1-1 Reprogrammability, multifunctionality.

1-2 **Forward kinematics:** Position and orientation of the end effector in terms of the joint variable

Inverse Kinematics: Joint variables in terms of position/orientation of the end effector.

Trajectory Planning: Planning the time history of the joint variables necessary for the robot to execute a given task.

Workspace: The total volume swept out by the end-effector as the manipulator executes all possible motions.

Accuracy: A measure of how close the manipulator can come to a given point within its workspace.

Repeatability: A measure of how close a manipulator can return to a previously taught point.

Resolution: The smallest increment of motion that can be sensed. The resolution is a function of the distance travelled and the number of bits of encoded accuracy.

Joint Variables: The relative displacement between adjacent links, denoted θ_i for revolute joins and d_i for prismatic joints.

Spherical Wrist: RRR wrist configuration with joint axes intersecting at a common point.

End Effector: A gripper or tool used to perform the robot's task.

1-3 Geometry, power source, application area, method of control.

1-4 Depending on the pattern to be followed, articulated, spherical, or cartesian manipulators may be used for applications such as welding, laying a bead of glue, cutting, grinding or sanding a surface, spray painting, auto assembly, and anthropomorphic tasks. Cartesian manipulators are also suited for table-top assembly and, as a gantry, for the transfer of material or cargo. SCARA manipulators are useful for table-top assembly and pick-and-place applications.

1-5 **Non-servo robots:** Materials handling, servicing a special purpose machine such as a press.

Point-to-point robots: Materials handling, spot welding, forging.

Continuous path: Arc welding, grinding and deburring, spray painting, assembly, sheep shearing.

1-6 Welding, painting, deburring, grinding, polishing.

1-7 Automating inspection of goods for defects, monitoring unknown terrain, sorting objects by color or shape, painting an object, picking up randomly placed objects.

1-8 Handling fragile objects (glass, eggs, etc.), grinding, assembly defusing explosives, machining.

1-9 Latest figures and projections may be found in the current edition of *World Robotics*, published by the International Federation of Robotics, or a similar publication.

1-10 The key point of this question is the rapidly of change. An overnight change would not allow time for workers to be retrained or to find other jobs, thus negating the beneficial effects of increased productivity. This point can be discussed at length in a classroom setting.

1-11 Here again the key point is in the extreme nature of allowing no robotic automation. While no one would be put out of work by robots, the productivity and quality of goods produced would soon lag behind that of other countries. The long term effect would likely be higher unemployment than would result by phased automation. This point can also be discussed in a classroom setting.

1-12 Applications involved reaching around or behind obstacles, assembly of a complex, intricate object, defusing explosives, artificial limbs.

1-13 Using the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$, let $a = \ell = b$ and $c = d$. Then $d^2 = 2\ell^2(1 - \cos\theta)$. Hence,

$$d \;=\; \ell\sqrt{2(1-\cos\theta)}.$$

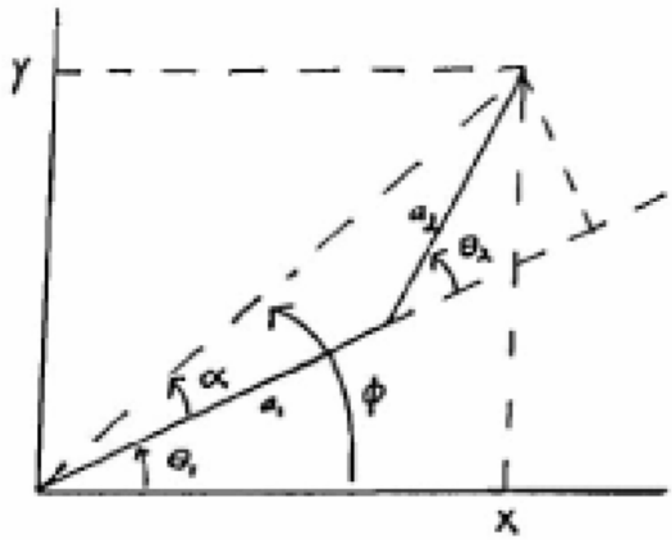
With $\ell = 1$ meter , $\theta = 90^\circ$, $d = \sqrt{2}$ meters $= 1.4142136$ meters. On the other hand, $s = \ell\theta = \frac{\pi}{2}$ meters $= 1.5707963$ meters. Resolution $= \frac{\text{Total distance}}{2^n}$ where $n =$ number of bits of encoder accuracy. The linear resolution is $\frac{\sqrt{2}}{2.2^{10}} = 0.003811 = 1.3811 \times 10^{-3}$ meters. The rotational resolution is $\frac{\pi}{2.2^{10}} = \frac{\pi}{2.2^{10}} = \frac{\pi}{2^{11}} = 0.001534 = 1.534 \times 10^{-3}$ meters.

1-14 Resolution $= \frac{\ell\theta}{2^n} = \frac{(50cm)(\pi)}{2^8} = 0.6136$cm.

1-15 $\ell = 0.5\text{meter} \quad \theta = \pi \qquad r = \frac{\text{total distance}}{2^n} = \frac{\frac{1}{2}\cdot\pi\cdot\frac{1}{5}}{2^8} m = 1.227 \times 10^{-2} cm.$

1-16 The position of the TCP is not measured directly but is computed from encoder measuring joint positions. Thus, the accuracy is affected by computational errors, machining accuracy in construction of robot parts, etc.

1-17 If direct end-pointing sensing were used, less uncertainty could enter the measurement of the end-effector position. Difficulties including introducing a vision system at the end-effector and feeding back the end-effector position which could cause the control system to become unstable.



1-18 We have $\theta_1 = \phi - a$ where $\phi = \tan^{-1}(y/x)$; $a = \tan^{-1}\left(\frac{a_2 \sin\theta_2}{a_1 + a_2 \cos\theta_2}\right)$

1-19 1.

$$
\begin{aligned}
x &= a_1 \cos\theta_1 + a_2 \cos(\theta_1+\theta_2) &=& \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{2\pi}{3}\right) \quad 0.366025 \\
y &= a_1 \sin\theta_1 + a_2 \sin(\theta_1+\theta_2) &=& \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{2\pi}{3}\right) \quad 1.3660254
\end{aligned}
$$

2.

$$
\begin{aligned}
\dot{x} &= -\sin\theta_1 - 3\sin(\theta_1+\theta_2) \\
\dot{y} &= \cos\theta_1 + 3\cos(\theta_1+\theta_2) \\
\text{At } \theta_1 &= \theta_2 = \frac{\pi}{4}, \\
\dot{x} &= -\left(\sin\frac{\pi}{4} + 3\sin\frac{\pi}{2}\right) = -3.7071068, \\
\dot{y} &= \cos\frac{\pi}{4} + 3\cos\frac{\pi}{2} = 0.7071068
\end{aligned}
$$

3. machine problem

4. machine problem

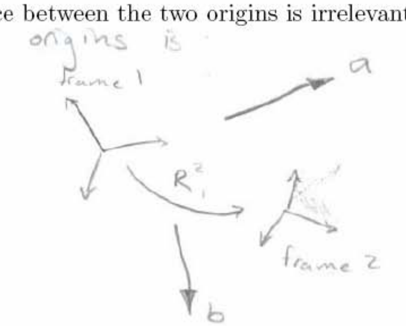
1-20 If both links are equal length then $x = 0$, $y = 0$ can be reached by infinitely many configurations, namely $\theta_2 = 180°$, $\theta_1 =$ arbitrary.

1-21 Moving a distal link with large mass will require more torque from all motors driving previous links. In addition, since momentum is the product of mass and velocity, a massive link far from the base may cause troublesome overshoot issues. We may reduce the mass of distal links in two ways:

1. driving distal joints from motors mounted on previous links, thereby eliminating the mass of the motor from the link mass. A consequence of this is the increased complexity of design due to trasmission of motion from motors to the joint they drive.

2. reducing the mass of the links themselves, either by selecting materials with less mass or by strategically boring holes in the link to reduce mass. The downside of this approach is seen in the control problem. For a robot whose links have small mass, picking up an object will drastically affect the dynamics of the manipulator. For more massive robots, manipulating small objects have less or negligible affect on the dynamics.

2-1 We are considering free vectors. Consequently, we do not need to know points in space — only direction and magnitude so we only need to know the rotation between the two coordinate frames; the distance between the two origins is irrelevant.



We write $a^2 = R_1^2 a^1$ and $b^2 = R_1^2 b^1$. Now,

$$\begin{aligned} a^2 \cdot b^2 &= (a^2)^T b^2 = (Ra^1)^T (Rb^1) = (a^1)^T R^T R b^1 \\ &= (a^1)^T R^{-1} R b^1 = (a^1)^T b^1 = a^1 \cdot b^1 \end{aligned}$$

2-2 Notice that $\|v\|^2 = v^T v \Rightarrow \|v\| = +\sqrt{v^T v}$. Therefore,

$$
\begin{aligned}
\|Rv\| &= +\sqrt{(Rv)^T Rv} = \sqrt{v^T R^T Rv} \\
&= \sqrt{v^T v} = \|v\|
\end{aligned}
$$

2-3 This follows from Problem 2-2 with $v = p_1 - p_2$.

2-4 Let $R = [r_1, r_2, r_3]$ where $r_i = \begin{pmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{pmatrix}$. Then $R^T R = I$ implies

$$\begin{bmatrix} r_1^T r_1 & r_1^T r_2 & r_1^T r_3 \\ r_1^T r_1 & r_2^T r_2 & r_2^T r_3 \\ r_3^T r_1 & r_2^T r_2 & r_3^T r_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Equating entries of the matrices shows that the column vectors of R are of unit length and mutually orthogonal.

2-5 **a)** For any matrices A and B, $\det(A^T) = \det(A)$ and $\det(AB) = \det(A)\det(B)$. Thus, if R is orthogonal

$$1 = \det(I) = \det(R^T R) = \det(R^T)\det(R) = (R)^2$$

which implies that

$$\det R \quad = \quad \pm 1.$$

b) For a right-handed coordinate system, $r_1 \times r_2 = r_3$. This implies that

$$r_{12}r_{23} - r_{13}r_{22} = r_{31}; \quad -r_{11}r_{23} + r_{13}r_{21} = r_{32}; \quad r_{11}r_{22} - r_{12}r_{21} = r_{33}.$$

Therefore, expanding $\det R$ about column 3 gives

$$\begin{aligned}
\det R &= \det \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \\
&= r_{31}(r_{12}r_{23} - r_{22}r_{13}) - r_{32}(r_{11}r_{23} - r_{21}r_{13}) + r_{33}(r_{11}r_{22} - r_{21}r_{12}) \\
&= r_{31}(r_{31}) + r_{32}(r_{32}) + r_{33}(r_{33}) \\
&= ||r_3||^2 = 1.
\end{aligned}$$

2-6 Equation (2.3) is obvious. Equation (2.4) follows from

$$
\begin{aligned}
R_{z,\theta}R_{z,\phi} &= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\phi - s_\theta s_\phi & -c_\theta s_\phi - c_\phi s_\theta & 0 \\ s_\theta c_\phi + c_\theta s_\phi & -s_\theta s_\phi + c_\theta c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & 0 \\ \sin(\theta+\phi) & \cos(\theta+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta+\phi}.
\end{aligned}
$$

Equation (2.5) follows from (2.3) and (2.4) since

$$R_{z,\theta}R_{z,-\theta} = R_{z,\theta-\theta} = R_{z,0} = I.$$

This can also be shown by noticing that

$$R_{z,\theta}^T = R_{z,-\theta}.$$

2-7 First, note that $x \in SO(n)$ means that $x^T x = x x^T = I$ and $\det x = 1$.

a) The first property follows from

$$(x_1 x_2)^T (x_1 x_2) = x_2^T x_1^T x_1 x_2 = x_2^T I x_2 = I$$

so

$$x_1 x_2 \in SO(n) \qquad \forall x_1, x_2 \in SO(n)$$

b) By the associative property of matrix multiplication,

$$(x_1, x_2) x_3 \quad = \quad x_1 (x_2 x_3).$$

for $x_1, x_2, x_3 \in SO(n)$

c) The $n \times n$ identity matrix satisfies the third property.

d) Since $x^T x = x x^T = I$, it follows that $x^T = x^{-1}$

2-8 For a rotation of θ about the x axis we have

$$\begin{aligned} x_0 \cdot x_1 &= 1 \\ y_0 \cdot y_1 &= \cos\theta \\ z_0 \cdot z_1 &= \cos\theta \\ z_0 \cdot y_1 &= \sin\theta \\ y_0 \cdot z_1 &= -\sin\theta \end{aligned}$$

and all other dot products are zero. Substituting into the rotation matrix in Section 2.2.2 gives

$$R_0^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

For a rotation of θ about the y axis we have

$$\begin{aligned} y_0 \cdot y_1 &= 1 \\ x_0 \cdot x_1 &= \cos\theta \\ z_0 \cdot z_i &= \cos\theta \\ z_0 \cdot x_1 &= -\sin\theta \\ x_0 \cdot z_1 &= \sin\theta \end{aligned}$$

and all other dot products are zero. Again using the rotation matrix gives

$$R = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

2-9 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2).$$

From Cramer's rule and the fact that $A \in SO(3)$ we have

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

which implies that $a = d$ and $b = -c$. Thus

$$A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$$

with $\det A = 1 = a^2 + c^2$. Define $\theta = \tan^{-1}(c/a)$. Then $\cos\theta = a$ and $\sin\theta = c$.

2-10

$$R = R_{y,\psi} R_{x,\phi} R_{z,\theta}$$

2-11

$$R = R_{z,\theta} R_{x,\phi} R_{x,\psi}$$

2-12

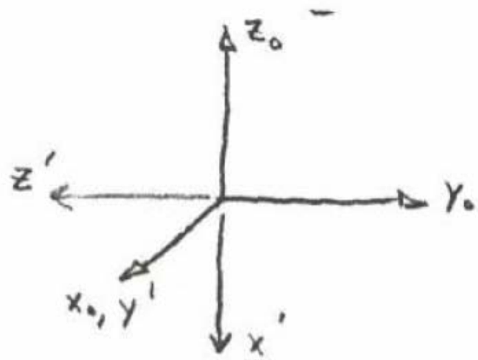
$$R = R_{z,\alpha} R_{x,\phi} R_{z,\theta} R_{x,\psi}$$

2-13

$$R = R_{z,\alpha}R_{z,\theta}R_{x,\phi}R_{x,\psi}$$

2-14

$$R = R_{y,\frac{\pi}{2}} R_{x,\frac{\pi}{2}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$



2-15

$$R_3^2 = R_1^2 R_3^1 \quad \text{where} \quad R_1^2 = (R_2^1)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Therefore,

$$R_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \end{bmatrix}$$

2-16 If r_{11}, r_{21} are not both zero, then

- $c_\theta \neq 0$ and $r_{31} = -s_\theta \neq \pm 1$
- r_{32}, r_{33} are not both zero.

so, $c_\theta = \pm\sqrt{1 - r_{31}^2}$ and $\theta =$Atan2$\big(\pm\sqrt{1 - r_{31}^2}, r_{31}\big)$.
Follow a development similar to that provided for the Euler angles to find ϕ, θ, and ψ.

2-17 Straightforward; follow directions given in sentence preceding the equation.

2-18 Straightforward. Substitute for r_{ij} in Equation (2.45) using the matrix elements given in Equation (2.43).

2-19 If λ is an eigenvalue of R and k is a unit eigenvector corresponding to λ then, $Rk = \lambda k$. Since R is a rotation $\|Rk\| = \|k\|$. This implies that $|\lambda| = 1$, i.e., the eigenvalues of R are on the unit circle in the complex plane. Since the characteristic polynomial of R is of degree three at least one eigenvalue of R must be real. Hence $+1$ or -1 is an eigenvalue of R. Now, since $+1 = \det R = \lambda_1\lambda_2\lambda_3$ where $\{\lambda_1, \lambda_2, \lambda_3\}$ is the set of eigenvalues of R, it is easy to see that if -1 is an eigenvalue then $\{\lambda_1, \lambda_2, \lambda_3\} = \{-1, -1, +1\}$. In any case $+1$ is always an eigenvalue of R.

The vector k defines the axis of rotation in the angle/axis representation of R.

2-20

$$
R_{k,\theta} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} \\ \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3} - \frac{1}{\sqrt{3}} \\ \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} \end{bmatrix}
$$

2-21 Straightforward.

2-22

$$R_{x,\theta}R_{y,\phi}R_{z,\pi}R_{y,-\phi}R_{x,-\theta}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -\cos(2\phi) & -2\cos(\phi)\sin(\phi)\sin(\theta) & \cos(\theta)\sin(2\phi) \\ -2\cos(\phi)\sin(\phi)\sin(\theta) & -\cos(\theta)^2 - \cos(2\phi)\sin(\theta)^2 & -\cos(\phi)^2\sin(2\theta) \\ \cos(\theta)\sin(2\phi) & -\cos(\phi)^2\sin(2\theta) & \cos(\phi)^2\cos(\theta)^2 - \cos(\theta)^2\sin(\phi)^2 - \sin(\theta)^2 \end{bmatrix}$$

2-23

$$R = R_{y,90}R_{z,45} \quad = \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$\theta \quad = \quad \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) = \cos^{1}\left(\frac{\frac{\sqrt{2}}{2}-1}{2}\right) = 98.42^\circ$$

$$k \quad = \quad \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} & - & r_{23} \\ r_{13} & - & r_{31} \\ r_{21} & - & r_{12} \end{bmatrix} = (0.5054481)\begin{bmatrix} 0.7071068 \\ 1.7071068 \\ 0.7071068 \end{bmatrix}$$

2-24

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

The direction of the x-axis is $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$.

2-25 Possible Euler angles:

XYZ	YZX	ZXY
XYX	YZY	ZXZ
XZY	YXZ	ZYX
XZX	YXY	ZYZ

We must be able to rotate about three different axes in order to specify an arbitrary rotation. Therefore, it is not possible to have ZZY Euler angles, since the consecutive Z rotations are rotations about the same axis.

2-26 For any two complex numbers $c_1,\, c_2 \in \mathbb{C}$,
$c_1 = a + ib = \|c_1\| \left(\cos\theta_1 + i\sin\theta_1\right)$
$c_2 = e + if = \|c_2\| \left(\cos\theta_2 + i\sin\theta_2\right)$
where $\theta_1 =$atan2(a, b) and $\theta_2 =$atan2(e, f).

$$\begin{aligned}
c_1c_2 &= \|c_1\|\,\|c_2\|\left(\cos\theta_1 + i\sin\theta_1\right)(\cos\theta_2 + i\sin\theta_2) \\
&= \|c_1\|\,\|c_2\|\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2)\right] \\
&= \|c_1\|\,\|c_2\|\left[cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]
\end{aligned}$$

$\Longrightarrow$multiplication of complex numbers corresponds to addition of angles.

2-27 Group: $\{\mathbb{C}, \cdot\}$
Using complex exponential notation, $c_1 = m_1 e^{j\theta_1}, c_2 = m_2 e^{j\theta_2}, c_3 = m_3 e^{j\theta_3}$.

1. Group is closed under group operation.
   For all c_1,$c_2 \in \mathbb{C}$,
   $$\begin{aligned} c_1 \cdot c_2 &= m_1 e^{j\theta_1} m_2 e^{j\theta_2} \\ &= m_1 m_2 e^{j(\theta_1+\theta_2)} = m_3 e^{j\theta_3} \end{aligned}$$
   where $m_3 = m_1 m_2$ and $\theta_3 = \theta_1 + \theta_2$.
2. Associativity
   For all c_1,c_2,$c_3 \in \mathbb{C}$,
   $$\begin{aligned} (c_1 c_2) c_3 &= (m_1 e^{j\theta_1} m_2 e^{j\theta_2}) m_3 e^{j\theta_3} \\ &= m_1 m_2 m_3 e^{j(\theta_1+\theta_2+\theta_3)} \\ &= m_1 e^{j\theta_1}(m_2 m_3 e^{j(\theta_2+\theta_3)}) \\ &= c_1(c_2 c_3). \end{aligned}$$
3. Identity element $I = 1 + j0 = 1e^{j0}$
   For all $c \in \mathbb{C}$,
   $$cI = c = Ic.$$
4. Inverse element
   For all $c_1 \in \mathbb{C}$, let inverse $c_2 \in \mathbb{C}$ be defined as $c_2 = \frac{1}{m_1} e^{-j\theta_1}$.
   $$c_1 c_2 = m_1 \frac{1}{m_1} e^{j\theta_1} e^{-j\theta_1} = c_2 c_1 = 1e^{j0} = I$$

2-28 Quaternion $Q = q_o + iq_1 + jq_2 + kq_3 = (q_0, q_1, q_2, q_3)$
$R_{k,\theta} \rightarrow Q = (\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2})$
Now, $\|k\| = \sqrt{n_x^2 + n_y^2 + n_z^2} = 1$ because $k = [n_x n_y n_z]^T$ is a unit vector.

$$
\begin{aligned}
\|Q\| &= \sqrt{\cos^2\frac{\theta}{2} + (n_x^2 + n_y^2 + n_z^2)\sin^2\frac{\theta}{2}} \\
&= \sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} \\
&= \sqrt{1} = 1
\end{aligned}
$$

2-29 $Q = (q_0, q_1, q_2, q_3) = (\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2})$.
Find rotation matrix $R_{k,\theta} \Rightarrow$ find k, θ.

1. $\theta = \cos^{-1}(2q_0)$
2. $k = [n_x, n_y, n_z]^T = \left[\frac{q_1}{\sin\frac{\theta}{2}}, \frac{q_2}{\sin\frac{\theta}{2}}, \frac{q_3}{\sin\frac{\theta}{2}}\right]^T$
3. Substitute values for k, θ into

$$R_{k,\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

where $v_\theta = \text{vers}\theta = 1 - c_\theta$.

2-30 Given R, find $Q = (q_0, q_1, q_2, q_3)$.

$$\theta = \cos^{-1}\left[\frac{Tr(R) - 1}{2}\right]$$

$$k = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

If $\|k\| \neq 1$, then $k' = \frac{k}{\|k\|}$.
$q_0 = \cos\frac{\theta}{2}, q_1 = n_x \sin\frac{\theta}{2}, q_2 = n_y \sin\frac{\theta}{2}, q_3 = n_z \sin\frac{\theta}{2}$

2-31 $X = x_0 + ix_1 + jx_2 + kx_3 = (x_0, x)$
$Y = y_0 + iy_1 + jy_2 + ky_3 = (y_0, y)$

$$\begin{aligned}
Z = XY &= (x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3) \\
&\vdots \\
&= x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 + x_0(iy_1 + jy_2 + ky_3) + y_0(ix_1 + jx_2 + kx_3) \\
&\quad +i(x_2y_3 - x_3y_2) - j(x_1y_3 - y_1x_3) + k(x_1y_2 - y_1x_2) \\
&= x_0y_0 - x^Ty + x_0y + y_0x + x \times y \\
&= \Big(x_0y_0 - x^Ty, (x_0y + y_0x + x \times y)\Big) \\
&= (z_0, z)
\end{aligned}$$

2-32 Given $Q = (q0, q)$ and $\|q\| = 1$,
show that $Q_I = (1, [0, 0, 0]^T) = (d_0, d)$ is the identity for unit quaternion multiplication.
We see that $d^T q = q^T d = 0$, and $d \times q = q \times d = [0, 0, 0]^T$.
Now, applying the result from problem 2-30,

$$\begin{aligned} QQ_I &= \Big(q_0 d_0 - d^T q, (q_0 d + d_0 q + q \times d)\Big) \\ &= (q_0 d_0, d_0 q) \\ &= (q_0, q) = Q. \end{aligned}$$

Similarly, we left-multiply by Q_I and find that $Q_I Q = Q$.

$$\Rightarrow QQ_I \quad = \quad Q_I Q = Q$$

Therefore Q_I is the identity element.

2-33 $Q^* = (q_0, q^*)$, where $q^* = [-q1, -q2, -q3]^T$.
Recall Q is a unit quaternion, so $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

$$q^T q^* = q^{*T} q = -q_1^2 - q_2^2 - q_3^2$$

$$q_0^2 - 1 = -q_1^2 - q_2^2 - q_3^2 = q^T q^*$$

$$\begin{aligned} q \times q^* &= i(-q_2 q_3 + q_2 q_3) - j(-q_1 q_3 + q_1 q_3) + k(q_1 q_2 + q_1 q_2) \\ &= [0, 0, 0]^T \\ &= q^* \times q \end{aligned}$$

$$\begin{aligned} QQ^* &= \Big(q_0 q_0 + 1 - q_0^2, (q_0 q^* + q_0 q + q \times q^*)\Big) \\ &= \Big(1, (0 + q \times q^*)\Big) \\ &= \Big(1, [0, 0, 0]^T\Big) = Q_I \end{aligned}$$

Similarly, $Q^* Q = \Big(q_0 q_0 + 1 - q_0^2, (q_0 q + q_0 q^* + q^* \times q)\Big) = \Big(1, [0, 0, 0]^T\Big)$.
$\Rightarrow QQ^* = Q^* Q = Q_I$.

2-34 Consider $\Big(0, [v_x, v_y, v_z]^T\Big) Q^* = X$

$$
\begin{aligned}
x_0 &= 0 - [v_x, v_y, v_z] \begin{bmatrix} -q1 \\ -q2 \\ -q3 \end{bmatrix} = v_x q_1 + v_y q_2 + v_z q_3
\end{aligned}
$$

$$
\begin{aligned}
x &= 0 + q_0 [v_x, v_y, v_z]^T + [v_x, v_y, v_z]^T \times [-q1, -q2, -q3]^T \\
&= i(q_0 v_x - q_3 v_y + q_2 v_z) + j(q_3 v_x + q_0 v_y - q_1 v_z) + k(-q_2 v_x + q_1 v_y + q_0 v_z)
\end{aligned}
$$

Now, consider $Q\Big(0, [v_x, v_y, v_z]^T\Big) Q^* = QX = Y$

$$
\begin{aligned}
y_0 &= q_0(v_x q_1 + v_y q_2 + v_z q_3) - [q1, q2, q3]^T x \\
&= q_0 q_1 v_x + q_0 q_2 v_y + q_0 q_3 v_z - q_0 q_1 v_x + q_1 q_3 v_y - q_1 q_2 v_z \\
&\quad - q_2 q_3 v_x - q_0 q_2 v_y + q_1 q_2 v_z + q_2 q_3 v_x - q_1 q_3 v_y - q_0 q_3 v_z \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
y &= q_0 x + x_0 q + q \times x \\
&= i(q_0^2 v_x + q_0 q_2 v_y + q_0 q_2 v_z + q_1^2 v_x + q_1 q_2 v_y + q_1 q_3 v_z) \\
&= +j(q_0 q_3 v_x + q_0^2 v_y - q_0 q_1 v_z + q_1 q_2 v_x + q_2^2 v_y + q_2 q_3 v_z) \\
&= +k(-q_0 q_2 v_x + q_0 q_1 v_y + q_0^2 v_x + q_1 q_3 v_x + q_2 q_3 v_y + q_3^2 v_z) + q \times x
\end{aligned}
$$

$$
\begin{aligned}
q \times x &= i(-q_2^2 v_x + q_1 q_2 v_y + q_0 q_2 v_z - q_3^2 v_x - q_0 q_3 v_y + q_1 q_3 v_z) \\
&\quad + j(q_0 q_3 v_x + q_3^2 v_y + q_2 q_3 v_z + q_1 q_2 v_x - q_1^2 v_y - q_0 q_1 v_z) \\
&\quad + k(q_1 q_3 v_x + q_0 q_1 v_y - q_1^2 v_z - q_0 q_2 v_x + q_2 q_3 v_y - q_2^2 v_z)
\end{aligned}
$$

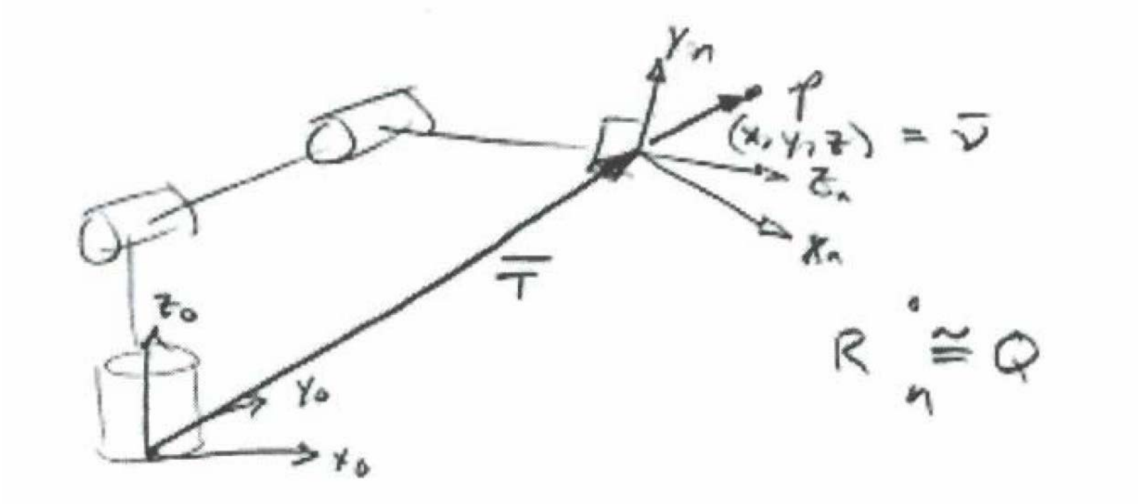
We now separate y by coefficients of i, j, k and v_x, v_y, v_z.

$$
\begin{aligned}
q_0^2 + q_1^2 - q_2^2 - q_3^2 &= (q_0^2 + q_1^2 + q_2^2 + q_3^2) - q_2^2 - q_3^2 - q_2^2 - q_3^2 \\
&= 1 - 2q_2^2 - 2q_3^2
\end{aligned}
$$

Similarly, $q_0^2 + q_2^2 - q_1^2 - q_3^2 = 1 - 2q_1^2 - 2q_3^2$
and $q_0^2 + q_3^2 - q_1^2 - q_2^2 = 1 - q_1^2 - 2q_2^2$.

$$
\Rightarrow y = \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2q_1 q_2 - 2q_0 q_3 & 2q_2 q_3 + 2q_0 q_2 \\ 2q_1 q_2 + 2q_0 q_3 & 1 - 2q_1^2 - 2q_3^2 & 2q_2 q_3 - 2q_0 q_1 \\ 2q_1 q_3 - 2q_0 q_2 & 2q_2 q_3 + 2q_0 q_1 & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = R_v
$$

Hence, $Y = Q(0, v_x, v_y, v_z) Q^* = (0, R_v)$ where R_v are the new rotated coordinates of v.



y_n
p
$(x, y, z) = \bar{v}$
z_n
x_n
$\bar{T}$
z_0
y_0
x_0
$R_n^0 \cong Q$

2-35 Suppose *point* p has been expressed in frame n as $p^n = [x, y, z]^T$. Ignoring quaternions, we know we can write the location of p in base frame coordinates as

$$p^0 = R_n^0 p^n + T.$$

Now, we apply the result from problem 2-33 which gives the following equivalence

$$(0, R_n^0 p^n = Q(0, p^n)Q^*.$$

Since T is just the vector between the two frames, we can now write the expression

$$(0, p^0) = (0, T) + Q(0, p^n)Q^*$$

2-36

$$H^{-1}H = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T R & R^T d - R^T d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$HH^{-1} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR^T & -RR^T d + d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} = I.$$

So H^{-1} is the inverse of H.

2-37

$$
\begin{aligned}
T &= T_{y,1} T_{x,3} T_{z,\pi/2} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

2-38

$$H_1^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2^1 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2-39

$$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2^0 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_3^0 = \begin{bmatrix} 0 & 1 & 0 & -.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2-40

$$H_2^1 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & .5 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^0 = H_1^0 H_2^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & .5 & \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = H_2^0 H_3^2 = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -.5 \\ 0 & -1 & 0 & 1.5 \\ 0 & 0 & -1 & 3.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2-41

$$H_3^2 = \begin{bmatrix} 1 & 0 & 0 & -.3 \\ 0 & -1 & 0 & .4 \\ 0 & 0 & -1 & 1.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The homogeneous transformation from the block frame to the base frame is

$$H_2^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & .8 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2-42 The earth rotates in the *ecliptic* plane about the sun, at a distance of approximately 150 million km. At the summer solstice ($t = 0$), the earth's axis of rotation z_{earth} is tilted 23.5° toward the sun. Let x_{earth} point in direction of the motion of the earth, always lying in the ecliptic plane and perpendicular to the vector from the sun to the earth. Let the z axis of the sun z_{sun} pass through the center of the sun and be perpendicular to the ecliptic plane. Noting that at $t = 0$ the earth's coordinate frame is coincident with the base frame, we write the homogeneous transformation between the base frame and the sun frame as follows.

$$H_{sun}^{base} = \begin{bmatrix} R_{x,23.5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix} 0 \\ 150 \times 10^6 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Suppose the units of time to be days. Let θ be the angle in degrees between x_{sun} and the ray from the center of the sun to the center of the earth. Since the earth makes a complete revolution about the sun in 365.25 days, we write

$$\theta = \frac{t}{365.25} 360° - 90°$$

where $-90°$ is the offset of θ when $t = 0$. We are now prepared to write the homogeneous transformation from the sun frame to the earth frame at any time t

$$H_{earth}^{sun} = \begin{bmatrix} I & \begin{bmatrix} 150 \times 10^6 \cos\theta \\ 150 \times 10^6 \sin\theta \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{x,-23.5} & 0 \\ 0 & 1 \end{bmatrix}.$$

The homogeneous transformation between the base frame and earth frame is given by

$$H_{earth}^{base} = H_{sun}^{base} H_{earth}^{sun}.$$

The instantaneous orientation of the earth frame w.r.t. the base frame is the product of the rotation matrices given above

$$R_{earth}^{base} = R_{x,23.5} I I R_{x,-23.5} = I.$$

This is as we expect, since the axis of the earth maintains the same tilt as the earth revolves around the sun.

$$H = Rot_{x,\alpha} \text{Trans}_{x,b} \text{Trans}_{z,d} Rot_{z,\theta}$$

Translation and Rotations about the same axis commute because the orientation of the axis is preserved.

Translations commute because the orientation of the reference axes is preserved.

$$H = \begin{cases} R_{x,\alpha} & T_{z,d} & T_{x,b} & R_{z,\theta} \\ T_{x,b} & R_{x,\alpha} & T_{z,d} & R_{z,\theta} \\ T_{x,b} & R_{x,\alpha} & R_{z,\theta} & T_{z,d} \\ R_{x,\alpha} & T_{x,b} & R_{z,\theta} & T_{z,d} \end{cases}$$

3-1 From Equation (3.13), we know that R has the form

$$R = \begin{bmatrix} c_\theta & r_{12} & r_{13} \\ s_\theta & r_{22} & r_{23} \\ 0 & s_\alpha & c_\alpha \end{bmatrix}$$

Since R is a rotation matrix the column vectors satisfy

$$\begin{aligned} r_{12}^2 + r_{22}^2 &= 1 - s_\alpha^2 = c_\alpha^2 \\ r_{13}^2 + r_{23}^2 &= 1 - c_\alpha^2 = s_\alpha^2 \end{aligned}$$

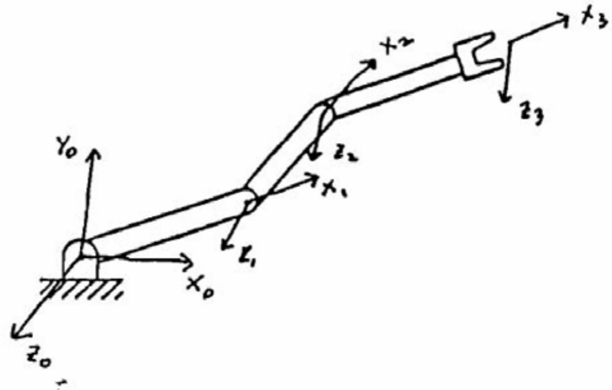
Therefore there is a unique angle θ such that

$$\begin{aligned} r_{12}/c_\alpha = -s_\theta; &\qquad r_{22}/c_\alpha = c_\theta \\ r_{13}/s_\alpha = s_\theta; &\qquad r_{23}/s_\alpha = -c_\theta \end{aligned}$$

and the results follows.

In each of the following problems 3-2 to 3-7, the figure shows the D-H coordinate frames and a table of D-H parameters that is used to generate the A matrices and the T matrix giving the transformation between the base frame and the end-effector frame.

3-2



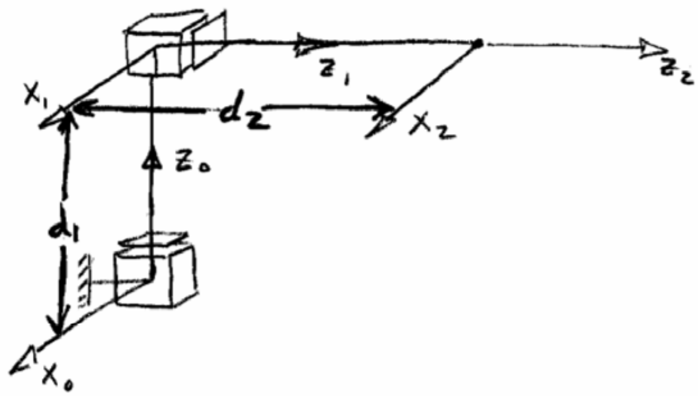
link	a_1	α_i	d_i	θ_i
1	α_1	0	0	θ_1
2	α_2	0	0	θ_2
3	α_3	0	0	θ_3

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} c_2 & -c_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = A_1A_2A_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-3

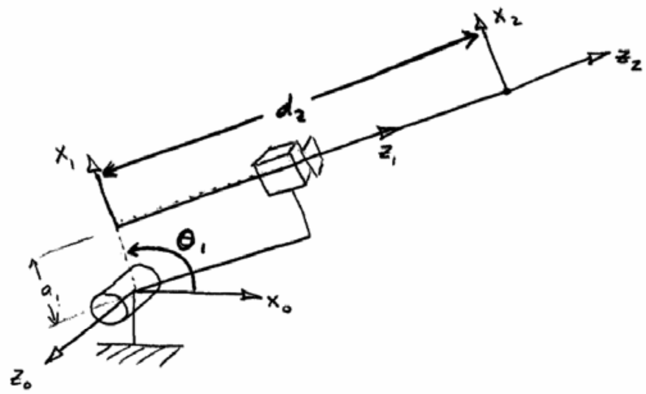


link	a_i	α_i	d_i	θ_i
1	0	-90°	d_1	0
2	0	0	d_2	0

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^2 = A_1 A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-4

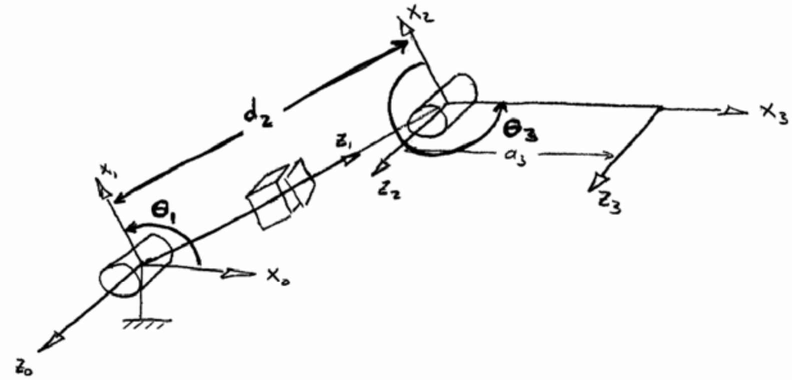


link	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	0	-90°	d_2	0
3	a_3	0	d_3	θ_3

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^2 = A_1 A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-5

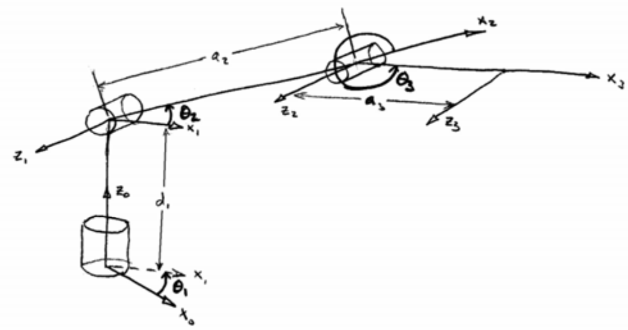


link	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	0	-90°	d_2	0
3	a_3	0	d_3	θ_3

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = A_1A_2A_3 = \begin{bmatrix} c_{13} & -s_{13} & 0 & s_1d_2 + a_3c_{13} \\ s_{13} & c_{13} & 0 & -c_1d_2 + a_3s_{13} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



link	a_i	α_i	d_i	θ_i
1	0	90	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

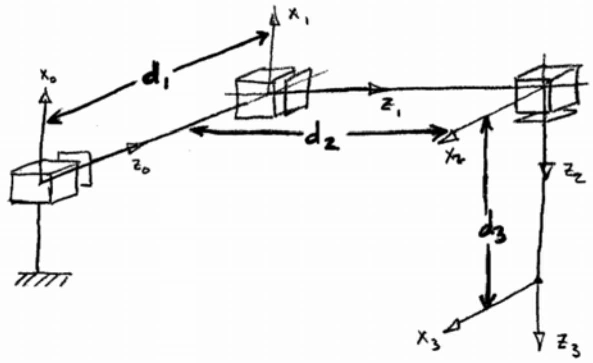
$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$\begin{aligned}
r_{11} &= c_1c_2c_3 - c_1s_2s_3 = c_1c_{23} \\
r_{12} &= -c_1c_2s_3 - c_1c_3c_2 = -c_1s_{23} \\
r_{13} &= s_1 \\
d_x &= a_2a_2c_1c_2 + a_3c_1c_2c_3 - a_3c_1s_2s_3 = a_2c_1c_2 + a_3c_1c_{23} \\
r_{21} &= c_2c_3s_1 - s_1s_2s_3 = x_1c_{23} \\
r_{22} &= -c_2s_1s_3 - c_3s_1s_2 = -s_1s_{23} \\
r_{23} &= -c_1 \\
d_y &= a_2c_2s_1 + a_3c_2c_3s_1 - a_3s_1s_2s_3 = a_2c_2s_1 + a_3s_1c_{23} \\
r_{31} &= c_2s_3 + c_3s_2 = s_{23} \\
r_{32} &= c_2c_3 - s_2s_3 = c_{23} \\
r_{33} &= 0 \\
d_z &= a_2s_2 + a_3c_2s_3 + a_3c_3s_2 = a_2s_2 + a_3s_{23}
\end{aligned}$$

3-7



link	a_i	α_i	d_i	θ_i
1	0	$-90°$	d_1	0
2	0	$90°$	d_2	$90°$
3	0	0	d_3	$-90°$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = A_1 A_2 A_3 = \begin{bmatrix} 0 & 0 & 1 & d_3 \\ -1 & 0 & 0 & d_2 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

link	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3
4	0	−90°	0	θ_4
5	0	0	0	θ_5
6	0	0	d_6	θ_6

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
r_{11} &= c_1[c_5c_6c_{234} - s_6s_{234}] - s_1s_5c_6 \\
r_{12} &= -c_1[c_5s_6c_{234} + c_6s_{234}] + s_1s_5s_6 \\
r_{13} &= c_1s_5c_{234} + s_1c_5 \\
d_x &= a_2c_1c_2 + a_3c_1c_{23} + d_6[c_1s_5c_{234} + s_1c_5] \\
r_{21} &= c_1s_5s_6 + s_1c_5c_6c_{234} - s_1s_6s_{234} \\
r_{22} &= -c_1s_5s_6 - s_1c_5s_6c_{234} \\
r_{23} &= -c_1c_5 + s_1s_5c_{234} \\
d_y &= a_2s_1c_2 + a_3s_1c_{23} - d_6[c_1c_5 + s_1s_5c_{234}] \\
r_{31} &= s_6c_{234} + c_5s_6s_{234} \\
r_{32} &= c_6s_{234} - c_5s_6s_{234} \\
r_{33} &= s_5s_{234} \\
d_z &= a_2s_2 + a_3c_2s_{23} + d_6s_5s_{234}
\end{aligned}$$

The matrix T_0^3 is given as in Problem 3-7. The matrix T_3^6 is given by Equation (3.15) of the text. Therefore

$$T_0^6 = \begin{bmatrix} -c_6s_5 & s_5s_6 & c_5 & d_3 + d_6c_5 \\ -c_4c_5c_6 + s_4s_6 & c_4c_5s_6 + c_6s_4 & -c_4s_5 & d_2 - d_6c_4s_5 \\ -c_4s_6 - c_5c_6s_4 & -c_4c_6 + c_5s_4s_6 & -s_4s_5 & d_1 - d_6s_4s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

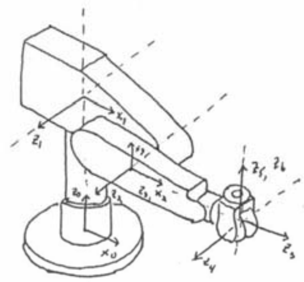
3-9 Attaching a spherical wrist to the robot of Problem 3-7 gives

$$T_0^5 = T_0^3 T_3^6$$

The matrix T_0^3 is given as in Problem 3-7. The matrix T_3^6 is given by Equation (3.15) of the text. Therefore

$$T_0^6 = \begin{bmatrix} -c_6 s_5 & s_5 s_6 & c_5 & d_3 + d_6 c_5 \\ -c_4 c_5 c_6 + s_4 s_6 & c_4 c_5 s_6 + c_6 s_4 & -c_4 s_5 & d_2 - d_6 c_4 s_5 \\ -c_4 s_6 - c_5 c_6 s_4 & -c_4 c_6 + c_5 s_4 s_6 & -s_4 s_5 & d_1 - d_6 s_4 s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3-10



link	a_i	α_i	d_i	θ_i
1	0	$90°$	$13''$	θ_1
2	$8''$	0	d_2	θ_2
3	$8''$	$90°$	0	θ_3
4	0	$-90°$	d_4	θ_4
5	0	$90°$	0	θ_5
6	0	0	d_6	θ_6

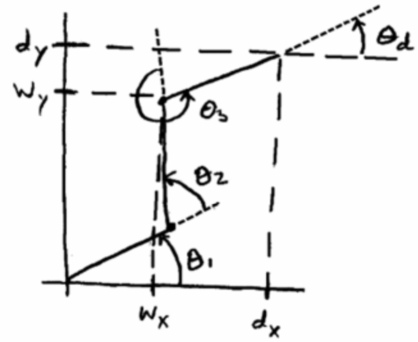
$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & 8c_2 \\ s_2 & c_2 & 0 & 8s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_3 = \begin{bmatrix} c_3 & 0 & s_3 & 8c_3 \\ s_3 & 0 & -c_3 & 8s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where:

$$\begin{aligned}
r_{11} &= c_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_4s_6s_{23}] + s_1[c_4s_6 + s_4c_5c_6] \\
r_{12} &= c_1[-c_{23}(c_4c_5s_6 + s_4c_6) + s_5s_6s_{23}] + s_1[c_4c_6 - s_4c_5s_6] \\
r_{13} &= c_1[c_4s_5c_{23} + c_5s_{23}] - s_1s_4s_5 \\
d_x &= d_2s_1 + d_4c_1s_{23} + d_6[c_1(c_4s_5c_{23} + c_5s_{23}) + s_1s_4 + s_5] + 8c_1[c_{23} + c_2] \\
r_{21} &= -c_1[c_4s_6 + s_4c_5c_6] + s_1[c_{23}(c_4c_5c_6 + s_4s_6) - s_5c_6s_{23}] \\
r_{22} &= c_1[s_4c_5s_6 - c_4c_6] + s_1[-c_{23}(c_4c_5s_6 + s_4c_6) + s_5s_6s_{23}] \\
r_{23} &= -c_1s_4s_5 + s_1[c_4s_5c_{23} + c_5s_{23}] \\
d_y &= -d_2c_1 + d_4s_1s_{23} + d_6[s_1(c_4s_5c_{23} + c_5s_{23}) - c_1s_4s_5] - c_1s_4s_5] + 8s_1[c_{23} + c_2] \\
r_{31} &= s_{23}(c_4c_5c_6 - s_4s_6) + s_5c_6c_{23} \\
r_{32} &= -s_{23}(c_4c_5s_6 + s_4c_6) - s_5s_6c_{23} \\
r_{33} &= -c_5c_{23} + c_4s_5s_{23} \\
d_z &= 13 - d_4c_{23} + d_6[-c_5c_{23} + c_4s_5s_{23}] + 8[s_{23} + s_2]
\end{aligned}$$

3-11



1. Given a desired position $d = [d_x, d_y]^T$ of the end-effector only, we can write the coordinates of the end-effector as two equations in three unknowns.

$$\begin{aligned} d_x &= a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) + a_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ d_y &= a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) + a_3 \sin(\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

Therefore, this problem is underconstrained. In general, there are infinitely many solutions to the inverse kinematics problem. More specifically,

$$\text{there are} \begin{cases} \infty & \text{solutions if } d \text{ is inside workspace} \\ 1 & \text{solution if } d \text{ is on workspace boundary} \\ 0 & \text{solutions if } d \text{ is outside workspace.} \end{cases}$$

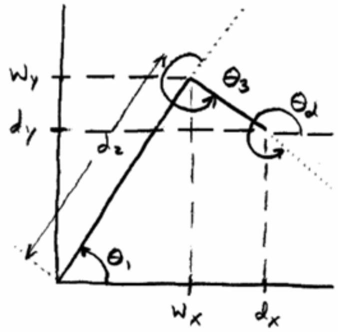
2. Given a desired position $d = [d_x, d_y]^T$ and orientation θ_d of the end-effector, we can write the coordinates of the wrist center $[w_x, w_y]^T$

$$\begin{aligned} w_x &= d_x - a_3 \cos(\theta_d) \\ w_y &= d_y - a_3 \sin(\theta_d). \end{aligned}$$

Now we have reduced the problem to finding a solution for the first two links that will reach the wrist center. The solutions for θ_1 and θ_2 are given in Equations (1.7-1.8).

$$\theta_3 = \theta_d - (\theta_1 + \theta_2)$$

$$\text{There are} \begin{cases} \infty & \text{solutions if the wrist center is the origin} \\ 2 & \text{solutions if the wrist center is inside the 2-link workspace} \\ 1 & \text{solution if the wrist center is on the 2-link workspace boundary} \\ 0 & \text{solutions if the wrist center is outside the 2-link workspace.} \end{cases}$$



1. Given a desired position $d = [d_x, d_y]^T$ of the end-effector only, we can write the coordinates of the end-effector as two equations in three unknowns.

$$\begin{aligned} d_x &= d_2 \cos(\theta_1) + a_3 \cos(\theta_1 + \theta_3) \\ d_y &= d_2 \sin(\theta_1) + a_3 \sin(\theta_1 + \theta_3) \end{aligned}$$

Therefore, this problem is underconstrained. In general, there are infinitely many solutions to the inverse kinematics problem. More specifically,

$$\text{there are} \begin{cases} \infty & \text{solutions if } d \text{ is inside workspace} \\ 1 & \text{solution if } d \text{ is on workspace boundary} \\ 0 & \text{solutions if } d \text{ is outside workspace.} \end{cases}$$

2. Given a desired position $d = [d_x, d_y]^T$ and orientation θ_d of the end-effector, we can write the coordinates of the wrist center $[w_x, w_y]^T$

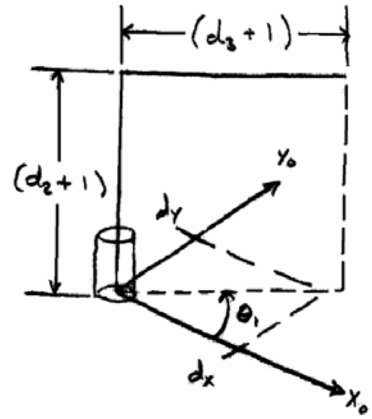
$$\begin{aligned} w_x &= d_x - a_3 \cos(\theta_d) \\ w_y &= d_y - a_3 \sin(\theta_d). \end{aligned}$$

Now we have reduced the problem to finding a solution for the first two links that will reach the wrist center. Solving the geometric problem, we find

$$\begin{aligned} \theta_1 &= \text{Atan2}(w_x, w_y) \\ d_2 &= \sqrt{(w_x^2 + w_y^2)} \\ \theta_3 &= \theta_d - \theta_1 \end{aligned}$$

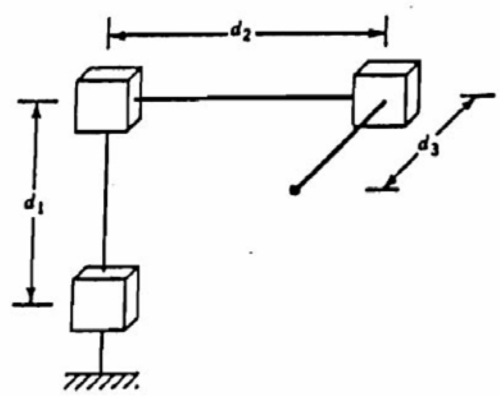
$$\text{There are} \begin{cases} \infty & \text{solutions if the wrist center is the origin} \\ 1 & \text{solution if the wrist center is on or inside the 2-link workspace boundary} \\ 0 & \text{solutions if the wrist center is outside the 2-link workspace.} \end{cases}$$

3-13



Given $d = (d_x, d_y, d_z)^T$, we have

$$\begin{aligned}
\theta_1 &= \text{Atan2}(d_x, d_y) \\
d_2 &= d_z - 1 \\
d_3 &= -1 + \sqrt{d_x^2 + d_y^2}.
\end{aligned}$$



3-14 Given $d = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$ can be reached by setting

$$\begin{bmatrix} d_z \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}$$

3-15

$$R_3^6 = (R_0^3)^T R = U = \begin{bmatrix} r_{31} & r_{32} & r_{33} \\ r_{11}c_1 + r_{21}s_1 & r_{12}c_1 + r_{22}s_1 & r_{13}c_1 + r_{23}s_1 \\ -r_{11}s_1 + r_{21}c_1 & -r_{21}s_1 + r_{22}c_1 & -r_{13}s_1 + r_{23}c_1 \end{bmatrix}$$

I. If not both $u_{13} + u_{23}$ are zero, then

$$\theta_5 = A\tan\left(-r_{13}s_1 + r_{23}c_1, \pm\sqrt{1 - (-r_{13}s_1 + r_{23}c_1)^2}\right)$$

a) If the positive square root is chosen

$$\theta_4 = A\tan(r_{33}, r_{13}c_1 + r_{23}s_1)$$
$$\theta_6 = A\tan(+r_{11}s_1 - r_{21}c_1, -s_1r_{12} + c_1r_{22})$$

b) If the negative square root is chosen

$$\theta_4 = A\tan(-r_{33}, -r_{13}c_1 + r_{23}s_1)$$
$$\theta_6 = A\tan(-r_{11}s_1 + r_{21}c_1, s_1r_{12} - c_1r_{22})$$

II. If $u_{13} = u_{23} = 0$

a) If $u_{33} = 1$

$$0 = r_{33} = r_{13}c_1r_{28}s_1 + r_{28}s_1 = c_4s_5 = s_4s_5 \to s_5 = 0 \quad \theta_5 = 0^\circ$$
$$\theta_4 + \theta_6 = A\tan(r_{31}, r_{11}c_1 + r_{21}s_1) = A\tan(r_{31}, -r_{32})$$

b) If $u_{33} = 1$ $\theta_4 = 0$; $c_5 = -1$ $s_5 = 0$ $\theta_5 = \pi$

$$\theta_4 - \theta_6 = A\tan(-r_{31}, -r_{32}) = A\tan(-r_{11}c_1 - r_{21}s_1, -r_{12}c_1 - r_{22}s_1)$$

3-16

link	a_i	α_i	d_i	θ_i
1	0	-90	d_1^*	0
2	0	90	d_2^*	90
3	0	0	d_3^*	0
4	0	90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* denotes variable

$$R_0^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Given d and R

$$p_c = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix}$$

$$r_3^6 = (R_0^6)^T R = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} -r_{31} & -r_{32} & -r_{33} \\ r_{21} & r_{22} & r_{23} \\ r_{11} & r_{12} & r_{13} \end{bmatrix}$$

Equate R_3^6 to matrix (4.4.1). Suppose that r_{33} and r_{23} are unzero, then $r_{13} \neq \pm 1$, so $c_\theta = r_{13}$, $s_\theta = \pm\sqrt{1 - r_{13}^2}$ and $\theta = A\tan\left(r_{13}, \sqrt{1 - r_{13}^2}\right)$, if $s_\theta > 0$, choose $\phi = A\tan(r_{33}, r_{23})$ and $\psi = A\tan(-4_{11}, r_{12})$. However, if $r_{33} = r_{23} = 0$, then $r_{13} = \pm 1$, if

$$r_{13} = +1 \quad - = 0, \phi + \psi = A\tan(-r_{31}, r_{21}) = A\tan(-r_{31}, r_{32})$$

if

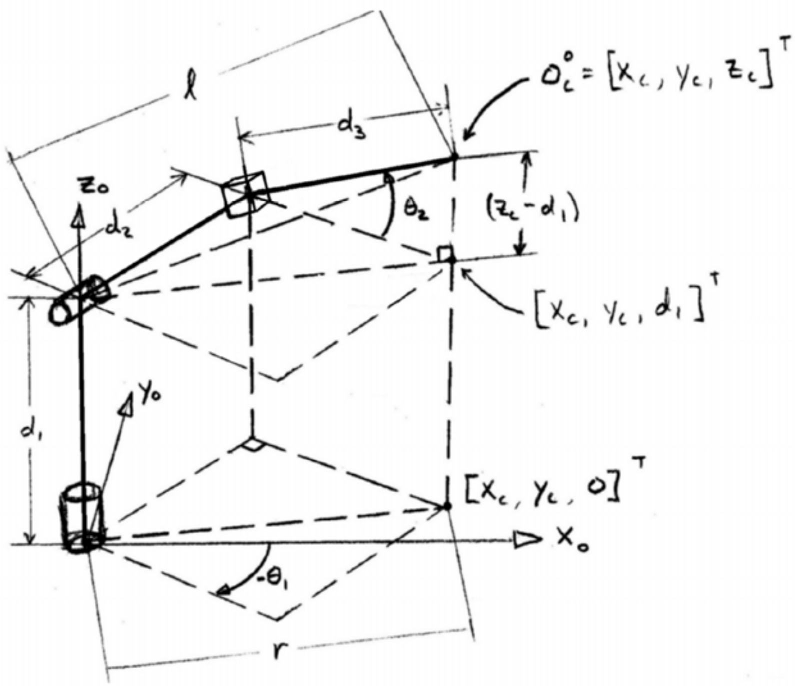
$$r_{13} = -1 \quad \theta = 0, \phi + \psi = A\tan(-r_{31}, r_{21}) = A\tan(-r_{31}, r_{32})$$

if

$$r_{13} = \pm 1,$$

there are an infinite number of solutions.

3-17 machine problem



We are given the desired position d and orientation R of the tool.

1. **desired coordinates of the wrist center**

$$o_c^0 = d - R \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix}$$

where d_6 is the distance from the wrist center to the origin of the tool frame.

2. **inverse position kinematics**
This problem is difficult to visualize; success will often depend on the quality of the sketch made of the first three links, especially the "cheese wedge" region formed by the upper arm, elbow, and lower arm. Making use of right triangles and the Pythagorean theorem, we have

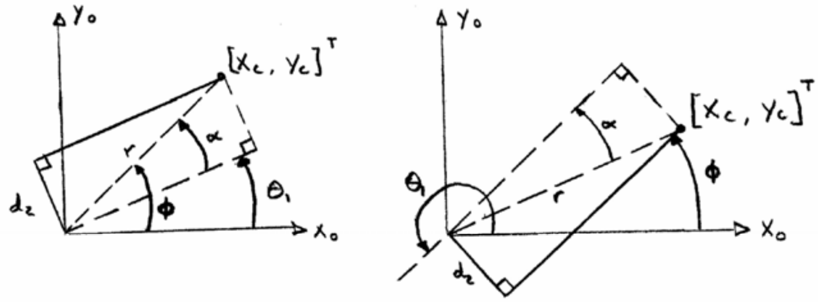
$$\begin{aligned} r^2 &= x_c^2 + y_c^2 \\ \ell^2 &= d_2^2 + d_3^2 \\ \text{and } \ell^2 &= (z_c - d_1)^2 + r^2. \end{aligned}$$

Solving these three equations simultaneously yields a solution for the prismatic joint.

$$d_3 = \sqrt{(z_c - d_1)^2 + x_c^2 + y_c^2 - d_2^2}$$

Again using a right triangle, we find a solution for θ_2.

$$\theta_2 = \begin{cases} \text{Atan2}\big(\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{left arm} \\ \text{Atan2}\big(-\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{right arm} \end{cases}$$



This results in a total of *two solutions.* Finally, we project the first three links of the manipulator onto the $x_0 - y_0$ plane to find solutions for waist angle θ_1.

$$\begin{aligned}
\phi &= \text{atan2}(x_c, y_c) \\
\alpha &= \text{atan2}(\sqrt{r^2 - d_2^2}, d_2)
\end{aligned}$$

$$\theta_1 = \begin{cases} \phi - \alpha & \text{left arm} \\ \phi + \alpha + \pi & \text{right arm} \end{cases}$$

3. **inverse orientation kinematics**

$$R_0^3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 \\ s_1c_2 & c_1 & s_1s_2 \\ -s_2 & 0 & c_2 \end{bmatrix}$$

found by multiplying $A_1A_2A_3$ and extracting first 3 rows and columns

$$R_3^6 = (R_0^3)^T R$$

$$= \begin{bmatrix} c_1c_2r_{11} + s_1c_2r_{21} - s_2r_{31} & c_1c_2r_{12} + s_1c_2r_{22} - s_2r_{32} & c_1c_2r_{13} + s_1c_2r_{23} - s_2r_{33} \\ -s_1r_{11} + c_1r_{21} & -s_1r_{12} + c_1r_{22} & -s_1r_{13} + c_1r_{23} \\ c_1s_2r_{11} + s_1s_2r_{21} + c_2r_{31} & c_1s_2r_{12} + s_1s_2r_{22}r_{32} & c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33} \end{bmatrix}$$

Assume $r_{13} \neq 0$ and $R_{23} \neq 0$ then

$$c_5 = c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33}$$

and

$$s_5 = \pm\sqrt{1 - (c_2s_2r_{13} + s_1s_2r_{23} + c_2r_{33})^2}$$

if $s_5 > 0$ then

$$\begin{aligned}
\theta_5 &= A\tan\left(c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33}, \sqrt{1 - (c_1s_2r_{13} + s_1s_2r_{23} + c_2r_{33})^2}\right) \\
\theta_4 &= A\tan(c_1c_2r_{13} + s_1c_2r_{23} - s_2r_{33}, -s_1r_{13} + c_1r_{23}) \\
\theta_6 &= A\tan(c_1s_2r_{11} + s_1s_2r_{21} + c_2r_{31}, -c_1s_2r_{12} - s_1s_2r_{22} - c_2r_{32})
\end{aligned}$$

if $s_5 < 0$ then

$$
\begin{aligned}
\theta_5 &= A\tan\left(c_1 s_2 r_{13} + s_1 s_2 r_{23} + c_2 r_{33}, -\sqrt{1 - (c_1 s_2 r_{13} + s_1 s_2 r_{23} + c_2 r_{33})^2}\right) \\
\theta_4 &= A\tan(-c_1 c_2 r_{13} - s_1 c_2 r_{23} + s_2 r_{33}, s_1 r_{13} - c_1 r_{23}) \\
\theta_6 &= A\tan(-c_1 s_2 r_{11} - s_1 s_2 r_{21} - c_2 r_{31}, c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32})
\end{aligned}
$$

if $r_{13} = r_{23} = 0$ then $r_{33} = \pm 1$
if $r_{33} = +1$ $\theta_5 = \theta_2$ abd $\theta_4 = \pi$
if $s_5 > 0$

$$
\theta_6 = A\tan(c_1 s_2 r_{11} + s_1 s_2 r_{21} + c_2 r_{31}, -c_1 s_2 r_{12} - s_1 s_2 r_{22} - c_2 r_{32})
$$

if $s_5 < 0$

$$
\theta_6 = A\tan(-c_1 s_2 r_{11} - s_1 s_2 r_{21} - c_2 r_{31}, c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32})
$$

if $r_{33} = -1$ $\theta_5 = \pi - \theta_2$ and $\theta_4 = 0$
if $s_5 > 0$

$$
\theta_6 = A\tan(c_1 s_2 r_{11} + s_1 s_2 r_{21} + c_2 r_{31}, -c_1 s_2 r_{12} - s_1 s_2 r_{22} - c_2 r_{32})
$$

if $s_5 < 0$

$$
\theta_6 = A\tan(-c_1 s_2 r_{11} - s_1 s_2 r_{21} - c_2 r_{31}, c_1 s_2 r_{12} + s_1 s_2 r_{22} + c_2 r_{32})
$$

link	a_1	α_i	d_i	θ_i
1	0	$90°$	d_1	θ_1^*
2	α_2	0	d_2	θ_2^*
3	α_3	0	0	θ_3^*

* denotes variable

1. **desired coordinates of the wrist center**

$$\begin{aligned} x_c &= x_0 - d_6 c_5 c_1 \\ y_c &= y_0 - d_6 c_5 s_1 \\ z_c &= z_0 - z_{0c} \end{aligned}$$

2. **inverse position kinematics**

$$\begin{aligned} \theta_1 &= \phi - \alpha \\ \phi &= \tan\left(\frac{y_c}{x_c}\right) \\ \alpha &= \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right) \end{aligned}$$

Elbow Right

$$\begin{aligned} \theta_1 &= \tan\left(\frac{y_c}{x_c}\right) - \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right) \\ \theta_1 &= \phi + \alpha \\ \phi &= \tan\left(\frac{y_c}{x_c}\right) \\ \alpha &= \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right) \end{aligned}$$

Elbow Left

$$\theta_1 = \tan\left(\frac{y_c}{x_c}\right) + \tan\left(\frac{a_3 c_{23} + a_2 c_2}{d_2}\right)$$

by the 2-link planar solution

$$\begin{aligned} \theta_3 &= A\tan\left(d, \pm\sqrt{1 - D^2}\right) \text{ where } D = \frac{s_c^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2 a_3} \\ \theta_2 &= A\tan(s_c, z_c - d_1) - A\tan(a_2 + a_3 c_{31} a_3 s_3) \end{aligned}$$

3. **inverse orientation kinematics**

$$R_0^3 = \begin{bmatrix} c_1c_2c_3 - c_1s_2s_3 & -c_1c_2s_3 - c_1s_2c_3 & s_1 \\ s_1c_2c_3 - s_1s_2s_3 & -s_1c_2s_3 - s_1s_2c_3 & -c_1 \\ s_2c_3 + c_2s_3 & -s_2s_3 + c_2c_3 & 0 \end{bmatrix}$$

$$u = (R_0^3)^T R = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$$

$$\begin{aligned}
u_{11} &= r_{11}(c_1c_2c_3 - c_1s_2s_3) + r_{21}(s_1c_2c_3 - s_1s_2s_3) + r_{31}(s_2c_3 + c_2s_3) \\
u_{21} &= -r_{12}(c_1c_2s_3 + c_1s_2c_3) - r_{21}(s_1c_2s_3 + s_1s_2c_3) + r_{31}(-s_2s_3 + c_2c_3) \\
u_{31} &= r_{11}s_1 - r_{21}c_1 \\
u_{12} &= r_{12}(c_1c_2c_3 - c_1s_2s_3) + r_{22}(s_1c_2c_3 - s_1s_2s_3) + r_{32}(s_2c_3 + c_2s_3) \\
u_{22} &= -r_{12}(c_1c_2s_3 + c_1s_2c_3) - r_{22}(s_1c_2s_3 + s_1s_2c_3) + r_{32}(-s_2s_3 + c_2c_3) \\
u_{32} &= r_{12}s_1 - r_{22}c_1 \\
u_{13} &= r_{13}(c_1c_2c_3 - c_1s_2s_3) + r_{23}(s_1c_2c_3 - s_1s_2s_3) + r_{33}(s_2c_3 + c_2s_3) \\
u_{23} &= -r_{13}(c_1c_2s_3 + c_1s_2c_3) - r_{23}(s_1c_2s_3 + s_1s_2c_3) + r_{33}(-s_2s_3 + c_2c_3) \\
u_{33} &= r_{13}s_1 - r_{23}c_1
\end{aligned}$$

Inverse orientation solutions

I. Suppose not both u_{13}, u_{23} are zero

$$\theta_5 = A\tan\left(u_{33} \pm \sqrt{I - u_{33}^2}\right)$$

a) If the positive square root is chosen

$$\begin{aligned}
\theta_4 &= A\tan(u_{13}, u_{23}) \\
\theta_6 &= A\tan(u_{31}, u_{32})
\end{aligned}$$

b) If the negative square root is chosen

$$\begin{aligned}
\theta_4 &= A\tan(-u_{13}, -u_{23}) \\
\theta_6 &= A\tan(u_{31}, -u_{32})
\end{aligned}$$

II. If $u_{13} = u_{23} = 0$

a) And if $u_{33} = 1$; $\theta_5 = 0$

$$\theta_4 + \theta_6 = A\tan(u_{11}, u_{21})$$

b) Or if $u_{33} = -1$; $\theta_5 = \pi$

$$\theta_4 - \theta_6 = A\tan(-u_{11}, -u_{12})$$

3-20

$$u = R_0^{3T} R$$

$$= \begin{bmatrix} c_1c_{23}r_{11} + s_1c_{23}r_{21} - s_{23}r_{31} & c_1c_{23}r_{12} + s_1c_{23}r_{22} - s_{23}r_{32} & c_1c_{23}r_{13} + s_1c_{23}r_{23} - s_{23}r_{33} \\ -c_1s_{23}r_{11} - s_1s_{23}r_{21} - c_{23}r_{31} & -c_1s_{23}r_{12} - s_1s_{23}r_{22} - c_{23}r_{32} & -c_1s_{23}r_{13} - s_1s_{23}r_{13} - s_1s_{23}r_{23} - c_2r_{33} \\ -s_1r_{11} + c_1r_{21} & -s_1r_{12} + c_1r_{22} & -s_1r_{13} + c_1r_{23} \end{bmatrix}$$

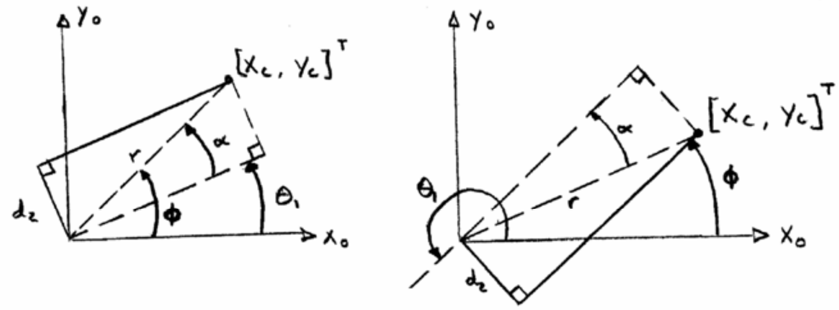
If $u_{13} = u_{23} = 0$ and $u_{33} = 1$

$$\theta_4 + \theta_6 = A\tan(c_1c_{23}r_{11} + s_1c_{23}r_{21} - s_{23}r_{31}, -c_1s_{23}r_{11} - s_1s_{23}r_{21} - c_{23}r_{31})$$

If $u_{33} = -1$

$$\theta_4 - \theta_6 = A\tan(-c_1c_{23}r_{11} - s_1c_{23}r_{21} + s_{23}r_{31}, -c_1c_{23}r_{12} - s_1c_{23}r_{22} + s_{23}r_{32})$$

3-21



Equation (3.47) for θ_1 would become

$$\theta_1 = \begin{cases} \phi - \alpha & \text{left arm} \\ \phi + \alpha + \pi & \text{right arm} \end{cases}$$

where

$$\begin{aligned} \phi &= \text{atan2}(x_c, y_c) \\ \alpha &= \text{atan2}(\sqrt{r^2 - d_2^2}, d_2) \end{aligned}$$

and Equation (3.49) for θ_2 would become

$$\theta_2 = \begin{cases} \text{Atan2}\big(\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{left arm} \\ \text{Atan2}\big(-\sqrt{r^2 - d_2^2}, (z_c - d_1)\big) & \text{right arm} \end{cases}$$

4-1 straightforward.

$$
\begin{aligned}
S(a)p &= \begin{bmatrix} 0 & -a_x & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -a_x p_y + a_y p_z \\ a_x p_z - a_x p_x \\ -a_y p_x + a_x p_y \end{bmatrix} \\
a \times p &= \begin{bmatrix} i & j & k \\ a_x & a_y & a_z \\ p_x & p_y & p_z \end{bmatrix} = i(a_y p_z - a_z p_y) - j(a_x p_z - a_z p_x) + k(a_z p_y - a_y p_x)
\end{aligned}
$$

Therefore $S(a)p = a \times p$.

4-3 Let $R = (r_1, r_2, r_3)$, where r_1, r_2, r_3 are the column vectors of R. Let $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ be vectors. Then

$$\begin{aligned} Ra &= a_1 r_1 + a_2 r_2 + a_3 r_3 \\ Rb &= b_1 r_1 + b_2 r_2 + b_3 r_3 \end{aligned}$$

Multiplying these together and using the properties of the cross product yields

$$\begin{aligned} Ra \times Rb &= (a_1 r_1 + a_2 r_2 + a_3 r_3) \times (b_1 r_1 + b_2 r_2 + b_3 r_3) \\ &= a_1 b_2 r_1 \times r_2 + a_1 b_3 r_1 \times r_3 \\ &+ a_2 b_1 r_2 \times r_1 + a_2 b_3 r_2 \times r_3 \\ &+ a_3 b_1 r_3 \times r_1 + a_3 b_2 r_3 \times r_2 \\ &= (a_1 b_2 - a_2 b_1) r_1 \times r_2 + (a_1 b_3 - a_3 b_1) r_1 \times r_3 + (a_2 b_3 - a_3 b_2) r_2 \times r_3 \end{aligned}$$

Since R is a rotation matrix, the column vectors satisfy

$$\begin{aligned} r_1 \times r_2 &= r_3 \\ r_1 \times r_3 &= r_2 \\ r_2 \times r_3 &= r_1 \end{aligned}$$

Making these substitutions yields

$$\begin{aligned} Ra \times Rb &= (a_2 b_3 - a_3 b_2) r_1 + (a_1 b_3 - a_3 b_1) r_2 + (a_1 b_2 - a_2 b_1) r_3 \\ &= R(a \times b) \end{aligned}$$

4-4 Set $Y = SX$. By commutativity of the inner product, $X^T Y = Y^T X$, or $X^T S X = X^T S^T X$. Since S is skew-symmetric, $S^T + S = 0$. Thus, for any vector X, we have

$$0 = X^T (S + S^T) X = X^T S X + X^T S^T X = 2 X^T S X$$

Therefore $X^T S X = 0$.

4-5

$$
\begin{aligned}
\frac{dR_{y,\theta}}{d\theta}R_{y,\theta}^T &= \begin{bmatrix} -s\theta & 0 & c\theta \\ 0 & 0 & 0 \\ -c\theta & 0 & -s\theta \end{bmatrix}\begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = S(j) \\
\frac{dR_{x,\theta}}{d\theta}R_{x,\theta}^T &= \begin{bmatrix} -s\theta & -c\theta & 0 \\ c_\theta & -s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S(k)
\end{aligned}
$$

4-6

$$
\begin{aligned}
R_{x,90} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; S(a) = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}; Ra = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \\
S(Ra) &= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

Then

$$
\begin{aligned}
RS(a)R^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = S(Ra)
\end{aligned}
$$

4-7

$$R_1^0 = R_{x,\theta} R_{y,\phi}$$

Then

$$\frac{\partial R_0^1}{\partial \phi} = R_{x,\theta} \frac{\partial R_{y,\phi}}{\partial \phi} = R_{x,\theta} S(j) R_{y,\phi} = \begin{bmatrix} -s\phi & 0 & c\phi \\ s\theta c\phi & 0 & s\theta s\phi \\ -c\theta c\phi & 0 & -s\phi c\theta \end{bmatrix}$$

$$\left.\frac{\partial R_0^1}{\partial \phi}\right|_{\substack{\theta=\phi/2 \\ \phi=\pi/2}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
I + S(k)s_\theta + S^2(k)v_\theta &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -k_z s_\theta & k_y s_\theta \\ k_z s_\theta & 0 & -k_x s_\theta \\ -k_y s_\theta & k_x s_\theta & 0 \end{bmatrix} \\
&+ \begin{bmatrix} (-k_z^2 - k_y^2)v_\theta & k_x k_y v_\theta & k_x k_z v_\theta \\ k_x k_y v_\theta & (-k_z^2 - k_x^2)v_\theta & k_y k_z v_\theta \\ k_x k_z v_\theta & k_y k_z v_\theta & (-k_y^2 - k_x^2)v_\theta \end{bmatrix}
\end{aligned}
$$

Adding the three matrices and using $k_x^2 + k_y^2 + k_z^2 = 1$ yields (2.2.16).

4-9 $S(k)^3 = -S(k)$ can be verified by direct multiplication. To show (2.5.20), we compute using Problem 2-25

$$\frac{dR}{d\theta} = S(k)\cos\theta + S^2(k)\sin\theta$$

also from Problem 2-25

$$\begin{aligned} S(k)R_{k,\theta} &= S(k) + S^2(k)\sin\theta + S^3(k)(1-\cos\theta) \\ &= S(k)\cos\theta + S^2(k)\sin\theta \end{aligned}$$

Using the fact that $S^3(k) = -S(k)$.

4-10 If $S \in so(3)$ then

$$(e^S)^T \quad = \quad e^{S^T} = e^{-S}$$

which can be verified using the series definition for e^S. Therefore

$$e^S(e^S)^T = e^S e^{-S} = e^{S-S} = e^0 = I$$

Also

$$\det(e^S) \quad = \quad e^{Tr(S)} = e^0 = 1$$

Hence $e^S \in S0(3)$.

4-11

$$
\begin{aligned}
e^{S(k)\theta} &= I + S\theta + \frac{\theta^2}{2!}S^2 + \frac{\theta^3}{3!}S^3 + \cdots \\
&= I + S\theta + \frac{\theta^2}{2!}S^2 + \frac{\theta^3}{4!}(-S^2) + \cdots \\
&= I + S\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) + S^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots\right) \\
&= I + S(k)\sin\theta + S^2(k)(1-\cos\theta) \\
&= I + S(k)\sin\theta + S^2(k)(\text{vers }(\theta)) = R_{k,\theta}
\end{aligned}
$$

4-12 From 4-11, we know that $R \in SO(3)$ satisfies $\frac{dR}{d\theta} = SR$. Therefore, a matrix S exists that satisfies $R = e^R$. To show $S \in so(3)$, we must show $S^T + S = 0$.

$$S = R^{-1}\frac{dR}{d\theta} = R^T\frac{dR}{d\theta}$$

$$S^T = \left(\frac{dR}{d\theta}\right)^T R = \frac{dR^T}{d\theta}R$$

where the final equality holds because matrix derivative is taken element by element.

$$\begin{aligned}
R^T R &= I \\
\frac{d}{d\theta}(R^T R) &= \frac{d}{d\theta}I \\
\frac{dR^T}{d\theta}R + R^T\frac{dR}{d\theta} &= 0 \\
S^T + S &= 0
\end{aligned}$$

So $S \in so(3)$.

4-13 For the Euler angle transformation, we have

$$R = R_{z,\psi} R_{y,\theta} R_{z,\phi}.$$

From Equation (4.18), we know that

$$\frac{dR}{d\theta} = S(k)R.$$

By the chain rule for differentiation, we have

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\theta}\frac{d\theta}{dt} = S(k)R\dot{\theta}.$$

Applying the product rule for differentiation to the Euler angle transformation, we have

$$\begin{aligned}
\dot{R} &= \dot{R}_z R_y R_z + R_z \dot{R}_y R_z + R_z R_y \dot{R}_z \\
&= [S(\dot{\psi}k)R_{z,\psi}]R_y R_z + R_z[S(\dot{\theta}j)R_{y,\theta}]R_z + R_z R_y[\dot{\phi}k)R_{z,\phi}] \\
&= S(\dot{\psi}k)R_z R_y R_z + S(R_{z,\theta}\dot{\theta}j)R_z R_y R_z + S(R_z R_y \dot{\phi}k)R_z R_y R_z \\
&= [S(\dot{\psi}k) + S(R_z\dot{\theta}j) + S(R_z R_y \dot{\phi}k)]R \\
&= S(\omega)R.
\end{aligned}$$

So

$$\begin{aligned}
\omega &= \dot{\psi}k + R_z\dot{\theta}j + R_z R_y \dot{\phi}k \\
&= (c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta})i + (s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta})j + (\dot{\psi} + c_\theta \dot{\phi})k.
\end{aligned}$$

4-14 For the Euler angle transformation, we have

$$R = R_{z,\phi} R_{y,\theta} R_{x,\psi}.$$

Following the derivation for Problem 4-13 yields

$$\begin{aligned} \dot{R} &= [S(\dot{\phi}k) + S(R_z\dot{\theta}j) + S(R_z R_y \dot{\psi}x)]R \\ &= S(\omega)R. \end{aligned}$$

Therefore,

$$\begin{aligned} \omega &= \dot{\phi}k + R_z\dot{\theta}j + R_z R_y \dot{\psi}x \\ &= (c_\phi c_\theta \dot{\psi} - s_\phi \dot{\theta})i + (c_\phi \dot{\theta} + c_\theta s_\phi \dot{\psi})j + (\dot{\phi} - s_\theta \dot{\psi})k. \end{aligned}$$

4-15

$$
\begin{aligned}
p_0 &= Rp_1 + d \\
\dot{p}_0 &= R\dot{p}_1 \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}
\end{aligned}
$$

4-16 Suppose a_c = distance from joint 2 to o_c, and a_e = length of link 1. Then $o_c = (x_c, y_c, z_c)^T$ where

$$\begin{aligned} x_c &= a_1c_1 + a_cc_{12} \\ y_c &= a_1s_1 + a_cc_{12} \\ z_c &= 0 \end{aligned}$$

Also

$$\begin{aligned} z_0 &= z_1 = (0,0,1)^T \\ o_0 &= (0,0,0)^T \\ o_c &= (a_1c_1 + a_cc_{12}, a_1s_1 + a_cs_{21}, 0)^T \\ o_1 &= (a_1c_1, a_1s_1, 0)^T \end{aligned}$$

$$\begin{aligned} z_0 \times (o_c - o_0) &= (-a_1s_1 - a_cs_{12}, a_cc_1 + a_cc_{12}, 0)^T \\ z_1 \times (o_c - o_1) &= (-a_cs_{12}, a_cc_{12}, 0)^T \end{aligned}$$

Therefore

$$J = \begin{bmatrix} -a_1s_1 - a_1s_{12} & -a_cs_{12} & 0 \\ a_1c_1 + a_cc_{12} & a_cc_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

4-17 Since all three joins are revolute,

$$J_{11} = [z_0 \times (o_3 - o_0) \quad z_1 \times (o_3 - o_1) \quad z_2 \times (o_3 - o_2)]$$

$$o_0 = o_1 = (0,0,0)^T; \quad o_2 = \begin{bmatrix} a_2c_1c_2 \\ a_2s_1c_2 \\ a_2s_2 \end{bmatrix}; \quad o_3 = \begin{bmatrix} a_2c_1c_2 + a_3c_1c_{23} \\ a_2s_1c_2 + a_3s_1c_{23} \\ a_2s_2 + a_3s_{23} \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

Therefore

$$\begin{aligned} x_0 \times (o_3 - o_0) &= \begin{bmatrix} -a_2s_1c_2 - a_3s_1c_{23} \\ a_2c_1c_2 + a_3c_1c_{23} \\ 0 \end{bmatrix}; \\ z_1 \times (o_3 - o_1) &= \begin{bmatrix} -c_1(a_2s_2 + a_3s_1c_{23} \\ -s_1(a_2s_2 + a_3s_{23}) \\ a_2c_2 + a_3c_{23} \end{bmatrix}; \\ z_2 \times (o_3 - o_2) &= \begin{bmatrix} -a_3c_1s_{23} \\ -a_3s_1s_{23} \\ a_3c_{23} \end{bmatrix} \end{aligned}$$

and hence

$$J_{11} = \begin{bmatrix} -a_2s_1c_2 - a_3s_1c_{23} & -a_2s_2c_1 - a_3s_{23}c_1 & -a_3c_1s_{23} \\ a_2c_1c_2 + a_3c_1c_{23} & -a_2s_1s_2 - a_3s_1s_{23} & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix}$$

which agrees with (5.3.14). Next,

$$\begin{aligned} \det J_{11} &= (-a_2s_1c_2 - a_3s_1c_{23})[(a_3c_{23})(-a_2s_1s_2 - a_3s_1s_{23}) + a_3s_1s_{23}(a_2c_2 + a_3c_{23})] \\ &\quad -(a_2c_1c_2 + a_3c_1c_{23})[(a_3c_{23})(-a_2s_2c_1 - a_3s_{23}c_1) + a_3c_1s_{23}(a_2c_2 + a_3c_{23})] \\ &= a_2^2a_3(s_2c_2c_{23} - s_{23}c_2^2) + a_2a_3^2(s_2c_{23}^2 - s_{23}c_2c_{23}) \\ &= -a_2^2a_3c_2s_3 - a_2a_3^2c_{23}s_3 \\ &= -a_2a_3s_3(a_2c_2 + a_3c_{23}) \end{aligned}$$

$$o_0 = 0; \quad o_1 = (0,0,d_1)^T; \quad o_2 = (0,0,d)1)^T; \quad o_3 = (-d_2 s_2 c_1, -d_2 s_2 s_1, d_1 + d_2 c_2)^T$$

$$z_0 = (0,0,1)^T; \quad z_1 = (s_1, -c_1, 0)^T; \quad z_2 = (-s_2 c_1, -s_2 s_1, c_2)^T$$

$$\begin{aligned} z_0 \times (o_3 - o_0) &= (s_1 s_2 d_2, c_1 s_2 d_2, 0)^T \\ z_1 \times (o_3 - o_1) &= (-c_1 c_2 d_2, s_1 c_2 d_2, s_2 d_2)^T \end{aligned}$$

Therefore

$$J = \begin{bmatrix} s_1 s_2 d_2 & -c_1 c_2 d_2 & -s_2 c_1 \\ c_1 s_2 d_2 & s_1 c_2 d_2 & -s_2 s_1 \\ 0 & s_2 d_2 & c_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

4-19 From (5.3.18), the singularities are given by $\alpha_1\alpha_4 - \alpha_2\alpha_3 = 0$. From (5.3.19), we have

$$\begin{aligned}\alpha_1\alpha_4 - \alpha_2\alpha_3 &= (-a_1s_1 - a_2s_{12})(a_1c_{12}) + (a_1s_{12})(a_1c_1 + a_2c_{12}) \\ &= a_1^2 s_2\end{aligned}$$

which agrees with (5.3.20).

4-20 From Figure 3.7,

$$o_0 = (0,0,0)^T; o_3 = (-d_3 s_1, d_3 c_1, 0)^T$$

$$z_0 = (0,0,0)^T; \quad z_1 = (0,0,1)^T; \quad z_2 = (-s_1, c_1, 0)^T$$

$$J = \begin{bmatrix} z_0 \times (o_3 - o_0) & z_1 & z_2 \\ z_0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -c_1 d_3 & 0 & -s_1 \\ -s_1 d_3 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore

$$\det \begin{bmatrix} -c_1 d_3 & 0 & -s_1 \\ -d_3 s_1 & 0 & c_1 \\ 0 & 1 & 0 \end{bmatrix} = c_1^2 d_3 + s_1^2 d_3 = d_3 \neq 0$$

4-21 For cartesian manipulator, all joints are prismatic and hence

$$J = \begin{bmatrix} z_0 & z_1 & z_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 3.

$$J = [J_1, \ldots, J_6]$$

where

$$J_1 = \begin{bmatrix} -d_y \\ d_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad J_2 = \begin{bmatrix} c_1 d_z \\ s_1 d_z \\ -s_1 d_y - c_1 d_x \\ -s_1 \\ c_1 \\ 0 \end{bmatrix}; \quad J_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} s_1 s_2 (d_z - o_{3,z}) + c2(d_y - o_{3y}) \\ -c_1 s_1 (d_z - o_{3z}) + c_2 (d_x - o_{3x}) \\ -c_1 c_2 s_4 - s_1 c_4 \\ s_2 s_4 \end{bmatrix}$$

$$J_5 = \begin{bmatrix} (-s_1 c_2 s_4 + c_1 c_4)(d_z - o_{3z}) - s_2 s_4 (d_y - o_{3y}) \\ (-c_1 c_2 s_4 + s_1 c_4)(d_z - o_{3z}) + s_2 s_4 (d_x - o_{3x}) \\ (-c_1 c_2 s_4 - s_1 c_4)(d_y - o_{3y}) + (s_1 c_2 s_4 - c_1 c_4)(d_x - o_{3x}) \\ -c_1 c_2 c_4 - s_1 c_4 \\ s_2 s_4 \end{bmatrix}$$

$$J_6 = \begin{bmatrix} (s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5)(d_y - o_{3y}) + (s_2 c_4 s_5 - c_2 c_5)(d_y - o_{3y}) \\ -(c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5)(d_z - o_{3z}) + (s_2 c_4 s_5 - c_2 c_5)(d_x - o_{3x}) \\ c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

where

$$o_6 = (d_x, d_y, d_z)^T$$

$$o_3 = \begin{bmatrix} o_{3x} \\ o_{3y} \\ o_{3z} \end{bmatrix} = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

4-23

$$\begin{bmatrix} R & SR \\ 0 & R \end{bmatrix} \begin{bmatrix} R^T & -R^T S \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} RR^T & -RR^T S + SRR^T \\ 0 & RR^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$\begin{bmatrix} R^T & -R^T S \\ 0 & R^T \end{bmatrix} \begin{bmatrix} R & SR \\ 0 & R \end{bmatrix} = \begin{bmatrix} R^T R & R^T SR - R^T SR \\ 0 & R^T R \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Therefore,

$$\begin{bmatrix} R & SR \\ 0 & R \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T S \\ 0 & R^T \end{bmatrix}.$$

$$det\big(B(\alpha)\big) = c_\psi^2 s_\theta + c_\psi^2 s_\theta = s_\theta$$

Therefore $B(\alpha)$ is invertible whenever $det\big(B(\alpha)\big) = s_\theta$ is nonzero.

4-25 1. Show that $\dot{q} = J^+\xi + (I - J^+J)b$ is a solution to Equation 4.110.

$$\begin{aligned} \xi &= J\dot{q} \\ &= J(J^+\xi + (I - J^+J)b) \\ &= JJ^+\xi + (J - JJ^+J)b \\ &= I\xi + (J - IJ)b \\ &= \xi \end{aligned}$$

2. Show that $b = 0$ minimizes the joint velocities.

$$\|\dot{q}\| = \left\|J^+\xi + (I - J^+J)b\right\|$$

By the triangle inequality, we have

$$\begin{aligned} \|\dot{q}\| &\leq \left\|J^+\xi\right\| + \left\|(I - J^+J)b\right\| \\ &= \left\|J^+\xi\right\| + \left\|(I - J^+J)\right\| \|b\| . \end{aligned}$$

Since $\|(I - J^+J)\| \geq 0$, choosing $b = 0$ minimizes $\|\dot{q}\|$.

4-26 Begin with the singular-value decomposition for J. Following the development in the appendix, we have

$$J = U\Sigma V^T = U\Sigma_m V_m^T.$$

Note that $\Sigma\Sigma^T = \Sigma_m^2$ is symmetric and that U and V are orthogonal matrices.

$$\begin{aligned}
J^+ &= J^T(JJ^T)^{-1} \\
&= (U\Sigma_m V_m^T)^T\left((U\Sigma_m V_m^T)(U\Sigma_m V_m^T)^T\right)^{-1} \\
&= V_m\Sigma_m U^T\left(U\Sigma_m V^T V\Sigma_m^T U^T\right)^{-1} \\
&= V_m\Sigma_m U^{-1}\left(U\Sigma_m^2 U^{-1}\right)^{-1} \\
&= V_m\Sigma_m U^{-1}U\Sigma_m^{-2}U^{-1} \\
&= V_m\Sigma_m^{-1}U^{-1} \\
&= V_m\Sigma_m^{-1}U^T \\
&= \left[V_m|V_{n-m}\right]\left[\Sigma_m^{-1}|0\right]^T U^T \\
&= V\Sigma^+U^T
\end{aligned}$$

4-27 To complete this problem we will use the fact that, for a square matrix A, $(A^T)^{-1} = (A^{-1})^T$.

$$
\begin{aligned}
\|\dot{q}\|^2 &= \dot{q}^T \dot{q} \\
&= (J^+\xi)^T (J^+\xi) \\
&= \left[J^T (JJ^T)^{-1}\xi\right]^T \left[J^T (JJ^T)^{-1}\xi\right] \\
&= \xi^T \left[(JJ^T)^{-1}\right]^T JJ^T (JJ^T)^{-1}\xi \\
&= \xi^T \left[(JJ^T)^{-1}\right]^T \xi \\
&= \xi^T \left[(JJ^T)^T\right]^{-1} \xi \\
&= \xi^T (JJ^T)^{-1}\xi
\end{aligned}
$$

4-28 Note that $\Sigma\Sigma^T = \Sigma_m^2$ is symmetric and that U and V are orthogonal matrices.

$$
\begin{aligned}
\xi^T(JJ^T)^{-1}\xi &= \xi^T\left(U\Sigma V^T)(U\Sigma V^T)^T\right)^{-1}\xi \\
&= \xi^T\left(U\Sigma V^T V\Sigma^T U^T\right)^{-1}\xi \\
&= \xi^T\left(U\Sigma\Sigma^T U^{-1}\right)^{-1}\xi \\
&= \xi^T\left(U\Sigma_m^2 U^{-1}\right)^{-1}\xi \\
&= \xi^T U\Sigma_m^{-2}U^{-1}\xi \\
&= \xi^T U\Sigma_m^{-2}U^T\xi \\
&= (U^T\xi)^T\Sigma_m^{-2}U^T\xi
\end{aligned}
$$

5-1 $\mathcal{Q} = \mathbb{R}^2 \times S^1$.

5-2 $Q = T^3$.

5-3 $\mathcal{Q} = \mathbb{R}^2$.

5-4 $\mathcal{Q} = \mathbb{R} \times S^1$.

5-5 $\mathcal{Q} = \mathbb{R} \times T^2$.

5-6 Assuming all joints are revolute, $\mathcal{Q} = T^6$.

5-7 Let $o_i(q) - o_i(f) = [x(q), y(q), z(q)]^T$. Now,

$$\|o_i(q) - o_i(f)\|^2 = x(q)^2 + y(q)^2 + z(q)^2.$$

and

$$U_{att,i}(q) = \frac{1}{2}\zeta_i \, \|o_i(q) - o_i(f)\|^2 = \frac{1}{2}\zeta_i [x(q)^2 + y(q)^2 + z(q)^2].$$

Therefore,

$$\begin{aligned} F_{att,i}(q) &= -\nabla U_{att,i}(q) \\ &= -\frac{1}{2}\zeta_i [2x(q), 2y(q), 2z(q)]^T \\ &= -\zeta_i [x(q), y(q), z(q)]^T \\ &= -\zeta_i \big(o_i(q) - o_i(f)\big) \end{aligned}$$

5-8 Let A denote the line *segment* passing through a_1 and a_2.
Let P denote the line passing through p that is perpendicular to A.
Define the point $a_\perp$ to be the intersection of A and P.

$$a_\perp = a_1 + t_\perp(a_2 - a_1) \text{ for some } t_\perp \in \mathbb{R}$$

Since line segment A does not extend beyond a_1 or a_2, $t_\perp$ is bounded.

$$t_\perp \in [0, 1]$$

Since $A \perp B$, we know the dot product of the two lines is zero.

$$\begin{aligned}(a_2 - a_1) \cdot (a_\perp - p) &= 0 \\ (a_2 - a_1) \cdot \big((a_1 - p) + t_\perp(a_2 - a_1)\big) &= 0\end{aligned}$$

We can solve for $t_\perp$ by choosing any of the n components of p, a_1, a_2. For example, using the first component, we have:

$$t_\perp = \frac{(p(1) - a_1(1))(a_2(1) - a_1(1))}{(a_2(1) - a_1(1))^2}.$$

– If $t_\perp \in [0, 1]$, then $a_\perp$ is on line segment A and the minimum distance to point p is $\|a_\perp - p\|$.

– If $t_\perp \notin [0, 1]$, then $a_\perp$ is not on line segment A. Therefore the minimum distance to point p is the smaller of the distances from p to a_1 and from p to a_2. That is,

$$min\{\|a_1 - p\|, \|a_2 - p\|\}.$$

5-9 For a polygon in the plane with vertices $a_i, i = 1 \ldots n$, let $A_i, i = 1 \ldots (n-1)$ be the line segment between vertices a_i and a_{i+1} and let A_n be the line segment between a_n and a_1. Repeat the algorithm given in problem 5-8 to determine the minimum distance d_i between point p and each line segment A_i.
$\Rightarrow$ The shortest distance between point p and the polygon is $min\{d_1, d_2, \ldots, d_n\}$.

5-10 Let G_i denote the ith flat face of the polygon and num_i the number of vertices that define face G_i. Let $a_{ij}, j = 1 \dots num_i$ denote the n_i vertices defining face G_i.

For each face G_i, at least three vertices are not colinear. We will call these three vertices v_1, v_2, v_3. These points define the plane p_i in which face G_i lies. We can find the equation of this plane by computing four determinants

$$\begin{vmatrix} 1 & & \\ 1 & v_2 & v_3 \\ 1 & & \end{vmatrix} x + \begin{vmatrix} & 1 & \\ v_1 & 1 & v_3 \\ & 1 & \end{vmatrix} x + \begin{vmatrix} & & 1 \\ v_1 & v_2 & 1 \\ & & 1 \end{vmatrix} x - \begin{vmatrix} v_1 & v_2 & v_3 \end{vmatrix} = 0$$

and the vector n_i normal to the plane by taking the cross product

$$n_i = (v_3 - v_1) \times (v_2 - v_1).$$

We will proceed according to the following algorithm.

1. Compute the perpendicular distances from p to each of the faces.
2. Compute the perpendicular distances from p to each of the edges.
3. Compute the distances from p to each of the vertices.

1. For each face G_i, solve for the point of intersection int_i of the plane p_i with the line defined by the normal vector n_i and passing through p. We must now check whether int_i is inside or outside face G_i. To do this, we may construct lines from int_i to each of the vertices a_{ij}.

   – If the sum of the angles of these lines (with a common reference of any line in the plane) is an integer multiple of 2π, the point int_i lies inside the face.
   – If the sum of the angles is not an integer multiple of 2π, the point lies outside the face.

   If any of the normal intersection points int_i lie inside their respective faces, we note the distance $\|p - int_i\|$.
2. For each edge, we follow the algorithm for Problem 5-8. If the perpendicular intersection of a line through p with the line containing the edge occurs within the edge's vertices, we note the distance.
3. Finally, we compute the distances between p and each of the vertices.

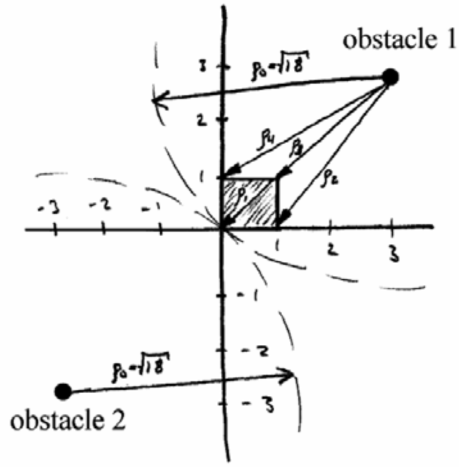
The minimum distance between point p and the polygon is the smallest of these distances.

5-11 Distance $\rho(o_i(q))$ is a function of the location in space $o_i(q)$. Consider writing this as $\rho(x)$, where x is a vector in three dimensions. Then we can treat the gradient ∇ as the partial derivative with respect to the vector x $\frac{\partial}{\partial x}$. The result now follows from the chain rule for differentiation.

$$\begin{aligned}
F &= -\frac{\partial}{\partial x}\left[\frac{1}{2}\eta_1\left(\rho^{-1}(x) - \frac{1}{\rho_0}\right)^2\right] \\
&= -\frac{1}{2}\eta_1 2\left(\rho^{-1}(x) - \frac{1}{\rho_0}\right)\frac{\partial}{\partial x}\left(\rho^{-1}(x)\right) \\
&= \eta_1\left(\rho^{-1}(x) - \frac{1}{\rho_0}\right)\rho^{-2}(x)\frac{\partial}{\partial x}\rho(x)
\end{aligned}$$

Therefore,

$$F_{rep,i}(q) = \eta_i\left(\frac{1}{\rho(o_i(q))} - \frac{1}{\rho_0}\right)\frac{1}{\rho^2(o_i(q))}\nabla\rho(o_i(q)).$$



– Assume the robot has three degrees of freedom, and thus three "joint variables" $q = \{x, y, \theta\}$. The robot is able to translate $\{x, y\}$ (think of this as two prismatic joints) and rotate $\{\theta\}$ (think of this as a single revolute joint).

1. **artificial workspace forces**
To avoid overlap of the regions of influence, we chose $\rho_0 = \sqrt{18}$ for both obstacles. In its given configuration, the robot is influenced only by obstacle 1. We construct the repulsive potential field and artificial workspace forces according to Equations (5.5) and (5.6).

$$U_{rep,i}(q) = \begin{cases} \frac{1}{2}\eta_i\left(\frac{1}{\rho(a_i(q))} - \frac{1}{\sqrt{18}}\right)^2 & \rho(a_i(q)) \leq \sqrt{18} \\ 0 & \rho(a_i(q)) > \sqrt{18} \end{cases}$$

$$F_{rep,i}(q) = \begin{cases} \eta_i\left(\frac{1}{\rho(a_i(q))} - \frac{1}{\sqrt{18}}\right)^2 \frac{1}{\rho^2(a_i(q))}\nabla\rho(a_i(q)) & \rho(a_i(q)) \leq \sqrt{18} \\ 0 & \rho(a_i(q)) > \sqrt{18} \end{cases}$$

where

$$\begin{aligned} \rho(a_i(q)) &= \|a_i(q) - b\| \\ \nabla\rho(a_i(q)) &= \frac{a_i(q) - b}{\|a_i(q) - b\|}. \end{aligned}$$

For obstacle 1, $b = [3,3]^T$ in the equations above; for obstacle 2, $b = [-3,-3]^T$. At time $t = 0$ we have $a_1(q) = [0,0]^T, a_2(q) = [1,0]^T, a_3(q) = [1,1]^T$, and $a_4(q) = [0,1]^T$. These yield

$$\begin{aligned} \rho(a_1(q)) &= \sqrt{18} & \nabla\rho(a_1(q)) &= \tfrac{[-3,-3]^T}{\sqrt{18}} \\ \rho(a_2(q)) &= \sqrt{13} & \nabla\rho(a_2(q)) &= \tfrac{[-2,-3]^T}{\sqrt{13}} \\ \rho(a_3(q)) &= \sqrt{8} & \nabla\rho(a_3(q)) &= \tfrac{[-2,-2]^T}{\sqrt{8}} \\ \rho(a_4(q)) &= \sqrt{13} & \nabla\rho(a_4(q)) &= \tfrac{[-3,-2]^T}{\sqrt{13}}. \end{aligned}$$

5-13 through 5-16 are machine problems.

5-13 through 5-16 are machine problems.

5-13 through 5-16 are machine problems.

5-13 through 5-16 are machine problems.

5-17

$$\det \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} = (t_f - t_0)^4$$

5-18 The problem is somewhat open-ended. Students should discuss issues related to matching the velocity of the conveyor, planning a straight line path, etc.

5-19 Using formulas given, the constants for the cubic polynomial are:

$$
\begin{aligned}
a_0 &= q_0 \\
a_1 &= 0 \\
a_2 &= \frac{(3q_1 - 3q_0 - 2)}{4} \\
a_3 &= \frac{(q_0 - q_1 + 1)}{4}
\end{aligned}
$$

The cubic polynomial for position is:

$$
\begin{aligned}
q_i^d(t) &= a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3 \\
&= q_0 + \frac{(3q_1 - 3q_0 - 2)}{4}(t - t_0)^2 + \frac{(q_0 - q_1 + 1)}{4}(t - t_0)^3
\end{aligned}
$$

5-20 For $t \in [2-t_b, 2]$, the *parabolic* trajectory and speed are given by:

$$\begin{aligned} q(t) &= b_0 + b_1 t + b_2 t^2 \\ \dot{q}(t) &= b_1 + 2b_2 t. \end{aligned}$$

Using the boundary condition $\dot{q}(2) = 1$, we find that

$$b_1 = 1 - 4b_2.$$

For $t \in [t_b, 2-t_b]$, the *linear* trajectory and speed are given by:

$$\begin{aligned} q(t) &= q(t_b) + Vt \\ \dot{q}(t) &= V. \end{aligned}$$

At time $t = 2 - t_b$, the *speeds* of the parabolic and linear segments must match

$$\dot{q}(2-t_b) = (1-4b_2) + 2b_2(t_b) = V$$

which implies

$$b_2 = \frac{V-1}{2t_b - 4}.$$

Using the boundary condition at $q(2) = q_1$ for the *parabolic* trajectory,

$$q(2) = b_0 - 4\frac{V-1}{2t_b-4} + 2 = q_1$$

which implies

$$b_0 = q_1 - 2 + 4\frac{V-1}{2t_b-4}.$$

At time $t = 2 - t_b$, the *trajectories* of the parabolic and linear segments must match

$$\begin{aligned} q(2-t_b) = b_0 + b_2\big[(2-t_b)^2 - 4(2-t_b)\big] + (2-t_b) &= q(t_b) + V(2-t_b) \\ \left(q_1 - 2 + 4\frac{V-1}{2t_b-4}\right) + \left(\frac{V-1}{2t_b-4}\right)\big[(2-t_b)^2 - 4(2-t_b)\big] &= q(t_b) + V(2-t_b) \end{aligned}$$

which may be solved to find slope t_b as a function of blend time V. From the development in the book, we know that

$$t_b = \frac{q_0 - q_1 + 2V}{V}.$$

Setting equal the two expressions for t_b, we can solve for V and then for t_b.

$$q(t) = \begin{cases} q_0 + \frac{V}{2t_b}t^2 & t \in [0, t_b] \\ \frac{q_0+q_1-2V}{2} + Vt & t \in [t_b, 2-t_b] \\ b_0 + b_1 t + b_2 t^2 & t \in [2-t_b, 2] \end{cases}$$

5-21 For $t_f - t_b < t \le t_f$, the desired *parabolic* trajectory and speed are given by:

$$\begin{aligned} q(t) &= b_0 + b_1 t + b_2 t^2 \\ \dot{q}(t) &= b_1 + 2b_2 t. \end{aligned}$$

Using the boundary condition $\dot{q}(t_f) = 0$ we find that

$$b_1 = -2b_2 t_f$$

For $t \in [t_b, 2 - t_b]$, the *linear* trajectory and speed are given by:

$$\begin{aligned} q(t) &= q(t_b) + Vt \\ \dot{q}(t) &= V. \end{aligned}$$

At time $t = t_f - t_b$, the *speeds* of the parabolic and linear trajectories must match

$$\dot{q}(t_f - t_b) = b_1 + 2b_2(t_f - t_b) = V$$

which gives us

$$\begin{aligned} b_2 &= \frac{-V}{2t_b} \\ b_1 &= \frac{Vt_f}{t_b}. \end{aligned}$$

Using the boundary condition $q(t_f) = q_f$ for the *parabolic* trajectory we find

$$q(t_f) = b_{at}\frac{Vt_f^2}{2t_b} = q_f$$

which implies

$$b_0 = q_f - \frac{Vt_f^2}{2t_b}.$$

Let $\alpha = \frac{V}{t_b}$.
Then for $t_f - t_b < t \le t_f$ the trajectory is given by

$$q(t) = q_f - \frac{\alpha}{2}t_f^2 + \alpha t_f t - \frac{\alpha}{2t^2}.$$

5-22 and 5-23 are machine problems.

5-22 and 5-23 are machine problems.

6-1 From the block diagram of Figure 6.6

$$\frac{\Theta_m}{V} = \left(\frac{1}{s}\right)\frac{\frac{K_m}{(Ls+R)(J_m s+B_m)}}{1+\frac{K_b K_m}{(Ls+R)(J_m s+B_m)}} = \frac{K_m}{s[(Ls+R)(J_m s+B_m)+K_b K_m]}$$

and

$$\frac{\Theta_m}{\tau_\ell} = \frac{\frac{-1/r}{s(J_m s+B_m)}}{1+\frac{K_b K_m}{(Ls+R)(J_m s+B_m)}} = \frac{-(Ls+R)/r}{s[(Ls+r)(J_m S+B_m)+K_b K_m]}$$

6-2 Divide Equations (6.11) and (6.12) by R and set the ratio $\frac{L}{R} = 0$ to get the reduced order system

$$\frac{\Theta_m}{V} = \frac{K_m/R}{s(J_m s + B_m + K_b K_m/R)} \; ; \; \frac{\Theta_m}{\tau_\ell} = \frac{-1/r}{s(J_m s + B_m + K_b K_m/R)}$$

6-3 Compute $\frac{\Theta(s)}{\Theta^d(s)}$ with $D(s) = 0$ and $\frac{\Theta(s)}{D(s)}$ with $\Theta^d(s) = 0$ and combine the resulting transfer functions using the Principle of Superposition.

6-4 The tracking error is computed as

$$\begin{aligned} E(s) &= \Theta^d(s) - \Theta(s) \\ &= \Theta^d(s) - \left[\frac{K_P + K_D s}{\Omega(s)}\Theta^d(s) - \frac{1}{\Omega(s)}D(s)\right] \\ &= \frac{Js^2 + Bs}{\Omega(s)}\Theta^d(s) + \frac{1}{\Omega(s)}D(s) \end{aligned}$$

The Final Value Theorem says that, if $F(s)$ is the Laplace transform of $f(t)$, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} F(s)$$

whenever both limits are well defined. Consult any textbook on control systems for a more detailed statement. The steady state error is defined as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} (\theta^d(t) - \theta(t))$$

Thus, from the Final Value Theorem, we have $e_{ss} = \lim_{s\to 0} E(s)$ where $E(s)$ is given by (6.20). Substituting (6.21) and (6.22) into (6.20) and computing the limit gives (6.23).

6-5 Equations (6.28) and (6.29) follow exactly as in Problem 6-3 using superposition and block diagram reduction from Figure 6.12.

6-6 The Routh-Hurwitz criterion can be used to derive the following general result. Any linear third-order system with characteristic polynomial $\Omega(s) = s^3 + a_2 s^2 + a_1 s + a_0$ is asymptotically stable if and only if a_0, a_1, a_2 are positive and $a_2 a_1 > a_0$. Applying this result to the characteristic polynomial (6.29) gives

$$\frac{(B + K_D)}{J}\frac{K_P}{J} > \frac{K_I}{J}$$

which reduces to (6.30) after multiplying through by J.

6-7 Use Matlab/Simulink to generate the system of Figure 6.14 and simulate the system with various parameter values.

6-8 From Figure 6.16 we have

$$\Theta = G(s)(H(s)(\Theta^d - \Theta) + F(s)\Theta^d)$$

Solving for Θ gives

$$\Theta = \frac{G(s)H(s) + G(s)F(s)}{1 + G(s)H(s)}$$

Substituting in the expressions for $G(s)$, $H(s)$, and $F(s)$ yields (6.32).

6-9 From Figure 6.17 we have, (suppressing the argument s),

$$\Theta = G\{D + H(\Theta^d - \Theta) + F\Theta^d\}$$

Solving for Θ gives

$$\Theta = \frac{G}{1+GH}D + \frac{GH+GF}{1+GH}\Theta^d$$

Therefore the error $E = \Theta^d - \Theta$ satisfies

$$\begin{aligned} E &= \Theta^d - \Theta \\ &= \Theta^d - \left[\frac{G}{1+GH}D + \frac{GH+GF}{1+GH}\Theta^d\right] \\ &= -\frac{G}{1+GH}D + \frac{1-GF}{1+GH}\Theta^d \\ &= -\frac{G}{1+GH}D \end{aligned}$$

since $1 - GF = 0$. Substituting the expressions for G and H into the above equation gives (6.35).

6-10 From block diagram (Figure 6.21)

$$\frac{\theta_\ell}{u} = \frac{\frac{k}{p_m p_\ell}}{1 - \frac{k^2}{p_m p_\ell}} = \frac{k}{p_m p_\ell - k^2}$$

The open-loop characteristic polynomial is

$$\begin{aligned} p_m p_\ell - k^2 &= (J_\ell s^2 + B_\ell s + k)(J_m s^2 + B_m s + k) - k^2 \\ &= J_\ell J_m s^4 + (J_\ell B_m + J_m B_\ell)s^3 + (k(J_m + J_\ell) + B_m B_\ell)s^2 + k(B_\ell + B_m)s \end{aligned}$$

If $B_m = B_\ell = 0$ the characteristic polynomial reduces to $J_\ell J_m s^4 + k(J_m + J_\ell)s^2$.

6-11 Using A, b, and c given by Equations (6.51) and (6.52), we have

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} adj(sI - A)$$

Carrying out the calculations give,

$$\det(sI - A) = s^4 + \left(\frac{B_m}{J_m} + \frac{B_\ell}{J_\ell}\right)s^3 + \left(\frac{k}{J_m} + \frac{B_\ell B_m}{J_m J_\ell} + \frac{k}{J_\ell}\right)s^2 + \left(\frac{kB_\ell}{J_\ell J_m} + \frac{kB_m}{J_\ell J_m}\right)s$$

and

$$c^T(sI - A)^{-1}b = \frac{k}{J_m J_\ell s^4 + (J_\ell B_m + J_m B_\ell)s^3 + (k(J_m + J_\ell) + B_\ell B_m)s^2 + k(B_\ell + B_m)s}$$

which is identical to (6.45).

There is no published solution for Problem 6.12.

6-13 Both (6.59) and (6.67) are found by direct calculation. It is instructive to write a *Mathematica* function to compute these terms symbolically. In the case of (6.59) we have

$$\det\begin{bmatrix} 0 & 0 & 0 & \frac{k}{J_m J_\ell} \\ 0 & 0 & \frac{k}{J_m J_\ell} & \frac{-B_\ell k}{J_m J_\ell^2} - \frac{B_m k}{J_\ell^2 J_m} \\ 0 & \frac{1}{J_m} & \frac{-B_m}{J_m^2} & \frac{-k}{J_m^2} + \frac{B_m^2}{J_m^3} \\ \frac{1}{J_m} & \frac{-B_m}{J_m^2} & \frac{-k}{J_m^2} + \frac{B_m^2}{J_m^3} & \frac{kB_m}{J_m^3} + \frac{kB_m}{J_m^3} - \frac{B_m^3}{J_m^4} \end{bmatrix} = \left(-\frac{1}{J_m}\right)\left(+\frac{1}{J_m}\right)\left(-\frac{k^2}{J_m^2 J_\ell^2}\right) = \frac{k^2}{J_m^4 J_\ell^2}$$

Equation (6.67) is derived similarly.

6-14 Integrator Wind up – If integral control is used, the integrator can build up large values when the actuator saturates.

Anti-wind up – Turn of the integral control when actuator saturates.

6-15 Adding the first-order dynamics of a permanent-magnet DC motor to the flexible-joint model (6.39)-(6.40) gives

$$\begin{aligned} J_\ell \ddot{\theta}_\ell + B_\ell \dot{\theta}_\ell + k(\theta_l - \theta_m) &= 0 \\ J_m \ddot{\theta}_m + B_m \dot{\theta}_m - k(\theta_l - \theta_m) &= u \\ u &= k_m I \\ L\dot{I} + rI &= V - k_b \dot{\theta}_m \end{aligned}$$

where I is the motor current, V is the input voltage, L is the armature inductance, R is the armature resistance and k_m, k_b are the torque and back-emf constants, respectively. The system is thus fifth-order. Defining state variables

$$x_1 \;=\; \theta_\ell \;;\; x_2 = \dot{\theta}_\ell \;; x_3 = \theta_m \;;\; x_4 = \dot{\theta}_m \;; x_5 = I$$

yields the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{J_\ell} & -\frac{B_\ell}{J_\ell} & \frac{k}{J_\ell} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m} & \frac{k_m}{J_m} \\ 0 & 0 & 0 & -\frac{k_b}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V$$

With output $y = q_\ell = x_1$, the output equation is

$$y = [1, 0, 0, 0, 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

It is now straightforward to compute the determinants

$$\det\,[b, Ab, A^2b, A^3b, A^4b] \quad \text{and} \quad \det \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \\ cA^4 \end{bmatrix}$$

and show that they are nonzero - hence the system is controllable and observable.

6-16 Choose state variables

$$x = \begin{bmatrix} I_a \\ \theta_m \\ \dot{\theta}_m \end{bmatrix}$$

Then the state equations can be written

$$\dot{x} = \begin{bmatrix} \dot{I}_a \\ \dot{\theta}_m \\ \ddot{\theta}_m \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_b}{L} \\ 0 & 0 & 1 \\ \frac{K_m}{J_m} & 0 & -\frac{B_m}{J_m} \end{bmatrix} x + \begin{bmatrix} \frac{V}{L} \\ 0 \\ \frac{-r\tau_\ell}{J_m} \end{bmatrix}$$

In state space, equation is a linear third order system.

6-17 (a) The open loop transfer function is given by Equation (6.45):

$$\frac{\theta_\ell(s)}{U(s)} = \frac{100}{20s^4 + 7s^3 + 1200.5s^2 + 150s}$$

There are 2 real poles at $s = 0$, $s = -0.125$ and a pair of complex poles at $s = -0.1125 \pm 7.7449j$.

6-18 For the system described by

$$\begin{aligned} J_1\ddot{q}_1 &= \tau \\ J_2\ddot{q}_2 &= \tau \end{aligned}$$

choose state variables

$$x_1 = q_1;\ x_2 = \dot{q}_1;\ x_3 = q_2;\ x_4 = \dot{q}_2$$

Then in state space, we have

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tau$$

It is easy to see that the matrix

$$[b, Ab, A^2b, A^3b] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

has rank 2 and, therefore, the system is uncontrollable.

6-19 Controllability follows from the calculation

$$\text{rank}\,[b\ Ab] = \quad \text{rank}\begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix} = 2$$

With $k = [k_1, k_2]$ and $u = kx$, the closed loop system matrix is given by

$$A + bk \quad = \quad \begin{bmatrix} 1 + k_1 & -3 + k_2 \\ 1 - 2k_1 & -2 - 2k_2 \end{bmatrix}$$

The characteristic polynomial is

$$\det(sI - A - bk) \quad = \quad s^2 + (1 + 2k_2 - k_1)s + (1 - 8k_1 - 3k_2) = s^2 + 4s + 4$$

Therefore equating coefficients gives

$$\begin{aligned} 1 - k_1 + 2k_2 &= 4 \\ 1 - 8k_1 - 3k_2 &= 4 \end{aligned}$$

Solving for k_1 and k_2 yields

$$k_1 = \frac{-15}{19}, \qquad k_2 = \frac{21}{19}$$

and therefore the state feedback control becomes:

$$u \quad = \quad \frac{-15}{19}x_1 + \frac{21}{19}x_2$$

6-20 In this case the closed loop system matrix is

$$A - bk \quad = \quad \begin{bmatrix} -1 & 0 \\ -k_1 & 2 - k_2 \end{bmatrix}$$

and so the characteristic equation is

$$\det(\lambda I - A + bk) \quad = \quad (\lambda + 1)(\lambda - 2 + k_2) = 0$$

Thus we see that $\lambda = -1$ is a closed loop pole for any choice of feedback gains k_1, k_2. The choice $k_2 = 4$ places one pole at $s = -2$ but it is not possible to place both poles at $s = -2$. However, the closed loop system is stable.

6-21 In this case, a similar calculation as above shows that there is always a pole at $s = +1$ for any choice of gains k_1 and k_2. Therefore, the system cannot be stabilized.

6-22 Choose the feedforward transfer $F(s)$ and PD compensator $C(s)$, respectively, as

$$F(s) \;=\; 2s^2 + s \;;\; C(s) = K_p + K_D s$$

The desired closed-loop characteristic polynomial, with $\omega = 10$ and $\zeta = 0.707$, is

$$s^2 + 2\zeta\omega s + \omega^2 = s^2 + 14.14s + 100$$

With $G(s) = \frac{1}{2s^2+s}$, and PD compensator, the closed loop characteristic polynomial is $2s^2 + (2 + K_D)s + K_P$. Thus, equating coefficients, leads to the PD gains

$$K_P = 200; \;\; K_D = 26.8$$

7-1 A direct calculation shows

$$
\begin{aligned}
& (r_1 + \delta r_1 - r_2 - \delta r_2)^T (r_1 + \delta r_1 - r_2 - \delta r_2) \\
= \;& (r_1 - r_2 + \delta r_1 - \delta r_2)(r_1 - r_2 + \delta r_1 - \delta r_2) \\
= \;& (r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2) + (\delta r_1 - \delta r_2)^T (\delta r_1 - \delta r_2) \\
= \;& (r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2)
\end{aligned}
$$

if we neglect the second-order terms in δr_1, δr_2. Therefore, from Equation (7.17) we have

$$\ell^2 = (r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2)$$

Equation (7.18) follows since $(r_1 - r_2)^T (r_1 - r_2) = \ell^2$

7-2 Euler's equation can be expressed as

$$I\dot{\omega} + \omega \times I\omega = 0$$

where

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}; \ \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Computing the cross product yields the three equations

$$\begin{aligned} I_{xx} + (I_{zz} - I_{yy}\omega_y\omega_z) &= 0 \\ I_{yy} + (I_{xx} - I_{zz}\omega_z\omega_x) &= 0 \\ I_{zz} + (I_{yy} - I_{xx}\omega_x\omega_y) &= 0 \end{aligned}$$

7-3 Referring to Figure 7.6 we have

$$\begin{aligned}\int (y^2+z^2)dm &= \int_0^c \int_0^b \int_0^a (y^2+z^2)\rho dx dy dz \\ &= \frac{1}{3}\rho abc(b^2+c^2)\end{aligned}$$

Computing the remaining terms similarly we have

$$\begin{aligned} I &= \begin{bmatrix} \int (y^2+z^2)dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (x^2+z^2)dm & -\int yzdm \\ -\int xzdm & -\int yzdm & -\int (x^2+y^2)dm \end{bmatrix} \\ &= \rho \begin{bmatrix} \frac{1}{3}abc(b^2+c^2) & -a^2b^2c/4 & -a^2bc^2/4 \\ -a^2b^2c/4 & \frac{1}{3}abc(a^2+c^2) & -ab^2c^2/4 \\ -a^2bc^2/4 & -ab^2c^2/4 & \frac{1}{3}abc(a^2+b^2) \end{bmatrix}\end{aligned}$$

7-4 From (7.83) we have $\det D = d_{11}d_{22} - d_{12}d_{21}$

$$
\begin{aligned}
&= (m_1\ell c_1^2 + m_2\ell_1^2 + m_2\ell c_2^2 + 2m_2\ell_1\ell c_2\cos q_2 + I_1 + I_2)(m_2\ell c_2^2 + I_2) \\
&\quad (m_2\ell c_2^2 + m_2\ell_1\ell c_2\cos q_2 + I_2)^2 \\
&= m_1m_2\ell c_2^2 + m_1\ell c_1^2 I_2 + m_2\ell_1^2 I_2 + m_2\ell_1^2 + I_1I_2 + m_2^2\ell_1^2\ell c_2^2(1-\cos^2 q_2) + m_2\ell c_2^2 I_1
\end{aligned}
$$

Since $0 \geq 1 - \cos^2 q_2 \geq 1$ we have $\det D > 0$.

7-5 One way to argue that the inertia matrix of an arbitrary n-DOF robot is positive definite is by a consideration of kinetic energy. The kinetic energy of an arbitrary robot is

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

The kinetic energy must be positive for nonzero velocities $\dot{q}$. If $D(q)$ were not sign definite there would be a nonzero velocity vector $\dot{q}$ such that $\dot{q}^T D(q)\dot{q} = 0$ which is a contradiction.

7-6 By symmetry of the inertia matrix we have

$$\sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} \right\} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{jk}}{\partial q_i} \right\} \dot{q}_i \dot{q}_j$$

Since the summation runs over all i, j we can interchange i and j in the second term to obtain the result

$$\sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} \right\} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} \right\} \dot{q}_i \dot{q}_j$$

7-7 (a) The inertia tensors are

$$J_1 = \begin{bmatrix} \frac{1}{192} & 0 & 0 \\ 0 & \frac{1}{192} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}; \quad J_2 = \begin{bmatrix} \frac{1}{192} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{192} \end{bmatrix}; \quad J_3 = \begin{bmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{192} & 0 \\ 0 & 0 & \frac{1}{192} \end{bmatrix}$$

(b) The inertia matrix is

$$D = \begin{bmatrix} m_3 & 0 & 0 \\ 0 & m_2 + m_3 & 0 \\ 0 & 0 & m_1 + m_2 + m_3 \end{bmatrix}$$

(c) Since the inertia matrix is constant, all Christoffel symbols are zero.

(d) From the Euler-Langrange equations, we have

$$\begin{bmatrix} m_3 & 0 & 0 \\ 0 & m_2 + m_3 & 0 \\ 0 & 0 & m_1 + m_2 + m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} g(m_1 + m_2 + m_3) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

For Problems 7-8 to 7-11, download the *Robotica* package from http://decision.csl.uiuc.edu/ spong/Robotica/.

Use *Robotica* to compute the Euler-Lagrange equations of motion in symbolic form.

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Use *Robotica* to compute the Euler-Lagrange equations of motion in symbolic form.

7-12 With the kinetic energy given by

$$K = \frac{1}{2}\sum_{ij}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j$$

we compute

$$p_k = \frac{\partial K}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}\dot{q}_j$$

Now

$$\sum_{k=1}^{n} \dot{q}_k p_k = \sum_{k=1}^{n} \dot{q}_k \sum_{j=1}^{n} d_{kj}\dot{q}_j = d_{kj}\dot{q}_j\dot{q}_k = 2K$$

7-13 (a)

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L = 2K - (K - V) = K + V$$

(b) From

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L$$

we have

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

and

$$\frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k} = u_k - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = u_k - \dot{p}_k$$

The last two inequalities coming from the Euler-Lagrange equations and the definition of p_k, respectively.

7-14 Proof 1) The total derivative dH/dt is given by

$$\frac{dH}{dt} = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k$$

From Hamilton's equation this becomes

$$\begin{aligned} \frac{dH}{dt} &= \sum_{k=1}^{n}(\tau_k - \dot{p}_k)\dot{q}_k\dot{p}_k \\ &= \sum_{k=1}^{n} \dot{q}_k\tau_k = \dot{q}^T\tau \end{aligned}$$

Proof 2) From

$$H = K + V = \frac{1}{2}\dot{q}^T D\dot{q} + V(q)$$

we have

$$\frac{dH}{dt} = \dot{q}^T D\ddot{q} + \frac{1}{2}\dot{q}^T\dot{D}\dot{q} + \frac{\partial V}{\partial q}$$

using the Euler-Lagrange equations, this becomes

$$\begin{aligned} \frac{dH}{dt} &= \dot{q}^T\tau + \frac{1}{2}\dot{q}^T(\dot{D} - 2C)\dot{q} \\ &= \dot{q}^T\tau \end{aligned}$$

by the skew-symmetry property. The units of dH/dt are power.

8-1 Since J is a constant, diagonal matrix, $M(q) = D(q) + J$ inherits the properties of $D(q)$. The skew-symmetry and passivity properties both follow from $\dot{M} = \dot{D}$. Likewise the additional term in the dynamic equations (8.6) is $J\ddot{q}$, which is linear in J; hence linearity in the parameters is preserved. M is positive definite since it is the sum of two positive definite matrices.

Write $J = diag\{J_1, \ldots, J_n\}$, where J_i are the positive, elements of the diagonal of J. Let λ_{J_1} and λ_{J_n} be the minimum and maximum values, respectively, of $J_1, \ldots, J_n$. Then

$$\lambda_{J_1} I_{n\times n} \leq J \leq \lambda_{J_n} I_{n\times n}$$

Therefore

$$(\lambda_1 + \lambda_{J_1}) I_{n\times n} \leq D(q) + J \leq (\lambda_n + \lambda_{J_n}) I_{n\times n}$$

8-2 From (8.15) and (8.16) the Lagrangian L is

$$L = \frac{1}{2}\dot{q}_1^T D(q_1)\dot{q}_1 + \frac{1}{2}\dot{q}_2^T J\dot{q}_2 - P(q_1) - \frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2)$$

Thus we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = D(q_1)\ddot{q}_1 + \dot{D}(q_1)\dot{q}_1$$

$$\frac{\partial L}{\partial q_1} = \frac{1}{2}\dot{q}_1^T \frac{\partial D}{\partial q_1} - \frac{\partial V_1}{\partial q_1} - K(q_1 - q_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = J\ddot{q}_2; \qquad \frac{\partial L}{\partial q_2} = K(q_1 - q_2)$$

Therefore, with

$$C(q_1, \dot{q}_1)\dot{q}_1 = \dot{D}\dot{q}_1 - \frac{1}{2}\dot{q}_1^T \frac{\partial D}{\partial q_1}; \qquad g(q_1) = \frac{\partial V}{\partial q_1}$$

the Euler-Lagrange equations for this system are

$$\begin{aligned} D(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1) + K(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - K(q_1 - q_2) &= \tau \end{aligned}$$

8-3 Computing $\dot{V}$ from (8.20) using the skew-symmetry property and (8.18) with the gravity term $g(q_1) = 0$, we obtain

$$\dot{V} = -\dot{q}_2^T K_d \dot{q}_2$$

Thus $\dot{V} < 0$ as long as $\dot{q}_2 \neq 0$. If $\dot{q}_2 \equiv 0$, then the second equation in (8.18) implies $K(q_2 - q_1) = K_p \tilde{q}_2$. By taking derivatives on both sides, since q_2^d is constant, we have $\dot{q}_1 \equiv 0$, $\ddot{q} \equiv 0$. Therefore from (8.18) we have $q_1 \equiv q_2$ and, hence, $\tilde{q}_2 = 0$. Asymptotic stability follows from Lasalle's Theorem.

8-4 In the steady state, $q_1 = q_2$ was shown in Problem 8-3. If gravity is present then the steady state equation becomes

$$g(q_1) + K(q_1 - q_2) = 0$$

from (8.18). Given a desired position q_1^d, we can modify the desired set point for the motor angle q_2 to satisfy the above equation as

$$q_2^d = q_1^d + \frac{1}{K} g(q_1^d)$$

8-5 The linear approximation of (8.18) is essentially a multivariable equivalent of the model (6.39)-(6.40) with the damping terms set to zero. As the root locus analysis shows in Figure 6.25, the system with PD-control using the link variables is unstable for all values of the gains.

8-6 Use Matlab/Simulink.

8-7 Use Matlab/Simulink.

8-8 Substituting (8.45) into (8.44) gives

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \hat{M}(q)a_q + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q)$$

Suppressing arguments for simplicity we have

$$\begin{aligned} M\ddot{q} &= \hat{M}a_q + \tilde{C}\dot{q} + \tilde{g} \\ &= \hat{M}a_q - Ma_q + Ma_q + \tilde{C}\dot{q} + \tilde{g} \\ &= Ma_q + \tilde{M}a_q + \tilde{C}\dot{q} + \tilde{g} \end{aligned}$$

Multiplying both sides by M^{-1} gives Equations (8.46) and (8.47).

8-9 Use Matlab/Simulink

8-10 Returning to the expression in Problem 8-8 above we have

$$M\ddot{q} = \hat{M}a_q + \tilde{C}\dot{q} + \tilde{g}$$

Adding and subtracting $\hat{M}\ddot{q}$ on the left-hand side gives

$$\hat{M}\ddot{q} - \tilde{M}\ddot{q} = \hat{M}a_q + \tilde{C}\dot{q} + \tilde{g}$$

Rearranging this equation and using linearity in the parameters yields

$$\begin{aligned} \hat{M}(\ddot{q} - a_q) &= \tilde{M}\ddot{q} + \tilde{C}\dot{q} + \tilde{g} \\ &= Y(q, \dot{q}, \ddot{q})\tilde{\theta} \end{aligned}$$

Multiplying both sides by $\hat{M}$ gives Equation (8.77).

8-11 From (8.78) and (8.82)

$$\begin{aligned}
\dot{e} &= Ae + B\Phi\tilde{\theta} \\
V &= e^T Pe + \tilde{\theta}^T \Gamma \tilde{\theta}
\end{aligned}$$

we have

$$\begin{aligned}
\dot{V} &= \dot{e}^T Pe + e^T P\dot{e} + 2\tilde{\theta}^T \Gamma \dot{\tilde{\theta}} \\
&= (Ae + B\Phi\tilde{\theta})^T Pe + e^T P(Ae + B\Phi\tilde{\theta}) + 2\tilde{\theta}^T \Gamma \dot{\tilde{\theta}} \\
&= e^T(A^T P + PA)e + 2\tilde{\theta}^T(\Phi^T B^T Pe + \Gamma\dot{\tilde{\theta}}) \\
&= -e^T Qe + 2\tilde{\theta}^T(\Phi^T B^T Pe + \Gamma\dot{\tilde{\theta}})
\end{aligned}$$

8-12 (a) The state space is four dimensional.

(b) Choose state and control variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}; \; u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Then

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ -3x_1x_3 - x_3^2 \\ x_4 \\ -x_4\cos x_1 - 3(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & x_3 \\ 0 & 0 \\ -3x_3\cos^2 x_1 & \end{bmatrix} u \\ &= f(x) + G(x)u \end{aligned}$$

for obvious definitions of $f(x)$, $G(x)$.

(c) The inverse dynamics control law is then

$$u = G^{-1}(v - F(x)$$

$$= \frac{1}{1 + x_3^2\cos^2 x_1} \begin{bmatrix} 3x_1x_3 + x_3^2 - x_3x_4\cos x_1 - cx_1x_3 + 3x_3^2 + v_1 - x_3v_2 \\ 3x_1x_3^2\cos^2 x_1 + x_3^2\cos^2 x_1 + x_4\cos x_1 + 3(x_1 - x_3) + x_3\cos^2 x_1 v_1 + v_2 \end{bmatrix}$$

where

$$\begin{aligned} v_1 &= -10x_2 - 100x_1 + r_1 \\ v_2 &= -10x_4 - 100x_3 + r_2 \end{aligned}$$

8-13 Mimic the proof of Theorem 3.

8-14 Thanks to Martin Corless for supplying this proof. Let

$$0 < \underline{M} \leq \lambda_{min}(M^{-1}) \qquad \text{and} \qquad \|M^{-1}\| = \lambda_{max}(M^{-1}) \leq \overline{M}\,.$$

where λ_{min} and λ_{max} denote minimum eigenvalue and maximum eigenvalue, respectively. Since

$$E = \frac{2}{\overline{M}+\underline{M}}M^{-1} - I\,,$$

its maximum eigenvalue satisfies

$$\lambda_{max}(E) = \frac{2\lambda_{max}(M^{-1})}{\overline{M}+\underline{M}} - 1 \leq \frac{2\overline{M}}{\overline{M}+\underline{M}} - 1 = \overline{\lambda}$$

where

$$\overline{\lambda} := \frac{\overline{M}-\underline{M}}{\overline{M}+\underline{M}} < 1\,.$$

In a similar fashion one can show that $\lambda_{min}(E) \geq -\overline{\lambda}$. Using the symmetry of E we now obtain that

$$||E||^2 = \lambda_{max}(E^TE) = \lambda_{max}(E^2) \leq \overline{\lambda}^2\,.$$

Hence $||E|| \leq \overline{\lambda} < 1$.

- Note that $\underline{M}$ and $\overline{M}$ can also be obtained from

$$\|M\| = \lambda_{max}(M) \leq 1/\underline{M} \qquad \text{and} \qquad \lambda_{min}(M) \geq 1/\overline{M}\,.$$

So, the above results can also be expressed in terms of bounds on $\lambda_{min}(M)$ and $\lambda_{max}(M)$.

9-1 The two-link RR Jacobian matrix is given by

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

The joint torque necessary to balance an end-effector force $F = (-1, -1)^T$ is given by

$$\begin{aligned} \tau &= J^T F \\ &= \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & a_1 c_1 + a_2 c_{12} \\ -a_2 s_{12} & a_2 c_{12} \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} a_1(s_1 - c_1) + a_2(s_{12} - c_{12}) \\ a_2(s_{12} - c_{12}) \end{bmatrix} \end{aligned}$$

9-2 The solution proceeds as in Problem 9-1. The torque required to balance an end-effector force F is given by $\tau = J^T F$, where now J represents the Jacobian of the robot with remote drive. The $x - y$ coordinates of the end-effector in terms of the absolute angles θ_1 and θ_2 are easily seen to be

$$\begin{aligned} x &= a_1 c_1 + a_2 c_2 \\ y &= a_1 s_1 + a_2 s_2 \end{aligned}$$

The velocities therefore are

$$\begin{aligned} \dot{x} &= -a_1 s_1 \dot{\theta}_1 - a_2 s_2 \dot{\theta}_2 \\ \dot{y} &= a_1 c_1 \dot{\theta}_1 + a_2 c_2 \dot{\theta}_2 \end{aligned}$$

Thus, the Jacobian is given by

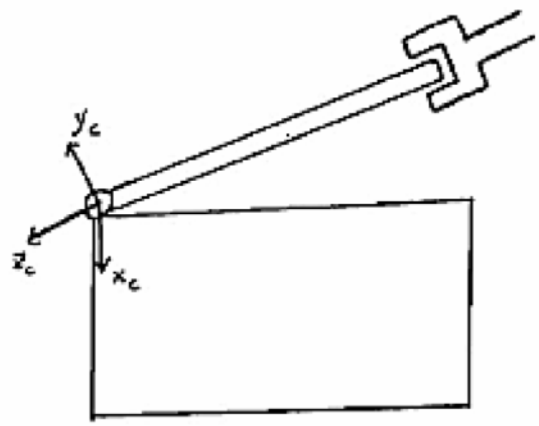
$$J = \begin{bmatrix} -a_1 s_1 & -a_2 s_2 \\ a_1 c_1 & a_2 c_2 \end{bmatrix}$$

9-3 Square peg in square hole

Natural constraints	Artificial constraints
$v_x = 0$	$f_x = 0$
$v_y = 0$	$f_y = 0$
$f_z = 0$	$v_z = v_d$
$w_x = 0$	$\tau_x = 0$
$w_y = 0$	$\tau_y = 0$
$w_z = 0$	$\tau_z = 0$

9-4 Opening a box with a hinged lid

Natural constraints	Artificial constraints
$v_x = 0$	$w_x = w_d$
$v_y = 0$	$\tau_x = I\dot{w}_d$
$v_z = 0$	$f_x = 0$
$w_y = 0$	$f_z = 0$
$w_z = 0$	$\tau_y = 0$
$f_y = 0$	$\tau_z = 0$



9-5 This is somewhat open ended and is a good question for classroom discussion. The notions of wedging and jamming are important to consider for this problem. The two manipulators should produce a straight-line motion and avoid rotating the drawer. Natural and artificial constraints are similar to the peg-in-hole problem.

9-6 Each task should be decomposed into single-DOF directions according to the compliance frame. In each direction the environment can be classified according to whether or not a significant inertia is to be moved, or significant compliance exits, and so on.

1. Turning a crank can be considered as inertial tangent to the circle defined by the crank rotation and capacitive along the crank direction.
2. Inserting a peg in a hole can be considered capacitive in directions with position constraints and inertial in other directions.
3. Polishing the hood of a car can be considered capacitive normal to the hood and inertial tangent to the hood.
4. Cutting cloth can be considered resistive in the cutting direction.
5. Shearing a sheep can be considered capacitive normal to the sheep and resistive in the shearing (tangent) direction.
6. Placing stamps on envelopes can be considered capacitive normal to the envelope.
7. Cutting meat can be considered resistive or capacitive in the cutting direction.

10-1 By definition,

$$L_g h = \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} g_j$$

Therefore, we have

$$\begin{aligned} L_f L_g h &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \frac{\partial h}{\partial x_j} g_j \right) f_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 h}{\partial x_i \partial x_j} g_j + \frac{\partial h}{\partial x_j} \frac{\partial g_j}{\partial x_i} \right) f_i \end{aligned}$$

Likewise, by interchanging f and g above we have

$$L_g L_f h = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 h}{\partial x_i \partial x_j} f_j + \frac{\partial h}{\partial x_j} \frac{\partial f_j}{\partial x_i} \right) g_i$$

Therefore, using the fact that

$$\frac{\partial^2 h}{\partial x_i d x_j} = \frac{\partial^2 h}{\partial x_j \partial x_i}$$

we have

$$L_f L_g h - L_g L_f h = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} \left(\frac{\partial g_i}{\partial x_i} f_i - \frac{\partial f_j}{\partial x_i} g_i \right) = L_{[f,g]} h$$

10-2 If $h = z - \phi(x, y)$, then

$$dh = \left(-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}, 1\right)$$

If X_1, X_2 are given by Equation (10.15) then

$$\begin{aligned} L_{X_1} h &= -\frac{\partial \phi}{\partial x} \cdot 1 - \frac{\partial \phi}{\partial y} \cdot 0 + 1 \cdot f(x, y, \phi(x, y)) \\ &= f(x, y, \phi) - \frac{\partial \phi}{\partial x} = 0 \end{aligned}$$

since ϕ satisfies Equation (10.10). Similarly,

$$L_{X_2} h = g(x, y, \phi) - \frac{\partial \phi}{\partial y} = 0$$

10-3 If $h(x,y,z)=0$ and $\partial h/\partial z \neq 0$ then, by the implicit function theorem we may solve for z as $x=\phi(x,y)$. Furthermore,

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= -\frac{\partial h}{\partial x}\Big/\frac{\partial h}{\partial z} \\
\frac{\partial \phi}{\partial y} &= -\frac{\partial h}{\partial y}\Big/\frac{\partial h}{\partial z}
\end{aligned}$$

Now

$$L_{X_1}h = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial z}f = -\frac{\partial h}{\partial z}\frac{\partial \phi}{\partial x} + \frac{\partial h}{\partial z}f = 0$$

which implies

$$\frac{\partial \phi}{\partial x} = f$$

since $\frac{\partial h}{\partial z} \neq 0$. The second equation is shown similarly.

10-4 By repeated application of Lemma 10.1, we have $L_{ad^i_f(g)}T_1 = (-1)^i L_g T_{i+1}$. Thus for $i < n-1$, $L_{ad^i_f(g)}T_1 = 0$ and $L_{ad^{n-1}_f(g)}T_1 = (-1)^{n-1} L_g T_n \neq 0$.

10-5 The vector fields f and g are given by

$$f = \begin{bmatrix} x_1^3 + x_2 \\ x_2^3 \end{bmatrix}; \; g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The dimension of the state space is $n = 2$ and therefore the necessary and sufficient conditions for local feedback linearizability are rank$\{g, [f, g]\} = 2$ and involutivity of the set $\{g\}$. Now, any distribution spanned by a single vector field is involutive and so for second order systems the rank condition alone is necessary and sufficient for local feedback linearizability. Since g is constant, the Lie Bracket $[f, g]$ is given by

$$[f, g] = -\frac{\partial f}{\partial x} g = -\begin{bmatrix} 3x_1^2 & 1 \\ 0 & 3x_2^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3x_2^2 \end{bmatrix}$$

Therefore

$$\{g, [f, g]\} = \begin{bmatrix} 0 & 1 \\ -1 & -3x_2^2 \end{bmatrix}$$

which has rank 2 for all x. Therefore the system is globally feedback linearizable since the rank condition holds globally. To find the change of coordinates we must solve the PDE's

$$L_g T_1 = 0$$

with the additional condition that $L_{[f,g]} \neq 0$. The first equation says, in effect that

$$\frac{\partial T_1}{\partial x_2} = 0$$

while the additional condition implies

$$\frac{\partial T_1}{\partial x_1} \neq 0$$

Thus we may take the simplest solution $T_1 = x_1$ and compute T_2 from

$$T_2 = L_f T_1 = x_1^3 + x_2$$

Therefore, the change of variables is

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_1^3 + x_2 \end{aligned}$$

The feedback linearizing control is found from

$$\begin{aligned} \dot{z}_2 &= 2x_1^2 \dot{x}_1 + \dot{x}_2 \\ &= 2x_1^2(x_1^3 + x_2) + x_2^3 + u = v \end{aligned}$$

Solving for u gives

$$u = v - [2x_1^2(x_1^3 + x_2) + x_2^3]$$

10-6 Choosing q_1 and q_2 as generalized coordinates, the kinetic energy is

$$K = \frac{1}{2}I\dot{q}_1^2 + \frac{1}{2}J\dot{q}_2^2$$

The potential energy is

$$V = MgL(1 - \cos q_1) + \frac{1}{2}k(q_1 - q_2)^2$$

The Lagrangian is

$$L = K - V = \frac{1}{2}I\dot{q}_2^2 + \frac{1}{2}J\dot{q}_2^2 - MgL(1 - \cos q_1) - \frac{1}{2}k(q_1 - q_2)^2$$

Therefore we compute

$$\frac{\partial L}{\partial \dot{q}_1} = I\dot{q}_1; \qquad \frac{\partial L}{\partial \dot{q}_2} = J\dot{q}_2$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = I\ddot{q}_1; \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = J\ddot{q}_2$$

$$\begin{aligned}\frac{\partial L}{\partial q_1} &= -MgL\sin q_1 - k(q_1 - q_2)\\ \frac{\partial L}{\partial q_2} &= k(q_1 - q_2)\end{aligned}$$

Therefore the equations of motion, ignoring damping, are given by

$$\begin{aligned}I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) &= 0\\ J\ddot{q}_2 - k(q_1 - q_2) &= u\end{aligned}$$

10-7 If there are damping terms $B_1\dot{q}_1$ and $B_2\dot{q}_2$ on the link and motor, respectively, the equations of motion are

$$\begin{aligned} I\ddot{q}_1 + B_1\dot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) &= 0 \\ J\ddot{q}_2 + B_2\dot{q}_2 - k(q_1 - q_2) &= u \end{aligned}$$

$$g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}; \qquad f = \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix}$$

Therefore, since g is a constant vector field

$$ad_f(g) \quad = \quad [f,g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g = -\frac{\partial f}{\partial x}g$$

Now,

$$\frac{\partial f}{\partial x} \quad = \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-MgL}{I}\cos x_1 - \frac{k}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix}$$

and so we have

$$ad_f(g) \quad = \quad \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{J} \\ 0 \end{bmatrix}$$

Similarly,

$$ad_f^2(f) \quad = \quad [f, ad_f(g)] = \frac{-\partial f}{\partial x}ad_f(g) = \begin{bmatrix} 0 \\ \frac{k}{IJ} \\ 0 \\ -\frac{k}{J^2} \end{bmatrix}$$

and

$$ad_f^3(g) \quad = \quad [f, ad_f^2(g)] = \frac{\partial f}{\partial x}ad_f^2(g) = \begin{bmatrix} -\frac{k}{IJ} \\ 0 \\ \frac{k}{J^2} \\ 0 \end{bmatrix}$$

10-9

$$
\begin{aligned}
L_g T_1 &= \frac{1}{J}\frac{\partial T_1}{\partial x_4} = 0 \Longrightarrow \frac{\partial T_1}{\partial x_4} = 0
\end{aligned}
$$

$$
\begin{aligned}
L_{[f,g]} T_1 &= -\frac{1}{J}\frac{\partial T_1}{\partial x_3} = 0 \Longrightarrow \frac{\partial T_1}{\partial x_3} = 0
\end{aligned}
$$

$$
\begin{aligned}
L_{ad_f^2(g)} T_1 &= \frac{k}{IJ}\frac{\partial T_1}{\partial x_2} - \frac{k}{J_2}\frac{\partial T_1}{\partial x_4} = 0 \Longrightarrow \frac{\partial T_1}{\partial x_2} = 0
\end{aligned}
$$

Since

$$
\frac{\partial T_1}{\partial x_4} = 0
$$

Finally

$$
\begin{aligned}
L_{ad_f^3(g)} T_1 &= \frac{-k}{IJ}\frac{\partial T_1}{\partial x_1} + \frac{k}{J^2}\frac{\partial T_1}{\partial x_3} \neq 0 \Longrightarrow \frac{\partial T_1}{\partial x_1} \neq 0
\end{aligned}
$$

since

$$
\frac{\partial T_1}{\partial x_3} = 0
$$

$$T_1 = x_1$$

$$T_2 = L_f T_1 = (1,0,0,0)f = x_2$$

$$T_3 = L_f T_2 = (0,1,0,0)f = \frac{-mgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)$$

$$\begin{aligned}
T_4 = L_f T_3 &= \left(\frac{-MgL}{I}\cos x_1 \frac{k}{I}, 0, \frac{k}{I}, 0, \frac{k}{I}, 0\right) f \\
&= \left(\frac{-MgL}{I}\cos x_1 - \frac{k}{I}\right) x_2 + \frac{k}{I} x_4 \\
&= \frac{-MgL}{I}\cos x_1 \cdot x_2 - \frac{k}{I}(x_2 - x_4)
\end{aligned}$$

$$
\begin{aligned}
L_g T_4 &= \frac{1}{J}\frac{\partial T_4}{\partial x_4} = \frac{k}{IJ} \\
\\
L_f T_4 &= \left(\frac{MgL}{I}\sin x_1 \cdot x_2, \frac{-MgL}{I}\cos x_1 - \frac{k}{I}, 0, \frac{k}{I}\right) f \\
&= \frac{MgL}{I}\sin x_1 \cdot x_2^2 - \left(\frac{MgL}{I}\cos x_1 + \frac{k}{I}\right)\left(\frac{-MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)\right) \\
&\quad + \frac{k^2}{IJ}(x_1 - x_3) \\
&= \frac{MgL}{I}\sin x_1\left[x_2^2 + \frac{MgL}{I}\cos x_1 + \frac{k}{I}\right] + \frac{k}{I}(x_1 - x_3)\left(\frac{k}{I} + \frac{k}{J} + \frac{MgL}{I}\cos x_1\right)
\end{aligned}
$$

10-12 The coordinate transformation is

$$
\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\
y_4 &= \frac{-MgL}{I}\cos x_1 \cdot x_2 - \frac{k}{I}(x_2 - x_4)
\end{aligned}
$$

Therefore the inverse transformation is

$$
\begin{aligned}
x_1 &= y_1 \\
x_2 &= y_2 \\
x_3 &= \frac{I}{k}\left(y_3 + \frac{MgL}{I}\sin x_1 + \frac{k}{I}x_1\right) \\
&= \frac{I}{k}y_3 + \frac{MgL}{k}\sin y_1 + y_1 \\
&= y_1 + \frac{I}{k}\left(y_3 + \frac{MgL}{I}\sin y_1\right) \\
x_4 &= \frac{I}{k}\left(y_4 + \frac{MgL}{I}\cos x_1 \cdot x_2 + \frac{k}{I}x_2\right) \\
&= y_2 + \frac{1}{k}\left(y_4 + \frac{MgL}{I}\cos y_1 \cdot y_2\right)
\end{aligned}
$$

10-13 This is an open-ended design problem. Use Matlab's pole placement or LQR routines to generate linear feedback gains.

10-14 In the case of a single-link rigid robot with a permanent-magnet DC motor we can write the dynamics equations of motion as

$$
\begin{aligned}
I\ddot{\theta} + Mg\ell\sin(\theta) &= u \\
L\dot{I} + RI &= V - K_b\dot{\theta} \\
u &= KI
\end{aligned}
$$

Define state and control variables as

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ I \end{bmatrix}; \; u = V
$$

and write the equations of motion as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{K}{I}x_3 - \frac{Mg\ell}{I}\sin(x_1) \\
\dot{x}_3 &= -\frac{K_b}{L}x_2 - \frac{R}{L}x_3 + \frac{1}{L}u
\end{aligned}
$$

The transformed state variables under which the above system can be feedback linearized are

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 \\
z_3 &= \frac{K}{I}x_3 - \frac{Mg\ell}{I}\sin(x_1)
\end{aligned}
$$

The feedback linearizing control law u is found from

$$
\begin{aligned}
\dot{z}_3 &= \frac{K}{I}\dot{x}_3 - \frac{Mg\ell}{I}\cos(x_1)\dot{x}_1 \\
&= -\frac{K}{I}(\frac{K_b}{L}x_2 - \frac{R}{L}x_3) - \frac{Mg\ell}{I}\cos(x_1)x_2 + \frac{K}{I}\frac{1}{L}u = v
\end{aligned}
$$

which results in

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= v
\end{aligned}
$$

10-15 From Equation (10.73) we see that, as $k \to \infty$,

$$x_3 \to y_1 = x_1$$
$$x_4 \to y_2 = x_2$$

The physical interpretation is that, as the joint stiffness tends to infinity, the link and motor position and velocity coincide. This makes sense because the shaft connecting the motor and link becomes a single DOF rigid body. Examining Equation (10.54), if we eliminate the spring torque $k(q_1 - q_2)$ and set $\ddot{q}_2 = \ddot{q}_1$ we recover the equations for a single-link rigid-joint robot

$$(I + J)\ddot{q} + Mg\ell \sin(q) = u$$

10-16 With spring force $F = \phi(q_1 - q_2)$ the equations of motion (10.54) become

$$\begin{aligned} I\ddot{q}_1 + Mg\ell\sin(q_1) + \phi(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - \phi(q_1 - q_2) &= u \end{aligned}$$

Thus the corresponding vector fields f, and g are

$$g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}; \qquad f = \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{1}{I}\phi(x_1 - x_3) \\ x_4 \\ \frac{1}{J}\phi(x_1 - x_3) \end{bmatrix}$$

Let ϕ' denote $\frac{\partial\phi}{\partial z}|_{z=x_1-x_3}$. Then a straightforward calculation shows that the distribution $[g, ad_f g, ad_f^2 g, ad_f^3 g]$ is modified as

$$[g, ad_f(g), ad_f^2(g), ad_f^3(g)] = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{IJ}\phi' \\ 0 & 0 & \frac{1}{IJ}\phi' & 0 \\ 0 & \frac{1}{J} & 0 & -\frac{1}{J^2}\phi' \\ \frac{1}{J} & 0 & -\frac{1}{J^2}\phi' & 0 \end{bmatrix}$$

which has rank four (and hence is feedback linearizable) provided $\phi' \neq 0$. The coordinate transformation and feedback linearizing control can be obtained by replacing $k(x_1 - x_3)$ in Equations (10.65)-(10.66) by $\phi(x_1 - x_3)$ and replacing k in Equations (10.68) and (10.69) by ϕ'.

10-17 With $y = q_1 = x_1$,we note that the vector field f in Equation (10.56) can be written as

$$
\begin{aligned}
f(x) &= \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin(x_1) - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{MgL}{I}\sin(x_1) \\ 0 \\ 0 \end{bmatrix} \\
&= Ax + \phi(y)
\end{aligned}
$$

where

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix} \quad \phi(y) = \begin{bmatrix} 0 \\ -\frac{MgL}{I}\sin(y) \\ 0 \\ 0 \end{bmatrix}
$$

Therefore, the flexible joint robot model can be written as

$$
\begin{aligned}
\dot{x} &= Ax + \phi(y) + bu \\
y &= Cx
\end{aligned}
$$

Therefore, with the observer equation defined as

$$
\dot{\hat{x}} = A\hat{x} + \phi(y) + bu + L(y - C\hat{x})
$$

the estimation error $e = x - \hat{x}$ is easily seen to satisfy

$$
\dot{e} = (A - LC)e
$$

and, therefore, observability of the pair (C, A) is sufficient to design an observer to estimate $\dot{q}_1, q_2, \dot{q}_2$ given only measurement of q_1.

10-18 With g_1 and g_2 given by

$$g_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 0 \\ 0 \end{bmatrix}$$

and with $q = (x, y, \theta, \phi)^T$ we have, since g_1 is constant, that

$$\begin{aligned} [g_1, g_2] &= \frac{\partial g_2}{\partial q} g_1 \\ &= \begin{bmatrix} 0 & 0 & -r\sin(\theta) & 0 \\ 0 & 0 & r\cos(\theta) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -r\sin\theta \\ r\cos(\theta) \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

10-19 A direct calculation shows that g_1 and g_2 are both orthogonal to w_1 and w_2. Since g_1 is constant, the Lie Bracket $[g_1, g_2]$ is

$$\begin{aligned}
[g_1, g_2] &= \frac{\partial g_2}{\partial q} g_1 \\
&= \begin{bmatrix} 0 & 0 & -\sin(\theta) & 0 \\ 0 & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & \frac{1}{d}\sec^2(\phi) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{d}\sec^2(\phi) \\ 0 \end{bmatrix}
\end{aligned}$$

which cannot be expressed as a linear combination of g_1 and g_2.

10-20 This follows simply from the definition of the cross product with some rearranging of terms.

$$
\begin{aligned}
\omega \times u &= \begin{bmatrix} \imath & \jmath & k \\ \omega_1 & \omega_2 & \omega_3 \\ u_1 & u_2 & u_3 \end{bmatrix} = \imath(\omega_2 u_3 - \omega_3 u_2) - \jmath(\omega_1 u_3 - \omega_3 u_1) + k(\omega_1 u_2 - \omega_2 u_1) \\
&= \begin{bmatrix} 0 \\ \omega_3 \\ -\omega_2 \end{bmatrix} u_1 + \begin{bmatrix} -\omega_3 \\ 0 \\ \omega_1 \end{bmatrix} u_2 + \begin{bmatrix} \omega_2 \\ -\omega_1 \\ 0 \end{bmatrix} u_3
\end{aligned}
$$

10-23 With $v_1 = a\sin(\omega t)$ and $v_2 = b\cos(\omega t)$, the first two equations in (10.119) can be explicitly integrated as

$$\begin{aligned} x_1(t) &= x_1(0) + \int_0^t a\sin(\omega\tau)d\tau = \frac{a}{\omega}(1 - \cos(\omega t)) \\ x_2(t) &= x_2(0) + \int_0^t b\cos(\omega\tau)d\tau = \frac{b}{\omega}\sin(\omega t) \end{aligned}$$

At $t = \frac{2\pi}{\omega}$ we have $x_1(\frac{2\pi}{\omega}) = x_1(0)$ and $x_2(\frac{2\pi}{\omega}) = x_2(0)$.

From the third equation in (10.119) we have

$$\dot{x}_3 = \frac{ab}{\omega}\sin(\omega t)(1 - cos(\omega t))$$

Hence, it is straightforward to compute by direct integration

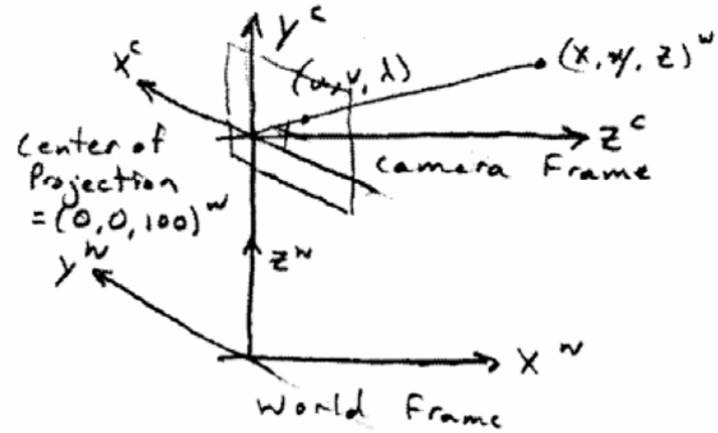
$$x_3(\frac{2\pi}{\omega}) = x_3(0) + \frac{ab\pi}{\omega^2}$$

and the result follows with $a = \omega = \pi$ and $b = 10$.

$$k \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} u \\ v \\ \lambda \end{bmatrix}$$

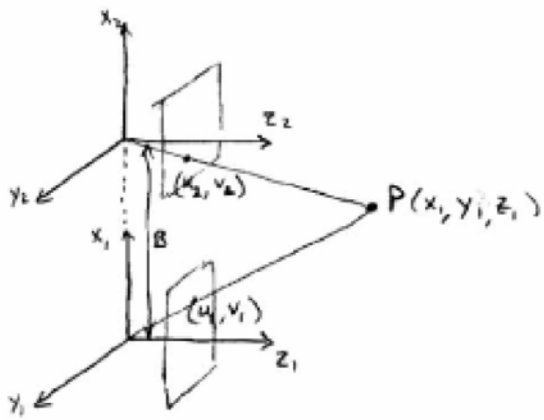
$$\begin{aligned} k &= \frac{\lambda}{z^c} \\ u &= kx^c \\ v &= ky^c \end{aligned}$$

a)	$(25, 25, 50)^c$	$\rightarrow$	$(5, 5) = (u, v)$
b)	$(-25, -25, 50)^c$	$\rightarrow$	$(-5, -5)$
c)	$(20, 5, -50)^c$		invisible (behind the image plane)
d)	$(15, 10, 25)^c$	$\rightarrow$	$(6, 4)$
e)	$(0, 0, 50)^c$	$\rightarrow$	$(0, 0)$
f)	$(0, 0, 100)^c$	$\rightarrow$	$(0, 0)$



$$\begin{aligned} x^c &= y^w \\ y^c &= z^2 - 100 \\ z^c &= x^w \end{aligned}$$

a)	$(25, 25, 50)^w$	$\rightarrow$	$(25, -50, 25)^c$	$\rightarrow$	$(10, -20) = (u, v)$
b)	$(-25, -25, 50)^w$	$\rightarrow$	$(-25, -30, -25)^c$		invisible (behind image plane)
c)	$(20, 5, -50)^w$	$\rightarrow$	$(5, -150, 20)^c$	$\rightarrow$	$(2.5, -75)$
d)	$(15, 10, 25)^w$	$\rightarrow$	$(10, -75, 15)^c$	$\rightarrow$	$\left(\frac{20}{3}, -50\right)$
e)	$(0, 0, 50)^w$	$\rightarrow$	$(0, -50, 0)^c$		invisible
f)	$(0, 0, 100)^w$	$\rightarrow$	$(0, 0, 0)^c$		invisible



From the transformation, we have

$$(x_2, y_2, z_2) \quad = \quad (x_1 - B, y_1, z_1)$$

We know that u_1, y_1, λ_1 and u_2, v_2, λ_2. We want to find *depth* z_1.

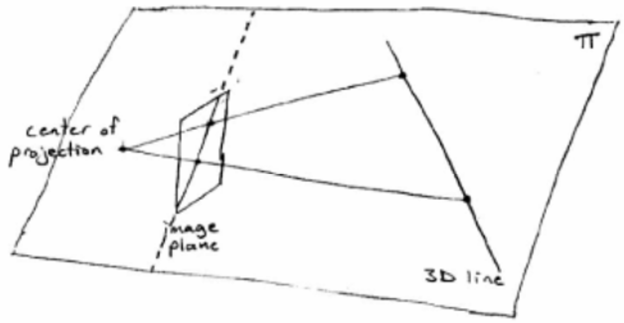
$$K_1 = \frac{\lambda_1}{z_1} \qquad K_2 = \frac{\lambda_2}{z_2} = \frac{\lambda_2}{z_1}$$

$$x_1 = \frac{u_1}{k_1} \qquad x_2 = \frac{u_2}{k_2}$$

$$x_1 = u_1\left(\frac{z_1}{\lambda_1}\right) \qquad x_1 - B = u_2\left(\frac{z_1}{\lambda_2}\right)$$

Setting equal:

$$\begin{aligned} x_1 &= u_1\left(\frac{z_1}{\lambda_1}\right) = B + u_2\left(\frac{z_1}{\lambda_2}\right) \\ B &= z_1\left[\frac{u_1}{\lambda_1} - \frac{u_1}{\lambda_2}\right] \\ \Rightarrow z_1 &= \frac{B}{\left(\frac{u_1}{\lambda_1} - \frac{u_2}{\lambda_2}\right)} \end{aligned}$$



Let Π be a plane defined by any two points on the 3D line and the center of the projection. The intersection of the image plane and plane Π is a straight line.

Degenerate cases:

1. Plane Π is *parallel* to and does not intersect the image plane. For this case to arise, the 3D line must be parallel to the image plane and lie behind the image plane.
2. If the 3D line passes through the center of projection, there are infinite possible planes Π. In this case, the 3D line appears as a single point in the image.

11-5 We are given two lines that are parallel in the camera frame.

$$L_1 : \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \gamma\overline{U}$$

$$L_2 : \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \gamma\overline{U}$$

where $\overline{U} = [U_x U_y U_z]^T$ is a unit vector.

From our equations relating points in the camera frame to coordinate in the image frame

$$k \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} u \\ v \\ \lambda \end{bmatrix}$$

we have

$$u = \frac{x^c}{z^c}\lambda \qquad v = \frac{y^c}{z^c}\lambda.$$

Substitute our lines:

$$u = \frac{(x_i + \gamma U_x)}{(z_i + \gamma U_2)}\lambda$$

Now take the limits as $\gamma \to \infty$.

$$\begin{aligned} u_\infty &= \lim_{\gamma\to\infty} \frac{(x_1 + \gamma U_x)}{(z_i + \gamma U_z)}\lambda = \lim_{\gamma\to\infty} \frac{\left(\frac{x_i}{\gamma} + U_x\right)}{\left(\frac{z_i}{\gamma} + U_z\right)}\lambda \\ &= \frac{U_x}{U_z}\lambda \end{aligned}$$

Similarly, we can find

$$v_\infty = \frac{U_y}{U_z}\lambda$$

The coordinate (u_∞, v_∞) is the vanishing point.

Remarks:

1. (u_∞, v_∞) does *not* depend on x_i, y_i, or z_i! This implies that *any* line with unit direction $\overline{U}$ will pass through (u_∞, v_∞).
2. (u_∞, v_∞) does not exist when $U_z = 0$. Why? When $U_z = 0$ the plane containing the 3D lines is parallel to the image plane. In this case, the two parallel lines remain parallel in the image; they never intersect.

11-6 All horizontal lines will have $U_y = 0$. From Problem 11-5, we substitute into the expressions for the vanishing point to see that all such lines converge to a point with $v_\infty = 0$. Therefore, all horizontal lines vanish at a point along the line $v = 0$ in the image.

11-7 From problem 11-5 we have

$$u_\infty = \lambda \frac{U_x}{U_z} \qquad v_\infty = \lambda . \frac{U_y}{U_z}$$

Since $\overline{U}$ is a unit vector, we also know

$$U_x^2 + U_y^2 + U_z^2 \quad = \quad 1.$$

Now substitute

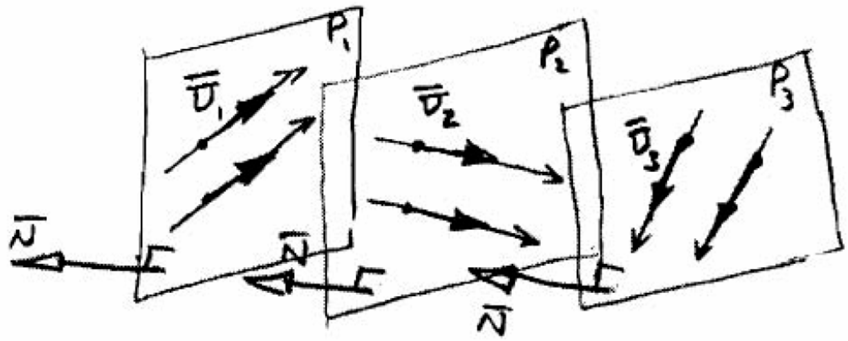
$$\begin{aligned} u_\infty^2 + v_\infty^2 + \lambda^2 &= \frac{\lambda^2}{U_z^2}(U_x^2 + U_y^2) + \lambda^2 \\ &= \frac{\lambda^2}{u_z^2}\underbrace{(U_x^2 + U_y^2 + U_z^2)}_{1} \\ &= \frac{\lambda^2}{U_z^2}. \end{aligned}$$

Rearranging terms yields

$$U_z \quad = \quad \frac{\lambda}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}}$$

Substituting this formula for U_z back into the expressions for u_∞ and v_∞, we get

$$\begin{aligned} U_x &= \frac{u_\infty}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}} \\ U_x &= \frac{v_\infty}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}} \end{aligned}$$



Let two parallel lines with unit vector U_i define plane P_i.

$$U_i = \begin{bmatrix} U_{xi} \\ U_{yi} \\ U_{zi} \end{bmatrix}$$

Consider three such pairs of parallel lines with planes P_1, P_2, P_3 all parallel. Since the planes are parallel, they share a common normal, $\overline{N}$.

$$N = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$$

From our formulas for vanishing points, we know

$$\begin{cases} u_{\infty i} &= \lambda \frac{U_{xi}}{U_{zi}} \\ v_{\infty i} &= \lambda \frac{U_{yi}}{U_{zi}} \end{cases} \Rightarrow \begin{cases} U_{xi} &= \frac{u_{\infty i} U_{zi}}{\lambda} \\ U_{yi} &= \frac{v_{\infty i} U_{zi}}{\lambda}. \end{cases}$$

Since $\overline{N}$ is normal to any of the lines, we know $U_i \cdot \overline{N} = 0$ for all i.

$$\begin{aligned} U_i \cdot \overline{N} = 0 \Rightarrow U_{xi}N_x + U_{yi}N_y + U_{zi}N_z &= 0 \\ \frac{u_{\infty i}U_{zi}}{\lambda}N_x + \frac{v_{\infty i}U_{zi}}{\lambda}N_y + U_{zi}N_z &= 0 \end{aligned}$$

When $\overline{U}_{zi} \neq 0$, we have

$$\left(\frac{Nx}{\lambda}\right)u_{\infty i} + \left(\frac{Ny}{\lambda}\right)v_{\infty i} + N_z = 0$$

which is the equation of a 2D line of the form $au_\infty + bv_\infty + c = 0$ (in the image plane). Therefore, all vanishing points $(u_{\infty i}, v_{\infty,i})$ lie along this line.
Remark: When $\overline{U}_{zi} = 0$, the plane is parallel to the image plane and the parallel lines do not converge in the image.

11-9 1. To show that $\angle V_a C V_b = \frac{\pi}{2}$, it is sufficient to show $C\vec{V}_a \cdot C\vec{V}_b = 0$. Since vectors $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), andc = (c_1, c_2, c_3)$ define the edges of the cube, we know they are perpendicular, thus $a \cdot b = b \cdot c = a \cdot c = 0$. From our formulas for vanishing points, we have that
$C\vec{V}_a = \lambda(\frac{a_1}{a_3}, \frac{a_2}{a_3}, 1)$ and $C\vec{V}_b = \lambda(\frac{b_1}{b_3}, \frac{b_2}{b_3}, 1)$.

$$\begin{aligned} C\vec{V}_a \cdot C\vec{V}_b &= \lambda^2 \left(\frac{a_1 b_1}{a_3 b_3} + \frac{a_2 b_2}{a_3 b_3} + 1 \right) \\ &= \frac{\lambda^2}{a_3 b_3}(a_1 b_2 + a_2 b_2 + a_3 b_3) \\ &= \frac{\lambda^2}{a_3 b_3}(a \cdot b) \\ &= 0. \end{aligned}$$

Therefore, $C\vec{V}_a \perp C\vec{V}_b$ which implies $\angle V_a C V_b = \frac{\pi}{2}$. Similar proofs hold for the other two angles in the problem statement.

2. To show $V_b\vec{V}_c$ is perpendicular to the plane, it is sufficient to show that two distinct vectors in the plane are perpendicular to $V_a\vec{V}_b$. That is, the dot products to two distinct vectors in the plane with $V_b\vec{V}_c$ are both zero.
One such vector is the altitude h_a, which is perpendicular to $V_b\vec{V}_c$ by definition. Another such vector is $C\vec{V}_a$.
By vector addition $C\vec{V}_b + V_b\vec{V}_c = C\vec{V}_c$, so $V_b\vec{V}_c = C\vec{V}_c - C\vec{V}_b$. We have

$$\begin{aligned} C\vec{V}_a \cdot V_b\vec{V}_C &= C\vec{V}_a \cdot \left(C\vec{V}_c - C\vec{V}_b \right) \\ &= C\vec{V}_a \cdot C\vec{V}_c - C\vec{V}_a \cdot C\vec{V}_b \\ &= 0 - 0. \end{aligned}$$

Therefore, $C\vec{V}_a$, another vector in the plane, is also perpendicular to $V_b\vec{V}_c$. We conclude that $V_b\vec{V}_c$ is perpendicular to the plane, and hence is the vector normal to the plane.

3. From the previous part, we know that $V_b\vec{V}_c$ is normal to plane P_a. The vector $\vec{N} = (0, 0, 1)$ is normal to the image plane. To show that plane P_a is perpendicular to the image plane, it is sufficient to show that the respective normal vectors are perpendicular.

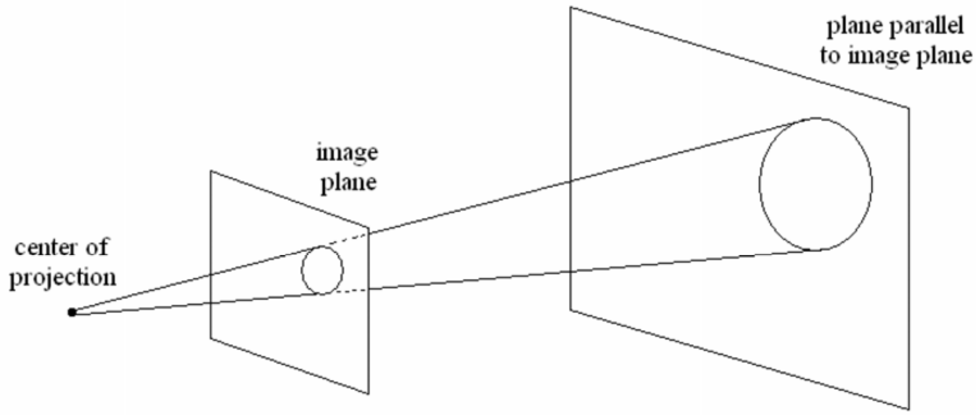
$$\begin{aligned} V_b\vec{V}_c \cdot \vec{N} &= (u_c - u_b)0 + (v_c - v_b)0 + (0)1 \\ &= 0 \end{aligned}$$

Therefore, the image plane is perpendicular to plane P_a.

4. The orthocenter of the triangle $V_a V_b V_c$ is $(0, 0, \lambda)$. It lies on the camera's z axis.

11-10 The fun solution.

11-11



Consider rays from the center of projection to every point on the circle. Taken together, these rays form a cone. The locus of the intersection of the cone with the image plane is a circle in the image.

$$
\begin{aligned}
&\sum_{i=1}^{N}(x_i-\mu)^2 p(x_i) \\
&\quad = \sum (x_i^2 - 2\mu x_i + \mu^2) P(x_i) \\
&\quad = \sum x_i^2 P(x_i) - \sum 2\mu x_i P(x_i) + \sum \mu^2 P(x_i) \\
&\quad = \sum x_i^2 P(x_i) - 2\mu \underbrace{\sum x_i P(x_i)}_{\mu} + \mu^2 \underbrace{\sum P(x_i)}_{1} \\
&\quad = \sum x_i^2 P(x_i) - 2\mu^2 + \mu^2 \\
&\quad = \sum x_i^2 P(x_i) - \mu^2
\end{aligned}
$$

11-13 Students simply need to know the formulas for the row and column centroids.

$$\bar{r} = \frac{m_{10}}{m_{00}} = \frac{\sum_{r,c} r\mathcal{I}(r,c)}{\sum_{r,c} \mathcal{I}(r,c)} \qquad \bar{c} = \frac{m_{01}}{m_{00}} = \frac{\sum_{r,c} c\mathcal{I}(r,c)}{\sum_{r,c} \mathcal{I}(r,c)}$$

Equation (11.21) is found by substituting $\bar{r}m_0 0$ for $\sum_{r,c} r\mathcal{I}(r,c)$ and $\bar{c}m_0 0$ for $\sum_{r,c} c\mathcal{I}(r,c)$.

11-14 $P_0(z)$ is the pdf for background pixels, which are low intensity (high z values).
$P_1(z)$ is the pdf for object pixels, which are high intensity (high z values).
Given a chosen threshold value t,

- the probability that a background pixel is misclassified as an object pixel is $\int_t^\infty P_0(z)dz$
- the probability that an object pixel is misclassified as a background pixel is $\int_{-\infty}^t P_1(z)dz$.

11-15 We can write the total probability of error as $E_{total} = \int_{-\infty}^{t} P_1(z)dz + \int_{t}^{\infty} P_0(z)dz$. To minimize E_{total}, so set its derivative with respect to t equal to zero. Using the Fundametnal Theorem of Calculus, we find

$$\begin{aligned}
0 &= \frac{d}{dt}E_{total} \\
&= \frac{d}{dt}\int_{-\infty}^{t} P_1(z)dz + \frac{d}{dt}\int_{t}^{\infty} P_0(z)dz \\
&= P_1(t) + \frac{d}{dt}\int_{\infty}^{t} -P_0(z)dz \\
&= P_1(t) - P_0(t).
\end{aligned}$$

Therefore, the total probability of error is minimized when $P_1(t) = P_0(t)$.
It is a fair assumption that the measured pixel intensity is the sum of a true intensity plus a noise term (which we assume to be additive Gaussian noise). Since all of the pixel values are manufactured by the same process, it's reasonable to assume that the same sort of noise will be introduced for each pixel. Therefore, we get two independent, identically distributed ($\sigma_0^2 = \sigma_1^2$) Gaussian random variables $P_0(t)$ and $P_1(t)$. So the pdfs are of the form

$$P(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}.$$

We know the total probability of error is minimized when $P_1(t) = P_0(t)$, so we substitute the above Gaussian pdf with variances equal. Working through the math leads to the conclusion $(t-\mu_1)^2 = (t-\mu_0)^2$. Due to the even power, this does not imply that $t-\mu_1 = t-\mu_0$. Rather, it implies $|t-\mu_1| = |t-\mu_0|$, so t is equidistant from μ_1 and μ_0. So we choose

$$t^* = \frac{\mu_1 + \mu_0}{2}.$$

12-1

$$L = \begin{bmatrix} -\dfrac{\lambda}{z_1} & 0 & \dfrac{u_1}{z_1} & \dfrac{u_1 v_1}{\lambda} & -\dfrac{\lambda^2 + u_1^2}{\lambda} & v_1 \\ 0 & -\dfrac{\lambda}{z_1} & \dfrac{v_1}{z_1} & \dfrac{\lambda^2 + v_1^2}{\lambda} & -\dfrac{u_1 v_1}{\lambda} & -u_1 \\ -\dfrac{\lambda}{z_2} & 0 & \dfrac{u_2}{z_2} & \dfrac{u_2 v_2}{\lambda} & -\dfrac{\lambda^2 + u_2^2}{\lambda} & v_2 \\ 0 & -\dfrac{\lambda}{z_2} & \dfrac{v_2}{z_2} & \dfrac{\lambda^2 + v_2^2}{\lambda} & -\dfrac{u_2 v_2}{\lambda} & -u_2 \end{bmatrix}$$

12-2 If the four rows of L are linearly independent, then L is rank 4, and its null space has rank 2. In general, the rank of the null space of L in problem 12-1 is given by $6 - \text{rank } L$.

12-3 For this problem, we shall let the coordinate frame of the left camera be the reference frame. Thus, the coordinates of the fixed point (x, y, z) are expressed relative to the (moving) left camera frame. It is common in stereo vision systems to choose a configuraion in which the position and orientation of the right camera w.r.t. the left camera is given by

$$H_r^l = \begin{bmatrix} 1 & 0 & 0 & B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in which B is called the *baseline distance* (see problem 11-3).

Suppose the left camera is moving with velocity $\xi_l = (v_l, \omega_l)$. The interaction matrix for u_l, v_l is just the usual interaction matrix for a point.

As we have seen in Chapter 4, if the two cameras are rigidly attached (i.e., their coordinate frames are rigidly attached), their angular velocities are the same, i.e., $\omega_l = \omega_r$. The velocity of the origin of the right camera frame is given by $v_r = \omega \times [B, 0, 0]^T + v_l$ which gives

$$\begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} = \begin{bmatrix} 0 \\ B\omega_z \\ -B\omega_y \end{bmatrix} + \begin{bmatrix} v_{lx} \\ v_{ly} \\ v_{lz} \end{bmatrix}$$

The coordinates of the fixed point w.r.t. the coordinate frame of the right camera are given by $(x - B, y, z)$, since the left and right frames are related by a pure translation along the x-axis. The velocity of the fixed point relative to the moving right camera frame is therefore given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} &= -\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} x - B \\ y \\ z \end{bmatrix} - \begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} \\ &= \begin{bmatrix} y\omega_z - z\omega_y - v_{lx} \\ -(x - B)\omega_z + z\omega_x - v_{ly} - B\omega_z \\ (x - B)\omega_y - y\omega_x - v_{lz} + B\omega_y \end{bmatrix} \end{aligned}$$

which can be written as the system of three equations

$$\begin{aligned} \dot{x} &= y\omega_z - z\omega_y - v_{lx} \\ \dot{y} &= -x\omega_z + z\omega_x - v_{ly} \\ \dot{z} &= x\omega_y - y\omega_x - v_{lz} \end{aligned}$$

There is no published solution for Problem 12.4.

12-5 Suppose the end effector frame is moving with velocity $\xi = (v, \omega)$. Then, the origin of the frame has velocity v, since angular velocity of the end-effector frame does not induce motion of the frame's orign. Therefore, we have

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \end{aligned}$$

We use the quotient rule for differentiation with the equations of perspective projection to obtain

$$\dot{u} = \frac{d}{dt}\frac{\lambda x}{z} = \lambda \frac{z\dot{x} - x\dot{z}}{z^2}$$

and

$$\dot{v} = \frac{d}{dt}\frac{\lambda y}{z} = \lambda \frac{z\dot{y} - y\dot{z}}{z^2}$$

By solving the perspective projection equations for x and y we obtain

$$x = \frac{uz}{\lambda}, \qquad y = \frac{vz}{\lambda}$$

and substituting thees results into the above derivatives we obtain

$$\begin{aligned} \dot{u} &= \lambda \frac{z\dot{x} - x\dot{z}}{z^2} = \frac{\lambda}{z^2}(zv_x - \frac{uz}{\lambda}v_z) = \frac{\lambda}{z}(v_x - \frac{u}{\lambda}v_z) \\ \dot{v} &= \frac{\lambda}{z}(v_y - \frac{u}{\lambda}v_z) \end{aligned}$$

Writing this in matrix form gives

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{z} & 0 & -\frac{u}{z} & 0 & 0 & 0 \\ 0 & \frac{\lambda}{z} & -\frac{v}{z} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

12-6 This problem is somewhat open ended. Derivations and discussion of this issue can be found in the following reference.

- F. Chaumette, "Image moments: a general and useful set of features for visual servoing," IEEE Trans. on Robotics, 20(4):713-723, August 2004.
- O. Tahri, F. Chaumette, "Point-based and region-based image moments for visual servoing of planar objects," IEEE Trans. on Robotics, 21(6), December 2005.

12-7 through 12-11 These problems are all simulation problems.

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