# ADVANCED ENGINEERING MATHEMATICS

SIXTH EDITION

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C.Ray Wylie
LouisC. Barrett

1995
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**C. Ray Wylie**
Professor Emeritus, Furman University

**Louis C. Barrett**
Professor Emeritus, Montana State University

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This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 0 VNH VNH 9 0 9 8 7 6 5

ISBN 0-07-072206-4

This book was set in Times Roman by York Graphic Services, Inc.
The editors were Maggie Lanzillo and Scott Amerman;
the designer was Jo Jones;
the production supervisor was Denise L. Puryear.
New drawings were done by Fine Line Illustrations, Inc.
Von Hoffmann Press, Inc., was printer and binder.

Library of Congress Cataloging-in-Publication Data

Wylie, Clarence Raymond, (date).
Advanced engineering mathematics / C. Ray Wylie, Louis C. Barrett. — 6th ed.
p. cm.
Includes index.
ISBN 0-07-072206-4
1. Mathematics. I. Barrett, Louis C. II. Title.
QA401.W9 1995
515—dc20 94-47667

**INTERNATIONAL EDITION**



When ordering this title, use ISBN 0-07-113543-X.

# ADVANCED ENGINEERING MATHEMATICS

# ABOUT THE AUTHORS

**C. Ray Wylie,** professor emeritus of Furman University in Greenville, South Carolina, received a B.A. degree (in mathematics) and a B.S. degree (in chemical engineering) from the College of the City of Detroit, now Wayne State University. He earned an M.S. degree in 1932 and a Ph.D. degree in 1934, both in mathematics, from Cornell University. In 1994 Furman University conferred an honorary D.Sc. degree on Professor Wylie.

From 1934 to 1946 Professor Wylie was a member of the Department of Mathematics of Ohio State University. During World War II, while on leave from Ohio State, he served as a civilian engineer in the Propeller Laboratory at Wright-Patterson Air Force Base. After the war, he served as chairman of the Department of Mathematics and acting dean of the College of Engineering at the base's Air Institute of Technology. He left the Air Institute in 1948 to become chairman of the Department of Mathematics at the University of Utah, in which capacity he served until 1967. Upon his retirement from the University of Utah, he became chairman of the Department of Mathematics at Furman University. From 1971 until his retirement from Furman he was William R. Kenan, Jr., Distinguished Professor of Mathematics.

Ray Wylie has published technical papers in applied mathematics and geometry. In addition to *Advanced Engineering Mathematics*, he is the author or co-author of twelve other books, including the McGraw-Hill texts *Differential Equations*, *Foundations of Geometry*, and *Introduction to Projective Geometry*. He is listed in *American Men and Women of Science* and *Who's Who in America*, and is a member of Phi Beta Kappa and Sigma Xi.

**Louis C. Barrett** received a B.A. in mathematics, an M.S. in mathematics and physics in 1951, and a Ph.D. in mathematics and physics from the University of Utah in 1956. He is currently professor emeritus of the Department of Mathematics at Montana State University, where he was professor and department head from 1967 until 1972. Prior to his service at Montana State, Professor Barrett was professor and chairman of the Department of Mathematics at Clarkson College of Technology (1965–1966); associate professor, then professor and department head of the Mathematics Department at South Dakota School of Mines and Technology (1957–1965); associate professor in the mathematics department of Arizona State College (1956–1957); and instructor in the mathematics department of the University of Utah (1953–1956).

In addition to his professorships, Louis Barrett has served as consultant to the Holloman Air Development Center in New Mexico and the Naval Weapons Center in China Lake, California. He has also been a lecturer in eight National Science Foundation Institutes and for the Mathematical Association of America.

Professor Barrett has written numerous technical reports in army and navy contract research. With C. Ray Wylie, he has co-authored the fifth edition of *Advanced Engineering Mathematics*.

# CONTENTS

# PREFACE

The first edition of this book was written to provide an introduction to those branches of postcalculus mathematics with which average analytical engineers or physicists need to be familiar in order to carry on their own work effectively and keep abreast of current developments in their fields. In the present edition, as in each of the preceding editions, much of the material has been rewritten, but the various additions, deletions, and refinements have been made only because they seem to contribute to the achievement of this goal.

## CONTENT OF THE BOOK

Because ordinary differential equations are probably the most immediately useful part of postcalculus mathematics for the student of applied science, and because the techniques for solving simple ordinary differential equations stem naturally from the techniques of calculus, this book begins with a chapter on ordinary differential equations of the first order and their applications. This chapter is followed by one which develops the theory and applications of linear differential equations, especially those with constant coefficients. In the present edition, this chapter has been augmented by two major sections, one dealing with the series solution of linear differential equations, the other covering the method of Frobenius. In earlier editions, this material was postponed until the chapter on Bessel functions. It is now included at this early stage so that the first four chapters can be used as a basis for a self-contained course in ordinary differential equations.

Next, in Chap. 3, to prepare for a discussion of linear differential systems with constant coefficients, there is an introduction to linear algebra. Although this material will be used extensively in Chap. 4, it can be omitted by students who are already familiar with matrices, determinants, and the solution of simultaneous linear algebraic equations. To this chapter there has now been added an introductory section on complex variables. This material will be used in many places throughout the rest of the book, but it can be omitted by students familiar with the properties of complex variables through De Moivre's theorem and the Euler formulas.

Chapter 5 is devoted to numerical methods, and it covers such topics as finite differences, interpolation formulas, numerical differentiation and integration, the numerical solution of ordinary differential equations, featuring the various Runge-Kutta methods and Milne's method, and difference equations. A section on least-squares has been restored in this edition and a new section on the $G$ and $Z$ transformations has been added. It is hoped that the material in this chapter will provide a useful background in classical finite differences on which a more extensive course in computer-oriented numerical analysis can be based. Chapter 6 is a new chapter that has not been included in any of the earlier editions of this text. It deals with the problem of determining such properties of the solutions of a differential equation as periodicity and stability without finding the solutions themselves.

Chapter 7 is devoted to the application to mechanical systems and electric circuits of the ideas developed in the first five chapters. As in the earlier editions, the mathematical identity of these fields is emphasized. The section on systems with more than one degree of freedom has been divided into a section on systems with several degrees of freedom and a section on systems with many degrees of freedom. This new section features the interplay between differential equations and difference equations, with emphasis on wave filters and wave traps. A final section on electromechanical analogies has been restored from the second edition.

Motivated by the work on periodic phenomena in Chap. 7, Fourier series and their applications are discussed in Chap. 8. In particular, in this edition more emphasis has been placed on the use of the jumps of a function and its derivatives to eliminate the need to integrate to determine the Fourier coefficients of a function. Chapter 9 is a new chapter containing in expanded form the material on Fourier integrals that was grouped with Fourier series in earlier editions. The Fourier integral is introduced as the limit of a Fourier series, and then a variety of Fourier transforms, with their basic properties obtained from it. This chapter contains a new section on the Gibbs phenomenon and the convergence of Fourier series and Fourier integrals at the jumps of a function. There is also a new section on singularity functions and their fundamental properties.

In Chapter 10, the Laplace transformation is introduced as a natural outgrowth of the Fourier integral and Fourier transforms. In this edition, the presentation of the requisite theory is a little less abrupt than it was in earlier editions, and examples of particular transforms are given very early. The chapter concludes with a new section in which the nature and properties of Laplace transforms, Fourier transforms, and Z- transforms are compared and contrasted.

Chapters 11 and 12 deal, respectively, with differential equations and boundary-value problems, and Bessel functions and Legendre polynomials. Here, Fourier series play a prominent role in satisfying initial and boundary conditions and provide motivation for the discussion and use of expansions in terms of more general systems of orthogonal functions. In this edition, a new section on the generating functions of $J_n$ and $I_n$ illustrate their use in obtaining many of the identities of these functions. New examples in these chapters include incomplete systems of orthogonal functions, interface Sturm-Liouville systems, and the use of Legendre polynomials in potential problems.

In Chaps. 13 and 14 we return to the subject of linear algebra and discuss vector spaces, linear transformations, the existence of Green's functions for systems of differential equations, and further properties of matrices and their eigenvalues and eigenvectors. An important addition to Chap. 14 is a section on the discrete and fast Fourier transforms, an important topic in the field of signal processing. This work is followed by a chapter on vector analysis developed in the traditional geometric way, much as it was in the fifth edition. New material here includes some interesting topics in differential geometry. Chapter 16 deals with the calculus of variations and its applications to dynamics. New material here includes a section on Hamilton's equations.

The last four chapters provide an introduction to the theory of functions of a complex variable, with applications to fluid mechanics and two-dimensional potential theory, the evaluation of real definite integrals, the complex inversion integral of Laplace transformation theory, stability criteria, conformal mapping, and the Schwarz-Christoffel transformation. The only significant difference between these chapters and the corresponding chapters in the last edition is that the introductory material through De Moivre's equation and the formal content of Euler's formulas has been moved ahead and now appears at the beginning of Chap. 3.

This book falls naturally into three main subdivisions. The first twelve chapters constitute a reasonably self-contained treatment of ordinary and partial differential equations and their applications. The next four chapters, 13 through 16, cover the related areas of linear algebra, vector analysis, and the calculus of variations. The last four chapters, 17 through 20, cover the elementary theory and applications of functions of a complex variable.

## FEATURES

This book contains enough material for a two-year course in applied mathematics. However, since we have tried to keep important subjects concentrated in specific chapters rather than diffused throughout the book, selected chapters are well-adapted for use as a text for any of several shorter courses. Following this preface, in the section headed "To the Instructor," there is a detailed Planning Guide showing how this text can be used for a number of courses.

In this edition, as in each of the others, every effort has been made to keep the presentation detailed and clear while at the same time maintaining acceptable standards of precision and accuracy. To achieve this goal, more than the usual number of worked examples and carefully drawn figures have been included, and in every development there has been a conscious attempt to make the transition from step to step so clear that a serious student, working with paper and pencil, should seldom be held up very long. Many new exercises have been added in this edition, and there are now more than 5000. Hints are given in many of the exercises, and answers to the odd-numbered ones are given at the end of the book. A manual containing the answers to the even-numbered ones is available for instructors using this text. More detailed solutions to the odd-numbered exercises are provided in a Student Solutions Manual. As in earlier editions, words and phrases defined informally in the body of the text are set in bold-faced type, and italics are used frequently for emphasis. Illustrative examples are consistently set in type of a different size than that used for the main body of the text.

## ACKNOWLEDGMENTS

The indebtedness of the authors to their colleagues, students, and former teachers is too great to catalog, and to all who have given help and encouragement through the years we can offer only a most inadequate acknowledgment of our appreciation. In particular, we are deeply grateful to those users of this book who have been kind enough to write us of their impressions and criticisms of the first five editions and their suggestions for an improved sixth edition and to the following McGraw-Hill reviewers: Barbara Bohannon, Hofstra University; Michael Bryant, University of Texas at Austin; Chung-wu Ho, Southern Illinois University; J. Lubliner, University of California at Berkeley; Gordon Melrose, Old Dominion University; Keith B. Olson, Montana College of Mineral Science and Technology; Mauro Pierucci, San Diego State University; Michael E. Ryan, State University of New York at Buffalo; Duane W. Storti, University of Washington; Mo Tavakoli, Chaffey College; and Arnold Villone, San Diego State University. We also appreciate greatly the invaluable advice and assistance that our editor, Maggie Lanzillo, gave so generously during both the preparation of the manuscript and the subsequent editorial process. Thanks, too, to the production team at McGraw-Hill who did such an admirable job of guiding this project from manuscript to bound book: Scott Amerman, editing supervisor; Denise Puryear, production supervisor; and Jo Jones, designer. Finally, we must express our gratitude to our wives, Ellen and Betty, not only for their assistance and encouragement in this project but for their patience and understanding during our long preoccupation with the manuscript.

C. Ray Wylie

Louis C. Barrett

# TO THE INSTRUCTOR

This book contains ample material for a two-year sequence in applied mathematics. It has been written so that important subjects are concentrated in specific chapters and are not covered partially in several different places. By the judicious selection of particular chapters, it is thus readily adaptable as a text for a number of short courses. To assist you in making maximum use of the book, we have prepared the accompanying Course Modules and Planning Guide. It identifies modules suitable for a variety of one-term courses, as well as combinations of modules on which several one-year sequences can be based. It also indicates prerequisite relations for instructors planning their own sequences.

One new feature of this edition is the inclusion in each chapter of an introductory, overview section alerting the student to the material to be covered in the chapter, pointing out portions that may be extensions of topics discussed earlier, and indicating where the new material will be used later in the book. We hope that you will encourage your students to orient themselves to the work in each chapter by reading these introductory statements carefully.

This book contains over 5000 exercises, many of them new. As in each of the other editions, many of these contain extensions of topics in the text or interesting new results that could not be included within the chapter because of space limitations. Since the difficulty of an exercise is often a subjective judgment, we have made no attempt to distinguish "hard" problems from "easy" ones, nor to arrange the exercises in an assumed order of increasing difficulty. Nonetheless, nearly every set begins with a few routine, practice problems. Answers to the odd-numbered exercises are listed in the back of the book. A manual containing the answers to the even-numbered exercises is available for the instructor, and a manual containing solutions to the odd-numbered exercises is available for the student.

C. Ray Wylie

Louis C. Barrett

## COURSE MODULES AND PLANNING GUIDE

| PREREQUISITES: Calculus | Ordinary Differential Equations | Complex Numbers | Fourier Analysis | Calculus-Based Physics | COURSE | 1. Ordinary Differential Equations of the First Order | 2. Linear Differential Equations | 3. Complex Numbers and Linear Algebra | 4. Simultaneous Linear Differential Equations | 5. Numerical Methods | 6. Descriptive Theory of Ordinary Differential Equations | 7. Mechanical Systems and Electrical Circuits | 8. Fourier Series |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 1 - Ordinary Differential Equations | | | | | Section 5.6 | | | |
| | | | | | 2 - Linear Algebra | | | | | | | | |
| | | | | | 3 - Mathematical Applications and Fourier Analysis | Sections 1.14 & 1.15 | Sections 2.11 & 2.12 | | | | | | |
| | | | | | 4 - Partial Differential Equations and Boundary Value Problems | | | | | | | | |
| | | | | | 5 - Numerical Methods | | | | | | | | |
| | | | | | 6 - Descriptive Theory of Ordinary Differential Equations | | | | | | | | |
| | | | | | 7 - Vector Analysis | | | Section 3.2 | | | | | |
| | | | | | 8 - Calculus of Variations | | | | | | | | |
| | | | | | 9 - Complex Variables | | | Section 3.1 | | | | | |

ACADEMIC YEAR SEQUENCES

Ordinary Differential Equations: Course 1
Applied Mathematics: Courses 3 & 4
Applied Analysis: Courses 5, 6, 7, & 8
Complex Variables: Course 9

| CHAPTER | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9. Fourier Integrals and Fourier Transforms | 10. The Laplace Transformation | 11. Partial Differential Equations | 12. Bessel Functions and Legendre Polynomials | 13. Vector Spaces and Linear Transformations | 14. Applications and Further Properties of Matrices | 15. Vector Analysis | 16. Calculus of Variations | 17. Analytic Functions of a Complex Variable | 18. Infinite Series in the Complex Plane | 19. Theory of Residues | 20. Conformal Mapping | OPTIONAL TOPICS |
| | | | | | | | | | | | | Sections: 1.14, 1.15, 2.11, 2.12, & 5.6 |
| | | | | | | | | | | | | Sections: 14.2 & 14.4 |
| | | | | | | | | | | | | |
| | | | | | Section 14.4 | | | | | | | Sections: 11.4 & 11.9 |
| | | Sections 11.4 & 11.9 | | | Section 14.2 | | | | | | | |
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# TO THE STUDENT

This book has been written to help you in your development as an applied scientist, whether an engineer, physicist, chemist, or mathematician. It contains material that will be of great use to you, not only in the technical courses you have yet to take, but also in your profession after graduation, as long as you deal with the analytical aspects of your field.

We have tried to write a book which you will find not only useful but also relatively easy, at least as easy as a book about advanced mathematics can be. There is a good deal of theory in it, for it is the theoretical portion of a subject which is the basis for the nonroutine applications of tomorrow. But nowhere will you find theory for its own sake, interesting and legitimate as this may be to a pure mathematician. Our theoretical discussions are designed to illuminate principles, to indicate generalizations, to establish limits within which a given technique may or may not be safely used, or to point out pitfalls into which one might otherwise stumble. On the other hand, there are many applications, illustrating with the material at hand the usual steps in the solution of a physical problem: formulation, manipulation, and interpretation. These examples are, without exception, carefully set up and completely worked, with all but the simplest steps included. Study them carefully, with paper and pencil at hand, for they are an integral part of the text. If you do this, you should find the exercises, though challenging, still within your ability to work.

A new feature in this edition is the inclusion in each chapter of an introductory section, giving an overview of the material to be covered, pointing out where we may have encountered some of it before, and indicating where and how it will be used later in the book. Be sure to orient yourself to the work and purpose of each new chapter by reading carefully these introductory sections. Another new feature is the inclusion of subtitles for many of the important examples. These will alert you to the main point of each example and, perhaps more importantly, help you to identify examples to which you may later wish to refer. You will find them listed inside the covers of this book.

Terms defined informally in the body of the text are always indicated by the use of **bold-faced type.** *Italic type* is used for emphasis, much as verbal stress is used when speaking. We suggest that you read each section through for the main ideas before you concentrate on filling in any of the details. You will probably be surprised at how many times a point which seems to hold you up in one paragraph will be explained in the next as the discussion unfolds.

Because this book is long and contains material suitable for various courses, your instructor may begin with any of a number of chapters. However, the overall structure of the book is this: The first twelve chapters are devoted to the general theme of ordinary and partial differential equations and related topics. Here you will find the basic analytical techniques for solving the equations in which physical problems must be formulated when continuously changing quantities are involved. Chapters 13 through 16 deal with the somewhat related topics of linear algebra and matrix theory, vector analysis, and the calculus of variations. Finally, Chaps. 17 through 20 provide an introduction to the theory and applications of functions of a complex variable.

It has been gratifying to receive letters from students who have used this book giving us their reactions to it, pointing out errors and misprints in it, and offering suggestions for its improvement. Should you be inclined to do so, we would be glad to hear from you also.

Finally, we hope that you will find this book in some sense a friendly book. It was written with you in mind, as someone with whom we would like to share not only our knowledge but our enthusiasm. We have written almost entirely in the first person plural. Never are you referred to obliquely and impersonally as "the student." Our use of the word "we" indicates that we feel we are exploring something interesting with you. And now good luck and every success.

C. Ray Wylie

Louis C. Barrett

# ADVANCED ENGINEERING MATHEMATICS

# CHAPTER 1

# ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

*Differential equations,* that is, equations involving rates of change, provide an indispensable tool for anyone studying continuously varying phenomena, such as velocities and accelerations or electric currents. In this chapter, after a review of the concepts of *variable* and *function* (Sec. 1.1), we identify the major types of differential equations (Sec. 1.2). Then, as a necessary foundation, we discuss the general notions of *solution* and *family of solutions* (Secs. 1.3–1.5) and the *existence* and *uniqueness* of solutions (Sec. 1.6).

With these ideas in mind, we begin the study of methods of solving differential equations by learning how to solve all the major forms of ordinary first-order equations. These include *exact equations* (Sec. 1.7), *equations solvable by integrating factors* (Sec. 1.8), *separable equations* (Sec. 1.9), *homogeneous equations* (Sec. 1.10), *linear equations* (Sec. 1.11), and several more special types of equations (Secs. 1.12 and 1.13).

Every section contains a number of examples, but significant practical applications we leave to Secs. 1.14 and 1.15. This organization makes it possible to use the bulk of this chapter for a portion of a course in the theory of differential equations. At the same time, the work on *orthogonal trajectories* in Sec. 1.14 and 8 examples and 120 exercises in Sec. 1.15 provide convincing evidence of the utility of differential equations and ample material for practice in using them to formulate and solve physical problems ranging from heat conduction and fluid flow to orbital motion.

Prerequisite for this chapter: single-variable calculus.

Prerequisite for Sec. 1.15: a calculus-based physics course and general chemistry.

## 1.1 VARIABLES AND FUNCTIONS

The variety and complexity of the problems which confront today's engineers and scientists have increased remarkably in recent years and, if anything, the increase seems to be accelerating. As a consequence, not only is there a continuing demand for more and more effective computers and better and better experimental facilities, but so too is there a growing need for more, and more thoroughly understood, mathematics to support the whole scientific enterprise. Mathematics demands clarity of thought and clarity of exposition, and so as we begin our study it seems proper that we review briefly the raw material of all our work, *variables* and *functions*.

A **variable** is a symbol identifying elements of a given set. A *function* can be thought of as a rule relating the elements of one set to the elements of a second set, possibly the same as the first. The rule defining the functional values is often a formula of some kind, although other modes of definition are possible. Variables that designate values for which a function is defined are called **independent variables,** and, collectively, these values form the **domain** of the function. Variables which identify values of a function are called **dependent variables** and, collectively, these values form the **range** of the function.

Functions are usually denoted by single letters. For each $x$ in the domain of a function $f$, the *value* of $f$ at $x$ is denoted by $f(x)$. As is customary, we shall often use the notation $f(x)$ not only to denote a value of $f$ but also to name the function itself, although this is notationally inaccurate. Depending on the domain of a function, which is never empty, the variable $x$ may stand for a number or any other object for which the function is defined.

**EXAMPLE 1**

Since for all real values of $x$, $2 + \cos \pi x \geq 1$, the expression $f(x) = \ln (2 + \cos \pi x)$ defines a function $f$ of the variable $x$ having the set of all real numbers as its domain. The value of $f$ at 1 is given by $f(1) = \ln (2 + \cos \pi) = \ln 1 = 0$, and $f(2) = \ln 3$ is the value of $f$ at 2. In this example, the independent variable $x$ may be replaced by any real number. The range of $f$ is made up of all values of the dependent variable $y = f(x)$, namely, all numbers $y$ such that $0 \leq y \leq \ln 3$.

**EXAMPLE 2**

Let the domain of a function $g$ defined by $g(x) = \int_0^1 x(t)\,dt$ be the set of all functions continuous on the closed interval $[0, 1]$. If $x_1(t) = t$ on $[0, 1]$, the value of $g$ at $x_1$ is given by $g(x_1) = \int_0^1 t\,dt = \frac{1}{2}$; while if $x_2(t) = \cos \pi t$ on $[0, 1]$, $g(x_2) = \int_0^1 \cos \pi t\,dt = 0$. Here $x$ can be replaced by any function that is continuous on $[0, 1]$. Once $x$ is chosen, the corresponding value of the dependent variable $y = g(x)$ is given by the integral of $x$ from 0 to 1.

Note that in Example 1 the independent variable $x$ stands for a real number, but that in Example 2 it stands for a function. A function whose domain is a set of functions is called a **functional.**

Frequently, values of a given function may be found by making the appropriate substitutions for variables in some analytic expression. Such an expression is commonly called a **representation** of the function on that part of the functional domain over which the representation yields the corresponding functional values. Often, as in Examples 1 and 2, all values of the function are determined by a single representation, but this need not be the case. In a variety of physical problems the very nature of the function requires different representations on different subsets of the domain.

**EXAMPLE 3**

At time $t = 0$, a unit voltage is suddenly introduced into an electric circuit, as suggested in Fig. 1.1*a*. This voltage is represented for $t < 0$ by the expression $E(t) = 0$ but represented for $0 \le t$ by the expression $E(t) = 1$. Thus on the domain $-\infty < t < \infty$ the voltage $E$ is given by the two representations

$$E(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \end{cases}$$

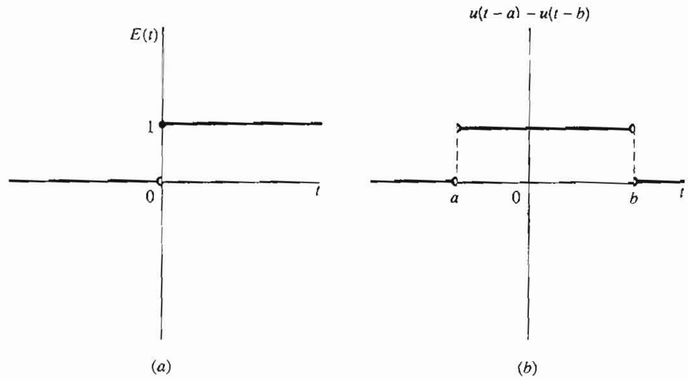



**Figure 1.1**
Graphs of (*a*) a unit step function voltage; (*b*) a unit pulse of duration $b - a$.

Example 3 illustrates the important fact that whereas the functions we dealt with in calculus were usually either continuous on their domains or had only removable discontinuities, in applied mathematics we must often consider functions with one or more nonremovable discontinuities. Typically, these discontinuities will be what are called *jumps*.

**DEFINITION 1** A function $f$ has a **jump** $J$ at $x = s$ if and only if the respective right- and left-hand limits $f(s^+)$ and $f(s^-)$ exist and

$$J = f(s^+) - f(s^-)$$

According as $J$ is positive or negative, the jump is said to be **upward** or **downward.** At a point where $f$ is continuous, the jump $J$ is of course zero. The function $E$ of Example 3 has an upward jump $J = 1$ at $t = 0$ and for all practical purposes is represented by the simple but very important **unit step function** $u(t)$, defined by

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases}$$

The unit step function will appear repeatedly in the work ahead of us.

A difference of two unit step functions

$$u(t-a) - u(t-b) = \begin{cases} 0 & t < a \\ 1 & a < t < b \qquad a < b \\ 0 & b < t \end{cases}$$

shown in Fig. 1.1$b$ as a unit pulse of duration $b - a$, is often used as an *analytic filter* to isolate a desired segment of a function. Hence, if $f(t)$ is a function whose domain includes the interval $(a, b)$, the product

$$f(t)[u(t-a) - u(t-b)]$$

represents the "filtered" function defined for all $t$ in the domain of $f$ by

$$\begin{cases} 0 & t < a \\ f(t) & a < t < b \\ 0 & b < t \end{cases}$$

In other words, the **filter function** $u(t-a) - u(t-b)$ reduces $f$ to zero outside the "passband" $a < t < b$ and reproduces $f$ exactly within the passband.

**EXAMPLE 4**

The absolute value function $|x|$ has the set of all real numbers as its domain. Its values are computed by using the definition

$$|x| = \begin{cases} -x & x \le 0 \\ x & 0 \le x \end{cases}$$

which expresses the function in terms of two alternative representations, one on $(-\infty, 0]$, the other on $[0, \infty)$. In particular, since $-3 < 0$, $|-3| = -(-3) = 3$.

The square root function $\sqrt{x}$ is always nonnegative and has the interval $[0, \infty)$ as its domain. Thus the domain of $\sqrt{x^2}$ is $(-\infty, \infty)$. In fact, $\sqrt{x^2} = |x|$ and of course $\sqrt{(-3)^2} = 3$.

The radical expression $\sqrt{x^4 - 2x^2}$ is undefined at $x = 1$, so it cannot by itself define a function. What is lacking is a suitable domain. By convention, this is to be taken as large as possible when not otherwise specified. The intended domain of $\sqrt{x^4 - 2x^2}$ is therefore the set of all real numbers for which the radicand $x^4 - 2x^2$ is nonnegative. Solving the inequality $x^2(x^2 - 2) \ge 0$, we find the domain to be the union of the intervals $(-\infty, -\sqrt{2}]$ and $[\sqrt{2}, \infty)$ and the isolated point $x = 0$. For every $x$ in this domain.

$$\sqrt{x^4 - 2x^2} = \sqrt{x^2(x^2-2)} = \sqrt{x^2}\sqrt{x^2-2} = |x|\sqrt{x^2-2}$$

hence our function is defined by the three representations

$$\sqrt{x^4 - 2x^2} = \begin{cases} -x\sqrt{x^2-2} & x \le -\sqrt{2} \\ 0 & x = 0 \\ x\sqrt{x^2-2} & \sqrt{2} \le x \end{cases}$$

Its functional values can now be computed by simply using the appropriate representation. For instance, its value at $x = -3$ is $-(-3)\sqrt{(-3)^2 - 2} = 3\sqrt{7}$. This is also its value at $x = 3$ because the value of the function remains unchanged if $x$ is replaced by $-x$.

**EXAMPLE 5**

A function $y$ has the set of positive integers $N$ as its domain, and for each $n$ contained in $N$, $y(n) = 2 + \cos n\pi - \sin[(2n-1)\pi/2]$. Among its values are $y(1) = 2 - 1 - 1 = 0$ and $y(2) = 2 + 1 - (-1) = 4$. The value of $y$ corresponding to any other positive integer can be computed using the same formula. However, there is a much easier way. It is to use the identities

$$\cos n\pi = (-1)^n \quad \text{and} \quad \sin[(2n-1)\pi/2] = \sin n\pi \cos\frac{\pi}{2} - \cos n\pi \sin\frac{\pi}{2} = (-1)^{n+1}$$

to transform the formula for $y(n)$ into the simpler representation

$$y(n) = 2 + (-1)^n - (-1)^{n+1} = 2[1 + (-1)^n]$$

and thence into the two different representations

$$y(n) = \begin{cases} 4 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

All values of $y$ are now known.

Various properties of a function are often evident from its *graph*. The **graph** of a function $f$ is the set of all points $(x, y)$ such that $x$ is in the domain of $f$ and $y = f(x)$. The voltage function $E(t)$ of Example 3 is partly graphed in Fig. 1.1.

The function of Example 1 and the two functions $|x|$ and $\sqrt{x^4 - 2x^2}$ of Example 4 are *even* functions.

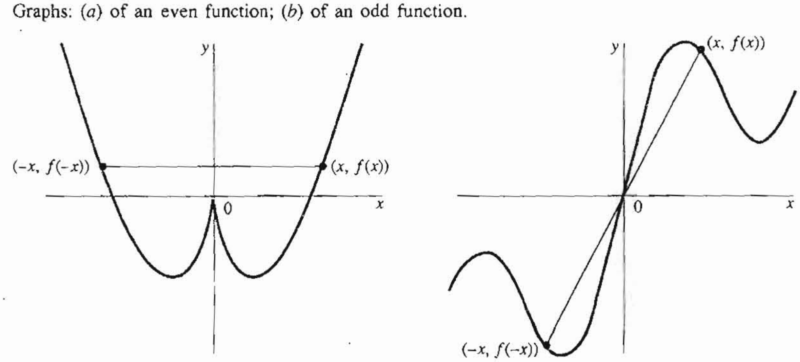
**DEFINITION 2** A function $f$ is **even** if and only if, for every $x$ in its domain, $f(-x) = f(x)$.

An important geometric property of every even function $f$ is that its graph is **symmetric in the $y$ axis;** i.e., for every $x$ in the domain of $f$, the $y$ axis is the perpendicular bisector of the line segment joining the points $(-x, f(-x))$ and $(x, f(x))$, as illustrated in Fig. 1.2*a*.

**DEFINITION 3** A function $f$ is **odd** if and only if, for every $x$ in its domain, $f(-x) = -f(x)$.

The graph of an odd function $f$ is **symmetric in the origin;** i.e., for every $x$ in its domain the origin is the midpoint of the line segment joining $(-x, f(-x))$ and $(x, f(x))$, as illustrated in Fig. 1.2*b*.

**Figure 1.2**
Graphs: (*a*) of an even function; (*b*) of an odd function.




(*a*) Symmetry in the $y$ axis (*b*) Symmetry in the origin

If $b > 0$ and $\int_{-b}^{b} f(x)\,dx$ exists, then

$$\begin{aligned}\int_{-b}^{b} f(x)\,dx &= \int_{-b}^{0} f(x)\,dx + \int_{0}^{b} f(x)\,dx = \int_{b}^{0} f(-x)\,d(-x) + \int_{0}^{b} f(x)\,dx \\ &= -\int_{b}^{0} f(-x)\,dx + \int_{0}^{b} f(x)\,dx = \int_{0}^{b} f(-x)\,dx + \int_{0}^{b} f(x)\,dx \\ &= \int_{0}^{b} [f(-x) + f(x)]\,dx\end{aligned}$$

According as $f$ is even or odd, the last *integrand* reduces to $2f(x)$ or 0. Thus we have

**THEOREM 1** If $f$ is even, $\int_{-b}^{b} f(x)\,dx = 2\int_{0}^{b} f(x)\,dx$.

**THEOREM 2** If $f$ is odd, $\int_{-b}^{b} f(x)\,dx = 0$.

**DEFINITION 4** A function $f$ is **trivial** if and only if for every $x$ in its domain $f(x) = 0$.

Our past work in mathematics frequently involved equations containing one or more variables whose solutions we had to find. Familiar examples might be

$$x^2 + 3x + 2 = 0 \qquad \tan\theta = \tfrac{2}{3} \qquad t = e^{-t} \qquad \{v = u^2, 8u = v^2\}$$

We are now going to consider **differential equations,** which are equations containing *derivatives* or *differentials* of one or more variables. Differential equations are of fundamental importance in many areas of pure and applied science and engineering, and much of this text will be devoted to their study.

Here are four examples of differential equations:

$$\frac{dy}{dx} = e^x + \sin x \tag{1}$$

$$y'' - 2y' + y = \cos x \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{3}$$

$$3x^2\,dx + 2y\,dy = 0 \tag{4}$$

In (1) and (2), only ordinary derivatives of $y$ with respect to $x$ occur. This signifies that $y$ must be a function of $x$. Thus $y$ is the *dependent* variable and $x$ is the *independent* variable. In (3), $u$ is the dependent variable and $x$, $y$, and $t$ are independent variables, as the partial derivatives of $u$ imply. In (4), either $x$ or $y$ can be thought of as the dependent variable, the other variable then being independent.

## EXERCISES

In Exercises 1–8 let $f$ be a function having the given representation.

1. $f(x) = \sqrt{x} + |1 - x| \ln(1 + x)^{1/3}$, $x \geq 0$; find (a) $f(0)$, (b) $f(1)$, (c) $f(4)$

2. $f(x) = \sin x + (\pi - x)\sinh x$, $x$ real; find (a) $f(0)$, (b) $f(\pi)$, (c) $f(-\pi)$

3. $f(x) = |x| + \mathrm{Tan}^{-1} x$, $x$ real; find (a) $f(0)$, (b) $f(1)$, (c) $f(-1)$

4. $f(x) = \pi \cos \pi x + \mathrm{Sin}^{-1} x$, $|x| \le 1$; find (a) $f(0)$, (b) $f(-1)$, (c) $f(\frac{1}{2})$
5. $f(x) = \sum_{n=0}^{\infty} [1 + (-1)^n]x^n/n!$, $x$ real; find (a) $f(0)$, (b) $f(1) - f(-1)$, (c) $f(\ln 2)$. *Hint:* Recall the *Maclaurin series*† for $e^x$.
6. $f(x) = \int_0^2 x^2(t)\,dt$, $x$ continuous on $[0, 2]$; if $x_1(t) = t^{1/2}$, $x_2(t) = t$, $x_3(t) = \sin \pi t$, find (a) $f(x_1)$, (b) $f(x_2)$, (c) $f(x_3)$
7. $f(x, y) = \int_{\pi}^{2\pi} x(t)y(t)\,dt$, $x$, $y$ continuous on $[\pi, 2\pi]$; if $x_1(t) = \cos t$, $y_1(t) = \sin 2t$, find (a) $f(1, y_1)$, (b) $f(x_1, 1)$, (c) $f(x_1, y_1)$
8. $f(x, y, z) = 2\int_0^x [3y(t) - z(t)]\,dt$, $y$, $z$ continuous for all real $x$; if $y_1(t) = 1/(1 + t^2)$ and $z_1(t) = \mathrm{Tan}^{-1} t$, find (a) $f(\infty, y_1, 0)$, (b) $f(1, y_1, z_1)$, (c) $f(-1, y_1, z_1)$
9. A function $f$ is represented on the set of real numbers by $f(t) = 2(1 - \cos^2 t) + \ln e^t$. Give three other representations of $f$ over the reals which do not contain the logarithmic function.
10. Every 40 min during an 8-h period, a dump truck reloads with roadbase. After $t$ hours, how many times has it been filled if it takes 40 min to get the first load?
11. A high-rise has 15 stories. The number of weeks it took to construct *each* floor is given in the table:

| Floors | 1–3 | 4–6 | 7–9 | 10–12 | 13–15 |
|---|---|---|---|---|---|
| Weeks | 3 | 4 | 5 | 6 | 7 |

    (a) Which floor was under construction during the forty-fifth week?
    (b) How many floors were completed at the end of 1 year?
    (c) What was the total construction time?
12. Determine whether the following functions are even, odd, or neither.
    (a) $x \sin x^2$  (b) $e^{|x|} + x \sinh x$
    (c) $x \ln x + \mathrm{Tan}^{-1} x$
13. Prove that if a function $f$ is both even and odd, then for every $x$ in the domain of $f$, $f(x) = 0$.
14. Assuming they have the same domain,
    (a) What can be said about the sum or difference of two even functions? Of two odd functions? Of an odd function and an even function?
    (b) What can be said about the product of two even functions? Of two odd functions? Of an odd function and an even function?
15. By considering the identity
$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$
show that any function defined over an interval which is symmetric with respect to the origin can be written as the sum of an even function and an odd function.
16. Express each of the following functions as the sum of an even function and an odd function.
    (a) $e^x + \ln|x|$  (b) $|x - 1|$
    (c) $f(x) = \begin{cases} 0 & x < 0 \\ 2 & 0 \le x \end{cases}$
17. Evaluate
    (a) $\displaystyle\int_{-1}^{1} \frac{\cos 2\pi - \cos \pi}{1 + x^2}\,dx$
    (b) $\displaystyle\int_{-\pi}^{\pi} \sin 2x \cos 3x \cosh 4x\,dx$
    (c) $\displaystyle\int_{-\ln 2}^{\ln 2} (e^{|x|} + \sin^5 \pi x)\,dx$
    (d) $\displaystyle\int_{-10}^{10} (6t^{99} - 13t \sin et^2 + 5)\,dt$

Identify the dependent and independent variables in each of the following differential equations.

18. $z'' + zy' + yx = \sec x$  19. $3xy'' + \tanh y' = y$
20. $u_{xx} + v_{yy} - xyuv = 0$

## 1.2 CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Various distinctive features are used to classify differential equations into a number of identifiable types. *Ordinary* and *partial* differential equations are characterized by the number of independent variables and the kind of derivatives they involve.

**DEFINITION 1** An **ordinary differential equation** is a differential equation in which all derivatives are ordinary derivatives of one or more dependent variables with respect to a single independent variable.

†Named for the Scottish mathematician Colin Maclaurin (1698–1746).

Clearly, Eqs. (1) and (2) of Sec. 1.1 are ordinary differential equations. The same is true of Eq. (4); for, according as $x$ or $y$ is chosen as independent variable and the equation is divided by the differential $dx$ or the differential $dy$, it involves one or the other of the ordinary derivatives $dy/dx$ or $dx/dy$.

**DEFINITION 2** A **partial differential equation** is a differential equation containing at least one partial derivative of some dependent variable.

Equation (3) of Sec. 1.1 is a partial differential equation.

Differential equations are also classified according to their *order.*

**DEFINITION 3** The **order** of a differential equation is the order of the highest-order derivative which appears in the equation.

Equations (1) and (4) of Sec. 1.1 are first-order differential equations; Eqs. (2) and (3) are second-order equations.

Another broad classification of differential equations is based on the way in which a dependent variable and its indicated derivatives appear in the terms of such an equation.

**DEFINITION 4** A differential equation is **linear in a set of one or more of its dependent variables** if and only if each term of the equation which contains a variable of the set or any of their derivatives is of the first degree in those variables and their derivatives.

A differential equation which is not linear in some dependent variable is said to be **nonlinear** in that variable. A differential equation which is not linear in the set of all of its dependent variables is simply said to be **nonlinear.**

**EXAMPLE 1**

The equation $y'' + 4xy' + 2y = \cos x$ is a linear ordinary differential equation of second order. The presence of the product $xy'$ and the term $\cos x$ does not alter the fact that the equation is linear because, by definition, linearity is determined solely by the way the *dependent* variable $y$ and its derivatives enter into combination among themselves within each term of the equation.

**EXAMPLE 2**

The equation $y'' + 4yy' + 2y = \cos x$ is a nonlinear equation because of the occurrence of the product of $y$ and one of its derivatives.

**EXAMPLE 3**

The equation $\partial^2u/\partial x^2 + \partial v/\partial t + u + v = \sin u$ is linear in the dependent variable $v$ but nonlinear in the dependent variable $u$ because $\sin u$ is a nonlinear function of $u$. The equation is also nonlinear.

**EXAMPLE 4**

The equation $d^2x/dt^2 + dy/dt + xy = \sin t$ is linear in each of the dependent variables $x$ and $y$. However, because of the term $xy$, it is not linear in the set of dependent variables $\{x, y\}$. As a consequence, the equation is nonlinear.

**EXAMPLE 5**

As written, the equation $3x^2\,dx + (\sin x)\,dy = 0$ is neither linear nor nonlinear. Division by $dx$ transforms it into the equation $3x^2 + (\sin x)y' = 0$ which is linear in $y$, but division by $dy$ gives $3x^2\,dx/dy + \sin x = 0$ which is nonlinear in $x$.

From Definition 4 it follows that the most general ordinary linear differential equation of order $n$ in a single dependent variable is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \tag{1}$$

where $a_0(x) \neq 0$ throughout some interval.

## EXERCISES

Describe each of the following equations, giving its order and telling whether it is ordinary or partial and linear or nonlinear ($a$, $b$ constants).

1. $y'' - 5y' + 3y = x^4$
2. $y' + (a + b\sin 3x)y = 0$
3. $y''' - 7y'' + 11y' - 13y = xe^x$
4. $y^{\text{iv}} + x^2y' + y^{1/2} = 0$
5. $x^{3/2}y'' - 9x^{1/2}y' + 5y = \text{Tan}^{-1} x$
6. $d(xy')/dx + xy = 0$
7. $d(axy)/dt + bxy = \ln t$
8. $a^2\,\partial^2u/\partial x^2 = b^2\,\partial^2u/\partial t^2$
9. $\partial^2(x^2\,\partial^2u/\partial x^2)/\partial x^2 = \partial^2u/\partial t^2$
10. $\partial^2u/\partial x^2 = u\,\partial u/\partial t$
11. $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = \phi(x, y)$
12. $\partial u/\partial t + xu = d^2x/dt^2$
13. $\partial u/\partial x + u\,\partial v/\partial y = v\,\partial^2v/\partial x\,\partial y$
14. $\sec t = t^3x + 2x^{1/2}\,d(x^{1/2})/dt$

For each of the following equations, determine whether or not it becomes linear when divided by $dx$ or $dy$.

15. $(x + y)\,dy = (x - y)\,dx$
16. $a\,dy + by\sin x\,dx = 0$
17. $3y\,dx + 2x\,dy = 0$
18. $e^x\,dy + xy^{1/3}\,dx = 0$

## 1.3 SOLUTIONS OF DIFFERENTIAL EQUATIONS

A **solution** of an algebraic or transcendental equation in a single variable $x$ is a *number* which satisfies the equation. On the other hand, solutions of differential equations are *functions,* rather than numbers, which satisfy the equation. Whereas all variables which appear in algebraic or transcendental equations are called "unknowns," only the dependent variables in a differential equation are referred to as "unknowns."

**EXAMPLE 1**

Under certain constraints, the motion of a spring-suspended mass is described by solutions of the differential equation $y'' + k^2y = 0$, where the dependent variable (or unknown) $y$ is the vertical displacement of the mass, the second derivative is taken with respect to the independent variable $t$ (or time) and $k$ is a positive constant. Let $f$ be the function represented by $\sin kt$ on the set of real numbers. Then $f''$ is represented by $-k^2\sin kt$. Upon replacing $y$ by $\sin kt$ and $y''$ by $-k^2\sin kt$, the given equation is transformed into $-k^2\sin kt + k^2\sin kt = 0$, which holds identically for all real values of $t$. The fact that this replacement process results in an identity over the reals is described by calling $f(t) = \sin kt$ a solution of $y'' + k^2y = 0$ on $(-\infty, \infty)$ or by saying that $f$ is a solution of the differential equation. In other words, the differential equation is satisfied when $f$ is substituted for the unknown $y$. Another solution is the function $g$ defined on $(-\infty, \infty)$ by $g(t) = \cos kt$.

**EXAMPLE 2**

The equation $u_x - v_y = 0$ has $u$ and $v$ as unknowns. Functions $f$ and $g$ are defined, for all real values of $x$ and $y$, by $f(x, y) = \sinh x \sinh y$ and $g(x, y) = \cosh x \cosh y$. Replacement of $u_x$ by $\partial f/\partial x = \cosh x \sinh y$, and of $v_y$ by $\partial g/\partial y = \cosh x \sinh y$, converts the given differential equation into $\cosh x \sinh y - \cosh x \sinh y = 0$. Since this is an identity for all real values of $x$ and $y$, we say that $f(x, y)$ and $g(x, y)$ are solutions for the respective unknowns $u$ and $v$ and that the set of functions $\{f, g\}$ is a solution of the given equation over the entire $xy$ plane.

The last two examples illustrate the concept of a solution of a differential equation. They also indicate how to substitute a set of functions for the unknowns of such an equation when testing to see if the set is a solution of the equation.

**DEFINITION 1** A **solution of a differential equation over a region** $R$ is a set of functions which, when they are substituted for the dependent variables in the differential equation, reduce the equation to an identity in the independent variables over $R$.

At present we shall be concerned primarily with ordinary differential equations in a single unknown. For such equations, the preceding definition may be rephrased as follows.

**DEFINITION 2** A **solution** of an ordinary differential equation in one dependent variable **on an interval** $I$ is a function which, when substituted for the dependent variable, reduces the equation to an identity in the independent variable over $I$.

A solution on $I$ whose values are all equal to 0 is said to be **trivial on** $I$.

If a differential equation of order $n$ is satisfied by a function $f$ on $I$, then the $n$th derivative $f^{(n)}$ of $f$ necessarily exists throughout $I$. Since a function must be continuous wherever its derivative exists, the existence of $f^{(n)}$ over $I$ implies that $f$ and its derivatives of all orders up to and including $n - 1$ are continuous on $I$. A function that is not continuously differentiable at least $n - 1$ times on $I$ cannot be a solution over $I$. In other words

**THEOREM 1** Every solution on an interval $I$ of an $n$th-order differential equation in one dependent variable must be continuously differentiable at least $n - 1$ times on $I$.

**EXAMPLE 3**

It is easy to verify that both $y_1 = 2 - x$ and $y_2 = x - 2$ are solutions of $y'' = 0$ on every interval $I$. Let $f$ be the function defined by $f(x) = |x - 2|$ on $(-\infty, \infty)$. Then $f$ is represented by $y_1(x) = 2 - x$ for $x \leq 2$ and by $y_2(x) = x - 2$ for $x \geq 2$, and $f$ is continuous at $x = 2$. However, $f$ is not a solution of $y'' = 0$ on any open interval containing $x = 2$ because $f'$ is not continuous at $x = 2$. In fact,

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{|h|}{h} = \begin{cases} -1 & \text{as } h \to 0^- \\ 1 & \text{as } h \to 0^+ \end{cases}$$

hence $f'(2)$ does not exist.

Some differential equations have no solutions; others have only a trivial solution.

**EXAMPLE 4**

The equation $|dy/dx| + 1 = 0$ has no solution. The equation $|dy/dx| + |y| = 0$ has only the trivial solution $y = 0$.

Differential equations for which every solution of each equation is a solution of all the others are said to be **equivalent.** If their solutions are the same only on some interval $I$, we say that the differential equations are **equivalent on** $I$.

The study of the existence, nature, and determination of solutions of differential equations is of fundamental importance not only to the pure mathematician but also to anyone engaged in the mathematical analysis of natural phenomena. In general, mathematicians consider it a triumph if they are able to prove that a given differential equation possesses a solution and if they can deduce a few of the more important properties of that solution. Engineers and applied scientists, on the other hand, are usually greatly disappointed if a specific expression for the solution cannot be exhibited. The usual compromise is to find some practical procedure by which the required solution can be approximated with satisfactory accuracy.