MATH3305 — Problem Sheet 2

Problems 2, 3 and 5 to be handed in **at the lecture** on Friday, 20 October 2017

1. Let $\mathcal{M}$ be a manifold. Let V^a be contravariant vector and let W_a be a covariant vector. Show that

$$\mu = V^a W_a$$

   is a scalar. (Hint: How does μ transform under coordinate transformations?)

2. You are given Euclidean 3-space with standard Cartesian coordinates $X^i = \{x, y, z\}$. Now introduce spherical polar coordinates $Y^i = \{r, \theta, \phi\}$ satisfying

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta . \end{aligned}$$

   (i) Find the line element $ds^2 = dx^2 + dy^2 + dz^2$ in spherical polar coordinates (Answer is in the lecture notes).
   (ii) Find the metric g_{ab} in spherical polar coordinates.
   (iii) Find the inverse metric g^{ab} in spherical coordinates.
   (iv) Show explicitly that $g_{ab} g^{bc} = \delta_a^c$.

3. Determine which of the following tensor equations are valid, and describe possible errors

$$\begin{aligned} K &= R_{abcd} R^{abcd} \\ T_{ab} &= F_{ac} F^c{}_c + \frac{1}{4} \eta_{ab} F_{ab} F^{ab} \\ R_{ab} - \frac{1}{2} R &= 8\pi\kappa T_{ab} \\ E_a{}^b &= F_{ac} H^{cb} . \end{aligned}$$

4. (Classical mechanics). Let $L(x(t), \dot{x}(t))$ be a smooth function of $x(t)$ and $\dot{x}(t) = dx(t)/dt$. What differential equation must L satisfy to extremise the following functional

$$S = \int L(x, \dot{x}) dt \,.$$

   (Keywords: Hamilton's/action principle, Euler-Lagrange equations, variational calculus)

5. Some 3-vector identities and index gymnastics. In index notation show that

$$\begin{aligned} &\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \,, \\ &\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ &\nabla \times (f\, \mathbf{a}) = \nabla f \times \mathbf{a} + f \nabla \times \mathbf{a} \,, \end{aligned}$$

   where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are 3-vectors and f is a smooth function.