VIP Refresher: Linear Algebra and Calculus

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General notations

❒ **Vector** – We note $x \in \mathbb{R}^n$ a vector with n entries, where $x_i \in \mathbb{R}$ is the i^{th} entry:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

❒ **Matrix** – We note $A \in \mathbb{R}^{m \times n}$ a matrix with m rows and n columns, where $A_{i,j} \in \mathbb{R}$ is the entry located in the i^{th} row and j^{th} column:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Remark: the vector x defined above can be viewed as a $n \times 1$ matrix and is more particularly called a column-vector.

❒ **Identity matrix** – The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix with ones in its diagonal and zero everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Remark: for all matrices $A \in \mathbb{R}^{n \times n}$, we have $A \times I = I \times A = A$.

❒ **Diagonal matrix** – A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is a square matrix with nonzero values in its diagonal and zero everywhere else:

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$$

Remark: we also note D as diag(d_1,...,d_n).

Matrix operations

❒ **Vector-vector multiplication** – There are two types of vector-vector products:

- inner product: for $x,y \in \mathbb{R}^n$, we have:

$$\boxed{x^T y = \sum_{i=1}^n x_i y_i \in \mathbb{R}}$$

- outer product: for $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, we have:

$$\boxed{xy^T = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{pmatrix} \in \mathbb{R}^{m \times n}}$$

❒ **Matrix-vector multiplication** – The product of matrix $A \in \mathbb{R}^{m \times n}$ and vector $x \in \mathbb{R}^n$ is a vector of size $\mathbb{R}^n$, such that:

$$\boxed{Ax = \begin{pmatrix} a_{r,1}^T x \\ \vdots \\ a_{r,m}^T x \end{pmatrix} = \sum_{i=1}^n a_{c,i} x_i \in \mathbb{R}^n}$$

where $a_{r,i}^T$ are the vector rows and $a_{c,j}$ are the vector columns of A, and x_i are the entries of x.

❒ **Matrix-matrix multiplication** – The product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ is a matrix of size $\mathbb{R}^{n \times p}$, such that:

$$\boxed{AB = \begin{pmatrix} a_{r,1}^T b_{c,1} & \cdots & a_{r,1}^T b_{c,p} \\ \vdots & & \vdots \\ a_{r,m}^T b_{c,1} & \cdots & a_{r,m}^T b_{c,p} \end{pmatrix} = \sum_{i=1}^n a_{c,i} b_{r,i}^T \in \mathbb{R}^{n \times p}}$$

where $a_{r,i}^T, b_{r,i}^T$ are the vector rows and $a_{c,j}, b_{c,j}$ are the vector columns of A and B respectively.

❒ **Transpose** – The transpose of a matrix $A \in \mathbb{R}^{m \times n}$, noted A^T, is such that its entries are flipped:

$$\boxed{\forall i,j, \qquad A_{i,j}^T = A_{j,i}}$$

Remark: for matrices A,B, we have $(AB)^T = B^T A^T$.

❒ **Inverse** – The inverse of an invertible square matrix A is noted A^{-1} and is the only matrix such that:

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Remark: not all square matrices are invertible. Also, for matrices A,B, we have $(AB)^{-1} = B^{-1}A^{-1}$

❒ **Trace** – The trace of a square matrix A, noted $\mathrm{tr}(A)$, is the sum of its diagonal entries:

$$\boxed{\mathrm{tr}(A) = \sum_{i=1}^n A_{i,i}}$$

Remark: for matrices A,B, we have $tr(A^T) = tr(A)$ and $tr(AB) = tr(BA)$

❒ **Determinant** – The determinant of a square matrix $A \in \mathbb{R}^{n \times n}$, noted $|A|$ or $\det(A)$ is expressed recursively in terms of $A_{\backslash i, \backslash j}$, which is the matrix A without its i^{th} row and j^{th} column, as follows:

$$\boxed{\det(A)=|A|=\sum_{j=1}^{n}(-1)^{i+j}A_{i,j}|A_{\backslash i,\backslash j}|}$$

Remark: A is invertible if and only if $|A|\neq 0$. Also, $|AB|=|A||B|$ and $|A^T|=|A|$.

Matrix properties

❒ **Symmetric decomposition** – A given matrix A can be expressed in terms of its symmetric and antisymmetric parts as follows:

$$\boxed{A=\underbrace{\frac{A+A^T}{2}}_{\textrm{Symmetric}}+\underbrace{\frac{A-A^T}{2}}_{\textrm{Antisymmetric}}}$$

❒ **Norm** – A norm is a function $N:V\longrightarrow[0,+\infty[$ where V is a vector space, and such that for all $x,y\in V$, we have:

- $N(x+y)\leqslant N(x)+N(y)$
- $N(ax)=|a|N(x)$ for a scalar
- if $N(x)=0$, then $x=0$

For $x\in V$, the most commonly used norms are summed up in the table below:

Norm	Notation	Definition	Use case
Manhattan, L^1	$\|\|x\|\|_1$	$\displaystyle\sum_{i=1}^{n}\|x_i\|$	LASSO regularization
Euclidean, L^2	$\|\|x\|\|_2$	$\displaystyle\sqrt{\sum_{i=1}^{n}x_i^2}$	Ridge regularization
p-norm, L^p	$\|\|x\|\|_p$	$\displaystyle\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}$	Hölder inequality
Infinity, L^∞	$\|\|x\|\|_\infty$	$\underset{i}{\max}\ \|x_i\|$	Uniform convergence

❒ **Linearly dependence** – A set of vectors is said to be linearly dependent if one of the vectors in the set can be defined as a linear combination of the others.
Remark: if no vector can be written this way, then the vectors are said to be linearly independent.

❒ **Matrix rank** – The rank of a given matrix A is noted $\textrm{rank}(A)$ and is the dimension of the vector space generated by its columns. This is equivalent to the maximum number of linearly independent columns of A.

❒ **Positive semi-definite matrix** – A matrix $A\in\mathbb{R}^{n\times n}$ is positive semi-definite (PSD) and is noted $A\succeq 0$ if we have:

$$\boxed{A=A^T}\quad\textrm{and}\quad\boxed{\forall x\in\mathbb{R}^n,\quad x^TAx\geqslant 0}$$

Remark: similarly, a matrix A is said to be positive definite, and is noted $A\succ 0$, if it is a PSD matrix which satisfies for all non-zero vector x, $x^TAx>0$.

❒ **Eigenvalue, eigenvector** – Given a matrix $A\in\mathbb{R}^{n\times n}$, λ is said to be an eigenvalue of A if there exists a vector $z\in\mathbb{R}^n\backslash\{0\}$, called eigenvector, such that we have:

$$\boxed{Az=\lambda z}$$

❒ **Spectral theorem** – Let $A\in\mathbb{R}^{n\times n}$. If A is symmetric, then A is diagonalizable by a real orthogonal matrix $U\in\mathbb{R}^{n\times n}$. By noting $\Lambda=\textrm{diag}(\lambda_1,...,\lambda_n)$, we have:

$$\boxed{\exists\Lambda\textrm{ diagonal},\quad A=U\Lambda U^T}$$

❒ **Singular-value decomposition** – For a given matrix A of dimensions $m\times n$, the singular-value decomposition (SVD) is a factorization technique that guarantees the existence of U $m\times m$ unitary, Σ $m\times n$ diagonal and V $n\times n$ unitary matrices, such that:

$$\boxed{A=U\Sigma V^T}$$

Matrix calculus

❒ **Gradient** – Let $f:\mathbb{R}^{m\times n}\rightarrow\mathbb{R}$ be a function and $A\in\mathbb{R}^{m\times n}$ be a matrix. The gradient of f with respect to A is a $m\times n$ matrix, noted $\nabla_A f(A)$, such that:

$$\boxed{\Big(\nabla_A f(A)\Big)_{i,j}=\frac{\partial f(A)}{\partial A_{i,j}}}$$

Remark: the gradient of f is only defined when f is a function that returns a scalar.

❒ **Hessian** – Let $f:\mathbb{R}^n\rightarrow\mathbb{R}$ be a function and $x\in\mathbb{R}^n$ be a vector. The hessian of f with respect to x is a $n\times n$ symmetric matrix, noted $\nabla_x^2 f(x)$, such that:

$$\boxed{\Big(\nabla_x^2 f(x)\Big)_{i,j}=\frac{\partial^2 f(x)}{\partial x_i\partial x_j}}$$

Remark: the hessian of f is only defined when f is a function that returns a scalar.

❒ **Gradient operations** – For matrices A,B,C, the following gradient properties are worth having in mind:

$$\boxed{\nabla_A\textrm{tr}(AB)=B^T}\qquad\boxed{\nabla_{A^T}f(A)=\left(\nabla_A f(A)\right)^T}$$

$$\boxed{\nabla_A\textrm{tr}(ABA^TC)=CAB+C^TAB^T}\qquad\boxed{\nabla_A|A|=|A|(A^{-1})^T}$$