3 ANALYSIS AND TRANSMISSION OF SIGNALS

Electrical engineers instinctively think of signals in terms of their frequency spectra and think of systems in terms of their frequency responses. Even teenagers know about audio signals having a bandwidth of 20 kHz and good-quality loud speakers responding up to 20 kHz. This is basically thinking in the frequency domain. In the last chapter we discussed spectral representation of periodic signals (Fourier series). In this chapter we extend this spectral representation to aperiodic signals.

3.1 APERIODIC SIGNAL REPRESENTATION BY FOURIER INTEGRAL

Applying a limiting process, we now show that an aperiodic signal can be expressed as a continuous sum (integral) of everlasting exponentials. To represent an aperiodic signal $g(t)$ such as the one shown in Fig. 3.1a by everlasting exponential signals, let us construct a new periodic signal $g_{T_0}(t)$ formed by repeating the signal $g(t)$ every T_0 seconds, as shown in Fig. 3.1b. The period T_0 is made long enough to avoid overlap between the repeating pulses. The periodic signal $g_{T_0}(t)$ can be represented by an exponential Fourier series. If we let $T_0 \to \infty$, the pulses in the periodic signal repeat after an infinite interval, and therefore

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t)$$

Thus, the Fourier series representing $g_{T_0}(t)$ will also represent $g(t)$ in the limit $T_0 \to \infty$. The exponential Fourier series for $g_{T_0}(t)$ is given by

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{3.1}$$

in which

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-jn\omega_0 t}\, dt \tag{3.2a}$$

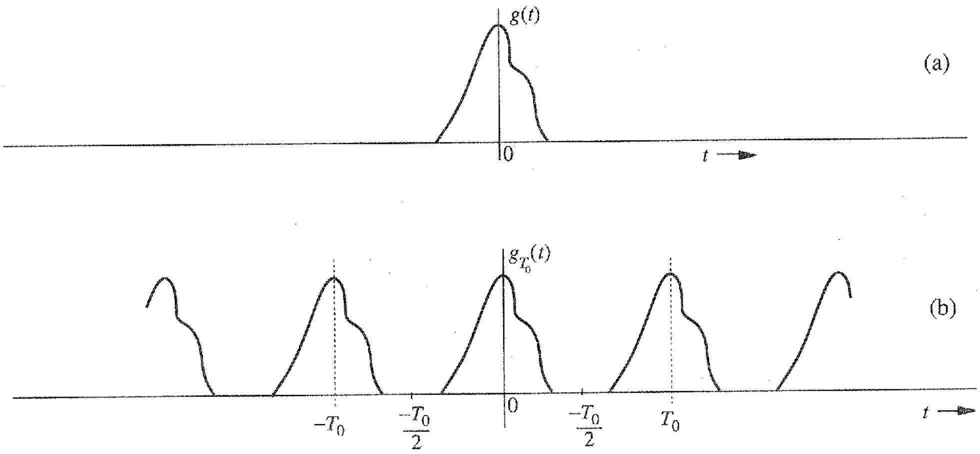

Figure 3.1 Construction of a periodic signal by periodic extension of $g(t)$.

and

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0 \tag{3.2b}$$

Observe that integrating $g_{T_0}(t)$ over $(-T_0/2, T_0/2)$ is the same as integrating $g(t)$ over $(-\infty, \infty)$. Therefore, Eq. (3.2a) can be expressed as

$$\begin{aligned} D_n &= \frac{1}{T_0} \int_{-\infty}^{\infty} g(t) e^{-jn\omega_0 t}\, dt \\ &= \frac{1}{T_0} \int_{-\infty}^{\infty} g(t) e^{-j2\pi n f_0 t}\, dt \end{aligned} \tag{3.2c}$$

It is interesting to see how the nature of the spectrum changes as T_0 increases. To understand this fascinating behavior, let us define $G(f)$, a continuous function of ω, as

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t}\, dt \tag{3.3}$$

$$= \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.4}$$

in which $\omega = 2\pi f$. A glance at Eqs. (3.2c) and (3.3) shows that

$$D_n = \frac{1}{T_0} G(nf_0) \tag{3.5}$$

This in turn shows that the Fourier coefficients D_n are $(1/T_0$ times) the samples of $G(f)$ uniformly spaced at intervals of f_0 Hz, as shown in Fig. 3.2a.*

Therefore, $(1/T_0)G(f)$ is the envelope for the coefficients D_n. We now let $T_0 \to \infty$ by doubling T_0 repeatedly. Doubling T_0 halves the fundamental frequency f_0, so that there are now twice as many components (samples) in the spectrum. However, by doubling T_0, the envelope $(1/T_0)G(f)$ is halved, as shown in Fig. 3.2b. If we continue this process of doubling T_0 repeatedly, the spectrum progressively becomes denser while its magnitude becomes smaller.

* For the sake of simplicity we assume D_n and therefore $G(f)$ in Fig. 3.2 to be real. The argument, however, is also valid for complex D_n [or $G(f)$].

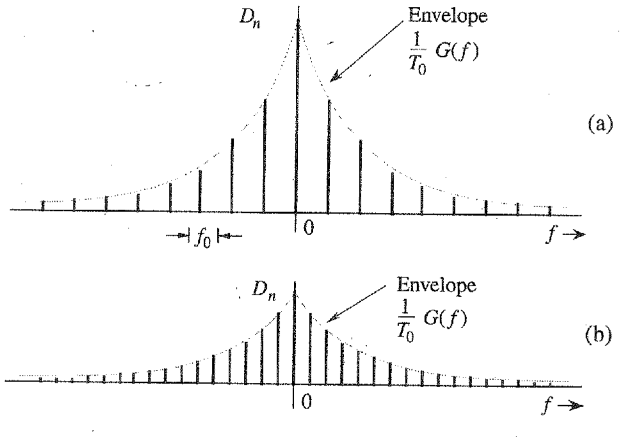

Figure 3.2 Change in the Fourier spectrum when the period T_0 in Fig. 3.1 is doubled.

Note, however, that the relative shape of the envelope remains the same [proportional to $G(f)$ in Eq. (3.3)]. In the limit as $T_0 \rightarrow \infty, f_0 \rightarrow 0$ and $D_n \rightarrow 0$. This means that the spectrum is so dense that the spectral components are spaced at zero (infinitesimal) interval. At the same time, the amplitude of each component is zero (infinitesimal). We have *nothing of everything, yet we have something!* This sounds like *Alice in Wonderland*, but as we shall see, these are the classic characteristics of a very familiar phenomenon.*

Substitution of Eq. (3.5) in Eq. (3.1) yields

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(nf_0)}{T_0} e^{jn2\pi f_0 t} \tag{3.6}$$

As $T_0 \rightarrow \infty, f_0 = 1/T_0$ becomes infinitesimal ($f_0 \rightarrow 0$). Because of this, we shall replace f_0 by a more appropriate notation, Δf. In terms of this new notation, Eq. (3.2b) becomes

$$\Delta f = \frac{1}{T_0}$$

and Eq. (3.6) becomes

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} [G(n\Delta f)\Delta f]\, e^{(j2\pi n\Delta f)t} \tag{3.7a}$$

Equation (3.7a) shows that $g_{T_0}(t)$ can be expressed as a sum of everlasting exponentials of frequencies $0, \pm\Delta f, \pm 2\Delta f, \pm 3\Delta f, \cdots$ (the Fourier series). The amount of the component of frequency $n\Delta f$ is $[G(n\Delta f)\Delta f]$. In the limit as $T_0 \rightarrow \infty$, $\Delta f \rightarrow 0$ and $g_{T_0}(t) \rightarrow g(t)$. Therefore,

$$g(t) = \lim_{T_0 \rightarrow \infty} g_{T_0}(t) = \lim_{\Delta f \rightarrow 0} \sum_{n=-\infty}^{\infty} G(n\Delta f) e^{(j2\pi n\Delta f)t} \Delta f \tag{3.7b}$$

The sum on the right-hand side of Eq. (3.7b) can be viewed as the area under the function $G(f)e^{j2\pi ft}$, as shown in Fig. 3.3. Therefore,

* You may consider this as an irrefutable proof of the proposition that 0% ownership of everything is better than 100% ownership of nothing!

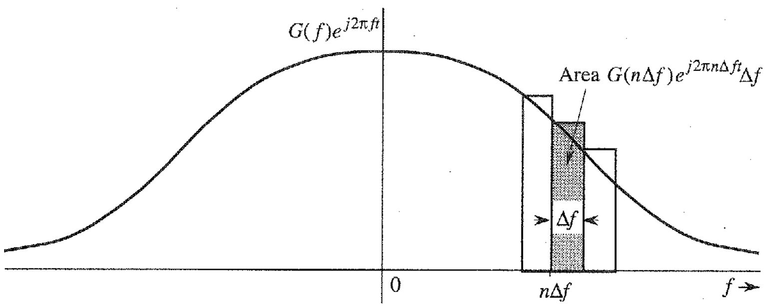

Figure 3.3 The Fourier series becomes the Fourier integral in the limit as $T_0 \to \infty$.

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df \tag{3.8}$$

The integral on the right-hand side is called the **Fourier integral**. We have now succeeded in representing an aperiodic signal $g(t)$ by a Fourier integral* (rather than a Fourier series). This integral is basically a Fourier series (in the limit) with fundamental frequency $\Delta f \to 0$, as seen from Eq. (3.7b). The amount of the exponential $e^{j2\pi n\Delta ft}$ is $G(n\Delta f)\Delta f$. Thus, the function $G(f)$ given by Eq. (3.3) acts as a spectral function.

We call $G(f)$ the **direct** Fourier transform of $g(t)$, and $g(t)$ the **inverse** Fourier transform of $G(f)$. The same information is conveyed by the statement that $g(t)$ and $G(f)$ are a Fourier transform pair. Symbolically, this is expressed as

$$G(f) = \mathcal{F}[g(t)] \qquad \text{and} \qquad g(t) = \mathcal{F}^{-1}[G(f)]$$

or

$$g(t) \Longleftrightarrow G(f)$$

To recapitulate,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\, dt \tag{3.9a}$$

and

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j\omega t}\, df \tag{3.9b}$$

where $\omega = 2\pi f$.

It is helpful to keep in mind that the Fourier integral in Eq. (3.9b) is of the nature of a Fourier series with fundamental frequency Δf approaching zero [Eq. (3.7b)]. Therefore, most of the discussion and properties of Fourier series apply to the Fourier transform as well. We can plot the spectrum $G(f)$ as a function of f. Since $G(f)$ is complex, we have both amplitude and angle (or phase) spectra:

$$G(f) = |G(f)|e^{j\theta_g(f)}$$

in which $|G(f)|$ is the amplitude and $\theta_g(f)$ is the angle (or phase) of $G(f)$. From Eq. (3.9a),

$$G(-f) = \int_{-\infty}^{\infty} g(t)e^{j2\pi ft}\, dt$$

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* This should not be considered as a rigorous proof of Eq. (3.8). The situation is not as simple as we have made it appear.[1]

f versus ω

Traditionally, we often use two equivalent notations of angular frequency ω and frequency f interchangeably in representing signals in the frequency domain. There is no conceptual difference between the use of angular frequency ω (in unit of radians per second) and frequency f (in units of hertz, Hz). Because of their direct relationship, we can simply substitute $\omega = 2\pi f$ into $G(f)$ to arrive at the Fourier transform relationship in the ω-domain:

$$\mathcal{F}[g(t)] = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt \tag{3.10}$$

Because of the additional 2π factor in the variable ω used by Eq. (3.10), the inverse transform as a function of ω requires an extra division by 2π. Therefore, the notation of f is slightly favored in practice when one is writing Fourier transforms. For this reason, we shall, for the most part, denote the Fourier transform of signals as functions of $G(f)$ in this book. On the other hand, the notation of angular frequency ω can also offer some convenience in dealing with sinusoids. Thus, in later chapters, whenever it is *convenient and nonconfusing,* we shall use the two equivalent notations interchangeably.

Conjugate Symmetry Property

From Eq. (3.9a), it follows that if $g(t)$ is a real function of t, then $G(f)$ and $G(-f)$ are complex conjugates, that is,*

$$G(-f) = G^*(f) \tag{3.11}$$

Therefore,

$$|G(-f)| = |G(f)| \tag{3.12a}$$

$$\theta_g(-f) = -\theta_g(f) \tag{3.12b}$$

Thus, for real $g(t)$, the amplitude spectrum $|G(f)|$ is an even function, and the phase spectrum $\theta_g(f)$ is an odd function of f. This property (the **conjugate symmetry property**) is valid only for real $g(t)$. These results were derived for the Fourier spectrum of a periodic signal in Chapter 2 and should come as no surprise. *The transform $G(f)$ is the frequency domain specification of $g(t)$.*

Example 3.1 Find the Fourier transform of $e^{-at}u(t)$.

By definition [Eq. (3.9a)],

$$G(f) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j2\pi ft}\,dt = \int_{0}^{\infty} e^{-(a+j2\pi f)t}\,dt = \frac{-1}{a+j2\pi f}e^{-(a+j2\pi f)t}\Big|_0^{\infty}$$

But $|e^{-j2\pi ft}| = 1$. Therefore, as $t \to \infty$, $e^{-(a+j2\pi f)t} = e^{-at}e^{-j2\pi ft} = 0$ if $a > 0$. Therefore,

$$G(f) = \frac{1}{a+j\omega} \qquad a > 0 \tag{3.13a}$$

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* *Hermitian symmetry* is the term used to describe complex functions that satisfy Eq. (3.11)

where $\omega = 2\pi f$. Expressing $a + j\omega$ in the polar form as $\sqrt{a^2 + \omega^2}\, e^{j \tan^{-1}(\frac{\omega}{a})}$, we obtain

$$G(f) = \frac{1}{\sqrt{a^2 + (2\pi f)^2}} e^{-j \tan^{-1}(\frac{2\pi f}{a})} \tag{3.13ba}$$

Therefore,

$$|G(f)| = \frac{1}{\sqrt{a^2 + (2\pi f)^2}} \quad \text{and} \quad \theta_g(f) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

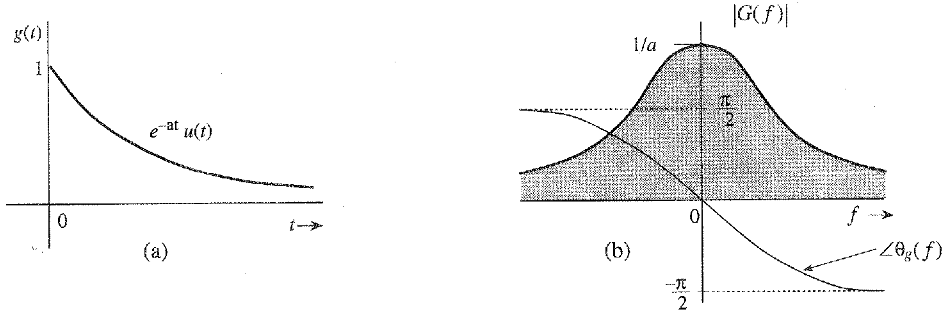


Figure 3.4 $e^{-at}u(t)$ and its Fourier spectra.

The amplitude spectrum $|G(f)|$ and the phase spectrum $\theta_g(f)$ are shown in Fig. 3.4b. Observe that $|G(f)|$ is an even function of f, and $\theta_g(f)$ is an odd function of f, as expected.

Existence of the Fourier Transform

In Example 3.1 we observed that when $a < 0$, the Fourier integral for $e^{-at}u(t)$ does not converge. Hence, the Fourier transform for $e^{-at}u(t)$ does not exist if $a < 0$ (growing exponentially). Clearly, not all signals are Fourier transformable. The existence of the Fourier transform is assured for any $g(t)$ satisfying the Dirichlet conditions, the first of which is*

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty \tag{3.14}$$

To show this, recall that $|e^{-j2\pi ft}| = 1$. Hence, from Eq. (3.9a) we obtain

$$|G(f)| \le \int_{-\infty}^{\infty} |g(t)|\, dt$$

This shows that the existence of the Fourier transform is assured if condition (3.14) is satisfied. Otherwise, there is no guarantee. We have seen in Example 3.1 that for an exponentially growing signal (which violates this condition) the Fourier transform does not exist. Although this condition is sufficient, it is not necessary for the existence of the Fourier transform of a signal.

* The remaining Dirichlet conditions are as follows: In any finite interval, $g(t)$ may have only a finite number of maxima and minima and a finite number of finite discontinuities. When these conditions are satisfied, the Fourier integral on the right-hand side of Eq. (3.9b) converges to $g(t)$ at all points where $g(t)$ is continuous and converges to the average of the right-hand and left-hand limits of $g(t)$ at points where $g(t)$ is discontinuous.

For example, the signal $(\sin at)/t$, violates condition (3.14), but does have a Fourier transform. Any signal that can be generated in practice satisfies the Dirichlet conditions and therefore has a Fourier transform. Thus, the physical existence of a signal is a sufficient condition for the existence of its transform.

Linearity of the Fourier Transform (Superposition Theorem)

The Fourier transform is linear; that is, if

$$g_1(t) \Longleftrightarrow G_1(f) \qquad \text{and} \qquad g_2(t) \Longleftrightarrow G_2(f)$$

then for all constants a_1 and a_2, we have

$$a_1 g_1(t) + a_2 g_2(t) \Longleftrightarrow a_1 G_1(f) + a_2 G_2(f) \tag{3.15}$$

The proof is simple and follows directly from Eq. (3.9a). This theorem simply states that linear combinations of signals in the time domain correspond to linear combinations of their Fourier transforms in the frequency domain. This result can be extended to any finite number of terms as

$$\sum_k a_k g_k(t) \Longleftrightarrow \sum_k a_k G_k(f)$$

for any constants $\{a_k\}$ and signals $\{g_k(t)\}$.

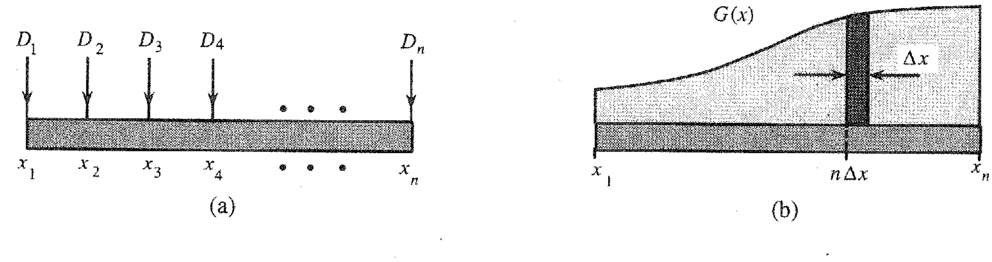
Physical Appreciation of the Fourier Transform

To understand any aspect of the Fourier transform, we should remember that Fourier representation is a way of expressing a signal in terms of everlasting sinusoids, or exponentials. The Fourier spectrum of a signal indicates the relative amplitudes and phases of the sinusoids that are required to synthesize that signal. A periodic signal's Fourier spectrum has finite amplitudes and exists at discrete frequencies (f and its multiples). Such a spectrum is easy to visualize, but the spectrum of an aperiodic signal is not easy to visualize because it has a continuous spectrum that exists at every frequency. The continuous spectrum concept can be appreciated by considering an analogous, more tangible phenomenon. One familiar example of a continuous distribution is the loading of a beam. Consider a beam loaded with weights $D_1, D_2, D_3, \ldots, D_n$ units at the uniformly spaced points $x_1, x_2, \ldots, x_n$, as shown in Fig. 3.5a. The total load W_T on the beam is given by the sum of these loads at each of the n points:

$$W_T = \sum_{i=1}^{n} D_i$$

Consider now the case of a continuously loaded beam, as shown in Fig. 3.5b. In this case, although there appears to be a load at every point, the load at any one point is zero. This does

Figure 3.5 Analogy for Fourier transform.



not mean that there is no load on the beam. A meaningful measure of load in this situation is not the load at a point, but rather the loading density per unit length at that point. Let $G(x)$ be the loading density per unit length of beam. This means that the load over a beam length Δx ($\Delta x \to 0$) at some point x is $G(x)\Delta x$. To find the total load on the beam, we divide the beam into segments of interval Δx ($\Delta x \to 0$). The load over the nth such segment of length Δx is $[G(n\Delta x)]\,\Delta x$. The total load W_T is given by

$$\begin{aligned} W_T &= \lim_{\Delta x \to 0} \sum_{x_1}^{x_n} G(n\Delta x)\,\Delta x \\ &= \int_{x_1}^{x_n} G(x)\,dx \end{aligned}$$

In the case of discrete loading (Fig. 3.5a), the load exists only at the n discrete points. At other points there is no load. On the other hand, in the continuously loaded case, the load exists at every point, but at any specific point x the load is zero. The load over a small interval Δx, however, is $[G(n\Delta x)]\,\Delta x$ (Fig. 3.5b). Thus, even though the load at a point x is zero, the relative load at that point is $G(x)$.

An exactly analogous situation exists in the case of a signal spectrum. When $g(t)$ is periodic, the spectrum is discrete, and $g(t)$ can be expressed as a sum of discrete exponentials with finite amplitudes:

$$g(t) = \sum_n D_n e^{j2\pi n f_0 t}$$

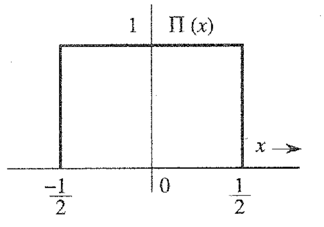
For an aperiodic signal, the spectrum becomes continuous; that is, the spectrum exists for every value of f, but the amplitude of each component in the spectrum is zero. The meaningful measure here is not the amplitude of a component of some frequency but the spectral density per unit bandwidth. From Eq. (3.7b) it is clear that $g(t)$ is synthesized by adding exponentials of the form $e^{j2\pi n\Delta f t}$, in which the contribution by any one exponential component is zero. But the contribution by exponentials in an infinitesimal band Δf located at $f = n\Delta f$ is $G(n\Delta f)\Delta f$, and the addition of all these components yields $g(t)$ in the integral form:

$$g(t) = \lim_{\Delta f \to 0} \sum_{n=-\infty}^{\infty} G(n\Delta f) e^{(jn2\pi f)t}\,\Delta f = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\,df$$

The contribution by components within the band df is $G(f)\,df$, in which df is the bandwidth in hertz. Clearly $G(f)$ is the **spectral density** per unit bandwidth (in hertz). This also means that even if the amplitude of any one component is zero, the relative amount of a component of frequency f is $G(f)$. Although $G(f)$ is a spectral density, in practice it is customarily called the **spectrum** of $g(t)$ rather than the spectral density of $g(t)$. Deferring to this convention, we shall call $G(f)$ the Fourier spectrum (or Fourier transform) of $g(t)$.

3.2 TRANSFORMS OF SOME USEFUL FUNCTIONS

For convenience, we now introduce a compact notation for some useful functions such as rectangular, triangular, and interpolation functions.



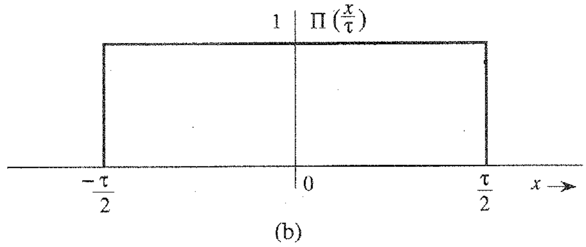
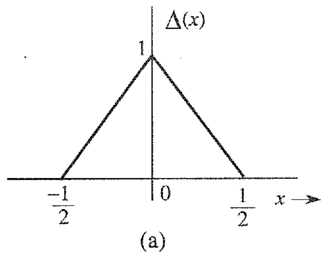


Figure 3.6 Rectangular pulse.



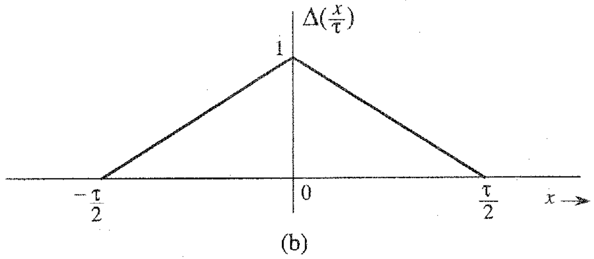


Figure 3.7 Triangular pulse.

Unit Rectangular Function

We use the pictorial notation $\Pi(x)$ for a rectangular pulse of unit height and unit width, centered at the origin, as shown in Fig. 3.6a:

$$\Pi(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0.5 & |x| = \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \tag{3.16}$$

Notice that the rectangular pulse in Fig. 3.6b is the unit rectangular pulse $\Pi(x)$ expanded by a factor τ and therefore can be expressed as $\Pi(x/\tau)$. Observe that the denominator τ in $\Pi(x/\tau)$ indicates the width of the pulse.

Unit Triangular Function

We use the pictorial notation $\Delta(x)$ for a triangular pulse of unit height and unit width, centered at the origin, as shown in Fig. 3.7a:

$$\Delta(x) = \begin{cases} 1 - 2|t| & |x| < \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \tag{3.17}$$

Observe that the pulse in Fig. 3.7b is $\Delta(x/\tau)$. Observe that here, as for the rectangular pulse, the denominator τ in $\Delta(x/\tau)$ indicates the pulse width.

Sinc Function sinc(x)

The function sin x/x is the "sine over argument" function denoted by sinc (x).*

* sinc (x) is also denoted by Sa (x) in the literature. Some authors define sinc (x) as

$$\text{sinc}\,(x) = \frac{\sin \pi x}{\pi x}$$

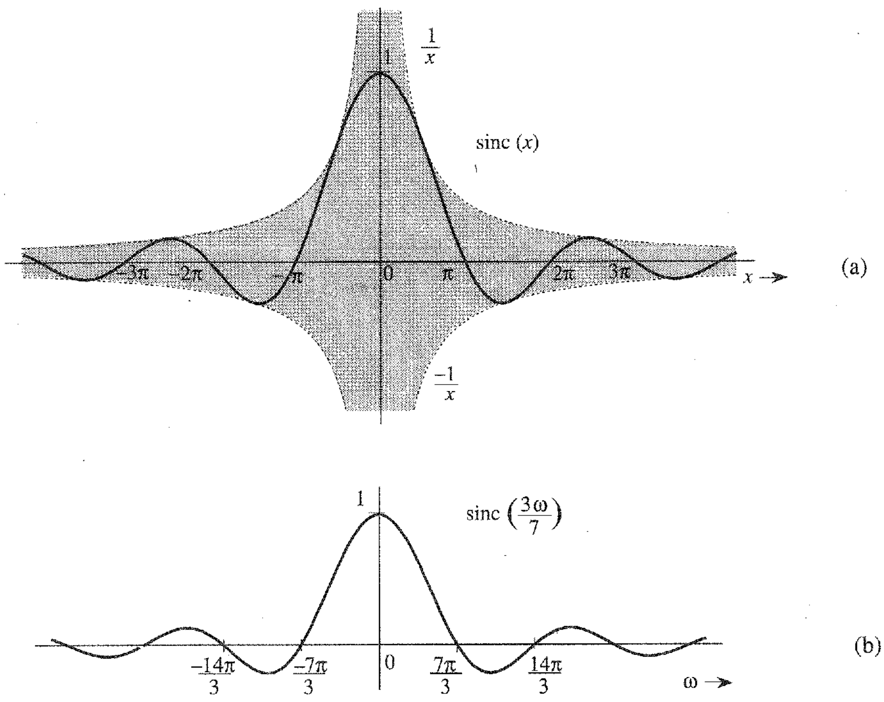


Figure 3.8
Sinc pulse.

This function plays an important role in signal processing. We define

$$\text{sinc}\,(x) = \frac{\sin x}{x} \tag{3.18}$$

Inspection of Eq. (3.18) shows that

1. sinc (x) is an even function of x.
2. sinc $(x) = 0$ when $\sin x = 0$ except at $x = 0$, where it is indeterminate. This means that sinc $(x) = 0$ for $t = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
3. Using L'Hôspital's rule, we find sinc $(0) = 1$.
4. sinc (x) is the product of an oscillating signal $\sin x$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, sinc (x) exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.
5. In summary, sinc (x) is an even oscillating function with decreasing amplitude. It has a unit peak at $x = 0$ and zero crossings at integer multiples of π.

Figure 3.8a shows sinc (x). Observe that sinc $(x) = 0$ for values of x that are positive and negative integral multiples of π. Figure 3.8b shows sinc $(3\omega/7)$. The argument $3\omega/7 = \pi$ when $\omega = 7\pi/3$ or $f = 7/6$. Therefore, the first zero of this function occurs at $\omega = 7\pi/3$ $(f = 7/6)$.

Example 3.2 Find the Fourier transform of $g(t) = \Pi(t/\tau)$ (Fig. 3.9a).

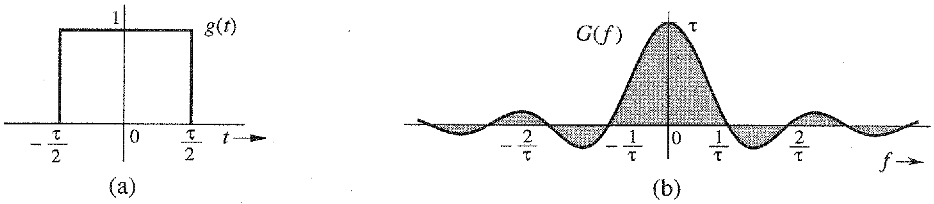


Figure 3.9 Rectangular pulse and its Fourier spectrum.

We have

$$G(f) = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j2\pi ft}\, dt$$

Since $\Pi(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$\begin{aligned} G(f) &= \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}\, dt \\ &= -\frac{1}{j2\pi f}(e^{-j\pi f\tau} - e^{j\pi f\tau}) = \frac{2\sin(\pi f\tau)}{2\pi f} \\ &= \tau\frac{\sin(\pi f\tau)}{(\pi f\tau)} = \tau \text{ sinc } (\pi f\tau) \end{aligned}$$

Therefore,

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \text{ sinc}\left(\frac{\omega\tau}{2}\right) = \tau \text{ sinc } (\pi f\tau) \tag{3.19}$$

Recall that sinc $(x) = 0$ when $x = \pm n\pi$. Hence, sinc $(\omega\tau/2) = 0$ when $\omega\tau/2 = \pm n\pi$; that is, when $f = \pm n/\tau$ $(n = 1, 2, 3, \ldots)$, as shown in Fig. 3.9b. Observe that in this case $G(f)$ happens to be real. Hence, we may convey the spectral information by a single plot of $G(f)$ shown in Fig. 3.9b.

Example 3.3 Find the Fourier transform of the unit impulse signal $\delta(t)$.

We use the sampling property of the impulse function [Eq. (2.11)] to obtain

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\, dt = e^{-j2\pi f\cdot 0} = 1 \tag{3.20a}$$

or

$$\delta(t) \Longleftrightarrow 1 \tag{3.20b}$$

Figure 3.10 shows $\delta(t)$ and its spectrum.

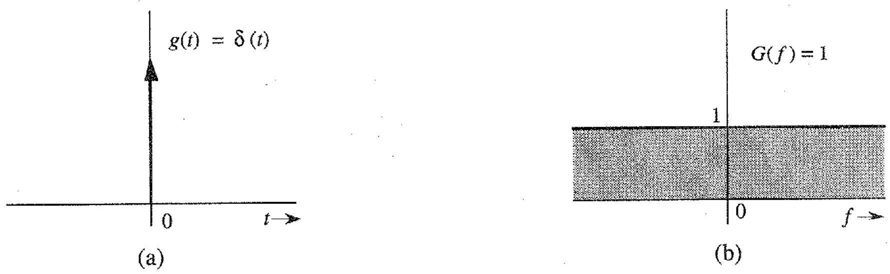


Figure 3.10 Unit impulse and its Fourier spectrum.

Example 3.4

Find the inverse Fourier transform of $\delta(2\pi f) = \frac{1}{2\pi}\delta(f)$.

From Eq. (3.9b) and the sampling property of the impulse function,

$$\mathcal{F}^{-1}[\delta(2\pi f)] = \int_{-\infty}^{\infty} \delta(2\pi f)e^{j2\pi ft}\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(2\pi f)e^{j2\pi ft}\,d(2\pi f)$$
$$= \frac{1}{2\pi}\cdot e^{-j2\pi f\cdot 0} = \frac{1}{2\pi}$$

Therefore,

$$\frac{1}{2\pi} \Longleftrightarrow \delta(2\pi f) \tag{3.21a}$$

or

$$1 \Longleftrightarrow \delta(f) \tag{3.21b}$$

This shows that the spectrum of a constant signal $g(t) = 1$ is an impulse $\delta(f) = 2\pi\delta(2\pi f)$, as shown in Fig. 3.11.

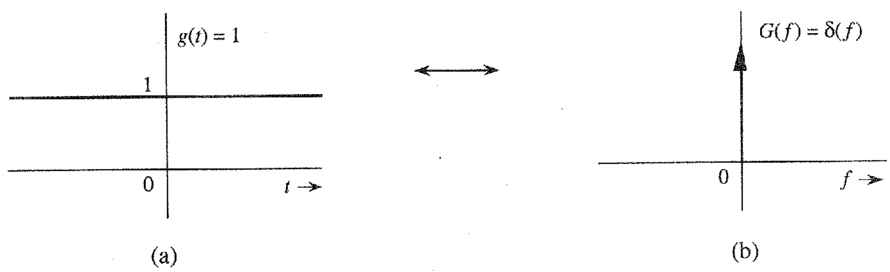


Figure 3.11 Constant (dc) signal and its Fourier spectrum.

The result [Eq. (3.21b)] also could have been anticipated on qualitative grounds. Recall that the Fourier transform of $g(t)$ is a spectral representation of $g(t)$ in terms of everlasting exponential components of the form $e^{j2\pi ft}$. Now to represent a constant signal $g(t) = 1$, we need a single everlasting exponential $e^{j2\pi ft}$ with $f = 0$. This results in a spectrum at a single frequency $f = 0$. We could also say that $g(t) = 1$ is a dc signal that has a single frequency component at $f = 0$ (dc).

If an impulse at $f = 0$ is a spectrum of a dc signal, what does an impulse at $f = f_0$ represent? We shall answer this question in the next example.

Example 3.5 Find the inverse Fourier transform of $\delta(f - f_0)$.

We the sampling property of the impulse function to obtain

$$\mathcal{F}^{-1}[\delta(f - f_0)] = \int_{-\infty}^{\infty} \delta(f - f_0)e^{j2\pi ft}\, df = e^{j2\pi f_0 t}$$

Therefore,

$$e^{j2\pi f_0 t} \Longleftrightarrow \delta(f - f_0) \qquad (3.22a)$$

This result shows that the spectrum of an everlasting exponential $e^{j2\pi f_0 t}$ is a single impulse at $f = f_0$. We reach the same conclusion by qualitative reasoning. To represent the everlasting exponential $e^{j2\pi f_0 t}$, we need a single everlasting exponential $e^{j2\pi ft}$ with $\omega = 2\pi f_0$. Therefore, the spectrum consists of a single component at frequency $f = f_0$.

From Eq. (3.22a) it follows that

$$e^{-j2\pi f_0 t} \Longleftrightarrow \delta(f + f_0) \qquad (3.22b)$$

Example 3.6 Find the Fourier transforms of the everlasting sinusoid $\cos 2\pi f_0$.

Recall the Euler formula

$$\cos 2\pi f_0 t = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

Adding Eqs. (3.22a) and (3.22b), and using the preceding formula, we obtain

$$\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)] \qquad (3.23)$$

The spectrum of $\cos 2\pi f_0 t$ consists of two impulses at f_0 and $-f_0$ in the f-domain, or, two impulses at $\pm\omega_0 = \pm 2\pi f_0$ in the ω-domain as shown in Fig. 3.12. The result also follows from qualitative reasoning. An everlasting sinusoid $\cos \omega_0 t$ can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore, the Fourier spectrum consists of only two components of frequencies ω_0 and $-\omega_0$.

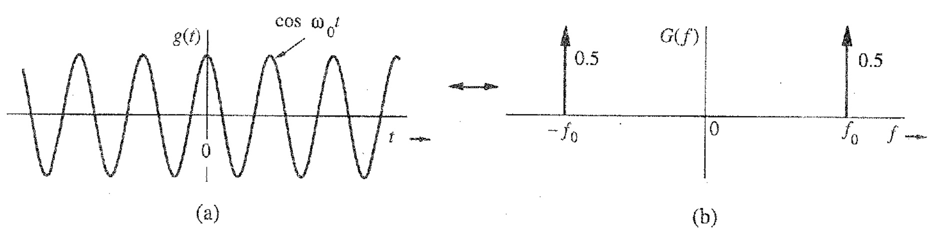


Figure 3.12 Cosine signal and its Fourier spectrum.

Example 3.7 Find the Fourier transform of the sign function sgn (t) (pronounced signum t), shown in Fig. 3.13. Its value is $+1$ or -1, depending on whether t is positive or negative:

$$\text{sgn}\,(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \tag{3.24}$$

We cannot use integration to find the transform of sgn (t) directly. This is because sgn (t) violates the Dirichlet condition [see E.g. (3.14) and the associated footnote]. Specifically, sgn (t) is not absolutely integrable. However, the transform can be obtained by considering sgn t as a sum of two exponentials, as shown in Fig. 3.13, in the limit as $a \to 0$:

$$\text{sgn}\,t = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

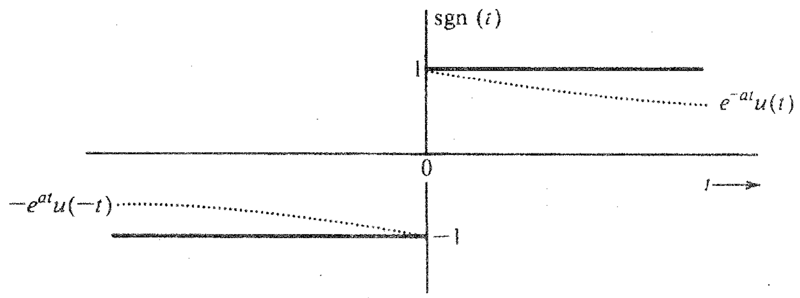


Figure 3.13 Sign function.

Therefore,

$$\begin{aligned}\mathcal{F}\,[\text{sgn}(t)] &= \lim_{a\to 0}\left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\} \\ &= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right) \qquad \text{(see pairs 1 and 2 in Table 3.1)} \\ &= \lim_{a\to 0}\left(\frac{-j4\pi f}{a^2+4\pi^2 f^2}\right) = \frac{1}{j\pi f} \end{aligned} \tag{3.25}$$

3.3 SOME PROPERTIES OF THE FOURIER TRANSFORM

We now study some of the important properties of the Fourier transform and their implications as well as their applications. Before embarking on this study, it is important to point out a pervasive aspect of the Fourier transform—the **time-frequency duality**.

TABLE 3.1
Short Table of Fourier Transforms

	$g(t)$	$G(f)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+(2\pi f)^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j2\pi f)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5\,[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5\,[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\dfrac{1}{j2\pi f}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j2\pi f}$	
13	$\cos 2\pi f_0 t\, u(t)$	$\dfrac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\dfrac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\, u(t)$	$\dfrac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\dfrac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\, u(t)$	$\dfrac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
16	$e^{-at}\cos 2\pi f_0 t\, u(t)$	$\dfrac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
17	$\Pi\left(\dfrac{t}{\tau}\right)$	$\tau \operatorname{sinc}(\pi f \tau)$	
18	$2B \operatorname{sinc}(2\pi B t)$	$\Pi\left(\dfrac{f}{2B}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\pi f \tau}{2}\right)$	
20	$B \operatorname{sinc}^2(\pi B t)$	$\Delta\left(\dfrac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	$f_0=\frac{1}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-2(\sigma\pi f)^2}$	

3.3.1 Time-Frequency Duality

Equations (3.9) show an interesting fact: the direct and the inverse transform operations are remarkably similar. These operations, required to go from $g(t)$ to $G(f)$ and then from $G(f)$ to $g(t)$, are shown graphically in Fig. 3.14. The only minor difference between these two operations lies in the opposite signs used in their exponential indices.

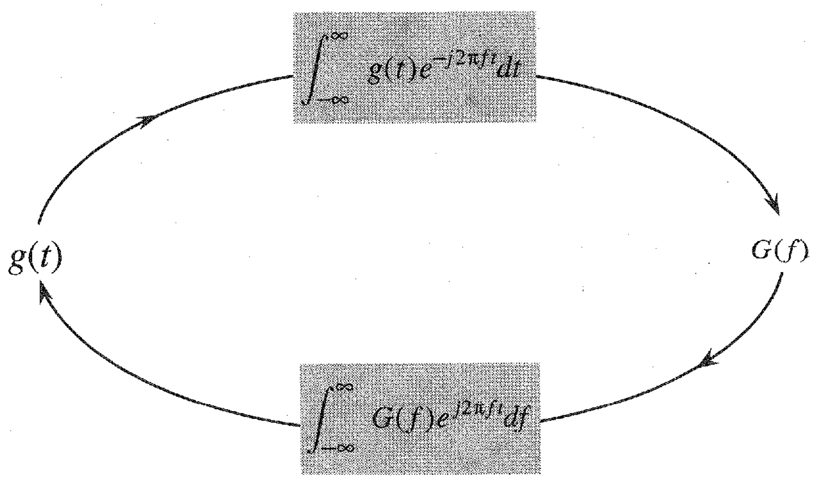


Figure 3.14 Near symmetry between direct and inverse Fourier transforms.

This similarity has far-reaching consequences in the study of Fourier transforms. It is the basis of the so-called duality of time and frequency. *The duality principle may be compared with a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, the negative can be obtained from the photograph.* For any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of the factor 2π and a sign change). For example, the time-shifting property, to be proved later, states that if $g(t) \Longleftrightarrow G(f)$, then

$$g(t - t_0) \Longleftrightarrow G(f)e^{-j2\pi ft_0}$$

The dual of this property (the frequency-shifting property) states that

$$g(t)e^{j2\pi f_0 t} \Longleftrightarrow G(f - f_0)$$

Observe the role reversal of time and frequency in these two equations (with the minor difference of the sign change in the exponential index). The value of this principle lies in the fact that *whenever we derive any result, we can be sure that it has a dual.* This knowledge can give valuable insights about many unsuspected properties or results in signal processing.

The properties of the Fourier transform are useful not only in deriving the direct and the inverse transforms of many functions, but also in obtaining several valuable results in signal processing. The reader should not fail to observe the ever-present duality in this discussion. We begin with the duality property, which is one of the consequences of the duality principle.

3.3.2 Duality Property

The duality property states that if

$$g(t) \Longleftrightarrow G(f)$$

then

$$G(t) \Longleftrightarrow g(-f) \tag{3.26}$$

The duality property states that if the Fourier transform of $g(t)$ is $G(f)$ then the Fourier transform of $G(t)$, with f replaced by t, is the $g(-f)$ which is the original time domain signal with t replaced by $-f$.

Proof: From Eq. (3.9b),

$$g(t) = \int_{-\infty}^{\infty} G(x)e^{j2\pi xt}\,dx$$

Hence,

$$g(-t) = \int_{-\infty}^{\infty} G(x)e^{-j2\pi xt}\,dx$$

Changing t to f yields Eq. (3.26). ■

Example 3.8 In this example we shall apply the duality property [Eq. (3.26)] to the pair in Fig. 3.15a.

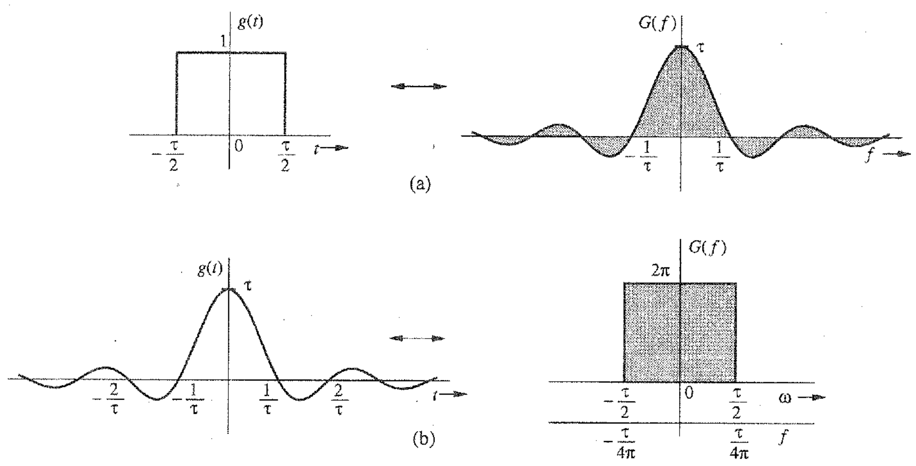


Figure 3.15 Duality property of the Fourier transform.

From Eq. (3.19) we have

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau\,\text{sinc}\,(\pi f \tau) \tag{3.27a}$$

$$\underbrace{\Pi\left(\frac{t}{\alpha}\right)}_{g(t)} \Longleftrightarrow \underbrace{\alpha\,\text{sinc}\,(\pi f \alpha)}_{G(f)} \tag{3.27b}$$

Also $G(t)$ is the same as $G(f)$ with f replaced by t, and $g(-f)$ is the same as $g(t)$ with t replaced by $-f$. Therefore, the duality property (3.26) yields

$$\underbrace{\alpha\,\text{sinc}\,(\pi\alpha t)}_{G(t)} \Longleftrightarrow \underbrace{\Pi\left(-\frac{f}{\alpha}\right)}_{g(-f)} = \Pi\left(\frac{f}{\alpha}\right) \tag{3.28a}$$

Substituting $\tau = 2\pi\alpha$, we obtain

$$\tau \text{ sinc}\left(\frac{\alpha t}{2}\right) \Longleftrightarrow 2\pi\ \Pi\left(\frac{2\pi f}{\tau}\right) \tag{3.28b}$$

In Eq. (3.8) we used the fact that $\Pi(-t) = \Pi(t)$ because $\Pi(t)$ is an even function. Figure 3.15b shows this pair graphically. Observe the interchange of the roles of t and $2\pi f$ (with the minor adjustment of the factor 2π). This result appears as pair 18 in Table 3.1 (with $\tau/2 = W$).

As an interesting exercise, generate a dual of every pair in Table 3.1 by applying the duality property.

3.3.3 Time-Scaling Property

If

$$g(t) \Longleftrightarrow G(f)$$

then, for any real constant a,

$$g(at) \Longleftrightarrow \frac{1}{|a|}G\left(\frac{f}{a}\right) \tag{3.29}$$

Proof: For a positive real constant a,

$$\mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft}\, dt = \frac{1}{a}\int_{-\infty}^{\infty} g(x)e^{(-j2\pi f/a)x}\, dx = \frac{1}{a}G\left(\frac{f}{a}\right)$$

Similarly, it can be shown that if $a < 0$,

$$g(at) \Longleftrightarrow \frac{-1}{a}G\left(\frac{f}{a}\right)$$

Hence follows Eq. (3.29). ∎

Significance of the Time-Scaling Property

The function $g(at)$ represents the function $g(t)$ compressed in time by a factor a ($|a| > 1$). Similarly, a function $G(f/a)$ represents the function $G(f)$ expanded in frequency by the same factor a. *The time-scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.* Intuitively, compression in time by a factor a means that the signal is varying **more rapidly** by the same factor. To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a. Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed. For instance, the signal $\cos 4\pi f_0 t$ is the same as the signal $\cos 2\pi f_0 t$ time-compressed by a factor of 2. Clearly, the spectrum of the former (impulse at $\pm 2f_0$) is an expanded version of the spectrum of the latter (impulse at $\pm f_0$). The effect of this scaling is demonstrated in Fig. 3.16.

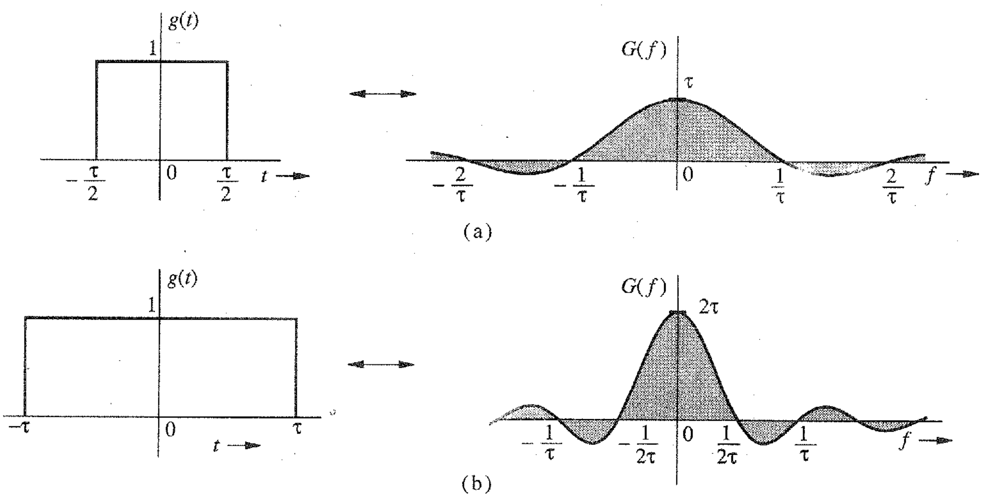


Figure 3.16 Scaling property of the Fourier transform.

Reciprocity of Signal Duration and Its Bandwidth

The time-scaling property implies that if $g(t)$ is wider, its spectrum is narrower, and vice versa. Doubling the signal duration halves its bandwidth, and vice versa. This suggests that the bandwidth of a signal is inversely proportional to the signal duration or width (in seconds). We have already verified this fact for the rectangular pulse, where we found that the bandwidth of a gate pulse of width τ seconds is $1/\tau$ Hz. More discussion of this interesting topic can be found in the literature.[2]

Example 3.9 Show that

$$g(-t) \Longleftrightarrow G(-f) \tag{3.30}$$

Use this result and the fact that $e^{-at}u(t) \Longleftrightarrow 1/(a + j2\pi f)$, to find the Fourier transforms of $e^{at}u(-t)$ and $e^{-a|t|}$.

Equation (3.30) follows from Eq. (3.29) by letting $a = -1$. Application of Eq. (3.30) to pair 1 of Table 3.1 yields

$$e^{at}u(-t) \Longleftrightarrow \frac{1}{a - j2\pi f}$$

Also

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

Therefore,

$$e^{-a|t|} \Longleftrightarrow \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2} \tag{3.31}$$

The signal $e^{-a|t|}$ and its spectrum are shown in Fig. 3.17.

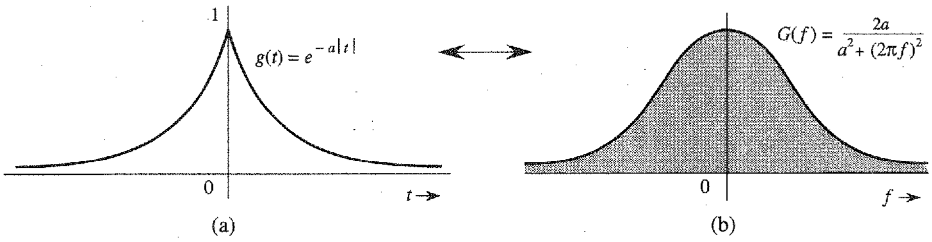


Figure 3.17 $e^{-a|t|}$ and its Fourier spectrum.

3.3.4 Time-Shifting Property

If

$$g(t) \Longleftrightarrow G(f)$$

then

$$g(t - t_0) \Longleftrightarrow G(f)e^{-j2\pi f t_0} \tag{3.32a}$$

Proof: By definition,

$$\mathcal{F}[g(t - t_0)] = \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft}\, dt$$

Letting $t - t_0 = x$, we have

$$\begin{aligned} \mathcal{F}[g(t - t_0)] &= \int_{-\infty}^{\infty} g(x)e^{-j2\pi f(x+t_0)}\, dx \\ &= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}\, dx = G(f)e^{-j2\pi f t_0} \end{aligned} \tag{3.32b}$$

This result shows that *delaying a signal by t_0 seconds does not change its amplitude spectrum. The phase spectrum, however, is changed by $-2\pi f t_0$.*

Physical Explanation of the Linear Phase

Time delay in a signal causes a linear phase shift in its spectrum. This result can also be derived by heuristic reasoning. Imagine $g(t)$ being synthesized by its Fourier components, which are sinusoids of certain amplitudes and phases. The delayed signal $g(t - t_0)$ can be synthesized by the same sinusoidal components, each delayed by t_0 seconds. The amplitudes of the components remain unchanged. Therefore, the amplitude spectrum of $g(t - t_0)$ is identical to that of $g(t)$. The time delay of t_0 in each sinusoid, however, does change the phase of each component. Now, a sinusoid $\cos 2\pi ft$ delayed by t_0 is given by

$$\cos 2\pi f(t - t_0) = \cos(2\pi ft - 2\pi f t_0)$$

Therefore, a time delay t_0 in a sinusoid of frequency f manifests as a phase delay of $2\pi f t_0$. This is a linear function of f, meaning that higher frequency components must undergo proportionately

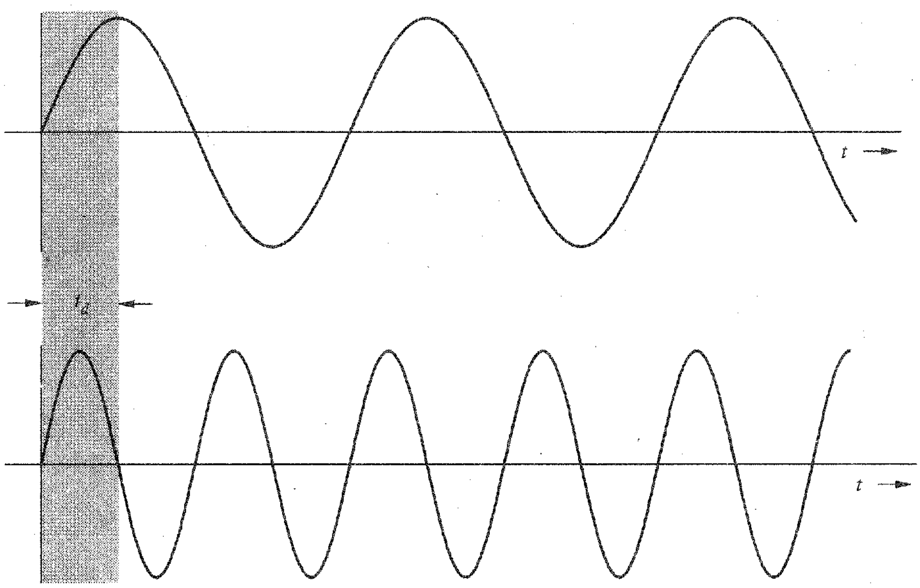


Figure 3.18 Physical explanation of the time-shifting property.

higher phase shifts to achieve the same time delay. This effect is shown in Fig. 3.18 with two sinusoids, the frequency of the lower sinusoid being twice that of the upper. The same time delay t_0 amounts to a phase shift of $\pi/2$ in the upper sinusoid and a phase shift of π in the lower sinusoid. This verifies that *to achieve the same time delay, higher frequency sinusoids must undergo proportionately higher phase shifts.*

Example 3.10

Find the Fourier transform of $e^{-a|t-t_0|}$.

This function, shown in Fig. 3.19a, is a time-shifted version of $e^{-a|t|}$ (shown in Fig. 3.17a). From Eqs. (3.31) and (3.32) we have

$$e^{-a|t-t_0|} \Longleftrightarrow \frac{2a}{a^2 + (2\pi f)^2} e^{-j2\pi f t_0} \tag{3.33}$$

The spectrum of $e^{-a|t-t_0|}$ (Fig. 3.19b) is the same as that of $e^{-a|t|}$ (Fig. 3.17b), except for an added phase shift of $-2\pi f t_0$.

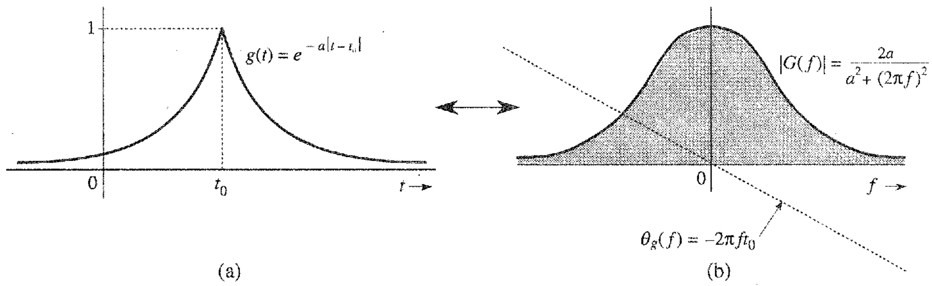


Figure 3.19 Effect of time shifting on the Fourier spectrum of a signal.

Observe that the time delay t_0 causes a **linear** phase spectrum $-2\pi f t_0$. This example clearly demonstrates the effect of time shift.

3.3.5 Frequency-Shifting Property

If

$$g(t) \Longleftrightarrow G(f)$$

then

$$g(t)e^{j2\pi f_0 t} \Longleftrightarrow G(f - f_0) \tag{3.34}$$

This property is also called the modulation property.

Proof: By definition,

$$\mathcal{F}[g(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} g(t)e^{j2\pi f_0 t}e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} g(t)e^{-j(2\pi f - 2\pi f_0)t}\,dt = G(f - f_0)$$

■

This property states that multiplication of a signal by a factor $e^{j2\pi f_0 t}$ shifts the spectrum of that signal by $f = f_0$. Note the duality between the time-shifting and the frequency-shifting properties.

Changing f_0 to $-f_0$ in Eq. (3.34) yields

$$g(t)e^{-j2\pi f_0 t} \Longleftrightarrow G(f + f_0) \tag{3.35}$$

Because $e^{j2\pi f_0 t}$ is not a real function that can be generated, frequency shifting in practice is achieved by multiplying $g(t)$ by a sinusoid. This can be seen from

$$g(t)\cos 2\pi f_0 t = \frac{1}{2}\left[g(t)e^{j2\pi f_0 t} + g(t)e^{-j2\pi f_0 t}\right]$$

From Eqs. (3.34) and (3.35), it follows that

$$g(t)\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[G(f - f_0) + G(f + f_0)] \tag{3.36}$$

This shows that the multiplication of a signal $g(t)$ by a sinusoid of frequency f_0 shifts the spectrum $G(f)$ by $\pm f_0$. Multiplication of a sinusoid $\cos 2\pi f_0 t$ by $g(t)$ amounts to modulating the sinusoid amplitude. This type of modulation is known as **amplitude modulation**. The sinusoid $\cos 2\pi f_0 t$ is called the **carrier**, the signal $g(t)$ is the **modulating signal**, and the signal $g(t)\cos 2\pi f_0 t$ is the **modulated signal**. Modulation and demodulation will be discussed in detail in Chapters 4 and 5.

To sketch a signal $g(t)\cos 2\pi f_0 t$, we observe that

$$g(t)\cos 2\pi f_0 t = \begin{cases} g(t) & \text{when } \cos 2\pi f_0 t = 1 \\ -g(t) & \text{when } \cos 2\pi f_0 t = -1 \end{cases}$$

Therefore, $g(t)\cos 2\pi f_0 t$ touches $g(t)$ when the sinusoid $\cos 2\pi f_0 t$ is at its positive peaks and touches $-g(t)$ when $\cos 2\pi f_0 t$ is at its negative peaks. This means that $g(t)$ and $-g(t)$ act as envelopes for the signal $g(t)\cos 2\pi f_0 t$ (see Fig. 3.20c). The signal $-g(t)$ is a mirror image of $g(t)$ about the horizontal axis. Figure 3.20 shows the signals $g(t)$, $g(t)\cos 2\pi f_0 t$, and their respective spectra.

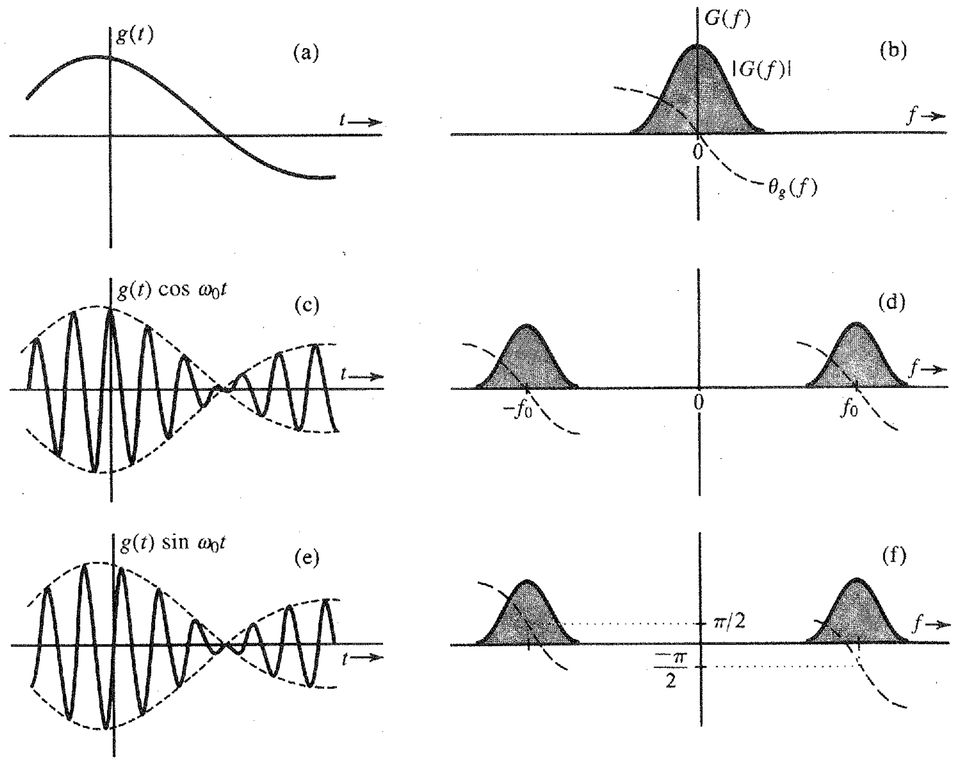


Figure 3.20 Amplitude modulation of a signal causes spectral shifting.

Shifting the Phase Spectrum of a Modulated Signal

We can shift the phase of each spectral component of a modulated signal by a constant amount θ_0 merely by using a carrier $\cos(2\pi f_0 t + \theta_0)$ instead of $\cos 2\pi f_0 t$. If a signal $g(t)$ is multiplied by $\cos(2\pi f_0 t + \theta_0)$, then we can use an argument similar to that used to derive Eq. (3.36), to show that

$$g(t)\cos(2\pi f_0 t + \theta_0) \Longleftrightarrow \frac{1}{2}\left[G(f - f_0)\, e^{j\theta_0} + G(f + f_0)\, e^{-j\theta_0}\right] \tag{3.37}$$

For a special case when $\theta_0 = -\pi/2$, Eq. (3.37) becomes

$$g(t)\sin 2\pi f_0 t \Longleftrightarrow \frac{1}{2}\left[G(f - f_0)\, e^{-j\pi/2} + G(f + f_0)\, e^{j\pi/2}\right] \tag{3.38}$$

Observe that $\sin 2\pi f_0 t$ is $\cos 2\pi f_0 t$ with a phase delay of $\pi/2$. Thus, shifting the carrier phase by $\pi/2$ shifts the phase of every spectral component by $\pi/2$. Figures 3.20e and f show the signal $g(t)\sin 2\pi f_0 t$ and its spectrum.

Modulation is a common application that shifts signal spectra. In particular, If several message signals, each occupying the same frequency band, are transmitted simultaneously over a common transmission medium, they will all interfere; it will be impossible to separate or retrieve them at a receiver. For example, if all radio stations decide to broadcast audio signals simultaneously, receivers will not be able to separate them. This problem is solved by using modulation, whereby each radio station is assigned a distinct carrier frequency. Each station transmits a modulated signal, thus shifting the signal spectrum to its allocated band, which is not occupied by any other station. A radio receiver can pick up any station by tuning to the

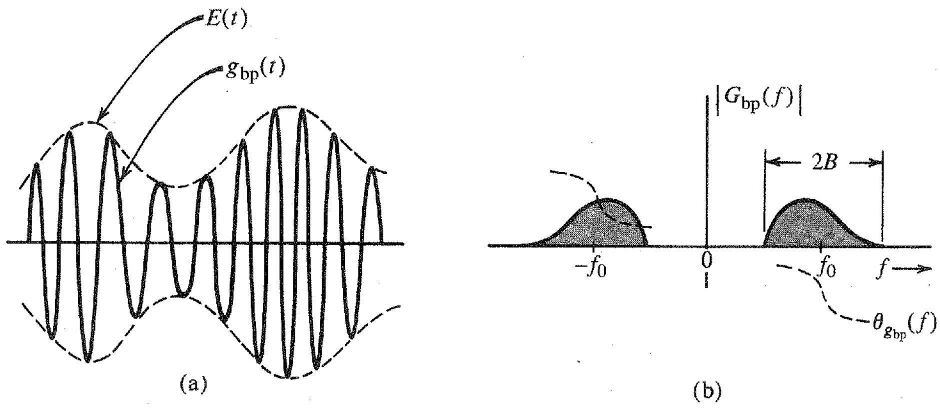


Figure 3.21 Bandpass signal and its spectrum.

band of the desired station. The receiver must now demodulate the received signal (undo the effect of modulation). Demodulation therefore consists of another spectral shift required to restore the signal to its original band.

Bandpass Signals

Figure 3.20(d)(f) shows that if $g_c(t)$ and $g_s(t)$ are low-pass signals, each with a bandwidth B Hz or $2\pi B$ rad/s, then the signals $g_c(t)\cos 2\pi f_0 t$ and $g_s(t)\sin 2\pi f_0 t$ are both bandpass signals occupying the same band, and each having a bandwidth of $2B$ Hz. Hence, a linear combination of both these signals will also be a bandpass signal occupying the same band as that of the either signal, and with the same bandwidth ($2B$ Hz). Hence, a general bandpass signal $g_{bp}(t)$ can be expressed as*

$$g_{bp}(t) = g_c(t)\cos 2\pi f_0 t + g_s(t)\sin 2\pi f_0 t \tag{3.39}$$

The spectrum of $g_{bp}(t)$ is centered at $\pm f_0$ and has a bandwidth $2B$, as shown in Fig. 3.21. Although the magnitude spectra of both $g_c(t)\cos 2\pi f_0 t$ and $g_s(t)\sin 2\pi f_0 t$ are symmetrical about $\pm f_0$, the magnitude spectrum of their sum, $g_{bp}(t)$, is not necessarily symmetrical about $\pm f_0$. This is because the different phases of the two signals do not allow their amplitudes to add directly for the reason that

$$a_1 e^{j\varphi_1} + a_2 e^{j\varphi_2} \neq (a_1 + a_2)e^{j(\varphi_1+\varphi_2)}$$

A typical bandpass signal $g_{bp}(t)$ and its spectra are shown in Fig. 3.21. We can use a well-known trigonometric identity to express Eq. (3.39) as

$$g_{bp}(t) = E(t)\cos\left[2\pi f_0 t + \psi(t)\right] \tag{3.40}$$

where

$$E(t) = +\sqrt{g_c^2(t) + g_s^2(t)} \tag{3.41a}$$

$$\psi(t) = -\tan^{-1}\left[\frac{g_s(t)}{g_c(t)}\right] \tag{3.41b}$$

* See Sec. 9.9 for a rigorous proof of this statement.

Because $g_c(t)$ and $g_s(t)$ are low-pass signals, $E(t)$ and $\psi(t)$ are also low-pass signals. Because $E(t)$ is nonnegative [Eq. (3.41a)], it follows from Eq. (3.40) that $E(t)$ is a slowly varying envelope and $\psi(t)$ is a slowly varying phase of the bandpass signal $g_{bp}(t)$, as shown in Fig. 3.21. Thus, the bandpass signal $g_{bp}(t)$ will appear as a sinusoid of slowly varying amplitude. Because of the time-varying phase $\psi(t)$, the frequency of the sinusoid also varies slowly* with time about the center frequency f_0.

Example 3.11 Find the Fourier transform of a general periodic signal $g(t)$ of period T_0, and hence, determine the Fourier transform of the periodic impulse train $\delta_{T_0}(t)$ shown in Fig. 3.22a.

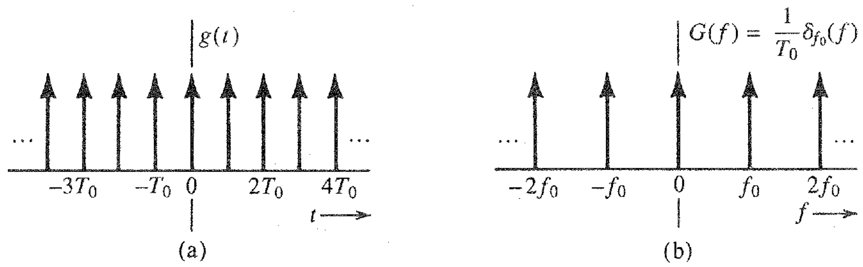


Figure 3.22 Impulse train and its spectrum.

A periodic signal $g(t)$ can be expressed as an exponential Fourier series as

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0}$$

Therefore,

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} \mathcal{F}[D_n e^{jn2\pi f_0 t}]$$

Now from Eq. (3.22a), it follows that

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} D_n \delta(f - nf_0) \tag{3.42}$$

Equation (2.67) shows that the impulse train $\delta_{T_0}(t)$ can be expressed as an exponential Fourier series as

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0}$$

* It is necessary that $B \ll f_0$ for a well-defined envelope. Otherwise the variations of $E(t)$ are of the same order as the carrier, and it will be difficult to separate the envelope from the carrier.

Here $D_n = 1/T_0$. Therefore, from Eq. (3.42),

$$\begin{aligned}\delta_{T_0}(t) &\Longleftrightarrow \frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f-nf_0)\\ &=\frac{1}{T_0}\delta_{f_0}(f) \qquad f_0=\frac{1}{T_0}\end{aligned} \tag{3.43}$$

Thus, the spectrum of the impulse train also happens to be an impulse train (in the frequency domain), as shown in Fig. 3.23b.

3.3.6 Convolution Theorem

The convolution of two functions $g(t)$ and $w(t)$, denoted by $g(t) * w(t)$, is defined by the integral

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau)w(t-\tau)\,d\tau$$

The time convolution property and its dual, the frequency convolution property, state that if

$$g_1(t) \Longleftrightarrow G_1(f) \qquad \text{and} \qquad g_2(t) \Longleftrightarrow G_2(f)$$

then (**time convolution**)

$$g_1(t) * g_2(t) \Longleftrightarrow G_1(f)G_2(f) \tag{3.44}$$

and (**frequency convolution**)

$$g_1(t)g_2(t) \Longleftrightarrow G_1(f) * G_2(f) \tag{3.45}$$

These two relationships of the convolution theorem state that convolution of two signals in the time domain becomes multiplication in the frequency domain, while multiplication of two signals in the time domain becomes convolution in the frequency domain.

Proof: By definition,

$$\begin{aligned}\mathcal{F}|g_1(t) * g_2(t)| &= \int_{-\infty}^{\infty} e^{-j2\pi ft}\left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau\right] dt\\ &= \int_{-\infty}^{\infty} g_1(\tau)\left[\int_{-\infty}^{\infty} e^{-j2\pi ft}g_2(t-\tau)dt\right]d\tau\end{aligned}$$

The inner integral is the Fourier transform of $g_2(t-\tau)$, given by [time-shifting property in Eq. (3.32a)] $G_2(f)e^{-j2\pi f\tau}$. Hence,

$$\begin{aligned}\mathcal{F}[g_1(t) * g_2(t)] &= \int_{-\infty}^{\infty} g_1(\tau)e^{-j2\pi f\tau}G_2(f)\,d\tau\\ &= G_2(f)\int_{-\infty}^{\infty} g_1(\tau)e^{-j2\pi f\tau}\,d\tau = G_1(f)G_2(f)\end{aligned}$$

■

The frequency convolution property (3.45) can be proved in exactly the same way by reversing the roles of $g(t)$ and $G(f)$.

Bandwidth of the Product of Two Signals

If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz. This result follows from the application of the width property of convolution[3] to Eq. (3.45). This property states that the width of $x * y$ is the sum of the widths of x and y. Consequently, if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^n(t)$ is nB Hz.*

Example 3.12 Using the time convolution property, show that if

$$g(t) \Longleftrightarrow G(f)$$

then

$$\int_{-\infty}^{t} g(\tau)d\tau \Longleftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \tag{3.46}$$

Because

$$u(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t \end{cases}$$

it follows that

$$g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} g(\tau)\,d\tau$$

Now from the time convolution property [Eq. (3.44)], it follows that

$$\begin{aligned} g(t) * u(t) &\Longleftrightarrow G(f)U(f) \\ &= G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \\ &= \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \end{aligned}$$

In deriving the last result we used pair 11 of Table 3.1 and Eq. (2.10a).

3.3.7 Time Differentiation and Time Integration

If

$$g(t) \Longleftrightarrow G(f),$$

* The width property of convolution does not hold in some pathological cases. It fails when the convolution of two functions is zero over a range even when both functions are nonzero [e.g., $\sin 2\pi f_0 t\, u(t) * u(t)$]. Technically the property holds even in this case if in calculating the width of the convolved function, we take into account the range in which the convolution is zero.

then **(time differentiation)***

$$\frac{dg(t)}{dt} \Longleftrightarrow j2\pi f G(f) \tag{3.47}$$

and **(time integration)**

$$\int_{-\infty}^{t} g(\tau)d\tau \Longleftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \tag{3.48}$$

Proof: Differentiation of both sides of Eq. (3.9b) yields

$$\frac{dg(t)}{dt} = \int_{-\infty}^{\infty} j2\pi f G(f) e^{j2\pi f t}\, df$$

This shows that

$$\frac{dg(t)}{dt} \Longleftrightarrow j2\pi f G(f)$$

Repeated application of this property yields

$$\frac{d^n g(t)}{dt^n} \Longleftrightarrow (j2\pi f)^n G(f) \tag{3.49}$$

The time integration property [Eq. (3.48)] already has been proved in Example 3.12. ■

Example 3.13 Use the time differentiation property to find the Fourier transform of the triangular pulse $\Delta(t/\tau)$ shown in Fig. 3.23a.

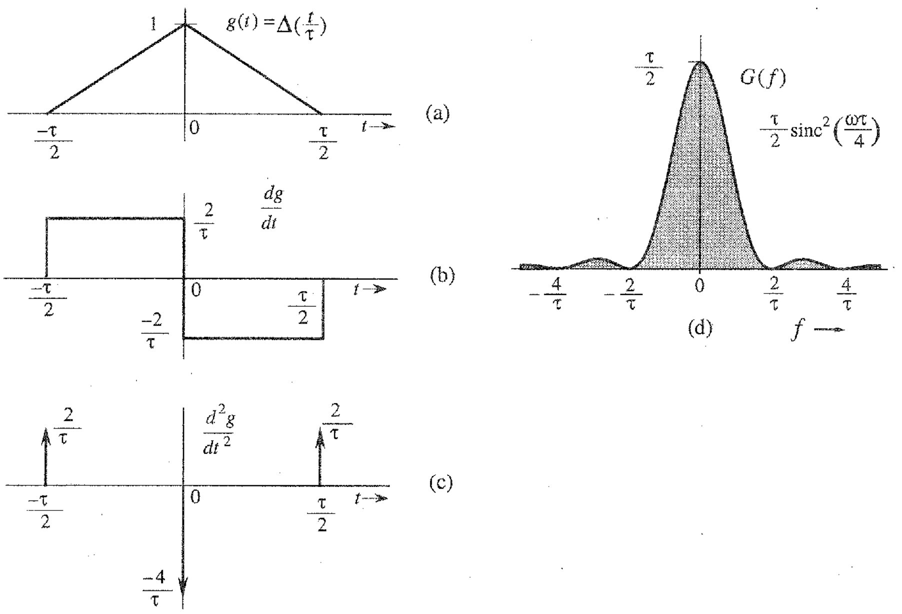


Figure 3.23 Using the time differentiation property to finding the Fourier transform of a piecewise-linear signal.

* Valid only if the transform of $dg(t)/dt$ exists.

To find the Fourier transform of this pulse, we differentiate it successively, as shown in Fig. 3.23b and c. The second derivative consists of a sequence of impulses (Fig. 3.23c). Recall that the derivative of a signal at a jump discontinuity is an impulse of strength equal to the amount of jump. The function $dg(t)/dt$ has a positive jump of $2/\tau$ at $t = \pm\tau/2$, and a negative jump of $4/\tau$ at $t = 0$. Therefore,

$$\frac{d^2g(t)}{dt^2} = \frac{2}{\tau}\left[\delta\left(t+\frac{\tau}{2}\right) - 2\delta(t) + \delta\left(t-\frac{\tau}{2}\right)\right] \tag{3.50}$$

From the time differentiation property [Eq. (3.49)],

$$\frac{d^2g}{dt^2} \Longleftrightarrow (j2\pi f)^2 G(f) = -(2\pi f)^2 G(f) \tag{3.51a}$$

Also, from the time-shifting property [Eqs. (3.32)],

$$\delta(t - t_0) \Longleftrightarrow e^{-j2\pi f t_0} \tag{3.51b}$$

Taking the Fourier transform of Eq. (3.50) and using the results in Eq. (3.51), we obtain

$$(j2\pi f)^2 G(f) = \frac{2}{\tau}\left(e^{j\pi f\tau} - 2 + e^{-j\pi f\tau}\right) = \frac{4}{\tau}(\cos \pi f\tau - 1) = -\frac{8}{\tau}\sin^2\left(\frac{\pi f\tau}{2}\right)$$

and

$$G(f) = \frac{8}{(2\pi f)^2\tau}\sin^2\left(\frac{\pi f\tau}{2}\right) = \frac{\tau}{2}\left[\frac{\sin(\pi f\tau/2)}{\pi f\tau/2}\right]^2 = \frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\pi f\tau}{2}\right) \tag{3.52}$$

The spectrum $G(f)$ is shown in Fig. 3.23d. This procedure of finding the Fourier transform can be applied to any function $g(t)$ made up of straight-line segments with $g(t) \to 0$ as $|t| \to \infty$. The second derivative of such a signal yields a sequence of impulses whose Fourier transform can be found by inspection. This example suggests a numerical method of finding the Fourier transform of an arbitrary signal $g(t)$ by approximating the signal by straight-line segments.

To provide easy reference, several important properties of Fourier transform are summarized in Table 3.2.

3.4 SIGNAL TRANSMISSION THROUGH A LINEAR SYSTEM

A linear time-invariant (LTI) continuous time system can be characterized equally well in either the time domain or the frequency domain. The LTI system model, illustrated in Fig. 3.24, can often be used to characterize communication channels. In communication systems and in signal processing, we are interested only in bounded-input–bounded-output (BIBO) stable linear systems. Detailed discussions on system stability can be found in the textbook by Lathi.[3]

TABLE 3.2
Properties of Fourier Transform Operations

Operation	$g(t)$	$G(f)$
Superposition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Scalar multiplication	$kg(t)$	$kG(f)$
Duality	$G(t)$	$g(-f)$
Time scaling	$g(at)$	$\frac{1}{\|a\|}G\left(\frac{f}{a}\right)$
Time shifting	$g(t-t_0)$	$G(f)e^{-j2\pi f t_0}$
Frequency shifting	$g(t)e^{j2\pi f_0 t}$	$G(f-f_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Frequency convolution	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x)\,dx$	$\frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$

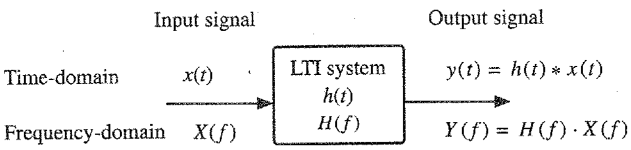


Figure 3.24 Signal transmission through a linear time-invariant system.

A stable LTI system can be characterized in the time domain by its impulse response $h(t)$, which is the system response to a unit impulse input, that is,

$$y(t) = h(t) \qquad \text{when} \qquad x(t) = \delta(t)$$

The system response to a bounded input signal $x(t)$ follows the convolutional relationship

$$y(t) = h(t) * x(t) \tag{3.53}$$

The frequency domain relationship between the input and the output is obtained by taking Fourier transform of both sides of Eq. (3.53). We let

$$x(t) \Longleftrightarrow X(f)$$
$$y(t) \Longleftrightarrow Y(f)$$
$$h(t) \Longleftrightarrow H(f)$$

Then according to the convolution theorem, Eq. (3.53) becomes

$$Y(f) = H(f) \cdot X(f) \tag{3.54}$$

Generally $H(f)$, the Fourier transform of the impulse response $h(t)$, is referred to as the **transfer function** or the **frequency response** of the LTI system. Again, in general, $H(f)$ is complex and can be written as

$$H(f) = |H(f)|e^{j\theta_h(f)}$$

where $|H(f)|$ is the amplitude response and $\theta_h(f)$ is the phase response of the LTI system.

3.4.1 Signal Distortion during Transmission

The transmission of an input signal $x(t)$ through a system changes it into the output signal $y(t)$. Equation (3.54) shows the nature of this change or modification. Here $X(f)$ and $Y(f)$ are the spectra of the input and the output, respectively. Therefore, $H(f)$ is the spectral response of the system. The output spectrum is given by the input spectrum multiplied by the spectral response of the system. Equation (3.54) clearly brings out the spectral shaping (or modification) of the signal by the system. Equation (3.54) can be expressed in polar form as

$$|Y(f)|e^{j\theta_y(f)} = |X(f)||H(f)|e^{j[\theta_x(f)+\theta_h(f)]}$$

Therefore, we have the amplitude and phase relationships

$$|Y(f)| = |X(f)|\,|H(f)| \tag{3.55a}$$

$$\theta_y(f) = \theta_x(f) + \theta_h(f) \tag{3.55b}$$

During the transmission, the input signal amplitude spectrum $|X(f)|$ is changed to $|X(f)| \cdot |H(f)|$. Similarly, the input signal phase spectrum $\theta_x(f)$ is changed to $\theta_x(f) + \theta_h(f)$.

An input signal spectral component of frequency f is modified in amplitude by a factor $|H(f)|$ and is shifted in phase by an angle $\theta_h(f)$. Clearly, $|H(f)|$ is the amplitude response, and $\theta_h(f)$ is the phase response of the system. The plots of $|H(f)|$ and $\theta_h(f)$ as functions of f show at a glance how the system modifies the amplitudes and phases of various sinusoidal inputs. This is why $H(f)$ is called the **frequency response** of the system. During transmission through the system, some frequency components may be boosted in amplitude, while others may be attenuated. The relative phases of the various components also change. In general, the output waveform will be different from the input waveform.

3.4.2 Distortionless Transmission

In several applications, such as signal amplification or message signal transmission over a communication channel, we require the output waveform to be a replica of the input waveform. In such cases, we need to minimize the distortion caused by the amplifier or the communication channel. It is therefore of practical interest to determine the characteristics of a system that allows a signal to pass without distortion (**distortionless transmission**).

Transmission is said to be distortionless if the input and the output have identical wave shapes within a multiplicative constant. A delayed output that retains the input waveform is also considered distortionless. Thus, in distortionless transmission, the input $x(t)$ and the output $y(t)$ satisfy the condition

$$y(t) = k \cdot x(t - t_d) \tag{3.56}$$

The Fourier transform of this equation yields

$$Y(f) = kX(f)e^{-j2\pi f t_d}$$

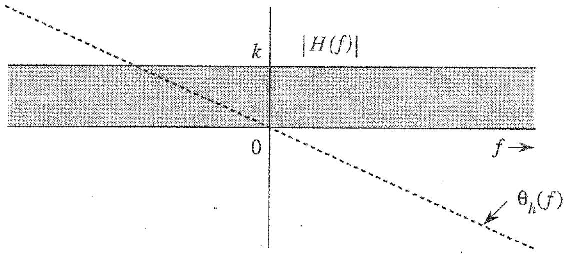


Figure 3.25 Linear time-invariant system frequency response for distortionless transmission.

But because

$$Y(f) = X(f)H(f)$$

we therefore have

$$H(f) = k\,e^{-j2\pi f t_d}$$

This is the transfer function required for distortionless transmission. From this equation it follows that

$$|H(f)| = k \tag{3.57a}$$

$$\theta_h(f) = -2\pi f t_d \tag{3.57b}$$

This shows that for distortionless transmission, the amplitude response $|H(f)|$ must be a constant, and the phase response $\theta_h(f)$ must be a linear function of f going through the origin $f = 0$, as shown in Fig. 3.25. The slope of $\theta_h(f)$ with respect to the angular frequency $\omega = 2\pi f$ is $-t_d$, where t_d is the delay of the output with respect to the input.*

All-Pass vs. Distortionless System

In circuit analysis and filter designs, we sometimes are mainly concerned with the gain of a system response. An all-pass system has a constant gain for all frequencies [i.e., $|H(f)| = k$], without the linear phase requirement. Note from Eq. (3.57) that a distortionless system is always an all-pass system, whereas the converse is not true. Because it is very common for beginners to be confused by the difference between all-pass and distortionless systems, now is the best time to clarify.

To see how an all-pass system may lead to distortion, let us consider an illustrative example. Imagine that we would like to transmit a recorded music signal from a violin-cello duet. The violin contributes to the high frequency part of this music signal, while the cello contributes to the bases part. When this music signal is transmitted through a particular *all-pass* system, both parts have the same gain. However, suppose that this all-pass system would cause a 1-second *extra* delay on the high-frequency content of the music (from the violin). As a result, the audience on the receiving end will hear a "music" signal that is totally out of sync even though *all signal components have the same gain and all are present.* The difference in transmission delay for components of different frequencies is contributed by the nonlinear phase of $H(f)$ in the all-pass filter.

* In addition, we require that $\theta_h(0)$ either be 0 (as shown in Fig. 3.25) or have a constant value $n\pi$ (n an integer), that is, $\theta_h(f) = n\pi - 2\pi f t_d$. The addition of the excess phase of $n\pi$ may at most change the sign of the signal.

To be more precise, the transfer function gain $|H(f)|$ determines the gain of each input frequency component, whereas $\angle H(f)$ determines the delay of each component. Imagine a system input $x(t)$ consisting of multiple sinusoids (its spectral components). For the output signal $y(t)$ to be distortionless, it should be the input signal multiplied by a gain k and delayed by t_d. To synthesize such a signal, $y(t)$ needs exactly the same components as those of $x(t)$, with each component multiplied by k and delayed by t_d. This means that the system transfer function $H(f)$ should be such that each sinusoidal component encounters the same gain (or loss) k and each component undergoes the same time delay of t_d seconds. The first condition requires that

$$|H(f)| = k$$

We have seen earlier (Sec. 3.3) that to achieve the same time delay t_d for every frequency component requires a linear phase delay $2\pi f t_d$ (Fig. 3.18) through the origin

$$\theta_h(f) = -2\pi f t_d$$

In practice, many systems have a phase characteristic that may be only approximately linear. A convenient method of checking phase linearity is to plot the slope of $\angle H(f)$ as a function of frequency. This slope can be a function of f in the general case and is given by

$$t_d(f) = -\frac{1}{2\pi} \cdot \frac{d\theta_h(f)}{df} \tag{3.58}$$

If the slope of θ_h is constant (that is, if θ_h is linear with respect to f), all the components are delayed by the same time interval t_d. But if the slope is not constant, then the time delay t_d varies with frequency. This means that different frequency components undergo different amounts of time delay, and consequently the output waveform will not be a replica of the input waveform (as in the example of the violin-cello duet). For a signal transmission to be distortionless, $t_d(f)$ should be a constant t_d over the frequency band of interest.*

Thus, there is a clear distinction between all-pass and distortionless systems. It is a common mistake to think that flatness of amplitude response $|H(f)|$ alone can guarantee signal quality. A system that has a flat amplitude response may yet distort a signal beyond recognition if the phase response is not linear (t_d not constant).

The Nature of Distortion in Audio and Video Signals

Generally speaking, a human ear can readily perceive amplitude distortion, although it is relatively insensitive to phase distortion. For the phase distortion to become noticeable, the

* Figure 3.25 shows that for distortionless transmission, the phase response not only is linear but also must pass through the origin. This latter requirement can be somewhat relaxed for bandpass signals. The phase at the origin may be any constant [$\theta_h(f) = \theta_0 - 2\pi f t_d$ or $\theta_h(0) = \theta_0$]. The reason for this can be found in Eq. (3.37), which shows that the addition of a constant phase θ_0 to a spectrum of a bandpass signal amounts to a phase shift of the carrier by θ_0. The modulating signal (the envelope) is not affected. The output envelope is the same as the input envelope delayed by

$$t_g = -\frac{1}{2\pi}\frac{d\theta_h(f)}{df}$$

called the **group delay** or **envelope delay**, and the output carrier is the same as the input carrier delayed by

$$t_p = -\frac{\theta_h(f)}{2\pi f}$$

called the **phase delay**, where f_0 is the center frequency of the passband.

variation in delay (variation in the slope of θ_h) should be comparable to the signal duration (or the physically perceptible duration, in case the signal itself is long). In the case of audio signals, each spoken syllable can be considered to be an individual signal. The average duration of a spoken syllable is of a magnitude on the order of 0.01 to 0.1 second. The audio systems may have nonlinear phases, yet no noticeable signal distortion results because in practical audio systems, maximum variation in the slope of θ_h is only a small fraction of a millisecond. This is the real reason behind the statement that "the human ear is relatively insensitive to phase distortion.[4] As a result, the manufacturers of audio equipment make available only $|H(f)|$, the amplitude response characteristic of their systems.

For video signals, on the other hand, the situation is exactly the opposite. The human eye is sensitive to phase distortion but is relatively insensitive to amplitude distortion. The amplitude distortion in television signals manifests itself as a partial destruction of the relative half-tone values of the resulting picture, which is not readily apparent to the human eye. The phase distortion (nonlinear phase), on the other hand, causes different time delays in different picture elements. This results in a smeared picture, which is readily apparent to the human eye. Phase distortion is also very important in digital communication systems because the nonlinear phase characteristic of a channel causes pulse dispersion (spreading out), which in turn causes pulses to interfere with neighboring pulses. This interference can cause an error in the pulse amplitude at the receiver: a binary **1** may read as **0**, and vice versa.

3.5 IDEAL VERSUS PRACTICAL FILTERS

Ideal filters allow distortionless transmission of a certain band of frequencies and suppress all the remaining frequencies. The ideal low-pass filter (Fig. 3.26), for example, allows all components below $f = B$ Hz to pass without distortion and suppresses all components above $f = B$. Figure 3.27 shows ideal high-pass and bandpass filter characteristics.

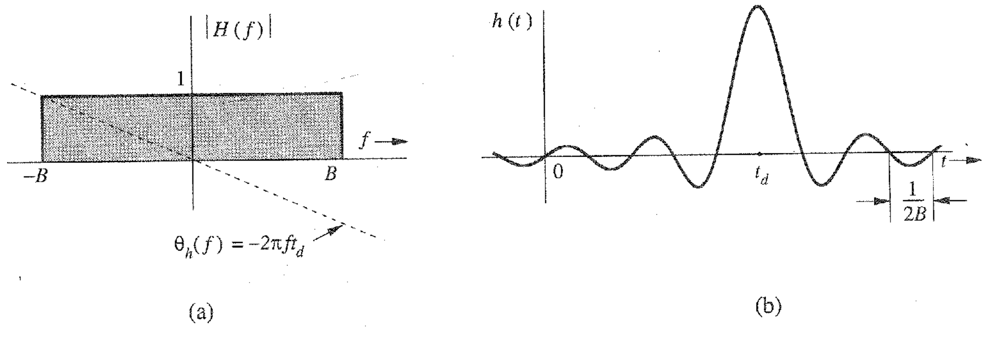
The ideal low-pass filter in Fig. 3.26a has a linear phase of slope $-t_d$, which results in a time delay of t_d seconds for all its input components of frequencies below B Hz. Therefore, if the input is a signal $g(t)$ band-limited to B Hz, the output $y(t)$ is $g(t)$ delayed by t_d, that is,

$$y(t) = g(t - t_d)$$

The signal $g(t)$ is transmitted by this system without distortion, but with time delay t_d. For this filter $|H(f)| = \Pi(f/2B)$, and $\theta_h(f) = -2\pi f t_d$, so that

$$H(f) = \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_d} \tag{3.59a}$$

Figure 3.26 Ideal low-pass filter frequency response and its impulse response.



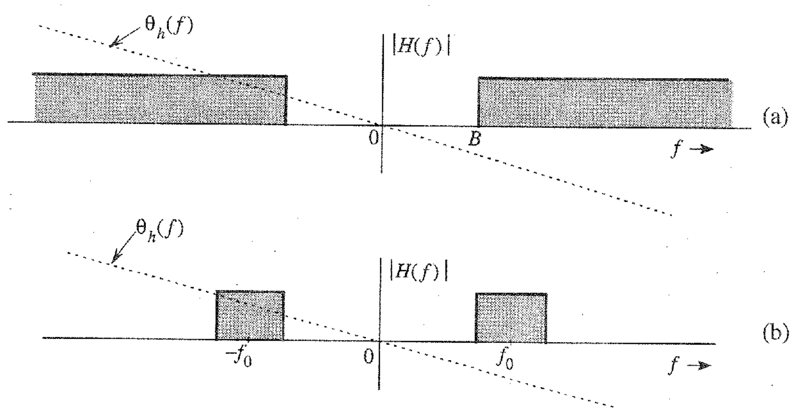


Figure 3.27 Ideal high-pass and bandpass filter frequency responses.

The unit impulse response $h(t)$ of this filter is found from pair 18 in Table 3.1 and the time-shifting property:

$$h(t) = \mathcal{F}^{-1}\left[\Pi\left(\frac{f}{2B}\right)e^{-j2\pi f t_d}\right]$$
$$= 2B \operatorname{sinc}[2\pi B(t - t_d)] \tag{3.59b}$$

Recall that $h(t)$ is the system response to impulse input $\delta(t)$, which is applied at $t = 0$. Figure 3.26b shows a curious fact: the response $h(t)$ begins even before the input is applied (at $t = 0$). Clearly, the filter is noncausal and therefore unrealizable; that is, such a system is physically impossible, since no sensible system can respond to an input **before** it is applied to the system. Similarly, one can show that other ideal filters (such as the ideal high-pass or the ideal bandpass filters shown in Fig. 3.27) are also physically unrealizable.

For a physically realizable system, $h(t)$ must be causal; that is,

$$h(t) = 0 \qquad \text{for } t < 0$$

In the frequency domain, this condition is equivalent to the **Paley-Wiener criterion**, which states that the necessary and sufficient condition for $|H(f)|$ to be the amplitude response of a realizable (or causal) system is*

$$\int_{-\infty}^{\infty} \frac{\big|\ln|H(f)|\big|}{1 + (2\pi f)^2} df < \infty \tag{3.60}$$

If $H(f)$ does not satisfy this condition, it is unrealizable. Note that if $|H(f)| = 0$ over any finite band, $\big|\ln|H(f)|\big| = \infty$ over that band, and the condition (3.60) is violated. If, however, $H(f) = 0$ at a single frequency (or a set of discrete frequencies), the integral in Eq. (3.60) may still be finite even though the integrand is infinite. Therefore, for a physically realizable system, $H(f)$ may be zero at some discrete frequencies, but it cannot be zero over any finite band. According to this criterion, ideal filter characteristics (Figs. 3.26 and 3.27) are clearly unrealizable.

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* $|H(f)|$ is assumed to be square integrable. That is,

$$\int_{-\infty}^{\infty} |H(f)|^2 \, df$$

is assumed to be finite.

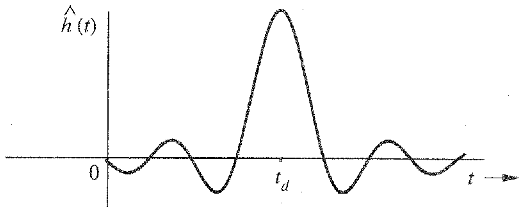


Figure 3.28 Approximate realization of an ideal low-pass filter by truncating its impulse response.

The impulse response $h(t)$ in Fig. 3.26 is not realizable. One practical approach to filter design is to cut off the tail of $h(t)$ for $t < 0$. The resulting causal impulse response $\widehat{h}(t)$, where

$$\widehat{h}(t) = h(t)u(t)$$

is physically realizable because it is causal (Fig. 3.28). If t_d is sufficiently large, $\widehat{h}(t)$ will be a close approximation of $h(t)$, and the resulting filter $\widehat{H}(f)$ will be a good approximation of an ideal filter. This close realization of the ideal filter is achieved because of the increased value of time delay t_d. This means that the price of close physical approximation is higher delay in the output; this is often true of noncausal systems. Of course, theoretically a delay $t_d = \infty$ is needed to realize the ideal characteristics. But a glance at Fig. 3.27b shows that a delay t_d of three or four times π/W will make $\widehat{h}(t)$ a reasonably close version of $h(t - t_d)$. For instance, audio filters are required to handle frequencies of up to 20 kHz (the highest frequency the human ear can hear). In this case a t_d of about 10^{-4} (0.1 ms) would be a reasonable choice. The truncation operation [cutting the tail of $h(t)$ to make it causal], however, creates some unsuspected problems of spectral spread and leakage, and which can be partly corrected by using a tapered window function to truncate $h(t)$ gradually (rather than abruptly).[5]

In practice, we can realize a variety of filter characteristics to approach ideal characteristics. Practical (realizable) filter characteristics are gradual, without jump discontinuities in the amplitude response $|H(f)|$. For example, Butterworth and Chebychev filters are used extensively in various applications including practical communication circuits.

Analog signals can also be processed by digital means (A/D conversion). This involves sampling, quantizing, and coding. The resulting digital signal can be processed by a small, special-purpose digital computer designed to convert the input sequence into a desired output sequence. The output sequence is converted back into the desired analog signal. A special algorithm of the processing digital computer can be used to achieve a given signal operation (e.g., low-pass, bandpass, or high-pass filtering). The subject of digital filtering is somewhat beyond our scope in this book. Several excellent books are available on the subject.[3]

3.6 SIGNAL DISTORTION OVER A COMMUNICATION CHANNEL

A signal transmitted over a channel is distorted because of various channel imperfections. The nature of signal distortion will now be studied.

3.6.1 Linear Distortion

We shall first consider linear time-invariant channels. Signal distortion can be caused over such a channel by nonideal characteristics of magnitude distortion, phase distortion, or both.

We can identify the effects these **nonidealities** will have on a pulse $g(t)$ transmitted through such a channel. Let the pulse exist over the interval (a, b) and be zero outside this interval. The components of the Fourier spectrum of the pulse have such a perfect and delicate balance of magnitudes and phases that they add up precisely to the pulse $g(t)$ over the interval (a, b) and to zero outside this interval. The transmission of $g(t)$ through an ideal channel that satisfies the conditions of distortionless transmission also leaves this balance undisturbed, because a distortionless channel multiplies each component by the same factor and delays each component by the same amount of time. Now, if the amplitude response of the channel is not ideal [i.e., if $|H(f)|$ is not equal to a constant], this delicate balance will be disturbed, and the sum of all the components cannot be zero outside the interval (a, b). In short, the pulse will spread out (see Example 3.14). The same thing happens if the channel phase characteristic is not ideal, that is, if $\theta_h(f) \neq -2\pi f t_d$. Thus, spreading, or **dispersion**, of the pulse will occur if either the amplitude response or the phase response, or both, are nonideal.

Linear channel distortion (dispersion in time) is particularly damaging to digital communication systems. It introduces what is known as intersymbol interferences (ISI). In other words, a digital symbol, when transmitted over a dispersive channel, tends to spread more widely than its allotted time. Therefore, adjacent symbols will interfere with one another, thereby increasing the probability of detection error at the receiver.

Example 3.14 A low-pass filter (Fig. 3.29a) transfer function $H(f)$ is given by

$$H(f) = \begin{cases} (1 + k\cos 2\pi fT)e^{-j2\pi f t_d} & |f| < B \\ 0 & |f| > B \end{cases} \tag{3.61}$$

A pulse $g(t)$ band-limited to B Hz (Fig. 3.29b) is applied at the input of this filter. Find the output $y(t)$.

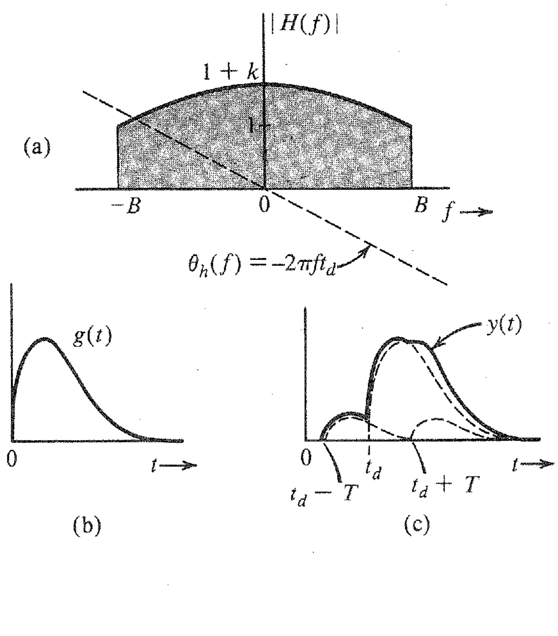


Figure 3.29 Pulse is dispersed when it passes through a system that is not distortionless.

This filter has ideal phase and nonideal magnitude characteristics. Because $g(t) \Leftrightarrow G(f)$, $y(t) \Leftrightarrow Y(f)$ and

$$\begin{aligned} Y(f) &= G(f)H(f) \\ &= G(f) \cdot \Pi\left(\frac{f}{2B}\right)(1 + k\cos 2\pi fT)e^{-j2\pi f t_d} \\ &= G(f)e^{-j2\pi f t_d} + k\,[G(f)\cos 2\pi fT]\;e^{-j2\pi f t_d} \end{aligned} \tag{3.62}$$

Note that in the derivation of Eq. (3.62) because $g(t)$ is band-limited to B Hz, we have $G(f) \cdot \Pi\left(\frac{f}{2B}\right) = G(f)$. Then, by using the time-shifting property and Eq. (3.32a), we have

$$y(t) = g(t - t_d) + \frac{k}{2}[g(t - t_d - T) + g(t - t_d + T)] \tag{3.63}$$

The output is actually $g(t) + (k/2)[g(t - T) + g(t + T)]$ delayed by t_d. It consists of $g(t)$ and its echoes shifted by $\pm t_d$. The dispersion of the pulse caused by its echoes is evident from Fig. 3.29c. Ideal amplitude but nonideal phase response of $H(f)$ has a similar effect (see Prob. 3.6-1).

3.6.2 Distortion Caused by Channel Nonlinearities

Until now we have considered the channel to be linear. This approximation is valid only for small signals. For large signal amplitudes, nonlinearities cannot be ignored. A general discussion of nonlinear systems is beyond our scope. Here we shall consider a simple case of a memoryless nonlinear channel where the input g and the output y are related by some (memoryless) nonlinear equation,

$$y = f(g)$$

The right-hand side of this equation can be expanded in a Maclaurin series as

$$y(t) = a_0 + a_1 g(t) + a_2 g^2(t) + a_3 g^3(t) + \cdots + a_k g^k(t) + \cdots$$

Recall the result in Sec. 3.3.6 (convolution) that if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^k(t)$ is kB Hz. Hence, the bandwidth of $y(t)$ is **greater than** kB Hz. Consequently, the output spectrum spreads well beyond the input spectrum, and the output signal contains new frequency components not contained in the input signal. In broadcast communication, we need to amplify signals at very high power levels, where high-efficiency (class C) amplifiers are desirable. Unfortunately, these amplifiers are nonlinear, and they cause distortion when used to amplify signals. This is one of the serious problems in AM signals. However, FM signals are not affected by nonlinear distortion, as shown in Chapter 5. If a signal is transmitted over a nonlinear channel, the nonlinearity not only distorts the signal but also causes interference with other signals on the channel because of its spectral dispersion (spreading).

For digital communication systems, the nonlinear distortion effect is in contrast to the time dispersion effect due to linear distortion. Linear distortion causes interference among signals within the same channel, whereas spectral dispersion due to nonlinear distortion causes interference among signals using different frequency channels.

Example 3.15 The input $x(t)$ and the output $y(t)$ of a certain nonlinear channel are related as

$$y(t) = x(t) + 0.000158x^2(t)$$

Find the output signal $y(t)$ and its spectrum $Y(f)$ if the input signal is $x(t) = 2000 \operatorname{sinc}(2000\pi t)$. Verify that the bandwidth of the output signal is twice that of the input signal. This is the result of signal squaring. Can the signal $x(t)$ be recovered (without distortion) from the output $y(t)$?

Since

$$x(t) = 2000 \operatorname{sinc}(2000\pi t) \quad \Longleftrightarrow \quad X(f) = \Pi\left(\frac{f}{2000}\right)$$

We have

$$y(t) = x(t) + 0.000158x^2(t) = 2000 \operatorname{sinc}(2000\pi t) + 0.316 \cdot 2000 \operatorname{sinc}^2(2000\pi t)$$

$$\Longleftrightarrow$$

$$Y(f) = \Pi\left(\frac{f}{2000}\right) + 0.316\,\Delta\left(\frac{f}{4000}\right)$$

Observe that $0.316 \cdot 2000\operatorname{sinc}^2(2000\pi t)$ is the unwanted (distortion) term in the received signal. Figure 3.30a shows the input (desired) signal spectrum $X(f)$; Fig. 3.30b shows the spectrum of the undesired (distortion) term; and Fig. 3.30c shows the received signal spectrum $Y(f)$. We make the following observations.

1. The bandwidth of the received signal $y(t)$ is twice that of the input signal $x(t)$ (because of signal squaring).
2. The received signal contains the input signal $x(t)$ plus an unwanted signal $632 \operatorname{sinc}^2(2000\pi t)$. The spectra of these two signals are shown in Fig. 3.30a and b. Figure 3.30c shows $Y(f)$, the spectrum of the received signal. Note that spectra of the desired signal and the distortion signal overlap, and it is impossible to recover the signal $x(t)$ from the received signal $y(t)$ without some distortion.
3. We can reduce the distortion by passing the received signal through a low-pass filter of bandwidth 1000 Hz. The spectrum of the output of this filter is shown in Fig. 3.30d. Observe that the output of this filter is the desired input signal $x(t)$ with some residual distortion.
4. We have an additional problem of interference with other signals if the input signal $x(t)$ is frequency-division-multiplexed along with several other signals on this channel. This means that several signals occupying nonoverlapping frequency bands are transmitted simultaneously on the same channel. Spreading the spectrum $X(f)$ outside its original band of 1000 Hz will interfere with the signal in the band of 1000 to 2000 Hz. Thus, in addition to the distortion of $x(t)$, we have an interference with the neighboring band.

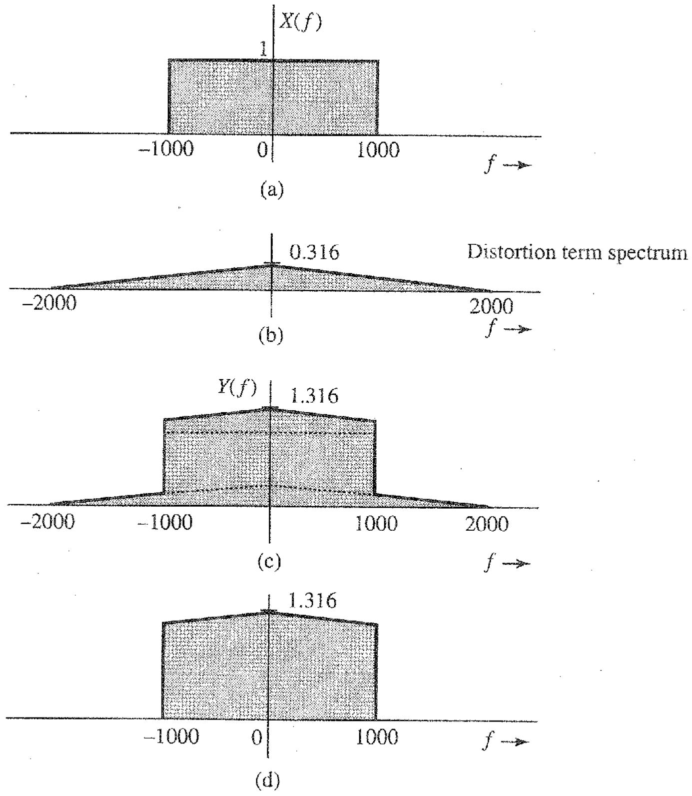


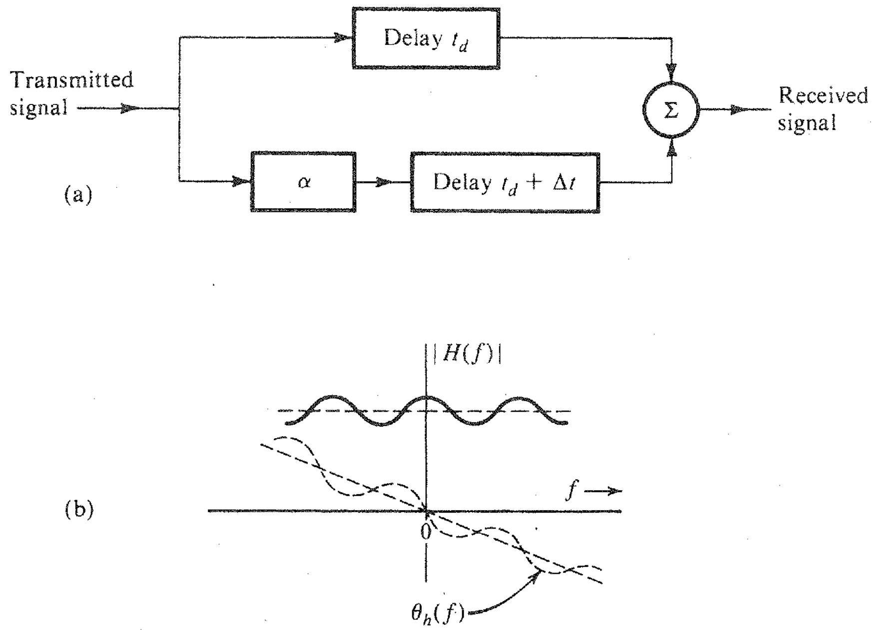
Figure 3.30 Signal distortion caused by nonlinear operation. (a) Desired (input) signal spectrum. (b) Spectrum of the unwanted signal (distortion) in the received signal. (c) Spectrum of the received signal. (d) Spectrum of the received signal after low-pass filtering.

5. If $x(t)$ were a digital signal consisting of a pulse train, each pulse would be distorted, but there would be no interference with the neighboring pulses. Moreover even with distorted pulses, data can be received without loss because digital communication can withstand considerable pulse distortion without loss of information. Thus, if this channel were used to transmit a time-division multiplexed signal consisting of two interleaved pulse trains, the data in the two trains would be recovered at the receiver.

3.6.3 Distortion Caused by Multipath Effects

A multipath transmission occurs when a transmitted signal arrives at the receiver by two or more paths of different delays. For example, if a signal is transmitted over a cable that has impedance irregularities (mismatching) along the path, the signal will arrive at the receiver in the form of a direct wave plus various reflections with various delays. In radio links, the signal can be received by direct path between the transmitting and the receiving antennas and also by reflections from other objects, such as hills and buildings. In long-distance radio links using the ionosphere, similar effects occur because of one-hop and multihop paths. In each of these cases, the transmission channel can be represented as several channels in parallel, each with a different relative attenuation and a different time delay. Let us consider the case of only two paths: one with a unity gain and a delay t_d, and the other with a gain α and a delay $t_d + \Delta t$, as shown in Fig. 3.31a. The transfer functions of the two paths are given by $e^{-j2\pi f t_d}$ and $\alpha e^{-j2\pi f(t_d+\Delta t)}$, respectively. The overall transfer function of such a channel is

Figure 3.31 Multipath transmission.



$H(f)$, given by

$$\begin{aligned} H(f) &= e^{-j2\pi f t_d} + \alpha e^{-j2\pi f (t_d+\Delta t)} \\ &= e^{-j2\pi f t_d}(1 + \alpha e^{-j2\pi f \Delta t}) \qquad (3.64a) \\ &= e^{-j2\pi f t_d}(1 + \alpha \cos 2\pi f \Delta t - j\alpha \sin 2\pi f \Delta t) \end{aligned}$$

$$= \underbrace{\sqrt{1 + \alpha^2 + 2\alpha \cos 2\pi f \Delta t}}_{|H(f)|} \exp\left[-j\underbrace{\left(2\pi f t_d + \tan^{-1} \frac{\alpha \sin 2\pi f \Delta t}{1 + \alpha \cos 2\pi f \Delta t}\right)}_{\theta_h(f)}\right] \qquad (3.64b)$$

Both the magnitude and the phase characteristics of $H(f)$ are periodic in f with a period of $1/\Delta t$ (Fig. 3.31b). The multipath channel, therefore, can exhibit nonidealities in the magnitude and the phase characteristics of the channel and can cause linear distortion (pulse dispersion), as discussed earlier.

If, for instance, the gains of the two paths are very close, that is, $\alpha \approx 1$, then the signals received from the two paths may have opposite phase (π radians apart) at certain frequencies. This means that at those frequencies where the two paths happen to result in opposite phases, the signals from the two paths will almost cancel each other. Equation (3.64b) shows that at frequencies where $f = n/(2\Delta t)$ (n odd), $\cos 2\pi f \Delta t = -1$, and $|H(f)| \approx 0$ when $\alpha \approx 1$. These frequencies are the multipath null frequencies. At frequencies $f = n/(2\Delta t)$ (n even), the two signals interfere constructively to enhance the gain. Such channels cause **frequency-selective fading** of transmitted signals. Such distortion can be partly corrected by using the tapped delay-line equalizer, as shown in Prob. 3.6-2. These equalizers are useful in several applications in communications. Their design issues are addressed later in Chapters 7 and 12.

3.6.4 Fading Channels

Thus far, the channel characteristics have been assumed to be constant with time. In practice, we encounter channels whose transmission characteristics vary with time. These include troposcatter channels and channels using the ionosphere for radio reflection to achieve long-distance communication. The time variations of the channel properties arise because of semi periodic and random changes in the propagation characteristics of the medium. The reflection properties of the ionosphere, for example, are related to meteorological conditions that change seasonally, daily, and even from hour to hour, much like the weather. Periods of sudden storms also occur. Hence, the effective channel transfer function varies semi periodically and randomly, causing random attenuation of the signal. This phenomenon is known as **fading**. One way to reduce the effects of slow fading is to use **automatic gain control (AGC)**.*

Fading may be strongly frequency dependent where different frequency components are affected unequally. Such fading, known as frequency-selective fading, can cause serious problems in communication. Multipath propagation can cause frequency-selective fading.

3.7 SIGNAL ENERGY AND ENERGY SPECTRAL DENSITY

The energy E_g of a signal $g(t)$ is defined as the area under $|g(t)|^2$. We can also determine the signal energy from its Fourier transform $G(f)$ through Parseval's theorem.

3.7.1 Parseval's Theorem

Signal energy can be related to the signal spectrum $G(f)$ by substituting Eq. (3.9b) in Eq. (2.2):

$$E_g = \int_{-\infty}^{\infty} g(t)g^*(t)\, dt = \int_{-\infty}^{\infty} g(t) \left[\int_{-\infty}^{\infty} G^*(f) e^{-j2\pi ft}\, df \right] dt$$

Here, we used the fact that $g^*(t)$, being the conjugate of $g(t)$, can be expressed as the conjugate of the right-hand side of Eq. (3.9b). Now, interchanging the order of integration yields

$$\begin{aligned} E_g &= \int_{-\infty}^{\infty} G^*(f) \left[\int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt \right] df \\ &= \int_{-\infty}^{\infty} G(f)G^*(f)\, df \\ &= \int_{-\infty}^{\infty} |G(f)|^2\, df \end{aligned} \tag{3.65}$$

This is the well-known statement of Parseval theorem. A similar result was obtained for a periodic signal and its Fourier series in Eq. (2.68). This result allows us to determine the signal energy from either the time domain specification $g(t)$ or the frequency domain specification $G(f)$ of the same signal.

* AGC will also suppress slow variations of the original signal.

Example 3.16 Verify Parseval's theorem for the signal $g(t) = e^{-at}u(t)$ $(a > 0)$.

We have

$$E_g = \int_{-\infty}^{\infty} g^2(t)\, dt = \int_0^{\infty} e^{-2at}\, dt = \frac{1}{2a} \tag{3.66}$$

We now determine E_g from the signal spectrum $G(f)$ given by

$$G(f) = \frac{1}{j2\pi f + a}$$

and from Eq. (3.65),

$$E_g = \int_{-\infty}^{\infty} |G(f)|^2\, df = \int_{-\infty}^{\infty} \frac{1}{(2\pi f)^2 + a^2}\, df = \frac{1}{2\pi a} \tan^{-1} \frac{2\pi f}{a} \Big|_{-\infty}^{\infty} = \frac{1}{2a}$$

which verifies Parseval's theorem.

3.7.2 Energy Spectral Density (ESD)

Equation (3.65) can be interpreted to mean that the energy of a signal $g(t)$ is the result of energies contributed by all the spectral components of the signal $g(t)$. The contribution of a spectral component of frequency f is proportional to $|G(f)|^2$. To elaborate this further, consider a signal $g(t)$ applied at the input of an ideal bandpass filter, whose transfer function $H(f)$ is shown in Fig. 3.32a. This filter suppresses all frequencies except a narrow band Δf $(\Delta f \to 0)$ centered at angular frequency ω_0 (Fig. 3.32b). If the filter output is $y(t)$, then its Fourier transform $Y(f) = G(f)H(f)$, and E_y, the energy of the output $y(t)$, is

$$E_y = \int_{-\infty}^{\infty} |G(f)H(f)|^2\, df \tag{3.67}$$

Because $H(f) = 1$ over the passband Δf, and zero everywhere else, the integral on the right-hand side is the sum of the two shaded areas in Fig. 3.32b, and we have (for $\Delta f \to 0$)

$$E_y = 2\,|G(f_0)|^2\, df$$

Thus, $2|G(f)|^2\, df$ is the energy contributed by the spectral components within the two narrow bands, each of width Δf Hz, centered at $\pm f_0$. Therefore, we can interpret $|G(f)|^2$ as the energy per unit bandwidth (in hertz) of the spectral components of $g(t)$ centered at frequency f. In other words, $|G(f)|^2$ is the energy spectral density (per unit bandwidth in hertz) of $g(t)$. Actually, since both the positive- and the negative-frequency components combine to form the components in the band Δf, the energy contributed per unit bandwidth is $2|G(f)|^2$. However, for the sake of convenience we consider the positive- and negative-frequency components to be independent. The **energy spectral density (ESD)** $\Psi_g(t)$ is thus defined as

$$\Psi_g(f) = |G(f)|^2 \tag{3.68}$$

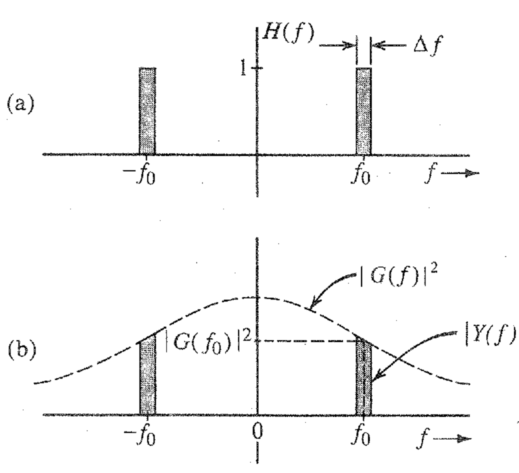


Figure 3.32 Interpretation of the energy spectral density of a signal.

and Eq. (3.65) can be expressed as

$$E_g = \int_{-\infty}^{\infty} \Psi_g(f)\, df \tag{3.69a}$$

From the results in Example 3.16, the ESD of the signal $g(t) = e^{-at}u(t)$ is

$$\Psi_g(f) = |G(f)|^2 = \frac{1}{(2\pi f)^2 + a^2} \tag{3.69b}$$

3.7.3 Essential Bandwidth of a Signal

The spectra of most signals extend to infinity. However, because the energy of a practical signal is finite, the signal spectrum must approach 0 as $f \to \infty$. Most of the signal energy is contained within a certain band of B Hz, and the energy content of the components of frequencies greater than B Hz is negligible. We can therefore suppress the signal spectrum beyond B Hz with little effect on the signal shape and energy. The bandwidth B is called the **essential bandwidth** of the signal. The criterion for selecting B depends on the error tolerance in a particular application. We may, for instance, select B to be that bandwidth that contains 95% of the signal energy.* The energy level may be higher or lower than 95%, depending on the precision needed. We can use such a criterion to determine the essential bandwidth of a signal. Suppression of all the spectral components of $g(t)$ beyond the essential bandwidth results in a signal $\hat{g}(t)$, which is a close approximation of $g(t)$.† If we use the 95% criterion for the essential bandwidth, the energy of the error (the difference) $g(t) - \hat{g}(t)$ is 5% of E_g. The following example demonstrates the bandwidth estimation procedure.

* Essential bandwidth for a low-pass signal may also be defined as a frequency at which the value of the amplitude spectrum is a small fraction (about 5–10%) of its peak value. In Example 3.16, the peak of $|G(f)|$ is $1/a$, and it occurs at $f = 0$.

† In practice the truncation is performed gradually, by using tapered windows, to avoid excessive spectral leakage due to the abrupt truncation.[5]

Example 3.17 Estimate the essential bandwidth W (in rad/s) of the signal $e^{-at}u(t)$ if the essential band is required to contain 95% of the signal energy.

In this case,

$$G(f) = \frac{1}{j2\pi f + a}$$

and the ESD is

$$|G(f)^2| = \frac{1}{(2\pi f)^2 + a^2}$$

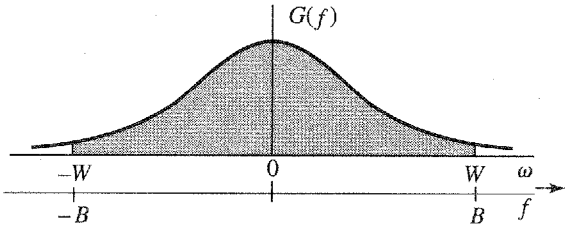


Figure 3.33 Estimating the essential bandwidth of a signal.

This ESD is shown in Fig. 3.33. Moreover, the signal energy E_g is the area under this ESD, which has already been found to be $1/2a$. Let W rad/s be the essential bandwidth, which contains 95% of the total signal energy E_g. This means $1/2\pi$ times the shaded area in Fig. 3.33 is $0.95/2a$, that is,

$$\frac{0.95}{2a} = \int_{-W/2\pi}^{W/2\pi} \frac{df}{(2\pi f)^2 + a^2}$$

$$= \frac{1}{2\pi a} \tan^{-1} \frac{2\pi f}{a} \Bigg|_{-W/2\pi}^{W/2\pi} = \frac{1}{\pi a} \tan^{-1} \frac{W}{a}$$

or

$$\frac{0.95\pi}{2} = \tan^{-1} \frac{W}{a} \Longrightarrow W = 12.7\, a \text{ rad/s}$$

In terms of hertz, the essential bandwidth is

$$B = \frac{W}{2\pi} = 2.02\, a \quad \text{Hz}$$

This means that in the band from 0 (dc) to $12.7 \times a$ rad/s ($2.02 \times a$ Hz), the spectral components of $g(t)$ contribute 95% of the total signal energy; all the remaining spectral components (in the band from $2.02 \times a$ Hz to ∞) contribute only 5% of the signal energy.*

* Note that although the ESD exists over the band $-\infty$ to ∞, the trigonometric spectrum exists only over the band 0 to ∞. The spectrum range $-\infty$ to ∞ applies to the exponential spectrum. In practice, whenever we talk about a bandwidth, we mean it in the trigonometric sense. Hence, the essential band is from 0 to B Hz (or W rad/s), not from $-B$ to B.

Example 3.18 Estimate the essential bandwidth of a rectangular pulse $g(t) = \Pi(t/T)$ (Fig. 3.34a), where the essential bandwidth is to contain at least 90% of the pulse energy.

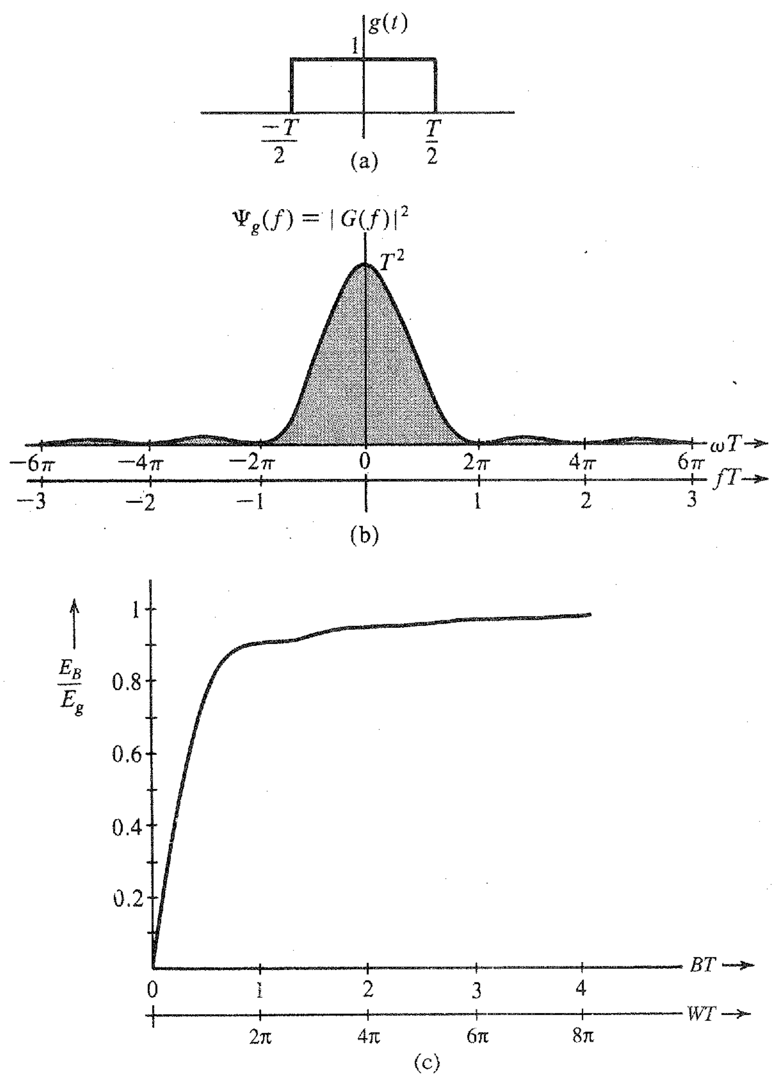
For this pulse, the energy E_g is

$$E_g = \int_{-\infty}^{\infty} g^2(t)\,dt = \int_{-T/2}^{T/2} dt = T$$

Also because

$$\Pi\left(\frac{t}{T}\right) \Longleftrightarrow T\ \text{sinc}\,(\pi f T)$$

Figure 3.34 (a) EX-FGN/FGC rectangular function, (b) its energy spectral density, and (c) fraction of energy inside B(H₂).



the ESD for this pulse is

$$\Psi_g(f) = |G(f)|^2 = T^2 \operatorname{sinc}^2(\pi f T)$$

This ESD is shown in Fig. 3.34b as a function of ωT as well as fT, where f is the frequency in hertz. The energy E_B within the band from 0 to B Hz is given by

$$E_B = \int_{-B}^{B} T^2 \operatorname{sinc}^2(\pi f T)\, df$$

Setting $2\pi f T = x$ in this integral so that $df = dx/(2\pi T)$, we obtain

$$E_B = \frac{T}{\pi} \int_0^{2\pi BT} \operatorname{sinc}^2\left(\frac{x}{2}\right) dx$$

Also because $E_g = T$, we have

$$\frac{E_B}{E_g} = \frac{1}{\pi} \int_0^{2\pi BT} \operatorname{sinc}^2\left(\frac{x}{2}\right) dx$$

The integral on the right-hand side is numerically computed, and the plot of E_B/E_g vs. BT is shown in Fig. 3.34c. Note that 90.28% of the total energy of the pulse $g(t)$ is contained within the band $B = 1/T$ Hz. Therefore, by the 90% criterion, the bandwidth of a rectangular pulse of width T seconds is $1/T$ Hz.

3.7.4 Energy of Modulated Signals

We have seen that modulation shifts the signal spectrum $G(f)$ to the left and right by f_0. We now show that a similar thing happens to the ESD of the modulated signal.

Let $g(t)$ be a baseband signal band-limited to B Hz. The amplitude-modulated signal $\varphi(t)$ is

$$\varphi(t) = g(t) \cos 2\pi f_0 t$$

and the spectrum (Fourier transform) of $\varphi(t)$ is

$$\Phi(f) = \frac{1}{2}[G(f + f_0) + G(f - f_0)]$$

The ESD of the modulated signal $\varphi(t)$ is $|\Phi(f)|^2$, that is,

$$\Psi_\varphi(f) = \frac{1}{4}|G(f + f_0) + G(f - f_0)|^2$$

If $f_0 \geq B$, then $G(f + f_0)$ and $G(f - f_0)$ are nonoverlapping (see Fig. 3.35), and

$$\begin{aligned}\Psi_\varphi(f) &= \frac{1}{4}\left[|G(f + f_0)|^2 + |G(f - f_0)|^2\right] \\ &= \frac{1}{4}\Psi_g(f + f_0) + \frac{1}{4}\Psi_g(f - f_0) \qquad (3.70)\end{aligned}$$

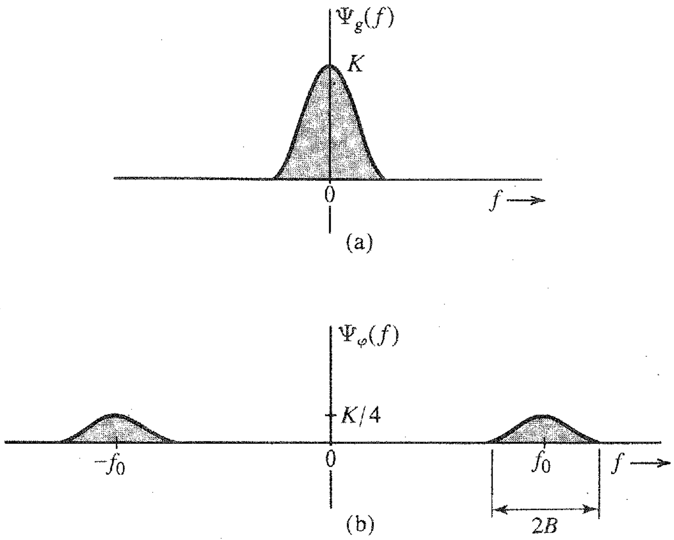


Figure 3.35 Energy spectral densities of modulating and modulated signals.

The ESDs of both $g(t)$ and the modulated signal $\varphi(t)$ are shown in Fig. 3.35. It is clear that modulation shifts the ESD of $g(t)$ by $\pm f_0$. Observe that the area under $\Psi_\varphi(f)$ is half the area under $\Psi_g(f)$. Because the energy of a signal is proportional to the area under its ESD, it follows that the energy of $\varphi(t)$ is half the energy of $g(t)$, that is,

$$E_\varphi = \frac{1}{2}E_g \qquad f_0 \geq B \tag{3.71}$$

It may seem surprising that a signal $\varphi(t)$, which appears so energetic in comparison to $g(t)$, should have only half the energy of $g(t)$. Appearances are deceiving, as usual. The energy of a signal is proportional to the square of its amplitude, and higher amplitudes contribute more energy. Signal $g(t)$ remains at higher amplitude levels most of the time. On the other hand, $\varphi(t)$, because of the factor $\cos 2\pi f_0 t$, dips to zero amplitude levels many times, which reduces its energy.

3.7.5 Time Autocorrelation Function and the Energy Spectral Density

In Chapter 2, we showed that a good measure of comparing two signals $g(t)$ and $z(t)$ is the cross-correlation function $\psi_{gz}(\tau)$ defined in Eq. (2.46). We also defined the correlation of a signal $g(t)$ with itself [the autocorrelation function $\psi_g(\tau)$] in Eq. (2.47). For a real signal $g(t)$, the autocorrelation function $\psi_g(\tau)$ is given by*

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)\,dt \tag{3.72a}$$

* For a complex signal $g(t)$, we define

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g^*(t-\tau)\,dt = \int_{-\infty}^{\infty} g^*(t)g(t+\tau)\,dt$$

Setting $x = t + \tau$ in Eq. (3.72a) yields

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(x)g(x - \tau)\, dx$$

In this equation, x is a dummy variable and could be replaced by t. Thus,

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t \pm \tau)\, dt \tag{3.72b}$$

This shows that for a real $g(t)$, the autocorrelation function is an even function of τ, that is,

$$\psi_g(\tau) = \psi_g(-\tau) \tag{3.72c}$$

There is, in fact, a very important relationship between the autocorrelation of a signal and its ESD. Specifically, the autocorrelation function of a signal $g(t)$ and its ESD $\Psi_g(f)$ form a Fourier transform pair, that is,

$$\psi_g(\tau) \Longleftrightarrow \Psi_g(f) \tag{3.73a}$$

Thus,

$$\Psi_g(f) = \mathcal{F}\{\psi_g(\tau)\} = \int_{-\infty}^{\infty} \psi_g(\tau)e^{-j2\pi f\tau}\, d\tau \tag{3.73b}$$

$$\psi_g(\tau) = \mathcal{F}^{-1}\{\Psi_g(f)\} = \int_{-\infty}^{\infty} \Psi_g(f)e^{-j2\pi f\tau}\, df \tag{3.73c}$$

Note that the Fourier transform of Eq. (3.73a) is performed with respect to τ in place of t.

We now prove that the ESD $\Psi_g(f) = |G(f)|^2$ is the Fourier transform of the autocorrelation function $\psi_g(\tau)$. Although the result is proved here for real signals, it is valid for complex signals also. Note that the autocorrelation function is a function of τ, not t. Hence, its Fourier transform is $\int \psi_g(\tau)e^{-j2\pi f\tau}\, d\tau$. Thus,

$$\mathcal{F}[\psi_g(\tau)] = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\left[\int_{-\infty}^{\infty} g(t)g(t+\tau)\, dt\right] d\tau$$
$$= \int_{-\infty}^{\infty} g(t)\left[\int_{-\infty}^{\infty} g(\tau + t)e^{-j2\pi f\tau} d\tau\right] dt$$

The inner integral is the Fourier transform of $g(\tau + t)$, which is $g(\tau)$ left-shifted by t. Hence, it is given by $G(f)e^{j2\pi ft}$, in accordance with the time-shifting property in Eq. (3.32a). Therefore,

$$\mathcal{F}[\psi_g(\tau)] = G(f)\int_{-\infty}^{\infty} g(t)e^{j2\pi ft} dt = G(f)G(-f) = |G(f)|^2$$

This completes the proof that

$$\psi_g(\tau) \Longleftrightarrow \Psi_g(f) = |G(f)|^2 \tag{3.74}$$

A careful observation of the operation of correlation shows a close connection to convolution. Indeed, the autocorrelation function $\psi_g(\tau)$ is the convolution of $g(\tau)$ with $g(-\tau)$ because

$$g(\tau) * g(-\tau) = \int_{-\infty}^{\infty} g(x)g[-(\tau - x)]\, dx = \int_{-\infty}^{\infty} g(x)g(x - \tau)\, dx = \psi_g(\tau)$$

Application of the time convolution property [Eq. (3.44)] to this equation yields Eq. (3.74).

ESD of the Input and the Output

If $x(t)$ and $y(t)$ are the input and the corresponding output of a linear time-invariant (LTI) system, then

$$Y(f) = H(f)X(f)$$

Therefore,

$$|Y(f)|^2 = |H(f)|^2|X(f)|^2$$

This shows that

$$\Psi_y(f) = |H(f)|^2\Psi_x(f) \tag{3.75}$$

Thus, the output signal ESD is $|H(f)|^2$ times the input signal ESD.

3.8 SIGNAL POWER AND POWER SPECTRAL DENSITY

For a power signal, a meaningful measure of its size is its power [defined in Eq. (2.4)] as the time average of the signal energy averaged over the infinite time interval. The power P_g of a real-valued signal $g(t)$ is given by

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)\, dt \tag{3.76}$$

The signal power and the related concepts can be readily understood by defining a truncated signal $g_T(t)$ as

$$g_T(t) = \begin{cases} g(t) & |t| \le T/2 \\ 0 & |t| > T/2 \end{cases}$$

The truncated signal is shown in Fig. 3.36. The integral on the right-hand side of Eq. (3.76) yields E_{g_T}, which is the energy of the truncated signal $g_T(t)$. Thus,

$$P_g = \lim_{T\to\infty} \frac{E_{g_T}}{T} \tag{3.77}$$

This equation describes the relationship between power and energy of nonperiodic signals. Understanding this relationship will be very helpful in understanding and relating all the power concepts to the energy concepts. Because the signal power is just the time average of energy, all the concepts and results of signal energy also apply to signal power if we modify the concepts properly by taking their time averages.

3.8.1 Power Spectral Density (PSD)

If the signal $g(t)$ is a power signal, then its power is finite, and the truncated signal $g_T(t)$ is an energy signal as long as T is finite. If $g_T(t) \Longleftrightarrow G_T(f)$, then from Parseval's theorem,

$$E_{g_T} = \int_{-\infty}^{\infty} g_T^2(t)\, dt = \int_{-\infty}^{\infty} |G_T(f)|^2\, df$$

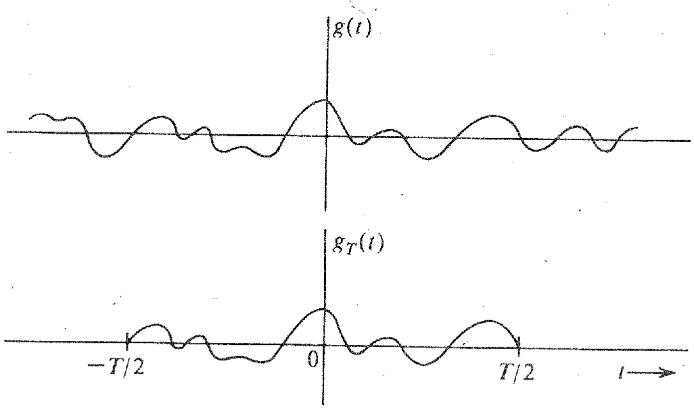


Figure 3.36 Limiting process in derivation of PSD.

Hence, P_g, the power of $g(t)$, is given by

$$P_g = \lim_{T\to\infty} \frac{E_{g_T}}{T} = \lim_{T\to\infty} \frac{1}{T}\left[\int_{-\infty}^{\infty} |G_T(f)|^2\, df\right] \tag{3.78}$$

As T increases, the duration of $g_T(t)$ increases, and its energy E_{g_T} also increases proportionately. This means that $|G_T(f)|^2$ also increases with T, and as $T \to \infty$, $|G_T(f)|^2$ also approaches ∞. However, $|G_T(f)|^2$ must approach ∞ at the same rate as T because for a power signal, the right-hand side of Eq. (3.78) must converge. This convergence permits us to interchange the order of the limiting process and integration in Eq. (3.78), and we have

$$P_g = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|G_T(f)|^2}{T}\, df \tag{3.79}$$

We define the **power spectral density (PSD)** $S_g(\omega)$ as

$$S_g(f) = \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} \tag{3.80}$$

Consequently,*

$$P_g = \int_{-\infty}^{\infty} S_g(f)\, df \tag{3.81a}$$

$$= 2\int_{0}^{\infty} S_g(f)\, df \tag{3.81b}$$

This result is parallel to the result [Eq. (3.69a)] for energy signals. The power is the area under the PSD. Observe that the PSD is the time average of the ESD of $g_T(t)$ [Eq. (3.80)].

As is the case with ESD, the PSD is also a positive, real, and even function of f. If $g(t)$ is a voltage signal, the units of PSD are volts squared per hertz.

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* One should be cautious in using a unilateral expression such as $P_g = 2\int_0^\infty S_g(f)\, df$ when $S_g(f)$ contains an impulse at the origin (a dc component). The impulse part should not be multiplied by the factor 2.

3.8.2 Time Autocorrelation Function of Power Signals

The (time) autocorrelation function $\mathcal{R}_g(\tau)$ of a real power signal $g(t)$ is defined as*

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t-\tau)\,dt \tag{3.82a}$$

We can use the same argument as that used for energy signals [Eqs. (3.72b) and (3.72c)] to show that $\mathcal{R}_g(\tau)$ is an even function of τ. This means that for a real $g(t)$,

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t+\tau)\,dt \tag{3.82b}$$

and

$$\mathcal{R}_g(\tau) = \mathcal{R}_g(-\tau) \tag{3.83}$$

For energy signals, the ESD $\Psi_g(f)$ is the Fourier transform of the autocorrelation function $\psi_g(\tau)$. A similar result applies to power signals. We now show that for a power signal, the PSD $S_g(f)$ is the Fourier transform of the autocorrelation function $\mathcal{R}_g(\tau)$. From Eq. (3.82b) and Fig. 3.36,

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} g_T(t)g_T(t+\tau)\,dt = \lim_{T\to\infty} \frac{\psi_{g_T}(\tau)}{T} \tag{3.84}$$

Recall from the Wiener-Khintchine theorem that $\psi_{g_T}(\tau) \Longleftrightarrow |G_T(f)|^2$. Hence, the Fourier transform of the preceding equation yields

$$\mathcal{R}_g(\tau) \Longleftrightarrow \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} = S_g(f) \tag{3.85}$$

Although we have proved these results for a real $g(t)$, Eqs. (3.80), (3.81a), (3.81b), and (3.85) are equally valid for a complex $g(t)$.

The concept and relationships for signal power are parallel to those for signal energy. This is brought out in Table 3.3.

Signal Power Is Its Mean Square Value

A glance at Eq. (3.76) shows that the signal power is the time average or mean of its squared value. In other words P_g is the mean square value of $g(t)$. We must remember, however, that this is a time mean, not a statistical mean (to be discussed in later chapters). Statistical means are denoted by overbars. Thus, the (statistical) mean square of a variable x is denoted by $\overline{x^2}$. To distinguish from this kind of mean, we shall use a wavy overbar to denote a time average. Thus, the time mean square value of $g(t)$ will be denoted by $\widetilde{g^2(t)}$. The time averages are conventionally denoted by angle brackets, written as $\langle g^2(t)\rangle$. We shall, however, use the wavy

* For a complex $g(t)$, we define

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g^*(t-\tau)\,dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^*(t)g(t+\tau)\,dt$$

TABLE 3.3

$E_g = \int_{-\infty}^{\infty} g^2(t)\,dt$	$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g^2(t)\,dt = \lim_{T\to\infty} \frac{E_{g_T}}{T}$
$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)\,dt$	$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)g(t+\tau)\,dt = \lim_{T\to\infty} \frac{\psi_{g_T}(\tau)}{T}$
$\Psi_g(f) = \lvert G(f)\rvert^2$	$S_g(f) = \lim_{T\to\infty} \frac{\lvert G_T(f)\rvert^2}{T} = \lim_{T\to\infty} \frac{\Psi_{g_T}(f)}{T}$
$\psi_g(\tau) \Longleftrightarrow \Psi_g(f)$	$\mathcal{R}_g(\tau) \Longleftrightarrow S_g(f)$
$E_g = \int_{-\infty}^{\infty} \Psi_g(f)\,df$	$P_g = \int_{-\infty}^{\infty} S_g(f)\,df$

overbar notation because it is much easier to associate means with a bar on top than with brackets. Using this notation, we see that

$$P_g = \widetilde{g^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g^2(t)\,dt \tag{3.86a}$$

Note that the rms value of a signal is the square root of its mean square value. Therefore,

$$[g(t)]\text{rms} = \sqrt{P_g} \tag{3.86b}$$

From Eqs. (3.82), it is clear that for a real signal $g(t)$, the time autocorrelation function $\mathcal{R}_g(\tau)$ is the time mean of $g(t)g(t \pm \tau)$. Thus,

$$\mathcal{R}_g(\tau) = \widetilde{g(t)g(t \pm \tau)} \tag{3.87}$$

This discussion also explains why we have been using "time autocorrelation" rather than just "autocorrelation". This is to distinguish clearly the present autocorrelation function (a time average) from the statistical autocorrelation function (a statistical average) to be introduced in Chapter 9 in the context of probability theory and random processes.

Interpretation of Power Spectral Density

Because the PSD is the time average of the ESD of $g(t)$, we can argue along the lines used in the interpretation of ESD. We can readily show that the PSD $S_g(f)$ represents the power per unit bandwidth (in hertz) of the spectral components at the frequency f. The amount of power contributed by the spectral components within the band f_1 to f_2 is given by

$$\Delta P_g = 2\int_{f_1}^{f_2} S_g(f)\,df \tag{3.88}$$

Autocorrelation Method: A Powerful Tool

For a signal $g(t)$, the ESD, which is equal to $|G(f)|^2$, can also be found by taking the Fourier transform of its autocorrelation function. If the Fourier transform of a signal is enough to determine its ESD, then why do we needlessly complicate our lives by talking about autocorrelation functions? The reason for following this alternate route is to lay a foundation for dealing with power signals and random signals. The Fourier transform of a power signal generally does not exist. Moreover, the luxury of finding the Fourier transform is available only for deterministic signals, which can be described as functions of time. The random message signals that occur in communication problems (e.g., random binary pulse train) cannot be described as functions of time, and it is impossible to find their Fourier transforms. However, the autocorrelation function for such signals can be determined from their statistical information. This allows us to determine the PSD (the spectral information) of such a signal. Indeed, we may consider the autocorrelation approach to be the generalization of Fourier techniques to power signals and random signals. The following example of a random binary pulse train dramatically illustrates the power of this technique.

Example 3.19 Figure 3.37a shows a random binary pulse train $g(t)$. The pulse width is $T_b/2$, and one binary digit is transmitted every T_b seconds. A binary **1** is transmitted by the positive pulse, and a binary **0** is transmitted by the negative pulse. The two symbols are equally likely and occur randomly. We shall determine the autocorrelation function, the PSD, and the essential bandwidth of this signal.

We cannot describe this signal as a function of time because the precise waveform, being random, is not known. We do, however, know its behavior in terms of the averages (the statistical information). The autocorrelation function, being an average parameter (time average) of the signal, is determinable from the given statistical (average) information. We have [Eq. (3.82a)]

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t-\tau)\,dt$$

Figure 3.37b shows $g(t)$ by solid lines and $g(t-\tau)$, which is $g(t)$ delayed by τ, by dashed lines. To determine the integrand on the right-hand side of the preceding equation, we multiply $g(t)$ with $g(t-\tau)$, find the area under the product $g(t)g(t-\tau)$, and divide it by the averaging interval T. Let there be N bits (pulses) during this interval T so that $T = NT_b$, and as $T \to \infty$, $N \to \infty$. Thus,

$$\mathcal{R}_g(\tau) = \lim_{N\to\infty} \frac{1}{NT_b} \int_{-NT_b/2}^{NT_b/2} g(t)g(t-\tau)\,dt$$

Let us first consider the case of $\tau < T_b/2$. In this case there is an overlap (shaded region) between each pulse of $g(t)$ and of $g(t-\tau)$. The area under the product $g(t)g(t-\tau)$ is $T_b/2 - \tau$ for each pulse. Since there are N pulses during the averaging interval, the total area under $g(t)g(t-\tau)$ is $N(T_b/2 - \tau)$, and

$$\begin{aligned}\mathcal{R}_g(\tau) &= \lim_{N\to\infty} \frac{1}{NT_b}\left[N\left(\frac{T_b}{2} - \tau\right)\right] \\ &= \frac{1}{2}\left(1 - \frac{2\tau}{T_b}\right) \qquad \tau < \frac{T_b}{2}\end{aligned}$$

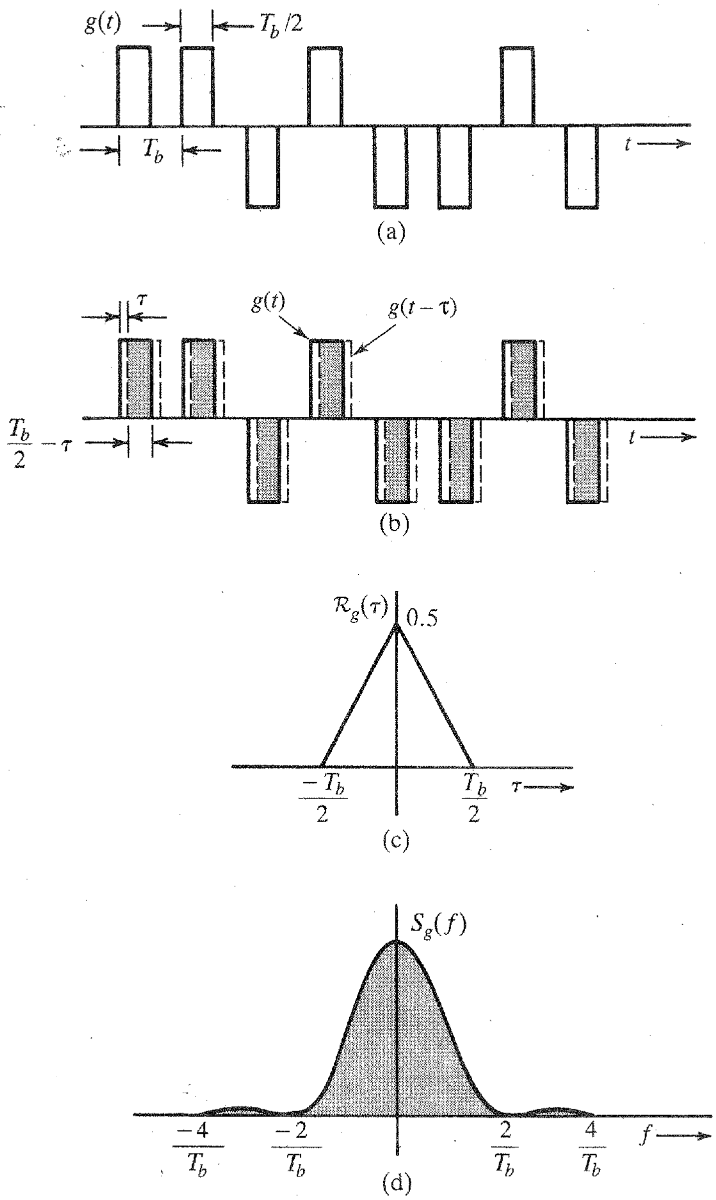

Figure 3.37 Autocorrelation function and power spectral density function of a random binary pulse train.

Because $\mathcal{R}_g(\tau)$ is an even function of τ,

$$\mathcal{R}_g(\tau) = \frac{1}{2}\left(1 - \frac{2|\tau|}{T_b}\right) \qquad |\tau| < \frac{T_b}{2} \tag{3.89a}$$

as shown in Fig. 3.37c.

As we increase τ beyond $T_b/2$, there will be overlap between each pulse and its immediate neighbor. The two overlapping pulses are equally likely to be of the same polarity or of opposite polarity. Their product is equally likely to be 1 or -1 over the overlapping interval. On the average, half the pulse products will be 1 (positive-positive or negative-negative pulse combinations), and the remaining half pulse products will be -1 (positive-negative or negative-positive combinations). Consequently, the area under $g(t)g(t-\tau)$ will be zero when averaged over an infinitely large time ($T \rightarrow \infty$), and

$$\mathcal{R}_g(\tau) = 0 \qquad |\tau| > \frac{T_b}{2} \tag{3.89b}$$

The two parts of Eq. (3.89) show that the autocorrelation function in this case is the triangular function $\frac{1}{2}\Delta(t/T_b)$ shown in Fig. 3.37c. The PSD is the Fourier transform of $\frac{1}{2}\Delta(t/T_b)$, which is found in Example 3.13 (or Table 3.1, pair 19) as

$$S_g(f) = \frac{T_b}{4}\operatorname{sinc}^2\left(\frac{\pi f T_b}{2}\right) \tag{3.90}$$

The PSD is the square of the sinc function, as shown in Fig. 3.37d. From the result in Example 3.18, we conclude that 90.28% of the area of this spectrum is contained within the band from 0 to $4\pi/T_b$ rad/s, or from 0 to $2/T_b$ Hz. Thus, the essential bandwidth may be taken as $2/T_b$ Hz (assuming a 90% power criterion). This example illustrates dramatically how the autocorrelation function can be used to obtain the spectral information of a (random) signal when conventional means of obtaining the Fourier spectrum are not usable.

3.8.3 Input and Output Power Spectral Densities

Because the PSD is a time average of ESDs, the relationship between the input and output signal PSDs of a linear time-invariant (LTI) system is similar to that of ESDs. Following the argument used for ESD [Eq. (3.75)], we can readily show that if $g(t)$ and $y(t)$ are the input and output signals of an LTI system with transfer function $H(f)$, then

$$S_y(f) = |H(f)|^2 S_g(f) \tag{3.91}$$

Example 3.20 A noise signal $n_i(t)$ with PSD $S_{n_i}(f) = K$ is applied at the input of an ideal differentiator (Fig. 3.38a). Determine the PSD and the power of the output noise signal $n_o(t)$.

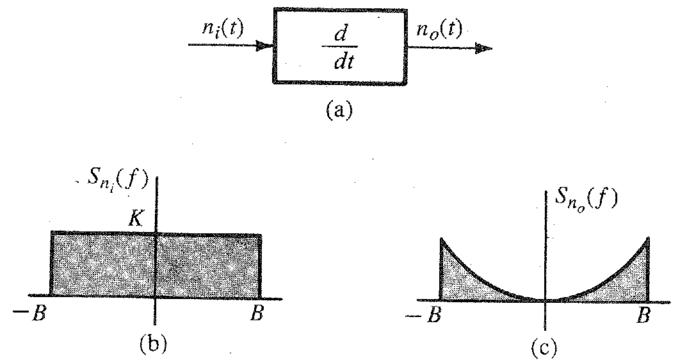

Figure 3.38 Power spectral densities at the input and the output of an ideal differentiator.

The transfer function of an ideal differentiator is $H(f) = j2\pi f$. If the noise at the demodulator output is $n_o(t)$, then from Eq. (3.91),

$$S_{n_o}(f) = |H(f)|^2 S_{\tilde{n}_i}(f) = |j2\pi f|^2 K$$

The output PSD $S_{n_o}(f)$ is parabolic, as shown in Fig. 3.38c. The output noise power N_o is the area under the output PSD. Therefore,

$$N_o = \int_{-B}^{B} K(2\pi f)^2\, df = 2K \int_0^B (2\pi f)^2\, df = \frac{8\pi^2 B^3 K}{3}$$

3.8.4 PSD of Modulated Signals

Following the argument in deriving Eqs. (3.70) and (3.71) for energy signals, we can derive similar results for power signals by taking the time averages. We can show that for a power signal $g(t)$, if

$$\varphi(t) = g(t) \cos 2\pi f_0 t$$

then the PSD $S_\varphi(f)$ of the modulated signal $\varphi(t)$ is given by

$$S_\varphi(f) = \frac{1}{4}\left[S_g(f + f_0) + S_g(f - f_0)\right] \tag{3.92}$$

The detailed derivation is provided in Sec. 7.8. Thus, modulation shifts the PSD of $g(t)$ by $\pm f_0$. The power of $\varphi(t)$ is half the power of $g(t)$, that is,

$$P_\varphi = \frac{1}{2} P_g \qquad f_0 \geq B \tag{3.93}$$

3.9 NUMERICAL COMPUTATION OF FOURIER TRANSFORM: THE DFT

To compute $G(f)$, the Fourier transform of $g(t)$, numerically, we have to use the samples of $g(t)$. Moreover, we can determine $G(f)$ only at some finite number of frequencies. Thus, we can compute only samples of $G(f)$. For this reason, we shall now find the relationships between samples of $g(t)$ and samples of $G(f)$.

In numerical computations, the data must be finite. This means that the number of samples of $g(t)$ and $G(f)$ must be finite. In other words, we must deal with time-limited signals. If the signal is not time-limited, then we need to truncate it to make its duration finite. The same is true of $G(f)$. To begin, let us consider a signal $g(t)$ of duration τ seconds, starting at $t = 0$, as shown in Fig. 3.39a. However, for reasons that will become clear as we go along, we shall consider the duration of $g(t)$ to be T_0, where $T_0 \geq \tau$, which makes $g(t) = 0$ in the interval $\tau < t \leq T_0$, as shown in Fig. 3.39a. Clearly, this makes no difference in the computation of $G(f)$. Let us take samples of $g(t)$ at uniform intervals of T_s seconds. There are a total of N_0 samples, where

$$N_0 = \frac{T_0}{T_s} \tag{3.94}$$