Iran TST 2016

**Test 1** Day 1

**1** Let $m$ and $n$ be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

**2** For a finite set $A$ of positive integers, a partition of $A$ into two disjoint nonempty subsets $A_1$ and $A_2$ is *good* if the least common multiple of the elements in $A_1$ is equal to the greatest common divisor of the elements in $A_2$. Determine the minimum value of $n$ such that there exists a set of $n$ positive integers with exactly 2015 good partitions.

**3** Let $ABCD$ be a convex quadrilateral, and let $P$, $Q$, $R$, and $S$ be points on the sides $AB$, $BC$, $CD$, and $DA$, respectively. Let the line segment $PR$ and $QS$ meet at $O$. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines $AC$, $PQ$, and $RS$ are either concurrent or parallel to each other.

**Test 1** Day 2

**4** Let $n$ be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n) x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

**5** Let $ABC$ be a triangle with $\angle C = 90^\circ$, and let $H$ be the foot of the altitude from $C$. A point $D$ is chosen inside the triangle $CBH$ so that $CH$ bisects $AD$. Let $P$ be the intersection point of the lines $BD$ and $CH$. Let $\omega$ be the semicircle with diameter $BD$ that meets the segment $CB$ at an interior point. A line through $P$ is tangent to $\omega$ at $Q$. Prove that the lines $CQ$ and $AD$ meet on $\omega$.

**6** In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

*Proposed by Russia*

**Test 2** Day 1

**1** Let $ABC$ be an acute triangle and let $M$ be the midpoint of $AC$. A circle $\omega$ passing through $B$ and $M$ meets the sides $AB$ and $BC$ at points $P$ and $Q$ respectively. Let $T$ be the point such that $BPTQ$ is a parallelogram. Suppose that $T$ lies on the circumcircle of $ABC$. Determine all possible values of $\frac{BT}{BM}$.

**2** Let $a, b, c, d$ be positive real numbers such that $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 2$. Prove that

$$\sum_{cyc} \sqrt{\frac{a^2+1}{2}} \geq (3.\sum_{cyc} \sqrt{a}) - 8$$

**3** Let $n$ be a positive integer. Two players $A$ and $B$ play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.
(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player $A$ takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

*Proposed by Finland*

**Test 2** Day 2

**4** Let $ABC$ be a triangle with $CA \neq CB$. Let $D$, $F$, and $G$ be the midpoints of the sides $AB$, $AC$, and $BC$ respectively. A circle $\Gamma$ passing through $C$ and tangent to $AB$ at $D$ meets the segments $AF$ and $BG$ at $H$ and $I$, respectively. The points $H'$ and $I'$ are symmetric to $H$ and $I$ about $F$ and $G$, respectively. The line $H'I'$ meets $CD$ and $FG$ at $Q$ and $M$, respectively. The line $CM$ meets $\Gamma$ again at $P$. Prove that $CQ = QP$.

*Proposed by El Salvador*

**6** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer $k$, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called [i]$k$-good[/i] if $\gcd(f(m)+n, f(n)+m) \le k$ for all $m \neq n$. Find all $k$ such that there exists a $k$-good function.

*Proposed by James Rickards, Canada*

**Test 3** Day 1

**2** Let $ABC$ be an arbitrary triangle and $O$ is the circumcenter of $\triangle ABC$.Points $X, Y$ lie on $AB, AC$,respectively such that the reflection of $BC$ WRT $XY$ is tangent to circumcircle of $\triangle AXY$.Prove that the circumcircle of triangle $AXY$ is tangent to circumcircle of triangle $BOC$.

**3** Let $p \neq 13$ be a prime number of the form $8k+5$ such that 39 is a quadratic non-residue modulo $p$. Prove that the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \equiv 0 \pmod p$$

has a solution in integers such that $p \nmid x_1x_2x_3x_4$.

**Test 3** Day 2

**4** Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \ge \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer $k$. Prove that $a_1 + a_2 + \ldots + a_n \ge n$ for every $n \ge 2$.

**5** Let $AD, BF, CE$ be altitudes of triangle $ABC$.$Q$ is a point on $EF$ such that $QF = DE$ and $F$ is between $E, Q$.$P$ is a point on $EF$ such that $EP = DF$ and $E$ is between $P, F$.Perpendicular bisector of $DQ$ intersect with $AB$ at $X$ and perpendicular bisector of $DP$ intersect with $AC$ at $Y$.Prove that midpoint of $BC$ lies on $XY$.