency matrix of X, smaller than k, then the graph X is called Ramanujan if and only if

$$\lambda(X) \leq 2\sqrt{k-1} \tag{4}$$

Note that $\lambda(X)$ is only defined for regular graphs X with 3 or more vertices. Writing n in the form $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ for some distinct primes $p_1 < p_2 < \cdots < p_k$, and positive integers $\alpha_1, \ldots, \alpha_k$, we can determine $\lambda(X_n)$ as follows. Since X_n is $\varphi(n)$-regular, we find the maximum absolute value of an eigenvalue $\lambda_m(n)$ of the adjacency matrix of X_n, smaller than $\varphi(n)$. This can be accomplished by looking at 1. Indeed, we see that if $n = 2^\alpha$ then the eigenvalues have absolute value of either 0 or $\varphi(n)$ (since the only values of $m, 0 \leq m \leq n-1$, which make $\frac{n}{(n,m)}$ square-free are $m = 0$ and $m = 2^{\alpha-1}$, resulting in eigenvalues $\varphi(n)$ and $-\varphi(n)$. Thus $\lambda(X_{2^\alpha}) = 0$ and so X_{2^α} satisfies 2 and thus is Ramanujan. It is known [1] that the graph X_n is Ramanujan if and only if n satisfies one of the following conditions for some distinct odd primes $p < q$ and natural α.

1. $n = 2^\alpha$, $\alpha \geq 2$.
2. $n = p$.
3. $n = 2^\alpha p$ with $\alpha \geq 1$, $p > 2^{\alpha-3} + 1$.
4. $n = p^2, 2p^2, 4p^2$.
5. $n = p, 2pq$ with $p < q \leq 4p - 5$.
6. $n = 4pq$ with $p < q \leq 2p - 3$.

3 Ramanujan complements of unitary Cayley graphs

Lemma 3.1 ([2]). *Let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, be the canonical factorization of an integer n into prime powers, where $p_1 < p_2 < \cdots < p_k$ are primes and each $\alpha_i \geq 1$. If $k \geq 3$ or $k = 2$ and $p_1 > 2$, then $2^{k-1}\varphi(n) > n$.*

Theorem 3.2. *Let $n \geq 2$ be a intager. Then $\bar{X}_n$ is Ramanujan if and only if n is one of the following forms:*

1. $n = p^a$ *with* p *a prime and* $a \geq 1$
2. $n = p^a q^b$ *with* $p = 2$, $q = 3$ *and* $1 \leq a \leq 3$, $b = 1$, *or* $a = 1$, $1 \leq b \leq 2$.
3. $n = pq$ *with* p *and* q *are primes and* $3 \leq p \leq 5$, $5 \leq q \leq 7$.
4. $n = 2 \cdot 5, 2 \cdot 3 \cdot 5$

Proof. By (**??**), $\bar{X}_{p^a}$ is Ramanujan.
Case 1: $p_1 = 2$ and $k \geq 2$. In this case $\lambda(\bar{X}_n) = \varphi(n) - 1$ and $\bar{X}_n$ is Ramanujan if and only if

$$\varphi(n) - 1 \leq 2\sqrt{n - \varphi(n) - 2} \tag{5}$$

This condition is satisfied io only $\varphi(n)^2 < 4n$. In particular, if $k \geq 4$, then by Lemma 3.1, $\varphi(n)^2 > 4n$ and so $\bar{X}_n$ is not Ramanujan. Assume $k \leq 3$.
Case 1.1: $k = 3$ and $n = 2^a p^b q^c$. Since $(p-1)^2 = p(p-2) + 1 > p(p-2)$, If $a \geq 3$, then

$$\frac{\varphi(n)^2}{n} = 2^{a-2} p^{b-1} q^{c-1} \frac{(p-1)^2}{p} \frac{(q-1)^2}{q} > 2(p-2)(q-2) > 4$$

and so $\bar{X}_n$ is not Ramanujan. It is easy to see that if $b \geq 2$ or $c \geq 2$, then $\varphi(n)^2 > 4n$, and so $\bar{X}_n$ is not Ramanujan. It remains to consider the case where $n = 2pq$ or $4pq$. If $n = 2pq$, then

$$\frac{\varphi(n)^2}{n} = \frac{(p-1)^2(q-1)^2}{2pq} > \frac{(p-2)(q-2)}{2}$$

If $p \neq 3$ or $p = 3$ and $q \geq 7$, then $\varphi(n)^2 > 4n$, and so $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 2 \cdot 3 \cdot 5$, then $\bar{X}_n$ is Ramanujan, whilst if $n = 2 \cdot 3 \cdot 7$, $\bar{X}_n$ is not Ramanujan. If $n = 4pq$, similar to earlier state if $p \neq 3$, or $p = 3$ and $q \neq 5$, then $\bar{X}_n$ is not Ramanujan. For $n = 4 \cdot 3 \cdot 5$, the condition (5) is not established and so $\bar{X}_n$ is not Ramanujan.
Case 1.2: $k = 2$ and $n = 2^a p^b$. In this case

$$\frac{\varphi(n)^2}{n} = \frac{2^{a-1} p^{b-1}}{2p}(p-1)$$

As case 1.1, If $a \geq 4$, $p \geq 7$ or $b \geq 3$, then $\varphi(n)^2 \geq 4n$ and so $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 2 \cdot 3, 2 \cdot 3^2, 2 \cdot 5, 2^2 \cdot 3, 2^3 \cdot 3$, then $\bar{X}_n$ is Ramanujan.
Case 2: $p_1 \geq 3$. . In this case $\lambda(\bar{X}_n) = \frac{\varphi(n)}{p_1 - 1)} - 1$ and so $\bar{X}_n$ is Ramanujan if and only if

$$\frac{\varphi(n)}{p_1 - 1)} - 1 \leq 2\sqrt{n - \varphi(n) - 2} \tag{6}$$

this inequality is equivalent to

$$\frac{\varphi(n)}{p_1 - 1)} \leq -(2p - 3) + \sqrt{4n - 9 + (2p-3)^2} \tag{7}$$

and this condition is not satisfied unless

$$\varphi(n)^2/n < 4(p-1)^2 \tag{8}$$

If $k \geq 4$, then by Lemma 3.1, $\varphi(n)^2/n > \varphi(n)/2^{k-1} > 4(p-1)$ and so $\bar{X}_n$ is not Ramanujan. Assume $k \leq 3$ and $n = p^a q^b r^c$.

Case 2.1: $k = 3$. In this case we have

$$\frac{\varphi(n)^2}{n} = p^{a-1}q^{b-1}r^{c-1}\frac{(p-1)^2(q-1)^2(r-1)^2}{pqr} \geq \frac{p^{a-1}}{p}(p-1)^2(q-2)(r-2)$$

Therefore, if $a \geq 3$, then $\varphi(n)^2/n \geq 4(p-1)^2$ and so $\bar{X}_n$ is not Ramanujan. Similarly, if $b \geq 2, c \geq 2$ or $p \geq 7$, or $n = 9pq, 25pq$, then $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 3 \cdot 5 \cdot 7$, then (7) is violated and again $\bar{X}_n$ is not Ramanujan. Moreover, If $n = 3pq$, whit $p \neq 3$ and $q \neq 7$, then then

$$\frac{\varphi(n)^2}{n} = \frac{4(p-1)^2(q-1)^2}{3pq} > (p-2)(q-2) \geq 16 = 4(3-1)^2$$

and so $\bar{X}_n$ is not Ramanujan.

Case 2.2: $k = 2$ and $n = p^a q^b$. In this case by Lemma 3.1,

$$\frac{\varphi(n)}{n} > \frac{1}{2} \Rightarrow \frac{\varphi(n)^2}{n} > \frac{\varphi(n)}{2} = \frac{p^{a-1}q^{b-1}(p-1)(q-1)}{2} > \frac{p^{a-1}q^{b-1}(p-1)^2}{2}$$

and so if $a \geq 3, b \geq 2$ or $a = b = 2$, then (8) is violated and $\bar{X}_n$ is not Ramanujan. If $n = pq$, then from (7), we have $(q-2)^2 \leq 4p$ and so if $q > 7$, then $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 3 \cdot 5, 3 \cdot 7, 5 \cdot 7$, then $\bar{X}_n$ is not Ramanujan. If $n = pq^2$, then

$$\frac{\varphi(n)^2}{n} = \frac{(p-1)^2(q-1)^2}{p} > 4(p-1)^2$$

and so $\bar{X}_n$ is not Ramanujan. As in previous cases, if $n = p^2 q$, then $\bar{X}_n$ is not Ramanujan. □

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Poster Edge group choosability of planar graphs with maximum degree at least 11 pp.: 1–4

Edge group choosability of planar graphs with maximum degree at least 11

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Abstract

A graph G is edge-k-group choosable if its line graph is k-group choosable. In this paper, we present an edge-group choosability version of Vizing's conjecture and we shall show that it is true for graphs with maximum degree less than 4 and for planar graphs with maximum degree at least 11.

Keywords: List coloring, Group choosability, Edge-group choosability
Mathematics Subject Classification [2010]: 05C15, 05C20

1 Introduction

We consider only simple graphs. For a graph G, we denote its vertex set, edge set, minimum degree, maximum degree, and line graph by $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$, and $\ell(G)$, respectively. Let $d_G(x)$, or simply $d(x)$, denote the degree of a vertex x in G. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. A *k-coloring* of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy. A graph G is *k-colorable* if it has a k-coloring. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is k-colorable. A mapping L is said to be a *list assignment* for G if it supplies a list $L(v)$ of possible colors to each vertex v. A *k-list assignment* of G is a list assignment L with $|L(v)| = k$ for each vertex $v \in V(G)$. If G has some k-coloring ϕ such that $\phi(v) \in L(v)$ for each vertex v, then G is *L-colorable* or ϕ is an *L-coloring* of G. We say that G is *k-choosable* if it is L-colorable for every k-list assignment L. The *choice number* or *list chromatic number* $\chi_l(G)$ is the smallest k such that G is k-choosable. By considering colorings for $E(G)$, we can define analogous notions such as *edge-k-colorability*, *edge-k-choosability*, the *chromatic index* $\chi'(G)$, the *choice index* $\chi'_l(G)$, etc. Clearly, we have $\chi'(G) = \chi(\ell(G))$ and $\chi'_l(G) = \chi_l(\ell(G))$. The notion of list coloring of graphs has been introduced by Erdős, Rubin, and Taylor [5] and Vizing [13]. The following conjecture, which first appeared in [1], is well-known as the List Edge Coloring Conjecture.

Conjecture 1. If G is a multi-graph, then $\chi'_l(G) = \chi'(G)$.

Although Conjecture 1 has been proved for a few special cases such as bipartite multigraphs [6], complete graphs of odd order [7], multicircuits [15], graphs with $\Delta(G) \geq 12$ that

*This is part of a joint work with G.R. Omidi.

can be embedded in a surface of non-negative characteristic [2], and outerplanar graphs [14], it is regarded as very difficult. Vizing proposed the following weaker conjecture (see [9]).

Conjecture 2. Every graph G is edge-$(\Delta(G)+1)$-choosable.

Assume A is an Abelian group and $F(G,A)$ denotes the set of all functions $f : E(G) \to A$. Consider an arbitrary orientation of G. The graph G is *A-colorable* if for every $f \in F(G,A)$, there is a vertex coloring $c : V(G) \to A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y. The *group chromatic number* of G, $\chi_g(G)$, is the minimum k such that G is A-colorable for any Abelian group A of order at least k. The notion of group coloring of graphs was first introduced by Jaeger et al. [8].

The concept of *group choosability* was introduced by Král and Nejedlý in [11] and some first results in this area was obtained in [3, 4, 12]. Let A be an Abelian group of order at least k and $L : V(G) \to 2^A$ be a list assignment of G. For $f \in F(G,A)$, an *(A,L,f)-coloring* under an orientation D of G is an L-coloring $c : V(G) \to A$ such that $c(x)-c(y) \neq f(xy)$ for every edge $e = xy$, e is directed from x to y. If for each $f \in F(G,A)$ there exists an (A,L,f)-coloring for G, then we say that G is *(A,L)-colorable*. The graph G is k-group choosable if G is (A,L)-colorable for each Abelian group A of order at least k and any k-list assignment $L : V(G) \to \binom{A}{k}$. The minimum k for which G is k-group choosable is called the *group choice number* of G and is denoted by $\chi_{gl}(G)$. It is clear that the concept of group choosability is independent of the orientation on G.

Graph G is called *edge-k-group choosable* if its line graph is k-group choosable. The *group-choice index* of G, $\chi'_{gl}(G)$, is the smallest k such that G is edge-k-group choosable, i.e. $\chi'_{gl}(G) = \chi_{gl}(\ell(G))$. It is easily seen that an even cycle is not edge-2-group choosable. This example shows that $\chi'_{gl}(G)$ is not generally equal to $\chi'(G)$. But we can extend the Vizing Conjecture as follows.

Conjecture 3. If G is a multi-graph, then $\chi'_{gl}(G) \leq \Delta(G)+1$.

Since $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, as a sufficient condition, we have the following weaker conjecture.

Conjecture 4. If G is a multi-graph, then $\chi'_{gl}(G) \leq \chi'(G)+1$.

In the next section, we prove that Conjecture 3 and consequently Conjecture 4 holds for graphs with $\Delta(G) \leq 3$ and for planar graphs with $\Delta(G) \geq 11$.

2 Main results

Theorem 2.1. *Let l be a natural number, v be a vertex of degree at most 2 of G and e be an edge adjacent to v. If $\chi'_{gl}(G-e) \leq \Delta(G)+l$, then $\chi'_{gl}(G) \leq \Delta(G)+l$.*

Proof. Let $\Delta = \Delta(G)$, D be an orientation of $\ell(G)$, A be an Abelian group of order at least $\Delta + l$, $L : V(\ell(G)) \to \binom{A}{\Delta+l}$ be a $(\Delta+l)$-list assignment and $f \in F(\ell(G),A)$. Suppose that $G' = G-e$. Then $\ell(G') = \ell(G)-e$ and since $\chi'_{gl}(G') \leq \Delta+l$, there exists an

(A, L, f)-coloring $c : V(\ell(G')) \to A$. For each $e' \in N_{\ell(G)}(e)$ we can consider, without loss of generality, ee' to be directed from e to e'. Then, since $|L(e)| = \Delta + l$ and $d_{\ell(G)}(e) \leq \Delta$, $|L(e) - \{c(e') + f(ee') : e' \in N_{\ell(G)}(e)\}| \geq 1$ and so there is now at least one color available for e. Thus we can color all edges of G. This completes the proof of lemma. □

An argument similar to the proof of Theorem 2.1 gives the following.

Theorem 2.2. *Let G be a graph with $\chi'_{gl}(G - e) < \chi'_{gl}(G)$ for each $e \in E(G)$. Then $\delta(\ell(G)) \geq \chi'_{gl}(G) - 1$.*

Lemma 2.3. *[3] Let P_n and C_n denote a path and a cycle of length n, respectively. Then*

(1) *$\chi_{gl}(P_n) = 2$ and $\chi_{gl}(C_n) = 3$,*

(2) *For any connected simple graph G, we have $\chi_{gl}(G) \leq \Delta(G) + 1$, with equality holds iff G is either a cycle or a complete graph.*

Immediately from Lemma 2.3, we have the following corollary.

Corollary 2.4. *$\chi'_{gl}(P_n) = \Delta(P_n) = 2$ and $\chi'_{gl}(C_n) = \Delta(C_n) + 1 = 3$.*

Theorem 2.5. *Let G be a graph with maximum degree $\Delta(G)$. If $\Delta(G) \leq 3$, then $\chi'_{gl}(G) \leq \Delta(G) + 1$ and if $\Delta(G) = 4$, then $\chi'_{gl}(G) \leq 6$.*

Proof. Clearly we can assume that G is connected. If $\Delta(G) = 1$, then $G = P_2$ and this theorem trivially holds. If $\Delta(G) = 2$, then $G = P_n$ or $G = C_n$ and the assertion holds by Corollary 2.4. It is clear that $\Delta(\ell(G)) \leq 4$ if $\Delta(G) \leq 3$ and $\Delta(\ell(G)) \leq 6$ if $\Delta(G) \leq 4$. The proof is completed by Lemma 2.3. □

We now turn our attention to planar graphs with maximum degree at least 11.

Lemma 2.6. *[10] For every planar graph G with minimum degree at least 3 there is an edge $e = uv$ with $d(u) + d(v) \leq 13$.*

Theorem 2.7. *If G is a planar graph with maximum degree Δ, then*

$$\chi'_{gl}(G) \leq \max\{\Delta + 1, 12\}.$$

Proof. Let G be a minimal counterexample to Theorem 2.7. By Theorem 2.1 and Lemma 2.6, there exists $e \in V(\ell(G))$ with $d_{\ell(G)}(e) \leq 11$, which is a contradiction by Theorem 2.2. □

The truth of Conjecture 3 for planar graphs with maximum degree at least 11 immediately follows from Theorem 2.7. In fact we have the following.

Corollary 2.8. *Let G be a planar graph with maximum degree Δ.*

(1) *If $\Delta \geq 11$, then $\chi'_{gl}(G) \leq \Delta + 1$,*

(2) *If $\Delta \geq 10$, then $\chi'_{gl}(G) \leq \Delta + 2$.*

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Independence Number of Fullerene Graph

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Abstract

A set $S \subseteq V(G)$ is independent if no two vertices from S are adjacent. The cardinality of any biggest independent set in $V(G)$ is called the independence number of G and denoted by $\alpha(G)$. In this paper, we compute independence number of infinite classes of fullerene graphs.

Keywords: Independent set, Independent number, Fullerene
Mathematics Subject Classification [2010]: 05C69

1 Introduction

We talk about one of the graph invariants. An independent set in a graph G is a set of vertices of G that are pairwise non-adjacent, and the independence number, $\alpha(G)$, is the order of the maximum independent set of G. Finding such a set is an NP-hard problem. In next section we discuss about independent number of special graph.
One of most important nano structures are Fullerenes. The discovery of the fullerene C_{60} by Kroto et al. in 1985. [7]. They are a trivalent plane graph with r -gon or s-gon faces. Values of r can be 3,4,5 and for s can be 6 so we named them as $[r, s]$-Fullerenes. The familiar of them are (5,6), (4,6) and (3,6) Fullerenes. It follows from Eulers formula that such graphs made up entirely of n vertices and having 12 pentagonal and $\frac{n}{2} - 10$ hexagonal rings. These graph theoretic fullerenes are simulated to model large carbon molecules, each vertex represents a carbon atom and the edges represent chemical bonds. Since a carbon atom has chemical valence 4, one edge at each of the graphs must represent a double chemical bond.
In [5] P.W. Fowler and et al. survey the independence numbers of fullerenes from C_{20} to C_{120}, a range that includes over 10 million isomers, Contrary to a literature proposal, stability and minimal independence number of fullerenes are poorly correlated.
In [2] T. Doslic present both upper and lower bound for independent number of fullerene graph. In this paper, we discuss independent number of (3,6)-fullerene graph with $4n$ and $8n$ vertices.

*Speaker

2 Initial discussion

In this section we see some of previous work on independent number of fullerene. At first there are some upper bounds on the independence number of a fullerene in [5].Then showing some of lower bounds on the independence number of a fullerene stated in [5].

Corollary 2.1. *[5] If G be a general cubic (= trivalent) polyhedral graph, then there is an obvious upper bound on the independence number of $\alpha(G) \leq \frac{n}{2}$, with equality only for bipartite G.*

Fullerenes have cycles of odd size, hence are non-bipartite.There was Conjecture of Graffiti [3] and is proved by a simple counting argument.

Corollary 2.2. *[3] If G be a general fullerene, the independence number has an upper bound of $\alpha(G) \leq \frac{n}{2} - 2$.*

Corollary 2.3. *[5] If G be an isolated-pentagon fullerene, then the independence number has an upper bound of $\alpha(G) \leq \frac{n}{2} - 4$. This upper bound is not sharp for the smaller values of n at which isolated-pentagon fullerenes exist.*

Now Lower bounds of independence number of fullerenes has been stated. From Brooks theorem [1] Fowler found a lower bound [5]. This theorem states that the vertices of a connected graph G with maximum degree $\Delta(G)$ can be coloured with $\chi(G) \leq \Delta(G) + 1$ colours, with equality only when G is either an odd cycle ($\Delta(G) = 2$) or the complete graph on $\Delta(G) + 1$ vertices. Fullerenes can therefore be coloured with three colours. Taking the largest colour class, an independent set of order at least $\frac{n}{3}$ is found. This bound hes been improved by W. Staton[8].

Corollary 2.4. *[8] If G be a triangle-free cubic graph, then: $\alpha(G) \geq \frac{5n}{14}$*

For triangle-free, planar, cubic graphs, a class that includes the fullerenes, a stronger result has recently been proved: $\alpha(G) \geq \frac{3n}{8}$[6]. Fullerenes C_{20}, C_{24}, C_{28} and C_{30} all have isomers with $\alpha(G) = \lceil \frac{3n}{8} \rceil$.

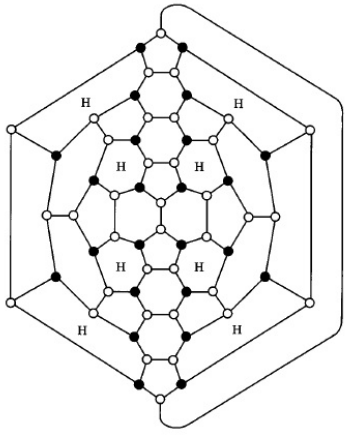

Figure 1: A maximum independent set of C_{60}

3 Main results

In this section, we discuss about independent number of (3,6)-fullerene with $4n$ and $8n$ vertices.

Corollary 3.1. *[2] For any fullerene graph G on n vertices we have:*

$$\frac{3}{2}n \leq \alpha(G) \leq \frac{n}{2} - 2 \tag{1}$$

Let n be positive integer number. We compute independent number of (3,6)-fullerene with $4n$ and $8n$ vertices with newgraph software and survey them:

Table 1: Independent number of C_{8n}

n	number of vertices	$\alpha(G)$
2	16	6
3	24	10
4	32	14
5	40	18
6	48	22
7	56	26

Proposition 3.2. *If G be fullerene graph with $8n$ vertices then:*

$$\alpha(G) = 4n - 2 \tag{2}$$

Table 2: Independent number of C_{4n}

n	number of vertices	$\alpha(G)$
4	16	7
5	20	9
6	24	11
7	28	13
8	32	15
9	36	17

Proposition 3.3. *If G be fullerene graph with $4n$ vertices then:*

$$\alpha(G) = 2n - 1 \tag{3}$$

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Majority domination in digraphs

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Abstract

The concept of majority domination in graphs has been defined in at least two different ways: As a function and as a set. In this work we extend the latter concept to digraphs, while we extended the former in another paper. Given a digraph $D = (V, A)$, a set $S \subseteq V$ is a *majority out-dominating set* (MODS) of D if $|N^+[S]| \geq \frac{n}{2}$. The minimum cardinality of a majority out-dominating set in D is the *set majority out-domination number* $\gamma_m^+(D)$ of D. In this work we introduce these concepts and prove some results about them, among which the characterization of minimal MODSs.

Keywords: Majority dominating set, majority out-dominating set.
Mathematics Subject Classification [2010]: 05C20, 05C69.

1 Introduction

This concept has interesting applications, specially related to democracy: The main idea of democracy is that of a representative group which is accepted by a majority of the population. In some way, this corresponds to majority dominating sets in undirected graphs. However, it is important to notice that the relation is actually directed: The representative group must be accepted by at least half of the population, but if the group itself accepts or not a particular sector of such population has no influence at all in the scope of simple democracy. Of course, more complex systems exist, with the aim that every important minority has some acceptance from the representative group, and those systems are better fit for large populations, like that of a country. Nevertheless, simple democracy is still the best option for small groups, like the members of a club or those of a small company. In the context of simple democracy, the concept of majority out-dominating set in digraphs works more accurately than that of majority dominating set in undirected graphs.

*Speaker

2 Majority out-dominating sets

Observation 1. If H is a spanning subdigraph of a digraph D, then $\gamma_m^+(D) \leq \gamma_m^+(H)$.

Observation 2. For the directed path P_n with $n \geq 1$ vertices, $\gamma^+(P_n) = \lceil \frac{n}{2} \rceil$ and for the directed cycle C_n with $n \geq 3$ vertices, $\gamma^+(C_n) = \lceil \frac{n}{2} \rceil$.

Observation 3. For the directed path P_n with $n \geq 1$ vertices, $\gamma_m^+(P_n) = \lceil \frac{n}{4} \rceil$ and for the directed cycle C_n with $n \geq 3$ vertices, $\gamma_m^+(C_n) = \lceil \frac{n}{4} \rceil$.

Observation 4. For any digraph D which has a hamiltonian circuit, $\gamma_m^+(D) \leq \lceil \frac{n}{4} \rceil$.

Proposition 2.1. *Let $l(D)$ denote the length of a longest directed path in D. Then $\gamma_m^+(D) \leq \lceil \frac{2n-l(D)-1}{4} \rceil$, and the bound is sharp.*

Proposition 2.2. *Let $c(D)$ denote the length of a longest directed cycle in D. Then $\gamma_m^+(D) \leq \lceil \frac{2n-c(D)}{4} \rceil$, and the bound is sharp.*

Theorem 2.3. *For any digraph D, $\gamma_m^+(D) = \gamma^+(D)$ if, and only if, $\Delta^+(D) = n - 1$.*

Theorem 2.4. *For any digraph D, $\gamma_m^+(D) \leq \lceil \frac{\gamma^+(D)}{2} \rceil$.*

Observation 5. Let D be a digraph of order $n \geq 1$, then $\gamma_m^+(D) = 1$ if, and only if, there exists one vertex $v \in D$ such that $d^+(v) \geq \lceil \frac{n}{2} \rceil - 1$.

Theorem 2.5. *For any digraph D,*

(i) $\lceil \frac{n}{2(\Delta^+(D)+1)} \rceil \leq \gamma_m^+(D)$.

(ii) Either $\gamma_m^+(D) = 1$ *or* $\gamma_m^+(D) \leq \lceil \frac{n}{2} \rceil - \Delta^+(D)$.

Corollary 2.6. *For every digraph D, $\gamma_m^+(D) \leq \frac{n-\Delta^+(D)+1}{2}$.*

Proposition 2.7. *Let D be a digraph which is not a totally disconnected digraph of odd order. If S is a minimal MODS of D, then $V \setminus S$ is a MODS of D.*

Theorem 2.8. *Let S be a MODS of a digraph $D = (V, A)$. Then S is minimal if, and only if, one of the following conditions hold:*

(i) $|N^+[S]| > \lceil \frac{n}{2} \rceil$ *and* $\forall\ v \in S,\ |pn^+[v, S]| > |N^+[S]| - \lceil \frac{n}{2} \rceil$.

(ii) $|N^+[S]| = \lceil \frac{n}{2} \rceil$ *and* $\forall\ v \in S$, *either v is an isolate in $D[S]$ or $pn^+(v, S) \neq \emptyset$.*

We now consider the effect on $\gamma_m^+(D)$ of the removal of a vertex or an arc from D.

Theorem 2.9. *Let D be any digraph with $\gamma_m^+(D) = k$. Let $v \in V(D)$ and $e \in A(D)$. Then*

(i) $k \leq \gamma_m^+(D - e) \leq k + 1$,

(ii) $k - 1 \leq \gamma_m^+(D - v) \leq \max\{k, k - 1 + d^+(v)\}$.

Now we consider the effect on $\gamma_m^+(D)$ of adding an arc to D.

Proposition 2.10. *Let D be any digraph with $\gamma_m^+(D) = k$, $e \in A(\overline{D})$. Then $\gamma_m^+(D) - 1 \leq \gamma_m^+(D + e) \leq \gamma_m^+(D)$.*

Proposition 2.11. *Let D be a digraph, and let D' be the digraph obtained by reversing the direction of a single arc of D. Then $|\gamma_m^+(D) - \gamma_m^+(D')| \leq 1$.*

Definition 2.12. Let $D = (V, A)$ be any digraph. An arc $e \in A(D)$ is γ_m^+-critical if $\gamma_m^+(D - e) = \gamma_m^+(D) + 1$.

Theorem 2.13. *An arc $e = uv$ of a digraph D is γ_m^+-critical if, and only if, for every $\gamma_m^+(D)$-set S we have that $u \in S$, $v \in pn^+(u, S)$, and $|N^+[S]| = \lceil \frac{n}{2} \rceil$.*

3 Oriented graphs

Definition 3.1. Let G be a graph. The lower orientable set majority domination number of G is $dom_m^+(G) = \min\{\gamma_m^+(D) \ : \ D \text{ is an orientation of } G\}$, and the upper orientable set majority domination number of G is $DOM_m^+(G) = \max\{\gamma_m^+(D) \ : \ D \text{ is an orientation of } G\}$.

These concepts are inspired in the notions of lower orientable domination number $dom(G)$ and upper orientable domination number $DOM(G)$, introduced by Chartrand et al. in [5].

Theorem 3.2. *For every graph G, $dom_m^+(G) = \gamma_m(G)$.*

We now proceed to determine the upper and lower orientable set majority domination numbers for several classes of graphs:

Proposition 3.3. *(i) For $n \geq 1$, we have $DOM_m^+(K_n) = dom_m^+(K_n) = 1$.*

(ii) For $n \geq 1$, $dom_m^+(P_n) = \lceil \frac{n}{6} \rceil$.

1. For $n \geq 3$, $dom_m^+(C_n) = \lceil \frac{n}{6} \rceil$.

(iii) For any two integers r, s with $r \leq s$, $dom_m^+(K_{r,s}) = 1$.

Theorem 3.4. *[5] For every integer $n \geq 3$, $DOM(P_n) = DOM(C_n) = \lceil \frac{n}{2} \rceil$.*

Proposition 3.5. *For every integer $n \geq 3$, $DOM_m^+(P_n) = DOM_m^+(C_n) = \lceil \frac{n}{4} \rceil$.*

Proposition 3.6. *For $n \geq 3$, $DOM_m^+(K_{1,n-1}) = \lfloor \frac{n-1}{2} \rfloor$.*

Theorem 3.7. *For every double star G, $dom_m^+(G) = 1$. Moreover, if $n \geq 5$ then $DOM_m^+(G) = 2 + \max\{0, \lceil \frac{n-8}{2} \rceil\}$.*

Observation 6. For every graph $G = (V, E)$ with $n \leq 4$ and such that $E \neq \emptyset$, $DOM_m^+(G) = 1$.

Proposition 3.8. *Take two positive integers r, s with $r \leq s$, then $DOM_m^+(K_{r,s}) = 1$ if, and only if, $r + s \leq 4$ or*

(i) $r = 2, \ s = 3$
(ii) $r = 2, \ s = 4$
(iii) $r = s = 3$

In general, it seems difficult to find $DOM_m^+(K_{r,s})$. However, we have the following conjecture:

Conjecture. Let $K_{r,s}$ be a complete bipartite graph with $r \leq s$, and such that $DOM_m^+(K_{r,s}) \neq 1$. Then:

$$DOM_m^+(K_{r,s}) = \begin{cases} 2 & \text{if } r \leq s \leq r+2, \\ \lceil \frac{s-r}{2} \rceil & \text{otherwise.} \end{cases}$$

Theorem 3.9. *For $n \geq 4$, $dom_m^+(W_n) = 1$ and $DOM_m^+(W_n) = \lceil \frac{n-2}{4} \rceil$.*

Finally, we note that an "Intermediate Value Theorem" for orientable majority out-domination holds:

Theorem 3.10. *For every graph G and every integer c with $dom_m^+(G) \leq c \leq DOM_m^+(G)$, there exists an orientation D of G such that $\gamma_m^+(D) = c$.*

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On incidence adjacent vertex-distinguishing total coloring of graphs

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Abstract

In this talk, we study total coloring (not necessarily proper) of graphs in which adjacent vertices are distinguished by their sets of colors. Zhang et al. in 2009 posed a conjecture regarding the upper bound for the minimum number of colors needed for such coloring of a graph in terms of maximum degree. We prove among some results that this conjecture is true for graphs with maximum degree 3.

Keywords: Graph, Total coloring, Incidence adjacent vertex-distinguishing total coloring, Incidence adjacent vertex-distinguishing total chromatic number.

Mathematics Subject Classification [2010]: 05C15

1 Introduction

All of the graphs considered in this paper are simple, finite and undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and edges of a graph G, respectively.

Definition 1.1. A semi total coloring c is a mapping from $V(G) \cup E(G)$ to $\mathbb{N}$ such that any two adjacent vertices and two adjacent edges receive distinct colors.

For any vertex x of G, let $S(x)$ denote the set of the colors of all edges incident to x together with the color assigend to x.

Definition 1.2. A semi total coloring is said to be an incidence adjacent vertex distinguishing total coloring if for every adjacent vertices x and y, $S(x) \neq S(y)$. The minimum number of colors required for an incidence adjacent vertex-distinguishing total coloring of G denote by $\chi^i_{at}(G)$ and is called the incidence adjacent vertex-distinguishing chromatic number of G.

Since an incidence adjacent vertex-distinguishing total coloring is a proper edge coloring, every graph satisfies $\chi^i_{at}(G) \geq \Delta(G)$. Moreover every graph G with two adjacent vertices of degree $\Delta(G)$ satisfies $\chi^i_{at}(G) \geq \Delta(G) + 1$.
After Burris and Schelp [3], Bazgan [2] and Balister et al. [1] discussed vertex-distinguishing proper edge coloring, Zhang et al. [5] presented the concept of adjacent vertex-distinguishing proper edge coloring of graphs.

*Speaker

Then Zhang et al. [4] proposed the new concept of incidence adjacent vertex-distinguishing total coloring of graphs. They completely determined the incidence adjacent vertex distinguishing chromatic number of paths, cycles, trees, completely graphs and complete bipartite graphs. They especially presented a meaningful conjecture.

Conjecture 1.3. *For a simple graph G, then $\chi^i_{at}(G) \leq \Delta(G) + 2$.*

2 Main results

In this section we prove among some results, that Conjecture 1.3 is true for graphs with maximum degree 3.

Lemma 2.1. *Let G be a graph with $\chi^i_{at}(G) \leq \Delta(G) + 2$. Also let H be the graph obtained from G by adding some new leaves. Then $\chi^i_{at}(H) \leq \Delta(H) + 2$.*

Lemma 2.2. *Let G be a graph with $\chi^i_{at}(G) \leq \Delta(G) + 2$. Also let H be the graph obtained from G by subdivision of some edges of G. Then $\chi^i_{at}(H) \leq \Delta(H) + 2$.*

Lemma 2.3. *Let G_1 be a graph with $\chi^i_{at}(G_1) \leq \Delta(G_1) + 2$ and G_2 be a graph with $\chi^i_{at}(G_2) \leq \Delta(G_2) + 2$. Also let H be the graph obtained from G_1 and G_2 by adding a bridge with them. Then $\chi^i_{at}(H) \leq \Delta(H) + 2$.*

Lemma 2.4. *If G is a 3-regular hamiltonian graph, then $\chi^i_{at}(G) \leq 5$.*

Theorem 2.5. *If G is a 3-regular graph containing a 1-factor, then $\chi^i_{at}(G) \leq 5$.*

Theorem 2.6. *If G is a graph with maximum degree 3, then $\chi^i_{at}(G) \leq 5$.*

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On symmetric hypergraphs

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Abstract

A hypergraph Γ is said to be symmetric if its automorphism group acts transitively on the set of its 1-arcs. In this paper we study some properties of the symmetric hypergraphs and then we connect the notions of symmetric hypergraphs with locally symmetric graphs. Besides, given a symmetric hypergraph, we prove that there are infinitely many symmetric hypergraphs that cover the given one.

KeyWords: Symmetric hypergraph, s-arc transitive, locally s-arc transitive.
Mathematics Subject Classification [2000]: 05C10, 05C25.

1 Introduction

For $s \geq 0$, an *s-arc* in a graph H is an $(s+1)$-tuple $(v_0, v_1, ..., v_s)$ of vertices such that each v_i is adjacent to v_{i+1} while $v_i \neq v_{i+2}$. Let G be a subgroup of $\mathrm{Aut}(H)$. We say that H is *(G, s)-arc transitive*, or just *s-arc transitive*, if G acts transitively on the set of s-arcs of H. H is said to be *symmetric* if G acts transitively on the set of 1-arcs of H.

Given $G \leq \mathrm{Aut}(H)$, we say that H is *locally* (G, s)-arc transitive, or just *locally* s-arc transitive, if for each vertex α, the stabilizer G_α acts transitively on the set of s-arcs starting at α. We say that H is *locally symmetric*, if it is locally 1-arc transitive. The study of s-arc transitive graphs and locally s-arc transitive graphs goes back to Tutte [6]. In this paper we consider a natural extension of symmetric graphs to the symmetric hypergraphs.

For $s \geq 0$, an *s-arc* in a hypergraph $\Gamma=(V, E)$ is an alternate sequence of vertices and edges,

$(v_0, e_1, v_1, e_2, v_2, ..., v_{s-1}, e_s, v_s)$

where each edge e_i is incident to the vertices v_{i-1}, v_i, $1 \leq i \leq s$ and two consecutive vertices or edges are distinct. The hypergraph Γ is *s-arc transitive* if it has an automorphism group which acts transitively on the set of s-arc. Γ is said to be *symmetric* if its automorphism group acts transitively on the set of 1-arcs.

*Speaker

2 Main results

We give the following results about symmetric hypergraphs:

Proposition 2.1. *Let* $\Gamma=(V,E)$ *be a symmetric hypergraph such that* $|e|\geq 2$ *for every* $e\in E$. *Then,*

1. Γ *is vertex transitive,*
2. Γ *is uniform,*
3. Γ *is edge transitive.*

To generalize simple graphs, we say that a hypergraph $\Gamma=(V,E)$ is *linear* if it is simple and for every pair $e,e'\in E$, $|e\cap e'|\leq 1$. It was shown in [5, Proposition 4] that 2-arc transitive hypergraphs, are linear. However, symmetric hypergraphs are not linear in general. Consider the following (counter)example.

Example 2.2. Let $\Gamma=(V,E)$ be a hypergraph, where $V=\{0,1,2,3\}$ and $E=\{\{0,1,2\},\{0,1,3\},\{0,2,3\},\{1,2,3\}\}$.
It is not difficult to see that this hypergraph is symmetric but not linear.

By using the notion of *dual* of a hypergraph we have,

Proposition 2.3. *Let* Γ *be a* 2*-regular and* r*-uniform symmetric hypergraph with* $r\geq 3$. *Then its dual* Γ^* *is a vertex transitive graph.*

The *incidence graph* of a hypergraph $\Gamma=(V,E)$ is a bipartite graph $\mathrm{IG}(\Gamma)$ with a vertex set $\mathrm{S}=V\cup E$, and where $x\in V$ and $e\in E$ are adjacent if and only if $x\in e$. The next proposition connects the notions of symmetric hypergraphs with locally symmetric graphs.

Theorem 2.4. *Let* $\Gamma=(V,E)$ *be a* d*-regular and* r*-uniform hypergraph with* $d,r\geq 3$ *and* $\Gamma':=IG(\Gamma)$ *its incidence graph. If* Γ *is symmetric, then* Γ' *is locally symmetric.*

We say that the hypergraph Γ_1 is a *cover* of the hypergraph Γ_2, with *covering map* f, if there is a positive integer h such that f is a h-to-one surjective hypergraph homomorphism of Γ_1 onto Γ_2, and f is locally bijective. We bring the following theorem from [5].

Proposition 2.5. *Let* Γ *be an* s*-arc transitive,* d*-regular and* r*-uniform hypergraph with* $d,r\geq 3$ *and* $s\geq 1$. *There are infinitely many* s*-arc transitive hypergraphs which cover* Γ.

Corollary 2.6. *Let* Γ *be a symmetric hypergraph,* d*-regular and* r*-uniform with* $d,r\geq 3$. *There are infinitely many symmetric hypergraphs which cover* Γ.

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On the Bipartite Ramsey Number $BR(C_6, C_6, mK_2)$

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Abstract

For given bipartite graphs $G_1, G_2, ..., G_t$, the multicolor bipartite Ramsey number $BR(G_1, G_2, ..., G_t)$ is the smallest positive integer b such that if the edges of the complete bipartite graph $K_{b,b}$ are partitioned into t disjoint color classes giving t graphs $H_1, H_2, ..., H_t$ then at least one H_i, $1 \leq i \leq t$ has a subgraph isomorphic to G_i. In this paper, the exact value of the bipartite Ramsey number $BR(C_6, C_6, mK_2)$ is provided for $m \geq 5$.

Keywords: Bipartite Ramsey number, Bipartite graphs, Cycles, Stripes
Mathematics Subject Classification [2010]: 05C55, 05D10

1 Introduction

In this paper, we only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. For a given graph G, we denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively, and for a vertex $v \in V(G)$, we use $\deg_G(v)$ (or simply $\deg(v)$) and $N_G(u)$ to denote the degree and neighbors of v in G, respectively. As usual, a cycle and a path on m vertices are denoted by C_m and P_m, respectively. Also the complete bipartite graph with partite set (X, Y), $|X| = m$ and $|Y| = n$ denoted by $K_{m,n}$. We use $[X, Y]$ to denote the set of edges between partite sets X and Y. Also by a *stripe* mK_2 we mean a graph on $2m$ vertices and m independent edges. The complement of a graph G, denoted by $\overline{G}$, is a graph with same vertices as G and contains those edges which are not in G. The neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) | uv \in E(G)\}$ and let $d(v) = |N(v)|$

Since the 1970's, Ramsey theory has grown into one of the most active areas of research within combinatorics, overlapping variously with graph theory, number theory, geometry and logic. Let $G_1, G_2, \ldots, G_t$ be bipartite graphs. The the multicolor bipartite Ramsey number $BR(G_1, G_2, ..., G_t)$ is the smallest positive integer b such that if the edges of the

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complete bipartite graph $K_{b,b}$ are partitioned into t disjoint color classes giving t graphs $H_1, H_2, ..., H_t$ then at least one H_i, $1 \leq i \leq t$ has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by a result of Erdős and Rado [2]. The bipartite Ramsey numbers has been studied extensively. The exact value of the bipartite Ramsey number of paths, $BR(P_n, P_m)$, follows from a special case of some results of Faudree and Schelp [3] and Gyárfás and Lehel [4]. Also the bipartite Ramsey number $BR(K_{1,n}, P_m)$ was determined by Hatting and Henning in [5]. In addition, in [6] the author study the multicolor bipartite Ramsey number $bR(G_1, G_2, \ldots, G_t)$, in the case that $G_1, G_2, \ldots, G_t$ being either stars and stripes or stars and a path. The aim of this paper is to provide the exact value of the bipartite Ramsey number $BR(C_6, C_6, mK_2)$ for every $m \geq 5$.

2 Main Result

In this section we prove that $BR(C_6, C_6, mK_2) = m + 4$, for every $m \geq 5$. We start with the following lemma.

Lemma 2.1. *For every $m \geq 1$, $BR(C_6, C_6, mK_2) \geq m + 4$.*

Proof. Consider $K_{m+3,m+3}$ with partite sets X, Y, $X = \{x_1, x_2, \ldots, x_{m+3}\}$, and decompose edges of $K_{m+3,m+3}$ into graphs G_1, G_2, G_3, where G_1 is a complete 2 by $m + 3$ bipartite graph, G_2 is a complete 2 by $m + 3$ bipartite graph and G_3 is a complete $m - 1$ by $m + 3$ bipartite graph. In fact,

$$V(G_1) = X_1 \cup Y, \quad X_1 = \{x_1, x_2\},$$

$$V(G_2) = X_2 \cup Y, \quad X_2 = \{x_3, x_4\},$$

$$V(G_3) = X_3 \cup Y, \quad X_3 = \{x_i : \ 5 \leq i \leq m + 3\}.$$

Clearly $E(K_{m+3,m+3}) = E(G_1) \cup E(G_2) \cup E(G_3)$ and $C_6 \nsubseteq G_1$, $C_6 \nsubseteq G_2$, and $mK_2 \nsubseteq G_3$ which means that $BR(C_6, C_6, mK_2) \geq m + 4$. □

Theorem 2.2. ([7]) *For the cycle of length 6 we have $BR(C_6, C_6) = 6$.*

Lemma 2.3. *For every $m \geq 5$, $BR(C_6, C_6, mK_2) \leq m + 4$.*

Proof. Let edges of $K_{m+4,m+4}$ are arbitrary colored red, blue and green and G^r, G^b and G^g denote the subgraphs of $K_{m+4,m+4}$ include by the edges of colors red, blue and green, respectively. We suppose that $mK_2 \nsubseteq G^g$ and we prove that $C_6 \subseteq G^r$, $C_6 \subseteq G^b$.

Let M be the Maximum matching in G^g. Then by the assumption, $E(M) \leq m - 1$. Now, we have the following claim.

Claim 1: Either $K_{6,6} \subseteq \overline{G}^g$ or $K_{5,9} \subseteq \overline{G}^g$.

By the claim 1, we have $K_{6,6} \subseteq \overline{G}^g$ or $K_{5,9} \subseteq \overline{G}^g$. If $K_{6,6} \subseteq \overline{G}^g$ then by Theorem 2.2 we have $C_6 \subseteq G^r$ or $C_6 \subseteq G^b$, which means that $BR(C_6, C_6, mK_2) \leq m+4$. Thus suppose that $K_{5,9} \subseteq \overline{G}^g$ i.e. $K_{5,9} \subseteq G^r \cup G^b$. In this case we have the following claim.

Claim 2: In any red-blue coloring of the edges of $K_{5,9}$, either $C_6 \subseteq G^r$ or $C_6 \subseteq G^b$.

Proof of the Claim. Let $X = \{x_1, x_2, ..., x_5\}$ and $Y = \{y_1, y_2, ..., y_9\}$ be the partite sets of $K_{5,9}$. By the Pigeonhole Principle there exist at least five vertices in Y, say $Z = \{y_1, y_2, \ldots, y_5\}$, such that for each $z \in Z$, either $\deg_{G^r}(z) \geq 3$ or $\deg_{G^b}(z) \geq 3$. Without loss of generality, we may assume that for each $z \in Z$, $\deg_{G^r}(z) \geq 3$. Since $|X| = 5$ then $|N(z) \cap N(z')| \geq 1$ for every $z, z' \in Z$. Now, we consider the following cases.

Case 1: For some $y_{i_1}, y_{i_2} \in Z$, $|N(y_{i_1}) \cap N(y_{i_2})| \geq 3$.

Let $N = N(y_{i_1}) \cap N(y_{i_2})$ and W.l.g, $N = \{x_1, x_2, x_3\}$. If there exists a vertex $z \in Z \setminus \{y_{i_1}, y_{i_2}\}$ such that $|N(z) \cap N| \geq 2$, then we have $C_6 \subseteq G$. Thus suppose that for all $z \in Z \setminus \{y_{i_1}, y_{i_2}\}$, $|N(z) \cap N| = 1$. Hence for all $z \in Z \setminus \{y_{i_1}, y_{i_2}\}$ we have $\{x_4, x_5\} \subseteq N(z)$. If there exist vertices $z, z' \in Z \setminus \{y_{i_1}, y_{i_2}\}$ such that these vertices have distinct neighbors in $\{x_1, x_2, x_3\}$ then obviously we have $C_6 \subseteq G^r$. Otherwise it can be easily that $C_6 \subset G^b$.

Case 2: For some $y_{i_1}, y_{i_2} \in Z$, $|N(y_{i_1}) \cap N(y_{i_2})| = 2$.

Let $N = N(y_{i_1}) \cap N(y_{i_2})$ and W.l.g, we may assume that $N = \{x_1, x_2\}$ and $x_3 \in N(y_{i_1})$, $x_4 \in N(y_{i_2})$. For each $z \in Z \setminus \{y_{i_1}, y_{i_2}\}$, $|N(z) \cap (Z \setminus \{x_5\})| \geq 2$. If there exist a vertex $z \in Z \setminus \{y_{i_1}, y_{i_2}\}$ such that $|N(z) \cap (Z \setminus \{x_5\})| \neq \{x_1, x_2\}$, then obviously we have $C_6 \subseteq G^r$. So let for each $z \in Z \setminus \{y_{i_1}, y_{i_2}\}$ we have $\{x_1, x_2, x_5\} = |N(z) \cap (Z \setminus \{x_5\})|$. Now, for any $z, z' \in Z \setminus \{y_{i_1}, y_{i_2}\}$ the set $\{y_{i_1}, z, z', x_1, x_2, x_5\}$ form a copy of $C_6 \subseteq G^r$.

Case 3: For some $y_{i_1}, y_{i_2} \in Z$, $|N(y_{i_1}) \cap N(y_{i_2})| = 1$.

W.l.g, we may assume that $N(y_{i_1}) = \{x_1, x_2, x_3\}$ and $N(y_{i_2}) = \{x_3, x_4, x_5\}$. so, for every $z \in Z \setminus \{y_{i_1}, y_{i_2}\}$ we have $|N(z) \cap N(y_{i_1})| \geq 2$ or $|N(z) \cap N(y_{i_2})| \geq 2$. Therefore by Case 2, we have $C_6 \subseteq G^r$ which completes the proof of the claim 2.

Since $K_{5,9} \subseteq G^r \cup G^b$, by Claim 2, either $C_6 \subseteq G^r$ or $C_6 \subseteq G^b$ which shows that $BR(C_6, C_6, mK_2) \leq m + 4$. □

Now, Combining Lemmas 2.1 and 2.4, we have the following theorem which determine the exact value of the bipartite Ramsey number $BR(C_6, C_6, mK_2)|$

Theorem 2.4. *For every $m \geq 5$, $BR(C_6, C_6, mK_2) = m + 4$.*

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On the rainbow connection number of certain graphs

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Abstract

The rainbow connection number $rc(G)$ of a connected graph G is the minimum number of colors needed to color its edges, so that every pair of its vertices is connected by at least one path in which no two edges are colored the same. In this paper, we study the rainbow connection number of some specific graphs. We apply our results to obtain the rainbow connection number of some graphs which are important in chemistry and nanoscience.

Keywords: Rainbow connection number, Dendrimer, Graph
Mathematics Subject Classification [2010]: 05C76 05C40

1 Introduction

A simple graph $G = (V, E)$ is a finite nonempty set V of objects called vertices together with a (possibly empty) set E of unordered pairs of distinct vertices of G called edges. In chemical graphs, the vertices of the graph correspond to the atoms of the molecule, and the edges represent the chemical bonds.

Edge coloring of a graph is a function from its edge set to the set of natural numbers (called colours). A path in an edge colored graph with no two edges sharing the same color is called a rainbow path. An edge colored graph is said to be rainbow connected if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a rainbow coloring of the graph. The minimum number of colors required to rainbow color a connected graph is called its rainbow connection number, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph K_n is 1, that of a path P_n is its length $n-1$, that of an even cycle C_{2n} is its diameter, that of an odd cycle of length at least 5 is one more than its diameter, and that of a tree is its number of edges. Topics related to rainbow problems were first introduced in a classical paper of Erdős et al. in 1975 [6] as a counterpart to Ramsey problems. Since then the development went in different directions. For the latest survey see [7]. Probably the latest such rainbow problem was introduced by Chartrand et al. in [4] and it is about "rainbow connection".

It was shown in [3] that computing the rainbow connection number of an arbitrary graph is NP-Hard. To rainbow color a graph, it is enough to ensure that every edge of

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some spanning tree in the graph gets a distinct color. Hence, the order of the graph minus one is an upper bound for rainbow connection number. There are some papers which find better upper bounds for the same in terms of other graph parameters such as connectivity, minimum degree, etc.

In this paper, we study the rainbow connection number of some specific graphs and obtain the rainbow connection number of some graphs which are important in chemistry and nanoscience.

2 Main results

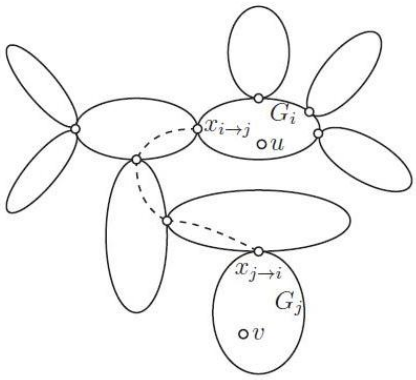

Figure 1: Graph G obtained by point-attaching from $G_1, ..., G_k$.

Let G be a connected graph constructed from pairwise disjoint connected graphs $G_1, ..., G_k$ as follows. Select a vertex of G_1, a vertex of G_2, and identify these two vertices. Then continue in this manner inductively. Note that the graph G constructed in this way has a tree-like structure, the G_i's being its building stones (see Figure 1). Usually say that G is obtained by point-attaching from $G_1, ..., G_k$ and that G_i's are the primary subgraphs of G. A particular case of this construction is the decomposition of a connected graph into blocks (see [5]). We obtain a formula for the rainbow connection number of this graph G.

As an example consider the graph $Q(m, n)$ constructed in the following manner: denoting by K_q the complete graph with q vertices, consider the graph K_m and m copies of K_n (see [5]). By definition, the graph $Q(m, n)$ is obtained by identifying each vertex of K_m with a vertex of a unique K_n. The graph $Q(6, 4)$ is shown in Figure 2.

Using our result for the point-attaching graph, we have the following result:

Theorem 2.1. *The rainbow connection number of* $Q(m, n)$ *is* $rc(Q(m, n)) = m + 1$.

Here we consider a special cases of point attaching of k graphs. Let $G_1, G_2, ..., G_k$ be a finite sequence of pairwise disjoint connected graphs and let $x_i, y_i \in V(G_i)$. By definition in [5], the chain G of the graphs $\{G_i\}_{i=1}^k$ with respect to the vertices $\{x_i, y_i\}_{i=1}^k$ is obtained by identifying the vertex y_i with the vertex x_{i+1} for $i \in \{1, 2, ..., k-1\}$ (see Figure 3 for $k = 4$). We have the following result for the rainbow connection number of the chain of graphs:

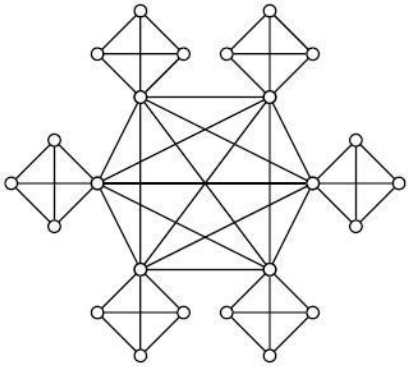

Figure 2: The graph $Q(6,4)$.

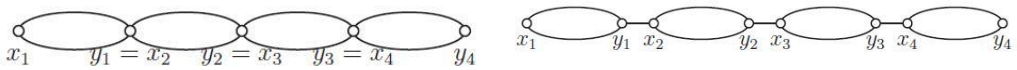

Figure 3: The chain and link graph of four graphs, respectively.

Theorem 2.2. *The rainbow connection number of the chain G of the graphs $\{G_i\}_{i=1}^k$ is* $rc(G) = \left(\sum_{i=1}^k rc(G_i)\right) - 1$

Here we consider another kind of graphs. Let $G_1, G_2, ..., G_k$ be a finite sequence of pairwise disjoint connected graphs and let $x_i, y_i \in V(G_i)$. By definition in [5], the link G of the graphs $\{G_i\}_{i=1}^k$ with respect to the vertices $\{x_i, y_i\}_{i=1}^k$ is obtained by adding an edge which connect the vertex y_i of G_i with the vertex x_{i+1} of G_{i+1} for all $i = 1, 2, ..., k-1$ (see Figure 3 for $k = 4$) ([5]). We obtain the rainbow connection number of the link of graphs.

Using our results we shall obtain the rainbow connection number of families of graphs which are important in chemistry. For example we compute the rainbow connection number of Spiro-chains ([5]), cactus chains ([2]), Polyphenylenes, and some dendrimer nanostars ([1, 5]).

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The restrained k-rainbow reinforcement numbers in graphs

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Abstract

for a positive integer k, a *restrained k-rainbow dominating function* (RkRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the conditions $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ and $|N(v) \cap \{u \in V(G) \mid f(u) = \emptyset\}| \geq 1$ are fulfilled, where $N(v)$ is the open neighborhood of v. The *weight* of an RkRDF f is the value $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The *restrained k-rainbow domination number* of a graph G, denoted by $\gamma_{rrk}(G)$, is the minimum weight of an RkRDF of G. The restrained k-rainbow reinforcement number $r_{rrk}(G)$ of a graph G is the minimum number of edges that must be added to G in order to decrease the restrained k-rainbow domination number. In this paper, we initiate the study of restrained k-rainbow reinforcement number in graphs and we present some sharp bounds on $r_{rrk}(G)$. In particular, we determine the restrained 2-rainbow reinforcement number of some classes of graphs.

Keywords: Retrained k-rainbow domination number, restrained k-rainbow reinforcement number.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write $K_{n,m}$ for the *complete bipartite graph* of order $n+m$, C_n for a *cycle* of length n and P_n for a path of order n.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of minimum cardinality of G is called a $\gamma(G)$-set. The *reinforcement number* $r(G)$ of a graph G is the minimum number of edges that must be added to G in order to decrease the domination number [13]. The reinforcement number is defined to be 0 when $\gamma(G) = 1$.

For a positive integer k, a *restrained k-rainbow dominating function*(RkRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$

*Speaker

such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ and $|N(v) \cap \{u \in V(G) \mid f(u) = \emptyset\}| \geq 1$ are fulfilled. The *weight* of a RkRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *restrained k-rainbow domination number* of a graph G, denoted by $\gamma_{rrk}(G)$, is the minimum weight of a RkRDF of G. A $\gamma_{rrk}(G)$-*function* is a restrained k-rainbow dominating function of G with weight $\gamma_{rrk}(G)$. Note that $\gamma_{rr1}(G)$ is the classical restrained domination number $\gamma_r(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [4] and has been studied by several authors (see for example [5, 6, 11]).

Our purpose in this paper is to initiate the study of restrained k-rainbow reinforcement number in graphs. We determine exact values of restrained 2-rainbow reinforcement number of some classes of graphs.

2 Main result

We will use the following results.

Proposition 2.1. If E is a $r_{rrk}(G)$-set, then

$$\gamma_{rrk}(G) - 2 \leq \gamma_{rrk}(G + E) \leq \gamma_{rrk}(G) - 1.$$

Proposition 2.2. $\max\{\gamma_{rk}(G), \gamma_r(G)\} \leq \gamma_{rrk}(G) \leq k\gamma_r(G)$.

Proposition 2.3. [1] For $n \geq 4$, $\gamma_{rr2}(P_n) = \lceil \frac{2n+1}{3} \rceil + 1$ and $\gamma_{rr2}(P_n) = n$, otherwise.

Proposition 2.4. [1] For $n \geq 6$,

$$\gamma_{rr2}(C_n) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1 & n \equiv 2 \pmod 3 \\ 2\lceil \frac{n}{3} \rceil & \text{otheriwise.} \end{cases}$$

Proposition 2.5. [2] For $1 \leq n \leq m$,

$$\gamma_{rr2}(K_{n,m}) = \begin{cases} m+1 & n = 1 \\ 4 & n \geq 2. \end{cases}$$

Theorem 2.6. For $n \geq 3$, $r_{rr2}(P_n) = 1$

Proof. Let $P_n := v_1 v_2 \ldots v_n$. If $3 \leq n \leq 6$, then it is not hard to see that $r_{rr2}(P_n) = 1$. So suppose that $n \geq 7$. We consider three cases.

Case 1. $n \equiv 1 \pmod 3$.

Define $f : V(P_n) \to \mathcal{P}(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod 3$ and $f(x) = \emptyset$ otherwise. Clearly f is a $\gamma_{rr2}(P_n)$-function of weight $\lceil \frac{2n+1}{3} \rceil + 1$. Then the function $g = (V_0^f, v_n, \emptyset, V_{1,2}^f - v_n)$ is an R2RDF on $P_n + v_1 v_{n-1}$ that implies $r_{rr2}(P_n) = 1$.

Case 2. $n \equiv 2 \pmod 3$.

Define $f : V(P_n) \to \mathcal{P}(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod 3$, $f(v_n) = \{1\}$ and $f(x) = \emptyset$ otherwise. Clearly f is a $\gamma_{rr2}(P_n)$-function of weight $\lceil \frac{2n+1}{3} \rceil + 1$. Then the function $g = (V_0^f \cup v_n, \emptyset, \emptyset, V_{1,2}^f)$ is an R2RDF on $P_n + v_2 v_n$ that implies $r_{rr2}(P_n) = 1$.

Case 3. $n \equiv 0 \pmod 3$.

Define $f : V(P_n) \to \mathcal{P}(\{1, 2\})$ by $f(v_i) = \{1, 2\}$ for $i \equiv 1 \pmod 3$, $f(v_n) = f(v_{n-1}) = \{1\}$ and $f(x) = \emptyset$ otherwise. Clearly f is a $\gamma_{rr2}(P_n)$-function of weight $\lceil \frac{2n+1}{3} \rceil + 1$. Then the function $g = (V_0^f \cup \{v_{n-1}, v_n\}, V_1^f - \{v_{n-1}, v_n\}, \emptyset, V_{1,2}^f)$ is an R2RDF on $P_n + v_1 v_n$ that implies $r_{rr2}(P_n) = 1$. □

Theorem 2.7. For $n \geq 6$,

$$r_{rr2}(C_n) = \begin{cases} 2 & n \equiv 0 \pmod 3 \\ 1 & \text{otheriwise.} \end{cases}$$

Proof. Let $C_n := (v_1, v_2, \ldots, v_n)$. We consider three cases.
Case 1. $n \equiv 0 \pmod 3$.

Define $f : V(C_n) \to \mathcal{P}(\{1,2\})$ by $f(v_i) = \{1,2\}$ for $i \equiv 1 \pmod 3$ and $f(x) = \emptyset$ otherwise. Clearly f is a $\gamma_{rr2}(C_n)$-function of weight $frac2n3$. Then the function $g = (V_0^f, v_{n-2}, \emptyset, V_{1,2}^f - v_{n-2})$ is an R2RDF on $C_n + \{v_1v_{n-1}, v_1v_{n-3}\}$, that implies $r_{rr2}(C_n) \leq 2$. It is not hard to see that for any $\gamma_{rr2}(C_n)$-function g, we have $|V_{1,2}^g| = \frac{n}{3}$ and $|V_1^g \cup V_2^g| = \emptyset$. Thus each vertex of $V_{1,2}^g$ has exactly two private neighbors. This implies that $r_{rr2}(C_n) \geq 2$. Therefore $r_{rr2}(C_n) = 2$ in this case.
Case 2. $n \equiv 1 \pmod 3$.

Define $f : V(C_n) \to \mathcal{P}(\{1,2\})$ by $f(v_i) = \{1,2\}$ for $i \equiv 1 \pmod 3$ and $f(x) = \emptyset$ otherwise. Clearly f is a $\gamma_{rr2}(C_n)$-function of weight $2\lceil \frac{n}{3} \rceil$. Then the function $g = (V_0^f, \emptyset, \emptyset, V_{1,2}^f - v_n)$ is an R2RDF on $C_n + v_1v_{n-1}$, that implies $r_{rr2}(C_n) = 1$ in this case.
Case 3. $n \equiv 2 \pmod 3$.

Define $f : V(C_n) \to \mathcal{P}(\{1,2\})$ by $f(v_i) = \{1,2\}$ for $i \equiv 1 \pmod 3$, $f(v_n) = \{1\}$ and $f(x) = \emptyset$ otherwise. Clearly f is a $\gamma_{rr2}(C_n)$-function of weight $2\lceil \frac{n}{3} \rceil + 1$. Then the function $g = (V_0^f \cup \{v_n\}, \emptyset, \emptyset, V_{1,2}^f)$ is an R2RDF on $C_n + v_{n-2}v_n$, that implies $r_{rr2}(C_n) = 1$. □

Theorem 2.8. For $1 \leq n \leq m$,

$$r_{rr2}(K_{n,m}) = \begin{cases} 1 & n = 1, 2, m \geq 2 \\ n-2 & n \geq 3. \end{cases}$$

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be the partite sets of $K_{n,m}$. We consider three cases.
Case 1. $n = 1, m \geq 2$.

Define function $f : V(K_{n,m}) \to \mathcal{P}(\{1,2\})$ by $f(y_i) = \{1\}$ for $1 \leq i \leq m$ and $f(x_1) = \{2\}$. Clearly f is a $\gamma_{rr2}(K_{n,m})$-function of weight $m+1$. Let $G = K_{n,m} + \{y_1y_2\}$ and define $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(x_1) = \{1,2\}$, $g(y_1) = g(y_2) = \emptyset$ and $g(x) = \{1\}$ otherwise. Obviously g is an R2RDF on G of weight m. Thus $r_{rr2}(K_{1,m}) = 1$ in this case.
Case 2. $n = 2$.

Define $f : V(K_{n,m}) \to \mathcal{P}(\{1,2\})$ by $f(x_1) = f(y_1) = \{1,2\}$ and $f(x) = \emptyset$ otherwise. Then f is a $\gamma_{rr2}(K_{n,m})$-function of weight 4. Join x_1 to x_2 and define $g : V(K_{n,m} + \{x_1x_2\}) \to \mathcal{P}(\{1,2\})$ by $g(x_1) = \{1,2\}$ and $g(x) = \emptyset$ otherwise. Obviously g is an R2RDF on $K_{n,m} + \{x_1x_2\}$ of weight 2. Therefore $r_{rr2}(K_{2,m}) = 1$.
Case 3. $n \geq 3$. Define $f : V(K_{n,m}) \to \mathcal{P}(\{1,2\})$ by $f(x_1) = f(y_1) = \{1,2\}$ and $f(x) = \emptyset$ otherwise. Let $G = K_{n,m} + \{x_1x_2, x_1x_3, \ldots, x_1x_{n-1}\}$. Define $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(x_1) = \{1,2\}$, $g(x_n) = \{1\}$ and $g(x) = \emptyset$ otherwise. Obviously g is an R2RDF on G of weight 3. So $r_{rr2}(K_{n,m}) \leq n-2$. Now let $\gamma_{rr2}(K_{m,n} + E) = 3$. Clearly, for an arbitrary graph G of order p, $\gamma_{rr2}(G) = 3$ if and only if either $|V_{1,2}^f| = 1$ and $\Delta(G) = p-2$ or $|V_{1,2}^f| = 0$ and $\Delta(G) \geq p-3$. Thus either $|E| \geq n-2$ or $|E| \geq n-3$. If $|E| \geq n-2$, we are done. Suppose that $\gamma_{rr2}(K_{m,n} + E) = 3$ and $|E| \geq n-3$. But in this case, by adding any $n-3$ edges to $K_{n,m}$ we have $\gamma_{rr2}(K_{m,n} + E) = 4$. So $r_{rr2}(K_{n,m}) = n-2$. The proof is complete. □

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A note on vague formal concept lattice
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A Note On Vague Formal Concept Lattice

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Abstract

In this paper, the concept of vague fuzzy sets is applied to the concept lattice theory. The notion of vague formal context is introduced and some related properties are investigated. The theory of vague concept lattice is useful in view of the complexity and fuzziness of information in real world.

Keywords: Formal Context, Formal Concept, Vague, Lattice.
Mathematics Subject Classification [2010]: 03G10, 03B52

1 Introduction

Rudolf Wille [4] introduced Formal Concept Analysis (FCA) in 1982. Formal Concept Analysis is a method for data analysis and knowledge representation that provides visualizations in the form of mathematical lattice diagrams for data stored in formal contexts. Formal Concept Analysis applied in many quite different realms like computer sciences, data mining, knowledge management, semantic web, mathematics and engineering.
A concept lattice is an ordered hierarchical structure of formal concepts that are defined by a binary relation between a set of objects and a set of attributes.
In recent years, many new achievements on these topics have been achieved on theories such as construction of concept lattice and acquisition of rules (see [1], [3]).
The notion of vague set theory introduced by W. L. Gau and D. J. Buehrer, as a generalizations of Zadeh's fuzzy set theory [2].

Definition 1.1. [1] A triplet (X, Y, I) is called formal context where X is a nonempty finite set of objects called universe of discourse, Y is a nonempty finite set of attributes and I is a binary relation between X and Y.

*Speaker

For a formal context (X, Y, I) a pair of dual operators for $A \subseteq X$ and $B \subseteq Y$ is defined as follows:

$$A^{\uparrow} = \{y \in Y | (x, y) \in I \text{ for all } x \in A\},$$
$$B^{\downarrow} = \{x \in X | (x, y) \in I \text{ for all } y \in B\}.$$

Definition 1.2. [1] Let (X, Y, I) be a formal context. A pair (A, B) is called formal concept, if $A^{\uparrow} = B$ and $B^{\downarrow} = A$.

Theorem 1.3. [3] *The set of all formal concept is a complete lattice.*

Definition 1.4. [5] A fuzzy set $A = \{(u, \mu(u)) | u \in U\}$ in the universe of discourse U is a characterized by a membership function μ as $\mu : U \longrightarrow [0, 1]$.

Definition 1.5. [2] A vague set A in the universe of discourse U is characterized by a true membership function t_A and a false membership function f_A, as follows:

$$t_A : U \longrightarrow [0, 1] \text{ and } f_A : U \longrightarrow [0, 1] \text{ such that } 0 \leq t_A(u) + f_A(u) \leq 1$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived by from the evidence for u, and $f_A(u)$ is lower bound on the grade of membership of the negation of U derived from the evidence against it. The vague set A is written as

$$A = \{(u, [t_A(u), 1 - f_A(u)]) | u \in U\}$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the vague value of u in A and denoted by $A(u)$. The set of all vague set in U is denoted by $VF(U) = \{A | A \text{ is a vague set in } U\}$.

Definition 1.6. [2] (1) A vague set A is contained in vague set B, written as $A \subseteq B$ if and only if $t_A(u) \leq t_B(u)$ and $f_A(u) \geq f_B(u)$ for all $u \in U$.
(2) The union of two vague set A and B is a vague set C, written as $C = A \cup B$ where $t_C(u) = \max(t_A(u), t_B(u))$ and $f_C(u) = \min(f_A(u), f_B(u))$ for all $u \in U$.
(3) The intersection of two vague set A and B is a vague set C, written as $C = A \cap B$ where $t_C(u) = \min(t_A(u), t_B(u))$ and $f_C(u) = \max(f_A(u), f_B(u))$ for all $u \in U$.

2 Vague Formal Concept Lattice

Definition 2.1. A vague formal context is a triplet (X, Y, I), where $X = \{x_1, x_2, ..., x_n\}$ is a nonempty finite set of objects called universe of discourse, $Y = \{y_1, y_2, ..., y_m\}$ is a nonempty finite set of attributes and I is a vague set in $X \times Y$, where

$$I = \{((x, y), [t_I((x, y)), 1 - f_I((x, y))]) | (x, y) \in X \times Y\}.$$

A vague formal context can be represent by cross table such that rows is objects and columns is attributes and in row i and column j write $[t_I(x_i, y_j), 1 - f_I(x_i, y_j)]$.

Example 2.2. Let $X = \{x_1, x_2, x_3\}$ be universe of discourse and $Y = \{y_1, y_2, y_3\}$ be the set of attributes where x_1= "Peter", x_2= "Robert", x_3= "Felix" and y_1= "Idealistic", y_2= "Pessimist", y_3= "Rational" and I represented in Table 1. Then (X, Y, I) is a vague formal context.

Table 1: A vague formal context

I	y_1	y_2	y_3
x_1	[0.0, 0.5]	[0.2, 0.3]	[0.1, 0.4]
x_2	[0.7, 0.9]	[0.0, 1.0]	[0.8, 0.9]
x_3	[0.4, 0.5]	[0.3, 0.4]	[0.1, 0.8]

Definition 2.3. Let (X, Y, I) be a vague formal context. Define two operators $\downarrow: VF(Y) \longrightarrow 2^X$ and $\uparrow: 2^X \longrightarrow VF(Y)$ by $B^{\downarrow} = \{x \in X | I(x, y) \geq B(y) \text{ for all } y \in Y\}$ and $A^{\uparrow} = \{(y, [t_{A^{\uparrow}}(y), 1 - f_{A^{\uparrow}}(y)]) | y \in Y\}$ where $t_{A^{\uparrow}}(y) = \wedge_{a \in A} t_I(a, y)$, $f_{A^{\uparrow}}(y) = \vee_{a \in A} f_I(a, y)$ and $\emptyset^{\uparrow} = \{(y, [1, 1]) | y \in Y\}$ for all $B \in VF(Y)$ and $A \in 2^X$.

Example 2.4. Consider B=$\{(y_1, [0.4, 0.5]), (y_2, [0.0, 0.4]), (y_3, [0.1, 0.8])\} \in VF(Y)$ and A=$\{x_1, x_2\} \in 2^X$ in Example 2.2. Then we get that $B^{\downarrow} = \{x_2, x_3\}$ and $A^{\uparrow} = \{(y_1, [0.0, 0.5]), (y_2, [0.0, 0.3]), (y_3, [0.1, 0.4])\}$ by Definition 2.3.

Proposition 2.5. *Let (X, Y, I) be a vague formal context and A, $A_1, A_2 \subseteq X$ and $B, B_1, B_2 \in VF(Y)$. Then*
(1) if $A_1 \subseteq A_2$ then ${A_2}^{\uparrow} \subseteq {A_1}^{\uparrow}$,
(2) if $B_1 \subseteq B_2$ then ${B_2}^{\downarrow} \subseteq {B_1}^{\downarrow}$,
(3) $A \subseteq A^{\uparrow\downarrow}$ and $B \subseteq B^{\downarrow\uparrow}$,
(4) $A^{\uparrow\downarrow\uparrow} = A^{\uparrow}$ and $B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$,
(5) $A \subseteq B^{\downarrow}$ if and only if $B \subseteq A^{\uparrow}$,
(6) $(A_1 \cup A_2)^{\uparrow} = {A_1}^{\uparrow} \cap {A_2}^{\uparrow}$ and $(A_1 \cap A_2)^{\uparrow} \supseteq {A_1}^{\uparrow} \cup {A_2}^{\uparrow}$,
(7) $(B_1 \cup B_2)^{\downarrow} = {B_1}^{\downarrow} \cap {B_2}^{\downarrow}$ and $(B_1 \cap B_2)^{\downarrow} \supseteq {B_1}^{\downarrow} \cup {B_2}^{\downarrow}$.

Corollary 2.6. *Let (X, Y, I) be a vague formal context. Then $\uparrow\downarrow$ is a closure operator on the set X and $fix(\uparrow\downarrow) = \{A \subseteq X | A^{\uparrow\downarrow} = A\}$ is a closure system on the set X.*

Definition 2.7. Let (X, Y, I) be a vague formal context. A pair (A, B) is called a vague formal concept, for short, a vague concept if $A^{\uparrow} = B$ and $B^{\downarrow} = A$. A is called the extension of the concept (A, B) and B is called the intension of the concept (A, B).

Example 2.8. The pair $(\{x_1, x_3\}, \{(y_1, [0.0, 0.5]), (y_2, [0.2, 0.3]), (y_3, [0.1, 0.4])\})$ is a vague formal concept but the pair $(\{x_1\}, \{(y_1, [0.0, 0.5]), (y_2, [0.2, 0.3]), (y_3, [0.1, 0.4])\})$ is not a vague formal concept in Example 2.2.

Proposition 2.9. *Let (A_1, B_1) and (A_2, B_2) be two vague formal concepts of a vague formal context (X, Y, I). Then $(A_1 \cap A_2, (B_1 \cup B_2)^{\downarrow\uparrow})$ and $((A_1 \cup A_2)^{\uparrow\downarrow}, B_1 \cap B_2)$ are also vague concept.*

Theorem 2.10. *Let (X, Y, I) be a vague formal context and $L(X, Y, I) = \{(A, B) \in 2^X \times VF(Y) | A^{\uparrow} = B, B^{\downarrow} = A\}$. Then*
(1) $(L(X, Y, I), \leq)$ is a partially order set where $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subseteq A_2$ iff $B_2 \subseteq B_1$ for all $(A_1, B_1), (A_2, B_2) \in L(X, Y, I)$.
(2) $(L(X, Y, I), \wedge, \vee)$ is complete lattice where

$$\wedge_i (A_i, B_i) = (\cap_i A_i, (\cup_i B_i)^{\downarrow\uparrow}) \text{ and } \vee_i (A_i, B_i) = ((\cup_i A_i)^{\uparrow\downarrow}, \cap_i B_i).$$

Definition 2.11. Let (X, Y, I) be a vague formal context. Then $(L(X, Y, I), \leq)$ is called a vague concept lattice.

Example 2.12. Consider Example 2.2. We have L(X, Y, I)=$\{C_1, C_2, ..., C_6\}$ where
$C_1 = (\emptyset, \{(y_1, [1.0, 1.0]), (y_2, [1.0, 1.0]), (y_3, [1.0, 1.0])\})$,
$C_2 = (\{x_2\}, \{(y_1, [0.7, 0.9]), (y_2, [0.0, 1.0]), (y_3, [0.8, 0.9])\})$,
$C_3 = (\{x_3\}, \{(y_1, [0.4, 0.5]), (y_2, [0.3, 0.4]), (y_3, [0.1, 0.8])\})$,
$C_4 = (\{x_1, x_3\}, \{(y_1, [0.0, 0.5]), (y_2, [0.2, 0.3]), (y_3, [0.1, 0.4])\})$,
$C_5 = (\{x_2, x_3\}, \{(y_1, [0.4, 0.5]), (y_2, [0.0, 0.4]), (y_3, [0.1, 0.8])\})$,
$C_6 = (\{x_1, x_2, x_3\}, \{(y_1, [0.0, 0.5]), (y_2, [0.0, 0.3]), (y_3, [0.1, 0.4])\}$.
Then the vague concept lattice is

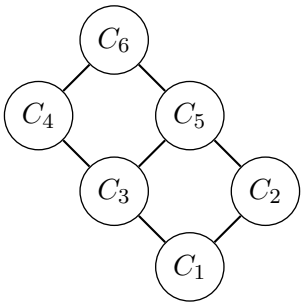

Figure 1: Concept lattice of example 2.2

Theorem 2.13. *Let (X, Y, I) be a vague formal context.*
(1) $(L(X, Y, I), \leq)$ is isomorphic to $(Ext(X, Y, I), \subseteq)$ where

$$Ext(X, Y, I) = \{A \in 2^X | (A, B) \in L(X, Y, I) \text{ for some } B \in VF(Y)\}.$$

(2) $(L(X, Y, I), \leq)$ is dual isomorphic to $(Int(X, Y, I), \subseteq)$ where

$$Int(X, Y, I) = \{B \in VF(Y) | (A, B) \in L(X, Y, I) \text{ for some } A \in 2^X\}.$$

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Adiabatic Approximation for the Matrix NLS Equation

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We develop a perturbation theory for matrix NLS equation. The formalism is based on using the Riemann-Hilbert problem and provides the means to analytically calculate evolution of the soliton parameters. Treating a small deviation from the integrability condition as a perturbation, we describe the rank-one soliton dynamics in the adiabatic approximation.

Keywords: Matrix NLS Equation, Perturbation soliton, Adiabetic approximation

Mathematics Subject Classification [2010]: 47H14, 70H11

1 Introduction

The matrix NLS equation, observed experimentally, are expected to be important for various applications in atom optics, including atom interferometry, atom lasers, and coherent atom transport. Recent experimental and theoretical advances in matrix NLS soliton dynamics are reviewed in Ref.[3].
Integrable models provide a very useful proving ground for testing new analytical and numerical approaches to study such a complicated system as the matrix NLS equation. As a step in this direction, in the present paper we develop a perturbation theory for the integrable the matrix NLS equation. Evidently, small disturbance of the integrability condition can be considered as a perturbation of the integrable model. Our formalism is based on the Riemann-Hilbert problem associated with the matrix NLS equation [1][2].

2 Model

We consider the integrable matrix NLS equation

$$i\partial_t \mathcal{Q} + \partial_x^2 \mathcal{Q} + 2\mathcal{Q}\mathcal{Q}^\dagger \mathcal{Q} = 0, \tag{1}$$

where

$$\mathcal{Q} = \begin{pmatrix} \phi_+ & \phi_0 \\ \phi_0 & \phi_- \end{pmatrix}. \tag{2}$$

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The matrix NLS equation (1) appears as a compatibility condition of the system of linear equations ,

$$\partial_x\psi = ik[\lambda,\psi] + \hat{\mathcal{Q}}\psi, \tag{3}$$
$$\partial_t\psi = 2ik^2[\lambda,\psi] + V\psi, \tag{4}$$

where $\lambda = diag(-1,-1,1,1)$,

$$\hat{\mathcal{Q}} = \begin{pmatrix} 0 & \mathcal{Q} \\ -\mathcal{Q}^\dagger & 0 \end{pmatrix}, \qquad V = 2k\hat{\mathcal{Q}} + i\begin{pmatrix} \mathcal{Q}\mathcal{Q}^\dagger & \mathcal{Q}_x \\ \mathcal{Q}_x^\dagger & -\mathcal{Q}^\dagger\mathcal{Q} \end{pmatrix}, \tag{5}$$

and k is a spectral parameter.

3 Rank-one Soliton Matrix NLS

To obtain the rank-one soliton solution of the matrix NLS equation (1), we consider the single pair k_1 and k_1^* of zeros and the eigenvector $|1\rangle$. The eigenvector takes the form

$$|1\rangle = (e^{-ik_1x-2ik_1^2t}n_1, e^{-ik_1x-2ik_1^2t}n_2, e^{ik_1x+2ik_1^2t}n_3, e^{ik_1x+2ik_1^2t}n_4)^T, \tag{6}$$

where $n_a, a = 1,..,4$ are complex numbers. We set $k_1 = \mu + i\nu$ and find the solution of the RH problem.
Indeed, the rank-one soliton (1) can be represented as

$$\mathcal{Q} = 2\nu\begin{pmatrix} e^{-i\chi}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & e^{i\chi}\sin^2\theta \end{pmatrix} e^{i\varphi}\ sech\ z, \tag{7}$$

where

$$\cos\theta = \frac{|n_1|}{\sqrt{(|n_1|^2+|n_2|^2)}} = \frac{|n_3|}{\sqrt{(|n_3|^2+|n_4|^2)}}, \quad \chi = arg(n_3) - arg(n_4), \tag{8}$$

$$\varphi = -2\mu x - 4(\mu^2-\nu^2)t + \varphi_0, \quad \varphi = arg(n_1) - arg(n_4) = arg(n_2) - arg(n_3), \tag{9}$$

$$z = 2\nu(x+4\mu t) + \rho, \quad e^{2\rho} = \frac{|n_1|^2+|n_2|^2}{|n_3|^2+|n_4|^2}, \tag{10}$$

The soliton amplitude is determined by the parameter ν, and its velocity is equal to 4μ . The parameters ρ and φ_0 give the initial position of the soliton center and its initial phase, respectively. The angle θ determines the normalized population of atoms in different spin states, while the phase factor $e^{i\chi}$ is responsible for the relative phases between the components $\phi_\pm$ and ϕ_0.

4 Perturbation Theory for the matrix NLS

In this section we perform a general analysis of the perturbed spinor BEC equations

$$i\frac{\delta\hat{\mathcal{Q}}}{\delta t} = \epsilon\hat{\mathcal{R}}, \quad \hat{R} = \begin{pmatrix} 0 & \mathcal{R} \\ \mathcal{R}^\dagger & 0 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} R_+ & R_0 \\ R_0 & R_- \end{pmatrix}. \tag{11}$$

We can write

$$\partial_t n_1 = -\epsilon(X_{11}n_1 + X_{12}n_2), \quad \partial_t n_2 = -\epsilon(X_{21}n_1 + X_{22}n_2),$$
$$\partial_t n_3 = -\epsilon(X_{31}n_1 + X_{32}n_2), \quad \partial_t n_4 = -\epsilon(X_{41}n_1 + X_{42}n_2), \tag{12}$$

where for simplicity we use the notation $X_{ab} = \Upsilon^{(reg)}_{+ab}(k_1)$, $a, b = 1, 2$, and $X_{ab} = \Upsilon^{(reg)}_{+ab}(k_1)e^{-8i\int dtk_1^3}$ for $a = 3, 4$ and $b = 1, 2$. Here $\Upsilon^{(reg)}_{+ab}$ is the regular part of Υ_+ in the point k_1,

$$\Upsilon^{(reg)}_{+}(k_1) = \int dx E^{-1}\{\hat{\mathcal{R}}(\mathbf{1} - p^{(1)}) + p^{(1)}\hat{\mathcal{R}}p^{(1)} + 2\nu x[\Lambda, p^{(1)}\hat{\mathcal{R}}(\mathbf{1} - p^{(1)})]\}E(k_1). \tag{13}$$

Now we can derive the evolution equation for the parameters θ and χ entering the polarization matrix of the soliton solution . Indeed, these parameters are defined in terms of na which in turn obey Eqs. 12. Simple calculation gives

$$\partial_t \cos\theta = \frac{i\epsilon}{2}[e^{\rho+i\varphi_0}(X_{31}e^{-i\chi}\cos\theta + X_{41}\sin\theta) - e^{\rho-i\varphi_0}(X^*_{31}e^{i\chi}\cos\theta + X^*_{41}\sin\theta)], \tag{14}$$

$$\begin{aligned}\partial_t\chi = \quad & \frac{\epsilon}{2}\{e^{\rho+i\varphi_0}[X_{31}e^{-i\chi} - X_{42}e^{i\chi} + (\tan\theta - \cot\theta)X_{41}] \\ & -e^{\rho-i\varphi_0}[X^*_{31}e^{i\chi} - X^*_{42}e^{-i\chi} + (\tan\theta - \cot\theta)X^*_{41}]\}.\end{aligned} \tag{15}$$

Just in the same way we obtain evolution equations for the parameters φ_0 and ρ which are also expressed in terms of n_i,

$$\begin{aligned}\partial_t\varphi_0 = \quad & \frac{\epsilon}{2}[X_{11} + X^*_{11} + (X_{12}e^{-i\chi} + X^*_{12}e^{i\chi})\tan\theta \\ & -e^{\rho+i\varphi_0}(X_{41}\cot\theta + X_{42}e^{i\chi}) - e^{\rho-i\varphi_0}(X^*_{41}\cot\theta + X^*_{42}e^{-i\chi}),\end{aligned} \tag{16}$$

$$\begin{aligned}\partial_t\rho = \quad & \frac{i\epsilon}{2}\{(X_{11} - X^*_{11})\cos^2\theta + (X_{22} - X^*_{22})\sin^2\theta + [(X_{12} - X^*_{21})e^{i\chi} \\ & -(X^*_{12} - X_{21})e^{-i\chi}]\sin\theta\cos\theta \ + e^{\rho+i\varphi_0}(X_{31}e^{-i\chi}\cos^2\theta \\ & +2X_{41}\sin\theta\cos\theta + X_{42}e^{i\chi}\sin^2\theta) - e^{\rho-i\varphi_0}(X^*_{31}e^{i\chi}\cos^2\theta \\ & +2X^*_{41}\sin\theta\cos\theta + X^*_{42}e^{-i\chi}\sin^2\theta)\},\end{aligned} \tag{17}$$

5 Adiabatic Approximation

As an important example, we consider a perturbation caused by a small disturbance of the integrability condition. In this case, while the functional form of the perturbations R has the form

$$R_{\pm,0} = (|\phi_+|^2 + 2|\phi_0|^2 + |\phi_-|^2)\phi_{\pm,0}. \tag{18}$$

Matrix elements of Υ_+ which are the main ingredients of the evolution equations for the soliton parameters are found from Eq. (13) , and the projector $p^{(1)}$ is calculated by means of the simple formula

$$p^{(1)} = \frac{|1\rangle\langle 1|}{\langle 1|1\rangle}, \quad \langle 1| = |1\rangle^\dagger, \tag{19}$$

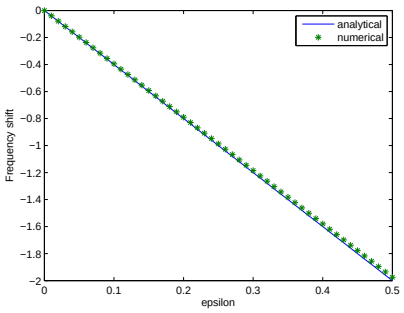

Figure 1: Comparison of the analytically predicted frequency shift with numerically

Calculation due to Eq. (13) gives

$$
\begin{aligned}
&X_{11} = X_{12} = X_{21} = X_{22} = 0,\\
&X_{31} = (2\nu^2) exp(-\rho - i\varphi_0 + i\chi)\cos^2\theta,\\
&X_{42} = (2\nu^2) exp(-\rho - i\varphi_0 - i\chi)\sin^2\theta,\\
&X_{41} = X_{32} = (2\nu^2) exp(-\rho - i\varphi_0)\sin\theta\cos\theta.
\end{aligned}
$$

Substituting these functions into Eqs. 14-17, we obtain

$$\theta = const,\ \chi = const,\quad \rho(t) = const,\quad \varphi_0(t) = \varphi_0(0) - 4\epsilon\alpha\nu^2 t \tag{20}$$

As a result, within the adiabatic approximation, the only manifestation of the perturbation caused by a small deviation from the consists in a small shift of the soliton frequency equal to $4\epsilon\nu^2$. Hence, a ferromagnetic soliton is a pretty robust object against a small disturbance of the integrability condition.
It is seen from the Fig.1 that there is a good agreement of the predicted linear dependence of the frequency shift on ϵ with that obtained numerically.

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Almost one-to-one factor maps on Smale spaces

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Abstract

Almost one-to-one maps between Smale spaces are factor maps whose the degrees are one. Putnam shows that any almost one-to-one factor map between Samle spaces can be decomposed as a composition of two resolving maps. In this paper, we investigate this subject and we show that two resolving maps are almost one-to-one, too.

Keywords: Almost one-to-one maps, Factor maps, Smale spaces
Mathematics Subject Classification [2010]: 37B10, 37D99

1 Introduction

Definition 1.1. [2, 3] Suppose that (X, f) is a compact metric space and f is a homeomorphism of X. Then (X, f) is called a Smale space if there exist constants ε_X and $0 < \lambda < 1$ and a continuous map from

$$\triangle_{\varepsilon_X} = \{(x, y) \in X \times X \quad | \quad d(x, y) \leq \varepsilon_X \quad \}$$

to X (denoted with $[,]$) such that:

$B\,1$ $[x, x] = x$,
$B\,2$ $[x, [y, z]] = [x, z]$,
$B\,3$ $[[x, y], z] = [x, z]$,
$B\,4$ $[f(x), f(y)] = [x, y]$,
$C\,1$ $d(f(x), f(y)) \leq \lambda\, d(x, y)$, whenever $[x, y] = y$,
$C\,2$ $d(f^{-1}(x), f^{-1}(y) \leq \lambda\, d(x, y)$, whenever $[x, y] = x$, whenever both sides of an equation are defined.

Definition 1.2. [2] Two points x and y in X are stably (or unstably) equivalent if

$$\lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0 \qquad (or \lim_{n\to-\infty} d(f^n(x), f^n(y)) = 0, \text{resp.}).$$

Let $X^s(x)$ and $X^u(x)$ denote the stable and unstable equivalence classes of x, respectively.

We recall that a factor map between two Smale spaces (Y, g) and (X, f) is a continuous function $\pi : Y \to X$ such that $\pi \circ g = f \circ \pi$. Of particular importance in this paper are factor maps which are s-bijective: that is, for each y in Y, the restriction of π to $Y^s(y)$ is a bijection to $X^s(\pi(y))$. There is obviously an analogous definition of a u-bijective factor map.[2]

*Speaker

2 Main results

Suppose (X, f) and (Y, g) be irreducible Smale spaces and $\pi : (X, f) \to (Y, g)$ be an almost one to one map, that is, there exists point $y \in Y$ such that $\sharp\pi^{-1}(y) = 1$. In [1], the author proves such map π can be decomposed as a composition of two resolving maps. In fact, he shows there exist irreducible Smale spaces $(\tilde{X}, \tilde{f})$, $(\tilde{Y}, \tilde{g})$ and factor maps α, β, $\tilde{\pi}$ such that the diagram below is commutative.

$$\begin{array}{ccc} (\tilde{X}, \tilde{f}) & \xrightarrow{\tilde{\pi}} & (\tilde{Y}, \tilde{g}) \\ \alpha \downarrow & & \downarrow \beta \\ (X, f) & \xrightarrow{\pi} & (Y, g) \end{array}$$

Moreover, maps α, β are u-bijective and map $\tilde{\pi}$ is s-bijective.

In the following, we show that all maps α, β, $\tilde{\pi}$ are almost one to one.
To prove, we mention all notations, definitions, etc from [1] that we will briefly mention some of them below.
i) $y_0 \in Y$ such that y_0 is periodic point, $\sharp\pi^{-1}(y_0) = 1$ and $x_0 \in X$ with $\pi(x_0) = y_0$. ii) $W = \cup_{n=0}^{\infty} V^u(f^n(x_0))$ and $\pi(W) = \cup_{n=0}^{\infty} V^u(g^n(y_0))$ that $V^u(t)$ denotes the unstable equivalence class of point t in the related space. We note that these unions are finite, because x_0 and y_0 are periodic points. iii)Let y_1, y_2 be in $\pi(W)$ with y_2 in $V^s(y_1, \varepsilon_Y/2)$. A compatibility map from y_1 to y_2 is a map $\upsilon : \pi^{-1}(y_1) \to \pi^{-1}(y_2)$ such that $\nu(x) \in V^s(x, \varepsilon_X/2)$ for all x in $\pi^{-1}(y_1)$. iv)Two points y_1, y_2 in $\pi(W)$ are compatible, if y_2 in $V^s(y_1, \varepsilon_Y/2)$ and there are compatibility map from y_1 to y_2 and from y_2 to y_1 v)Metrics δ_X (δ_X^0, δ_X^k) and δ_Y (δ_Y^0, δ_Y^k) defined on W and $\pi(W)$, respectively. Spaces $\tilde{X}$, $\tilde{Y}$ that are the completion of W and $\pi(W)$ with respect to δ_X and δ_Y, respectively and $0 < r < (1 - \lambda)$ in definitions of δ_X and δ_Y.

Definition 2.1. [1] For each positive integer k and $y_1, y_2 \in \pi(W)$,

$$\delta_Y^k(y_1, y_2) = r^k \delta_Y^0(g^{-k}(y_1), g^{-k}(y_2)) \quad \textit{and} \quad \delta_Y(y_1, y_2) = \textstyle\sum_{k=0}^{\infty} \delta_Y^k(y_1, y_2).$$

Lemma 2.2. *[1] 1) For all $y_1, y_2 \in Y$, $\delta_Y^0(y_1, y_2) \geq d_Y(y_1, y_2)$.*
2)If $y_2 \in V^u(y_1, \varepsilon_Y/2)$, then $\delta_Y^0(y_1, y_2) = d_Y(y_1, y_2)$.
3) If $y_2 \in V^u(y_1, \varepsilon_Y/2)$ and y_1 and y_2 are ρ-compatible, then $\delta_Y^0(y_1, y_2) = d_Y(y_1, y_2)$.
4) If $y_2 \in V^u(y_1, \varepsilon_Y/2)$, then $\delta_Y(y_1, y_2) \leq (1 - r\lambda)^{-1} d_Y(y_1, y_2)$.

Lemma 2.3. *[1] Let $x_1, x_2 \in W$. 1) If $x_2 \in V^u(x_1, \rho_X)$, then $\delta_X^0(x_1, x_2) = d_X(x_1, x_2)$.*
2) If $x_2 \in V^s(x_1, \rho_X)$ and $\pi(x_1)$ and $\pi(x_2)$ are ρ-compatible, then $\delta_X^0(x_1, x_2) = d_X(x_1, x_2)$.
3) If $x_2 \in V^u(x_1, \rho_X)$, then $\delta_X(x_1, x_2) \leq (1 - r\lambda)^{-1} d_X(x_1, x_2)$.

Lemma 2.4. *[1] If y_n is a sequence in $\pi(W) \cap V^s(y_0, \varepsilon_y/2)$ converging to y_0 with respect to d_Y, then there is $N \geq 1$ such that, for all $n \geq N$, y_n and y_0 are ρ-compatible.*

The proof of above lemma shows that y_0 can be replaced by any point $y \in \pi(W)$ with $\pi^{-1}(y) = 1$ while we have still its result. Also we can obtain more result with these conditions. In other words, we have:

Lemma 2.5. *If $y \in \pi(W)$ with $\pi^{-1}(y) = 1$ and y_n is a sequence in $\pi(W) \cap V^s(y, \varepsilon_y/2)$ converging to y with respect to d_Y, than y_n convers to y with respect to δ_Y.*

Proof. For a given $0 < \varepsilon < \varepsilon_{Y/2}$, we find $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, $\delta_Y(y_n, y) \leq \varepsilon$. Suppose D is the diameter of metric space (Y, d_Y). Since $0 < r < 1$, therefore there is $K_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=K_\varepsilon}^{\infty} r^k \leq \frac{\varepsilon}{4} D. \tag{1}$$

For $k = 0, ..., K_\varepsilon$, there is $0 < \delta \leq \varepsilon$ such that for all $y_1, y_2 \in Y$:

$$\begin{aligned} &d_Y(y_1, y_2) < \delta \Rightarrow \\ &\quad g^{-k}[y_1, y_2] = [g^{-k}(y_1), g^{-k}(y_2)] \text{ and } d_Y(g^{-k}(y_1), g^{-k}(y_2)) \leq \frac{\varepsilon}{4(K_\varepsilon + 1)}. \end{aligned} \tag{2}$$

Let $N' \in \mathbb{N}$ such that for all $n \geq N'$, $d_Y(y_n, y) \leq \delta$. Thererfore (2) implies for all $k = 0, ..., K_\varepsilon$ and $n \geq N'$:

$$d_Y(g^{-k}(y_n), g^{-k}(y)) \leq \frac{\varepsilon}{4(K_\varepsilon + 1)} \tag{3}$$

and

$$g^{-k}(y_n) = g^{-k}[y, y_n] = [g^{-k}(y_1), g^{-k}(y_2)]. \tag{4}$$

The first statement follows from $y_n \in V^s(y, \varepsilon_Y/2)$. But (4) means for all $k = 0, ..., K_\varepsilon$ and $n \geq N'$: $g^{-k}(y_n) \in V^s(g^{-k}(y), \varepsilon_Y/2)$. Applying lemma (2.4) for each $k = 0, ..., K_\varepsilon$, we get $N_k \geq N'$ such that for all $n \geq N_k$, $g^{-k}(y_n)$ and $g^{-1}(y)$ are ρ-compatible. if $N_\varepsilon = \max\{N_k \ : \ k = 0, ..., K_\varepsilon\}$, then for all $k = 0, ..., K_\varepsilon$ and $n \geq N_\varepsilon$, $g^{-k}(y_n) \in V^s(g^{-k}(y), \varepsilon_Y/2)$ and $g^{-k}(y_n)$ and $g^{-k}(y)$ are ρ-compatible. Therefore

$$\delta_Y^k(y_n, y) = r^k \delta_Y^0(g^{-k}(y_n), g^{-k}(y)) = r^k \ d_Y(g^{-k}(y_n), g^{-k}(y))$$

for all $k = 0, ..., K_\varepsilon$ and $n \geq N_\varepsilon$ by the third part of lemma (2.2).

Finally (1), (3) and the definition of δ_Y imply for all $n \geq N_\varepsilon$

$$\begin{aligned} \delta_Y(y_n, y) &= \sum_{k=0}^{\infty} \delta_Y^k(y_n, y) = \sum_{k=0}^{K_\varepsilon} \delta_Y^k(y_n, y) + \sum_{k=K_\varepsilon+1}^{\infty} \delta_Y^k(y_n, y) \\ &= \sum_{k=0}^{K_\varepsilon} r^k \delta_Y^0(g^{-1}(y_n), g^{-1}(y)) + \frac{\varepsilon}{4} \leq \sum_{k=0}^{K_\varepsilon} \frac{\varepsilon}{4(K_\varepsilon + 1)} + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

□

Corollary 2.6. *If $y \in Y$ with $\pi^{-1}(y) = x$, x_n is a sequence in $W \cap V^s(x, \rho_X)$ converging to x with respect to d_X, than there is $N \geq 1$ such that, for all $n \geq N$, $\pi(x_n)$ and $\pi(x)$ are ρ-compatible.*

Corollary 2.7. *If $y \in \pi(W)$ with $\pi^{-1}(y) = x$, x_n is a sequence in $W \cap V^s(x, \rho_X)$ converging to x with respect to d_X, than x_n convers to x with respect to δ_X.*

Lemma 2.8. *If $y \in \pi(W)$ with $\pi^{-1}(y) = 1$ and $\tilde{y} \in \tilde{Y}$ with $\beta(\tilde{y}) = y$, then $\tilde{y} = y$.*

Proof. We claim that for any $\varepsilon > 0$, $\delta_Y(\tilde{y}, y) < \varepsilon$ and this implies $\tilde{y} = y$.
For simpility, we let $\xrightarrow{\delta_Y}$ and $\xrightarrow{d_Y}$ denote convergence in $\tilde{X}$ and X with respect to δ_Y and d_Y, respectivly. Since $\tilde{y} \in \tilde{X}$ and $\tilde{X}$ is the completion of $\pi(W)$ with respect to δ_Y, hence we can choose $y_n \in \pi(W)$ such that $y_n \xrightarrow{\delta_Y} \tilde{y}$. Since $[\tilde{y}, y_n] \xrightarrow{\delta_Y} \tilde{y}$ and $[\tilde{y}, y_n]$ and y_n have the same convergence point, therefore we can find $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$\delta_Y(\tilde{y}, [\tilde{y}, y_n]) \leq \varepsilon/6, \quad \delta_Y([\tilde{y}, y_n], y_n) \leq \varepsilon/6. \tag{5}$$

On the other hand, according to the definition of β, $y = \beta(\tilde{y}) = \beta(\lim y_n) = \lim y_n$, that is, $y_n \xrightarrow{d_Y} y$ that this implies $[y, y_n] \xrightarrow{d_Y} y$. Therefore by lemma 2.5, $[y, y_n] \xrightarrow{\delta_Y} y$. Let $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\delta_Y([y, y_n], y) \leq \varepsilon/6 \tag{6}$$

Also since both sequences $\{y_n\}$ and $\{[y, y_n]\}$ have the same convergence point in the space (Y, g), we can choose N_3 such that for all $n \geq N_3$, $d_Y(y_n, ([y, y_n]) \leq ((1 - r\lambda)\varepsilon)/6$ and $y_n \in V^u([y, y_n], \varepsilon_Y/2)$. Therefore by the fourth part of lemma 2.2 for all $n \geq N_3$,

$$\delta_Y(y_n, ([y, y_n]) \leq (1 - r\lambda)^{-1} d_Y(y_n, ([y, y_n]) \leq \varepsilon/6. \tag{7}$$

Putting all (5), (6), (7), we see that for all $n \geq N = \max\{N_1, N_2, N_3\}$,

$$\begin{aligned}
&\delta_Y(\tilde{y}, y) \leq \\
&\quad \delta_Y(\tilde{y}, [\tilde{y}, y_n]) + \delta_Y([\tilde{y}, y_n], y_n) + \delta_Y(y_n, [y, y_n]) + \delta_Y([y, y_n], y) \\
&\quad \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \varepsilon.
\end{aligned}$$

□

By the similar proof and corollaries 2.6 and 2.9, we can get the same result about α.

Lemma 2.9. *If $y \in \pi(W)$ with $\pi^{-1}(y) = x$, $\tilde{x} \in \tilde{X}$ with $\alpha(\tilde{x}) = x$, then $\tilde{x} = x$.*

So far, our results shows that if $y \in Y$ with $\pi^{-1}(y) = x$, then $\sharp\beta^{-1}(y) = \sharp\alpha^{-1}(\pi^{-1}(y)) = 1$. But these imply that $\sharp\tilde{\pi}^{-1}(\beta^{-1}(y)) = 1$, because if $\tilde{\pi}(\tilde{x}) = \tilde{\pi}(\tilde{x'}) = \beta^{-1}(y)$, then

$$\begin{aligned}
\beta\circ\tilde{\pi}(\tilde{x}) = \beta \circ \tilde{\pi}(\tilde{x'}) = y \quad &\Rightarrow \quad \pi \circ \alpha(\tilde{x}) = \pi \circ \alpha(\tilde{x'}) \Rightarrow \\
\tilde{x},\ \tilde{x'} \in \alpha^{-1} \circ \pi^{-1}(y) = \alpha^{-1}(x) \quad &\Rightarrow \quad \tilde{x} = \tilde{x'}.
\end{aligned}$$

Finally these computations show that maps $\pi \circ \alpha$ and $\beta \circ \tilde{\pi}$ are almost one to one, too.

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Application of coupled fixed point theorems in partially ordered sets to a boundary value problem of fractional order

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Abstract

In this paper, we discuss the existence and uniqueness of solutions for nonlinear boundary value problem of differential equations of fractional order. Our analysis relies on the coupled fixed point theorems in partially ordered metric spaces.

Keywords: Partially coupled fixed point, Caputo fractional differential equation, Existence, Uniqueness.
Mathematics Subject Classification [2010]: 26A33, 34A08, 34A12.

1 Introduction

Fractional differential equations have attracted huge attention in the past few years because of their unique physical properties and their potential in the modeling of many physical phenomena and also in various field of science and engineering [3, 2, 5]. During last years, the study of such kind of problems have received much attention from both theoretical and applied point of view. We will mention the following recent works on this topic [3, 2, 5, 6, 7].

In this paper, we study the following boundary value problems (BVP) for fractional differential equations involving the Caputo derivative

$$\begin{cases} (D_*^{\alpha} y)(x) = f(x, y(x)), \quad (0 < \alpha < 1, x \in [0, T]), \\ ay(0) + by(T) = c, \end{cases} \tag{1}$$

where D_*^{α} is the Caputo fractional derivative of order α, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later and $a, b, c \in \mathbb{R}$ are real constants with $a + b \neq 0$.

The existence of solutions for this kind of BVP has been studied by several authors, see [2, 4]. We will present the new existence and uniqueness results for the fractional BVP (1) using partially couple fixed point theorems. The advantage and importance of this method arises from the fact that it is a constructive method that yields monotone sequences that converge to the unique solution of BVP (1).

*Speaker

2 Preliminaries

In this section we state some definitions and lemmas which are essential in our analysis.

Definition 2.1. ([2]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : [0, T] \to \mathbb{R}$ is defined by

$$I^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t) dt,$$

provided the right-hand side is defined for $x \in [0, T]$.

Definition 2.2. ([2]). The Caputo fractional derivative $D_*^{\alpha} y$ of order $0 < \alpha \leq 1$ is defined by

$$D_*^{\alpha} y(x) = I^{1-\alpha} y'(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} y'(t) dt,$$

provided the right-hand side is defined for $x \in [0, T]$.

Now, we present the fixed point theorems which will be used later.

Bhaskar and Lakshmikantham [1] introduced the following notions of mixed monotone mappings and coupled fixed points.

Definition 2.3. ([1]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y.

Definition 2.4. ([1]). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \quad \text{and} \quad F(y, x) = y.$$

The main results of Bhaskar and Lakshmikantham in [1] are the following two coupled fixed point theorems.

Theorem 2.5. ([1]). *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \text{ for each } x \succeq u \text{ and } y \preceq v.$$

Suppose either F is continuous or X has the following property:
(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*
If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

then F has a coupled fixed point in X.

We define the following partial order on the product space $X \times X$:

$$(x, y), (\tilde{x}, \tilde{y}) \in X \times X, \quad (x, y) \preceq (\tilde{x}, \tilde{y}) \Leftrightarrow x \preceq \tilde{x}, \ \tilde{y} \preceq y.$$

Theorem 2.6. ([1]). *In addition to the hypothesis of Theorem 2.5, suppose that for every $(x, y), (z, t) \in X \times X$, there exists an element $(u, v) \in X \times X$ that is comparable to (x, y) and (z, t), then F has a unique coupled fixed point.*

Theorem 2.7. ([1]). *In addition to the hypothesis of Theorem 2.5, suppose that every pair of elements of X has an upper bound or a lower bound in X. Then $x = y$.*

3 Existence and Uniqueness

In this section, we will prove our main results.

Lemma 3.1. ([2]). *The function $y \in C[0, T]$ is a solution of the BVP (1) if and only if it is a solution of the nonlinear mixed Fredholm-Volterra integral equation*

$$y(t) = \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, y(\tau))}{(t-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \frac{b}{a+b} \int_0^T \frac{f(\tau, y(\tau))}{(T-\tau)^{1-\alpha}} d\tau. \tag{2}$$

Definition 3.2. An element $(\underline{y}, \overline{y}) \in C[0, T] \times C[0, T]$ is called a coupled lower and upper solution of (1) if

$$\underline{y}(t) \leq \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, \underline{y}(\tau))}{(t-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \frac{b}{a+b} \int_0^T \frac{f(\tau, \overline{y}(\tau))}{(T-\tau)^{1-\alpha}} d\tau,$$

and

$$\overline{y}(t) \geq \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, \overline{y}(\tau))}{(t-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \frac{b}{a+b} \int_0^T \frac{f(\tau, \underline{y}(\tau))}{(T-\tau)^{1-\alpha}} d\tau,$$

for all $t \in [0, T]$.

Theorem 3.3. *Assume that $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(t, y(t)) \in C[0, T]$ for every $y \in C[0, T]$ and there exists $\lambda > 0$ such that $0 \leq f(t, y(t)) - f(t, \tilde{y}(t)) \leq \lambda(y(t) - \tilde{y}(t))$ for all $y(t) \geq \tilde{y}(t)$ on $[0, T]$. Then the existence of a coupled lower and upper solution of (1) provides the existence unique solution of (1).*

Proof. We choose $0 < \delta < T$ such that the inequality $\frac{\lambda \delta^\alpha}{\Gamma(\alpha+1)} < \frac{1}{2}$ holds. Now we define $F : C[0, \delta] \times C[0, \delta] \to C[0, \delta]$ by

$$F(x, y)(t) = \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} d\tau - \frac{1}{\Gamma(\alpha)} \frac{b}{a+b} \int_0^T \frac{f(\tau, y(\tau))}{(T-\tau)^{1-\alpha}} d\tau.$$

First, we know from [2, Theorem 6.1 in §6.1] if $(x, y) \in C[0, \delta] \times C[0, \delta]$, then $F(x, y) \in C[0, \delta]$. We now show that all the conditions of Theorem 2.5 are satisfied. For every $x, \tilde{x} \in C[0, \delta]$, we define $x \preceq \tilde{x}$ if and only if $x(t) \leq \tilde{x}(t)$ for all $t \in [0, \delta]$. Furthermore, we now $(C[0, \delta], d)$ is a complete metric space with $d(x, \tilde{x}) = \|x - \tilde{x}\| = \sup_{t \in [0, \delta]} |x(t) - \tilde{x}(t)|$. It

is easy to see that F is mixed monotone map. Now, for $x, y, \tilde{x}, \tilde{y} \in C[0, \delta]$ with $x \succeq \tilde{x}, y \preceq \tilde{y}$, we have

$$\begin{aligned} \|F(x,y) - F(\tilde{x},\tilde{y})\| &= \Big\| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, x(\tau)) - f(\tau, \tilde{x}(\tau))}{(t-\tau)^{1-\alpha}} d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)} \frac{b}{a+b} \int_0^T \frac{f(\tau, \tilde{y}(\tau)) - f(\tau, y(\tau))}{(T-\tau)^{1-\alpha}} d\tau. \Big\| \\ &\leq \frac{\lambda \delta^\alpha}{\Gamma(\alpha+1)} \|x - \tilde{x}\| + \frac{\lambda \delta^\alpha}{\Gamma(\alpha+1)} \frac{|b|}{|a+b|} \|y - \tilde{y}\| \\ &\leq \frac{\lambda \delta^\alpha}{\Gamma(\alpha+1)} (\|x - \tilde{x}\| + \|y - \tilde{y}\|) \end{aligned}$$

This proves that F satisfies the corresponding hypothesis in Theorem 2.5. The uniqueness of solution comes from the application of Theorems 2.6 and 2.7. □

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Application of Chebyshev Collocation Method for Numerical Solution of Volterra-Fredholm Equations

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Abstract

In this article, the chebyshev collocation method is presented for the solutions of Volterra-Fredholm integral equations. This method is based on approximating unknown function with shifted Chebyshev polynomials. The method is using a simple computations manner to obtain a quite acceptable approximate solution. We also get an upper bound for the error of this algorithm. Finally, one example is presented to show the applicability of our method with compare to the four well known algorithms in the literature

Keywords: shifted Chebyshev polynomials, Volterra-Fredholm integral equations, Numerical method
Mathematics Subject Classification [2010]: 45GXX, 65M12, 45BXX, 45DXX, 65RXX

1 Introduction

In recent years, many different basic functions have used to estimate the solution of linear and nonlinear Volterrae-Fredholm integral equations. Our aim in this article is to propose a method to approximate solution of a class of Volterra-Fredholm integral equations on the interval $[0, 1]$ by using the shifted Chebyshev polynomials. The problems under consideration are nonlinear Volterra-Fredholm integral equations defined as follows:

$$\begin{aligned}\sum_{j=0}^{m}(A_j(x)\,y(x)+B_j(x)\,y(h(x))) =& f(x)\\ &+\lambda_1\int_0^x k_1(x,\,t)\,y(t)\,dt\\ &+\lambda_2\int_0^1 k_2(x,\,t)\,y(h(t))\,dt,\end{aligned}\tag{1}$$

where$k_1(x,\,t)$ and $k_2(x,\,t)$ are known kernel functions on the interval $[0,\,1]\times\,[0,\,1]$ also $u(x)$ and $f(x)$ are known functions defined on the interval $[0,\,1]$ and $0\le h(x)<\infty$. $y(x)$. is unknown function and $\lambda_1,\ \lambda_2$ are real constants such that ${\lambda_1}^2+{\lambda_2}^2\neq 0$.. When $h(x)$ is

*Speaker

a first-order polynomial, Eq. (1) is a functional integral equation with proportional delay. Recently, numerical methods such as Taylor collocation method [1], Taylor polynomial method [2], Lagrange collocation method [3] and Legendre collocation method [4] for solving differential, integral and integro differential equations have attracted much attention. In this article, we will apply an elegant way named the shifted Chebyshev collocation method to approximate solution of this equation. Our presented method is based on the discretization of the integral equations (1) and then using the shifted Chebyshev collocation. Also an upper bound in order to show the error incurred by using this method will be given. The obtained upper bound confirmed the convergence of this algorithm. Finally, we apply this method to an example to show the e fficiency of this algorithm.

2 The Chebyshev approximation and Basic concepts

In this section, we will introduce the shifted Chebyshev collocation method and explain some required Definitions, Theorems and Lemmas.

Definition 2.1 (5)**.** The Chebyshev polynomials of the first kind with orthogonality property in interval $[-1,\ 1]$ are defined as follows

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n = 1,\ 2,\ 3,\ \ldots, \tag{2}$$

where, $T_0(x) = 1$ and $T_1(x) = x$.

The orthogonality property of this polynomials is given as

$$\int_{-1}^{1} T_n(x)\, T_m(x) w(x)\, dx = \begin{cases} 0 & if\ n \neq m, \\ \pi & if\ n = m = 0, \\ \frac{\pi}{2} & if\ n = m \neq 0, \end{cases}$$

where, $w(x) = \frac{1}{\sqrt{1-x^2}}$ is the weight function.

In order to express the whole analysis in the interval $[0,\ 1]$, we first replace x by $2x - 1$ in the Chebyshev polynomial recurrence relation as defined in Eq. (2) and then, the so-called shifted Chebyshev polynomials of degree n will be defined as

$$C_{n+1}(x) = T_{n+1}(2x - 1), \qquad 0 \leq x \leq 1,$$

where, $C_n(x)$ are the shifted Chebyshev orthonormal polynomials in the interval [0, 1]. The orthogonality property of shifted Chebyshev polynomials is given by

$$\int_{0}^{1} C_i(x)\, C_j(x)\, w(x) = \begin{cases} 0 & if\ i \neq j, \\ 1 & if\ i = j. \end{cases}$$

A function $y(x) \in L^2[0,\ 1]$, may be expressed in terms of shifted Chebyshev polynomials as

$$y(x) = \lim_{i \to \infty} \sum_{i=0}^{\infty} a_i C_i(x), \tag{3}$$

where, $a_i = \langle y,\ C_i \rangle_w$ and $\langle\ ,\ \rangle_w$ is the standard inner product on $L^2[0,\ 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-(2x-1)^2}}$. $C_j(x)$ denotes the shifted Chebyshev polynomials of the first kind, a_i are unknown shifted Chebyshev coefficients and n is any chosen positive integer.

3 Description of the method

In this section, we approximate solution of this equation by using the shifted Chebyshev polynomials. Besides the convergence analysis of the presented method will also be discussed. To approximate $y(x)$ in Eq. (1), substituting $y(h(x))$ instead of $h(x)$ in this equation, the integral equation (1) can be written as

$$\begin{aligned}\sum_{j=0}^{m}(A_j(x)\sum_{i=0}^{n}a_iC_i(x)+B_j(x)\sum_{i=0}^{n}a_iC_i(h(x)))=f(x)\\ +\lambda_1\int_0^x k_1(x,\ t)(\sum_{i=0}^{n}a_iC_i(x))dt\\ +\lambda_2\int_0^1 k_1(x,\ t)(\sum_{i=0}^{n}a_iC_i(h(x)))dt. \qquad (4)\end{aligned}$$

Replacing x with $n+1$ roots of the shifted Chebyshev polynomial $C_{n+1}(x)$, the unknown coefficients a_i can be computed.

Theorem 3.1. *Suppose that $H^k(0,1)$ is a Sobolev space and $y(x)\in H^k(0,1)$. Let $y_n(x)=\sum_{i=0}^{n}a_iC_i(x)$ be the best approximation polynomial of $y(x)$ in L^2_w-norm and $y'_n(x)=\sum_{i=0}^{n}a'_iC_i(x)$ be an approximate solution obtained by the proposed method, then one obtaion*

$$\|y(x)-y'_n(x)\|\le r_1n^{-k}\|y(x)\|+r_2(n+1)^{\frac{1}{2}}n^{-(k-1)}, \qquad (5)$$

where r_1 and r_2 are positive constants. Also r_2 is independent of n and r_1 is dependent on the selected norm and independent from $y(x)$ and n.

Example 3.2. Consider the Volterra-Fredholm integral equation

$$x^2y(x)+e^xy(2x)=f(x)+\int_0^{2x}e^{x+t}y(t)dt-\int_0^1 e^{x-2t}y(2t)\,dt,$$

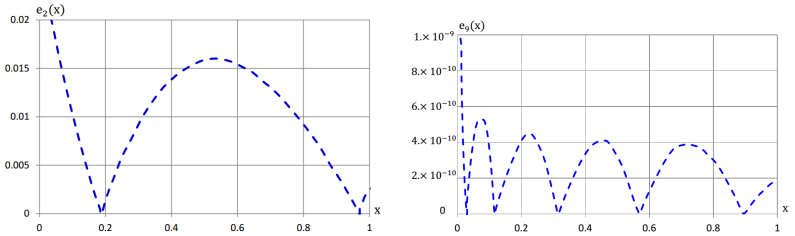
where

$$f(x)=-\frac{e^x}{4}-\frac{e^{-2+x}}{4}cos2+\frac{e^{3x}}{2}cos2x-\frac{e^{-2+x}}{4}sin2+x^2sinx+e^xsin2x-\frac{e^{3x}}{2}sin2x.$$

The exact solution is $y(x)=sin\ x$. Numerical results for this example are displayed in Tables 2 and Figure 3. In Table 2, numerical results for value of ρ_n in methods 1, 2, 3 and 4 with $n=2,\ 5,\ 8$ and 9 have been recorded.

Table 2: Comparison of errors for Example 2.

n	Method 1	Method 2	Method 3	Method 4	Method 5
2	1.23×10^{-2}	1.46×10^{-2}	7.87×10^{-2}	3.41×10^{-2}	7.87×10^{-2}
5	2.64×10^{-5}	2.93×10^{-5}	6.23×10^{-5}	3.68×10^{-4}	6.23×10^{-5}
8	1.78×10^{-8}	3.94×10^{-8}	1.89×10^{-8}	1.24×10^{-5}	1.77×10^{-7}
9	1.32×10^{-9}	2.29×10^{-9}	2.35×10^{-8}	3.46×10^{-7}	7.21×10^{-6}



(a) Plot of $e_2(x)$ (b) Plot of $e_4(x)$

Figure 1: Plot of $e_n(x)$ with $n = 2$ and 9 for Example 3.2.

4 Conclusion

In this paper, The shifted Chebyshev collocation method as a reliable algorithm was presented for solved a special class of Volterra-Fredholm integral equations. In this method we approximate unknown function with shifted Chebyshev polynomials. It was found that the proposed method had very satisfactory stability properties, as n increases, the error reduces initially and then finally stabilizes. The comparison results with other well known methods showed that the presented method was a powerful tool for finding numerical solutions of such equations. The convergence analysis of this algorithm was discussed and numerical examples supported our claims.

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Bifurcation analysis of a heavy water reactor system
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Bifurcation analysis of a heavy water reactor system

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Abstract

In this paper a heavy water reactor system has been studied by using dynamical techniques and numerical continuation algorithms. First, we obtain equilibrium point of the system and then we determine codim 1 and codim 2 bifurcation points and their corresponding critical normal form coefficients by using numerical techniques implemented in Matlab software Matcont. We also compute several bifurcation curves including a curve of hopf points and limit cycles.

Keywords: bifurcations of equilibrium points, limit cycles, normal form
Mathematics Subject Classification [2010]: 34C23, 34K18

1 Introduction

We consider a heavy water reactor system, which is given by a planer system of differential equation [**?**]

$$Y : \begin{cases} \dfrac{dx_1}{dt} = -x_1(t) + x_2(t) + a_v x_4(t) + a_f x_3(t)x_1(t) + a_v x_4(t)x_1(t) \\ \dfrac{dx_2}{dt} = b(x_1(t) - x_2(t)) \\ \dfrac{dx_3}{dt} = p(x_1(t) - x_3(t)) \\ \dfrac{dx_4}{dt} = qx_3(t) \end{cases} \tag{1}$$

where x_1, x_2, x_3 and x_4 variable are the fluctuations in neutron density, delayed neutron precursor density, average fuel temperature and coolant void fraction respectively around their steady state values. Non-dimensional time in the model is represented by t. The a_f and a_v are linearly proportional to the fuel temperature coefficient of reactivity and void coefficient reactivity respectively while b, p and q represents the combination of neutron generation time, delayed neutron fraction, heat capacities, coolant temperature. We study this system based on the theory of bifurcations and computing bifurcation diagram and computing curve of different codim 1 bifurcations. We show that for some parameters an equilibrium point undergoes to a limit point bifurcation and a hopf bifurcation. We study complicated behaviors of the system such as codim 1 bifurcation points and curve limit cycles by MATCONT[**?**].

*Speaker

Definition 1.1. Consider a general autonomous vector field

$$x^{'} = f(x, \alpha) \tag{2}$$

$x \in \Re^n$ and $\alpha \in \Re^m$ An equilibrium solution of (**??**) is a point $x^{'} \in \Re^n$ such that $f(x^{'}) = 0$. The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation. The Hopf bifurcation occur when the pair of conjugate eigenvalues crosses the imaginary axis transversally. The Hopf point is the point at which the system losses its stability via Hopf bifurcation.[**?**]

2 Main results

To reveal more complicated behaviors of the system we use numerical approach based on numerical continuation method by fixing all the parameters, except one which is called bifurcation parameter. First we fix the parameters $p = 0.0522$, $q = 0.039$, $v = -218.83$, $b = 0.0055$ and consider $f = -7.211$ as bifurcation parameter and consider the equilibrium $E_0 = (0, 0, 0, 0)$. Now we use the software MATCONT [**?**], to compute a curve of equilibria by starting from E_0. This curve along with two codim 1 bifurcations are depicted in Fig.1. The numerical results are given as:

```
label = H , x = ( 0.000000 0.000000 0.000000 0.000000 -7.211729 )
First Lyapunov coefficient = 3.420757e-007
```

The first lyapunov coefficient at the hopf point is positive which indicates that the hopf point is supercritical. This means that the limit cycles bifurcating from the hopf point is stable.
In Fig.2 two orbits of the system are depicted. One convergent to E_0 (red color) and other one (blue color) diverge from E_0.
We choose the hopf point as initial point and consider f and b as free parameters and compute a curve of hopf point is presented in Fig.3.

```
label = GH, x = ( 0.000000 0.000000 -0.000000 0.000000 -7.211701 0.005498 0.423149 )
l2=1.130002e-001
label = ZH, x = ( -0.000000 0.000000 -0.000000 0.000000 -7.110994 0.000000 0.423394 )
Degenerate Zero-Hopf
Zero-Neutral Saddle
label = HH, x = ( 0.000000 0.000000 0.000000 -0.000000 -8.110977 -1.052200 0.926567 )
```

We detected some codim 2 bifurcations on the hopf curve along with computation of their normal forms. A generalized hopf bifurcation and a zero hopf bifurcation and hopf-hopf bifurcation observed in the hopf curve. The generalized hopf point represents the transition point from subcritical to supercritical hopf bifurcation.
We also compute a curve of limit cycle by straining for the hopf point. The numerical results are given as fallows and the curve of limit cycle is depicted in Fig.4.

```
Limit point cycle (period = 9.659212e+000, parameter = -7.211729e+000)
Normal form coefficient = 4.274562e-005
Neimark-Sacker (period = 1.073440e+001, parameter = -7.370698e+000)
```

We detected a Neimark-Sacker bifurcation along the curve of limit cycles. This bifurcation occurs when one pair of complex multiplier crosses the unit circle.

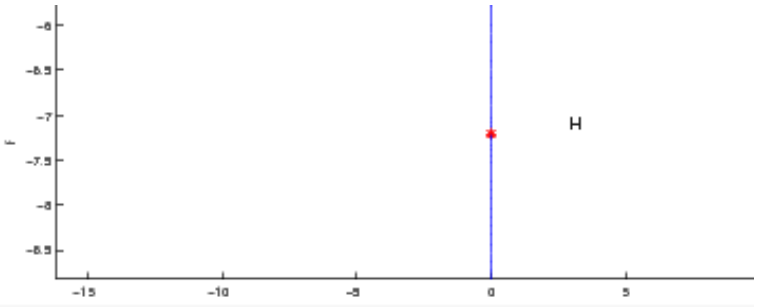

Figure 1: A equilibrium curve

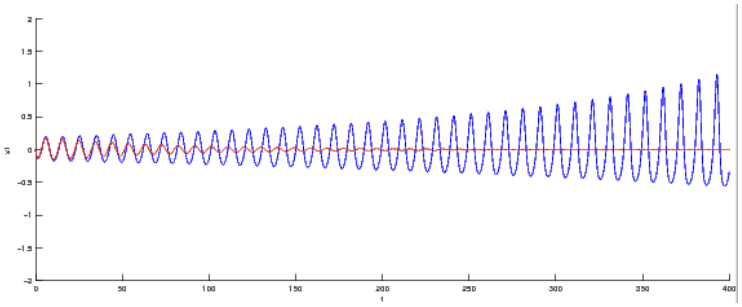

Figure 2: orbit curve

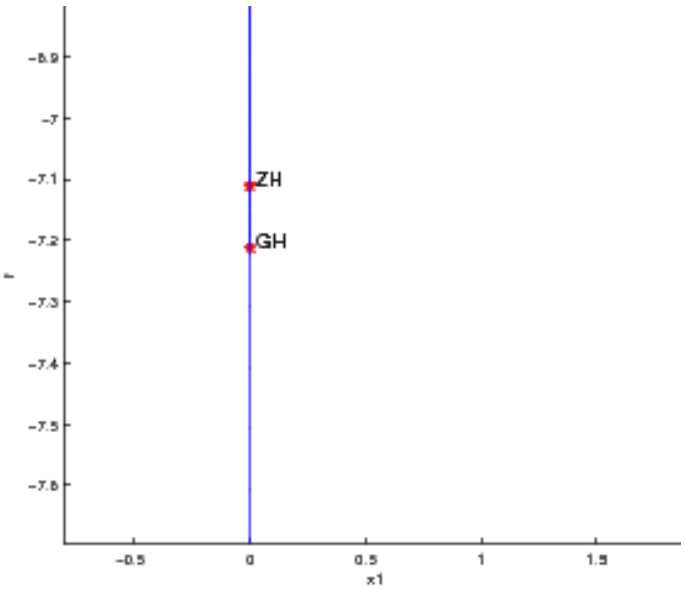

Figure 3: A Hopf curve

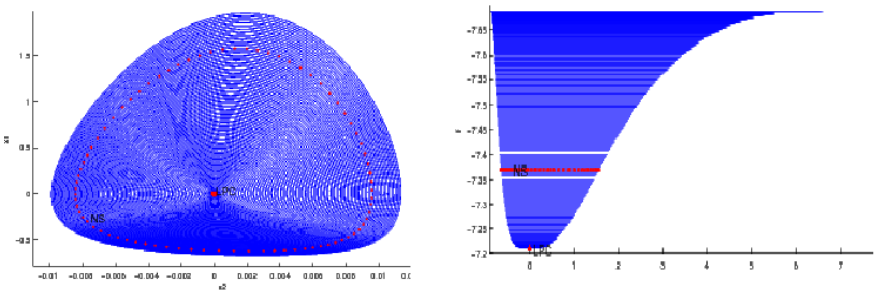

Figure 4: A limit cycle

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Exact traveling wave solution of the Zoomeron equation by (F/G) -expansion method

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Abstract

In this work, the new (F/G)-expansion method is proposed for obtaining traveling wave solutions of non linear evolution equations. This method is more powerful than the method $(G^{'}/G)$-expansion method. The efficiency of the method is demonstrated on a variety of nonlinear PDEs such as, Zoomeron equation. As a result, more traveling wave solutions are obtained including not only all the known solutions but also the computation burden is greatly decreased compared with the existing method. Abundant exact traveling wave solutions of these equations are expressed by the hyperbolic functions the trigonometric functions.

Keywords: (F/G)-expansion method, Traveling wave solutions,Zoomeron equation, Exact solutions.
Mathematics Subject Classification [2010]: 35A09, 35A25,

1 Introduction

Nonlinear partial differential equations $(NLPDEs)$ have been widely applied in many branches of applied sciences such as fluids dynamics, bio-mechanics and chemical physics etc.The solutions of nonlinear equations play a crucial role in applied mathematics and physics, because; solutions of nonlinear partial differential equations provide a very significant contribution to people about the exact solutions of nonlinear evolution equations have been established and developed, such as the sub-ODE method [1],the homogeneous balance method [2] and so on.

Recently, Wang et al. [3] interoduced a new direct method called the $(G^{'}/G)$-expansion method. Motivated by work in [3], the main purpose of this paper is to introduce a new technique called (F/G)-expansion method is that the traveling wave solutions of a nonlinear evolution equation can be expressed by a polynomial in (F/G), where $G = G(\xi)$ and $F = F(\xi)$ satisfy the first order linear ordinary differential system $(FLODS)$ as follows: $F^{'}(\xi) = \lambda G(\xi)$, $G^{'}(\xi) = \mu F(\xi)$, where μ, λ are constants. This new method will play an important role in expressing the traveling wave solutions for Zoomeron equation.

*Speaker

2 Description of the (F/G)-expansion method

Suppose that a nonlinear equation is given by

$$p(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function and p is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (F/G)-expansion method.
Step 1. Combining the independent variables x and t into one variable $\xi = x - wt$, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x - wt, \tag{2}$$

where w is a nonzero constant. The traveling wave variable ξ permits us to reducing Eq. (1) to an ODE for $u = u(\xi)$,

$$p(u, u^{'}, u^{''}, u^{'''}, \ldots) = 0. \tag{3}$$

Step 2. Suppose that the solution of ODE Eq. (3) can be expressed by a polynomial in (F/G) as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (\frac{F}{G})^i, \tag{4}$$

where $G = G(\xi)$ and $F = F(\xi)$ satisfy the FLODS in the form

$$F^{'}(\xi) = \lambda G(\xi), \quad G^{'}(\xi) = \mu F(\xi). \tag{5}$$

$a_0, a_1, \cdots, a_m, \lambda$ and μ are constants to be determined later, $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE equation (3).
with the aid of (5), we can find the following solutions $F(\xi)$ and $G(\xi)$, which are listed as follows:

Case 1. If $\lambda > 0$ and $\mu > 0$, then equation (5) has the following hyperbolic function solutions:

$$\begin{cases} F(\xi) = C_1 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \dfrac{\sqrt{\lambda}}{\sqrt{\mu}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi), \\ G(\xi) = C_1 \dfrac{\sqrt{\mu}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi). \end{cases} \tag{6}$$

Case 2. If $\lambda < 0$ and $\mu < 0$, then (5) has the following hyperbolic function solutions:

$$\begin{cases} F(\xi) = C_1 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2 \dfrac{\sqrt{-\lambda}}{\sqrt{-\mu}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi), \\ G(\xi) = -C_1 \dfrac{\sqrt{-\mu}}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi). \end{cases} \tag{7}$$

Case 3. If $\lambda > 0$ and $\mu < 0$, then Eq. (5) has the following trigonometric function solutions:

$$\begin{cases} F(\xi) = C_1 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \dfrac{\sqrt{\lambda}}{\sqrt{-\mu}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi), \\ G(\xi) = -C_1 \dfrac{\sqrt{-\mu}}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi). \end{cases} \tag{8}$$

Case 4. If $\lambda < 0$ and $\mu > 0$, then Eq. (5) has the following trigonometric function solutions:

$$\begin{cases} F(\xi) = C_1 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2 \dfrac{\sqrt{-\lambda}}{\sqrt{\mu}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi), \\ G(\xi) = C_1 \dfrac{\sqrt{\mu}}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi). \end{cases} \tag{9}$$

Step 3. Substituting (4) and (5) into equation (3) separately yields a set of algebraic equations for $(F/G)^i (i = 1, 2, \ldots, m)$. Setting the coefficients of $(F/G)^i$ to zero yields a set of nonlinear algebraic equations in $a_i (i = 0, 1, \ldots, m)$ and w. Solving the nonlinear algebraic equations by Maple and Mathematica, we obtain many exact solutions of equation (1) according to equation (2) to (9).

3 Application to the Zoomeron equation

In this section, we will apply the (F/G)-expansion method to construct the traveling solutions for Zoomeron equation

$$(\frac{u_{xy}}{u})_{tt} - (\frac{u_{xy}}{u})_{xx} + 2(u^2)_{xt} = 0, \tag{10}$$

where $u(x, y, t)$ is the amplitude of the relevant wave mode. We know that this equation was introduced by Calogero and Degasperis [4]. The travelig wave variable below,

$$u(x, y, t) = u(\xi), \quad \xi = x - cy - wt, \tag{11}$$

permits us to convert (10) into an ODE for $u(x, y, t) = u(\xi)$ in the form

$$c(1 - w^2)u'' - 2wu^3 - Ru = 0, \tag{12}$$

where R is a constant of integration and $w \neq \{0, 1\}$.
By balancing between u'' with u^3 in equation (12),we get $m = 1$. Consequently, we get

$$u(\xi) = a_0 + a_1(\frac{F}{G}) \qquad a_1 \neq 0, \tag{13}$$

where a_0, a_1 are constants to be determined later.

By substituting (13) into (12), collecting the coefficients of each power of $(\frac{F}{G})$, and solve the system of algebraic equations using Maple, we obtain the set of solution:

$$a_0 = 0, \quad a_1 = \sqrt{\frac{c(1 - w^2)}{w}}, \quad R = 2cw^2\mu\lambda - 2c\lambda\mu. \tag{14}$$

Substituting (14) into (15) we can obtain four types of traveling wave solutions of the Zoomeron equation (10) as follows:
When $\lambda > 0$ and $\mu > 0$, we obtain the hyperbolic function solutions

$$u_1(x,y,t) = \sqrt{\frac{c(1-w^2)}{w}} \left(\frac{C_1 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{\mu}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi)} \right), \qquad (15)$$

where $\xi = x - cy - wt$, C_1 and C_2 are arbitrary constants.
Other cases of λ and μ are calculate based on equations (7) to (9).

4 conclusion

In this paper, (F/G) -expansion method is used to obtain more general exact solution of the Zoomeron equation. The advantages of the (F/G) -expansion method is that it is possible to obtain mor travelling wave solutions with distinct physical structures. Form our results, some results previousl known as traveling wave solutions and soliton-like solutions can be recovered. Moreover, the proposed method is capable of greatly can be minimizing the size of computational work compared to the existing technique. Finally, it is worth to mention that the implementation of this proposed method is very simple and straightforward.

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Existence and uniqueness of positive solutions for a class of singular nonlinear fractional differential equations with integral boundary value conditions

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Abstract

In this paper, we prove the existence and uniqueness of positive solutions for the following singular nonlinear fractional boundary value problem

$$^cD^\alpha u(t) + f(t, u(t), u(t), u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u''(0) = u'''(0) = 0, \qquad u'(0) + u'(1) = \lambda \int_0^1 u(s)ds,$$

where $3 < \alpha \leq 4$, $0 < \lambda < 4$, $^cD^\alpha$ is the Caputo fractional derivative and $f : (0, 1] \times [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, $\lim_{t\to 0^+} f(t, ., ., .) = +\infty$ (i.e. f is singular at $t = 0$). Our analysis is based on a tripled fixed point theorem in partially ordered metric spaces. An example is presented to illustrate the main results.

Keywords: Caputo fractional derivative, Positive solutions, Singular fractional equations, Fixed point
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Fractional calculus is an extended concept of integral ones and fractional differential equations are widely used in various fields of sciences. There are some papers dealing with the existence of positive solutions for nonlinear fractional differential equations, see ([1], [2], [3], [4], [5]).

In this paper, we investigate the existence and uniqueness of positive solutions for the following singular nonlinear fractional boundary value problem

$$\begin{cases} ^cD^\alpha u(t) + f(t, u(t), u(t), u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = u'''(0) = 0, \qquad u'(0) + u'(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $3 < \alpha \leq 4$, $0 < \lambda < 4$, $^cD^\alpha$ is the Caputo fractional derivative and $f : (0, 1] \times [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, $\lim_{t\to 0^+} f(t, ., ., .) = +\infty$ (i.e. f is singular at $t = 0$).

Our analysis is based on a new tripled fixed point theorem in partially ordered metric spaces.

*Speaker

2 Preliminaries

In this section we introduce preliminary facts and some basic results, which are used throughout this paper.

Lemma 2.1. *Let $3 < \alpha \leq 4$ and $\lambda \neq 4$ and $y \in C(0,1]$, then the unique solution of the problem*

$$\begin{cases} {}^cD^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = u'''(0) = 0, \qquad u'(0) + u'(1) = \lambda \int_0^1 u(s)ds \end{cases}$$

is $u(t) = \int_0^1 G(t,s)y(s)ds$, *where $G(t,s)$ is the Green's function defined as*

$$G(t,s) = \begin{cases} \frac{2t[\alpha(\alpha-1)(1-s)^{\alpha-2} - \lambda(1-s)^{\alpha}] - \alpha(4-\lambda)(t-s)^{\alpha-1}}{\Gamma(\alpha+1)(4-\lambda)}, & 0 \leq s \leq t \leq 1, \\ \frac{2t[\alpha(\alpha-1)(1-s)^{\alpha-2} - \lambda(1-s)^{\alpha}]}{\Gamma(\alpha+1)(4-\lambda)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 2.2. *By assumption that $\lambda \in [0,4)$, the function $G(t,s)$ defined in Lemma 2.1 is a continuous function and $G(t,s) \geq 0$, for all $t,s \in [0,1]$.*

Definition 2.3. ([5]) Let X be a nonempty set and $F : X \times X \times X \to X$ be a map. An element $(x,y,z) \in X \times X \times X$ is called a tripled fixed point of F if

$$F(x,y,z) = x, \quad F(y,z,x) = y, \quad F(z,x,y) = z.$$

Theorem 2.4. ([6]) *Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. Assume there exist nondecreasing functions $\psi_i : [0,\infty) \to [0,\infty)$, $i = 1,2,3$ such that $\psi = \psi_1 + \psi_2 + \psi_3$ is convex, $\psi(0) = 0$ and $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$. Let $F : X \times X \times X \to X$ be a mapping which is nondecreasing in each of its variables and satisfying*

$$d(F(x,y,z), F(u,v,w)) \leq \psi_1(d(x,u)) + \psi_2(d(y,v)) + \psi_3(d(z,w)),$$

for each $x \geq u$, $y \geq v$, $z \geq w$.

Suppose either

(a) F is continuous;

Or

(b) if a nondecreasing sequence $(x_n,y_n,z_n) \to (x,y,z)$, then $(x_n,y_n,z_n) \preceq (x,y,z)$, for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ with

$$x_0 \leq F(x_0,y_0,z_0), \quad y_0 \leq F(y_0,z_0,x_0) \quad \text{and} \quad z_0 \leq F(z_0,x_0,y_0),$$

then there exist $\bar{x}, \bar{y}, \bar{z} \in X$ such that $F(\bar{x},\bar{y},\bar{z}) = \bar{x}$, $F(\bar{y},\bar{z},\bar{x}) = \bar{y}$, $F(\bar{z},\bar{x},\bar{y}) = \bar{z}$. Furthermore,

if for each $(x,y,z),(r,s,t) \in X \times X \times X$, there exists $(u,v,w) \in X \times X \times X$ that is comparable to (x,y,z) and (r,s,t), then the tripled fixed point $(\bar{x},\bar{y},\bar{z})$ of F is unique and $\bar{x} = \bar{y} = \bar{z}$.

3 Main results

In this section, we establish the existence and uniqueness of positive solutions for the singular nonlinear fractional boundar value problem (1).

Lemma 3.1. *Let $0 < \sigma < 1$, $3 < \alpha \leq 4$ and $F : (0,1] \to \mathbb{R}$ is continuous function with $\lim_{t\to 0^+} F(t) = \infty$. Suppose that $t^\sigma F(t)$ is a continuous function on $[0,1]$. Then the function defined by*

$$H(t) = \int_0^1 G(t,s)F(s)ds$$

is continuous on $[0,1]$, where $G(t,s)$ is the Green function appearing in Lemma 2.1.

Definition 3.2. We say that $(\alpha, \beta, \gamma) \in C([0,1]) \times C([0,1]) \times C([0,1])$ is a tripled lower solution of the problem (1) if

$$\begin{aligned}
\alpha(t) &\leq \int_0^1 G(t,s)f(s,\alpha(s),\beta(s),\gamma(s))ds, \quad \forall t \in [0,1],\\
\beta(t) &\leq \int_0^1 G(t,s)f(s,\beta(s),\gamma(s),\alpha(s))ds, \quad \forall t \in [0,1],\\
\gamma(t) &\leq \int_0^1 G(t,s)f(s,\gamma(s),\alpha(s),\beta(s))ds, \quad \forall t \in [0,1].
\end{aligned}$$

Theorem 3.3. *Let $0 < \sigma < 1$, $3 < \alpha \leq 4$, $f : (0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, $\lim_{t\to 0^+} f(t,.,.,.) = \infty$ and $t^\sigma f(t,x,y,z)$ is continuous on $[0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty)$. Assume that there exists $0 < \eta \leq \frac{\Gamma(\alpha-\sigma+2)(4-\lambda)}{2[(\alpha-\sigma+1)(\alpha-\sigma)-\lambda]\Gamma(1-\sigma)}$ such that for all $x,y,z,u,v,w \in [0,\infty)$ with $x \geq u$, $y \geq v$, $z \geq w$ and all $t \in [0,1]$, we have*

$$0 \leq t^\sigma \Big(f(t,x,y,z) - f(t,u,v,w) \Big) \leq \eta[\psi_1(x-u) + \psi_2(y-v) + \psi_3(z-w)],$$

where $\psi_i : [0,\infty) \to [0,\infty)$, $i = 1,2,3$ are nondecreasing functions such that $\psi = \psi_1 + \psi_2 + \psi_3$ is convex, $\psi(0) = 0$ and $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$. Then the existence of a tripled lower solution for the problem (1) provides the existence of a unique positive solution for the problem (1) in $C[0,1]$.

4 Example

Example 4.1. Consider the following singular fractional boundary value problem.

$$\begin{cases}
{}^cD^{\frac{7}{2}}u(t) + \frac{(t-\frac{1}{2})^2 e^{-t}}{\sqrt{t}(9+e^t)}\left[\frac{\cos^2 t}{\sqrt{1+t^2}} + \alpha u(t) + \beta\frac{u(t)}{1+u(t)} + \gamma(u(t) - \ln(1+u(t)))\right] = 0,\\
u(0) = u''(0) = u'''(0) = 0, \quad u'(0) + u'(1) = \frac{8\cos(1)}{e^2+1}\int_0^1 u(s)ds,
\end{cases} \tag{2}$$

where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma < 1$.
In this case, $f(t,x,y,z) = \frac{(t-\frac{1}{2})^2 e^{-t}}{\sqrt{t}(9+e^t)}\left[\frac{\cos^2 t}{\sqrt{1+t^2}} + \alpha x + \beta\frac{y}{1+y} + \gamma(z - \ln(1+z))\right]$, for $(t,x,y,z) \in (0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty)$. Note that f is continuous on $(0,1] \times [0,\infty) \times [0,\infty) \times$

$[0,\infty)$ and $\lim_{t\to 0^+} f(t,.,.,.) = +\infty$. Moreover, $\sigma = \frac{1}{2}$, $\alpha = \frac{7}{2}$ and $\lambda = \frac{8\cos(1)}{e^2+1}$. For all $x, y, z, u, v, w \in [0,\infty)$ with $x \geq u$, $y \geq v$, $z \geq w$ and $t \in [0,1]$, we have

$$\begin{aligned}
0 &\leq t^{\frac{1}{2}}\Big(f(t,x,y,z) - f(t,u,v,w)\Big) \\
&= \frac{(t-\frac{1}{2})^2 e^{-t}}{9+e^t}\Big[\alpha(x-u) + \beta(\frac{y}{1+y} - \frac{v}{1+v}) + \gamma\Big((z-w) - (\ln(1+z) - \ln(1+w))\Big)\Big] \\
&\leq \frac{1}{40}\Big[\alpha(x-u) + \beta(y-v) + \gamma\Big((z-w) - \ln(1+\frac{z-w}{1+w})\Big)\Big] \\
&\leq \frac{1}{40}\Big[\alpha(x-u) + \beta(y-v) + \gamma(z-w)\Big] = \eta\Big[\psi_1(x-u) + \psi_2(y-v) + \psi_3(z-w)\Big],
\end{aligned}$$

where, $\eta = \frac{1}{40}$. It is easily to checked that ψ_i $(i = 1,2,3)$ is nondecreasing and $\psi = \psi_1 + \psi_2 + \psi_3$ is convex, $\psi(0) = 0$ and $\lim_{n\to\infty} \psi^n(t) = 0$, for each $t > 0$. Note that $(\alpha = 0, \beta = 0, \gamma = 0)$ is a tripled lower solution of the problem (2) and

$$\eta = \frac{1}{40} < \frac{\Gamma(\alpha - \sigma + 2)(4-\lambda)}{2[(\alpha-\sigma+1)(\alpha-\sigma) - \lambda]\Gamma(1-\sigma)} \simeq 1.6547.$$

Finally, by Theorem 3.3, problem (2) has a unique positive solution.

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Existence of nontrivial solutions for a nonlocal elliptic system of p-Kirchhoff type with nonlinear boundary conditions

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Abstract

This paper is concerned with the existence of nontrivial solutions to a class of p-Kirchhoff type systems under nonlinear boundary conditions. By Mountain Pass Lemma, we establish conditions ensuring the existence of solutions for the system.

Keywords: p-Kirchhoff type system; Mountain Pass lemma; Nonlinear boundary conditions.
Mathematics Subject Classification [2010]: 35J60; 35J20; 35J25

1 Introduction

In this paper we deal with the nonlocal elliptic system of the p-Kirchhoff type given by

$$\begin{cases} -M\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f_1(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & x\in\Omega,\\ -M\Big(\int_\Omega |\nabla v|^p dx\Big)\Delta_p v = f_2(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x\in\Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = g(x,u), & x\in\partial\Omega,\\ |\nabla v|^{p-2}\frac{\partial v}{\partial n} = h(x,v), & x\in\partial\Omega, \end{cases} \tag{1}$$

where $M:\mathbb{R}^+\to\mathbb{R}$, is continuous function defined by $M(s)=a+bs^k$, $a,b>0$, $k>0$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\alpha>1$, $\beta>1$, $1<p<q<\alpha+\beta<p^*=\frac{Np}{N-p}$ and $f_1,f_2\in C(\bar\Omega)$ and g,h are Carathéodory functions.

Recently in [2], The authors studied the following system

$$\begin{cases} -M\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = g(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & x\in\Omega,\\ -M\Big(\int_\Omega |\nabla v|^p dx\Big)\Delta_p v = h(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x\in\Omega,\\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{2}$$

*Speaker

The goal of our study is to extend their results to nonlinear boundary conditions. We make the following assumptions to state our main result:

(H1) There exist $C > 0$, such that

$$\Phi(u,v) \leq \frac{1}{r}\frac{\partial}{\partial t}\Phi(tu,tv)|_{t=1} \leq C(u^r + v^r)$$

where $p < r < p^*$ for all $x \in \partial\Omega$ and $u, v \in \mathbb{R}^+$.

(H2) there exist $\delta > 0$, such that for $0 \leq t \leq \delta$, we have $G(x,t) \leq 0,\ H(x,t) \leq 0$.

(H3) there exist $t_0 \geq 0$, such that $G(x,t_0) > 0,\ H(x,t_0) > 0$.

(H4) $\lim_{|t|\to\infty} \frac{G(x,t)}{|t|^p} = 0$, uniformly for any x.

(H5) $\lim_{|t|\to\infty} \frac{H(x,t)}{|t|^p} = 0$, uniformly for any x.

Definition 1.1. Let X be a Banach space and $I \in C^1(X,R)$. We say that I satisfies the $(PS)_\theta$-condition if any sequence $\{u_n, v_n\} \subset X$ that $\{I(u_n,v_n)\}$ be bounded and $\{I'(u_n,v_n)\} \longrightarrow 0$ as $n \longrightarrow \infty$, possesses a convergent subsequence.

Definition 1.2. A pair of functions $(\varphi_1,\varphi_2) \in X$ is said to be a weak solutions of problem (1) if for all $(u,v) \in X$

$$\begin{aligned} &M\Big(\int_\Omega |\nabla u|^p dx\Big)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi_1 dx + M\Big(\int_\Omega |\nabla v|^p dx\Big)\int_\Omega |\nabla v|^{p-2}\nabla v\nabla\varphi_2 dx \\ &-\int_\Omega \Big(g(x)|u|^{q-2}u\varphi_1 + h(x)|v|^{q-2}v\varphi_2\Big)dx - \frac{\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta\varphi_1 dx - \frac{\beta}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^{\beta-2}v\varphi_2 dx \\ &-\int_{\partial\Omega}\Big(g(x,u)\varphi_1 + h(x,v)\varphi_2\Big)dx = 0, \end{aligned}$$

Seeking a weak solution of problem (1) is equivalent to finding a critical point of the C^1 functional

$$\begin{aligned} I(u,v) &= \frac{1}{p}\widehat{M}\Big(\int_\Omega |\nabla u|^p dx\Big) + \frac{1}{p}\widehat{M}\Big(\int_\Omega |\nabla v|^p dx\Big) - \frac{1}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^\beta dx \\ &- \frac{1}{q}\int_\Omega \Big(g(x)|u|^q + h(x)|v|^q dx\Big) - \Phi(u,v), \end{aligned}$$

where

$$\Phi(u,v) = \int_{\partial\Omega}\big(G(x,u) + H(x,v)\big)dx,$$

and $\widehat{M}(t) = \int_0^t M(s)ds = at + \frac{b}{k+1}t^{k+1}$, $G(x,u) = \int_0^u g(x,t)dt$ and $H(x,v) = \int_0^v h(x,t)$. Then

$$\begin{aligned} & \langle I'(u,v), (\varphi_1,\varphi_2)\rangle \\ = \; & M\big(\int_\Omega |\nabla u|^p dx\big)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi_1 dx + M\big(\int_\Omega |\nabla v|^p dx\big)\int_\Omega |\nabla v|^{p-2}\nabla v\nabla\varphi_2 dx \\ - \; & \frac{\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta\varphi_1 dx - \int_\Omega (g(x)|u|^{q-2}u\varphi_1 + h(x)|v|^{q-2}v\varphi_2)dx \\ - \; & \frac{\beta}{\alpha+\beta}\int_\Omega |u|^\alpha|v|^{\beta-2}v\varphi_2 dx - \int_{\partial\Omega}\big(g(x,u)\varphi_1 + h(x,v)\varphi_2\big)dx. \end{aligned}$$

As I is not bounded below on X, we consider the behaviors of I on the Nehari manifold

$$N = \{(u,v) \in X \;\; \langle I'(u,v),(u,v)\rangle = 0\}.$$

Theorem 1.3. *(Mountain Pass Lemma). Let X be a real Banach space and $I \in C^1(X,R^1)$ satisfying $(PS)_\theta$-condition. Suppose*

(L1) *there are constants $a, r > 0$ such that for any $(u,v) \in X$ that $\|(u,v)\| = r$, we have*

$$I(u,v) \geq a > 0;$$

(L2) *there exists $(\overline{u},\overline{v}) \in X \setminus \{(0,0)\}$ such that $\|(\overline{u},\overline{v})\| > r$ and $I(\overline{u},\overline{v}) < 0$;*

Then I possesses a critical value as

$$C = \inf_{g\in\Gamma}\max_{t\in[0,1]} I(g(t)),$$

where

$$\Gamma = \{g \in C([0,1],X) : g(0) = 0,\;\; g(1) = e\}.$$

Lemma 1.4. *The Euler functional I is bounded from below on N.*

Lemma 1.5. *There are $\rho, r_0 > 0$ such that $I(u,v) \geq r_0$ for $\|(u,v)\| = \rho$.*

Lemma 1.6. *There exists $(\overline{u},\overline{v}) \in X \setminus \{(0,0)\}$ such that $\|(\overline{u},\overline{v})\| > \rho$ and $I((\overline{u},\overline{v})) < 0$.*

Lemma 1.7. *There exists a $(PS)_\theta$-sequence $\{(u_n,v_n)\} \subset N$ for I.*

Lemma 1.8. *I satisfies the $(PS)_\theta$-condition in X.*

Theorem 1.9. *Let $(H1)-(H4)$ hold. Then the problem (1) has at least one nontrivial solution in X.*

2 Main results

Let $X = W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ be the Sobolev space endowed with the norm

$$\|(u,v)\| = \Big(\int_\Omega (|\nabla u|^p + |\nabla v|^p) dx \Big)^{\frac{1}{p}}$$

and $|u|_r$ denotes the norm in $L^r(\Omega)$, i.e.

$$|u|_r = \Big(\int_\Omega |u|^r dx \Big)^{\frac{1}{r}}.$$

Following we consider the well-known inequality

$$\Big(\int_\Omega |u|^{p^*} dx \Big)^{\frac{1}{p^*}} \leq \frac{1}{S^{1/p}} \Big(\int_\Omega |\nabla u|^p dx \Big)^{\frac{1}{p}}, \forall u \in W_0^{1,p},$$

where, S is the best constant in the Sobolov embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.
The base of our work is finding critical points by using the Mountain Pass Lemma (see [1]).
By using the lemma1-4, we have that I is bounded from below on the Nehari manifold N,and we define

$$\theta = \inf_{(u,v)\in N} I(u,v).$$

Now, we complete the proof of Theorem 1.9. By proof of Lemmas 1.5-1.8 the conditions of Mountain Pass Lemma are satisfied. Therefore, I has a nontrivial critical point as

$$C = \inf_{g\in\Gamma} \max_{t\in[0,1]} I(g(t)),$$

that

$$\Gamma = \{g \in C([0,1], X); g(0) = 0,\ g(1) = (t_0 u, t_0 v)\}.$$

Then the problem (1) has a nontrivial solution and also lemma 1.5 implies that C is positive.

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Mittag-Leffler identity for half-Hermite transform

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Abstract

In this paper in view of the Fourier series of a periodic function on interval $(0, \infty)$, we obtain a Mittag-Leffler type identity for the half-Hermite transform of order n.

Keywords: Mittag-Leffler identity, Fourier series, Half-Hermite transform
Mathematics Subject Classification [2010]: 42A16, 44A.

1 Introduction and Preliminaries

We consider the periodic function $f(x)$ and approximate it by a Fourier series with period $2T$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{2n\pi x}{T}) + b_n \sin(\frac{2n\pi x}{T})], \tag{1}$$

where a_n and b_n are the Fourier coefficients as follows

$$a_n = \frac{1}{T}\int_0^{2T} f(x)\cos(\frac{2n\pi x}{T})dx, \quad n = 0, 1, 2, \cdots, \tag{2}$$

$$b_n = \frac{1}{T}\int_0^{2T} f(x)\sin(\frac{2n\pi x}{T})dx, \quad n = 1, 2, \cdots. \tag{3}$$

Related to the theory of integral transforms, by applying the suitable integral transform on relation (1), the Mittag-Leffler identity can be written. For example, using the Laplace transform this identity is obtained as [4]

$$\frac{a_0}{2s} + T\sum_{n=1}^{\infty}\frac{sTa_n + 2\pi n b_n}{s^2T^2 + 4\pi^2 n^2} = \frac{1}{1 - e^{-sT}}\int_0^T e^{-su} f(u), \tag{4}$$

and using the Meijer transform, we get [3]

$$\frac{a_0\pi}{4s} + \sum_{n=1}^{\infty}[\frac{\pi T a_n}{2\sqrt{n^2\pi^2 + s^2T^2}} + \frac{Tb_n}{\sqrt{n^2\pi^2 + s^2T^2}}\ln\left(\frac{n\pi}{Ts} + \sqrt{\frac{n^2\pi^2}{T^2s^2} + 1}\right)] =$$

$$\int_0^T f(x)[K_0(sx) + \int_0^{\infty}\frac{1}{\sqrt{t^2 + s^2}}\frac{e^{-xt}}{e^{Tt} - 1}dt]dx, \tag{5}$$

where K_0 is the modified Bessel function of second kind.

*Speaker

2 Mittag-Leffler Identity

Now, in view of the Hermite function of order m

$$H_m(x) = (-1)^n e^{-x^2} \frac{d^m}{dx^m}(e^{-x^2}), \tag{6}$$

and the half-Hermite Transform of order m [1]

$$\mathcal{H}_e(f(x); s) = \int_0^\infty e^{-x^2} f(x) H_m(x) dx, \tag{7}$$

we intend to get the associated Mittag-Leffler identity. For this purpose, first we assume that $f(x)$ satisfy the Dirichlet conditions with period T and apply the half-Hermite transform on the function $f(x)$, therefore we get

$$\begin{aligned} \int_0^\infty e^{-x^2} H_m(x) f(x) dx &= \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-x^2} H_m(x) f(x+nT) dx \\ &= \int_0^T \sum_{n=0}^\infty e^{-(u+nT)^2} H_m(u+nT) f(u) du. \end{aligned} \tag{8}$$

At this point, by using the following integral representation for the Hermite function $H_m(x)$ [2, page 998]

$$H_m(u+nT) = 2^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} H_{m-k}(u\sqrt{2}) H_k(nT\sqrt{2}), \tag{9}$$

the relation (7) is changed to

$$\int_0^T \sum_{n=0}^\infty e^{-(u^2+n^2T^2+2unT)} H_m(u+nT) f(u) du =$$

$$\int_0^T \sum_{n=0}^\infty e^{-(u^2+n^2T^2+2unT)} 2^{-\frac{m}{2}} \sum_{k=0}^m \binom{m}{k} H_{m-k}(u\sqrt{2}) H_k(nT\sqrt{2}) =$$

$$\int_0^T \sum_{n=0}^\infty \sum_{k=0}^m 2^{-\frac{m}{2}} e^{-u^2-n^2T^2-2unT} \binom{m}{k} H_{m-k}(u\sqrt{2}) H_k(nT\sqrt{2}) du. \tag{10}$$

Also, by applying the half-Hermite transform on the left hand side of Fourier series (1), we have (with period T)

$$\mathcal{H}_e(f(x); s) = \sum_{n=1}^\infty \Big((-1)^m 2^{m-1} \pi^{\frac{1}{2}} (\frac{n\pi}{\sqrt{2}T})^{2m} e^{-\frac{1}{2}(\frac{n\pi}{\sqrt{2}T})^2} a_n + (-1)^m 2^{m-\frac{1}{2}} \pi^{\frac{1}{2}} (\frac{n\pi}{\sqrt{2}T})^{2m+1} e^{-\frac{1}{2}(\frac{n\pi}{\sqrt{2}T})^2} b_n \Big), \tag{11}$$

where we used the following facts for the Hermite function [2, page 996]

$$\int_0^\infty e^{-x^2} H_{2m+1}(x) \sin(\sqrt{2}\alpha x) dx = (-1)^m 2^{m-\frac{1}{2}} \pi^{\frac{1}{2}} \alpha^{2m+1} e^{-\frac{1}{2}\alpha^2}, \quad \alpha > 0, \tag{12}$$

$$\int_0^\infty e^{-x^2} H_{2m+1}(x)\cos(\sqrt{2}\alpha x)dx = 0, \quad \alpha > 0, \tag{13}$$

$$\int_0^\infty e^{-x^2} H_{2m}(x)\cos(\sqrt{2}\beta x)dx = (-1)^m 2^{m-1}\pi^{\frac{1}{2}}\beta^{2m}e^{-\frac{1}{2}\beta^2}, \quad \beta \geq 0, \tag{14}$$

$$\int_0^\infty e^{-x^2} H_{2m}(x)\sin(\sqrt{2}\beta x)dx = 0, \quad \beta \geq 0. \tag{15}$$

Finally, after evaluations the relation (11) and (10), we obtain the Mittag-Leffler identity for the half-Hermite transform as follows

$$\sum_{n=1}^{\infty}\Big((-1)^m 2^{m-1}\pi^{\frac{1}{2}}(\frac{n\pi}{\sqrt{2}T})^{2m}e^{-\frac{1}{2}(\frac{n\pi}{\sqrt{2}T})^2}a_n + (-1)^m 2^{m-\frac{1}{2}}\pi^{\frac{1}{2}}(\frac{n\pi}{\sqrt{2}T})^{2m+1}e^{-\frac{1}{2}(\frac{n\pi}{\sqrt{2}T})^2}b_n\Big) =$$

$$\int_0^T \sum_{n=0}^{\infty}\sum_{k=0}^{m} 2^{-\frac{m}{2}}e^{-u^2-n^2T^2-2unT}\binom{m}{k}H_{m-k}(u\sqrt{2})H_k(nT\sqrt{2})du. \tag{16}$$

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Multisoliton Perturbation Theory for the Manakov Equation

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Abstract

he effect of small perturbations on the collision of vector solitons in the Manakov equations is studied in this paper. The evolution equations for the soliton parameters (amplitude, velocity, polarization, position, and phases) throughout collision are derived. The method is based on the completeness of the bounded eigenstates of the associated linear operator in L_2 space and a multiple-scale perturbation technique.

Keywords: Manakov equation, Soliton solution, IST method, Soliton perturbation theory
Mathematics Subject Classification [2010]: 42.65.Tg, 41.20.Jb

1 Introduction

Nonlinear pulse propagation in optical fibers has been studied over 30 years. The idea of using optical solitons as information bits in high-speed telecommunication systems was first proposed in 1973, and then demonstrated experimentally in 1980.
In an ideal fiber, optical solitons can be modeled approximately by the nonlinear Schrödinger (NLS) equation, whose solution behaviors are completely known. But in reality, optical fibers are birefringent. Pulses travel at slightly different speeds along the two orthogonal polarization axes. If the birefringence randomly varies along the fiber due to bending, twisting, and the environmental perturbations, the pulses evolve according to the Manakov equations with corrections caused by polarization mode dispersion.
The collision of vector solitons is critical in many optical switching devices and nonlinear optical telecommunication networks. A rigorous analytical theory describing the collision process has still been lacking. In this paper, we present such an analytical theory. We study the collision of two vector solitons, based on the perturbed
Manakov equations:

$$iA_t + A_{xx} + (|A|^2 + |B|^2)A = \varepsilon M(A, B, \partial_x, \partial_t), \quad (1)$$
$$iB_t + B_{xx} + (|A|^2 + |B|^2)B = \varepsilon N(A, B, \partial_x, \partial_t). \quad (2)$$

Here A and B are complex functions, and $\varepsilon \ll 1$. When $\varepsilon = 0$ Eqs. (**??**-**??**) are the integrable Manakov equations. Vector solitons collide with each other elastically, except

*Speaker

that their polarizations may change after collision. If the incoming vector solitons have the same or orthogonal polarizations, such change will not occur. When $\varepsilon \ll 1$, Eqs. (**??-??**) are the perturbed Manakov equations. Generally, all the soliton parameters will change after collision.
In the present paper, we study the soliton collision in the perturbed Manakov equations. Our mthod is based on the closure of the bounded eigenstates of the associated linear operator and a direct perturbation technique. We first construct the exact two-soliton solution of the Manakov equations by the Hirota method. Then, we employ this perturbation technique to the colliding vector solitons under perturbations, and derive the evolution equations for the amplitudes, velocities, polarizations, positions, and phases of the two colliding solitons. Integration of these evolution equations will give these soliton parameters through out the collision. Such information is valuable for understanding the collision process of vector solitons in the presence of perturbations.

2 Exact soliton solution for Manakov equation

When ε is zero, Eqs. (**??-??**) are the Manakov equations, which allow exact N-soliton solutions. The one-soliton solution of the Manakov equations is given by

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} cos(\theta)e^{i\delta} \\ sin(\theta)e^{-i\delta} \end{bmatrix} \frac{e^{\eta}}{1+e^{\eta+\eta^*}} \tag{3}$$

where

$$\eta = ax + ia^2t + \eta_0, \tag{4}$$

$$a = r + iv, \quad \eta_0 = \xi_0 + i\zeta_0, \tag{5}$$

and $r, v, \theta, \delta, \xi_0$, and ζ_0 are real constants. This soliton can be rewritten as

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} cos(\theta)e^{i\delta} \\ sin(\theta)e^{-i\delta} \end{bmatrix} \sqrt{2}re^{i\{vx+(r^2-v^2)t+\zeta_0\}}sech\{r(x-2vt)+\xi_0\} \tag{6}$$

We can see that it has amplitude $\sqrt{2}r$, velocity $2v$, polarization θ, initial central position $-\xi_0/r$, in-phase constant ζ_0, and opposite phase constant δ.

3 Colliding vector solitons under perturbation

When $\varepsilon \ll 1$, the colliding soliton solution can be expanded into a perturbation series,

$$A = A_h(r_k, v_k, \xi_k, \zeta_k, \theta_k, \delta_k, \quad k = 1, 2) + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \tag{7}$$

$$B = B_h(r_k, v_k, \xi_k, \zeta_k, \theta_k, \delta_k, \quad k = 1, 2) + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots \tag{8}$$

where

$$\xi_k = \int_0^t r_k v_k dt + \xi_{k0}, \tag{9}$$

$$\zeta_k = \int_0^t (r_k^2 - v_k^2)dt + \zeta_{k0}. \tag{10}$$

Due to the small perturbations, the soliton parameters $r_k, v_k, \xi_{k0}, \zeta_{k0}, \theta_k$, and δ_k will be forced to vary. In the following, we derive the evolution equations for the soliton parameters throughout collision. The method we will use is based on the completeness of the bounded eigenstates of the associated linear operator and a multiple-scale perturbation procedure.
When Eqs. (**??**-**??**) are substituted into Eqs. (**??**-**??**), the zerothorder equations are trivially satisfied since A_h and B_h are the exact soliton solutions of the Manakov equations. At order ε, we get

$$L\Phi = R - W, \tag{11}$$

where

$$L = i\partial_t + \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix} H, \tag{12}$$

$$H = \begin{pmatrix} 2|A_h|^2 + |B_h|^2 & A_h^2 & A_h B_h^* & A_h B_h \\ A_h^{*^2} & 2|A_h|^2 + |B_h|^2 & A_h^* B_h^* & A_h^* B_h \\ A_h^* B_h & A_h B_h & 2|A_h|^2 + |B_h|^2 & B_h^2 \\ A_h^* B_h^* & A_h B_h^* & B_h^{*^2} & 2|A_h|^2 + |B_h|^2 \end{pmatrix} H, \tag{13}$$

which is a Hermitian matrix,

$$\Phi = (A_1, A_1^*, B_1, B_1^*)^T, \tag{14}$$
$$R = (M, -M^*, N, -N^*)^T, \tag{15}$$
$$W = \sum_{k=1}^{2} \{\Psi_{r_k} r_{kT} + \Psi_{v_k} v_{kT} + \Psi_{\xi_k} \xi_{k0T} + \Psi_{\theta_k} \theta_{kT} + \Psi_{\delta_k} \delta_{kT}\}, \tag{16}$$
$$\Psi = (A_h, A_h^*, B_h, B_h^*)^T, \tag{17}$$

Here, $\sigma_3 = diag(1, -1)$ is the third Pauli spin matrix, the subscript T is the derivative with respect to the slow time εt, and the superscript T represents the transpose of a matrix.
Even though the linear operator L in Eq. (**??**) is a partial differential operator with variable coefficients, Eq. (**??**) can still be solved. Here the key idea is to establish the completeness of the bounded eigenstates of L in L_2 space, and define an appropriate inner product. We first study the null space of L. Recall that the soliton solution has 12 free parameters. The derivatives of Ψ with respect to each of these parameters, i.e.,

$$\{\tilde{\Psi}_{r_k}, \tilde{\Psi}_{v_k}, \Psi_{\xi_k}, \Psi_{\zeta_k}, \Psi_{\theta_k}, \Psi_{\delta_k}, \quad k = 1, 2\} \tag{18}$$

span the discrete subspace of this null space. Here,

$$\tilde{\Psi}_{r_k} = \Psi_{r_k} - 2t(v_k \Psi_{\xi_k} - r_k \Psi_{\zeta_k}), \tag{19}$$
$$\tilde{\Psi}_{v_k} = \Psi_{v_k} - 2t(r_k \Psi_{\xi_k} + v_k \Psi_{\zeta_k}). \tag{20}$$

The continuous subspace consists of eigenfunctions $\Phi_c(x, t, \lambda)$, which are oscillatory at infinity. Here the parameter λ is the wave number of the function at infinity, which characterizes the continuous eigenfunction.
The above discrete and continuous eigenfunctions form a complete set. We define the inner product as

$$(\psi_1, \psi_2) = \int_{-\infty}^{\infty} \psi_1^{*^T} \begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix} \psi_2 dx. \tag{21}$$

With respect to this inner product, we can easily find inner product between discrete and continuous eigen-functions of L. we can show that the discrete and continuous eigenfunctions in the null space of L are orthogonal to each other. Furthermore, the nonzero inner products of the discrete eigenfunctions can be easily find.
Notice that the basis $\tilde{\Psi_{r_k}}$ and $\tilde{\Psi_{v_k}}$ have a secular term proportional to t. To avoid such an undesirable behavior, we use instead the equivalent set

$$\{\Psi_{r_k}, \Psi_{v_k}, \Psi_{\xi_k}, \Psi_{\zeta_k}, \Psi_{\theta_k}, \Psi_{\delta_k}, \quad k = 1, 2\} \tag{22}$$

which also spans the discrete subspace.
Now, we are ready to solve the linear equation (**??**) by expanding the solution Φ and the forcing term $R - W$ into this complete set of Ls eigen-functions:

$$\begin{aligned} \Phi &= \sum_{k=1}^{2}\{c_{1k}\Psi_{r_k} + c_{2k}\Psi_{v_k} + c_{3k}\Psi_{\xi_k} + c_{4k}\Psi_{\zeta_k} + c_{5k}\Psi_{\theta_k} + c_{6k}\Psi_{\delta_k}\} \\ &+ \int C_\lambda \Phi_c(x,t,\lambda)d\lambda \end{aligned} \tag{23}$$

$$\begin{aligned} R - W &= \sum_{k=1}^{2}\{d_{1k}\Psi_{r_k} + d_{2k}\Psi_{v_k} + d_{3k}\Psi_{\xi_k} + d_{4k}\Psi_{\zeta_k} + d_{5k}\Psi_{\theta_k} + d_{6k}\Psi_{\delta_k}\} \\ &+ \int D_\lambda \Phi_c(x,t,\lambda)d\lambda \end{aligned} \tag{24}$$

Finally we can find evolution equations of soliton parameters as

$$r_{kt} = \frac{\varepsilon}{4}(R, \Psi_{\zeta_k}) \tag{25}$$

$$v_{kt} = \frac{\varepsilon}{4}(R, \Psi_{\xi_k}) \tag{26}$$

$$\xi_{kt} = -2r_k v_k - \frac{\varepsilon}{4}(R, \Psi_{v_k}) \tag{27}$$

$$\zeta_{kt} = r_k^2 - v_k^2 - cos2\theta_k\delta_{kt} - \frac{\varepsilon}{4}(R, \Psi_{r_k}) \tag{28}$$

$$\theta_{kt} = \frac{4cos\theta_k r_{kt} - \varepsilon(R, \Psi_{\delta_k})}{8r_k sin2\theta_k}, \tag{29}$$

$$\delta_{kt} = \frac{\varepsilon(R, \Psi_{\theta_k})}{8r_k sin2\theta_k}. \tag{30}$$

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New rational approximation of Mittag-Leffler function with orthogonal polynomials

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Abstract

In this paper we drive a uniform rational approximation for the Mittag-Leffler function using the Chebyshev polynomials and asymptotic series. Next, we use this approximation to find the solution of the fractional diffusion equation.

Keywords: Mittag-Leffler function, global rational approximation, Time- Fractional Diffusion Equation
Mathematics Subject Classification [2010]: 26A33,33E12

1 Introduction and Preliminaries

The Mittag-Leffler functions arise naturally as the solution of fractional differential and integral equations. The Mittag-Leffler function of order α is stated as the following series [3]

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)}, \tag{1}$$

where α is an arbitrary real number. For computational works, one have to truncate the above series which yields truncated error cost in computation. So it is important to substitute a good approximation instead of the Mittag-Leffler function expansion (1). The Pade approximations for the Mittag-Leffler function are discussed in [4]. Atkinson et. al. used both Taylor and asymptotic series to find good approximations for the Mittag-Leffler function [1]. In this paper we introduce a new method based on [1] to approximate the Mittag-Leffler function. In this method we substitute the Chebyshev polynomial expansion instead of (1) to obtain a better approximation for $E_\alpha(-x)$ in two cases.

Case 1. Expanding $E_\alpha(-x)$ by the Chebyshev polynomials of the first kind. In this case we have

$$\Gamma(1-\alpha)xE_\alpha(-x) = \Gamma(1-\alpha)x\sum_{k=0}^{m-2} a_k T_k(x-1) + O(x^m) \equiv a(x) + O(x^m), x \in [0,a]. \tag{2}$$

*Speaker

Also the Mittag-Leffler function admits the following asymptotic series:

$$\begin{aligned}\Gamma(1-\alpha)xE_\alpha(-x) &= -\Gamma(1-\alpha)x\sum_{k=1}^{n}\frac{(-x)^k}{\Gamma(1-\alpha k)}+O(x^{-n})\\ &\equiv b(x^{-1})+O(x^{-n}), x\longrightarrow\infty. \end{aligned}\tag{3}$$

Now we want to find a rational approximation of the form

$$\Gamma(1-\alpha)xE_\alpha(-x)\approx\frac{p(x)}{q(x)}\equiv\frac{p_0+p_1x+\cdots+p_\nu x^\nu}{q_0+q_1x+\cdots+q_\nu x^\nu}.\tag{4}$$

The problem is to determine the coefficients p_i and q_i such that (4) has the correct expansions at the interval $[0,a]$ and $x=\infty$. According to (4), the leading term of fraction at $x=\infty$ is $\frac{p_\nu}{q_\nu}$, therefore we can set $p_\nu=q_\nu=1$. The unknown coefficients p_i and q_i can be found from the system of linear equations

$$p(x)-q(x)a(x)=O(x^m) \text{ at } [0,a],\tag{5}$$

$$\frac{p(x)}{x^\nu}-\frac{q(x)}{x^\nu}b(x^{-1})=O(x^{-n}) \text{ at } x=\infty.\tag{6}$$

Now we work out the case $\nu=2, m=3, n=2$.

$$\begin{aligned}a(x)&=\Gamma(1-\alpha)x((a_0-a_1)+a_1x),\\ b(x^{-1})&=1-\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)x}.\end{aligned}$$

Collecting equal powers of x via $O(x^m)$ and $O(x^{-n})$, respectively, yields a system of four equations with four unknowns as follows:

$$\begin{aligned}p_0=0, p_1&=-\frac{(a_0-a_1)(\Gamma(1-\alpha)^2a_0-\Gamma(1-\alpha)^2a_1-\Gamma(1-2\alpha))}{(\Gamma(1-2\alpha)(\Gamma(1-\alpha)a_0^2-2\Gamma(1-\alpha)a_0a_1+\Gamma(1-\alpha)a_1^2+a_1)},\\ q_0&=-\frac{\Gamma(1-\alpha)^2a_0-\Gamma(1-\alpha)^2a_1-\Gamma(1-2\alpha)}{\Gamma(1-2\alpha)\Gamma(1-\alpha)(\Gamma(1-\alpha)a_0^2-2\Gamma(1-\alpha)a_0a_1+\Gamma(1-\alpha)a_1^2+a_1)},\\ q_1&=\frac{\Gamma(1-\alpha)a_1+\Gamma(1-2\alpha)a_0-\Gamma(1-2\alpha)a_1}{(\Gamma(1-2\alpha)(\Gamma(1-\alpha)a_0^2-2\Gamma(1-\alpha)a_0a_1+\Gamma(1-\alpha)a_1^2+a_1)}.\end{aligned}$$

Case 2. Using the Chebyshev polynomials of the second kind for the series expansion instead of (1). Similar to Case 1, we have

$$\begin{aligned}p_0=0, p_1&=-\frac{(c_0-2c_1)(\Gamma(1-\alpha)^2c_0-2\Gamma(1-\alpha)^2c_1-\Gamma(1-2\alpha))}{\Gamma(1-2\alpha)(\Gamma(1-\alpha)c_0^2-4\Gamma(1-\alpha)c_0c_1+4\Gamma(1-\alpha)c_1^2+2c_1)},\\ q_0&=-\frac{\Gamma(1-\alpha)^2c_0-2\Gamma(1-\alpha)^2c_1-\Gamma(1-2\alpha)}{\Gamma(1-2\alpha)\Gamma(1-\alpha)(\Gamma(1-\alpha)c_0^2-4\Gamma(1-\alpha)c_0c_1+4\Gamma(1-\alpha)c_1^2+2c_1)},\\ q_1&=\frac{2\Gamma(1-\alpha)c_1+\Gamma(1-2\alpha)c_0-2\Gamma(1-2\alpha)c_1}{\Gamma(1-2\alpha)(\Gamma(1-\alpha)c_0^2-4\Gamma(1-\alpha)c_0c_1+4\Gamma(1-\alpha)c_1^2+2c_1)}.\end{aligned}$$

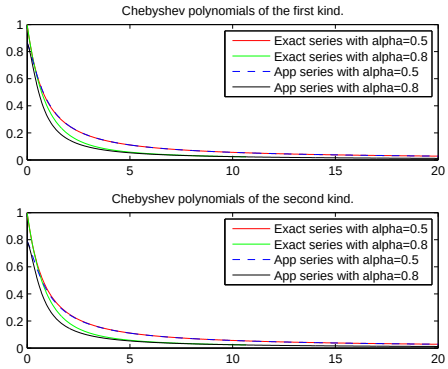

Figure 1: The Mittag-Leffler function and approximation by Chebyshev polynomials.

2 Rational solutions of the time-fractional diffusion equation

To obtain a rational approximation for the solution of the time-fractional diffusion equation,

$$\frac{\partial^\alpha}{\partial t^\alpha} p_t^\alpha(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} p_t^\alpha(x), \tag{7}$$

we apply the rational approximation for the Mittag-Leffler function the integral representation of the solution of the time-fractional diffusion [1].

$$p_t^\alpha(x) = \frac{1}{2\pi}\sqrt{\frac{2}{t^\alpha}} \int_{\Re} E_\alpha(-p^2) e^{-ip\sqrt{\frac{2}{t^\alpha}}x} dp. \tag{8}$$

To apply the rational approximation to the function $E_\alpha(-p^2)$ and compute the integral, first split the approximation as follows:

$$\begin{aligned} E_\alpha(-x) &\approx \frac{1}{\Gamma(1-\alpha)} \frac{p_1 + x}{q_0 + q_1 x + x^2} = \frac{1}{\Gamma(1-\alpha)} \frac{p_1 + x}{(x+Q_1)(x+Q_2)} \\ &= \frac{\frac{Q_1 - p_1}{Q_1 - Q_2}}{\Gamma(1-\alpha)} \frac{1}{x+Q_1} + \frac{\frac{p_1 - Q_2}{Q_1 - Q_2}}{\Gamma(1-\alpha)} \frac{1}{x+Q_2}, \end{aligned} \tag{9}$$

where Q_1 and Q_2 are the roots of the equation $q_0 + q_1 x + x^2 = 0$. By substituting into the formula we obtain

$$p_t^\alpha(x) = \frac{1}{\sqrt{2t^\alpha}\pi} \frac{1}{(Q_1 - Q_2)\Gamma(1-\alpha)} [(Q_1 - p_1) \int_{\Re} \frac{e^{-ip\sqrt{\frac{2}{t^\alpha}}x}}{p^2 + Q_1} dp - (Q_2 - p_1) \int_{\Re} \frac{e^{-ip\sqrt{\frac{2}{t^\alpha}}x}}{p^2 + Q_2} dp].$$

The integrals inside the brackets are solved using the following fact of the Fourier transforms. Finally we have

$$p_t^\alpha(x) = \frac{1}{\sqrt{2t^\alpha}} \frac{1}{(Q_1 - Q_2)\Gamma(1-\alpha)} [\frac{Q_1 - p_1}{\sqrt{Q_1}} e^{-\sqrt{\frac{2Q_1}{t^\alpha}}|x|} - \frac{Q_2 - p_1}{\sqrt{Q_2}} e^{-\sqrt{\frac{2Q_2}{t^\alpha}}|x|}]. \tag{10}$$

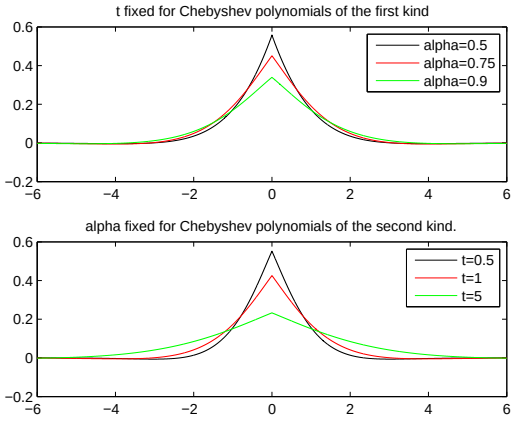

Figure 2: The approximate solution $p_t^\alpha(x)$ of the time-fractional diffusion equation plotted against x.where $t = 1$ fixed and $\alpha = 0.75$ fixed.

3 Bounds for the approximation error

The rational approximation $\frac{p(x)}{\Gamma(1-\alpha)xq(x)}$ is shown by $R_\alpha(x)$ discussed above. use [2],for the Chebyshev polynomials of the first kind:

$$|E_\alpha(-p^2) - R_\alpha(p^2)| = \frac{1}{2^n(n+1)!}|E_\alpha^{(n+1)}(x)| \leq \frac{1}{2^n}, \tag{11}$$

for the Chebyshev polynomials of the second kind:

$$|E_\alpha(-p^2) - R_\alpha(p^2)| = \frac{n+2}{2^{n+1}(n+1)!}|E_\alpha^{(n+1)}(x)| \leq \frac{n+2}{2^{n+1}}. \tag{12}$$

So approximation of the time-fractional can be made arbitrarily accurate.

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Nonwandering flows of some spaces

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Abstract

In this paper, Our effort is studying nonwandering flows, Planar flows, and their properties. We have shown that the set of periodic (noncritical) points is open. When S is connected, such a flow has a simple characterization; namely, it is nonwandering. We have given conditions on some spaces by their flows that proves when a space is disconnected.

Keywords: Wandering,Flow, Connected Space
Mathematics Subject Classification [2010]: 37Axx , 28D99

1 Introduction

The theory of prolongation, introduced by T. Ura [1], has proven to be a rather useful apparatus in studying the structure of dynamical systems. In [2], the first author studied planar flows in which the positive prolongation of each point coincided with the closure of the positive semitrajectory through the point. Such flows were referred to as flows of characteristic 0^+. Such flows were subsequently studied over more general phase spaces in [3], [4], [5], [6], and [7]. Knight [8] carried on a similar study for planar flows of characteristic 0; these are flows where the prolongation of each point coincides with the closure of the trajectory through the point. The structure of these flows turned out to be surprisingly simple. In this paper, we study nonwandering flows, planar flows and their properties. An interesting characterization, which is somewhat surprising, is that if the phase space is Hausdorff then the flow is nonwandering if and only if the positive prolongation of each point coincides with its negative prolongation.We have shown that the set of periodic (noncritical) points is open. When S is connected, such a flow has a simple characterization; namely, it is nonwandering if and only if every point of $R^2 - S$ lies on a cycle surrounding S ; then we prove that if S is unbounded and the flow is nonwandering, S is disconnected.

Definition 1.1. Dynamic System(Continues Flow) Let X be a topological space and let R denote the additive group of real num- bers with the usual topology. The pair (X, π) is called a dynamical system or a continuous flow if $\pi : X \times R \to X$ is a continuous mapping such that for each $x \in X$ and $s, t \in R$, $\pi(x, 0) = x$ and $\pi(\pi(x, s), t) = \pi(x, s+t)$.

*Speaker

For convenience we shall denote $\pi(x,t)$ by x.t. For each $x \in X$, we let $C^+(x)(C^-(x))(C(x))$ denote the positive trajectory (negative trajectory) (trajectory) through x. We let $K^+(x) = \overline{C^+(x)}$, $K^-(x) = \overline{C^-(x)}$, and $K(x) = \overline{C(x)}$. The positive limit set (negative limit set) (limit set) of x is denoted by $L^+(x)(L^-(x))(L(x))$. We denote the positive prolongation (negative prolongation) (prolongation) of $mathrmx$ by $D^+(x)(D^-(x))(D(x))$. Similarly, the positive prolongational limit set (negative prolongational limit set) (prolongational limit set) of x shall be denoted by $J^+(x)(J^-(x))(J(x))$.

Definition 1.2. positively stable subset For a point x of X (a subset M of X) we let $\eta(x)(\eta(M))$ denote the neighborhood filter of x (of M). A subset M of X is said to be positively stable if to each $U \in \eta(M)$ corresponds $V \in \eta(M)$ such that $C^+(V) \subset U$. The negative and bilateral versions are defined similarly.

Definition 1.3. Poisson stable point A point x of X is said to be Poisson stable (positively Poisson stable) (negatively Poisson stable) if $x \in L^+(x) \cap L^-(x)(x \in L^+(x))(x \in L^-(x))$. A point x of X is said to be nonwandering if $x \in J^+(x)$. A flow (X,π) is said to be nonwandering if each of its points is nonwandering.

We let int(M) denote the interior of a subset M of $X, \overline{M}$ the closure of M, and ∂M the boundary of M. In particular, if x is a cyclic (i.e. periodic but not critical) point of a planar flow, then int $(C(x))(\text{ext}(C(x)))$ shall denote the bounded (unbounded) component of the complement of the Jordan curve $C(x)$. The trajectory of a cyclic point is called a cycle. Throughout this paper the phase space X of any dynamical system will be assumed to be Hausdorff.

2 Nonwandering and Plana flows

Lemma 2.1. *For any flow (X,π) , the following two conditions are equivalent:*
(a) $D^+(x) \subset D^-(x)$ for all $x \in X$;
(b) $D^+(x) = D^-(x)$ for all $x \in X$.

Theorem 2.2. *For any flow (X,π) , the following statements are mutually equivalent:*
(i) for each $x \in X$, $D^+(x) = D^-(x)$;
(ii) the flow is nonwandering;
(iii) the set of nonwandering points is dense in X.

Theorem 2.3. *If the phase space ofa nonwandering dynamical system is metric and either locally compact or complete, then the set of Poisson stable points is dense in X.*

we assume a given planar flow (R^2,π) . The set of critical points of this flow shall be denoted by S, and that of cyclic points by P . For a point $x \in P$, we shall let $M_x = C(x) \cup$ int $(C(x))$, so that $M_x = \overline{\text{int}(C(x))}$. We say that a cyclic trajectory $C(x)$ surrounds a set F if $F \subset$ int $(C(x))$. We further assume that $R^2 \neq S$. We note that periodic points are necessarily nonwandering. In planar flows a point is Poisson stable if and only if it belongs to $P \cup S$.

Theorem 2.4. *The planar flow (R^2,π) is nonwandering if and only if the set $P \cup S$ is dense in R^2 .*

Lemma 2.5. *For each $x \in R^2$, $L^{\pm}(x)$ is a cycle if and only if $x \in P$.*

Lemma 2.6. *For each $x_0 \in P$, there exists an invariant neighborhood U of M_{x_0} such that for each $x \in U - M_{x_0}$, $C(x)$ is a cycle surrounding M_{x_0} with int $(C(x)) \cap S =$ int $(C(x_0)) \cap S$.*

Lemma 2.7. *For any $x_0 \in P$, there exists an invariant neighborhood W of $C(x_0)$ such that for each $x \in W \cap$ int $(C(x_0))$, $C(x)$ is a cycle with int $(C(x)) \cap S =$ int $(C(x_0)) \cap S$*

Theorem 2.8. *For any $x_0 \in P$, there exists an invariant neighborhood V of $C(X_0)$ such that for each $x \in V$, $C(x)$ is a cycle with int $(C(x)) \cap S =$ int $(C(x_0)) \cap S$.*

We note that The set P is open.

Lemma 2.9. *Let S_0 be the set of critical points surrounded by the cycle $C(x_0)$. Then all other points within $C(x_0)$ are cyclic provided every cycle within $C(x_0)$ surrounds S_0 .*

3 Main results

Definition 3.1. A nonempty subset S_0 of S_0 is said to be a central set if it has a neighborhood N such that for each $x \in N - S_0$, $C(x)$ is a cycle surrounding S_0 . Further, S_0 is called global if we can have $N = R^2$; otherwise, it is called local.

Theorem 3.2. *Let $x \in P$. If $\mathrm{S_x} =$ int $(C(x)) \cap S$ has only a finite number of components, then at least one of them is a central set.*

Theorem 3.3. *If S has only a finite number of components, then at least one of them is a central set.*

Notation: If S is finite, then some point of S is a Poincare center.
Our effort is proving the following theorem.

Theorem 3.4. *Let (R^2, π) be a planar flow in which the set S, $S \neq R^2$, of critical points is connected. Then (R^2, π) is nonwandering if and only if S is a global central set which is a simply connected continuum, and $P \cup S = R^2$.*

Proof. Suppose that (R^2, π) is nonwandering. Then it follows from notation that S is a central set. In order to show that S is a global center it suffices to show that $P \cup S = R^2$. We note that $S \subset$ int $(C(x))$ for every $x \in P$, as S is connected and int $(C(x)) \cap S \neq \emptyset$. It follows from Lemma 2.9 that for each $x \in P$, int $(C(x)) - S \subset P$. Set $\mathrm{A} = P \cup S$. Then, A is open since $\mathrm{A} = \cup$ int $(C(x))|x \in P$ and P is open by previous notations. Next, we wish to show that $\partial \mathrm{A} = \emptyset$. Assume $\partial \mathrm{A} \neq \emptyset$, and let $x_0 \in \partial \mathrm{A}$. Since A is open, $\partial \mathrm{A}$ does not contain any cyclic or critical points. Furthermore, $L(x_0) \subset \partial \mathrm{A}$, as $\partial \mathrm{A}$ is invariant and closed. Therefore, we must have $L(x_0) = \emptyset$ (see [6, p.184]). But this implies that $C(x_0)$ separates the plane into two open invariant regions V_1 and V_2 (see e.g. [8, 1.7]). Assume, without loss of generality, that $S \subset V_1$. By Theorem 2.4, the nonempty open set V_2 contains a point $\tilde{z} \in P$. But then $\overline{\mathrm{int}(C(\tilde{z}))} \cap S \neq \emptyset$ implies that $\overline{\mathrm{int}(C(\tilde{z}))} \cap V_1 \neq \emptyset \neq \overline{\mathrm{int}(C(\tilde{z}))} \cap V_2$. Further, $\overline{\mathrm{int}(C(\tilde{z}))} \cap \partial \mathrm{A} = \emptyset$, as $\overline{\mathrm{int}(C(\tilde{z}))} \subset P \cup S$. Therefore, $\overline{\mathrm{int}(C(\tilde{z}))} \subset V_1 \cup V_2$, contradicting the fact that $\overline{\mathrm{int}(C(\tilde{z}))}$ is connected. The fact that S is a simply connected continuum is clear, since $S = \cap\{\overline{\mathrm{int}(C(x))}|x \in P$. The converse implication is obvious from Theorem 2.4. □

Notation: Let (R^2, π) be a planar flow with one critical point s_0 . Then (R^2, π) is nonwandering if and only if s_0 is a global Poincare center.

Result: If S is unbounded and the flow is nonwandering, S is disconnected.

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Numerical solution of fractional Fokker-Planck equation by using of radial basis functions

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Abstract

In this paper, we propose a numerical method which is coupled of the radial basis functions (RBFs) and finite difference scheme for solving time fractional Fokker-Planck equation defined by Caputo sense for $(0 < \alpha < 1)$. It uses the collocation method and approximates the solution using thin plate splines (TPS) RBFs.

Keywords: Fractional differential equation, Fokker-Planck equation, Radial Basis Functions(RBFs), Collocation method
Mathematics Subject Classification [2010]: 35Q84, 35K28

1 Introduction

The Fokker-Planck equation (FPE) was first introduced by Fokker and Planck to describe the Brownian motion of particles. Phenomena such as anomalous diffusion, continuous random walk, wave propagation and etc. are modeled by space and time fractional FPE (see [4] and references therein.).

Consider the following time fractional FPE of order α $(0 < \alpha < 1)$ with the initial and boundary conditions:

$$D_t^\alpha u - u_{xx} + p(x)u_x + p'(x)u = f, \qquad 0 < x < L, 0 < t \le T, \tag{1}$$

$$u(x,0) = g(x), \qquad 0 \le x \le L, \tag{2}$$

$$u(0,t) = h_1(t), \qquad u(L,t) = h_2(t), \qquad 0 < t \le T. \tag{3}$$

where D_t^α is the Caputo fractional derivative operator of order $\alpha \ge 0$, which is defined as

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} \frac{\partial^k u(x,s)}{\partial s^k} ds, \qquad k-1 < \alpha < k. \tag{4}$$

The fractional FPE has been solved in several ways including (high–order) finite difference methods [1, 5] and finite element method [2].

Considering a finite set of interpolation points $\chi = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^d$ and a function $u : \chi \to \mathbb{R}$, the interpolant of u using radial basis functions (RBFs) is constructed as

$$u(x) \simeq \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|) + \sum_{j=N+1}^{N+\ell} \lambda_j q_j(x), \qquad x \in \mathbb{R}^d, \ \ell = \binom{m+d-1}{d} \tag{5}$$

*Speaker

where $\|.\|$ is the Euclidean norm and $\phi(\|.\|)$ is a radial function. In addition to the N equations resulting from collocating (5) at the N points, an extra ℓ equations are required. This is insured by the ℓ conditions for (5),

$$\sum_{i=1}^{N} \lambda_i q_j(x_i) = 0, \qquad \text{for all } \; q_j \in \Pi_{m-1}^d. \tag{6}$$

where Π_{m-1}^d denotes the space of all polynomials on $\mathbb{R}^d$ of total degree at most $m-1$. We will use the generalized thin plate splines (TPS) as RBF in the following form:

$$\phi(\|x - x_i\|) = \phi(r_i) = r_i^{2m} \ln(r_i), \qquad i = 1, 2, 3, \ldots, \qquad m = 1, 2, 3, \ldots, \tag{7}$$

where $r_i = \|x - x_i\|$. We note that ϕ in Eq. (7) is $\mathbf{C}^{2m-1}$ continuous. So higher order TPSs must be used for higher order partial differential operators. For the FPE, $m = 3$ is used for TPS in numerical computations. In a similar representation as (5), for any linear differential operator $\mathcal{L}$, $\mathcal{L}u$ can be approximated by

$$\mathcal{L}u(x) = \sum_{i=1}^{N} \lambda_i \mathcal{L}\phi(\|x - x_i\|) + \sum_{j=N+1}^{N+\ell} \lambda_j \mathcal{L}q_j(x), \qquad x \in \mathbb{R}^d. \tag{8}$$

2 Description of the method

Consider the fractional FPE (1)-(3) with $0 < \alpha < 1$. In order to discretize the problem in the time direction, we substitute t^{n+1} into Eq. (4), then the integral can be partitioned as

$$D_t^\alpha u(x, t^{n+1}) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{t^k}^{t^{k+1}} (t^{n+1} - s)^{-\alpha} \frac{\partial u(x, s)}{\partial s} ds, \qquad 0 < \alpha < 1. \tag{9}$$

where $t^{n+1} = t^n + \delta t, \; n = 0, 1, 2, \ldots, M$. Now, we approximate the first order derivative with the forward finite difference formulae

$$\frac{\partial u(x, \sigma)}{\partial t} = \frac{u(x, t^{n+1}) - u(x, t^n)}{\delta t} + o(\delta t), \tag{10}$$

where $\sigma \in [t^n, t^{n+1}]$. Replacement of Eq. (10) into Eq. (9), gives

$$D_t^\alpha u^{n+1} = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{u^{k+1} - u^k}{\delta t} \int_{t^k}^{t^{k+1}} (t^{n+1} - s)^{-\alpha} ds, \tag{11}$$

where $u^k = u(x, t^k)$. The right-hand side integral is easily obtained as

$$\int_{t^k}^{t^{k+1}} (t^{n+1} - s)^{-\alpha} ds = \frac{1}{1-\alpha} \delta t^{1-\alpha} \big[(n-k+1)^{1-\alpha} - (n-k)^{1-\alpha} \big]. \tag{12}$$

Rearrangement of Eq. (11) and assumption $b_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}$ lead to

$$D_t^\alpha u^{n+1} = \frac{\delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} b_k (u^{n-k+1} - u^{n-k}) = a_0 \Big\{ u^{n+1} - u^n + \sum_{k=1}^{n} b_k (u^{n-k+1} - u^{n-k}) \Big\} \tag{13}$$

where $a_0 = \delta t^{-\alpha}/\Gamma(2-\alpha)$ and $n = 0, 1, 2, \ldots, M$. By discretize the Eq. (1) with the finite difference method

$$D_t^\alpha u - \nabla^2 u^{n+1} + p(x)\nabla u^{n+1} + p'(x)u^{n+1} = f^{n+1}, \tag{14}$$

and substitute Eq. (13) into Eq. (14), we obtain

$$\{a_0 - \nabla^2 + p(x)\nabla + p'(x)\}u^1 = a_0 u^0 + f^1, \tag{15}$$

and

$$\{a_0 - \nabla^2 + p(x)\nabla + p'(x)\}u^{n+1} = a_0 u^n - a_0 \sum_{k=1}^{n} b_k (u^{n-k+1} - u^{n-k}) + f^{n+1}, \tag{16}$$

at $n = 0$ and $n \geq 1$, respectively. Now we approximate $u^n(x_i)$ by the RBFs on the N collocation points as follows:

$$u_i^n = u^n(x_i) \simeq \sum_{j=1}^{N} \lambda_j^n \phi(r_{ij}) + \lambda_{N+1}^n x_i + \lambda_{N+2}^n, \qquad i = 1, 2, \ldots, N. \tag{17}$$

where $r_{ij} = \|x_i - x_j\|$. The additional conditions can be described as

$$\sum_{j=1}^{N} \lambda_j^{n+1} = \sum_{j=1}^{N} \lambda_j^{n+1} x_j = 0. \tag{18}$$

By considering Eq. (17) together with Eq. (18) in a matrix form, we obtain

$$[u]^{n+1} = A[\lambda]^{n+1}, \tag{19}$$

where $[u]^{n+1} = [u_1^{n+1} u_2^{n+1} \ldots u_N^{n+1} 0\ 0]^T$ and $[\lambda]^{n+1} = [\lambda_1^{n+1} \lambda_2^{n+1} \ldots \lambda_{N+2}^{n+1}]^T$ and A is

$$A = \begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix}_{(N+2)\times(N+2)}, \quad \Phi = [\phi_{ij}]_{N\times N}, \quad P = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix}_{N\times 2} \tag{20}$$

Reconstruction of Eq. (15) in the matrix form can be illustrated as follows:

$$[c]^1 = B[\lambda]^1, \tag{21}$$

in which

$$B = \begin{bmatrix} \mathcal{L}\Phi & \mathcal{L}P \\ P^T & 0 \end{bmatrix}_{(N+2)\times(N+2)} \tag{22}$$

where $\mathcal{L}$ is an operator defined by Eq. (15) as

$$\mathcal{L}(*) = \begin{cases} (a_0 - \nabla^2 + p(x_i)\nabla + p'(x_i))(*), & 1 < i < N \\ (*), & i = 1 \text{ or } i = N \end{cases}, \tag{23}$$

and $[c]^1 = [c_1^1, c_2^1, \cdots, c_N^1, 0, 0]^T$, where $c_1^1 = h_1^1$, $c_N^1 = h_2^1$ and

$$c_i^1 = a_0 u_i^0 + f_i^1, \qquad i = 2, 3, \ldots, N-1.$$

Also, for $n \geq 1$,

$$[c]^{n+1} = B[\lambda]^{n+1}, \tag{24}$$

where $[c]^{n+1} = [c_1^{n+1}, c_2^{n+1}, \cdots, c_N^{n+1}, 0, 0]^T$ are obtained by Eq. (16) as

$$c_i^{n+1} = a_0 u_i^n - a_0 \sum_{k=1}^{n} b_k (u_i^{n-k+1} - u_i^{n-k}) + f_i^{n+1}, \qquad i = 2, 3, \ldots, N-1$$

and finally considering boundary conditions $c_1^{n+1} = h_1^{n+1}$ and $c_N^{n+1} = h_2^{n+1}$.

3 Numerical results

Example 3.1. Consider the following problem from [3]

$$D_t^\alpha u - u_{xx} - e^{(x-0.5)^2} u_x - 2(x-0.5)e^{(x-0.5)^2} u = f(x,t), \qquad 0 < x < 1,\ 0 < t \leq 1, \tag{25}$$

with exact solution $u = t^2 sin(\pi x)$. The initial and boundary functions g, h_1, h_2 and right-hand side function f can be obtained from the exact solution. The L_∞, L_2 and RMS of errors are obtained in Table 1 for different α. The numerical results demonstrate the good accuracy of this scheme.

Table 1: L_∞,L_2 and RMS errors, with $n = 100$, $N = 100$

α	$L_\infty - error$	$L_2 - error$	RMS
0.2	3.9104×10^{-6}	2.6019×10^{-5}	2.6019×10^{-6}
0.5	4.2431×10^{-5}	2.9644×10^{-4}	2.9644×10^{-5}
0.8	2.9348×10^{-4}	2.1000×10^{-3}	2.0604×10^{-4}

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Numerical solution of functional nonlinear Feredholm integral equations by using RBF interpolation

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Abstract

The main aim of this study is to obtain numerical solution of functional nonlinear Fredholm integral equations using meshless Radial Basis Function (RBF) interpolation which is based on linear combinations of terms. Applying RBF in functional integral equation, a linear system $\Psi C = G$ will be obtain which by defining coefficients vector C, target function will be approximiated. Finally, validity of the method is illustrated by some examples.

Keywords: Functional nonlinear Fredholm integral equations, Radial basis functions, Multi quadric functiones, Meshless method
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

RBFs are computationally means to approximate functions which are complicated or have many variables, by other simpler functions which are easier to understand and readily evaluated. One of the outstanding advantages of interpolation by RBF, unlike multivariable polynomial interpolation or splines [1], is applicability in scattered data aspect of existence and uniqueness results since there is little restrictions on dimension and also high accuracy or fast convergence to the target function. As another advantage of RBF there are not required to triangulations of the data points, while other numerical methods such as finite element or multivariate spline methods need triangulations [1, 2]. This requirement computationally cost, especially in more than two dimensions. In this paper, we consider functional nonlinear integral equations of Fredholm type with unknown function $y(x)$. To approximate the target function, we employ RBF interpolation in distinct grids from a definite domain. To this purpose, consider N distinct points as $(x_1, x_2, \ldots, x_N) \in R^d$, in an d dimensional Euclidean space at which the function to be approximated is known

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and real scalars $(g(x_1), g(x_2), \ldots, g(x_N))$ and a continuous function $s : R^d \to R$ is considered to construct so that $s(x_j) = g_j$ for $j = 1, 2, \ldots, N$. Radial basis function method is based on choosing a continuous function such as $\phi : R^+ \to R^+$ and a norm $\| \ . \ \|$ in R^d, then s can be such as $s(x) = \sum_{j=1}^{N} c_j \phi(\| \ x - x_j \ \|)$, where c_j's are unknown scalars for $j = 0, 1, \ldots, N$ which should be defined. Gaussian (GA) $\phi(r) = \exp(-\beta r^2)$, Multiquadric (MQ) $\phi(r) = \sqrt{(r^2 + \beta^2)}$ and Inverse quadric (IQ) $\phi(r) = (r^2+\sigma^2)^{-1}$ are some well-known functions that generate RBF which is infinitely globally supported differentiable, and depend on a free parameter β which is a real constant known as the shape parameter or width of the RBF that must be specied by the user and $r = \|x - x_i\|$. In this study, shape parameter is considered $\beta = 1$. In the next Section, the application of RBF to functional integral equations to approximate the solution of functional nonlinear Fredholm integral equations will be described and efficiency of the method is shown by two examples.

2 Main results

Consider the general functional nonlinear integral equation of Fredholm type:

$$y(x) + A(x)y(h(x)) + \lambda \int_a^b k(x, t, y(t))dt = g(x), \quad a \le x \le b. \tag{1}$$

where λ is a real number, the kernel $k(x, t)$ is a continuous function in $[a, b] \times [a, b]$ and $A(x)$, $h(x)$ and $g(x)$ are analytical known functions. By applying RBF interpolation to approximate $y(x)$ as the solution of (1), linear combination of functions ϕ_j is replased in $y(x)$ as the following form:

$$y(x) \approx \sum_{i=0}^{N} c_i \phi_i(x), \qquad y(h(x)) \approx \sum_{i=0}^{N} c_i \phi_i(h(x)), \tag{2}$$

$$\int_a^b k(x, t, y(t))dt \approx \int_a^b k\left(x, t, \sum_{i=0}^{N} c_i \phi_i(t)\right) dt. \tag{3}$$

Replacing Eqs. (2) and (3) into Eq. (1) and considering $x = x_j$, the following equivalence will yeild for $j = 1, \ldots N$:

$$g(x_j) = A(x_j) \sum_{i=0}^{N} c_i \phi_i(h(x_j)) + \sum_{i=0}^{N} c_i \phi_i(x_j) + \lambda \int_a^b k\left(x_j, t, \sum_{i=0}^{N} c_i \phi_i(t)\right) dt \tag{4}$$

According to interpolation conditions a nonlinear system will be defined as $\Psi C = G$, where $\Psi = \phi_i(x_j)$ is called a distance matrix, and given by $\Psi_{ij} = \phi(\| \ x_i - x_j \ \|)$ and also $C = (c_1, \ldots, c_N)^T$ and $G = (g_1, \ldots, g_N)^T$. To solve this nonlinear system, Newton method is used which is briefly described here [3, 4]: Let we have $F_j(c_1, c_2, \ldots, c_N) = 0$, where $F = \Psi C - G$. The aim is definig unknown vector of parameters C by appliying Newton method. Given $C^n = (c_1^{(n)}, c_2^{(n)}, \ldots, c_N^{(n)})$ we want to find $C^{n+1} = (c_1^{(n+1)}, c_2^{(n+1)}, \ldots, c_N^{(n+1)})$ as the following:

$$C^{(n+1)} = C^{(n)} + H^{(n)}. \tag{5}$$

$H^{(n)}$ would be obtained, by solving the linear equations $\Gamma^{(n)}H^{(n)} = B^{(n)}$, where $\Gamma^{(n)} = \gamma_{jl}^{(n)}$ is a $N \times N$ matrix and $B^{(n)} = b_j^{(n)}$ is a $N \times 1$ matrix in which:

$$\gamma_{jl}^{(n)} = \frac{\partial}{\partial c_l} F_j(C^{(n)}), \quad b_j^{(k)} = -F_j(C^{(n)}) \tag{6}$$

Let $E = C^{(n+1)} - C^{(n)}$, we consider $\|E\| \leq \epsilon$ as stop condition for a defined ϵ. To illustrate the efficiency of the proposed technique, we consider two examples in which the error is calculated from $E_{RMS} = \sqrt{\frac{\sum_{i=1}^{N}(y(x_i) - s(x_i))^2}{N}}$.

Example 2.1. Consider the following functional nonlinear integral equation:

$$y(x) + e^{-x}y(\frac{x^2}{2}) + \int_{-1}^{1} t(y(t))^2 dt = e^x - 2e^{2x} + e^{-x+\frac{x^2}{2}}$$

where the analytical solution is $y(x) = e^x$. Gaussian function, $\phi(r) = \exp(-r^2)$ is used in RBF interpolation in which $N = 20$ and $\epsilon = 10^{-5}$. The error is $E_{RMS} = 1.66533 \times 10^{-15}$. The solution of analytical and the proposed methods are analytical and proposed method and also accuracy of the method are shown in Figure 1.

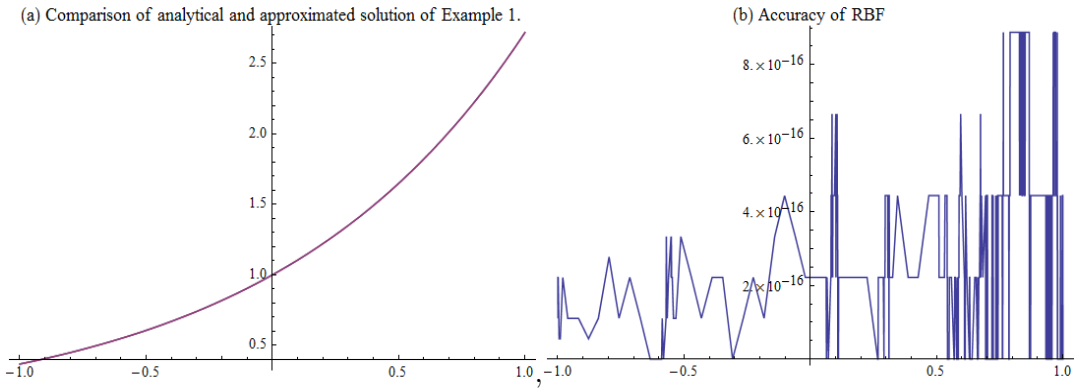

Figure 1: (a) Plot of both analytical solution and RBF solution, (b) error estimation of proposed method for functional integral equatin of Example 1.

Example 2.2. Consider the following functional nonlinear integral equation:

$$y(x) + 2x\sin(\pi\frac{x^2}{2}) - \frac{1}{5}\int_0^1 e^x y(t)dt = \sin(\pi x) - 0.2e^x\sin(\pi x) + 2x\sin(\pi\frac{x^2}{2})$$

where the analytical solution is $y(x) = \sin(\pi x)$. Gaussian function, $\phi(r) = \exp(-r^2)$ is used in RBF interpolation in which $N = 20$ and $\varepsilon = 10^{-5}$. The error is $E_{RMS} = 2.7526 \times 10^{-7}$

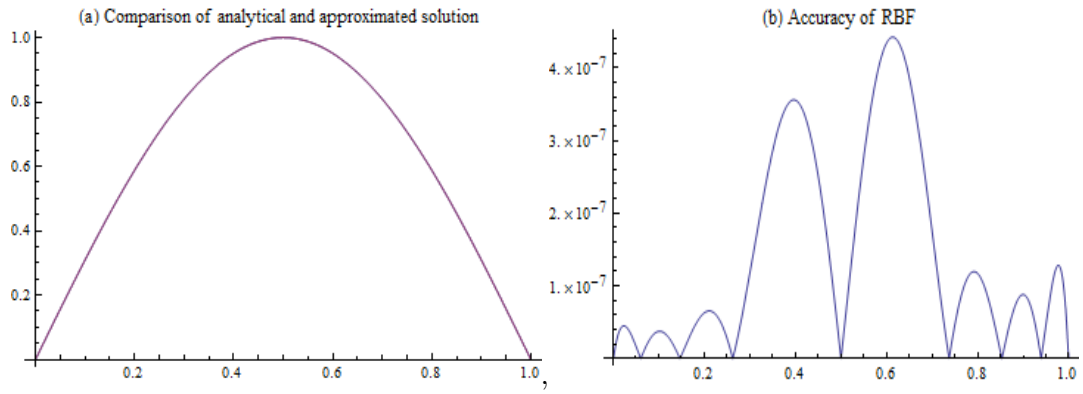

Figure 2: (a) Plot of both analytical solution and RBF solution, (b) error estimation of proposed method for functional integral equatin of Example 2.

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Poster On entropy for fuzzy sets and discrete dynamical system pp.: 1–4

On entropy for fuzzy sets and discrete dynamical system

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Abstract

In this paper a random variable on a probability space which is a fuzzy set is introduced and the entropy of the fuzzy set with respect to a finite partition is defined. As follow a discrete dynamical system on a fuzzy set is presented and its entropy has been defined.

Keywords: Dynamical system, Random variable, fuzzy set, Entropy
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

One of the most important characterizations can attach to a random variable and to a stochastic process is its entropy. The notion of entropy for discrete random variables as well as absuloutly continuous random variables is well defined. The discrete random variable is a random variable with at most countable image. Absoultly continuous random variable is a random variable with uncountable image that has a non- negative density function[1, 4]. A stochastic process is a mathematical model for the occurrence of random phenomena as time goes on. This is the case, for example, when a random experiment is repeated over and over again [1].
In this paper, we consider a probability space, Ω, which is fuzzy set with function $m : \Omega \longrightarrow [0, 1]$[2].
In the next section, we define the entropy of the random variable X^α, which α is a finite partition on Ω.
We introduce random variables on the fuzzy set with respect to a discrete dynamical system on Ω and define the entropy of the dynamical system in the last section.

2 Entropy of a fuzzy set

Let Ω be a probability measure space with $\sigma-$ algebra β and probability measure μ. Also consider (Ω, m) be a fuzzy set $(m : \Omega \longrightarrow [0, 1])$ and $\alpha = \{A_1, A_2, ..., A_{|\alpha|}\}$ be a finite partition on Ω.
Define $\chi_0^\alpha : \Omega \longrightarrow \{1, 2, ..., |\alpha|\}$, $\chi_0^\alpha(\omega) = i$ where $\omega \in A_i$.
Consider the random variable $X^\alpha : \Omega \longrightarrow \{1, 2, ..., |\alpha|\}\times[0, 1]$ which $X^\alpha(\omega) = (\chi_0^\alpha(\omega), m(\omega))$, so two different cases will be happened:

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Case 1) $Im\ m$ is at most countable, so the image of X^α is at most countable. Hence by the following process, the entropy of the fuzzy set can be computed.
χ_0^α and m are discrete random variables and the joint probability function defined by [1, 3]

$$p(x,y) = \mu\{\omega \in \Omega : \chi_0^\alpha(\omega) = x, m(\omega) = y\} \tag{1}$$

the joint entropy of χ_0^α and m is defined as follow

$$H(\chi_0^\alpha, m) = -\sum_{i\in S_1}\sum_{j\in S_2} p(x,y)logp(x,y) \tag{2}$$

where $S_1 = \{1, 2, ..., |\alpha|\}$ and $S_2 = Im\ m$.

Remark 2.1. If $m(x) = k$ on Ω for $k \in [0, 1]$ then the entropy of the random variable (χ_0^α, m) is equal to the Shannon entropy.

$$p(i,k) = \mu\{\omega \in \Omega : \chi_0^\alpha(\omega) = i, m(\omega) = k\} = \mu\{\omega \in \Omega : \chi_0^\alpha(\omega) = i\} = \mu(A_i). \tag{3}$$

and for $y \neq k$

$$p(i,y) = \mu\{\omega \in \Omega : \chi_0^\alpha(\omega) = i, m(\omega) = y\} = 0. \tag{4}$$

Therefore

$$H(\chi_0^\alpha, m) = -\sum_i \mu(A_i)log\mu(A_i) = H(\chi_0^\alpha). \tag{5}$$

Case 2) $Im\ m$ is uncountable, in this case we assume that m is absoulotly continuous, so X^α is joint of a discrete and absoultly continous random variables.
In these conditions, we compute the entropy of the fuzzy set. We denote the random variable X^α with $X^\alpha = (\chi_0^\alpha, m)$, we call X^α a mixed- pair random variable if χ_0^α is a discrete random variable and m is an absoultly continouos random variable.
Observe that $X^\alpha = (\chi_0^\alpha, m)$ induces measures $\{\mu_1, \mu_2, ..., \mu_\alpha\}$ that are absoulutely continnous with respect to the lebesgue measure on $[0, 1]$ where $\mu_i(A) = prob(\chi_0^\alpha = i, m \in A)$ for any $A \in B[0, 1]$, ($B[0, 1]$ borel $\sigma-$algebra) [3].
Hence $\mu_i : B[0, 1] \longrightarrow [0, 1]$ is lebesgue measure on $[0, 1]$ where $\mu_i(A) = \mu(m^{-1}(A) \bigcap A_i)$. μ_i is absoultly continuous with respect to lebesgue measure, so there are $g_i : [0, 1] \longrightarrow \mathbb{R}$ such that $\int_A g_i(x)dx = \mu_i(A)$, ($g_i$ is called a probability density function with respect μ_i's) that satisfy $\sum_i \int_{\mathbb{R}} g_i(y)dy = 1$.

Definition 2.2. The entropy of the random variable X^α is defined by

$$H(X^\alpha) = -\sum_i \int_{\mathbb{R}} g_i(y)logg_i(y)dy \tag{6}$$

Example 2.3. Let $([0, 1], \beta, \mu)$ be a probability measure space and $\alpha = \{A_1, A_2, ..., A_{|\alpha|}\} \subseteq \beta$ be a finite partition on $[0, 1]$. Consider $\mu(A_i) = p_i$ and m is an independent continuous random variable distributed uniformely on the interval $[0, 1]$. We have $g_i(y) = p_i$ for every $y \in [0, 1]$. Therefore

$$H(X^\alpha) = -\sum_i \int_{\mathbb{R}} g_i(y)logg_i(y)dy = -\sum_i \int_{\mathbb{R}} p_i logp_i dy = -\sum_i p_i logp_i = H(\chi_0^\alpha) < \infty. \tag{7}$$

3 Entropy of a dynamical system on a fuzzy set

Let Ω be a probability space with $\sigma-$ algebra β and probability measure μ. Consider a measure- perserving transformation $f : \Omega \longrightarrow \Omega$ which induces a discrete dynamical system on Ω. Also we assume that (Ω, m) is a fuzzy set.
Let $\alpha = \{A_1, A_2, ..., A_{|\alpha|}\}$ be a finite partition on Ω where $A_i \in \beta$. Define the random variable

$$\chi_n^\alpha : \Omega \longrightarrow \{1, 2, ..., |\alpha|\}$$
$$\chi_n^\alpha(\omega) = i$$

where $f^n(\omega) \in A_i$.

Definition 3.1. The random variable of a discrete dynamical system on the fuzzy set (Ω, m) with respect to the finite partition α on Ω is defined as follow

$$X_n^\alpha : \Omega \longrightarrow \{1, 2, ..., |\alpha|\} \times [0, 1]$$

$X_n^\alpha(\omega) = (\chi_n^\alpha(\omega), m_n(\omega))$ where $m_n(\omega) = m(f^n(\omega))$. And $X^\alpha = \{X_n^\alpha\}_{n=0}^\infty$ is a stochastic process.

Consider $Im\ m_n$ be at most countable, so χ_n^α and m_n are discrete random variables. The joint probability function defined by

$$p(x, y) = \mu\{\omega \in \Omega : \chi_n^\alpha(\omega) = x, m_n(\omega) = y\} \tag{8}$$

and the joint entropy of χ_n^α and m_n is as follow

$$H(X_n^\alpha) = H(\chi_n^\alpha, m_n) = -\sum_{i \in S_1} \sum_{j \in S_2} p(x, y) log p(x, y) \tag{9}$$

where $S_1 = \{1, 2, ..., |\alpha|\}$ and $S_2 = Im\ m_n$.

Theorem 3.2. *Let χ_n^α and m_n be discrete random variables. Then $H(\chi_n^\alpha, m_n) = H(\chi_n^\alpha) + H(m_n|\chi_n^\alpha)$.*
Moreover, if χ_n^α and m_n are independent, therefore

$$H(X_n^\alpha) = H(\chi_n^\alpha) + H(m_n).$$

Corollary 3.3. $H(X_0^\alpha, X_1^\alpha, ..., X_{n-1}^\alpha) = H(\chi_0^\alpha, \chi_1^\alpha, ..., \chi_{n-1}^\alpha) + H(m_0, m_1, ..., m_{n-1})$.

Definition 3.4. Entropy of the stochastic process X^α is defined by

$$h_\alpha(f, m) := H(X^\alpha) = lim_{n\to\infty} \frac{1}{n} H(X_0^\alpha, X_1^\alpha, ..., X_{n-1}^\alpha).$$

Definition 3.5. Entropy of the fuzzy set (X, m) with respect to the discrete dynamical system f is

$$h(f, m) = sup_\alpha h_\alpha(f, m)$$

where α is a finite partition on ω.

Example 3.6. Let $f : [0,1] \longrightarrow [0,1]$ be a measure preserving function ($i.e. \mu(f^{-1}(A)) = \mu(A)$) and $\alpha = \{[\frac{k}{2^m}, \frac{k+1}{2^m}) : k = 0, 1, ..., 2^m - 1\}$ be a finite partition on $[0,1]$. Consider $\chi_n^\alpha(\omega) = i$ where $f^n(\omega) \in [\frac{i-1}{2^m}, \frac{i}{2^m})$ and $m_n(\omega) = \frac{k}{2^m}$ where $f^n(\omega) \in [\frac{k}{2^m}, \frac{k+1}{2^m})$.
$p(i, \frac{j}{2^m}) = \mu\{\omega : \chi_n^\alpha(\omega) = i, m_n(\omega) = \frac{j}{2^m}\} = \mu\{\omega : \frac{i-1}{2^m} \leqslant f^n(\omega) < \frac{i}{2^m}, \frac{j}{2^m} \leqslant f^n(\omega) < \frac{j+1}{2^m}\} = \mu(f^{-n}([\frac{i-1}{2^m}, \frac{i}{2^m}) \bigcap [\frac{j}{2^m}, \frac{j+1}{2^m})))$.
If $i = j$, so $p(i, \frac{i}{2^m}) = \mu([\frac{i-1}{2^m}, \frac{i}{2^m}) = \frac{1}{2^m}$. If $i \neq j$ then $p(i, \frac{j}{2^m}) = 0$.
Therefore
$H(X_n^\alpha) = H(\chi_n^\alpha, m_n) = -\sum_i p(i, \frac{i}{2^m}) log p(i, \frac{i}{2^m}) = -\sum_{i=1}^{2^m} \frac{1}{2^m} log \frac{1}{2^m} = \sum_{i=1}^{2^m} \frac{m}{2^m} log 2 = m log 2$.

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Robust ergodicity of expanding transitive actions

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Abstract

In this talk, we present that every expanding transitive group (or semigroup) action of $C^{1+\alpha}$ conformal diffeomorphisms of a compact manifold is robustly ergodic with respect to the Lebesgue measure.

Keywords: Robust ergodicity, Expanding, Robust Transitivity, semigroup (group) actions.
Mathematics Subject Classification [2010]: 37A05, 37C05

1 Introduction

Ergodic theory is one of the parts of theory of dynamical systems. The theory deals with measure preserving actions of measurable maps on a measure space. A measure-preserving map is called ergodic if the measure of every invariant measurable set be either 0 or 1. For ergodicity of group (or semigroup) actions, the notion of measure-preserving map can be extended to *quasi-invariant* group (or semigroup) action. A group (or semigroup) action is quasi-invariant with respect to a measure μ if the puch-forward of μ, by the generators of action, be absolutely continuous with respect to μ.

Authors in [1] showed that every expanding minimal semigroup action of $C^{1+\alpha}$ conformal diffeomorphisms of a compact manifold is robustly ergodic with respect to Lebesgue measure. They used the tools of Lebesgue density point and Lebesgue number. We obtian the ergodicity of group (or semigroup) actions by weaker assumptions. We show that every expanding transitive group (or semigroup) action of $C^{1+\alpha}$ conformal diffeomorphisms of a compact manifold is robustly ergodic with respect to Lebesgue measure. We also present an example for showing different our work from [1]

Minimality and so transitivity, in general, does not imply ergodicity. See [2, 3].

1.1 Notations and definitions

Consider a collection of diffeomorphisms $\{f_1, f_2, \cdots, f_k\}$ on a compact manifold M. Let us denote by $\mathcal{F}$ (or $\mathcal{F}^+$) the group (or semigroup) action generated by $f_1, \cdots, f_k$.

Consider the group (or semigroup) action $\mathcal{F}$ (or $\mathcal{F}^+$). Let $\sum_k$ (or $\sum_k^+$) be the space of two-sided (or one-sided) infinite sequences of elements of the set $\{1, \cdots, k\}$. For $\omega =$

*Speaker

$(\omega_0\,\omega_1\cdots)\in\sum_k^+$ we introduce the notation, $f_\omega^0=\mathrm{id}$, $f_\omega^n=f_{\omega_{n-1}}\circ\cdots\circ f_{\omega_0}$ and for $\omega=(\cdots\omega_{-1}\,;\omega_0\,\omega_1\cdots)\in\sum_k$

$$f_\omega^n=\begin{cases} f_{\omega_{n-1}}\circ\cdots\circ f_{\omega_0} & \text{if } n\geq 1\\ \mathrm{id} & \text{if } n=0\\ f_{\omega_n}^{-1}\circ\cdots\circ f_{\omega_{-1}}^{-1} & \text{if } n\leq -1.\end{cases}$$

A set $A\subset M$ is *invariant* for $\mathcal{F}$ (or $\mathcal{F}^+$) if $g(A)=A$ (or $g(A)\subset A$) for all $g\in\mathcal{F}$ (or $g\in\mathcal{F}^+$).

Let us denote both group action $\mathcal{F}$ and semigroup action $\mathcal{F}^+$ by $\mathcal{G}$. A probability measure μ is quasi-invariant for an action $\mathcal{G}$ if $g_*\mu\ll\mu$ for all $g\in\mathcal{G}$, where $g_*\mu$ is the puch-forward of μ. We consider the Lebesgue measure m which is quasi-invariant for C^1 diffeomorphisms.

An action $\mathcal{G}$ is *ergodic* with respect to a quasi-invariant probability measure μ if $\mu(A)=0$ or $\mu(A)=1$, for all $\mathcal{G}$-invariant set $A\subset M$.

Let $m(T)$ be the co-norm of a linear transformation T, i.e., $m(T)=\|\,T^{-1}\,\|^{-1}$.

Definition 1.1. An action $\mathcal{G}$, generated by C^1 diffeomorphisms $f_1,f_2,\cdots,f_k$, is expanding if for every $x\in M$ there is $g\in\mathcal{G}$ such that

$$m(Dg^{-1}(x))>1.$$

A diffeomorphism f is *confomal* if the derivative expands or contracts distances by the same amount in all directions. In other word, there exists a map $\alpha:M\to\mathbb{R}$ such that $Df(x)=\alpha(x)Ism(x)$, for all $x\in M$, where $Ism(x)$ denotes an isometry of T_xM. So, for every $x\in M$,

$$\|\,Df(x)\,\|=m(Df(x))=\alpha(x).$$

For $x\in M$, we write the orbit of x as $\mathcal{O}_\mathcal{G}(x)=\{f(x):f\in\mathcal{G}\}$. A branch orbit of x corresponding to $\omega\in\sum_k$ (or $\sum_k^+$) is the sequence of $\mathcal{O}_\omega(x)=\{f_\omega^n(x)\}_{n=-\infty}^\infty$ (or $\mathcal{O}_\omega^+(x)=\{f_\omega^n(x)\}_{n=0}^\infty$).

Definition 1.2. The action of $\mathcal{G}$ is transitive if there exists a dense orbit $\mathcal{O}_\mathcal{G}(x)\subset M$, for some $x\in M$.

A property is said to be C^r-robust for $\mathcal{G}$ if it hold for the action $\tilde{\mathcal{G}}$, for which generators are C^r-perturbations of generators of $\mathcal{G}$. Our main result is as follows.

Theorem 1.3. *Every expanding transitive group (or semigroup) action generated by $C^{1+\alpha}$ conformal diffeomorphisms of a compact manifold is robustly ergodic with respect to the Lebesgue measure.*

2 Main results

The following lemma is obtained straightforward from the compactnees of M.

Lemma 2.1. *The action of $\mathcal{G}$ is expanding if and only if there are maps $g_1,\cdots,g_k\in\mathcal{G}$, open balls $B_1,\cdots,B_k$ in M and a constant $\eta>1$ such that $M=B_1\cup\cdots\cup B_k$ and*

$$m(Dg_i^{-1}(x))>\eta\quad \textit{for all } x\in B_i.$$

As a consequence of the lemma we have:

Remark 2.2. The set of expanding actions of C^1-diffeomorphisms of a compact manifold is open. Indeed, expanding property of an action is robust under perturbation of the generators.

Theorem 2.3. *Every expanding transitive action $\mathcal{G}$ generated by C^1-diffeomorphisms of a compact manifold is C^1-robustly transitive.*

Proof. Consider an expanding transitive action $\mathcal{G}$ of C^1-diffeomorphisms of a compact manifold M. So $\mathcal{G}$ has a dense orbit of x, for some $x \in M$. By lemma 2.1 there exsit a constant $\eta > 1$, an open cover $\{B_1, \cdots, B_k\}$ of M and maps $g_1, \cdots, g_k \in \mathcal{G}$ such that

$$d(g_i(x), g_i(y)) < \eta^{-1} d(x, y) \quad \text{for all } x, y \in g_i^{-1}(B_i).$$

Let L be the Lebesgue number of cover $\{B_1, \cdots, B_k\}$. We consider a sufficiently small C^1-perturbation of the generators of $\mathcal{G}$ such that we have

- a finite open cover $\{B_i\}$ with Lebesgue number greater than $L/2$,
- maps $\{g_i\}$ in the perturbed action $\tilde{\mathcal{G}}$ such g_i restricted to $g_i^{-1}(B_i)$ is a contraction of rate η^{-1},
- the ε-density of the orbit of x with $\varepsilon \leq L/2$.

M is compact and so locally connected. We also consider ε small enough so that every open ball of radius $r \leq \varepsilon$ is a connected set.

Let $z \in B_i$ and $r \leq \eta^{-1}\varepsilon$ such that $B(z, r) \subset B_i$. Therefore $B(g_i^{-1}(z), \eta\, r)$ is connected. On the other hand, the restriction of g_i to $g_i^{-1}(B_i)$ is a contraction of rate η^{-1}. Hence

$$g_i(B(g_i^{-1}(z), \eta\, r)) \subset B(z, r). \tag{1}$$

Now we show that the orbit of x is dense under action $\tilde{\mathcal{G}}$. We have $\eta^{-1}\varepsilon < L/2$ and so for any $z \in M$, $B(z, \eta^{-1}\varepsilon) \subset B_i$ for some i. By the ε-density of the orbit of x, there is $g \in \tilde{\mathcal{G}}$ such that

$$g(x) \in B(g_i^{-1}(z), \varepsilon).$$

By (2.1)

$$g_i \circ g(x) \in B(z, \eta^{-1}\varepsilon).$$

Since z is arbitrary, this shows That the orbit of x is $\eta^{-1}\varepsilon$-dense. By induction, the orbit of x is $\eta^{-n}\varepsilon$-dense for every $n \in \mathbb{N}$ and so is dense. □

Let A and C be sets in M. We write $C \overset{\circ}{\subset} A$ and say C is contained (mod 0) in A if $m(C \setminus A) = 0$.

The following proposition is the main tool to proof of Theorem 1.3.

Proposition 2.4. *Consider an expanding group (resp. semigroup) action $\mathcal{G}$ generated by $C^{1+\alpha}$ conformal diffeomorphisms of a compact manifold M. Then, there exists $r > 0$ such that for every invariant set $A \subset M$ with positive Lebesgue measure (resp. A^c has positive Lebesgue measure) there exsit an open ball B of radios $r > 0$ such that*

$$B \overset{\circ}{\subset} A \quad (resp.\ B \overset{\circ}{\subset} A^c).$$

Proof. See [1], proposition 4.7. □

proof of theorem 1.3. Consider an expanding transitive action $\mathcal{G}$ generated by $C^{1+\alpha}$ conformal diffeomorphisms of a compact manifold M. Let $A \subset M$ be a $\mathcal{G}$-invariant measurable set. By contradiction, Assume that $0 < m(A) < 1$. As a consequence, A^c is invariant and $0 < m(A^c) < 1$. By proposition 2.4 there are the balls B_1 and B_2 such that $m(B_1 \setminus A) = 0$ and $m(B_2 \setminus A^c) = 0$. By transitivity of $\mathcal{F}$ (or $\mathcal{F}^+$), there exist $\omega \in \sum_k$ (or $\sum_k^+$) such that

$$m(f_\omega^n(B_1) \cap B_2) > 0 \qquad \text{for some } n \in \mathbb{N}.$$

Hence

$$m(f_\omega^n(B_1) \cap A^c) > 0.$$

On the other hands, since A is invariant we have

$$m(f_\omega^n(B_1) \setminus A) = m(f_\omega^n(B_1 \setminus A)) = 0.$$

The last equality holds because f_ω^n is a diffeomorphism. This gives a contradiction and so $\mathcal{G}$ is ergodic. The robustness is concluded by theorem 3.2 and remark 2.2. □

The following example show that every expanding transitive action is not necessarily an expanding minimal action.

Example 2.5. Consider the action of $\mathcal{G} = (M; f_1, f_2)$ where M is a compact manifold, f_1 an expanding transitive $C^{1+\alpha}$ conformal diffeomorphism which is not minimal and f_2 an identity map. then $\mathcal{G}$ is an expanding transitive action and so is robustly ergodic. Note that $\mathcal{G}$ is not minimal.

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Poster Shadowing property in nonautonomous discrete dynamical systems pp.: 1–4

shadowing property in nonautonomous discrete dynamical systems

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Abstract

In this paper we study shadowing property for sequences of mappings on compact metric spaces, i.e. nonautonomous discrete dynamical systems. We investigate the relations of expansivity and weak expansivity with shadowing and h-shadowing property.

Keywords: Nonautonomous, weak expansivity, shadowing, locally expanding
Mathematics Subject Classification [2010]: 37B99, 54H20

1 Introduction

Let (X, d) be a compact metric space, and f be a continuous map on X. We consider the associated autonomous difference equation of the following form:

$$x_{i+1} = f(x_i) \tag{1}$$

A finite or infinite sequence $\{x_0, x_1, ...\}$ of points in X is called a δ-pseudo-orbit ($\delta > 0$) of (1.1) if $d(f(x_{i-1}), x_i) < \delta$ for all $i \geq 1$. We say that equation (1.1), (or f) has usual shadowing property if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, ...\}$, there exists $y \in X$ with $d(f^i(y), x_i) < \varepsilon$ for all $i \geq 0$. The notion of pseudo-orbits appeared in several branches of dynamical systems theory, and various types of the shadowing property were presented and investigated extensively, see [1,7].
In this paper we study shadowing property of nonautonomous discrete systems. We consider the compact metric space X and a sequence $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ in which each $f_i : X \to X$ is continuous. We call the pair $(X, f_{1,\infty})$ a nonautonomous discrete system (on X). For further simplicity we use only $f_{1,\infty}$ in the sequel. The associated nonautonomous difference equation has the following form:

$$x_{i+1} = f_i(x_i) \tag{2}$$

For every $n \geq i \geq 1$, we write $f_i^n = f_n \circ f_{n-1} \circ ... \circ f_i$.
Orbit of a nonautonomous system $f_{1,\infty}$ in a point x is the following sequence:

$$O(x) = \{x, f_1(x), f_2 \circ f_1(x), ..., f_n \circ ... \circ f_1(x), ...\}$$

On the other hand a pseudo-orbit of the system is as follows:

*Speaker

Definition 1.1. A finite or infinite sequence $\{x_0, x_1, ...\}$ of points in X is called a δ-pseudo-orbit ($\delta > 0$) of (1.2), if $d(f_i(x_{i-1}), x_i) < \delta$ for all $i \geq 1$.

In the nonautonomous case the standard definition of shadowing has the following form, see [4, 6]:

Definition 1.2. We say that $f_{1,\infty}$ has shadowing property if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, ...\}$, there exists $y \in X$ with $d(y, x_0) < \varepsilon$ and $d(f_1^i(y), x_i) < \varepsilon$, for all $i \geq 1$.

2 Shadowing and expansivity

First we prove the following simple lemma.

Lemma 2.1. *The sequence $f_{1,\infty}$ has shadowing property if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every finite δ-pseudo-orbit is ε-shadowed.*

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be such that every finite δ-pseudo-orbit, $\frac{\varepsilon}{2}$-shadowed. Let $\{x_i\}_{i=1}^{\infty}$ be a δ-pseudo-orbit. For every $n \geq 1$, $\{x_0, x_1, ..., x_n\}$, $\frac{\varepsilon}{2}$-shadowed by $y_n \epsilon X$ and there is a subsequence $\{y_{n_k}\}_{k\geq 0}$ and a point $y \epsilon X$ such that $y_{n_k} \to y$ as $k \to \infty$. Now for each $i \geq 1$, there is a $n_k > i$ such that $d(f_1^i(y_{n_k}), f_1^i(y)) < \frac{\varepsilon}{2}$. Therefore

$$d(f_1^i(y), x_i) \leq d(f_1^i(y), f_1^i(y_{n_k})) + d(f_1^i(y_{n_k}), x_i) < \varepsilon$$

and hence $f_{1,\infty}$ has the shadowing property. □

There are several variants of shadowing property, we define a stronger form which is called h-shadowing, see [1, 3, 4].

Definition 2.2. The sequence $f_{1,\infty}$ has h-shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, ..., x_n\} \subseteq X$ there is $y \in X$ with $d(y, x_0) < \varepsilon$ and,

$$d(f_1^i(y), x_i) < \varepsilon \ for\ all\ 1 \leq i < n \qquad and \qquad f_1^n(y) = x_n.$$

In the case of an autonomous difference equation various notions of expansivity such as positively expansive, locally expanding,... have been introduced and their properties studied extensively, see [2, 5, 7]. We consider a nonautonomous form of expansivity and a modified form of equicontinuouity.

Definition 2.3. We say that the sequence $f_{1,\infty}$ is positively expansive, with expansive constant $e > 0$, if $x \neq y$, then for every $N \in \mathbb{N}$ there is $n \geq N$ such that $d(f_N^n(x), f_N^n(y)) > e$.

Theorem 2.4. *Suppose that the sequence $f_{1,\infty}$ is positively expansive and has shadowing property then it has h-shadowing property.*

Definition 2.5. The sequence $f_{1,\infty}$ called inverse equicontinuous if for every $x \in X$ and for every $\varepsilon > 0$ there exists $\delta(x) > 0$ such that:

$$B_{\delta(x)}(f_i(x)) \subseteq f_i(B_\varepsilon(x)) \ for\ all\ i$$

Remark 2.6. Suppose that $f_i : X \to X$ is one to one and surjective, for all i. Then the sequence $f_{1,\infty}$ is inverse equicontinuous if and only if the sequence $\{f_i^{-1}\}_{i=1}^{\infty}$ is equicontinuous.

Definition 2.7. We say that $f_{1,\infty}$ is weakly expanding small distances if there exists $\gamma > 0$ such that for every $x, y \in X$ and every i,

$$d(x, y) < \gamma \qquad \Longrightarrow \qquad d(f_i(x), f_i(y)) > d(x, y).$$

Definition 2.8. We say that $f_{1,\infty}$ is locally expanding if there exists $\lambda > 1$ such that for every $x \in X$, $i \geq 1$ and $\varepsilon > 0$, $B_{\lambda\varepsilon}(f_i(x)) \subseteq f_i(B_\varepsilon(x))$

Definition 2.9. We say that $f_{1,\infty}$ is weakly locally expanding if there exists $\gamma > 0$ such that for every $x \in X$, $i \geq 1$ and $\varepsilon < \gamma$, $B_\varepsilon(f_i(x)) \subseteq f_i(B_\varepsilon(x))$.

Lemma 2.10. *Suppose that the sequence $f_{1,\infty}$ is inverse equicontinuous and weakly expanding small distance then it has weakly locally expanding property.*

Definition 2.11. We say that $f_{1,\infty}$ is uniformly expanding if there exist $\lambda > 1$ and $\gamma > 0$ such that for every $x, y \in X$ and $i \geq 1$:

$$d(f_i(x), f_i(y)) < \gamma \quad \Rightarrow \quad d(f_i(x), f_i(y)) > \lambda d(x, y)$$

Definition 2.12. We say that $f_{1,\infty}$ is weakly uniformly expanding if there exists $\gamma > 0$ such that for every $x, y \in X$ and $i \geq 1$:

$$d(f_i(x), f_i(y)) < \gamma \quad \Rightarrow \quad d(f_i(x), f_i(y)) > d(x, y)$$

Remark 2.13. If $f_{1,\infty}$ is weakly uniformly expanding and for all $i \geq 1$, f_i is surjective, then $f_{1,\infty}$ is weakly locally expanding.

Proof. Let $\gamma > 0$ be as in the weakly uniformly expanding definition. It is enough to prove that for each $\epsilon < \gamma$, $B_\varepsilon(f_i(x)) \subseteq f_i(B_\varepsilon(x))$. If $z \in B_\varepsilon(f_i(x))$ then there is $y \in X$ such that $f_i(y) = z$. Since $d(f_i(x), f_i(y)) < \epsilon$, we obtain $d(x, y) < d(f_i(x), f_i(y)) < \epsilon$. So $z \in f_i(B_\varepsilon(x))$. □

Now we investigate the relation of h-shadowing and the expansivity notions mentioned above.

Theorem 2.14. *Suppose that there is a continuous map f such that $f_i \to f$ pointwise. If the sequence $f_{1,\infty}$ is inverse equicontinuous and weakly expanding small distances, and f is weakly expanding small distances then it has h-shadowing property.*

As a consequence, in the case of a single map we have the following result.

Corollary 2.15. *Suppose that $f : X \to X$ is a continuous and an open map. If f is weakly expanding small distances then f has h-shadowing property.*

Theorem 2.16. *The following conditions hold:*

(1) If the sequence $f_{1,\infty}$ is locally expanding, then it has h-shadowing property.

(2) If the sequence $f_{1,\infty}$ is uniformly expanding, and for all $i \geq 1$, f_i is surjective, then $f_{1,\infty}$ has h-shadowing property.

Theorem 2.17. *Suppose there is a continuous map f such that $f_i \to f$ pointwise. If both $f_{1,\infty}$ and f are weakly uniformly expanding, and for all $i \geq 1$, f_i is surjective then $f_{1,\infty}$ has h-shadowing property.*

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The infinite product representation of solutions of indefinite Sturm-Liouville problems with three turning points.

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Abstract

We study the infinite product representation of solutions of second order differential equation of Sturm-Liouville type on a finite interval having three turning points under the assumption that the turning points are types IV, II, III , respectively. Such representations are useful in the associated studies of inverse spectral problems for such equations.

Keywords: Turning point; Sturm-Liouville, Nondefinite problem; Infinite products, Hadamard Factorization Theorem; Spectral theory
Mathematics Subject Classification [2010]: 34E20, 34E05

1 Introduction

The main purpose of the paper is to consider the infinite product representation of solutions of second order differential equation of Sturm-Liouville type on a finite interval of the form

$$y'' + (\lambda\phi^2(x) - q(x))y = 0, \qquad 0 \le x \le 1, \tag{1}$$

The functions $\phi^2(x)$ and $q(x)$ are referred to as the coefficients of the problem, the function $\phi^2(x)$ as the weight; they are real valued on the interval $(0, 1)$. The zeros of $\phi^2(x)$ (assumed to be a discrete set) are called the *turning points* or *transition points* (TP) of ((2)). The parameter λ is real.
The nature of the solutions of such Sturm-Liouville equation in the neighborhood of the turning points have been the object of humerous investigations. Readers interested in a historical survey on linear turning point theory are referred to the survey article of MCHUGH [13] .The results of Doronidcyn [2], McKelvey [7], Langer [5], Dyachenko [3], and Tumanov [11] bring important innovations to the asymptotic approximation of solutions of Sturm-Liouville equations with two turning points.

The representation of solutions of Sturm-Liouville equations by means of an infinite product is a direct consequence of the fact that any solution $y(x, \lambda)$ defined by a fixed set of initial conditions (as we have seen above) is necessarily an entire function of λ for each fixed $x \in [-1, 1]$, whose order does not exceed $1/2$ (see [1]). It follows from the

*Speaker

classical Hadamard Factorization Theorem that such solutions are expressible as an infinite product, and so this gives an alternate description that has not been used as of yet for approximation purposes in the various applications. Such infinite product representations have been used effectively by Trubowitz [10] . In theoretical considerations revolving around the *inverse spectral problem* associated with ((2)) in *definite cases* that is, cases where the coefficient of the parameter λ in ((2)) is of a *fixed* sign in $[-1,1]$, as opposed to indefinite cases as considered here where clearly this is not the case.
In a previous article [14] we considered the real second order differential

$$y'' + (\lambda\phi^2(x) - q(x))y = 0, \qquad 0 \le x \le 1, \tag{2}$$

where λ is a real parameter and the functions $q(x)$ and $\phi^2(x)$ satisfy:

(i) $\phi^2(x)$ is real and has two zeros x_ν, $\nu = 1,2$ in $I \equiv [0,1]$, where the x_ν are of order ℓ_ν, where ℓ_1 is odd and ℓ_2 is even. In the terminology of [4], x_1 is of Type IV while x_2 is of Type II.

(ii) The function $\phi_0 : I \to R - \{0\}$ defined by setting $\phi_0(x) = \phi^2(x)\prod_{\nu=1}^{2}(x - x_\nu)^{-\ell_\nu}$ is twice continuously differentiable.
(iii) $q(x)$ is bounded and integrable in I.
We obtained the following results:

Theorem 1.1. *Let $U(x,\lambda)$ be the solution of ((2)) satisfying the initial conditions $U(0,\lambda) = 0, \frac{\partial U}{\partial x}(0,\lambda) = 1$. Then for $0 < x < x_1$,*

$$U(x,\lambda) = R_-(x) \mid \phi(x)\phi(0) \mid^{-\frac{1}{2}} \prod_{m\ge 1} \frac{\lambda - \lambda_m(x)}{z_m^2}$$

where $z_m = \frac{m\pi}{R_-(x)}$,$R_-(x) = \int_0^x \sqrt{max\{0,-\phi^2(t)\}}dt$, the sequence $\lambda_m(x), m \ge 1$, represents the sequence of negative eigenvalues on the Dirichlet problem associated with ((2)) on $[0,x]$.

Theorem 1.2. *For, $x_1 < x < x_2$,*

$$U(x,\lambda) = \frac{\pi}{8}\csc\frac{\pi\mu_1}{2}R_+^{1/2}(x)R_-^{1/2}(x) \mid \phi(x)\phi(0) \mid^{-\frac{1}{2}} \prod_{k\ge 1}\frac{(\lambda - r_{1k}(x))R_-^2(x)}{\tilde{j}_k^2}\prod_{k\ge 1}\frac{R_+^2(x)(u_{1k}(x)-\lambda)}{\tilde{j}_k^2},$$

where $R_+(x) = \int_0^x \sqrt{max\{0,\phi^2(t)\}}dt$, $R_-(x) = \int_0^x \sqrt{max\{0,-\phi^2(t)\}}dt$, the sequence $\{u_{1k}(x)\}$ represents the sequence of positive eigenvalues and $\{r_{1k}(x)\}$ the sequence of negative eigenvalues of the Dirichlet problem associated with ((2)) on [0,x].

Theorem 1.3. *For $x = x_1$,*

$$U(x_1,\lambda) = \frac{\mid \phi(0) \mid^{-\frac{1}{2}}}{2\mu_1}\psi(x_1)R_-(x_1)^{\frac{1}{2}+\mu_1}\prod_{n\ge 1}\frac{(\lambda - \lambda_n(x_1))R_-^2(x_1)}{j_n^2}.$$

Where $R_-(x) = \int_0^x \sqrt{max\{0,-\phi^2(t)\}}dt$, $j_n, n = 1,2,\ldots$ is the sequence of positive zeros of the Bessel functions of order μ_1, the sequence $\lambda_n(x_1)$ represents the sequence of negative eigenvalues of the Dirichlet problem associated with ((2)) on $[0,x_1]$ and $\psi(x_1) = \lim_{x\to x_1}\phi^{-\frac{1}{2}}(x)\{\int_{x_1}^x \phi(t)dt\}^{\frac{1}{2}-\mu_1}$.

Theorem 1.4. *For $x_2 < x \leq 1$,*

$$U(x,\lambda) = \frac{\pi}{8}(\csc\frac{\pi\mu_1}{2})(\csc\pi\mu_2)R_+^{1/2}(x)R_-^{1/2}(x) \mid \phi(x)\phi(0) \mid^{-\frac{1}{2}} \prod_{k\geq 1}\frac{(\lambda - r_{2k}(x))R_-^2(x)}{\tilde{j}_k^2}\prod_{k\geq 1}\frac{R_+^2(x)(u_{2k}(x)-\lambda)}{\tilde{j}_k^2},$$

where $R_+(x) = \int_0^x \sqrt{max\{0,\phi^2(t)\}}dt$, $R_-(x) = \int_0^x \sqrt{max\{0,-\phi^2(t)\}}dt$, the sequence $\{u_{2k}(x)\}$ represents the sequence of positive eigenvalues and $\{r_{2k}(x)\}$ the sequence of negative eigenvalues of the Dirichlet problem associated with ((2)) on [0,x].

Theorem 1.5. *For $x = x_2$,*

$$U(x_2,\lambda) = \frac{\sqrt{\pi}\Gamma(\mu_2) \mid \phi(0) \mid^{-\frac{1}{2}}}{4\Gamma(\mu_2+\frac{1}{2})}\psi(x_2)\csc\frac{\pi\mu_1}{2}R_+^{\mu_2}(x_2)R_-^{\frac{1}{2}}(x_1)\prod_{k\geq 1}\frac{(\lambda - r_{2k}(x))R_-^2(x)}{\tilde{j}_k^2}$$
$$\prod_{m\geq 1}\frac{(u_{m2}(x_2)-\lambda)R_+^2(x_2)}{\tilde{r}_m^2}$$

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The solution of two-dimensional inverse heat conduction problems by using two methods: Finite difference method and Duhamel Integral method

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Abstract

In this paper we present a new simple method consists the matrix form of Duhamel's principle for solving two-dimensional IHCP using temperature data containing significant noise and comparative this method with the numerical algorithm based on finite-difference method and the least-squares scheme for solving the inverse problem. The measurements ensure that the inverse problem has a unique solution, but both of these methods solution is unstable hence the problem is ill-posed. This instability is overcome using the Tikhonov regularization method with the gcv criterion for the choice of the regularization parameter.

Keywords: Two-dimensional inverse heat conduction problem, Duhamel's theorem, Tikhonov regularization method, SVD method

Mathematics Subject Classification [2010]: 65M32, 35K05

1 Introduction

Inverse heat conduction problems (IHCPs) have been extensively studied over the last 60 years. They have numerous applications in many branches of science and technology. The problem consists in determining the temperature and flux heat at inaccessible parts of the boundary of a 2 or 3-dimensional body from corresponding data on accessible parts of the boundary. It is well-known that IHCPs are severely ill-posed which means that small perturbations in the data may cause extremely large errors in the solution.

The inverse problem is to find one part of the boundary conditions in two-dimensional body while the temperature measurements at the other part are given.

For $Q = \{(x, y, t) : x \in (0, 1), y \in (0, 1), t \in (0, t_M)\}$, the dimensionless mathematical formulation of two-dimensional IHCP may be expressed as follows:

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$$U_t(x,y,t) = U_{xx}(x,y,t) + U_{yy}(x,y,t), \quad \text{in } Q, \tag{1a}$$
$$U(0,y,t) = q(y,t), \quad 0 \le y \le 1, 0 \le t \le t_M, \tag{1b}$$
$$U(1,y,t) = \mu(y,t), \quad 0 \le y \le 1, 0 \le t \le t_M, \tag{1c}$$
$$U(x,0,t) = \psi(x,t), \quad 0 \le x \le 1, 0 \le t \le t_M, \tag{1d}$$
$$U(x,1,t) = p(x,t), \quad 0 \le x \le 1, 0 \le t \le t_M, \tag{1e}$$
$$U(x,y,0) = h(x,y), \quad 0 \le x \le 1, 0 \le y \le 1, \tag{1f}$$

where $\mu(y,t)$, $\psi(x,t)$, $p(x,t)$ an d $h(x,y)$ are known functions and t_M represents the final time of interest for the time evolution of the problem while $q(y,t)$ is unknown.

2 Main results

2.1 Description of Duhamel's Method

First, for a fixed point (x_1, y_1) where $0 < x_1 < 1$ and $0 < y_1 < 1$,we suppose the overspecified condition

$$U(x_1,y_1,t) = g(x_1,y_1,t), \quad 0 \le t \le t_M, \tag{2}$$

The solution of the problem (1) can can be written as follows

$$U(x,y,t) = \sum_{i=1}^{5} U_i(x,y,t),$$

where $U_i(x,y,t)$, for $i = 1,2,3,4,5$, satisfy the following problem:

$$\frac{\partial T_i}{\partial t} = \frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2}, \quad \text{in } Q, \tag{3a}$$
$$U_i(0,y,t) = \begin{cases} q(y,t), & i = 1 \\ 0, & \text{otherwise} \end{cases} \quad 0 \le y \le 1, 0 \le t \le t_M, \tag{3b}$$
$$U_i(1,y,t) = \begin{cases} \mu(y,t), & i = 2 \\ 0, & \text{otherwise} \end{cases} \quad 0 \le y \le 1, 0 \le t \le t_M, \tag{3c}$$
$$U_i(x,0,t) = \begin{cases} \psi(x,t), & i = 3 \\ 0, & \text{otherwise} \end{cases} \quad 0 \le x \le 1, 0 \le t \le t_M, \tag{3d}$$
$$U_i(x,1,t) = \begin{cases} p(x,t), & i = 4 \\ 0, & \text{otherwise} \end{cases} \quad 0 \le x \le 1, 0 \le t \le t_M, \tag{3e}$$
$$U_i(x,y,0) = \begin{cases} h(x,y), & i = 5 \\ 0, & \text{otherwise} \end{cases} \quad 0 \le x \le 1, 0 \le y \le 1. \tag{3f}$$

In a linear problem a linear dependence exists between the input (in this case $q(y,t)$) and the response (at $x = 1$). This dependence can be expressed analytically by the Duhamel integral

$$U_1(x,y,t) = \int_0^t q(s) \frac{\partial \phi}{\partial t}(x,y,t-s)ds + U_1(x,y,0), \tag{4}$$

where $\phi(x,y,t)$ represents the temperature response at location (x,y) for a unit step change (of flux) in the input, and $T_1(x,y,0)$ is the initial condition for problem (3) for $i = 1$ (in this case it is 0). Considering that the objective in the inverse problem (3), for $i = 1$, is the estimate of $q(y,t)$ in a discrete form equation (4) can be approximated at time t_M as

$$(U_1)_M = \sum_{n=1}^{M} q_n \Delta \phi_{M-n}, \tag{5}$$

where subscripts denote the time instant considered. Note that $\Delta\phi$ represents the temperature response to a unit pulse in the input so that $\Delta\phi_k = \phi_{k+1} - \phi_k$ for $k = 1, ..., M$. As it is evident that $\Delta\phi_{k-j} = \frac{\partial U_{1k}}{\partial q_j}$, consequently it represents the sensitivity coefficient measured at time t_k with respect to component q_j. Obviously, the sensitivity coefficients will be zero when $k < j$.

Considering the expression (5), for $M = 1, 2, \ldots$, we obtain the following matrix equation

$$U_1 = Xq \tag{6}$$

where $q = [q_1, \ldots, q_M]^T$, $q_k = q(t_k)$ and

$$X = \begin{pmatrix} \Delta\phi_0 & 0 & \ldots & 0 & 0 & 0 \\ \Delta\phi_1 & \Delta\phi_0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta\phi_{M-2} & \Delta\phi_{M-3} & \ldots & \Delta\phi_1 & \Delta\phi_0 & 0 \\ \Delta\phi_{M-1} & \Delta\phi_{M-2} & \Delta\phi_{M-3} & \ldots & \Delta\phi_1 & \Delta\phi_0 \end{pmatrix}.$$

If the time history covers a large period of time, this matrix and the corresponding vector can be of a considerable dimensional.

By solving the direct problem (3), for $i = 2, 3, 4, 5$, and using the overspecified condition (2) and the equation (1), we have

$$U^*(x_1, y_1, t) = g(x_1, y_1, t) - \sum_{i=2}^{5} U_i(x_1, y_1, t) = U_1(x_1, y_1, t). \tag{7}$$

Considering the Duhamel'theorem, for $M = 1, 2, \ldots$, we obtain the following equation

$$U^* = Xq, \tag{8}$$

where $U^* = [U_1^*, \ldots, U_M^*]^U$, $U_k^* = U^*(x_1, y_1, t_k)$.

In this study we consider a possible method in order to get a stable algorithm, so that the Tikhonov regularization method must be used to control this measurement errors.

2.2 Overview of finite difference Method

We start by dividing the domain $[0, 1]^2 \times [0, T]$ into an $M^2 \times N$ mesh with spatial step size $\Delta h = 1/M$ in both x-and y-directions and the time-step size $\Delta t = T/N$, respectively. The grid points (x, y, t) are given by:

$$x_i = i\Delta h, \quad i = 0, 1, 2, ..., M, \quad y_j = j\Delta h, \quad j = 0, 1, 2, ..., M, \quad t_n = n\Delta t, \quad n = 0, 1, 2, ..., N, \tag{9}$$

where M and N are integers. Note that $u_{i,j,n}$ used to denote the finite difference approximation of $u(i\Delta h, j\Delta h, n\Delta t)$. We assume that:

$$\Delta x = x_{i+1} - x_i, \quad \Delta y = y_{j+1} - y_j, \quad s_x = \frac{\Delta t}{\Delta x^2}, \quad s_y = \frac{\Delta t}{\Delta y^2}. \tag{10}$$

In this work, the spatial step size is Δx and the time step size is Δt. The process of stepping from time t_n, to t_{n+1} is carried out in two stages.

In the first half-time interval of this ADI procedure, with each $i = 1, 2, ..., (M-1)$, and for each$j = 1, 2, ..., (M-1)$,we have

$$-s_x U_{i-1,j,n+1/2} + 2(1 + s_x)U_{i,j,n+1/2} - s_x U_{i+1,j,n+1/2} =$$

$$s_y U_{i,j-1,n} + 2(1 - s_y)U_{i,j,n} + s_y U_{i,j+1,n}, \tag{11}$$

where the notation $U_{i,j,n+1/2}$ refers to values of $U_{i,j,n}$ computed at the intermediate stage.

In the second half-time interval, the following formula is used with $i = 1, 2, ..., (M-1)$, and for each$j = 1, 2, ..., (M-1)$:

$$-s_y U_{i,j-1,n+1} + 2(1+s_y)U_{i,j,n+1} - s_y U_{i,j+1,n+1} =$$
$$s_x U_{i-1,j,n+1/2} + 2(1-s_x)U_{i,j,n+1/2} + s_x U_{i+1,j,n+1/2}. \quad (12)$$

In the case $\Delta x = \Delta y = \Delta h$, we have $s_x = s_y = r$, and the formulae to be used in the two half-time steps of a time-split procedure are

$$-rU_{i-1,j,n+1/2} + 2(1+r)U_{i,j,n+1/2} - rU_{i+1,j,n+1/2} =$$
$$rU_{i,j-1,n} + 2(1-r)U_{i,j,n} + rU_{i,j+1,n}. \quad (13)$$

The equation (13) for $i = 1, 2, ..., (M-1)$, can be written as

$$AX_1 = D_1, \quad (14)$$

and

$$-rU_{i,j-1,n+1} + 2(1+r)U_{i,j,n+1} - rU_{i,j+1,n+1} =$$
$$rU_{i-1,j,n} + 2(1-r)U_{i,j,n} + rU_{i+1,j,n}. \quad (15)$$

The equation (15) for $j = 1, 2, ..., (M-1)$, can be written as

$$AX_2 = D_2, \quad (16)$$

Remark 2.1. In this work the polynomial form proposed for the unknown function $q(y,t)$ before performing the calculation. Therefore $q(y,t)$ approximated as

$$q(y,t) = \sum_{i=0}^{\gamma} \sum_{j=0}^{\iota} a_{i,j} y^i t^j, \quad (17)$$

where $a_{i,j}$ are constants which remain to be determined simultaneously for each interval.

Substitution the $q(y,t)$ into (13), therefore the solution of this equation is

$$U_{i,j,n}; i, j = 1, 2, \ldots, (M-1),\ n = 1, \ldots, N.$$

Note that, the unknown function $q(y,t)$ is difficult to be approximated by a polynomial function for the whole time domain considered. Therefore the time domain $0 \leq t \leq T$ will be divided into W subintervals where W is integer . Each of the subintervals is assumed to be $\mu\Delta t$.(μ is integer,$\mu \leq N$).

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Poster The stability of Predator-Prey model with disease infection pp.: 1–4

The Stability of Predator-Prey Model With Disease Infection

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Abstract

The stability of ecological and biological modeling has special important. In this paper a predator-prey model with disease infection in both populations is proposed. By using local analysis of various equilibria, we obtained several threshold parameters which determine the stability of the existing equilibria. We also considered disease infection in both populations, and so the model yields more complex dynamics. Finally, we analyzed the locally and globally stability.

Keywords: Ecological threshold parameter; Basic reproduction number; Stability.
Mathematics Subject Classification [2010]: 92D40,92D30,93D05.

1 Introduction

Infectious diseases have been known to be an important regulating factor for human and animal population sizes. In particular, for predator-prey ecosystems, infectious diseases coupled with predator-prey interaction to produce a complex combined effect as regulators of predator and prey sizes. Most of these previous studies focussed mainly on parasite infection and in prey only, although some studies did consider infection of predator through eating prey [1, 2]. In this present work, we begin by describing a predator-prey system with infection, based on the work C.F. McQuaid and N.F. Britton. We briefly discuss the disease-free model and analysis of the full model will be given[5].

2 Main Results

We propose a general theoretical model for a trophically transmitted parasite, where the parasite requires both an intermediate (prey) host and a definitive (predator) host, and there is no intraspecies infection. The parasite is transmitted from prey to predator by consumption of infected prey species, and from predator to prey environmentally through routes such as faecal contamination. Infected individuals immediately become infectious, and remain so for life. Here, we describe a model where Y represents the population of the prey species, and P the predator. The presence of a pathogen leads to infected classes

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y and p of prey and predator species, respectively.

$$\begin{aligned}
\frac{dY}{dt} &= (Y+y)(b-dY) - \tau_y Yp - \frac{\gamma Y(P+p)}{H+h\gamma Y+\nu h\gamma y},\\
\frac{dP}{dt} &= \frac{\varepsilon\gamma(P+p)Y}{H+h\gamma Y+\nu h\gamma y} - \delta P + \frac{\varepsilon\nu\gamma(P+p-\tau_p P)y}{H+h\gamma Y+\nu h\gamma y},\\
\frac{dy}{dt} &= \tau_y Yp - d(Y+y)y - \omega_y y - \frac{\nu\gamma(P+p)y}{H+h\gamma Y+\nu h\gamma y},\\
\frac{dp}{dt} &= \frac{\tau_p\varepsilon\nu\gamma yP}{H+h\gamma Y+\nu h\gamma y} - (\omega_p+\delta)p.
\end{aligned} \tag{1}$$

2.1 The Disease-Free Case

We first consider the disease-free case. That is, we consider system (1) where $y = p = 0$ which is given as follows:

$$\begin{aligned}
\frac{dY}{dt} &= Y(b-dY) - \frac{\gamma YP}{H+h\gamma Y}\\
\frac{dP}{dt} &= \frac{\varepsilon\gamma PY}{H+h\gamma Y} - \delta P.
\end{aligned} \tag{2}$$

Analytical results for this system are described below. We assume the local stability of each equilibrium. The Jacobian matrix of the system (2) is given by

$$J(Y,P) = \begin{bmatrix} \frac{\partial f_1}{\partial Y}(Y,P) & \frac{\partial f_1}{\partial P}(Y,P) \\ \frac{\partial f_2}{\partial Y}(Y,P) & \frac{\partial f_2}{\partial P}(Y,P) \end{bmatrix}$$

The equilibrias of the system are discussed below:

(1) Trivial Equilibrium: $E_0 = (0,0)$. Since the multiply of eigenvalues of the Jacobian matrix at E_0 is negative, it is a saddle point.

(2) Boundary Equilibrium: $E_B = (\frac{b}{d}, 0)$. It is easy to show that if we let $R_1 = \dfrac{\varepsilon\gamma\frac{b}{d}}{\delta(H+h\gamma\frac{b}{d})}$, then E_B is locally asymptotically stable if and only if $R_1 < 1$.

(3) Interior Equilibrium: $E^* = (Y^*, P^*)$,

where $(Y^*, P^*) = (\dfrac{H\delta}{\gamma(\varepsilon-\delta h)}, \dfrac{1}{\gamma}(H + \dfrac{hH\delta}{\varepsilon-\delta h})(b - \dfrac{dH\delta}{\gamma(\varepsilon-\delta h)}))$.

We have the following global result regarding the stability of these equilibrias.

Lemma 2.1. *(i) If $R_1 < 1$, then E_B is globally asymptotically stable for system(2).*
(ii) If $R_1 > 1$, then E_B is unstable for system(2) and the positive equilibrium E^* exists.
(iii) If $R_1 = 1$, E_B is globally asymptotically stable for system(2).

Lemma 2.2. *(i) If $1 < R_1 < \frac{h\gamma b}{(b\gamma h - dH)}$, then E^* is locally asymptotically stable for system(2). (ii) If $R_1 > \frac{h\gamma b}{(b\gamma h - dH)}$, then E^* is unstable for system(2).*
(iii) If $R_1 = \frac{h\gamma b}{(b\gamma h - dH)}$, then E^* may be either a centre or a spiral point for system(2).

We now give two theorems pertaining to the global stability of the positive interior equilibrium E^*.

Theorem 2.3. *If $1 < R_1 < \frac{h\gamma b}{(b\gamma h - dH)}$, then E^* is globally asymptotically stable for system(2).*

Theorem 2.4. *If $R_1 > \frac{h\gamma b}{(b\gamma h - dH)}$, then E^* is unstable for system(2) and this system has a unique limit cycle which is globally orbitally stable.*

2.2 The Model With Disease

We now proceed to consider the full $4D$ model in system (1), which has as many as five equilibrias, depending on the parameter values. We can divide these five equilibrias into three types: trivial, boundary and positive interior equilibria. First, we consider the local stability of the trivial equilibrium and the boundary equilibria. Note that all equilibrias in the $4D$ system are boldfaced to distinguish them from the equilibrias of $2D$ disease-free model. From the Jacobian matrix of system(1) we have the following results on the equilibrias:

(1) Trivial Equilibrium: $E_0 = (0, 0, 0, 0)$. It is easy to show that $E_0 = (0, 0, 0, 0)$ always exists but is unstable for system(1).

(2) Three Boundary Equilibria. Subcase (i) $E_B = (K, 0, 0, 0)$ is the axial equilibrium on Y-axis with healthy prey only, which always exists.
Subcase (ii) $E_B = (Y, y, 0, 0)$ is the boundary equilibrium on Yy -plane with endemic prey population only.
Subcase (iii) $E_B^* = (Y^*, 0, P^*, 0)$ is the boundary equilibrium on YP-plane with disease-free coexistence of predators and prey.

(3) Positive Interior Equilibrium $\tilde{E} = (\tilde{Y}, \tilde{y}, \tilde{P}, \tilde{p})$ with endemic coexistence.

Lemma 2.5. *Let $R_0 = \frac{-kd}{w_y}$,*

(i) If $R_0 < 1$ and $R_1 < 1$, then E_B is locally asymptotically stable for system(1).
(ii) If $R_0 > 1$ or $R_1 > 1$, then E_B is unstable for system(1).

Lemma 2.6. *Let* $R_0^* = \dfrac{-dY^*}{\dfrac{\nu\gamma p^*}{H+h\gamma Y^*}}$,

(i) If $R_0^* < 1$ *and* $1 < R_1 < \dfrac{h\gamma b}{(h\gamma b - dH)}$, *then* E_B^* *is locally asymptotically stable for system(1).*

(ii) If $R_0^* > 1$ *or* $R_1 > \dfrac{h\gamma b}{(h\gamma b - dH)}$, *then* E_B^* *is unstable for system(1).*

3 Conclusion

We obtained that R_0 and R_0^* are disease basic reproduction numbers which determine the local stability of the two disease-free equilibrias E_B and E_B^*, while R_1, $\bar{R}_1$ and $\tilde{R}_1$ are the average numbers of prey converted to predator biomass in a course of the predators life span. Note that if we define a function $R_1(Y)$, then it follows that $R_1 = R_1(\frac{b}{d})$, $\bar{R}_1 = R_1(\bar{Y})$ and $\tilde{R}_1 = R_1(\tilde{Y})$, which are the respective threshold parameters or ecological basic reproduction numbers for the predator prey system at E_B, $\bar{E}_B$ and $\tilde{E}$ that determine the coexistence of prey and predators at these equilibrias. One of the condition always determine the coexistence of the predator-prey system, the other condition dictates whether the disease will be eradicated. This phenomenon of dual threshold parameters has previously been observed in [3, 4].

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Poster The stability and Bifurcation of Food Chain Model, Holling Type II pp.: 1–4

The Stability and Bifurcation of Food Chain Model; Holling Type II

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Abstract

Food chains in the environment can be modeled by systems of differential equations, that approximate species with functional responses. In this paper, an ecological model with Holling type-II functional response in order to describe the dynamical behavior of a three-species food chain is investigated. The local stability and the existence of Hopf bifurcations are established. Finally, numerical simulations are carried out to illustrate the theoretical results.

Keywords: Stability, Bifurcation, Food chain.
Mathematics Subject Classification [2010]: 34D20, 34C23

1 Introduction

Food chains in the environment are very important systems in many different fields such ecological science, applied mathematics, and etc. Food chains can be modeled by systems of differential equations which approximate species with different functional responses. In the history of population ecology, both mathematicians and ecologists have a great interest in the Holling type predator-prey models including Holling types I-III. The Hastings-Powells food chain was analyzed in two different ways [2]. First, the asymptotic states were obtained by direct numerical integration of the dynamical system, varying a key parameter. Second, some relevant features of the whole system were identified by using the delay coordinate embedding from a time series [5].

2 Main Results

We will describe three species Hastings-Powells food chain model revisited given by,

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{a_1 x}{1+b_1 x} y \\ \frac{dy}{dt} &= \frac{a_1 x}{1+b_1 x} y - \frac{a_2 y}{1+b_2 y} z - d_1 y \\ \frac{dz}{dt} &= \frac{a_2 y}{1+b_2 y} z - d_2 z. \end{aligned} \tag{1}$$

*speaker

Where x, y and z are the dimensionless population variables; t is the dimensionless time variable; a_1, a_2, b_2, d_1, d_2, $b_1 = \frac{K_0}{B_1}$, are dimensionless parameters [2].

2.1 Analysis of The Model

We study the existence and local stability of the positive equilibrium point of system (1) and investigate the conditions which Hopf bifurcation occur's. There exist four equilibrium points $E_0 = (0,0,0)$, $E_1 = (1,0,0)$, $E_2 = (\tilde{x},\tilde{y},0)$ and $E_3 = (x^*,y^*,z^*)$ where $\tilde{x}$ and $\tilde{y}$ are given as follows:

$$\tilde{x} = \frac{d_1}{a_1 - b_1 d_1}, \qquad \tilde{y} = \frac{b_1\bar{x}^2 + (1-b_1)\bar{x} - 1}{a_1}.$$

The interior equilibrium point is $E_3 = (x^*,y^*,z^*) = (\frac{-p}{2b_1}, \frac{d_2}{q}, \frac{r}{b_1(2-b_1)q})$ where

$$p = 1 - b_1\sqrt{(1+b_1)^2 - \frac{4a_1b_1d_1}{a_2 - b_2d_2}}, \quad q = a_2 - b_2d_2,$$

$$r = b_1d_1(p-2) - a_1p.$$

In order to study the behavior of solution near the equilibrium points, we need to compute the Jacubian matrix of the system (1) .

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$J_{11} = 1 - 2x_0 + \frac{a_1b_1x_0y_0}{(1+b_1x_0)^2} - \frac{a_1y_0}{1+b_1x_0}, \quad J_{12} = -\frac{a_1x_0}{1+b_1x_0}, \quad J_{13} = 0,$$

$$J_{21} = \frac{a_1y_0}{(1+b_1x_0)^2}, \quad J_{22} = -d_1 + \frac{a_1x_0}{1+b_1x_0} - \frac{a_2z_0}{(1+b_2y_0)^2}, \quad J_{23} = -\frac{a_2y_0}{1+b_2y_0},$$

$$J_{31} = 0, \quad J_{32} = \frac{a_2z_0}{(1+b_2y_0)^2}, \quad J_{33} = -d_2 + \frac{a_2y_0}{1+b_2y_0}.$$

The equilibrium point $E_0 = (0,0,0)$ has eigen values $1, -d_1, -d_2$, Thus E_0 is a saddle point [4].
The roots of the characteristic equation $p_2(\lambda) = 0$ of matrix J at point $E_1 = (1,0,0)$ satisfy as follows:

$$\lambda_1 + \lambda_2 = \frac{a_1}{(1+b_1)} - d_1, \quad \lambda_1\lambda_2 = \frac{a_1^2}{(1+b_1)^3} \text{ and } \quad \lambda_3 = -d_3.$$

It is stable in $xy-$plane, since z-direction $\lambda_3 = -d_2$ is negative.
The equilibrium point $(\tilde{x},\tilde{y},0)$ is stable if the conditions $d_2 > \frac{a_2\tilde{y}}{(1+b_2y)^2}$ and $d_1 > \frac{a_2\tilde{x}}{(1+b_2x)}$ are true.

The Jacubian matrix at equilibrium point E_3 is given by,

$$J^* = \begin{bmatrix} \frac{(p-2)^2q(p+b_1) - 4a_1b_1d_2}{q(p-2)^2b_1} & \frac{a_1p}{2b_1(1-\frac{p}{2})} & 0 \\ \frac{4a_1d_2}{q(p-2)^2} & \frac{-b_2d_2r}{a_2b_1(p-2)} & -d_2 \\ 0 & \frac{-qr}{a_2b_1(p-2)} & 0 \end{bmatrix}$$

By using the Routh-Hurwitz criterion, we see that $E_3(x^*, y^*, z^*)$ is locally asymptotically stable provided the following conditions $A_1 > 0, \quad A_3 > 0, \quad A_1A_2 > A_3$.

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0.$$

Where $A_1 = -(a_{11} + a_{22}), \quad A_2 = a_{11}a_{22} - a_{23}a_{32} - a_{12}a_{21}, \quad A_3 = a_{11}a_{23}a_{32}$.
We show that, $A_1 > 0$ iff the following condition is satisfied,

$$\frac{(p-2)^2q(p+b_1) - 4a_1b_1d_2}{q(p-2)^2b_1} < \frac{rb_2d_2}{(p-2)a_2b_1}. \tag{2}$$

Also $A_1A_2 - A_3 = (a_{11} + a_{22})(a_{12}a_{21} - a_{11}a_{22}) + a_{22}a_{23}a_{32}$.
By substituting the $E_3 = (x^*, y^*, z^*)$ we get

$$A_1A_2 - A_3 = \frac{1}{\alpha^5\beta^4}(M_1M_2 - \frac{\alpha^4(-d_2\alpha\beta^5 + a_1x^*\beta^5 - a_2z^*\alpha\beta^3)(-a_2^2y^*z^*\alpha\beta^2)}{\beta}).$$

$\alpha = 1 + b_1x^*, \quad \beta = 1 + b_2y^*,$

$M_1 = \alpha^2\beta^2 - 2x^*\alpha^2\beta^2 + a_1b_1x^*y^*\beta^2 - d_1\alpha^2\beta^2 + a_1x^*\alpha\beta^2 - a_2z^*\alpha^2,$

$M_2 = -a_1^2\beta^2 - (\alpha^3\beta^2 - 2x^*\alpha^3\beta^2 + a_1b_1\alpha\beta^2x^*y^* - a_y\alpha^2\beta)(-d_1\alpha^3\beta^2 + a_1\alpha^3\beta^2 - a_2\alpha^3z^*).$

The necessary and sufficient condition for $A_1A_2 - A_3 > 0$ is

$$d_2 < \frac{M_1M_2\beta + a_1a_2^2x^*y^*z^*\alpha^5\beta^7 - a_2^3\alpha^6\beta^5y^*z^2}{\alpha^6\beta^7a_2^2y^*Z^*}. \tag{3}$$

Theorem 2.1. *Suppose that the positive equilibrium point $E_3 = (x^*, y^*, z^*)$ of system (1) exists. Then equilibrium point E_3 is locally asymptotically stable if and only if conditions (2) and (3) hold.*

In order to investigate the Hopf bifurcation of the system (1), we follow the technique given by Liu [3]. The simple Hopf bifurcation at $\mu = \mu^*$ can occur provided $A_1(\mu)$, $A_3(\mu)$ and $\psi(\mu) = A_1(\mu)A_2(\mu) - A_3(\mu)$ are smooth functions of μ in an open interval which includes $\mu_* \in R$ such that,

$A_1(\mu_*) > 0, A_3(\mu_*) > 0, \quad \psi(\mu_*) = A_1(\mu_*)A_2(\mu_*) - A_3(\mu_*) = 0$ and $\frac{D\psi(\mu)}{D\mu} \mid_{\mu=\mu*} \neq 0.$

Now, let the decay rate of the top predator d_2 be the bifurcation parameter

$$d_* = \frac{M_1M_2\beta + a_1a_2^2x^*y^*z^*\alpha^5\beta^7 - a_2^3\alpha^6\beta^5y^*z^2}{\alpha^6\beta^7a_2^2y^*Z^*} \tag{4}$$

Then $A_1(d_*) > 0$, $A_3(d_*) > 0$ and $\psi(d_*) = A_1(d_*)A_2(d_*) - A_3(d_*) = 0$.
Furthermore, it is easy to verify that,

$$\frac{D\psi(d)}{Dd} \mid_{d=d*} = a_2^2\beta^2y^*z^* \neq 0.$$

Theorem 2.2. *Under the conditions (2) and (4) there is a simple Hopf bifurcation of the positive equilibrium point $E_3 = (x^*, y^*, z^*)$ for the system (1) at some critical value of the parameter d_2 given by (4).*

2.2 Numerical Results

We get the time series corresponding to the variables of system (1).

$$a_1 = 5, \quad a_2 = 0.1, \quad b_2 = 2, \quad d_1 = 0.4, \quad d_2 = 0.01, \quad b_1 = 2.75.$$

Starting with small d_2, the population sizes move towards a stable equilibrium point; as d_2 increases, a Hopf bifurcation occurs; and, as d_2 increases further, the limit cycles to period doubles, and the system undergoes to a sequence of period-doubling bifurcations.

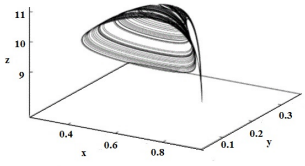

Figure 1: Attractor in the xyz space corresponding to $d_2 = 0.01$.

3 Conclusion

As regording to figure 1, the period-doubling phenomenon leading to chaos is a well known feature of a range of nonlinear systems of biological populations. Having used the set of parameters HP [2], we are able to show that the model could exhibit chaotic dynamics.

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Poster Upper bound for the number of limit cycles in a Lienard system pp.: 1–4

Upper bound for the number of limit cycles in a Lienard system

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Abstract

Lienard system forms one of the important class of differential equations which is considered widely in recent years. An interesting problem studied about this equations is to obtain an upper bound for the number of limit cycle. In this paper we study hopf bifurcation for special polynomial Lienard system and find a maximal number of limit cycle near the origin which named Hopf cyclicity.

Keywords: Lienard system, Limit cycle, Hopf bifurcation
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Consider the Lienard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, which has a equivalent form

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x) \tag{1}$$

where $F(x) = \int_0^x f(x)dx$. Depending of F and g, this system has been widely studied by mathematicans and scientists and many conclusions about the number of limit cycles for this system are obtained. For example in [1] it is proved that the system

$$\dot{x} = y - \frac{\Sigma_{i=0}^{n} a_i x^i}{1 + \Sigma_{i=1}^{m} b_i x^i}, \qquad \dot{y} = -g(x)$$

has Hopf cyclicity $[\frac{n+m-1}{2}]$ at the point $(0, a_0)$, where $g(-x) = -g(x)$, $g(0) = 0$ and $g\prime(0) > 0$. In [2] author gives the number $[\frac{4n+2m-4}{3}] - [\frac{n-m}{3}]$ ($n \geqslant m$), as an upper bound for the maximum number of limit cycle in neighborhod of the point $(0, a_0)$, for above system with $gx) = x(x + 1)$, where this number is Hopf cyclicity in case $n = m$. Now suppose F and g be polynomials. Let $\widehat{H}(i, j)$ denote the maximal number of small-amplitude limit cycle bifurcated from a focus of system (1), where i and j are degree of f and g. Yu and Han in [3] gave a table on $\widehat{H}(i, j)$ which summarizes the existing result for some i and j. In particular if g is quadratic then $\widehat{H}(i, 2) = [\frac{2i+1}{3}]$ for $i \geqslant 2$, and if g is cubic then $\widehat{H}(i, 3) = 2[\frac{3i+6}{8}]$ for $2 \leqslant i \leqslant 50$. Yu and Lynch in [4] considered two type of symmetric Lienard systems and proved that the system

$$\dot{x} = y - \Sigma_{i=0}^{m} a_i x^{2i+1}, \quad \dot{y} = -x(x^2 - 1)$$

*Speaker

has Hopf cyclicity m at the point $A(1, y_0)$ and $B(-1, -y_0)$ and the system

$$\dot{x} = y - \Sigma_{i=0}^{m} a_i x^{2i}, \quad \dot{y} = -x(x^2 - 1)$$

has Hopf cyclicity $[\frac{m-1}{2}]$ at the point A and B, where $y_0 = \Sigma_{i=0}^{m} a_i$. In this paper we study a polynomial Lienard system and by using some analysis technics find Hopf cyclicity for this system. In order to prove main result we will apply a theorem given by Han in [5]. To state this theorem consider a Lienard system of the form

$$\dot{x} = p(y) - F(x, a), \qquad \dot{y} = -g(x), \tag{2}$$

where F, p and g are c^∞ function near the origin with $g(0) = 0, g\prime(0) > 0, p(0) = 0, p\prime(0) > 0, F(0, a) = 0, a \in R^n$. Let $\alpha(x) = -x + O(x^2)$ satisfying $G(\alpha(x)) = G(x)$ for $|x| \ll 1$, where $G(x) = \int_0^x g(x)dx$.

Theorem 1.1. *(Han [5]). Suppose for $|x| \ll 1$,*

$$F(\alpha(x), a) - F(x, a) = \Sigma_{i \geqslant 1} B_i(a) x^i, \quad a = (a_1, ..., a_n).$$

If $B_{2j+1}(a_0) = 0$ and $B_{2k+1}(a_0) \neq 0$ for $j = 0, 1, ..., k-1$, and $rank \frac{\partial(B_1, B_3, ..., B_{2k-1})}{\partial(a_1, a_2, ..., a_n)}(a_0) = k$ for some $a_0 \in R^n$, then Eq.(2) has Hopf cyclicity k at the origin for all a near a_0.

The proof of the above theorem given in [5] directly implies a generalized form of the following.

Theorem 1.2. *If there exists $k \geqslant 1$ such that for $j \geqslant k+1$, $B_{2j+1} = O(B_1, B_3, ..., B_{2k+1})$ as $|B_1|$, $|B_3|$, $|B_{2k+1}|$ are sufficiently small, then for any $N > 0$, there exists a neigborhood U of the origin such that Eq.(2) has at most k limit cycle in U for all $|B_{2j+1}| \leqslant N$, $j = 0, 1, ..., k$.*

2 Main results

In this section we prove below theorem as the main result of this paper.

Theorem 2.1. *The Lienard system*

$$\dot{x} = y - q_n(x), \qquad \dot{y} = -x(x + 1) \tag{3}$$

has Hopf cyclicity $[\frac{2n-1}{3}]$ at the point $(0, a_0)$, where $q_n(x) = \Sigma_{i=0}^{n} a_i x^i$.

Note that for $g(x) = x(x + 1)$ we suppose $G(x) = \int_0^x g(x)dx = \frac{x^2}{2} + \frac{x^3}{3}$. Because of $G(\alpha(x) = G(x)$ for $|x| \ll 1$, so we can take

$$\alpha(x) = \frac{-2x - 3 + \sqrt{-12x^2 - 12x + 9}}{4} = -x - \frac{2}{3}x^2 - (\frac{2}{3})^2 x^3 - 2(\frac{2}{3})^3 x^4 + O(x^5).$$

Let $I_i(x) = \alpha^i(x) - x^i$. To prove the theorem we first give some lemma without proof which given in [2].

Lemma 2.2. *For any integer $n > 0$, we have $I_{3n}(x) = -\Sigma_{i=1}^{n} C_n^i (\frac{3}{2})^i I_{3n-i}(x)$.*

We introduce a new variable θ by $x = \frac{(-1-\sqrt{3}\sin\theta+\cos\theta)}{2} = \xi(\theta) \quad for|\theta| \ll 1$. Then it follows that $\alpha(x) = \frac{-1+\sqrt{3}\sin\theta+\cos\theta}{2} = \xi(-\theta)$. Suppose that $I_n(x) = I_n(\xi(\theta)) = \widehat{I}_n(\theta)$. Then we have $\widehat{I}_n(\theta) = [\xi(-\theta)]^n - [\xi(\theta)]^n$. Thus the periodic function $\widehat{I}_n$ is odd in θ. Further for its Furier expansion we have the following lemma.

Lemma 2.3. *For any integer n, the function $\widehat{I}_n(\theta)$ has the following Furier expansion*

$$\widehat{I}_n(\theta) = \Sigma_{i\in S(n)} b_{n,i}\sin i\theta,$$

where $S(n) = \{k|k \neq 0(mod3), \quad 1 \leqslant k \leqslant n\}$ and $b_{n,i}$ are cofficients independent of θ with $b_{n,n} = 2^{-n+2}\sin\frac{n\pi}{3}$.

By lemma 2.3 we have $b_{k,k} \neq 0$ for $k \neq 0 \pmod 3$. Hence, noting that different functions $\sin(k\theta)$ for $k \neq 0 \pmod 3$ are linearly independent, lemma 2.3 implise that the functions $\widehat{I}_i(\theta), i \in S(n)$, are linearly independent for any positive integer $n \geqslant 1$. In other words, we have the following lemma.

Lemma 2.4. *For any integer $n \geqslant 1$, the $[\frac{2n-1}{3}] + 1$ functions $I_i(x), i \in S(n)$, are linearly independent.*

Then by lemmas 2.2 and 2.3 we see the fact that the n functions $I_i(x), i = 1, ..., n$, can span a linear space of dimension $[\frac{2n-1}{3}] + 1$. Now we prove theorem (2.1).

Proof. Firstly we prove that Eq.(3) has at most $[\frac{2n-1}{3}]$ limit cycle near the origin by theorem 1.2 . We have

$$q_n(\alpha(x)) - q_n(x) = \Sigma_{i=1}^n a_i I_i(x) = \Sigma_{i\geqslant 1} B_i x^i = Q_n(x). \tag{4}$$

Then by lemma 2.3

$$Q_n(x) = \Sigma_{i=1}^n a_i \widehat{I}_i(\theta) = \Sigma_{i=1}^n a_i \Sigma_{j\in S(i)} b_{i,j}\sin j\theta = \widehat{Q}_n(\theta).$$

Noting that $S(j) = \cup_{i=1}^j \widehat{S}_i$, where $\widehat{S}_i = \{i\}$ for $i \neq 0(\text{mod } 3)$ and $\widehat{S}_i = \emptyset$ for $i = 0(\text{mod } 3)$, one has

$$\widehat{Q}_n(\theta) = \Sigma_{j\in S(n)} \Sigma_{i=j}^n a_i b_{i,j}\sin j\theta = \Sigma_{j\in S(n)} c_j \sin j\theta,$$

where $c_j = \Sigma_{i=j}^n a_i b_{i,j}$. Then

$$\widehat{Q}_n(\theta) = \Sigma_{j\in S(n)} c_j \Sigma_{i\geqslant 0} \frac{(-1)^i}{(2i+1)!} j^{2i+1}\theta^{2i+1} = \Sigma_{i\geqslant 0} \frac{(-1)^i}{(2i+1)!}\widehat{c}_{2i+1}\theta^{2i+1} \tag{5}$$

where $\widehat{c}_{2i+1} = \Sigma_{j\in S(n)} j^{2i+1} c_j, \quad i \geqslant 0$. From last equation we can obtain $C_{l+1} = A_{l+1}^{-1}\widehat{C}_{l+1}$, where $\det A_{l+1} \neq 0$. Hence we have

$$\widehat{c}_{2j+1} = O(|\widehat{c}_1, \widehat{c}_3, ..., \widehat{c}_{2l+1}|) \quad forj \geqslant l+1. \tag{6}$$

Further noting that $\theta = -\frac{2\sqrt{3}}{3}x + O(x^2)$, from (4) and (5) we have

$$B_{2j+1} = \frac{(2\sqrt{3})^{2j+1}(-1)^{j+1}}{3^{2j+1}(2j+1)!}\widehat{c}_{2j+1} + O(|\widehat{c}_1, \widehat{c}_3, ..., \widehat{c}_{2j-1}|), \quad forj \geqslant 0 \tag{7}$$

which gives

$$\widehat{c}_{2j+1} = \frac{3^{2j+1}(2j+1)!}{(2\sqrt{3})^{2j+1}(-1)^{j+1}} B_{2j+1} + O(|B_1, B_3, ..., B_{2j-1}|), \quad for \quad 0 \leqslant j \leqslant l. \qquad (8)$$

By (6), (7) and (8)

$$B_{2j+1} = O(|B_1, B_3, ..., B_{2l+1}|) \quad for j \geqslant l+1.$$

In particular, $Q_n(x) = 0$ when $B_{2j+1} = 0, 0 \leqslant j \leqslant l$. It follow by theorem 1.2 that Eq. (3) has at most l limit cycle near the origin. Finally we prove that l limit cycle can apear near the origin. For simplicity, take $a_{3j} = 0, 1 \leqslant j \leqslant [\frac{n}{3}]$. In this case, from (4) we have

$$(B_1, B_3, ..., B_{2l+1})^T = S_l(a_{k_0}, a_{k_1}, ..., a_{k_l})^T,$$

where S_l is a constant matrix of order $l+1$. On the one hand, from lemma 2.4, the functions $I_{k_0}(x), I_{k_1}(x), ..., I_{k_l}(x)$ are linearly independent, and hence from (4) it is easy to see that $a_{k_j} = 0, 0 \leqslant j \leqslant l$, if and only if $q_n(\alpha(x)) - q_n(x) \equiv 0$. On the other hand, from the above proof, we see that $B_{2j+1} = 0, 0 \leqslant j \leqslant l$, if and only if $q_n(\alpha(x)) - q_n(x) \equiv 0$. Therefore, we have $\det S_l \neq 0$, and the conclusion follows by theorem 1.1. The proof is completed. □

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Poster ADI application in solution of problem option pricing under the HHW model pp.: 1–4

ADI Application in Solution of Problem Option Pricing under the HHW Model

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Abstract

we introduce a very important application of PDE in financial markets.for this purpose a european option pricing under underling asset with volatility and interest rate is stochastic. for estimating the option pricing of the european model.here we obtained approximation PDE and affine PDE, we solution this approximation PDE with Alternatig Directio Implicit (ADI) time Discretization scheme then the estimate eropean call and put option under HHW model.

Keywords: HHW model, ADI scheme,T- forward

1 Introduction

we describe the triple Heston-Hull-White (HHW) Model .In section 2 the HHW model is combined Heston model with stochastic volatility and hull-White for a stochastic interest rates process,as described by Grzelak and oosterlee (2009,2011),wich is three factor model.these dynamic are formed by three correlated standard Brownian motions.in this paper we briefly describe how to model PDE HHW with construct a portfolio of assets .By applying Ito's formula on the dynamics of the portfolio we will reach to the PDE HHW. unfortunately ,the HHW model and that's PDE are not affine ,not even apply to the log transform .so in section 3 PDE approximation is obtained by removal of non affine PDE HHW with this change.It has been possible to accept numerical model will be affine.In section 4 we using a ADI scheme to solve PDE and we will get to a solution wich is highly efficient

2 Heston-Hull-White Model

Consider the following system of the stochastic diffrential equation subject to the filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and finite time $[0, t]$ .the model is defined ,under the risk natural measure $\mathbb{Q}$,the dynamics of these are presented as follow:

$$\begin{cases} dS(t)/S(t) = r(t)dt + \sqrt{v(t)}dW_x^Q(t) & S(0) > 0 \\ dv(t) = \kappa(\bar{v} - v)dt + \gamma\sqrt{v(t)}dW_v^Q(t) & S(0) > 0 \\ dr(t) = \lambda(\theta(t) - r(t))dt + \eta dW_v^Q(t) & S(0) > 0 \end{cases} \tag{1}$$

*Speaker

Where $S(t)$, $v(t)$ and $r(t)$ $(t > 0)$ are the stock price process,instantaneous volatility and short term rate of interest ,respectively.the correlation of the Brownian motions are given in the following way:

$$\begin{cases} dW_x(t)dW_v(t) & = \rho_{x,y}dt \\ dW_x(t)dW_r(t) & = \rho_{x,}dt \\ dW_v(t)dW_r(t) & = 0 \end{cases} \tag{2}$$

Where λ ,$\theta(t)$ and η are parameters from Hull-White model and $\lambda > 0$ means the speed of mean reversion of the short rate , $\theta(t)$ is the interest rate term-structure ,η is defind as volatility of the interest rate.Glazark and Oosterlee interest is to price, under the risk-free measure $\mathbb{Q}$ associated to the bank account as numeraire .

3 Mesure change to move T-forward

In the Heston-Hull-White model ,by switching between two pricing measures ,i.e we change to T-forward stock price . the dynamics for the zero-coupon bond (ZCB) expiring at time T , under the spot measure $\mathbb{Q}$ in the Heston-Hull-white for the forward asset price is defind as :

$$F(t) = \frac{S(t)}{P(t,T)} = \frac{S(t)}{exp[A_r(t,T) - B_r(t,T)r(t)]} \tag{3}$$

Forward exchange rate satisfies the following PDE under the forward measure.

$$\begin{cases} \dfrac{dF_t}{F_t} = \sqrt{v_t}dW_F^T - _r\,(t,T)dW_r^T \\ dv(t) = \kappa(\bar{v} - v)dt + \gamma\sqrt{v(t)}dW_v^T(t) \end{cases} \tag{4}$$

With $dW_F^T dW_v^T = \rho_{F,v}dt, dW_F^T dW_r^T = \rho_{F,r}dt$. Wich implies the following result

$$\begin{aligned} -\frac{\partial V}{\partial t} &= \kappa(\bar{v} - v)\frac{\partial V}{\partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2 V}{\partial v^2} + F^2(\frac{1}{2}v + \frac{1}{2}\eta^2 B_r^2(t,T) - \cdots \\ &\quad - \rho_{F,r}\eta B_r(t,T)\sqrt{v})\frac{\partial^2 V}{\partial F^2} + \rho_{F,v}\gamma F v\frac{\partial^2 V}{\partial F \partial v} \end{aligned} \tag{5}$$

System (1) and PDE (5) does not fit in the class of affine diffusion process(AD), as it contains the non-affine factor $\rho_{F,r}\eta B_r(t,T)\sqrt{v}$,with a non zero correlation,not even when we make the log transform of the asset price.we cannot determine the characteristic function by standard procedures due to the non-affine form.hence , accurate approximation are needed.we therefor define an approximation to PDE (5) for a highly efficient computation of an approximation solution.PDE affine by the replace the term none-affine $\sqrt{v}$ in (5) by time-dependent function ,$\phi(t, v(0))$ wich gives us ,approximating pricing PDE, we denote the solution by$\tilde{V}$:

$$\begin{aligned} -\frac{\partial \tilde{V}}{\partial t} &= \kappa(\bar{v} - v)\frac{\partial \tilde{V}}{\partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2 \tilde{V}}{\partial v^2} + F^2(\frac{1}{2}v + \frac{1}{2}\eta^2 B_r^2(t,T) - \cdots \\ &\quad - \rho_{F,r}\eta B_r(t,T)\sqrt{v})\frac{\partial^2 \tilde{V}}{\partial F^2} + \rho_{F,v}\gamma F v\frac{\partial^2 \tilde{V}}{\partial F \partial v} \end{aligned} \tag{6}$$

With the same boundary and final condition as (5).

4 Alternative Differential Implicit Scheme

with discretizing PDE (5),its approximating PDE (6).we are following final value problem for a system of ordinary differential equations(ODEs):

$$\frac{V^{h+1} - V_h}{\triangle t} = A_h(t)V_h(t) + R \qquad V_h(T) = V_T \tag{7}$$

where $V_h(t)$ is a vector of approximation of option value and its unknown.that estimate with boudary condition.$A_h(t)$ $0 \leq t \leq T$ is a given square matrix,and $g_h(t)$ is a given real vector that is estimated with boundary condition.splitting a matrix in sub matrix : $A_h(t) = A_0 + A_1(t) + A_2$ where $A_0(t)$ corresponds to the mixed derivatives,$A_1(t)$ to the derivatives in the s direction and $A_2(t)$ to the derivatives in the v direction .similary the vector R is split up: $R = R_0 + R_1 + R_2$ .we will obtain the first-order for solve the ODEs,the Douglas scheme is following:

$$\begin{aligned}
&\mathrm{Y}_0 = V_n + \triangle t(A_h(t)V_h(t) + R)\\
&\mathrm{Y}_1 = Y_0 + \theta \triangle tA_2(Y_1(t) - V_n)\\
&\mathrm{Y}_2 = V_1 + \theta \triangle t(A_1(t_{n-1}) - A_1(t_n)V_n + R)\\
&\mathrm{V}_{n-1} = Y_2
\end{aligned}$$

wich generate the succsessive approximations,V_n to the solutions $V(t_n, F_i, v_j)$ and θ is given a real parameter. and the Craig-Sneyd scheme :

$$\begin{aligned}
&\mathrm{Y}_0 = V_n + \triangle t(A_h(t)V_h(t) + R)\\
&\mathrm{Y}_1 = Y_0 + \theta \triangle t(A_1Y_1 - A_1V_h)\\
&\mathrm{Y}_2 = Y_1 + \theta \triangle t(A_2Y_2 - A_2V_h)\\
&\mathrm{Y}_3 = Y_0 + \frac{1}{2} \triangle tA_0(Y_2 - V_h)\\
&\mathrm{Y}_4 = Y_3 + \theta \triangle t(A_1Y_4 - A_1V_h)\\
&\mathrm{Y}_5 = Y_4 + \theta \triangle t(A_2Y_5 - A_2V_h)\\
&\mathrm{V}_{h+1} = Y_5
\end{aligned}$$

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Existence of Solution for G-BSDE with Quadratic Growth and Unbounded Terminal Value

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Abstract

In this paper, we present the existence of solutions for G-backward stochastic differential equations with quadratic growth and unbounded terminal value, under some assumptions.

Keywords: G-expectation, G-Brownian motion, G-Backward stochastic differential equations, quadratic growth, unbounded terminal value .
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

We consider the following G-backward stochastic differential equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{1}$$

where K is a decreasing G-martingale. The terminal value ξ and the generator f are given. B_t is the G-Brownian motion. We present the existence of a solution (Y, Z, K) for (1) (see Theorems 3.1) in the G-framework.

2 Preliminaries

We briefly recall some basic notions of G-expectation. Let $(\Omega, \mathcal{H}, \mathbb{E})$ be the G-expectation space. We denote by $lip(\mathbb{R}^n)$ the space of all bounded and Lipschitz real functions on $\mathbb{R}^n$. In this paper we set $G(a) = \frac{1}{2}(a^+ - \sigma_0^2 a^-)$, where $a \in \mathbb{R}$ and $\sigma_0 \in [0, 1]$ is fixed.
Let $\Omega = \mathbb{R}$ and $\mathcal{H} = lip(\mathbb{R})$, in [1], X with G-normal distribution (with mean at $x \in \mathbb{R}$ and variance $t > 0$), is defined by

$$\mathbb{E}[\varphi(x + \sqrt{t}X)] = P_G^t(\varphi(x)) := u(t, x),$$

Where $\varphi \in lip(\mathbb{R})$ and $u = u(t, x)$ is a bounded continuous function on $[0, \infty) \times \mathbb{R}$ which is the solution of the following G-heat equation

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x).$$

*Speaker

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$with $\omega_0 = 0$. We set, for each $t \in [0,\infty)$

$$\begin{aligned} W_t &:= \{\omega_{.\wedge t} : \omega \in \Omega\}, \\ F_t &:= \mathcal{B}_t(W) = \mathcal{B}(W_t), \\ F_{t^+} &:= \mathcal{B}_{t^+}(W) = \bigcap_{s>t} \mathcal{B}_s(W), \\ F &:= \bigvee_{s>t} F_s. \end{aligned}$$

Then (Ω, F) is the canonical space with the natural filtration and $B_t(\omega) = \omega_t$ is the canonical process. This space is used throughout the rest of this paper.

For each fixed $T \geq 0$, we consider the following space of random variables

$$l^0_{ip}(F_T) := \{X(\omega) = \varphi(\omega_{t_1}, \ldots, \omega_{t_m}), \quad \forall m \geq 1,\ t_1, \ldots, t_m \in [0,T], \forall \varphi \in lip(\mathbb{R}^m)\}.$$

We further define $l^0_{ip}(F) := \bigcup_{n=1}^{\infty} l^0_{ip}(F_n)$.

Definition 2.1. [1] The canonical process $B_t(\omega) = \omega_t$ is called a G-Brownian motion under a nonlinear expectation $\mathbb{E}$ defined on $l^0_{ip}(F)$ if for each $T > 0$, $m = 1, 2, \ldots$, and for each $\varphi \in lip(\mathbb{R}^m)$, $0 \leq t_1 < \ldots < t_m \leq T$, we have

$$\mathbb{E}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] = \varphi_m,$$

where $\varphi_m \in \mathbb{R}$ is obtained via the following procedure:

$$\begin{aligned} &\varphi_1(x_1, \ldots, x_{m-1}) = P_G^{t_m - t_{m-1}}(\varphi(x_1, \ldots, x_{m-1}, .)), \\ &\varphi_2(x_1, \ldots, x_{m-2}) = P_G^{t_{m-1} - t_{m-2}}(\varphi_1(x_1, \ldots, x_{m-2}, .)), \\ &\vdots \\ &\varphi_{m-1}(x_1) = P_G^{t_2 - t_1}(\varphi_{m-2}(x_1, .)), \\ &\varphi_m = P_G^{t_1}(\varphi_{m-1}(.)). \end{aligned}$$

It is proved in [3] that $\mathbb{E}[.]$ consistently defines a nonlinear expectation on the vector lattice $l^0_{ip}(F_T)$ as well as on $l^0_{ip}(F)$, It follows that $\mathbb{E}[|X|]$ where $X \in l^0_{ip}(F_T)$ (resp. $l^0_{ip}(F)$) forms a norm and that $l^0_{ip}(F_T)$ (resp. $l^0_{ip}(F)$) can be continuously extended to a Banach space, denoted by $L^1_G(F_T)$ (resp. $L^1_G(F)$). For a given $p > 1$, we also denote $L^p_G(F) = \{X \in L^1_G(F), |X|^p \in L^1_G(F)\}$. $L^p_G(F)$ is also a Banach space under the norm $||X||_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$.

Definition 2.2. [1] Let $M^{p,0}_G(0,T)$ be the collection of processes in the following form: for a given partition $\pi_T = \{t_0, \ldots, t_N\}$ of $[0,T]$

$$\mu_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

Where $T \in \mathbb{R}^+$, $p \geq 1$ and $\xi_j \in L^p_G(F_{t_j})$, are given.

let $||\eta||_{H_G^p} = \left[\mathbb{E}\left[\left(\int_0^T |\eta_s|^2 ds\right)^{\frac{p}{2}}\right]\right]^{\frac{1}{p}}$, $||\eta||_{M_G^p} = \left[\mathbb{E}\left[\int_0^T |\eta_s|^p ds\right]\right]^{\frac{1}{p}}$ and denote by $H_G^p(0,T)$, $M_G^p(0,T)$ the completions of $M_G^{p,0}(0,T)$ under the norms $||\eta||_{H_G^p}$, $||\eta||_{M_G^p}$ respectively.
Let $S_G^{p,0}(0,T) = \{h(t, B_{t_1\wedge t}, \ldots, B_{t_n\wedge t}) : t_1, \ldots, t_n \in [0,T], h \in lip(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^{p,0}(0,T)$, set $||\eta||_{S_G^p} = \left[\mathbb{E}\left[sup_{t\in[0,T]}|\eta_t|^p\right]\right]^{\frac{1}{p}}$. Denote by $S_G^p(0,T)$ the completion of $S_G^{p,0}(0,T)$ under the norm $||\eta||_{S_G^p}$.
We call $L_G^p(F_T)$, $M_G^p(0,T)$, $H_G^p(0,T)$ and $S_G^p(0,T)$ the spaces of the G-framework.

Definition 2.3. [2] For each $\eta \in M_G^{2,0}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

we define

$$\mathcal{I}(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j})$$

The mapping $M_G^{2,0}(0,T) \to L_G^2(F_T)$ is a linear continuous mapping and thus can be continuously extended to $\mathcal{I} : M_G^2(0,T) \to L_G^2(F_T)$.

Definition 2.4. [2] We define, for a fixed $\eta \in M_G^2(0,T)$, the stochastic integral

$$\int_0^T \eta(s) dB_s := \mathcal{I}(\eta).$$

We now introduce the generator of our G-BSDE. We assume that f is a function defined on $[0,T] \times \Omega \times \mathbb{R}^2$ with values in $\mathbb{R}$ and has a linear growth in y and a quadratic growth in z. We make the following assumptions:

H1 There exist $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$ such that
$\forall t \in [0,T]$, $(y,z) \to f(t,y,z)$ is continuous,
$\forall t, y, z \in [0,T] \times \mathbb{R} \times \mathbb{R}$, $|f(t,y,z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$.

H2 we will assume that

$$\mathbb{E}\left[e^{\gamma e^{\beta T}|\xi|}\right] < \infty.$$

H3 $\forall y, z$, $f(.,.,y,z) \in M_G^2(0,T)$,

H4 $\exists \lambda > \gamma e^{\beta T}$, $\mathbb{E}\left[e^{\lambda|\xi|}\right] < \infty$.

3 Main Results

The goal of this section is to study the G-BSDE (1) under the assumptions (H1)–(H4).

3.1 Existence of the solutions

Theorem 3.1. *Let the assumptions (H1)–(H4) hold. Then the BSDE* (1) *has at least a solution* $(Y, Z, K) \in S_G^2 \times H_G^2 \times L_G^2$.

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A covering property of pullback fibrations

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Abstract

In this talk, we show that the total space of the pullback fibration of a given fibration $f : E \longrightarrow X$ by a covering map $c : \widetilde{X} \longrightarrow X$ is a covering space of E. Also, we study conditions in which give us universal property of these covering maps.

Keywords: Covering map, Fibration, Pullback fibration
Mathematics Subject Classification [2010]: 57M10, 55R05

1 Introduction

We recall that a *fibration* $f : E \longrightarrow X$ is a continuous map in which has homotopy lifting property with respect to an arbitrary space. If $f : E \longrightarrow X$ is a fibration and $g : Y \longrightarrow X$ is a continuous map, then

$$g^*E := \{(y, e) \in Y \times E | g(y) = f(e)\},$$

equipped by the subspace topology and projection map $p_1 := pr_1 : g^*E \longrightarrow Y$ defines a fibration over Y, named *pulback* of f by g and denoted by g^*f. Also, recall that a *covering map* is a continuous map $c : \widetilde{X} \longrightarrow X$ such that for every $x \in X$ there exists an open subset U of X with $x \in U$ for which U is *evenly covered* by c, that is, $c^{-1}(U)$ is a disjoint union of open subsets of $\widetilde{X}$ each of which is mapped homeomorphically onto U by c. When $\widetilde{X}$ is simply connected, c is called universal covering and it is called categorical universal covering if for every covering map $q : \widetilde{Y} \longrightarrow X$ with a path connected total space $\widetilde{Y}$, there exists a covering map $f : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ f = c$.

In this note, we consider the following commutative diagram:

$$\begin{array}{ccc} c^*E & \xrightarrow{p_2} & E \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{X} & \xrightarrow{c} & X, \end{array}$$

where c is covering map, f is fibration and p_1 is pullback fibration of f by c.

It is well-known that p_1 is a fibration [4, page 98]. Here we prove that if c is a covering map, then p_2 is also a covering map and hence we have a diagram such that vertical maps are fibration and horizontal maps are covering map. Next, we show that if c is universal covering map, p_2 is not necessarily universal unless f has unique path lifting property.

*Speaker

2 Main results

Theorem 2.1. *If $f : E \longrightarrow X$ is a fibration and $c : \widetilde{X} \longrightarrow X$ is a covering map, then $p_2 : c^*E \longrightarrow E$ is a covering map.*

Proof. For e given point $e \in E$, we would find an evenly covered open neighborhood. Let $x = f(e)$. Since c is a covering map, there exist an evenly covered neighborhood U of x such that $c^{-1}(U) = \bigsqcup_{\alpha\in J} V_\alpha$ and $c|_{V_\alpha} : V_\alpha \longrightarrow U$ is homeomorphism.

Let $W := f^{-1}(U)$ which is an open neighborhood of e by continuity of f. Also,

$$p_2^{-1}(W) = p_1^{-1} \circ c^{-1}(U) = p_1^{-1}(\bigsqcup V_\alpha) = \bigsqcup_{\alpha\in J} p_1^{-1}(V_\alpha).$$

Since V_α's are disjoint, $W_\alpha = p_1^{-1}(V_\alpha)$'s are disjoint. it suffices to show that $p_2|_{W_\alpha} : w_\alpha \longrightarrow W$ is homeomorphism, for every $\alpha \in J$. Note that $W_\alpha = \{(y, g) \in V_\alpha \times f^{-1}(U) | c(y) = f(g)\}$ and hence:
If $p_2(y, g) = p_2(y', g')$, then $g = g'$ which implies that $c(y) = f(g) = f(g') = c(y')$ and since $c|_{V_\alpha}$ is injective, $y = y'$. So $p_2|_{W_\alpha}$ is injective.
If $g \in W$ is arbitrary, then $x' = f(g) \in U$. Let $\widetilde{x'} = V_\alpha \cap c^{-1}(x')$. Then $(\widetilde{x'}, g) \in W_\alpha$ and $p_2(\widetilde{x'}, g) = g$ which implies that $p|_{W_\alpha}$ is onto. Continuity and openness of $p_2|_{W\alpha}$ comes from openness of projection maps and openness of W_α in c^*E. □

Now, there is a question: If $c : \widetilde{X} \longrightarrow X$ is universal covering, then, is $p_2 : c^*E \longrightarrow E$ a universal covering? In the sequel, we answer this question in the negative sense generally and find a sufficient condition for the positive case.

Example 2.2. Let $c := exp : \mathcal{R} \longrightarrow S^1$ be the exponential map and $f := p_1 : S^1 \times S^1 \longrightarrow S^1$ be projection, which are covering map and fibration, respectively. We can show easily that $c^*(S^1 \times S^1) \approx \mathcal{R} \times S^1$ and hence $p_2 : c^*(S^1 \times S^1) \longrightarrow S^1 \times S^1$ cannot be universal since $\mathcal{R} \times S^1$ is not simply connected, while c is universal covering.

Definition 2.3. [4] Let $p : E \to B$ be a map and $\widetilde{\alpha}$ *and* $\widetilde{\beta}$ be paths in E, then we say that p has **unique path lifting** if

$$\widetilde{\alpha}(0) = \widetilde{\beta}(0),\ p \circ \widetilde{\alpha} = p \circ \widetilde{\beta} \Rightarrow \widetilde{\alpha} = \widetilde{\beta}.$$

Proposition 2.4. *[4, Theorem 2.3.4] Let $f : E \longrightarrow X$ be a fibration with unique path lifting property and $e \in f^{-1}(x)$, then $f_* : \pi_1(E, e) \longrightarrow \pi_1(X, x)$ is a monomorphism.*

In the next theorem, we show that unique path lifting property of fibrations is sufficient to universal property of covering maps preserved, when we are pulling back them by a fibration.

Theorem 2.5. *If $f : E \longrightarrow X$ is a fibration with unique path lifting and $c : \widetilde{X} \longrightarrow X$ is a universal covering map, then $p_2 : c^*E \longrightarrow E$ is a universal covering map.*

Proof. Assume that c^*E is not simply connected and use injectivity of $(p_1)_*, (p_2)_*, f_*$ to have contradiction. □

Remark 2.6. Note that in the above Theorem, by the universal covering we mean simply connected covering in which the existence of them is strongly depended on the local property of the base space. In fact, the base space of covering map must be locally path connected and semi-locally simply connected. For the spaces that haven't got this nice local behavior, there is a big challenge. Fortunately, for some families of spaces, existence of the categorical universal covering is studied by the first author et.al [2, 3]. They proved that a given space X has categorical universal covering if and only if X is semi-locally Spanier space, that means every point $x \in X$ has a neighborhood in which homotopy class of every loop belongs to the Spanier group $\pi_1^{sp}(X, x)$[1]. Since for a covering $c : \widetilde{X} \longrightarrow X$, $\pi_1^{sp}(X, x) \leq p_*\pi_1(\widetilde{X}, \tilde{x})$, by a bit change in the proof, we can prove Theorem 2.5 for the categorical universal covering.

Now, we consider the problem from another point of view. As an easy exercise, it can be shown that pullback of a covering map by a continuous map, which is defined as same as pullback fibration, is a covering map. In Theorem 2.1, we have seen that the total space of pullback of fibration by a covering map is covering space of the total space of the original fibration. Here, we show that the total space of pullback of a covering map by a fibration give us a new fibration over the total space of the original covering.

Theorem 2.7. *Let $c : \widetilde{X} \longrightarrow X$ be a covering map and $f : E \longrightarrow X$ be a fibration. Then*
i) $p_1 : f^*\widetilde{X} \longrightarrow E$ is a covering map.
ii) $p_2 : f^*\widetilde{X} \longrightarrow \widetilde{X}$ is a fibration.
iii) p_2 has unique path lifting property if f has unique path lifting property.

Proof. For (i), use definitions and universal property of pullbacks.
For (ii), assume that $F : Y \times I \longrightarrow \widetilde{X}$ is a homotopy, for a given space Y in which $F \circ J_0 = p_2 \circ g_0$, where $j_0 : Y \longrightarrow Y \times I$ is $j_0(y) = (y, 0)$ and $g_0 : Y \longrightarrow f^*\widetilde{X}$ is continuous. We have the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{p_1 \circ g_0} & E \\ {\scriptstyle J_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y \times I & \xrightarrow{c \circ F} & X, \end{array}$$

which is commutative by commutativity of the pullback diagram. Since f is fibration, there exist $\widetilde{c \circ F} : Y \times I \longrightarrow E$ in which commute two triangles of the diagram. Now, we have the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g_0} & f^*\widetilde{X} \\ {\scriptstyle J_0}\downarrow & & \downarrow{\scriptstyle p_1} \\ Y \times I & \xrightarrow{\widetilde{c \circ F}} & E. \end{array}$$

Since covering maps are fibration, there exists $\widetilde{F} : Y \times I \longrightarrow f^*\widetilde{X}$ such that $p_1 \circ \widetilde{F} = \widetilde{c \circ F}$. Therefore,

$$c \circ p_2 \circ \widetilde{F} = f \circ p_1 \circ \widetilde{F} = f \circ \widetilde{c \circ F} = c \circ F.$$

An application of diagram show that $p_2 \circ \widetilde{F} = F$, as desired.
For (iii), use the fact that fibers of a fibration with unique path lifting has no nonconstant path.

□

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A generalization of paracontact psudo-metric manifolds
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A generalization of paracontact psudo-metric manifolds

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Abstract

In this paper, we give a characterization of a paracontact psudo-metric and K-paracontact psudo-metric manifolds as a special almost paracontact psudo-metric manifold and introduce a quasi paracontact psudo-metric manifold according to quasi para Kähler psudo-metric manifold which is a natural generalization of the paracontact psudo-metric manifolds.

Keywords: Almost paracontact, Paracontact, Psudo-metric, Quasi para Kähler
Mathematics Subject Classification [2010]: 32C15, 53B35

1 Introduction

A systematic study of general paracontact pseudo-metric structures was undertaken by Simeon Zamkovoy in 2009 [8]. Cruceanu, Fortuny and Gadea gave the survey article on paracomplex geometry in 1996 [2] and Mykhaylo Chursin, Lars Schafer and Knut Smoczyk defined quasi para Kähler manifolds in 2010 [3]. In this paper, we introduce a quasi paracontact psudo-metric manifold according to quasi para Kähler psudo-metric manifold which is a natural generalization of the paracontact psudo-metric manifolds. Now, let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost paracontact pseudo-metric manifold and $\overline{M} = M \times \mathbb{R}$ be the product manifold of M and a real line $\mathbb{R}$ equipped with the following almost para Hermitian structure $(\overline{J}, \overline{g})$ defined by

$$\begin{aligned} &\overline{J}X = \phi X + \eta(X)\frac{\partial}{\partial t}, \quad \overline{J}\frac{\partial}{\partial t} = \xi, \\ &\overline{g}(X,Y) = e^{-2t}g(X,Y), \ \ \overline{g}(X,\frac{\partial}{\partial t}) = 0, \ \ \overline{g}(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = -e^{-2t}, \end{aligned} \tag{1}$$

for $X, Y \in \chi(M)$ and $t \in \mathbb{R}$. Now, we denote by $\overline{\nabla}$ the covariant derivative with respect to the metric $\overline{g}$ on $\overline{M}$. Then, from (1) by direct calculation, we have

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)\frac{\partial}{\partial t}, \quad \overline{\nabla}_{\frac{\partial}{\partial t}} X = -X, \\ &\overline{\nabla}_X \frac{\partial}{\partial t} = -X, \quad \overline{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = -\frac{\partial}{\partial t}, \end{aligned} \tag{2}$$

*Speaker

for $X, Y \in \chi(M)$. we shall introduce a class of almost paracontact pseudo-metric manifolds as the class of almost paracontact pseudo-metric manifolds corresponding to the class of quasi para Kähler manifolds, which is regarded as a generalization of the class of paracontact pseudo-metric manifolds.

Definition 1.1. An almost paracontact psudo-metric manifold $M = (M, \phi, \xi, \eta, g)$ is called a quasi paracontact pseudo-metric manifold if the corresponding almost para Hermitian psudo-metric manifold $\overline{M} = (\overline{M}, \overline{J}, \overline{g})$ defined by (1) is a quasi para Kähler pseudo-metric manifold.

Thus, from (1) and (2), we have further

$$(\overline{\nabla}_X \overline{J})Y = (\nabla_X \phi)Y + g(X,Y)\xi - \eta(Y)X - g(X,\phi Y)\frac{\partial}{\partial t} + (\nabla_X \eta)(Y)\frac{\partial}{\partial t}, \tag{3}$$

$$(\overline{\nabla}_X \overline{J})\frac{\partial}{\partial t} = \nabla_X \xi + \phi X, \tag{4}$$

$$(\overline{\nabla}_{\frac{\partial}{\partial t}} \overline{J})X = 0, \quad (\overline{\nabla}_{\frac{\partial}{\partial t}} \overline{J})\frac{\partial}{\partial t} = 0. \tag{5}$$

Now, we shall derive the condition for an almost paracontact psudo-metric manifold to be a quasi paracontact psudo-metric manifold. Again, from (3)-(5), we see that $\overline{M} = (\overline{M}, \overline{J}, \overline{g})$ is quasi para Kähler if and only if

$$\begin{aligned} 0 =& (\overline{\nabla}_{\overline{J}X} \overline{J})\overline{J}Y - (\overline{\nabla}_X \overline{J})Y \\ =& (\nabla_{\phi X}\phi)\phi Y - (\nabla_X \phi)Y - 2g(X,Y)\xi + 2\eta(Y)X + \eta(Y)\nabla_{\phi X}\xi \\ &+ ((\nabla_{\phi X}\eta)(\phi Y) - (\nabla_X \eta)(Y) + 2g(X,\phi Y))\frac{\partial}{\partial t}, \end{aligned} \tag{6}$$

$$0 = (\overline{\nabla}_{\overline{J}X} \overline{J})\overline{J}\frac{\partial}{\partial t} - (\overline{\nabla}_X \overline{J})\frac{\partial}{\partial t} = -\nabla_X \xi - \phi(\nabla_{\phi X})\xi - 2\phi X, \tag{7}$$

$$0 = (\overline{\nabla}_{\overline{J}\frac{\partial}{\partial t}} \overline{J})\overline{J}X - (\overline{\nabla}_{\frac{\partial}{\partial t}} \overline{J})X = (\nabla_\xi \phi)(\phi X) + \eta(X)\nabla_\xi \xi + (\nabla_\xi \eta)(\phi X)\frac{\partial}{\partial t}, \tag{8}$$

for $X, Y \in \chi(M)$.

Proposition 1.2. *An almost paracontact psudo-metric manifold $M = (M, \phi, \xi, \eta, g)$ is a quasi paracontact psudo-metric manifold if and only if the following equalities*

$$(\nabla_{\phi X}\phi)\phi Y - (\nabla_X \phi)Y = 2g(X,Y)\xi - 2\eta(Y)X - \eta(Y)\nabla_{\phi X}\xi, \tag{9}$$

$$(\nabla_{\phi X}\eta)(\phi Y) - (\nabla_X \eta)(Y) = -2g(X,\phi Y), \tag{10}$$

$$\nabla_X \xi + \phi(\nabla_{\phi X})\xi + 2\phi X = 0, \tag{11}$$

$$(\nabla_\xi \phi)(\phi X) + \eta(X)\nabla_\xi \xi = 0, \quad (\nabla_\xi \eta)(\phi X) = 0 \tag{12}$$

hold everywhere on M.

Proposition 1.3. *Let $M = (M, \phi, \xi, \eta, g)$ be an almost paracontact psudo-metric manifold satisfying the following condition*

$$(\nabla_{\phi X}\phi)\phi Y - (\nabla_X \phi)Y = 2g(X,Y)\xi - 2\eta(Y)X - \eta(Y)\nabla_{\phi X}\xi, \tag{13}$$

then, the following equalities are derived from the above equality

$$(\nabla_{\phi X}\eta)(\phi Y) - (\nabla_X\eta)(Y) = -2g(X,\phi Y), \tag{14}$$
$$\nabla_\xi\phi = 0, \quad \nabla_\xi\xi = 0, \tag{15}$$

for $X, Y \in \chi(M)$.

Proposition 1.4. *Let* $M = (M, \phi, \xi, \eta, g)$ *be an almost paracontact psudo-metric manifold satisfying the following condition*

$$(\nabla_{\phi X}\phi)\phi Y - (\nabla_X\phi)Y = 2g(X,Y)\xi - \eta(Y)(X + \eta(X)\xi + hX), \tag{16}$$

for any $X, Y \in \chi(M)$. *Then, the equalities (14) and (15) in Proposition 1.3 are derived from (16), where we define a* $(1,1)$*-tensor field* h *on* M *by* $h = \frac{1}{2}\ell_\xi\phi$.

The tensor field h plays an important role in the geometry of almost paracontact psudo-metric manifolds.

Lemma 1.5. *On a quasi paracontact psudo-metric manifold* M^{2n+1}, h *has the following formulas*

$$\eta \circ h = 0, \quad \phi^2 h = h, \quad h\xi = 0. \tag{17}$$

2 Main results

Lemma 2.1. *On a quasi paracontact psudo-metric manifold* M^{2n+1}, *we have the following formulas:*

$$h \text{ isn}'\text{t a symmetric operator}, \tag{18}$$
$$\nabla_X\xi = -\phi X + \phi hX, \tag{19}$$
$$g((\nabla_X\phi)\xi, Y) = -g(\phi X, \phi Y) + g(\phi hX, \phi Y), \tag{20}$$
$$h \text{ anti commutes with } \phi, \tag{21}$$
$$(\nabla_X h)\xi = h(\phi X - \phi hX), \tag{22}$$
$$d\eta(X,Y) = g(X,\phi Y) - \frac{1}{2}g(hX,\phi Y) - \frac{1}{2}g(X,\phi hY), \tag{23}$$

for any $X, Y \in \chi(M)$.

Theorem 2.2. *A paracontact psudo-metric manifold is characterized as a quasi paracontact psudo-metric manifold satisfying* $N^2 = 0$.

Remark 2.3. A quasi paracontact psudo-metric manifold isn$'$t cosymplectic manifold.

Theorem 2.4. *A* K*-paracontact psudo-metric manifold is characterized as a quasi paracontact psudo-metric manifold satisfying* $N^3 = 0$.

Theorem 2.5. *A quasi paracontact psudo-metric manifold is characterized as an almost paracontact psudo-metric manifold* $M = (M, \phi, \xi, \eta, g)$ *satisfying the following condition*

$$(\nabla_{\phi X}\phi)\phi Y - (\nabla_X\phi)Y = 2g(X,Y)\xi - \eta(Y)(X + \eta(X)\xi - hX), \tag{24}$$

for any $X, Y \in \chi(M)$.

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Poster A report of a project: Equivalence spaces pp.: 1–3

A report of a project : Equivalence spaces

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Abstract

In this paper induced equivalence spaces and $\mathcal{U}$-products are introduced and discussed. also the notion of equivalently open subspace of a equivalence space and equivalently open functions are studied.

Keywords: equivalence space, equivalently open subspace, equivalently continuous, topology
Mathematics Subject Classification [2010]: 54H99

1 Introduction

In this paper we deal with equivalence spaces. An equivalence space is a structure close to uniform spaces(Uniform spaces are somewhere the mid way points between metric spaces on one hand and abstract topological spaces on the other hand). These spaces have been introduced first in 2014 by F. Omidi and M.R. molaee [1].There are however a few aspects of metric spaces that are lost in general topological spaces. For example, since the notion of nearness is not defined for a general topological space, we cannot define the notion of uniform continuity in abstract topological spaces. The same can be said about notions such as total boundedness. An equivalence space is a mathematical construction in which such uniform concepts are still available.
An equivalence space $(X,\mathcal{U})$ is a set X along with a collection $\mathcal{U}$ of equivalence relations on X such that $\mathcal{U}$ is closed under finite intersections. We refer to $\mathcal{U}$ as equivalence collection on X.
A function $f : X \to Y$ where $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are two equivalence spaces, is called equivalently continuous if $(f \times f)^{-1}(V) \in \mathcal{U}$ whenever $V \in \mathcal{V}$, where

$$(f \times f)^{-1}(V) = \{(x, y) \in X \times X \mid (f(x), f(y)) \in V\}.$$

Moreover, if $(X,\mathcal{U})$ is an equivalence space, then the collection $\mathcal{T}_{\mathcal{U}} = \{G \subseteq X \mid$ for each x $\in G$, there exists U $\in \mathcal{U}$ such that U$[x] \subseteq G\}$ is a topology on X with the base $\{U[x] \mid U \in \mathcal{U}, x \in X\}$ where $U[x] = \{y \in X \mid (x, y) \in U\}$. We refer to $\mathcal{T}_{\mathcal{U}}$ as the $\mathcal{U}$-induced topology.

We are going to consider induced equivalence spaces and $\mathcal{U}$-products. Also, we will introduce and discuss equivalently open subspaces.

*Speaker

2 Induced equivalence spaces

Let $\{\phi_i : X \to X_i\}_{i\in I}$ be an indexed family of functions where X is a set and for each $i \in I$, $(X_i,\mathcal{U}_i)$ is an equivalence space. The idea is to induce an equivalence collection on X for which each ϕ_i is equivalently continuous without making equivalence collection on X unnecessarily strong.

Definition 2.1. Let $\mathcal{S}$ be a family of equivalence relations on a set X. Then the collection of all finite intersections of members of $\mathcal{S}$ (that forms an equivalence collection on X) called the equivalence collection generated by $\mathcal{S}$ and is denoted by$< \mathcal{S} >$.
Note that $< \mathcal{S} >$ is the smallest equivalence collection on X which contains $\mathcal{S}$.

Proposition 2.2. *Let $\{\phi_i : X \to X_i\}_{i\in I}$ be an indexed family of functions where X is a set and for each $i \in I$, $(X_i,\mathcal{U}_i)$ is an equivalence space. Then there exists a smallest equivalence collection on X for which each ϕ_i is equivalently continuous.*

Proof. Let $\mathcal{S}^{\leftarrow} = \{(\phi_i \times \phi_i)^{-1}(U_i) \mid U_i \in \mathcal{U}_i, i \in I\}$. Let $\mathcal{U}^{\leftarrow}$ be equivalence collection generated by $\mathcal{S}^{\leftarrow}$. It is easy to see that $\mathcal{U}^{\leftarrow}$ is an equivalence collection on X. □

The equivalence collection $\mathcal{U}^{\leftarrow}$ in the last proposition is called induced equivalence collection.

Corollary 2.3. *Let $\phi : X \to Y$ be a function where X is a set and $(Y,\mathcal{V})$ is an equivalence space. Then $\mathcal{V}^{\leftarrow} = \{(\phi \times \phi)^{-1}(V) \mid V \in \mathcal{V}\}$.*

The following property is a characteristic of the equivalence collections.

Proposition 2.4. *Let $\phi : X \to Y$ and $\psi : Y \to Z$ be functions where $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are equivalence spaces. If Y has the induced equivalence collection, then ϕ is equivalently continuous if and only if $\psi\phi$ is equivalently continuous.*

Let $(X,\mathcal{U})$ be a equivalence space and let $A \subseteq X$. By corolarry 2.3, (let ϕbe the inclusion map) the collection $\{(A \times A) \cap U \mid U \in \mathcal{U}$ is the equivalence collection on A that is called relative equivalence collection and denoted by $\mathcal{U}/A$.

Definition 2.5. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two equivalence spaces. A function $\phi : X \to Y$ is said to be $\mathcal{U}$ - equivalence if ϕ and ϕ^{-1} are equivalently continuous. A function $\phi : X \to Y$ is said to be $\mathcal{U}$ - embedding if it is one to one and $\mathcal{U}$ - equivalence when regarded as a function from $(X,\mathcal{U})$ to $(f(X), \mathcal{V}/f(X)$.

Theorem 2.6. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be to equivalence spaces and let $\phi : X \to Y$ be a function.Then the following statements are equivalent :*
(a) ϕ is $\mathcal{U}$ - embedding.
(b) ϕ is one to one, equivalently continuous and $\mathcal{U} = \mathcal{V}^{\leftarrow}$.

Proposition 2.7. *Let $\phi : X \to Y$ be an equivalently continuous function where $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are equivalence spaces. Suppose ϕ admits a $\mathcal{U}$-equivalently left inverse. Then ϕ is a $\mathcal{U}$ - embedding.*

3 equivalently open subspaces

Among the subspaces of an equivalence space, special attention should be given to those which are equivalently open, in the sense that the inclusion map is equivalently open. Roughly speaking, every equivalently open function is locally surjective(see [1]). For example ϕ is always equivalently open when $\mathcal{V}$ is discrete i.e $\mathcal{V}$ is the collection of all equivalence relations on Y.
Let $(X,\mathcal{U})$ be a equivalence space. Then $\mathcal{U}$ is called rich if $X^2 \in \mathcal{U}$.

Proposition 3.1. *Let $(X,\mathcal{U})$ be a rich equivalence space and $A \subseteq X$. Then the following statements are the same:*
(a) A is equivalently open.
(b) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V[x] \subseteq U[x] \cap A$ for all points $x \in A$.
(c) there exists $V_0 \in \mathcal{U}$ such that $V_0[x] \subseteq A$ for all points $x \in A$.

Suppose $\alpha, \beta : X \to Y$ are maps from X into Y. The coincidence set of α and β is the set $\mathcal{C}(\alpha,\beta) = \{x \in X \mid \alpha(x) = \beta(x)\}$. Also, if $\phi : X \to Y$ is a function from the equivalence space $(X,\mathcal{U})$ into the set Y, then we say that ϕ is transverse to X whenever $(\phi \times \phi)^{-1}(\triangle Y) \cap U = \triangle X$ for some $U \in \mathcal{U}$. Roughly speaking, ϕ is transverse to X if ϕ is one to one on a regoin of X^2.

Proposition 3.2. *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ be equivalence spaces, $\alpha, \beta : X \to Y$ be two equivalently continuous functions and $\phi : Y \to Z$ be transitive toY. If $\phi \circ \alpha = \phi \circ \beta$ and $\mathcal{U}$ is rich, then the coincidence set $\mathcal{C}(\alpha,\beta)$ is equivalently open in X.*

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Causality conditions and cosmological time function

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Abstract

In this paper the concept of dual cosmological time function and its regularity is introduced. It is shown that the regularity of cosmological and dual cosmological time functions are independent of each other. It is proved that if the cosmological time function of the spacetime (M, g) is continuous, $\tau \to 0$ along every past inextendible causal curve and $\tau \to \infty$ along every future inextendible causal curve then (M, g) is globally hyperbolic.

Keywords: Spacetime, Globally hyperbolic, Cosmological time function, Lorentzian metric
Mathematics Subject Classification [2010]: 83C99, 83C75, 83C20

1 Introduction

In this paper we investigate the concept of cosmological time function and its relation with the causal hierarchy of the spacetime. So let us recall some ladders in the causal hierarchy which are needed in this paper.

Definition 1.1. [2] A spacetime is non-total future imprisoning if no future inextendible causal curve is totally future imprisoned in a compact set. A spacetime is non-partially future imprisoning if no future inextendible causal curve is partially future imprisoned in a compact set. Analogue definitions hold in the past case. A spacetime is non-total imprisoning if it is bout non total future and non total past imprisoning.

Definition 1.2. [2] A spacetime (M, g) is globally hyperbolic if it is causal and the intersections $J^+(p) \cap J^-(q)$ are compact for all p, $q \in M$.

The domain of dependence of A is defined as $D(A) = D^+(A) \cup D^-(A)$, where $D^+(A)$ (resp. $D^-(A)$) is defined as the set of points $p \in M$ such that every past (resp. future) inextendible causal curve through p intersects A.

Definition 1.3. [2] A Cauchy hypersurface is a subset $S \subset M$ which is crossed exactly once by any inextendible timelike curve $(D(S) = M)$.

Equivalently it is proved that (M, g) is globally hyperbolic if it admits a Cauchy hypersurface [2].

*Speaker

Remark 1.4. The continuous function $t : M \to R$ is a time function if it strictly increases on every causal curve. We also recall that a temporal function t is a smooth time function t with past directed timelike gradient. In addition t is called a Cauchy temporal function if its level sets be Cauchy hypersurfaces. It is proved by Sanchez [3] that if M admits a Cauchy temporal function then it is equivalent to the existence of a global orthogonal splitting, $M \equiv (R \times S, g)$, where g can be written as $g = -dt^2 + \beta g_t$,

β is a function on $R \times S$ and g_t is a Riemannian metric on each slice $\{t\} \times S$.

2 Cosmological time function and dual cosmological time function

Let (M, g) be a spacetime and $d : M \times M \to [0, \infty]$ be the Lorentzian distance function. We recall that the cosmological time function $\tau : M \to [0, \infty)$ is defined by:

$$\tau(q) := sup_{p<q} d(p, q)$$

We can also define dual cosmological time function by:

$$\tau_d(p) := sup_{p<q} d(p, q)$$

Definition 2.1. [1] The cosmological time function τ of (M, g) is regular if and only if:

- $\tau(q) < \infty$, for all $q \in M$,
- $\tau \to 0$ along every past inextendible causal curve.

Remark 2.2. If $p < q$ then $\tau(p) + d(p, q) \leq \tau(q)$.

Definition 2.3. The dual cosmological time function τ_d is regular if and only if:

- $\tau_d(q) < \infty$, for all $q \in M$,
- $\tau_d \to 0$ along every future inextendible causal curve.

The first condition shows that for each point q any particle that passes through it will be in existence for at most a time of $\tau_d(q)$. If we believe that $\tau_d = 0$ is a singularity the second condition shows that the end of history of every particle is in a singularity.

Example 2.4. 1) Minkowski spacetime is an example of a spacetime that $\tau, \tau_d = \infty$.

2) (M, g), $M = (a, b) \times H$ with $a > -\infty$, $b = \infty$, $g = -dt^2 \oplus fh$, and (H, h) a homogeneous Riemannian manifold is an example of a spacetime that its dual cosmological time function is not regular but its cosmological time function is regular.

2) (M, g), $M = (a, b) \times H$ with $a = \infty$, $b < \infty$, $g = -dt^2 \oplus fh$, and (H, h) a homogeneous Riemannian manifold is an example of a spacetime that its dual cosmological time function is regular but its cosmological time function is not regular.

3) (M, g), $M = (a, b) \times H$ with $a = \infty$, $b = \infty$, $g = -dt^2 \oplus fh$, and (H, h) a homogeneous Riemannian manifold is an example of a spacetime that its dual cosmological time and cosmological time function are regular.

Lemma 2.5. *If the cosmological time function of (M, g) is finite but its dual cosmological time function is not then there is a sequence p_n such that $\tau(p_n) \to \infty$.*

Proof. Since τ_d is not finite, there is $p \in M$ that $\tau_d(p) = \infty$. Hence there is a sequence p_n such that $d(p_n, p) \to \infty$ and consequently $\tau(p_n) \to \infty$. □

Corollary. If $\tau < a$ then τ_d is finite.

Lemma 2.6. *Let (M, g) be a spacetime that its cosmological time function is continuous. If $\tau \to 0$ along every past inextendible causal curve then (M, g) is non total imprisoning.*

Proof. Suppose by contradiction that $\gamma : [a, b) \to M$ is a past inextendible causal curve which is imprisoned in a compact set C. Let $\{p_n\}$ be an increasing sequence that converges to p. Since C is compact and τ is continuous $\tau(p_n) \to \tau(p) > 0$, which is a contradiction. □

Lemma 2.7. *If (M, g) is globally hyperbolic spacetime there is $\omega > 0$ such that the cosmological time function of $(M, \omega g)$ is equal to ∞.*

Proof. Let $M = R \times S$, $g \equiv dt^2 \oplus \beta g_t$, as in remark 1.4. For every $p = (s, a) \in R \times S$ let $\gamma(t) = (t, a)$. γ is a past inextendible causal curve with length ∞ that shows that $\tau((s, a)) = \infty$. □

Theorem 2.8. *Let (M, g) be a spacetime that its cosmological time function is continuous. If $\tau \to 0$ along every past inextendible causal curve and $\tau \to \infty$ along every future inextendible causal curve then (M, g) is a globally hyperbolic spacetime.*

Proof. Let $S_t = \{p \in M : \ \tau(p) = t\}$. S_t is closed, achronal and edgeless. $D^+(S_t) = \{q \in M : \tau(q) \geq t\}$ and $D^-(S_t) = \{q \in M : \tau(q) \leq t\}$. Hence $M = D(S_t) = D^+(S_t) \cup D^-(S_t)$ and consequently S_t is a Cauchy hypersurface. □

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Poster Characterization of representation up to homotopy of double groupoids pp.: 1–4

Characterization of Representation up to Homotopy of Double Groupoids

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Abstract

In this paper, we introduce the concept of representation of double Lie groupoids and characterize them by Lie groupoid cohomology and using this result to introduce representations up to homotopy of double Lie groupoids.

Keywords: Double Lie groupoid, Representation up to homotopy, Lie groupoid cohomology, gauge-equivalent.
Mathematics Subject Classification [2010]: 20L05, 22A22, 18D05.

1 Introduction

The theory of representation of Lie groupoids is extend by Gracia-Saz and Mehta, [2] and we would like to extend their results on double Lie groupoids.

Let $G \underset{s}{\overset{t}{\rightrightarrows}} M$ be a Lie groupoid, let $G^{(0)} := M$, and $G^{(p)}$ be the manifold consisting of composable p-tuples of elements of G, where $p > 0$, i.e.

$$G^{(p)} := \{(g_1, \cdots, g_p) : s(g_i) = t(g_{i+1})\}.$$

There is a coboundary operator $\sigma : C^p(G) \to C^{p+1}(G)$ on the space of $\mathbb{R}$-valued smooth groupoid cochains $C^p(G) := C^\infty(G^{(p)})$, which introduced by C. Arias Abad and M. Crainic, [3]. We know that $\sigma^2 = 0$, and then the cohomology of the complex $(C^\bullet(G), \sigma)$ is known as the smooth groupoid cohomology of G.

For $E \to M$ and $F \to M$ as a vector bundles, the space of smooth groupoid p-cochains with values in E and the space of transformation p-cochains from E to F is introduced in [2, 3].

Definition 1.1. [2] A E-valued cochain is called normalized if it vanishes whenever at least one of its arguments is a unite.

*Speaker

2 Representation of double Lie Groupoids

The square

$$\begin{array}{ccc} A & \overset{t_B}{\underset{s_B}{\rightrightarrows}} & B \\ {\scriptstyle t_C}\downarrow\downarrow{\scriptstyle s_C} & & {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ C & \underset{t}{\overset{s}{\rightrightarrows}} & D \end{array} \tag{2.1}$$

is a double Lie groupoid if the following conditions hold:

1. The horizontal and vertical source and target maps commute,
2. The multiplication maps respect source and target,
3. The interchange law

$$(\alpha_{11}.{}_C\alpha_{12}).{}_B(\alpha_{21}.{}_C\alpha_{22}) = (\alpha_{11}.{}_B\alpha_{21}).{}_C(\alpha_{12}.{}_B\alpha_{22})$$

holds for all $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \in A$ such that $s_B(\alpha_{i1}) = t_B(\alpha_{i2})$ and $s_C(\alpha_{1i}) = t_C(\alpha_{2i})$, for $i = 1, 2$.

4. The double-source map $(s_B, s_C) : A \to B_{\mathrm{s}}\times_{\mathrm{s}}C$ is a submersion.

Now we define the representation of double Lie groupoid.

Definition 2.1. A representation of a double Lie groupoid (2.1) is a bi-vector bundle E over B and C, together with a double Lie groupoid morphism (ρ, Id_B, Id_C, Id_D), which ρ is a map from A to $Gl_B(E) \times Gl_C(E)$.

Definition 2.2. A smooth double groupoid cohomology of double Lie groupoid (2.1) is two smooth groupoid cohomologies, together with two degree 1 coboundary operators σ_B and σ_C.

Let E be a bi-vector bundle over B and C. There are right $C_B(A)$-module structure on $C_B(A; E)$ and $C_C(A)$-module structure on $C_C(A; E)$, given by:

1. for $p, q > 0$

$$(\omega \star f)(g_1, \cdots, g_{p+q}) := \omega(g_1, \cdots, g_p)f(g_{p+1}, \cdots, g_{p+q}),$$

2. for $p = 0$

$$(\omega \star f)(g_1, \cdots, g_q) := \omega(t(g_1))f(g_1, \cdots, g_q),$$

3. for $q = 0$

$$(\omega \star f)(g_1, \cdots, g_p) := \omega(g_1, \cdots, g_p)f(s(g_p)),$$

for $\omega \in C^p_{\mathcal{V}}(A; E)$, $f \in C^q_{\mathcal{V}}(A)$ and $\mathcal{V} \in \{B, C\}$.

Given a representation ρ of double Lie groupoid on bi-vector bundle E, we can construct two operators Δ_B and Δ_C with degree 1 on $C_B(A; E)$ and $C_C(A; E)$ respectively, whose action of 0-forms is given by

$$(\Delta_{\mathcal{V}}x)(g) := \pi_{\mathcal{V}} \circ \rho_g x_{s_{\mathcal{V}}(g)} - x_{t_{\mathcal{V}}(g)},$$

for $\mathcal{V} \in \{B, C\}$, $x \in \Gamma_{\mathcal{V}}(E)$ and $g \in A$, and for $p > 0$ it is given by

$$\begin{aligned}(\Delta_{\mathcal{V}}\omega)(g_0, \cdots, g_p) &:= (\pi_{\mathcal{V}} \circ \rho)_{g_0}\omega(g_1, \cdots, g_p) \\ &+ \Sigma_{k=1}^{p}(-1)^k\omega(g_0, \cdots, g_{k-1}g_k, \cdots, g_p) \\ &+ (-1)^{p+1}\omega(g_0, \cdots, g_{p-1}),\end{aligned}$$

for $\mathcal{V} \in \{B, C\}$, $\omega \in C^p_{\mathcal{V}}(G; E)$. It is easy to show that $\Delta_B^2 = \Delta_C^2 = 0$ and

$$\Delta_{\mathcal{V}}(\omega \star f) = (\Delta_{\mathcal{V}}\omega) \star f + (-1)^p\omega \star (\sigma_{\mathcal{V}} f). \tag{2.2}$$

We conclude this section by the following theorem.

Theorem 2.3. There is a one-to-one correspondence between representations of double Lie groupoid (2.1) on E and continuous degree 1 operators Δ_B and Δ_C on $C_B(A; E)$ and $C_C(A; E)$, respectively, satisfying (2.2), preserving the space of normalized cochains and $\Delta_B^2 = \Delta_C^2 = 0$.

3 Representation up to homotopy of double Lie groupoids

Let $\varepsilon = \bigoplus E_i$ be a graded bi-vector bundle over B and C. We consider $C_{\mathcal{V}}(A; \varepsilon)$ to be a graded right $C_{\mathcal{V}}(A)$-module with respect to the total grading:

$$C^p_{\mathcal{V}}(A; \varepsilon) = \bigoplus_{q-r=p} C^q_{\mathcal{V}}(A; E_r),$$

where $\mathcal{V} \in \{B, C\}$.

Definition 3.1. A representation up to homotopy of a double Lie groupoid A on a graded bi-vector bundle ε is two continuous degree 1 operators Δ_B and Δ_C on $C_B(A; \varepsilon)$ and $C_C(A; \varepsilon)$, respectively, satisfying (2.2), preserving the space of normalized cochains and $\Delta_B^2 = \Delta_C^2 = 0$.

Know, we consider representation up to homotopy on double groupoids, in one and two term graded bi-vector bundles.

Case 1. In the case where $\varepsilon = E$ is concentrated in degree 0, a representation up to homotopy is the same as a representation, and there are no nontrivial gauge transformations.

Case 2. In this case, we use the notation $\varepsilon = E \oplus F[1]$, where E is the part with degree 0 part and F is the part with degree 1 part. Then for $\mathcal{V} \in \{B, C\}$

$$C^p_{\mathcal{V}}(A; E \oplus F[1]) = C^p_{\mathcal{V}}(A; E) \oplus C^{p+1}_{\mathcal{V}}(A; F).$$

Any degree 1 operator $\Delta_{\mathcal{V}}$ on $C_{\mathcal{V}}(A; E \oplus F[1])$ decomposes as the sum of the following four homogeneous components:

$$\begin{aligned}\Delta^F_{\mathcal{V}} &: C^\bullet_{\mathcal{V}}(A; F) \to C^{\bullet+1}_{\mathcal{V}}(A; F), \\ \Delta^E_{\mathcal{V}} &: C^\bullet_{\mathcal{V}}(A; E) \to C^{\bullet+1}_{\mathcal{V}}(A; E), \\ \hat{\sigma}_{\mathcal{V}} &: C^\bullet_{\mathcal{V}}(A; F) \to C^\bullet_{\mathcal{V}}(A; E), \\ \hat{\Omega}_{\mathcal{V}} &: C^\bullet_{\mathcal{V}}(A; E) \to C^{\bullet+2}_{\mathcal{V}}(A; F).\end{aligned}$$

The Leibniz rule (2.2) for $\Delta_{\mathcal{V}}$ is equivalent to:

1. $\Delta^F_{\mathcal{V}}$ and $\Delta^E_{\mathcal{V}}$ satisfy (2.2),
2. $\hat{\sigma}_{\mathcal{V}}$ and $\hat{\Omega}_{\mathcal{V}}$ are right $C(A)$-module morphisms.

Part (1) allows us to define quasi-actions $\rho^F_{\mathcal{V}}$ and $\rho^E_{\mathcal{V}}$ on F and E over $\mathcal{V}$, respectively, as follows:

$$(\rho^F_{\mathcal{V}}(g))(\alpha) = -(\Delta^F_{\mathcal{V}}\alpha)(g) + \alpha_{t_{\mathcal{V}}(g)} \quad (3.1)$$
$$(\rho^E_{\mathcal{V}}(g))(\epsilon) = (\Delta^E_{\mathcal{V}}\epsilon)(g) + \epsilon_{t_{\mathcal{V}}(g)} \quad (3.2)$$

for $\alpha \in \Gamma_{\mathcal{V}}(C)$ and $\epsilon \in \Gamma_{\mathcal{V}}(E)$.

Part (2) implies that $\hat{\sigma}_{\mathcal{V}}$ corresponds to a linear map $\sigma_{\mathcal{V}} \in Hom_{\mathcal{V}}(F, E) = C^0_{\mathcal{V}}(A; F \to E)$, and $\hat{\Omega}_{\mathcal{V}}$ corresponds to a transformation 2-cochain $\Omega_{\mathcal{V}} \in C^2_{\mathcal{V}}(A; E \to F)$. Also the equation $\Delta^2_{\mathcal{V}} = 0$ is equivalent to:

$$\begin{aligned}
\Delta^E_{\mathcal{V}}\hat{\sigma} + \hat{\sigma}\Delta^F_{\mathcal{V}} &= 0,\\
(\Delta^F_{\mathcal{V}})^2 + \hat{\Omega}_{\mathcal{V}}\hat{\sigma}_{\mathcal{V}} &= 0,\\
(\Delta^E_{\mathcal{V}})^2 + \hat{\sigma}_{\mathcal{V}}\hat{\Omega}_{\mathcal{V}} &= 0,\\
\Delta^F_{\mathcal{V}}\hat{\Omega}_{\mathcal{V}} + \hat{\Omega}\Delta^E_{\mathcal{V}} &= 0.
\end{aligned}$$

The total operator $\Delta_{\mathcal{V}}$ preserves normalized cochains if and only if for every components $\rho^F_{\mathcal{V}}$, $\rho^E_{\mathcal{V}}$ and $\Omega_{\mathcal{V}}$, we obtain the following conditions:

$$\rho^F_{\mathcal{V}} \quad \text{and } \rho^E_{\mathcal{V}} \text{ are unital,} \quad (3.3)$$
$$\Omega_{\mathcal{V}} \quad \text{is normalized.} \quad (3.4)$$

The results of this section is summarized in the following theorem.

Theorem 3.2. There is a one-to-one correspondence between double representations up to homotopy on a 2-term graded vector bundle $E \oplus F[1]$ and two 4-tuples $(\rho^F_B, \rho^E_B, \sigma_B, \Omega_B)$ and $(\rho^E_C, \rho^E_C, \sigma_C, \Omega_C)$, where

1. $\rho^F_{\mathcal{V}}$ and $\rho^E_{\mathcal{V}}$ are unital quasi-actions on $F_{\mathcal{V}}$ and $E_{\mathcal{V}}$, respectively,
2. $\sigma_{\mathcal{V}} : F_{\mathcal{V}} \to E_{\mathcal{V}}$ is a linear map,
3. $\Omega_{\mathcal{V}}$ is a normalized element of $C^2(A; E_{\mathcal{V}} \to F_{\mathcal{V}})$,

satisfying (3.1-3.4), for $\mathcal{V} \in \{B, C\}$.

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Electromagnetic Field Tensor in Kerr-Newman Geometry

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Abstract

An application of Riemannian geometry in theoretical physics is considered. As we will see by a suitable geometric structure on a rotating charged 4-manifold by spacetime coordinate (t, r, θ, ϕ) called *Kerr-Newman geometry* we will discuss an important tensor on a given Riemannian 4-manifold which is so applicable in theoretical physics. We consider a rotating charged black hole as a smooth Riemannian 4-manifold structure by the spacetime coordinate (t, r, θ, ϕ). Then, by using differential forms and exterior derivative we will find the corresponding electromagnetic field tensor.

Keywords: Riemannian manifold, Electromagnetic field tensor, Differential forms, Spacetime.
Mathematics Subject Classification [2010]: 58J70, 35A30.

1 Introduction

One of the most important tools in differential geometry are tensors on smooth manifolds. These objects could discuss a vast field of physical phenomena specially those are not reachable such as cosmological phenomena, also they are very applicable in theoretical physics, quantum mechanics, string theory and etc. This article illustrates Kerr-Newman geometry, a geometrical structure on a rotating charged black hole by considering the spacetime coordinate (t, r, θ, ϕ) that induces a four-dimensional smooth structure on the black hole. Rotating charged black hole electromagnetic field analysis could effect our perception about black hole and it's region on.

In the first section we introduce a special case of spacetime coordinate called *Boyer-Lindquist* coordinate (generalization of Schwarzschild coordinate) together with its corresponding metric to make the associated Riemannian manifold. In section two we define the expression for component of electromagnetic tensor field. In section three we define an equation for electromagnetic field from rotating charge body by using the associated potential of Kerr-Newman metric. Finally the calculation of corresponding electromagnetic field tensor in Kerr-Newman geometry is made in the last section.

2 Kerr-Newman Geometry

Consider an $n-$dimensional smooth manifold (with boundary) M. By a *covariant k-tensor* on a point $p \in M$ we mean a $k-$linear map $F : \underbrace{T_pM \times \cdots \times T_pM}_{k-\text{times}} \to \mathbb{R}$ defines on the

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tangent space T_pM. Suppose $x = (x^1, ..., x^n)$ is a coordinate chart on M. It in not hard to see that in this coordinate we have [5],

$$F = \sum_{i_1,...,i_k=1}^{n} F_{i_1\cdots i_k}(p) dx^{i_1} \otimes \cdots \otimes dx^{i_k}, \tag{1}$$

where $\otimes$ is tensor product and $F_{i_1\cdots i_k}(p)$'s are smooth real-valued functions on M. One of the most important case of these tensors are called *alternating tensors*, those are anti-symmetric on an odd permutation of their components.

Definition 2.1. A covariant two-tensor $g : T_pM \times T_pM \to \mathbb{R}$ is called a *Riemannian metric* if g is symmetric and non-degenerate. Thus, g induces an inner product on the vector space T_pM. A manifold with a Riemannian metric is called *Riemannian manifold*.

Suppose we have a spherical symmetric mass with a given charge and rotation. Written in the coordinate (t, r, θ, ϕ) in Boyer-Lindquist form [1], the Kerr-Newman metric [2], is a Riemannian metric of the form

$$g = -\frac{\Delta}{\rho^2}\left[dt - a\sin^2\theta d\phi\right]^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - adt\right] + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \tag{2}$$

where $\Delta \equiv r^2 - 2mr + a^2 + Q^2$, $\rho^2 \equiv r^2 + a^2\cos^2\theta$ and $a \equiv S/m$ is angular momentum per unit mass. In these equations m and Q are mass and charge respectively. As we see the Kerr-Newman geometry has a horizon, and therefore describes a black hole, if and only if $m^2 \geq Q^2 + a^2$.

3 Electro Magnetic Field Tensor

First we begin with a definition for a kind of covariant $k-$tensor called differential forms. Consider a smooth manifold with coordinate chart $x = (x^1, ..., x^n)$. A covariant $k-$tensor (1) is called an *alternating tensor* if

$$F(X_1, ..., X_i, ..., X_j, ..., X_k) = -F(X_1, ..., X_j, ..., X_i, ..., X_k),$$

i.e., they are anti-symmetric on an odd permutation of their components.

Definition 3.1. In any smooth coordinate chart $x = (x^1, ..., x^n)$, a $k-$diffenetial form ω is a covariant $k-$tensor is written locally as:

$$\omega = \sum_{i_1,...,i_k=1}^{n} \omega_{i_1,\cdots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{3}$$

where the wedge product $\wedge$, is defined by the tensor product in (1), [5].

We know that the electromagnetic fields are made up of the electric field vector E_i and magnetic field and the magnetic field vector B_i. In fact they are components of a covariant two-tensor $F_{\mu\nu}$ defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = -F_{\nu\mu}. \tag{4}$$

here $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ are electric and magnetic field respectively. This matrix equation demonstrates the unity of the electric and magnetic fields.

The electromagnetic field tensor [3], Faraday= $\mathbf{F}$, is an antisymmetric second-rank tensor. Instead of expanding it in terms of the tensor product we have,

$$\mathbf{F} = F_{\alpha\beta} dx^\alpha \otimes dx^\beta, \tag{5}$$

The exterior calculus prefers to expand in terms of antisymmetrized tensor products (recall: $dx^\alpha \wedge dx^\beta = dx^\alpha \otimes dx^\beta - dx^\beta \otimes dx^\alpha$) thus, $\mathbf{F}$ is a differential 2-forms such as:

$$\mathbf{F} = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \tag{6}$$

consequently, the differential 2-form for the general electromagnetic field can be written as:

$$\mathbf{F} = E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy. \tag{7}$$

4 Kerr-Newman Electro Magnetic Field Tensor

According to the Kerr-Newman metric (4) the non-zero components of g_{ij} of Kerr-Newman metric are given as follow:

$$\begin{array}{ll} g_{tt} = \rho^{-2}\left[a^2 \sin^2\theta - \Delta\right], & g_{rr} = \frac{\rho^2}{\Delta}, \\ g_{\theta\theta} = \rho^2, & g_{\phi\phi} = \rho^{-2}\left(\sin^2\theta (r^2+a^2)\right)^2 - \Delta a^2 c \sin^4\theta, \\ g_{\phi t} = \rho^{-2}\left[2a(r^2+a^2+\Delta)\sin^2\theta\right]. & \end{array}$$

and the associated potential of the Kerr-Newman metric are expressed as:

$$\begin{array}{l} A_t = \rho^{-2}\left[Qr - \rho a\cos\theta\right], \quad A_r = 0, \quad A_\theta = 0, \\ A_\phi = \rho^{-2}\left[\rho(r^2+a^2)\cos\theta - Qar\sin^2\theta\right]. \end{array}$$

Thus, the electromagnetic field tensor ($F_{\alpha\beta}$) can be expressed in terms of potential as:

$$\begin{array}{lcl} F_{\alpha\beta} & = & \nabla_\alpha A_\beta - \nabla_\beta A_\alpha, \\ \nabla_\alpha A_\beta & = & \partial_\alpha A_\beta - \Gamma^\rho_{\alpha\beta} A_\rho, \\ \Gamma^\rho_{\alpha\beta} A_\rho & = & \frac{1}{2} g^{\rho\alpha}\left(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}\right), \end{array}$$

where $\Gamma_{\alpha\beta}$ is the Christofel symbol of the second line [4].

By using potential, we can calculate the electromagnetic field of a rotating charged black hole with Kerr-Newman geometry such as:

$$\mathbf{F} = F_{t\phi} dx^t \wedge dx^\phi \quad \text{where} \quad F_{t\phi} = \partial_t A_\phi - \Gamma^\lambda_{t\phi} A_\lambda - \partial_\phi A_t + \Gamma^\lambda_{\phi t} A_\lambda, \tag{8}$$

then, for $\lambda = t, r, \theta, \phi$ we have

$$\begin{array}{lcl} \Gamma^r_{t\phi} & = & -\frac{\Delta}{2\rho^2}\left(\frac{(2acr^2+a^2+\Delta)\sin^2\theta}{\rho^2}\partial r\right), \\ \Gamma^\theta_{t\phi} & = & -\frac{1}{\rho^2}\left(\frac{(2a(r^2+a^2+\Delta)\sin^2\theta}{\rho^2}\partial\theta\right), \\ \Gamma^t_{t\phi} & = & 0, \\ \Gamma^\phi_{t\phi} & = & 0. \end{array}$$

For these non-zero Christofel symbols the electro magnetic field tensor (8) turns to:

$$\begin{aligned}\partial_\phi A_\theta &= 0,\\ \partial_\theta A_\phi &= -2a^2\cos\theta\sin\theta\left[\rho(r^2+a^2)\cos\theta - Qar\sin^2\theta\right]\\ &\quad +\rho^{-2}\left[-\rho(r^2+a^2)\sin\theta - 2Qar\sin\theta\cos\theta\right].\end{aligned}$$

Finally the corresponding electromagnetic field tensor for the Kerr-Newman black hole is written as the following 2-form:

$$\mathbf{F} = \frac{Q}{\rho^4}(r^2 - a^2\cos^2\theta)dr\wedge[dt - a\sin^2\theta d\phi + 2Q\rho^4 ar\cos\theta\sin\theta d\theta]\wedge\left[(r^2+a^2)d\phi - adt\right].$$

Acknowledgment

As we have seen for an application differential geometry specially tensors play a very important role in theoretical physics and cosmology for illustrating cosmological phenomena. But it is not only the applications of differential geometry, there is a lots of literatures such as papers and books for the applications in engineering, biology and etc.

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Poster | Erratum to: The configuration space integral for links in $\mathbb{R}^3$ | pp.: 1–3

Erratum to: The Configuration Space Integral for Links in $\mathbb{R}^3$

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Abstract

In this paper we give some corrections to a mistake happened in a paper about a link invariant called Configuration Space Integral. The Configuration Space Integral can be seen as a generalisation of the Gauss formula for the linking number of two knots. This invariant is a strong finite type invariant for links and knots. We will correct the mistake and reprove the concerning theorems.

Keywords: Knots, Finite type invariants, Configuration Space Integral
Mathematics Subject Classification [2010]: 51H20, 51H30

1 Introduction

In this paper we give some corrections to a mistake happened in the paper "The Configuration Space Integral for Links in $\mathbb{R}^3$" by Sylvain Poirier [1]. The Configuration Space Integral can be seen as a generalisation of the Gauss formula for the linking number of two knots. This invariant is a strong finite type invariant for links and knots. In section 2 we are giving some definitions in this field and in section 3 we will correct the mistake. In the last section we give a proof for compactification result using the new diffeomorphism defined in section 3.

2 Definitions

Let M be a compact one-dimensional manifold with boundary. Let L denote an embedding of M into $\mathbb{R}^3$. We say that L is a *link* if we moreover have the condition that the boundary of M is empty. And a link L it is a *knot* when $M = S^1$.

The configuration space integral is a linear combination of integrals on configuration spaces of a link. In [1] this integral is defined. For proving that this integrals converge, the author chose a compactification of a configuration space which has a natural structure of a smooth manifold. In this process Poirier constructed the compactified space $\mathcal{H}(G)$ of a graph G, that is defined as follows:

Definition 2.1. If A is a finite set with at least two elements, let C^A denote the space of non-constant maps from A to $\mathbb{R}^3$ quotiened by the translation-dilations group (that is the group of translations and positive homotheties of $\mathbb{R}^3$).

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Let G be a graph defined as a pair $G = (V, E)$ where V is the set of "vertices" and E is the set of "edges": V is a finite set and E is a set of pairs of elements of V. We suppose that $\#V \geq 2$ and that G is connected. Let R be the set of connected subsets A of V such that $\#A \geq 2$.

Definition 2.2. Let $\mathcal{H}(G)$ be the subset of $\prod_{A\in R} C^A$ made of the $x = (x_A)_{A\in R}$ such that for all $A, B \in R$, $A \subseteq B$ implies that the restriction of x_B to A is either the constant map or the map x_A.

Now we are ready to correct the mistake.

3 The Diffeomorphism

The mistake is that in [1] the author asserted that the space C^A is diffeomorphic to the sphere $S^{3\#A-4}$. But in fact C^A is diffeomorphic to the space

$$\{0\} \times S^2 \times D^3 \times \cdots \times D^3,$$

where D^3 denote the 3-dimensional disk in $\mathbb{R}^3$, that is

$$D^3 = \{x \in \mathbb{R}^3 \mid ||x|| \leq 1\},$$

and the number of factors D^3 in the product above is $\#A - 2$.

Proof. Intuitively, for translations choose an element $a \in A$ to be at the origin. In this way the images of other elements of A are to be considered modulo the dilations with with center the origin. Choose an other element $B \in A$ so that it's image in $\mathbb{R}^3$ has largest distance from origin. We can choose such an element $b \in A$ because $\#A \geq 2$ and elements of C^A are not constant maps. Then for dilations normilize this biggest distance, i. e. divide all vectors by the largest distance, so the image of b will be in S^2. Now the image of other elements of A can be any point in the disk D^3.

More precisely consider the following diffeomorphism:

$$\varphi : C^A \to \{0\} \times S^2 \times D^3 \times \cdots \times D^3$$

defined by

$$[f] \mapsto \left(\frac{f(x) - f(a)}{\|f(b) - f(a)\|} \right)_{x\in A},$$

where $[f]$ denote all the maps $f : A \to \mathbb{R}^3$ modulo translations and dilations in $\mathbb{R}^3$. The map φ is well-defined, because for all $x \in A$, the term $\frac{f(x)-f(a)}{\|f(b)-f(a)\|}$ is invariant under all translations and dilations.

The map φ is one-to-one. Let $\varphi([f]) = \varphi([g])$ for non-constant maps $f, g : A \to \mathbb{R}^3$. So there are elements $b, b' \in A$ so that for all $x \in A$ we have

$$\frac{f(x) - f(a)}{\|f(b) - f(a)\|} = \frac{g(x) - g(a)}{\|g(b') - g(a)\|}.$$

Hence for all $x \in A$,

$$f(x) = \frac{\|f(b) - f(a)\|}{\|g(b') - g(a)\|} \cdot (g(x) - g(a)) + f(a).$$

This means that the value of f at each point of A is gained by some translations and dilations of the value of g in that point. So $[f] = [g]$.

The map φ is onto. For each element

$$(c_x)_{x\in A} \in \{0\} \times S^2 \times D^3 \times \cdots \times D^3 \subseteq \prod \mathbb{R}^3$$

define the map f by $f(x) = c_x$. So we have $[f] \in C^A$ and $\varphi([f]) = (c_x)_{x\in A}$.

Moreover, φ is smooth and have a smooth inverse, clearly. □

The space $S^2 \times D^3 \times \cdots \times D^3$ is a closed set and it is bounded, hence it is compact. On the other hand the homology groups of this space are not equal to homology groups of the sphere $3(\#A - 2) + 2 = 3\#A - 4$, so they are not diffeomorphic.

4 Compactness of $\mathcal{H}(G)$

We can see that $\mathcal{H}(G)$ is compact, as a closed subset of the compact manifold $\prod_{A\in R} C^A$, for it is defined as an intersection of closed sets. For all $A, B \in R$ such that $A \subseteq B$, let us see why $\{(x_A, x_B) \in C^A \times C^B \mid$ the restriction of x_B to A is either the constant map or the map $x_A\}$ is closed, in the following way: by identifying each of C^A, C^B as a compact spaces in section 3 a convenient way (fixing one vertex in A to the origin) in the respective linear space $\mathcal{L}$, $\mathcal{L}'$ with the canonical linear projection π from $\mathcal{L}'$ onto $\mathcal{L}$, then the above set is the projection by $(x_A, x_B, \lambda) \mapsto (x_A, x_B)$ of the closed thus compact set of $(x_A, x_B, \lambda) \in C^A \times C^B \times [0, 1]$ such that $\pi(x_B) = \lambda x_A$.

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Integrating of Ordinary Differential Equations via a One-Parameter Symmetry Group

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Abstract

The main goal of this article is concerned with finding first integrals for first-order and higher-order ODEs and using one-parameter symmetries to reduce the order of a given ODE. All techniques, which are expressed in this article, are illustrated by examples.

Keywords: Lie groups, Differential equations, Symmetry.
Mathematics Subject Classification [2010]: 58J70, 35A30.

1 Introduction

There are many techniques for integrating of differential equations, but most works are only for very limited class of problems. Surprisingly, most well-known techniques have a common feature: they exploit symmetries of differential equations. The most basic type of symmetry is a group of point transformations acting on the space of independent and dependent variables. Lie's fundamental observation was that knowledge of a sufficiently large group of symmetries of a system of ODEs allow one to integrate the system by quadratures and thereby deduce the general solution. This approach unifies and significantly extends the various special methods introduced for the integration of ODEs. In this article, a survey of these methods is presented.

We begin the first section by a brief definition of a system of differential equations with the total space of dependent and independent variables, then the prolong formulation for finding the Lie algebra of symmetries including the invariance condition theoremare given. These two ones together construct a computational method for finding the Lie algebra of the symmetry group which is a vector space spanned by some vector fields called *infinitesimal generators* correspond to the transformations in the Lie group of symmetries. Finally in the main section we apply the symmetries to integrate a given ODE.

2 Mathematical Formulation

This section starts with a geometrical definition of a system of differential equations.

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Definition 2.1. An $n-$th order system of $\ell-$differential equations in p independent and q dependent variables with total space E is given as a system of equations

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \cdots, \ell, \tag{1}$$

involving $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and the derivatives of u with respect to x up to order n. The functions $\Delta(x, u^{(n)}) = \big(\Delta_1(x, u^{(n)}), \ldots, \Delta_\ell(x, u^{(n)})\big)$ will be assumed to be smooth in their arguments, so Δ can be viewed as a smooth map from the $n-$th jet space $\mathbf{J}^n(E)$ to some $\ell-$dimensional Euclidean space $\mathbb{R}^\ell$, such as $\Delta : \mathbf{J}^n(E) \to \mathbb{R}^\ell$.

Let us $\mathbf{v} = \sum_{i=1}^p \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \varphi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$ be a vector field on E, a computational theorem gives an important formula to obtain the prolongation of a vector filed;

Definition 2.2. Suppose $\mathbf{v}$ is a vector field on the Euclidean space E. The n-th prolongation of $\mathbf{v}$, denoted $\mathrm{pr}^{(n)}\mathbf{v}$, will be a vector field on the n-th jet space $\mathbf{J}^n(E)$.

Theorem 2.3. *The vector field* $\mathbf{v}$ *has the* $n-$*th prolongation*

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J} \varphi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{2}$$

where $\varphi_J^\alpha = D_J\left(\varphi_\alpha - \sum_{i=1}^p \xi^i \frac{\partial u^\alpha}{\partial x_i}\right) + \sum_{i=1}^p \xi^i u^\alpha_{J,i}$, $J = (j_1, ..., j_k)$ *is a multi-indices* $D_J = D_{j_1} D_{j_2} \cdots D_{j_k}$ *is the* $J-$*th total derivative.*

Proposition 2.4. *(Infinitesimal condition) Suppose (1) is a system of differential equations of maximal rank defined over* E*. If* G *is a local group of transformations acting on* E*, and*

$$\mathrm{pr}^{(n)}\mathbf{v}\left[\Delta_\nu(x, u(n))\right] = 0, \quad \nu = 1, \cdots, \ell, \quad \textit{whenever} \quad \Delta(x, u^{(n)}) = 0, \tag{3}$$

for every infinitesimal generator $\mathbf{v}$ *of* G*, then* G *is a symmetry group of the system.*

For example an straightforward calculation [2], shows that the equation for the conduction of heat in a one-dimensional rod $u_t = u_{xx}$ has a six-dimensional symmetry group with a Lie algebra spanned by the generators: $\mathbf{v}_1 = \frac{\partial}{\partial x}, \mathbf{v}_2 = \frac{\partial}{\partial t}, \mathbf{v}_3 = u\frac{\partial}{\partial u}, \mathbf{v}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \mathbf{v}_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \mathbf{v}_6 = 4t\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}$.

3 First Order Equations

We begin by considering a single first order ordinary differential equation

$$\frac{du}{dx} = F(x, u) \tag{4}$$

It will be shown that if this equation is invariant under a one-parameter group of transformations, then it can be integrated by quadrature. If G is a one-parameter group of transformations on an open subset $E \simeq \mathbb{R}^2$, let $\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u}$ be its infinitesimal generator. The first prolongation of $\mathbf{v}$ is tbe vector field $\mathrm{pr}^{(1)}\mathbf{v} = \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial u} + \varphi^x\frac{\partial}{\partial u_x}$,

where $\varphi^x = D_x\varphi - u_x D_x\xi = \varphi_x + (\varphi_u - \xi_x)u_x - \xi_u u_x^2$. Thus the infinitesimal condition that G be a symmetry group of (4) is

$$\frac{\partial\varphi}{\partial x} + \left(\frac{\partial\varphi}{\partial u} - \frac{\partial\xi}{\partial x}\right)F - \frac{\partial\xi}{\partial u}F^2 = \xi\frac{\partial F}{\partial x} + \varphi\frac{\partial F}{\partial u}, \tag{5}$$

and any solution $\xi(x,u)$, $\varphi(x,u)$ of the partial differential equation (5) generates a one-parameter symmetry group of our ODE.

Once we have found a symmetry group G, there are several different methods we can employ to integrate (4). Suppose $\mathbf{v}$ is the infinitesimal generator of the symmetry group, and assume that $\mathbf{v}|_{(x_0,u_0)} \neq 0$, we introduce new coordinates

$$y = \eta(x,u), \quad w = \zeta(x,u), \tag{6}$$

near (x_0,u_0) so in this new coordinate we can rectify $\mathbf{v}$ to $\mathbf{v} = \frac{\partial}{\partial w}$. Thus in the new coordinate system, in order to be invariant, the differential equation must be independent of w, so (4) is equivalent to elementary equation

$$\frac{dw}{dy} = H(y),$$

for some function H This equation is trivially integrated by quadrature, with $w = \int H(y)dy+ c$ for some constant c. Re-substituting the expression (6) for w and y, we obtain a solution $u = f(x)$ of our original system in implicit form.

We change variable (6) by using the methods for finding group invariants. So $\mathbf{v}$ is transformed into the form $\frac{\partial}{\partial w}$ provided η and ξ satisfy the linear partial differential equations

$$\mathbf{v}(\eta) = \xi\frac{\partial\eta}{\partial x} + \varphi\frac{\partial\eta}{\partial u} = 0, \tag{7a}$$

$$\mathbf{v}(\xi) = \xi\frac{\partial\zeta}{\partial x} + \varphi\frac{\partial\zeta}{\partial u} = 1. \tag{7b}$$

The first of these equations just says that $\eta(x,u)$ is an invariant of the group generated by $\mathbf{v}$. We can thus find η by solving the associated characteristic ordinary differential equation

$$\frac{dx}{\xi(x,u)} = \frac{du}{\varphi(x,u)}. \tag{8}$$

Often the corresponding solution ξ of (7b) can be found by inspection.

The equation in the new coordinates will be invariant if and only if it has the form

$$\frac{\varphi(x,u)}{\xi(x,u)} = F(x,u), \tag{9}$$

then we have a solution of the determining equation (3), so such a vector field $\mathbf{v} = \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial u}$ is always a symmetry of the equation.

Example 3.1. Consider the first order ODE $u_x = \frac{x^4+2u^4}{x^2u^2}$. This is a homogeneous ODE which is invariant under the one-parameter group of scaling transformation $(x,u) \mapsto (\lambda x, \lambda u)$, $\lambda > 0$ with infinitesimal generator $\mathbf{v} = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$. The new coordinates are $y = u/x$ and $w = \ln x$. In this coordinates the equation reduces to $w_y = y^{-2}+2y^2$, thus, the solution is $w = -y^{-1} + \frac{2}{3}y^3$, or, in terms of the original variables $\exp\left\{-\frac{x}{u} + \frac{2}{3}\left(\frac{u}{x}\right)^3\right\} = cx$ for a constant c.

4 Higher Order Equations

Symmetry groups can be used to aid in the solution of higher order ODEs. The integration method based on the invariants of the group extends straightforwardly. Let

$$\Delta(x, u^{(n)}) = \Delta(x, u, u_x, \dots, u_n) = 0, \tag{10}$$

where $u_n \equiv d^n u/dx^n$, be a single n-th order differential equation involving the single dependent variable u. The basic result in this case is that if we know a one-parameter symmetry group of this equation, then we can reduce the order of the equation by one.

Example 4.1. Consider a homogeneous second order linear equation

$$u_{xx} + p(x)u_x + q(x)u = 0. \tag{11}$$

This is clearly invariant under the group of scale transformations

$$(x, u) \mapsto (x, \lambda u),$$

with infinitesimal generator $\mathbf{v} = u\frac{\partial}{\partial u}$. Coordinates (y, w) which straighten out $\mathbf{v}$ are given by $y = x$, $w = \ln u$ (provided $u \neq 0$), with $\mathbf{v} = \frac{\partial}{\partial w}$ in these coordinates. we have

$$u = e^w, \quad u_x = w_x e^w, \quad u_{xx} = (w_{xx} + w_x^2)e^w,$$

so the equation becomes

$$w_{xx} + w_x^2 + p(x)w_x + q(x) = 0,$$

which is independent of w. We have thus reconstructed the well-known transformation between a linear second order equation and a first order Riccati equation; namely $z = w_x = u_x/u$ changes (11) into the Riccati equation

$$z_x = -z^2 - p(x)z - q(x).$$

Acknowledgment

Symmetries of differential equations is a very important to geometric analysis of differential equations. There is a lots of papers and useful books about this subject. In this article we introduce a simple method for reduction of ODE by a one-parameter Lie algebra of associated symmetry Lie group. Also it is applicable for a PDE system [1].

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On the Whisker Topology on Fundamental Group

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Abstract

In this talk, after reviewing concepts of compact-open topology, Whisker topology and Lasso toplogy on fundamental groups, we present some topological properties for the Whisker topology on a fundamental group.

Keywords: Whisker Topology, Fundamental Group, Topological Group
Mathematics Subject Classification [2010]: 57M05, 54D10, 54D15, 54H11

1 Introduction

The concept of a natural topology on the fundamental group appears to have originated with Hurewicz [8] in 1935. The topology inherited from the loop space by quotient map, where equipped with compact-open topology, on fundamental group is denoted by $\pi_1^{qtop}(X, x_0)$. Spanier [10, Theorem 13 on page 82] introduced a different topology that Dydak et al. [4] called it the Whisker topology and denoted by $\pi_1^{wh}(X, x_0)$. They also introduced a new topology on $\pi_1(X, x_0)$ and called it the Lasso topology to characterize the unique path lifting property which is denoted by $\pi_1^{l}(X, x_0)$ and showed that this topology makes the fundamental group a topological group [3]. However Biss [2] claimed that $\pi_1^{qtop}(X, x_0)$ is a topological group, but it is shown that the multiplication map is not continuous, in general, hence $\pi_1^{qtop}(X, x_0)$ is a quasitopological group (see [6]). In this talk, we show that $\pi_1^{wh}(X, x_0)$ is not a topological group, in general. In addition, it is not even a semitopological group, but it has some properties similar to toplogical groups. For instance, every open subgroup of $\pi_1^{wh}(X, x_0)$ is also a closed subgroup of $\pi_1^{wh}(X, x_0)$ and $\pi_1^{wh}(X, x_0)$ is T_0 if and only if it is T_2. Moreover, $\pi_1^{wh}(X, x_0)$ is a homogeous and regular space, and it is totally seperated if and only if is T_0.

2 Notation and Preliminaries

Definition 2.1. Let H be a subgroup of $\pi_1(X, x_0)$ and $P(X, x_0) = \{\alpha : (I, 0) \to (X, x_0) | \, \alpha$ *is a path*$\}$ be a path space. Then $\alpha_1 \sim \alpha_2$ *mod* H if $\alpha_1(1) = \alpha_2(1)$ and $\left[\alpha_1 * {\alpha_2}^{-1}\right] \in H$. It is easy to check that this is an equivalence relation on $P(X, x_0)$. The equivalence class of α is denoted by $\langle\alpha\rangle_H$. Now one can define the quotient space $\tilde{X}_H = \frac{P(X,x_0)}{\sim}$ and the

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map $p_H : (\tilde{X}_H, e_H) \to (X, x_0)$ by $p_H(\langle\alpha\rangle_H) = \alpha(1)$ where e_H is the class of constant path at x_0.

For $\alpha \in P(X, x_0)$ and an open neighborhood U of $\alpha(1)$, a continuation of α in U is a path $\beta \in P(X, x_0)$ of the form $\beta = \alpha * \gamma$, where $\gamma(0) = \alpha(1)$ and $\gamma(I) \subseteq U$. Thus we can define a set $\langle U, \langle\alpha\rangle_H\rangle = \{\langle\beta\rangle_H \in X_H |\ \beta \textit{ is a continuation of } \alpha \textit{ in } U\}$ where U is an open neighborhood of $\alpha(1)$ in X. It is shown that the subsets $\langle U, \langle\alpha\rangle_H\rangle$ as defined above form a basis for a topology on $\tilde{X}_H$ for which the function $p_H : \tilde{X}_H \to X$ is continuous [9, Theorem 10.31]. Moreover, if X is path connected, then p_H is surjective. This topology on $\tilde{X}_H$ is called the Whisker topology [4].

Definition 2.2. Let $p_e : \tilde{X}_e \to X$ be the defined end point projection map for $\{e\} \leq \pi_1(X, x_0)$ and put $p_e^{-1}(x_0)$ as a subspace of $(\tilde{X}_e, \tilde{x}_0)$ with its default Whisker topology. One can transfer this topology by the bijection $f : \pi_1(X, x_0) \to p_e^{-1}(x_0)$ into $\pi_1(X, x_0)$ with $[\alpha] \mapsto \langle\alpha\rangle_H$. The fundamental group with Whisker topology is denoted by $\pi_1^{wh}(X, x_0)$. Fishcer and Zastrow [7, Lemma 2.1.] have shown that the Whisker topology is finer than the inherited topology from loop space with compact-open topology on $\pi_1(X, x_0)$ which is denoted by $\pi_1^{qtop}(X, x_0)$.

3 Main results

In this section we are going to present some interesting properties of $\pi_1^{wh}(X, x_0)$. At first, it seems necessary to characterize the open subsets and subgroups of $\pi_1^{wh}(X, x_0)$. Let $[\alpha] \in \pi_1(X, x_0)$, for every open subset U of x_0 there is a bijection $\varphi_\alpha : i_*\pi_1(U, x_0) \to (U, [\alpha]) \bigcap p_e^{-1}(x_0)$ defined by $\varphi_\alpha([\gamma]) = [\alpha * \gamma]$. It is easy to check that φ_α is a well defined bijection.

The collection $\{[\alpha]i_*\pi_1(U, x_0) \mid [\alpha] \in \pi_1(X, x_0) \textit{ and } U \textit{ open subset of } x_0\}$ form a basis for the Whisker topology on $\pi_1(X, x_0)$. Moreover, these basis elements are closed and hence they are clopen subsets.

The left (right) topological group is a group equipped with a topology that makes all of the left (right) translations continuous. A semitopological group is a left topological group which is also a right topological group [1, Section 1.2.]. $\pi_1^{wh}(X, x_0)$ is not a right topological group in general, hence it is not a semitopological group. For example see the Hawaiian earring is not a topological group since the inverse map in $\pi_1^{wh}(HE, *)$ is not continous [4]. Recall that a non-empty topological space X is called a $G-$ space, for a group G, if it is equipped with an action of G on X . A homogeneous space is a $G-$ space on X which G acts transitively.

Proposition 3.1. $\pi_1^{wh}(X, x_0)$ *is a homogenous space.*

Proof. Clearly $\pi_1^{wh}(X, x_0)$ acts on itself. To show that this action is transitive, it is enough to prove that left translation map in $\pi_1^{wh}(X, x_0)$ is homeomorphism. It is known that every left topological group is a homogenous space. Hence $\pi_1^{wh}(X, x_0)$ is a homogenous space.

Corollary 3.2. *Every open subgroup of* $\pi_1^{wh}(X, x_0)$ *is a closed subgroup.*

Recall that a topological space is called totally separated if for every pair of disjoint points there exists a clopen subset which contains one of points and does not contain another. The following proposition state some separation axioms for $\pi_1^{wh}(X, x_0)$.

Proposition 3.3. *For a connected and locally path connected space X, the following statement are equivalent:*

1. $\pi_1^{wh}(X, x_0)$ *is* T_0.
2. $\pi_1^{wh}(X, x_0)$ *is* T_1.
3. $\pi_1^{wh}(X, x_0)$ *is* T_2.
4. $\pi_1^{wh}(X, x_0)$ *is* T_3 $(T_3 = regular + T_1)$.
5. $\pi_1^{s}(X, x_0) = 1$, *where* $\pi_1^{s}(X, x_0)$ *is the collection of small loops at* x_0.
6. $\pi_1^{wh}(X, x_0)$ *is totally separated.*
   Moreover, $\pi_1^{wh}(X, x_0)$ *is regular.*

Corollary 3.4. *If the right translation in* $\pi_1^{wh}(X, x_0)$ *are continuous, then* $\pi_1^{wh}(X, x_0)$ *is a topological group.*

It seems interesting to know that when $\pi_1^{wh}(X, x_0)$ *has the countable axiom properties.*

Proposition 3.5. *If X is a first countable space, then* $\pi_1^{wh}(X, x_0)$ *is also first countable.*

Proof. Let β_{x_0} be a countable neighborhood basis at x_0 and let $[f] \in \pi_1^{wh}(X, x_0)$. Then the colection $\beta_f = \{[f]i_*\pi_1(V, x_0) \mid V \in \beta_{x_0}\}$ form a countable neighborhood basis at $[f]$.

Proposition 3.6. *The closure of trivial element in* $\pi_1^{wh}(X, x_0)$ *is equals to* $\pi_1^{s}(X, x_0)$.

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On (Semi) Topological BCC-algebras

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Abstract

In this paper, we introduce the notion of (semi) topological BCC-algebras and derive here conditions that imply a BCC-algebra to be a (semi) topological BCC-algebra. We prove that for each cardinal number α there is at least a (semi) topological BCC-algebra of order α. Also we study separation axioms on (semi) topological BCC-algebras and show that for any infinite cardinal number α there is a Hausdorff (semi) topological BCC-algebra of order α with nontrivial topology.

Keywords: BCC-algebra, (semi)topological BCC-algebra, ideal, preideal, Hausdorff space, Uryshon space
Mathematics Subject Classification [2010]: 06B10 , 03G10

1 Introduction

In 1966, Y. Imai and K. Iséki in [6] introduced a class of algebras of type $(2, 0)$ called BCK-algebras which generalizes on one hand the notion of algebra of sets whit the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of impliction algebra. K. Iséki posed an interesting problem whether the class of BCK-algebras form a variety. In connection with this problem Y. Komori in [7] introduced a notion of BCC-algebras which is a generalization of notion BCK-algebras and proved that class of all BCC-algebras is not a variety. W. A. Dudek in [5] redefined the notion of BCC-algebras by using a dual form of the ordinary definition. Further study of BCC-algebras was continued [5]. In recent years some mathematicians have endowed algebraic structures associated with logical systems with a topology and have studied some their propertises. For example, Borzooei et.al in [2] introduced (semi) topological BL-algebras and in [3] and [4] studied metrizability and separation axioms on them. In [8] Kouhestani and Borzooei introduced (semi) topological residuated lattices and studied separation axioms T_0, T_1, and T_2 on them. In this paper, in section 3 we will define (left, right, semi) topological BCC-algebras and show that for each cardinal number α there is at least a topological BCC-algebra of order α. In section 4, we study some topological results on BCC-algebras endowed with a topology. In section 5, we will study connection between (semi) topological BCC-algebras and T_i spaces, when $i = 0, 1, 2$. We prove that for any infinte cardinal number α there is Hausdorff topological BCC-algebra of order α which its topology is non trivial.

*Speaker

Definition 1.1. A BCC-algebra is a non empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms, for all $x, y, z \in X$:
(1) $((x * y) * (z * y)) * (x * z) = 0$,
(2) $0 * x = 0$,
(3) $x * 0 = x$,
(4)$x*y = 0$ and $y*x = 0$ imply $x = y$. On any BCC-algebra X one define $x \leq y \Leftrightarrow x*y = 0$.

Definition 1.2. Let $(X, *, 0)$ be a BCC-algebra and $I \subseteq X$. I is called:
(i) *ideal* if $0 \in I$, and for each $x, y \in X$, $x * y \in I$ and $y \in I$ imply $x \in I$,
(ii) *BCC-ideal* if $0 \in I$, and $y \in I$, $(x * y) * z \in I$, imply $x * z \in I$.

Definition 1.3. Let $\mathcal{T}$ be a topology on a BCC-algebra $(X, *, 0)$. Then:
(i) $(X, *, \mathcal{T})$ is *(right) left topological* BCC-algebra if $x * y \in U \in \mathcal{T}$, then there is a (V) $W \in \mathcal{T}$ such that $(x \in V)$ $y \in W$ and $(V * y \subseteq U)$ $x * W \subseteq U$. In this case, we also say that $*$ is *continous in (first)second variable* ,
(ii) $(X, *, \mathcal{T})$ is *semi topological* BCC-algebra if it is left and right topological BCC-algebra, i.e. if $x * y \in U \in \mathcal{T}$, then there are $V, W \in \mathcal{T}$ such that $x \in V, y \in W$ and $x * W \subseteq U$ and $V * y \subseteq U$. In this case we also say that $*$ is continuous in each variable separately,
(iii) $(X, *, \mathcal{T})$ is *topological* BCC-algebra if $*$ is continous , i.e. if $x * y \subseteq U \in \mathcal{T}$, then there are two neighborhoods V, W of x, y, respectively, such that $V * W \subseteq U$.

Definition 1.4. Let $(X, *, 0)$ be a BCC-algebra. Then:
(i) a family Ω of subsets X is *prefilter* if for each $U, V \in \Omega$, there exists a $W \in \Omega$ such that $W \subseteq U \cap V$,
(ii) for each $V \subseteq X$ and $x \in X$, we denote $V[x] = \{y \in X : y * x \in V\}$ $V(x) = \{y \in X : y * x, x * y \in V\}$.
(iii) a non empty subset V on X is *preideal* if for each $x, y \in X$, $x \leq y$, $y \in V$ imply $x \in V$.

Theorem 1.5. *Let $\mathcal{I}$ be a prefilter of BCC-ideals in a BCC-algebra $(X, *, 0)$. Then there is a topology $\mathcal{T}$ on X such that $(X, *, \mathcal{T})$ is a topological BCC-algebra.*

Theorem 1.6. *Let I be an ideal in BCC-algebra $(X, *, 0)$. Then there is a topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is right topological BCC-algebra. Moreover, if for each $x, y, z \in X$, $(x * y) * z = (x * z) * y$, then $(X, *, 0, \mathcal{T})$ is a topological BCC-algebra.*

Theorem 1.7. *Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra and $a \notin X$. Suppose $X_a = X \cup \{a\}$ and $\mathcal{T}^* = \mathcal{T} \setminus \{\phi\}$. If $0 \in \cap\mathcal{T}^*$, then there are an operation $\otimes$ and a topology $\mathcal{T}_a$ on X_a such that $(X_a, \otimes, \mathcal{T}_a)$ is a topological BCC-algebra and $0 \in \cap\mathcal{T}_a^*$.*

Theorem 1.8. *For any ineger $n \geq 4$ there exists a topological BCC-algebra of order n.*

Theorem 1.9. *Let α be an infinite cardinal number. Then there is a topological BCC-algebra of order α.*

Theorem 1.10. *Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra and α be a cardinal number. If $\alpha \geq |X|$, then there is a topological BCC-algebra $(Y, \circ, 0, \mathcal{U})$ such that $\alpha \leq |Y|$ and X is a subalgebra of Y.*

Theorem 1.11. *Let α be an infinite cardinal number. Then there is a right topological BCC-algebra of order α which is not a topological BCC-algebra.*

Theorem 1.12. *Let Ω be a family of preideals in BCC-algebra $(X, *, 0)$ such that is closed under intersection. If for each $x \in V \in \Omega$, there is a $U \in \Omega$ such that $U[x] \subseteq V$, then there is a topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a right topological BCC-algebra.*

Theorem 1.13. *Let Ω be a family of preideals in BCC-algebra $(X, *, 0)$ such that is closed under intersection. Let for each $x \in V \in \Omega$, there is a $U \in \Omega$ such that $U(x) \subseteq V$. If for each $x, y, z \in X$, $(x * y) * z = (x * z) * y$, then there is a topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a semi topological BCC-algebra.*

Proposition 1.14. *Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra. If $0 \in \cap\mathcal{T}$, then $B \subseteq X$ is open iff, 0 is an interior point of B and $W * x \subseteq U$.*

Proposition 1.15. *Let $(X, *, 0, \mathcal{T})$ be a left topological BCC-algebra and I be an ideal in X. Then I is closed if 0 is an interior point of I, or $0 \in \{x_j : j \in J\}$, for each net $\{x_j : j \in J\}$ which converges to 0.*

Proposition 1.16. *Let $(X, *, 0, \mathcal{T})$ be a semi topological BCC-algebra and I be an ideal in X. Then I is open and closed if 0 is an interior point of I.*

Proposition 1.17. *Let $(X, *, 0, \mathcal{T})$ be a right topological BCC-algebra. If all of elements of X are atoms, then $0 \in \overline{B}$, or B is closed, for each $B \subseteq X$ which $0 \notin B$.*

Proposition 1.18. *Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra and I be a BCC-ideal in X. Then $\overline{I}$ is a BCC-ideal.*

2 Main results

Theorem 2.1. *Let $\mathcal{T}$ be a topology on BCC-algebra $(X, *, 0)$. If for any $a \in X$ the map $l_a : X \hookrightarrow X$, by $l_a(x) = a * x$, is an open map, then $(X, \mathcal{T})$ is a T_0 space.*

Theorem 2.2. *Let $(X, *, 0, \mathcal{T})$ be a right (left) topological BCC-algebra. Then $(X, \mathcal{T})$ is a T_0 space iff, for any $x \neq 0$, there is a $U \in \mathcal{T}$ such that $x \in U$ and $0 \notin U$.*

Theorem 2.3. *Let X be a BCC-algebra such that for any $a \in X \setminus \{0\}$, there is a $b \in X$ such that $0 < b < a$. Then there exists a non trivial topology $\mathcal{T}$ on X such that $(X, *, 0, \mathcal{T})$ is a T_0 right topological BCC-algebra.*

Theorem 2.4. *If α is an infinite cardinal number, then there is a T_0 right topological BCC-algebra of order α.*

Theorem 2.5. *If α is an infinite cardinal number, then there is a T_0 topological BCC-algebra of order α which its topology is nontrivial.*

Theorem 2.6. *Let $(X, *, 0, \mathcal{T})$ be a semi topological BCC-algebra. Then $(X, \mathcal{T})$ is a T_1 space if and only if for any $x \neq 0$, there are two open neighborhoods U and V of x and 0, respectively, such that $0 \notin U$ and $x \notin V$.*

Theorem 2.7. *Let $(X, *, 0, \mathcal{T})$ be a semi topological BCC-algebra. Then $(X, \mathcal{T})$ is a T_1 space if and only if it is T_0 space.*

Theorem 2.8. *Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra. Then $(X, \mathcal{T})$ is Hausdorff if and only if for each $x \neq 0$, there are two disjoint open neighborhoods U and V of x and 0, respectively.*

Theorem 2.9. *Let $(X, *, 0, \mathcal{T})$ be a topological BCC-algebra. Then $(X, \mathcal{T})$ is a T_1 space if and only if it is Hausdorff space if and only if $\{0\}$ is closed.*

Corollary 2.10. *If α is an infinite cardinal number, then there is a $T_1(T_2)$ topological BCC-algebra of order α which its topology is nontrivial.*

Theorem 2.11. *Let $\mathcal{N}$ be a fundamental system of neighborhoods of 0 in topological BCC-algebra $(X, *, 0, \mathcal{T})$. The following conditions are equivalent.*
(i) $(X, \mathcal{T})$ is T_0 space,
(ii) $(X, \mathcal{T})$ is T_1 space,
(iii) $(X, \mathcal{T})$ is Hausdorff space,
(iv) $\cap\mathcal{N} = \{0\}$.

Theorem 2.12. *Topological BCC-algebra $(X, *, 0, \mathcal{T})$ is Uryshon space if and only if for any $x \neq 0$, there are two open sets U and V containing x and 0, respectively, such that $\overline{U} \cap \overline{V} = \phi$.*

Theorem 2.13. *Topological BCC-algebra $(X, *, 0, \mathcal{T})$ is Uryshon space if and only if it is Hausdorff.*

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On Benz-Planes admitting automorphisms with exactly two fix points

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Abstract

The axiom of Miquel with 8 points plays the same role in Benz-planes as Pappus axiom in projective planes. In this paper we show a Benz plane (i.e. a Möbius, Laguerre or Minkowski plane) that admits automorphisms with two arbitrary fix points satisfies a degenerate form of miquel axiom with 6 points. The converse of this assertion is a part of authors investigations.

Mathematics Subject Classification (2000) . 14N05; 14N15;
Keywords. Benz plane, Möbius plane, Laguerre plane, Minkowskie plane, Miquel axiom, point reflection.

1 Introduction

The geometry of circles is as old as Euclidean geometry. Appollonius problem is the most famous elementary problem in this area. The classical version of Appollonius problem is:

"find all circles tangent to three given circles in the Euclidian plane".

in order to find all solutions we should consider lines as circles with a point at infinity. This idea is well known since Gaus because of his elegant model for Euclidean geometry based on complex plane. Adding an extra point ∞ to all lines in Euclidean plane we find a uniform representation of lines and circles.The objects of this nonlinear geometric structure are points and circles and it is more homogenous than Euclidean plane. This geometry is called classical "*inversive geometry*" or "*Möbius plane*". With stereographic projection this geometry can be consider as the geometry of plane sections (circles) on a sphere in a three dimensional Euclidean space. The geometry of plane sections of a conic in 3-dimensional space [1] gives general structures called Benz planes. Besides Möbius planes, Laguerre planes and Minkowski planes are other types of Benz planes. Walter Benz find in 1970 a uniform analytic definition for them [1]. In Laguerre planes and Minkowski planes we have another type of objects called "generator". We recall an axiomatic definition for all of them.

Let $\mathcal{P}$ be a nonempty set which we call it's elements **"point"**, denoted by capital letters and $\mathfrak{C}$ a nonempty subset of the power set of $\mathcal{P}$ which we call it's elements **" cycle"**, denoted by small letters.

*Speaker

In a Möbius planes each triple of distinct points determine a unique cycle but this is not the case for Laguerre or Minkowski planes. On the point set of a Laguerre plane we define an equivalence relation called "parallel" relation. Two points P, Q are called parallel, denoted by $P||Q$, if $P = Q$ or there is no cycle containing P, Q. On the point set of a Minkowski plane we define two equivalence relation of parallel points.

The common axioms for Möbius, Laguerre and Minkowski planes are the following three axioms:

(B_1) For any three pairwise not parallel points there is exactly one cycle which contains them.

If $A_1, \cdots, A_i$ are points of a cycle we write $(A_1, \cdots, A_i) \in \mathfrak{C}$. The unique cycle contains three pairwise not parallel points A, B, C will be showed with $(A, B, C)^\circ$.

(B_2) For any cycle z and any points $P \in z$, $Q \notin z$ and P nonparallel with Q there exists exactly one cycle z' with $P \in z'$ and $z \cap z' = \{P\}$ (we say z and z' are touching in P).

(B_3) Each cycle contains at least three points and there are at least four points not on a cycle.

More precisely we describe the uniform definition of Benz-planes with the notion of net. As before Let $\mathcal{P}$ be a nonempty set which we call its elements "points" and $\mathfrak{G}$ a subset of powerset of $\mathcal{P}$ which we call its elements generators. The pair $(\mathcal{P}, \mathfrak{G})$ is called a net if there is a partition $\mathfrak{G} = \bigcup_{i \in I} \mathfrak{G}_i$ such that:

G_1. For each $P \in \mathcal{P}$ and $i \in I$ there is exactly one $G_i \in \mathfrak{G}_i$ with $P \in G_i$ (we set $\overline{P} = G_1 \cup G_2$).

G_2. For $i, j \in I$ with $i \neq j$, $G \in \mathfrak{G}_i$, $H \in \mathfrak{G}_j$ we have $|G \cap H| = 1$ and $|G| \geq 1$.

Let $(\mathcal{P}, G)$ be a net and $\mathfrak{C}$ be another subset of $\mathcal{P}$ with $\phi \notin \mathfrak{C}$. The elements of $\mathfrak{C}$ will be called "*cycles*". Let $\mathcal{P}^P := \mathcal{P} \setminus \overline{P}$, $\mathfrak{C}^P := \{c \setminus P \mid c \in \mathfrak{C}, P \in c\}$, $\mathfrak{G}^P := \{G \setminus \overline{P} \mid G \in \mathfrak{G}, P \notin G\}$and $\mathcal{A}(P) := (\mathcal{P}^P, \mathfrak{C}^P \cup \mathfrak{G}^P)$. The triple $(\mathcal{P}, \mathfrak{C}, \mathfrak{G})$ is called a Benz plane if for each point $P \in \mathcal{P}$, $\mathcal{A}(P)$ is an affine plane.

For a Benz plane, $\mathcal{A}(P)$ is called **"the residue at P"**. For the classical model, the residue at each point is the underlying real affine plane.

2 Miquel axiom

We call a Benz plane *Miquelian*, if the following property is satisfied[4].

$\mathbf{M_8}$. Let $A_1, A_2, A_3, A_4, A'_1, A'_2, A'_3, A'_4$ be distinct points of a Benz plane $(L, \mathfrak{C}, G)$. Then from each five assertions of six assertions $(A_1, A'_1, A_2, A'_2) \in C$, $(A_2, A'_2, A_3, A'_3) \in \mathfrak{C}$, $(A_3, A'_3, A_4, A'_4) \in \mathfrak{C}$, $(A_4, A'_4, A_1, A'_1) \in \mathfrak{C}$, $(A_1, A_2, A_3, A_4) \in \mathfrak{C}$ and $(A'_1, A'_2, A'_3, A'_4) \in \mathfrak{C}$ follows the sixth one (figure 1).
For simplicity we associate the eight points $A_1, A_2, A_3, A_4, A'_1, A'_2, A'_3, A'_4$ of $\mathbf{M_8}$ the vertexes of a cube and correspond to each four points of a cycle a face of the cube (figure 1).

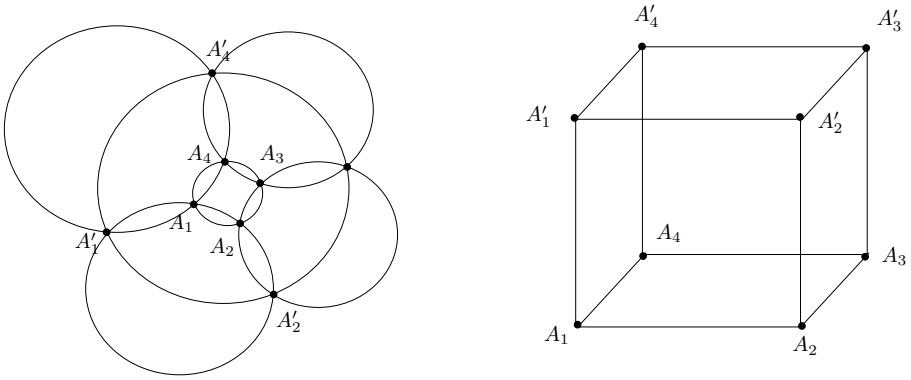

figure 1. Representaion of $\mathbf{M_8}$.

In a degenerate version of $\mathbf{M_8}$ called $\mathbf{M_6}$ we have only six distinct, pairwise non parallel points A, B, C, D, E, F (figure 2)[3].

$\mathbf{M_6}$. Let A, B, C, D, E, F be six distinct pairwise non parallel points of a Benz plane. Then from each three assertions of four assertions $(A, C, D, F) \in \mathfrak{C}$, $(A, B, C)^\circ(C)(C, E, D)^\circ$, $(A, B, F)^\circ(F)(F, D, E)^\circ$ and $(B, C, E, F) \in \mathfrak{C}$ follows the fourth one.

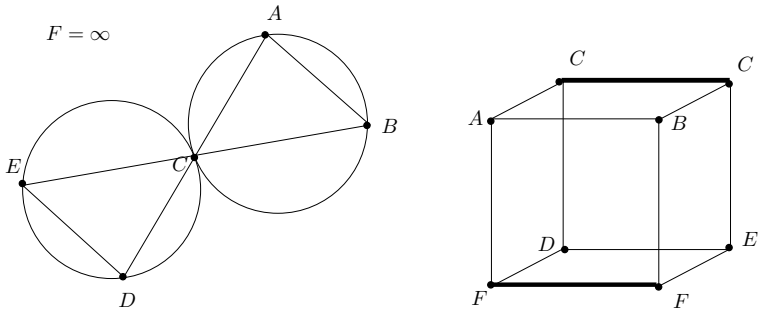

Figure 2. ($\mathbf{M_6}$)

3 Main results

Theorem 3.1. *If the automorphism group of a Benz-plane contains for each four pairwise non parallel points* $\{X, Y, Z, W\}$ *an involution* α *with* $\mathrm{Fix}(\alpha) = \{X, Y\}$ *and* $\alpha(Z) = W$ *then it satisfies* $\mathbf{M_6}$.

Proof. Let A, B, C, D, E, F be six distinct pairwise non parallel points. We show from $(A, C, D, F) \in \mathfrak{C}$, $(A, B, C)^\circ(C)(C, E, D)^\circ$ and $(B, C, E, F) \in \mathfrak{C}$ follows $(A, B, F)^\circ(F)(F, D, E)^\circ$.

By assumption there exists an automorphism α with $\mathrm{Fix}(\alpha) = \{F, C\}$ and $\alpha(A) = D$. Then α induces an involution with exactly one fix point C in the residue $\mathcal{A}(F)$. Therefore $\widehat{\alpha} := \alpha|_{\mathcal{A}(F)}$ is a point reflection in the affine plane $\mathcal{A}(F)$ with $\widehat{\alpha}(A) = D$. From $(A, B, C)^\circ(C)(C, E, D)^\circ$ and $\widehat{\alpha}(A) = D$ follows that $\alpha(A, B, C)^\circ = (E, D, C)^\circ$. Hence $\widehat{\alpha}(B) = E$ since $\widehat{\alpha}(\overline{B, E}) = \overline{B, E}$ in $\mathcal{A}(F)$. Hence $\widehat{\alpha}(\overline{A, B}) = \overline{E, D}$, i.e. $\overline{A, B} || \overline{E, D}$ in $\mathcal{A}(F)$. This means $(A, B, F)^\circ(F)(F, D, E)^\circ$. □

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On the Tangent Space of an $n-$Surface

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Abstract

We are supposed to characterize the tangent space of an $n-$surface $S = f^{-1}(c)$ for some $f : U \to R$, U open in R^{n+1}, and f is a smooth function with the property that $\nabla f(p) \neq 0$ for all $p \in S$, in the case that the whole space admits the general form of the inner product. Finally we introduce a vector field X with integral curve α through p such that the covariant derivative of f with respect to $\acute{\alpha}(0)$ at p, i.e., $\nabla_{\acute{\alpha}(0)} f$ has maximum value.

Keywords: Flow, Inner product, Positive definite, Surface, Vector field.
Mathematics Subject Classification [2010]: 37C10, 46C99, 14J29, 43A35.

1 Introduction

The idea of the definition of a regular surface is to introduce a set S, that is, in a certain sense, two dimensional and that also is smooth so that the usual notions of calculus can be extended to it [1, 9]. for example, if $x : U \subseteq R^2 \to S$ be a parameterization of a regular surface S and $q \in U$, then the vector subspace $dx_q R^2 \subseteq R^3$ of dimension 2, coincides with the set of tangent vectors of S at $x(q)$. In the case that $f : U \subseteq R^3 \to R$ is a smooth function and $a \in f(U)$ is a regular point of f, then $S = f^{-1}(a)$ is a regular surface in R^3. As a result, in this case, the tangent space of S at p consides with $\nabla f(p)^{\perp}$,i.e., the set of vectors at p which are perpendicular with respect to the usual inner product of R^3 to $\nabla f(p)$ [2, 10]. In this note we are supposed to characterized the tangent space T_pS for an $n-$surface S in the case that R^{n+1} admits a general inner product $\alpha_A(u, v) = uAv^t$, in which A is a symmetric positive definite $(n + 1) \times (n + 1)$ real matrix [3].

2 Preliminaries

Definition 2.1. [1] Let $f : U \to R$ be a smooth function, where $U \subseteq R^{n+1}$ is an open set, let $c \in R$ be such that $f^{-1}(c)$ is non-empty and let $p \in f^{-1}(c)$. A vector is said to be a tangent to the level set $f^{-1}(c)$ if it is the velocity vector of a parameterized curve in R^{n+1} whose image is contained in $f^{-1}(c)$.

Definition 2.2. A parameterized curve is a smooth function $\alpha : I \to R^{n+1}$ for some open interval I, and an $n-$surface is a non empty subset $S \subseteq R^{n+1}$ for some $n \in N$ of the form $S = f^{-1}(c)$ where $f : U \to R$, U open in R^{n+1}, is a smooth function with the property that $\nabla f(p) \neq 0$ for all $p \in S$.

*Speaker

Definition 2.3. [1, 9] A vector field χ on an $n-$surface $S \subseteq R^{n+1}$ is a function which assigns to each point $p \in S$ a vector $\chi(p) \subseteq T_pR^{n+1}$. A tangent vector field on an $n-$surface $S \subseteq R^{n+1}$ is a function which assigns to each point $p \in S$ an element of T_pS.

Definition 2.4. [2, 4] A parametrized curve $\alpha : I \to R^{n+1}$ is said to be an integral curve of the vector field χ on the open set $U \subseteq R^{n+1}$, if $\alpha(t) \in U$ and $\dot{\alpha}(t) = \chi(\alpha(t))$ for all $t \in I$. If χ is a smooth tangent vector field on an $n-$surface S, then $\alpha : I \to S$ satisfying previous conditions is called an integral curve of χ.

Definition 2.5. Let S be an $n-$surface in R^{n+1}. A function $g : S \to R^k$ is smooth if it is the restriction to S of a smooth function $\widetilde{g} : V \to R^k$ defined on some open set V consisting S. A smooth vector field is defined similarly.

Definition 2.6. Let U be an open set in R^{n+1} and $f : U \to R$ be smooth. A point $p \in U$ such that $\nabla f(p) \neq 0$ is called a regular point of f.

3 Main results

Theorem 3.1. *Let A be a positive definite symmetric matrix and $p \in f^{-1}(c)$. Then the vector $\nabla f(p)A^{-1}$, is orthogonal to all vectors tangent to $f^{-1}(c)$.*

Proof. Each vector tangent to $f^{-1}(c)$ at p is of the form $\dot{\alpha}(t_0)$ for some parameterized curve $\alpha : I \to R^{n+1}$ with $\alpha(t_0) = p$ and $Im\alpha \subseteq f^{-1}(c)$. But $Im\alpha \subseteq f^{-1}(c)$ implies that $f(\alpha(t)) = c$ for all $t \in I$. So the chain rule implies that

$$\alpha_A(\nabla f(p)A^{-1}, \dot{\alpha}(t_0)) = \nabla f(\alpha(t_0)) \cdot \dot{\alpha}(t_0) = \frac{d}{dt}(f \circ \alpha)(t)_{|t=t_0} = 0$$

□

Theorem 3.2. *Let U be an open set in R^{n+1} and $f : U \to R$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Then the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $(\nabla f(p)A^{-1})^{\perp}$.*

Proof. It suffices to show that, if $V = (p, v) \in (\nabla f(p)A^{-1})^{\perp}$, then $V = \dot{\alpha}(0)$ for some parametrized curve α with $Im\alpha \subseteq f^{-1}(c)$. Consider the constant vector field χ on U defined by $\chi(q) = (q, v)$. Let

$$Y(q) = \chi(q) - \frac{\alpha_A(\nabla f(q)A^{-1}, \chi(q))}{\alpha_A(\nabla f(q)A^{-1}, \nabla f(q))}\nabla f(q) \tag{1}$$

Y is defined on an open set U where $\nabla f(q) \neq 0$ for all $q \in U$. Obviously $p \in U$, $\chi(p) = V \in (\nabla f(p)A^{-1})^{\perp}$ and (1) implies that $Y(p) = \chi(p)$ and

$$\begin{aligned}\alpha_A(\nabla f(q)A^{-1}, Y(q)) = &\alpha_A(\nabla f(q)A^{-1}, \chi(q)) \\ &- \frac{\alpha_A(\nabla f(q)A^{-1}, \chi(q))}{\alpha_A(\nabla f(q)A^{-1}, \nabla f(q))}\alpha_A(\nabla f(q)A^{-1}, \nabla f(q)) = 0\end{aligned}$$

for all $q \in U$ and $Y(p) = V$. Thus $Y(q) \perp \nabla f(q)A^{-1}$ for all $q \in U$. Let α be an integral curve [5, 6] of Y through p, then

$$\alpha(0) = p, \acute{\alpha}(0) = Y(\alpha(0)) = Y(p) = \chi(p) = V$$

and

$$\frac{d}{dt}(f\circ\alpha)(t)=\nabla f(\alpha(t))\cdot\dot{\alpha}(t)=\alpha_A(\nabla f(\alpha(t))A^{-1},Y(\alpha((t))=0$$

for all $t\in I$, so $f(\alpha(t))=c$. Since $f(\alpha(0))=f(p)=c$, so $Im\alpha\subseteq f^{-1}(c)$. □

Corollary 3.3. *Let S be an $n-$surface in R^{n+1}, let χ be a smooth tangent vector field on S. Then there exists an open interval I containing 0 and a parametrized curve $\alpha:I\to S$ such that, (1) $\alpha(0)=p$, (2) $\dot{\alpha}(t)=X(\alpha(t))$ for all $t\in I$, (3) and if $\beta:\tilde{I}\to S$ is any other parameterized curve in S satisfying (1) and (2), then $\tilde{I}\subseteq I$ and $\beta(t)=\alpha(t)$ for all $t\in\tilde{I}$.*

Theorem 3.4. *Let $f:U\to R$ be a smooth function and let $\alpha:I\to U$ be an integral curve of $(\nabla f)A^{-1}$, then for each $t_0\in I$, the function f is increasing faster along α at $\alpha(t_0)$ than along any other curve passing through $\alpha(t_0)$ with the same speed, i.e., if $\beta:\tilde{I}\to U$ is such that $\beta(s_0)=\alpha(t_0)$ for some $s_0\in\tilde{I}$ and $\|\dot{\beta}(s_0)\|=\|\dot{\alpha}(t_0)\|$ then $\frac{d}{dt}(f\circ\beta)(s_0)\leq\frac{d}{ds}(f\circ\alpha)(t_0)$.*

Proof. There exists a real number $k\in[-1,1]$ such that

$$\begin{aligned}\frac{d}{dt}(f\circ\beta)(s_0)&=\alpha_A(\nabla f(\beta(s_0))A^{-1},\dot{\beta}(s_0))\\&=k\|\nabla f(\beta(s_0))A^{-1}\|\|\dot{\beta}(s_0)\|\\&=k\|\nabla f(\alpha(t_0))A^{-1}\|\|\dot{\alpha}(t_0)\|\\&=k\|\nabla f(\alpha(t_0))A^{-1}\|\|\nabla f(\alpha(t_0))A^{-1}\|\\&=k\alpha_A(\nabla f(\alpha(t_0))A^{-1},\nabla f(\alpha(t_0))A^{-1})\\&\leq\alpha_A(\nabla f(\alpha(t_0))A^{-1},\nabla f(\alpha(t_0))A^{-1})\\&=\alpha_A(\nabla f(\alpha(t_0))A^{-1},\dot{\alpha}(t_0))=\frac{d}{dt}(f\circ\alpha)(t_0)\end{aligned}$$

□

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Some results on fuzzy topological generalized groups
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Some results on fuzzy topological generalized groups

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Abstract

In this paper, we find a condition that the right and left translation of fuzzy topological generalized group are relatively fuzzy homeomorphism and then we investigate some properties of fuzzy topological generalized group with respet to this condition.

Keywords: Generalized group, Topological generalized group, Fuzzy topology, Fuzzy group.
Mathematics Subject Classification [2010]: 20N25, 20Nxx, 54A40

1 Introduction

The concepts of generalized group is defined by Prof Molaei in [3] and the concept of fuzzy topological group is defined in [1]. In this paper some results on fuzzy topological generalized group that is defined in [2], are investigated.
Let X be a non empty set and $I = [0, 1]$. A fuzzy set A in X is characterized by a membership function μ_A which associates with each $x \in X$ its grade of membership $\mu_A(x) \in I$.

Definition 1.1. Let A and B be fuzzy sets in X. Then:

- $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, for all $x \in X$,
- $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, for all $x \in X$,
- $C = A \cup B \Leftrightarrow \mu_C(x) = max\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$,
- $D = A \cap B \Leftrightarrow \mu_D(x) = min\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.

Remark 1.2. For a family of fuzzy sets $\{A_i, i \in I\}$, the union $C = \bigcup_{i\in I} A_i$ and the intersection $D = \bigcap_{i\in I} A_i$, are defined by

$$\mu_C(x) = sup\mu_{A_i}(x), \quad x \in X,$$

$$\mu_D(x) = inf\mu_{A_i}(x), \quad x \in X.$$

We denote by k_c the fuzzy set in X with membership function $\mu_{k_c}(x) = c$ for all $x \in X$. The fuzzy set k_1 and k_0 correspond to X and $\emptyset$, respectively.

*Speaker

Definition 1.3. [1] Let f be a mapping from a set X to a set Y. Let B be a fuzzy set in Y with the membership function μ_B. then the inverse image of B, $f^{-1}[B]$, is the fuzzy set in X with the membership function defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)),$$

for all $x \in X$.

If A is a fuzzy set in X then the image of A, $f[A]$, is a fuzzy set in Y with the membership function defined by

$$\mu_{f[A]}(y) = sup_{z \in f^{-1}(y)} \mu_A(z),$$

if $f^{-1}(y)$ is nonempty and 0 otherwise.

Definition 1.4. [1] A fuzzy topology on a set X is a family τ of fuzzy sets which satisfies the following conditions:

- For all $c \in I$, $k_c \in I$,
- If A, $B \in \tau$, then $A \cap B \in \tau$,
- If $A_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} A_i \in \tau$.

The pair (X, τ) is called a fuzzy topological space or FTS for short.

Definition 1.5. [1] Let A be a fuzzy set in X and τ a fuzzy topology on X. Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersection with A of τ- open fuzzy sets in X. The induced fuzzy topology is denoted by τ_A.

Definition 1.6. Let (A, τ_A) and $(B, \mathcal{U}_B)$ be fuzzy subspace of FTS (X, τ), $(Y, \mathcal{U})$, respectively. Then a map $f : (A, \tau_A) \to (B, \mathcal{U}_B)$ is relatively fuzzy continuous iff for each fuzzy set $V \in \mathcal{U}_B$, $f^{-1}(V) \cap A \in \tau A$. $f : (A, \tau_A) \to (B, \mathcal{U}_B)$ is ralatively fuzzy open iff for open fuzzy set $W \in \tau_A$, $f(W) \in \mathcal{U}$. A bijective map $f : (X, \tau) \to (Y, \mathcal{U})$ is a fuzzy homeomorphism iff it is a fuzzy continuous and fuzzy open. A bijective map $f : (A, \tau_A) \to (B, \mathcal{U}_B)$ is relatively fuzzy homeomorphism iff $f(A) = B$ and f is relatively fuzzy continuous and relatively fuzzy open.

Definition 1.7. [3] A generalized group is a non-empty set G admitting an operation called multiplication, subject to the set of rules given below:

- $(xy)z = x(yz)$, for all $x, y, z \in G$;
- for each x in G there exists a unique z in G such that $xz = zx = x$ (we denote z by $e(x)$);
- For each $x \in T$ there exists $y \in T$ such that $xy = yx = e(x)$ (we denote y by x^{-1}).

2 main results

Definition 2.1. Let X be a generalized group and G be a fuzzy set in X with membership function μ_G. Then G is called a fuzzy generalized group if and only if the following conditions satisfied:
i) $\mu_G(xy) \geq min\{\mu_G(x), \mu_G(y)\}$, for all $x,\ y \in X$.
$\mu_G(x^{-1}) \geq \mu_G(x)$, for all $x \in X$.

Remark 2.2. It is an immediate consequence of the above definition that $\mu_G(x) = \mu_G(x^{-1})$ and $\mu_G(e(x)) \geq \mu_G(x)$ for all $x \in X$.

Definition 2.3. Let X be a generalized group and τ a fuzzy topology on X. Let G be a fuzzy generalized group in X and let G be endowed with the induced fuzzy topology τ_G. G is called a fuzzy topological generalized group if the multiplication and inverse maps are relatively fuzzy continuous.

Lemma 2.4. *Let X be a generalized group and τ a fuzzy topology on X. Let G be a fuzzy topological generalized group in X and $e(G) = \{e_1, ..., e_n\}$ is finite set. Let $\mu_G(e_k) = max\{\mu_G(e_i) : i = 1, ..., n\}$ and $a \in \{x : \mu_G(x) = \mu_G(e_k)\}$. Then $\mu_G(e_a) = \mu_G(e_k)$.*

Proof. Clearly $\mu_G(e_a) \leq \mu_G(e_k)$. Since $\mu_G(a) = \mu_G(e_k)$, according to Remark 2.2, we have $\mu_G(e_k) \leq \mu_G(e_a)$. So $\mu_G(e_a) = \mu_G(e_k)$. □

Proposition 2.5. *Let X be a generalized group and τ a fuzzy topology on X. Let G be a fuzzy topological generalized group in X and $e(G) = e_1, ..., e_n$ is finite set. Then $f : G \to G$ be an inversion map defined by $f(x) = x^{-1}$, $r_a : G \to G$ be a right translation defined by $r_a(x) = xa$, $l_a(x) : G \to G$ be a left translation defined by $l_a(x) = ax$ and the inner automorphism $h : G \to G$ defined by $h(g) = aga^{-1}$ are all relative fuzzy homeomorphism, where $a \in \{x : \mu_G(x) = max\{\mu_G(e_i)\} i = 1, .., n\}$.*

Proof. Let $f : G \to G$ be inversion map. Clearly f is one-to-one. Since

$$\mu_{f(G)}(y) = \sup_{z \in f^{-1}(y)} \mu_G(z) = \mu_G(y)$$

for all $y \in G$, $f(G) = G$. Since $f^{-1}(x) = x^{-1}$ is relatively fuzzy continuous, f is relatively fuzzy open. Thus f is a relative fuzzy homeomorphism. Set $\mu_G(e_k) = max\{\mu_G(e_i) : i = 1, ..., n\}$. Let $r_a : G \to G$ be a right translation. Then

$$\mu_{r_a(G)}(x) = \sup_{z \in f^{-1}(y)} \mu_G(z) = \mu_G(xa^{-1})$$

$$\geq min(\mu_G(x), \mu_G(a^{-1})) = min(\mu_G(x), \mu_G(e_k))$$

$$= \mu_G(x) = \mu_G(xa^{-1}a) \geq min(\mu_G(xa^{-1}), \mu_G(a))$$

$$= \mu_G(xa^{-1}) = \mu_{r_a(G)}(x).$$

Thus $r_a(G) = G$.

Let $\Phi : G \to G \times G$ be map defined by $\Phi(x) = (x, a)$ and $\Psi : G \times G \to G$ be a map defined by $\Psi(x, y) = xy$. Then $r_a = \Psi \circ \Phi$. Since Φ and Ψ are relatively fuzzy continuous, r_a is a relatively fuzzy continuous. Since $r_a^{-1} = r_{a^{-1}}$, r_a is a relative fuzzy homeomorphism. Similary l_a is a relative fuzzy homeomorphism. Since $h = r_{a^{-1}} \circ l_a$, h is a relative fuzzy homeomorphism. □

Corollary 2.6. *Let F be a fuzzy closed subset, U an fuzzy open subset, and A any fuzzy subset of a fuzzy topological generalized group G. Suppose $a \in \{x : \mu_G(x) = max\{\mu_G(e_i)\} : i = 1, .., n\}$. Then aU, Ua, U^{-1}, AU, UA are relatively open and aF, Fa, F^{-1} are relatively closed.*

Let $f : G \to G$ be a map defined by $f(x) = ax$. According to Proposition 2.5, f is a a relative homomorphism, so $f(U) = aU$ is relatively open. Similary we may prove th remaining parts of the corollary.

Proposition 2.7. *Let G be a fuzzy topological generalized group in a group X and $e(G) = \{e_1, ..., e_n\}$ be a finite set of identity elements. If $a \in \{x : \mu_G(x) = max\{\mu_G(e_i)\} : i = 1, .., n\}$ and W is a neighborhood of e_a such that $\mu_W(e_a) = 1$, then aW is a neighborhood of a such that $\mu_{aW}(a) = 1$.*

Proof. Since W is a neighborhood of e_a such that $\mu_W(e_a) = 1$, there exist a fuzzy open set U such that $U \subseteq W$ and $\mu_U(e_a) = \mu_W(e_a) = 1$. Let $l_a : G \to G$ be a left translation . By Propsition 2.5, l_a is a fuzzy homeomorphism. Thus aU is a fuzzy open set. So

$$\mu_{aU}(a) = \mu_U(a^{-1}a) = \mu_U(e_a) = 1$$

and

$$\mu_{aW}(x) = \mu_W(a^{-1}x) \geq \mu_U(a^{-1}x) = \mu_{aU}(x),$$

for all $x \in X$. So $\mu_{aW}(a) = 1$ and there exist an fuzzy open set aU such that $aU \subseteq aW$. □

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The asymptotic average shadowing property for IFS

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Abstract

In this present paper we defined the asymptotic average shadowing property for itrated function systems (IFS) and show that if itrated function system $\mathbb{F}$ has the asymptotic average shadowing property then $\mathbb{F}^k$ has the asymptotic average shadowing property for all $k \geq 0$. Also, if $\mathbb{F}$ is an IFS with the asymptotic average shadowing property (on $\mathbb{Z}_+$), then so $\mathbb{F}^{-1}$ has the asymptotic average shadowing property.

Keywords: Shadowing property, Asymptotic average shadowing property, Asymptotic-average pseudo orbit, IFS.

1 Introduction

The notion of shadowing is an important tool for studying prperties of discrete dynamical systems. From numerical point of view , if dynamical system has the shadowing property, then numerically obtained orbits reect the real behavior of trajectories of the systems. [1 , 4]
Iterated function systems(IFS), are used for the costruction of deterministic fractals and have found numerous applications, in particular to image compression and image processing. Important notions in dynamics like attractors, minimality, and shadowing can be extended to IFS. [2, 3, 6, 7]
Let (X , d) be a complete metric space. Let us recall that a parametrized Iterated Function system (IFS) $\mathbb{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is any family of continuous mappings $f_\lambda : X \longrightarrow X, \lambda \in \Lambda$ where Λ is a finite nonempty set. [5]
Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$ and$\Lambda^{\mathbb{Z}_+}$ denote the set of all infinite sequences $\{\lambda_i\}_{i\in T}$ of symbols belonging to Λ. A typical element of $\Lambda^{\mathbb{Z}_+}$ can be denoted as $\sigma = \{\lambda_0, \lambda_1, ...\}$ and we use the shorted notation

$$\mathbb{F}_{\sigma_n} = f_{\lambda_0} o f_{\lambda_1} o ... o f_{\lambda_n}.$$

Definition 1.1. A sequence $\{x_n\}_{n\in T}$ is called an orbit of the IFS $\mathbb{F}$ if there exist $\sigma \in \Lambda^T$ Such that $x_{n+1} = f_{\lambda_n}(x_n)$, for $\lambda_n \in \sigma$.
Given $\delta > 0$, a sequence $\{x_n\}_{n\in T}$ in X is called a δ-pseudo orbit of $\mathbb{F}$ if there exist $\delta \in \Lambda^T$ such that for every $\lambda_n \in \sigma$, we have $d(x_{n+1}, f_\lambda(x_n)) < \delta$.
One says that the parameterized IFS has the shadowing property (on T), if given $\varepsilon > 0$,

*Speaker

there exists $\delta > 0$ such that for any δ-pseudo orbit $\{x_n\}_{n\in T}$ there exist an orbit $\{y_n\}_{n\in T}$, satisfying the inequality $d(x_n, y_n) \leq \varepsilon$ for all $n \in T$. In this case one says that $\{y_n\}_{n\in T}$ or point y_0 , $\varepsilon-$shadows the $\delta-$pseudo orbit $\{x_n\}_{n\in T}$.
Please note that if Λ is a set whit one member then parameterized IFS $\mathbb{F}$ is an ordinary discrete dynamical system. [5]

2 The asymptotic average shadowing property for IFS

In this section we investigate the structure of parameterized IFS whit the asymptotic average shadowing property.

Definition 2.1. A sequence $\{x_n\}_{n\in T}$ in X is called an asymptotic-average pseudo orbit of the parameterized IFS $\mathbb{F}$, if there exists the $\delta \in \Lambda^T$ that

$$\lim_{n\longrightarrow\infty} \frac{1}{n}\Sigma_{i=1}^{n-1} d(f_{\lambda_i}(x_i), x_{i+1}) = 0.$$

Definition 2.2. A system IFS $\mathbb{F}$ is said to have the asymptotic average shadowing property (AASP), if there exists the $\delta \in \Lambda^T$ that every asymptotic-average pseudo orbit $\{x_n\}_{n\in T}$ of $\mathbb{F}$ can be asymptotically shadowed in the average by the orbit $\{y_n\}_{n\in T}$.

$$\lim_{n\longrightarrow\infty} \frac{1}{n}\Sigma_{i=1}^{n-1} d(f_{\lambda_i}(y_n), x_n) = 0, \forall \lambda_i \in \sigma.$$

Theorem 2.3. *Let Λ be a finite set , $\mathbb{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is an IFS and let $k > 0$ be an integer . set $\mathbb{F}^k = \{g_\mu \ | \mu \in \Pi\} = \{f_{\lambda_k} o f_{\lambda_{k-1}} o ... o f_{\lambda_1} \mid \lambda_1, ..., \lambda_k \in \Lambda\}$.*
i) If the parameterized IFS $\mathbb{F}$ has the asymptotic-average shadowing property (AASP) then so dose $\mathbb{F}^k$ for every positive integer K.
ii) If the parameterized IFS $\mathbb{F}^k$ has the asymptotic-average shadowing property (AASP) then so dose $\mathbb{F}$.

Theorem 2.4. *Let X be a compact metric space. If $\mathbb{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is an IFS with the asymptotic average shadowing property (on $\mathbb{Z}_+$), so then $\mathbb{F}^{-1} = \{X; g_\lambda | \lambda \in \Lambda\}$ has the asymptotic average shadowing property where $f_\lambda : X \longrightarrow X$ is homeomorphism and $g_\lambda = f_\lambda^{-1}$ for all $i \geq 0$.*

Example 2.5. Let $X = \{x_1, x_2, \ ... \ , x_n\}$ be a finite set with the discrete metric d. Suppose $\{f_\lambda\}_{\lambda\in\Lambda}$ is the family of all surjective functions on X. $\mathbb{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ has the asymptotic average shadowing property.

The following example shows that if for every $\lambda \in \Lambda$, $f_\lambda : X \longrightarrow X$, as a discrete dynamical system, has the asymptotic average shadowing property then $\mathbb{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is necessarily an IFS whit the asymptotic average shadowing property.

Example 2.6. Let $X = [0, 1]$. This is clear that the constant function $f_1(x) = 1$ has the asymptotic-average shadowing property and tent map

$$f_2(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ -2x + 2 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

has the asymptotic-average shadowing property. But $\mathbb{F} = \{X; f_1, f_2\}$ does not have the asymptotic-average shadowing property.

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Totally Umbilical foliations

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Abstract

In this paper some characterization of totally umbilical foliations are given and some properties of these are investigated.

Keywords: Foliation, Distribution,Totally Umbilical Foliation, Connection.
Mathematics Subject Classification [2010]: 53C12, 53B05

1 Introduction

The theory of foliations of manifolds was created in 1994 by Ehresmann and Reeb [2]. In recent years the study of this subject has become One of the most elegant and fruitful areas of research in mathematics and physics. one of the most important class of foliations is totally umbilical foliations. Let $(M, g, \mathcal{F})$ be an $(n+p)$-dimensional foliated semi-Riemannian manifold, where $\mathcal{F}$ is non degenerate n-foliation on M whose tangent distribution (structural distribution) is $\mathcal{D}$. Denote by $\mathcal{D}^{\perp}$ the complementary orthogonal distribution to $\mathcal{D}$ in the tangent bundle TM of M with respect to g,and call it the transversal distribution to $\mathcal{F}$. Throughout the paper, $F(M)$ stands for the algebra of smooth functions on M, while $\Gamma(TM)$, $\Gamma(\mathcal{D})$ and $\Gamma(\mathcal{D}^{\perp})$ are the $F(M)$-modules of smooth sections of TM, $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. The projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to the decomposition

$$TM = \mathcal{D} + \mathcal{D}^{\perp}$$

are denoted by $\mathcal{T}$ and $\mathcal{N}$ respectively. Some operators will play an important role in the paper. First, for any $\mathcal{N}X \in \Gamma(\mathcal{D}^{\perp})$ we define the shape operator

$$A_{\mathcal{N}X} : \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}); A_{\mathcal{N}X}(\mathcal{T}Y) = -\mathcal{T}(\widetilde{\nabla}_{\mathcal{T}Y}\mathcal{N}X),$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on (M, g). Also, we define the following $F(M)$ bilinear operator:

$$h : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \longrightarrow (D^{\perp}); h(\mathcal{T}X, \mathcal{T}X) = \mathcal{N}(\widetilde{\nabla}_{\mathcal{T}X}\mathcal{T}Y). \quad (1)$$

*Speaker

we call h the second fundamental form of the foliation $\mathcal{F}$, which is symmetric. Taking into account that g is parallel with respect to $\widetilde{\nabla}$ we obtain

$$g(h(\mathcal{T}X,\mathcal{T}Y),\mathcal{N}Z) = g(A_{\mathcal{N}Z}\mathcal{T}X), \tag{2}$$

$\forall X, Y, Z \in \Gamma(TM)$.
And choose an orthonormal frame field $\{E_1, ..., E_n\}$ in $\Gamma(\mathcal{D})$ of sighature $\{\varepsilon_1, ..., \varepsilon_n\}$, that is $\varepsilon_i = g(E_i, E_i)$.

2 Mean curvature

Definition 2.1. We define the mean curvature vector field H of $\mathcal{F}$ by the formula

$$H = \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i h(E_i, E_i). \tag{3}$$

It is easy to check that H does not depend on orthonormal basis $\{E_i\}$, so it is a global section of the transversal distribution $(\mathcal{D}^{\perp})$. We denote by A_α the shape operators of $\mathcal{F}$ with respect to E_α.

Lemma 2.2. *[3]*

i)Any semi-Euclidean space (V, g) with $V \neq 0$ has an orthonormal basis $B = \{e_1, ..., e_m\}$.

ii)Any vector v has a unique expression

$$v = \Sigma_{i=1}^{n}\varepsilon_i g(v, e_i)e_i,$$

where $\varepsilon_i = g(e_i, e_i)$.

by using Lemma (2.2) and (2) we express H as follows

$$H = \frac{1}{n}\Sigma_{\alpha=n+1}^{n+p}\Sigma_{i=1}^{n}\varepsilon_\alpha\varepsilon_i g(A_\alpha E_i, E_i)E_\alpha. \tag{4}$$

The mean curvature form of the foliation $\mathcal{F}$ on $(M, g)$is a 1-form k on M defined by

$$k(X) = g(X, H), \quad \forall X \in \Gamma(TM). \tag{5}$$

Thus we have $k(\mathcal{T}X) = 0$ and

$$k(\mathcal{N}X) = g(\mathcal{N}X, H). \tag{6}$$

By using (3), (4) in (6) we deduce that

$$k(\mathcal{N}X) = \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i g(h(E_i, E_i), \mathcal{N}X) = \frac{1}{n}\Sigma_{\alpha=n+1}^{n+p}\Sigma_{i=1}^{n}\varepsilon_\alpha\varepsilon_i g(A_\alpha E_i, E_i)g(E_\alpha, \mathcal{N}X). \tag{7}$$

Now, let $\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\}$ be a semi-holonomic frame field on the foliated semi-Riemannian manifold $(M, g, \mathcal{F})$. Then we put

$$H = H^\alpha\frac{\delta}{\delta x^\alpha} \quad and \quad k_\alpha = k(\frac{\delta}{\delta x^\alpha}).$$

Thus on the domain of a foliated chart on M we have

$$k_\alpha = g_{\alpha\beta}H^\beta. \tag{8}$$

3 Totally Umbilical Foliation

Definition 3.1. we say $\mathcal{F}$ is a totally umbilical foliation if its second fundamental form h satisfies

$$h(\mathcal{T}X, \mathcal{T}Y) = g(\mathcal{T}X, \mathcal{T}Y)H, \tag{9}$$

$\forall X, Y \in \Gamma(TM)$, where H is the mean curvature vector field of $\mathcal{F}$.

The condition (9) can also be expressed by using the shape operator of the foliation. Indeed, by using (2) and (9) we obtain

$$g(A_{\mathcal{N}Z}\mathcal{T}X, \mathcal{T}Y) = g(h(\mathcal{T}X, \mathcal{T}Y), \mathcal{N}Z) = g(g(H, \mathcal{N}Z)\mathcal{T}X, \mathcal{T}Y).$$

Thus $\mathcal{F}$ is totally umbilical if and only if its shape operators satisfy

$$A_{\mathcal{N}Z}\mathcal{T}X = k(\mathcal{N}Z)\mathcal{T}X, \qquad \forall X, Z \in \Gamma(TM). \tag{10}$$

Now, we put $A_\alpha = A_{\frac{\delta}{\delta x^\alpha}}, \alpha = \{n+1, ..., n+p\}$ and by using (10) obtain the following.

Theorem 3.2. *A non-degenerate foliation $\mathcal{F}$ on a semi-Riemannian manifold (M, g) is totally umbilical if and only if its shape operators A_α satisfy*

$$A_\alpha = k_\alpha I, \quad \alpha \in \{n+1, ..., n+p\},$$

where I is the identity on $\Gamma(\mathcal{D})$ and k_α are the local components of the mean curvature form given by (8).

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Numerical Analysis

A Bernoulli pseudo-spectral method for solving nonlinear fractional integro-differential equations

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Abstract

In this paper, a Bernoulli pseudo-spectral method for solving nonlinear Volterra integro-differential equations of fractional order is considered. The fractional derivative is described in the Caputo sense. The suggested technique transform these types of equations to the solution of a system of algebraic equations. The technique is applied to some problems to show the validity and applicability of the proposed method.

Keywords: Fractional calculus, Caputo derivative, Bernoulli polynomials, Volttera integro-differential equations.
Mathematics Subject Classification [2010]: 34A08, 45D05

1 Introduction

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to an arbitrary order. A history of the development of fractional differential operators can be found in [1].

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\nu \geq 0$ is defined as [2]

$$I^{\nu} f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^t \frac{f(s)}{(t-s)^{1-\nu}} ds, & \nu > 0, t > 0, \\ f(t), & \nu = 0. \end{cases} \tag{1}$$

For the Riemann-Liouville fractional integral we have [2]:

$$I^{\nu} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)} t^{\nu+\beta}, \qquad \beta > -1. \tag{2}$$

Definition 1.2. Caputo's fractional derivative of order ν is defined as [2]

$$D^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\nu+1-n}} ds, \qquad n-1 < \nu \leqslant n, n \in \mathbb{N}, t > 0. \tag{3}$$

For the Caputo derivative we have the following two basic properties[2]:
$(i) D^{\nu} I^{\nu} f(t) = f(t), \qquad (ii) I^{\nu} D^{\nu} f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0) \frac{t^i}{i!}$.

*Speaker

Definition 1.3. Bernoulli polynomials of order m can be defined with the following formula [3]

$$B_m(t) = \sum_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} B_{m-i} t^i. \tag{4}$$

Theorem 1.4. Let $D^\nu y(x)$ be approximated by the Bernoulli polynomials ($D^\nu y(x) = \sum_{k=0}^{m-1} c_k B_k(x)$), and also suppose $n-1 < \nu \leqslant n$. Then

$$y(x) = \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k b_{k,r} x^{r+\nu} + \sum_{i=0}^{n-1} y^{(i)}(0) \frac{x^i}{i!}, \tag{5}$$

where $b_{k,r} = \begin{pmatrix} k \\ r \end{pmatrix} \frac{\Gamma(r+1)}{\Gamma(r+1+\nu)} B_{k-r}$.

Proof. Applying operator I^ν, on both sides of $D^\nu y(x) = \sum_{k=0}^{m-1} c_k B_k(x)$, we have

$$\begin{aligned} y(x) - \sum_{i=0}^{n-1} y^{(i)}(0) \frac{x^i}{i!} = I^\nu \Big(\sum_{k=0}^{m-1} c_k B_k(x) \Big) = I^\nu \Big(\sum_{k=0}^{m-1} c_k \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} B_{k-r} x^r \Big) = \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k \begin{pmatrix} k \\ r \end{pmatrix} B_{k-r} I^\nu (x^r) \\ = \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k \begin{pmatrix} k \\ r \end{pmatrix} B_{k-r} \frac{\Gamma(r+1)}{\Gamma(r+1+\nu)} x^{r+\nu} = \sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k b_{k,r} x^{r+\nu}. \end{aligned}$$

□

Example 1.5. In this paper, we consider the following equation

$$D^\nu y(x) - \lambda \int_0^x k(x,t) F(y(t)) dt = f(x), \qquad 0 \leqslant x < 1, n-1 < \nu \leqslant n, \tag{6}$$

$$y^{(i)}(0) = \delta_i, \qquad i = 0, 1, ..., n-1, n \in \mathbb{N}. \tag{7}$$

Solution. We approximate $D^\nu y(x)$ as:

$$D^\nu y(x) = \sum_{k=0}^{m-1} c_k B_k(x). \tag{8}$$

From Eqs. (6), (7), (8) and Theorem 1, we have

$$\sum_{k=0}^{m-1} c_k B_k(x) - \lambda \int_0^x k(x,t) F\Big(\sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k b_{k,r} t^{r+\nu} + \sum_{i=0}^{n-1} y^{(i)} \frac{t^i}{i!} \Big) dt = f(x). \tag{9}$$

Now, we collocate (9) at the zeros $x_p, p = 0, 1, ..., m-1$ of Legendre polynomial $P_m(t)$

$$\sum_{k=0}^{m-1} c_k B_k(x_p) - \lambda \int_0^{x_p} k(x_p,t) F\Big(\sum_{k=0}^{m-1} \sum_{r=0}^{k} c_k b_{k,r} t^{r+\nu} + \sum_{i=0}^{n-1} y^{(i)} \frac{t^i}{i!} \Big) dt = f(x_p). \tag{10}$$

Then, we transfer the t−interval $[0, x_p]$ into τ −interval $[-1, 1]$ by change of variable $\tau = \frac{2}{x_p}t - 1$,

$$\sum_{k=0}^{m-1} c_k B_k(x_p) - \lambda \frac{x_p}{2} \int_{-1}^{1} k(x_p, \frac{x_p}{2}(\tau+1)) F(\sum_{k=0}^{m-1}\sum_{r=0}^{k} c_k b_{k,r} (\frac{x_p}{2}(\tau+1))^{r+\nu} + \sum_{i=0}^{n-1} y^{(i)} \frac{(\frac{x_p}{2}(\tau+1))^i}{i!}) d\tau = f(x_p). \tag{11}$$

By using the Gauss −Legendre integration formula [4], for $p = 0, 1, ..., m-1$, we have:

$$\sum_{k=0}^{m-1} c_k B_k(x_p) - \lambda \frac{x_p}{2} \sum_{q=1}^{m} \omega_q k(x_p, \frac{x_p}{2}(\tau_q+1)) F(\sum_{k=0}^{m-1}\sum_{r=0}^{k} c_k b_{k,r} (\frac{x_p}{2}(\tau_q+1))^{r+\nu} + \sum_{i=0}^{n-1} y^{(i)} \frac{(\frac{x_p}{2}(\tau_q+1))^i}{i!}) = f(x_p), \tag{12}$$

where τ_q, $q = 1, 2, ..., m$, are zeros of Legendre polynomial $p_m(x)$ and $\omega_j = \frac{-2}{(n+1)p'_m(x_j)p_{m+1}(x_j)}$, $j = 1, 2, ..., m$. Eq. (12), give m nonlinear algebraic equations which can be solved, for the unknowns c_k, $k = 0, 1, ..., m-1$, using Newton's iterative method. Finally, $y(x)$ given in (5) can be calculated.

2 Main results

First, we consider the following equation [5]

$$D^\nu y(x) - \int_0^x [y(t)]^3 dt = e^x - \frac{1}{3}e^{3x} + \frac{1}{3}, \qquad 0 \leqslant x < 1, 0 < \nu \leqslant 1, \tag{13}$$

subject to the initial condition $y(0) = 1$. The exact solution of this problem, when $\nu = 1$, is $y(x) = e^x$. Table 1 shows the approximate solutions obtained for different values of t by using the present method for $m = 4, 6, 8$ and $\nu = 1$, the second Chebyshev wavelet method [5] for $k = 5, M = 2$ and $\nu = 1$, toghether with the exact solutions. Also, the numerical results for $y(x)$ with $m = 8$ and $\nu = 0.7, 0.8, 0.9, 1$ are plotted in Fig. 1.

Table 1: Comparison of numerical solutions with the other methods for $\nu = 1$

t	*Exact solution*	*Present method*			*Ref*[5]
		$m = 4$	$m = 6$	$m = 8$	
0	1	1	1	1	1.000122
0.2	1.221403	1.220677	1.221408	1.221403	1.221645
0.4	1.491825	1.490795	1.491828	1.491825	1.492295
0.6	1.822119	1.821352	1.822117	1.822119	1.823061
0.8	2.225541	2.224992	2.225543	2.225541	2.227565

Then, consider the following equation [5]

$$D^\nu y(x) - \int_0^x [y(t)]^2 dt = -1, \qquad 0 \leqslant x < 1, 0 < \nu \leqslant 1, \tag{14}$$

subject to the initial condition $y(0) = 0$. Table 2, shows the numerical for $\nu = 0.8, 0.9, 1$, by using the present method, when $m = 8$, and the second Chebyshev wavelet method [5], for $k = 6, M = 2$.

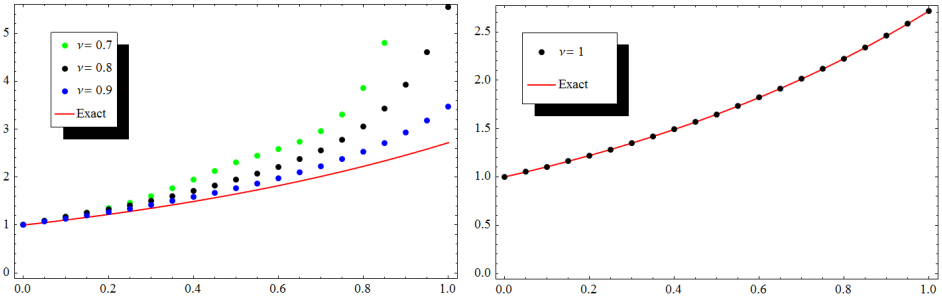

Figure 1: Comparison of $y(x)$ for $m = 8$, with $\nu = 0.7, 0.8, 0.9, 1$ and exact solution

Table 2: The numerical results for variouse values of ν

t	*Exact solution*	*Present method*			*Ref*[5]		
		$\nu = 1$	$\nu = 0.9$	$\nu = 0.8$	$\nu = 1$	$\nu = 0.9$	$\nu = 0.8$
0	0	0	0	0	0	− 0.00017	− 0.00055
0.1250	−0.12498	−0.12498	− 0.15997	− 0.20339	−0.12498	− 0.16003	− 0.20344
0.2500	−0.24968	−0.24968	− 0.29791	− 0.35281	−0.24968	− 0.29794	− 0.35281
0.3750	−0.37336	−0.37336	− 0.42702	− 0.48422	−0.37336	− 0.42702	− 0.48420
0.5000	−0.49482	−0.49482	− 0.54829	− 0.60159	−0.49483	− 0.54828	− 0.60156
0.6250	−0.61243	−0.61243	− 0.66089	− 0.70527	−0.61245	− 0.66090	− 0.70527
0.7500	−0.72415	−0.72415	− 0.76327	− 0.79452	−0.72418	− 0.76332	− 0.79457
0.8750	−0.82767	−0.82767	− 0.85360	− 0.86834	−0.82770	− 0.85365	− 0.86838

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A Meshless Method Using the Radial Basis Functions for Numerical Solution of the Gilson-Pickering Equation

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Abstract

In this article, thin plate splines radial basis function method is presented for solutions of Gilson-Pickering equation. This scheme works in a similar form as finite difference methods and we use collocation points for basis nodes in radial basis function. A numerical example is studied to demonstrate the accuracy and efficiency of the presented method.

Keywords: Gilson-Pickering (GP) equation, Radial basis functions (RBFs), Thin plate splines radial basis functions(TPS-RBFs)
Mathematics Subject Classification [2010]: 65M50, 65N35

1 Introduction

We consider a class of fully nonlinear third-order partial differential equations for studying by name Gilson and Pickering equation as follows [1]:

$$u_t - \epsilon u_{xxt} + 2\kappa u_x - u u_{xxx} - \alpha u u_x - \beta u_x u_{xx} = 0, \tag{1}$$

where ϵ, κ, α and β are arbitrary constants. Three special cases of equation have appeared in the literature, up to some resealings. If $\epsilon = 1, \alpha = -1, \beta = 3$, and $\kappa = \frac{1}{2}$, then (1) is the Fornberg-Whitham equation, for $\epsilon = 0, \alpha = 1, \beta = 3$, and $\kappa = 0$, (1) is Rosenau- Hyman equation and (1) is the Fuchssteiner-Fokas-Camassa-Holm equation for the parameters $\epsilon = 1, \alpha = -3$, and $\beta = 2$.
Irshad and Tauseef [1] applied tanh-coth method for obtaining numerical solutions of GP equation. Also, Fan and other authors [4] used the $\frac{G'}{G}$-expansion method for solving this equation. Fronberg and Flyer [5] obtained accuracy of radial basis function interpolation. The purpose of this paper is to study numerical results of thin plate splines radial basis function methods to GP equation. TPS-RBF-methods for solving the GP equation is a new work.

*Speaker

2 Thin plate splines radial basis function approximation

In the interpolation of different data using RBFs, we write the approximation of a distribution $u(X)$ as a linear combination form of N radial functions φ_j with N distinct centers $\xi_1, \xi_2, ..., \xi_N$. This approximation usually takes the following form:

$$u(X) \simeq \sum_{j=1}^{N} \lambda_j \varphi(r_j) + \psi(X), \quad \forall \ X \in \Omega \subset \mathbb{R}^d, \tag{2}$$

where $X = (x_1, x_2, ..., x_d)$, d is the dimension of the problem, λ's are coefficients to be determined and $r_j = ||X - \xi_j||$ in the Euclidean norm.
Equation (2) can be written without the additional polynomial ψ. However, ψ is usually required when φ is conditionally positive definite and φ has a polynomial growth toward infinity.
One of the well-known RBFs is thin plate spline method [2] which we see in applying this kind of RBFs for GP equation, ψ is exist and nonzero because φ is conditionally positive definite in TPS-RBF method.
The generalized thin plate splines (TPS) defined as:

$$\varphi(x, \xi_j) = \varphi(rj) = r_j^{2m} \log(r_j), \qquad m = 1, 2, 3, ...$$

If $\mathcal{P}_d^q$ represents the space of d-variate polynomials of order not exceeding q, and the polynomials $P_1, ..., P_m$ are the basis of $\mathcal{P}_d^q$ in $\mathbb{R}^d$, then the polynomial $\psi(x)$ in Eq. (2) is usually written in the following form:

$$\psi(x) = \sum_{i=1}^{m} \eta_i P_i(x),$$

where $m = (q - 1 + d)!/(d!(q - 1)!)$. Therefore, we obtain $m = 2$ for TPS-RBF method in GP equation.
We use collocation method for determinating the coefficients $(\lambda_1, ..., \lambda_N)$ and $(\eta_1, ..., \eta_m)$. However, in addition to the N equations resulting from collocating (2) at the N points $\xi_1, \xi_2, ..., \xi_N$, an extra m equations are required. This is insured by the m conditions for (2),

$$\sum_{j=1}^{N} \lambda_j P_i(x_j) = 0, \quad i = 1, ..., m.$$

In a similar representation as (2), for any linear partial differential operator L, Lu can be approximated by

$$Lu(x) \simeq \sum_{j=1}^{N} \lambda_j L\phi(r_j) + L\psi(x).$$

3 Numerical results

In this section, we give some computational results of the TPS-RBF method on the GP equation to support our discussion in the previous sections. Accuracy of the estimated

solutions can be worked out by measuring with the L_2 and L_∞ error norms and RMS invariant which are defined by

$$L_2 = ||U_{exact} - U_{numerical}||_2 = \sqrt{(\delta x)\sum_{i=1}^{N}(U_i^{exact} - U_i^{numerical})^2},$$

$$L_\infty = ||U_{exact} - U_{numerical}||_\infty = \max_i |U_i^{exact} - U_i^{numerical}|,$$

$$RMS = \frac{L_2}{\sqrt{N}}.$$

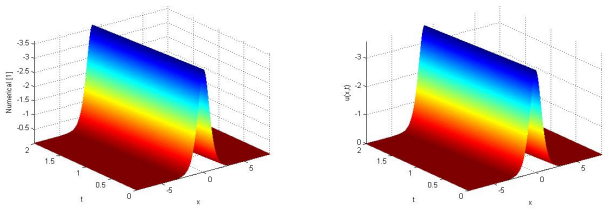
In [1], one of the numerical solutions of GP equation is

$$u(x,t) = \frac{3(c-2\kappa)\epsilon c}{\alpha\epsilon c - c + 2\kappa}(-1 + \tan[\frac{1}{2}\sqrt{\frac{-2\kappa - c}{\epsilon c}}(x - ct)]^2), \tag{3}$$

We apply TPS-RBF method for GP equation for $\epsilon = 1.5, \alpha = 1.75, \beta = -2, c = -0.5$ and $\kappa = 1.5$. To compare the new technique with numerical solution (3), the spatial grid spacing $\delta x = 0.01$ and time step $t = 0.001$ are taken.

Table 1: Error norms of TPS-RBF method for GP equation with $\delta t = 0.001$, and $\delta x = 0.01$.

t	L_2	L_∞	RMS
0.5	2.0102×10^{-5}	1.8386×10^{-6}	5.0209×10^{-7}
1	9.4144×10^{-5}	7.2845×10^{-6}	2.3514×10^{-6}
1.5	2.8283×10^{-4}	2.1934×10^{-5}	7.0642×10^{-6}
2	8.9844×10^{-4}	6.8264×10^{-5}	2.2440×10^{-5}

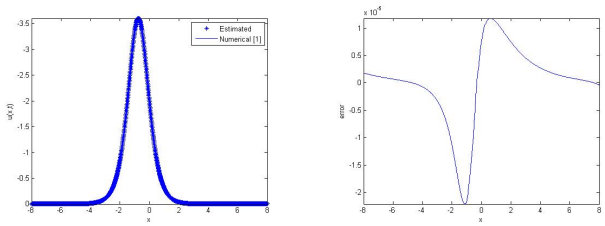


(a) (b)

Figure 1: (a) Numerical solution (3) for $t \in [0, 2]$, (b) TPS-RBF numrical solution for $t \in [0, 2]$

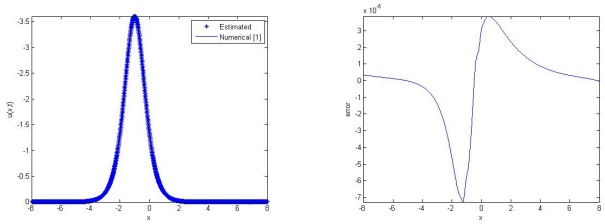
Errors in L_2, L_∞ norms and RMS invariant taken at several times are tabulated in Table 1. In Figure 1, the plot of estimated solution (3) and estimated solution of TPS-RBF are shown at different times. Also, the plot of estimated solution and distributions of the errors at $t = 1.5$ and $t = 2$ are shown in Figures 2 and 3, respectively.

The numerical results are demonstrated the good accuracy of the scheme proposed in this research. The method proposed in this work can be extended to solve the other important nonlinear partial differential equations.



(a) (b)

Figure 2: (a) Numerical solution (3) and TPS-RBF numrical solution , (b) Error= TPS-RBF numrical solution- numrical solution (3) for $t = 1.5$ with $\delta t = 0.001$, and $\delta x = 0.01$.



(a) (b)

Figure 3: (a) Numerical solution (3) and TPS-RBF numrical solution , (b) Error= TPS-RBF numrical solution- numrical solution (3) for $t = 2$ with $\delta t = 0.001$, and $\delta x = 0.01$.

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A Modification of Adomian Decomposition Method to Delay Differential Equations Using Padé Approximation

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Abstract

In this paper we present an application of technique combining Adomian Decomposition Method (ADM), Laplace transform and Padé approximant to find the analytical solutions for Delay Differential Equations (DDE). Solutions to DDEs are first obtained in convergent series form using the ADM. Then obtained from ADM's truncated series, we apply Laplace transform to it, then convert the transformed series into a meromorphic function by forming its Padé approximant. Finally, we take the inverse Laplace transform of the Padé approximant to obtain the analytical solution.

Keywords: Adomian decomposition method, Delay differential equation, Laplace transform, Padé approximant, Laplace-Padé resummation method
Mathematics Subject Classification [2010]: 44A10, 65Qxx, 74H15

1 Introduction

In this section we will explain the basic definitions of DDE, ADM, Padé approximant, Laplace-Padé resummation.

Definition 1.1: We define the nth order delay differential equations (DDE) of the form as follows:

$$u^{(n)}(x) = f(x, u(x), u(\eta_1(x)), u(\eta_2(x)), \cdots, u(\eta_m(x))), \quad x \in I = [0, a], \tag{1}$$

where $u : I \to R$, $f : I \times R^2 \to R$, $\eta_i : I (i = 1, 2, \cdots, m)$ and $\eta_i(x) < x$ for $x \in I$.

Definition 1.2: To introduce the basic idea of the ADM [**?**], we consider the operator equation $Fu = G$, where F represents a general nonlinear ordinary differential operator and G is a given function. Then F can be decomposed as:

$$Lu + Ru + Nu = G, \tag{2}$$

where N is a nonlinear operator, L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order less than L and G is the nonhomogeneous term. The method is based by applying the operator L^{-1} formally to the expression (2) we obtain:

$$u = h + L^{-1}G - L^{-1}Ru - L^{-1}Nu, \tag{3}$$

where h is the solution of the homogeneous equation $Lu = 0$, with the initial-boundary conditions. The problem now is the decomposition of the nonlinear term Nu. To do this,

*Speaker

Adomian developed a very elegant technique as follows:
Define the decomposition parameter λ as $u = \sum_{n=0}^{\infty} \lambda^n u_n$ then $N(u)$ will be a function of $\lambda, u_0, u_1, \cdots$, next expanding $N(u)$ in Maclurian series with respect to λ we obtain $N(u) = \sum_{n=0}^{\infty} \lambda^n u_n$ where: $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{n=0}^{\infty} \lambda^k u_k)]_{\lambda=0}$, (4)

where the components of A_n are so called the Adomian polynomials, they are generated for each nonlinearity, for example, for $N(u) = f(u)$ the Adomian polynomials, are given as:

$$\begin{cases} A_0 = f(u_0) \\ A_1 = f(u_1)f'(u_0) \\ A_2 = u_2 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0) \\ \vdots \end{cases} \quad (5)$$

Now, we parameterize (3) in the form:

$$u = h + L^{-1}G - \lambda L^{-1}Ru - \lambda L^{-1}Nu, \quad (6)$$

where λ is just an identifier for collecting terms in a suitable way such that u_n depends on $u_0, u_1, \cdots, u_n$ and we will later set $\lambda = 1$,

$$\sum_{n=0}^{\infty} \lambda^n u_n = h + L^{-1}G - \lambda L^{-1}Ru - \lambda L^{-1}Nu. \quad (7)$$

Equating the coefficients of equal powers of λ, we obtain:

$$\begin{cases} u_0 = h + L^{-1}G \\ u_1 = -L^{-1}(Ru_0) - L^{-1}(A_0) \\ u_2 = -L^{-1}(Ru_1) - L^{-1}(A_1) \\ \vdots \end{cases} \quad (8)$$

and in general:

$$u_n = -L^{-1}(Ru_{n_1}) - L^{-1}(A_{n-1}), \quad n \geq 1.$$

Finally, an N-terms that approximate solution is given by:

$$\phi_N(T) = \sum_{n=0}^{N-1} u_n(T), \quad N \geq 1,$$

and the exact solution is $u(t) = \lim_{n\to\infty} \phi_N(t)$.

Definition 1.3: Let $u(t)$ be an analytical function with the Maclaurin's expansion

$$u(t) = \sum_{n=0}^{\infty} u_n t^n, \quad 0 \leq t \leq T. \quad (9)$$

Then the Padé approximant to $u(t)$ of order $[L/M](t)$ which we denote by $[L/M]_u(t)$ is defined by [**?**]

$$[L/M]_u(t) = \frac{p_0 + p_1 t + \cdots + p_L t^L}{1 + q_1 t + \cdots + q_M t^M}, \quad (10)$$

where we considered $q_0 = 1$, and the numerator and denominator have no common factors.

The numerator and the denominator in (10) are constructed so that $u(t)$ and $[L/M]_u(t)$ and their derivatives agree at $t = 0$ up to $L + M$. That is

$$u(t) - [L/M]_u(t) = O\left(t^{L+M+1}\right). \tag{11}$$

From (11), we have

$$u(t)\sum_{n=0}^{M} q_n t^n - \sum_{n=0}^{L} p_n t^n = O\left(t^{L+M+1}\right). \tag{12}$$

From (12), we get the following algebraic linear systems

$$\begin{cases} u_L q_1 + \cdots + u_{L-M+1} q_M = -u_{L+1} \\ u_{L+1} q_1 + \cdots + u_{L-M+2} q_M = -u_{L+2} \\ \vdots \\ u_{L+M-1} q_1 + \cdots + u_L q_M = -u_{L+M} \end{cases} \tag{13}$$

and

$$\begin{cases} p_0 = u_0 \\ p_1 = u_1 + u_0 q_1 \\ \vdots \\ p_L = u_L + u_{L-1} q_1 + \cdots + u_0 q_L \end{cases} \tag{14}$$

From (13), we calculate first all the coefficients q_n, $1 \le n \le M$. Then, we determine the coefficients p_n, $0 \le n \le L$ from (14).

Definition 1.4: Several approximate methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lacks of large domains of convergence. Therefore, Laplace-Padé resummation method [**?**] is used in literature to enlarge the domain of convergence of solutions or inclusive to find exact solutions. The Laplace-Padé method can be explained as follows:
First, Laplace transformation is applied to power series (1). Next s is substituted by $1/t$ in the resulting equation. After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller values than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence, then t is substituted by $1/s$ and by using the inverse Laplace s transformation, we obtain the exact or approximate solution.

2 Main results

In this section, we will demonstrate the effectiveness and accuracy of the LPADM described in the previous section through one DDE system.

Example 2.1: we consider the following nonlinear diffrential equation of the first order:

$$\frac{du(t)}{dt} = 1 - 2u^2(t/2), \quad u(0) = 1, \quad 0 \le t \le 1, \tag{15}$$

which $u(t) = \sin(t)$ is the exact solution of equation (14). Using the ADM, we get

$$u_{n+1}(t) = -2\int_0^t A_n dt, \quad n \geq 0, \quad u_0(x) = 1, \tag{16}$$

where A_n, $n \geq 0$ are the domain polynomials that represent the nonlinear terms. We list the set of Adomian polynomials [**?**] as follows:

$$\begin{cases} A_0(t) = u_0^2(t/2), \\ A_1(t) = u_0(t/2)u_1(t/2) \\ \vdots \end{cases} \tag{17}$$

The solution in a series form is given by

$$u_3(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} + \cdots . \tag{18}$$

We get the fifth order approximation solution

$$u_3(t) \simeq t - \frac{t^3}{6} + \frac{t^5}{120}. \tag{19}$$

Applying the Laplace transforms to $u_3(t)$ yields

$$\ell[u_3(t)] = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6}. \tag{20}$$

For simplicity let $s = 1/t$, then

$$\ell[u_3(t)] = t^2 - t^4 + t^6. \tag{21}$$

All of the $[L/M]$ t-Padé approximants of (21) with $L \geqslant 1$, $M \geqslant 1$ and $L + M \leqslant 4$ yields

$$[L/M]_{u_3}(t) = \frac{t^2}{1+t^2}. \tag{22}$$

Let $t = 1/s$, applying the inverse Laplace transform to the Padé approximant yields

$$\ell^{-1}[\frac{1}{s^2+1}] = \sin(t), \tag{23}$$

as an approximate solution which in this case is the exact solution.

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A Modified Reduced Differential Transform Method to Delay Differential Equations Using Padé Approximation

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Abstract

In this paper we present an application of technique combining Reduced Differential Transform Method (RDTM), Laplace transform and Padé approximant to find the analytical solutions for DDEs. Solutions to DDEs are first obtained in convergent series form using the RDTM. To improve the solution obtained from RDTM's truncated series, we apply Laplace transform to it, then convert the transformed series into a meromorphic function by forming its Padé approximant. Finally, we take the inverse Laplace transform of the Padé approximant to obtain the analytical solution.

Keywords: Reduced differential transform method, Delay differential equation, Laplace transform, Padé approximant, Laplace-Padé resummation method
Mathematics Subject Classification [2010]: 44A10, 65Qxx, 74H15

1 Introduction

In this section we will explain the basic definitions of DDE, RDTM, Padé approximant, Laplace-Padé resummation.

Definition 1.1: We define the nth order delay differential equations (DDE) of the form as follows:

$$u^{(n)}(x) = f(x, u(x), u(\eta_1(x)), u(\eta_2(x)), \cdots, u(\eta_m(x))), \quad x \in I = [0, a], \tag{1}$$

where $u : I \to R$, $f : I \times R^2 \to R$, $\eta_i : I (i = 1, 2, \cdots, m)$ and $\eta_i(x) < x$ for $x \in I$.
The basic definitions of RDTM [**?**] are defined as follows.

Definition 1.2: If function $u(x, t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k} u(x, t)]_{t=0}, \tag{2}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x, t)$ represents the original function while the uppercase $U_k(x)$ stands for the transformed function.

Definition 1.3: The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x, t) = \sum U_k(x) t^k, \tag{3}$$

*Speaker

then combining equation (2) and (3) we can write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}\,t^k. \tag{4}$$

From the above definitions, it is easy to verify that the concept of the RDTM is obtained from the power series expansion. inverse transformation [**?**] of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives approximation solution in the following form

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k, \tag{5}$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,t)=\lim_{n\to\infty}\tilde{u}_n(x,t). \tag{6}$$

The solutions series obtained from RDTM may have limited regions of convergence, even if we take a large number of terms. Therefore, we propose to apply the Laplace-Padé resummation method to RDTM truncated series to enlarge the convergence region as depicted in the next definition.

Definition 1.4: Let $u(t)$ be an analytical function with the Maclaurin's expansion

$$u(t)=\sum_{n=0}^{\infty}u_nt^n,\quad 0\le t\le T, \tag{7}$$

then the Padé approximant to $u(t)$ of order $[L/M](t)$ which we denote by $[L/M]_u(t)$ is defined by [**?**]

$$[L/M]_u(t)=\frac{p_0+p_1t+\cdots+p_Lt^L}{1+q_1t+\cdots+q_Mt^M}, \tag{8}$$

where we considered $q_0=1$, and the numerator and denominator have no common factors.

The numerator and the denominator in (8) are constructed so that $u(t)$ and $[L/M]_u(t)$ and their derivatives agree at $t=0$ up to $L+M$. That is

$$u(t)-[L/M]_u(t)=O\left(t^{L+M+1}\right). \tag{9}$$

From (9), we have

$$u(t)\sum_{n=0}^{M}q_nt^n-\sum_{n=0}^{L}p_nt^n=O\left(t^{L+M+1}\right). \tag{10}$$

From (10), we get the following algebraic linear systems

$$\begin{cases}u_Lq_1+\cdots+u_{L-M+1}q_M=-u_{L+1}\\ u_{L+1}q_1+\cdots+u_{L-M+2}q_M=-u_{L+2}\\ \vdots\\ u_{L+M-1}q_1+\cdots+u_Lq_M=-u_{L+M}\end{cases} \tag{11}$$

and

$$\begin{cases} p_0 = u_0 \\ p_1 = u_1 + u_0 q_1 \\ \vdots \\ p_L = u_L + u_{L-1} q_1 + \cdots + u_0 q_L \end{cases} \tag{12}$$

From (11), we calculate first all the coefficients q_n, $1 \leq n \leq M$. Then, we determine the coefficients p_n, $0 \leq n \leq L$ from (12).

Definition 1.5: Several approximate methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lacks of large domains of convergence. Therefore, Laplace-Padé resummation method [**?**] is used in literature to enlarge the domain of convergence of solutions or inclusive to find exact solutions.

The Laplace-Padé method can be explained as follows:

1. First, Laplace transformation is applied to power series (1).
2. Next, s is substituted by $1/t$ in the resulting equation.
3. After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller values than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.
4. Then, t is substituted by $1/s$.
5. Finally, by using the inverse Laplace s transformation, we obtain the exact or approximate solution.

2 Main results

In this section, we will demonstrate the effectiveness and accuracy of the LPRDTM described in the previous section through one DDE system.
We consider the following example:

Example 2.1: We consider the following pantograph delay [**?**] equation:

$$u'(t) = \frac{1}{2} exp \frac{t}{2} u(\frac{t}{2}) + \frac{1}{2} u(t), \quad u(0) = 1, \tag{13}$$

which $u(t) = e^t$ is the exact solution of equation (13).
Using the differential transformation method, the differential transform version of equation (13) is

$$(k+1)U(k+1) = \frac{1}{2} \sum_{l=0}^{k} \frac{1}{2^l l!} \frac{1}{2^{k-l}} + \frac{1}{2} U(k), \quad k \geq 0. \tag{14}$$

Using the inverse transformation, we obtain the following series solution:

$$u(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^k}{k!} + \cdots . \tag{15}$$

We get the fourth order approximation solution

$$u(t) \cong \sum_{k=0}^{6} U(k)t^k = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}. \tag{16}$$

Applying Laplace transforms to $u(t)$ yields

$$\ell[u(t)] = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \frac{1}{s^5}. \tag{17}$$

For simplicity let $s = 1/t$, then

$$\ell[u(t)] = t + t^2 + t^3 + t^4 + t^5. \tag{18}$$

All of the $[L/M]$ t-Padé approximants of (18) with $L \geqslant 1$, $M \geqslant 1$ and $L + M \leqslant 4$ yields

$$[L/M]_u(t) = \frac{t}{1-t}. \tag{19}$$

Now since $t = 1/s$, we obtain in terms of s as follows

$$[L/M]_u(t) = \frac{1}{s-1}. \tag{20}$$

Finally, applying the inverse Laplace transform to the Padé approximants (20) yields

$$\ell^{-1}[\frac{1}{s-1}] = e^t, \tag{21}$$

an approximate solution which in this case is the exact solution.

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A new approach for numerical solution of Fokker-Plank Equation using the Chelyshkov cardinal Function

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Abstract

In this paper a numerical method is presented for the solution of Fokker-Plank equation. The main idea of this method is expanding the approximate solution by the Chelyshkov cardinal function. At the end, using the operator derivative matrix the problem turns into a system of algebraic equations.

Keywords: Fokker-Plank equation, Chelyshkov polynomials, cardinal functions.

1 Introduction

In statistical mechanics, the Fokker-Plank is a partial differential equation that describe the time volution of the probability density function of the velocity of a particle under the influence of day forces and random forces as in Brownian motion. Fokker-Plank equation occurs in many different fields such as solid state physics, quantum optics, theoretical biology, ect. The general Fokker-Plank equation for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the form

$$\frac{\partial u}{\partial t} = [-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)]u. \tag{1}$$

$$u(x,,0) = f(x) \qquad x \in (-\infty, \infty) \tag{2}$$

where $u(x,t)$ is unknown, $B(x,t) > 0$ is the diffusion coefficient and $A(x,t) > 0$ is the drift coefficient. We know that in one variable case the nonlinear Fokker-Plank equation is

$$\frac{\partial u}{\partial t} = [-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)]u. \tag{3}$$

These polynomials are orthogonal respect to weight function 1 over the interval $[0,1]$. The explicit definition of these polynomials are as follow

$$P_{N,k}(x) = \sum_{j=0}^{N-k}(-1)^j \left(\begin{array}{c} N-k \\ j \end{array} \right) \left(\begin{array}{c} N-k+1+j \\ N-k \end{array} \right) x^{k+j}, \qquad k = 0, 1, ..., N. \tag{4}$$

In the present paper, we consider these polynomials for the case $k = 0$, then (4) becomes to

$$P_{N,0}(x) = \sum_{j=0}^{N}(-1)^j \left(\begin{array}{c} N \\ j \end{array} \right) \left(\begin{array}{c} N+1+j \\ N \end{array} \right) x^{j}, \tag{5}$$

*Speaker

$P_{N,0}$ is a polynomial of order N that has N zeros, so another representation of this polynomial can be written as follow

$$P_{N,0} = \prod_{i=1}^{N}(x - x_i), \tag{6}$$

where x_i, $i = 1, ..., N$ are zeros of $P_{N,0}$. Hereinafter we show $P_{N,0}$ by P_N (see[5]).

2 Cardinal function of Chelyshkov polynomial

For any orthogonal function $\Psi_N(x)$ of order N, the cardinal function corresponding to it can be written (see[6])

$$C_j(x) = \frac{\Psi_N(x)}{\Psi_{N,x}(x_j)(x - x_j)}, \qquad j = 1, ..., N. \tag{7}$$

where the subscript x_j denotes $x-$ differentiantion and x_j, $j = 1, ..., N$ are roots of $\Psi_N(x)$. Chelyshkov cardinal function is built as follow

$$C_j(x) = \frac{P_N(x)}{P_{N,x}(x_j)(x - x_j)} = \frac{\prod_{\substack{i=1 \\ i\neq j}}^{N}(x - x_i)}{P_{N,x}(x_j)} \tag{8}$$

Any function $h(x)$ can be approximated by Chelyshkov cardinal function as follow

$$h(x) \simeq \sum_{i=1}^{N} h(x_i)C_i(x) = H^T\Psi_N(x), \tag{9}$$

where H and $\Psi_N(x)$ are vectors that respectively given by

$$H = [h(x_1), ..., h(x_N)]^T, \qquad \Psi_N(x) = [C_1(x), ..., C_N(x)]^T,$$

and we can write derivative of $\Psi_N(x)$ as follow

$$\Psi'_N(x) = [C'_1(x), ..., C'_N(x)]^T, \tag{10}$$

the above vector can be expressed as

$$\Psi'_N(x) = D\Psi_N(x), \tag{11}$$

D is $N \times N$ operator derivative matrix. To compute the entries of matrix D we use (9), any function $C'_j(x)$ can be expressed as

$$C'_j(x) \simeq \sum_{i=1}^{N} C'_j(x_i)C_i(x). \tag{12}$$

By using (11) ,(12), we obtain

$$D = \begin{bmatrix} C'_1(x_1) & \cdots & C'_1(x_N) \\ \vdots & \vdots & \vdots \\ C'_N(x_1) & \cdots & C'_N(x_N) \end{bmatrix}. \tag{13}$$

Hence, $u(x, t)$ is approximated by

$$u(x, t) \simeq \Psi_N^T(t)U\Psi_N(x), \tag{14}$$

such that U is $N \times N$ unknown matrix.

Theorem 2.1. *Let $L^2[\Omega]$ be a Hilbert space with the inner product $\langle f,g\rangle = \int_\Omega f(x)\,g(x)\,dx$, and $u \in L^2[\Omega]$. Then, we find the unique vector $c_j = [c_{1,j}, c_{2,j}, \cdots, c_{m,j}]^T, \quad j = 1, ..., n$ such that*

$$u(x,t) \approx \sum_{i,j=0}^{m,n} c_{i,j}\, C_i(x) C_j(t) = \Psi_N^T(t) U \Psi_N(x). \tag{15}$$

Proof. We improve theorems of [6] and [5]. □

3 Description of Numerical Method and Experimental Results

Consider the Fokker-Plank equation (3) that we can write as

$$\frac{\partial u}{\partial t} = -u\frac{\partial}{\partial x}A - A\frac{\partial u}{\partial x} + u\frac{\partial^2}{\partial x^2}B + 2\frac{\partial u}{\partial x}\frac{\partial}{\partial x}B + B\frac{\partial^2 u}{\partial x^2}. \tag{16}$$

Where $A := A(x,t,u)$ and $B := B(x,t,u)$.

$$\frac{\partial}{\partial x}A(x,t,u) = \frac{\partial A}{\partial x} + \frac{\partial A}{\partial u}.\frac{\partial u}{\partial x} \tag{17}$$

$$\frac{\partial}{\partial x}B(x,t,u) = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial u}.\frac{\partial u}{\partial x} \tag{18}$$

$$\frac{\partial^2}{\partial x^2}B = \frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial x \partial u}.\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\frac{\partial B}{\partial u} + \frac{\partial^2 B}{\partial u^2}(\frac{\partial u}{\partial x})^2 + \frac{\partial^2 B}{\partial u \partial x}.\frac{\partial u}{\partial x} \tag{19}$$

By using (11) , (14) we have:

$$\frac{\partial u}{\partial t} = \Psi_M^T(t) D^T U \Psi_M(x), \quad \frac{\partial u}{\partial x} = \Psi_M^T(t) U D \Psi_M(x), \quad \frac{\partial^2 u}{\partial x^2} = \Psi_M^T(t) U D^2 \Psi_M(x). \tag{20}$$

Replacing (17)-(20) into (16) and then by collocating the obtained equation in $N \times (N-1)$ points (x_i, t_j) , $i = 1, ..., N$, $j = 1, ..., N-1$ on $[0,1] \times [0,1]$ we obtain a system with $N-1$ equation and N unknowns. To obtain N other equations, we use initial condition as

$$\Psi_N^T(0) U D \Psi_M(x_i) = f(x_i),\ i = 1, ..., N.$$

So we get a system of $N \times N$ equations an unknown that be solved respect to $u_{i,j}$.

Example 1. Consider equation (1) with boundary condition (2) where $f(x) = x, \qquad x \in [0,1]$, $A(x) = x$ and $B(x) = \frac{x^2}{2}$. We observe that the exact solution is $u(x,t) = x\exp(t)$.

Example 2. Consider equations (3), (2) with $f(x) = x^2, \qquad x \in [0,1]$. $A(x,t,u) = 4u/x - x/3$

Table 1: Absolute error of the method for example 1 with $N = 3$

	x=0.1	x=0.2	x=0.3	x=0.4	x=0.5	x=0.6
$t = 0.1$	1.5e-18	1.2e-18	2.0e-18	1.5e-18	2.3e-18	2.0e-18
$t = 0.2$	2.9e-18	2.4e-18	3.9e-18	3.0e-18	4.9e-18	3.9e-18
$t = 0.3$	4.4e-18	3.6e-18	5.9e-18	5.3e-18	1.3e-17	2.1e-17

and $B(x,t,u) = u$, the exact solution is $u(x,t) = x^2 exp(t)$.

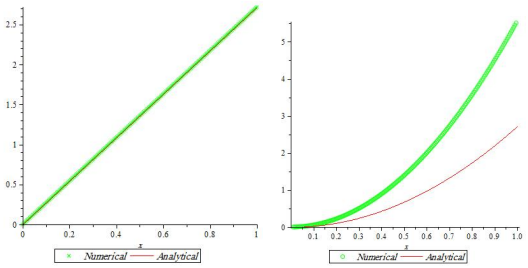

Figure 1:
(left) Comparison of numerical solution and analytical solution for $u(x,1)$ of Example 1 with $N=3$.
(right) Comparison of numerical solution and analytical solution for $u(x,1)$ of Example 2 with $N=3$.

Table 2: Absolute error of the method for example 2 with $N=3$

	x=0	x=0.1	x=0.2	x=0.3	x=0.4	x=0.5	x=0.6
$t=0$	1.4e-17	1.3e-15	6.9e-16	4.2e-15	2.8e-15	1.8e-14	2.1e-14
$t=0.1$	2.3e-18	13e-3	12e-3	13e-3	16e-3	17e-3	17e-3
$t=0.2$	9.7e-19	51e-3	47e-3	51e-3	62e-3	69e-3	69e-3
$t=0.3$	6.7e-19	11e-2	11e-2	11e-2	14e-2	15e-2	16e-2

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A new four-step explicit method with vanished phase-lag and its derivatives for the numerical solution of radial Schrödinger equation

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Abstract

In this paper, we present a new method for the numerical solution of the time-independent Schrödinger equation for one spatial dimension and related problems. A technique, based on the phase-lag and its derivatives, is used, in order to calculate the parameters of the new Numerov-type algorithm. We illustrate the accuracy and computational efficiency of the new developed method via numerical examples.

Keywords: Multistep methods, Oscillating solution, Phase-lag, Initial value problems, Schrödinger equation
Mathematics Subject Classification [2010]: 65L05, 65L06

1 Introduction

The radial time- independent Shorödinger equation can be written as

$$y''(x) = \left(\frac{l(l+1)}{x^2} + V(x) - E\right) y(x), \tag{1}$$

where $\frac{l(l+1)}{x^2}$ is the centrifugal potential, $V(x)$ is the potential, E is the energy and $W(x) = \frac{l(l+1)}{x^2} + V(x)$ is the effective potential. It is valid that $\lim_{x\to\infty} V(x) = 0$ and therefore $\lim_{x\to\infty} W(x) = 0$. We consider $E > 0$ and divide $[0, \infty)$ into subintervals $[a_i, b_i)$ so that $W(x)$ is a constant with value $\overline{W}$. After this the problem (1) can be expressed by the approximation: $y_i'' = (\overline{W} - E)y_i$, whose theoretical solution is $y_i = A_i \exp(\sqrt{\overline{W} - E}x) + B_i \exp(\sqrt{\overline{W} - E}\,x)$, where $A_i,\ B_i \in \mathbb{R}$. Many numerical methods have been developed for the efficient solution of the Schrödinger equation and related problems [1 - 5].

2 Phase-lag analysis of symmetric multistep methods

For the numerical solution of the initial value problem

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{2}$$

*Speaker

the multistep methods of the form

$$\sum_{i=0}^{m} c_i q_{n+i} = h^2 \sum_{i=0}^{m} b_i f(x_{n+i}, q_{n+i}), \tag{3}$$

with m steps can be used over the equally spaced intervals $\{x_i\}_{i=0}^{m} \in [a,b]$ and $h := |x_{i+1} - x_i|$, $i = 0\,(1)\,m-1$. If the method is symmetric then $c_i = c_{m-i}$ and $b_i = b_{m-i}$, $i = 0\,(1)\,\lfloor \frac{m}{2} \rfloor$. Method (3) is associated with the operator $L(x) = \sum_{i=0}^{m} c_i\, u(x+ih) - h^2 \sum_{i=0}^{m} b_i\, u''(x+ih)$, where $u \in \mathbb{C}^2$.

Definition 2.1. The multistep method (3) is called algebraic of order p if the associated linear operator L vanishes for any linear combination of the linearly independent functions $1, x, x^2, ..., x^{p+1}$.

When a symmetric $2k$-step method, that is for $i = -k\,(1)\,k$, is applied to the scalar test equation $y'' = -\omega^2 y$, a difference equation of the form

$$\sum_{i=1}^{k} A_i(\nu)\,(y_{n+i} + y_{n-i}) + A_0(\nu) y_n = 0, \tag{4}$$

is obtained, where $\nu = \omega h, h$ is the step length and $A_0(\nu)$, $A_1(\nu)$, $\cdots$, $A_k(\nu)$ are polynomials of ν. The characteristic equation associated with (4) is

$$\sum_{i=1}^{k} A_i(\nu)\left(\lambda^i + \lambda^{-i}\right) + A_0(\nu) = 0. \tag{5}$$

From Lambert and Watson [15] we have the following definitions.

Definition 2.2. A symmetric $2k$-step method with characteristic equation given by (5) is said to have an interval of periodicity $(0, \nu_0^2)$ if, for all $\nu \in (0, \nu_0^2)$, the roots s_i , $i = 1(1)2m$ of Eq. (5) satisfy $\lambda_1 = e^{i\,\theta(\nu)}$, $\lambda_2 = e^{-i\,\theta(\nu)}$, and $|\lambda_i| \leqslant 1$, $i = 3(1)2m$, where $\theta(\nu)$ is a real function of ν.

Definition 2.3. For any method corresponding to the characteristic equation (5) the phase-lag is defined as the leading term in the expansion of $t = \nu - \theta(\nu)$. Then if the quantity $t = O(\nu^{r+1})$ as $\nu \to \infty$, the order of phase-lag is r.

Theorem 2.4. *The symmetric $2k$-step method with characteristic equation given by (5) has phase-lag order r and phase-lag constant c given by $-c\nu^{r+2} + O(\nu^{r+4}) = \dfrac{D_1}{D_2}$ where $D_1 = \sum_{i=1}^{k} 2A_i(\nu)\cos(i\nu) + A_0(\nu)$ and $D_2 = \sum_{i=1}^{k} 2k^2 A_k(\nu)$.*

Proof. See [4]. □

3 Introduction

We write the explicit symmetric $2\,m$-step method as

$$\begin{aligned} & q_{n+m} + \Sigma_{i=0}^{m-1} c_i (q_{n+i} + q_{n-i}) + q_{n-m} \\ & = h^2 \Sigma_{i=1}^{m-1} b_i [f(x_{n+i}, q_{n+i}) + f(x_{n-i}, q_{n-i})] + b_0 f(x_n, q_n) \end{aligned} \tag{6}$$

From the form (6) with$m = 2$ we get the following form of the explicit symmetric four-step methods:

$$q_{n+2} + c_1(q_{n+1} + q_{n-1}) + c_0 q_n + q_{n-2} = h^2[b_1(f_{n+1} + f_{n-1}) + b_0 f_n] \tag{7}$$

Where $f_i = y''(x_i, q_i), i = n - 1(1)n + 1$. Considering (7), We choose $c_1 = -\frac{1}{10}$. The free parameter c_1 is based on the paper. In this study it has been proved that the above value of c_1 gives for the method (7) the higher accuracy. Requesting the above method to have the phase-lag and its first and second derivatives vanished. The phase-lag of this method is $PL = \frac{T_1}{\frac{39}{5} + 2\upsilon^2 b_1}$. By solving the system of equations $PL = 0$, $PL' = 0$ and $PL'' = 0$, we obtain the coefficients of the new proposed method:

$$b_0 = \frac{T_1}{10cos(\upsilon)\upsilon^3 + 30\upsilon^2 sin(\upsilon)}, \quad b_1 = \frac{T_2}{10cos(\upsilon)\upsilon^3 + 30\upsilon^2 sin(\upsilon)},$$

$$c_0 = \frac{T_3}{10\upsilon cos(\upsilon) + 30sin(\upsilon)},$$

where

$$T_1 = 30\upsilon^2 \sin(\upsilon) - 10\upsilon^2 \sin(3\upsilon) + \upsilon \cos(2\upsilon) + 80\upsilon \cos(\upsilon) - 20 \sin(\upsilon) + \sin(2\upsilon) - 20 \sin(3\upsilon) - 3\upsilon,$$

$$T_2 = -40\upsilon \cos(2\upsilon) + \upsilon \cos(\upsilon) + 20 \sin(2\upsilon) - \sin(\upsilon),$$

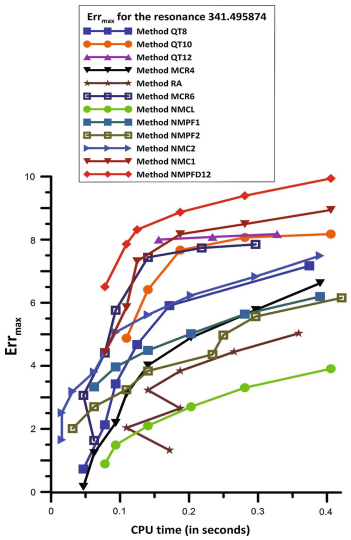
$$\begin{aligned} T_3 \;=\; & -30\upsilon^2 \sin(\upsilon) + 10\upsilon^2 \sin(3\upsilon) - \upsilon \cos(2\upsilon) - 50\upsilon \cos(\upsilon) + 30\upsilon \cos(3\upsilon) + 30 \sin(\upsilon) \\ & + 3 \sin(2\upsilon) - 30 \sin(3\upsilon) + 3\upsilon \end{aligned}$$

The new obtained method with the coefficients given by$(10) - (11)$ has a local truncation error which is given by: $LTE = \frac{161h^6}{2400} \left(q_n^{(6)} + 3\omega^2 q_n^{(4)} + 3\omega^4 q_n^{(2)} + \omega^6 q_n \right) + o(h^8)$.

4 Numerical resales

In order to apply the new methods to the radial Schrödinger equation the value of parameter υ is needed. For every problem of the radial Schrödinger equation given by (1.1), the parameter υ is given by $\upsilon = \sqrt{|V(x) - E|}$, where $V(x)$ is the potential and E is the energy. In our example the well known Woods-Saxon potential given by $V(x) = \frac{u_0}{1+z} - \frac{u_0 z}{a(1+z)^2}$ is used, with $z = \exp[\frac{(x-X_0)}{a}]$, $u_0 = -50$, $a = 0:6$, and $X_0 = 7:0$. The boundary conditions for this problem are $y(0) = 0$, $y(x) = \cos(\sqrt{E}x)$ for large x. We compute the approximate positive eigenenergies of the Woods Saxon resonance problem using:

The eighth order multi-step method developed by Quinlan and Tremaine [4], which is indicated as Method QT8. The tenth order multi-step method developed by Quinlan



and Tremaine [4], which is indicated as Method QT10. The twelfth order multi-step method developed by Quinlan and Tremaine [4], which is indicated as Method QT12. The fourth algebraic order method of Chawla and Rao with minimal phase-lag [2], which is indicated as Method MCR4. The exponentially-fitted method of Raptis and Allison, which is indicated as Method MRA. The hybrid sixth algebraic order method developed by Chawla and Raowith minimal phase-lag, which is indicated as Method MCR6. The classical form of the fourth algebraic order four-step method, which is indicated as Method NMCL2. The Phase-Fitted Method (Case 1), which is indicated as Method NMPF1. The phase-fitted method (Case 2), which is indicated as Method NMPF2. The four-step method with vanished phase-lag and its first derivative (Case 2), which is indicated as Method NMC2. The four-step method with vanished phase-lag and its first derivative (Case 1), which is indicated as Method NMC1. The new obtained method, which is indicated as Method NMPFD12.

In the following figure, we present the maximum absolute error $Err_{max} = |log_{10}(Err)|$ where $Err = |E_{calculated} - E_{accurate}|$ of the eigenenergies $E = 341.495874$, for several values of CPU time (in seconds). We note that the CPU time (in seconds) counts the computational cost for each method.

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A new method for error analysis in generalized Volterra integral equations in L^p space

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Abstract

In this paper, with a new simple proof, an important inequality for a contraction integral equation is obtained. From a practical programming point of view, this inequality allows to express iterative algorithm with a "for loop" rather than a "while loop". The main tool used in this paper is the fixed point theorem in the Lebesgue space.

Keywords: Integral operator; Successive approximation method; Approximation error.
Mathematics Subject Classification [2010]: 34A12, 65R10, 65R20.

1 Introduction

The solutions of integral equations play a major role in the fields of science and engineering. Usually, physical events are modeled by a differential equation, an integral or an integro-differential equation, or a system of these. Since few of these equations can not be solved explicitly, it is often necessary to resort to numerical techniques [3]. There are several numerical methods for solving integral equations, such as the method of Galerkin, Collocation, Taylor series, Legendre wavelets, Jacobi polynomials, homotopy perturbation, expansion, and recently, Chebyshev polynomials. On the other hand, investigations on existence theorems for diverse functional-integral equations have been presented in other references such as [1, 2, 4]. It seems that the presented method used in our study is the best stopping rule for iterative algorithm in integral equation comparison with other researches.

At the first we need to some preliminaries, and so in the next Section, by using the weighted norm method, a contraction mapping is obtained. Thereafter at the continue, by a simple technique, the stopping rule for our iterative algorithm has been introduced. Finally, we report numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical example. In this paper, we intend to prove the existence and uniqueness of solutions of the nonhomogeneous nonlinear

*Speaker

Volterra integral equation

$$x(t) = f(t) + \varphi\Big(\int_0^t F(t,s,x(s))ds\Big). \tag{1}$$

Here,
(i) $f \in L^p(I,\mathbb{R})$ for $I := \{t \in \mathbb{R} : 0 \le t \le 1\}$ and $p > 1$,
(ii) $F : T \times \mathbb{R} \to \mathbb{R}$ is measurable, where $T := \{(t,s) \in I \times I : t \le s\}$.
We further assume that:
(iii) the function $t \mapsto \int_0^t F(t,s,f(s))ds$ belongs to $L^p(I,\mathbb{R})$;
(iv) $|F(t,s,x) - F(t,s,y)| \le L(t,s)|x-y|$, $x,y \in \mathbb{R}, (t,s) \in T$, where L is a nonnegative and measurable function for which

$$M(t) := \Big(\int_0^t L^q(t,s)ds\Big)^{\frac{p}{q}}, t \in I, \frac{1}{p} + \frac{1}{q} = 1$$

exists and is integrable over I.
(v) φ is Lipschitz, that is, there exists $\alpha > 0$ such that for all $x,y \in \mathbb{R}$, $|\varphi(x) - \varphi(y)| \le \alpha|x-y|$.

We extend the Volterra integral equation and discuss it's solutions in L^p spaces. To this end, we use the weighted norm method instead of the successive approximation method. In the reminder of this section, we recall some basic results which we will need in this paper. Let $\omega : I \to \mathbb{R}_+, \mathbb{R}_+ = (0,+\infty)$, be a continuous function.
Put

$$\|u\|_{p,\omega} = \Big(\sup\Big\{\omega^{-1}(x)\int_0^x |u(s)|^p ds; x \in I\Big\}\Big)^{\frac{1}{p}}. \tag{2}$$

Note that for $\omega \equiv 1$ we obtain the classical norm $\|u\|_p$ which makes $L^p(I,\mathbb{R})$ into a Banach space. In general, it is easy to see that (2) defines a norm for any ω. Indeed, multiplying the Minkowski inequality by $\Big(\omega(x)\Big)^{\frac{-1}{p}}$ we obtain

$$\Big(\omega^{-1}(x)\int_0^x |u(s)+v(s)|^p ds\Big)^{\frac{1}{p}} \le \Big(\omega^{-1}(x)\int_0^x |u(s)|^p ds\Big)^{\frac{1}{p}} + \Big(\omega^{-1}(x)\int_0^x |v(s)|^p ds\Big)^{\frac{1}{p}}.$$

2 The Main Result

In this section, we prove that $\mathcal{F}$ defined by the right hand side of equation (1), is a contraction with $\|.\|_{p,\omega_\lambda}$, when λ is sufficiently large and ω_λ is defined as equation (3), and so give the main results.

Theorem 2.1. *Under the assumed conditions, the operator $\mathcal{F}$ defined by the right hand side of equation (1) is a contraction in $L^p(I,\mathbb{R})$ with respect to the norm $\|.\|_{p,\omega_\lambda}$, when λ is sufficiently large and ω_λ is defined by*

$$\omega_\lambda(x) = exp\Big(\lambda\int_0^x M(s)ds\Big), M(s) = \Big(\int_0^s L^q(s,t)dt\Big)^{\frac{p}{q}}, \lambda > 1. \tag{3}$$

The equation (1) has a unique solution $u^* \in L^p(I,\mathbb{R})$, which is the limit in $L^p(I,\mathbb{R})$ of the sequence of iteration $\{\mathcal{F}^n u_0\}$, for any u_0 in $L^p(I,\mathbb{R})$.

Let $p > 1$ be arbitrary. Suppose λ is the smallest positive integer number for which $\mathcal{F}$ is a contraction with respect to $\|.\|_{p,\omega_\lambda}$.
Now, by using the above theorem for $m \geq 1$, we have:

$$\|\mathcal{F}^m u_1 - \mathcal{F}^m u_2\|_{p,\omega_\lambda} \leq K_\lambda^m . \|u_1 - u_2\|_{p,\omega_\lambda}.$$

By the triangle inequality, we have

$$\|u_1 - u_2\|_{p,\omega_\lambda} \leq \|u_1 - \mathcal{F}u_1\|_{p,\omega_\lambda} + \|\mathcal{F}u_1 - \mathcal{F}u_2\|_{p,\omega_\lambda} + \|\mathcal{F}u_2 - u_2\|_{p,\omega_\lambda}.$$

Thus

$$\|u_1 - u_2\|_{p,\omega_\lambda} \leq \frac{1}{1-K_\lambda}\Big(\|u_1 - \mathcal{F}u_1\|_{p,\omega_\lambda} + \|u_2 - \mathcal{F}u_2\|_{p,\omega_\lambda}\Big). \tag{4}$$

In particular, if u_1 and u_2 are the fixed points of $\mathcal{F}$, we get $\|u_1 - u_2\|_{p,\omega_\lambda} = 0$. This shows that the contraction mapping $\mathcal{F}$ has at most one fixed point. For any $u \in (L^p(I,\mathbb{R}), \|.\|_{p,\omega_\lambda})$, by letting $u_1 = \mathcal{F}^n u$ and $u_2 = \mathcal{F}^m u$ in (4) we find that

$$\begin{aligned}\|\mathcal{F}^n u - \mathcal{F}^m u\|_{p,\omega_\lambda} &\leq \frac{1}{1-K_\lambda}\Big(\|\mathcal{F}^n u - \mathcal{F}^n(\mathcal{F}u)\|_{p,\omega_\lambda} + \|\mathcal{F}^m u - \mathcal{F}^m(\mathcal{F}u)\|_{p,\omega_\lambda}\Big)\\ &\leq \frac{K_\lambda^n + K_\lambda^m}{1-K_\lambda}\|\mathcal{F}u - u\|_{p,\omega_\lambda},\end{aligned}$$

and since $K_\lambda < 1$, $K_\lambda^n \to 0$ as n tends to infinity. So $\|\mathcal{F}^n u - \mathcal{F}^m u\|_{p,\omega_\lambda} \to 0$ as n and m tend to infinity. Since $(L^p(I,\mathbb{R}), \|.\|_{p,\omega_\lambda})$ is a Banach space, this Cauchy sequence converges to some $u^* \in L^p(I,\mathbb{R})$, and this u^* is clearly a fixed point of $\mathcal{F}$.
The stopping rule : Now letting m tend to infinity in the last inequality, the following important inequality is obtained

$$\|\mathcal{F}^n u - u^*\|_{p,\omega_\lambda} \leq \frac{K_\lambda^n}{1-K_\lambda}\|\mathcal{F}u - u\|_{p,\omega_\lambda}. \tag{5}$$

Now, let us explain the importance of the inequality (5). Suppose we are willing to accept an error of ϵ, i.e., instead of the actual fixed point u^* of $\mathcal{F}$, we will be satisfied with a point $\mathcal{F}^n u$ satisfying $\|\mathcal{F}^n u - u^*\|_{p,\omega_\lambda} < \epsilon$, and suppose also that we start our iteration at some point u_0 in $L^p(I,\mathbb{R})$. Since we want $\|\mathcal{F}^n u_0 - u^*\|_{p,\omega_\lambda} < \epsilon$, we just have to pick N_λ so large that $\frac{K_\lambda^{N_\lambda}}{1-K_\lambda}\|\mathcal{F}u_0 - u_0\|_{p,\omega_\lambda} < \epsilon$. Now the quantity $\|\mathcal{F}u_0 - u_0\|_{p,\omega_\lambda}$ is something that we can compute after the first iteration and we can then compute how large N_λ has to be by taking the log of the above inequality and solving for N_λ (remembering that $\log(K_\lambda)$ is negative). The result is that if $\beta_\lambda := \|\mathcal{F}u_0 - u_0\|_{p,\omega_\lambda}$ and

$$N_\lambda > \frac{\log(\epsilon) + \log(1-K_\lambda) - \log(\beta_\lambda)}{\log(K_\lambda)},$$

then $\|\mathcal{F}^{N_\lambda} u_0 - u^*\|_{p,\omega_\lambda} < \epsilon$. From a practical programming point of view, this inequality allows us to express our iterative algorithm with a "for loop" rather than a "while loop", but it has another interesting interpretation. Suppose we take $\epsilon = 10^{-m}$ in our stopping

rule inequality. What we see is that the growth of N_λ with m is a constant plus $\frac{m}{|\log(K_\lambda)|}$, or in other words, to get one more decimal digit of precision we have to do (approximately) $\frac{1}{|\log(K_\lambda)|}$ more iteration steps. Stated a little differently, if we need N_λ iterative steps to get m decimal digits of precision, then we need another N_λ to double the precision to $2m$ digits.

Example 2.2. Consider the following linear Volterra integral equation

$$u(t) = f(t) - \int_0^t \sin(2(t-s))u(s)ds, \quad t \in I. \tag{6}$$

In this integral equation, the exact solution is

$$u(t) = f(t) - \frac{2}{\sqrt{6}} \int_0^t \sin(\sqrt{6}(t-s))f(s)ds.$$

In particular, for $f(t) = \cos(t)$, this solution becomes $u^*(t) = 0.6\cos(t)+0.4\cos(\sqrt{6}t)$. Now by taking $\epsilon = 10^{-m}$, we guess that after N_λ iterative steps, m decimal digits of precision must be obtained. In Table 1, for some initial guesses u_0, the value of the parameters are calculated.

U_0	ϵ	p	q	λ	K_λ	β_λ	N_λ	$\|U^* - U_{N\lambda}\|_{p,\omega_\lambda}$
$\cos(t)$	10^{-6}	2	2	50	0.1414	0.0050	5	$2.5003e-010$
t	10^{-4}	2	2	40	0.1581	0.0739	4	$3.3858e-006$
1	10^{-8}	2	2	30	0.1825	0.0939	10	$6.4212e-017$

Table 1: Numerical results for Example 2.2

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A smooth approximation for numerical solution of nonlinear Schrödinger equations

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Abstract

We present a numerical method based on exponential splines for solving the nonlinear Schrödinger equations with variable coefficients. The error analysis, stability and convergence properties of the method are investigated. The efficiency of the method is demonstrated by test problems. The numerical simulations validate and demonstrate the advantages of the method.

Keywords: Nonlinear Schrödinger equation, Exponential spline, Convergence
Mathematics Subject Classification [2010]: 65L10, 65M06, 65M12

1 Introduction

We consider the following nonlinear Schrödinger

$$\imath\frac{\partial u}{\partial t}+\alpha(t)\frac{\partial^2 u}{\partial x^2}+F(x,t)u+\beta(t)|u|^2u=0,\quad a<x<b,\quad 0<t\leq T, \tag{1}$$

with the boundary conditions

$$u(a,t)=f_0(t),\quad u(b,t)=f_1(t),\quad 0<t\leq T, \tag{2}$$

and the initial condition

$$u(x,0)=\phi(x),\quad x\in[a,b], \tag{3}$$

where $\alpha(t)$, $F(x,t)$ and $\beta(t)$ are bounded real functions and also $u(x,t)$ is the complex-valued wave function and $\alpha(t)$ is related to the second order dispersion coefficient.This equation is one of the most universal models that describes many physical nonlinear systems. This problem has been studied by several authors such as [1, 2].

The purpose of this paper is to give a numerical method, based on a uniform mesh using exponential splines for the nonlinear Schrödinger equation, which leads to the recurrence relation. We give the truncation error of the method, convergence and stability analysis. The analysis will be illustrated by investigating some examples. The numerical simulations validate and demonstrate the advantages of the method.

*Speaker

2 Exponential spline functions

We set up a grid in the x, t plane with grid points (x_i, t_j) and uniform grid spacing h and k, where $x_i = a + ih, h = (b-a)/N, i = 0, 1, 2, ..., N$, and $t_j = jk, k = T/M, j = 0, 1, 2, ..., M$.

A function $S(x,t)$ of class $C^4[a,b]$ which interpolates $u(x,t)$ at the mesh points (x_i, t_j), depends on a parameter τ, reduces to quintic spline $S(x,t)$, in $[a,b]$ as $\tau \to 0$, is termed an exponential spline function. For each segment $[x_i, x_{i+1}], i = 0, ..., N-1$ the function $S(x,t)$, can be defined in the following form

$$\begin{gathered} S(x,t) = a_i^*(t_j) + b_i^*(t_j)(x-x_i) + c_i^*(t_j)(x-x_i)^2 + d_i^*(t_j)(x-x_i)^3 + \\ e_i^*(t_j)e^{\imath\tau(x-x_i)} + f_i^*(t_j)e^{-\imath\tau(x-x_i)}, \quad i = 0, ..., N, \end{gathered} \tag{4}$$

where $a_i^*(t_j)$, $b_i^*(t_j)$, $c_i^*(t_j)$, $d_i^*(t_j)$, $e_i^*(t_j)$ and $f_i^*(t_j)$ are unknown coefficients, τ is a free parameter and $\imath = \sqrt{-1}$. We first develop the explicit expressions for the six coefficients in (4) in terms of u_i^j, u_{i+1}^j, M_i^j, M_{i+1}^j, Z_i^j and Z_{i+1}^j, where:

$$\begin{array}{lll} S(x_i,t_j) = u_i^j, & S''(x_i,t_j) = M_i^j, & S^{(4)}(x_i,t_j) = Z_i^j, \\ S(x_{i+1},t_j) = u_{i+1}^j, & S''(x_{i+1},t_j) = M_{i+1}^j, & S^{(4)}(x_{i+1},t_j) = Z_{i+1}^j. \end{array} \tag{5}$$

Now using (5), we can determine the four unknown coefficients in (4). By using the continuity of the first and third derivatives of $S(x,t_j)$ at $x = x_i$ and the above expressions for coefficients, we obtain the following useful relation

$$\begin{gathered} \frac{-3h^2\tau^2(h\tau\csc h\tau - 1)}{-6+2h^2\tau^2+6h\tau\cot h\tau}(u_{i-2}^j + u_{i+2}^j) + \frac{3h^2\tau^2(h\tau\cot(h\tau/2)-2)}{-3+h^2\tau^2+3h\tau\cot h\tau}(u_{i-1}^j + u_{i+1}^j) + \\ \frac{-3h^2\tau^2(h\tau(\csc h\tau + 2\cot h\tau)-3)}{-3+h^2\tau^2+3h\tau\cot h\tau}u_i^j = h^2\{\frac{(6+h\tau(h^2\tau^2-6)\csc h\tau)}{4(3-h^2\tau^2-3h\tau\cot h\tau}(M_{i-2}^j + M_{i+2}^j) + \\ \frac{(-12+h\tau(6-h^2\tau^2)\cot h\tau + 2h\tau(3+h^2\tau^2)\csc h\tau)}{6-2h^2\tau^2-6h\tau\cot h\tau}(M_{i-1}^j + M_{i+1}^j) + \\ \frac{(18-4h\tau(3+h^2\tau^2)\cot h\tau + h\tau(-6+h^2\tau^2)\csc h\tau)}{6-2h^2\tau^2-6h\tau\cot h\tau}M_i^j, \quad i = 2, ..., N-2. \end{gathered} \tag{6}$$

In the limit for τ going to 0, (6) reduces to:

$$\begin{gathered} u_{i-2}^j + 2u_{i-1}^j - 6u_i^j + 2u_{i+1}^j + u_{i+2}^j = \\ \frac{h^2}{20}(M_{i-2}^j + 26M_{i-1}^j + 66M_i^j + 26M_{i+1}^j + M_{i+2}^j), \quad i = 2, ..., N-2, \end{gathered} \tag{7}$$

and the truncation error can be written as

$$t_i = \frac{1}{32}h^6\frac{\partial^6 u_i^j}{\partial x^6} + \frac{1}{896}h^8\frac{\partial^8 u^j}{\partial x^8}(\xi), \quad i = 2, ..., N-2, \tag{8}$$

whereby $x_{i-2} < \xi < x_{i+2}$. Eq. (7) gives $N-3$ linear algebraic equations in the unknowns $u_i^j, i = 1, ..., N-1$, since u_0^j and u_N^j are known from the boundary conditions. The two missing equations, one at each end of the range of integration, can be derived as follows:

$$\begin{gathered} 4u_0^j - 7u_1^j + 2u_2^j + u_3^j = \frac{h^2}{12}(4M_0^j + 41M_1^j + 14M_2^j + M_3^j) - \frac{1}{48}h^6\frac{\partial^6 u_0^j}{\partial x^6} + \mathcal{O}(h^8), \\ u_{N-3}^j + 2u_{N-2}^j - 7u_{N-1}^j + 4u_N^j = \frac{h^2}{12}(M_{N-3}^j - 14M_{N-2}^j + 41M_{N-1}^j + 4M_N^j) - \\ \frac{1}{48}h^6\frac{\partial^6 u_N^j}{\partial x^6} + \mathcal{O}(h^8). \end{gathered} \tag{9}$$

3 The Numerical method

At first, we discrete the problem in time variable by means of the θ-finite difference method, $\theta \in [\frac{1}{2}, 1]$. In this case, we get a system of ordinary differential equations with boundary conditions. Discretization by the proposed method yields the following system of differential equations:

$$\begin{gathered} \imath u^{j+1}(x) + k\theta(\alpha^{j+1}u_{xx}^{j+1}(x) + \beta^{j+1}|u^{j+1}(x)|^2 u^{j+1}(x) + F(x,t_{j+1})u^{j+1}(x)) = \\ G(x,t_j), \\ G(x,t_j) = \imath u^j(x) - k(1-\theta)(\alpha^j u_{xx}^j(x) + \beta^j|u^j(x)|^2 u^j(x) + F(x,t_j)u^j(x)), \\ j = 0, ..., M-1, \end{gathered} \tag{10}$$

with

$$\begin{aligned} &u^0 = \phi(x), \\ &u(a,t_{j+1}) = f_0(t_{j+1}), \quad u(b,t_{j+1}) = f_1(t_{j+1}). \end{aligned} \tag{11}$$

At the grid point (x_i, t_j), the proposed differential Eq. (10) may be discretized by:

$$\imath u_i^{j+1} + k\theta(\alpha^{j+1}M_i^{j+1} + \beta^{j+1}|u_i^{j+1}|^2 u_i^{j+1} + F_i^{j+1}u_i^{j+1}) = G_i^j, \tag{12}$$

where $M_i^{j+1} = S''(x_i, t_{j+1}) \simeq u_{xxi}^{j+1}$. From (12) we have

$$M_i^{j+1} = \frac{1}{k\theta\alpha^{j+1}}(G_i^j - \imath u_i^{j+1} - k\theta\beta^{j+1}|u_i^{j+1}|^2 u_i^{j+1} - k\theta F_i^{j+1}u_i^{j+1}). \tag{13}$$

Substituting $M_{i\pm2}^{j+1}$, $M_{i\pm1}^{j+1}$ and M_i^{j+1} into (7) and (9) we obtain the following system:

$$\mathcal{A}u^{j+1} + \frac{h^2\beta^{j+1}}{\alpha^{j+1}}Q(u^{j+1}) = \frac{h^2}{k\theta\alpha^{j+1}}G + \mathcal{T}, \tag{14}$$

with $u^{j+1} = [u_1^{j+1}, u_2^{j+1}, ..., u_{N-1}^{j+1}]^T$, $\mathcal{T} = [t_1^{j+1}, t_2^{j+1}, ..., t_{N-1}^{j+1}]^T$, $\mathcal{A} = \mathcal{A}_0 + \frac{h^2}{k\theta\alpha^{j+1}}\mathcal{A}_1$ and

$$\mathcal{A}_0 = \begin{pmatrix} -7 & 2 & 1 & & & \\ 2 & -6 & 2 & 1 & & \\ 1 & 2 & -6 & 2 & 1 & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & 1 & 2 & -6 & 2 & 1 \\ & & 1 & 2 & -6 & 2 \\ & & & 1 & 2 & -7 \end{pmatrix},$$

$$G = \begin{pmatrix} \frac{4G_0^j+41G_1^j+14G_2^j+G_3^j+4(-\imath-k\theta\beta^{j+1}|f_0(t_{j+1})|^2-k\theta F_0^{j+1})f_0(t_{j+1})}{12} \\ \frac{G_0^j+26G_1^j+66G_2^j+26G_3^j+G_4^j+(-\imath-k\theta\beta^{j+1}|f_0(t_{j+1})|^2-k\theta F_0^{j+1})f_0(t_{j+1})}{20} \\ \vdots \\ \frac{G_{i-2}^j+26G_{i-1}^j+66G_i^j+26G_{i+1}^j+G_{i+2}^j}{20} \\ \vdots \\ \frac{G_{N-4}^j+26G_{N-3}^j+66G_{N-2}^j+26G_{N-1}^j+G_4^j+(-\imath-k\theta\beta^{j+1}|f_1(t_{j+1})|^2-k\theta F_N^{j+1})f_1(t_{j+1})}{20} \\ \frac{G_{N-3}^j+14G_{N-2}^j+41G_{N-1}^j+4G_N^j+4(-\imath-k\theta\beta^{j+1}|f_1(t_{j+1})|^2-k\theta F_N^{j+1})f_1(t_{j+1})}{12} \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} \frac{41(\imath+k\theta F_1^{j+1})}{12} & \frac{7(\imath+k\theta F_2^{j+1})}{6} & \frac{(\imath+k\theta F_3^{j+1})}{12} & & & \\ \frac{13(\imath+k\theta F_1^{j+1})}{10} & \frac{33(\imath+k\theta F_2^{j+1})}{10} & \frac{13(\imath+k\theta F_3^{j+1})}{10} & \frac{(\imath+k\theta F_4^{j+1})}{20} & & \\ \frac{(\imath+k\theta F_1^{j+1})}{20} & \frac{13(\imath+k\theta F_2^{j+1})}{10} & \frac{33(\imath+k\theta F_3^{j+1})}{10} & \frac{13(\imath+k\theta F_4^{j+1})}{10} & \frac{(\imath+k\theta F_5^{j+1})}{20} & \\ & \ddots & \ddots & \ddots & \ddots & \\ & \frac{(\imath+k\theta F_{N-5}^{j+1})}{20} & \frac{13(\imath+k\theta F_{N-4}^{j+1})}{10} & \frac{33(\imath+k\theta F_{N-3}^{j+1})}{10} & \frac{13(\imath+k\theta F_{N-2}^{j+1})}{10} & \frac{(\imath+k\theta F_{N-1}^{j+1})}{20} \\ & & \frac{(\imath+k\theta F_{N-4}^{j+1})}{20} & \frac{13(\imath+k\theta F_{N-3}^{j+1})}{10} & \frac{33(\imath+k\theta F_{N-2}^{j+1})}{10} & \frac{13(\imath+k\theta F_{N-1}^{j+1})}{10} \\ & & & \frac{(\imath+k\theta F_{N-3}^{j+1})}{12} & \frac{7(\imath+k\theta F_{N-2}^{j+1})}{6} & \frac{41(\imath+k\theta F_{N-1}^{j+1})}{12} \end{pmatrix},$$

and

$$Q(u^{j+1}) = \begin{pmatrix} \frac{1}{12}(41|u_1^{j+1}|^2u_1^{j+1} + 14|u_2^{j+1}|^2u_2^{j+1} + |u_3^{j+1}|^2u_3^{j+1}) \\ \frac{1}{20}(26|u_1^{j+1}|^2u_1^{j+1} + 66|u_2^{j+1}|^2u_2^{j+1} + 26|u_3^{j+1}|^2u_3^{j+1} + |u_4^{j+1}|^2u_4^{j+1}) \\ \vdots \\ \frac{1}{20}(|u_{i-2}^{j+1}|^2u_{i-2}^{j+1} + 26|u_{i-1}^{j+1}|^2u_{i-1}^{j+1} + 66|u_i^{j+1}|^2u_i^{j+1} + 26|u_{i+1}^{j+1}|^2u_{i+1}^{j+1} + |u_{i+2}^{j+1}|^2u_{i+2}^{j+1}) \\ \vdots \\ \frac{1}{20}(|u_{N-4}^{j+1}|^2u_{N-4}^{j+1} + 26|u_{N-3}^{j+1}|^2u_{N-3}^{j+1} + 66|u_{N-2}^{j+1}|^2u_{N-2}^{j+1} + 26|u_{N-1}^{j+1}|^2u_{N-1}^{j+1}) \\ \frac{1}{12}(|u_{N-3}^{j+1}|^2u_{N-3}^{j+1} + 14|u_{N-2}^{j+1}|^2u_{N-2}^{j+1} + 41|u_{N-1}^{j+1}|^2u_{N-1}^{j+1}) \end{pmatrix}.$$

4 Stability and Convergence

Theorem 4.1. *The time semi-discrete method (10)-(11) is unconditionally stable for all values of* $\theta \in [\frac{1}{2}, 1]$.

Theorem 4.2. *The exponential spline approximation* u^{j+1} *converges to the exact solution* U^{j+1} *of the boundary value problem defined by Eqs. (10)-(11) with order three by the* $\|.\|_\infty$ *norm, i.e.,* $\|U^{j+1} - u^{j+1}\|_\infty = \mathcal{O}(h^4)$.

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An implicit finite difference method for solving integro-partial time fractional diffusion equation with weakly singular kernel

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Abstract

In this paper we develop an implicit finite difference method to solve an one-dimensional linear integro-partial time fractional diffusion equation with weakly singular kernel, formulated with Caputos fractional derivative. The numerical test is performed and comparative results are provided to illustrate the usefulness of the proposed method.

Keywords: Implicit finite difference method, Fractional calculus, Numerical methods, Time fractional diffusion equation
Mathematics Subject Classification [2010]: 65N06, 65R10, 35R11

1 Introduction

In this paper we introduced an method for the numerical solution of the following linear integro-partial time fractional diffusion equation with weakly singular kernel:

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = \mu \frac{\partial^2 u}{\partial x^2}(x,t) + \int_0^t (t-s)^{-1/2} \frac{\partial^2 u}{\partial x^2}(x,s)\mathrm{d}s, \qquad x \in [0,1], t \in [0,T], \tag{1}$$

where $\mu \geqslant 0, 0 < \alpha \leqslant 1$ and the unknown real function $u(x,t)$ is sought for $0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T$, with the boundary and initial conditions:

$$u(0,t) = u(1,t) = 0, \qquad t \geqslant 0, \qquad u(x,0) = g(x), \qquad 0 \leqslant x \leqslant 1. \tag{2}$$

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $f(t)$ with respect to point $t = 0$ is defined as:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad \alpha > 0, t > 0, \qquad I^0 f(t) = f(t).$$

Definition 1.2. The Caputo fractional derivative of order α of function$f(t)$ is defined as:

$$D_*^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau, \quad n \in \mathbb{N}, \qquad n-1 < \alpha \leqslant n.$$

*Speaker

Note that for $n-1<\alpha\leqslant n, n\in\mathbb{N}$ and $t>0$, $\beta>0$,

$$I^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{d^{k}f}{dt^{k}}(0^{+})\frac{t^{k}}{k!},\qquad I^{\alpha}I^{\beta}f=I^{\alpha+\beta}f. \tag{3}$$

Properties of the operators I^{α} and D_{*}^{α} can be found in [1].

2 Description of the method

Suppose that we are working on a uniform grid $\{x|x_i=ih, i=0,1,\cdots,M\}$, and $\{t|t_j=jk, j=0,1,\cdots,N\}$, where M, N, are positive integers and $\triangle x=h=\frac{1}{M}$, $\triangle t=k=\frac{T}{N}$. Let $u_{i,j}\simeq u(x_i,t_j)$, where $i=0,1,\cdots,M$, $j=0,1,\cdots,N$. First, consider the Eq.(1) and evaluate it in $x=x_i$, then by approximate $\frac{\partial^2 u}{\partial x^2}\Big|_{x_i,t}$ using the following finite difference formula:

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x_i,t}\simeq\frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{h^2},$$

we have:

$$\begin{aligned}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}(x_i,t)&=\mu\frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{h^2}\\&+\frac{\Gamma(1/2)}{h^2}I_t^{1/2}u(x_{i+1},t)-2\frac{\Gamma(1/2)}{h^2}I_t^{1/2}u(x_i,t)+\frac{\Gamma(1/2)}{h^2}I_t^{1/2}u(x_{i-1},t),\end{aligned} \tag{4}$$

by applying the operator I_t^{α} to both sides of Eq.(4) and using the properties (3) we obtain:

$$\begin{aligned}u(x_i,t)&=\frac{\mu}{h^2}I_t^{\alpha}u(x_{i+1},t)-2\frac{\mu}{h^2}I_t^{\alpha}u(x_i,t)+\frac{\mu}{h^2}I_t^{\alpha}u(x_{i-1},t)\\&+\frac{\Gamma(1/2)}{h^2}[I_t^{\alpha+1/2}u(x_{i+1},t)-2I_t^{\alpha+1/2}u(x_i,t)+I_t^{\alpha+1/2}u(x_{i-1},t)].\end{aligned} \tag{5}$$

Now, by replacing $t=t_{n+1}$ $(n=0,1,\cdots,N-1)$ in Eq.(5), we need to calculate the integrals in. For this purpose we use the following integration rule:

$$\int_0^{t_{n+1}}(t_{n+1}-z)^{\alpha-1}g(z)\mathrm{d}z\simeq\int_0^{t_{n+1}}(t_{n+1}-z)^{\alpha-1}\tilde{g}_{n+1}(z)\mathrm{d}z, \tag{6}$$

where $\tilde{g}$ is the piecewise linear interpolant for g in nodes t_j and t_{j+1}:

$$\tilde{g}_{n+1}(z)=\frac{z-t_{j+1}}{t_j-t_{j+1}}g(t_j)+\frac{z-t_j}{t_{j+1}-t_j}g(t_{j+1}),$$

where $t_j\leqslant z\leqslant t_{j+1}$ $(j=0,1,\cdots,N-1)$.

We can write the integral on the right-hand side of Eq.(6) as [2]:

$$\int_0^{t_{n+1}}(t_{n+1}-z)^{\alpha-1}\tilde{g}_{n+1}(z)\mathrm{d}z=\frac{k^{\alpha}}{\alpha(\alpha+1)}\sum_{j=0}^{n+1}a_{j,n+1}g(t_j),$$

where:

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{if} \quad j = 0, \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, & \text{if} \quad 1 \leqslant j \leqslant n, \\ 1, & \text{if} \quad j = n+1, \end{cases} \tag{7}$$

and also:

$$\int_0^{t_{n+1}} (t_{n+1} - z)^{\alpha - 1/2} g(z) \mathrm{d}z \simeq \int_0^{t_{n+1}} (t_{n+1} - z)^{(\alpha - 1/2)} \tilde{g}_{n+1}(z) \mathrm{d}z,$$

furthermore:

$$\int_0^{t_{n+1}} (t_{n+1} - z)^{\alpha - 1/2} \tilde{g}_{n+1}(z) \mathrm{d}z = \frac{k^{\alpha+1/2}}{(\alpha+1/2)(\alpha+3/2)} \sum_{j=0}^{n+1} b_{j,n+1} g(t_j),$$

where:

$$b_{j,n+1} = \begin{cases} n^{\alpha+3/2} - (n-\alpha-1/2)(n+1)^{\alpha+1/2}, & \text{if} \quad j = 0, \\ (n-j+2)^{\alpha+3/2} - 2(n-j+1)^{\alpha+3/2} + (n-j)^{\alpha+3/2}, & \text{if} \quad 1 \leqslant j \leqslant n, \\ 1, & \text{if} \quad j = n+1. \end{cases} \tag{8}$$

By using the above integrals approximations in Eq.(5) we have:

$$\begin{aligned} u_{i,n+1} = {} & \psi \sum_{j=0}^{n} a_{j,n+1} u_{i+1,j} + \psi u_{i+1,n+1} - 2\psi \sum_{j=0}^{n} a_{j,n+1} u_{i,j} - 2\psi u_{i,n+1} + \psi \sum_{j=0}^{n} a_{j,n+1} u_{i-1,j} \\ & + \psi u_{i-1,n+1} + \phi \sum_{j=0}^{n} b_{j,n+1} u_{i+1,j} + \phi u_{i+1,n+1} - 2\phi \sum_{j=0}^{n} b_{j,n+1} u_{i,j} - 2\phi u_{i,n+1} \\ & + \phi \sum_{j=0}^{n} b_{j,n+1} u_{i-1,j} + \phi u_{i-1,n+1}, \qquad i = 1, 2, \cdots, M-1, \quad n = 0, 1, \cdots, N-1, \end{aligned} \tag{9}$$

where $\psi := \frac{\mu}{h^2} \frac{k^{\alpha}}{\Gamma(\alpha+2)}$, $\phi := \frac{\Gamma(1/2)}{h^2} \frac{k^{\alpha+1/2}}{\Gamma(\alpha+1/2)(\alpha+1/2)(\alpha+3/2)}$.

In general, Eq.(9) can be written as:

$$-\lambda u_{i-1,n+1} + (1+2\lambda) u_{i,n+1} - \lambda u_{i+1,n+1} = \sum_{j=0}^{n} p_j (u_{i-1,j} - 2u_{i,j} + u_{i+1,j}), \tag{10}$$

where $1 \leqslant i \leqslant M-1$, $0 \leqslant n \leqslant N-1$, $\lambda := \psi + \phi$, $p_j = \psi a_{j,n+1} + \phi b_{j,n+1}$, $(j = 0, 1, \cdots, n)$. For each $1 \leqslant i \leqslant M-1, 0 \leqslant n \leqslant N-1$, Eq.(10) is system a of $M-1$ equations and $M-1$ unknowns which has the following matrix form:

$$AU_{n+1} + R_{n+1} = \sum_{j=0}^{n} (B_j U_j + Q_j),$$

where:

$$AU_{n+1}+R_{n+1}=\begin{pmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & -\lambda & 1+2\lambda & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1+2\lambda \end{pmatrix}\begin{pmatrix} u_{1,n+1} \\ u_{2,n+1} \\ \vdots \\ u_{M-2,n+1} \\ u_{M-1,n+1} \end{pmatrix}+\begin{pmatrix} -\lambda u_{0,n+1} \\ 0 \\ \vdots \\ 0 \\ -\lambda u_{M,n+1} \end{pmatrix},$$

$$\sum_{j=0}^{n}(B_jU_j+Q_j)=\sum_{j=0}^{n}(\begin{pmatrix} -2p_j & p_j & 0 & \cdots & 0 \\ p_j & -2p_j & p_j & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & p_j & -2p_j & p_j \\ 0 & \cdots & 0 & p_j & -2p_j \end{pmatrix}\begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-2,j} \\ u_{M-1,j} \end{pmatrix}+\begin{pmatrix} p_ju_{0,j} \\ 0 \\ \vdots \\ 0 \\ p_ju_{M,j} \end{pmatrix}),$$

for known square matrices A and B_j, and a known vectors R, Q, where the details of the boundary conditions have been fully incorporated. Matrix A is a invertible matrix, therefore for $n=0,1,\cdots,N-1$ the above system can be solved by:

$$U_{n+1}=A^{-1}(\sum_{j=0}^{n}(B_jU_j+Q_j)-R_{n+1}).$$

Example 2.1. Numerical results for the problem (1),(2) with $g(x)=\sin(\pi x)$ and $u(x,t)=\sum_{n=0}^{\infty}(-1)^n\Gamma(\frac{3}{2}n+1)^{-1}(\pi^{\frac{5}{2}}t^{\frac{3}{2}})^n\sin(\pi x)$, when $t\in[0,1]$ with $h=k=0.1$ and $\alpha=1$, are shown in Table 1.

Table 1: Absolute errors for equation (1) with $\mu=1$

x	Errors
0.1	$3.4240e-4$
0.2	$6.5129e-4$
0.3	$8.9643e-4$
0.4	$1.0538e-3$
0.5	$1.1080e-3$
0.6	$1.0538e-3$

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An investigation on some properties of special Hankel matrices

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Abstract

Let a, b and c be fixed complex numbers and $H_n(a, b, c)$ be the $n \times n$ Hankel matrix, all of whose entries above the anti diagonal are b, all of whose entries below the anti diagonal are a and all of whose entries on the anti diagonal are c. In this paper new explicite formulae for determinant and eigenvalues of this matrix are suggested . Then we represent some intresting examples about them.

Keywords: Matrix theory, Hankel matrix, recursive relation, determinant.
Mathematics Subject Classification [2010]: 65F40; 15B05, 11B37

1 Introduction

Due to various applications of Hankel matrices, many authors have studied Hankel matrices. A Hankel Matrix is a square matrix with constant skew-diagonal. Hankel Matrices like Toeplitz Matrices have important applications, for instance in the Theory of Moment, and Pade approximation. The well - known relation of Hankel matrices to orthogonal polynomials yields a combinatorial application of the famous Berlekamp-Massey algorithm in Coding Theory, which can be applied in order to calculate the coefficients in the three-terms recurrence of the family of orthogonal polynomials related to the sequence of Hankel matrices.Their connection to orthogonal polynomials often yields useful applications in Combinatorics. They have considerable applicatins in System Theory, Control Engineering, some branches of Computer Sciences and Numerical Analysis (see [1]-[5]).

For each positive integer n and for all Complex numbers a, b, c, let $H_n(a, b, c)$ be the $n \times n$ Hankel matrix, all of whose entries above the anti diagonal are b, all of whose entries below the anti diagonal are a, and all of whose entries on the anti diagonal are c. For example,

$$H_3(a, b, c) = \begin{bmatrix} b & b & c \\ b & c & a \\ c & a & a \end{bmatrix}.$$

*Speaker

In section 2 we show that, determinant of $H_n(a,b,c)$ satisfies a linear recursive relation. We solve that recursive relation to obtain a simple formula for the determinant of $H_n(a,b,c)$. Then we give some results about eigenvalues of this matirx. Finally we represent some examples about these results.

2 Main results

In this section $det(H_n)$ will denote the determinant of H_n, $det(H_{n-1})$ will denote the determinant of the matrix that resulted from H_n by deleting of first row and last column of (H_n), also $det(H_{n-2})$ will denote the determinant of the matrix that resulted from H_n by deleting of two first rows and two last columns of (H_n).

Lemma 2.1. *Let a,b,c be complex numbers. For each positive integer n, let $H_n = H_n(a,b,c)$, then*

$$det(H_1) = c, det(H_2) = ab - c^2,$$

and for $n \geq 3$ we have

$$det(H_n) = (-1)^{n+1}[(2c-a-b)det(H_{n-1})] + (a-c)(b-c)det(H_{n-2}).$$

Theorem 2.2. *Let a,b,c be complex numbers. For each positive integer n, let $H_n = H_n(a,b,c)$. If $a=b=c$, then $det(H_1)=c$, and $det(H_n)=0$ for $n \geq 2$.*
otherwise we have

$$det(H_n) = (\frac{c(\beta+c)-ab}{\beta-\alpha})\alpha^{n-1} + (\frac{ab-c(\alpha+c)}{\beta-\alpha})\beta^{n-1},$$

where

$$\alpha = \frac{(-1)^{n+1}(2c-a-b) + \sqrt{2(2c-a-b)^2-(a-b)^2}}{2},$$

$$\beta = \frac{(-1)^{n+1}(2c-a-b) - \sqrt{2(2c-a-b)^2-(a-b)^2}}{2}.$$

Corollary 2.3. *If we set $a = x, b = x-1$ and $c = x$, then for all positive integer $n \geq 2$ and for all complex variable x, determinants of all (H_n) are equal to $(-x)$.*
In exact we have

$$det(H_n) = (2x - x - (x-1))det(H_{n-1}) - (x-x)((x-1)-x)det(H_{n-2})$$

$$= det(H_{n-1}) = det(H_{n-2}) = det(H_{n-3}) = \ldots = det(H_3)$$

$$= det(H_2) = x(x-1) - x^2 = -x.$$

Theorem 2.4. *Let a,b and c be complex numbers. For each positive integer n, if λ is an eigenvalue of $H_n(a,b,c)$. Then by well-known Gershgorin circles theorem we have*

$$|\lambda - u| \leq (n-1)max\{|a|,|b|,|c|\},$$

where $u = a$, $u = b$ and $u = c$.

Corollary 2.5. *Let a, b and c be complex numbers. For each positive integer n, if λ is an eigenvalue of $H_n(a, b, c)$, then we have*

$$|\lambda| \leq (n-1)k + |u|\,,$$

where
$k = max\{|a|, |b|, |c|\}$ *and* $u = a$, $u = b$ *and* $u = c$.

Corollary 2.6. *Let a, b and c be complex numbers and $H_n = H_n(a, b, c)$. Then we have the following upper bounds for the spectral radius of H_n, in exact we have*

$$\rho(H_n) \leq (n-1)k + |u|\,,$$

where
$k = max\{|a|, |b|, |c|\}$ *and* $u = a$, $u = b$ *and* $u = c$.

Example 2.7. If $a = b = 1, c = -1$, in exact if we set

$$H_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & 1 \\ 1 & 1 & 1 & \cdots & -1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 1 & 1 & -1 & \cdots & 1 & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

Then we have

$$det(H_n) = \frac{2^{n-2}}{\sqrt{2}}\left[((-1)^n - \sqrt{2})^{n-2} - ((-1)^n + \sqrt{2})^{n-2}\right].$$

Corollary 2.8. *If $a = b = 1, c = -1$, then by example (2.7) and Corollary(2.6) we have*

$$\rho(H_n) = n.$$

Example 2.9. If $a = 1, b = -1, c = 0$, in exact if we set

$$H_n = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 & -1 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & 1 \\ -1 & -1 & -1 & \cdots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ -1 & -1 & 0 & \cdots & 1 & 1 & 1 \\ -1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}.$$

Then we have

$$det(H_n) = \frac{1}{4i}(-2i)^{n-1} - \frac{1}{4i}(2i)^{n-1}$$

$$= \frac{(\sqrt{2})^{n-1}}{2} Sin(\frac{(n-1)\pi}{2}).$$

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Fast approximate method for solving nonlinear system of Fredholm-Volterra integral equations

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Abstract

A numerical method for solving nonlinear system of Fredholm-Volterra Hammerstain integral equations of second kind is presented. This method is based on replacement of the unknown functions by truncated series of well known Chebyshev expansion of functions. The quadrature formula which we use to calculate integral terms can be estimated by Fast Fourier Transform (FFT). Also convergence and rate of convergence are given.

Keywords: Nonlinear system of Fredholm-Volterra integral equation, Chebyshev polynomials, error analysis.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In this paper we present a computational method for solving a system of nonlinear Fredholm-Volterra integral equations of Hammerstain type:

$$x_i(s) = y_i(s) + \lambda_1 \sum_{j=1}^{n} \int_0^s K_{ij}(s,t)F(x_j(t))dt + \lambda_2 \sum_{j=1}^{n} \int_0^1 K'_{ij}(s,t)G(x_j(t))dt, \quad i = 1, ..., n, \qquad 0 \le s, t \le 1. \tag{1}$$

Consider the nonlinear system of integral equation (1). At first we approximate $x_i(t)$ for $i = 1, .., n$, as

$$x_i(t) \simeq \mathbf{C}_i^T \mathbf{T}(t), \tag{2}$$

then we substitute this approximation into eq. (1) to get

$$\mathbf{C}_i^T \mathbf{T}(s) = y_i(s) + \lambda_1 \sum_{j=1}^{n} \int_0^s K_{ij}(s,t)F(\mathbf{C}_j^T \mathbf{T}(t))dt + \lambda_2 \sum_{j=1}^{n} \int_0^1 K'_{ij}(s,t)G(\mathbf{C}_j^T \mathbf{T}(t))dt, \quad i = 1, ..., n, \qquad 0 \le s, t \le 1. \tag{3}$$

*Speaker

In order to use Gaussian integration formula for eq. (6), we transfer the intervals $[0, s_l]$ and $[0, 1]$ into interval $[-1, 1]$ by transformations

$$\tau_1 = \frac{2}{s_l}t - 1, \quad \tau_2 = 2t - 1.$$

For Chebyshev polynomials we consider the collocation points

$$s_l = \cos(\frac{l\pi}{N}), \qquad l = 0, 1, \ldots, N. \tag{4}$$

Let

$$H^f_{ij}(s,t) = K_{ij}(s,t)F(\mathbf{C}^T_j\mathbf{T}(t)), \qquad H^g_{ij}(s,t) = K'_{ij}(s,t)G(\mathbf{C}^T_j\mathbf{T}(t)).$$

Using collocation points (7) in transformed eq. (6) we get

$$\mathbf{C}^T_i\mathbf{T}(s_l) = y(s_l) + \lambda_1\frac{s_l}{2}\int_{-1}^{1}\sum_{j=1}^{n}H^f_{ij}(s_l, \frac{s_l(\tau_1+1)}{2})d\tau_1 + \frac{\lambda_2}{2}\int_{-1}^{1}\sum_{j=1}^{n}H^g_{ij}(s_l, \frac{(\tau_2+1)}{2})d\tau_2, \tag{5}$$

for $i = 1, .., n$. Now we use Clenshaw-Curtis quadrature formula [10], to get

$$\mathbf{C}^T_i\mathbf{T}(s_l) = y(s_l) + \sum_{j=1}^{n}{\sum_{k=0}^{N}}''w_k[\lambda_1\frac{s_l}{2}H^f_{ij}(s_l, \frac{s_l(s_k+1)}{2}) + \frac{\lambda_2}{2}H^g_{ij}(s_l, \frac{(s_k+1)}{2})], \tag{6}$$

for $l = 0, 1, \ldots, N$, and where

$$w_k = \frac{4}{N}\sum_{even\ n=0}^{N}{}''\frac{1}{1-n^2}\cos(\frac{nk\pi}{N}), \tag{7}$$

and double prime means that the first and the last terms are halved. The system (9) consist of nonlinear equations with unknown vector with elements of $\mathbf{C}_i$ as $\mathbf{C} = [c_{00}, c_{01}, \ldots c_{0N}, \ldots, c_{N0}, c_{N1}, \ldots c_{NN}]$, which can be solved by usual iterative method such as Newton's method or simplex method. The Fast Fourier Transform (FFT) technique can be used to evaluate the summation part in (9) in $O(N \log N)$ operations. In fact eq. (10) for weights w_k can also be viewed as the discrete cosine transformation of the vector v with entries:

$$v_n = \begin{cases} 2/(1-n^2), & n\ even \\ 0, & n\ odd. \end{cases}$$

2 Main results

Theorem 1. The solution of nonlinear system of Fredholm-Volterra Hammerstain Integral equation (1) by using Chebyshev polynomials converge if $0 < \alpha < 1/n$; in other word for $i = 1, ., n$, $\lim_{N\to\infty}\|x_i(s) - x_{iN}(s)\| = 0$.

Proof:

$$\begin{aligned}\|x_i(s)-x_{iN}(s)\|_\infty &= \max_{s\in[0,1]}|x_i(s)-x_{iN}(s)|\\ &= \max_{s\in[0,1]}|\lambda_1\sum_{j=1}^n\int_0^s K_{ij}(s,t)(F(x_j(t)-F(x_{jN}(s)))dt|+\\ &\quad \max_{s\in[0,1]}|\lambda_2\sum_{j=1}^n\int_0^s K'_{ij}(s,t)(G(x_j(t)-G(x_{jN}(s)))dt|\\ &\quad \leq \sum_{j=1}^n|\lambda_1|M_1L_1s\|x_j(s)-x_{jN}(s)\|_\infty+\\ &\quad \sum_{j=1}^n|\lambda_2|M_2L_2\|x_j(s)-x_{jN}(s)\|_\infty\\ &\quad = \sum_{j=1}^n(|\lambda_1|M_1L_1s+|\lambda_2|M_2L_2)\|x_j(s)-x_{jN}(s)\|_\infty\\ &\quad \leq \sum_{j=1}^n(|\lambda_1|M_1L_1+|\lambda_2|M_2L_2)\|x_j(s)-x_{jN}(s)\|_\infty\end{aligned}$$

$$\Rightarrow \|x_i(s)-x_{iN}(s)\|_\infty \leq \sum_{j=1}^n \alpha\|x_j(s)-x_{jN}(s)\|_\infty.$$

if we write the last relation for $i=1,..,n$, and add up them, we obtain

$$\sum_{i=1}^n\|x_i(s)-x_{iN}(s)\|_\infty \leq \sum_{j=1}^n n\alpha\|x_j(s)-x_{jN}(s)\|_\infty.$$

so

$$\sum_{i=1}^n(1-n\alpha)\|x_i(s)-x_{iN}(s)\|_\infty \leq 0.$$

According to this equation if we choose $0<\alpha<1/n$ we have:

$$\lim_{N\to\infty}\|x_i(s)-x_{iN}(s)\|=0,$$

so the proof is completed.

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Local Gaussian radial basis function method for solution of 2D steady convection diffusion equation

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Abstract

A local collocation method based on Gaussian radial basis function is developed for solution of steady state two dimensional convection- diffusion equation. consequently the arising system is sparse, which is an effort to reduce the condition number of the system and to overcome the ill- conditioning . This approach is suitable for solving problems in high dimensional too.

Keywords: Gaussian radial basis function, Local collocation method, Convection-diffusion equation.

1 Introduction

There are many methods for solving a partial differential equations, but these methods need to a mesh on domain like Finite Difference (FD), Finite Element (FE), Finite volume (FV) and other approaches. Therefore many scientists prefer to use meshless methods. One of these methods is Radial Basis Function (RBF). RBF interpolation has been shown to work in many cases where polynomial interpolation has failed [1]. Kansa showed RBFs are an effective way for solving partial differential equations [2, 3]. This meshless method has high accuracy, but it is ill-condition. There are many approaches to overcome this ill-conditioning, the local collocation method is one of such approaches.

Definition 1.1. A function $\Phi : R^d \to R$ is called radial if there exists a univariate function $\varphi : [0, \infty) \to R$ such that

$$\Phi(x) = \varphi(r),$$

where $r = \|x\|$ and $\|.\|$ is a norm on R^d ($\|.\|$ is typically the Euclidean norm.)[4].

Definition 1.2. The scattered data interpolation problem: given data $(\mathrm{x}_j, f_j), with$ j=1,...,N, $\mathrm{x}_j \in R^d$, and $f_j \in R$, find a smooth function u such that $u(x_j) = f_j$, for $j = 1, ..., N$.
For a set of N centers, $x_1^c, ..., x_N^c$ in R^d, a radial basis function interpolant is of the form

$$u(x) = \sum_{j=1}^{N} \alpha_j \phi(\|x - x_j^c\|_2, c). \tag{1}$$

The α_j coefficients in the RBF are determined by enforcing the interpolation condition $u(x_j) = f(x_j)$ at a set of points that usually coincides with the N centers and ϕ is a radial basis function.

*Speaker

2 Main results

2.1 Kansa's method

Consider steady state problem:

$$Lu(x) = f(x), \qquad x \in \Omega \subset R^d \tag{2}$$

$$Bu(x) = g(x), \qquad x \in \partial\Omega \tag{3}$$

where L is a linear differential operator in Ω and B is a boundary operator on $\partial\Omega$. By using interpolation formula and collocation method in centers, following systems of equation is obtained:

$$\begin{bmatrix} L\phi \\ B\phi \end{bmatrix} \alpha = \begin{bmatrix} f \\ g \end{bmatrix} \tag{4}$$

This method is known as Kansas method [5, 6].

2.2 Local method

In this method around of each strictly-interior point which are selected in domain, a stencil is formed which this node is called centerpoint of its stencil. Each system collocates the solution value in the boundaries of stencil (solution center) and collocates PDE operator in its internal nodes (PDE center). In this paper stencils are chosen in the form of square in sizes of 3×3 (Figure1). An RBF collocation is formed for each system that yield N RBF local collocation system

$$A^{(s)}\lambda^{(s)} = d^{(s)} \qquad s = 1, 2, ...N, \tag{5}$$

A^s is collocation matrix as explained by equation(4) , α^s is interpolation coefficients and d^s is called data vectors for each system. The solution of PDE, can be approximated by equation(1). so that for each stencil we have:

$$u^{(s)}(x) = Q^{(s)}(x)\lambda^{(s)}, \tag{6}$$

which $Q^{(s)}$ is called a reconstruction vector for stencil s. At each one of the centerpoint of system s, we have:

$$\begin{aligned} u^{(s)}(x_c^{(s)}) &= Q^{(s)}(x_c^{(s)})\lambda^{(s)}, \\ &= Q^{(s)}(x_c^{(s)}) \left[A^{(s)}\right]^{-1} d^{(s)} \\ &= W^{(s)}(x_c^{(s)})d^{(s)}, \end{aligned}$$

where $W^{(s)}(x_c^{(s)})$ is weight vector. The obtained N simultaneous equations yield to a sparse global system [7].

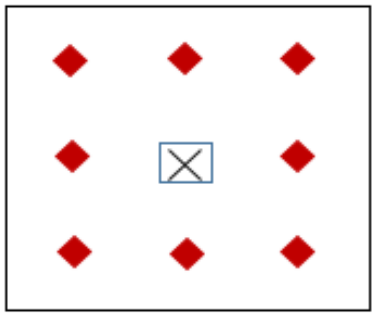

Figure 1: A stencil in size 3×3, Red diamonds represent solution centers, Blue square show PDE centers and black cross show center point.

3 Numerical illustration

In this section, we applied our approach based on Gaussian RBF, $exp(-r^2/c^2)$, that $r = \|x - x_i\|$ and $c = 50/N$.
Consider 2D steady convection diffusion equation:

$$D\frac{\partial^2 u}{\partial x_i^2} - q_i\frac{\partial u}{\partial x_i} = 0, \tag{7}$$

where $x_1 \in [0, 1]$, $x_2 \in [0, 0.2]$ and $q_i = (0, 1)$ exact solution of 7 is

$$\phi_e = 2 - \frac{1 - e^{(x_1-1)pe}}{1 - e^{-pe}}.$$

the Dirichlet boundary conditions can be obtained by using the exact solution and Peclet

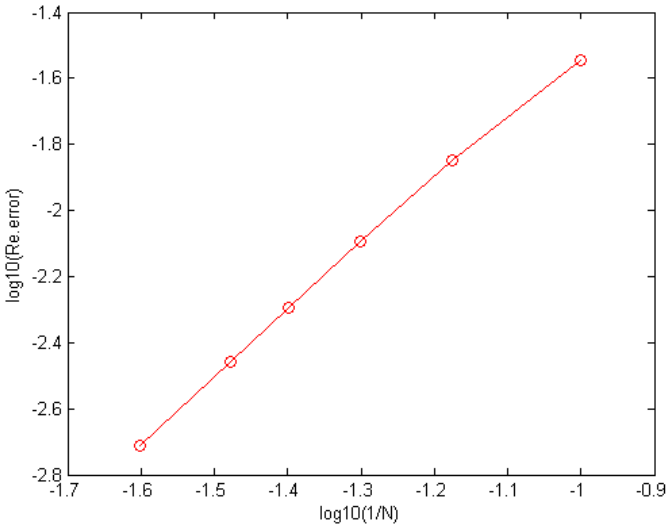

Figure 2: Relative error for stencil 3×3

number is defined as $pe = \frac{1}{D}$ that $pe = 25$. Dataset is selected $N = 10, 15, 20, 25, 30, 40$.

This dataset is discretized with N+1 piont in x_1 direction and $N/5+1$ point in x_2 direction.

Relative error is calculated by following formula:

$$RE.error = \sqrt{\frac{\sum_{j=1}^{k}(u_{app}(x_j) - u_{exa}(x_j))^2}{k(u_{max} - u_{min})^2}},$$

where x_j are values u within the domain, u is the computed solution, u_{exa} is exact solution, u_{max}, u_{min} are the maximum and minimum values in the domain. Results is obtained in Figure2 that show relative error which is reducing exponentially.

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Maximum Principle Theorems For Forth Order Differential Equations

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Abstract

In this paper first we present continuous maximum principle theorem for fourth order differential equations. Then we express discrete maximum principle theorem for matrix form of discrete problem(by finite element method or finite difference method). At the end we make an example to find maximum of u.

1 Introduction

The early development of numerical analysis of partial differential equations was dominated by functional analysis. In such a method an approximate solution is sought at the end points of a finite grid of points, and the approximation of differential equation is accomplished by replacing appropriate difference quotients and a finite linear system of algebraic equations. The maximum principle theorem is used to show uniqueness and continuous dependence on data for solution and approximate solution of partial equation. In this paper we consider the discrete approximation to the fourth order boundary value problem

$$u^{(4)}(x) = f(x, u(x), u^{'}(x), u^{''}(x), u^{'''}(x)), \qquad a \leq x \leq b, \tag{1}$$

subject to the boundary conditions

$$u(a) = g_0, \tag{2}$$

$$u(b) = g_1, \tag{3}$$

$$u'(a) = \alpha_1, \qquad u'(b) = \alpha_2. \tag{4}$$

Some applications of involving population dynamics with spatial migration, chemical reaction and control systems are given by some authors e.g. [1, 2, 3] and the references therein. In all these studies, we observe that a result of employing a special type of (1). By using the maximum principle theorem and discrete maximum principle we have proved uniqueness both in (1) and the finite difference method respectively.
Therefore,the paper is organized as follows.
In the next section we give the continuous maximum principle theorem. This problem is implemented in section 3 and we give a discrete maximum principle theorem. At the end to illustrate this principle we give an example.

*Speaker

2 Existence and uniqueness solution (1)-(4) by the continuous maximum principle

If u is a real valued function of $C^2([a,b])$ and $u''(x) \geq 0$ in the interior of $I = [a,b]$, then u satisfies the maximum principle. Moreover if u attains its maximum at an interior point of I then u is constant on I. As an application of this principle we prove the existence of non negative solution of a fourth order boundary value problem.

Theorem 2.1. *(Continuous maximum principle) Let u be a real valued function of $C^4([a,b])$. Suppose u satisfies the inequalities*

$$u^{(4)}(x) \geq 0 \quad , \quad x \in (a,b) \tag{5}$$

$$u'(a) \geq 0 \quad , \quad u'(b) \leq 0, \tag{6}$$

and moreover , attains its minimum at a point $x_0 \in (a,b)$, then u is constant on $[a,b]$.

Proof. Since u attains its minimum at $x_0 \in (a,b)$, we consider $w(x) = u''(x)$ that $u'(x) = 0$,$w(x_0) \geq 0$ and

$$-u'(a) = \int_a^{x_0} w(x)dx \leq 0 \qquad , u'(b) = \int_{x_0}^{b} w(x)dx \leq 0, \tag{7}$$

now we assume two possible cases:

(i) There is a point $x_1 \in (a,x_0) \cup (x_0,b)$ s.t. $w(x_1) < 0$. or

(ii) $w(x) \geq 0$ on (a,b).

If the second one holds , then by (7) we have $w(x) = u''(x) \geq 0$ on $[a,b]$ and $u'(x_0) = 0$ hence u is constant on$[a,b]$. So we prove that the first one is impossible. We assume that $x_1 \in (a,x_0)$, the second inequality in (7) implies that there is a point $x_2 \in (x_0,b)$, s.t $w(x_2) \leq 0$ thus we have $w(x_1) < 0$, $w(x_0) \geq 0$, $w(x_2) \leq 0$, $x_1 < x_0 < x_2$, and consequently, there is a point $x_3 \in (x_1,x_0)$ such that $w'(x_3) > 0$ and a point $x_4 \in (x_0,x_2)$ s.t. $w'(x_4) \leq 0$. On the other hand we have using (5)

$$w'(x_4) - w'(x_3) = \int_{x_3}^{x_4} w''(x)dx = \int_{x_3}^{x_4} w^{(4)}(x)dx \geq 0,$$

that is a contradiction. □

Let u be a real valued function of $C^4([a,b])$, and u satisfies (5), then u attains its minimum at $x = a$ or $x = b$ (see[1, 2]).

3 Implementation of the finite difference method and the discrete maximum principle

For our analysis we first show a discrete maximum principle similar to that in the continuous case of Theorem 2.1. A matrix $A = (a_{i,j})$ $1 \leq i,j \leq N$ is reducible if there exist a permutation matrix P such that

$$PAP^t = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{8}$$

A_{11} is an $r \times r$ $(1 \leq r < N)$ sub matrix. A_{22} is an $(N-r) \times (N-r)$ sub matrix. A is irreducible if there isn't P . A is diagonally dominate if

$$| a_{ii} | \geq \sum_{j=1_j \neq i}^{N} | a_{ij} | \qquad 1 \leq i \leq N \tag{9}$$

Moreover , A is irreducibly diagonally dominate if it is irreducible and diagonally dominate and for at least one $i_0 \quad (1 \leq i_0 \leq N)$

$$| a_{i_0,i_0} | > \sum_{j=1_{j\neq i_0}}^{N} | a_{i_0,j} | \tag{10}$$

Lemma 3.1. *An $N \times N$ matrix A is irreducible if and only if for any two distinct indicas $1 \leq i, j \leq N$,there is a sequence of non zero element of A*

$$\{a_{i,i_1}, a_{i_1,i_2}, \cdots, a_{i_s,j}\}$$

where $i, i_1, i_2, ..., i_s, j$ are distinct.

Before we represent discrete maximum principle theorem, define some useful sets.
$\wp_N = \{1, 2, \ldots, N\}, \quad \xi_M = \{N+1, \ldots, N+M\} \backslash \ell_M, \quad \ell_M = \bigcup_{i=1}^{N} \Im_{i,M},$

$$\Im_{i,M} = \{j; N+1 \leq j \leq N+M, \quad a_{i,j} \neq 0\}, \qquad 1 \leq i \leq N,$$

ℓ_M is the set of indices of the boundary point that are connected with some interior point $(a_{ij} \neq 0)$
ξ_M is the set of indices of the boundary points which are not connected with any interior points. generally ξ_M is the set of indices of the corner points.

Theorem 3.2. *(Discrete maximum principle) Suppose that $a_{ii} > 0$, $a_{ij} \leq 0$, $i \neq j \quad 1 \leq i \leq N$ $1 \leq j \leq N+M$*

$$\sum_{j=1}^{N+M} a_{ij} \geq 0,$$

and $A = (a_{ij})$ $1 \leq i, j \leq N$ is irreducibly diagonally dominate. Let $(w_1, w_2, \ldots, W_{N+M})$ satisfy

$$\sum_{j=1}^{N+M} a_{ij} w_j < 0 \qquad 1 \leq i \leq N,$$

if there exists r $(1 \leq r \leq N)$ such that

$$\max_{1 \leq j \leq N+M} w_j = w_r \geq 0.$$

Then

$$w_j = w_r, \qquad j \in \wp_N \cup \ell_M,$$
$$w_j \leq w_r, \qquad j \in \xi_M.$$

Proof. Because of A is irreducible for any k ,$(1 \leq k \leq N)$ according lemma (3.1) a sequence of nonzero elements of A there is $\{a_{r,i_1}, a_{i_1,i_2}, \ldots, a_{i_s,k}\}$. Thus we have

$$\begin{aligned} 0 \geq \sum_{j=1}^{N+M} a_{r,j} w_j &= a_{r,r} w_r + \sum_{\substack{j=1 \\ j\neq r}}^{N+M} a_{r,j} w_j \\ &\geq -\sum_{\substack{j=1 \\ j\neq r}}^{N+M} a_{r,j} w_r + \sum_{\substack{j=1 \\ j\neq r}}^{N+M} a_{r,j} w_j = \sum_{\substack{j=1 \\ j\neq r}}^{N+M} = \sum_{\substack{j=1 \\ j\neq r}}^{N+M} a_{r,j}(w_r, w_j) \\ &= -a_{r,i_1}(w_r - w_{i_1}) - \sum_{j \in \Im_{r,M}} a_{r,j}(w_r - w_j) - \sum_{\substack{j=1 \\ j\neq r \\ j\neq i_1}}^{N} a_{r,j}(w_r - w_j) \geq 0. \end{aligned}$$

By considering $a_{r,i_1} < 0$,$a_{r,j} < 0$, $j \in \Im_{r,M}$,we get

$$w_{i_1} = w_r$$

$$w_j = w_r, \qquad j \in \Im_{r,M}.$$

The same arguments yield

$$w_{i_1} = w_{i_2} = \cdots = w_{i_s} = w_k = w_r,$$

$$w_j = w_r \qquad j \in \Im_{r,M}, i = i_1, i_2, \ldots, i_s, k.$$

Since k $(1 \leq k \leq N)$ is arbitrary, we obtain

$$w_j = w_r, \qquad j \in \wp_N \cup \ell_M,$$

it is clear that $w_j \leq w_r \qquad j \in \xi_M$. □

Now, we give an example to illustrate the discrete maximum principle. For a given domain $[a, b]$, we consider a uniform discretization with mesh size $h = 1$ such that

$$x_i = ih, \qquad i = 0, 1, \cdots, n, \quad a = x_0 < x_1 < \cdots < x_n = b.$$

We consider problem (1) as $u^{(4)} = -4u''$ and then we apply the central finite difference method for this equation. For $n = 6$, we write the matrix form of this discretization as follow

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -\alpha_1 \\ g_0 \\ \alpha_2 - g_1 \\ g_1 \end{bmatrix}.$$

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Numerical analysis of fractional differential equation by wavelets

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Abstract

In this paper, we introduce methods baised on operational matrix of fractional integration for solving a typical n-term non-homogeneous fractional differential equation (FDE). We use Block-puls, Haar wavletes and Hybrid of Block-pulse functions and shifted Legendre polynomials matrices of fractional integration where a fractional derivative is defined in the Caputo sense. By uses these methods we translate an FDE to an algebric liear equations. Methods has been tested by some numerical examples.

Keywords: FDE, Haar, Blok-pulse, Hybrid function, operational matrices
Mathematics Subject Classification [2010]: 26A33, 97N40

1 Introduction

In recent years, study on application of the FDE in science has attracted increasing attention [5, 6, 7]. For instance, Bagley and Torvik formulated the motion of a rigid plate immersing in a Newtonian fluid[1]. It shoulld be mentioned that the main reasons for the theoretical development are mainly the wide use of polymers in various fields of engineering [8, 9, 11, 12]. An FODE in time domain can be described as the following form,

$$a_n({}_aD_t^{\alpha_n}y(t)) + \cdots + a_1({}_aD_t^{\alpha_1}y(t)) + a_0({}_aD_t^{\alpha_0}y(t)) = u(t), \tag{1}$$

subject to the initial conditions $y^{(i)}(a) = d_i, i = 0, ..., n$, where $a_i \in R, n < \alpha \leq n+1, 0 < \beta_1 < \beta_2 < ... < 1 < \alpha$, and ${}_aD_t^{\alpha_n}y(t)$ denotes the caputo fractional derivative of order α. We begin by introducing some necessary definitions and theorems of the fractional calculus theory. In 1.1 the operational matrices of fractional order integration for some wavelets are obtained. Section 2 is devoted to applying the operational matrices of fractional order integration for solving FODE. Also in 2.2 the proposed methods are applied to an example.

1.1 Definitions and theorems

Definition 1.1. The Riemann-Liouville fractional integral of order α is

$$I^{\alpha}(f(x)) =_a D_t^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}f(\tau)d\tau \quad ;\alpha > 0 \tag{2}$$

*Speaker

Definition 1.2. The fractional derivative of $f(x)$ by means of Caputo sense is defined as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau; n-1 < \alpha \leq n, n \in N, x > 0, f \in C_{-1}^n \quad (3) $$

The relation between the RiemannLiouville operator and Caputo operator is given by the following expressions[2]:

$$ I^\alpha {}_aD_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!}, x > 0. \quad (4) $$

Definition 1.3. The m-set of block-pulse functions is defined as:

$$ b_i(t) = \begin{cases} 1 & ; \frac{\eta i}{m} \leq t \leq \frac{\eta(i+1)}{m} \\ 0 & ; otherwise \end{cases} \quad Where \quad i = 0, 1, 2, \cdots, m-1. \quad (5) $$

Definition 1.4. (The Haar Wavelet Function) Let $[0, \eta)$ be an interval, we define $h_0(t)$ and $h_1(t)$ on $[0, \eta)$ as follows

$$ h_0(t) = \frac{1}{\sqrt{\eta}} \begin{cases} 1 & ; 0 \leq t < \eta \\ 0 & ; otherwise \end{cases} \quad h_1(t) = \frac{1}{\sqrt{\eta}} \begin{cases} 1 & ; 0 \leq t < \frac{\eta}{2} \\ -1 & ; \frac{\eta}{2} \leq t < 1 \\ 0 & ; otherwise \end{cases}, \quad (6) $$

and for $i = 2^j + k$, $j \geqslant 0$, $0 \leq k \leq 2^j - 1$, we define $h_i(t) = \frac{2^{\frac{j}{2}}}{\sqrt{\eta}} h_1(2^j t - k)$.

Definition 1.5. The shifted Legendre polynomials are defined on the interval [0,1] and can be determined with the aid of the following recurrence formulae[4]:

$$ P_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} P_i(x) - \frac{i}{i+1} P_{i-1}(x), \qquad i = 1, 2, ... $$

Theorem 1.6. *A function $f(x) \in L^2([0, T_1))$ may be expanded by the Block-puls functions as:*

$$ f(x) \simeq F^T B_m(x); F = \begin{pmatrix} f_1 & \cdots & f_m \end{pmatrix}, \; B_m(x) = \begin{pmatrix} b_1(x) & \cdots & b_m(x) \end{pmatrix}. \quad (7) $$

The Block-pulse coefficients f_i are obtained as $f_i = \frac{1}{h} \int_{(i-1)h}^{ih} f(x) dx$.

Proof. [3] □

Theorem 1.7. *Any function $y(t) \in L^2[0, \eta)$ can be decomposed as*

$$ y(t) \simeq C^T H_m(x); C = \begin{pmatrix} f_1 & \cdots & f_m \end{pmatrix}, \; H_m(x) = \begin{pmatrix} h_1(x) & \cdots & h_m(x) \end{pmatrix}. \quad (8) $$

$c_i's$ are determined by $c_i = 2^j \int_0^\eta y(t) h_i(t) dt$ $i = 2^j + k, j \geqslant 0, 0 \leq k \leq 2^j - 1$.

Proof. [3] □

Definition 1.8. Hybrid function of Block-pulse and shifted legender $hy_{i,j}(x)$, $i = 0, ...m-1$ *and* $j = 0, ..., n-1$ are defined on the interval $[0, 1)$ as

$$ hy_{i,j}(x) = \begin{cases} P_j(mx - i); \; \frac{i}{m} \leq x < \frac{i+1}{m} \\ 0 \quad ; otherwise \end{cases}, \quad (9) $$

Now for approximate the functione $f(x)$ we can set $f(x) \simeq C^T H y_{nm}(x)$ where $C^T = \begin{pmatrix} c_{0,0} & \cdots & c_{0,n-1} c_{(m-1),(n-1)} \end{pmatrix}$, $Hy_{mm}(x) = \begin{pmatrix} hy_{0,0}(x) & \cdots & hy_{(m-1),(n-1)}(x) \end{pmatrix}$, and $c_{i,j} = \frac{< f(x), hy_{i,j} >}{< hy_{i,j}, hy_{i,j} >}$ where $< u(x), v(x) >= \int_0^1 u(x) v(x) dx$ [10]

2 Main results

In this Section we intriduse operational matrix methods baised on Block-pulse and Haar wavelets and Hybrid of Block-pulse and shifted Legendre. Afterward we apply it to the FDE as Eq. 1. Firstly we construct operational matrices.

2.1 Operational matrices of fractional order integration

Let F^α be the operational matrix of fractional integration for Block-pluse wavelet [2], Then the Haar wavelet operational matrix derive as following

$$P^\alpha_{m\times m} = H_{m\times m}F^\alpha H^{-1}_{m\times m}; H_{m\times m} = \begin{pmatrix} H_m(t_0) & \cdots & H_m(t_{m-1}) \end{pmatrix}, t_i = \frac{2i+1}{2m}. \quad (10)$$

Now let $Hy_{nm} \simeq \Phi B_{mn}(x)$ and $I^\alpha Hy_{nm}(x) = Q^\alpha Hy_{nm}(x)$ then we can construct operational matrix of fractional order integration for Hybrid functions as:

$$Q^\alpha = \Phi F^\alpha \Phi^{-1}. \quad (11)$$

In order to show the efficiency of operational matrix, we apply it to solve an example.

2.2 numerical examples

Example 2.1. Consider the following initial value problems Bagley-Torvik equation

$$D^2y(x) + D^{\frac{3}{2}}y(x) + y(x) = 1 + x \quad ; \quad y(0) = 1, \quad y'(0) = 1. \quad (12)$$

The exact solution is $y(x) = 1 + x$, [4]. The integral representation of Eq. (21) is:

$$y(x) - x - 1 + I^{\frac{1}{2}}(y(x)) + I^{\frac{1}{2}}(-x - 1) + I^2(y(x)) = I^2(1 + x). \quad (13)$$

Now We consider three operational matrix of fractional order integration method to solve numerical solution of eq.(19).Set

$$y(x) = C_b^T B_m(x) = C_h^T H_m(x) = C_{hy}^T Hy_{nm}(x) \quad (14)$$

$$1 + x = C_{1b}^T B_m(x) = C_{1h}^T H_m(x) = C_{1hy}^T Hy_{nm}(x). \quad (15)$$

Then Eq. (22) can be represented in matrix form as:

$$C^T(I_m + O^{\frac{1}{2}} + O^2) = C_1^T(I_m + O^{\frac{1}{2}} + O^2), \quad (16)$$

where O denote F, P and Q in Block-pulse, Haar and Hybrid operational matrix respectivelly.This shows that $C^T = C_1^T$. Thus we have $C^T B_m(x) = C_1^T B_(x) = 1 + x$ which is the exact solution.

Conclusion

This article uses Block-pulse, Haar and Hybrid operational matrix method to solve FDE by combining these functions with operational matrix of fractional order integration. We translate the initial equation into a linear algebric equation which is easily to solve. The example shows that the mothod can be more efficient.

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Numerical solution of Hammerstein Volterra integral equations by using of polynomials

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Abstract

In this paper, the solution of nonlinear Volterra integral equations is approximated with polynomials. These approximate solutions are obtained based on a minimization method. In addition, the existence and convergence of these approximate solutions are investigated, Also, it is shown which are very near to the best approximation. In order to use Newton's method for minimization, a suitable initial point is introduced.
Keywords: Nonlinear Volterra integral equations, Minimization, Initial point.

1 Introduction

In this paper, author intend to solve a large class of these equations which are called as Hammerstein Volterra integral equations and their general form is as follows:

$$y(x) = \int_0^x k(x,t)F(y(t))dt + g(x), \;\; 0 \leq x \leq 1 \tag{1}$$

where functions $k(x,t), F(y(t))$ and $g(x)$ are known and $y(x)$ is the unknown function to be determined. Here, the kernel k and the functions F and g are assumed to be continuous and also, it is supposed that the equation (1) has a unique continuous solution on $[0,1]$. There are many numerical methods for solving Hammerstein Volterra integral equations , for example, Maleknejad and his co-authors obtained a numerical solution of these equations by using Bernsteins approximation[1] and using Chebyshev polynomials[2]. In [3] J. Saberi-Nadjafi, M. Mehrabinezhad and H. Akbari Solved Volterra integral equations of the second kind by wavelet-Galerkin scheme. Most of these methods are based on the appropriate linear combinations of some basic functions such as Chebyshev polynomials, Bernstein polynomials, Taylor polynomials, wavelets. In order to increase the rate of the convergence for approximate solutions, we approximate the solution by some linear combinations of polynomials. In order to obtaining this polynomials for the solution approximation of integral equation, author is developed and innovated exists methods in the work [4] in which leads to the highly significant results. Based on the proposed method, the problem of solving a nonlinear integral equation is converted to a minimization problem of unconstrained nonlinear programming. This new approach obtained polynomial solutions in which are very close to best polynomial approximations.

*Speaker

2 Method description and fundamental theorems

Let me consider the operator R corresponding to equation (1) as the following:

$$R[z](x) = z(x) - \int_0^x k(x,t)F(z(t))dt - g(x), \tag{2}$$

where, z is a function defined on $[0,1]$. Suppose that the approximate solution of Eq.(1) is presented as the following form

$$y_n(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + ... + a_n\varphi_n(x), \tag{3}$$

where the parameters a_i's are unknown constants and $\varphi_i(x) = x^i, (i = 0, 1, ..., n)$.

Definition 2.1. Let M be a positive number and sufficiently large, $y_n^* = a_0^*\varphi_0 + a_1^*\varphi_1 + \cdots + a_n^*\varphi_n$ is called the approximate solution for equation (1) if

$$M^n\|R[y_n^*]\|_\infty + \|y_n^*\|_\infty = \min_{(a_0,a_1,...,a_n)}(M^n\|R[y_n]\|_\infty + \|y_n\|_\infty), \tag{4}$$

where $y_n = a_0\varphi_0 + a_1\varphi_1 + \cdots + a_n\varphi_n$, and $\|R[y_n]\|_\infty = \max_{0\le x\le 1}|R[y_n](x)|$.

Lemma 2.2. *[Existence] For each positive integer n, there exists the approximate solution y_n^* introduced as in the above definition.*

Lemma 2.3. *The sequence $\{y_n^*\}$ generated by the proposed method is uniformly bounded.*

Theorem 2.4. *The sequence $\{y_n^*\}$ is convergent to the exact solution of Equation* (1).

Conclusion I: It concludes the sequence $\{y_n^*\}$ is the approximate solution of the minimization problem following:

$$\min_{y_n\in A}\|R[y_n]\|_\infty \tag{5}$$

where $A = \{y_n \mid \ \|y_n\|_\infty \le N\}$ and N is sufficiently large. It is clear that equation (5) has optimum solution $\tilde{y_n}$ and the sequence $\{\tilde{y_n}\}$ is convergent to y^*. Since y_n^* is the optimum solution of the problem (4) then

$$|\ \|R[y_n^*]\|_\infty - \|R[\tilde{y_n}]\|_\infty\ | \le (\frac{1}{M})^n(\|\tilde{y_n}\|_\infty - \|y_n^*\|_\infty). \tag{6}$$

By considering M is very large and $\|\tilde{y_n}\|_\infty - \|y_n^*\|_\infty$ is convergent to zero then concluded

$$\|R[y_n^*]\|_\infty \cong \|R[\tilde{y_n}]\|_\infty. \tag{7}$$

Applying Gronwall's inequality on $R[y_n^*] - R[\tilde{y_n}]$, definition $R[y_n^*]$, using of theorem 2.4.6 in [5] and Eq.(7) lead to

$$\|y_n^* - \tilde{y_n}\|_\infty = O(\frac{1}{2^n(n+1)!}),\ \|\tilde{y_n} - y^*\|_\infty = O(\frac{1}{2^n(n+1)!}),\ \|y_n^* - y^*\|_\infty = O(\frac{1}{2^n(n+1)!}). \tag{8}$$

From (8) is concluded that $\tilde{y_n}$ and y_n^* are very close to each other and their convergence rate is very high.

3 More Computational Operations

In this section, it is illustrated the calculation method for the determination of the unknown parameters of the approximate solution. According to conclusion I, it is sufficient to solve the following mathematics programming problem:

$$\begin{array}{l} min \;\; \sigma \\ \quad s.t \\ |L[y_n](x)| \leq \sigma, \;\; \forall x \in [0,1] \\ \quad \sigma \geq 0. \end{array} \tag{9}$$

Of course the bounded condition of the solution of problem (9) must be considered which by the choosing of suitable initial point in the Newton's iterative method is guaranteed $y_n^* \in A$. To solve the above problem, a finite number of points $x_i, \;\; i = 1, 2, ..., m$ is selected in interval $[0, 1]$ and solve the following mathematical programming problem:

$$\begin{array}{l} min \;\; \sigma \\ \quad s.t \\ |L[y_n](x_i)| \leq \sigma, \quad i = 1, 2, ..., m \\ \quad \sigma \geq 0. \end{array} \tag{10}$$

The numbers and positions of these points are selected such that the optimum value of Eq.(10) is approached to that of Eq.(9). But for obtaining the optimum solution of problem (10), it is sufficient to solve problem as following in which theirs optimum solutions is the same $\min_{(a_0,a_1,...,a_n)} \sum_{i=1}^{m} (L[y_n](x_i))^2$. The Newton's iterative method is used for minimizing of this expression. It is obvious that the rate of convergence of this method depends on a suitable initial point which will be discussed in the next section.

3.1 Initial point

It is well known that the initial guesses for Newtons iterative method are very important. To choose the initial guesses, in the first stage, the solution of (1) is approximated by the constant polynomial $y_0(x) = a_0$. Since y^* is continuous and usually does not have large oscillations and especially $y^*(0) = g(0)$, then solving the problem $\min_{a_0 \in B} \|R[y_0]\|_\infty$ where $B = \{g(0) - 1, g(0) - 1 + \frac{2}{m}, \cdots, g(0) + 1 - \frac{2}{m}, g(0) + 1\}$ leads to obtaining polynomial approximation of degree zero for y^*. Next, the solution of (1) is approximated by a polynomial of degree at most one and use the approximate solution in the first stage as initial guess in this stage. In the way same, this approach is continued until polynomials of upper degree yield for the approximation of y^*.

4 Numerical examples

In this section, it is used the method discussed of the previous sections for solving some examples, at first introduce maximum absolute error:

$$e_n = \max_{x \in [a,b]} |y_n^*(x) - y^*(x)|, \tag{11}$$

where y_n^* is the approximate solution obtained by the presented method.

Example 4.1. Consider the equations :

$$y(x) = e^x - (x+1)sin(x) + \int_{-1}^{x} sin(x)e^{-2t}y^2(t)dt, \quad x \in [-1,1] \tag{12}$$

$$y(x) = -x^3(-1+e^{sin(x)}) + sin(x) + \int_{0}^{x} x^3\cos(t)e^{y(t)}dt, \quad x \in [0,1] \tag{13}$$

whose the theirs exact solutions are $y(x) = e^x$ and $y(x) = \sin(x)$, respectively. Maximum absolute errors by current method are shown in table 1. Also, table 1 indicates that the results of this method having a rapid rate of convergence.

Table 1: Maximum absolute errors based on proposed method

n	2	4	6	8	10	12
e_n in (12)	0.10	$0.10e-2$	$0.67e-5$	$0.22e-7$	$0.44e-10$	$0.65e-13$
e_n in (13)	$0.24e-1$	$0.12e-3$	$0.40e-5$	$0.74e-7$	$0.77e-9$	$0.43e-11$

5 Conclusion

In this paper, the approximate solution of nonlinear Volterra integral equations is presented as a linear combination of some basic monomials. The unknown coefficients are calculated based on the minimization of the residual function. In addition, the existence and convergence of approximate solutions are investigated. This problem was solved by using Newton's method with a suitable initial point. It was observed that the approximate solutions based on the proposed method are very near to the best approximation.

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Numerical solution of Volterra integral equations of the second kind on Adomian decomposition method on the Taylor series with step length of h

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Abstract

In this paper, a new method for solving Volterra integral equations of the second kind on Adomian decomposition method on Taylor series expansion is evaluated with step length h ,then this method is used for the numerical solution of second kind Volterra equation integral ,the proposed method compared to the approximate solution of the problem have much less error.

Keywords: Numerical solution; Taylor series; Step length; Volterra integral equations; Adomian decomposition.
Mathematics Subject Classification [2010]: 45G05

1 Introduction

In this function

$$u(x) = f(x) + \int_0^x k(x,t)u(t)dt \tag{1}$$

that is a Volterra equation integral of the second kind resolved into component that will determine , in which the $u(x)$ is unknown function, $f(x)$ is known function and $k(x,t)$ is integral equation kernel.

Assume the functions $k(x,t), k_x(x,t), ...f, f_x$,...for $0 \leq x \leq T$ and $0 \leq t \leq x$ is continues. Then the equation (1) has uniqueness of a solution. Assume that $u(x) = \sum_0^\infty u_j(x)$ by utilize finite number of terms of the series($x = x_{i+1}$and $x_{i+1} - x_i = h$)which $u_0(x_{i+1}) = f(x_{i+1})$, $u(x_{i+1}) = \sum_0^n u_j(x_{i+1})$ approximation is heading. At each step of Taylor series expansion we get function $u_j(x_{i+1})$ to return $j = 1, 2, \ldots, n$.In this method, the focus is on non- homogeneous Volterra integral equation of the second kind (1) in which $k(x,t)$ is the kernel of equation.

This equation try using an decomposition method on Taylor expansion by step length h as a number of the equation (1) are deliberation .

As respected to the Adomian decomposition for solving Volterra equations:

$$u_0(x) = f(x), \quad u_n(x) = \int_0^x k(x,t)u(t)dt \tag{2}$$

*Speaker

Lemma 1.1. *Suppose F is a continuous function if $u(x) = f(x) + \int_0^x k(x,u)du$ In this case, the derivative with respect to x is obtained from the following equation.*

$$u^{'}(x) = f^{'}(x) + F(x,u) + \int_0^x \frac{\partial}{\partial x} F(x,u)du$$

Proof: See [1]

Example 1.2. consider the answer of Volterra integral equation of the second kind , the length step $h = 0.1$

$$u(x) = 1 + \int_0^x u(t)dt$$

The exact solution to the equation $u(x) = e^x$ and for the$x = 0.2$ to 8 decimal

$$u(0.2) = e^{0.2} \cong 1.22140275$$

And results obtained from the mentioned method

$$u_0(x) = f(x), \quad u_0(0.2) = 1$$

$u_1(0.2) = 0.2$, $u_2(0.2) = 0.02$, $u_3(0.2) = 0.00140267$ $u(0.2) \cong u_0 + u_1 + u_2 \cong 1.22140267$

The error of this method is $error = 8.81601698 \times 10^{-8}$.

Example 1.3. consider the answer of Volterra integral equation the second kind , with the length step$h = 0.02$,

$$u(x) = x^3 - x^5 + 5\int_0^x tu(t)dt$$

that the exact answer is $u(x) = x^3$ And for$x = 0.12$ we have

$$u(0.12) = (0.12)^3 \cong 0.001728$$

The proposed method for$x_0 = 0.1$ and $x_1 = 0.12$ is as follows

$$u(x) \cong u_0(x_1) + u_1(x_1) + u_2(x_1) \cong 0.0017276$$

The error is $error = 4 \times 10^{-7}$.

2 Main results

From Taylor expansion of the relationship (2) and return $n = 1$around $x = x_i$

$$u_1(x_{i+1}) = u_1(x_i) + hu_1^{'}(x_i) + \frac{h^2}{2!}u_1^{''}(x_i) + \cdots + \frac{h^j}{j!}u^j(x_i). \tag{3}$$

If we use the lemma (1.1) and relationships (1), (3).

$$u_1^{(j)}(x_i) = \sum_{r=1}^{j} \binom{j}{r} \left(\left(\frac{\partial^{j-1}}{\partial x_i^{j-1}} k(x_i, x_i) \right) u_0^{(r-1)}(x_i) + \int_0^{x_i} \frac{\partial^j}{\partial x_i^j} k(x_i, t) u_0(t) dt \right). \tag{4}$$

By substituting the equation(4) in equation (3):

$$u_1(x_{i+1}) = \int_0^{x_i} k(x_i, t) u_0(t) dt. + \sum_{j=1}^{n} \left(\sum_{r=1}^{j} \binom{j}{r} \frac{h^j}{j!} \left(\frac{\partial^{j-1}}{\partial x_i^{j-1}} k(x_i, x_i) \right) u_0^{(r-1)}(x_i) + \frac{h^j}{j!} \int_0^{x_i} \frac{\partial^j}{\partial x_i^j} k(x_i, t) u_0(t) dt \right). \tag{5}$$

$$\tag{6}$$

Where $u_0(x_i) = f(x_i)$.With going on this approach by $j = n$ and with the use of Taylor series expansion for the relationship (3)around $x = x_i$

$$u_n(x_{i+1}) = \int_0^{x_i} k(x_i, t) u_0(t) dt. + \sum_{j=1}^{n} \left(\sum_{r=1}^{j} \binom{j}{r} \frac{h^j}{j!} \left(\frac{\partial^{j-1}}{\partial x_i^{j-1}} k(x_i, x_i) \right) u_{n-1}^{(r-1)}(x_i) + \frac{h^j}{j!} \int_0^{x_i} \frac{\partial^j}{\partial x_i^j} k(x_i, t) u_{n-1}(t) dt \right). \tag{7}$$

If the value of h in equation (6)is neighbor to zero , we can waive then h third powers ($O(h3)$) to function $1 \leq j \leq n$, $u_j(x_{i+1})$

$$u_n(x_{i+1}) = \int_0^{x_i} k(x_i, t) u_{n-1}(t) dt + h \left(k(x_i, x_i) u_{n-1}(x_i) + \int_0^{x_i} k(x_i, t) u_{n-1}(t) dt \right) + \frac{h^2}{2} \begin{pmatrix} 2\frac{\partial}{\partial x_i} k(x_i, x_i) u_{n-1}(x_i) + k(x_i, x_i) u'_{n-1}(x_i) \\ + \int_0^{x_i} \frac{\partial^2}{\partial x_i^2} k(x_i, t) u_{n-1}(t) dt \end{pmatrix} + O(h^3). \tag{8}$$

We get the answer

$$u(x_{i+1}) = u_0(x_i) + u_1(x_i) + u_2(x_i) + ...$$

3 Result

The results suggest that to find the value of Volterra equations with an decomposition method based on Taylor series expansion by step length h using any more words it will apply with high accuracy, and it can be applied as a good approximation for the numerical solution of integral equations

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Poster On solving of the matrix equation $AX = B$ with respect to semi-tensor product pp.: 1–4

On solving of the matrix equation $AX = B$ with respect to semi-tensor product

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Abstract

This paper studies the solutions of the matrix equation $AX = B$ with respect to semi-tensor product. Firstly, the matrix-vector equation $AX = B$ with semi-tensor product is discussed. Compatible conditions are established for the matrices, and a necessary and sucient condition for the solvability of the matrix-vector equation is proposed and several examples are presented to illustrate the eciency of the results.

Keywords: Matrix equation, matrix-vector equation, semi-tensor product, Kronecker product
Mathematics Subject Classification [2010]: 65-XX, 65FXX, 65F10, 65N22

1 Introduction

In this paper, we study the solutions of the matrix equation $AX = B$ with respect to semi-tensor product, where $A \in M_{m\times n}$, $B \in M_{h\times k}$ are known, and X is to be solved. The semi-tensor product of matrices is proposed by Daizhan Cheng in order to solve linearization problem of nonlinear systems, and a detailed introduction can be found in [1]. Classical matrix theory is good at dealing with bilinear functions, but it can hardly be used for multilinear functions. However, using the semi-tensor product method, a multilinear function can be easily described in a matrix expression. Besides this, the semi-tensor product of matrices is proved to be a powerful tool in many other elds. By semitensor product, a logical system can be converted into an algebraic equation with the same form as a discrete system, and kinds of control problems of logical systems are studied [2]. Moreover semitensor product is well used in game theory [3], nonlinear systems, graph coloring [7] and fuzzy logic systems [4]. During the research, some matrix equations with semi-tensor product are involved.

Definition 1.1. let $A = [a_{ij}] \in M_{m\times n}$ and $B = [b_{ij}] \in M_{p\times q}$. The Kronecker product of A and B is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} \in M_{mp\times nq} \tag{1}$$

*Speaker

Lemma 1.2. *let $A \in M_{m\times n}$ and $B \in M_{m\times k}$ be given, and $X \in M_{n\times k}$ be unknown. The solvability of the matrix equation $AX = B$ is quivalent to the solvability of the matrix-vector equation*

$$(I_k \otimes A)V_c(X) = V_c(B). \tag{2}$$

Next we give a brief introduction of the semi-tensor product of matrices.

Definition 1.3. A Toeplitz matrix or diagonal-constant matrix is a matrix in which each descending diagonal from left to right is constant. For instance, the following matrix is a Toeplitz matrix:

$$\begin{pmatrix} a & b & c & d \\ e & a & b & c \\ f & e & a & b \end{pmatrix}$$

In this section, we discuss the solvability of the matrix-vector equation with semi-tensor product

$$AX = B \tag{3}$$

where $A \in M_{m\times n}$ and $B \in M_{h\times k}$ are known. The problem is to nd a vector X satisfying matrix-vector equation (3). Firstly, we start from the simple case $m = h$. Then the general case is studied.

Theorem 1.4. *The matrix equation $AX = B$, $X \in M_{p\times q}$, with semi-tensor product is equivalent to the following matrix-vector equation with conventional matrix product*

$$(I_q \otimes \overline{A})V_c(X) = V_c(B),$$

where

$$\overline{A} = \begin{pmatrix} V_c(\widehat{A_1}) & V_c(\widehat{A_2}) & \dots & V_c(\widehat{A_{\overline{q}}}) \end{pmatrix} = \begin{pmatrix} A_1 & A_{\overline{\alpha}+1} & \dots & A_{(\overline{p}-1)\overline{\alpha}+1} \\ A_2 & A_{\overline{\alpha}+2} & \dots & A_{(\overline{p}-1)\overline{\alpha}+2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\overline{\alpha}} & A_{2\overline{\alpha}} & \dots & A_{\overline{p}\overline{\alpha}} \end{pmatrix},$$

and A_i is the i-th column of A.

Lemma 1.5. *If matrix-vector equation (3) has a solution, then $\frac{n}{k}$ must be a positive integer, and the solution must belong to $\mathbb{C}^p$,where $p = \frac{n}{k}$.*

Lemma 1.6. *(1) If matrix equation (3) has a solution, $\frac{h}{m}$ must be a positive integer; (2) If a matrix with certain size $p \times q$ is a solution of matrix equation (3), we have that $p = \frac{n}{\alpha}\frac{h}{m}$, $q = \frac{k}{\alpha}$, where α is a common divisor of n and k, and satises $gcd\{\alpha, \frac{h}{m}\} = 1$.*

Corollary 1.7. *Matrix equation (3) has a solution belonging to $M_{\overline{p}\times\overline{q}}$ if and only if the following rank condition holds*

$$rank\overline{A} = rank \begin{pmatrix} \overline{A} & V_c(\widehat{B_1}) & V_c(\widehat{B_2}) & \dots & V_c(\widehat{B_{\overline{q}}}) \end{pmatrix}.$$

Example 1.8. 1) Take

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 & 2 & 2 \\ 4 & 1 & -1 & 2 \\ -1 & 4 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 \end{pmatrix}$$

Solution. By Lemma 1.5, the admissible sizes are 3×1, 6×2, 12×4, and it is easy to verify that $X_a = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}^T$ is a solution of matrix equation (3). Hence $X_a \otimes I_2$,$X_a \otimes I_4$ are also solutions of the matrix equation(3). By Corollary 3, matrix equation (3) does not have the unique solution for admissible size 12×4, but$X_a \otimes I_2$ is the unique solution for admissible size 6×2.By Remark 3, X_a is the unique solution for admissible size 3×1. 2) Take

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & & & & & & & & \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & & & & & & & & \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & & & & & & & & \\ 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & & & & & & & & \end{pmatrix}^T$$

By Lemma 5, the admissible sizes are 3×1, 6×2, 12×4. We can verify that A is invertible and

$$X_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & & & & & & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & & & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & & & & & & \end{pmatrix}^T$$

is the unique solution for admissible size 12×4,so $X_b = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ is the unique solution for admissible size 3×1 and $X_c = X_b \otimes I_2$ is the unique solution for admissible size 6×2. In the following, we assume that $m \mid h$ and investigate the solvability of the equation. Similarly, we give a necessary condition for the solvability of the matrix equation.

Theorem 1.9. *Suppose that matrix equation (3) has a solution belonging to $M_{p\times q}$ Split B into blocks of size $\frac{h}{m}$ by $\frac{k}{q}$, then each block is a Toeplitz matrix. Actually, matrix B is required to be in the following form:*

$$B = \begin{pmatrix} Block_{11}(B) & \dots & Block_{1q}(B) \\ \dots & \dots & \dots \\ Block_{m1}(B) & \dots & Block_{mq}(B) \end{pmatrix}, \tag{4}$$

where $Block_{ij}(B) \in M_{\frac{h}{m}\times\frac{k}{q}}, i = 1, ..., m,\ j = 1, ..., q$ *are Toeplitz matrices.*

To obtain all the solutions of matrix equation (3), rstly we gure out the admissible sizes satisfying the condition in Theorem 1.8. Then for each size p, the solutions of matrix equation (3) can be obtained via solving q matrix-vector equations with semi-tensor product. The rest is to solve the matrix-vector equations with semi-tensor product, which is the same as last section. We do not repeat it. Besides this, for each size p, matrix equation (3) can be transformed into the simple case $m = h$, and the solutions can be derived according to Theorem 1.4.

Remark 1.10. For a admissible size, even though it satises the condition in Theorem 1.9, there may be no solution for the matrix equation.

2 Main results

In this paper, the solvability of the matrix-vector equation $AX = B$ with semi-tensor product has been considered. For this case, compatible conditions of matrices have been derived, and a necessary and sucient condition for the solvability has been established. Furthermore, concrete solving methods have been provided. Based on this, the solvability of the matrix equation $AX = B$ with semi-tensor product has been studied. Similarly, compatible conditions, solvability conditions, and concrete solving methods of the matrix equation have been developed as well. For each part, several examples have been presented to illustrate the eciency of the results.

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Poster Real orthogonal eigenvalue decomposition of symmetric normal matrices pp.: 1–4

Real orthogonal eigenvalue decomposition of symmetric normal matrices

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Abstract

We propose an algorithm for eigenvalue decomposition of symmetric normal complex matrices via real orthogonal transformations. This algorithm answers positively to the open question which is raised in [M. Ferranti, R. Vandebril, *Computing eigenvalues of normal matrices via complex symmetric matrices*, J. Comput. Appl. Math., vol. 259, (2014), part A, 281-293].

Keywords: normal matrix, eigenvalue decomposition, real orthogonal transformation, common eigenvector.
Mathematics Subject Classification [2010]: 65F15, 65F30.

1 Introduction

There are various well-known methods for finding eigenpairs of complex matrices. Most of these methods are based on a two-step approach, first the original matrix is transformed to a unitary similar matrix of suitable shape, e.g. tridiagonal or Hessenberg matrix and then using standard methods like QR-methods, divide-and-conquer, etc. (see[3]) to compute the eigenvalue of a matrix. Though these two-step methods reduced the cost, but some of the properties of the original matrix can be neglected in these procedure. For example, when a symmetric normal matrix transformed to a tridiagonal matrix, the transformed matrix may not be normal anymore. In fact, a matrix is normal and symmetric if and only if it admits a real orthogonal eigenvalue decomposition [4], i.e. there are a real orthogonal matrix Q and a diagonal matrix Λ for a symmetric normal matrix A such that $A = Q\Lambda Q^T$. In this paper, we propose an algorithm for eigenvalue decomposition of any symmetric normal matrix A using only real orthogonal transformations.

Theorem 1.1. *[?] Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ be given and $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2 > 0$. If $\boldsymbol{x} = e^{i\theta}\boldsymbol{y}$ for some θ, let $U(\boldsymbol{x}, \boldsymbol{y}) = e^{i\theta} I_n$; otherwise, let $\phi \in [0, 2\pi)$ be such that $\boldsymbol{y}^*\boldsymbol{x} = e^{i\phi}|\boldsymbol{y}^*\boldsymbol{x}|$ (take $\phi = 0$ if $\boldsymbol{y}^*\boldsymbol{x} = 0$), let $\omega = e^{i\phi}\boldsymbol{y} - \boldsymbol{x}$ and let $U(\boldsymbol{x}, \boldsymbol{y}) = e^{i\phi} U_\omega$, in which $U_\omega = I - 2(\omega^*\omega)^{-1}\omega\omega^*$ is a Housholder matrix. . Then U is unitary and $U(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{y} = \boldsymbol{x}$.*

Theorem 1.2. *[?] Let $A \in M_n(\mathbb{C})$ be partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, in which A_{11} and A_{22} are square. Then A is normal if and only if A_{11} and A_{22} are normal and $A_{12} = 0$.*

Lemma 1.3. *[?] Let $\mathcal{N} \subset M_n(\mathbb{C}^n)$ be a commuting family of matrices, then some nonzero vector in $\mathbb{C}^n$ is an eigenvalue of every $A \in \mathcal{N}$.*

Let $A \in M_n$ be a symmetric and normal matrix. If $A = B + iC$ in which B and C are real then it is easy to check that B and C are symmetric matrices too and $\{B, C\}$ is a commuting real normal family.

Theorem 1.4. *[?] Let $A \in M_n(\mathbb{R})$ be a normal matrix. There is a real orthogonal $Q \in M_n(\mathbb{R})$ such that $Q^T AQ$ is a real quasidiagonal matrix*

$$A_1 \bigoplus \cdots \bigoplus A_m \in M_n(\mathbb{R}),$$

each A_i is 1-by-1 or 2-by-2. The 1-by-1 direct summands display all the real eigenvalues of A and each 2-by-2 direct summand has special form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ in which $a, b \in \mathbb{R}$, $b > 0$; it is normal and has eigenvalues $a \pm ib$.

In this theorem if in addition, A is a symmetric real matrix then it can not admit summand of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Therefore there are an orthogonal real matrix Q and an diagonal matrix Λ in which its diagonal entries are eigenvalues of A, such that $Q^T AQ = \Lambda$.

2 Main results

We want to construct an real orthogonal matrix Q such that $Q^T AQ = \Lambda$ be a diagonal matrix. For this aim, we use the Schur theorem and the construction which has been used in its proof as follows. Assuming A be a real symmetric matrix and $\lambda_1, \cdots, \lambda_n$ be its eigenvalues in any prescribed order and $\mathbf{x} \in \mathbb{R}^n$ be a unit eigenvector of A. Let $U_1 = sign(x_1)I_n$ if $\mathbf{x} = \pm \mathbf{e}_1$ and $U_1 = sign(x_1)U_\omega$ otherwise, where $\omega = sign(x_1)\mathbf{e}_1 - \mathbf{x}$, see Theorem 2.1 and x_1 is the first component of $\mathbf{x}$. Therefore $U_1 = [\mathbf{x}\, \mathbf{u}_2 \cdots \mathbf{u}_n]$ and

$$\begin{aligned} U_1^T AU_1 &= U_1^T [A\mathbf{x}\ A\mathbf{u}_2 \cdots A\mathbf{u}_n] \\ &= U_1^T [\lambda_1 \mathbf{x}\ A\mathbf{u}_2 \cdots A\mathbf{u}_n] \\ &= \begin{bmatrix} \mathbf{x}^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} [\lambda_1 \mathbf{x}\ A\mathbf{u}_2 \cdots A\mathbf{u}_n] \end{aligned}$$

since U_1 is an orthogonal matrix hence $U_1^T AU = \begin{bmatrix} \lambda_1 & \bigstar \\ 0 & A_1 \end{bmatrix}$. In the other hand, since $\begin{bmatrix} \lambda_1 & \bigstar \\ 0 & A_1 \end{bmatrix}$ is symmetric and real matrix so we have $U_1^T AU = \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{bmatrix}$ by Theorem 1.2. We have $A_1 = [\mathbf{u}_i^T A\, \mathbf{u}_j]_{i,j=2}^n$ and $\lambda_2, \cdots, \lambda_n$ are its eigenvalues. Similarly we can find V_2 such that $V_2^T A_1 V_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & A_2 \end{bmatrix}$, let $U_2 = [1] \bigoplus V_2$ so $(U_1 U_2)^T A(U_1 U_2) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & A_2 \end{bmatrix}$. Finally, this procedure yields to matrices $U_i \in M_{n-i+1}(\mathbb{R}), i = 1, \cdots, n-1$. Now, let $Q = U_1 U_2 \cdots U_{n-1}$. Suppose that A be a symmetric and normal matrix, one can write $A = B + iC$ where B and C are real and symmetric matrices and $BC = CB$. Let $\lambda_1, \cdots, \lambda_n$

and $\mu_1, \cdots, \mu_n$ be the eigenvalues of B and C respectively, and $\mathbf{x}$ be a common eigenvector of B and C. As we saw, we can write $U_1^T B U_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & B_1 \end{bmatrix}$ and $U_1^T C U_1 = \begin{bmatrix} \mu_1 & 0 \\ 0 & C_1 \end{bmatrix}$ for (n-1)-by-(n-1) matrices B_1 and C_1, we have $B_1 C_1 = C_1 B_1$ since $BC = CB$. Therefore

$$Q^T B Q = diag(\lambda_1, ..., \lambda_n) \quad \text{and} \quad Q^T C Q = diag(\mu_1, ..., \mu_n$$

hence

$$Q^T A Q = Q^T (B + iC) Q = diag(\lambda_1 + i\mu_1, ..., \lambda_n + i\mu_n)$$

We saw that there exists a real orthogonal matrix Q such that $Q^T B Q = \Lambda_1$ and $Q^T C Q = \Lambda_2$ are diagonal matrices, hence we have the eigenvalue decomposition $A = Q \Lambda Q^T$. For getting this decomposition, we need a common eigenvector of B and C which we know that it exists, see Lemma 1.3. Suppose $C\mathbf{y} = \mu \mathbf{y}$ for an eigenvector $\mathbf{y} \in \mathbb{R}^n$ and eigenvalue $\mu \in \mathbb{R}$ of the matrix C. Let k be the greatest integer such that the set $\{\mathbf{y}, By, \cdots, B^k \mathbf{y}\}$ is an linearly independent set. Let $\mathcal{W} = span\{\mathbf{y}, B\mathbf{y}, \cdots, B^k \mathbf{y}\}$ be the linear space of these vectors. Obviously, the vector space $\mathcal{W}$ is B-invariant and each non-zero vector in $\mathcal{W}$ is an eigenvector of C, because

$$C(B^j \mathbf{y}) = B^j (C\mathbf{y}) = \mu (B^j \mathbf{y}), \;\; j = 0, 1, \cdots, k.$$

Now, we find an eigenvector of B in $\mathcal{W}$. Using Gram-Schmidt method to construct an orthogonal basis $\mathcal{B}' = \{\mathbf{q}_1, \cdots, \mathbf{q}_k\}$ for $\mathcal{W}$, then we extend $\mathcal{B}'$ to an orthogonal basis $\mathcal{B} = \{\mathbf{q}_1, \cdots, \mathbf{q}_k, \mathbf{q}_{k+1}, \cdots, \mathbf{q}_n\}$ for $\mathbb{R}^n$. The matrix B is represented in the basis $\mathcal{B}$ by $B' = [\mathbf{q}_i^T B \mathbf{q}_j]_{1 \le i,j \le n}$. Since $\mathcal{W}$ is B-invariant, we have $\mathbf{q}_i^T B \mathbf{q}_j = 0$ for $i = k+1, \cdots, n$, $j = 1, \cdots, k$.

Therefore $B' = \begin{bmatrix} B'' & \bigstar \\ 0 & \bigstar \end{bmatrix}$ where $B'' = [\mathbf{q}_i^T B \mathbf{q}_j]_{1 \le i,j \le k}$ is a k-by-k matrix. Let λ be an eigenvalue of B'' and $\mathbf{z} = (z_1, \cdots, z_k)^T \neq 0$ be its associated eigenvector. Let $\mathbf{x} = z_1 \mathbf{q}_1 + \cdots + z_k \mathbf{q}_k$. We claim that $\mathbf{x} \in \mathcal{W}$ is a common eigenvector of B and C as we desired. The i-th entry in the identity $B''\mathbf{z} = \lambda \mathbf{z}$ is

$$\sum_{j=1}^{k} (\mathbf{q}_i^T B \mathbf{q}_j)(z_j) = \lambda z_i, \;\; i = 1, \cdots, k.$$

Multiplying these relations by the vector $\mathbf{q}_i$, we have

$$\sum_{j=1}^{k} \mathbf{q}_i^T B \mathbf{q}_j z_j \mathbf{q}_i = \lambda z_i \mathbf{q}_i, \;\; i = 1, \cdots, k.$$

Hence, we have

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \mathbf{q}_i^T B \mathbf{q}_j z_j \mathbf{q}_i = \lambda \sum_{i=1}^{k} z_i \mathbf{q}_i,$$

and by definition of $\mathbf{x}$ we have

$$\sum_{i=1}^{k} (\mathbf{q}_i^T B \mathbf{x}) \mathbf{q}_i = \lambda \mathbf{x}. \tag{1}$$

On the other hand $B\mathbf{x} \in \mathcal{W}$, so $B\mathbf{x} = \sum_{i=1}^{k} a_i \mathbf{q}_i$ for some scalars $a_i \in \mathbb{R}$, $i = 1, \cdots, k$. Multiplying this by $\mathbf{q}_i^T$ we have $\mathbf{q}_i^T B\mathbf{x} = a_i$, $i = 1, \cdots, k$. Finally

$$B\mathbf{x} = \sum_{i=1}^{k} (\mathbf{q}_i^T B\mathbf{x})\mathbf{q}_i \tag{2}$$

By (1) and (2) we have $B\mathbf{x} = \lambda\mathbf{x}$. Note that we know that $\mathbf{x}$ is eigenvector of C since $\mathbf{x} \in \mathcal{W}$, so $\mathbf{x}$ is a common eigenvector of B and C.

Example 2.1. Consider the following symmetric normal matrix A,

$$A = \begin{bmatrix} -1+5i & 2 & 4-14i \\ 2 & 2+3i & -2+8i \\ 4-14i & -2+8i & -1+i \end{bmatrix},$$

we can write $A = B + iC$ where

$$B = \begin{bmatrix} -1 & 2 & 4 \\ 2 & 2 & -2 \\ 4 & -2 & -1 \end{bmatrix} \quad \textbf{and} \quad C = \begin{bmatrix} 5 & 0 & -14 \\ 0 & 3 & 8 \\ -14 & 8 & 1 \end{bmatrix}.$$

A common eigenvector of B and C is $\mathbf{x} = \begin{bmatrix} -0.4884 \\ -0.8710 \\ -0.0528 \end{bmatrix}$ and $Q = \begin{bmatrix} -0.4884 & -0.5631 & -0.6667 \\ -0.8710 & 0.3608 & 0.3333 \\ -0.0528 & -0.7435 & -0.7435 \end{bmatrix}$.

Therefore

$$Q^T BQ = \begin{bmatrix} 3.0000 & -0.0000 & -0.0000 \\ -0.0000 & 3.0000 & 0.0000 \\ -0.0000 & 0.0000 & -6.0000 \end{bmatrix} \quad \textbf{and} \quad Q^T CQ = \begin{bmatrix} 3.4853 & -0.0000 & 0.0000 \\ -0.0000 & -13.4853 & -0.0000 \\ 0.0000 & -0.0000 & 19.0000 \end{bmatrix}.$$

Hence

$$A = Q \begin{bmatrix} 3.0000 + 3.4853i & 0 & 0 \\ 0 & 3.0000 - 13.4853i & 0 \\ 0 & 0 & -6.0000 + 19.0000i \end{bmatrix} Q^T.$$

The eigenpair in this example have been computed by Matlab's eig command.

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Reproducing kernel method for solving a class of Fredholm integro-differential equations

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Abstract

In this paper, we present a new algorithm in the reproducing kernel Hilbert space (RKHS) to solve integro-differential equations. The approximation solution is expressed by n-term summation of reproducing kernel functions. Some examples are displayed to illustrate the effectiveness and stability of the present method. Results obtained by the proposed method imply that it can be considered as a simple and accurate method for solving such integro-differential equation.

Keywords: Reproducing kernel, Integro-differential, Exact solution.
Mathematics Subject Classification [2010]: 47G20, 33F05, 41A10.

1 Introduction

Numerical modeling of integral and integro-differential equations have been paid attention by many scholars. Several numerical methods have been developed for the solution of the integro-differential equations. Particularly, in [1-5]. We study a class of Fredholm integro-differential equations in the reproducing kernel Hilbert space

$$u^{(n)}(x) = f(x) + \int_a^b k(x,t)(Nu(t))\,dt, \qquad u^{(k)}(x) = \alpha_k, \qquad 0 \le k \le n-1, n \ge 0, \quad (1)$$

where $u^{(n)}(x)$ is the nth derivative of the unknown function $u(x)$ that will be determined, $k(x,t)$ is the kernel of the integral equation, $f(x)$ is an analytic function, $N(u)$ is a linear function of u. Our aim in this paper is to obtain the analytical solutions by using the reproducing kernel method.

*Speaker

2 Reproducing kernel Hilbert space

In this section, to solve Eq. (1), first, we define some useful reproducing kernel spaces ${}^oW^{n+1}[a,b]$.

Definition 2.1. ${}^oW^{n+1}[a,b] = \{u^{(n)}(x)$ is an absolutely continuous real value function, $u^{(n+1)}(x) \in L^2[a,b]\}, u(a) = \alpha_0, u'(a) = \alpha_1, \cdots u^{(n)}(a) = \alpha_n\}$. The inner product and norm in ${}^oW^{n+1}[a,b]$ are given respectively by

$$\langle u, v\rangle = \sum_{i=0}^{n} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(n+1)}(x)v^{(n+1)}(x)\,\mathrm{d}x, \tag{2}$$

and

$$\|u\|_m = \sqrt{\langle u, u\rangle}_m, \qquad u, v \in {}^oW^{n+1}[a,b]. \tag{3}$$

It easy to see that $\langle u, v\rangle_{{}^oW^{n+1}[a,b]}$ satisfies all the requirements for the inner product. The space ${}^oW^{n+1}[a,b]$ is a reproducing kernel Hilbert space [6]. There exists $R_y(x) \in {}^oW^{n+1}[a,b]$, for any $u(y) \in {}^oW^{n+1}[a,b]$ and each fixed $x \in [a,b]$, $y \in [a,b]$, such that $\langle u(y), R_x(y)\rangle = u(x)$. We subsequently obtain a representation of the reproducing kernel in ${}^oW^{n+1}[a,b]$.

The reproducing kernel $R_y(x)$ can be denoted by

$$R_y(x) = \begin{cases} R_1(x,y) = \sum_{i=1}^{2n+2} c_i(y)x^{i-1}, & y \le x, \\ R_2(x,y) = \sum_{i=1}^{2n+2} d_i(y)x^{i-1}, & y > x, \end{cases} \tag{4}$$

where coefficients $c_i(y), d_i(y)$, $\{i = 1, 2, \cdots, 2n+2\}$, could be obtained by solving the following equations

$$\frac{\partial^i R_y(x)}{\partial x^i}|_{x=y+0} = \frac{\partial^i R_y(x)}{\partial x^i}|_{x=y-0}, \qquad i = 0, 1, 2, \cdots, 2n, \tag{5}$$

$$(-1)^{n+1}\left(\frac{\partial^{2n+1} R_y(x)}{\partial x^{2n+1}}|_{x=y+0} - \frac{\partial^{2n+1} R_y(x)}{\partial x^{2n+1}}|_{x=y-0}\right) = 1, \tag{6}$$

$$\begin{cases} \frac{\partial^i R_y(a)}{\partial x^i} - (-1)^{n-i}\frac{\partial^{2n-i+1} R_y(a)}{\partial x^{2n-i+1}} = 0, \\ \frac{\partial^{2n-i+1} R_y(b)}{\partial x^{2n-i+1}} = 0, \qquad i = 0, 1, \cdots, n, \\ R_y^{(i)}(a) = 0, \qquad i = 0, 1, \cdots n. \end{cases} \tag{7}$$

The definition of the spaces ${}^oW^m[a,b]$, $(m \ge n+1)$ are convenient for our numerical experiment and we consider the RKHS ${}^oW^6[a,b]$.

3 The analytical solution

3.1 Definition of operators

We define the operator $\mathbb{L} : {}^oW^m[a,b] \longrightarrow C[a,b], (m \ge n+1)$, as

$$\mathbb{L}(u) = u^{(n)}(x) - \int_a^b k(x,t)Nu(t)\mathrm{d}t, \qquad (n \ge 0), \tag{8}$$

then equation (1) can be written as

$$\mathbb{L}(u) = f(x). \tag{9}$$

It is clear that $\mathbb{L}$ is a bounded linear operator and $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$.

3.2 Solution of Eq. (1)

To obtain the solutions of Eq. (1), we define

$$\psi_i(x) = [\mathbb{L}_y R_y(x)](x_i) = \mathbb{L}^* R_y(x_i), \qquad i = 1, 2, \cdots, \tag{10}$$

where $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[a, b]$.

$$\psi_i(x) = \frac{\partial^n R_x(t)}{\partial t^n}|_{t=x_i} - \int_a^b k(x,t) NR(x,t) \mathrm{d}t, \qquad (n \geq 0). \tag{11}$$

The orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ of ${}^oW^m[a, b]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \qquad (\beta_{ii} > 0, \quad i = 1, 2, \cdots), \tag{12}$$

where β_{ik} are orthogonal coefficients.

Lemma 3.1. *The function system $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of the space ${}^oW^m[a, b]$.*

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$ and the solution of (1) is unique, then the solution of (1)*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k) \bar{\psi}_i(x), \tag{13}$$

and the approximate solution of equation (1) can be obtained by the n-term intercept of (13) and $u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k) \bar{\psi}_i(x)$.

4 Numerical experiments

Example 4.1. In this example, we solve the integro-differential equation

$$u'''(x) = \sin(x) - x - \int_0^{\frac{\pi}{2}} xtu'(t)\mathrm{d}t, \qquad u(0) = 1, \qquad u'(0) = 0, \qquad u''(0) = -1,$$

where the exact solution is $u_{ex}(x) = \cos(x)$. By the present method, taking $n = 10$ and $n = 20$, $x_i = \frac{\pi}{2(n+1)} \times i, i = 1, 2, \ldots, n$. The approximate solution, the absolute errors $|u_n(x) - u(x)|$ for $n = 10$ and $n = 20$ are graphically shown in figure 1, respectively. However, by increasing n, the behavior improves.

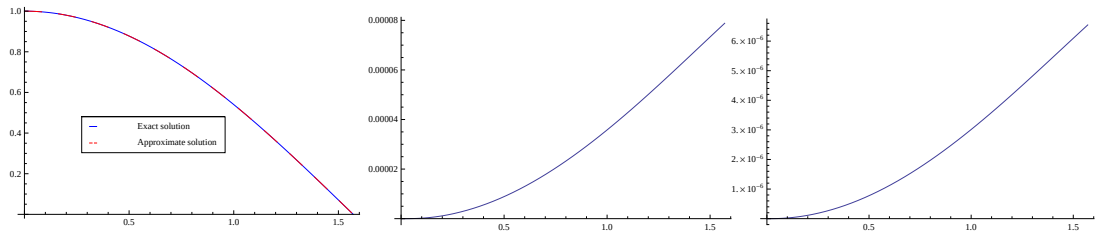

Figure 1: The approximate solution, the absolute errors for $n = 10$ and 20, respectively.

Example 4.2. Take

$$u''(x)+xu'(x)-xu(x) = e^x-2\sin(x)+\int_{-1}^{1}\sin(x)e^{-t}u(t)\mathrm{d}t, \qquad u(0)=1, \qquad u'(0)=1,$$

with the exact solution $u_{ex}(x) = e^x$. To solve this example nodes Example 1 were chosen with the same descriptions.

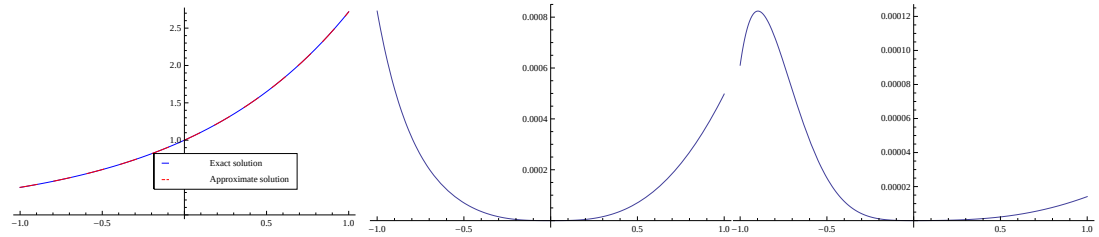

Figure 2: The approximate solution, the absolute errors for $n = 10$ and 20, respectively.

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Spectral properties of the packing matrix of fully binary tree
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Spectral Properties of The Packing Matrix of Fully binary tree

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Abstract

The packing matrix, is proposed by this paper, is designed as a means for uniquely representing the structure of a full binary tree (FBT). Then we establish a number of its spectral properties. Some results for the energy of the packing matrix are also obtained.

Keywords: Full binary tree, Packing Matrix, eigenvalues.
Mathematics Subject Classification [2010]: 42A05, 42A10

1 Introduction and Preliminary

For notation and graph theory and linear algebra terminology, we in general follow [1] AND [2]. Let $T = (V, E)$ be a rooted tree and m is a nonnegative integer. T is called an $m - ray$ tree, if $deg^+(v) \leq m$ for all $v \in V$. When $m = 2$, the tree is called a binary tree. if $deg^+(v) \leq 0$ or m for all $v \in V$, then T is called a complete $m - ray$ tree. The special case of $m = 2$ results in a complete binary tree. In a complete $m - ray$ tree, each internal vertex has exactly m children. (Each leaf of this tree still has no children.) If T is a rooted tree and h is the largest level number achieved by a leaf of T, then T is said to have height h. If T is a complete binary tree of height h, then T is called a full binary tree (FBT) if all the leaves in T are at level h. If $|V| = n$ we denoted T by the notation $FBT(n)$. As examples $FBT(3)$ and $FBT(7)$ are taken in figure 1. Its obvious that there are only $FBT(3)$, $FBT(7)$, $FBT(15)$, $FBT(31)$, $FBT(63)$ and in general case $FBT(\sum_{k=0}^{n} 2^k)$, for $n = 1, 2, 3, \cdots$ Let A be an $n \times n$ matrix. The determinant $det(A - \lambda I)$ is a polynomial in the (complex) variable λ of degree n and is called the characteristic polynomial of A: The equation $det(A - \lambda I) = 0$ is called the characteristic equation of A. By the fundamental theorem of algebra the equation has n complex roots and these roots are called the eigenvalues of A. The eigenvalues might not all be distinct. The number of times an eigenvalue occurs as a root of the characteristic equation is called the algebraic multiplicity of the eigenvalue. The energy of the graph G whose eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$ is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. A square matrix A is called symmetric if $A = A'$. The eigenvalues of a symmetric matrix are real. Furthermore, if A is a symmetric $n \times n$ matrix, then according

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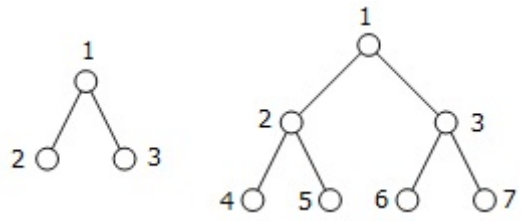

Figure 1: $FBT(3)$ and $FBT(7)$

to the spectral theorem there exists an orthogonal matrix P such that

$$PA\acute{P} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

In the case of a symmetric matrix the algebraic and the geometric multiplicities of any eigenvalue coincide. Also, the rank of the matrix equals the number of nonzero eigenvalues, counting multiplicities. Finally in terms of "Gershgorins Theorem", every eigenvalue of matrix $A_{n\times n}$ satisfies in the following inequality:

$$|\lambda - a_{ii}| \leq \sum_{i\neq j} |a_{ij}|, \quad i \in \{1, 2, \cdots, n\}$$

2 Main results.

Definition 2.1. Let T is a $FBT(n)$, then we define *Packing matrix* $A_{n\times n}$ such that

$$a_{ij} = \begin{cases} 1, & \text{if } v_i(v_j) \text{ is the left children of } v_j(v_i); \\ -1, & \text{if } v_i(v_j) \text{ is the right children of } v_j(v_i); \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Example 2.2. The packing matrices corresponding to $FBT(3)$, $FBT(7)$ and $FBT(15)$ are denoted with A_3, A_7 and A_{15} respectively and are given by following matrices.

$$A_3 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{15} = \begin{pmatrix}
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

It is immediately seen that $A_m = A(FBT(m))$ for $m = \sum_{k=0}^{n} 2^k, \quad n = 1, 2, 3, \cdots$ is a symmetric matrix with all diagonal elements equal to zero. Therefore its eigenvalues are real, and their sum is equal to zero. The eigenvalues of A_m form the spectrum of the packing matrix and may be ordered as

$$\lambda_1(A_m) \geq \lambda_2(A_m) \geq \lambda_3(A_m) \geq \cdots \geq \lambda_m(A_m)$$

For instance, with use "MATLAB" (numbers have been written with 4 decimal digits):
The characteristic polynomial of A_3 is $\lambda^3 - 2\lambda = 0$ and its eigenvalues are $\lambda_1 = 1.4142$, $\lambda_2 = 0$ and $\lambda_3 = -1.4142$.
The characteristic polynomial of A_7 is $\lambda^7 - 6\lambda^5 + 8\lambda^3 = 0$ and its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1.4142$, $\lambda_3 = \lambda_4 = \lambda_5 = 0$, $\lambda_6 = -1.4142$ and $\lambda_7 = -2$.
The characteristic polynomial of A_{15} is

$$\lambda^{15} - 14\lambda^{13} + 72\lambda^{11} - 168\lambda^9 + 176\lambda^7 - 64\lambda^5 = 0$$

and its eigenvalues are $\lambda_1 = 2.2882$, $\lambda_2 = 2$, $\lambda_3 = \lambda_4 = 1.4142$, $\lambda_5 = 0.8740$, $\lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 0$, $\lambda_{11} = -0.8740$, $\lambda_{12} = \lambda_{13} = -1.4142$, $\lambda_{14} = -2$ and $\lambda_{15} = -2.2882$,
The eigenvalues of A_{31} are $\lambda_1 = 2.4495$, $\lambda_2 = 2.2882$, $\lambda_3 = \lambda_4 = 2$, $\lambda_5 = \lambda_6 = \cdots = \lambda_9 = 1.4142$, $\lambda_{10} = 0.8740$, $\lambda_{11} = \lambda_{12} = \cdots = \lambda_{21} = 0$, $\lambda_{22} = -0.8740$, $\lambda_{23} = \lambda_{24} = \cdots\cdots = \lambda_{27} = -1.4142$, $\lambda_{28} = \lambda_{29} = -2$, $\lambda_{30} = -2.2882$, $\lambda_{31} = -2.4495$

Proposition 2.3. *For every $m = \sum_{k=0}^{n} 2^k, n = 1, 2, 3, \cdots$, if λ is a eigenvalues of A_m, then $-3 \leq \lambda \leq 3$.*

Proof. With definition of Packing matrix $A_m = A(FBT(m))$ (see difinition 2.1)and structure of FBT(m) (see Figure 1.) it is immediately seen in every rows of A_m there are at most 3 non zero numbers included 1 or -1. Now use the "Gershgorins Theorem", the result obtained. □

Definition 2.4. If T is a FBT(m) and $A_m = A(FBT(m))$ is its Packing matrix, then energy of A_m is called "packing energy" and calculated by $PE(T) = \sum_{i=1}^{m} |\lambda_i|$.

Lemma 2.5. *If $m = \sum_{k=0}^{n} 2^k, n = 1, 2, 3, \cdots$ then $m = 3t$ such that t is odd or $m = 3t+1$ such that t is even.*

Lemma 2.6. *For $A_m = A(FBT(m))$ such that $m = \sum_{k=0}^{n} 2^k, n = 1, 2, 3, \cdots$*
(a) The algebraic multiplicity of "zero" is $t + 1$ if $m = 3t + 1$.
(b) The algebraic multiplicity of "zero" is t if $m = 3t$.
Therefore the algebraic multiplicity of "zero" always is a odd number.

Proposition 2.7. *For every $m = \sum_{k=0}^{n} 2^k, n = 1, 2, 3, \cdots,$*
(a) If $m = 3t + 1$ then $PE(T) \leq 3(m - t - 1)$.
(b) If $m = 3t$ then $PE(T) \leq 3(m - t)$.

Proof. Use the Proposition 2.3 and Lemma 2.6. □

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The Combination of Collocation and Multigrid Methods in Solution of the 1-D Telegraph Equation

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Abstract

In this article, we introduce a high-order accurate method for solving one-space dimensional linear hyperbolic Telegraph equation. We apply a compact finite difference approximation of two and four orders for discretizing spatial derivative of linear telegraph equation and collocation method for the time component.

Keywords: collocation technique; compact finite difference schemes; high accuracy; multigrid method; telegraph equation

1 INTRODUCTION

In the present work we are dealing with the numerical approximation of the following second-order hyperbolic problem:

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \alpha \frac{\partial u}{\partial t}(x,t) + \beta u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t) \tag{1}$$

$$(x,t) \in [0,L] \times [0,\infty) \qquad \alpha, \beta > 0$$

with the initial conditions:

$$u(x,0) = f_1(x) \qquad \frac{\partial u}{\partial t}(x,0) = f_2(x) \tag{2}$$

and boundary conditions:

$$u(0,t) = h_1(t)) \qquad u(L,t) = h_2(t) \qquad t \geq 0 \tag{3}$$

Equation (1), referred to as second-order telegraph equation with constant coefficients.

2 DERIVATION OF THE METHOD

For a positive integer n let $h = \frac{1}{n}$denote the step size of spatial variable x, and Δt for step size of time variable t. So we define

$$x_i = ih \qquad i = 0, 1, ..., n \tag{4}$$

$$t_k = k\Delta t \qquad k = 0, 1, 2, ... \tag{5}$$

For derivation of the method, we first discretize Eq. (1) in space to obtain a system of ordinary differential equations with unknown function at each spatial grid point. Then in each spatial grid point, we construct a polynomial of r degree which depends on time and obtain unknown coefficients with collocation approach. For Eq. (1) if we discretize spatial derivatives in each grid point by the second order schemes (SO) we can obtain:

$$u_i''(t) + \alpha u_i'(t) + \beta u_i(t) - f_i(x,t) = \frac{1}{h^2}[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)] \tag{6}$$

Similarly if we discretize spatial derivatives in each grid point by the fourth order schemes (FO) [2], we can obtain:

$$\frac{1}{12}[(u_{i+1}''(t) + \alpha u_{i+1}'(t) + \beta u_{i+1}(t) - f_{i+1}(t)) + (u_{i-1}''(t) + \alpha u_{i-1}'(t) + \tag{7}$$
$$\beta u_{i-1}(t) - f_{i-1}(t)) + 10(u_i''(t) + \alpha u_i'(t) + \beta u_i(t) - f_i(t))] =$$
$$\frac{1}{h^2}[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)]$$

Where $u_i(t) = u(x_i,t), u_i'(t) = \frac{\partial u}{\partial t}(x_i,t), u_i''(t) = \frac{\partial^2 u}{\partial t^2}(x_i,t)$ and $f_i(t) = f(x_i,t)$. the initial and boundary conditions (2) and (3) for Eqs. (6) and (7) as follows:

$$u_i(0) = f_1(x_i), \qquad u_i'(0) = f_2(x_i), \tag{8}$$
$$u_0(t) = h_1(t), \qquad u_0'(t) = \frac{dh_1}{dt}(t), \qquad u_0''(t) = \frac{d^2h_1}{dt^2}(t),$$
$$u_n(t) = h_2(t), \qquad u_n'(t) = \frac{dh_2}{dt}(t), \qquad u_n''(t) = \frac{d^2h_2}{dt^2}(t)$$

If we write Eqs. (6) , (7) for each grid point, we obtain a system of (n + 1) ordinary differential equations of second-order that should be solved. We employ the collocation method in order to solve the resulted second order system of ordinary differential equations. We give this concept and other results for the system of ordinary differential equations obtained from (7) and can be easily stated for (6).
Let $P_i(t)$ be a polynomial of degree r which approximates the solution at grid point x_i and is as follows:

$$P_i(t) = a_{i,r}t^r + a_{i,r-1}t^{r-1} + ... + a_{i,1}t + a_{i,0} \tag{9}$$

For each i we have:

$$P_i(t_k) = a_{i,r}t_k^r + a_{i,r-1}t_k^{r-1} + ... + a_{i,1}t_k + a_{i,0} \qquad k = 0,1,...,r-2 \tag{10}$$

By forcing the initial conditions (2) we have the following relations for $a_{i,0}$ and $a_{i,1}$:

$$a_{i,0} = P_i(0) = u_i(0) = f_1(x_i), \tag{11}$$

$$a_{i,1} = \frac{dP_i(t)}{dt}|_{t=0} = \frac{\partial u}{\partial t}|_{(x_i,0)} = f_2(x_i) \tag{12}$$

So, for any given r, we have the (r-1) unknown coefficients $a_{i,r}, a_{i,r-1}, ..., a_{i,2}$ in each $P_i(t)$ that should be determined. To obtain these unknown coefficients, we can construct (r-1)

equations at grid point x_i by replacing $P_i(t)$ in (7) and using collocation points t_k , k = 0, 1, . . . , (r - 2).
If we apply the above procedure for each grid point x_i , i = 1, 2, . . . , (n-1), we obtain a linear system of$(r-1)\times(n-1)$ equations AX = B Where A is a block-tridiagonal matrix given by $A = tri[A_{i-1}, A_i, A_{i+1}]$, and A_{i-1}, A_i, A_{i+1} are $(r-1)\times(r-1)$ matrices and B is a vector consists of known values $a_{i,0}$ and $a_{i,1}$ for internal nodes and combination of $a_{i,0}$ and $a_{i,1}$ and boundary values of u , u_t and u_{tt} for near boundaries nodes. the vector X is consists of unknown values $a_{i,2}, a_{i,3}, ..., a_{i,r}$.
We present the multigrid method for solving the above linear system of equations. Now we want to use ICM method to obtain solution of Eq. (1) at any arbitrary time t = T. We first describe how one can obtain the solution of system of ordinary differential equations obtained from (7) on the interval $[t_0, t_0 + (r-2)\Delta t]$ in which $t_0 \neq 0$.
One approach that is given in [4] is to employ translation $u_i(t_0 + t) = P_i(t)$, i.e. for obtaining the approximate solution at any point in the interval $[t_0, t_0 + (r-2)\Delta t]$ it is sufficient to calculate $P_i(t)$ in which $t \in [0, (r-2)\Delta t]$. In this case the boundary values of u , u_t and u_{tt} and in right hand side vector B should be calculated at $t_0 + t_k$ where t_k are collocation points. Once the solution was obtained in the interval $[t_0, t_0 + (r-2)\Delta t]$, the above procedure can be successfully implemented to obtain solution at $[t_0 + (r-2)\Delta t, t_0 + 2(r-2)\Delta t]$.
Therefore, at each interval, the coefficient matrix is invariant and the right hand side vector B is updated. It should be noted that the approximate solution and its derivative respect to t which were calculated at the current interval are used as initial conditions for the next interval. The above procedure proceeds until the interval containing target time t = T is reached.

3 conclusion

We applied finite difference approximations of orders two and four for discretizing the spatial derivatives of telegraph equation. Also we used the collocation method for the time component. We introduced the procedure that can incorporate multigrid method with the mentioned schemes. numerical results show that compact finite difference approximations of fourth order, collocation and multigrid methods give a very efficient technique for solving the telegraph equation. The results of this paper can be extended to solve the two and three dimensional telegraph equation.

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The Construction of Fractional Integration Operational Matrix for Generalized Fractional order Legendre Functions

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Abstract

In this article, a general formulation for the generalized fractional-order Legendre functions (GFLFs) on the interval $[0, h]$ is constructed to obtain the numerical solution of the fractional integration of a given function. Numerical example illustrate the validity and applicability of the method.

Keywords: Generalized fractional-order Legendre functions, Operational matrix, Fractional calculus, Numerical integration
Mathematics Subject Classification [2010]: 65D32, 26A33

1 Introduction

In this paper, we intend to expand fractional Legender functions in interval $[0, h]$ to obtain the fractional integration operational matrix.

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $u \in \mathrm{C}_\mu$, $\mu \geqslant -1$, is defined as:

$$\mathrm{I}^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t)\, dt, \quad \alpha > 0, \tag{1}$$

where $\Gamma(.)$ is gamma function.

Definition 1.2. The Caputo definition of fractional derivative operator is given by:

$$\mathrm{D}^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{u^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt, & m-1 \leqslant \alpha < m, \\ \frac{d^m u(x)}{dx^m}, & \alpha = m, \quad x > 0, \end{cases} \tag{2}$$

The properties of the operators I^α and D^α, can be find in [1, 3]. For example two useful properties are as:

$$\mathrm{I}^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}, \tag{3}$$

$$\mathrm{I}^\alpha \left(\sum_{i=0}^m c_i u_i(x) \right) = \sum_{i=0}^m c_i \mathrm{I}^\alpha u_i(x), \quad \text{where } \{c_i\}_{i=0}^m \text{ are constants.} \tag{4}$$

*Speaker

2 Generalized fractional-order Legendre function

The fractional-order Legendre functions (FLFs) [2] can be defined by introducing the change of variable $t = x^\alpha$ for $\alpha > 0$ on shifted Legendre polynomials. These functions are denoted by $Fl_i^\alpha(x)$, $i = 1, 2, \ldots$. The fractional-order Legendre functions are a particular solution of the following Sturm-Liouville problem:

$$\left(\left(x - x^{1+\alpha}\right) Fl_i'^{\alpha}(x)\right)' + \alpha^2 i\,(i+1)\, x^{\alpha-1} Fl_i^\alpha(x) = 0, \quad x \in [0,1].$$

The functions $Fl_i^\alpha(x)$ can be obtained by the following recursive form:

$$Fl_{i+1}^\alpha(x) = \frac{(2i+1)\,(2x^\alpha - 1)}{i+1} Fl_i^\alpha(x) - \frac{i}{i+1} Fl_{i-1}^\alpha(x), \quad i = 1, 2, \ldots, \tag{5}$$

where $Fl_0^\alpha(x) = 1$, $Fl_1^\alpha(x) = 2x^\alpha - 1$.

Also, the analytic form of $Fl_i^\alpha(x)$ of degree α can be given by:

$$Fl_i^\alpha(x) = \sum_{s=0}^{i} b_{s,i} x^{s\alpha}, \quad i = 0, 1, 2, \ldots, \tag{6}$$

where $b_{s,i} = \frac{(-1)^{i+s}(i+s)!}{(i-s)!(s!)^2}$ and $Fl_i^\alpha(0) = (-1)^i$, $Fl_i^\alpha(1) = 1$.

The FLFs are orthogonal functions with respect to the weight function $\omega_l^\alpha(x) = x^{\alpha-1}$ on the interval [0,1], i.e:

$$\int_0^1 Fl_n^\alpha(x) Fl_m^\alpha(x) \omega_l^\alpha(x)\, dx = \frac{1}{(2n+1)\alpha} \delta_{nm}, \tag{7}$$

where δ_{nm} is the Kronecker delta.

By introducing the change of variable $t = xh$, we define the generalized fractional-order Legendre functions on the interval $[0, h]$. If these GFLFs are devoted by $Fl_i^{h\alpha}(x), i = 1, 2, \ldots$, then $Fl_i^{h\alpha}(x)$ has a recurrence formula as follows:

$$Fl_{i+1}^{h\alpha}(t) = \frac{(2i+1)(2(\frac{t}{h})^\alpha - 1)}{i+1} Fl_i^{h\alpha}(t) - \frac{i}{i+1} Fl_{i-1}^{h\alpha}(t), \quad i = 1, 2, \ldots, \tag{8}$$

where $Fl_0^{h\alpha}(t) = 1$, $Fl_1^{h\alpha}(t) = 2(\frac{t}{h})^\alpha - 1$, and analytic form of the $Fl_i^{h\alpha}(t)$ of degree α is given by:

$$Fl_i^{h\alpha}(t) = \sum_{s=0}^{i} b_{s,i} \frac{t^{s\alpha}}{h^{s\alpha}}, \quad i = 1, 2, \ldots, \tag{9}$$

The FLFs are orthogonal with the weight function $\omega_l^\alpha(t) = t^{\alpha-1}$ on the interval $[0, h]$, i.e:

$$\int_0^h Fl_n^{h\alpha}(t) Fl_m^{h\alpha}(t) t^{\alpha-1}\, dt = \frac{h^\alpha}{(2n+1)\alpha} \delta_{nm}. \tag{10}$$

2.1 Functions approximation

Suppose function $u(x)$ defined over the interval $[0, h]$, it can be expanded by GFLFs as:

$$u(x) = \sum_{i=0}^{\infty} c_i Fl_i^{h\alpha}(x), \tag{11}$$

where the coefficients c_i are obtained by:

$$c_i = \frac{(2i+1)\alpha}{h^\alpha} \int_0^h Fl_i^{h\alpha}(x) u(x) \omega_l^\alpha(x)\, dx, \quad i = 0, 1, 2, \ldots. \tag{12}$$

In practice, only the first m-terms GFLFs are consider. Then we have:

$$u(x) \simeq u_m(x) = \sum_{i=0}^{m-1} c_i Fl_i^{h\alpha}(x) = \mathrm{C}^{\mathrm{T}} \Phi(x),$$

where the GFLFs coefficients vector C and the GFLFs vector $\Phi(x)$ are given by:

$$\mathrm{C} = [c_0, c_1, \ldots, c_{m-1}]^{\mathrm{T}}, \quad \Phi(x) = [Fl_0^{h\alpha}(x), Fl_1^{h\alpha}(x), \ldots, Fl_{m-1}^{h\alpha}(x)]^{\mathrm{T}}. \tag{13}$$

Also, we can also approximate the arbitrary function $u(x,t) \in \mathrm{L}^2([0,h] \times [0,l])$ as follows:

$$u(x,t) \simeq \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_{ij} Fl_i^{h\alpha}(x) Fl_j^{l\beta}(t) = \Phi^{\mathrm{T}}(x) \mathrm{U} \Phi(t), \tag{14}$$

where U is an $m \times n$ matrix, with coefficients:

$$u_{ij} = (2i+1)(2j+1)\alpha\beta \times h^{-\alpha} l^{-\beta} \int_0^h \int_0^l u(x,t) Fl_i^{h\alpha}(x) Fl_j^{l\beta}(t) \omega_l^\alpha(x) \omega_l^\beta(t)\, dxdt, \tag{15}$$

where $i = 0, 1, \ldots, m-1$; $j = 0, 1, \ldots, n-1$, and

$$\Phi(x) = \left[Fl_0^{h\alpha}(x), Fl_1^{h\alpha}(x), \ldots, Fl_{m-1}^{h\alpha}(x)\right]^{\mathrm{T}}; \quad \Phi(t) = \left[Fl_0^{l\beta}(t), Fl_1^{l\beta}(t), \ldots, Fl_{n-1}^{l\beta}(t)\right]^{\mathrm{T}}.$$

3 The fractional integration operational matrix of GFLFs

Lemma 3.1. *The GFLFs Rimann-Liouville fractional integral of order $\beta > 0$ are the following form:*

$$\mathrm{I}^\beta Fl_i^{h\alpha}(x) = \sum_{s=0}^{i} \frac{b_{s,i}}{h^{s\alpha}} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha+\beta+1)} x^{s\alpha+\beta}. \tag{16}$$

Lemma 3.2. *Let $\alpha, \beta > 0$, then:*

$$\int_0^h \mathrm{I}^\beta Fl_i^{h\alpha}(x) Fl_j^{h\alpha}(x) \omega_l^\alpha(x)\, dx = \sum_{s=0}^{i} \sum_{r=0}^{j} \frac{b_{s,i} b_{r,j}}{(s+r+1)\alpha+\beta} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha+\beta+1)} h^{\alpha+\beta}. \tag{17}$$

In the following theorem we introduce the operational matrix of fractional integral of GFLFs.

Theorem 3.3. *Let $\Phi(x)$ be the GFLFs vector and $\beta > 0$, then:*

$$\mathrm{I}^{\beta}\Phi(x) \simeq \mathrm{P}^{\beta}\Phi(x), \tag{18}$$

where P^{β} is the $m \times m$ operational matrix of fractional integral of order β and the elements of the matrix $\mathrm{P}^{\beta} = [p_{ij}]$ are obtained as:

$$\{p_{ij}\}_{i,j=0}^{m-1} = (2j+1)\alpha h^{\beta} \sum_{s=0}^{i} \sum_{r=0}^{j} \frac{b_{s,i} b_{r,j}}{(s+r+1)\alpha + \beta} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha+\beta+1)}. \tag{19}$$

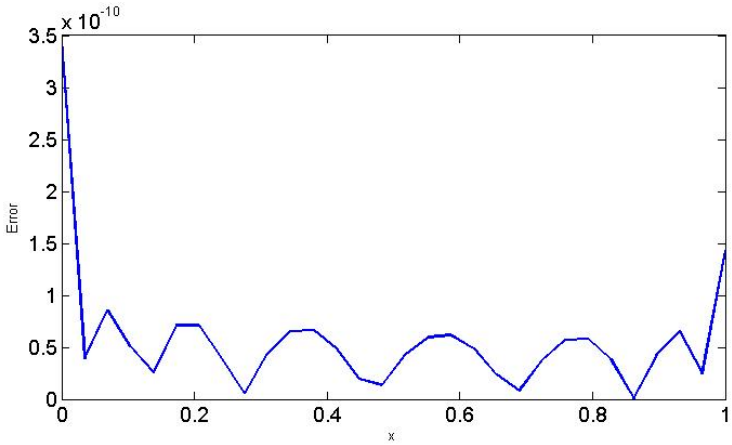

Figure 1: The error function with m=10

Remark 3.4. Suppose that $f(t) \in L^2([0,1])$. If $f(t) \cong \mathrm{C}^{\mathrm{T}}\Phi(\mathrm{t})$, then:

$$\mathrm{I}^{\beta}\mathrm{f}(\mathrm{t}) \cong \mathrm{I}^{\beta}\mathrm{C}^{\mathrm{T}}\Phi(\mathrm{t}) \cong \mathrm{C}^{\mathrm{T}}\mathrm{P}^{\beta}\Phi(\mathrm{t}).$$

Example 3.5. For $f(t) = t^2$, Fig.(1) shows the absolute error for the above approximation.

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Uniformly local biorthogonal wavelet constructions on intervals by extension operators

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Abstract

We construct a basis for a range of Sobolev spaces on interval $(-1,1)$ from corresponding bases on $(-1,0)$ and $(0,1)$ by the application of extension operators. Two examples of Hestenes extensions (as extension operators) are presented for constructing wavelets that are in $C^0(-1,1)$ and $C^1(-1,1)$.

Keywords: Wavelets, extension operators, Sobolev spaces
Mathematics Subject Classification [2010]: 42C40, 20E22, 46E35

1 Introduction

For $t \in [0,\infty) \setminus (\mathbb{N}_0 + \{\frac{1}{2}\})$ and $\vec{\sigma} = (\sigma_\ell, \sigma_r) \in \{0,\dots,\lfloor t+\frac{1}{2}\rfloor\}^2$, let

$$H^t_{\vec{\sigma}}(\mathcal{I}) := \{v \in H^t(\mathcal{I}) : v(0) = \cdots = v^{(\sigma_\ell-1)}(0) = 0 = v(1) = \cdots = v^{(\sigma_r-1)}(1)\}.$$

For t and $\vec{\sigma}$ as above, and for $\tilde{t} \in [0,\infty) \setminus (\mathbb{N}_0 + \{\frac{1}{2}\})$ and $\vec{\tilde{\sigma}} = (\tilde{\sigma}_\ell, \tilde{\sigma}_r) \in \{0,\dots,\lfloor \tilde{t}+\frac{1}{2}\rfloor\}^2$, let *univariate wavelet* collections $\Psi_{\vec{\sigma},\vec{\tilde{\sigma}}} := \{\psi_\lambda^{(\vec{\sigma},\vec{\tilde{\sigma}})} : \lambda \in \nabla_{\vec{\sigma},\vec{\tilde{\sigma}}}\}$, $\tilde{\Psi}_{\vec{\sigma},\,\vec{\tilde{\sigma}}} := \{\tilde{\psi}_\lambda^{(\vec{\sigma},\vec{\tilde{\sigma}})} : \lambda \in \nabla_{\vec{\sigma},\vec{\tilde{\sigma}}}\}$ be Riesz bases for $H^t_{\vec{\sigma}}(\mathcal{I})$ and $H^{\tilde{t}}_{\vec{\tilde{\sigma}}}(\mathcal{I})$, after renormalizing, that satisfy some properties in [1]. We assume to have available a univariate extension operator

$$\breve{G}_1 \in B(L_2(0,1), L_2(-1,1)) \text{ with } \begin{cases} \breve{G}_1 \in B(H^t(0,1), H^t(-1,1)), \\ \breve{G}_1^* \in B(H^{\tilde{t}}(-1,1), H^{\tilde{t}}_{(\lfloor \tilde{t}+\frac{1}{2}\rfloor,0)}(0,1)). \end{cases} \tag{1}$$

Let η_1 and η_2 denote the extensions by zero of functions on $(0,1)$ and on $(-1,0)$ to functions on $(-1,1)$, respectively, with R_1 and R_2 denoting their adjoints. We assume that $\breve{G}_1$ and its "adjoint extension", i. e., $\breve{G}_2 := (\mathrm{Id} - \eta_1\breve{G}_1^*)\eta_2$ are local. For $\breve{G}_1$, we will consider the Hestenes extension which is of the form $\breve{G}_1 v(-x) = \sum_{l=0}^L \gamma_l(\zeta v)(\beta_l x) \quad (v \in L_2(\mathcal{I}),\, x \in \mathcal{I})$, where $\gamma_l \in \mathbb{R}$, $\beta_l > 0$, and $\zeta : [0,\infty) \to [0,\infty)$ is a smooth cut-off function. Its adjoint reads as $\breve{G}_1^* w(x) = w(x) + \zeta(x)\sum_{l=0}^L \frac{\gamma_l}{\beta_l} w\Big(\frac{-x}{\beta_l}\Big)$ where $w \in L_2(-1,1)$ and $x \in \mathcal{I}$. A Hestenes extension satisfies (1) if and only if

$$\sum_{l=0}^L \gamma_l \beta_l^i = (-1)^i \,(\mathbb{N}_0 \ni i \le \lfloor t-\tfrac{1}{2}\rfloor),\ \sum_{l=0}^L \gamma_l \beta_l^{-(j+1)} = (-1)^{j+1} \,(\mathbb{N}_0 \ni j \le \lfloor \tilde{t}-\tfrac{1}{2}\rfloor).$$

Following [1] we will apply our construction using the modified, *scale-dependent* univariate extension operator

$$G_1 : u \mapsto \sum_{\lambda\in\bar\nabla^{(\ell)}_{0,0}} \langle u, \tilde\psi^{(\vec 0,\vec 0)}_\lambda\rangle_{L_2(\mathcal I)} \breve G_1\psi^{(\vec 0,\vec 0)}_\lambda + \sum_{\lambda\in\hat\nabla^{(\ell)}_{0,0}\cup\nabla^{(I)}\cup\nabla^{(r)}_{0,0}} \langle u, \tilde\psi^{(\vec 0,\vec 0)}_\lambda\rangle_{L_2(\mathcal I)}\eta_1\psi^{(\vec 0,\vec 0)}_\lambda. \tag{2}$$

such that $\nabla_{\vec\sigma,\vec{\tilde\sigma}}$ is the disjoint union of $\nabla^{(\ell)}_{\sigma_\ell,\tilde\sigma_\ell}$ (index set of left boundary wavelets), $\nabla^{(I)}$ (index set of interior wavelets), and $\nabla^{(r)}_{\sigma_r,\tilde\sigma_r}$ (index set of right boundary wavelets) and also $\nabla^{(\ell)}_{0,0} = \bar\nabla^{(\ell)}_{0,0}\cup\hat\nabla^{(\ell)}_{0,0}$ with $\bar\nabla^{\ell}_{0,0} = \{\lambda : \lambda\in\nabla_{\vec 0,\vec 0}, \psi^{\vec 0,\vec 0}_\lambda(0)\neq 0\}$. For simplicity, let us consider $L=0$, $\gamma_0 = 1$ (reflection), $\frac12 < t < \frac32$, $0 < \tilde t < \frac12$. So $\vec{\tilde\sigma} = \vec 0$, and $\sigma_\ell, \sigma_r \in \{0,1\}$. By definition of G_1, for $\sigma_\ell = 0$ we have

$$G_1\psi^{(\vec\sigma,\vec 0)}_\mu = \begin{cases} \eta_1\psi^{(\vec\sigma,\vec 0)}_\mu & \text{when } \mu\in\hat\nabla^{(\ell)}_{0,0}\cup\nabla^{(I)}\cup\nabla^{(r)}_{\sigma_r,0},\\ \breve G_1\psi^{(\vec\sigma,\vec 0)}_\mu & \text{when } \mu\in\bar\nabla^{(\ell)}_{0,0}.\end{cases} \tag{3}$$

Proposition 1.1. *For all $\mu\in\nabla_{\vec\sigma,\vec 0}$ the adjoint extension $G_2 := (\mathrm{Id} - \eta_1 G_1^*)\eta_2$ satisfies*

$$\operatorname{diam}(\operatorname{supp} G_2\tilde\psi^{(\vec\sigma,\vec{\tilde\sigma})}_\mu) \lesssim \operatorname{diam}(\operatorname{supp}\tilde\psi^{(\vec\sigma,\vec{\tilde\sigma})}_\mu).$$

To construct Riesz bases for Sobolev spaces on $(-1,1)$, we fix some $\sigma_r,\tau_\ell\in\{0,\cdots,\lfloor t+\frac12\rfloor\}$, and $\tilde\sigma_r,\tilde\tau_\ell\in\{0,\cdots,\lfloor\tilde t+\frac12\rfloor\}$. Then from [1, Corollary 4.6], we have that

$$\eta_2\Psi_{(\tau_\ell,\lfloor t+\frac12\rfloor),(\tilde\tau_\ell,0)}(1+\cdot)\cup G_1\Psi_{(0,\sigma_r),(\lfloor\tilde t+\frac12\rfloor,\tilde\sigma_r)}$$

is, properly scaled, a Riesz basis for $[L_2(-1,1), H^t_{(\tau_\ell,\sigma_r)}(-1,1)]_s$. Furthermore, its dual collection is $G_2\tilde\Psi_{(\tilde\ell,\lfloor t+\frac12\rfloor),(\tilde\tau_\ell,0)}(1+\cdot)\cup\eta_1\tilde\Psi_{(0,\sigma_r),(\lfloor\tilde r+\frac12\rfloor,\tilde\sigma_r)}$, where $G_2 := (\mathrm{Id}-\eta_1G_1^*)\eta_2$.

2 Biorthogonal multi-resolution analyses and wavelets

In order to construct wavelets that, properly scaled, generate Riesz bases for a range of Sobolev spaces, we use the following well-known theorem and proposition [3]. For simplicity, we replace $(\vec\sigma,\vec 0)$ with $\vec\sigma$.

Theorem 2.1. *Let $V_0^{\vec\sigma}\subset V_1^{\vec\sigma}\subset\cdots\subset L_2(\mathcal I)$, $\tilde V_0^{\vec{\tilde\sigma}}\subset\tilde V_1^{\vec{\tilde\sigma}}\subset\cdots\subset L_2(\mathcal I)$, be sequences of primal and dual spaces such that $\dim V_j^{\vec\sigma} = \dim\tilde V_j^{\vec{\tilde\sigma}} < \infty$. Let $\Phi_j^{\vec\sigma}$ and $\tilde\Phi_j^{\vec{\tilde\sigma}}$ be biorthogonal uniform $L_2(\mathcal I)$-Riesz bases of subspaces $V_j^{\vec\sigma}$ and $\tilde V_j^{\vec{\tilde\sigma}}$, respectively. In addition, Jackson estimate and Bernstein estimate hold for $V_j^{\vec\sigma}$ and $\tilde V_j^{\vec{\tilde\sigma}}$. Then with $\Phi_0^{\vec\sigma}$ and $\Psi_j^{\vec\sigma} = \{\psi^{\vec\sigma}_{j,k} : k\in J_j\}$ $(j\in\mathbb N)$, being uniform $L_2(\mathcal I)$-Riesz bases for $V_j^{\vec\sigma}\cap(\tilde V_{j-1}^{\vec{\tilde\sigma}})^{\perp_{L_2(\mathcal I)}}$ (wavelets), for some s the collection $\Psi_{\vec\sigma} := \Phi_0^{\vec\sigma}\cup\cup_{j\in\mathbb N}2^{-sj}\Psi_j^{\vec\sigma}$, is a Riesz basis for $[L_2(\mathcal I), H^d(\mathcal I)\cap H^1_{\vec\sigma}(\mathcal I)]_{s/d}$ when $s\geq 0$ and $([L_2(\mathcal I), H^{\tilde d}(\mathcal I)]_{-s/\tilde d})'$ when $s\leq 0$.*

Proposition 2.2. *Let $\Xi_{j+1}^{\vec\sigma}\subset V_{j+1}^{\vec\sigma}$ be such that $\Phi_j^{\vec\sigma}\cup\Xi_{j+1}^{\vec\sigma}$ is a uniform $L_2(\mathcal I)$-Riesz basis for $V_{j+1}^{\vec\sigma}$, and such that the basis transformations from $\Phi_j^{\vec\sigma}\cup\Xi_{j+1}^{\vec\sigma}$ to $\Phi_{j+1}^{\vec\sigma}$ and from $\Phi_{j+1}^{\vec\sigma}$ to $\Phi_j^{\vec\sigma}\cup\Xi_{j+1}^{\vec\sigma}$ are uniformly sparse. Then $\Psi_{j+1}^{\vec\sigma} := \Xi_{j+1}^{\vec\sigma} - \langle\Xi_{j+1}^{\vec\sigma},\tilde\Phi_j^{\vec{\tilde\sigma}}\rangle_{L_2(\mathcal I)}\Phi_j^{\vec\sigma}$, and its unique dual collection $\tilde\Psi_{j+1}^{\vec\sigma}$ in $\tilde V_{j+1}^{\vec{\tilde\sigma}}\cap V_j^{(\sigma)^{\perp_{L_2(\mathcal I)}}}$ are biorthogonal, uniformly local, uniform $L_2(\mathcal I)$-Riesz bases for $V_{j+1}^{\vec\sigma}\cap\tilde V_j^{\vec{\tilde\sigma}^{\perp_{L_2(\mathcal I)}}}$ and $\tilde V_{j+1}^{\vec{\tilde\sigma}}\cap V_j^{\vec\sigma^{\perp_{L_2(\mathcal I)}}}$, respectively.*

The relations Span $\Phi_{j+1}^{\vec{\sigma}} =$ Span $\Phi_j^{\vec{\sigma}} \cup$Span $\Psi_{j+1}^{\vec{\sigma}}$ and Span $\tilde{\Phi}_{j+1}^{\vec{\sigma}} =$ Span $\tilde{\Phi}_j^{\vec{\sigma}} \cup$Span $\tilde{\Psi}_{j+1}^{\vec{\sigma}}$ imply that any function in $\Psi_j^{\vec{\sigma}}$ and $\Phi_j^{\vec{\sigma}}$ and/or $\tilde{\Psi}_j^{\vec{\sigma}}$ and $\tilde{\Phi}_j^{\vec{\sigma}}$ can be expressed as a linear combination of the functions in $\Phi_{j+1}^{\vec{\sigma}}$ and $\tilde{\Phi}_{j+1}^{\vec{\sigma}}$, respectively. So matrices $M_j^{\vec{\sigma}}$ and $\tilde{M}_j^{\vec{\sigma}}$ exist such that $[(\Phi_j^{\vec{\sigma}})^\top \quad (\Psi_{j+1}^{\vec{\sigma}})^\top] = (\Phi_{j+1}^{\vec{\sigma}})^\top M_j^{\vec{\sigma}}$, $[(\tilde{\Phi}_j^{\vec{\sigma}})^\top \quad (\tilde{\Psi}_{j+1}^{\vec{\sigma}})^\top] = (\tilde{\Phi}_{j+1}^{\vec{\sigma}})^\top \tilde{M}_j^{\vec{\sigma}}$. Moreover

$$(\Phi_{j+1}^{\vec{\sigma}})^\top = (\Phi_j^{\vec{\sigma}})^\top G_{j,0}^{\vec{\sigma}} + (\Psi_{j+1}^{\vec{\sigma}})^\top G_{j,1}^{\vec{\sigma}}$$

where $\tilde{M}_j^{\vec{\sigma}} = (M_j^{\vec{\sigma}})^{-\top} := \begin{bmatrix} G_{j,0}^{\vec{\sigma}} \\ G_{j,1}^{\vec{\sigma}} \end{bmatrix}$. Regarding to the locality of $\Phi_j^{\vec{\sigma}}$, $\Psi_j^{\vec{\sigma}}$ and their corresponding dual collections, $M_j^{\vec{\sigma}}$ and $\tilde{M}_j^{\vec{\sigma}}$ are uniformly local. To build dual wavelet collection, we need to determine $M_j^{\vec{\sigma}}$ or $\tilde{M}_j^{\vec{\sigma}}$. On account of biorthogonality between dual and primal scaling functions collections, we conclude that $M_j^{\vec{\sigma}} = \langle \tilde{\Phi}_{j+1}^{\vec{\sigma}}, [(\Phi_j^{\vec{\sigma}})^\top \quad (\Psi_{j+1}^{\vec{\sigma}})^\top]^\top \rangle_{L_2(\mathcal{I})}$. To avoid the use of the $L_2(\mathcal{I})$-inner product, we define the operator $\boldsymbol{\pi}_j^{\vec{\sigma}} : V_j^{\vec{\sigma}} \to \mathbb{R}^{\dim V_j^{\vec{\sigma}}}$,

$$\begin{aligned}\boldsymbol{\pi}_j^{\vec{\sigma}}(v_j) := [&v_j(0), v_j^{(\sigma_\ell)}(0), v_j(x_1), v_j(x_2), v_j'(x_2), v_j(x_3), v_j(x_4), v_j'(x_4),\\ &\ldots, v_j(x_{m-3}), v_j(x_{m-2}), v_j'(x_{m-2}), v_j(x_{m-1}), v_j(1), v_j^{(\sigma_r)}(1)],\end{aligned}$$

where $x_k := k2^{-(j+2)}, k = 0, \ldots, m$ and $m = 2^{j+2}$. By applying $\boldsymbol{\pi}_{j+1}^{\vec{\sigma}}$ to $[(\Phi_j^{\vec{\sigma}})^\top \quad (\Psi_{j+1}^{\vec{\sigma}})^\top] = (\Phi_{j+1}^{\vec{\sigma}})^\top M_j^{\vec{\sigma}}$, we deduce that $M_j^{\vec{\sigma}} = (\boldsymbol{\pi}_{j+1}^{\vec{\sigma}}(\Phi_{j+1}^{\vec{\sigma}}))^{-1} \boldsymbol{\pi}_{j+1}^{\vec{\sigma}}([(\Phi_j^{\vec{\sigma}})^\top \quad (\Psi_{j+1}^{\vec{\sigma}})^\top])$.

3 Examples of Hestenes extensions

As the univariate wavelet construction, we apply the C^1, piecewise quartic (so $d = 5$) (multi-) wavelets, with (discontinuous) piecewise quartic duals as constructed in [2] on interval $(0, 1)$. The relevant Hestenes extension with $\beta_0 = 1$, $L = 0$, $\gamma_0 = 1$ is formed by $\breve{G}_1 v := v(-x)$ when $x \in (-1, 0)$ and $v(x)$ when $x \in \mathcal{I}$ and it's dual reads as $\breve{G}_1^* v(x) = v(x) + v(-x)$, $x \in \mathcal{I}$ and $\breve{G}_1^* \in B(H^{\tilde{t}}(0,1), H^{\tilde{t}}(-1,1))$ for $\tilde{t} < \frac{1}{2}$ ($v \in L_2(\mathcal{I})$). The adjoint

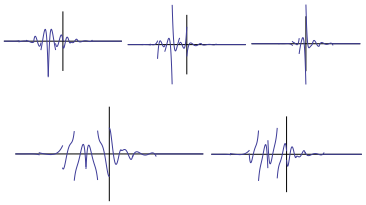

Figure 1: The five $G_2\tilde{\psi}_\mu^{(1,1)}$ on $(-1, 1)$. The vertical line is $x = 0$.

$\breve{G}_1$, namely $\breve{G}_2$, for all $v \in L_2(-1, 1)$ can be written as $\breve{G}_2 v(x) := v(x)$ when $-1 < x < 0$ and $v(-x)$ when $x \in \mathcal{I}$.

For some $\mu \in \nabla_{\sigma_r, \tilde{\sigma}_r}^{(r)}$, the operator G_2 expands $\tilde{\psi}_\mu^{(1,1)}(1 + \cdot)$ to $(-1, 1)$ for $\mu \in \nabla_{1,0}^{(r)}$ and the set of expanded dual wavelets per level $1 < |\mu|$ has a cardinality of "5 ". The corresponding dual wavelets on $(-1, 1)$ are illustrated in Figure 1. Now we consider a Hestense extension with $\beta \notin \{2^\ell : \ell \in \mathbb{N}_0\}$, e.g., $\beta = \frac{1}{3}$ and $L = 0, \gamma_0 = 1$. Figure 2 shows that $G_2\tilde{\psi}_\mu^{(1,1)}(1+\cdot)$ for some $\mu \in \nabla_{1,0}^{(\ell)}$ are not local. Now we would like to construct

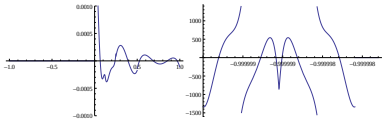

Figure 2: Left: One dual wavelet function $G_2\tilde{\psi}_\mu^{(1,1)}(1+\cdot)$ on $(-1,1)$ when $\mu \in \nabla_{1,0}^{(\ell)}$ and $|\mu| = 20$. Right: Zoom in the left figure around right side $x = -1$.

wavelet functions by Hestense extension that their derivatives are continues at interface $x = 0$. The relevant Hestense extension is given by $\breve{G}_1 v := 3v(-x) - 2v(-2x)$ when $x \in (-1, 0)$ and $v(x)$ when $x \in \mathcal{I}$. We apply G_1 to a few wavelet functions $\psi_\mu^{\vec{0}}$ where $\psi_\mu^{\vec{0}}(0) \neq 0$ or $\psi_\mu'^{\vec{0}}(0) \neq 0$. In our case, just two wavelet functions will be extended from $\mathcal{I}$ to $(-1, 1)$. The corresponding C^1-wavelet functions are illustrated in Figure 3.

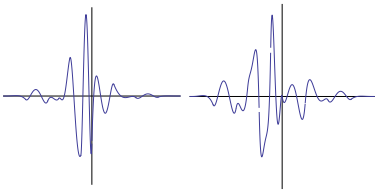

Figure 3: The extended C^1-wavelet functions $G_1\psi_\mu^{(\vec{\sigma},\vec{\sigma})}$.

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Operation Research & Control Theory

A New active queue management based on the prediction of the packet arrival rate

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Abstract

In this paper,we predict the change in the packet arrival rate at the link through the analysis of the network congestion control mechanism. An appropriate expression for dropping probability is derived based on this prediction to stabilize the queue length to the desired value. Its analysis of the stability is also carried out, and the necessary and sufficient condition for the system to be stable is presented

Keywords: Network congestion control, Active Queue Management(AQM), Packet arrival rate, Prediction, Stability
Mathematics Subject Classification [2010]: 34B15, 76A10

1 Introduction

With the growth of computer networks,excessive request for the limited network resources results in more and more serious congestion. Network congestion avoidance and control [1] gathers increasing attention in the past three decades. Transmission Control Protocol (TCP) and Active Queue Management (AQM) are the effective congestion control mechanisms at the end hosts and links,respectively.
In this paper,the prediction of packet arrival rate is derived from the analysis of the network congestion control mechanism. A new AQM algorithm named as Straightforward AQM ($SFAQM$) is proposed based on such a prediction.

2 Prediction of the change of packet arrival rate

Consider a system where there is a single congested router with a transmission capacity of C. Let N TCP flows(compliant with protocol of TCP Reno)traverse the router,labeled $i = 1, \ldots, N$$W_i(t)$, $V_i(t)$ and $R_i(t)$ denote the congestion window size,packets ending rate and Round Trip Time(RTT)of flow $TCP_i(i = 1, \ldots, N)$ at time $t > 0$, respectively.Let $\lambda(t)$ denote the packets arrival rate at the router at time $t > 0$, then

$$V_i(t) = W_i(t)/R_i(t), \tag{1}$$

*Speaker

$$\lambda(t) = \sum_{i=1}^{n} V_i(t), \qquad R_i(t) = r_i + q(t)/C, \tag{2}$$

where r_i is the Round Trip Propagation Time($RTPT$) of TCP_i, $q(t)$ is the queue length at the congested link,and $q(t)/C$ models the queuing delay.
The TCP strategy has the characteristic of Additive Increase and Multiplicative Decrease($AIMD$) [**?**]. Corresponding to TCP's multiplicative decrease phase,if a packet is dropped,the congestion window size of TCP_i will decrease to $W_i(t)/2$, and the sending rate becomes $V_i(t)/2$. Corresponding to its additive increase phase,if a packet is acknowledged,the TCP_i will increase its window size $1/W_i(t)$, and the sending rate increases with $1/(R_i(t) * W_i(t))$. Thus,the expectation of the increment of the arrival rate at the router will be

$$\Delta\lambda(t) = \frac{1}{W_i(t)R_i(t)}(1 - p(t-\tau)) - (V_i(t)/2)p(t-\tau), \tag{3}$$

where $\tau > 0$ represents the time delay from the moment that the packets are dropped or acknowledged to the moment that the host receives the information.
the expressions for the expectations of the sending rate $V_i(t)$ and window size $W_i(t)$ will be developed.The proportion of the packets that are generated by TCP_i is $V_i(t)/\lambda(t)$. Hence,the expectations of the send rate and window size are respectively

$$\overline{V}(t) = \sum_{i=1}^{n} V_i(t)\frac{V_i(t)}{\lambda(t)} \geq \frac{(\sum_{i=1}^{n} V_i(t))^2}{N\lambda(t)} = \frac{\lambda(t)}{N}, \tag{4}$$

$$\overline{W}(t) = \sum_{i=1}^{n} W_i(t)\frac{V_i(t)}{\lambda(t)} \geq \frac{(\sum_{i=1}^{n} V_i(t))^2 R}{N\lambda(t)} = \frac{\lambda(t)R}{N}, \tag{5}$$

the variables $W_i(t)$, $V_i(t)$ and $R_i(t)$ have been estimated for (3). Let the packet dropping probability be updated once in every time interval and the number of arriving packets and the dropping probability at the congested router be $m(t)$ and $p(t)$, respectively.Note that the dropping process of each packet is independent of the process of other packets during each sample interval.Recalling $Eqs.(4)(6)$, the expectation of the increment of arrival rate at the router will be

$$\overline{\lambda}(t+\tau) \geq \sum^{m(t)} W_i \frac{N(1-p(t))}{\lambda(t+\tau)R_i^2(t+\tau)} - \frac{\lambda(t+\tau)p(t)m(t)}{2N} \approx \frac{N(1-p(t))m(t)}{\lambda(t+\tau)R^2} - \frac{\lambda(t+\tau)p(t)m(t)}{2N}. \tag{6}$$

3 New AQM algorithm

Assume that the number of packets that will arrive at the router in the interval related to time $t > 0$ is $m(t)$. The arrival rate is $\lambda(t+\tau)$ when the congestion information arrive at the hosts at time $t+\tau > 0$, the desired arrival rate at the link is $\lambda_{ref}(t+\tau)$, and then we obtain

$$\lambda(t+\tau) + \Delta\lambda(t+\tau) = \lambda_{ref}(t+\tau). \tag{7}$$

To achieve the desired queue length q_{ref}, the desired arrival rate at the router should be

$$\lambda_{ref}(t+\tau) = C + (q_{ref} - q(t))/\alpha. \tag{8}$$

Here, α is a parameter related to the time cost to control queue length to the desired value.Using (6) in (7), we obtain

$$p(t) = \frac{\lambda(t+\tau) + \frac{m(t)N}{\lambda(t+\tau)R^2} - \lambda_{ref}(t+\tau)}{(\frac{\lambda(t+\tau)}{2N} + \frac{N}{\lambda(t+\tau)R^2})m(t)}. \tag{9}$$

We estimate the arrival rate $\lambda(t)$ by counting the number of arriving packets $m(t)$ at time $t > 0$ as follows:

$$\lambda(t) = m(t)\delta, \tag{10}$$

where δ is the length of the sampling period. Considering this, (9) can be expressed as

$$p(t) = \frac{m(t+\tau) + \frac{m(t)N\delta^2}{m(t+\tau)R^2} - mref(t+\tau)}{(\frac{m(t+\tau)}{2N} + \frac{N\delta^2}{m(t+\tau)R^2})m(t)}, \tag{11}$$

where $m(t)$ and $m(t+\tau)$ are the number of arriving packets during the interval related to time t and $t+\tau$ respectively,and $m_{ref}(t+\tau)$ can be expressed as

$$m_{ref}(t+\tau) = C\delta + (q_{ref} - q(t))\delta\alpha. \tag{12}$$

The last task is to estimate $m(t)$ and $m(t+\tau)$. In this paper we predict $m(t)$ and $m(t+\tau)$ through simple analysis.The value of $m(t)$ is predicted as the exponential weighted moving average ($EWMA$) of $m(t)$. Since $m(t+\tau)$ should change from $m(t)$ to $m_{ref}(t+\tau)$, $m(t+\tau)$ is predicted as a value between $m(t)$ and $m_{ref}(t+\tau)$. Hence, $m(t)$ and $m(t+\tau)$ are expressed as

$$m(t) = (1-\omega_1)m(t-\delta) + \omega_1 m_0 \tag{13}$$

$$m(t+\tau) = (1-\omega_2)m(t) + \omega_2 m_{ref}(t+\tau). \tag{14}$$

4 Stability

The fluid-flow models have been widely used to describe the TCP and queue dynamics, such as [2, 3]. In this paper the model introduced in Low et al[4]. is used because it has the same value of dropping probability as $SFAQM$ at the operating point, which will be mentioned later. According to the demands of the analysis in this paper,we consider a network with one bottleneck link, with only one TCP flow from each endhost.Furthermore, the variation of RTT is ignored. The model is as follows:

$$\begin{cases} \dot{W}_i(t) = \frac{W_i(t-R)}{R}(1-p(t-R))1/W_i(t) - \frac{W_i(t-R)}{R}\frac{W_i(t)}{2}p(t-R) \\ \dot{q}(t) = -C + \sum_{i=1}^{N}\frac{W_i(t)}{R} \end{cases} \tag{15}$$

where $W_i(t)$ is the window size of TCP_i at $t > 0$, R is the RTT defined in (4), $p(t)$ is the packet dropping probability,and $q(t)$ is the queue length at the link.As we are interested in the average behavior of the flows instead of any specific one, $V_i(t)$ is approximated to the expected packets ending rate $V(t)$ of flows.Using (1) in this model,(15) can be rewritten as follows:

$$\begin{cases} \dot{V}(t) = \frac{V(t-R)}{R^2V(t)}(1 - p(t-R)) - 1/2V(t-R)V(t)p(t-R) \\ \dot{q}(t) = -C + NV(t) \end{cases} \tag{16}$$

Taking (V, q) as a state and p as the input,the operating point (V_0, q_0, p_0) is defined by $\dot{V}(t) = 0$ and $\dot{q}(t) = 0$, hence

$$\dot{V}(t) = 0 \Longrightarrow p_0 = \frac{1}{V_0^2R^2/(2) + 1} \tag{17}$$

$$\dot{q}(t) = 0 \Longrightarrow V_0 = C/N. \tag{18}$$

Recall the $SFAQM$ algorithm. Rewrite (11) by substitution of (10),(12), and (14) as

$$p(t) = \frac{[(1-\omega_2)\lambda(t) + \omega_2\lambda_{ref}(t+\tau)]^2 + N\delta\lambda(t)/(R^2) - \lambda(t)\lambda_{ref}(t+\tau)}{(\frac{[(1-\omega_2)\lambda(t)+\omega_2\lambda_{ref}(t+\tau)]^2}{2N} + N/R^2)\lambda(t)\delta}. \tag{19}$$

When the system is at the steady state, the queue occupation is stabilized at the reference value q_{ref}. The packet arrival rate is stabilized at link capacity C, and the desired packet arrival rate λ_{ref} is also stabilized at C according to (8). Then, packet dropping probability will be

$$p_0 = \frac{1}{C^2R^2/2N^2 + 1}. \tag{20}$$

Linearize (16) and (19) at the operating point,then we obtain

$$\begin{cases} \dot{V}(t) = -\frac{C}{N(C^2R^2/2N^2+1)}\delta V(t) - C^2/2N^2\delta p(t-R) \\ \dot{q}(t) = N\delta V(t) \\ \delta p(t) = \frac{\partial p}{\partial \lambda}\delta\dot{q}(t) - 1/\alpha\frac{\partial p}{\partial \lambda_{ref}}\delta q(t) \end{cases} \tag{21}$$

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Poster A new applicable method for solving fractional differential equations pp.: 1–4

A New Applicable Method for Solving Fractional Differential Equations

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Abstract

This paper presents a new idea to solve fractional differential equation based on the linear programming problem. Indeed, by using the first concept of fractional derivative, we will suggest a method where an equation with fractional derivative is changed to linear programming and by solving it, the fractional derivative will be obtained. Actually this suggested method is based on the minimization of total error. Also some numerical examples are provided to confirm the accuracy of the proposed method.

Keywords: Fractional calculus, fractional differential equation, Linear programming
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

It is difficult to solve a differential equation exactly, so recently some approximated methods were presented to solve them, for instance Homotopy Interfrence method, Homotopi analytical method, and numerical methods. Most of the researches present different methods to formulate and model the physical structures. In last years, science and engineering scientists paid attention to fraction calculations.Usages of fractional differential equations in physics and engineering significantly have increased. Many phenomena, in various fields, can be modeled by fractional differential equations, like some in control problems, statistical models, economic issues, electromagnetism, electrochemistry, telecommunication lines and so on.

Fractional calculation is generalization of ordinary derivative and integral which has non- integer arbitrary order. History of these problems like differential calculus back to the time when Leibniz and Newton invented differential calculus. Fraction calculus has introduced in September 30 of 1695 for the first time [1]. In 1812, Laplace defined a fraction integral as a fractional derivative [2]. In 1819, the first description of a fractional derivative was written by Lacroix in computational version [3]. First step in expanding differential arbitrary functions was by Fourier in 1822 [4]. But the first theorem was written by Liouville in 1823. The famous formula which is useful today named RiemannLiouville integral inserted by Reimann in 1847. Next we will describe some definitions in fractional derivatives concept.

*Speaker

In the next, a brief description of initial definition of fractional calculus which is needed in this paper is described.

Definition 1.1. $\Gamma : (0, \infty) \to \mathbb{R}$ is known as the Euler-Gamma function (or Euler integral of the second kind) and is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-} e^t dt. \tag{1}$$

Definition 1.2. Suppose that $n \in \mathbb{R}^+$. The operator $J_a^n f(x)$ is defined on $L_[a, b]$ as the following form:

$$J_a^n f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt, \tag{2}$$

which is Riemann-Liouville fractional operator of order n. For $n = 0$, it is identity operator $J_a^0 = I$.

Definition 1.3. Let $n \in \mathbb{R}^+$ and $m = [n]$. The operator D_a^n is defined as:

$$D_a^n f = D^m J_a^{m-n} f, \tag{3}$$

which is named RiemannLiouville fractional derivativeoperator of order n.

In classical computation, finite difference function is used to define derivative. As an example n^{th} order of backward difference by hstep size which is defined as:

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} f(x - kh). \tag{4}$$

It can be useful to the following classical result.

Theorem 1.4. *Suppose that $n \in \mathbb{N}$, $f \in C^m[a, b]$ that denote the space of functions with continuous partial derivatives in $[a, b]$ of order less than or equal to $m \in \mathbb{N}$ by $C^m[a, b]$ and $a < x$. Then*

$$D^n f(x) = \lim_{h \to 0} \frac{\Delta_h^n f(x)}{h^n}. \tag{5}$$

Proof. See [5]. □

2 Numerical Scheme for Fractional Differential Formulation

In this section, a new method is presented to solve the equation with fractional derivative and Grunwald-Letnikov fractional derivative is used. Linear programming (LP; also called linear optimization) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. Its feasible region is a convex polytope, which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality. Its objective function is a real-valued affine function defined on this polyhedron.

In the continue, you will see how a fractional differential equation changed to linear programming. According to advantage of linear programming, solving a linear programming is easier than fractional differential equation. The following theorem is used in suggested method:

Theorem 2.1. *If $h(x)$ be a real, non-linear and non-negative function on $[a,b]$, necessary and sufficient condition for $\int_a^b h(x)dx = 0$ is that $h(x)$ be 0 on $[a,b]$.*

Proof. See [6]. □

Consider the following fractional differential equation:

$$D_a^\alpha x(t) = f(x(t)), \quad 0 < \alpha < 1. \tag{6}$$

According to equation (6) we have $D_a^\alpha x(t) - f(x(t)) = 0$ and then

$$|D_a^\alpha x(t) - f(x(t))| = 0. \tag{7}$$

But according to the Theorem 2.1, the necessary and sufficient condition which cause the equation (6) has a solution, is that the following problem has the zero objective function:

$$\min \int_0^1 |D_a^\alpha x(t) - f(x(t))|dt. \tag{8}$$

To solve this problem, first we divided the integral interval into n suninterval and use the midpoint method in each suninterval for approximating the integral. Then we have

$$\begin{aligned} \min \int_0^1 \Big|D_a^\alpha x(t) - f(x(t))\Big|dt &= \min \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \Big|D_a^\alpha x(t) - f(x(t))\Big|dt \\ &= \min \frac{1}{n} \sum_{i=1}^n \Big|D_a^\alpha x(\frac{i}{n} - \frac{1}{2n}) - f(x(\frac{i}{n} - \frac{1}{2n}))\Big| \end{aligned} \tag{9}$$

But according to definition of Grunwald-Letnikov fractional derivative, equation (9) will be in the form:

$$\min \frac{1}{n} \sum_{i=1}^n \Big(\Big|\frac{1}{h^\alpha} \sum_{j=0}^i w_j^\alpha x(\frac{i-j}{n} - \frac{1}{2n}) - f(x(\frac{i}{n} - \frac{1}{2n}))\Big|\Big), \tag{10}$$

where $w_j^\alpha = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}$. In equation (10) there exists an absolute value function, so we can change this problem to a linear programming problem with the following form:

$$\begin{aligned} &\min \frac{1}{n} \sum_{i=1}^n u_i + v_i \\ &subjectto \\ &\frac{1}{h^\alpha} \sum_{j=0}^i w_j^\alpha x(\frac{i-j}{n} - \frac{1}{2n}) - f(x(\frac{i}{n} - \frac{1}{2n})) = u_i - v_i \\ &u_i, v_i \geq 0, \qquad i = 1, 2, \cdots, n. \end{aligned} \tag{11}$$

3 Numerical Example

Consider the fractional derivative equation $D_0^\alpha x(t) = -x^2(t) + 1$ with $0 < \alpha \leq 1$ and the initial condition $x(0) = 0$. According to the suggested method, we used numerical results of linear programming (11). In Figure (1), the state variable $x(t)$ is plotted for different values of α. This equation have been solved by other numerical solutions such as

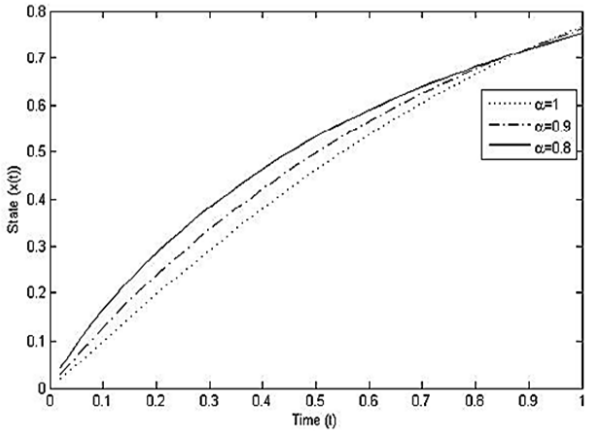

Figure 1: Approximate solution of $x(t)$ for $n = 100$.

numerical solution by Baleanu and et al. (2013), Odibat and et al. (2008), Raja and et al. (2010) and Batiha and et al. (2007). It can be seen that the proposed method of this paper provide relatively the better results. Indeed, as the number of discrete points (n) was increased and α approaches to 1, the solutions convergend.

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A New Approach for Eigenvalue Assignment in Descriptor Systems with Output Feedback Matrix

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Abstract

In this paper, we present a new method for eigenvalue assignment in descriptor systems based on matrix inverse eigenvalue problem. First a descriptor system is changed to standard system with output feedback by defining the input vector as a multiple of the output-derivative feedback, then using the matrix inverse eigenvalue problem, output feedback matrix K is calculated such that eigenvalues of closed-loop system are desirable and prescribed. A simple algorithm and an example is given to illustrate the results.

Keywords: Eigenvalue assignment, Matrix inverse eigenvalue problem, Descriptor systems, Null space
Mathematics Subject Classification [2010]: 93C05, 93B60

1 Introduction

Eigenvalue assignment techniques to improve the dynamic response of linear systems are one of the most important problems in the modern control theory. Many approaches have been proposed for this problem like [2, 5, 6]. Descriptor systems describe a physical system more than other models of linear systems. Applications of descriptor systems can be found in various fields such as electrical circuit networks, robotic systems, chemical processes, economics. Bunse (1992), Duan and Wang (2005) and Darouach (2006) studied on descriptor systems respectively in [1, 4, 3].

In this paper we investigate a new method for eigenvalue assignment in descriptor systems with output-derivative feedback matrix. Many authors investigate approaches to solve this problem, but the method that we have in this paper based on the matrix inverse eigenvalue problem is much useful. The first superiority of this method is, we do not need some certain conditions like in [5] or any restrictions for eigenvalues like amount or multiplicity. Also we do not deal with nonlinear equations which solving them is so difficult and time-consuming, especially for large system and it is the next advantage of this method.

Consider the linear time invariant controllable and observable sysytem of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0 \tag{1a}$$
$$y(t) = Cx(t) \tag{1b}$$

*Speaker

where open-loop, input and output matrices A, B and C have dimensions $n \times n$, $n \times m$ and $l \times n$ respectively and $x(t) \in \mathbb{R}^n$ is state vector, $u(t) \in \mathbb{R}^m$ is input vector and $y(t) \in \mathbb{R}^l$ is output vector which $1 \leq m \leq n$.

The aim of this paper is finding the output feedback matrix K in (1a), so that the closed-loop matrix of system (1a) has desirable and prescribed eigenvalues. First we convert the descriptor linear system to standard system using output-derivative feedback. Next according to new standard system and method of solving matrix inverse eigenvalue problem, the output feedback matrix K can be calculated. To establish the proposed results, consider following assumptions: $(I) rank[E|B] = n$, $(II) rank[A] = n$, $(III) rank[B] = m$. We define control low as

$$u(t) = K\dot{y}(t) = KC\dot{x}(t). \tag{2}$$

It is clear that if assumption (I) holds, then there exists K such that [1]

$$rank[E - BKC] = n \tag{3}$$

for K such that (3) holds, from (2) system (1a) with output feedback (1b) can be rewrite such as standard linear system, given by: $E\dot{x}(t) = Ax(t) + BKC\dot{x}(t)$, So $\dot{x}(t) = (E - BKC)^{-1}Ax(t)$.

Lemma 1.1. *consider a matrix $M \in \mathbb{R}^{n\times m}$ with $rank(M) = n$ and the eigenvalues equal to $\lambda_1, \lambda_2, ..., \lambda_n$. Then, the eigenvalues of M^{-1} are the following: $\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}$. [2]*

Theorem 1.2. *define the matrices:*

$$\bar{A} = A^{-1}E, \quad \bar{B} = -A^{-1}B \tag{4}$$

and suppose $(\bar{A}, \bar{B})$ is controllable. Let K be output feedback matrix, such that $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ are the eigenvalues of the closed-loop system (5a) *with* (5b) *and control law* (5c)

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \tag{5a}$$
$$\bar{y}(t) = C\bar{x}(t) \tag{5b}$$
$$\bar{u}(t) = KC\bar{x}(t) \tag{5c}$$

where $\lambda_i \in \mathbb{C}$ and $\lambda_i \neq 0$, $i = 1, 2, ..., n$, are arbitrarily specified. Then for this K, the desired spectrum $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ is the eigenvalues of the controlled system (1a) *with* (1b) *and control law* (2) *and also, the condition* (3) *holds.*

Proof. Considering that $(\bar{A}, \bar{B})$ is controlled, then one can find an output feedback matrix K such that the controlled system (5a) and (5b) with output feedback (5c) given by $\dot{\bar{x}} = (\bar{A} + \bar{B}KC)\bar{x}(t)$ has the poles equals to $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$. Now by (4) note that:

$$(\bar{A} + \bar{B}KC)^{-1} = (A^{-1}(E - BKC))^{-1} \Longrightarrow (\bar{A} + \bar{B}KC)^{-1} = (E - BKC)^{-1}A \tag{6}$$

According to lemma 1.1 and equation (6), the spectrum $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ are the eigenvalues of closed-loop matrix $(E - BKC)^{-1}A$. Therefore (3) holds and the eigenvalue of closed-loop system (1a) with (1b) and controller (2) are equal to $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. □

Definition 1.3. The matrix inverse eigenvalue problem is that given four linearly independent sets of real n-vectors $\{z_1, z_2, ..., z_p\}$, $\{z_{p+1}, z_{p+2}, ..., z_{p+q}\}$, $\{w_1, w_2, ..., w_p\}$ and $\{w_{p+1}, w_{p+2}, ..., w_{p+q}\}$ with $p+q \leq n$ and a set of complex numbers $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$, find a

real matrix $\Gamma_{n\times n}$ such that $\Gamma z_i = w_i$, $i = 1, 2, ..., p$ and $\Gamma z_j = w_j$, $j = p+1, p+2, ..., p+q$ and the spectrum of Γ is $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$, where we assume that the set L^{-1} is closed under complex conjucation, i.e., $\lambda \in L^{-1}$ if and only if $\bar{\lambda} \in L^{-1}$.

Let $X_r = [z_1, z_2, ..., z_p]$, $X_l = [z_{p+1}, z_{p+2}, ..., z_{p+q}]$, $Y_r = [w_1, w_2, ..., w_p]$ and $Y_l = [w_{p+1}, w_{p+2}, ..., w_{p+q}]$, clear that if the matrix Γ of the problem exists, the following consistency condition must be satisfied

$$X_l^t Y_r = Y_l^t X_r \tag{7}$$

Theorem 1.4. *If the matrix inverse eigenvalue problem satisfies the consistency condition* (7)*, then the necessary and sufficient condition for the existence of the matrix* Γ *is that there are vectors* $u_i \in \int_u^i$ *and* $v_i \in \int_v^i$, $i = 1, 2, ..., n$ *such that*

$$u_i v_j = \sigma_{ij}, i, j = 1, 2, ..., n \tag{8}$$

where $\int_u^i$ *and* $\int_v^i$ *are the null spaces* $(\lambda_i^{-1} X_l^t - Y_l^t)$ *and* $(\lambda_i^{-1} X_r^t - Y_r^t)$ *respectively. if such* u_i *exist, then* Γ *can be obtained using the equation*

$$\Gamma = T^{-1} diag\{\ \lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\} T, \tag{9a}$$

$$T = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix}. \tag{9b}$$

Let the base vectors of $\int_u^i$ *and* $\int_v^i$ *be the form of matrices* S_u^i *and* S_u^i *respectively, then vectors* u_i *and* v_i, $i = 1, 2, ..., n$, *can be expressed as* $u_i = S_u^i z_i$, $v_i = S_v^i w_i$. *Thus from Equation* (8) *we have:* $z_i^t (S_u^i)^t S_v^j w_j$, $i, j = 1, 2, ..., n$. *This equation can be solved with the iterative method. Briefly, first we assign some initial values to all* w_i, *then the* n*-systems of linear equations can be solved easily. [8]*

2 Main results

In this paper, we consider $\Gamma = \bar{A} + \bar{B}KC$ that $\bar{A} + \bar{B}KC$ is the closed-loop matrix of system (5a) and also U_1 and V_1 that are the matrices formed by the base vectors of the null spaces of $\bar{B}$ and C respectively. Then we have $\Gamma V_1 = (\bar{A} + \bar{B}KC)V_1 = \bar{A}V_1$, $U_1^t \Gamma = U_1^t(\bar{A} + \bar{B}KC) = U_1^t \bar{A}$. Let $X_l = U_1$, $X_r = V_1$, $Y_l = \bar{A}^t U_1$ and $Y_r = \bar{A}V_1$, now according to theorem 1.4 we can find Γ. If such Γ exist, the matrix K can be computed through the equation

$$K = \bar{B}^{\dagger}(\Gamma - A)C^{\dagger}. \tag{10}$$

where $\bar{B}^{\dagger}$ and $C^{\dagger}$ are the Moore-Penrose generalized inverse of $\bar{B}$ and C respectively. This method is generally solved when $\bar{B}$ and C are full rank and $rank(\bar{B}) + rank(C) \geq rank(\bar{A})$, so we can expect a solution with probability 1 for a given set L. (For more details about Moore-Penrose see [7])

Now we display briefly calculation that we have had in this paper by below algorithm:
Subject. Finding output feedback matrix K in (2) for eigenvalue assignment in descriptor system (1a) with (1b).
Step 1. Inter A, B, C, E and eigenvalues $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$.
Step 2. Calculate $\bar{A}$, $\bar{B}$ and $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ by (4) and lemma 1.1.
Step 3. Obtain $X_l = U_1$ and $X_r = V_1$ that are null space of $\bar{B}^t$ and C respectively, then calculate $Y_l = \bar{A}^t U_1$, $Y_r = \bar{A}V_1$ and then obtain the null spaces $(\lambda_i^{-1} X_l^t - Y_l^t)$ and $(\lambda_i^{-1} X_r^t - Y_r^t)$.
Step 4. Obtain $Z = \begin{bmatrix} z_1 & z_2 & ... & z_n \end{bmatrix}, W = \begin{bmatrix} w_1 & w_2 & ... & w_n \end{bmatrix}$. [8]
Step 5. Calculate matrices T, Γ and obtain K from (9b), (9a) and (10) respectively.

Conclusions

In this research, we investigate a new method for the eigenvalue assignment in descriptor systems by output feedback matrix, which is based on the matrix inverse eigenvalue problem. There are many approaches for this problem and our method is much simpler than other methods, for example in [5] static output feedback matrix can be obtained from state feedback matrix under certain conditions. Yheir method requaires solving non-linear equations and for large systems it is quite cumbersome and time-consuming to derive the non-linear system of equations. Removal of non-linear equations is an important advantage of this method over other existing methods. This method does not require prior knowledge of the open-loop eigenvalues and the controller does not impose any restrictions on the position of the desired eigenvalues or their nature and multiplicity. The error of method will be zero when $\bar{B}$ and C are invertible.

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A new approach for two-objective inverse scheduling single machine problems based on fuzzy distance minimization

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Abstract

Acoording to the importance of inverse scheduling problems in industry and commerce, in this paper the model of single machine inverse scheduling problem, with fuzzy cost cofficient has been presented. Based on fuzzy distance minimization, the model for solving this type of problems has been introduced. this model could optimally solve problems for planners or decission making.

Keywords: Two-objective inverse scheduling problem, Fuzzy cost coefficients, Fuzzy distance minimization
Mathematics Subject Classification [2010]: 86A22, 90C70, 90B36

1 Introduction

Consider inverse scheduling as a single machine, that should process n-job [3]:

- p_j : Each job j, j = 1, ..., n has a processing time, p_j associated with it. and p'_j is the new minimally perturbed processing time of j job.
- d_j : Each job j, j = 1, ..., n has a delivery time, d_j associated with it. and d'_j is the new minimally perturbed delivery time of j job.
- C_j : Completion time of job j, so where: $C_j = \sum_{k=1}^{j}$.

Definition 1.1. [3] The amount of time a job takes more than delivery times (d_j), we call delay and it shows as $C_j - d_j$ and the time that shows the difference between completion and delivery of job j, has been called tradiness $T_j = max\{0, L_j\}$.

Theorem 1.2. *[3] The sequence of operations SPT ($p_1 \leq p_2 \leq \cdots \leq p_n$), minimize the mean time of completion time.*

Theorem 1.3. *[3] The sequence of operations EDD ($d_1 \leq d_2 \leq \cdots \leq d_n$), minimize the maximum tardiness.*

Definition 1.4. [1] The value of a fuzzy number $\tilde{a}$, in relation to the reducing function $s(\alpha) = \alpha$, $V(\tilde{a})$, is defined as $V(\tilde{a}) = \int_0^1 \alpha[a_\alpha^L + a_\alpha^U]\mathrm{d}\alpha$.

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Definition 1.5. [1] The ambiguity of a fuzzy number $\tilde{a}$, in relation to the reducing function $s(\alpha) = \alpha$, $A(\tilde{a})$ is defined as $A(\tilde{a}) = \int_0^1 \alpha[a_\alpha^U - a_\alpha^L]\mathrm{d}\alpha$.

Definition 1.6. [1] The fuzziness of a fuzzy number $\tilde{a}$ in relation to the reducing function $s(\alpha) = \alpha$, $F(\tilde{a})$, is defined as $F(\tilde{a}) = \int_0^{\frac{1}{2}}[a_\alpha^U - a_\alpha^L]\mathrm{d}\alpha + \int_{\frac{1}{2}}^1[a_\alpha^L - a_\alpha^U]\mathrm{d}\alpha$ The above indices are linear if nonnegative scalars are considered.

2 Problem formulation

In this paper two objectives has been considered, The first objective functions is discussed in [2] with fuzzy cost coefficients $\tilde{\alpha}_j$, problem is solved by determining the minimum total perturbation as the job processing times so the given sequence is converted to the SPT sequence, in another objective function, it has been tried to minimize the delivery perturbations cost coefficients $\tilde{k}_j$ and at the end change the scheduling sequence to the EDD sequence.

$$\begin{array}{lll} p1 & \min & \sum_{j=1}^n \tilde{c}_j|p'_j - p_j| \\ & \min & \sum_{j=1}^n \tilde{k}_j|d'_j - d_j| \\ & s.t & 0 \le p'_1 \le p'_2 \le \cdots \le p'_n \\ & & 0 \le d'_1 \le d'_2 \le \cdots \le d'_n \end{array} \tag{1}$$

p1 is not a linear programming problem but can be converted into one by using a standard transformation.

$$\begin{array}{lll} \min & \sum_{j=1}^n \tilde{\alpha}_j x_j + \tilde{\alpha}_j y_j & \\ \min & \sum_{j=1}^n \tilde{k}_j a_j + \tilde{k}_j b_j & \\ s.t & 0 \le p'_1 \le p'_2 \le \cdots \le p'_n & \\ & 0 \le d'_1 \le d'_2 \le \cdots \le d'_n & \\ & x_j - y_j = p'_j - p_j \quad \forall j = 1, \cdots, n & \\ & a_j - b_j = d'_j - d_j \quad \forall j = 1, \cdots, n & \\ & x_j, y_j, a_j, b_j \ge 0 \quad \forall j = 1, \cdots, n & \end{array} \tag{2}$$

As it clear in the article titled,[1], the fuzzy distance minimization has been studied for problem (2), a distance metric approach is defined as one that minimizes the distance of a solution to some ideal or desired point which can be provided by the DM as a point composed of desired achievement levels In the space of the objective functions, this point will be noted by $(\tilde{f}_1^*, \tilde{f}_2^*)$. Now, the definition of the L_p distance allows us to formulate the following model:

$$\begin{array}{lll} \min & L_p = \left[\sum_{k=1}^2 w_k^p \left(\dfrac{\tilde{f}_k(x) - \tilde{f}_k^*}{r_{\tilde{f}_k}}\right)^p\right]^{\frac{1}{p}} & \\ s.t & 0 \le p'_1 \le p'_2 \le \cdots \le p'_n & \\ & 0 \le d'_1 \le d'_2 \le \cdots \le d'_n & \\ & x_j - y_j = p'_j - p_j \quad \forall j = 1, \cdots, n & \\ & a_j - b_j = d'_j - d_j \quad \forall j = 1, \cdots, n & \\ & x_j, y_j, a_j, b_j \ge 0 \quad \forall j = 1, \cdots, n & \end{array} \tag{3}$$

where $r_{\tilde{f}_j} = f_{j0}^{*U} - f_{j*0}^{L}$ is a normalization constant attached to j-th objective.
Let us use the degree of discrepancy corresponding to the j-th objective

$$\tilde{d}_1(x) = \frac{\tilde{f}_1(x) - \tilde{f}_1^*}{r_{\tilde{f}_1}} \qquad \tilde{d}_2(x) = \frac{\tilde{f}_2(x) - \tilde{f}_2^*}{r_{\tilde{f}_2}} \tag{4}$$

3 Solving L_1-problem

The model with two steps has been considered: Step 1 calculates the fuzzy minimum distance under the L_1 metric. In Step 2, it finds an optimum decision order to the fuzzy minimum distance obtained in first step.

3.1 Calculating the fuzzy minimum distance

Setting p = 1 in (3), let us define the following problem:

$$\begin{array}{ll} \min & \sum_{k=1}^{2} w_k \tilde{d}_k(x) \\ s.t & 0 \le p'_1 \le p'_2 \le \cdots \le p'_n \\ & 0 \le d'_1 \le d'_2 \le \cdots \le d'_n \\ & x_j - y_j = p'_j - p_j \quad \forall j = 1, \cdots, n \\ & a_j - b_j = d'_j - d_j \quad \forall j = 1, \cdots, n \\ & x_j, y_j, a_j, b_j \ge 0 \qquad \forall j = 1, \cdots, n \end{array} \tag{5}$$

In order to solve (5), for each $\alpha \in [0, 1]$ two ordinary problems are introduced:

$$\left.\begin{array}{ll} \min & d_\alpha^L = \sum_{k=1}^{2} w_k d_{k\alpha}^L(x) \\ s.t & \dfrac{f_{1\alpha}^L(x)}{r_{\tilde{f}_1}} - d_{1\alpha}^L = \dfrac{f_{1\alpha}^{*U}(x)}{r_{\tilde{f}_1}} \\ & \dfrac{f_{2\alpha}^L(x)}{r_{\tilde{f}_2}} - d_{2\alpha}^L = \dfrac{f_{2\alpha}^{*U}(x)}{r_{\tilde{f}_2}} \\ & 0 \le p'_1 \le p'_2 \le \cdots \le p'_n \\ & 0 \le d'_1 \le d'_2 \le \cdots \le d'_n \\ & x_j - y_j = p'_j - p_j \quad \forall j = 1, \cdots, n \\ & a_j - b_j = d'_j - d_j \quad \forall j = 1, \cdots, n \\ & x_j, y_j, a_j, b_j \ge 0 \qquad \forall j = 1, \cdots, n \end{array}\right\} (P - left_1(\alpha)) \tag{6}$$

and

$$\left.\begin{array}{ll} \min & d_\alpha^U = \sum_{k=1}^{2} w_k d_{k\alpha}^U(x) \\ & \dfrac{f_{1\alpha}^U(x)}{r_{\tilde{f}_1}} - d_{1\alpha}^U = \dfrac{f_{1\alpha}^{*L}(x)}{r_{\tilde{f}_1}} \\ & \dfrac{f_{2\alpha}^U(x)}{r_{\tilde{f}_2}} - d_{2\alpha}^U = \dfrac{f_{2\alpha}^{*L}(x)}{r_{\tilde{f}_2}} \\ & 0 \le p'_1 \le p'_2 \le \cdots \le p'_n \\ & 0 \le d'_1 \le d'_2 \le \cdots \le d'_n \\ & x_j - y_j = p'_j - p_j \quad \forall j = 1, \cdots, n \\ & a_j - b_j = d'_j - d_j \quad \forall j = 1, \cdots, n \\ & x_j, y_j, a_j, b_j \ge 0 \qquad \forall j = 1, \cdots, n \end{array}\right\} (P - right_1(\alpha)) \tag{7}$$

where $f_{j\alpha}^{L}$ denotes the value in x of j-th objective function when the coefficients are equal to the left bound of the α-cut, and analogously $f_{j\alpha}^{U}$ is for the case when the coefficients are equal to the right bound of the α-cut. Now, the following theorem can be formulated:

Theorem 3.1. *[1] Let $\tilde{d}$ be the fuzzy set defined through the following family of ordinary sets: $\{[d_\alpha^L, d_\alpha^U], \alpha \in [0,1]\}$ where d_α^L and d_α^U are the solutions of problems (6) and (7) respectively. Then, $\tilde{d}$ is a fuzzy number which represents the minimum L_1-distance with respect to desired point.*

Theorem 3.2. *[1] Assume that the desired point, $(\tilde{f}_1^*, \tilde{f}_2^*)$, is better than or equal to the ideal point in each α-cut, and let $\tilde{d}$ be defined in the theorem 3.1; then, the value of $\tilde{d}$, $V(\tilde{d})$, is greater than or equal to zero.*

Theorem 3.3. *[1] Assume that the desired point is better than or equal to the ideal point in each α-cut, and let $\tilde{d}$ be defined in the above theorem; then the value of $\tilde{d}$, $V(\tilde{d})$, is equal to 0 if and only if in each α-cut each individual objective achieves its desired level.*

3.2 Model for optimum decision under L_1-metric

In this subsection the way to find a decision vector has been introuduced such that its distance to the desired point is the most like a fuzzy number $\tilde{d}$. An method defuzzyfication has been choosed based on the value, the ambiguity, and the fuzziness of a fuzzy number. These real indices lead to the following modelling:

$$\begin{array}{ll} & Find\ x \\ s.t & V\left(\sum_{j=1}^{2} w_j \tilde{d}_j(x)\right) \approx V(\tilde{d}) \\ & A\left(\sum_{j=1}^{2} w_j \tilde{d}_j(x)\right) \approx A(\tilde{d}) \\ & F\left(\sum_{j=1}^{2} w_j \tilde{d}_j(x)\right) \approx F(\tilde{d}) \\ & 0 \le p_1' \le p_2' \le \cdots \le p_n' \\ & 0 \le d_1' \le d_2' \le \cdots \le d_n' \\ & x_j - y_j = p_j' - p_j \quad \forall j = 1, \cdots, n \\ & a_j - b_j = d_j' - d_j \quad \forall j = 1, \cdots, n \\ & x_j, y_j, a_j, b_j \ge 0 \qquad \forall j = 1, \cdots, n \end{array} \tag{8}$$

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A numerical scheme for two-dimensional optimal control problems with Grünwald-Letnikov for Riesz Fractional Derivatives

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Abstract

In this paper, we study control systems containing a Riesz fractional derivative and solve this problem by a numerical method which is so called Grünwald-Letnikov approximation scheme . A two-dimensional fractional optimal control problem is studied as an example to demonstrate the performance of this method.

Keywords: Calculus of variations, Riesz fractional derivative, Grünwald-Letnikov
Mathematics Subject Classification [2010]: 49K05, 26A33

1 Introduction

Fractional calculus (FC) generalizes integrals and derivatives to non-integer orders. During the last decade, FC was found to play a fundamental role in the modeling of a considerable number of phenomena, in particular, the modeling of memory dependent phenomena and complex media such as porous media. first we define a fractional derivative, and then formulate a fractional optimal control problem (FOCPs) and find the necessary conditions for optimality. The left and right Riemann-Liouville fractional derivatives of order α are defined respectively:

$$
\begin{aligned}
{}_aD_t^{\alpha}y(t) &= \frac{1}{\Gamma(1-\alpha)}(\frac{d}{dt})\int_a^t (t-\tau)^{-\alpha}y(\tau)d\tau,\\
{}_tD_b^{\alpha}y(t) &= \frac{1}{\Gamma(1-\alpha)}(\frac{d}{dt})\int_t^b (\tau-t)^{-\alpha}y(\tau)d\tau,
\end{aligned}
$$

where $n-1<\alpha<n$. The usual definitions of the derivatives are obtained when α is an integer. Note that for $\alpha\in(0,1)$, the fractional operators are non-local. One space needed Riesz fractional derivative ${}_a^RD_b^{\alpha}y(t)$ is given by

$$
\begin{aligned}
{}_a^RD_b^{\alpha}y(t) &= \frac{1}{2}({}_aD_t^{\alpha}y(t) -_t D_b^{\alpha}y(t)),\\
{}_b^RD_t^{\alpha}y(t) &= \frac{1}{2}({}_tD_b^{\alpha}y(t) -_a D_t^{\alpha}y(t)).
\end{aligned}
\tag{1}
$$

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The main point in the FOCPs is to find the optimal control $u(t)$ which minimizes the performance index [4]

$$J(u) = \int_a^b f(y, u, t)dt,$$

subjected to the system dynamic constraints

$${}_a^R D_b^\alpha y(t) = g(y, u, t),$$

satisfying the terminal conditions $y(a) = c$ and $y(b) = d$. Here f and g are a scalar and ${}_a^R D_b^\alpha y(t)$ is the right Riesz fractional derivative of order α. following expression defines a modified performance index

$$\bar{J}(u) = \int_a^b [H(y, u, \lambda, t) - \lambda^T {}_a^R D_b^\alpha y(t)]dt, \tag{2}$$

where $H(y, u, \lambda, t)$ is the following Hamiltonian

$$H(y, u, \lambda, t) = f(y, u, t) - \lambda^T g(y, t), \tag{3}$$

and λ^T is the transpose of a $n_y \times 1$ vector of Lagrange multipliers. From (2) and (3) we have

$${}_b^R D_t^\alpha \lambda(t) = \frac{\partial H}{\partial y}, \tag{4}$$

$$\frac{\partial H}{\partial u} = 0, \tag{5}$$

$${}_a^R D_b^\alpha y(t) = \frac{\partial H}{\partial \lambda}, \tag{6}$$

and it is also required that

$$\lambda(b) = 0. \tag{7}$$

equations (4)-(6) coincide with the classical ones as α goes to 1. equations in (4)-(7) describe the necessary conditions in terms of the Hamiltonian for the FOCP defined above. For the numerical solution of the system of FDEs with some terminal conditions, the Gr*ü*nwald-Letnikov approximation (GLA)[4] can be used. The Gr*ü*nwald-Letnikov definition of the left Riemann-Liouville and right Riemann-Liouville fractional derivatives is given as follows:

$${}_aD_t^\alpha y(t) = \lim_{h\to 0} \frac{\sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} y(t - jh)}{h^\alpha},$$

$${}_tD_b^\alpha y(t) = \lim_{h\to 0} \frac{\sum_{j=0}^{[\frac{b-t}{h}]} (-1)^j \binom{\alpha}{j} y(t + jh)}{h^\alpha},$$

respectively. the GLA of the left Riemann-Liouville and the right Riemann-Liouville fractional derivatives at the ith node of the time interval $[a, b]$, where the time interval $[a, b]$ is discretized by $(N + 1)$ equally-spaced grid points, are defined as

$$\begin{aligned} {}_aD_t^\alpha y(t_i) &= \tfrac{1}{h^\alpha}\sum_{j=0}^{i}\zeta_j^\alpha y_{i-j}, \quad i = 0, 1, \cdots, N, \\ {}_aD_t^\alpha y(t_i) &= \tfrac{1}{h^\alpha}\sum_{j=0}^{N-i}\zeta_j^\alpha y_{i+j}, \quad i = N, N-1, \cdots, 0. \end{aligned} \tag{8}$$

respectively, where $y_i \approx y(t_i),\ h = \frac{b-a}{N},\ t_i = a + ih$, and

$$\zeta_j^\alpha = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix}.$$

2 The two-dimensional example

Consider the following two-dimensional system in order to apply the fractional optimal control (FOC) formulation described in Section 1. The aim is to minimize the following quadratic performance index

$$J = \frac{1}{2}\int_0^1 [y_1^2 + y_2^2 + u^2]dt, \tag{9}$$

subjected to the dynamic constraints,

$$\begin{aligned} {}_0^RD_1^\alpha y_1(t) &= -y_1(t) + y_2(t) + u(t), & (10)\\ {}_0^RD_1^\alpha y_2(t) &= -2y_2(t). & (11) \end{aligned}$$

Note that for this example $a = 0,\ b = 1$, and

$$\begin{aligned} f(y(t), u(t), t) &= \frac{1}{2}[y_1^2 + y_2^2 + u^2], \\ g(y(t), t) &= \begin{pmatrix} -y_1(t) + y_2(t) + u(t) \\ -2y_2(t) \end{pmatrix}, \end{aligned} \tag{12}$$

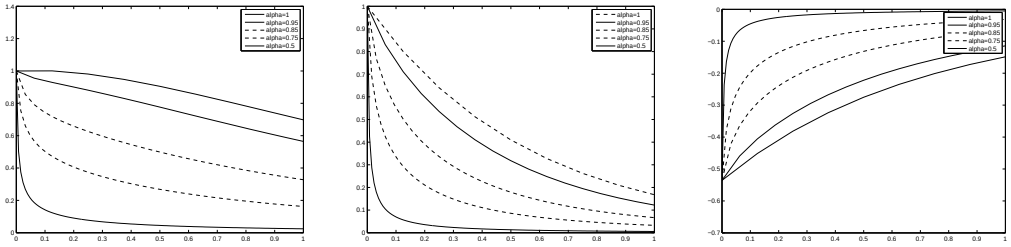
The necessary conditions (3)-(6) of the FOC formulation result the following FDEs,

$$\begin{aligned} {}_0^RD_1^\alpha y_1(t) &= -y_1(t) + y_2(t) + u(t), \\ {}_0^RD_1^\alpha y_2(t) &= -2y_2(t), \\ u(t) + \lambda_1(t) &= 0, \\ {}_t^RD_0^\alpha \lambda_1(t) &= y_1(t) - \lambda_1(t), \\ {}_t^RD_0^\alpha \lambda_1(t) &= y_2(t) - 2\lambda_2(t) + \lambda_1(t), \end{aligned} \tag{13}$$

together with the condition $u(1) = 0$. Moreover, the initial conditions $y_1(0) = 1,\ y_2(0) = 1$ are imposed. firstly is substitutions the left and right Riesz Fractional derivatives (1) in (13), next we apply the direct numerical scheme to solve the system of FDE's in (13) numerically using GLA.

2.1 Numerical Results

The formulation presented in the above sections is applied to the multi-dimensional problem described in Section 1. It is solved numerically by the direct numerical scheme for different values of α and N. When we take the number of nodes N as 128 and analyze the behavior of the state variable $y(t)$ and control variable $u(t)$ as functions of t for the values of $\alpha = 0.5, 0.75, 0.85, 0.95$ and $\alpha = 1$, we obtain the results given in the Figures(1).



(a) State $y_1(t)$ as a function of t for different α

(b) State $y_2(t)$ as a function of t for different α

(c) State $u(t)$ as a function of t for different α

Figure 1: State $y_1(t),\ y_2(t),\ u(t)$ as a function of t for different α

3 conclusion

A two-dimensional example was studied in details to apply the formalism. The set of FDEs obtained from the formulation were approximated by Grünwald-Letnikov definition and solved numerically using the direct numerical scheme. It is seen that as the sizes of time sub-domains are decreased, the solutions converge slowly. Analytical solutions for integer order systems are reached as the order of fractional derivatives goes to an integer value.

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An interior-point algorithm with a new iterative scheme for semidefinite optimization problems

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Abstract

In this paper, a new interior-point algorithm is presented for semidefinite optimization (SDO) problems. The algorithm is based on a new class of search directions and Ai-Zhang's wide neighborhood for linear complementarity problems. Although, the algorithm belongs to the class of large-step interior-point algorithms, its complexity is coincide with the best iteration bound of short-step ones for SDO problems.

Keywords: Semidefinite optimization, Wide neighborhood, Interior-point method, Polynomial complexity
Mathematics Subject Classification [2010]: 90C51, 90C22

1 Introduction

Semidefinite optimization (SDO) problems are one of the most important classes of optimization problems which has become a popular research area in mathematical programming. Among various methods for solving this class of problems, interior-point methods (IPMs) are one of the most efficient and applicable classes of iterative algorithms which solve SDO problems in polynomial time complexity.
The first IPMs for SDO problems were developed by Alizadeh [1] and Nesterov et al. [2]. After that, several authors such as Helmberg et al. [3], Vandenberghe et al. [4], Wang et al. [5] and Mansouri et al. [6, 7] have proposed some interior-point algorithms for solving the SDO problems. Most of these mentioned interior-point algorithms are based on a small neighborhood of the central path (short-step IPMs) and their complexities coincide with the best obtained iteration bound for solving the SDO problems.
In this paper, using Ai-Zhang's wide neighborhood for linear complementarity problems [8], we propose a large-step interior-point algorithm for SDO problems. Although, the algorithm belongs to the class of large-step algorithms, we prove that its complexity coincides with the best iteration bound obtained by the short-step interior-point algorithms for SDO problems.

*Speaker

2 Interior-point methods for SDO problems

We consider the standard form of the SDO problem:

$$\min\Big\{\mathbf{Tr}\,(CX) \quad s.t. \quad \mathbf{Tr}\,(A_iX)=b_i, \quad i=1,2,...,m, \quad X\succeq 0\Big\}, \tag{1}$$

and its dual

$$\max\Big\{b^Ty \quad s.t. \quad \sum_{i=1}^m y_iA_i+S=C, \quad S\succeq 0\Big\}, \tag{2}$$

where $C,X,A_i\in S^n$ for $i=1,2,...,m$ and $y\in\mathbb{R}^m$. Denoting $\mathcal{F}^0$ as the strictly feasibility set of the primal-dual pair of SDO problem, we assume that $\mathcal{F}^0\neq\emptyset$. Therefore, both problems (1) and (2) are solvable and the perturbed KKT conditions for SDO problem can be written as follows:

$$\begin{aligned} \langle A_i,X\rangle &= b_i, \quad i=1,2,...,m,\\ \sum_{i=1}^m y_iA_i+S &= C,\\ H_P(XS) &= \tau\mu I, \qquad X,S\succeq 0, \end{aligned} \tag{3}$$

where the last equality is called the perturbed complementarity equation and

$$H_P(XS)=\frac{1}{2}\left(PXSP^{-1}+P^{-T}SXP^T\right),$$

in which the matrix P belongs to the specific class

$$\mathcal{C}(X,S):=\{P\in S^n_{++}|\ PXSP^{-1}\in S^n\}. \tag{4}$$

In this paper, we consider the wide neighborhood

$$\mathcal{N}(\tau,\beta)=\left\{(X,y,S)\in\mathcal{F}^0:\quad \left\|\left(\tau\mu I-X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^+\right\|_F\leq\beta\tau\mu\right\},$$

where $\beta,\tau\in(0,1)$ are given constants. Using Newton's method on System (3), we can derive the following linearized Newton search direction system:

$$\begin{aligned} \langle A_i,\Delta X\rangle &= 0, \quad i=1,2,...,m,\\ \sum_{i=1}^m \Delta y_iA_i+\Delta S &= 0,\\ H_P(X\Delta S+\Delta XS) &= \tau\mu I-H_P(XS). \end{aligned} \tag{5}$$

Defining the new matrix

$$V:=(\tau\mu I-H_P(XS))^+ +\eta\,(\tau\mu I-H_P(XS))^- +H_P(XS), \tag{6}$$

where

$$\eta:=-\frac{\mathbf{Tr}\,(\tau\mu I-H_P(XS))^+}{\mathbf{Tr}\,(\tau\mu I-H_P(XS))^-}, \tag{7}$$

and replacing the right hand side of the third equation in System (5) by the term $tV - H_P(XS)$ with $t \in [0,1]$, this system can be rewritten as follows:

$$\begin{aligned} \langle A_i, \Delta X \rangle &= 0, \quad i = 1, 2, ..., m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S &= 0, \\ H_P(X\Delta S + \Delta X S) &= tV - H_P(XS). \end{aligned} \tag{8}$$

Taking a full Newton step along $(\Delta X, \Delta y, \Delta S)$, the new iterate is given by

$$(X(t), y(t), S(t)) = (X, y, S) + (\Delta X, \Delta y, \Delta S). \tag{9}$$

Below, we describe more precisely the wide neighborhood feasible interior-point algorithm for SDO problems.

The wide neighborhood feasible algorithm for SDO problems

• **Input parameters:** Required precision $\varepsilon > 0$, neighborhood parameters $\beta, \tau \in (0, \frac{1}{3}]$ and the initial iterate $\left(X^0, y^0, S^0\right) \in \mathcal{N}(\tau, \beta)$.

• **step 0:** Set $k := 0$.

• **step 1:** If $n\mu^k \leq \varepsilon$, then stop. Otherwise go to **step 2**.

• **step 2:** Let $(X, y, S) = (X^k, y^k, S^k)$ and $\mu = \mu^k$. Compute $V = V^k$ from (6) and $(\Delta X, \Delta y, \Delta S)(t)$ from (8).
Let $(X(t), y(t), S(t)) = (X, y, S) + (\Delta X, \Delta y, \Delta S)(t)$ and find the smallest $\bar{t}$ such that $(X(t), y(t), S(t)) \in \mathcal{N}(\tau, \beta)$, for any $t \in [\bar{t}, 1]$.

• **step 3:** Let $t^k = \bar{t}$ and set $(X^{k+1}, y^{k+1}, S^{k+1}) = (X(t^k), y(t^k), S(t^k))$ and $\mu^{k+1} = t^k \mu^k$. Then, go to **step 1**.

3 Convergence analysis

In this section, we investigate the proposed feasible algorithm is well-defined and its complexity is $O(\sqrt{n})$. The following two lemmas play an important role in convergence analysis.

Lemma 3.1. *After a full Newton-step one has*
(i) $H_P(X(t)S(t)) = tV + H_P(\Delta X \Delta S)$,
(ii) $\mu(t) = t\mu$.

Lemma 3.2. *Let* $(X, y, S) \in \mathcal{N}(\tau, \beta)$. *Then*

$$\boldsymbol{Tr}\left(\tau\mu I - X^{\frac{1}{2}} S X^{\frac{1}{2}}\right)^{-} \leq (\tau - 1)n\mu, \tag{10}$$

$$\boldsymbol{Tr}\left(\tau\mu I - X^{\frac{1}{2}} S X^{\frac{1}{2}}\right)^{+} \leq \sqrt{n}\beta\tau\mu. \tag{11}$$

In following, we present the main result of the paper as follows.

Lemma 3.3. *Assume that the current iterate* $(X, y, S) \in \mathcal{N}(\tau, \beta)$. *Let* $0 < \tau \leq \beta \leq \frac{1}{3}$ *and* $1 - \frac{\alpha}{\sqrt{n}} \leq t \leq 1$ *such that* $\alpha \leq \frac{1}{4}\sqrt{\beta\tau}$. *Then, the new iterate* $(X(t), y(t), S(t))$ *generated by the feasible wide neighborhood algorithm belongs to* $\mathcal{N}(\tau, \beta)$.

The following theorem states the complexity bound of the proposed algorithm .

Theorem 3.4. *The proposed feasible wide neighborhood algorithm terminates in at most* $O(\sqrt{n}L)$ *iterations where* $L = \frac{1}{\alpha}\log\frac{\mathbf{Tr}(X^0S^0)}{\varepsilon}$ *and* $\alpha = \frac{1}{4}\sqrt{\beta\tau}$.

4 Concluding remarks

We proposed a new path-following wide neighborhood feasible interior-point algorithm for SDO problems. The algorithm is based on using a wide neighborhood and a new class of search directions. We proved that the proposed feasible wide neighborhood algorithm terminates in at most $O(\sqrt{n}L)$ iterations.

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An inverse linear fractional modeling

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Abstract

In this paper we consider the inverse fractional programming. Let x_0 be a feasible points of the model. We present a method to change the right hand side and coefficients of objective function which is a linear fraction function in order to make the point x_0 optimal or at least close to the optimal set.

Keywords: Inverse optimization; Fractional; linear programming
Mathematics subject classification [2010]: 90B90, 90B06.

1 Introduction

An inverse optimization is to change the values of parameters of the given problem as little as possible such that an feasible known solution becomes an optimal solution of the problem. For more information about inverse optimization see [1, 7]. In this paper we consider the inverse linear fractional programming with changing the right hand side and coefficients of objective function to make a given feasible point optimal or close as possible as to the optimal set. Jain and Arya [5, 6] presented a model for inverse fractional programming with adjust the coefficients of objective function to make a feasible given point optimal.

Consider the following linear fractional programming.

$$\begin{aligned} \max \quad & f(x) = \frac{\sum_{j=1}^{n} c_j x_j + c_0}{\sum_{j=1}^{n} d_j x_j + d_0} \\ & \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, ..., m \\ & x_j \geq 0, \quad j = 1, ..., n \end{aligned} \tag{1}$$

We want to use the method of Dampe and Loshe [4] which is presented for linear programming, for changing the right hand side and coefficients of objective function in order to make a given point optimal or at least close to the optimal set.

*Speaker

2 The inverse model

By using the Charnes and Cooper [3] transformation (also see [2]), let

$$t_j = \frac{x_j}{\sum_{i=1}^{n} d_i x_i + d_0} \qquad for j = 1, ..., n$$

and $t_0 = \frac{1}{\sum_{i=1}^{n} d_i x_i + d_0}$. Also by setting

$$E = \begin{bmatrix} d_0 & d_1 & ... & d_n \\ -b_1 & a_{11} & ... & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_m & a_{m1} & ... & a_{mn} \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad t = \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_n \end{bmatrix}$$

we get the following problem

$$\begin{aligned} \max \quad & ct \\ & Et = e \\ & t \geq 0. \end{aligned} \tag{2}$$

Lemma 1. By transforming model (1) using Charnes and Cooper [3] to model (2), if vector e in the model (2) is replaced by vector

$$g = \begin{bmatrix} 1 \\ g_1 \\ \vdots \\ g_n \end{bmatrix}$$

then in the corresponding fractional model, only the values of right hand side vector b will be changed.

Now let $\varphi_t(g, c) = argmax\{c^T t : Et = g, t \geq 0\}$ be the set of optimal solutions of the following linear parametric optimization problem,

$$\max\{c^T t : Et = g, t \geq 0\} \tag{3}$$

where the parameters of the right hand side and in the objective function are elements of given sets $\beta = \{g : Bg = \hat{g}\}$ and $\zeta = \{c : Cc = \hat{c}\}$, respectively.

Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times m}$, $C \in \mathbb{R}^{q \times n}$, $\hat{g} \in \mathbb{R}^p$ and $\hat{c} \in \mathbb{R}^q$ be fixed. Let $t^0 \in \mathbb{R}^n$ also be fixed. We want to find values $\bar{g}$ and $\bar{c}$ for the parameters, such that $t^0 \in \varphi_t(\bar{g}, \bar{c})$ or, if this is not possible, t^0 is at least close to $\varphi_t(\bar{g}, \bar{c})$. Therefore we consider the following problem

$$\min\{||t - t^0|| : t \in \varphi_t(g, c), g \in \beta, c \in \zeta\} \tag{4}$$

The problem (4) can be solved by the method of Dampe and Loshe [4] in the polynomial time.

After solving the problem (4), let the optimal parameters for g and c be $\tilde{g}$ and $\tilde{c}$, respectively, and $\tilde{t}$ be the optimal value of t. Also let $m = n$, E^{-1} and A^{-1} exist then in following theorem we show the optimal solution of model (1) can be found by solving model (4).

Theorem 1. Let $m = n$, E^{-1} and A^{-1} exist. The point $\tilde{x}$ where $\tilde{x_j} = \frac{\tilde{t_j}}{\tilde{t_0}}$ for $j = 1, ..., n$ is an optimal solution of model (1) by setting $c = \tilde{c}$ or

$$||\tilde{x} - x^0|| = \min\{||x - x^0|| : x \in \varphi_x(b, c), b \in \beta', c \in \zeta\} \tag{5}$$

where $\varphi_x(b, c) = argmax\{c^T x : Ax = b, x \geq 0\}$ and $\beta' = \{b : Bb = \hat{b}\}$.

Proof. By setting $\tilde{g}$ and $\tilde{c}$ in the model (3) we get the following problem;

$$\begin{aligned} \max \quad & \sum_{j=0}^{n} \tilde{c_j} t_j \\ & \sum_{j=0}^{n} d_j t_j = \tilde{g_0} \\ & \sum_{j=1}^{n} a_{ij} t_j = \tilde{g_i} + b_i t_0 \quad i = 0, ..., m \\ & t_j \geq 0, \quad j = 0, ..., n. \end{aligned} \tag{6}$$

By reversing transform t_j to x_j, we get $t_j = x_j t_0$ for $j = 1, ..., n$. And by setting $t_0 = \frac{1}{\sum_{j=1}^{n} d_j x_j + d_0}$;

$$\begin{aligned} \max \quad & \frac{\sum_{j=1}^{n} \tilde{c_j} x_j + \tilde{c_0}}{\sum_{j=1}^{n} d_j x_j + d_0} \\ & \sum_{j=0}^{n} d_j x_j = \tilde{g_0}(\sum_{j=1}^{n} d_j x_j + d_0) \\ & \sum_{j=1}^{n} a_{ij} x_j = \tilde{g_i}(\sum_{j=1}^{n} d_j x_j + d_0) + b_i \quad i = 1, ..., m \\ & x_j \geq 0, \quad j = 1, ..., n \end{aligned} \tag{7}$$

If $\tilde{g_0} = 1$ and $\tilde{g_i} = 0$ for $j = 1, ..., n$ then the model will be changed to the model (1). Other wise we show the relation (5) holds.

$$||\tilde{x} - x^0||_1 = \sum_{j=1}^{n} |\tilde{x_j} - x_j^0| = \sum_{j=1}^{n} |\frac{\tilde{t_j}}{\tilde{t_0}} - \frac{t_j^0}{t_0^0}|.$$

□

3 Conclusion

In this paper we presented a method for solving fractional linear programming with changing the right hand side and coefficients of objective function in order to make a given point optimal or at least close to the optimal set.

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Descriptor systems controller, with minimizing the norm of state feedback matrix

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Abstract

In this paper, we describe the similarity transformation of the state space of a descriptor system and then using state-derivative feedback. Then we determine parametric state feedback matrix for a linear descriptor system. First, we define the input as a multiple of the state-derivative feedback, and when the descriptor system was changed a standard system with state feedback, consider that the similarity transformations. Using the system closed-loop matrix graph, we find the parametric state feedback matrix. Finally we get the controlled optimal matrix with minimum norm.

Keywords: Descriptor system, State-derivative feedback, Closed-loop matrix graph
Mathematics Subject Classification [2010]: 93C05, 93B60

1 Introduction

Minimum norm of feedback matrix in controllable descriptor systems in recent years have had a favorable and certain effect on types of human life. Many researcheres have performed extensive amounts of researches on eigenstructure assignment in descriptor system like Bunse (1992), Duan and Wang (2005) in [1, 2]. Many authors like Karbassi and Bell worked on minimization of the norm of feedback controllers, too [3, 4, 5, 6, 7].

The first advantago of this paper is, using the state-derivative feedback to convert the descriptor to a standard system. Then using similarity transformation and parameterization, we transform the state space will facilitate the calculation of the system feedback matrix and the feedback matrix using graph theory. Also, by identifying the parameters location of the feedback matrix, we can count them in their numbers. The most important superiority of this paper is that, makes it possible to calculate the state space similarity transformation and parametric state feedback matrix for a descriptor system.

In this paper it is assumed that the descriptor system (1) is a linear time-invariant system with state-derivative feedback (2). First we convert the descriptor system (1) with (2) to standard system (5b) with state feedback control (5c), then we obtain state feedback matrix for assigning desired eigenvalues to system (5b), finally we obtain minimum norm of state feedback controller. Consider a controllable linear descriptor system described by

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0 \tag{1}$$

*Speaker

where $E \in \mathbb{R}^{n\times n}$, $x(t) \in \mathbb{R}^n$ is state feedback, $u(t) \in \mathbb{R}^m$ is input vector and $y(t) \in \mathbb{R}^l$ is output vector. It is assuemed that $1 \leq m \leq n$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are time-invariant. Now consider the state-derivative feedback control

$$u(t) = K_d\dot{x}(t) \tag{2}$$

The program is to obtain a state-derivative feedback K_d, using the state feedback techniques, such that the poles of the controlled system (1) and (2) are arbitrarily specified by $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$, where $\lambda_i \in \mathbb{C}$ and $\lambda_i \neq 0$, $i = 1, 2, ..., n$, such that this closed-loop systems presents a suitable performance. To establish the proposed results, consider following assumptions: $I) rank[E|B] = n$, $II) rank[A] = n$, $III) rank[B] = m$

Remark 1.1. It is clear that if assumption (I) holds, then there exists K_d such that:

$$rank[E - BK_d] = n \tag{3}$$

For K_d such that (3) holds, then from (2) it follows that (1) can be rewrite such as a standard linear system, given by:

$$E\dot{x}(t) = Ax(t) + BK_d\dot{x}(t) \Longrightarrow \dot{x}(t) = (E - BK_d)^{-1}Ax(t) \tag{4}$$

From (4) note that if $rank(A) < n$ then the controlled system (1), with (2) given by (4) is unstable, because it presents at least one pole equal to zero. [1]

Lemma 1.2. *consider a matrix $M \in \mathbb{R}^{n\times m}$ with $rank(M) = n$ and the eigenvalues equal to $\lambda_1, \lambda_2, ..., \lambda_n$. Then, the eigenvalues of M^{-1} are the following: $\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}$. [3, 4]*

Remark 1.3. Consider that $\lambda = a + jb$ is an eigenvalue of M, then from lemma 1.2 $\lambda^{-1} = (a + jb)^{-1} = \frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2}$

Theorem 1.4. *define the matrices N, M as* (5a) *and suppose (N, M) is controllable. Also let K_d be state feedback matrix, such that $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ are the eigenvalues of the closed-loop system* (5b) *with control law* (5c)*,*

$$N = A^{-1}E, M = -A^{-1}B \tag{5a}$$
$$\dot{z}(t) = Nz(t) + Mw(t) \tag{5b}$$
$$w(t) = K_dz(t) \tag{5c}$$

where $\lambda_i \in \mathbb{C}$ and $\lambda_i \neq 0$, $i = 1, 2, ..., n$, are arbitrarily specified. Then for this K_d, the desired spectrum $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ is the eigenvalues of the controlled system (1) *with state-derivative feedback* (2) *and also, the condition* (3) *holds.*

Proof. Considering that (N, M) is controlled, then one can find a state feedback matrix K_d such that the controlled system (5b) with state feedback (5c) given by $\dot{z} = (N+MK_d)z(t)$. has poles equal to $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$. Now by (5a) note that:

$$(N + MK_d)^{-1} = (A^{-1}(E - BK_d))^{-1} \Longrightarrow (N + MK_d)^{-1} = (E - BK_d)^{-1}A \tag{6}$$

By equation (6), the spectrum $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ is the eigenvalues of closed-loop matrix $(E-BK_d)^{-1}A$. Therefore (3) holds and the eigenvalue of closed-loop system (1) and (2) are equal to $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. □

2 Main results

Consider the following controllable system (5b) corresponding to the descriptor system (1). First assign $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ to system (5b) with (5c) by [5]. According theorem 1.4 it is assigned $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ to system (1) with (2). Then minimize the norm of the primary state feedback matrix for the pair (N, M) instead of state feedback matrix for the pair of (B, A) by parameterizations of [6]. Now briefly display the method of assigning and parameterizations of [6] by below algorithm. Pay attention to some basic formula that we need in algorithm.

Consider the state transformation $z(t) = T\tilde{z}(t)$ where T can be obtained by elementary similarity operations as described in [3, 4]. Substituting (6) into (5b) yields $\dot{\tilde{z}} = T^{-1}NT\tilde{z} + T^{-1}Mw(t)$. It is noted that the transformation matrix T is invertible. In this way, $\tilde{N} = T^{-1}NT$ and $\tilde{M} = T^{-1}M$ are in a compact canonical form know as vector companion form [6]:

$$\tilde{N} = \begin{bmatrix} & G_0 & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}, \tilde{M} = \begin{bmatrix} M_0 \\ 0_{n-m,m} \end{bmatrix}$$

Now follow this algorithm:

Object. To assign desired eigenvalue $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ to system (1) and obtain the state feedback matrix with minimum norm.

Input. The pair (B, A), matrices $\tilde{N}$, $\tilde{M}$, G_0, M_0 and T^{-1}.

Step 1. Obtain the state feedback matrix $\tilde{F}_d$ and F_d, that assigns zero eigenvalues to system $(\tilde{N}, \tilde{M})$ and (N, M) respectively, which $\tilde{F}_d = -M_0^{-1}G_0$, $F_d = \tilde{F}_d T^{-1}$.

Step 2. Consider the transformed closed-loop matrix $\tilde{\Gamma} = \tilde{N} + \tilde{M}\tilde{F}_d$ which $\tilde{\Gamma} = \begin{bmatrix} & 0_{m,n} & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}$ and assumes a compact Jordan form with zero eigenvalues.

Step 3. Add a diagonal matrix $D = diag\{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ for an arbitrary set of self-conjugate eigenvalues to $\tilde{\Gamma}$. Then the closed-loop system matrix $(\tilde{N} + \tilde{M}\tilde{F}_d)$ becomes $V_p = (\tilde{N} + \tilde{M}\tilde{F}_d) + D$

Step 4. Use simple elementary similarity operations of [3, 4] to obtain the matrix $\tilde{V}$ from V_p such that $\tilde{V} = \begin{bmatrix} & G_\lambda & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}$

Step 5. Obtain the primary feedback matrix K_d that gives rise to the assignment of eigenvalues $L^{-1} = \{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}$ to system (5b) by $K_d = F_d + M_0^{-1}G_\lambda T^{-1}$.

Step 6. Obtain $F_\alpha = M_0^{-1}G_\alpha T^{-1}$ that G_α is a $m \times n$ matrix containing free parameters. Use state transition graph (STG) method to locate free parameter in F_α. (For more detail about determining parameter in G_α, see algorithm of [5])

Step 7. Obtain the parametric feedback matrix is defined as $F = K_d + F_\alpha$ which assigns $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ with minimum the norm of state feedback matrix of system (1).

Example 2.1. consider the following linear descriptor system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

In this example, the suitable closed-loop poles are for the controlled systems (1) and (2) are: $\lambda_1 = -0.5$, $\lambda_{2,3} = -0.5 \pm 0.5i$. One can obtain the feedback matrix K_d and F

below that their norms are 4.2426 and 3.7417 respectively.

$$K_d = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Conclusions

Considering in this paper, using the state-derivative feedback, the descriptor system becomes converted to a standard system, explains the advantages of this method. Because working with a standard system, it is much easier than the descriptor mode. Then using similarity transformation, we transform the state space will facilitate the calculation of the system feedback matrix. Also we do the parameterization, the feedback matrix using graph theory. Moreover, using this method, we identified the parameters location of the feedback matrix, can also be counted in their numbers. This will be advantage of this paper is that, makes it possible to calculate the state space similarity transformation and parametric state feedback matrix for a descriptor system.

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Efficiency measure by interval data envelopment analysis model and its application

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Abstract

Data envelopment analysis (DEA) is a non-parametric technique to measure the efficiencies of a set of decision making units (DMUs) with common crisp inputs and outputs. In real-world problems, however, inputs and outputs are interval. To analyze a DMU with interval input/output data, this paper proposed an associated evaluating approach. Nonetheless, numerous deficiencies must be improved in mentioned models.

Keywords: Data envelopment analysis, Efficiency, Interval data.
Mathematics Subject Classification [2010]: 90B50

1 Introduction

DEA is a non-parametric method for evaluating the efficiency of DMUs like bank branches, schools, transport sectors etc. on the basis of multiple inputs and outputs. Charnes, Cooper and Rhodes (CCR) [1] developed the DEA approach in 1978. After the paper of CCR, there was an exponential growth in number of publications on DEA. In more general cases, the data for evaluation are often collected from investigations employing a polling approach, where in natural language, such as good, medium, and bad, are used to represent a type of general situation of the examined entities rather than a specific case. Thus, several studies proposed the interval DEA model for input and output data [2, 3]. In this paper, DEA model is extended to be an interval model for evaluating efficiency and ranking of DMUs with interval data. At last a numerical presentation of real data from a commercial bank of Iran is considered.

2 DEA model with interval data

This paper is proposing a model which is the extension of CCR model to an interval framework. Let a set of n DMUs has m interval inputs $[X_{ij}^L, X_{ij}^U]$ and s interval outputs $[Y_{ij}^L, Y_{ij}^U]$ i.e., inputs and outputs are approximately known and not precisely measured. Thus, interval CCR model is given by interval linear programming problem(LPP) as

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follows:

$$\begin{aligned} E_k = \max\ & \frac{\sum_{r=1}^{s} u_{rk}\tilde{Y}_{rk}}{\sum_{i=1}^{m} v_{ik}\tilde{X}_{ik}} \\ s.t\ & \frac{\sum_{r=1}^{s} u_{rk}\tilde{Y}_{rj}}{\sum_{i=1}^{m} v_{ik}\tilde{X}_{ij}} \leq 1, \quad \forall j = 1, ..., n, \\ & u_{rk} \geq \varepsilon \quad \forall r = 1, ..., s, \\ & v_{ik} \geq \varepsilon \quad \forall i = 1, ..., m. \end{aligned} \tag{1}$$

The efficiency score evaluated from the model should be interval because this model contains interval parameters.

$$\begin{aligned} E_k = \max\ & \frac{\sum_{r=1}^{s} u_{rk}(Y_{rk})^L}{\sum_{i=1}^{m} v_{ik}(X_{ik})^U} \\ s.t\ & \frac{\sum_{r=1}^{s} u_{rk}(Y_{rk})^L}{\sum_{i=1}^{m} v_{ik}(X_{ik})^U} \leq 1 \\ & \frac{\sum_{r=1}^{s} u_{rk}(Y_{rj})^U}{\sum_{i=1}^{m} v_{ik}(X_{ij})^L} \leq 1, \quad \forall j = 1, ..., n, j \neq k \\ & u_{rk} \geq \varepsilon \quad \forall r = 1, ..., s, \\ & v_{ik} \geq \varepsilon \quad \forall i = 1, ..., m. \end{aligned} \tag{2}$$

Similarly we can obtain$(E_k)^U$.
The theory fo fractional linear programming [5] make them possible to replace model (2) with an equivalent LPP.

$$\begin{aligned} (E_k)^L = \max\ & \sum_{r=1}^{s} u_{rk}(Y_{rk})^L \\ s.t.\ & \sum_{i=1}^{m} v_{ik}(X_{ik})^U = 1, \\ & \sum_{r=1}^{s} u_{rk}(Y_{rk})^L - \sum_{i=1}^{m} v_{ik}(X_{ik})^U \leq 0 \\ & \sum_{r=1}^{s} u_{rk}(Y_{rj})^U - \sum_{i=1}^{m} v_{ik}(X_{ij})^L \leq 0, \quad \forall j = 1, ..., n, \\ & u_{rk} \geq \varepsilon \quad \forall r = 1, ..., s, \\ & v_{ik} \geq \varepsilon \quad \forall i = 1, ..., m. \end{aligned} \tag{3}$$

$$
\begin{aligned}
(E_k)^U = \max\ & \sum_{r=1}^{s} u_{rk}(Y_{rk})^U \\
s.t.\ & \sum_{i=1}^{m} v_{ik}(X_{ik})^L = 1, \\
& \sum_{r=1}^{s} u_{rk}(Y_{rk})^U - \sum_{i=1}^{m} v_{ik}(X_{ik})^L \leq 0 \\
& \sum_{r=1}^{s} u_{rk}(Y_{rj})^L - \sum_{i=1}^{m} v_{ik}(X_{ij})^U \leq 0, \quad \forall j = 1, ..., n, \\
& u_{rk} \geq \varepsilon \quad \forall r = 1, ..., s, \\
& v_{ik} \geq \varepsilon \quad \forall i = 1, ..., m.
\end{aligned} \tag{4}
$$

3 Ranking of DMUS

Ranking of DMUs plays an important part in DEA interpretations. The final efficiency of a DMU in DEA model with interval data is no longer a crisp number. The ranking index for the jth DMU as

$$I_j = \frac{\sum_{j=0}^{n} E_j^U}{\sum_{j=0}^{n} E_j^U - \sum_{j=0}^{n} E_j^L}$$

4 numerical example

To illustrate the proposed interval DEA model consider this data.

DMU	Input	Output
1	$[11, 14]$	$[10, 10]$
2	$[30, 30]$	$[12, 16]$
3	$[40, 40]$	$[11, 11]$
4	$[45, 52]$	$[12, 22]$

We obtain the efficiency for first of interval and end of interval by proposed models.

Efficiency	A	B	C	D
height	$[1, 1]$	$[0.440, 0.747]$	$[0.302, 0.385]$	$[0.240, 0.684]$

conclusion

DEA has wide application to evaluate the relative efficiency in a set of DMUs by using multiple to common crisp inputs and outputs. The existing DEA models are usually limited to common crisp inputs and outputs. In some cases, input and output data of

DMUs cant be precisely measured, for example, quality of service, quality of input resource, degree of satisfaction etc. so, the uncertain theory has played an important role in DEA. In these cases, the data with crisp number will not satisfy the real needs and this restriction will diminish the practical flexibility of DEA in application. Thus, the data can be represented by interval sets. This paper attempts to extend the traditional DEA model to an interval framework.

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Necessary and Sufficient Conditions for Weak Efficinecy on K-subdifferentiable functions

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Abstract

In this paper a multiobjective problem (MP) with a feasible set defined by inequality and equality constraints and a set constraint are considered. Then, by using the concept of K-directional derivatives, we obtain necessary and sufficient conditions for local weak efficiency on a new class of functions.

Keywords: K-directional derivative, Local cone approximation, Constraint qualification, Efficiency, Weak Efficiency.
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

The subject of optimality conditions for optimization problems, which are not necessarily smooth, has been extensively researched for the past decade. Central to this study has been the development of the appropriate cone approximations for generalizing the concepts of directional derivatives and subdifferentials. In this way, Elster and Thierfelder [5, 4] and independently Ward [8] exploiting a general and axiomatic definition of local cone approximation of a set, introduced a general definition of directional derivative for a function $f : X \to R$ where X is a finite dimensional space or also a topological linear space. Also ,Nobakhtian [7] by using the concept of K-directional derivative proved general optimality conditions for a multiobjective problem with a feasible set defined by equality and inequality constraints. In this paper, we introduce a new class of functions and prove that under a suitable constraint qualification, it is a both necessary and sufficient condition in order to K-strongly efficient stationary points, K-weakly efficient stationary points, local efficient, local weak efficient, efficient and weak efficient be equivalent.

2 Notations and Preliminaries

Given the function $f : X \to R$, the epigraph of f is epi $f = \{(x, r) \in X \times R : f(x) \leq r\}$. The set epi f will be locally approximated at the point $(x, f(x))$ by a local cone approximation K and a positively homogenous function $f^K(x, .)$ will be uniquely determined.

Definition 2.1. ([3]) Let $f : X \to R, x \in X$ be a local cone approximation; the positively homogeneous function $f^K(x; .) : R^\ell \to [-\infty, +\infty]$ defined by $f^K(x; d) := inf\{\xi \in R : (d, \xi) \in K(epif; (x, f(x)))\}$ is called the K-directional derivative of f at x.

We introduce a type of generalized directional derivatives used in literature; the upper Dini directional derivative of f at x

$$f^Z(x;v) = \limsup_{t\to 0^+} \frac{f(x+tv)-f(x)}{t}$$

is associated to the cone of the feasible directions

$$Z(Q,x) := \{v \in R^\ell : \forall\{t_k\} \to 0^+, x + t_k v \in Q,\ \forall k\}.$$

Definition 2.2. ([3]). Let $f : X \to R$ and K be a local cone approximation, the function f is said to be K-subdiffrensiable at x if there exists a convex compact set $\partial^K f(x)$ such that

$$f^K(x;v) = \max_{x^* \in \partial^K f(x)} \langle x^*, v\rangle \quad \forall v \in R^\ell$$

which $\partial^K f(x) := \{x^* \in R^\ell : \langle x^*, v\rangle \le f^K(x,v),\ \forall v \in R^\ell\}$ is called the K-subdifferential of f at x.

3 Optimality Conditions

In this section, we study the problem (MP).

$$\begin{array}{lll} \textbf{(MP)} & \min & f(x) = (f_1(x), \dots, f_m(x)) \\ & \text{s.t.} & g(x) = (g_1(x), \dots, g_n(x)) \leqq 0, \\ & & h(x) = (h_1(x), \dots, h_p(x)) = 0, \\ & & x \in Q, \end{array}$$

where $f_i, i \in I = \{1,\dots,m\}, g_j, j \in J = \{1,\dots,n\}, h_s, s \in E = \{1,\dots,p\}$ are K-subdifferentiable and real valued functions which are defined from R^l and Q is an arbitrary set. Let us introduce some notations which are used in the sequel.

$$\begin{aligned} S &:= \Big\{x \in R^\ell \Big| g(x) \leqq 0, h(x) = 0, x \in Q\Big\} \\ S^l &:= \Big\{x \in R^\ell \Big| f_i(x) \le f_i(x_0), \forall i \ne \ell, g(x) \leqq 0, h(x) = 0, x \in Q\Big\} \\ F &:= \bigcup_{i\in I} \partial^K f_i(x_0), \quad F^l := \bigcup_{i\in I-\{\ell\}} \partial^K f_i(x_0), \\ G &:= \bigcup_{j\in J(x_0)} \partial^K g_j(x_0), \quad H := \bigcup_{s\in E} \partial^K h_s(x_0). \end{aligned}$$

where $J(x_0)$ denote the index set of active constraints at the given point x_0.

Definition 3.1. The following constraint qualifications are considered:

$$(CQ1): \quad F^- \bigcap G^- \bigcap H^- \bigcap K(Q,x_0) \subseteq \bigcap_{i=1}^m WF(S^i, x_0).$$

$$(CQ2): \quad F^s \bigcap G^- \bigcap H^- \bigcap K(Q,x_0) \subseteq WF(S, x_0).$$

Definition 3.2. Let $x_0 \in S$, and K be a local cone approximation. The point x_0 is said to be

- K-strongly efficient stationary point for (MP) if the following system is impossible:

$$\begin{aligned} f_i^K(x_0;v) &\leq 0 \quad \text{for all } i \in I, \\ f_i^K(x_0;v) &< 0 \quad \text{for some } i \in I, \\ g_j^K(x_0;v) &\leq 0 \quad \text{for all } j \in J(x_0), \\ h_s^K(x_0;v) &= 0 \quad \text{for all } s \in E, \\ v &\in K(Q,x_0). \end{aligned} \tag{1}$$

- K-weakly efficient stationary point for (MP) if the following system is impossible:

$$\begin{aligned} f_i^K(x_0;v) &< 0 \quad \forall i \in I, \\ g_j^K(x_0;v) &\leq 0 \quad \forall j \in J(x_0), \\ h_s^K(x_0;v) &= 0 \quad \forall s \in E, \\ v &\in K(Q,x_0). \end{aligned} \tag{2}$$

It is always possible to choose suitable local cone approximation K such that every efficient solution $\bar{x}$ is a K-weakly or K-strongly efficient stationary point [1]. For differentiable functions we can see in [6], that every efficient solution for the problem (MP) under some constraint qualification is strongly efficient stationary point. We will prove under suitable constraint qualification, it is possible to deduce necessary and sufficient optimality conditions directly from impossibility of the system (1) or (2).

Theorem 3.3. *Let (CQ1) be satisfied at $x_0 \in S$ and K be a local cone approximation such that $K(Q,x_0) \subseteq Z(Q,x_0)$. If x_0 is a local efficient solution for (MP), then system (1) is impossible.*

Theorem 3.4. *Let (CQ2) be satisfied at $x_0 \in S$ and K be a local cone approximation such that $K(Q,x_0) \subseteq Z(Q,x_0)$. If x_0 is a local weak efficient solution for (MP), then the system (2) is impossible.*

In the sequel we define a new class of functions in order to prove under some constraint qualification it is both necessary and sufficient condition for K-strongly efficient stationary points, K-weakly efficient stationary points, local efficient, local weak efficient, efficient and weak efficient solution of (MP) be equivalent.

Definition 3.5. Let $x_0 \in S$. Problem (MP) is said to be KT-K-strictly pseudoinvex-infine at x_0 w.r.t η on S, if for any $x \in S$ with $f(x) \leq f(x_0)$, there exists $\eta(x,x_0) \in K(Q,x_0)$ such that the following system holds

$$\begin{aligned} f_i^K(x_0;\eta(x,x_0)) &< 0 \quad \forall i \in I, \\ g_j^K(x_0;\eta(x,x_0)) &\leq 0 \quad \forall j \in J(x_0), \\ h_s^K(x_0;\eta(x,x_0)) &= 0 \quad \forall s \in E. \end{aligned}$$

From definition 3.5 we obtain that the KT-K-pseudoinvex-infineness is a generalization of the KT-pseudoinvexity-II given by Arana [2] for a nondiferentiable multiobjective problem when the local cone approximation K is Clarke's tangent cone and $f_i, i \in I$, $g_j, j \in J$, $h_s, s \in E$ are locally Lipschitz functions.

Theorem 3.6. *Let (CQ1) be satisfied at $x_0 \in S$ and K be a local cone approximation such that $K(Q, x_0) \subseteq Z(Q, x_0)$. If problem (MP) is KT-K-strictly pseudoinvex-infine at x_0 w.r.t η on S, then the following statement are equivalent.*

(1) x_0 is an efficient solution for (MP),

(2) x_0 is a local efficient solution for (MP),

(3) x_0 is a K-strongly efficient stationary point for (MP),

(4) x_0 is a K-weakly efficient stationary point for (MP),

(5) x_0 is a local weak efficient solution for (MP),

(6) x_0 is a weak efficient solution for (MP).

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Parameterization of the State Feedback Controller for Linear Time-invariant Systems with Disturbance

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Abstract

In this paper, we obtain a state feedback matrix by elementary similarity operations such that the eigenvalues of the close-loop system lie in the self-conjugate eigenvalue spectrum Λ and for rejected disturbance a arbitrary system controllable without disturbance is considered, then with use augmented matrix, the two systems Merged in a new system without disturbance, by transformations similarity compute optimal controller, so that disturbance has no effect on the system. Since, relation between the parameters of state feedback matrix is nonlinear in general therefore, many different constraint may be imposed by the designer to obtain desired performance criteria. A illustrative example is presented.

Keywords: Parameterization, Continuous-time, State feedback matrix, Disturbance rejection,
Mathematics Subject Classification [2010]: 35A17, 60J28,15A18

1 Introduction

Different methods of parametric eigenvalue assignment for systems have been proposed [1]-[2]. Also, different ways to describe disturbances and to analyze their effect on a system are discussed in [3]-[5]. An overwive of different ways to eliminate disturbances include use of feedback, feedforward, and prediction methods, which may be through classical control, advanced control and adaptive control to be performed. Disturbance can be based on calling control, remove the source and measurement and or estimated.
The classic disturbance models, impulse, step, ramp, and sinusoid, were discussed in [3].

2 Main results

Consider a linear system defined by the state equation

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t) \tag{1}$$

Where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^n$ is the disturbance input and the matrices A and B are real constant matrices of dimensions $n \times n$ and $n \times m$,

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respectively, with $rank(B) = m$ and control rule $u(t) = F_1 x(t)$. First, we define the following arbitrary system controllable:

$$\dot{z}(t) = A_2 z(t) + B_2 u(t) \tag{2}$$

Where $u \in \mathbb{R}^m$ is the control input, matrices A_2 and B_2 are real constant matrices of dimensions $n \times n$ and $n \times m$, respectively, with $rank(B_2) = m$ and control rule $u(t) = F_2 z(t)$.
Provided that $z(0) \neq 0$,. Now we defind

$$d(t) = Cz(t) \tag{3}$$

Where C is diagonal matrix with $C_{ii} = \dfrac{d_i}{z_i},\ i = 1, 2, .., n$. Therefore, with replacement (3) in (1) and with equation (2) we have:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_1 & C \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \tag{4}$$

we will

$$\dot{y}(t) = Ay(t) + Bu(t) \tag{5}$$

Where, matrices A and B are real constant matrices of dimensions $2n \times 2n$ and $2n \times m$ However, with definition F for control rule $u(t) = Fy(t)$ so that the without disturbance system is stable. with using a similarity transformations described in [2] we have:

$$\tilde{A}_\lambda = \begin{bmatrix} & G_\lambda & \\ I_{n-m} & , & O_{n-m,m} \end{bmatrix} \tag{6}$$

where G_λ is the first $m \times n$ sub-matrix of $\tilde{A}_\lambda$. Obviously, $\tilde{A}_\lambda$ possesses the desired set of eigenvalues and is in the same canonical form as $\tilde{A}$.
Thus, the primary feedback matrix which gives rise to the assignment of eigenvalues becomes

$$\tilde{F} = \tilde{F}_p + {B_0}^{-1} G_\lambda = {B_0}^{-1}(-G_0 + G_\lambda) \tag{7}$$

The above experssion leads to a general framework for obtaining the parametric controllers in general. Thus, let

$$det(\tilde{A}_\lambda - \lambda I) = P_n(\lambda) = (-1)^n(\lambda^n + C_1\lambda^{n-1} + \cdots + C_{n-1}\lambda + C_n), \tag{8}$$

That is the characteristic polynomial of the closed-loop system. Since it is required that the zeros of this polynomial lie in the set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, it is clear that

$$P_n(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n). \tag{9}$$

By equating these two equations the coefficients $C_i,\ (i = 1, 2, \cdots, n)$ can be obtained as follows [2]:

$$\begin{aligned} &C_1 = -\textstyle\sum_{i=1}^{n} \lambda_i \\ &C_2 = \textstyle\sum_{i,j=1\ i\neq j}^{n} \lambda_i\lambda_j \\ &\vdots \\ &C_n = (-1)^n \Pi_{i=1}^{n} \lambda_i \end{aligned} \tag{10}$$

Now by direct computation of $det(\tilde{A}_\lambda - \lambda I)$ in parametric form and equating the coefficients of the characteristic polynomial with (10), the following nonlinear system of equations is obtained:

$$\begin{aligned} &f1(g_{11}, g_{12}, \ldots, g_{1n}, g_{21}, \ldots, g_{2n}, \ldots, g_{m1}, g_{m2}, \ldots, g_{mn}) = C_1 \\ &f2(g_{11}, g_{12}, \ldots, g_{1n}, g_{21}, \ldots, g_{2n}, \ldots, g_{m1}, g_{m2}, \ldots, g_{mn}) = C_2 \\ &\vdots \\ &fn(g_{11}, g_{12}, \ldots, g_{1n}, g_{21}, \ldots, g_{2n}, \ldots, g_{m1}, g_{m2}, \ldots, g_{mn}) = C_n, \end{aligned} \tag{11}$$

where $g_{ij},\ (i = 1, \ldots, m,\ j = 1, \ldots, n)$, are the elements of G_λ.
In this way, a nonlinear system of n equations with $n \times m$ unknowns is obtained. By choosing $N = n(m-1)$ unknowns arbitrarily it is then possible to solve the system. Thus, different selections can be made to obtain different solutions. Clearly, some of these choices lead to linear parameters. The controller gain matrix for the original pair (A, B) can then be obtained by (7). In other words, the nonlinear system of equations (11) is uniquely defined for any given pair (A, B) of fixed dimensions and regular Kronecker invariants [2].

Example 2.1. We consider a continuse-time linear system given by

$$\dot{x}(t) = Ax(t) + Bu_1(t) + d(t) \tag{12}$$

Where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ -1 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 0 \end{bmatrix} \tag{13}$$

We suppose in the first step $x(0) = (1, 0, 1)^t$ and $d(0) = (1, 6, 1)^t$, then for solve the system (12), should consider the Controllable system following:

$$\dot{z} = Az(t) + Bu_2(t), \quad u = F_2 z(t) \tag{14}$$

$$A = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix},\ Z(0) = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \tag{15}$$

With the help of similarity transformations expressed in [1]-[2], and equations (4) and (8) we have,

$$P_6(\lambda) = \lambda^6 + 6\lambda^5 + 15\lambda^4 + 20\lambda^3 + 15\lambda^2 + 6\lambda + 1, \tag{16}$$

By equating the coefficients of equations (10) and (11) we obtain the nonlinear system of equations

$$\begin{cases} g_{15}g_{26} - g_{16}g_{25} = 1 \\ g_{13}g_{26} - g_{14}g_{25} + g_{15}g_{24} - g_{16}g_{23} = 6 \\ g_{11}g_{26} - g_{12}g_{25} + g_{13}g_{24} - g_{14}g_{23} + g_{15}g_{22} - g_{16}g_{21} = 15 \\ -g_{15} - g_{26} + g_{11}g_{24} - g_{12}g_{23} + g_{13}g_{22} - g_{14}g_{21} = 20 \\ -g_{13} - g_{24} + g_{11}g_{22} - g_{12}g_{21} = 15 \\ -g_{11} - g_{22} = 6 \end{cases}$$

Here, there are 6 equations with 12 unknowns. the nonlinear parametric controller matrix F for the pair (A, B) is $F = \tilde{F}T^{-1}$.
Different options can be considered for parameters g_{ij} and achieved different results. For example, suppose $g_{11} = -4,\ g_{21} = g_{24} = 2,\ g_{23} = -g_{25} = 1,\ g_{26} = 0$, therefore we have:

$$F = \begin{bmatrix} 0.2134 & -4.2620 & -0.7415 & 0.0006 & 1.1601 & 4.7243 \\ -0.7560 & -0.7893 & 0.4762 & 0.6518 & 1.1268 & -2.6908 \end{bmatrix}$$

Conclusions

In this paper, we compute a nonlinear parametic matrix state feedback of linear systems with disturbance input. We first by introduce a augmented matrix, rejected disturbance. Also for obtain feedback parameter nonlinear matrix to form a parallel operation of the system by taking into our companion, the closed-loop system matrix in the new space are we the first element of the m row to put parameters and using equation obtains characteristic parameters of the relationship between nonlinear relationships are obtained. The advantage of this parameter matrix the designer can impose constraints on the system and the fact that in the case of nonlinear parameter is the number of parameters on the behavior of the system designer may impose other constraints, since in most physical systems and nonlinear engineering parameters appear, the importance of the matrix is determined.

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Solving inverse scheduling single machine problems by Karush-Kuhn-Tucker condition optimally

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Abstract

The importance of scheduling problems leads us to pay more attention in using solution methods. due to high costs of prossesing and job transfer, inverse scheduling problem has been used. In this paper inverse scheduling single machine has been considered in order to minimize the cost coefficient of time parameters setting. So Karush-Kuhn-Tucker condition has been used to represent a set of equations. The proposed solution method for these equations, could be solved by MATLAB.

Keywords: Scheduling, Inverse scheduling, Single machine, Karush-Kuhn-Tucker condition
Mathematics Subject Classification [2010]: 90B36, 68M20

1 Introduction

Consider single machine scheduling problem with n job.

- p_j: Processing time of job j
- C_j: Completion time of job j which is $\sum_{k=1}^{j} p_k$

Suppose scheduling problem is intended to minimize $\overline{C}$ which will be done with Shortest Processing Times [1]. In this sequence job has been sorted based on non-decrease processing time.

Theorem 1.1. *[1] The sequence job SPT($p_1 \leq p_2 \leq \cdots \leq p_n$), minimize the mean time of compeletion time*

Now, consider the problem discussed in [3] is solved by determining the minimum total setting of the job processing times so that a given sequence is converted into an SPT sequence. We formulate this problem as a mathematical programming:

$$\begin{array}{ll} min & \sum_{i=1}^{n} \alpha_i |p'_i - p_i| \\ s.t. & \\ & 0 \leq p'_1 \leq p'_2 \leq \cdots \leq p'_n \end{array} \tag{1}$$

*Speaker
[1]SPT

where for $i = 1, 2, \ldots, n$, p_i^* is the new value of processing time after regulation and deviation from initial value and α_i is the cost unit of deviation from p_j. This problem has been linearly stated at [2].

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^{n} \alpha_i + x_i + \sum_{i=1}^{n} \alpha_i + y_i && && (2)\\ s.t \quad & 0 \le p_1^\star \le p_2^\star \le \cdots \le p_n^\star && && (3)\\ & p_i^\star - p_i = x_i - y_i && i = 1, 2, \ldots, n && (4)\\ & x_i \ge 0, y_i \ge 0, && i = 1, 2, \ldots, n && (5) \end{aligned}$$

Theorem 1.2. *[3](Karush-Kuhn-Tucker condition) Let f and $g_i, i = 1, \ldots, n$, be real valued functions efined on $\mathbb{R}^n$. We consider the following optimization problem*

$$\begin{aligned} min \quad & f(x) = f(x_1, \ldots, x_n)\\ s.t \quad & g_i(x) \le 0, \quad i = 1, \ldots, m \end{aligned} \tag{6}$$

Assume that constraint $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex on $\mathbb{R}$ for $i = 1, \ldots, m$. Let $X = \{x \in \mathbb{R}^n : g_i(x) \le 0, i = 1, \ldots, m\}$ be a feasible and a point $x^* \in X$. Suppose that the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex at x^*, andf, $g_i, i = 1, \ldots, m$, are continuously differentiable at x^*. If there exist (Lagrange) multipliers $0 \le \mu_i \in \mathbb{R}, i = 1, \ldots, m$, such that

(i) $\nabla f(x^*) + \sum_{i=1}^{m} \nabla g_i(x^*) = 0$;

(ii) $\mu_i g_i(x^*) = 0 \quad \forall i = 1, \ldots, m$

Then x^* is an optimal solution of problem (6).

The objective function is linear so it is convex. Also all constraints are linear so are continuously differentiable. Thus theorem 1.2 could be applied.
Constraint (4) can be wrote in the form $\mathrm{p}_j^* = x_j - y_j + p_j, so:$

$$0 \le p_1^* \le \cdots \le p_n^* \longrightarrow \begin{cases} -p_1^* \le 0\\ p_1^* - p_2^* \le 0\\ \vdots\\ p_{n-1}^* - p_n^* \le 0 \end{cases} \tag{7}$$

is g_1 to g_n,

$$x_i \ge 0 \longrightarrow -x_i \le 0 \tag{8}$$

is g_{n+1} to g_{2n},

$$y_i \ge 0 \longrightarrow -y_i \le 0 \tag{9}$$

is g_{2n+1} to g_{3n} .

For each $i = 1, \ldots, n$ for constraints in (7) coefficient λ_i, has been placed, for constraints in (8) coefficient μ_i, has been placed, for constraints in (9) coefficient γ_i has been placed. Trough the indisvication of Karush-Kuhn-Tucker condition:

(i) in 1.2 :

$$\begin{cases} 2\alpha_1 - \mu_1 - \gamma_1 = 0 \\ 2\alpha_2 - \mu_2 - \gamma_2 = 0 \\ \vdots \\ 2\alpha_n - \mu_n - \gamma_n = 0 \end{cases}$$

(ii) in 1.2:

$$\begin{cases} \lambda_1(-x_1 + y_1 - p_1) = 0 \\ \lambda_2(x_1 - y_1 + p_1 - x_2 + y_2 - p_2) = 0 \\ \vdots \\ \lambda_n(x_{n-1} - y_{n-1} + p_{n-1} - x_n + y_n - p_n) = 0 \end{cases}$$

$$\begin{cases} \mu_1(-x_1) = 0 \\ \mu_2(-x_2) = 0 \\ \vdots \\ \mu_n(-x_n) = 0 \end{cases}$$

$$\begin{cases} \gamma_1(-y_1) = 0 \\ \gamma_2(-y_2) = 0 \\ \vdots \\ \gamma_n(-y_n) = 0 \end{cases}$$

That

$$\begin{cases} \lambda_1 \geq 0 \\ \lambda_2 \geq 0 \\ \vdots \\ \lambda_n \geq 0 \end{cases}, \quad \begin{cases} \mu_1 \geq 0 \\ \mu_2 \geq 0 \\ \vdots \\ \mu_n \geq 0 \end{cases}, \quad \begin{cases} \gamma_1 \geq 0 \\ \gamma_2 \geq 0 \\ \vdots \\ \gamma_n \geq 0 \end{cases}$$

Finally have:

$$\Longrightarrow \begin{cases} 2\alpha_i - \mu_i - \gamma_i = 0, & \forall i = 1, \cdots, n \\ 2\lambda_i(x_{i-1} - y_{i-1} + p_{i-1} - x_i + y_i - p_i) = 0, & \forall i = 1, \cdots, n, \quad x_0, y_0, p_0 = 0 \\ \mu_i(-x_i) = 0, & \forall i = 1, \cdots, n \\ \gamma_i(-y_i) = 0, & \forall i = 1, \cdots, n \\ \gamma_i, \lambda_i, \mu_i \geq 0 & \forall i = 1, \cdots, n \end{cases}$$

By solving the equations, the unknown of systems will be find$(x_i, y_i, \lambda_i, \mu_i, \gamma_i)$ by MATLAB and the problem will be solved.

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Solving mixed optimal control problems using a feed forward neural network model

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Abstract

This paper gives a survey of the various forms of Pontryagins maximum principle for optimal control problems with state variable inequality constraints. Furthermore, the application of these maximum principle conditions is demonstrated by solving one illustrative example.

Keywords: optimal control, maximum principles, state constraints, control constraints, mixed constraints
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

Optimal control problems with state variable inequality constraints (SVICs) arise frequently not only in mechanics and aerospace engineering, but also in the areas of management science and economics.
This paper gives a survey of the various forms of the maximum principle for deterministic continuous-time optimal control problems with SVICs and explains the connection between these approaches; see also Hartl[1], [2], Hartl and Sethi [3], and Arutyunov [1] for earlier such attempts.

2 Problem statement and transformation

Let us consider the following optimal control problem with state constraints:

$$\text{maximize} \qquad \int_0^T F(x(t), u(t), t)\, dt + S(x(T), T), \tag{1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \qquad x(0) = x_0, \tag{2}$$

$$g(x(t), u(t), t) \geq 0, \tag{3}$$

$$h(x(t), t) \geq 0, \tag{4}$$

$$a(x(T), T) \geq 0, \tag{5}$$

$$b(x(T), T) = 0. \tag{6}$$

*Speaker

3 Direct adjoining approach

In this method, the Hamiltonian H and Lagrangian L are defined as follows:

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 F(x,u,t) + \lambda f(x,u,t), \tag{7}$$
$$L(x,u,\lambda_0,\lambda,\mu,\nu,t) = H(x,u,\lambda_0,\lambda,t) + \mu g(x,u,t) + \nu h(x,t) \tag{8}$$

where

$$\lambda_0 \geq 0$$

is a constant, $\lambda \in E^n$ is the adjoint variable, and $\mu \in E^s$ and $\nu \in E^q$ are multipliers.

Theorem 3.1. *Let $\{x^*(.), u^*(.)\}$ be an optimal pair problem (1)-(6) over a fixed interval $[0,T]$, such that $u^*(.)$ is right-continuous with left-hand limits and the constraint qualification (2) holds for every triple $\{t, x^*(t), u\}$, $t \in [0,T]$ with $u \in \Omega(x^*(t),t)$. Assume that $x^*(.)$ has only finitely many junction times. Then there exist a constant $\lambda_0 \geq 0$, a piecewise absolutely continuous costate trajectory $\lambda(.)$ mapping $[0,T]$ into E^n, piecewise continuous multiplier functions $\mu(.)$ and $\nu(.)$ mapping $[0,T]$ into E^s and E^q, respectively, a vector $\eta(\tau_i) \in E^q$ .for each point τ_i of discontinuity of $\lambda(.)$, and $\alpha \in E^\ell, \beta \in E^{\ell'}, \gamma \in E^q$ such that $(\lambda_0, \lambda(t), \mu(t), \nu(t), \alpha, \beta, \gamma, \eta(\tau_1), \eta(\tau_2), \ldots) \neq 0$ for every t and the following conditions hold almost everywhere:*

$$u^*(t) = \underset{u\in\Omega(x^*(t),t)}{argmax}\ H(x^*(t), u, \lambda_0, \lambda(t), t), \tag{9}$$
$$L_u^*[t] = H_u^*[t] + \mu g_u^*[t] = 0, \tag{10}$$
$$\dot{\lambda}(t) = -L_x^*[t], \tag{11}$$
$$\mu(t) \geq 0, \quad \mu(t) g^*[t] = 0, \tag{12}$$
$$\nu(t) \geq 0, \quad \nu(t) h^*[t] = 0, \tag{13}$$
$$dH^*[t]/dt = dL^*[t]/dt = L_t^*[t] \overset{\triangle}{=} \partial L^*[t]/\partial t. \tag{14}$$

4 Neural network construction

It can calculate the output value of feed–forward nwural network by the following formulation:

$$\begin{cases} \text{output} = \sum_{i=1}^{I} v_i \sigma(z_i), \quad \sigma(x) = \dfrac{1}{1+e^{-x}}. \\ z_i = w_i x + b_i, \end{cases} \tag{15}$$

We consider three neural networks for each function: state (its neural network is n_x), costate (its neural network is n_p) and the control (its neural network is n_u) function.
The trial solutions (for state, costate and control function) can be defined in the following structures:

$$\begin{cases} x_T = x_0 + t(t-t_0) n_x, & n_x = \sum_{i=1}^{I} v_x^i \sigma(z_x^i), & z_x^i = w_x^i \tau + b_x^i, \\ p_T = n_p, & n_p = \sum_{i=1}^{I} v_p^i \sigma(z_p^i), & z_p^i = w_p^i \tau + b_p^i, \\ u_T = n_u, & n_u = \sum_{i=1}^{I} v_u^i \sigma(z_u^i), & z_u^i = w_u^i \tau + b_u^i, \end{cases} \tag{16}$$

Note that we may have $p(.) = 0$ for free end points. For example, if $x(t_f)$ is free, we must have $p(t_f) = 0$, and thus, we can define p_T in (16) as: $p_T = (t - t_f)n_p$. For other initial (or boundary) conditions, we can construct appropriate trial functions.
The trial solutions (16) are the universal approximation and must satisfy conditions (9)-(14). Thus we have

$$u_T^*(t) = \underset{u\in\Omega(x_T^*(t),t)}{argmax}\ H(x_T^*(t), u_T, \lambda_0, \lambda_T(t), t), \tag{17}$$

$$L_{u_T}^*[t] = H_{u_T}^*[t] + \mu g_{u_T}^*[t] = 0, \tag{18}$$

$$\dot{\lambda_T}(t) = -L_{x_T}^*[t], \tag{19}$$

$$\mu(t) \geq 0, \quad \mu(t)g^*[x_T(t), u_T(t), t] = 0, \tag{20}$$

$$\nu(t) \geq 0, \quad \nu(t)h^*[x_T(t), t] = 0, \tag{21}$$

$$dH^*[t]/dt = dL^*[t]/dt = L_t^*[t] \stackrel{\triangle}{=} \partial L^*[t]/\partial t. \tag{22}$$

where

$$L_T = H(x_T, u_T, \lambda_0, \lambda_T, t) + \mu g(x_T, u_T, t) + \nu h(x_T, t).$$

In order to reformulate (17)-(22) as an unconstrained minimization problem, we first collocate the optimality system (17)-(22) on the m points $t_k, k = 1, ..., m$ of the interval $[t_0, t_f]$ and then define an optimization problem as

$$\text{minimize}_y E(y) = \frac{1}{2}\sum_{k=1}^{m}\{E_1(t_k, y) + E_2(t_k, y) + E_3(t_k, y) + E_4(t_k, y) + E_5(t_k, y)\}, \tag{23}$$

where $y = (w_x, w_p, w_u, b_x, b_p, b_u, v_x, v_p, v_u)^T \in \mathbb{R}^{3I(2n+m)}$ and

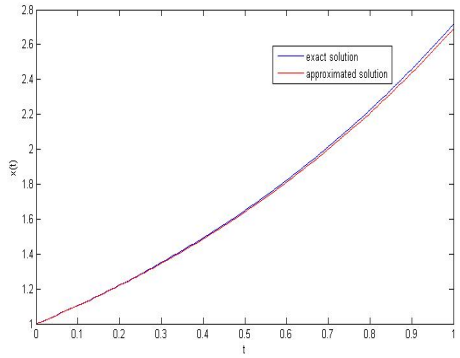
$$\begin{cases} E_1(t_k, y) = \left[\dfrac{\partial L_T}{\partial x_T} + \dot{\lambda_T}\right]^2, & k = 1, 2, \ldots, m, \\ E_2(t_k, y) = \left[\dfrac{\partial L_T}{\partial \lambda_T} - \dot{x_T}\right]^2, & k = 1, 2, \ldots, m, \\ E_3(t_k, y) = \left[\dfrac{\partial L_T}{\partial u_T}\right]^2, & k = 1, 2, \ldots, m \\ E_4(t_k, y) = \left[\mu(t)g_T^*(t)\right]^2, & k = 1, 2, \ldots, m \\ E_5(t_k, y) = \left[\nu(t)h_T^*(t)\right]^2, & k = 1, 2, \ldots, m. \end{cases} \tag{24}$$

5 Numerical example

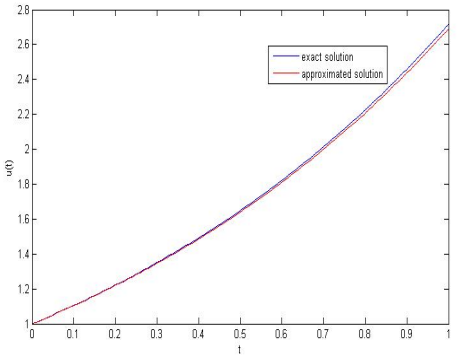
In this section, we try to implement the proposed algorithm to solve some problem.

Example 5.1. Consider the following optimization problem:

$$\begin{aligned} &\text{maximize} && \int_0^1 u(t)\,dt, \\ &\text{subject to} && \\ &\dot{x}(t) = u(t), && x(0) = 1,\ x(1) = isfree, \\ &u(t) \geq 0, && x(t) - u(t) \geq 0 \end{aligned}$$



(a) exacte and approximated state function



(b) exacte and approximated control function

Figure 1: state and control function for Example (5.1)

The exact state and control functions are as follows:

$$\begin{cases} x(t) = exp(t), \\ u(t) = exp(t). \end{cases}$$

because $x(1)$ is free, we have $p(1) = 0$. Considering this condition and the initial condition $x(0) = 1$, we can choose the trial solutions as:

$$\begin{cases} x_T = 1 + tn_x, \\ p_T = (t-1)n_p, \\ u_t = n_u. \end{cases}$$

We can see the approximate solutions for $x(t)$ and $u(t)$ in figure (1), respectively.

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Stabilizaton of fractional discrete-time linear systems
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Stabilizaton of fractional discrete -time linear systems

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Abstract

In this article a new notion of the practical stability of fractional discrete-time linear systems is introduced. Necessary and sufficient conditions for this kind of systems are established. It is shown that the fractional systems are practically unstable if corresponding standard fractional systems are asymptotically unstable.

Keywords: practical stability, fractional, discrete-time, linear system
Mathematics Subject Classification [2010]: 93B55,93B52,93D15

1 Introduction

Development of models based on fractional-order differential systems has recently gained popularity in the investigation of dynamical systems [4-8]. Fractional derivatives provide an excellent instrument to describe memory and hereditary properties of various materials and processes. The advantages or the real objects of the fractional-order systems are that we have more degrees of freedom in the model and that a memory is included in the model (fractional-order systems have an unlimited memory).Recently, studying fractional-order systems has become an active research area. In this paper a new concept of the practical stability of fractional discrete-time linear systems will be introduced and necessary and sufficient conditions for the practical stability will be established. In this paper, we recall and present some stability results for linear fractional-order systems.

2 Preliminaries and definitions

2.1 Fractional-order derivatives

Definition 2.1. The disctere-time fractional derivative defined by Grunwald−Letnikov is

$$ {}_{G}D^{\alpha}x(t_k) = \lim_{\mathrm{h}\to 0} \frac{1}{h^{\alpha}} \sum_{i=0}^{k} (-1)^i \begin{pmatrix} \alpha \\ i \end{pmatrix} x(t_{k-i}) \tag{1} $$

where

$$ \begin{pmatrix} \alpha \\ i \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-i)\times\Gamma(i+1)} \tag{2} $$

*Speaker

The generalization of the integer-order difference to a non-integer order (or fractional-order) difference with zero initial time is defined as follows [6].

$$\Delta^{\alpha} x_k = \Delta^{\alpha} x(t_k) = \sum_{i=0}^{k} (-1)^i \begin{pmatrix} \alpha \\ i \end{pmatrix} x(t_{k-i}) \tag{3}$$

2.2 Fractional-order discrete-time linear systems

In this section we consider the commensurate fractional discrete-time linear system

$$\Delta^{\alpha} x_{k+1} = Ax_k + Bu_k \tag{4}$$

$$x_{k+1} = (A + \alpha I_n)x_k + \sum_{i=1}^{k} c_i x_{k-i} + Bu_k, \;\; c_i = (-1)^i \begin{pmatrix} \alpha \\ i+1 \end{pmatrix} \tag{5}$$

Stability of this kind of systems is tested by practical stability [3].

3 Stability of fractional discrete-time linear systems

By (5) the sequence c_i converges to zero. Getting $c_i = 0$ for $i > L$ (greater L is better) the system (5) will be a time delay system with L delays [1]

$$x_{k+1} = (A + \alpha I_n)x_k + \sum_{i=1}^{L} c_i x_{k-i} + Bu_k \tag{6}$$

$$X_{k+1} = \overline{A} X_k + \overline{B} u_k \tag{7}$$

where

$$X_k = \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \cdot \\ \cdot \\ \cdot \\ x_{k-L} \end{bmatrix}, \overline{A} = \begin{bmatrix} A + \alpha I_n & c_1 I & c_2 I & \cdots & c_{L-1} I & c_L I \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} \tag{8}$$

$$\overline{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad , \overline{C} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \end{bmatrix}, \overline{D}_k = D \tag{9}$$

3.1 New special form of state feedback law

With a state feedback law of the form

$$u(k) = \sum_{i=0}^{L} F x_{k-i} \tag{10}$$

where $F_k(i)$ is a feedback gain, applied to the system (6). The closed-loop system is

$$x_{k+1} = (A + \alpha I_n + BF)x_k + \sum_{i=1}^{L}(c_i I_n + BF)x_{k-i} \tag{11}$$

defining

$$\overline{\Gamma} = \begin{bmatrix} A+\alpha I_n + BF & c_1 I + BF & \cdots & c_L I + BF \\ I & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \tag{12}$$

the system (11) changes to a standard closed-loop system $X_{k+1} = \overline{\Gamma} X_k$.

3.2 Decoupling and pole assignment of the closed-loop matrix

Defining

$$\overline{F} = [F \quad F \quad \cdots F], \overline{\Gamma} = \overline{A} + \overline{B}\,\overline{F} \tag{13}$$

The feedback matrix can be obtained by the algorithm given by Karbassi and Bell [4]

4 Numerical examples

In this section, we give two examples to show the success of the proposed method.

Example Check the practical stability of the fractional system

$$\Delta^{0.8} x_{k+1} = Ax_k + Bu_k \tag{14}$$

$$A = \begin{bmatrix} -0.625 & 0.8 & 0.9 \\ 0.7 & 0 & 0.2 \\ 1 & 1.2 & -0.8 \end{bmatrix}, A_\alpha = \begin{bmatrix} .175 & 0.8 & .9 \\ 0.7 & 0.8 & .2 \\ 1 & 1.2 & 0 \end{bmatrix}, B = \begin{bmatrix} 3.2 & 0.8 \\ 4.1 & 1 \\ 0 & 0 \end{bmatrix} \tag{15}$$

Creating $\overline{A}, \overline{B}$ as we said we otain the feedback matrix so that the eigenalues of matrix $\overline{A} + \overline{BF}$ changes to

$$\{\lambda_1 = .1, \lambda_2 = .1, \lambda_3 = .1, \lambda_4 = .2, \lambda_5 = .3, \lambda_6 = -.4, \lambda_7 = .5, \lambda_8 = .3, \lambda_9 = -.5\} \tag{16}$$

that

$$\overline{F} = [FFF] = \begin{bmatrix} -2.72 & 2.01 & -14.05 & 13.48 & 16.68 & 2.78 & -0.25 & -1.31 & 2.40 \\ 10.95 & -8.55 & 56.33 & -54.67 & -67.64 & -11.35 & 1.01 & 5.27 & -9.70 \end{bmatrix} \tag{17}$$

By Fig.1 the input variables $x_i(t)$ converged to zero.

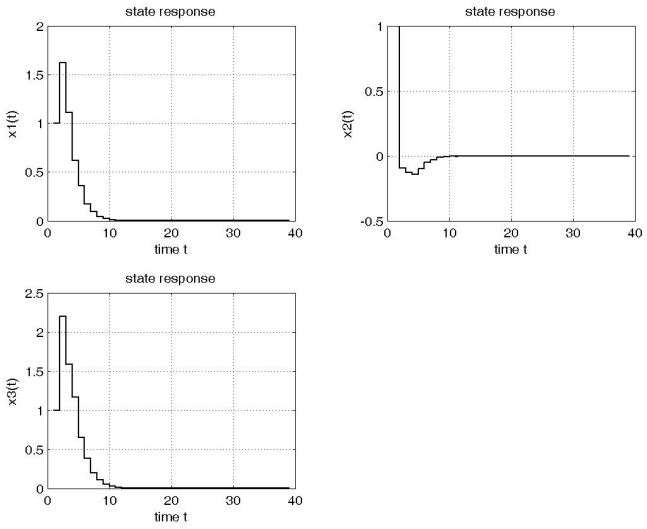

Figure 1: $x_i(t)$ of Example

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Poster State feedback matrix with minimum norm in linear systems with disturbance pp.: 1–4

State feedback matrix with minimum norm in linear systems with disturbance

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Abstract

In this paper, we presented a new method for control discrete-time linear systems with disturbance, meaning that, for a discrete-time linear controllable with disturbance, a arbitrary system controllable without disturbance is considered, then with use augmented matrix, the two systems Merged in a new system without disturbance, by transformations similarity compute optimal controller, so that disturbance has no effect on the system.

Keywords: Discrete-time, State feedback matrix, Disturbance rejection, Parameterization, Norm minimization.
Mathematics Subject Classification [2010]: 93B52, 15A60,15A18

1 Introduction

The disturbance-accommodation problem that has been studied extensively in the literature [1]-[3] is concerned with designing a feedback control law which ensures that the effect of some or all disturbances acting on a linear system are completely rejected or reduced to an acceptable level in steady state. Mathematical problems concerning disturbance rejection controllers have received somewhat more attention than those concerning disturbance minimization.
Different ways to describe disturbances and to analyze their effect on a system are discussed in [2]-[3]. An overview of different ways to eliminate disturbances include use of feedback, feedforward, and prediction methods, which may be through classical control, advanced control and adaptive control to be performed, see [1].

2 Main results

Consider a controllable linear discrete-time-invariant system defined by the state equation

$$x(k+1) = Ax(k) + Bu_1(k) + d(k) \tag{1}$$

Where $x \in \mathbb{R}^n$ is the state vector, $u_1 \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^n$ is the disturbance input and the matrices A and B are real constant matrices of dimensions $n \times n$ and $n \times m$,

*Speaker

respectively, with $rank(B) = m$ and control rule $u_1(k) = F_1x(k)$. First, we define the following arbitrary system controllable:

$$z(k+1) = A_2z(k) + B_2u_2(k) \tag{2}$$

Where $u_2 \in \mathbb{R}^m$ is the control input, matrices A_2 and B_2 are real constant matrices of dimensions $n \times n$ and $n \times m$, respectively, with $rank(B_2) = m$ and control rule $u_2(k) = F_2z(k)$.
Provided that $z(0) \neq 0$,. Now we defind

$$d(k) = Cz(k) \tag{3}$$

Where C is diagonal matrix with $C_{ii} = \dfrac{d_i}{z_i},\ i = 1, 2, .., n$. Therefore, with replacement (3) in (1) and with equation (2) we have:

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & C \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{4}$$

Now, Put $y(k) = \begin{bmatrix} x(k) \\ z(k) \end{bmatrix}$, we will

$$y(k+1) = Ay(k) + Bu(k) \tag{5}$$

Where, matrices A and B are real constant matrices of dimensions $2n \times 2n$ and $2n \times m$ However, with definition F for control rule $u(k) = Fy(k)$ so that the system is stable, the system controls that can be used to control the system without disturbance.

$$y(k+1) = (A + BF)y(k) \tag{6}$$

Using a similarity transformations and find a good F, can $y(k)$ desire to zero.
So, our goal find the matrix $F = [F_1, F_2]$ so that the system without disturbance (6) be stable.
On the other hand, The state feedback matrix F in (6) Can be achieved so that, the norm in addition to maintaining the stability of the system (6) be minimized [6].

Example 2.1. To demonstrate the advantages of the proposed approach, we consider a discrete-time linear system given by

$$x(k+1) = A_1x(k) + B_1u_1(k) + d(k) \tag{7}$$

Where

$$A_1 = \begin{bmatrix} 3 & 3 & 3 & 4 \\ 1 & 5 & 8 & 8 \\ 1 & 1 & 8 & 8 \\ 6 & 6 & 5 & 6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 8 & 3 \\ 6 & 5 \\ 3 & 7 \\ 2 & 3 \end{bmatrix} \tag{8}$$

We want feedback matrix is defined in such a way that in addition to the disturbance rejection system stability is to be maintained.
We suppose in the first step $x(0) = (1, 2, 3, 4)^t$ and $d(0) = (-1, 0, 1, 1)^t$, then consider the Controllable system following:

$$z(k+1) = A_2 z(k) + B_2 u_2(k), \quad u_2 = F_2 z(k) \tag{9}$$

$$A_2 = \begin{bmatrix} 5 & 6 & 4 & 8 \\ 2 & 0 & 5 & 7 \\ 8 & 3 & 6 & 7 \\ 3 & 4 & 1 & 8 \end{bmatrix}, \; B_2 = \begin{bmatrix} 6 & 5 \\ 2 & 0 \\ 2 & 0 \\ 6 & 2 \end{bmatrix}, \; Z(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{10}$$

with definitions $u(k) = Fy(k)$, replacement (7), (9) in equation (4) and with solving equation (4) With the help of similarity transformations expressed in [4]-[5], we obtain,

$$F = \begin{bmatrix} 0.5081 & 0.6659 & 1.0628 & 1.1474 & -1.6023 & -1.4728 & -1.4416 & -3.0064 \\ -0.9402 & -1.3243 & -1.9134 & -2.0911 & 0.5320 & 1.1399 & 0.5809 & 2.5932 \end{bmatrix} \tag{11}$$

With norm 6.19. Also, state feedback matrix F Can be achieved so that, the norm in addition to maintaining the stability of the system (4) be minimized [6]. now let us follow the algorithm in [6] step by step.
According to Step 1 of the algorithm, two independent sets of parameters can be selected. Either $[g_{21}, g_{23}]$ is the effective parameters which produce parametric feedback matrix. According to Step 1 of the algorithm, column 2 of ${B_o}^{-1}$ and rows 1 ans 3 of T^{-1} are stored in V and W, respectively. That is,
Step 1.

$$V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad W = \begin{bmatrix} -0.0428 & -0.0403 & -0.0507 & -0.0610 & 0.1229 & 0.0860 & 0.0946 & 0.1267 \\ -0.0069 & 0.0006 & -0.0022 & -0.0034 & 0.0093 & -0.0035 & 0.0127 & -0.0016 \end{bmatrix}$$

Step 2.

$$P = V^T V = 1, \quad Q = WW^T = \begin{bmatrix} 0.0572 & 0.0024 \\ 0.0024 & 0.0003 \end{bmatrix}$$

and

$$C = V^T K W^T = \begin{bmatrix} 0.8651 & 0.0212 \end{bmatrix}$$

Step 3.

$$\alpha = -P^{-1} C Q^{-1} = \begin{bmatrix} -18.0584 & 69.4165 \end{bmatrix}$$

and therefore,

$$G_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18.0584 & 0 & 69.4165 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad F_\alpha = {B_0}^{-1} G_\alpha T^{-1}$$

Step4.

$$K = F + F_\alpha = \begin{bmatrix} 0.5081 & 0.6659 & 1.0628 & 1.1474 & -1.6023 & -1.4728 & -1.4416 & -3.0064 \\ -0.6461 & -0.5523 & -1.1534 & -1.2266 & -1.0409 & -0.6569 & -0.2432 & 0.1976 \end{bmatrix}$$

with the Frobenius norm 4.9164. Obviously, norm of the state feedback matrix with method presented in [6] is reduced.

In the event that disturbance in every step change(for example $d(k) = (-1, sin(\frac{\pi}{6}k), cos(\frac{\pi}{6}k), 1)^t$), can be used as a proposed method to control this systems.

Conclusions

In this peaper, a method for control discrete-time linear systems with disturbance with useing augmented matrix and similarity transformations presented. This method has simpler calculation of the existing methods and In fact, a system with disturbance turns into the system without disturbance. Also with use theory graph and linear parametric matrix presented method in [6] reduction in the norm of feedback matrix.

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The overall efficiency of decision making units with undesirable outputs *

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Abstract

In real applications, desirable and undesirable outputs are usually produced in a production process. Furthermore, the majority of data envelopment analysis (DEA) models evaluate the efficiency of decision making units (DMUs) from optimistic point of view. In the current paper, DEA models are introduced for evaluating the performance of DMUs with undesirable outputs from two viewpoints, optimistic and pessimistic viewpoints. Afterwards, the overall efficiency of each DMU is calculated by using the geometric average of two efficiencies.

Keywords: Data envelopment analysis, Efficiency, Undesirable outputs, Geometric average
Mathematics Subject Classification [2010]: 90B50, 90C05

1 Introduction

The conventional data envelopment analysis (DEA) models usually evaluate the efficiency of decision making units (DMUs) from optimistic point of view in which inputs are minimized and outputs are maximized. Nevertheless, there are studies with calculating the performance from two aspects, optimistic and pessimistic [1]-[8]. Moreover, in many situations desirable and undesirable outputs are produced simultaneously. In DEA literature, numerous papers exist with considering undesirable outputs. Some studies consider undesirable outputs with strong disposability while others take weak disposability. For instance, readers can refer to [2]-[4]-[5]-[6]-[7]-[9]. Wang et al. [8] calculated the overall performance via calculating the geometric average of two efficiencies, optimistic and pessimistic efficiencies. In the current paper, optimistic and pessimistic efficiencies of DMUs are evaluated in the presence of undesirable outputs with different disposability assumptions. Then, the overall efficiency of each DMU is evaluated by using the geometric average of efficiencies. Actually, Wang's models [8] are extended for obtaining the overall efficiency when undesirable outputs with different disposability, strong or weak disposability, exist.

*Will be presented in English
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2 The proposed approach

Assume there are n DMUs, $DMU_j\,(j=1,...,n)$, with using m inputs, producing s desirable outputs, and emitting k undesirable outputs. Inputs, desirable outputs and undesirable outputs are indicated as $x_{ij}\ (i=1,...,m)$, $y_{rj}\ (r=1,...,s)$ and $z_{kj}\ (k=1,...,K)$, respectively. In this study, strong and weak disposability assumptions are deemed for undesirable outputs. "Strong disposability of outputs implies that given an input vector x, if an output vector y can be produced, then y^* can also be produced as long as $y^* \leq y$. Weak disposability of outputs means that if y can be produced, then $\theta y\,(0 \leq \theta \leq 1)$ can also be produced proportionally." according to [3]-[9]. At first, suppose undesirable outputs are strongly disposable. Therefore, the following models are introduced for evaluating the performance of DMUs from two viewpoints. Model (1) calculates the efficiency from optimistic point of view as follows:

$$\begin{array}{ll} Max & e_o^{best} = \sum_{r=1}^{s} u_r y_{ro} \\ s.t. & \sum_{i=1}^{m} v_i x_{io} + \sum_{k=1}^{k} w_k z_{ko} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{k} w_k z_{kj} \leq 0, \\ & v_i, u_r, w_k \geq \varepsilon. \end{array} \tag{1}$$

From pessimistic viewpoint, the performance is calculated as follows:

$$\begin{array}{ll} Min & e_o^{worst} = \sum_{r=1}^{s} u_r y_{ro} \\ s.t. & \sum_{i=1}^{m} v_i x_{io} + \sum_{k=1}^{k} w_k z_{ko} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{k} w_k z_{kj} \geq 0, \\ & v_i, u_r, w_k \geq \varepsilon. \end{array} \tag{2}$$

Then, the overall efficiency is calculated by using the geometric average of efficiencies, i.e.

$$e_j^{Overall} = \sqrt{e_j^{best} * e_j^{worst}} \tag{3}$$

If we assume undesirable outputs are weakly disposable, models (1) and (2) are substituted with the following problems:

$$\begin{array}{ll} Max & e_o^{best} = \sum_{r=1}^{s} u_r y_{ro} \\ s.t. & \sum_{i=1}^{m} v_i x_{io} + \sum_{k=1}^{k} w_k z_{ko} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{k} w_k z_{kj} \leq 0, \\ & v_i, u_r \geq \varepsilon,\ w_k free. \end{array} \tag{4}$$

$$\begin{array}{ll} Min & e_o^{worst} = \sum_{r=1}^{s} u_r y_{ro} \\ s.t. & \sum_{i=1}^{m} v_i x_{io} + \sum_{k=1}^{k} w_k z_{ko} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{k=1}^{k} w_k z_{kj} \geq 0, \\ & v_i, u_r \geq \varepsilon, w_k\, free. \end{array} \tag{5}$$

Afterwards, the overall efficiency can be obtained by using (3). It is clear, the aforementioned models can be extended when undesirable outputs with strong and weak disposability assumptions present simultaneously.

3 Example

Assume the aim is to evaluate the performance of 10 branches of a bank with two inputs, labor (I1) and deposits (I2), one desirable output, performing loans (O1), and one undesirable output, non-performing loans (O2). Suppose the undesirable output is strongly disposable. Data can be seen in Table 1. Models (1) and (2) are calculated. Then, overall efficiencies are obtained by using (3). Results can be found in Table 2.

Table 1: Bank branches data

#Branch	I1	I2	O1	O2
1	32	515578	1277833	446698
2	19	187679	102808	22585
3	14	150026	106734	12830
4	5	88358	14628	161
5	18	124349	75509	21035
6	18	127370	149860	39525
7	16	95288	55757	9632
8	17	89304	84631	13955
9	9	160138	102353	7153
10	13	148755	38375	7806

Table 2: Performance results

#Branch	e_j^{best}	e_j^{worst}	$e_j^{Overall}$
1	1	1	1
2	0.661478	1.214058	0.896143
3	0.952636	2.023167	1.388287
4	1	1	1
5	0.634907	1	0.796811
6	0.926137	1.17842	1.044691
7	0.735916	1.179178	0.931545
8	1	1.449636	1.204008
9	1	2.622405	1.619384
10	0.390432	1	0.624846

The column 4 of Table 2 shows branch 9 is generally the most efficient unit while branch 10 has the least overall efficiency score.

4 Conclusions

In this study, efficiencies of DMUs with desirable and undesirable outputs have been evaluated from optimistic and pessimistic viewpoints. Then, the geometric average has been utilized for calculating the overall efficiency. Finally, the efficiencies of 10 branches of a bank have been measured for indicating the application of the approach.

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Autoregressive Hilbertian Processes

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Abstract

In this article, We consider autoregressive processes in Hilbert spaces. We present here existence, the strong law of large numbers and estimation of autocovariance operators.

Keywords: Hilbertian white noise, Hilbertian autoregressive process, autocovariance operators
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

functional data often arise from measurements obtained by separating an almost continuous time record into natural consecutive intervals, for examples days. many important examples of data that can be naturally treated as functional come from financial records. The price of an asset exists only when the asset is traded. A great deal of financial research has been done using the closing daily price, i.e. the price in the last transaction of a trading day. However many assets are traded so frequently that one can practically think of a price curve that is defined at any moment of time.[2]
The Hilbertian autoregressive model of order 1 (ARH(1)) generalizes the classical AR(1) model to random elements with values in Hilbert spaces. This model was introduced by Bosq (2000), then studied by several authors, as Mourid (1993), Besse and Cardot (1996), Pumo (1999), Mas (2002, 2007), Horvath et al. (2010). Bosq in his fundamental work (2000) provides basic results on Hilbertian strongly second order autoregressive and moving average processes. The existence, covariance structure, parameter estimation, strong law of large numbers and central limit theorem, are the topics that are covered by Bosq (2000).

For writing definition, theorem, proof, throughout this paper, we consider H as a real separable Hilbert space equipped with scalar product $\langle .,. \rangle$, norm $\| \, . \, \|$ and Borel σ-field $\mathcal{B}$. The H-valued random variables considered below are defined over the same probability space $(\Omega, \mathcal{F}, P)$ supposed to be rich enough and complete.

Definition 1.1. A sequence $X = \{X_n, n \in \mathcal{Z}\}$ of H-random variables is called an autoregressive Hilbertian process of order 1 (ARH(1)) associated with (μ, ϵ, ρ) if it is stationary and such that

$$X_n - \mu = \rho(X_{n-1} - \mu) + \epsilon_n, \quad n \in \mathcal{Z} \tag{1}$$

where $\epsilon = \{\epsilon_n, n \in \mathcal{Z}\}$ is an H-white noise, $\mu \in H$, and $\rho \in \mathcal{L}$.

In order to study existence of x, consider the following conditions:
(C_0) There exists an integer $j_0 \geq 1$ such that $\| \rho^{j_0} \|_{\mathcal{L}} < 1$.
(C_1) There exist $a > 0$ and $0 < b < 1$ such that $\| \rho^j \|_{\mathcal{L}} \leq ab^j, \;\; j \geq 0$.
Bosq shows that C_0 and C_1 are equivalent.[1]
We may now give a statement concerning existence and uniqueness of X.

Theorem 1.2. *If C_0 holds, then (1) has a unique stationary solution given by*

$$X_n = \mu + \sum_{j=0}^{\infty} \rho^j(\epsilon_{n-j}), \;\; n \in \mathcal{Z} \tag{2}$$

where the series converges in $\mathcal{L}^2(\omega, \mathcal{A}, P)$ and almost surely. Moreover, ϵ is the innovation process of $(X_n - \mu)$.[1]

Now, we state laws of large number.

Theorem 1.3. *Let X be a standard ARH(1). Then, as $n \to \infty$,*

$$E \| \frac{S_n}{n} \|^2 = O(\frac{1}{n}) \tag{3}$$

and, for all $\beta > 0.5$,

$$\frac{n^{\frac{1}{4}}}{(Logn)^{\beta}} \frac{S_n}{n} \to 0 \quad a.s \tag{4}$$

[1]

For estimation of autocovariance operator, Mas(2007) use classical moment method and provides the following normal equation

$$\Delta = \rho\Gamma \tag{5}$$

where

$$\Gamma = E(X_1 \otimes X_1), \;\; \Delta = E(X_1 \otimes X_2).$$

If (5) is the starting point in estimation procedure, replacing the unknown operators by the empirical counterparts gives:

$$\Delta_n = \rho_n \Gamma_n$$

where

$$\Gamma_n = \frac{1}{n} \sum_{k=1}^{n} X_k \otimes X_k$$

$$\Delta_n = \frac{1}{n-1} \sum_{k=1}^{n-1} X_k \otimes X_{k+1}$$

$$\Gamma_n^{\dagger} = \sum_{l \leq k_n} \frac{1}{\hat{\lambda}_l} \hat{e}_l \otimes \hat{e}_l$$

where $\hat{\lambda}_l$ and $\hat{e}_l$ are the empirical counterparts of eigenvalues and eigenvectors of Γ_n.

Definition 1.4. The estimate of ρ is ρ_n given by $\rho_n = \Delta_n \Gamma_n^{\dagger}$.

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Clustering of longitudinal data based on a random change-point model using Dirichlet processes

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Abstract

A dynamic regression model which switches between two models with different slopes at random time points is considered. Model-based clustering of longitudinal data is conducted by assuming the Dirichlet process (DP) as a prior for the distribution of random change points. The discreteness nature of the DP is utilized to cluster subjects according to the time of changing slopes. Markov chain Monte Carlo simulation methods are adopted to achieve the parameter estimates. Performance of the proposed model is illustrated by conducting a simulation study.

Keywords: Bayesian approach, Change-point model, Dirichlet processes, Gibbs sampler.
Mathematics Subject Classification [2010]: 62M99, 62J05, 62H30

1 Introduction

In recent years increasing interest has been shown in the problem of clustering of longitudinal data. These data sets are available by repeatedly measuring subjects through time. Usually in Econometrics and Biostatistics, it frequently happens that effect of occurring an event on changing slope of response variable against time will appear in different times lag for different subjects. Thus, it is usually important that different subjects can be categorized based on these change-point times. For example, when an economical event leads to profitability declines for companies in the stock market, different firms show different tolerances such that face values of some companies start decreasing sooner than others. Thus, clustering of firms based on the time of standing against the decline is useful. Addressing this issue, we propose the use of the Dirichlet process (DP) in the structure of a dynamic change-point model.

The rest of this paper is organized as follows. In Sections 2, we briefly introduce the DP prior. Section 3 specifies the proposed dynamic change-point model. In Section 4, we implement the Gibbs sampling scheme. The last section presents a simulation study.

*Speaker

2 The Dirichlet process prior

The random measure G is distributed according to a DP [1] with a scaling parameter M and a base distribution G_0 which is a probability measure defined on a measurable space $(\tau;\Omega)$, denoted by $G\sim DP\left(M,G_0\right)$, if for any partition of Ω such as $A_1,\cdots,A_k$:

$$\left(G(A_1),\cdots,G(A_k)\right)\sim Dir(MG_0(A_1),\cdots,MG(A_k)), \tag{1}$$

where Dir indicates the Dirichlet distribution. The DP generates a discrete random probability measure G as $G\left(\cdot\right)=\sum_{j=1}^{\infty}\pi_j\Delta_{\xi_j}\left(\cdot\right)$, where $\Delta\left(\cdot\right)$ denotes a degenerated distribution of unit mass centered at ξ_j with $\xi_j\overset{\text{iid}}{\sim}G_0$ and $\pi_j=\gamma_j\prod_{i=1}^{j-1}\left(1-\gamma_i\right)$ with $\gamma_j\overset{\text{iid}}{\sim}Beta\left(1,M\right)$ (we say $\pi_j\sim SB\left(M\right)$). This representation is called stick-breaking. Variation of G around G_0 is controlled by $M(>0)$ such that, if M gets large then G becomes close to G_0. Being applicable, the above infinite summation is truncated to a finite integer C.
In a Bayesian framework, the DP is assumed as a prior for the distribution of random effects. Let $Y_i|\xi_i\overset{\text{iid}}{\sim}f(.|\xi_i)$, $\xi_i|G\overset{\text{iid}}{\sim}G$ and $G\sim DP(M,G_0)$ for $i=1,\cdots,n$. The realizations of the DP are discrete, thus several ξ_i's prone to take similar values, such that the number of distinct values of ξ_i, each constructing a cluster, being less than or equal to n.

3 The dynamic change-point model

Let Y_{it} denotes the t-th measurement taken on the i-th subject, $t=1,\cdots,T$, $i=1,\cdots,n$. The proposed dynamic change-point model for longitudinal data is given by

$$Y_{it}=\boldsymbol{x}_{it}'\boldsymbol{\beta}+\delta_1 t+\delta_{2_i}(t-c_i)\varpi(t-c_i)+\gamma y_{i,t-1}+\alpha_i+\varepsilon_{it}, \tag{2}$$

where δ_1, δ_2 and $\boldsymbol{\beta}$ are regression coefficients. The $\alpha_i\overset{\text{iid}}{\sim}N(0,\sigma_\alpha^2)$, $\delta_{2_i}\overset{\text{iid}}{\sim}N(\mu_{\delta_2},\sigma_{\delta_2}^2)$ and $\epsilon_{it}\overset{\text{iid}}{\sim}N(0,\sigma_\varepsilon^2)$. The covariate $y_{i,t-1}$ is the lagged-response variable for the i-th subject which represents state dependence. The y_{i0}'s are assumed to be fixed observed values. The $\varpi(t-c_i)=1$ for $t\leq c_i$ and is set to zero otherwise. The c_i's are random-change points. To have a flexible modeling structure along with being able to cluster subjects, we consider the DP as a prior for unknown distribution of c_i's. More specifically, we assume $c_i|G\overset{\text{iid}}{\sim}G$ where $G\sim DP(M,G_0)$ and G_0 is a discrete uniform distribution on $\{1,\cdots,T\}$. For simplicity, we assume all random effects be independent.

4 Bayesian Estimation

Now, consider the following vector representation of the model

$$\begin{gathered}\mathbf{Y}_i|c_{\lambda_i},\alpha_i,\boldsymbol{\theta},\sigma_\varepsilon^2,\delta_{2_i}\overset{\text{ind}}{\sim}N_T\left(\widetilde{\mathbf{X}}_i\boldsymbol{\theta}+\alpha_i\mathbf{1}_T+\delta_{2_i}\mathbf{a}_{i,c_{\lambda_i}},\sigma_\varepsilon^2\mathbf{I}_T\right),\\ \lambda_i\overset{\text{ind}}{\sim}G\left(\lambda_i\right)=\textstyle\sum_{j=1}^{C}\pi_j\Delta_j\left(\lambda_i\right),\ where\ \ \pi_j\overset{\text{ind}}{\sim}SB(M),\\ c_j\overset{\text{iid}}{\sim}DU\{1,\cdots,T\},\ \alpha_i\overset{\text{iid}}{\sim}N(0,\sigma_\alpha^2)\ and\ \ \delta_{2_i}\overset{\text{iid}}{\sim}N(\mu_{\delta_2},\sigma_{\delta_2}^2),\end{gathered} \tag{3}$$

where $\mathbf{Y}_i=(Y_{i1},\cdots,Y_{iT})'$, $\boldsymbol{\theta}=\left(\delta_1,\gamma,\boldsymbol{\beta}'\right)'$, $\mathbf{a}_{i,c_j}=(\boldsymbol{b}_{c_j}'-c_j\mathbf{1}_{c_j}',\mathbf{0}_{T-c_j}')'$, $\boldsymbol{b}_{c_j}=(1,\cdots,c_j)'$ and model matrix $\widetilde{\mathbf{X}}_i$ is defined according to $\boldsymbol{\theta}$. In a Bayesian framework, the following

distributions are adopted as priors: Inverse gamma priors, $IG(\tau_1,\tau_2)$ for σ^2_ε, $IG(\eta_1,\eta_2)$ for σ^2_α and $IG(\kappa_1,\kappa_2)$ for $\sigma^2_{\delta_2}$, the $N(\boldsymbol{\theta}_0,\boldsymbol{\Lambda})$ for $\boldsymbol{\theta}$ and $N(\mu_0,\sigma^2_0)$ for μ_{δ_2}. Data analysis are conducted by using the Gibbs sampler which simulates iteratively from the complete conditional posterior (CCPs) distributions derived in below. Then, average of samples for each parameter is used as its Bayes estimate. After some algebra is done, we have

- $\boldsymbol{\theta} \mid \sigma^2_\varepsilon, \boldsymbol{\delta}_{2_i}, \boldsymbol{\alpha}, \mathbf{y} \sim N_p(\mu_\theta, \boldsymbol{\Sigma}_\theta)$, with the mean vector and the covariance matrix

$$\mu_\theta = \boldsymbol{\Sigma}_\theta \left(\boldsymbol{\theta}_0' \boldsymbol{\Lambda}^{-1} + \frac{1}{\sigma^2_\varepsilon} \sum_{i=1}^{n} \left(\mathbf{y}_i - \alpha_i \mathbf{1}_T - \delta_{2_i} \mathbf{a}_{i,c_{\lambda_i}} \right)' \widetilde{\mathbf{X}}_i \right), \quad \boldsymbol{\Sigma}_\theta = \left(\boldsymbol{\Lambda}^{-1} + \frac{1}{\sigma^2_\varepsilon} \sum_{i=1}^{n} \widetilde{\mathbf{X}}_i' \widetilde{\mathbf{X}}_i \right)^{-1}.$$

- $\sigma^2_\varepsilon \mid \boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\delta}_{2_i}, \mathbf{y} \sim IG(\tau_1^*, \tau_2^*)$, where $\tau_1^* = \tau_1 + nT/2$ and $\tau_2^* = \tau_2 + \frac{1}{2}\sum_{i=1}^{n} \tilde{\mathbf{r}}_i' \tilde{\mathbf{r}}_i$ where $\tilde{\mathbf{r}}_i = \mathbf{r}_i - \alpha_i \mathbf{1}_T - \delta_{2_i} \mathbf{a}_{i,c_{\lambda_i}}$ and $\mathbf{r}_i = \mathbf{Y}_i - \widetilde{\mathbf{X}}_i \boldsymbol{\theta}$.

- $\alpha_i \mid \sigma^2_\varepsilon, \boldsymbol{\delta}_{2_i}, \sigma^2_\alpha, \boldsymbol{\theta}, \mathbf{y} \sim N\left(\mu_i^*, \sigma^{2*}_\alpha\right)$, where $\mu_i^* = \psi \mathbf{1}' \left(\mathbf{r}_i - \delta_{2_i} \mathbf{a}_{i,c_{\lambda_i}} \right)$ and $\sigma^{2*}_\alpha = \sigma^2_\varepsilon \psi$ for $\psi = \sigma^2_\alpha / \sigma^2_c$ with $\sigma^2_c = \sigma^2_\varepsilon + T\sigma^2_\alpha$.

- $\sigma^2_\alpha \mid \alpha_i, \mathbf{y} \sim IG(\eta_1^*, \eta_2^*)$ where $\eta_1^* = \eta_1 + \frac{n}{2}$ and $\eta_2^* = \eta_2 + \frac{1}{2}\sum_{i=1}^{n} \alpha_i^2$.

- $\lambda_i \mid \boldsymbol{\theta}, \sigma^2_\varepsilon, \alpha_i, c_{\lambda_i}, \boldsymbol{\delta}_{2_i}, \boldsymbol{\pi}, \mathbf{y} \sim \sum_{j=1}^{C} \pi_j \varphi_T \left(\mathbf{y}_i | \widetilde{\mathbf{X}}_i \boldsymbol{\theta} - \alpha_i \mathbf{1}_T - \delta_{2_i} \mathbf{a}_{i,c_{\lambda_i}}, \sigma^2_\varepsilon \mathbf{I}_T \right) \Delta_j(\lambda_i)$,

- For each random effect, let λ_r^*'s, $r = 1, \cdots, m$ be m unique values of λ_i's.

  $$f(c^*_{\lambda_r} \mid \boldsymbol{\alpha}, \boldsymbol{\delta}_{2_i}, \sigma^2_\varepsilon, \boldsymbol{\theta}, \mathbf{y}) \propto exp\{ -\frac{1}{2\sigma^2_\varepsilon} \sum_{\{i:\lambda_i = \lambda_r^*\}} (\mathbf{r}_i - \alpha_i \mathbf{1}_t - \delta_{2_i} \mathbf{a}_{i,c_{\lambda_i}})' (\mathbf{r}_i - \alpha_i \mathbf{1}_t - \delta_{2_i} \mathbf{a}_{i,c_{\lambda_i}}) \},$$

  for $c^*_{\lambda_r} \in \{1, \cdots, T\}$. We simulate from this CCP by Metropolis-Hastings algorithm.

- $\delta_{2_i} \mid \boldsymbol{\theta}, \boldsymbol{\alpha}, \mathbf{c}, \boldsymbol{\lambda}, \sigma^2_\varepsilon, \mathbf{y} \sim N(\mu^*_{\delta_2}, \sigma^{2*}_{\delta_2})$, where $\mu^*_{\delta_2} = \sigma^{2*}_{\delta_2} \left(\frac{\mu_{\delta_2}}{\sigma^2_{\delta_2}} + \frac{1}{\sigma^2_\varepsilon} (\mathbf{r}_i - \alpha_i \mathbf{1}_T)' \mathbf{a}_{i,c_{\lambda_i}} \right)$, and $\sigma^{2*}_{\delta_2} = \left(\frac{1}{\sigma^2_{\delta_2}} + \frac{1}{\sigma^2_\varepsilon} \mathbf{a}'_{i,c_{\lambda_i}} \mathbf{a}_{i,c_{\lambda_i}} \right)^{-1}$.

- $\mu_{\delta_2} \mid \boldsymbol{\delta}_{2_i}, \mathbf{y} \sim N\left(\mu_0^*, \sigma^{2*}_0\right)$, where $\mu_0^* = \frac{1}{\sigma^{2*}_0} \left(\frac{\mu_0}{\sigma^2_0} + \frac{1}{\sigma^2_{\delta_2}} \sum_{i=1}^{n} \delta_{2_i} \right)$ and $\sigma^{2*}_0 = \frac{\sigma^2_0 \sigma^2_{\delta_2}}{n\sigma^2_0 + \sigma^2_{\delta_2}}$.

- $\sigma^2_{\delta_2} \mid \boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\delta}_{2_i}, \mathbf{y} \sim IG(\kappa_1^*, \kappa_2^*)$, where $\tau_1^* = \kappa_1 + n/2$ and $\kappa_2^* = \kappa_2 + \frac{1}{2}\sum_{i=1}^{n} (\delta_{2_i} - \mu_{\delta_2})^2$.

5 Simulation study

The data generating process is organized to the mixed model (2) by assuming the only parameter of $\boldsymbol{\beta}$ be the intercept β_0 and considering the sample size of $n = 200$ and $T = 50$. We assume $\delta_{2_i} \overset{\text{iid}}{\sim} N(\mu_{\delta_2}, \sigma^2_{\delta_2})$, $\alpha_i \overset{\text{iid}}{\sim} N(0, \sigma^2_\alpha)$ and $\varepsilon_{it} \overset{\text{ind}}{\sim} N(0, \sigma^2_\varepsilon)$. We set $\beta_0 = 0$, $\delta_1 = 0$, $\mu_{\delta_2} = -16$ and $\gamma = 0.5$, and variance components $\sigma^2_\alpha = 400$, $\sigma^2_\varepsilon = 900$ and $\sigma^2_{\delta_2} = 64$. For each subject, c_i is randomly generated form the set $\{10, 20, 30, 40\}$ and therefore, each

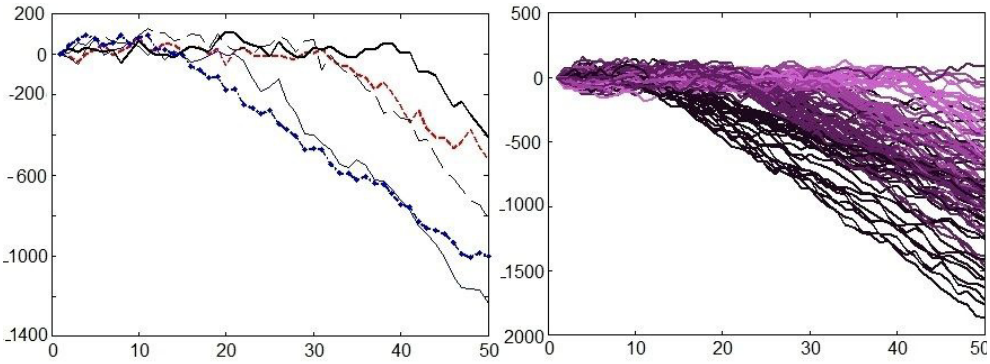

Figure 1: Profiles of generated data points for 5 subjects (left) and for all subjects (right panel).

subject is randomly assigned to a cluster. The y_{i0}'s are set to zero. Configuration of generated data points are depicted in Figure 1. We fit model (3) by assuming G_0 be $DU\{1, \cdots, 50\}$. We use the OpenBUGs software [2] with 100000 samples generated after 300000 burn-in. Non-informative independent priors with large variances are also adopted.

Results of estimated values of change points, c_j^*, $j = 1, \cdots, 7$ and estimate of size of each cluster are reported in Table 1. As is seen, the estimated change points are close to the real values, i.e. $\{10, 20, 30, 40\}$, except that the first cluster with the change-point equals to 10 is divided to two clusters with change points equal to 9.12 and 13.18. Also, it should be noted that the last two estimated change-points are inactive (empty) in most runs of Gibbs sampler such that their estimated cluster sizes are 0.10 and 0.03.

Furthermore, we cluster the data set by using the estimated values. The likelihood of each data point is computed for active clusters. The average of c_1^* and c_2^* are considered as the change point of the first cluster. Then, each data point is assigned to the cluster with larger likelihood. The ratio of true clustered subjects is obtained 0.8750.

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Table 1: Bayesian estimates of change points and sizes of each cluster are reported.

	c_1^*	c_2^*	c_3^*	c_4^*	c_5^*	c_6^*	c_7^*
c_j^*	9.12 (0.23)	13.82 (0.31)	19.77 (0.20)	29.62 (0.25)	38.96 (0.26)	40.95 (2.19)	45.38 (2.88)
size	44.58 (2.26)	14.26 (2.16)	60.68 (2.52)	37.27 (2.04)	43.08 (1.95)	0.10 (0.42)	0.03 (0.19)

∗ Bayesian standard deviations are given in parentheses.

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Income inequality indicies based on distance measures
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Income inequality indicies based on distance measures

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Abstract

One of the important issues in many applications of income inequality indices is finding an appropriate function of distance to measure disparity in population. A natural way of viewing an income inequality index is in terms of the distance function between an actual distribution of incomes and an appropriately normative distribution. Based on Lorenz curve and line of equality, real analysis offers a number of distance functions to construct some inequality measures. In this paper, we introduce some income economic indices from the mathematical point of view.

Keywords: Distance measure, Income inequality index, Lorenz curve
Mathematics Subject Classification [2010]: 62A99, 62P20, 91B02

1 Introduction

A natural way of viewing an income inequality measure is in terms of the distance between an actual distribution of incomes and an appropriately normative distribution (i.e. reflecting a perfectly equal distribution of incomes). With this idea, Lorenz (1905) proposed a graphical representation based on the percentage of total income earned by cumulative percentage of the population for representing inequality in the wealth distribution. In a perfectly equal society, the poorest $\%p$ of the population would earn exactly $\%p$ of the total income and the Lorenz curve would follow the path of the 45 (line of equality). As inequality increases, the Lorenz curve deviates from the egalitarian line. Based on the discussion about Lorenz curve and line of equality, real analysis offers a number of distance functions to construct some inequality measures. Cha (2007)[1] provides a particularly useful review on distance measures and some of these have been employed in the economics measurement literature. One of the important issues in many applications of income inequality measure is finding an appropriate measure of distance to measure disparity in population. The 20th century witnessed tremendous efforts to exploit new distance measures for a variety of applications. A number of distance measures for this purpose have been proposed and extensively studied by statisticians and econometrists.

*Speaker

2 Lorenz curve

The Lorenz curve is a graphical representation of the cumulative income distribution. It was developed by Max O. Lorenz (1905) [4] for representing inequality in the wealth distribution. The Lorenz curve can usually be represented by a function $L(p)$, where p, the cumulative portion of the population, is represented by the horizontal axis, and L, the cumulative portion of the total wealth or income, is represented by the vertical axis. Let $\mathcal{L}$ be the class of all non-negative income random variables with positive finite expectations. For a random variable X in $\mathcal{L}$ with cumulative distribution function F, we define its inverse distribution function by ${F^{-1}}_X(y) = \inf\{x : F_X(x) \geq y\}$.

According to Zenga (2007)[5], the Lorenz curve L_X corresponding to X is defined by

$$L_X(p) = \frac{\int_0^p {F_X}^{-1}(t)dt}{\int_0^1 {F_X}^{-1}(t)dt} \qquad 0 \leq p \leq 1. \tag{1}$$

The numerator sums the incomes of the bottom $\%p$ proportion of the population. The denominator sums the incomes of all the population. $L(p)$ thus indicates the cumulative percentage of total income held by a cumulative proportion $\%p$ of the population. To visualize proportions (1), like Figure 1, we plot the points $(p, L(p))$.

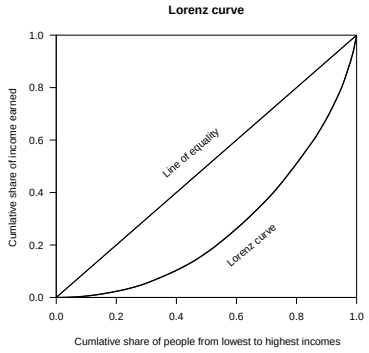

Figure 1: Lorenz curve

It can be noted that the Lorenz curve is always below the diagonal

$$I(p) = p \qquad 0 \leq p \leq 1.$$

The diagonal I, on the other hand, is also a Lorenz curve. Indeed, assuming that all the incomes are equal. Thus the interpretation of I (the straight line) represents perfect equality and any departure from this 45^0 line represents inequality. Based on the discussion about Lorenz curve and line of equality, it now becomes natural to measure the economic inequality by using some distance $d(I, L)$ between the egalitarian Lorenz curve(I) and the actual one L, then we can consider $d(I, L)$ as a measure of economic inequality in the population. The main idea behind the construction of $d(.,.)$ is based on the fact that we are merely interested in measuring the distance between I and L. This implies that we are really interested only in the functional $\mathcal{D} = d(I, L)$ defined on the set of all Lorenz curves.

3 Inequality measure as a distance function

Here, we introduce some income inequality measures according to distance functionals.

3.1 Robin Hood index

The Robin Hood index is an inequality measure, which admits a simple description in terms of the geometry of the Lorenz curve. The Robin Hood is defined as the maximal vertical distance between the Lorenz curve and the line of equality.

$$I_{Robin} = d(p, L(p)) = \max_{0 \le p \le 1} (p - L(p)) .$$

The Robin Hood index is based on the Lorenz Curve and is closely tied to the better known inequality measure the Gini coefficient which is also based on the Lorenz curve.

3.2 Gini Index

The most well-known member of the income inequality family is the Gini coefficient. It is widely used to measure income inequality, mainly because of its clear economic interpretation. This measure can be defined in various ways. The best known definition of the Gini index of inequality is as twice the area between the equality line and the Lorenz curve. Therefore, it can be expressed as

$$I_{Gini} = d(p, L(p)) = 2 \int_0^1 (p - L(p)) dp.$$

Several generalizations of the classical Gini index, placing smaller or greater weights on various portions of income distribution, have been proposed by a number of authors. Kakwani (1980) [3] introduced the S- Gini Index as a generalization of Gini index. He proposed a one-parameter family of generalized Gini indices by introducing different weighting functions for the area under the Lorenz curve,

$$d(p, L(p)) = (\int_0^1 (p - L(p))^{\alpha} w(p) dp)^{\frac{1}{\alpha}},$$

where $w(p)$ is some increasing function which allows value judgments about in equality to be incorporated.

3.3 Bonferroni index

The Bonferroni index has been revalued and studied for its features and for its interesting applications in social and economic contexts. This index is based on comparison between partial means of lower group and total mean. The Bonferroni index as a distance function can be defined as

$$I_{Bonf} = d(p, L(p)) = \int_0^1 (1 - \frac{L(p)}{p}) dp.$$

This index is sensitive to low income values.

3.4 Zenga index

In 2007, Michele Zenga [5] introduced a new inequality measure based on ratio between lower and upper group means of incomes in population. Zenga's index of inequality is defined as a distance function by the formula

$$I_{Zenga} = d(p, L(p)) = \int_0^1 [1 - \frac{L(p)/p}{(1 - L(p))/(1 - p)}]dp,$$

This index is more sensitive to extreme income values rather than other indices.

3.5 Canberra index

There are metrics which have not received much attention even thought they have many advantages. One such metric is the Canberra metric, which is based on the function of the partial means and the general mean of income distribution. The Canberra distance is a weighted version of the classic L1 distance family which naturally extends to a metric on symmetric groups. This measure may be defines as

$$I_{Canberra} = d(p, L(p)) = \int_0^1 \frac{p - L(p)}{p + L(p)}\, dp, \qquad \forall p \in (0, 1).$$

The Canberra measure such as the bonferroni index is sensitive to low income values.

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On the estimation of the shape parameter of the Topp-Leone Distribution

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Abstract

In this paper, we describe and present results on the parameter estimation for the Topp-Leone distribution. Three estimating methods have been investigated, namely, the maximum likelihood method, the method of moments and the probability weighted moments method. A simulation study has shown that the maximum likelihood estimator outperforms the estimators obtained by other methods.

Keywords: Maximum likelihood estimation, Probability weighted moment method, Simulation
Mathematics Subject Classification [2010]: 62F10, 68U20

1 Introduction

The Topp-Leone distribution is a univariate continuous two parameter distribution with bounded support which was first proposed and used as a model for some failure data by [2]. Reference [1] provided a motivation for this distribution based on its hazard rate function and then studied the moments and some other main properties of the Topp-Leone distribution. The random variable X with the range $(0, 1)$ has one-parameter Topp-Leone distribution if its probability density function (pdf) can be stated as

$$f(x;\nu) = 2\nu(1-x)x^{\nu-1}(2-x)^{\nu-1}, \quad 0 < x < 1, \quad \nu > 0, \tag{1}$$

and we write $X \sim TL(\nu)$. The corresponding cumulative distribution function (cdf) is given by

$$F(x;\nu) = [x(2-x)]^{\nu}, \quad 0 < x < 1. \tag{2}$$

Moreover, the quantile function of the Topp-Leone distribution is

$$Q(u) = 1 - \sqrt{1-u^{\frac{1}{\nu}}}, \quad 0 < u < 1. \tag{3}$$

In what follows, we investigate the estimation of the shape parameter of the Topp-Leone distribution with pdf (1) using the maximum likelihood (ML) method, the method of moments and the probability weighted moments (PWM) method. We also provide a simulation study to compare the mentioned methods numerically.

*Speaker

2 Main results

We derive and present three different estimators for the shape parameter of the Topp-Leone distribution using three considered methods. We start with maximum likelihood method.

2.1 Maximum likelihood estimation

Let $X_1, \cdots, X_n$ be a random sample of size n coming from the Topp-Leone distribution with pdf (1). Then the log-likelihood function for the parameter ν becomes

$$\ell(\nu) = n \log 2\nu + \sum_{i=1}^{n} \log(1 - x_i) + (\nu - 1) \sum_{i=1}^{n} \log\{x_i(2 - x_i)\}.$$

The maximum likelihood estimator (MLE) of ν is obtained by maximizing $\ell(\nu)$ with respect to ν. Upon differentiating $\ell(\nu)$ with respect to ν and equating the result with zero, the MLE of ν, denoted as $\widehat{\nu}^M$, will be obtained as

$$\widehat{\nu} = \frac{-n}{\sum_{i=1}^{n} \log\{X_i(2 - X_i)\}}. \tag{4}$$

2.2 Method of moments

One of the simplest and oldest methods of estimation is the method of moments. In this method, we obtain the estimators of the unknown parameters by equating the population moments with sample moments, i.e.

$$E(X^r) = \frac{1}{n} \sum_{i=1}^{n} X_i^r.$$

Reference [1] derived the r-th moment of the Topp-Leone distribution as follows

$$E(X^r) = 2^{r+2\nu}\nu[Be(0.5, r + \nu, \nu) - 2Be(0.5, r + \nu + 1, \nu)],$$

where $Be(\cdot, \cdot, \cdot)$ is the incomplete beta function, defined as

$$Be(x, \alpha, \beta) = \int_0^x t^{\alpha-1}(1 - t)^{\beta-1} \mathrm{d}t.$$

Considering the first population moment, (setting $r = 1$), the moment estimator of ν, denoted as $\widehat{\nu}^m$, can be obtained by solving the following equation

$$2^{2\nu+1}\nu[Be(0.5, \nu + 1, \nu) - 2Be(0.5, \nu + 2, \nu)] = \overline{X},$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean (sample first moment).

2.3 PWM method

For an arbitrary random variable X with pdf $f(x)$ and cdf $F(x)$, the population probability weighted moment of order (j, r, s) is defined as

$$M_{j,r,s} = E(X^j[F(X)]^r[1 - F(X)]^s). \tag{5}$$

A logical estimator for $M_{j,r,s}$ will be the sample probability weighted moment of order (j, r, s), i.e.

$$\widehat{M}_{j,r,s} = m_{j,r,s} = \frac{1}{n}\sum_{i=1}^{n} X_i^j[F(X_i)]^r[1 - F(X_i)]^s). \tag{6}$$

One can obtain estimators of the unknown parameters of the model by equating (5) with (6). Here, we use the usual orders $j = r = 1$ and $s = 0$ for the Topp-Leone distribution, therefore we get

$$M_{1,1,0} = 2^{4\nu+1}\nu[Be(0.5, 2\nu + 1, 2\nu) - 2Be(0.5, 2\nu + 2, 2\nu)].$$

Consequently, from (2), the PWM estimator of ν, denoted as $\widehat{\nu}^A$, will be obtained by solving the following non-linear equation with respect to ν

$$2^{4\nu+1}\nu[Be(0.5, 2\nu + 1, 2\nu) - 2Be(0.5, 2\nu + 2, 2\nu)] = \frac{1}{n}\sum_{i=1}^{n} X_i^{\nu+1}(2 - X_i)^\nu.$$

3 A simulation study

In this section, we compare the performance of estimators using a simulation. We consider three values for ν, i.e. $\nu = 1, 2$ and 3 and consider the sample sizes to be 10, 20 and 30. We can use the quantile function given in (3) in order to generate a Topp-Leone random variable. The following algorithm has been applied to perform our simulation.

Step 1: Generate a random sample of size n, $U_1, \cdots, U_n$, from the uniform distribution on (0,1).

Step 2: Set $X_i = 1 - \sqrt{1 - U_i^{\frac{1}{\nu}}}$ for $i = 1, \cdots, n$ to attain a random sample of size n from $TL(\nu)$.

Step 3: Obtain the ML, moment and PWM estimators of ν based on the random sample generated in Step 2.

Step 4: Repeat Steps 1–3 $N = 1000$ times and obtain the estimated mean squared errors of the estimators.

Let $\widehat{\nu}(i)$ be the estimator (MLE, moment estimator or PWM estimator) of ν obtained in Step 3 and in the i-th iteration. Then the estimated mean squared error (EMSE) of the estimator is given by

$$EMSE(\widehat{\nu}) = \frac{1}{N}\sum_{i=1}^{N}[\widehat{\nu}(i) - \nu]^2.$$

Table 1: Numerical Results.

	ν	1	2	3
n	method			
10	ML	0.17955	0.65066	1.60353
	Moment	0.22257	0.76345	1.68954
	PWM	0.36563	1.23059	2.45521
20	ML	0.06833	0.28730	0.61134
	Moment	0.09130	0.35022	0.70822
	PWM	0.12330	0.49930	1.01141
30	ML	0.035961	0.14235	0.37223
	Moment	0.050963	0.17017	0.43632
	PWM	0.069264	0.22635	0.60825

All the computations of the simulation have been performed using MAPLE 17 and the results are presented in Table 1. From Table 1, we observe that the ML method performs better than the other methods as the EMSEs of the MLEs are less than the corresponding EMSEs of Moment and PWM estimators. In addition the EMSEs are increasing with respect to ν. It can also be seen that the EMSEs are decreasing with respect to n, in other words, as the sample size increases, the estimators become exacter (get nearer to the true value of the parameter), as expected. Summing up, one may prefer to estimate the shape parameter of the Topp-Leone distribution using ML method if s/he considers the results of this paper.

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Poster Some properties and aplication of the Weibull-G distribution pp.: 1–4

Some Properties and Aplication of The Weibull-G Distribution

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Abstract

In this article, a new distribution, namely, weibull-G distribution is defined and studied.We introduce a new generator based on the Weibull random variable called the new Weibull-G family. Various properties and a characterization of the weibull-G family are obtained. we discuss the estimation of the model parameters by maximum likelihood and illustrate the potentiality of the extended family with applications to real data.

Keywords: weibull distribution; maximum likelihood estimation; generator
Mathematics Subject Classification [2010]: 62E15, 62F10

1 Introduction

Fit a suitable model to the data is the goal of many researchers. To add more flexibility to Weibull distribution, many researchers developed many generalizations of the distribution.These generalizations include the generalized Weibull distribution by Mudholkar and Kollia(1994), the exponentiated-Weibull distribution by Mudholkar et al.,(1995) . Extended weibull Gurvich et al.,(1997) and gamma Zografos and Balakrishnan(2009) families. Consider a continuous distribution G with density g and therefore $\frac{G(x;\xi)}{1-G(x;\xi)}$ is called odds ratio. Moreover we assume that random variable X has weibull distrbution [1]. In this case, weibull-G family distribution will be defined as follows.

Definition 1.1. we define as the cdf of the weibull-G family distribution by replacing x by $\frac{G(x;\xi)}{1-G(x;\xi)}$ in the weibull cdf

$$F(x;\alpha,\beta,\xi)=\int_0^{\frac{G(x;\xi)}{1-G(x;\xi)}}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx=1-exp\{-\alpha[\frac{G(x;\xi)}{1-G(x;\xi)}]^{\beta}\},\qquad x\epsilon R;\alpha,\beta>0, \tag{1}$$

The pdf corresponding to (1) is given by

$$f(x;\alpha,\beta,\xi)=\ \alpha\beta g(x;\xi)\frac{G(x;\xi)^{\beta-1}}{\overline{G}(x;\xi)^{\beta+1}}exp\{-\alpha[\frac{G(x;\xi)}{\overline{G}(x;\xi)}]^{\beta}\},\qquad x>0,\alpha>0,\beta>0. \tag{2}$$

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Example 1.2. suppose that the parent distribution is BURXII with pdf and cdf given by $g(x) = cks^{-c}x^{c-1}[1+(\frac{x}{s})^c]^{-k-1}$, $s,k,c>0$ and $G(x) = 1-[1+(\frac{x}{s})^c]^{-k}$respectively.Then, the Weibull-BURRXII distribution has pdf given by

$$f_{WBBXII}(x;\alpha,\beta,s,k,c) = \frac{\alpha\beta cks^{-c}x^{c-1}}{(1+(\frac{x}{s})^c)^{1-k}}exp\{-\alpha[(1+(\frac{x}{s})^c)^k-1]^\beta\}. \quad (3)$$

The following Lemma gives the relation between weibull-G and weibull, exponential, and Gumbel distributions.

Lemma 1.3. *(a) If a random variable Y follows the weibull distribution with parameters α and β , then the random variable $X = G^{-1}(\frac{Y}{Y+1})$ follows wei-G(α,β,ξ).*
(b)If a random variable Y follows the standard exponential distribution then the random variable $X= G^{-1}(\frac{Y^{\frac{1}{\beta}}}{Y^{\frac{1}{\beta}}+1})$ follows wei-G(α,β,ξ).
(c)If a random variable Y follows the Gumble distribution with scale parameter β, then the random variable $X = G^{-1}(\frac{e^{-Y}}{\alpha^{\frac{1}{\beta}}+e^{-Y}})$ follows wei-G(α,β,ξ).

Theorem 1.4. *Let (Ω,Σ,P) be a given probability space and let $H=[a,b]$ be an interval for some b ($a=-\infty$, $b=\infty$ might as well be allowed). Let $X:\Omega\to H$ be a continuous random variable with distribution function $F(x)$ and let q_1 and q_2 be two real function defined on H such that*

$$E[q_1(X) \mid X\geq x] = E[q_2(X) \mid X\geq x]\eta(x), \quad x\epsilon H,$$

Is defined with some real function η. Assume that q_1, $q_2\epsilon C^1(H)$, $\eta\epsilon C^2(H)$ and $G(x)$ is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $q_2\eta = q_1$ has no real solution in the interior of H. Then G is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(X) = \int_a^x C \mid \frac{\eta^{'}(x)}{\eta(u)q_2(x)-q_1(u)} \mid e^{-s(u)}du,$$

Where the function s is a solution of the differential equation $s^{'} = \frac{\eta^{'}q_2}{\eta q_2-q_1}$ and C is a constant, chosen to make $\int_H df = 1$ [2].

Proposition 1.5. *Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x) = \alpha(\frac{G(x;\xi)}{1-G(x;\xi)})^\beta - 1$ and $q_2(x)=1$ for $x>0$. The pdf of X is if and only if the function η defined in Theorem 1.4 has the form*

$$\eta = \alpha(\frac{G(x;\xi)}{1-G(x;\xi)})^\beta, \quad x>0.$$

Proof. Let X have density (2) , then clearly F, q_1, q_2 and η satisfy both the hypotheses of 1.4 and the above differential equation. Conversely, assume that there are function q_2 and η satisfying the differential equation then

$$s^{'} = \frac{\eta^{'}q_2(x)}{\eta(x)q_2(x)-q_1(x)} = \alpha\beta g(x)\frac{G(x;\xi)^{\beta-1}}{\bar{G}(x;\xi)^{\beta+1}}, \quad x>0$$

$$s(x)=\alpha(\frac{G(x;\xi)}{\bar{G}(x;\xi)})^{\beta},\quad x>0,\qquad or\qquad e^{-s(x)}=\ e^{-\alpha(\frac{G(x;\xi)}{\bar{G}(x;\xi)})^{\beta}},\quad x>0.$$

Now, in view of Theorem1.4, X has density function (2). □

2 Maximum Likelihood Estimation and Application

Here, we determine the maximum likelihood estimates ($MLEs$) of the parameters of the new family of distributions from complete samples only. Let $x_1, x_2, ..., x_n$ be observed values from the Wei-G distribution with parameters α , β and ξ. Let $\Theta=(\alpha,\beta,\xi)^T$ be theparameter vector. The total log-lielihood function for Θ is given by

$$\ell(\theta)=\ nlog(\alpha)+\ nlog(\beta)+\ \Sigma_{i=1}^{n}log[g(x_i;\xi)]-\ \alpha\Sigma_{i=1}^{n}H(x_i;\xi)^{\beta}+$$

$$\beta\Sigma_{i=1}^{n}log[H(x_i;\xi)]-\ \Sigma_{i=1}^{n}log[G(x_i;\xi)]-\ \Sigma_{i=1}^{n}log[1-G(x_i;\xi)],$$

Where $H(x;\xi)=\ \frac{G(x;\xi)}{\bar{G}(x;\xi)}$ .The components of the score function $U(\Theta)=\ (U_\alpha,U_\beta,U_\xi)$ are

$$U_\alpha=\ \frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\Sigma_{i=1}^{n}H(x_i;\xi)^{\beta},$$

$$U_\beta=\ \frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-\alpha\Sigma_{i=1}^{n}H(x_i;\xi)^{\beta}log[H(x_i;\xi)]+\Sigma_{i=1}^{n}log[H(x_i;\xi)],$$

$$U_{\xi\kappa}=\ \frac{\partial\ell}{\partial\xi}=-\alpha\beta\Sigma_{i=1}^{n}H(x_i;\xi)^{\beta-1}\partial H(x_i;\xi)/\partial\xi_\kappa+\beta\Sigma_{i=1}^{n}\frac{\partial H(x_i;\xi)/\partial\xi_\kappa}{H(x_i;\xi)}$$

$$+\Sigma_{i=1}^{n}\frac{\partial g(x_i;\xi)/\partial\xi_\kappa}{g(x_i;\xi)}-\Sigma_{i=1}^{n}\frac{\partial G(x_i;\xi)/\partial\xi_\kappa}{G(x_i;\xi)}-\Sigma_{i=1}^{n}\frac{\partial\overline{G}(x_i;\xi)/\partial\xi_\kappa}{\overline{G}(x_i;\xi)}.$$

Setting U_α, U_β and U_ξ equal to zero and solving the equations simultaneosly yields the MLE of These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative method the Newton-Raphson type algorithm.
***Application**.The data set were used by (Birnbaum and Saunders 1969) and correspond to the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). The data are:*
70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112,113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134,134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142,142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157,157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212.
For these data, we fit the WBXII distribution defined in (3) and compare it with the Weibull-log-logistic (WLL) (for $x>0$) and the beta Burr XII (BBXII))(for $x>0$) (Paranaba et al., 2011) models with corresponding densities:

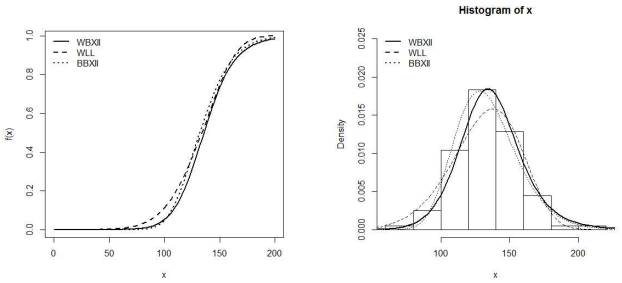
$$f_{WLL}(x;\alpha,\beta,s,c)=\ \frac{\alpha\beta cs^{-c}x^{c-1}exp\{-\alpha[(1+(\frac{x}{s})^c)-1]^{\beta}\}[(1+(\frac{x}{s})^c)-1]^{\beta-1}}{1+(\frac{x}{s})^c},$$

$$f_{BBXII}(x;a,b,s,k,c)=\ \frac{cks^{-c}x^{c-1}}{B(a,b)}[1+(\frac{x}{s})^c]^{-kb+1}\{1-[1+(\frac{x}{s})^c]^{-k}\}^{a-1},$$

where a, b, s, k, c, α, β > 0 and B(a,b) is the beta function [3]. The MLEs of the model parameters and the Akaike information criterion (AIC) for the WBXII and the other models are listed in table1.The ftted densities for the data sets aredisplayed in figures 1 ,These results illustrate the potentiality of the WBXII distributions and the importance of the two additional parameters.

Table 1: MLEs of the parameters and AIC of the WBXII models for the Data sets

Model	Estimation	AIC
$WBXII(\alpha,\beta,s,k,c)$	103.5 0.95 141.42 0.04 9.68	877.97
$WLL(\alpha,\beta,s,c)$	4174.28 0.99 566.53 6.09	924.37
$BBXII(a,b,s,c,k)$	102.13 119.74 61.54 0.53	924.8



(a) (b)

Figure 1: Estimated (a) cdf and (b) pdf for the WBXII,WLL,BBXII models

We fit the Weibull-G distributions to the real data sets to demonstrate the potentiality of this family.

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Poster The half generalized normal distribution and its properties pp.: 1–4

The half generalized normal distribution and its properties

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Abstract

In this paper, we introduce a new distribution called as the half generalized normal distribution. This distribution contains the half normal distribution in a special case. Some mathematical properties of this distribution such as pdf, cdf, hazard rate, rth moment and the moment generating function is studied in this paper.

Keywords: Generalized normal distribution, Half normal distribution, Hazard rate function, Moments.
Mathematics Subject Classification [2010]: 60E05, 62H10, 62H12.

1 Introduction

A random variable X is said to have the generalized normal distribution with parameters μ, σ and s if its density function is given by

$$f(x) = K \exp\{-|\frac{x-\mu}{\sigma}|^s\}, \qquad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0, s > 0,$$

where $K = \dfrac{s}{2\sigma\Gamma(1/s)}$. We denote this as $X \sim GN(\mu, \sigma, s)$. The generalized normal distribution is bell-shaped and unimodal with mode at $x = \mu$.
The cdf of X can be written as

$$F(x) = \begin{cases} \dfrac{\Gamma(1/s, ((\mu - x)/\sigma)^s)}{2\Gamma(1/s)} & if x \leqslant \mu \\ 1 - \dfrac{\Gamma(1/s, ((x - \mu)/\sigma)^s)}{2\Gamma(1/s)} & if x > \mu, \end{cases}$$

where incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t) dt.$$

Nadarajah [2] introduced this distribution as the generalization of normal and Laplace distributions where $s = 2$ and $s = 1$, respectively. A generalization of the half normal distribution is studied by Cooray and Ananda [1].

*Speaker

2 Half generalized normal distribution

In this section we introduce the random variable Y and study some of its distributional properties.
We consider that the random variable Y has a half generalized normal distribution with parameters σ and s if it can be represented as

$$Y = |X|, \tag{1}$$

where $X \sim GN(0, \sigma, s)$, $\sigma > 0$ and $s > 0$. We denote this as $Y \sim HGN(\sigma, s)$.
A distribution that plays an important role in this paper is the gamma distribution with pdf given by

$$g(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

where $x > 0$, $\alpha > 0$ and $\beta > 0$ and also its cdf given by

$$G(x; \alpha, \beta) = \int_0^x g(t; \alpha, \beta) dt.$$

Theorem 2.1. *let $Y \sim HGN(\sigma, s)$. Then the density function of Y is given by*

$$f_Y(y; \sigma, s) = 2K \exp\{-\frac{y^s}{\sigma^s}\},$$

where $K = \dfrac{s}{2\sigma\Gamma(\frac{1}{s})}$, $\sigma > 0$, $s > 0$ and $y > 0$.

Proof. One can use the formula $f_Y(y; \sigma, s) = f_X(y) + f_X(-y)$ to obtain the pdf of $Y = |X|$. □

Remark 2.2. If $Y \sim HGN(\sigma', s)$, $\sigma'^2 = 2\sigma^2$ and $s = 2$, we obtain the half normal distribution with parameter σ, and hence the density function of Y is given by

$$f_Y(y) = \frac{2}{\sqrt{2\pi}\sigma} \exp\{-\frac{y^2}{2\sigma^2}\}, \quad y > 0, \sigma > 0.$$

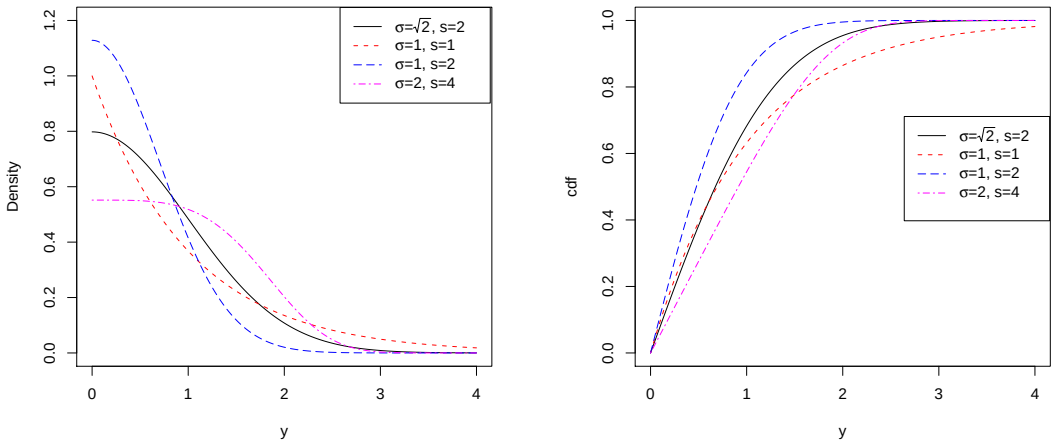
The cdf of random variable Y with HGN distribution can be written as

$$F(y) = G(y^s, \frac{1}{s}, \frac{1}{\sigma^s}),$$

where G is cdf of gamma distribution.
Figure 1 plots some possible shapes of pdf and cdf of HGN distribution for distinct values σ and s.
The survival function is defined as $S(y) = 1 - F(y)$, where $F(y)$ is the cdf of Y.
The hazard function is given by

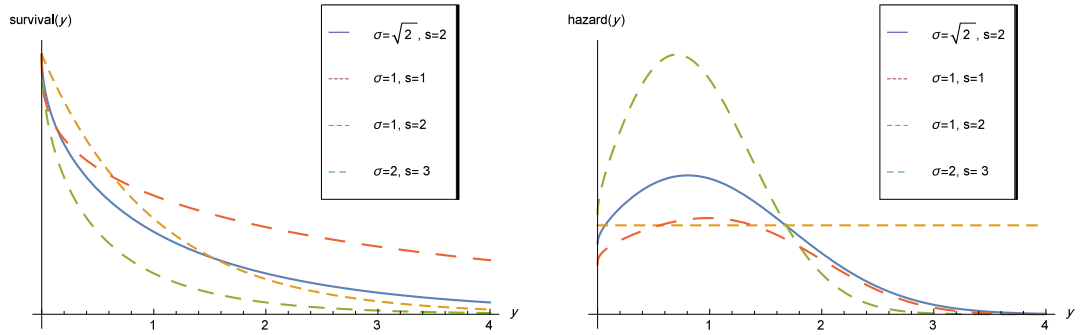
$$h(y) = \frac{2K \exp\{-\frac{y^s}{\sigma^s}\}}{1 - G(y^s, \frac{1}{s}, \frac{1}{\sigma^s})}.$$

Figure 2 plots some possible shapes of survival and hazard function of HGN distribution for distinct values σ and s.



(a) Plots of pdf of HGN distribution. (b) Plots of cdf of HGN distribution.

Figure 1: Graphs of pdf and cdf of HGN distribution for diffrent values σ and s.



(a) Plots of survival function. (b) Plots of hazard function.

Figure 2: Graphs of survival and hazard functions of HGN distribution for diffrent values σ and s.

3 Moments of HGN distribution

The rth moment and moment generating function of half generalized normal distribution is derived in this section.

Theorem 3.1. *Let $Y \sim HGN(\sigma, s)$. Then for $r = 1, 2, \cdots$ it follows that the rth moment is given by*

$$E[Y^r] = \frac{\sigma^r \Gamma(\frac{r+1}{s})}{\Gamma(1/s)}.$$

Proof. Using the definition of expectation, we have

$$E[Y^r] = \frac{s}{\sigma\Gamma(1/s)} \int_0^\infty y^r \exp\{\frac{y^s}{\sigma^s}\} dy,$$

and making the variable transformation $z = y^s$, we have that

$$E[Y^r] = \frac{1}{\sigma\Gamma(1/s)} \int_0^\infty z^{\frac{r+1}{s}-1} \exp\{-\frac{z}{\sigma^s}\} dz,$$

thuse the result is obtained after identifying the gamma function. □

Theorem 3.2. *Let $Y \sim HGN(\sigma, s)$. Then the moment generating function of the random variable Y is given by*

$$\begin{aligned} M_Y(t) &= \frac{s}{\sigma\Gamma(1/s)} \int_0^\infty \exp\{-(\frac{y^s}{\sigma^s} - ty)\} dy \\ &\simeq \frac{1}{\Gamma(1/s)} \sum_{n=0}^{\infty} (t\sigma)^n \Gamma(\frac{n+1}{s}). \end{aligned}$$

4 Conclusion

In this paper, we have introduced a new distribution called as the half generalized normal distribution. This distribution contains the half normal distribution in a special case. This new distribution can be written as the absolute value of generalized normal random variable.

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Poster A post quantum (n, n)-threshold secret sharing scheme using AD cryptosystem pp.: 1–4

A post quantum (n, n)-threshold secret sharing scheme using AD cryptosystem

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Abstract

The existing secret sharing schemes either require integer numbers or require DLP (Discrete Logarithm Problem) for verification. In addition, they use secure channel for transmission of secret. In this paper we present a (n, n)-threshold secret sharing scheme using AD cryptosystem through insecure channel in which the floating numbers can be used. The proposed scheme doesn't need DLP for verification. In addition, it is secure against quantum algorithms.

Keywords: Secret Sharing Scheme, Ajtai and Dwork (AD) Cryptosystem, Lattice, Post Quantum Cryptography.
Mathematics Subject Classification [2010]: 94A62, 94A62

1 Introduction

Secret sharing scheme (SSS) is a cryptographic primitive that allows a secret to be shared among a set of participants such that only a qualified subset (or even the whole set) can recover the secret [6]. SSSs are ideal for sensitive and highly important systems such as encryption keys, missile launch codes, numbered bank accounts access control systems, e-voting, authentication protocols and etc [6]. Classical constructions for (k,n)-threshold secret-sharing schemes include the polynomial based Shamir scheme [1], the nonparallel hyper planes-based Blakley scheme [5] and the integer-based Chinese remainder Theorem (CRT) scheme [2]. In fact Blakley and Shamir invented threshold sharing schemes independently [1, 5]. However, the existing schemes either use DLP for verification of secret or require a secure channel for secret transmission. In 1994, Shor discovered a quantum algorithm for solution of DLP [7]. Therefore, SSSs, which use DLP, are not resistant against quantum attacks.

Lattice-based cryptosystems are resistant against quantum attacks. Steinfeld et al. has introduced Lattice-based threshold-changeability for standard Shamir secret-sharing schemes in [8]. They proved the security of their works by lattice reduction techniques but their schemes require secure channel for secret transmission.

The Ajtai and Dwork (AD) cryptosystem is one of the post-quantum cryptosystems [4]. Post-quantum means that they are resistant against quantum attacks. In this paper, we propose a (n,n)-threshold SSS based on the AD cryptosystem. Ajtai and Dwork uses hard

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Lattice problems to prove the security of AD-cryptosystem. Lattice-based cryptographic constructions hold a great promise for post-quantum cryptography. Therefore, we use a lattice based cryptosystem for SSS which is believed secure against quantum computers.

The outline of this article is organized as follows: First, we present the Ajtai-Dwork Error-Free Construction in Section 2. Then we introduce verifiable AD-based SSS in section 3.

2 The Ajtai-Dwork Error-Free Construction

In this section we recall the error-free construction of Ajtai and Dwork [3]. Ajtai and Dwork have introduced a public-key encryption scheme which is secure under the assumption that a certain computational problem on lattices is hard on the worst-case [4].

To simplify the exposition, we present the scheme in terms of real numbers, but we always mean numbers with some fixed finite precision. For given security parameter n, we let $m = n^3$, and $\rho_n = 2^{nlogn}$. We show the n-dimensional cube of side-length ρ_n by B_n. In addition, we show the n-dimensional sphere of radius n^{-8} by S_n (for Small or Sphere). Namely, we have $B_n = \{x \in R^n : 0 \leqslant x_i < \rho_n, i = 1, 2, ..., n\}$ and $S_n = \{x \in R^n : ||x|| \leqslant n^{-8}\}$.

Private-key: For a given security parameter n, the private-key is a uniformly chosen vector in the n-dimensional unit sphere. We denote this vector by u.

For a private key u, define H_u (the distribution on points in B_n) by the following process: 1) Pick arbitrary point a from the set $\{x \in B_n : \langle x, u\rangle \in Z\}$, 2)Select random numbers $\delta_1, \delta_2, ..., \delta_n$ from S_n, 3) Output the point $v = a + \sum_i \delta_i$.

Public-key: The public-key is consist of the sequence of points $w_1, w_2, ..., w_n, v_1, v_2, ..., v_m$ and the integer i_1 where the arbitrary points $w_1, w_2, ..., w_n, v_1, v_2, ..., v_m$ are from the distribution H_u and i_1 is a random number from all the indices i so that $\langle a_i, u\rangle \in 2Z+1$.

Encryption. The encryption works in a bit-by-bit fashion. Namely, to encrypt a string $\sigma_1, \sigma_2, ..., \sigma_l$, each bit σ_i, is encrypted separately. To encrypt a '0', we uniformly select $b_1, ..., b_m$ in $\{0, 1\}$, and reduce the vector $\sum_{i=1}^m b_i v_i$ modulo the parallelepiped $P(w_1, w_2, ..., w_n)$. The vector $x = \sum_{i=1}^m b_i v_i \bmod P(w_1, w_2, ..., w_n)$ is the ciphertext which corresponds to the bit '0'. To encrypt a '1' we uniformly select $b_1, ..., b_m$ in $\{0, 1\}$, and reduce the vector $\frac{1}{2}v_{i_1} + \sum_{i=1}^m b_i v_i$ modulo the parallelepiped $P(w_1, w_2, ..., w_n)$ which is denoted by $v' = \sum_{i=1}^n c_j w_j$. The vector $x = \frac{1}{2}v_{i_1} + \sum_{i=1}^m b_i v_i \bmod P(w_1, w_2, ..., w_n)$ is the ciphertext which corresponds to the bit '1'.

Decryption (Error-Frees): For a given ciphertext x, and the private-key u, we compute $\tau = \langle v, u\rangle$ and decrypt the ciphertext as a '0' if τ is within $\frac{1}{4}$ of an integer and decrypt it as a '1' otherwise.

Goldreich et al have proved the correctness of encryption-decryption process of AD cryptosystem in [3].

3 The proposed scheme

The dealer divides the secret in a bit-by-bit fashion. Our scheme is a (n,n)-threshold scheme, so in order to recover the secret, n participants have to pool their shares.

3.1 Initialization phase

Common Parameters: The common parameters $n, m, P, B_n, S_n, w_1, w_2, ..., w_n, v_1, ..., v_m$, i_1, v' and u are determined in exactly the same manner as in the section 2.

3.2 Construction phase

Suppose that $W_i = (w_{1i}, w_{2i}, ..., w_{ni})^t$ and $U = (u_1, u_2, ..., u_n)^t$. In order to compute the shares, the dealer executes the following procedure.

1. Choose random numbers $k_i \neq 0$ and $d_i \neq 0$ for $1 \leq i \leq n-1$ such that $\sum_{i=1}^{n} k_i = 0$. Note that the values d_is are selected for more security because without using d_is, if $k_i = 1$ then u_i is detected by P_i.
2. Compute $Z_i = (z_{1i}, z_{2i}, ..., z_{ni})^t = (k_i d_1 u_1, k_i d_2 u_2, ..., k_i d_n u_n)$ for $1 \leq i \leq n-1$. Also, $Z_n = (z_{1n}, z_{2n}, ..., z_{nn})^t = ((k_n d_1 + 1)u_1, (k_n d_2 + 1)u_2, ..., (k_n d_n + 1)u_n)$.
3. Distribute Z_i and W_i to P_i for $i = 1, ..., n$.
4. Compute and publish C to P_i (for $i = 1, 2, ..., n$) where $C = \sum_{i=1}^{m} b_i v_i$ if the i-th bit of the secret is '0' and $C = \frac{1}{2} v_{i_1} + \sum_{i=1}^{m} b_i v_i$ otherwise. Note that $b_1, ..., b_m$ are uniformly select in $\{0, 1\}$.
5. Compute and publish $G = (\sum_{j=1}^{n} z_{1j} w_{1j}, \sum_{j=1}^{n} z_{2j} w_{2j}, ..., \sum_{j=1}^{n} z_{nj} w_{nj})^t$.
6. Distribute G_i to P_i (for $i = 1, ..., n$) where $G = (\sum_{j=1, j\neq i}^{n} z_{1j} w_{1j}, \sum_{j=1, j\neq i}^{n} z_{2j} w_{2j}, ..., \sum_{j=1, j\neq i}^{n} z_{nj} w_{nj})^t$.

3.3 Verification and recovery phase

According to the section 2, each P_i receives W_i, Z_i, G and G_i. We use the following proposition in our scheme.

Proposition 3.1. *If $k_i \neq 0$ and $d_i \neq 0$ such that $\sum_{i=1}^{n} k_i = 0$, $Z_i = (z_{1i}, z_{2i}, ..., z_{ni})^t = (k_i d_1 u_1, k_i d_2 u_2, ..., k_i d_n u_n)$ for $1 \leq i \leq n-1$ and $Z_n = (z_{1n}, z_{2n}, ..., z_{nn})^t = ((k_n d_1 + 1)u_1, (k_n\ d_2 + 1)u_2, ..., (k_n d_n + 1)u_n)$ then $\sum_{i=1}^{n} Z_i = U$.*

Proof.

$$(\sum_{i=1}^{n} Z_i)_j = \sum_{i=1}^{n} (Z_i)_j = \sum_{i=1}^{n-1} k_i d_j u_j + (k_n d_j + 1)u_j = d_j u_j \sum_{i=1}^{n} k_i + u_j = u_j$$

. □

To recover the secret, all the shares are needed. In fact, only n participants can recover the secret. The recovery with verification is made as follows:

1. Each P_i for $i = 1, ..., n$ can verify the validation of his share w_i: If $G \neq G_i + (z_{1i} w_{1i}, z_{2i} w_{2i}, ..., z_{ni} w_{ni})^t$ then P_i has received a wrong share else his share is valid.
2. If all the shares are valid then recover the vector U by $U = \sum_{i=1}^{n} Z_i$ (see proposition 3.1), and compute $x = \langle C \bmod P(w_1, w_2, ..., w_n), U \rangle$.
3. If x is within $\frac{1}{4}$ of some integer then the secret is '0' else it is '1'.

4 Conclusion and future works

We constructed a simple, efficient unanimous consent SSS based on the famous AD cryptosystem. We showed that the scheme doesn't allow recovering the secret if at least one participant is missing and this situation reduces to the break AD cryptosystem which is believed to be hard. The scheme offers the possibility for the participants to check if all the shares distributed by the dealer are valid. There is still a lot of work to be done in order to improve the capabilities of the scheme: it would be good to find a (k, n) variant of the scheme with $k \neq n$ and a way to make it multisecret (to allow sharing several secrets instead of one secret shared on each round).

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Algorithmic Construction of Equal Norm-Angle Tight Frames in R^2 of a given vector

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Abstract

The objective of this paper is to produce an implementable algorithm for construct equal norm equangular tight frame (ENATF) in R^2 of a given vector. This structure have some application in some atomic structure in crystal physics and filter bank theory in comminication.

Keywords: frame, egual norm, algoritmic, Hilbert space.
Mathematics Subject Classification [2010]: 57R25, 42C15

1 Introduction

Frames were introduced by Duffin and Shaefer in 1952 [4] as componet in the development of non harmonic Fourier series and a paper by Daubechies Grossman Meyer in 1986 initated the use of frame theory in signal processing [6].
The paper is organized as follows: section 2 contain preliminary definition on frames and the basic notation used throughout the paper. In section 3 we reviewsome basic elements concerning the notion of equal norm-angle tight frame (ENATF) in the form suitable for the algorithm structure ENATF and we present throughout the review and several examples, and suggest some possible applications.

2 Finite tight frames

Definitin a sequense $\{w_i\}_{i=1}^m$ of elements of a Hilbert space n-dimensional H over C or R, is called finite frame, if there are constans $A, B > 0$ such that

$$Av \leq \sum_{i=1}^{m} < v, w_i > w_i \leq Bv.$$

for all $v \in H$. the numbers A, B are called frame bounds respectively. The frame is called tight frame if $A = B$. The tight frame is called Parseval frame if $A = 1$. The frame is

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called equal norm frame if $\| w_i \| = \| w_j \|$ for all i, j . the frame is called equal angle if the angle betwen for w_i, w_{i-1} or w_i, w_{i+1} are equal for all i. The frame is called ENATF if it be a tight, equal norm and angular frame.
Some frames can be defined in a natural way by using group repersentation [5] [2] Let H be a real(complex) $n-$ dimensional Hilbert space and G a finite group such that $g : H \to H$ be an unitary and irreducible reperesentation and let $w \in H$be a fixed vector. we defined the subgroup G_w of G as follows:

$G_w = \{g \in G | g(w) = \alpha(w)\}$

where α is a scaler depending on g. if $\{g_i\}_{i=1}^m$ is a system of the left cosets of G on G_w then

$w_1 = g_1(w), w_2 = g_2(w), \ldots, w_m = g_m(w)$
form an equal norm tight frame in H , namely for all $v \in H$ we have

$$\sum_{i=1}^{m} < v, w_i > w_i = \frac{m}{n} \| w \|^2 v.$$

The orbit G

$$G(w) = \{g(w) | g \in G\}$$

is a tight frame with bound $\frac{m'}{n} \|v\|^2$ where m' is a order the finite group G.

Proof. We refer to Theorem 2 we have for all $v \in H$

$$\sum_{i=1}^{m} < v, w_i > w_i = \frac{m}{n} \| w \|^2 v.$$

The repersentation being unitary we have for any $g \in G$

$$\langle v, w_i \rangle w_i = \langle v, g(w) \rangle g(w) \tag{1}$$

Let k be the number of elements of G_w. Then G has $m' = km$ elements, let $\{g^1, \ldots, g^{m'}\}$ be an elements of the group G by equation 1 we have:

$$\frac{m}{n} \|v\|^2 = m \langle v, g(w) \rangle g(w)$$

so $\frac{1}{n} \|w\|^2 v = \langle v, g(w) \rangle g(w)$ consequently

$$\sum_{i=1}^{m'} \langle v, g^i(w) \rangle g^i(w) = \frac{m'}{n} \|w\|^2 v$$

□

let $H = R^2$ and $w^T = (\alpha_1, \alpha_2) \in H$ then a sequense $\{w_i\}_{i=0}^{m-1}$ $(m > 2)$ is a ENATF with angle $\frac{2\pi}{m}$ such that

$$w_0 = P^0 w, w_1 = P^1 w, \ldots, w_{m-1} = P^{m-1} w$$

where $P^0 = I_{R^2}$,

$$P = \begin{bmatrix} cos\frac{2\pi}{m} & -sin\frac{2\pi}{m} \\ sin\frac{2\pi}{m} & cos\frac{2\pi}{m} \end{bmatrix}$$

namely for all $v \in H$ we have

$$\sum_{i=0}^{m-1} < v, w_i > w_i = \frac{m}{2} \parallel w \parallel^2 v.$$

Proof. we consider the map $g : R^2 \to R^2$ where

$(\alpha_1, \alpha_2) \mapsto (\alpha_1 cos\frac{2\pi}{m} - \alpha_2 sin\frac{2\pi}{m}, \alpha_1 sin\frac{2\pi}{m} + \alpha_2 cos\frac{2\pi}{m}) = Pw$

the cyclic group $C_m = \langle g | g^m = e \rangle = \{e, g, \ldots, g^{m-1}\}$ defines a unitary and irreducible reperesentstion on R^2. so by 2 the vectors

$$w_0 = w, w_1 = g(w), \ldots w_{m-1} = g^{m-1}(w)$$

are tight frame with bound $\frac{m}{2}\|w\|^2$. under hand $g^i(w) = P^i w$, since P is rotation of the R^2 so $\{w_i\}_{i=0}^{m-1}$ is a ENATF frame with angle $\frac{2\pi}{m}$.

□

3 Main results

now we present the bellow algoritm for prouduct ENATF frame with a favorate vector in R^2.

---

algoritm1. ENATF

---

parameter:
m number of desired frame vector
, $f_0 = x_0 e_0 + y_0 e_1$ is a given vector
algorithm:

1) $For\ i = 1, \ldots, M-1 \quad do$
2) $\qquad x_i = x_{i-1} cos\frac{2\pi}{m} - y_{i-1} sin\frac{2\pi}{m}$
3) $\qquad y_i = x_{i-1} sin\frac{2\pi}{m} + y_{i-1} sin\frac{2\pi}{m}$
4) $\qquad f_i = x_i e_1 + y_i e_2$
5) $end.$

output :
$ENATF\{f_i\}_{i=0}^{m-1}$.

---

for $m = 3$ and $f_0 = (1, 0)$ we have

$$w_0^T = (1, 0),\ w_1^T = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}),\ w_2^T = (-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2})$$

is a ENATF with bound $\frac{3}{2}$ and angle $\frac{2\pi}{3}$.
for $m = 4$ and $f_0 = (1, 0)$ we have

$$w_0^T = (1, 0),\ w_1^T = (0, 1),\ w_2^T = (-1, 0)\ ,\ w_4 = (0, -1)$$

is a ENATF with bound 2 and angle $\frac{\pi}{2}$.
so for any m the algorithm prouduct a $m-$regular hedral and such
$m \longrightarrow \infty$ this algoritm will prouduct a circle that is a continuous frame also for $m = 3$ the frames of algorithm product a Merseds Benz frame for use filter bank [1] and structer Honycomb lattic[3] . If $m \parallel v \parallel^2 = 2$ then the frames pruduct for algorithm is Parseval frame.

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Efficient Encoding of APM-LDPC Codes

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Abstract

Low-density parity-check codes from Affine permutation matrices, called APM-LDPC codes, are a class of LDPC codes whose parity-check matrices consist of zero matrices or affine permutation matrices of the same orders. In this paper, using the similar approach proposed for encoding of LDPC codes introduced by Richardson and Urbanke, we propose an efficient algorithm for encoding APM-LDPC codes with a linear complexity.

Keywords: LDPC codes, Efficient encoding, Girth, Affine permutation.

Mathematics Subject Classification [2010]: 11T71, 68P30

1 Introduction and preliminaries

Low density parity check (LDPC) codes [1] are excellent error correcting codes with performance close to the Shannon Capacity limit under the assumption of having long codeword length. LDPC codes can be algebraically constructed in several ways [2] based on circulant matrices; these codes belong to a class of quasi-cyclic (QC) codes. In accordance with their quasi-cyclic structure, some efficient encoding methods [6] that involve circulant matrices and division techniques have been proposed.

One interesting method for algebraic construction of LDPC codes is based on affine permutation matrices [3] which are a generalization of QC-LDPC codes. APM-LDPC codes may be a good candidate to solve the memory problem due to their algebraic structures. In fact, the required memory for storing their parity-check matrices can be reduced by a factor $1/m$, when $m \times m$ affine permutation matrices are employed.

For some non-negative integers m, s, $0 \leq s \leq m-1$, the circulant permutation matrix (CPM) I_m^s, or briefly I^s when m is certain, is defined as the matrix $I^s = \left(\begin{array}{c|c} 0 & I_s \\ \hline I_{m-s} & 0 \end{array}\right)$, where I_{m-s} and I_s are the identity matrices of orders $m-s$ and s, respectively. It is clear that $I^0 = I$ is the identity matrix of order m. Moreover, we accept this convention that for $s = \infty$, I^s is the $m \times m$ zero matrix. Let $\mathbb{Z}_m = \{0, 1, \cdots, m-1\}$ be the ring of integers modulus of m and $\mathbb{Z}_m^* = \{a \in \mathbb{Z}_m | \gcd(a, m) = 1\}$ be the set of elements in $\mathbb{Z}_m$ which are prime relative to m. For non-empty bijection f on $\mathbb{Z}_m$, define I_m^f, or I^f when m is certain, as $m \times m$ binary matrix $(e_{i,j})_{m\times m}$ in which $e_{i,j} = 1$ if and only if $i = f(j)$. For empty function $\emptyset$ on $\mathbb{Z}_m$, we accept this convention that $I^\emptyset$ is the $m \times m$ zero matrix.

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In this case, I^f is briefly denoted by $I^{s,a}$ which is permutation matrix (with row and column indices from 0 to $m-1$) for which the only non-zero element in the first column is happen in position s, and each other column has shifted down within a positions, regard to the previous column. $I^{s,a}$ is called a *affine permutation* (APM) matrix with slope s and shift a. Clearly, the transpose of the matrix I^f is $I^{f^{-1}}$, where f^{-1} is the inverse of function f. The inverse of the function $f(j) = aj + s \pmod m$ is $f^{-1}(j) = a^{-1}j - a^{-1}s \pmod m$.

Now, for some positive integers v and k, $v \leq k$, let $S = (s_{i,j})$ and $A = (a_{i,j})$ be two $v \times k$ arrays of some elements belong to $\mathbb{Z}_m \cup \{\infty\}$ and $\mathbb{Z}_m^*$, respectively. By $I^{S,A}$, with slope matrix S and shift matrix A, we mean the following matrix.

$$I^{S,A} = \begin{pmatrix} I^{s_{1,1},a_{1,1}} & I^{s_{1,2},a_{1,2}} & \dots & I^{s_{1,k},a_{1,k}} \\ I^{s_{2,1},a_{2,1}} & I^{s_{2,2},a_{2,2}} & \dots & I^{s_{2,k},a_{2,k}} \\ \vdots & \vdots & \ddots & \vdots \\ I^{s_{v,1},a_{v,1}} & I^{s_{v,2},a_{v,2}} & \dots & I^{s_{v,k},a_{v,k}} \end{pmatrix} \tag{1}$$

where each $I^{s_{i,j},a_{i,j}}$ is an APM matrix of order m. The matrix $I^{S,A}$ can be considered as the parity-check matrix of a (v,k) LDPC code, called (v,k) APM-LDPC code, with CPM size m.

In the case of general LDPC codes, a significant amount of memory is needed to store their parity-check matrices. One of the advantage of APM-LDPC codes over other types of LDPC codes in hardware implementation is memory efficiency in encoding and decoding. In fact, by Lemma 1.1, if the locations of non zero elements in first two rows of each block matrix are determined, then the location of the remaining non zero elements will be determined. Therefore, like QC LDPC codes, the required memory for storing APM-LDPC codes with CPM size m can be reduced by a factor of $1/m$.

Lemma 1.1. If two first rows (or columns) of an APM matrix $I^{s,a}$ are known, then s and a (subsequently, the whole matrix $I^{s,a}$) can be determined uniquely.

2 Encoding Scheme for a Class of APM-LDPC Codes

In this section, we propose an efficient algorithm for encoding a class of APM-LDPC codes with a linear complexity. For convenience, we divide the parity-check matrix of APM-LDPC codes (1) into two parts: the information part H_I and the parity part H_P, i.e., $H = [H_I H_P]$. For efficient encoding, we restrict the parity part H_P of the parity-check matrix to an almost lower triangular matrix with additional constraints. Consider the parity-check of APM-LDPC codes in the following form.

$$H = \left(\begin{array}{c|c|cccccc} & I^{s_1,a_1} & I^{0,1} & 0 & 0 & \cdots & 0 & 0 \\ & 0 & I^{s_2,a_2} & I^{0,1} & 0 & \cdots & 0 & 0 \\ & \vdots & 0 & I^{s_3,a_3} & I^{0,1} & \cdots & 0 & 0 \\ H_I & I^{y,q} & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & \cdots & I^{0,1} & \vdots \\ & 0 & 0 & 0 & 0 & \cdots & I^{s_{v-1},a_{v-1}} & I^{0,1} \\ \hline & I^{x,p} & 0 & 0 & 0 & \cdots & 0 & I^{s_v,a_v} \end{array}\right) \tag{2}$$

where $I^{y,q}$ is located in the lth row block, for an integer l, $l \neq 1, v$, and H_I is the information part of H. Moreover, $I^{y,q}$, $I^{x,p}$, $I^{0,1}$ and I^{s_i,a_i}, $1 \leq i \leq v$, are $m \times m$ APM matrices. Using a modified Richardson-Urbank encoding method [5], this approach is derived from [6] for encoding of a class of QC LDPC codes having the parity-check matrices with a similar form as (2). Now, H is divided into the following form

$$H = \left(\begin{array}{c|c|c} A & B & T \\ \hline C & D & E \end{array}\right) \tag{3}$$

where A is $(v-1)m \times km$, B is $(v-1)m \times m$, T is $(v-1)m \times (v-1)m$, C is $m \times km$, $D = I^{x,p}$ is $m \times m$ and E is $m \times (v-1)m$. Using the Gaussian elimination, H is changed as follows.

$$H = \begin{pmatrix} A & B & T \\ ET^{-1}A + C & ET^{-1}B + D & 0 \end{pmatrix}$$

Now, $c = \begin{pmatrix} s & p_1 & p_2 \end{pmatrix}$, with systematic part s and parity parts p_1 and p_2, is a codeword if and only if $Hc^T = 0$. This implies the following equations.

$$\begin{cases} As^T + Bp_1^T + Tp_2^T &= 0 \\ (ET^{-1}A + C)s^T + (ET^{-1}B + D)p_1^T &= 0 \end{cases}$$

By the second equation, we have $p_1^T = \phi^{-1}(ET^{-1}A + C)s^T$, where $\phi = ET^{-1}B + D$ is a $m \times m$ nonsingular matrix. Now, p_2^T can be obtained by substitution p_1^T in the first equation. If $n = (k+v)m$ is the code length, then the overall complexity of computing p_1 is $O(n) + O(m^2)$ which decreases to the linear complexity $O(n)$, if ϕ is chosen as the identity matrix and n is large compared to m^2. Therefore, we must choice (x, p) and (y, q) such that ϕ becomes the identity matrix. However, it can be seen easily that the inverse of T in relation 3 is as follows.

$$T^{-1} = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 & 0 \\ I^{s_2,a_2} & I & 0 & \cdots & 0 & 0 & 0 \\ I^{(2,3)} & I^{s_3,a_3} & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ I^{(2,v-2)} & I^{(3,v-2)} & I^{(4,v-2)} & \cdots & I^{s_{v-2},a_{v-2}} & I & 0 \\ I^{(2,v-1)} & I^{(3,v-1)} & I^{(4,v-1)} & \cdots & I^{(v-2,v-1)} & I^{s_{v-1},a_{v-1}} & I \end{pmatrix}$$

in which, $I^{(i,j)}$, $i < j$, is defined as $I^{(i,j)} = I^{s_j,a_j} I^{s_{j-1},a_{j-1}} \cdots I^{s_i,a_i} = I^{s^{(i,j)},a^{(i,j)}}$, where $a^{(i,j)} = \prod_{k=i}^{j} a_k \bmod m$ and $s^{(i,j)} = s_j + \sum_{k=i}^{j-1} s_k a^{(k+1,j)} \bmod m$. Hence,

$$\begin{aligned} ET^{-1}B &= I^{s_v,a_v} \left[\begin{array}{ccccccc} I^{(2,v-1)} & I^{(3,v-1)} & I^{(4,v-1)} & \cdots & I^{(v-2,v-1)} & I^{s_{v-1},a_{v-1}} & I \end{array}\right] B \\ &= I^{s_v,a_v} I^{(2,v-1)} I^{s_1,a_1} + I^{s_v,a_v} I^{(l+1,v-1)} I^{y,q} \\ &= I^{(1,v)} + I^{(l+1,v)} I^{y,q} = I^{s^{(1,v)},a^{(1,v)}} + I^{s^{(l+1,v)},a^{(l+1,v)}} I^{y,q} \\ &= I^{s^{(1,v)},a^{(1,v)}} + I^{s^{(l+1,v)}+ya^{(l+1,v)},qa^{(l+1,v)}} \end{aligned}$$

and

$$\phi = ET^{-1}B + D = I^{s^{(1,v)},a^{(1,v)}} + I^{s^{(l+1,v)}+ya^{(l+1,v)},qa^{(l+1,v)}} + I^{x,p}.$$

Now, a proper choice of (x, p) and (y, q) can be summarized as the following theorem.

Theorem 2.1. If (x, p) and (y, q) in the parity-check matrix H, given by (2), are chosen in the following form, then matrix $\phi = ET^{-1}B + D$ becomes the identity matrix.

1. $(x, p) = (s^{(1,v)}, a^{(1,v)}) \bmod m$ and $(s^{(l+1,v)} + ya^{(l+1,v)}, qa^{(l+1,v)}) = (0, 1) \bmod m$, which implies $(y, q) = (a^{(l+1,v)})^{-1}(-s^{(l+1,v)}, 1) \bmod m$.
2. $(s^{(1,v)}, a^{(1,v)}) = (0, 1)$ and $(x, p) = (s^{(l+1,v)} + ya^{(l+1,v)}, qa^{(l+1,v)}) \bmod m$.

Consequently, the encoding algorithm for APM-LDPC codes having parity-check matrices in form (2) with (x, p) and (y, q) satisfied in Theorem 2.1, are summarized as follows.

Algorithm. Encoding for a Class of APM-LDPC Codes

S1. Compute As^T and Cs^T.

S2. Compute $ET^{-1}As^T = \left[\begin{array}{cccc} I^{(2,v)} & I^{(3,v)} & \cdots & I^{(v-1,v)} \end{array}\right.$ $\left. I^{s_v,a_v} \right] As^T$.

S3. Compute p_1^T by $p_1^T = ET^{-1}As^T + Cs^T$.

S4. Compute p_2^T by $Tp_2^T = As^T + Bp_1^T$.

The complexity of the above Algorithm is $O(n)$, where n is the length of APM-LDPC codes.

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Interpreting Absorbtion of radiation by affecting the Bohr operator on Meijer's G-function

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Abstract

The Mellin-Barnes integral which is a special type of Mellin inversion integral is often used to analyze the behavior of special functions, here Meijer's G-function. By using properties of Meijer's G-function, we demonstrate that absorbtion or emission of spectral lines in Hydrogen atom may be described by Meijer's G-functions.

Keywords: Radial function, Meijer's G-function, Integral transform, Mellin transform, Special function.
Mathematics Subject Classification [2010]: 33C60, 44Axx

1 Introduction

The Mellin transform and its inverse are useful tool in mathematics. It is used in many diverse areas of mathematics, such as the study of special functions, the study of difference equations, asymptotic expansions, and analytic number theory. In this paper to be familiar with Meijer's G-function and to prove one theorem related to this function one needs to pay attention the form of inverse Mellin transform for two examples. Since they are integral representations in which gamma functions appear as main part of integrand, Mellin-Barnes integrals and Meijer's G-functions are their generalized form.

Definition 1.1. The Mellin transform, of a function f(x) defined on the interval $[0, \infty)$ is given by

$$\mathcal{M}\{f(x)\} = \int_0^\infty f(x)x^{s-1}dx, \tag{1}$$

and its inverse integral is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\mathcal{M}\{f(x)\}ds \tag{2}$$

Example 1.2. Consider the function $f(x) = e^{-x}$

$$\mathcal{M}\{e^{-x}\} = \int_0^{+\infty} x^{s-1}e^{-x}dx = \Gamma(s)$$

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Solution. So for this function the Mellin transform is just the definition of the Γ function

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}ds \tag{3}$$

Example 1.3. Consider the function $f(x) = (1+x)^{-1}$

Solution.

$$\mathcal{M}\{(1+x)^{-1}\} = \int_0^{+\infty} x^{s-1}(1+x)^{-1}dx.$$

By the substitution $u = \frac{x}{1+x}$

$$\mathcal{M}\{(1+x)^{-1}\} = \int_0^{+\infty} x^{s-1}(1+x)^{-1}dx = \int_0^1 u^{s-1}(1-u)^{1-s-1}du = B(s,1-s)$$

which $B(s,1-s)$ is Beta function, we note that the Beta function can be represented in terms of the Gamma function $B(s,1-s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1)}$

$$(1+x)^{-1} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1)}x^{-s}ds$$

A special type of an inverse Mellin integral called the Mellin-Barnes integral is often used to analyze the behavior of special functions. A Mellin-Barnres integral is an inverse Mellin integral involving Gamma functions in the integrand. Integrals of this form lead to a surprising variety of special functions. Most generally it is of the form

$$\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\ \prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\ \prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s ds \tag{4}$$

Definition 1.4. A definition of the Meijer G-function is given by the following path integral in the complex plane, called Mellin-Barnes type integral [1]:

$$G_{p,q}^{m,n}\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}|z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\ \prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\ \prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s ds. \tag{5}$$

The Meijer's G-functions (MGFs) are an admirable family of functions which each of them is determined by finitely many indices. Analytic manipulations and numerical computations involving Meijer's G-functions have been provided by the software packages such as *Mathematica, Maple,* and *Matlab* [2]. Choosing $m = 1, n = 0, p = 0$, $q = 1$, and $b_1 = 0$ we have

$$G_{0,1}^{1,0}\left(\begin{matrix}-\\ 0\end{matrix}|z\right) = \frac{1}{2\pi i}\int_L \Gamma(-s)z^s ds. \tag{6}$$

Changing variable $s \to -s$ and comparing (3) with (6) gives

$$G_{0,1}^{1,0}\left(\begin{matrix}-\\ 0\end{matrix}|x\right) = e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}ds \tag{7}$$

Based on the definition, the following basic property is easily derived:

$$z^{\alpha}\, G_{p,q}^{m,n}\left(\begin{matrix}\mathbf{a_p}\\ \mathbf{b_q}\end{matrix}|z\right) = G_{p,q}^{m,n}\left(\begin{matrix}\mathbf{a_p}+\alpha\\ \mathbf{b_q}+\alpha\end{matrix}|z\right), \tag{8}$$

where the multiplying term z^{α} changes the parameters of the G-function. In [3,4] we deduce some useful relations upon which the radial functions of the Hydrogen atom may be represented by summation of a number of Meijer's G-functions.

2 Main results

An operator acting on a function generally maps it into another function.

Definition 2.1. The Bohr operator is represented by $\hat{B}$ and defined as follows [4]:

$$\hat{B}[(\frac{1}{r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{r_B})] = [(\frac{1}{2r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{2r_B})] \tag{9}$$

The meaning of square $\hat{B}^2$ of the Bohr operator $\hat{B}$ is obvious:

$$\hat{B}^2[(\frac{1}{r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{r_B})] = \hat{B}[(\frac{1}{2r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{2r_B})] = [(\frac{1}{3r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{3r_B})]$$

The operator $\hat{B}^n$ is formed by $\hat{B}$ acting n times:

$$\hat{B}^n[(\frac{1}{r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{r_B})] = \hat{B}\hat{B}\hat{B}...\hat{B}[(\frac{1}{r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{r_B})] = \hat{B}^{n-1}(\hat{B}[(\frac{1}{r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{r_B})])$$

with $r_B = \frac{4\pi\epsilon_0\hbar^2}{me^2}$, where r_B is the Bohr radius.

Theorem 2.2. *Let* $R_{10} = 2(\frac{1}{r_B})^{\frac{3}{2}} e^{-\frac{r}{r_B}} = 2(\frac{1}{r_B})^{\frac{3}{2}} G_{0,1}^{1,0}(\begin{matrix}-\\ 0\end{matrix}|\frac{r}{r_B})$ *be the ground radial state of the Hydrogen atom, then* $\hat{B}^n(\frac{r}{r_B})^n$ *maps the ground radial state* R_{10} *into multiple of radial state* $R_{n+1,n}$ *such that*

$$\hat{B}^n(\frac{r}{r_B})^n R_{10} = \hat{B}^n[(\frac{r}{r_B})^n 2(\frac{1}{r_B})^{\frac{3}{2}} e^{-\frac{r}{r_B}}] = 2[\frac{1}{(n+1)r_B}]^{\frac{3}{2}}[\frac{r}{(n+1)r_B}]^n e^{-\frac{r}{(n+1)r_B}} \tag{10}$$

Proof. The above statement holds when n=1

$$\hat{B}[(\frac{r}{r_B})R_{10}] = \hat{B}[2(\frac{1}{r_B})^{\frac{3}{2}}(\frac{r}{r_B})e^{-\frac{r}{r_B}}] = 2(\frac{1}{2r_B})^{\frac{3}{2}}(\frac{r}{2r_B})e^{-\frac{r}{2r_B}} = \sqrt{3}R_{21}$$

Now by induction we may easily show that if $\hat{B}^n(\frac{r}{r_B})^n$ maps the ground radial state R_{10} into multiple of radial state $R_{n+1,n}$, then $\hat{B}^{n+1}(\frac{r}{r_B})^{n+1}$ maps the ground radial state R_{10} into multiple of radial state $R_{n+2,n+1}$.

□

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The authors declare that they have no competing interests. Minoo Nasiri Hamed is currently a M.Sc. student under supervision of the Dr. Amir Pishkoo.

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