بسم الله الرحمن الرحیم

Decidability

Ali Shakiba
ali.shakiba@vru.ac.ir
Vali-e-Asr University of Rafsanjan

What we are going to discuss?

- Decidable problems concerning
  - Regular languages
  - Context-free languages
- Undecidability
  - Diagonalization method
  - An unrecognizable language

An Undecidable Problem

The general problem of software verification **is not** solvable by computers.

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and accepts } w\}$$

Undecidable

Theorem 4.9

The proof is by diagonalization.

Some Terminologies

- Let $f: A \rightarrow B$ be a function. It is said to be …
  - … one-to-one, or injective, if $f(a) = f(b)$ implies $a = b$.
  - … onto, or surjective, if for every $b \in B$, there is an $a \in A$ such that $f(a) = b$.
  - … correspondence, or bijection, if f is both onto and one-to-one.

The sets A and B are **the same size** if there is a correspondence, or bijection, function $f: A \rightarrow B$.

A set A is **countable** if either it is finite or it has the same size as $\mathbb{N}$.

Warm up with DIAGONALIZATION Technique

Theorem 4.17: $\mathbb{R}$ is uncountable.

Proof: (*Proof by contradiction*)

- Suppose there is a correspondence $f: \mathbb{N} \to \mathbb{R}$.
- We will construct an $x \in \mathbb{R}$ such that it differs from all paired real numbers, i.e. $x \neq f(n)$ for every $n \in \mathbb{N}$.
- An we use diagonalization to construct that $0 < x < 1$ by choosing the i^{th} decimal of x different from the i^{th} decimal of $f(i)$.

The fact that “$\mathbb{R}$ is uncountable” leads to …

Corollary 4.18: Some languages are not Turing-recognizable.

- Observe that Σ^* is countable for any alphabet Σ.
    - How?
- The set of all Turing machines is countable.
    - How?
- The set of all languages is uncountable.
    - How?
- Sum it up! ☺

$$A_{TM} = \{\langle M, w\rangle | M \text{ is a TM and accepts } w\}$$

Proof: (By contradiction)

- Suppose there is a decider H

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

- We construct D as follows

$D =$ "On input $\langle M\rangle$, where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$.
2. Output the opposite of what H outputs. That is, if H accepts, *reject*; and if H rejects, *accept*."

$A_{TM} = \{\langle M, w\rangle | M \text{ is a TM and accepts } w\}$

Proof: (By contradiction)

- Suppose there is a decider H

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

- We construct D as follows

$$D(\langle M\rangle) = \begin{cases} accept & \text{if } M \text{ does not accept } \langle M\rangle \\ reject & \text{if } M \text{ accepts } \langle M\rangle. \end{cases}$$

- Then, what about $D(\langle D\rangle)$?

$$A_{TM} = \{\langle M, w\rangle | M \text{ is a TM and accepts } w\}$$

Proof: (By contradiction)

- S

$$D(\langle D\rangle) = \begin{cases} accept & \text{if } D \text{ does not accept } \langle D\rangle \\ reject & \text{if } D \text{ accepts } \langle D\rangle. \end{cases}$$

- V

$reject$ if M accep

Undecidable

Where is diagonalization?

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*		*accept*		
M_2	*accept*	*accept*	*accept*	*accept*	
M_3					$\cdots$
M_4	*accept*	*accept*			
$\vdots$			$\vdots$		

$$
\begin{array}{c|ccccc}
 & \langle M_1 \rangle & \langle M_2 \rangle & \langle M_3 \rangle & \langle M_4 \rangle & \cdots \\
\hline
M_1 & accept & reject & accept & reject & \\
M_2 & accept & accept & accept & accept & \cdots \\
M_3 & reject & reject & reject & reject & \\
M_4 & accept & accept & reject & reject & \\
\vdots & & & \vdots & &
\end{array}
$$

$$
\begin{array}{c|ccccccc}
 & \langle M_1\rangle & \langle M_2\rangle & \langle M_3\rangle & \langle M_4\rangle & \cdots & \langle D\rangle & \cdots \\
\hline
M_1 & \underline{accept} & reject & accept & reject & & accept & \\
M_2 & accept & \underline{accept} & accept & accept & \cdots & accept & \cdots \\
M_3 & reject & reject & \underline{reject} & reject & & reject & \\
M_4 & accept & accept & reject & \underline{reject} & & accept & \\
\vdots & & & \vdots & & \ddots & & \\
D & reject & reject & accept & accept & & \underline{?} & \\
\vdots & & & \vdots & & & & \ddots
\end{array}
$$

Co-Turing recognizable

A language L is co-Turing recognizable if it is the complement of Turing recognizable language.

Theorem 4.22: A language is decidable if and only if it is Turing-recognizable and co-Turing recognizable.

Corollary 4.23: Language $\overline{A_{TM}}$ is not Turing-recognizable.

That’s all for now!

Chapter 4 is finished.

Assignments are on their way…