# An Introduction to Homotopy Type Theory

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# Outline

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- A special year on univalent foundation of Mathematics was held in 2012-2013 at the Institute for Advanced Study, school of mathematics.
- Recently, philosopher of science, James Ladyman, argues that HoTT can be considered as an autonomous foundation of mathematics, independent of Homotopy Theory.

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- If $\theta(x)$ is a predicate, what is the place from which $x$ may be taken s.t. $\theta(x)$ is a sensible assertion?
  For example, if $\theta(x)$ stands for " $x$ is either true or false", then $x$ should be a proposition.

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  For example, if $\theta(x)$ stands for " $x$ is either true or false", then $x$ should be a proposition.
- The basic notion in modern type theory is that each object is assigned a type, and this type is something to which the object is explicitly linked.

# Curry-Howard Isomorphism

**Curry-Howard (1934)**:
Corresponding between Computations in type theory and Natural Deduction proofs

$$\frac{f : A \to B \qquad x : A}{f(x) : B}$$

# Proposition as types

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- Propositions are syntactic objects, like $2+3=5$. On the other hand, "$2+2=4$ is true" is a judgement.
- A proposition is true if we have a token of the corresponding type.

# Logical operators

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$$A \to B = B^A$$
$$A \wedge B = A \times B$$
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**Identity Type**: For $a, b$ of type $U$, we have the type $Id_U(a, b)$.

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  A proof of $\forall x : A\ \varphi(x)$ is a function with domain $A$ s.t. for $x : A$ we have $f(x) : \varphi(x)$.

$$\forall x : A\ \varphi(x) = \prod_{x \in A} \varphi(x)$$

$$\underbrace{\exists x : A\ \varphi(x)}_{\text{a pair } (x,y) \text{ s.t. } x:A \text{ and } y:\varphi(x)} \qquad = \qquad \underbrace{\coprod_{x:A} \varphi(x)}_{\text{coproduct or disjoint union}}$$

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- "$exp_1 = exp_2$" means expressions $exp_1$ and $exp_2$ name the same tokens or types.
- "$exp_1 := exp_2$" means "by definition, expression $exp_1$ names the same tokens or types as $exp_2$".

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$$a = b \not\Leftarrow Id_U(a,b) \neq \emptyset$$

This failure is definition of intentionality for a type theory

# Path Induction

For any type $U$ and property $P$ that can be asserted of identifications between tokens of $U$, if we can prove that $P$ holds of all trivial self-identifications $refl_a$ for all $a \in U$, then it holds of all identifications in $U$.

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- $A \simeq B$ is the type of equivalences between $A$ and $B$, i.e. types of functions $f : A \to B$ for which there exists a quasi-invense.

# Isomorphisms are Identities

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- why is $UA$ plausible?
- Two structures $A$ and $B$ are isomorphic if for each structural property $P(x)$, if $P(A)$ then $P(B)$, i.e. $\dfrac{A \cong B \quad P(A)}{P(B)}$

Now Consider $P(x) := Id_U(A, X)$. We have

$$\frac{A \cong B \quad Id_U(A, A)}{Id_U(A, B)}$$

That means isomorphic objects are identical.

# Consistency of HoTT

**Voevodsky:** HoTT and Univalent Axiom are consistent relative to $ZFC$ (it has a model in the category of Kan complexes).

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- The type of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ can be constructed using type constructors of HoTT similar to the usual constructions of such numbers.

# Homotopy Theory

The study of topological spaces and functions between them up to continuous distortion.
That is if there is a continuous deformation that transforms one topological space into another or one continuous function into another, then in homotopy theory they are regarded as equivalent.

# Homotopy Theory

H1: Let $A, B$ be topological spaces. Two continuous functions $f, g : A \to B$ are homotopic if there exists a continuous function $h : [0, 1] \times A \to B$ s.t. $h(0, x) = f(x)$ and $h(1, x) = g(x)$. We denote this by $f \sim g$.

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H2: Let $x, y \in X$. A path between $x, y$ is a continuous function $\gamma : [0, 1] \to X$ s.t. $\gamma(0) = x, \gamma(1) = y$.

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H4: Example: (i) Any two paths between any two points in Euclidean plane are homotopic.
(ii) A disk and a single point are homotopy equivalent. But, a circle and its root are not.

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- **Interpretation of Path induction**: Any path between $x$ and $y$ is homotopic to a constant path at $x$.

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2- **Semantics:** A type is a mathematical concept and its tokens are instances of the concept (e.g. Number-2) Intentionality (e.g. even divisor of $9 \neq$ even divisor of $11$). When we work formally in HoTT, we construct expressions according to formula rules. They are names of tokens and types.

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3- **Metaphysics:** The only metaphysical commitment required is to the existence of concepts.

4- **Epistemology:** The truth of a proposition is demonstrated by exhibiting a certificate, and a proof is a step-by-step construction of a certificate to the conclusion from certificates to the premises.

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5- **Methodology:** We begin by formulating types corresponding to the kinds of mathematical entities under discussion. Then from these we form types expressing the proposition to be proved and the premises to be assumed.

# References


- Awodey, Steve; Structuralism, invariance, and univalence. Philos. Math. (3) 22 (2014), no. 1, 1 - 11
- Martin-Löf, Per; On the meanings of the logical constants and the justifications of the logical laws. Nordic J. Philos. Logic 1 (1996), no. 1, 11 - 60
- Ladyman, James; Presnell, Stuart; Does homotopy type theory provide a foundation for mathematics? British J. Phil. Sci. 69 (2018), no. 2, 377 - 420.
- Homotopy Type Theory: Univalent Foundations of Mathematics. The Univalent Foundations Program, Institute for Advanced Study (IAS), Princeton, NJ, 2013.

Thanks for your attention