Radmila Bulajich Manfrino
José Antonio Gómez Ortega
Rogelio Valdez Delgado

Topics in Algebra and Analysis

Preparing for the Mathematical Olympiad

Birkhäuser

Birkhäuser

Radmila Bulajich Manfrino
José Antonio Gómez Ortega
Rogelio Valdez Delgado

Topics in Algebra and Analysis

Preparing for the Mathematical Olympiad

Birkhäuser

Radmila Bulajich Manfrino
Facultad de Ciencias
Universidad Autónoma del Estado de Morelos
Cuernavaca, Morelos, México

José Antonio Gómez Ortega
Facultad de Ciencias
Universidad Nacional Autónoma de México
Distrito Federal, México

Rogelio Valdez Delgado
Facultad de Ciencias
Universidad Autónoma del Estado de Morelos
Cuernavaca, Morelos, México

ISBN 978-3-319-11945-8 ISBN 978-3-319-11946-5 (eBook)
DOI 10.1007/978-3-319-11946-5

Library of Congress Control Number: 2015930195

Mathematics Subject Classification (2010): 00A07, 11B25, 11B65, 11C08, 39B22, 40A05, 97Fxx

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

Introduction

Topics in algebra and analysis have become fundamental for the mathematical olympiad. Today, the problems in these topics that appear in the contests are frequent, and the problems from other areas that use algebra and analysis in their solutions are also frequent. In this book, we want to point out the principal algebra and analysis tools that a student must assimilate and learn to use gradually in training for mathematical contests and olympiads. Some of the topics that we study in the book are also part of the mathematical syllabus in high school courses, but there are other topics that are presented at the college level. That is, the book can be used as a reference text for undergraduates in the first year of college who will be facing algebra and analysis problems and will be interested in learning techniques to solve them.

The book is divided in ten chapters. The first four correspond to topics from high school and they are basic for the students that are training for the mathematical olympiad contest, at a local and national level. The next four chapters are usually studied in the first year of college, but they have become fundamental tools, for the students competing in an international level. The last two chapters contain the problems and solutions of the theory studied in the book.

The first chapter covers the basic algebra, as are the numerical systems, absolute value, notable products, and factorization, among others. We expect that the reader gain some skills for the manipulation of equations and algebraic formulae to carry them in equivalent forms, which are easier to understand and work with them.

In Chapter 2 the study of the finite sums of numbers is presented, for instance, the sum of the squares of the first n natural numbers. The telescopic sums, arithmetic and geometric progressions are analyzed, as well as some of its properties.

Chapter 3 talks about the mathematical technique to prove mathematical statements that involve natural numbers, known as the principle of mathematical induction. Its use is exemplified with several problems. Many equivalent statements of the principle of mathematical induction are presented.

To complete the first part of the book, in Chapter 4 the quadratic and cubic polynomials are studied, with emphasis in the study of the discriminant of a quadratic polynomial and Vieta's formulas for these two classes of polynomials.

The second part of the text begins with Chapter 5, where the complex numbers are studied, as well as its properties and some applications are given. All these with examples related to mathematical olympiad problems. In addition, a proof of the fundamental theorem of algebra is included.

In Chapter 6, the principal properties of functions are studied. Also, there is an introduction to the functional equations theory, its properties and a series of recommendations are given to solve the problems where appear functional equations.

Chapter 7 talks about the notion of sequence and series. Special sequences are studied as bounded, periodic, monotone, recursive, among others. In addition, the concept of convergence for sequences and series is introduced.

In Chapter 8, the study of polynomials from the first part of the book is generalized. The theory of polynomials of arbitrary degree is presented, as well as several techniques to analyze properties of the polynomials. At the end of the chapter, the polynomials of several variables are studied. Most of the sections of these first eight chapters have at the end a list of exercises for the reader, selected and suitable to practice the topics in the corresponding sections. The difficulty of the exercises vary from being a direct application of a result seen in the section to being a contest problem that with the technique studied is possible to solve.

Chapter 9 is a collection of problems, each one of them close to one or more of the topics seen in the book. These problems have a degree of difficulty greater than the exercises. Most of the problems have appeared in some mathematical contests around the world or olympiads. In the solution of each problem is implicit the knowledge and skills that are need to manipulate algebraic expressions.

Finally, Chapter 10 contains the solutions to all exercises and problems presented in the book. The reader can notice that at the end of some sections there is a $\star$ symbol, this means that the level of the section is harder than others sections. In a first lecture, the reader can skip these sections; however, it is recommended that the reader have them in mind for the techniques used in them.

We thank Leonardo Ignacio Martínez Sandoval and Rafael Martínez Enríquez for his always-helpful comments and suggestions, which contribute to the improvement of the material presented in this book.

Contents

Chapter 1

Preliminaries

1.1 Numbers

We will assume that the reader is familiar with the notion of the set of numbers that we usually use to count. This set is called the set of natural numbers and it is usually denoted by $\mathbb{N}$, that is,

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

In this set we have two operations, the sum and the product, that is, if we add or multiply two numbers in the set we obtain a natural number. In some books 0 is considered a natural number, however, in this book it is not, but we will suppose that 0 is such that $n + 0 = n$, for every natural number n.

Now, suppose that we want to solve the equation $x + a = 0$, with $a \in \mathbb{N}$, that is, we want to find an x such that the equality is true. This equation does not have a solution in the set of natural numbers $\mathbb{N}$, therefore we need to define another set which includes the set of numbers $\mathbb{N}$ but also the negative numbers. In other words, we need to extend the set of numbers $\mathbb{N}$ in order that such equation can be solved in the new set. This new set is called the set of integers and is denoted by $\mathbb{Z}$, that is,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

In this set we also have two operations, the sum and the product, which satisfy the following properties.

Properties 1.1.1.

(a) *The sum and the product of integers are commutative. That is, if* a, $b \in \mathbb{Z}$, *then*

$$a + b = b + a \quad \textit{and} \quad ab = ba.$$

(b) *The sum and the product of integers are associative. That is, if* a, b *and* $c \in \mathbb{Z}$, *then*

$$(a + b) + c = a + (b + c) \quad \textit{and} \quad (ab)c = a(bc).$$

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_1

(c) *There exists in* $\mathbb{Z}$ *a neutral element for the sum, the number* 0*. That is, if* $a \in \mathbb{Z}$*, then*

$$a + 0 = 0 + a = a.$$

(d) *There exists in* $\mathbb{Z}$ *a neutral element for the multiplication, the number* 1*. That is, if* $a \in \mathbb{Z}$*, then*

$$a1 = 1a = a.$$

(e) *For each* $a \in \mathbb{Z}$ *there exists an inverse element under the sum which is denoted by* $-a$*. That is,*

$$a + (-a) = (-a) + a = 0.$$

(f) *In* $\mathbb{Z}$ *a distributive law holds, in which addition and multiplication are involved. That is, if* a, b *and* $c \in \mathbb{Z}$*, then*

$$a(b + c) = ab + ac.$$

Note that the existence of the additive inverse allows us to solve equations of the type mentioned above, that is, $x + a = b$, where a and b are integers. However, there is not necessarily an integer number x that solves the equation $qx = p$, with p and q integers. Therefore the necessity arises to extend the set of integers. Consider the set of rational numbers, which is denoted by $\mathbb{Q}$, defined as

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z}\backslash\{0\} \right\}.$$

In general, when working with the rational number $\frac{p}{q}$ we ask that p and q do not have common prime factors, that is, the numbers are relative primes, which is denoted by $(p, q) = 1$. In the set of rational numbers we also have two operations, the sum and the product, which satisfy all the properties valid in the set of integers, but in the case of the product we have an extra property, the multiplicative inverse element.

Property 1.1.2. *If* $\frac{p}{q} \in \mathbb{Q}$*, with* $p \neq 0$ *and* $(p, q) = 1$*, then there exists a unique number,* $\frac{q}{p} \in \mathbb{Q}$*, which is called the multiplicative inverse of* $\frac{p}{q}$*, such that*

$$\frac{p}{q} \cdot \frac{q}{p} = 1.$$

Using this property we can solve equations of the form $qx = p$, however, there are numbers that we cannot write as the quotient of two integers. For example, if we want to solve the equation $x^2 - 2 = 0$, this equation does not have a solution in the set $\mathbb{Q}$. We write the solutions of the equation as $x = \pm\sqrt{2}$ and proceed to prove that $\sqrt{2}$ is not in $\mathbb{Q}$.

Proposition 1.1.3. *The number* $\sqrt{2}$ *is not a rational number.*

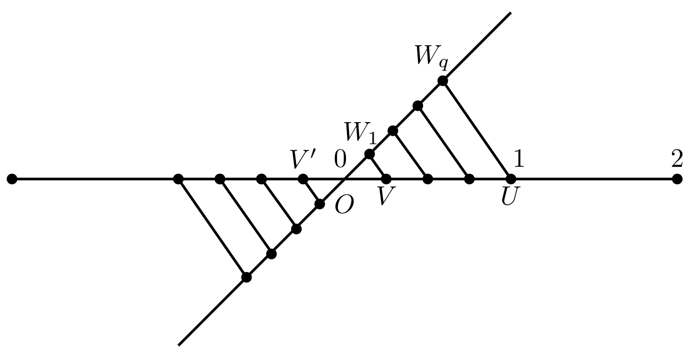
Proof. Suppose the contrary, that is, $\sqrt{2}$ is a rational number. Therefore, it can be written as $\sqrt{2} = \frac{p}{q}$, where p and q do not have common factors. Squaring both sides of the equation we get $2 = \frac{p^2}{q^2}$, that is, $2q^2 = p^2$. This means that

p^2 is an even number, but then p is also even. But if p is even, p is of the form $p = 2m$, then $2q^2 = (2m)^2 = 4m^2$. Dividing by 2 both sides of the equation, give us $q^2 = 2m^2$, that is, q^2 is even and therefore q is also even. Hence, p and q are even numbers, which contradicts the fact that p and q were assumed as not having common factors. Thus, $\sqrt{2}$ is not a rational number. $\square$

We can give a geometric representation of the rational numbers as points on a straight line, which in this case is called the **number line**. A straight line can be travelled in two directions, one which we call the **positive direction** and the other the **negative direction**. Once we have agreed about which of the directions is to be taken as positive we talk of an **oriented straight line**. For example, we can decide that the positive direction goes from left to right. If we consider two points O and U on the straight line, we will give the same orientation to a line segment contained on the straight line. That is, if we let the point O be 0 and U be on the right of it, we will say that the line segment OU is traversed in the positive direction. If U represents the point 1, we will call OU an unitary oriented straight line segment. In this way, we place, to the right of 0, what we take to be all the positive integers equally spaced along the line, that is, two consecutive integers are spaced a distance equal to the length of the segment OU. To represent the negative numbers it is sufficient to do the same, starting at O and traversing the straight line in the opposite direction.

The rational number $\frac{p}{q}$ is defined as the oriented segment $\frac{p}{q}OU$. This segment is obtained when we sum p times the qth part of the segment OU. More precisely, we do the following:

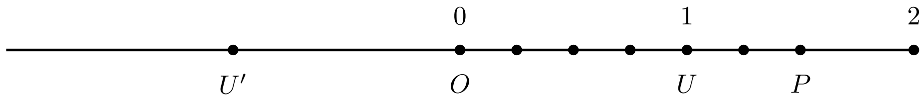
(a) Divide the segment OU into q equal parts. To do this, we make use of an extra straight line passing through O and not perpendicular to OU, and in this line we take q points $W_1, \ldots, W_q$, where two consecutive points are separated a distance OW_1. Now, we draw the line segment from W_q to U and for each W_j we draw a parallel straight line to UW_q, the intersection points of the parallels with OU will divide OU in q equal parts. If V is the intersection point of the parallel to UW_q through W_1, we have that V is the point that represents the number $\frac{1}{q}$ (note that OV has the same orientation as OU). We also consider V' the symmetric point to V, with respect to O. In the following figure, we took $q = 4$.



(b) If p is a non-negative integer, take

$$OP = \underbrace{OV + OV + \cdots + OV}_{p \text{ times}} = p \cdot OV.$$

The segment OP is, by definition, $\frac{p}{q}OU$. In the next figure we have marked the point P, with $p = 6$ and $q = 4$.

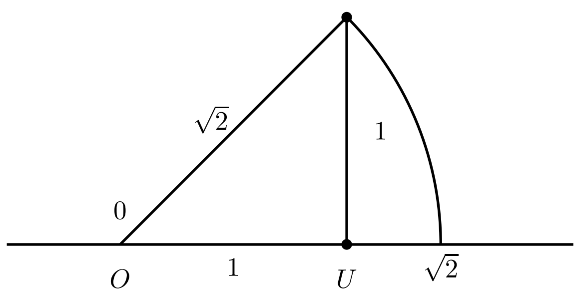


(c) If p is negative, let p' be the positive integer such that $p = -p'$. Then

$$OP = \underbrace{OV' + OV' + \cdots + OV'}_{p' \text{ times}} = p'OV' = (-p')OV = p \cdot OV.$$

The segment OP is, by definition, $\frac{p}{q}OU$. Since OU is the unitary segment, this point is simply denoted by $\frac{p}{q}$.

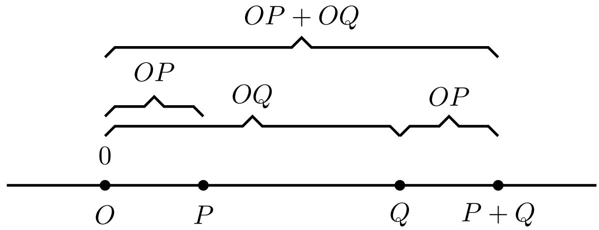
With this representation of the rational numbers, we have that every rational number defines a point on the number line, but there are points on the number line that are not represented by a rational number. For example, we want to determine, on the number line, which point represents the number $\sqrt{2}$, which we proved above is not a rational number. To do this, take the right triangle whose legs are each equal to 1. Then, by the Pythagorean theorem, the hypotenuse of the triangle is equal to $\sqrt{2}$. If we take a compass and draw a circle of radius $\sqrt{2}$ and center at 0, the point where the circle intersects the positive part of the number line corresponds to $\sqrt{2}$.



A point on the number line that does not correspond to a rational number represents an irrational number, and the set of irrational numbers is denoted by $\mathbb{I}$.

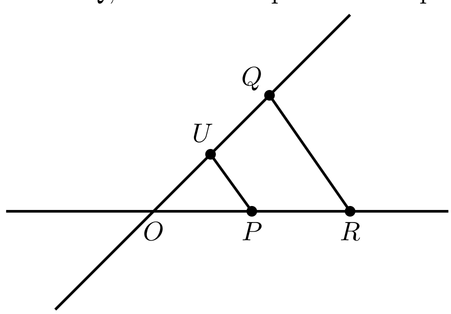
The union of these two sets is called the set of real numbers and is denoted by $\mathbb{R}$, that is, $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
The set of real numbers $\mathbb{R}$ contains the set of natural numbers, the set of integers and the set of rational numbers. In fact, we have the following chain of inclusions $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Given two points on the number line that we know represent two real numbers, we can find the point which represents the sum of these two numbers in the following way: if P and Q are two points on the straight line and O is the origin, the sum will be the addition of the oriented segments OP and OQ, as we can see in the following figure.



We can also find the point that represents the product of two points P and Q which are on the number line. In order to do that we consider an extra straight line passing through the origin O and not perpendicular to the line OP. We mark on the extra straight line the unity U and the point Q. Through Q we draw a parallel line to UP which will intersect the real line at a point R.

Since the triangles ORQ and OPU are similar it follows that $\frac{OR}{OP} = \frac{OQ}{OU}$, therefore $OR \cdot OU = OP \cdot OQ$, hence OR represents the product of P and Q.



With this geometric representation of the numbers it is easy to find, in the real line, the sum and the product of any two real numbers, no matter whether they are rational or irrational numbers.

Similarly, as it happens in the set of integers, the operations in the set of real numbers have the same properties.

Properties 1.1.4.

(a) *The sum of two real numbers is a real number.*
(b) *The sum of two real numbers is commutative.*
(c) *The sum is associative.*
(d) *The number* 0 *is called the additive neutral element. That is,* $x + 0 = x$ *for all* $x \in \mathbb{R}$.
(e) *Every real number* x *has an additive inverse. That is, there is a real number, which we denote by* $-x$, *such that* $x + (-x) = 0$.

(f) *The product of two real numbers is a real number.*
(g) *The product of two real numbers is commutative.*
(h) *The product is associative.*
(i) *The number* 1 *is the multiplicative neutral element. That is,* $x \cdot 1 = x$ *for all* $x \in \mathbb{R}$.
(j) *Every real number* x *different from* 0, *has a multiplicative inverse. That is, there exists a real number, which we denote by* x^{-1}, *such that* $x \cdot x^{-1} = 1$.
(k) *For any three real numbers* x, y, z *it follows that*

$$x\,(y+z) = x \cdot y + x \cdot z.$$

This property is called the distributive law.

In the set of integers there is an order. With this we want to point out that given two integers a and b we can say which one is greater. We say that a **is greater than** b if $a - b$ is a natural number. In symbols we have that

$$a > b \ \text{ if and only if } \ a - b \in \mathbb{N}.$$

This is equivalent to saying that $a - b > 0$.
In general, the notation $a > b$ is equivalent to $b < a$. The expression $a \geq b$ means that $a > b$ or $a = b$. Similarly, $a \leq b$ means that $a < b$ or $a = b$.

Properties 1.1.5. *If* a *is an integer number, one and only one of the following relations holds:*

(a) $a > 0$, (b) $a = 0$, (c) $a < 0$.

In the set of rational numbers and in the set of real numbers, we also have the order properties. The order of the real numbers enables us to compare two numbers and to decide which one of them is greater or whether they are equal. Let us assume that the real number system contains a set P, which we will call the set of positive numbers, and we will express in symbols $a > 0$ if a belongs to P.

In the geometric representation of the real numbers, the set P in the number line is, of the two pieces in which O has divided the straight line, the piece which contains U (the number 1). The following properties are satisfied.

Properties 1.1.6. *Every real number* x *has one and only one of the following properties:*

(a) $x = 0$. (b) $x \in P$, *that is,* $x > 0$. (c) $-x \in P$, *that is,* $-x > 0$.

Properties 1.1.7.

(a) *If* x, $y \in P$, *then* $x + y \in P$ *(in symbols* $x > 0$, $y > 0$, *then* $x + y > 0$*).*
(b) *If* x, $y \in P$, *then* $xy \in P$ *(in symbols* $x > 0$, $y > 0$, *then* $xy > 0$*).*

We will denote by $\mathbb{R}^+$ the set P of positive real numbers.

Now we can define the relation x **is greater than** y, by saying that it holds $x - y \in P$ (in symbols $x > y$). Similarly, x **is smaller than** y, if $y - x \in P$ (in symbols $x < y$). Observe that $x < y$ is equivalent to $y > x$. We can also define that x **is smaller than or equal to** y, if $x < y$ or $x = y$, (using symbols $x \leq y$).

Example 1.1.8.

(a) *If $x < y$ and z is any number, then $x + z < y + z$.*
(b) *If $x < y$ and $z > 0$, then $xz < yz$.*

In fact, to prove (a) we see that $x + z < y + z$ if and only if $(y + z) - (x + z) > 0$ if and only if $y - x > 0$ if and only if $x < y$. To prove (b), we proceed as follows: $x < y$ implies that $y - x > 0$, and since $z > 0$, then $(y - x)z > 0$, therefore $yz - xz > 0$, hence $xz < yz$.

Exercise 1.1. *Prove the following statements:*

(i) *If $a < 0$, $b < 0$, then $ab > 0$.*
(ii) *If $a < 0$, $b > 0$, then $ab < 0$.*
(iii) *If $a < b$, $b < c$, then $a < c$.*
(iv) *If $a < b$, $c < d$, then $a + c < b + d$.*
(v) *If $a > 0$, then $a^{-1} > 0$.*
(vi) *If $a < 0$, then $a^{-1} < 0$.*

Exercise 1.2. *Let a, b be real numbers. Prove that, if $a + b$, $a^2 + b$ and $a + b^2$ are rational numbers, and $a + b \neq 1$, then a and b are rational numbers.*

Exercise 1.3. *Let a, b be real numbers such that $a^2 + b^2$, $a^3 + b^3$ and $a^4 + b^4$ are rational numbers. Prove that $a + b$, ab are also rational numbers.*

Exercise 1.4.

(i) *Prove that if p is a prime number, then $\sqrt{p}$ is an irrational number.*
(ii) *Prove that if m is a positive integer which is not a perfect square, then $\sqrt{m}$ is an irrational number.*

Exercise 1.5. *Prove that there are an infinite number of pairs of irrational numbers a, b such that $a + b = ab$ is an integer number.*

Exercise 1.6. *If the coefficients of*

$$ax^2 + bx + c = 0$$

are odd integers, then the roots of the equation cannot be rational numbers.

Exercise 1.7. *Prove that for real numbers a and b, it follows that*

$$\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

Exercise 1.8. *For positive numbers a and b, find the value of:*

(i) $\sqrt{a\sqrt{a\sqrt{a\sqrt{a\dots}}}}$ (ii) $\sqrt{a\sqrt{b\sqrt{a\sqrt{b\dots}}}}$

Exercise 1.9 (Romania, 2001)**.** *Let x, y and z be non-zero real numbers such that xy, yz and zx are rational numbers. Prove that:*

(i) *$x^2 + y^2 + z^2$ is a rational number.*
(ii) *If $x^3 + y^3 + z^3$ is a rational number different from zero, then x, y and z are rational numbers.*

Exercise 1.10 (Romania, 2011)**.** *Let a, b be different real positive numbers, such that $a-\sqrt{ab}$ and $b-\sqrt{ab}$ are both rational numbers. Prove that a and b are rational numbers.*

The decimal system is a positional system in which every digit takes a value based on its position with respect to the decimal point. That is, the digit is multiplied by a power of 10 according to the position it occupies. For the units digit, that is, the digit which is just to the left-hand side of the decimal point, it is multiplied by 10^n, with $n = 0$. Accordingly, the digit in the tens position must be multiplied by $10^1 = 10$. The exponent increases one by one when we move from right to left and decreases one by one when we move in the other direction. For example,

$$87325.31 = 8 \cdot 10^4 + 7 \cdot 10^3 + 3 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0 + 3 \cdot 10^{-1} + 1 \cdot 10^{-2}.$$

In general, every real number can be written as an infinite decimal expansion in the following way

$$b_m \dots b_1 b_0.a_1 a_2 a_3 \dots,$$

where b_i and a_i belong to $\{0, 1, \dots, 9\}$, for every i. The symbol $\dots$ means that on the right-hand side of the number we can have an infinite number of digits, in this way the number $b_m \dots b_1 b_0.a_1 a_2 a_3 \dots$, represents the real number

$$b_m \cdot 10^m + \cdots + b_1 \cdot 10 + b_0 \cdot 10^0 + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + \cdots .$$

For example,

$$\tfrac{1}{3} = 0.3333\dots, \qquad \tfrac{3}{7} = 0.428571428571\dots,$$

$$\tfrac{1}{2} = 0.50000\dots, \quad \sqrt{2} = 1.4142135\dots.$$

With this notation we can distinguish between rational and irrational numbers. The rational numbers are those in which the decimal expansion is finite or infinite, but there is, always, a certain number of digits which, after a certain point, repeat periodically, for example the number $\frac{34}{275} = 0.123636\dots$ is periodic of period 2 after the third digit. Meanwhile, for the irrational numbers, the decimal expansion is infinite and never becomes periodic.

This manner of representing numbers is essential to solve some of the problems that appear in the mathematical olympiad.
In the same way, as in the decimal representation of the numbers in base 10, we can represent the integers in any base.
If m is a positive integer, we can find its representation in base b if we write the number as a sum of powers of b, that is, $m = a_r b^r + \cdots + a_1 b + a_0$. The integers which appear as coefficients of the powers of b in the representation must be smaller than b.

Observation 1.1.9. *To identify a number which is not written in base* 10*, we will use a subindex indicating the base, for example,* 1204_7 *means that the number* 1204 *is a number expressed using base* 7.

Let us analyze the following example.

Example 1.1.10. *In which base is the number* 221 *a factor of* 1215*?*

The number 1215 in base a is written as a^3+2a^2+a+5 and the number 221 in base a is $2a^2+2a+1$. Therefore, if we divide a^3+2a^2+a+5 by $2a^2+2a+1$ we obtain that

$$a^3+2a^2+a+5 = (2a^2+2a+1)\left(\frac{1}{2}a+\frac{1}{2}\right)+\left(-\frac{1}{2}a+\frac{9}{2}\right).$$

Therefore, since 1215_a has to be a multiple of 221_a, the remainder $\left(-\frac{1}{2}a+\frac{9}{2}\right)$ has to be 0 and $\frac{1}{2}a+\frac{1}{2}$ has to be an integer number. Both conditions are satisfied if $a = 9$.

Exercise 1.11. *Write the following numbers as* $\frac{m}{n}$*, where* m *and* n *are positive integers:*
(i) $0.11111\ldots$ (ii) $1.14141414\ldots$.

Exercise 1.12.

(i) *Prove that* 121_b *is a perfect square in any base* $b \geq 2$.
(ii) *Determine the smallest value of* b *such that* 232_b *is a perfect square.*

Exercise 1.13 (IMO, 1970)**.** *Let* a*,* b *and* n *be positive integers greater than* 1*. Let* A_{n-1} *and* A_n *be numbers expressed in the numerical system in base* a *and* B_{n-1} *and* B_n *be two numbers in the numerical system in base* b*. These numbers are related in the following way,*

$$\begin{aligned} A_n &= x_n x_{n-1}\ldots x_0, & A_{n-1} &= x_{n-1}x_{n-2}\ldots x_0,\\ B_n &= x_n x_{n-1}\ldots x_0, & B_{n-1} &= x_{n-1}x_{n-2}\ldots x_0,\end{aligned}$$

with $x_n \neq 0$ *and* $x_{n-1} \neq 0$*. Prove that* $a > b$ *if and only if*

$$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}.$$

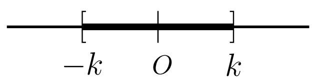
1.2 Absolute value

We define the **absolute value** of a real number x as

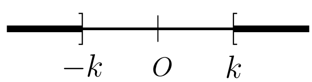
$$|x| = \begin{cases} x, & \text{if} \quad x \geq 0 \\ -x, & \text{if} \quad x < 0. \end{cases} \tag{1.1}$$

For a real non-negative number k, the identity $|x| = k$ is satisfied only by $x = k$ and $x = -k$.

The inequality $|x| \leq k$ is equivalent to $-k \leq x \leq k$. This can be seen as follows: If $x \geq 0$, then $0 \leq x = |x| \leq k$. On the other hand, if $x \leq 0$, then $-x = |x| \leq k$, therefore, $x \geq -k$. As a consequence of the previous discussion, we observe that $x \leq |x|$. In the next figure we show the values of x which satisfy the inequality. These being the values lying between $-k$ and k, and including the two numbers. The set $[-k, k] = \{x \in \mathbb{R} \mid -k \leq x \leq k\}$ is called a **closed interval**, since it contains the numbers k and $-k$. The numbers $-k$, k are called the **endpoints** of the interval.



Similarly, the inequality $|x| \geq k$ is equivalent to $x \geq k$ or $-x \geq k$. In the next figure, the values of x that satisfy the inequalities are the values falling before $-k$ and $-k$ itself, or k and those values to the right of k. The set $(-k, k) = \{x \in \mathbb{R} \mid -k < x < k\}$ is called an **open interval**, since it does not contain either k or $-k$, that is, an open interval is an interval that does not contain its endpoints. With this definition, we see that the set of values x that satisfy $|x| \geq k$ are the values $x \notin (-k, k)$.



Example 1.2.1. *Find in the Cartesian plane*[1] *the area enclosed by the graph of the relation* $|x| + |y| = 1$.

For $|x| + |y| = 1$ we have to consider four cases:

(a) $x \geq 0$ and $y \geq 0$; this implies $x + y = 1$, that is, $y = 1 - x$.

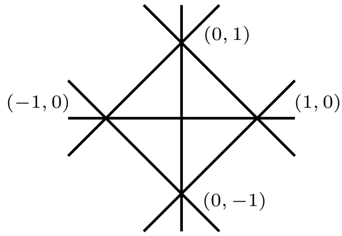
(b) $x \geq 0$ and $y < 0$; this implies $x - y = 1$, that is, $y = x - 1$.

(c) $x < 0$ and $y \geq 0$; this implies $-x + y = 1$, that is, $y = x + 1$.

(d) $x < 0$ and $y < 0$; this implies $-x - y = 1$, that is, $y = -x - 1$.

[1]The **Cartesian plane** is defined as $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R},\, y \in \mathbb{R}\}$.

We can now draw the graph of the four straight lines.



The area enclosed by the four straight lines is formed by four isosceles right triangles, each of which has two sides of length equal to 1. Since the area of each of these triangles is $\frac{1\times 1}{2} = \frac{1}{2}$, the area of the square is $4\left(\frac{1}{2}\right) = 2$.

Example 1.2.2. *Solve the equation* $|2x - 4| = |x + 5|$.

We have that

$$|2x-4| = \begin{cases} 2x-4, & \text{if} \quad x \geq 2 \\ -2x+4, & \text{if} \quad x < 2. \end{cases}$$

We also have that

$$|x+5| = \begin{cases} x+5, & \text{if} \quad x \geq -5 \\ -x-5, & \text{if} \quad x < -5. \end{cases}$$

If $x \geq 2$, then $2x-4 = x+5$, that is, $x = 9$. If $x < -5$, then $-2x+4 = -x-5$, hence $x = 9$, but this is impossible since $x < -5$. The last case that we have to consider is $-5 \leq x < 2$. Then, the equation that we have to solve is $-2x + 4 = x + 5$. Solving for x we get $x = -\frac{1}{3}$. Therefore, the numbers which satisfy the equation are $x = 9$ and $x = -\frac{1}{3}$.

Sometimes it is easier to solve these equations without using the explicit form of the absolute value, just by observing that $|a| = |b|$ if and only if $a = \pm b$ and making use of the absolute value properties.

Observation 1.2.3. *If x is any real number then the relation between the square root and the absolute value is given by $\sqrt{x^2} = |x|$. The identity follows from $|x|^2 = x^2$ and $|x| \geq 0$.*

Properties 1.2.4. *If x and y are real numbers the following relations hold:*

(a) $|xy| = |x||y|$. *This implies also that* $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, *if* $y \neq 0$.

(b) $|x + y| \leq |x| + |y|$, *where the equality holds if and only if* $xy \geq 0$.

Proof. (a) The proof follows directly from $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2$, and taking the square root on both sides gives the result.

(b) Since both sides of the inequality are positive numbers, it is enough to verify that $|x+y|^2 \leq (|x|+|y|)^2$.

$$\begin{aligned} |x+y|^2 &= (x+y)^2 = x^2 + 2xy + y^2 = |x|^2 + 2xy + |y|^2 \\ &\leq |x|^2 + 2\,|xy| + |y|^2 = |x|^2 + 2\,|x|\,|y| + |y|^2 = (|x|+|y|)^2 . \end{aligned}$$

In the previous chain of relations there is only one inequality, it follows trivially from the fact that $xy \leq |xy|$. Moreover, the equality holds if and only if $xy = |xy|$, which is true only when $xy \geq 0$. □

Inequality (b) in Properties 1.2.4 can be extended in a general form as

$$|\pm x_1 \pm x_2 \pm \cdots \pm x_n| \leq |x_1| + |x_2| + \cdots + |x_n|,$$

for real numbers x_1, x_2, ..., x_n. The equality holds when all the $\pm x_i$'s have the same sign.

This last inequality can be proved in a similar way, and also using induction[2].

Exercise 1.14. *If a and b are any real numbers, prove that $||a| - |b|| \leq |a-b|$.*

Exercise 1.15. *Find, in each case, the numbers x that satisfy the following:*

(i) $|x-1| - |x+1| = 0$.
(ii) $|x-1||x+1| = 1$.
(iii) $|x-1| + |x+1| = 2$.

Exercise 1.16. *Find all the triplets (x, y, z) of real numbers satisfying*

$$\begin{aligned} |x+y| &\geq 1, \\ 2xy - z^2 &\geq 1, \\ z - |x+y| &\geq -1. \end{aligned}$$

Exercise 1.17 (OMM, 2004)**.** *Find the largest number of positive integers that can be found in such a way that any two of them, a and b (with $a \neq b$), satisfy the next inequality*

$$|a-b| \geq \frac{ab}{100}.$$

1.3 Integer part and fractional part of a number

Given any number $x \in \mathbb{R}$, sometimes it is useful to consider the integer number $\max\{k \in \mathbb{Z} \mid k \leq x\}$, that is, the greatest integer less than or equal to x. This number is denoted by $\lfloor x \rfloor$ and it is known as **the integer part of** x.

From this definition, the following properties hold.

[2]See Section 3.1.

Properties 1.3.1. *Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then it follows that:*

(a) $x - 1 < \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.
(b) *x is an integer if and only if $\lfloor x \rfloor = x$.*
(c) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.
(d) $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$.
(e) $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. The proof of the first three properties follows immediately from the definition.

(d) If we divide $\lfloor x \rfloor$ by n, we have that $\lfloor x \rfloor = an + b$, for an integer number a and for an integer number b such that $0 \leq b < n$.

On the one hand, we have that $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{an+b}{n} \right\rfloor = a + \left\lfloor \frac{b}{n} \right\rfloor = a$. On the other hand, since $x = \lfloor x \rfloor + c$, with $0 \leq c < 1$, and $b + c < n - 1 + 1 = n$, we get $\left\lfloor \frac{x}{n} \right\rfloor = \left\lfloor \frac{an+b+c}{n} \right\rfloor = a + \left\lfloor \frac{b+c}{n} \right\rfloor = a$. Then the equality holds.

(e) Since $x = \lfloor x \rfloor + a$ and $y = \lfloor y \rfloor + b$ with $0 \leq a, b < 1$, then $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor a + b \rfloor$ by property (c). The inequalities follow if we observe that if $0 \leq a, b < 1$, then $0 \leq \lfloor a + b \rfloor \leq 1$. $\square$

Example 1.3.2. *For any real number x, it follows that*

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor - \lfloor 2x \rfloor = 0.$$

If we let $n = \lfloor x \rfloor$, then x can be expressed as $x = n + a$ with $0 \leq a < 1$, hence

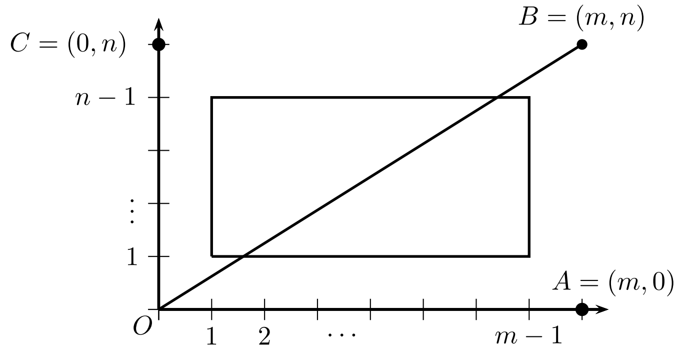
$$\begin{aligned}
\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor - \lfloor 2x \rfloor &= n + \left\lfloor n + a + \frac{1}{2} \right\rfloor - \lfloor 2(n + a) \rfloor \\
&= n + n + \left\lfloor a + \frac{1}{2} \right\rfloor - 2n - \lfloor 2a \rfloor \\
&= \left\lfloor a + \frac{1}{2} \right\rfloor - \lfloor 2a \rfloor ,
\end{aligned}$$

where the second equality follows by property (c). Now, if $0 \leq a < \frac{1}{2}$, then $\left\lfloor a + \frac{1}{2} \right\rfloor = \lfloor 2a \rfloor = 0$, meanwhile, if $\frac{1}{2} \leq a < 1$, it follows that $\left\lfloor a + \frac{1}{2} \right\rfloor = \lfloor 2a \rfloor = 1$.

Example 1.3.3. *If n and m are positive integers without common factors, then*

$$\left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + \cdots + \left\lfloor \frac{(m-1)n}{m} \right\rfloor = \frac{(m-1)(n-1)}{2}.$$

Consider, in the Cartesian plane, the straight line passing through the origin and the point (m, n). Since m and n are relative primes, then on the segment where points $(0,0)$ and (m,n) lie, there is no other point with integer coordinates.



The equation of the straight line is $y=\frac{n}{m}x$ and passes over the points $(j,\frac{n}{m}j)$, with $j=1,\dots,(m-1)$, and such that $\frac{n}{m}j$ is not an integer. The number $\lfloor\frac{n}{m}j\rfloor$ is equal to the number of points with integer coordinates lying on the straight line $x=j$ and between the straight lines $y=\frac{n}{m}x$ and $y=1$ included. It follows that the sum is equal to the number of points with integer coordinates that lie in the interior of the triangle OAB, and by symmetry it is equal to half of the points with integer coordinates inside the rectangle $OABC$. The number of points with integer coordinates in the rectangle is $(n-1)(m-1)$, therefore

$$\left\lfloor\frac{n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor+\cdots+\left\lfloor\frac{(m-1)n}{m}\right\rfloor=\frac{(m-1)(n-1)}{2}.$$

Observation 1.3.4. *Since the right-hand side of the last inequality is symmetric in m and n, then*

$$\left\lfloor\frac{n}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor+\cdots+\left\lfloor\frac{(m-1)n}{m}\right\rfloor=\left\lfloor\frac{m}{n}\right\rfloor+\left\lfloor\frac{2m}{n}\right\rfloor+\cdots+\left\lfloor\frac{(n-1)m}{n}\right\rfloor.$$

For a number $x\in\mathbb{R}$ we can also consider the number $\{x\}=x-\lfloor x\rfloor$, which we call the **fractional part of** x, and for which the following properties are fulfilled.

Properties 1.3.5. *Let $x,y\in\mathbb{R}$ and $n\in\mathbb{Z}$, then it follows that:*

(a) $0\le\{x\}<1$.
(b) $x=\lfloor x\rfloor+\{x\}$.
(c) $\{x+y\}\le\{x\}+\{y\}\le\{x+y\}+1$.
(d) $\{x+n\}=\{x\}$.

Exercise 1.18. *For any real numbers a, $b>0$, prove that*

$$\lfloor 2a\rfloor+\lfloor 2b\rfloor\ge\lfloor a\rfloor+\lfloor b\rfloor+\lfloor a+b\rfloor.$$

Exercise 1.19. *Find the values of x that satisfy the following equation:*

(i) $\lfloor x\lfloor x\rfloor\rfloor=1$.
(ii) $||x|-\lfloor x\rfloor|=\lfloor |x|-\lfloor x\rfloor\rfloor$.

Exercise 1.20. *Find the solutions of the system of equations*

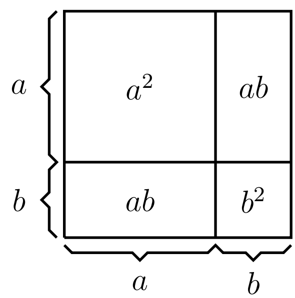
$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 1.1, \\ \lfloor x \rfloor + \{y\} + z &= 2.2, \\ \{x\} + y + \lfloor z \rfloor &= 3.3. \end{aligned}$$

Exercise 1.21 (Canada, 1987)**.** *For any natural number n, prove that*

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4n+3} \rfloor.$$

1.4 Notable products

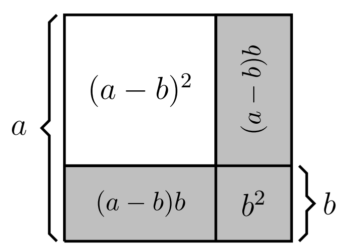
The area of a square is the square of the length of its side. If the length of the sides is $a + b$ then the area is $(a + b)^2$. However, the area of the square can be divided in four rectangles as shown in the figure.



Hence, the sum of the areas of the four rectangles will be equal to the area of the square, that is,

$$(a + b)^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2. \tag{1.2}$$

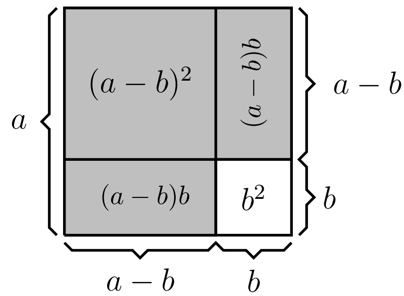
Now we give a geometric representation of the square of the difference of two numbers a, b, where $b \leq a$. The problem now is to find the area of the square of side $a - b$.



In the figure we observe that the area of the square of side a is equal to the sum of the areas of the square of sides $(a-b)$ and b, respectively, plus the area of two equal rectangles with sides of length b and $(a-b)$. That is, $a^2 = (a-b)^2 + b^2 + 2b(a-b)$, hence

$$(a-b)^2 = a^2 - 2ab + b^2. \tag{1.3}$$

Now, in order to find the area of the shaded region of the following figure,



observe that the sum of the areas of the rectangles covering the regions is $a(a-b) + b(a-b)$, and factorizing this sum we get

$$a(a-b) + b(a-b) = (a+b)(a-b), \tag{1.4}$$

which is equivalent to the area of the large square minus the area of the small square, that is,

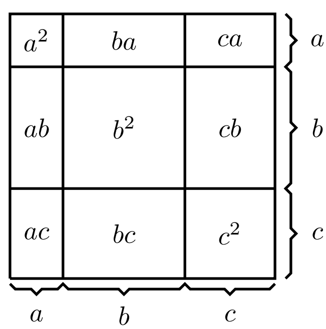
$$(a+b)(a-b) = a^2 - b^2. \tag{1.5}$$

Another notable product, but now dealing with three variables, is given by

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc. \tag{1.6}$$

The geometric representation of this product is given by the equality between the area of the square with side length $a+b+c$ and the sum of the areas of the nine rectangles in which the square is partitioned, that is,

$$(a+b+c)^2 = a^2+b^2+c^2+ab+ac+ba+bc+ca+cb = a^2+b^2+c^2+2ab+2ac+2bc.$$



Next, we provide a series of identities, some of them very well known and some others less known, but all of them very helpful for solving many problems.

Exercise 1.22. *For every x and $y \in \mathbb{R}$, verify the following second-degree identities:*

(i) $x^2 + y^2 = (x+y)^2 - 2xy = (x-y)^2 + 2xy$.

(ii) $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$.

(iii) $(x+y)^2 - (x-y)^2 = 4xy$.

(iv) $x^2 + y^2 + xy = \dfrac{x^2+y^2+(x+y)^2}{2}$.

(v) $x^2 + y^2 - xy = \dfrac{x^2+y^2+(x-y)^2}{2}$.

(vi) *Prove that* $x^2+y^2+xy \geq 0$ *and* $x^2+y^2-xy \geq 0$.

Exercise 1.23. *For any real numbers x, y, z, verify that*

(i) $x^2+y^2+z^2+xy+yz+zx = \dfrac{(x+y)^2+(y+z)^2+(z+x)^2}{2}$.

(ii) $x^2+y^2+z^2-xy-yz-zx = \dfrac{(x-y)^2+(y-z)^2+(z-x)^2}{2}$.

(iii) *Prove that* $x^2+y^2+z^2+xy+yz+zx \geq 0$ *and* $x^2+y^2+x^2-xy-yz-zx \geq 0$.

Exercise 1.24. *For all real numbers x, y, z, verify the following identities:*

(i) $(xy+yz+zx)(x+y+z) = (x^2y+y^2z+z^2x) + (xy^2+yz^2+zx^2) + 3xyz$.

(ii) $(x+y)(y+z)(z+x) = (x^2y+y^2z+z^2x) + (xy^2+yz^2+zx^2) + 2xyz$.

(iii) $(xy+yz+zx)(x+y+z) = (x+y)(y+z)(z+x) + xyz$.

(iv) $(x-y)(y-z)(z-x) = (xy^2+yz^2+zx^2) - (x^2y+y^2z+z^2x)$.

(v) $(x+y)(y+z)(z+x) - 8xyz = 2z(x-y)^2 + (x+y)(x-z)(y-z)$.

(vi) $xy^2+yz^2+zx^2-3xyz = z(x-y)^2 + y(x-z)(y-z)$.

Exercise 1.25. *For any real numbers x, y, z, verify that:*

(i) $x^2+y^2+z^2+3(xy+yz+zx) = (x+y)(y+z)+(y+z)(z+x)+(z+x)(x+y)$.

(ii) $xy+yz+zx-\left(x^2+y^2+z^2\right) = (x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)$.

Exercise 1.26. *For any real numbers x, y, z, verify that:*

$$(x-y)^2+(y-z)^2+(z-x)^2 = 2\left[(x-y)(x-z)+(y-z)(y-x)+(z-x)(z-y)\right].$$

1.5 Matrices and determinants

A 2×2 **matrix** is an array

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where a_{11}, a_{12}, a_{21} and a_{22} are real or complex numbers[3]. The **determinant** of the above matrix, which is denoted by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is the real number defined by $a_{11}a_{22} - a_{12}a_{21}$.

A 3×3 **matrix** is an array

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where, again, every a_{ij} is a number. The subindex indicates the position of the number in the array. Then, a_{ij} is the number located in the ith row and in the jth column. We define the **determinant of a** 3×3 **matrix** by the rule

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

That is, we move along the first row, multiplying a_{1j} by the determinant of a 2×2 matrix obtained by eliminating the first row and the jth column, and then adding all this together, and keeping in mind to place the minus sign before a_{12}. The result of the determinant is not modified if instead of choosing the first row as the first step we chose the second or third row. In case we choose the second row, we begin with a negative sign and if we choose the third row the first sign is positive, that is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

The signs alternate according to the following diagram:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

There are many properties of the determinants which follow immediately from the definitions. These properties become rules of a sort and the following are the most frequently used.

[3]Complex numbers will be treated in Chapter 5.

Properties 1.5.1.

(a) *If we interchange two consecutive rows or two consecutive columns, the sign of the determinant does change, for example,*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

(b) *We can factorize the common factor of any row or column of a matrix and the corresponding determinants are related in the following way, for example,*

$$\begin{vmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

(c) *If to a row (or column) we add another row (or column), the value of the determinant does not change, as in the following example:*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+a_{21} & a_{12}+a_{22} & a_{13}+a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \left(or \quad \begin{vmatrix} a_{11}+a_{12} & a_{12} & a_{13} \\ a_{21}+a_{22} & a_{22} & a_{23} \\ a_{31}+a_{32} & a_{32} & a_{33} \end{vmatrix}, \right).$$

(d) *If a matrix has two equal rows (or two equal columns) the determinant is zero.*

Example 1.5.2. *Using determinants we can establish the identity*

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca). \tag{1.7}$$

Note that

$$D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a \begin{vmatrix} a & b \\ c & a \end{vmatrix} - b \begin{vmatrix} c & b \\ b & a \end{vmatrix} + c \begin{vmatrix} c & a \\ b & c \end{vmatrix}$$
$$= a^3 - abc - abc + b^3 + c^3 - abc = a^3 + b^3 + c^3 - 3abc. \tag{1.8}$$

On the other hand, adding to the first column the other two, we have

$$D = \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

By properties (b) and (c), the determinants are equal.

Observe that the expression $a^2+b^2+c^2-ab-bc-ca$ can be written as[4]

$$\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right].$$

From this we obtain another version of identity (1.7), that is,

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]. \tag{1.9}$$

Observe that if the above identity satisfies the condition $a+b+c=0$ or the condition $a=b=c$, then the following identity holds:

$$a^3+b^3+c^3=3abc. \tag{1.10}$$

Reciprocally, if identity (1.10) is satisfied, then it follows that either $a+b+c=0$ or $a=b=c$.

Exercise 1.27. *Prove that* $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ *is a rational number.*

Exercise 1.28. *Find the factors of the expression* $(x-y)^3+(y-z)^3+(z-x)^3$.

Exercise 1.29. *Find the factors of the expression* $(x+2y-3z)^3+(y+2z-3x)^3+(z+2x-3y)^3$.

Exercise 1.30. *Prove that if* x, y, z *are different real numbers, then*

$$\sqrt[3]{x-y}+\sqrt[3]{y-z}+\sqrt[3]{z-x}\neq 0.$$

Exercise 1.31. *Let* r *be a real number such that* $\sqrt[3]{r}-\frac{1}{\sqrt[3]{r}}=1$. *Find the values of* $r-\frac{1}{r}$ *and* $r^3-\frac{1}{r^3}$.

Exercise 1.32. *Let* a, b, c *be digits different from zero. Prove that if the integers (written in decimal notation)* abc, bca *and* cab *are divisible by* n *then also* $a^3+b^3+c^3-3abc$ *is divisible by* n.

Exercise 1.33. *How many ordered pairs of integers* (m,n) *are there such that the following conditions are satisfied:* $mn\geq 0$ *and* $m^3+99mn+n^3=33^3$*?*

Exercise 1.34. *Find the locus of points* (x,y) *such that* $x^3+y^3+3xy=1$.

Exercise 1.35. *Find the real solutions* x, y, z *of the equation,*

$$x^3+y^3+z^3=(x+y+z)^3.$$

[4] See Exercise 1.23.

1.6 Inequalities

We begin this section with one of the most important inequalities. For any real number x, we have that

$$x^2 \geq 0. \tag{1.11}$$

This follows from the equality $x^2 = |x|^2 \geq 0$.
From this result, we can deduce that the sum of n squares is non-negative,

$$\sum_{i=1}^{n} x_i^2 \geq 0 \tag{1.12}$$

and it will be zero, if and only if all the x_i's are zero.
If we make the substitution $x = a-b$, where a and b are non-negative real numbers, in equation (1.11), we get

$$(a-b)^2 \geq 0.$$

Simplifying, the previous inequality leads to

$$a^2 + b^2 \geq 2ab. \tag{1.13}$$

Since

$$a^2 + b^2 \geq 2ab \text{ if and only if } 2a^2 + 2b^2 \geq a^2 + 2ab + b^2 = (a+b)^2,$$

we also have the inequality

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{(a+b)}{2}. \tag{1.14}$$

In case both a and b are positive numbers, the inequality (1.13) guarantees that

$$\frac{a}{b} + \frac{b}{a} \geq 2. \tag{1.15}$$

If we take $b = 1$ in the previous inequality, then we have that $a + \frac{1}{a} \geq 2$, that is, the sum of $a > 0$ and its reciprocal is greater than or equal to 2, and it will be 2 if and only if $a = 1$.

Replacing a, b by $\sqrt{a}$, $\sqrt{b}$ in (1.13), we obtain that

$$a + b \geq 2\sqrt{ab} \text{ if and only if } \frac{a+b}{2} \geq \sqrt{ab}. \tag{1.16}$$

Multiplying the last inequality by $\sqrt{ab}$ and reordering, we obtain

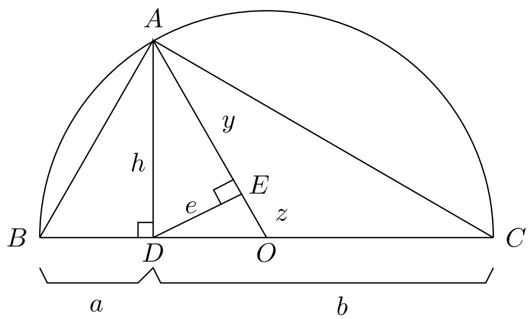
$$\sqrt{ab} \geq \frac{2ab}{a+b}. \tag{1.17}$$

Summarizing, the inequalities (1.14), (1.16) and (1.17), which we have just proved, imply that

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}. \tag{1.18}$$

The first expression is known as the **harmonic mean** (HM), the second is the **geometric mean** (GM), the third is the **arithmetic mean** (AM) and the last one is known as the **quadratic mean** (QM).

Now, we will present a geometric and a visual proof of the previous inequalities. Consider a semicircle with center in O, radius $\frac{a+b}{2}$ and right triangles ABC, DBA and DAC, as shown in the following figure:



These triangles are similar triangles and therefore the following equality holds:

$$\begin{aligned} \frac{AD}{DB} &= \frac{DC}{DA} \\ \frac{h}{a} &= \frac{b}{h} \\ h^2 &= ab, \end{aligned}$$

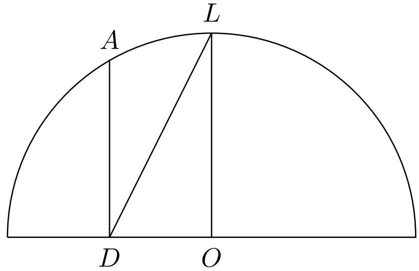
hence, the height h of the triangles is given by $h = \sqrt{ab}$, which according to the diagram is clearly smaller than the radius. Therefore, $\sqrt{ab} \leq \frac{a+b}{2}$.

To prove the first inequality in (1.18), observe that the triangles DAE and OAD are similar, hence

$$\begin{aligned} \frac{AD}{AE} &= \frac{AO}{AD}, \\ h^2 &= y(y+z), \\ \frac{2ab}{a+b} &= y, \end{aligned}$$

that is, y represents the harmonic mean. Clearly we have that $y \leq h$, therefore $\frac{2ab}{a+b} \leq \sqrt{ab}$.

To prove, geometrically, the last inequality in (1.18), consider the next figure.



We have that $OD = \frac{a+b}{2} - a = \frac{b-a}{2}$ and using the Pythagorean theorem, we obtain

$$DL^2 = OD^2 + OL^2 = \left(\frac{b-a}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 = \frac{a^2+b^2}{2},$$

that is, $DL = \sqrt{\frac{a^2+b^2}{2}}$ which is clearly greater than $\frac{a+b}{2}$.

Using Example 1.5.2, we can provide a proof of the arithmetic mean and the geometric mean inequality for three non-negative real numbers. In fact, using the next identity

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right],$$

it is clear that if a, b and c are non-negative numbers, then $a^3+b^3+c^3-3abc \geq 0$, that is, $a^3+b^3+c^3 \geq 3abc$. Moreover, if $a+b+c=0$ or $(a-b)^2+(b-c)^2+(c-a)^2 = 0$ the equality holds, and this happens only when $a = b = c$. Now, if x, y and z are non-negative numbers, define $a = \sqrt[3]{x}$, $b = \sqrt[3]{y}$ and $c = \sqrt[3]{z}$, then

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} \tag{1.19}$$

with equality if and only if $x = y = z$.

Example 1.6.1. *For every real number x, it follows that $\frac{x^2+2}{\sqrt{x^2+1}} \geq 2$.*

In fact,

$$\frac{x^2+2}{\sqrt{x^2+1}} = \frac{x^2+1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}} \geq 2.$$

The inequality now follows if we apply inequality (1.15).

Example 1.6.2. *If a, b, c are non-negative numbers, then*

$$(a+b)(b+c)(a+c) \geq 8abc.$$

As we have seen, $\frac{(a+b)}{2} \geq \sqrt{ab}$, $\frac{(b+c)}{2} \geq \sqrt{bc}$ and $\frac{(a+c)}{2} \geq \sqrt{ac}$, then

$$\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{a+c}{2}\right) \geq \sqrt{a^2b^2c^2} = abc.$$

Example 1.6.3. *If $x_1 > x_2 > x_3$ and $y_1 > y_2 > y_3$, which sum is greater?*

$$\begin{aligned} S &= x_1y_1 + x_2y_2 + x_3y_3, \\ S' &= x_1y_2 + x_2y_1 + x_3y_3. \end{aligned}$$

Consider the difference,

$$\begin{aligned} S' - S &= x_1y_2 - x_1y_1 + x_2y_1 - x_2y_2 \\ &= x_1(y_2 - y_1) + x_2(y_1 - y_2) \\ &= -x_1(y_1 - y_2) + x_2(y_1 - y_2) \\ &= (x_2 - x_1)(y_1 - y_2) < 0, \end{aligned}$$

then, $S' < S$.
In general, for any permutation $\{y_1', y_2', y_3'\}$ of $\{y_1, y_2, y_3\}$, we have that

$$S \geq x_1y_1' + x_2y_2' + x_3y_3', \tag{1.20}$$

which is known as the **rearrangement inequality**[5].

Exercise 1.36. *Let a, b be real numbers with $0 \leq a \leq b \leq 1$, prove that:*

(i) $0 \leq \dfrac{b-a}{1-ab} \leq 1.$

(ii) $0 \leq \dfrac{a}{1+b} + \dfrac{b}{1+a} \leq 1.$

Exercise 1.37 (Nesbitt Inequality)**.** *If a, b, $c \geq 0$, prove that*

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Exercise 1.38. *If a, b, c are the lengths of the sides of a triangle, prove that*

$$\sqrt[3]{\frac{a^3 + b^3 + c^3 + 3abc}{2}} \geq \max\{a, b, c\}.$$

Exercise 1.39. *Let p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that:*

(i) $\dfrac{1}{3} \leq \dfrac{1}{p(p+1)} + \dfrac{1}{q(q+1)} \leq \dfrac{1}{2}.$ (ii) $\dfrac{1}{p(p-1)} + \dfrac{1}{q(q-1)} \geq 1.$

[5]To see a general version, consult Example 7.3.6.

Exercise 1.40. *Find the smallest positive number k such that, for any $0 < a$, $b < 1$, with $ab = k$, it follows that*

$$\frac{a}{b} + \frac{b}{a} + \frac{a}{1-b} + \frac{b}{1-a} \geq 4.$$

Exercise 1.41. *Let a, b, c be non-negative real numbers, prove that*

$$(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)(ab+bc+ca).$$

Exercise 1.42. *Let a, b, c be positive real numbers that satisfy the equality $(a+b)(b+c)(c+a) = 1$. Prove that*

$$ab + bc + ca \leq \frac{3}{4}.$$

Exercise 1.43. *Let a, b, c be positive real numbers that satisfy $abc = 1$. Prove that*

$$(a+b)(b+c)(c+a) \geq 4(a+b+c-1).$$

Exercise 1.44 (APMO, 2011)**.** *Let a, b, c be positive integers. Prove that it is impossible for all three numbers $a^2 + b + c$, $b^2 + c + a$ and $c^2 + a + b$ to be perfect squares.*

1.7 Factorization

One of the most important forms of algebraic manipulation is known as factorization. In this section we study some examples and problems whose solutions depend on factorization formulas. Many of the problems that involve algebraic expressions can be easily solved using algebraic transformations in which the fundamental strategy is to find appropriate factors.

We start with some elemental formulas of factorization, where x, y are real numbers:

(a) $x^2 - y^2 = (x+y)(x-y)$.
(b) $x^2 + 2xy + y^2 = (x+y)^2$ and $x^2 - 2xy + y^2 = (x-y)^2$.
(c) $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x+y+z)^2$.

These algebraic identities are cataloged as second-order identities. In fact, we studied these four identities in the section of notable products. However, now we would like, given an algebraic expression, to reduce it to a product of simpler algebraic expressions.

Example 1.7.1. *For real numbers a, b, x, y, with x and y different from zero, it follows that*

$$\frac{a^2}{x}+\frac{b^2}{y}-\frac{(a+b)^2}{x+y}=\frac{(ay-bx)^2}{xy(x+y)}.$$

To obtain the equality, we start with the sum on the left side of the identity and follow the ensuing equalities

$$\begin{aligned}\frac{a^2}{x}+\frac{b^2}{y}-\frac{(a+b)^2}{x+y}&=\frac{a^2y(x+y)+b^2x(x+y)-xy(a+b)^2}{xy(x+y)}\\&=\frac{a^2y^2+b^2x^2-2xyab}{xy(x+y)}\\&=\frac{(ay-bx)^2}{xy(x+y)}.\end{aligned}$$

An application of the previous identity helps us to prove, in a straightforward manner, the so-called **helpful inequality**[6] of degree 2. This inequality assures that for real numbers a, b and real positive numbers x, y, it follows that

$$\frac{a^2}{x}+\frac{b^2}{y}\geq\frac{(a+b)^2}{x+y}.$$

The following identities are known as third-degree identities, with x, y, $z \in \mathbb{R}$:

(a) $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$.
(b) $x^3-y^3=(x-y)^3+3xy(x-y)$.
(c) $(x+y)^3-(x^3+y^3)=3xy(x+y)$.
(d) $x^3-xy^2+x^2y-y^3=(x+y)(x^2-y^2)$.
(e) $x^3+xy^2-x^2y-y^3=(x-y)(x^2+y^2)$.

To prove the validity of these identities, it is enough to expand one of the sides of the equalities or use the Newton binomial theorem, which we will study in Section 3.2.

Another quite important identity of degree 3, previously given as (1.7), is

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx),$$

for all x, y, z real numbers. A proof of this identity can be obtained directly expanding the right-hand side of the identity. Different proofs of this equality will be given throughout this book.

An equivalent form of the above identity is

$$x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right].$$

[6] See [6] or [7].

The identities $x^2 - y^2 = (x+y)(x-y)$ and $x^3 - y^3 = (x-y)\left(x^2 + xy + y^2\right)$ are particular cases of the nth degree identity,

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}), \tag{1.21}$$

for any x, y real numbers. If n is odd, we can replace y by $-y$ in the last formula to obtain the factorization for the sum of two nth odd power numbers,

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}). \tag{1.22}$$

In general, when n is even, the sum of two nth powers cannot be factored, however there are some exceptions when it is possible to complete squares. Let us see the following example.

Example 1.7.2 (Sophie Germain identity)**.** *For any x, y real numbers, it follows that*

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy).$$

Completing the square, we have

$$\begin{aligned} x^4 + 4y^4 &= x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 = (x^2 + 2y^2)^2 - (2xy)^2 \\ &= (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy). \end{aligned}$$

Another example, with even powers is the following.

Example 1.7.3. *For any x, y real numbers, it follows that*

$$x^{2n} - y^{2n} = (x+y)(x^{2n-1} - x^{2n-2}y + x^{2n-3}y^2 - \cdots + xy^{2n-2} - y^{2n-1}).$$

To prove this we only have to divide $x^{2n} - y^{2n}$ by $x+y$ or do the product on the right-hand side and simplify.

Example 1.7.4. *The number $n^4 - 22n^2 + 9$ is a composite number for any integer n.*

The idea is to try to factorize the expression. We do it by completing squares, and the common way to do it is as follows:

$$n^4 - 22n^2 + 9 = (n^4 - 22n^2 + 121) - 112 = (n^2 - 11)^2 - 112.$$

While doing this a problem arises: 112 is not a perfect square, therefore the factorization is not immediate. However, we can try the following strategy to complete squares, which is less usual,

$$\begin{aligned} n^4 - 22n^2 + 9 &= (n^4 - 6n^2 + 9) - 16n^2 = (n^2 - 3)^2 - 16n^2 \\ &= (n^2 - 3)^2 - (4n)^2 = (n^2 - 3 + 4n)(n^2 - 3 - 4n) \\ &= ((n+2)^2 - 7)((n-2)^2 - 7), \end{aligned}$$

and observe that none of the factors is equal to ± 1.

The following is another example of how to solve problems using basic factorizations.

Example 1.7.5. *Find all the pairs (m, n) of positive integers such that $|3^m - 2^n| = 1$.*

When $m = 1$ or $m = 2$, it is easy to find the solutions $(m, n) = (1, 1)$, $(1, 2)$, $(2, 3)$. Now, we will prove that these are the only solutions of the equation. Suppose that (m, n) is a solution of $|3^m - 2^n| = 1$, with $m > 2$ and therefore $n > 3$. We analyze both cases: $3^m - 2^n = 1$ and $3^m - 2^n = -1$.
Suppose that $3^m - 2^n = -1$ with $n > 3$, then 8 divides $3^m + 1$. However, if we divide 3^m by 8 we obtain as a remainder 1 or 3, depending on whether n is odd or even, therefore in this case we do not have a solution.
Suppose now that $3^m - 2^n = 1$ with $m \geq 3$, therefore $n \geq 5$, since $2^n + 1 = 3^m \geq 27$. Then $3^m - 1$ is divisible by 8, hence m is even. Write $m = 2k$, with $k > 1$. Then $2^n = 3^{2k} - 1 = (3^k + 1)(3^k - 1)$, therefore $3^k + 1 = 2^r$, for some $r > 3$. But the previous case tells us we know that this is impossible, therefore in this case there are no solutions.

The following formulas are also helpful to factorize. For real numbers x, y, z we have the following equalities:

$$(x + y)(y + z)(z + x) + xyz = (x + y + z)(xy + yz + zx), \tag{1.23}$$

$$(x + y + z)^3 = x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x). \tag{1.24}$$

To convince ourselves of the validity of these equalities, just expand both sides in each equality. From these equalities the following observation arises.

Observation 1.7.6.

(a) *If x, y, z are real numbers, with $xyz = 1$, then*

$$(x + y)(y + z)(z + x) + 1 = (x + y + z)(xy + yz + zx). \tag{1.25}$$

(b) *If x, y, z are real numbers with $xy + yz + zx = 1$, then*

$$(x + y)(y + z)(z + x) + xyz = x + y + z. \tag{1.26}$$

Exercise 1.45. *For all real numbers x, y, z, the following identities hold:*

(i) $(x + y + z)^3 - (y + z - x)^3 - (z + x - y)^3 - (x + y - z)^3 = 24xyz$.

(ii) $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.

(iii) $(x - y)(y + z)(z + x) + (y - z)(z + x)(x + y) + (z - x)(x + y)(y + z)$
$= -(x - y)(y - z)(z - x)$.

Exercise 1.46. *For all real numbers x, y, z, prove that*

(i) *If $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$, then*

$$f(x, y, z) = \frac{1}{2} f(x + y, y + z, z + x) = \frac{1}{4} f(-x + y + z, x - y + z, x + y - z).$$

(ii) *If $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$, then $f(x, y, z) \geq 0$ if and only if $x + y + z \geq 0$, and $f(x, y, z) \leq 0$ if and only if $x + y + z \leq 0$.*

Exercise 1.47. *Prove that for any real numbers x, y, the following identities are satisfied:*

(i) $(x+y)^5 - (x^5+y^5) = 5xy(x+y)(x^2+xy+y^2)$.
(ii) $(x+y)^7 - (x^7+y^7) = 7xy(x+y)(x^2+xy+y^2)^2$.

Exercise 1.48. *Let x, y and z be real numbers such that $x \neq y$ and*

$$x^2(y+z) = y^2(x+z) = 2.$$

Find the value of $z^2(x+y)$.

Exercise 1.49. *Find the real solutions x, y, z and w of the system of equations*

$$x+y+z = w,$$
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{w}.$$

Exercise 1.50. *Let x, y and z be real numbers different from zero such that $x+y+z \neq 0$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{x+y+z}$. Prove that for any odd integer n it follows that*

$$\frac{1}{x^n}+\frac{1}{y^n}+\frac{1}{z^n} = \frac{1}{x^n+y^n+z^n}.$$

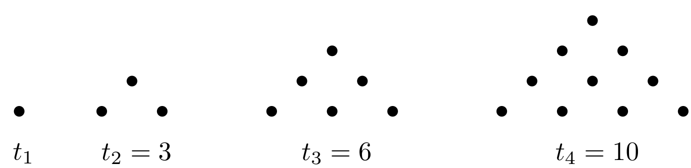
Chapter 2

Progressions and Finite Sums

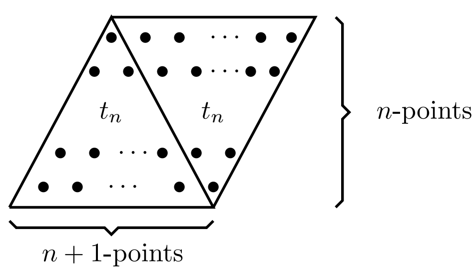
2.1 Arithmetic progressions

In antiquity, patterns of points played an important role in the use of numbers and cosmological conceptions. The Pythagoreans used to represent some integers as a set of points arranged in polygonal or polyhedral forms. These integers, presented as spatial arrays, are known as figurate numbers. In this section we will study some of these numbers.

Suppose that we want to add the natural numbers $1 + 2 + 3 + \cdots + n$, where n is any natural number. Call t_n the sum of these numbers, for example, $t_1 = 1$, $t_2 = 1 + 2 = 3$, $t_3 = 1 + 2 + 3 = 6$, $t_4 = 1 + 2 + 3 + 4 = 10$. We will represent them as patterns of points in the following form:



We can obtain the sum t_n from the following figure, where the geometrical arrangement shows that $2t_n = n(n + 1)$.



These numbers are known as **triangular numbers**.

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_2

This sum is also known as the **Gauss sum**, named after Carl Friedrich Gauss who, when he was only a child, calculated this sum using the following trick. Let t_n be the sum of the n numbers that we want to add. Since the sum is commutative, we can arrange the numbers starting from the higher value and ending with the lower. If now we sum, term by term, both arrangements, we get

$$\begin{array}{rcccccccccc} & t_n & = & 1 & + & 2 & + & \cdots & + & n \\ \text{and} & t_n & = & n & + & n-1 & + & \cdots & + & 1 \\ \hline & 2t_n & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1), \end{array} \tag{2.1}$$

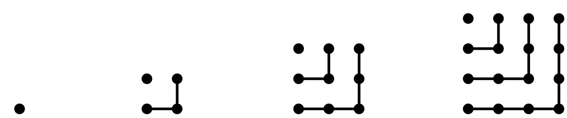
therefore $2t_n = n(n+1)$, that is, $t_n = \frac{n(n+1)}{2}$.

Remember that **even numbers** can be represented as $2n$, where $n = 1, 2, \ldots$ and **odd numbers** can be written as $2n - 1$, where $n = 1, 2, \ldots$.

If we want to sum the first n odd numbers, that is, the ones that run from 1 to $2n - 1$, and we call c_n the sum of these numbers, then we have, following Gauss, that

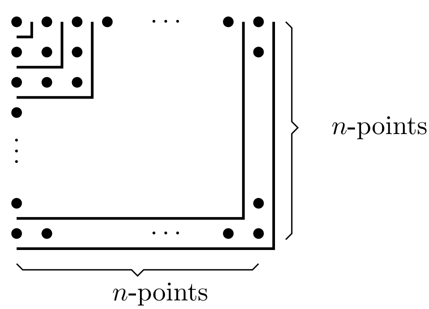
$$\begin{array}{rcccccccccc} & c_n & = & 1 & + & 3 & + & \cdots & + & 2n-1 \\ \text{and} & c_n & = & 2n-1 & + & 2n-3 & + & \cdots & + & 1 \\ \hline & 2c_n & = & 2n & + & 2n & + & \cdots & + & 2n \end{array}, \tag{2.2}$$

therefore, $2c_n = n \cdot 2n = 2n^2$, where we have multiplied by n because we have exactly n odd numbers. Therefore, $c_n = n^2$. We can represent these sums using patterns of points in the following way:



c_1 $c_2 = 4$ $c_3 = 9$ $c_4 = 16$

Then the sum of the first n odd natural numbers corresponds to the representation of the perfect square n^2, that is,



Every corridor represents the corresponding odd number we are adding, therefore, the sum of the first n odd numbers is the number of points in the square, that is, $n \cdot n = n^2$. These numbers are known as the **square numbers**.

An **arithmetic progression** is a collection or sequence of numbers such that each term of the sequence can be obtained from the preceding number adding a fixed quantity. That is, a collection $\{a_0, a_2, \dots\}$ is an arithmetic progression if, for each $n \geq 0$, $a_{n+1} = a_n + d$, where d is a constant. This common constant is called the **difference of the progression** and the collection or sequence will be represented by $\{a_n\}$.

Proposition 2.1.1. *If $\{a_n\}$ is an arithmetic progression with difference d, it follows that:*

(a) *The term a_n is equal to $a_0 + nd$, for $n = 0, 1, 2, \dots$.*
(b) $a_0 + a_1 + \cdots + a_n = \frac{a_0+a_n}{2}(n+1) = \frac{2a_0+nd}{2}(n+1)$, *for $n = 0, 1, 2, \dots$.*
(c) $a_n = \frac{a_{n-1}+a_{n+1}}{2}$, *for $n = 1, 2, 3, \dots$, that is, each term is the arithmetic mean of its two neighbours.*

Proof. (a) $a_n - a_0 = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_1 - a_0) = d + d + \cdots + d = nd$.
(b) Similarly as we did with the Gauss sum, if $S = a_0 + a_1 + \cdots + a_n = a_n + a_{n-1} + \cdots + a_0$, then

$$2S = (a_0 + a_n) + (a_1 + a_{n-1}) + \cdots + (a_n + a_0) = (a_0 + a_n)(n + 1),$$

a result of the identity

$$a_j + a_{n-j} = a_{j+1} + a_{n-j-1},$$

since $a_{j+1} - a_j = a_{n-j} - a_{n-j-1} = d$.
(c) It follows from $a_n - a_{n-1} = a_{n+1} - a_n$. □

Example 2.1.2. *The following figures are formed with toothpicks and they are composed by equilateral triangles.*

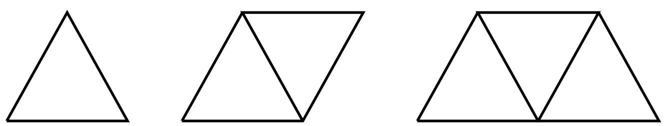


Figure 1 Figure 2 Figure 3

How many toothpicks are needed to construct the figure with n triangles?

To build the first figures we need 3, 5 and 7 toothpicks, respectively. The fourth figure will have 4 triangles, that is, one more than the third figure; but we only need 2 more toothpicks to make an additional triangle. In general, this happens always as we go from figure j to figure $j+1$, that is, a new triangle is constructed, but only 2 additional toothpicks are needed. Therefore, the difference of toothpicks going from one figure to the next is 2, that is, if a_j and a_{j+1} are the number of toothpicks necessary to construct the jth figure and the next one, respectively, we have that $a_{j+1} - a_j = 2$. Then,

$$a_n = a_{n-1} + 2 = a_{n-2} + 2 \cdot 2 = \cdots = a_1 + 2 \cdot (n - 1) = 3 + 2n - 2 = 2n + 1.$$

Proposition 2.1.3. *The sequence $\{a_n\}$ is an arithmetic progression if and only if there exist real numbers m and b such that $a_n = mn + b$, for every $n \geq 0$.*

Proof. If $\{a_n\}$ is an arithmetic progression with difference d, part (a) of the previous proposition, $a_n = a_0 + nd$, leads to $m = d$ and $b = a_0$ been the numbers we were searching for.
Reciprocally, if $a_n = mn + b$, then $a_{n+1} - a_n = m$ is constant and therefore $\{a_n\}$ is an arithmetic progression, with difference m. $\square$

An **harmonic progression** is a sequence $\{a_n\}$ which satisfies that the sequence $\left\{\frac{1}{a_n}\right\}$ is an arithmetic progression. That is, $\frac{1}{a_{n+1}} - \frac{1}{a_n}$ is constant, for every natural number n.

For example, the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$ is an harmonic progression since $\{1, 2, 3, 4, \dots\}$ is an arithmetic progression.

Exercise 2.1. *Calculate the sum of the first n even numbers. Can you exhibit a geometric representation of them?*

Exercise 2.2.

(i) *If $\{a_n\}$ and $\{b_n\}$ are arithmetic progressions, then $\{a_n + b_n\}$ and $\{a_n - b_n\}$ are arithmetic progressions, this can be shortened saying that $\{a_n \pm b_n\}$ are arithmetic progressions.*
(ii) *If $\{a_n\}$ is an arithmetic progression, then $\left\{b_n = a_{n+1}^2 - a_n^2\right\}$ is an arithmetic progression.*

Exercise 2.3. *If $\{a_n\}$ is an arithmetic progression with $a_j \neq 0$, for every $j = 0, 1, 2, \dots$, then*

$$\frac{1}{a_0\, a_1} + \frac{1}{a_1\, a_2} + \cdots + \frac{1}{a_{n-1}\, a_n} = \frac{n}{a_0\, a_n}.$$

Exercise 2.4. *Prove that a sequence $\{a_n\}$ is an arithmetic progression if and only if there exist real numbers A and B such that $S_n = a_0 + a_1 + \cdots + a_{n-1} = An^2 + Bn$, for every $n \geq 0$.*

Exercise 2.5. *A sequence $\{a_n\}$ is an* **arithmetic progression of order** 2 *if $\{a_{n+1} - a_n\}$ is an arithmetic progression.*
Prove that $\{a_n\}$ is an arithmetic progression of order 2 if and only if there exists a degree 2 polynomial $P(x)$ such that $P(n) = a_n$, for every $n \geq 0$.

Exercise 2.6. *If $\{a_n\} \subset \mathbb{R}^+$ is an arithmetic progression, then*

$$\sqrt{a_1 a_n} \leq \sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_n}{2}.$$

Exercise 2.7. *Prove that there are* 5 *prime numbers which are in arithmetic progression with difference* 6*. Is the progression unique?*

Exercise 2.8. *For any natural number* n*, let* S_n *be the sum of the integers* m*, with* $2^n < m < 2^{n+1}$*. Prove that* S_n *is a multiple of* 3*, for all* n*.*

Exercise 2.9. *If* $a < b < c$ *are real numbers which are in harmonic progression, then*

$$\frac{1}{b-c} + \frac{4}{c-a} + \frac{1}{a-b} = \frac{1}{c} - \frac{1}{a}.$$

Exercise 2.10. *If* a*,* b*,* c *and* d *are in harmonic progression, then* $a + d > b + c$*.*

Exercise 2.11. *If* a*,* b *and* c *are real numbers, prove that* $b+c$*,* $c+a$ *and* $a+b$ *are in harmonic progression if and only if* a^2*,* b^2 *and* c^2 *are in arithmetic progression.*

Exercise 2.12. *An increasing arithmetic progression satisfies that the product of any two terms of the progression is also an element of the progression. Prove that each term of the progression is an integer number.*

Exercise 2.13. *In the following arrangement, all the odd numbers were placed in such a way that in the* j*th row there are* j *consecutive odd numbers,*

$$\begin{array}{llll} 1 & & & \\ 3 & 5 & & \\ 7 & 9 & 11 & \\ 13 & 15 & 17 & 19 \\ \vdots & & & \vdots \end{array}$$

(i) *Which is the first number (on the left) in the* 100*th row?*
(ii) *Which is the sum of the numbers in the* 100*th row?*

Exercise 2.14. *Consider the following array, in which the numbers from* 1 *to* 9 *are placed as indicated.*

1	2	3
4	5	6
7	8	9

Observe that the sum of the integers in the two main diagonals is 15*. If we construct a similar array with the numbers from* 1 *to* 10000*, what is the value of the sum of the numbers in the two main diagonals?*

Exercise 2.15. *Fill the following board in such a way that the numbers in the rows and columns are in arithmetic progression.*

	74			
				186
		103		
0				

2.2 Geometric progressions

A **geometric progression** is a sequence of numbers related in such a way that each number can be obtained by multiplying the previous one by a fixed constant, different from zero, which we call the common ratio or the **ratio of the progression**. That is, the sequence $\{a_n\}$ is a geometric progression if the ratio $\frac{a_{n+1}}{a_n}$ is constant. Let this ratio be denoted by r. Note that $a_0 = 0$ and $r = 0$ are not included.

Proposition 2.2.1. *If $\{a_n\}$ is a geometric progression with ratio r, then:*

(a) *The nth term is $a_n = a_0 r^n$, for $n = 0, 1, 2, \ldots$.*

(b) *The sum $a_0 + a_1 + \cdots + a_n = a_0 \dfrac{1 - r^{n+1}}{1 - r}$, for $n = 0, 1, 2, \ldots$.*

(c) *If a_n are positive numbers then $a_n = \sqrt{a_{n-1}a_{n+1}}$, for $n = 1, 2, 3, \ldots$.*

Proof. (a) Since $\frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \cdots \cdot \frac{a_1}{a_0} = r^n$ we have, after simplifying, that $a_n = a_0 r^n$.
(b) Using part (a) we get

$$\begin{aligned} S = a_0 + a_1 + \cdots + a_n &= a_0 + a_0 r + \cdots + a_0 r^n \\ &= a_0(1 + r + \cdots + r^n) = \frac{a_0(1 - r^{n+1})}{1 - r}. \end{aligned}$$

The last equality follows from identity (1.21).
(c) Since $\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}}$, it follows that $a_n = \sqrt{a_{n-1}a_{n+1}}$. □

Example 2.2.2. *If $\{a_n\}$ and $\{b_n\}$ are geometric progressions, then $\{a_n \cdot b_n\}$ is a geometric progression.*

If $\{a_n\}$ and $\{b_n\}$ are geometric progressions, then $a_n = a_0 r^n$ and $b_n = b_0 s^n$, for some real numbers r and s. Then, $a_n b_n = (a_0 b_0)(rs)^n$ is a geometric progression with ratio rs.

Exercise 2.16. *If $\{a_n\}$ and $\{b_n\}$ are geometric progressions, with $b_n \neq 0$ for all n, prove that $\left\{\frac{a_n}{b_n}\right\}$ is a geometric progression.*

Exercise 2.17. *Find the geometric progressions* $\{a_n\}$ *satisfying that* $a_{n+2} = a_{n+1} + a_n$*, for all* $n \geq 0$.

Exercise 2.18. *If* $\{a_n\}$ *is a geometric progression with ratio* r*, and if the product of* a_0, a_1, $\dots$, a_{n-1} *is* P_n*, prove that:*

(i) $P_n = a_0^n r^{n(n-1)/2}$.

(ii) $(P_n)^2 = (a_0 a_{n-1})^n$.

Exercise 2.19. *If* $\{a_n\} \subset \mathbb{R}^+$ *is a geometric progression with ratio* r*, prove that* $\{b_n = \log a_n\}$ *is an arithmetic progression with difference* $\log r$.

Exercise 2.20. *If* a, b *and* c *are in geometric progression, then*

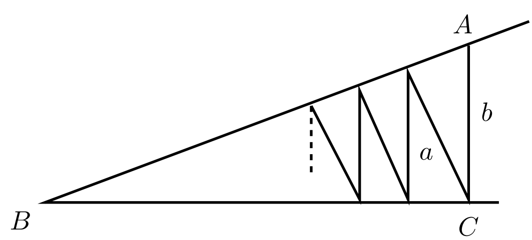
$$a^3b^3 + b^3c^3 + c^3a^3 = abc(a^3 + b^3 + c^3).$$

Exercise 2.21. *Prove that it is possible to eliminate terms of an arithmetic progression of positive integers in such a way that the remaining terms form a geometric progression.*

Exercise 2.22. *The lengths of the sides of a right triangle, given by* $a < b < c$*, are in geometric progression. Find the ratio of the progression.*

Exercise 2.23 (Slovenia, 2009)**.** *Let* $\{a_n\}$ *be a non-constant arithmetic progression with initial term* $a_1 = 1$*. The terms* a_2, a_5, a_{11} *form a geometric progression. Find the sum of the first* 2009 *terms.*

Exercise 2.24. *In the next figure the polygonal line has been constructed between the sides of the angle* ABC *as follows: the first segment* AC *of length* b *is perpendicular to* BC*, the second segment, of length* a*, starts where the previous segment ended and it is perpendicular to* AB*; proceeding in the same manner, the following segments start where the previous one ended, and keeping the perpendicularity to* BC *and to* AB *alternately.*



(i) *What is the length of the* n*th segment?*

(ii) *What is the length of the polygonal line of* n *sides?*

(iii) *What is the length of the polygonal line if it has an infinite number of sides?*

2.3 Other sums

Now, we proceed to study more complicated sums. For example, let us see how to sum the squares of the first natural numbers,

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

Consider the following array of numbers:

$$\left.\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & \cdots & k & \cdots & n \\
1 & 2 & 3 & 4 & 5 & \cdots & k & \cdots & n \\
1 & 2 & 3 & 4 & 5 & \cdots & k & \cdots & n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & 4 & 5 & \cdots & k & \cdots & n
\end{array}\right\} \quad n\text{-rows}$$

We will calculate the sum of the numbers on the array in two different ways. First, add the numbers in each of the rows of the array which, according to the Gauss sum formula, is a triangular number, that is,

$$R_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

But since we have n rows, then the sum of all the numbers in the array is

$$S_T = n\frac{n(n+1)}{2}.$$

On the other hand, if we add each of the corridors marked, we have that the sum of the first corridor is only 1. The sum of the second corridor is $1 + 2 \cdot 2 = 1 + 2^2$. The sum of the third corridor is $1 + 2 + 3 \cdot 3 = 1 + 2 + 3^2$. Continuing in the same fashion, the sum of the kth corridor is

$$C_k = 1 + 2 + \cdots + (k-1) + k \cdot k = \frac{k(k-1)}{2} + k^2 = \frac{3}{2}k^2 - \frac{k}{2}.$$

Now, if we sum all the corridors, we get the sum of all the numbers in the array

$$\begin{aligned}
S_T = C_1 + C_2 + \cdots + C_n &= \frac{3}{2}(1^2 + 2^2 + \cdots + n^2) - \frac{1}{2}(1 + 2 + \cdots + n) \\
&= \frac{3}{2}(1^2 + 2^2 + \cdots + n^2) - \frac{1}{2} \cdot \frac{n(n+1)}{2}.
\end{aligned}$$

Equating both sums S_T, we get

$$n\frac{n(n+1)}{2} = \frac{3}{2}(1^2 + 2^2 + \cdots + n^2) - \frac{1}{2} \cdot \frac{n(n+1)}{2}.$$

Then

$$\frac{3}{2}(1^2+2^2+\cdots+n^2)=\frac{1}{2}\cdot\frac{n(n+1)}{2}+n\frac{n(n+1)}{2},$$

therefore

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

Now, we will study the sum of the cubes of the first n natural numbers,

$$1^3+2^3+\cdots+n^3.$$

To that purpose, consider the following array of numbers:

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & \cdots \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & \cdots \\
4 & 8 & 12 & 16 & 20 & 24 & 28 & \cdots \\
5 & 10 & 15 & 20 & 25 & 30 & 35 & \cdots \\
\vdots & & & & & & \vdots &
\end{array}$$

If we add the numbers in each of the squares, we obtain

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1+2+2(1+2) \\
S_3 &= 1+2+3+2(1+2+3)+3(1+2+3) \\
\vdots\quad & \quad\vdots \\
S_n &= (1+2+\cdots+n)+2(1+2+\cdots+n)+\cdots+n(1+2+\cdots+n) \\
&= \left(\frac{n(n+1)}{2}\right)^2.
\end{aligned}$$

If we now sum all the corridors

$$\begin{aligned}
C_1 &= 1 \\
C_2 &= 2\cdot 2+2\cdot 2=2^2+2^2=2^2(1+1)=2^3 \\
C_3 &= 2(3+2\cdot 3)+3^2=2\cdot 3(1+2)+3^2=2\cdot 3^2+3^2=3^2(2+1)=3^3 \\
\vdots\quad & \quad\vdots \\
C_n &= 2(n+2n+3n+\cdots+(n-1)n)+n^2 \\
&= 2n\left(\frac{(n-1)n}{2}\right)+n^2=n^3-n^2+n^2=n^3.
\end{aligned}$$

But, since $C_1 + C_2 + C_3 + \cdots + C_n = S_n$, we get

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

The above sum can also be written as

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Exercise 2.25. *Using the corridor technique in the following array*

$$\left.\begin{array}{ccccccccc}
1^2 & 2^2 & 3^2 & 4^2 & 5^2 & \cdots & k^2 & \cdots & n^2 \\
1^2 & 2^2 & 3^2 & 4^2 & 5^2 & \cdots & k^2 & \cdots & n^2 \\
1^2 & 2^2 & 3^2 & 4^2 & 5^2 & \cdots & k^2 & \cdots & n^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1^2 & 2^2 & 3^2 & 4^2 & 5^2 & \cdots & k^2 & \cdots & n^2
\end{array}\right\} \quad n\text{-rows}$$

prove the following equality,

$$\left(\frac{n(n+1)}{2}\right)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

2.4 Telescopic sums

When we develop a sum of the form

$$\sum_{k=1}^{n} \left[f(k+1) - f(k)\right],$$

we have that

$$\sum_{k=1}^{n} \left[f(k+1) - f(k)\right] = f(2) - f(1) + f(3) - f(2) + \cdots + f(n+1) - f(n),$$

then the terms $f(k)$, for k between 2 and n, cancel out and we obtain that the sum is equal to $f(n+1) - f(1)$. This type of sums are called telescopic sums. Let us see some examples.

Example 2.4.1. *Evaluate the sum*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

Observe that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, therefore

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}.\end{aligned}$$

Example 2.4.2 (Canada, 1969)**.** *Calculate the sum* $\sum_{k=1}^{n} k! \cdot k$.

Observe that $k! \cdot k = k!(k+1-1) = (k+1)! - k!$, then we have that

$$\begin{aligned}\sum_{k=1}^{n} k! \cdot k &= \sum_{k=1}^{n} ((k+1)! - k!) \\ &= (2! - 1!) + (3! - 2!) + \cdots + ((n+1)! - n!) \\ &= ((n+1)! - n!) + (n! - (n-1)!) + \cdots + (2! - 1!) \\ &= (n+1)! - 1.\end{aligned}$$

Example 2.4.3. *Evaluate the product*

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right).$$

We have that

$$\begin{aligned}&\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right) \\ &\quad = \left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 + \frac{1}{2011}\right)\left(1 - \frac{1}{2011}\right) \\ &\quad = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{2011}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{2011}\right) \\ &\quad = \left(\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{2012}{2011}\right)\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{2010}{2011}\right) \\ &\quad = \left(\frac{2012}{2}\right)\left(\frac{1}{2011}\right) = \frac{1006}{2011}.\end{aligned}$$

Exercise 2.26. *Find the sum*

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{2998 \cdot 3001}.$$

Exercise 2.27. *Calculate the following sums:*

$$\text{(i)} \sum_{k=1}^{n} \frac{1}{k(k+2)} \qquad \text{(ii)} \sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2}.$$

Exercise 2.28. *Calculate the following sums:*

$$\text{(i)} \sum_{k=1}^{n} \frac{k}{(k+1)!} \qquad \text{(ii)} \sum_{k=1}^{n} \frac{k+1}{(k-1)!+k!+(k+1)!}.$$

Exercise 2.29. *Find the sum*

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2011^2}+\frac{1}{2012^2}}.$$

Exercise 2.30. *Find the following product*

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}\right)\cdots\left(1+\frac{1}{2^{2^n}}\right).$$

Chapter 3

Induction Principle

3.1 The principle of mathematical induction

In Chapter 2 we deduced several formulas for finite sums of numbers. Thus we learned that the sum of the first n natural numbers is given by the identity

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{3.1}$$

In fact, this formula is a collection of statements, $\mathcal{P}(n)$, which we have proved using algebra, that are valid for every positive integer n.

The validity of these series of statements can be proved using what is known as the principle of mathematical induction, which will be developed in this chapter.

The principle of mathematical induction claims that a sequence of propositions $\mathcal{P}(1)$, $\mathcal{P}(2)$, $\mathcal{P}(3)$, $\ldots$ are valid if:

1. The statement $\mathcal{P}(1)$ is true.
2. The statement "$\mathcal{P}(k)$ implies $\mathcal{P}(k+1)$" is true.

We can guarantee that this last statement is true assuming that $\mathcal{P}(k)$ is true and proving the validity of $\mathcal{P}(k+1)$[7].

Observation 3.1.1. *Statement* 1 *of the principle of mathematical induction is called the* **induction basis** *and statement* 2 *is known as the* **inductive step**.

We will prove using this principle the validity of the identity (3.1) for every natural number.

Example 3.1.2. *The identity* $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ *is valid for every positive integer* n.

[7]To prove 2. is equivalent to prove that: "not $P(k+1)$ implies not $P(k)$", is true.

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_3

If $n = 1$ the left-hand side of the identity has a unique term, that is, 1. The right-hand side is $\frac{1(1+1)}{2}$, since this last term is also 1. We have that the formula is valid for $n = 1$.
Now, suppose that the statement $\mathcal{P}(k)$ is true, that is, the equality

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \tag{3.2}$$

is valid. We will prove the corresponding identity for $k + 1$.
According to the formula, the terms on the left-hand side, for $k + 1$, are $1 + 2 + 3 + \cdots + k + (k + 1)$, and if we use (3.2), we have that

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{(k+1)(k+2)}{2}, \end{aligned}$$

which is the same as the right-hand side of the formula for $k + 1$. This proves the validity of $\mathcal{P}(k + 1)$. Therefore, by the principle of mathematical induction the identity is valid for every positive integer n

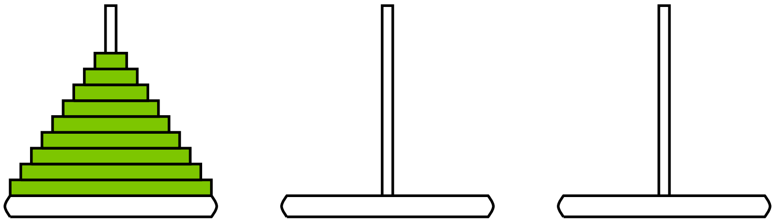
Example 3.1.3. *For every natural number n, the number $n^3 - n$ is a multiple of* 6.

If $n = 1$, then $n^3 - n = 0$ is a multiple of 6, since $0 = 6 \cdot 0$.
Suppose that $k^3 - k$ is a multiple of 6, that is, $k^3 - k = 6\,r$, for some integer number r. We will prove that the statement is true for $k+1$, that is, we will show that $(k + 1)^3 - (k + 1)$ is a multiple of 6.
Now, we have that

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3\,k^2 + 3\,k + 1 - k - 1 \\ &= (k^3 - k) + 3(k^2 + k) = 6\,r + 3(k^2 + k). \end{aligned}$$

Here we have used the induction hypothesis that $\mathcal{P}(k)$ holds, that is, $k^3 - k = 6r$. Since $k^2+k = k(k+1)$ is the product of consecutive numbers, one of them has to be even, then $k^2 + k$ is even. Then, $3(k^2 + k)$ is a multiple of 6. Thus, $(k+1)^3 - (k+1)$ is a sum of multiples of 6. Therefore, by the principle of mathematical induction the result is valid for every positive integer n.

Example 3.1.4 (Hanoi Towers)**.** *A beautiful legend of the creation of the world tells us that Brahma placed on the Earth three bars of diamond and* 64 *golden discs. The discs had all different sizes and at the beginning they were located in the first diamond bar following a decreasing order of the diameters from bottom to top. Also Brahma created a monastery where the monks had the fixed task of moving all the discs from the first bar to the third, following some rules. The allowed rules were to move one disc from one bar to any other bar but under the condition that a disc with a greater diameter could never be placed on top of a smaller disc. The* *legend also tells us that when the monks would finish their duty, the world will end. What is the smallest number of necessary moves that the monks have to make to accomplish the work?*



$1°$ *bar* $2°$ *bar* $3°$ *bar*

Let x_n be the minimum number of necessary moves to move n discs. If $n = 1$, there is only one disc and therefore just one move is needed, that is, $x_1 = 1$. With two discs in the first bar, we move the small disc to the second bar, then the big one from the first bar to the third one, and finally the small one from the second bar to the third. Therefore, $x_2 = 3$. With three discs the problem starts to become interesting.

Expressing as $i \to j$ the move of the top disc from bar i to bar j, the discs can move in the following sequence of moves $1 \to 3$, $1 \to 2$, $3 \to 2$, $1 \to 3$, $2 \to 1$, $2 \to 3$, $1 \to 3$, that is, $x_3 = 7$. Observing the obtained sequence 1, 3, 7, we can note that if we sum 1 to each term we obtain consecutive powers of 2. Therefore, we conjecture that $x_n = 2^n - 1$. To prove this conjecture by induction, we only need to prove the inductive step. Suppose that $x_n = 2^n - 1$ moves and let us see what happens with $n + 1$ discs. By the inductive hypothesis, the n superior discs can be moved to the second bar in $2^n - 1$ moves. After this, the largest disc moves from the first to the third bar in one move, and finally the n discs in the second bar can be moved to the third bar in $2^n - 1$ moves. Therefore, in total we made $(2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1$ moves.

Therefore, $2^n - 1$ moves are enough, but is this the minimum? We will prove by induction that in fact this is the minimum. For $n = 1$, this is obvious. Suppose that it is true for n. Now, we want to move $n + 1$ discs from the first to the third bar, that is, in some moment we have to move the largest disc to the third bar. At this moment the remaining n discs have to be all together in another bar and, by the induction hypothesis, they could not have arrived there in less than $2^n - 1$ moves. In the same way, when the largest disc moves to its final position in the third bar, the n remaining discs have to be all in another bar, and to move them to the third bar we need at least $2^n - 1$ moves.

Adding, we see that we cannot move $n+1$ discs in less than $(2^n-1)+1+(2^n-1) = 2^{n+1} - 1$ moves, which completes the proof.

With 64 discs, as in the legend, it is necessary to do $2^{64} - 1$ moves, and if we suppose that the monks can do one move every second, to finish their task they will need 500 thousand millions of years.

Observation 3.1.5. *In many cases, the basis of induction is not for $n = 1$ but for $n = k$, for some natural number k. Then, if we prove the induction step, we can conclude that $\mathcal{P}(n)$ is true for all natural numbers $n \geq k$.*

Example 3.1.6. *For x, y real numbers and for $n \geq 2$, it follows that*

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

If $n = 2$ the result follows because $x^2 - y^2 = (x - y)(x + y)$.
Suppose that $x^k - y^k$ can be written as

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}).$$

We will prove the result for $k + 1$. We start from the following identity,

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - xy^k + xy^k - y^{k+1} \\ &= x(x^k - y^k) + y^k(x - y). \end{aligned}$$

Using the induction hypothesis, we have that

$$\begin{aligned} x(x^k - y^k) + y^k(x - y) &= x(x - y)(x^{k-1} + x^{k-2}y + \cdots + y^{k-1}) + y^k(x - y) \\ &= (x - y)(x(x^{k-1} + x^{k-2}y + \cdots + y^{k-1}) + y^k) \\ &= (x - y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k). \end{aligned}$$

Therefore, $x^{k+1} - y^{k+1} = (x - y)(x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k)$, as we wanted to prove.

Example 3.1.7. *The sum of the interior angles of an n-sided convex polygon is $180°(n - 2)$.*

The statement makes sense for $n \geq 3$, that is, when we have a triangle. For $n = 3$, the statement holds since the sum of the interior angles of a triangle is $180° = 180°(3 - 2)$.
Suppose the statement is valid for any convex n-gon. Given a convex $(n + 1)$-gon with vertices $A_1, \ldots, A_{n+1}$, the diagonal A_nA_1 divides the polygon in a convex n-gon $A_1A_2 \ldots A_n$ and a triangle $A_nA_{n+1}A_1$. The sum of the interior angles of the $(n+1)$-gon will be the sum of the angles of the n-gon $A_1 \ldots A_n$ plus the sum of the interior angles of the triangle $A_nA_{n+1}A_1$, that is, $180°(n-2)+180° = 180°(n-1)$.

To verify a series of statements $\mathcal{P}(n)$, in some cases it is better to work with more general propositions $\mathcal{P}'(n)$, that is, propositions such that the validity of $\mathcal{P}'(n)$ will guarantee the validity of $\mathcal{P}(n)$. Let us see some examples.

Example 3.1.8. *For any positive integer $n \geq 2$, it follows that*

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{3}{4}.$$

To use the principle of mathematical induction to prove the inequality can be complicated, since if we suppose that $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{3}{4}$, we would have to prove that $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{3}{4}$, but the margin of manoeuvre is limited. Then we will try to work with the following stronger result

$$S_n \leq \frac{3}{4} - a_n, \quad \text{for all} \quad n \geq 2,$$

where S_n is the sum on the left-hand side of the inequality that we want to prove, and $\{a_n\}$ are positive numbers that we have to discover. In order to produce a proof by mathematical induction we need to show that the basis of induction is valid, that is

$$\frac{1}{4} \leq \frac{3}{4} - a_2 \quad \text{or} \quad a_2 \leq \frac{1}{2}.$$

The inductive step consists in showing that the condition $S_n \leq \frac{3}{4} - a_n$ implies that $S_{n+1} \leq \frac{3}{4} - a_{n+1}$. Since $S_{n+1} = S_n + \frac{1}{(n+1)^2}$, the above will be true if a_n and a_{n+1} satisfy

$$\frac{1}{(n+1)^2} - a_n + a_{n+1} \leq 0,$$

which is equivalent to

$$a_n - a_{n+1} \geq \frac{1}{(n+1)^2}, \quad \text{for all} \quad n \geq 2.$$

Now, the numbers $a_n = \frac{1}{n}$ satisfy the above inequality, since

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \geq \frac{1}{(n+1)^2}.$$

Also, $a_2 = \frac{1}{2}$ satisfies the condition of the basis of induction. Therefore, the numbers $a_n = \frac{1}{n}$ are good candidates to make possible the induction. In fact we have proved that, for every $n \geq 2$, it follows that $S_n \leq \frac{3}{4} - \frac{1}{n}$, thus

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{3}{4}.$$

A similar example is the following.

Example 3.1.9. *For any positive integer n, it follows that*

$$\frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \cdots + \frac{1}{(n+1)\sqrt{n}} < 2.$$

As in the previous example, it is sufficient to prove that

$$\frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \cdots + \frac{1}{(n+1)\sqrt{n}} < 2 - \frac{2}{\sqrt{n+1}}.$$

For $n = 1$, we have $\frac{1}{2\sqrt{1}} < 2 - \frac{2}{\sqrt{2}}$, which is true, that is, the induction basis holds.

For the inductive step, it suffices to prove that

$$\frac{1}{(n+2)\sqrt{n+1}} < \frac{2}{\sqrt{n+1}} - \frac{2}{\sqrt{n+2}},$$

which can be reduced to prove that

$$\frac{1}{\sqrt{n+2}} < 2(\sqrt{n+2} - \sqrt{n+1}) = \frac{2}{\sqrt{n+2}+\sqrt{n+1}},$$

but this last inequality is obviously true.

Another frequently used version of the principle of mathematical induction is the following.

Strong induction principle. The set of propositions $\mathcal{P}(1)$, $\mathcal{P}(2)$, $\mathcal{P}(3)$, $\dots$, $\mathcal{P}(n)$, $\dots$ are true if:

1. The statement $\mathcal{P}(1)$ is true.
2. The statement "$\mathcal{P}(1), \dots, \mathcal{P}(n)$ implies that $\mathcal{P}(n+1)$" is true.

Observations 3.1.10. (a) *In the strong induction principle, the inductive step is valid if each time that $\mathcal{P}(k)$ is valid for $k \leq n$ then, starting with this hypothesis, we prove that $\mathcal{P}(n+1)$ is valid.*

(b) *It is clear that the strong induction principle implies the simple principle of induction. But in fact both are equivalent, since strong induction is a consequence of simple induction. To see this, it is enough to consider the logical conjunction*[8] *$\mathcal{Q}(n)$ of the propositions $\mathcal{P}(1), \dots, \mathcal{P}(n)$. If $\mathcal{P}(1)$ is true also $Q(1)$ is true (since they are exactly the same). If $\mathcal{Q}(n)$ is valid so are $\mathcal{P}(1)$, $\mathcal{P}(2)$, $\dots$, $\mathcal{P}(n)$, and by the strong induction hypothesis also $\mathcal{P}(n+1)$ is valid, this implies that $\mathcal{Q}(n+1)$ is true. Then, by simple induction $\mathcal{Q}(n)$, is true for any natural number n and the same is true for $\mathcal{P}(n)$.*
Although strong induction and the simple one are logically equivalent, in some cases it is easier to use one rather than the other.

The strong induction is implicit in the definition of sequences by recurrence relations.

Example 3.1.11. *If a is a real number such that $a + \frac{1}{a}$ is an integer, then $a^n + \frac{1}{a^n}$ is an integer for all $n \geq 1$.*

For $n = 1$, the statement is true by the hypothesis that $a + \frac{1}{a}$ is an integer number. Let us see how to solve for $n = 2$; sometimes this case gives us ideas about how to justify the inductive step.
Note that $\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2$; then $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$ is an integer number and then our statement for $n = 2$ is true.

[8]The logical conjunction of $\mathcal{P}(1), \dots, \mathcal{P}(n)$ means that all the propositions are true at the same time.

Let us analyze again our formula

$$\left(a+\frac{1}{a}\right)^2=\left(a+\frac{1}{a}\right)\left(a+\frac{1}{a}\right)=a^2+\frac{1}{a^2}+a\cdot\frac{1}{a}+\frac{1}{a}\cdot a=a^2+\frac{1}{a^2}+a^0+\frac{1}{a^0},$$

then we get, $a^2+\frac{1}{a^2}=\left(a+\frac{1}{a}\right)\cdot\left(a+\frac{1}{a}\right)-(a^0+\frac{1}{a^0})$, but this gives us another idea, the statement for $n=2$ depends on the statements for $n=1$ and for $n=0$ (which by the way are also valid).
To obtain the statement for $n=3$, we will work following the previous idea,

$$\left(a^2+\frac{1}{a^2}\right)\cdot\left(a+\frac{1}{a}\right)=a^3+\frac{1}{a^3}+a+\frac{1}{a}.$$

It follows that

$$a^3+\frac{1}{a^3}=\left(a^2+\frac{1}{a^2}\right)\cdot\left(a+\frac{1}{a}\right)-\left(a+\frac{1}{a}\right),$$

the right-hand side is an integer number if $a+\frac{1}{a}$ and also $a^2+\frac{1}{a^2}$ are integers, but this is already known. It is now clear how we can prove the inductive step. Suppose that the statement is valid for integers less than or equal to n, then from the identity

$$a^{n+1}+\frac{1}{a^{n+1}}=\left(a^n+\frac{1}{a^n}\right)\cdot\left(a+\frac{1}{a}\right)-\left(a^{n-1}+\frac{1}{a^{n-1}}\right), \tag{3.3}$$

it follows that the statement for $n+1$ is also valid.

There are other ways to prove statements inductively.

Cauchy's induction principle. The set of propositions $\mathcal{P}(1)$, $\mathcal{P}(2)$, ..., $\mathcal{P}(n)$, ... are all valid if:

1. The statement $\mathcal{P}(2)$ is true.
2. The statement "$\mathcal{P}(n)$ implies $\mathcal{P}(n-1)$" is true.
3. The statement "$\mathcal{P}(n)$ implies $\mathcal{P}(2n)$" is true.

Observation 3.1.12. *Let us see why Cauchy's induction principle implies the principle of mathematical induction. First, note that* 1 *and* 2 *guarantee that* $\mathcal{P}(1)$ *is true, that is, the induction basis holds. Since* 3 *holds, the validity of* $\mathcal{P}(n)$ *implies that* $\mathcal{P}(2n)$ *is true. Now, applying* $n-1$ *times the condition* 2*, we get that* $\mathcal{P}(2n-1)$, $\mathcal{P}(2n-2)$, ..., $\mathcal{P}(n+1)$ *are all true. In particular,* $\mathcal{P}(n+1)$ *is true. Thus, we have proved the inductive step of the principle of mathematical induction. Therefore all* $\mathcal{P}(n)$ *are true.*

We apply this inductive process to prove the inequality between the geometric mean and the arithmetic mean.

Example 3.1.13. *For real non-negative numbers $x_1, \ldots, x_n$ the following inequality holds for all n:*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}. \tag{3.4}$$

Denote by $A = \frac{x_1+x_2+\cdots+x_n}{n}$ and $G = \sqrt[n]{x_1 x_2 \ldots x_n}$, the arithmetic mean and the geometric mean of the numbers $x_1, \ldots, x_n$, respectively. The proof will be done by induction over n, using the inductive principle just described.

1. For $n = 1$ we have the equality, and the case $n = 2$ was proved in Section 1.6.
2. Let $x_1, x_2, \ldots, x_n$ be non-negative numbers and let $g = \sqrt[n-1]{x_1 \cdots x_{n-1}}$. If we add this number to the numbers $x_1, \ldots, x_{n-1}$, we obtain n numbers to which we apply $\mathcal{P}(n)$ and obtain

$$\frac{x_1 + \cdots + x_{n-1} + g}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_{n-1} g} = \sqrt[n]{g^{n-1} \cdot g} = g.$$

   Then, $x_1 + \cdots + x_{n-1} + g \geq ng$, which in turn leads to $\frac{x_1+\cdots+x_{n-1}}{n-1} \geq g$, and therefore $\mathcal{P}(n-1)$ is valid.
3. Let $x_1, x_2, \ldots, x_{2n}$ be non-negative numbers, then

$$\begin{aligned} x_1 + x_2 + \cdots + x_{2n} &= (x_1 + x_2) + (x_3 + x_4) + \cdots + (x_{2n-1} + x_{2n}) \\ &\geq 2\left(\sqrt{x_1 x_2} + \sqrt{x_3 x_4} + \cdots + \sqrt{x_{2n-1} x_{2n}}\right) \\ &\geq 2n\left(\sqrt{x_1 x_2}\sqrt{x_3 x_4} \cdots \sqrt{x_{2n-1} x_{2n}}\right)^{\frac{1}{n}} = 2n\left(x_1 x_2 \cdots x_{2n}\right)^{\frac{1}{2n}}. \end{aligned}$$

In the previous sequence we applied several times the statement $\mathcal{P}(2)$, which we know is true, and after that the same was done with statement $\mathcal{P}(n)$ to the numbers $\sqrt{x_1 x_2}, \sqrt{x_3 x_4}, \ldots, \sqrt{x_{2n-1} x_{2n}}$.

Example 3.1.14. *Let us see another way to prove the previous inequality, but in this case using another variant of the induction principle. Observe first that if some $x_i = 0$, then the inequality is clear.*
Suppose then that every $x_i > 0$, and that $x_1 x_2 \ldots x_n = 1$. Prove the statements $\mathcal{P}(n)$: $x_1 + x_2 + \cdots + x_n \geq n$.

Clearly the basis of induction is true, that is, $\mathcal{P}(1) : x_1 \geq 1$ is true, in fact $x_1 = 1$.

Suppose that $\mathcal{P}(n)$ is valid for any n positive numbers whose product is 1, and let $x_1, \ldots, x_n, x_{n+1}$ be positive numbers whose product is 1. Then, there will be some $x_i \geq 1$ and some $x_j \leq 1$. Without loss of generality, we can suppose that $x_1 \geq 1$ and $x_2 \leq 1$. Then, by the previous statement, $(x_1 - 1)(x_2 - 1) \leq 0$, then $x_1 x_2 + 1 \leq x_1 + x_2$. Therefore, $x_1 + x_2 + \cdots + x_n + x_{n+1} \geq 1 + x_1 x_2 + x_3 + \cdots + x_{n+1}$. Now apply the induction hypothesis to the n numbers $x_1 x_2, x_3, \ldots, x_{n+1}$ to show that $x_1 + x_2 + \cdots + x_n + x_{n+1} \geq 1 + n$, that is, the statement $\mathcal{P}(n+1)$ is true.

For the general case, if $a_1, \ldots, a_n$ are positive and letting $G = \sqrt[n]{a_1 a_2 \ldots a_n}$, if we consider the identities $x_1 = \frac{a_1}{G}, \ldots, x_n = \frac{a_n}{G}$, we have that $x_1 \ldots x_n = 1$. Then, $x_1 + \cdots + x_n \geq n$, which is equivalent to $\dfrac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}$.

Example 3.1.15. *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be natural numbers. Suppose that $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_m < mn$. Then it is possible to cancel out some terms (not all of them) from both sides of the above equality while always preserving the equality.*

We use induction over $k = m + n$. Since $n \leq x_1 + x_2 + \cdots + x_n < mn$, then $m > 1$, and similarly $n > 1$, then m, $n \geq 2$ and $k \geq 4$. For $m + n = 4$, we have that $m = n = 2$, and the only possible cases are $1 + 1 = 1 + 1$ and $1 + 2 = 1 + 2$ (maybe in a different order) and the result is immediate.
Suppose that $k = m + n > 4$ and consider

$$s = x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_m < mn.$$

Without loss of generality, we can suppose that x_1 is the largest term of the set of x_i's, with $i = 1, 2, \ldots, n$, and y_1 is the largest term of the set of y_j's, with $j = 1, 2, \ldots, m$. We can also assume that $x_1 > y_1$, because if $x_1 = y_1$ the problem is solved. Then we have

$$(x_1 - y_1) + x_2 + \cdots + x_n = y_2 + \cdots + y_m.$$

We need to prove that the sum $s' = y_2 + \cdots + y_m$ satisfies the required condition, that is, $s' < n(m-1)$. Since $y_1 \geq y_2 \geq \cdots \geq y_m$, it follows that $y_1 \geq \frac{s}{m}$, hence

$$s' = s - y_1 \leq s - \frac{s}{m} = s\frac{m-1}{m} < mn\frac{m-1}{m} = n(m-1),$$

and now we can apply the principle of mathematical induction to reach the desired conclusion.

Exercise 3.1.

(i) *Prove by induction that for $q \neq 1$,*

$$1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$

(ii) *Prove that $1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.*

Exercise 3.2. *For the* **Fibonacci sequence**, *defined by $a_1 = 1$, $a_2 = 1$ and, for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$, prove that*

$$a_{n+2} = 1 + a_1 + a_2 + \cdots + a_n.$$

Exercise 3.3. *Prove that 3^{n+1} divides $2^{3^n} + 1$, for any integer number $n \geq 0$.*

Exercise 3.4. *There are 3^n coins with identical aspect, but one of the coins is false and its weight is less than the weight of the real coins. Prove how, with a plate weighing scale[9], in n weighings we can identify the false coin.*

[9]A plate weighing scale is a balance with two plates that will be at the same level if the weight of the objects placed in each one of the plates is the same.

Exercise 3.5.

(i) *For which integers n does it follow that 7 divides $2^n - 1$?*
(ii) *For which positive integers n does it follow that 7 divides $2^n + 1$?*

Exercise 3.6. *Find the values of a_n, if $a_1 = 1$ and for each $n \geq 2$, it follows that*

$$a_1 + a_2 + \cdots + a_n = n^2.$$

Exercise 3.7. *For $n \geq 1$, prove that*

$$\left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{3}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Exercise 3.8. *Find the values of a_n, if $a_1 = 1$ and for each $n \geq 2$*

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} = \frac{n\sqrt{a_{n+1}}}{2}.$$

Exercise 3.9. *Prove the next inequality relating the geometric mean and the arithmetic mean,*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

by following the indicated steps:

(i) *Use induction to prove that*

$$x^{n+1} - (n+1)x + n = (x-1)^2(x^{n-1} + 2x^{n-2} + 3x^{n-3} + \cdots + n).$$

Therefore, for $x > 0$, it follows that $x^{n+1} - (n+1)x + n \geq 0$.
(ii) *Apply the previous inequality to $x = \frac{a}{b}$, where $a = \frac{x_1+\cdots+x_{n+1}}{n+1}$ and $b = \frac{x_1+\cdots+x_n}{n}$, and conclude that*

$$\left(\frac{x_1 + \cdots + x_{n+1}}{n+1} \right)^{n+1} \geq x_{n+1} \left(\frac{x_1 + \cdots + x_n}{n} \right)^n = x_{n+1} b^n.$$

(iii) *Now use induction again in order to finish the proof.*

Exercise 3.10. *Let $0 < a_1 < a_2 < \cdots < a_n$ and $e_i = \pm 1$. Prove that $\sum_{i=1}^n e_i a_i$ has at least $\binom{n+1}{2}$ different values when the e_i vary over the 2^n possible elections of the signs.*

Exercise 3.11. *A sequence $a_1, a_2, \ldots, a_{2n}$ of numbers 0 or 1 is said to be even-balanced if $a_1 + a_3 + \cdots + a_{2n-1} = a_2 + a_4 + \cdots + a_{2n}$. Prove that an arbitrary sequence of numbers 0 or 1, with $2n + 1$ elements, has a subsequence*[10] *even-balanced with $2n$ elements.*

[10]See Subsection 7.2.7, for the definition of subsequence.

Exercise 3.12. *Prove that the only infinite sequence $\{a_n\}$ of positive numbers such that for each positive integer number n, the equality*

$$a_1^3 + a_2^3 + \cdots + a_n^3 = (a_1 + a_2 + \cdots + a_n)^2$$

holds, is the sequence given by $a_n = n$, for $n = 1, 2, \ldots$.

Exercise 3.13. *If $a_1 < a_2 < \cdots < a_n$ are positive integers, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2,$$

with equality if and only if $a_k = k$, for each $k = 1, 2, \ldots, n$.

Exercise 3.14.

(i) *Prove that, for $n \geq 1$, it follows that*

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right)\cdots\left(1 + \frac{1}{2^n}\right) < \frac{5}{2}.$$

(ii) *Prove that, for $n \geq 1$, it follows that*

$$\left(1 + \frac{1}{1^3}\right)\left(1 + \frac{1}{2^3}\right)\cdots\left(1 + \frac{1}{n^3}\right) < 3.$$

Exercise 3.15. *Let $a_1, a_2, \ldots, a_n$ be real numbers greater than or equal to 1. Prove that*

$$\sum_{i=1}^{n} \frac{1}{1 + a_i} \geq \frac{n}{1 + \sqrt[n]{a_1 \cdots a_n}}.$$

3.2 Binomial coefficients

The **factorial** of an integer number $n \geq 0$, denoted by $n!$, can be defined by induction as follows:

(a) $0! = 1$,
(b) $n! = n(n-1)!$, for $n \geq 1$.

Observation 3.2.1. *If $n \geq 1$, then*

$$n! = n(n-1)\cdots 2 \cdot 1.$$

For all integers n and m, with $0 \leq m \leq n$, we define the **binomial coefficient**[11] $\binom{n}{m}$ as

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}. \tag{3.5}$$

[11] For a combinatorial meaning of the binomial coefficients and the factorial, see [19].

Properties 3.2.2. *For $0 \leq m \leq n$, it follows that:*

(a) $\binom{n}{0} = 1$ *and* $\binom{n}{n} = 1$.

(b) $\binom{n}{m} = \binom{n}{n-m}$.

(c) *For each $m = 1, 2, \ldots, n-1$, it follows that*

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}. \tag{3.6}$$

This identity is known as **Pascal's formula**.

(d) $\binom{n}{m}$ *is a positive integer.*

Proof. To prove (a) and (b) we just apply the definition of binomial coefficient.

(c) To prove this property, observe that

$$\begin{aligned}\binom{n-1}{m-1} + \binom{n-1}{m} &= \frac{(n-1)!}{(m-1)!(n-1-m+1)!} + \frac{(n-1)!}{m!(n-1-m)!} \\ &= \frac{m(n-1)! + (n-m)(n-1)!}{m!(n-m)!} = \frac{n!}{m!(n-m)!} = \binom{n}{m}.\end{aligned}$$

(d) This statement can be justified by induction over n. For $n = 0$ and $n = 1$, it is clear since $\binom{0}{0} = 1$ and $\binom{1}{0} = \binom{1}{1} = 1$.

Suppose that for $n-1$ the assumption is true, and let us prove that $\binom{n}{m}$ is an integer, for $m = 0, \ldots, n$. By (a), we have that $\binom{n}{0} = \binom{n}{n} = 1$ are integers. By (c), for $m = 1, \ldots, n-1$, we have that $\binom{n}{m}$ is a sum of two integers and therefore it is an integer. □

Theorem 3.2.3 (Binomial theorem)**.** *Let a and b be real numbers and let n and m be integers with $0 \leq m \leq n$. The numbers $\binom{n}{m}$ are the binomial coefficients in the binomial development $(a+b)^n$, that is,*

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{i}a^{n-i}b^i + \cdots + \binom{n}{n}b^n. \tag{3.7}$$

Using the sum notation we can write the previous equality as

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

This identity is known as **Newton's binomial formula**.

Proof. Induction over n yields a prove of this identity.
If $n = 0$, then $(a+b)^0 = 1$ and $\binom{0}{0}a^0b^0 = 1$.
Suppose that $n > 0$ and that the identity is valid for $n-1$, that is,

$$(a+b)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-1-i}b^i$$

is true, then

$$\begin{aligned}
(a+b)^n &= (a+b)(a+b)^{n-1} = (a+b)\sum_{i=0}^{n-1}\binom{n-1}{i}a^{n-1-i}b^i \\
&= \binom{n-1}{0}a^n + \sum_{i=1}^{n-1}\binom{n-1}{i}a^{n-i}b^i + \sum_{i=1}^{n-1}\binom{n-1}{i-1}a^{n-i}b^i + \binom{n-1}{n-1}a^0b^n \\
&= \binom{n}{0}a^n + \sum_{i=1}^{n-1}\left[\binom{n-1}{i} + \binom{n-1}{i-1}\right]a^{n-i}b^i + \binom{n-1}{n-1}b^n \\
&= \binom{n}{0}a^n + \sum_{i=1}^{n-1}\binom{n}{i}a^{n-i}b^i + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i.
\end{aligned}$$

In the last step we have used Pascal's formula. □

Example 3.2.4. *If n is a positive integer, then*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0.$$

Apply the binomial theorem with $a = 1$ and $b = -1$, to obtain

$$0 = (1-1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i = \sum_{i=0}^{n}(-1)^i\binom{n}{i}.$$

Exercise 3.16. *Prove the following equalities:*

(i) $\displaystyle\sum_{j=0}^{n}\binom{n}{j} = 2^n.$ (ii) $\displaystyle\sum_{j=0}^{n}\binom{n}{j}2^j = 3^n.$

Exercise 3.17. *Prove the following equalities:*

(i) $\displaystyle\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}.$

(ii) $\displaystyle\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}.$

Exercise 3.18. *Prove the following equalities:*

(i) $\displaystyle\sum_{j=0}^{n}\binom{n}{j}^2=\binom{2n}{n}$.

(ii) $\displaystyle\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}=\binom{n+m}{r}$.

(iii) $\displaystyle\sum_{k=0}^{n}\binom{m+k}{k}=\binom{m+n+1}{n}$.

(iv) $\displaystyle\sum_{k=m}^{n}\binom{k}{m}=\binom{n+1}{m+1}$.

Exercise 3.19. *Calculate the following sums:*

(i) $\displaystyle\sum_{j=1}^{n}j\binom{n}{j}$. (ii) $\displaystyle\sum_{j=0}^{n}\frac{1}{j+1}\binom{n}{j}$.

Exercise 3.20. *Prove the following equalities:*

(i) $\displaystyle\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j}\binom{n}{j}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

(ii) $\displaystyle\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j(j+1)}\binom{n}{j}=\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

(iii) $\displaystyle\sum_{j=0}^{n}\frac{(-1)^{j}}{(j+1)^2}\binom{n}{j}=\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$.

Exercise 3.21. *Prove that the following relations hold:*

(i) $\displaystyle\sum_{j=1}^{n}(-1)^j j\binom{n}{j}=0$. (ii) $\displaystyle\sum_{j=1}^{n}(-1)^j j^2\binom{n}{j}=0$.

Exercise 3.22. *Prove that the number of odd integers in the following list*

$$\binom{n}{0},\binom{n}{1},\ldots,\binom{n}{n}$$

is a number which is a power of 2.

Exercise 3.23. *For each prime number* $p\geq 3$, *prove that the number* $\binom{2p-1}{p-1}-1$ *is divisible by* p^2.

3.3 Infinite descent

The method of infinite descent was frequently used by Pierre Fermat (1601–1665), therefore it is also known as Fermat's method. In general, it is used to prove that something does not happen. For instance, Fermat used it to prove that there are no integer solutions of the equation $x^4 + y^4 = z^2$, with $xyz \neq 0$.

The theoretical basis of his method rests on the fact that there is no such thing as an infinite decreasing collection of positive integers. In other words, we cannot find an infinite collection of positive integers such that $n_1 > n_2 > n_3 > \cdots$.

There are two ways to use this idea in order to prove a statement. The first is to start with a statement $\mathcal{P}(n_1)$ which we suppose is true. If from this statement, we can find a positive integer number $n_2 < n_1$ such that $\mathcal{P}(n_2)$ is valid and if from this last statement we can find a positive integer number $n_3 < n_2$ such that in turn $\mathcal{P}(n_3)$ is valid, and so on and so forth, then an infinite number of positive integers is generated satisfying $n_1 > n_2 > n_3 > \cdots$, but this is not possible, so $\mathcal{P}(n_1)$ is not true. Let us see an example in order to illustrate this method.

Example 3.3.1. *The number $\sqrt{2}$ is not a rational number.*

Suppose that $\sqrt{2}$ is a rational number, then $\sqrt{2} = \frac{m_1}{n_1}$, with m_1 and n_1 positive integers. Since $\sqrt{2} + 1 = \frac{1}{\sqrt{2}-1}$, we have that

$$\sqrt{2} + 1 = \frac{1}{\frac{m_1}{n_1} - 1} = \frac{n_1}{m_1 - n_1}, \quad \text{therefore} \quad \sqrt{2} = \frac{n_1}{m_1 - n_1} - 1 = \frac{2n_1 - m_1}{m_1 - n_1}.$$

Since $1 < \sqrt{2} < 2$, substituting the suppose rational value $\sqrt{2}$ leads to $1 < \frac{m_1}{n_1} < 2$, hence $n_1 < m_1 < 2n_1$. From here we have that, $2n_1 - m_1 > 0$ and $m_1 - n_1 > 0$. Then, if we define $m_2 = 2n_1 - m_1$ and $n_2 = m_1 - n_1$, we have that $m_2 < m_1$ and $n_2 < n_1$, since $n_1 < m_1$ and $m_1 < 2n_1$, respectively. Then, $\sqrt{2} = \frac{m_1}{n_1} = \frac{m_2}{n_2}$, with $m_2 < m_1$ and $n_2 < n_1$. Continuing this process we can generate an infinite number of positive integers m_i and n_i such that

$$\sqrt{2} = \frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m_3}{n_3} = \cdots,$$

with $m_1 > m_2 > m_3 > \cdots$ and $n_1 > n_2 > n_3 > \cdots$, but this is not possible. Therefore, $\sqrt{2}$ is not a rational number.

Example 3.3.2. *Find all the pairs of positive integers a, b that satisfy the equation*

$$a^2 - 2b^2 = 0. \tag{3.8}$$

Suppose that there exist positive integers a_1, b_1 such that

$$a_1^2 - 2b_1^2 = 0.$$

This implies that a_1 is an even number, that is, $a_1 = 2a_2$ for some positive integer a_2. Then, since

$$(2a_2)^2 - 2b_1^2 = 0, \quad \text{it follows that} \quad 2a_2^2 - b_1^2 = 0.$$

Therefore b_1 is even, that is, $b_1 = 2b_2$ with b_2 a positive integer. Substitution in the last equation leads to

$$2a_2^2 - (2b_2)^2 = 0, \quad \text{or} \quad a_2^2 - 2b_2^2 = 0. \tag{3.9}$$

This implies that a_2 and b_2 is another pair of positive integers satisfying equation (3.8). Since $a_1 = 2a_2$, we have that $a_1 > a_2$. Moreover, the above equations imply that $a_1 > b_1 > a_2 > b_2$. Equation (3.9) proves that a_2 is even, that is, $a_2 = 2a_3$ for some positive integer a_3.

Repeating the above arguments we obtain an infinite sequence of natural numbers in which each term is smaller than the previous one, that is,

$$a_1 > b_1 > a_2 > b_2 > a_3 > b_3 > \cdots .$$

However, this sequence cannot exist. Therefore, there is no pair of natural numbers that satisfy the equation (3.8). Note that this example proves also that $\sqrt{2}$ is not a rational number.

The other way to use the infinite descent method has a more positive character. It can be used to show that a series of propositions $\mathcal{P}(a)$ are valid, where a is an element of a set $A \subset \mathbb{N}$. To do this we use the following argument: suppose that $\mathcal{P}(a)$ is not valid for some $a \in A$ and define the set $B = \{a \in A \,|\, \mathcal{P}(a) \text{ is not true}\}$. Since $B \neq \emptyset$, in B there is a first element, say b. Now, using the hypothesis of the problem, we can find a positive integer number $c < b$ such that $\mathcal{P}(c)$ is not valid. This leads to a contradiction since b was the minimum element of B. Then, $\mathcal{P}(a)$ has to be true for all $a \in A$.

Example 3.3.3 (Putnam, 1973). *Let $a_1, a_2, \ldots, a_{2n+1}$ be integers such that if we take one out, then the remaining numbers can be divided in two sets of n integers which have the same sum. Prove that all numbers are equal.*

We can suppose that $a_1 \leq a_2 \leq \cdots \leq a_{2n+1}$. If we subtract the smallest number from all these numbers, the new numbers also satisfy the inequality and the conditions of the problem. Then, without loss of generality, we can suppose that $a_1 = 0$.

The sum of the $2n$ remaining numbers different from a_1 satisfies the condition of being congruent to 0 modulo 2. Now, let us see that if we choose any two numbers the pair will have the same parity. Let a_i and a_j be any two such numbers and $S = a_1 + \cdots + a_{2n+1}$. Since $S - a_i \equiv S - a_j \equiv 0 \mod 2$, we have that $a_i \equiv a_j \mod 2$.

If we divide by 2 all the numbers, the new collection has the same properties. Using the same arguments we can conclude that

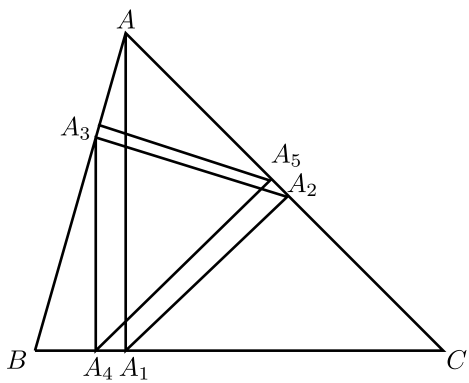
$$a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \equiv 0 \mod 2^2.$$

We can continue this argument to conclude that

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \equiv 0 \mod 2^k,$$

for each $k \geq 1$, but this is possible only in the case in which all the numbers are equal to zero, and therefore the original numbers are equal.

Example 3.3.4. *Let ABC be an acute triangle. Let A_1 be the foot of the altitude from A, A_2 the foot of the altitude from A_1 over CA, A_3 the foot of the altitude from A_2 over AB, A_4 the foot of the altitude from A_3 over CB, and so on and so forth. Prove that all the A_i's are different.*



Observe that each A_n is a point in one of the sides of the triangle, that is, it cannot be in the extension of the side and it cannot be a vertex of the triangle. This follows from the fact that if A_n is in the extension of one side of the triangle, then the triangle is obtuse, and if it is a vertex, then the triangle is either obtuse or a right triangle.

Note also that A_n and A_{n+1} do not coincide, because they belong to different sides of the triangle and they are not vertices.

Suppose now that A_n coincides with A_m, with $n < m$, and suppose also that n is the smallest index with the property that A_n coincides with some A_m. We now see that $n = 1$, since otherwise we would have that A_{n-1} coincides with A_{m-1}, and therefore n does not satisfy the property of being the smallest index n such that $A_n = A_m$, for some $m > n$.

Now if A_1 coincides with A_m, with $m \geq 3$, then A_{m-1} has to be the vertex A, but we already saw that no A_m is vertex of a triangle. Then, A_1 cannot coincide with any A_m.

Exercise 3.24 (Hungary, 2000). *Find the prime numbers p for which it is impossible to find integers a, b and n, with n positive, such that $p^n = a^3 + b^3$.*

Exercise 3.25. *The equation $x^2 + y^2 + z^2 = 2xyz$ has no integer solutions except when $x = y = z = 0$.*

Exercise 3.26. *Find all the pairs of positive integers (a, b) such that $ab + a + b$ divides $a^2 + b^2 + 1$.*

Exercise 3.27. *Prove that for $n \neq 4$, it does not exist a regular polygon with n sides such that the vertices are points with integer coordinates.*

3.4 Erroneous induction proofs

In this section we present some examples showing the necessity of verifying all the steps required in a proof which uses the induction mathematical principle. Sometimes, if we miss proving one detail this can lead us to some absurd situations, as we will see.

Example 3.4.1. *All non-negative powers of 2 are equal to 1.*

The induction basis is $n = 0$. The statement is true because $2^0 = 1$.
Suppose this is true for all $k \leq n$, that is, $2^0 = 2^1 = \cdots = 2^n = 1$. Let us now verify the statement for 2^{n+1}. Follow the identities and see that

$$2^{n+1} = \frac{2^{2n}}{2^{n-1}} = \frac{2^n \cdot 2^n}{2^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$

Then the proof is complete.

Error. The inductive step is not valid for $n = 0$, that is, $P(0) \Rightarrow P(1)$ is a false statement.

Example 3.4.2. *All integers greater than or equal to 2 are even.*

The induction base for $n = 2$ is clearly valid, since 2 is even.
Suppose that for each integer number k with $2 \leq k \leq n$, the statement is true, that is, such numbers k are even. Let us see now that $n+1$ is also an even number. We write $n + 1$ as $n + 1 = k_1 + k_2$, with $k_1, k_2 \leq n$. By the induction hypothesis k_1 and k_2 are even numbers, then the sum $n + 1$ is also even. This completes the proof.

Error. The inductive step is erroneous. Numbers k_1 and k_2 have to be greater than or equal to 2. But this is not always true, for example, for $n = 3$. Also, we can justify that in the inductive step we require two previous numbers satisfying the statement. But it turns out that the statement is not valid for the first two numbers, 2 and 3, since 3 is not an even number.

Example 3.4.3. *Consider the statement*

$$I_n : n(n+1) \quad \textit{is odd for all positive integers } n.$$

Where is the mistake in the following proof?

Suppose that the statement I_n is valid for n and we will show that it is true for $n+1$. We start from the identity

$$(n+1)(n+2) = n(n+1) + 2(n+1).$$

On the right-hand side of the identity we have, by the induction hypothesis, that $n(n+1)$ is odd and, if we add to this last number the even number $2(n+1)$, we have that $n(n+1)+2(n+1)$ is also odd, and therefore the statement I_{n+1} is also valid.

Error. We have not verified the induction basis.

Example 3.4.4. *Consider the following statement:*

R_n : *if there are n straight lines, not two of them parallel,*
then all the lines have a common point.

Where is the error in the following proof?

The statement R_1 is true. Also R_2 is true, since two non-parallel straight lines meet in one point.
Suppose the statement valid for $n-1$ straight lines, and consider now n straight lines $l_1, l_2, \dots, l_n$, where no two of them are parallel. By the induction hypothesis the $n-1$ straight lines $l_1, \dots, l_{n-1}$ have a common point, call it P. Now, instead of taking out the straight line l_n, take out the line l_{n-1}. Then, by the induction hypothesis, the $n-1$ straight lines $l_1, \dots, l_{n-2}, l_n$ have a common point, call it Q. But l_1 and l_2 have just one common point, then $P = Q$. Then the n straight lines $l_1, \dots, l_n$ have P as a common point.

Error. The induction basis has to be proved for $n = 3$, but the statement R_3 is false. Also note that the proof of the inductive step from R_2 to R_3 does not hold since l_2 is eliminated in the second part.

Exercise 3.28. *Consider the statement: Every function defined on a finite set is constant.*

Find the error in the following proof.

Let $f : A \to B$ be a function defined on a finite set A. We will perform the proof by induction over n, the number of elements of the set A. If $n = 1$, it is clear that f is constant. Suppose that the statement is valid for all the functions defined on sets with n elements and let f be a function defined on a set $A = \{a_1, \dots, a_{n+1}\}$, with $n+1$ elements. Consider $C = A \setminus \{a_{n+1}\}$, which is a set with n elements, by the induction hypothesis, the function f restricted to C is constant. If we now *consider* $D = A \setminus \{a_n\}$ *, which also has* n *elements, the function* f *restricted to* D *is constant, but these two constants coincide with* $f(a_1)$*. Then the function* f *is constant in all the set* A *and therefore the proof is complete.*

Exercise 3.29. *Consider the statement: For any* $n \geq 2$*, if there are* n *coins, one of which is false and lighter than the other coins, then the false coin can be identified in at most* 4 *weighing of the coins in a balance with two plates.*

Find the error in the following proof.

Of course, if we just have 2 *coins, weighing the coins just once is enough to identify the false coin, we just put one in each plate of the balance and the plate that raises up has the false coin.*

Suppose that the result is valid for k *coins and consider* $k+1$ *coins, where just one of them is lighter than the others. Now leave one coin out, if in the rest* k *coins we cannot identify the false coin weighing* 4 *times we are done; the false coin is the one we took out, otherwise weighing* 4 *times we found the false coin among the* k *coins.*

Exercise 3.30. *Consider the statement: Every non-negative integer is equal to* 0*.*

Find the error in the following proof.

For $n \geq 0$*, let* $\mathcal{P}(n)$ *be the statement: "*$n = 0$*".*

Of course $\mathcal{P}(0)$ *is true. Suppose that for* $k \geq 0$ *the statements* $\mathcal{P}(0), \mathcal{P}(1), \ldots, \mathcal{P}(k)$ *are true. The veracity of* $\mathcal{P}(k+1)$ *follows from the fact that* $(k+1) = k+1$*, because if* $\mathcal{P}(k)$ *and* $\mathcal{P}(1)$ *are true, "*$k = 0$*", "*$1 = 0$*", then* $k+1 = 0+0 = 0$ *and therefore* $\mathcal{P}(k+1)$ *is true. By the strong induction principle "*$n = 0$*" for all* $n \geq 0$*.*

Chapter 4

Quadratic and Cubic Polynomials

4.1 Definition and properties

Consider an expression of the form

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$

where a_0, a_1, a_2 and a_3 are constant numbers. We say that $P(x)$ is a **cubic polynomial** or a polynomial of degree 3 in the variable x if $a_3 \neq 0$; if $a_3 = 0$ and $a_2 \neq 0$ we say that $P(x)$ is a **quadratic polynomial** or a **polynomial of degree** 2; in case that $a_3 = a_2 = 0$ and $a_1 \neq 0$, we say that $P(x)$ is a **linear polynomial** or a polynomial of degree 1; finally, if $a_3 = a_2 = a_1 = 0$ and $a_0 \neq 0$, we say that $P(x)$ is a **constant polynomial** and its degree is 0. The degree of the polynomial $P(x)$ is denoted by $\deg(P)$. The constants a_3, a_2, a_1 and a_0 are called the **coefficients** of the polynomial. A polynomial with all the coefficients equal to zero is known as the **zero polynomial**[12]. If in a polynomial, the coefficient of the highest power of x is 1, we say that the polynomial is a **monic polynomial**.

We say that $a_3x^3 + a_2x^2 + a_1x + a_0$ and $b_3x^3 + b_2x^2 + b_1x + b_0$, are two **equal polynomials** if $a_i = b_i$, for $i = 0, 1, 2, 3$.

We can evaluate the polynomials by replacing the variable x by a number t, the value of the polynomial $P(x)$, in $x = t$, is $P(t)$.

A zero of a polynomial $P(x)$ is a number r such that $P(r) = 0$. We also say that r is a **root of the polynomial** or a **solution** of the equation $P(x) = 0$.

If the coefficients of a polynomial $P(x)$ are integers, we say that $P(x)$ is a **polynomial over the integers** or a polynomial with integer coefficients; similarly, if the coefficients are rational numbers, we say that the polynomial is a **polynomial over the rationals**, etc.

[12]In this book, the zero polynomial has no degree.

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_4

In many aspects, the polynomials are like the integers, they can be added, subtracted, multiplied and divided. To see how this is done consider the polynomials

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3,$$
$$Q(x) = b_0 + b_1x + b_2x^2 + b_3x^3.$$

We define the **sum of polynomials** as

$$(P+Q)(x) = P(x) + Q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3;$$

the **subtraction** as

$$(P-Q)(x) = P(x) - Q(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + (a_3 - b_3)x^3,$$

and the product of a polynomial by a constant c as

$$(cP)(x) = cP(x) = ca_0 + ca_1x + ca_2x^2 + ca_3x^3.$$

The **product of two polynomials** is defined as

$$\begin{aligned}(PQ)(x) = (P(x))(Q(x)) &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\ &+ (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\ &+ (a_2b_3 + a_3b_2)x^5 + a_3b_3x^6.\end{aligned}$$

Finally, we define the **polynomial division**. Given the above two polynomials, with the following restrictions

$$\begin{aligned}P(x) &= a_3x^3 + a_2x^2 + a_1x + a_0 \quad \text{with} \quad a_3 \neq 0, \\ Q(x) &= b_3x^3 + b_2x^2 + b_1x + b_0, \quad \text{with some coefficient different from zero,}\end{aligned}$$

there are always $S(x)$ and $R(x)$ such that

$$P(x) = S(x)Q(x) + R(x), \quad \text{with } \deg(R) < \deg(Q) \quad \text{or} \quad R(x) = 0.$$

We call the polynomial $S(x)$ **the quotient** and $R(x)$ **the remainder** of the division of $P(x)$ by $Q(x)$.

If $R(x) = 0$, we say that $Q(x)$ **divides** (exactly) $P(x)$ and we write $Q(x)|P(x)$. In the next section, we will see that a number a is a zero of a polynomial $P(x)$ if and only if $x - a$ divides $P(x)$.

A polynomial $H(x)$ is the **greatest common divisor** of $P(x)$ and $Q(x)$ if and only if

1. $H(x)$ divides both $P(x)$ and $Q(x)$,
2. If $K(x)$ is any other polynomial which divides both $P(x)$ and $Q(x)$ then $K(x)$ also divides $H(x)$.

It can be proved that $H(x)$ is unique up to a multiplication by a constant number.

4.1.1 Vieta's formulas

(a) If a monic polynomial $P(x) = x^2 + px + q$ has roots a and b, then

$$x^2 + px + q = (x - a)(x - b) = x^2 - (a + b)x + ab,$$

comparing the coefficients, it follows that

$$p = -(a + b) \quad \text{and} \quad q = ab. \tag{4.1}$$

(b) If a monic polynomial $P(x) = x^3 + px^2 + qx + r$ has roots a, b and c, then

$$x^3 + px^2 + qx + r = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc$$

and comparing the coefficients we have that

$$p = -(a + b + c), \quad q = ab + bc + ca, \quad r = -abc. \tag{4.2}$$

Formulas (4.1) and (4.2) are known as **Vieta's formulas**.

Let us see the following examples.

Example 4.1.1 (USSR, 1986). *The roots of the polynomial $x^2 + ax + b + 1 = 0$ are natural numbers. Prove that $a^2 + b^2$ is not a prime number.*

If r and s are roots of the polynomial, formula (4.1), assures us that $r+s = -a$ and $rs = b + 1$. Then, a and b are integers and $a^2 + b^2 = (r + s)^2 + (rs - 1)^2 = (r^2 + 1)(s^2 + 1)$ is the product of two numbers greater than 1.

Example 4.1.2 (Germany, 1970). *Let p and q be real numbers, with $p \neq 0$, and let a, b, c be roots of the polynomial $px^3 - px^2 + qx + q$.*

Prove that $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = -1$.

Since $px^3 - px^2 + qx + q = p(x^3 - x^2 + \frac{q}{p}x + \frac{q}{p}) = p(x - a)(x - b)(x - c)$, by formula (4.2), we have that

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = (a + b + c)\left(\frac{ab + bc + ca}{abc}\right) = \frac{\frac{q}{p}}{-\frac{q}{p}} = -1.$$

The generalization of Vieta's formulas for polynomials of degree greater than 3, will be presented in Chapter 8, dedicated to the theory of polynomials.

Exercise 4.1. *Find all solutions of $m^2 - 3m + 1 = n^2 + n - 1$, where m and n are positive integers.*

Exercise 4.2. *Let $P(x) = ax^2 + bx + c$ be a quadratic polynomial with real coefficients. Suppose that $P(-1)$, $P(0)$ and $P(1)$ are integers. Prove that $P(n)$ is an integer number for every integer number n.*

Exercise 4.3. *If a, b, c, p and q are integers, with $q \neq 0$, $(p,q)=1$ and $\frac{p}{q}$ a root of the equation $ax^2+bx+c=0$, prove that p divides c and q divides a.*

Exercise 4.4. *Let a, b, c be real numbers, with a and c non-zero. Let α and β be the roots of the polynomial ax^2+bx+c and let α' and β' be the roots of the polynomial cx^2+bx+a. Prove that if $\alpha, \beta, \alpha', \beta'$ are positive numbers then $(\alpha+\beta)(\alpha'+\beta') \geq 4$.*

Exercise 4.5. *Find all real numbers a such that the sum of the squares of the roots of $P(x) = x^2-(a-2)x-a-1$ is a minimum.*

Exercise 4.6. *If p, q and r are solutions of the equation $x^3-7x^2+3x+1=0$, find the value of $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$.*

Exercise 4.7. *The solutions of the equation $x^3+bx^2+cx+d=0$ are p, q and r. Find a quadratic equation with roots $p^2+q^2+r^2$ and $p+q+r$, in terms of b, c and d.*

Exercise 4.8. *For which positive real numbers m, the roots x_1 and x_2 of the equation*

$$x^2-\left(\frac{2m-1}{2}\right)x+\frac{m^2-3}{2}=0,$$

satisfy the condition $x_1 = x_2 - \frac{1}{2}$?

Exercise 4.9 (Kangaroo, 2003)**.** *Let $P(x)$ be a polynomial such that $P(x^2+1) = x^4+4x^2$. Find $P(x^2-1)$.*

Exercise 4.10. *The natural numbers a, b, c and d satisfy that $a^3+b^3=c^3+d^3$ and $a+b=c+d$. Prove that two of these numbers are equal.*

4.2 Roots

If we divide a degree 3 polynomial $P(x)$ by $x-a$ we get

$$P(x) = (x-a)Q(x)+r, \quad \text{with} \quad r \in \mathbb{R} \quad \text{and} \quad \deg(Q)=2.$$

Let $x=a$, then it follows that $P(a)=r$, therefore

$$P(x) = (x-a)Q(x)+P(a). \tag{4.3}$$

It follows, from equation (4.3), that

$$P(a)=0 \quad \text{if and only if} \quad P(x)=(x-a)Q(x), \tag{4.4}$$

for some polynomial $Q(x)$. This result is known as the factor theorem.

If a_1 and a_2 are two different zeros of $P(x)$, then, by equation (4.4), it follows that $P(x) = (x - a_1)Q(x)$. Since, $P(a_2) = (a_2 - a_1)Q(a_2) = 0$ and $a_2 \neq a_1$, then $Q(a_2) = 0$, hence $Q(x) = (x - a_2)Q_1(x)$, for some polynomial $Q_1(x)$. Then,

$$P(x) = (x - a_1)(x - a_2)Q_1(x) \quad \text{with} \quad \deg(Q_1) = 1.$$

If $\deg(P) = 3$ and $P(a_i) = 0$, for a_1, a_2, a_3, then

$$P(x) = c(x - a_1)(x - a_2)(x - a_3) \quad \text{with} \quad c \in \mathbb{R}.$$

If there exists $m \in \mathbb{N}$ and a polynomial $Q(x)$ such that

$$P(x) = (x - a)^m Q(x) \quad \text{with} \quad Q(a) \neq 0, \tag{4.5}$$

we say that the **multiplicity** of the root a is m.

Example 4.2.1. *If $P(x) = ax^2 + bx + c$ is a quadratic polynomial, then its roots are*

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \textit{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{4.6}$$

Since $a \neq 0$, we can rewrite the polynomial $P(x)$ as

$$\begin{aligned}
P(x) = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left(x^2 + \frac{2bx}{2a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{4c}{4a}\right) \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{1}{4a^2}\left(b^2 - 4ac\right)\right] \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right] \\
&= a\left[\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2 - 4ac}}{2a}\right]\left[\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2 - 4ac}}{2a}\right] \\
&= a\left[x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right]\left[x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right].
\end{aligned}$$

Then,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{are its roots.}$$

The number $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic polynomial $P(x) = ax^2 + bx + c$.

Observation 4.2.2. *If the discriminant,* b^2-4ac *of the quadratic polynomial* $P(x) = ax^2 + bx + c$ *is zero, the polynomial* $P(x)$ *can be written as a constant multiplied by the square of a linear polynomial. In this case the polynomial* $P(x)$ *has only one real root, which is equal to* $-\frac{b}{2a}$.

In the case where the discriminant is greater than zero, that is, $b^2 - 4ac > 0$ *then the polynomial* $P(x)$ *has two different real roots.*

Finally, when the discriminant is negative, there are two distinct complex roots. This case will be analyzed in Chapter 5.

Summarizing, if r and s are the roots of $P(x)$, then the discriminant is zero if and only if $r = s$. In case the discriminant is different from zero, then $r \neq s$, and $P(x) = a(x-r)(x-s)$.

Now, we shall see what is the geometric meaning of the discriminant of a quadratic polynomial. Remember that $P(x) = ax^2 + bx + c$ can be written as

$$P(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{1}{4a^2}\left(b^2 - 4ac\right)\right] = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}.$$

To construct the graph of the previous equation, that is, to locate the set of pairs of points $(x, y) = (x, P(x))$ in the Cartesian plane, we let $y = P(x)$ and obtain the equation

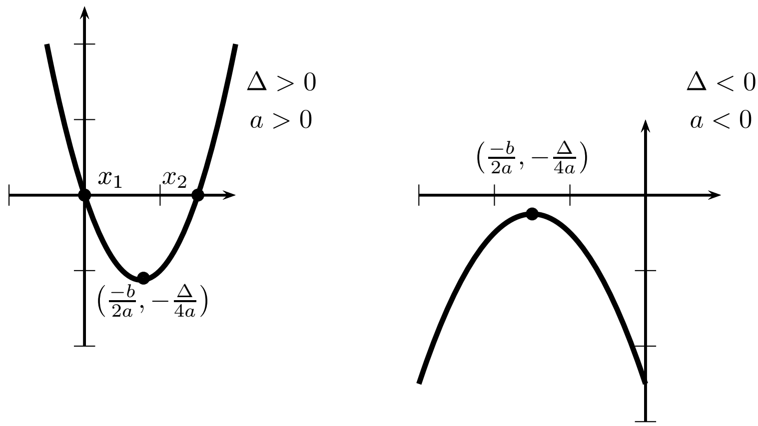
$$y + \frac{\Delta}{4a} = a\left(x + \frac{b}{2a}\right)^2, \tag{4.7}$$

which represents the parabola with vertex at the point $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$, where the sign of the coefficient a determines if the parabola opens up ($a > 0$), or down ($a < 0$).

In fact, the equation (4.7) tells us much more about the quadratic polynomial $P(x)$. Suppose that $a > 0$, that is, the parabola opens up. The second coordinate of the vertex, $-\frac{\Delta}{4a}$, is positive if and only if $-\Delta > 0$, that is, if the discriminant of $P(x)$ is negative, which means that the graph of the parabola does not intersect the X-axis. Then $P(x)$ has no real roots.

If $-\frac{\Delta}{4a}$ is negative, the graph of the parabola intersects the X-axis in two points x_1 and x_2, which are the roots of $P(x)$. Observe that in this case, we have that Δ is positive, which agrees with the fact that $P(x)$ has two real roots. Moreover, the polynomial $P(x)$ reaches the minimum value at the point $x = -\frac{b}{2a}$, and $-\frac{\Delta}{4a}$ is the minimum value of $P(x)$.

Making a similar analysis if $a < 0$, we can conclude that when the parabola opens down it will intersect the X-axis if and only if $\Delta \geq 0$. Here, the polynomial $P(x)$ reaches its maximum value at the point $x = -\frac{b}{2a}$ where the maximum value of $P(x)$ is $-\frac{\Delta}{4a}$.



When the graph of the parabola is tangent to the X-axis, we are in the case in which the discriminant is zero, that is, when both roots are equal.

As an application for the quadratic polynomial theory we can prove the following inequality.

Example 4.2.3 (Cauchy–Schwarz inequality). *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, it follows that*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

The expression $P(x) = \sum_{i=1}^{n}(a_i x + b_i)^2$ is a quadratic polynomial in x. Since $P(x) \geq 0$, we have that $P(x)$ cannot have two different real roots, therefore the discriminant cannot be positive. Now, in order to calculate the discriminant of this polynomial, we expand each term of the sum $(a_i x + b_i)^2 = a_i^2 x^2 + 2a_i b_i x + b_i^2$, from where the polynomial takes the form

$$P(x) = \left(\sum_{i=1}^{n} a_i^2\right) x^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right) x + \left(\sum_{i=1}^{n} b_i^2\right),$$

therefore, the discriminant is

$$4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0.$$

Rewriting the above expression and taking the square root, we obtain the desired result, that is,

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

Note that the equality holds if the discriminant is zero, that is, when the polynomial has just one real root. Observe that this holds if $x = \frac{-b_i}{a_i}$ for all $i = 1, 2, \ldots, n$, and this means $\frac{-b_i}{a_i}$ is constant.

It is also possible to prove the inequality between the geometric mean and the arithmetic mean of two numbers, by analyzing the discriminant of a certain quadratic polynomial.

Example 4.2.4. *For a, $b \geq 0$, it follows that $\frac{a+b}{2} \geq \sqrt{ab}$ and the equality holds if and only if $a = b$.*

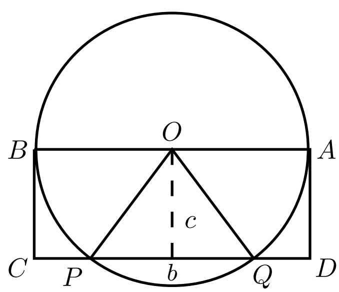
Consider the quadratic polynomial $P(x) = (x - \sqrt{a})(x - \sqrt{b})$, which has two real roots, so its discriminant is positive or zero. But the discriminant of $P(x) = (x - \sqrt{a})\left(x - \sqrt{b}\right) = x^2 - \left(\sqrt{a} + \sqrt{b}\right)x + \sqrt{ab}$ is

$$\left(\sqrt{a} + \sqrt{b}\right)^2 - 4\sqrt{ab} = a + b - 2\sqrt{ab}.$$

Since the discriminant is positive or zero, then $a + b \geq 2\sqrt{ab}$. Also, the equality holds if and only if $\sqrt{a} = \sqrt{b}$ and then $a = b$.

Example 4.2.5. *The following construction dates from the Greek period and it is the procedure to find geometrically what in modern terms are the roots of a quadratic polynomial $x^2 + bx - c^2$, with b and c positive numbers.*

Construct first an isosceles triangle OPQ with base PQ of length b, and the altitude from O of length c. We draw a circumference through the vertices P and Q of the triangle and with center O.



Let AB be the diameter of the circumference parallel to PQ and construct the rectangle $ABCD$. The positive root of $x^2 + bx - c^2$ is equal to the length of the segment CP and the other root (the negative root) is equal to the negative length of the segment CQ.

Let us see how to prove this statement. By Vieta's formulas (4.1) it follows that if α and β are the roots of the polynomial, $\alpha + \beta = -b$ and $\alpha\beta = -c^2$, then we necessarily have a negative root. Let us consider the right triangles BCP

and QCB. Since $\angle PBC = \angle BQC$, these triangles are similar. Then we have that $CP \cdot CQ = BC^2 = c^2$, therefore

$$CP(-CQ) = -c^2.$$

On the other hand, since $CQ = CP + b$, we have that $CP - CQ = -b$. Then, CP and $-CQ$ fulfill Vieta's relations, therefore these last numbers are the roots of the equation.

Example 4.2.6. *Factorize $a^3 + b^3 + c^3 - 3abc$.*

Let us consider the polynomial

$$P(x) = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc,$$

with roots a, b and c. That is,

$$\begin{aligned} a^3 - (a+b+c)a^2 + (ab+bc+ca)a - abc = 0, \\ b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc = 0, \\ c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc = 0. \end{aligned}$$

Adding these three equalities and factoring $a + b + c$, we get

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca). \tag{4.8}$$

Remember that the expression $a^2 + b^2 + c^2 - ab - bc - ca$ can be written as

$$\frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right].$$

From this we get the following factorization,

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]. \tag{4.9}$$

Example 4.2.7 (Czechoslovakia, 1969)**.** *Let a, b and c be real numbers such that $a + b + c > 0$, $ab + bc + ca > 0$ and $abc > 0$. Prove that a, b, c are positive.*

Let us consider the cubic and monic polynomial with roots a, b and c, that is,

$$P(x) = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc.$$

For $x \leq 0$, we have that $P(x) < 0$, then we can guarantee that the roots are positive.

Exercise 4.11. *For what values λ the polynomial $\lambda x^2 + 2x + 1 - \frac{1}{\lambda}$ does have two equal roots?*

Exercise 4.12. *Let a, b and c be positive real numbers. Is it possible, for each of the following polynomials $P(x) = ax^2+bx+c$, $Q(x) = bx^2+cx+a$, $R(x) = cx^2+ax+b$, to have both real roots?*

Exercise 4.13. *For what integer values of k are the solutions of the equation*

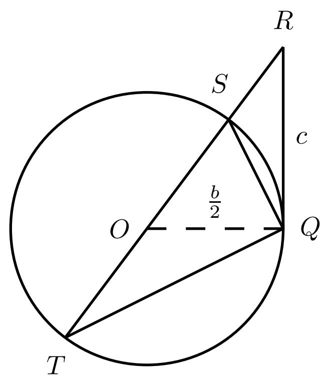
$$kx^2 - (1-2k)x + k - 2 = 0$$

rational numbers?

Exercise 4.14 (Czech-Slovakia, 2006)**.** *Find all pairs of integers (a, b) such that $a + b$ is a root of the polynomial $x^2 + ax + b$.*

Exercise 4.15. *Find all integer values of x for which the polynomial $x^2 - 5x - 1$ is a perfect square.*

Exercise 4.16. *Solve, using geometry, the equation $x^2 + bx - c^2$, with b and c positive numbers, using the following construction due to R. Descartes. Draw a circumference with center O and radius $\frac{b}{2}$. Draw QR, a tangent to the circumference through Q, with $QR = c$. Let S and T be the points where the straight line through R and O cuts the circumference.*



The quadratic polynomial has one positive root and one negative root. In the figure, the length of the segment RS is equal to the positive root and the negative root is equal to the negative length of the segment RT.

Exercise 4.17. *Let $P(x) = ax^2+bx+c$ be a quadratic polynomial such that $P(x) = x$ does not have real solutions. Prove that $P(P(x)) = x$ has no real solutions either.*

Exercise 4.18 (Poland, 2007)**.** *Let $P(x)$ be a polynomial with integer coefficients. Prove that if $P(x)$ and $P(P(P(x)))$ have a common root, then they have a common integer root.*

Exercise 4.19 (Russia, 2009). *Let a, b and c be non-zero real numbers such that $ax^2 + bx + c > cx$, for all real numbers x. Prove that $cx^2 - bx + a > cx - b$, for all real numbers x.*

Exercise 4.20 (Russia, 2009). *Two different real numbers a and b are such that the equation $(x^2 + 20ax + 10b)(x^2 + 20bx + 10a) = 0$ has no real solutions. Prove that $20(b - a)$ cannot be an integer number.*

Exercise 4.21 (Russia, 2011). *Let $P(x)$ be a monic quadratic polynomial such that $P(x)$ and $P(P(P(x)))$ have a common root. Prove that $P(0)P(1) = 0$.*

Exercise 4.22. *Let a, b, c, d, e and f be positive integers such that they satisfy the relation $ab + ac + bc = de + df + ef$, and let $N = a + b + c + d + e + f$. Prove that if N divides $abc + def$ then N is a composite number.*

Exercise 4.23. *Let $P(x)$ and $Q(x)$ be two quadratic polynomials with integer coefficients. If both polynomials have an irrational number as a common zero, prove that one of them is a multiple of the other.*

Exercise 4.24. *Determine if there exist polynomials $x^2 - b_1x + c_1 = 0$ and $x^2 - b_2x + c_2 = 0$, with b_1, c_1, b_2 and c_2 different, such that the four roots are b_1, c_1, b_2 and c_2.*

Exercise 4.25. *Let a, b and c be real numbers. Prove that at least one of the following equations has a real solution:*

$$\begin{aligned}
x^2 + (a - b)x + (b - c) &= 0,\\
x^2 + (b - c)x + (c - a) &= 0,\\
x^2 + (c - a)x + (a - b) &= 0.
\end{aligned}$$

Exercise 4.26. *Let a, b and c be real numbers such that $a + b + c = 0$. Prove that*

$$\frac{a^5 + b^5 + c^5}{5} = \left(\frac{a^2 + b^2 + c^2}{2}\right)\left(\frac{a^3 + b^3 + c^3}{3}\right).$$

Exercise 4.27. *Let a, b and c be real numbers such that $a+b+c = 3$, $a^2+b^2+c^2 = 5$, $a^3 + b^3 + c^3 = 7$. Find $a^4 + b^4 + c^4$.*

Exercise 4.28 (OMCC, 2001). *Let a, b and c be real numbers such that the equation $ax^2+bx+c = 0$ has two different real solutions p_1, p_2 and the equation $cx^2+bx+a = 0$ has two different real solutions q_1, q_2. Also the numbers p_1, q_1, p_2, q_2, in this order, form an arithmetic progression. Prove that $a + c = 0$.*

Exercise 4.29. *Let a, b and c be real numbers different from zero, with $a+b+c=0$ and $a^3+b^3+c^3=a^5+b^5+c^5$. Prove that $a^2+b^2+c^2=\frac{6}{5}$.*

Exercise 4.30 (Russia, 2010)**.** *Let $P(x)$ be a cubic polynomial with integer coefficients such that there exist different integers a, b, and c such that $P(a)=P(b)=P(c)=2$. Prove that no integer number d satisfying $P(d)=3$ exists.*

Chapter 5

Complex Numbers

5.1 Complex numbers and their properties

The roots of a quadratic equation are not always real numbers. For instance, the roots of the equation $x^2 + 2x + 10 = 0$, which can be calculated using directly equation (4.6) of Section 4.2, are

$$\frac{-2+\sqrt{-36}}{2} \qquad \text{and} \qquad \frac{-2-\sqrt{-36}}{2}.$$

For this reason it is necessary to consider "imaginary" numbers, like $\sqrt{-36}$.

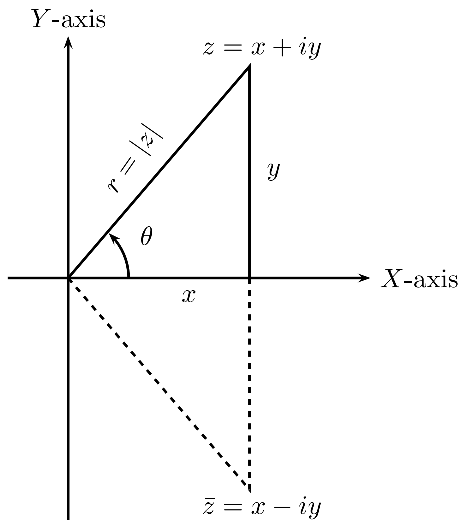
A **complex number** z is an expression of the form $x + iy$, where x and y are real numbers, and $i^2 = -1$. At this moment we will not worry about the meaning of i, for the time being we will be interested only in the fact that its square is -1. The **real part** of z, which will be denoted by Re z, is the number x, and the **imaginary part** of z, denoted by Im z, is the number y.

The set $\mathbb{C}$ of all complex numbers $x + iy$ can be identified with the set of points (x, y) in the Cartesian plane, and when this is done, due to this representation it is called the complex plane. The X-axis is called the real axis and the Y-axis is known as the imaginary axis.

In order to work with complex numbers, we need the following three definitions: the **complex conjugate** of z, denoted by $\bar{z}$, is the complex number $x - iy$; the **module** or **norm** of z, denoted by $|z|$, is the real number $\sqrt{x^2 + y^2}$, which is the distance from the origin to the point (x, y) representing z. Finally, the **argument** of $z \neq 0$ is the angle between the positive real axis and the line through 0 and z, taken in the counterclockwise direction. The argument of z is denoted by $\arg z$ and generally it is assigned a value between 0 and 2π.

There is another form to write a complex number z, which is known as the **polar form** of the complex number z. Let r be the norm of z, $r = |z|$, and $\theta = \arg z$, then $z = r(\cos\theta + i\sin\theta)$.

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_5



The set of complex numbers is similar to the set of real numbers, in the sense that there are two operations that can be applied to its elements, the sum and the product of complex numbers.
Let $z = x + iy$ and $w = u + iv$ be two complex numbers. Then the sum of these complex numbers is given by

$$z + w = (x + iy) + (u + iv) = (x + u) + i(y + v),$$

and the product by

$$\begin{aligned} z \cdot w &= (x + iy)(u + iv) = xu + xiv + iyu + i^2 yv \\ &= (xu - yv) + i(xv + yu). \end{aligned}$$

The set of real numbers can be seen as a subset of the complex numbers, if we identify each real number x with the complex number $(x, 0)$.

Observe that to number i corresponds the number $(0, 1)$ and that

$$(x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1) \cdot (y, 0),$$

and this is also represented as $x + iy$, that is, to (x, y) corresponds the complex number $x + iy$.

The operations of sum and product of complex numbers satisfy the same properties that these operations satisfy in the set of real numbers, as being commutative, associative and share also the existence of neutral elements for both operations, these being $0 = (0, 0)$ and $1 = (1, 0)$, respectively. The sum of complex numbers is exactly the same operation as the sum of vectors in the Cartesian plane; the operation that can be a novelty is the product of complex numbers. Let us see how to find the inverse of a complex number. For that purpose we need an important relation that exists between the norm of a complex number z and its conjugate, that is

$$z\bar{z} = |z|^2. \tag{5.1}$$

In order to see that, note that if $z = x+iy$, then $z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$. From identity (5.1), if $z \neq 0$, tells us that its multiplicative inverse, $\frac{1}{z}$, is equal to $\frac{\bar{z}}{|z|^2}$.

Exercise 5.1. *Let z, w be two complex numbers. Prove that:*

(i) $\overline{z+w} = \bar{z}+\bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$, $\bar{\bar{z}} = z$.
(ii) $|\bar{z}| = |z|$, $|zw| = |z|\,|w|$, $\left|\frac{1}{z}\right| = \frac{1}{|z|}$, *if* $z \neq 0$.
(iii) $Re\ z = \frac{1}{2}(z+\bar{z}) \leq |z|$.
(iv) $Im\ z = \frac{1}{2i}(z-\bar{z}) \leq |z|$.
(v) $|z+w| \leq |z|+|w|$.
(vi) $|z+w|^2 + |z-w|^2 = 2(|z|^2+|w|^2)$.

Exercise 5.2. *Find the complex numbers z such that* $Im\big(z+\frac{1}{z}\big) = 0$.

Exercise 5.3. *If z and w are complex numbers with $|z+w| = |z-w|$ and $w \neq 0$, then $\dfrac{iz}{w}$ is a real number.*

Exercise 5.4. *If z and w are complex numbers, prove that:*

(i) $|1-\bar{z}w|^2 - |z-w|^2 = (1+|zw|)^2 - (|z|+|w|)^2$.
(ii) $|1+\bar{z}w|^2 - |z+w|^2 = (1-|z|^2)(1-|w|^2)$.

Exercise 5.5. *If z and w are complex numbers such that $(1+|w|^2)z = (1+|z|^2)w$, prove that $z = w$ or $\bar{z}w = 1$.*

Exercise 5.6. *Let z_1, z_2, z_3 be complex numbers such that*

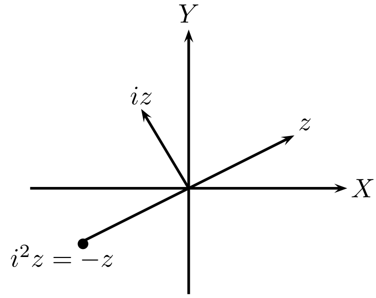
$$z_1 + z_2 + z_3 = 0 \quad \textit{and} \quad |z_1| = |z_2| = |z_3| = 1.$$

Prove that $z_1^2 + z_2^2 + z_3^2 = 0$.

Exercise 5.7. *Prove that if z_1, z_2 are complex numbers with $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$, then $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.*

Exercise 5.8. *Let a, b, c and d be complex numbers with the same norm, and such that $a+b+c = d$. Prove that d is equal to a, b or c.*

Now, we present the geometric meaning of the complex product. Let us see two examples that will help us to understand such meaning. First, let us consider the transformation $z \to iz$, that is, $x+iy \to i(x+iy) = ix-y$, which in vectorial form means that the vector (x,y) becomes the vector $(-y,x)$, and note that both vectors are perpendicular. Then, the transformation $z \to iz$ corresponds to the rotation of the complex plane in the counterclockwise direction, around zero, with angle $\frac{\pi}{2}$.



If we apply the previous transformation twice, we obtain $z \to iz \to i(iz) = i^2 z = -z$, which is a rotation with angle π.

The previous examples show that complex multiplication implicitly carries a rotation of the Cartesian plane. If instead of taking the product of a complex number z with i, we take the product with another complex number w, certain rotation appears in a natural way. We will see this now.

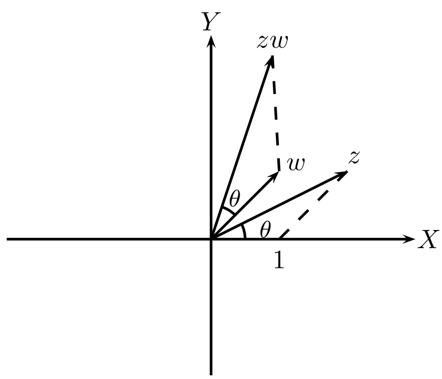
Let $z = x+iy = r(\cos\theta + i\sin\theta)$ and $w = u+iv = t(\cos\phi + i\sin\phi)$ be two complex numbers written in polar form. Taking its product, we obtain

$$\begin{aligned} z \cdot w &= r(\cos\theta + i\ \sin\theta) \cdot t(\cos\phi + i\ \sin\phi) \\ &= r\,t(\cos\theta\cos\phi + i\ \cos\theta\ \sin\phi + i\ \sin\theta\cos\phi + i^2\ \sin\theta\ \sin\phi) \\ &= r\,t\,(\cos\theta\cos\phi - \ \sin\theta\ \sin\phi + i(\cos\theta\ \sin\phi + \sin\theta\cos\phi)) \\ &= r\,t\,(\cos(\theta+\phi) + i(\sin\theta + \phi))\,, \end{aligned} \tag{5.2}$$

where we used the sine and cosine formula for the sum of two angles.

Thus, $\arg(z \cdot w) = \arg z + \arg w$, modulo 2π.

By identity (5.2), we conclude that geometrically the product of two complex numbers, z and w, is the complex number whose norm is the product of the norms of z and w, and its argument is the sum of the arguments of the same two numbers.



Using formula (5.2) repeatedly, for $z = \cos\theta + i\sin\theta$, we obtain the so-called **de Moivre's formula**, where for every integer n we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{5.3}$$

Exercise 5.9. *Prove that, for complex numbers a, b and c, the following are equivalent:*

(i) *The points a, b and c are collinear.*

(ii) $\dfrac{c-a}{b-a} \in \mathbb{R}$.

(iii) $c\bar{b} - c\bar{a} - a\bar{b} \in \mathbb{R}$.

(iv) $\begin{vmatrix} 1 & a & \bar{a} \\ 1 & b & \bar{b} \\ 1 & c & \bar{c} \end{vmatrix} = 0.$

Conclude that the equation of the line through b and c is $Im(\frac{z-c}{z-b}) = 0$.

Exercise 5.10. *Find the complex numbers z such that z, i, iz are collinear.*

Exercise 5.11. *Let z, w be two vertices of a square, find the other two vertices in terms of z and w.*

Exercise 5.12. *Prove by induction de Moivre's formula* (5.3).

Exercise 5.13. *Prove that if $z + \frac{1}{z} = 2\cos\theta$, then $z^n + \frac{1}{z^n} = 2\cos n\theta$, for every integer $n \geq 1$.*

5.2 Quadratic polynomials with complex coefficients

Now, let us see that it is possible to obtain explicitly the roots of every polynomial of degree 2 with complex coefficients. In order to do that, first we need to be able to solve the following type of equations

$$(x + iy)^2 = a + ib, \tag{5.4}$$

where a and b are real numbers. That is, we need to find the values of x and y, which means to find the square root of $a + ib$.

First, observe that since $(x+iy)^2 = x^2 - y^2 + i2xy$, equation (5.4) is equivalent to

$$a = x^2 - y^2 \quad \text{and} \quad b = 2xy, \tag{5.5}$$

the second equation implies that the sign of b determines if the signs of x and y are equal or different. Now, taking the norm on both sides of equation (5.4), it follows that

$$x^2 + y^2 = \sqrt{a^2 + b^2}. \tag{5.6}$$

Adding this last equation to the first equality of (5.5), we obtain $2x^2 = a + \sqrt{a^2 + b^2}$, that is,

$$x^2 = \frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right) \quad \text{or} \quad x = \pm\sqrt{\frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right)}.$$

Similarly, we obtain $y = \pm\sqrt{\frac{1}{2}\left(-a + \sqrt{a^2 + b^2}\right)}$. Note that x, y are well defined since $a + \sqrt{a^2 + b^2} \geq a + |a| \geq 0$ and $-a + \sqrt{a^2 + b^2} \geq -a + |a| \geq 0$.

Once we know how to calculate the square roots of complex numbers, we can calculate the zeros of every quadratic polynomial. In fact, if we consider the quadratic polynomial $P(z) = az^2 + bz + c$, with a, b, $c \in \mathbb{C}$, then, by using formula (4.6) we can always find both zeros of the polynomial. For instance, if the discriminant $b^2 - 4ac$ is a complex number, we can obtain its square root and then calculate the two roots, which are now complex numbers too.

As a consequence, every quadratic polynomial with complex coefficients has at least one complex root, and therefore can be written as the product of two linear factors with complex coefficients.

Observe also that if $P(z)$ is a polynomial with real coefficients, then for every complex number w we have that $P(\bar{w}) = \overline{P(w)}$, since the conjugate of each coefficient is the same number, just because it is a real number. Thus, if w is a zero of $P(z)$, then $\bar{w}$ is also a zero. Hence, for a polynomial with real coefficients, the complex roots, if they exist, appear in conjugate pairs.

Example 5.2.1. *Solve the equation $z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = 0$.*

Dividing the expression by z^4, we obtain that

$$\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) - 10 = \left(z + \frac{1}{z}\right)^4 - 6 - 10 = 0. \tag{5.7}$$

Making the change of variable $u = z + \frac{1}{z}$ we get equation $u^4 = 16$, which has solutions $u_1 = 2$, $u_2 = -2$, $u_3 = 2i$, $u_4 = -2i$.

From equation $u = z + \frac{1}{z}$ we have that $z = \frac{u}{2} \pm \sqrt{u^2/4 - 1}$. By substituting the four values of u we obtain the eight solutions: $z_{1,2} = 1$, $z_{3,4} = -1$, $z_{5,6} = i(1 \pm \sqrt{2})$, $z_{7,8} = -i(1 \pm \sqrt{2})$.

Example 5.2.2 (Romania, 1999). *Let p and q be complex numbers with $q \neq 0$. Prove that if the roots of the quadratic polynomial $x^2 + px + q^2 = 0$ have the same norm, then $\frac{p}{q}$ is a real number.*

Let x_1, x_2 be the roots of the given equation and let $r = |x_1| = |x_2|$. Then

$$\frac{p^2}{q^2} = \frac{(x_1+x_2)^2}{x_1x_2} = \frac{x_1}{x_2} + \frac{x_2}{x_1} + 2 = \frac{x_1\bar{x}_2}{r^2} + \frac{x_2\bar{x}_1}{r^2} + 2 = 2 + \frac{2}{r^2}\operatorname{Re}(x_1\bar{x}_2)$$

is a real number. Moreover, $\operatorname{Re}(x_1\bar{x}_2) \geq -|x_1\bar{x}_2| = -r^2$ and then $\frac{p^2}{q^2} \geq 0$.

Thus, $\frac{p}{q}$ is a real number.

Exercise 5.14. *Find all the complex numbers z such that $|z| = 1$ and $|z^2 + \bar{z}^2| = 1$.*

Exercise 5.15. *Let a, b, c be complex numbers with $|a| = |b| = |c| \neq 0$.*

(i) *Prove that if a root of the equation $az^2 + bz + c = 0$ has norm 1, then $b^2 - ac = 0$.*

(ii) *If each of the equations $az^2 + bz + c = 0$ and $bz^2 + cz + a = 0$ have a root of norm 1, then $|a - b| = |b - c| = |c - a|$.*

Exercise 5.16 (Romania, 2003). *If the complex numbers z_1, z_2, z_3, z_4, z_5 all have norm 1 and satisfy $\sum_{i=1}^{5} z_i = \sum_{i=1}^{5} z_i^2 = 0$, prove that these numbers are the vertices of a regular pentagon.*

Exercise 5.17 (Romania, 2007). *Let a, b, c be complex numbers of norm 1. Prove that there exist numbers $\alpha, \beta, \gamma \in \{-1, 1\}$, such that $|\alpha a + \beta b + \gamma c| \leq 1$.*

Exercise 5.18 (Romania, 2008). *Let a, b, c be complex numbers that satisfy $a\,|bc| + b\,|ca| + c\,|ab| = 0$. Prove that $|(a-b)(b-c)(c-a)| \geq 3\sqrt{3}\,|abc|$.*

Exercise 5.19 (Romania, 2009). *Find the complex numbers a, b, c, all with the same norm, such that $a + b + c = abc = 1$.*

5.3 The fundamental theorem of algebra

One of the goals of polynomial theory is to find the roots or the factors in which a polynomial can be decomposed. In this spirit we have the result known as the fundamental theorem of algebra.

Theorem 5.3.1 (The fundamental theorem of algebra). *Every polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $n \geq 1$, $a_i \in \mathbb{C}$ and $a_n \neq 0$, has at least one root in $\mathbb{C}$.*

For the proof of this theorem, see Section 5.5.

Let us first remember the factor theorem.

Theorem 5.3.2 (Factor theorem)**.** *If a is a zero of a polynomial $P(z)$, then $z - a$ is a factor of $P(z)$.*

This theorem has the following interpretation: to know the zeros of the polynomial is to know the polynomial. This is made precise in the following result.

Corollary 5.3.3. *Every polynomial $P(z)$ of degree n with coefficients in $\mathbb{C}$, can be written in the form*

$$P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n), \quad z_i \in \mathbb{C}, \ c \in \mathbb{C}.$$

That is, the polynomial has exactly n complex roots. The numbers z_1, z_2, ..., z_n are not necessarily distinct.

If it is difficult to find the roots of the polynomial, then it is a good idea to find another polynomial for which it could be easier to find the roots. Let us see an example.

Example 5.3.4 (USA, 1975)**.** *Let $P(x)$ be a polynomial of degree n such that $P(k) = k/(k+1)$, for $k = 0, 1, 2, \ldots, n$. Find $P(n+1)$.*

The condition $P(k) = k/(k+1)$ does not say anything about the roots of $P(x)$. Then, we can consider the polynomial of degree $n+1$,

$$Q(x) = (x+1)P(x) - x.$$

Clearly, the roots of $Q(x)$ are 0, 1, 2, ..., n, therefore we can write

$$(x+1)P(x) - x = Cx(x-1)(x-2) \cdots (x-n),$$

where C is a constant that is going to be determined. Evaluating in $x = -1$, we obtain that $1 = C(-1)(-2)(-3) \cdots (-n)(-(n+1))$, from where $C = \frac{(-1)^{n+1}}{(n+1)!}$.

Finally, if $x = n+1$, we get $(n+2)P(n+1) - n - 1 = (-1)^{n+1}$, hence $P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}$.

5.4 Roots of unity

Using de Moivre's formula (5.3) it follows immediately that the polynomial $z^n - 1 = 0$ has roots 1, w, w^2, ..., w^{n-1}, where $w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. These roots are known as the **nth roots of unity** and in the complex plane they can be identified as the vertices of a regular n-sided polygon inscribed in the unit circle with center at the origin[13]. By the factor Theorem 5.3.2, we have the decomposition

$$z^n - 1 = (z-1)(z-w)(z-w^2) \cdots (z-w^{n-1}).$$

[13]More generally, the equation $z^n - a = 0$, for any complex number $a \neq 0$, has n different complex solutions called the nth roots of a. These solutions are $\sqrt[n]{|a|}w^j$, for $j = 0, 1, \ldots, n-1$ and $w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

For instance, for the case $n = 3$, the roots of $z^3 - 1 = (z-1)(z^2+z+1) = 0$ are 1, $w = \frac{-1+i\sqrt{3}}{2}$ and $w^2 = \frac{1}{w} = \bar{w} = \frac{-1-i\sqrt{3}}{2}$, and they are known as the cubic roots of unity. Note that w satisfies $w^3 = 1$ and $1 + w + w^2 = 0$.

We have seen that $w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ generates all the nth roots of unity, that is, $U_n = \{w, w^2, \ldots, w^{n-1}, w^n = 1\}$. We say that one element $u \in U_n$ is a **primitive root of unity** if $u^m \neq 1$ for all positive integers $m < n$. Now, we can state the following result.

Theorem 5.4.1.

(a) *If n divides q, then any root of $z^n - 1 = 0$ is a root of $z^q - 1 = 0$.*

(b) *The common roots of $z^m - 1 = 0$ and $z^n - 1 = 0$ are the roots of $z^d - 1 = 0$, where d is the greatest common divisor of m and n, which is denoted by $d = (m, n)$.*

(c) *The primitive roots of $z^n - 1 = 0$ are w^k, where $0 \leq k \leq n$ and $(k, n) = 1$.*

Proof. (a) If $q = pn$ and w is a root of $z^n - 1$, it follows that $w^q - 1 = w^{pn} - 1 = (w^n)^p - 1 = (1)^p - 1 = 0$, then w is root of $z^q - 1$.

(b) If w is a root of $z^m - 1$ and $z^n - 1$, and $d = (m, n)$, we have that $d = am+bn$ for some integers a and b. Hence, $w^d - 1 = w^{am+bn} - 1 = w^{am}w^{bn} - 1 = (w^m)^a(w^n)^b - 1 = 1^a 1^b - 1 = 0$, therefore, w is a root of $z^d - 1$.
Conversely, since d divides m and n, by property (a), if w is a root of $z^d - 1$ then it is root of $z^m - 1$ and $z^n - 1$.

(c) To prove this part, let us see the following lemma.

Lemma 5.4.2. *Let $w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. The smallest positive integer m such that $(w^k)^m = 1$ is $m = \frac{n}{(k,n)}$, where (k, n) is the greatest common divisor of k and n.*

As a consequence of this lemma part (c) follows and w^k is a primitive root if and only if $(k, n) = 1$.

Proof of the lemma. First, note that $w^s = 1$ if and only if $s = an$. Let m be the smallest positive integer such that $(w^k)^m = 1$, then $km = an$. If $d = (k, n)$, then $k = k_1 d$ and $n = n_1 d$, with $(k_1, n_1) = 1$, hence $a = \frac{km}{n} = \frac{k_1 dm}{n_1 d} = \frac{k_1 m}{n_1}$ is an integer. Thus $n_1 | m$, and since $w^{kn_1} = w^{k_1 dn_1} = w^{k_1 n} = 1$, it follows that $m = n_1$. Finally, we have that $n = n_1 d = md$, therefore $m = \frac{n}{d} = \frac{n}{(k,n)}$. □

Example 5.4.3. *Prove the identity*

$$\sin\frac{\pi}{n} \cdot \sin\frac{2\pi}{n} \cdot \cdots \cdot \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

Consider the polynomial $P(z) = (1-z)^n - 1$, which can be written as $w^n - 1$, where $w = 1 - z$. The roots of $w^n = 1$ are the nth roots of unity, $w_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, for $k = 0, 1, \ldots, n-1$. Then, the roots of $P(z)$ are $z_k = 1 - w_k$.

Looking at the polynomial $P(z)$, we observe that it can be written as $P(z) = z(-n+Q(z))$, where $Q(z)$ is a polynomial of degree $n-1$. Hence, if we let $P(z) = 0$ we see that the roots should satisfy, by Vieta's formula (8.4) Section 8.3, that $(-1)^n n = \prod_{k=1}^{n-1} z_k$, and from this $n = \prod_{k=1}^{n-1} |z_k|$.

Now, we need to calculate $|z_i|$.

$$\begin{aligned}
|z_k| = |1 - w_k| &= \sqrt{\left(1 - \cos\left(\frac{2\pi k}{n}\right)\right)^2 + \left(\sin\left(\frac{2\pi k}{n}\right)\right)^2} \\
&= \sqrt{1 - 2\cos\left(\frac{2\pi k}{n}\right) + \cos^2\left(\frac{2\pi k}{n}\right) + \sin^2\left(\frac{2\pi k}{n}\right)} \\
&= \sqrt{2 - 2\cos\left(\frac{2\pi k}{n}\right)} = \sqrt{4\,\sin^2\left(\frac{\pi k}{n}\right)} = 2\,\sin\left(\frac{\pi k}{n}\right),
\end{aligned}$$

where we used that $\cos^2 x + \sin^2 x = 1$ and $1 - \cos(2x) = 2\,\sin^2 x$. The identity now follows.

Example 5.4.4 (AMC, 2002)**.** *Find the number of ordered pairs of real numbers (a, b) such that $(a + ib)^{2002} = a - ib$.*

Let $z = a + ib$, $\bar{z} = a - ib$, and $|z| = \sqrt{a^2 + b^2}$. The relation given above is equivalent to $z^{2002} = \bar{z}$. Note that $|z|^{2002} = |z^{2002}| = |\bar{z}| = |z|$, thus

$$|z|(|z|^{2001} - 1) = 0.$$

Hence, $|z| = 0$ and therefore $(a, b) = (0, 0)$ or $|z| = 1$. In the latter case we have that $z^{2002} = \bar{z}$, which is equivalent to $z^{2003} = \bar{z} \cdot z = |z|^2 = 1$. Since the equation $z^{2003} = 1$ has 2003 different solutions, then there are $1 + 2003 = 2004$ ordered pairs that satisfy the equation.

Exercise 5.20. *Solve the equation*

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

Exercise 5.21. *Find the solutions of the equation*

$$x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0.$$

Exercise 5.22 (Romania, 2007)**.** *Let n be a positive integer. Prove that there exists a complex number with norm 1 that is a solution of the equation $z^n + z + 1 = 0$ if and only if $n = 3m + 2$, for some positive integer m.*

Exercise 5.23. *If $w \neq 1$ is an nth root of unity, prove that:*

(i) $1 + w + w^2 + \cdots + w^{n-1} = 0$.

(ii) $1 + 2w + 3w^2 + \cdots + nw^{n-1} = \frac{n}{w-1}$.

Exercise 5.24. *If $w \neq 1$ is a nth root of unity:*

(i) *Prove that* $(1-w)(1-w^2)\dots(1-w^{n-1}) = n$.
(ii) *Find the value of* $\frac{1}{1-w} + \frac{1}{1-w^2} + \cdots + \frac{1}{1-w^{n-1}}$.

Exercise 5.25.

(i) *Prove that if $\omega \neq 1$ is a cubic root of unity (that is, $\omega^3 = 1$), then for a, b, $c \in \mathbb{C}$, it follows that*

$$a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega).$$

(ii) *Use* (i) *and equation* (4.8) *to obtain the following identity:*

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega). \tag{5.8}$$

Exercise 5.26. *Two regular polygons are inscribed in the same circle. The first polygon has* 1982 *sides and the second one has* 2973 *sides. If the polygons have some vertices in common, how many vertices in common do they have?*

Exercise 5.27. *Find the positive integers n for which x^2+x+1 divides $x^{2n}+x^n+1$.*

Exercise 5.28. *Let S be the set of integers x that can be written in the form $x = a^3 + b^3 + c^3 - 3abc$, for some integers a, b, c. Prove that if x, $y \in S$, then $xy \in S$.*

Exercise 5.29 (USA, 1976)**.** *If $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are polynomials such that*

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.

Exercise 5.30. *Find all the polynomials $P(z)$ of degree at most* 2 *with coefficients in $\mathbb{C}$, that satisfy $P(z)P(-z) = P(z^2)$.*

5.5 Proof of the fundamental theorem of algebra $\star$

As we have seen, the fundamental theorem of algebra states that every non-constant polynomial with complex coefficients has at least one complex root. This theorem includes polynomials with real coefficients.
The fundamental theorem of algebra can also be stated saying that every polynomial of degree n, with complex coefficients, has exactly n roots, counting the multiple roots as many times as they appear. The equivalence of both theorems can be proved using repeatedly the factor theorem.

In spite of the name of the theorem, there are no purely algebraic elementary proofs, since these proofs use the fact that the real numbers are complete and this is not an algebraic concept. This theorem was considered fundamental for algebra when the study of this discipline was concentrated in finding the roots of the polynomial equations with real or complex coefficients.

Some proofs of the theorem only show that every non-constant polynomial, with real coefficients, has a complex root. This is sufficient to establish the theorem in the general case, since given a non-constant polynomial $P(z)$ with complex coefficients, the polynomial $Q(z) = P(z)\overline{P(\bar{z})}$ has real coefficients and if z is a root of $Q(z)$, then z or its conjugate $\bar{z}$ is also a root of $P(z)$.

The proof that we present here uses a result known as the **growth lemma**.

Lemma 5.5.1 (Growth lemma)**.** *Given a polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ of degree $n \geq 1$, with complex coefficients, there is a real number $R > 0$ such that if $|z| \geq R$ then*

$$\frac{1}{2}|a_n||z^n| \leq |P(z)| \leq 2|a_n||z^n|.$$

Proof. Let $r(z) = \sum_{k=0}^{n-1} |a_k||z|^k$. By the triangle inequality, it follows that for every complex number z,

$$|a_n||z|^n - r(z) \leq |P(z)| \leq |a_n||z|^n + r(z).$$

Now, if $|z| \geq 1$ and $m < n$, it follows that $|z|^m \leq |z|^{n-1}$, therefore $r(z) \leq M|z|^{n-1}$, where $M = \sum_{k=0}^{n-1} |a_k|$. Taking $R =$max $\{1, 2M|a_n|^{-1}\}$, for $|z| \geq R$, it follows that

$$\begin{aligned}
|P(z)| &\leq |a_n||z|^n + r(z) \leq |a_n||z|^n + M|z|^{n-1} \\
&= |z|^{n-1}(|a_n||z| + M) \leq |z|^{n-1}(|a_n||z| + |a_n||z|) \\
&= 2|a_n||z|^n,
\end{aligned}$$

where the last inequality follows from the fact that $|z| \geq R \geq \frac{2M}{|a_n|} > \frac{M}{|a_n|}$.

Again, for $|z| \geq R$, we have that

$$\begin{aligned}
|P(z)| &\geq |a_n||z|^n - r(z) \geq |a_n||z|^n - M|z|^{n-1} \\
&\geq |a_n||z|^n - \frac{1}{2}|a_n||z|^n \\
&= \frac{1}{2}|a_n||z|^n,
\end{aligned}$$

where the last inequality holds if and only if $\frac{1}{2}|a_n||z|^n \geq M|z|^{n-1}$, which is true if and only if $|z| \geq \frac{2M}{|a_n|}$, and the proof is complete. □

The proof of the fundamental theorem of algebra that we present here is based only on advanced calculus. Calculus provide a very useful result presented as the following lemma, which can be consulted in [17].

Lemma 5.5.2. *If $f : D \to \mathbb{R}$ is a continuous function on D, a closed and bounded subset of $\mathbb{R}^2$, then f attains its minimum and maximum values in points of D.*

Theorem 5.5.3 (The fundamental theorem of algebra)**.** *Every polynomial*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $n \geq 1$, $a_i \in \mathbb{C}$ and $a_n \neq 0$, has at least one root in $\mathbb{C}$.

The proof is based on the following two lemmas.

Lemma 5.5.4. *Let $P(z)$ be a polynomial with complex coefficients. Then $|P(z)|$ attains its minimum value in some point $z_0 \in \mathbb{C}$.*

Proof. Let $s = |P(0)| = |a_0|$ and $R_1 = \max\left\{R, \sqrt[n]{2s|a_n|^{-1}}\right\}$, where the number R is given by the growth lemma. If $|z| > R_1$, then it follows that

$$|P(z)| \geq \frac{1}{2}|a_n||z^n| > \frac{1}{2}|a_n||R_1^n| \geq \frac{1}{2}|a_n|\frac{2s}{|a_n|} = s.$$

Thus, for all z such that $|z| > R_1$, it follows that $|P(z)| > |P(0)|$. In particular, if $R_2 > R_1$, then for every z such that $|z| \geq R_2$, the same inequality holds.

In this way we have found a closed disk D of radius R_2 with center at 0 such that $|P(z)| > |P(0)|$, for all $|z| > R_2$. Since $|P(z)|$ is a continuous function with real values, it follows by Lemma 5.5.2 that $|P(z)|$ attains its minimum value in D. $\square$

Lemma 5.5.5. *Let $P(z)$ be a non-constant polynomial with complex coefficients. If $P(z_0) \neq 0$, then $|P(z_0)|$ is not the minimum value of $|P(z)|$.*

Proof. Let $P(z)$ be a non-constant polynomial with complex coefficients, and let z_0 be a point such that $P(z_0) \neq 0$. Making the change of variable $z + z_0$ for z moves z_0 to the origin, and then we can assume that $P(0) \neq 0$. Now, we multiply $P(z)$ by $P(0)^{-1}$ so that we may assume that $P(0) = 1$. Then, we must show that 1 is not the minimum value of $|P(z)|$.

Let k be the smallest non-zero power of z in $P(z)$. Then we can assume that $P(z)$ has the form $P(z) = 1 + az^k + g(z)$, with $g(z)$ a polynomial of degree greater than k.

Let α be a kth root of $-a^{-1}$. Then, making one last change of variable, αz by z, we obtain that $P(z)$ has the form

$$P(z) = 1 - z^k + z^{k+1}g(z), \quad \text{for some polynomial } g(z).$$

For real positive values of z we obtain, using the triangle inequality, that

$$|P(z)| \leq |1 - z^k| + z^{k+1}|g(z)|.$$

Since $z^k < 1$, for $|z| < 1$, then

$$|P(z)| \leq 1 - z^k + z^{k+1}|g(z)| = 1 - z^k(1 - z|g(z)|).$$

For z small, $z|g(z)|$ is also small, then we can choose z_1 so that $z_1|g(z_1)| < 1$. It follows that $z_1^k(1 - z_1|g(z_1)|) > 0$, and then $|P(z_1)| < 1 = |P(0)|$, and this finishes the proof. □

Using these two lemmas we obtain the proof of the fundamental theorem of algebra.

Proof. Let $P(z)$ be a non-constant polynomial with complex coefficients. By Lemma 5.5.4, $|P(z)|$ has a minimum value in some point $z_0 \in \mathbb{C}$. Then, by Lemma 5.5.5, it follows that $|P(z_0)| = 0$. Therefore, $P(z)$ has a complex root. □

Chapter 6

Functions and Functional Equations

6.1 Functions

The concept of function is one of the most important in mathematics. A function is a relation between elements of two sets X and Y, which we denote by $f : X \to Y$, that satisfies:

(a) Every element $x \in X$ is related to some element $y \in Y$, and we write $y = f(x)$.

(b) Every element $x \in X$ is related to one and only one element of Y, that is, if $f(x) = y_1$ and $f(x) = y_2$, then necessarily $y_1 = y_2$.

The set X is called the **domain**, the set Y the **codomain** and the relation $y = f(x)$ the **correspondence rule**. We define the **image** or the **range** of a function $f : X \to Y$ as

$$\operatorname{Img} f = \{y \in Y \mid \text{there exists}\, x \in X \,\text{with}\, f(x) = y\}.$$

We say that f and g are **equal functions** if they have the same domain X, the same correspondence rule ($f(x) = g(x)$, for all $x \in X$) and the same codomain.

We define the **graph** of a function $f : X \to Y$ as

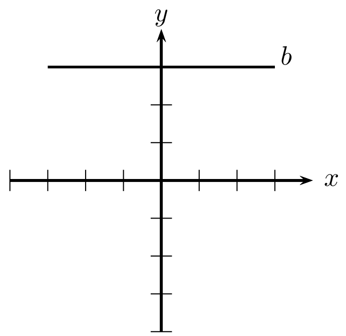
$$\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\},$$

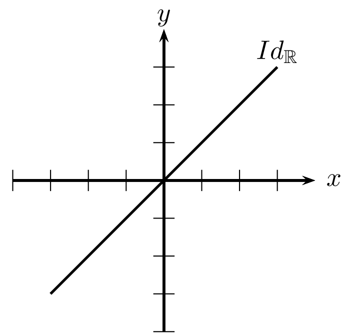
where $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ is the Cartesian product of X and Y.

Two simple but important functions are the constant function and the identity function, which are defined as follows: if $f : X \to Y$ is such that $f(x) = b$ for all $x \in X$, with $b \in Y$ fixed, then f is called the **constant function** equal to b, while the image is the set with only one element $\{b\}$; the **identity function** has the same domain and codomain, that is, $Id : X \to X$, and it is defined as $Id(x) = x$ for all $x \in X$.

For functions $f : \mathbb{R} \to \mathbb{R}$, the geometric representation of the graph in the Cartesian plane is useful. For instance, the geometric representation of the constant function b and the identity function $Id_{\mathbb{R}}$ are the following:

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_6





Some of the problems that appear in the mathematical olympiad contests, which make reference to functions, ask to find all the functions that satisfy a given property, or to find a specific value of some function. Often, these are difficult tasks, therefore it is important to understand the general behavior of the function before, so as to be able to decide correctly which functions satisfy or not the property. In this first section we will offer several examples of the kind of problems that can appear, and we will point out a series of facts that we can ask about a function.

Example 6.1.1. *Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = n(n+1)$. Let us find the values of m and n such that $4f(n) = f(m)$, where m and n are natural numbers.*

Suppose that $4f(n) = f(m)$, then $4n^2 + 4n = m^2 + m$. If we complete the square on the left side of the equation, we obtain

$$\begin{aligned} 4n^2 + 4n + 1 &= m^2 + m + 1, \\ (2n+1)^2 &= m^2 + m + 1, \end{aligned}$$

but $m^2 + m + 1$ cannot be the square of an integer because $m^2 < m^2 + m + 1 < (m+1)^2$. Therefore, there are no natural numbers m and n satisfying the condition.

Example 6.1.2. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(3n) = n + f(3n-3)$, for every positive integer n greater than 1 and such that $f(3) = 1$. Find the value of $f(12)$.*

It is natural to use the fact that $12 = 3 \cdot 4$ in order to find $f(12) = f(3 \cdot 4)$. Using the formula or relation of the hypothesis, we have that $f(12) = f(3 \cdot 4) = 4 + f(3 \cdot 3)$. We can repeatedly apply these substitutions to get

$$\begin{aligned} f(12) &= 4 + f(3 \cdot 3) = 4 + 3 + f(3 \cdot 2) \\ &= 4 + 3 + 2 + f(3 \cdot 1) = 4 + 3 + 2 + 1 = 10. \end{aligned}$$

Observe that we can find $f(3n)$ for all n, in the following way:

$$\begin{aligned} f(3n) - f(3(n-1)) &= n \\ f(3(n-1)) - f(3(n-2)) &= n - 1 \\ \vdots \quad & \quad \vdots \end{aligned}$$

$$\begin{aligned} \vdots \quad & \quad \vdots \\ f(6) - f(3) &= 2 \\ f(3) &= 1. \end{aligned}$$

Adding these equations results in $f(3n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Functions can be combined to form new functions. For instance, if we have functions with the same domain and codomain, we can add, subtract or multiply them to obtain new functions. For two functions f, $g : X \to Y$, with $Y \subset \mathbb{C}$, we define the **sum** and the **difference** of the functions f and g as

$$(f \pm g)(x) = f(x) \pm g(x), \text{ for all } x \in X.$$

We define the **product** of the functions f and g as

$$(f \cdot g)(x) = f(x) \cdot g(x), \text{ for all } x \in X.$$

Finally, if $g(x) \neq 0$ for all x, we define the **quotient** of the functions f and g as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ for all } x \in X.$$

Example 6.1.3. *Find all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(y + x) - f(y - x) = 4yx, \textit{ for all } x, y \in \mathbb{R}. \tag{6.1}$$

If we let $y = x$, we get that $f(2y) - f(0) = 4y^2$ and taking $f(0) = c$, we obtain $f(2y) = 4y^2 + c$. Now, if we let $2y = x$, we get that the solution of the functional equation is of the form $f(x) = x^2 + c$. It is easy to see that these functions satisfy equation (6.1).

Example 6.1.4. *Find all the functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$f(xy) = f\left(\frac{x}{y}\right), \textit{ for all } x, y \in \mathbb{R}^+.$$

In this case, if we let $y = x$ we have that $f(x^2) = f(1)$. This makes sense since the function is defined on the positive real numbers. On the other hand, if we let $y = x^2$, then $f(y) = f(1)$ for all $y \in \mathbb{R}^+$. Hence, the functions solving the functional equation are the constant functions. It is easy to see that these functions satisfy the functional equation.

In the last two examples we made the substitution $y = x$, which directly gave us the solution of the given functional equations. Next we present an example where, after substituting the variables for numerical values, we get information about the function.

Example 6.1.5. *Find all the functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x+y) + f(x-y) = f(x) + 6xy^2 + x^3, \text{ for all } x, y \in \mathbb{R}.$$

Let $y = 0$ to see that $2f(x) = f(x) + x^3$. Then $f(x) = x^3$ is the function we are looking for. We can directly check that $f(x) = x^3$ satisfies the equation.

Example 6.1.6. *Find all the functions* $f : \mathbb{N} \to \mathbb{R}$ *such that* $f(1) = 3$, $f(2) = 2$ *and*

$$f(n+2) + \frac{1}{f(n)} = 2, \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.2}$$

Observe that the original equation gives us

$$f(3) = 2 - \frac{1}{f(1)} = 2 - \frac{1}{3} = \frac{5}{3} \quad \text{and} \quad f(4) = 2 - \frac{1}{f(2)} = 2 - \frac{1}{2} = \frac{3}{2} = \frac{6}{4}.$$

Hence, we can conjecture that $f(n) = \frac{n+2}{n}$ is true, for all natural numbers n. It is true by hypothesis for the cases $n = 1$ and $n = 2$, and we already verified the result for $n = 3$ and $n = 4$. We will finish the proof using induction, that is, assuming the result holds for n, and then proving it for $n + 2$. Suppose that $f(n) = \frac{n+2}{n}$, then

$$\begin{aligned} f(n+2) &= 2 - \frac{1}{f(n)} = 2 - \frac{1}{\frac{n+2}{n}} = 2 - \frac{n}{n+2} \\ &= \frac{2n+4-n}{n+2} = \frac{n+4}{n+2}. \end{aligned}$$

Hence, we have the result for all $n \in \mathbb{N}$ and, in fact, the function satisfies the condition.

Example 6.1.7 (India, 2010)**.** *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *satisfying,*

$$f(x+y) + xy = f(x)f(y), \quad \text{for all} \quad x, y \in \mathbb{R}. \tag{6.3}$$

Let $x = y = 0$ in equation (6.3), then $f(0) = f(0)^2$, hence $f(0) = 0$ or $f(0) = 1$.
If $f(0) = 0$, we let $y = 0$ in equation (6.3) to get $f(x) = 0$, for all $x \in \mathbb{R}$. But the function $f(x) = 0$, for all $x \in \mathbb{R}$, (the constant function zero) does not satisfy condition (6.3) when $xy \neq 0$.
Suppose that $f(0) = 1$. If we let $x = 1$, $y = -1$, we get that $f(1)f(-1) = f(1-1) - 1 = f(0) - 1 = 0$, then $f(1) = 0$ or $f(-1) = 0$.
If $f(-1) = 0$, letting $y = -1$ we get $f(x-1) - x = 0$, then $f(x-1) = x$, and using $y = x - 1$ in this last equality, we get $f(y) = y + 1$.
If $f(1) = 0$, letting $y = 1$ we obtain $f(x+1) + x = f(x)f(1) = 0$ so that $f(x+1) = -x$. Finally, if we take $y = x + 1$ we obtain $f(y) = 1 - y$.
In this way, the only solutions are $f(x) = x + 1$ and $f(x) = 1 - x$. It is easy to check that these functions satisfy the functional equation (6.3).

Another way to create a new function given two functions $f : X \to Y$ and $g : Y \to Z$ is the **composition** of f and g, which is the function $g \circ f : X \to Z$ defined for all $x \in X$ by

$$(g \circ f)(x) = g(f(x)).$$

Observe that the composition $g \circ f$ is defined only if the codomain of f is contained in the domain of g.

Example 6.1.8 (IMO, 1977)**.** *Consider $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > f(f(n))$, for every positive integer n. Prove that $f(n) = n$, for every $n \in \mathbb{N}$.*

Let $A = \{f(1), f(2), \dots\}$ be the image of f. By hypothesis, note that for every $n \geq 2$, it follows that $f(n) > f(f(n-1))$. Hence for $n \geq 2$, $f(n)$ cannot be the minimum of the image of f. It follows that the minimum of A is $f(1)$ and that $f(n) > f(1)$ for $n \geq 2$.

Observe that if $m \geq p$, then $f(m) \geq p$. For $p = 2$, the result follows from the discussion above. Suppose the result true for $p \geq 2$ and let us show it holds for $p+1$. Let $m \geq p+1$, then $m-1 \geq p$ and $f(m-1) \geq p$; now, by hypothesis, $f(m) > f(f(m-1)) \geq p$, hence $f(m) \geq p+1$.

Now, let $A_p = \{f(p), f(p+1), f(p+2), \dots\}$. For every $n \geq p+1$, it follows that $f(n) > f(f(n-1))$. Since the observation guarantees that $f(f(n-1))$ belongs to the set A_p, then $f(n)$ cannot be the minimum of A_p, hence the minimum must be $f(p)$. Therefore, $f(n) > f(p)$ for all $n \geq p+1$.

Finally, from the last paragraph, it follows that

$$f(1) < f(2) < \cdots < f(p) < f(p+1) < \dots. \tag{6.4}$$

Now, let us show that $f(n) \geq n$. We have that $f(1) \geq 1$ and $f(2) > f(1)$, hence $f(2) \geq 2$. Similarly, from $f(2) \geq 2$ and $f(3) > f(2)$, it follows that $f(3) \geq 3$, and using induction we can prove that $f(n) \geq n$. Finally, if for some n we have that $f(n) > n$, then $f(n) \geq n+1$ and, by (6.4), $f(f(n)) \geq f(n+1)$, contradicting the hypothesis. Thus, $f(n) = n$, for every $n \in \mathbb{N}$.

Exercise 6.1. *Find all functions $f : \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$ satisfying the functional equation*

$$f(x) + f\left(\frac{1}{1-x}\right) = x, \text{ for all } x \neq 0, 1.$$

Exercise 6.2 (Canada, 1969)**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a function with the following properties:*

(i) $f(2) = 2$.

(ii) $f(mn) = f(m)f(n)$, *for all m and n.*

(iii) $f(m) > f(n)$, *for $m > n$.*

Prove that $f(n) = n$.

Exercise 6.3. *Find all functions* $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *such that*

$$xf(x)+2xf(-x)=-1,\ \text{for } x\neq 0.$$

Exercise 6.4. *Find all functions* $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *that satisfy the equation*

$$\frac{1}{x}f(-x)+f\left(\frac{1}{x}\right)=x,\ \text{for } x\neq 0.$$

Exercise 6.5 (Ireland, 1995)**.** *Find all functions* $f:\mathbb{R}\to\mathbb{R}$ *for which*

$$xf(x)-yf(y)=(x-y)f(x+y),\quad \text{for all } x,y\in\mathbb{R}.$$

Exercise 6.6 (Ukraine, 1997)**.** *Find all functions* $f:\mathbb{Q}^{+}\cup\{0\}\to\mathbb{Q}^{+}\cup\{0\}$*, that satisfy the following conditions:*

(a) $f(x+1)=f(x)+1$*, for all* $x\in\mathbb{Q}^{+}\cup\{0\}$.
(b) $f(x^2)=f(x)^2$*, for all* $x\in\mathbb{Q}^{+}\cup\{0\}$.

Exercise 6.7. *Find all functions* $f:\mathbb{R}\to\mathbb{R}$ *such that*

$$xf(y)+yf(z)+zf(x)=yf(x)+zf(y)+xf(z),\ \text{for } x,y,z \text{ real numbers.}$$

6.2 Properties of functions

In this section, we will study important properties that a function may or may not have. A good knowledge of these properties can help us to detect what kind of function we have.

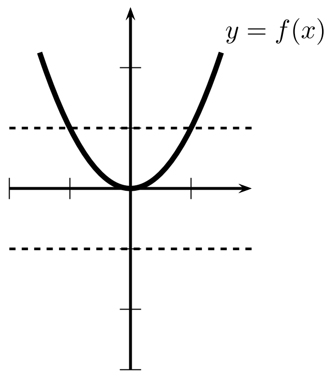
6.2.1 Injective, surjective and bijective functions

We say that a function $f:X\to Y$ is **injective** (also known as one-to-one) if for any x_1, $x_2\in X$, with $x_1\neq x_2$, it follows that $f(x_1)\neq f(x_2)$. The condition is equivalent to saying that if $f(x_1)=f(x_2)$, then $x_1=x_2$.
We say that a function $f:X\to Y$ is **surjective** (also known as onto) if $\operatorname{Img} f=Y$, that is, if for every $y\in Y$ there exists $x\in X$ such that $f(x)=y$.

Finally, we say that a function is **bijective** if it is injective and surjective.

For functions from the real numbers to the real numbers knowing the graph of the function can be very useful. The graph can help us to determine if the function is injective, surjective or both.

More precisely, a function is injective if any parallel line to the x-axis intersects the graph of the function in at most one point. A function is surjective, if any horizontal line $y=y_0$, with y_0 in the codomain of the function, intersects the graph in at least one point. The function shown in the following graph is not injective, since any parallel line above the x-axis, intersects the graph in two points. Moreover, it is not surjective since a horizontal line below the x-axis never intersects the graph of the function.



Example 6.2.1. *A function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $f(f(m) + f(n)) = m + n$ for all m, $n \in \mathbb{N}$, is injective.*

The function is injective, because if $f(m) = f(n)$, then $f(m)+f(n) = f(n)+f(n)$, and from here it follows that $m + n = f(f(m) + f(n)) = f(f(n) + f(n)) = n + n$, thus $m = n$.

Example 6.2.2. *The functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the condition*

$$f(xf(y)) + f(yf(x)) = 2xy, \text{ for all } x, y \in \mathbb{R}^+$$

are injective.

Letting $x = y$, we have that $f(xf(x)) = x^2$, in particular $f(f(1)) = 1$. Letting $x = f(1)$ in the last equation, we get that

$$f(1)^2 = f(f(1)f(f(1))) = f(f(1)) = 1,$$

hence $f(1) = 1$. If one takes $y = 1$ in the original equation, we obtain that

$$f(x) + f(f(x)) = 2x.$$

With this last equality we can show that f is injective. If $f(x) = f(y)$, then $2x = f(x) + f(f(x)) = f(y) + f(f(y)) = 2y$, hence $x = y$.

Example 6.2.3. *A function $f : \mathbb{R} \to \mathbb{R}$ that satisfies*

$$f(f(x) + y) = 2x + f(f(y) - x), \text{ for all } x, y \in \mathbb{R}$$

is surjective.

If we take $y = -f(x)$, it follows that $f(f(x) - f(x)) = 2x + f(f(-f(x)) - x)$, that is, $f(f(-f(x)) - x) = f(0) - 2x$. Now, if $y \in \mathbb{R}$ we need to find x_0 such that $f(0) - 2x_0 = y$, but $x_0 = \frac{f(0)-y}{2}$ satisfies $f(f(-f(x_0)) - x_0) = y$, thus f is surjective.

Example 6.2.4. *A function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ that satisfies*

$$f(xf(y)) = \frac{f(x)}{y}, \textit{ for all } x, y \in \mathbb{Q}^+$$

is bijective.

If $x = 1$, then $f(f(y)) = \frac{f(1)}{y}$, this will help us to show that f is injective. If $f(y_1) = f(y_2)$, then $f(f(y_1)) = f(f(y_2))$, that is, $\frac{f(1)}{y_1} = \frac{f(1)}{y_2}$. Hence $y_1 = y_2$, and therefore f is injective. It is left to show that f is surjective. Let $\frac{m}{n} \in \mathbb{Q}^+$, we want to show that there is $x \in \mathbb{Q}^+$ such that $f(x) = \frac{m}{n}$. If this happens, then $f(f(x)) = f\left(\frac{m}{n}\right)$, hence $\frac{f(1)}{x} = f\left(\frac{m}{n}\right)$, and solving for x we get $x = \frac{f(1)}{f\left(\frac{m}{n}\right)}$, which is a rational number. That is, f is surjective, and therefore f is bijective.

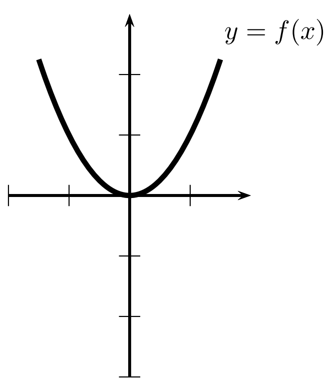
Observation 6.2.5. *In the previous examples of this section, we were faced with a situation where*

$$f(g(x)) = h(x).$$

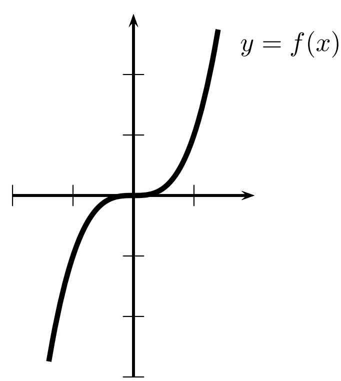
Notice that if h is injective then g is injective. Also, if h is surjective then f is surjective.

6.2.2 Even and odd functions

The functions that satisfy $f(x) = f(-x)$ are called **even functions** and the functions such that $f(x) = -f(-x)$ are called **odd functions**. A function $f : \mathbb{R} \to \mathbb{R}$ is even if its graph is symmetric with respect the y-axis, whereas the graph of an odd function is symmetric with respect to the origin.



Graph of an even function.



Graph of an odd function.

Example 6.2.6. *Find all the functions $f : \mathbb{Q} \to \mathbb{Q}$, that satisfy*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad \textit{for all } x, y \in \mathbb{Q}.$$

Letting $x = y = 0$, we have $f(0) = 0$. If $x = y$, then $f(2x) = 4f(x)$, and by induction $f(nx) = n^2 f(x)$ for every positive integer n. With $x = 0$, we have that $f(y) + f(-y) = 2f(y)$, then $f(y) = f(-y)$, hence f is even and $f(nx) = n^2 f(x)$ for every integer number n and every rational number x. If $r = \frac{p}{q}$ is a rational number, with $p \geq 1$, then $p^2 f(1) = f(p) = f\left(q\frac{p}{q}\right) = q^2 f\left(\frac{p}{q}\right)$, and then $f\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right)^2 f(1)$. Hence $f(r) = cr^2$, for all $r \in \mathbb{Q}$ with $c = f(1)$. It is easy to check that these functions satisfy the given condition.

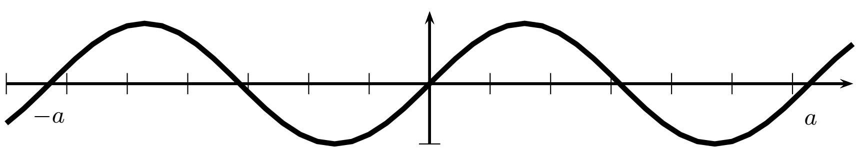
6.2.3 Periodic functions

Periodicity plays an important role in mathematics and for this reason we include some examples about this topic.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **periodic** if there exists $a \neq 0 \in \mathbb{R}$ such that

$$f(x+a) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

The number a is called a **period** of f. It is clear that for all $n \neq 0$, the number na is also a period of f.



Example 6.2.7. *A function $f : \mathbb{R} \to \mathbb{R}$ is periodic, if for some $a \in \mathbb{R}$ and every $x \in \mathbb{R}$, it is true that*

$$f(x+a) = \frac{1+f(x)}{1-f(x)}.$$

From the equation $f(x+a) = \frac{1+f(x)}{1-f(x)}$, evaluating in $x - a$, we obtain that $f(x) = \frac{1+f(x-a)}{1-f(x-a)}$. After solving for $f(x)$ from the original equation, we get that $f(x) = \frac{f(x+a)-1}{f(x+a)+1}$. Equating the last two equations, we get that $\frac{1+f(x-a)}{1-f(x-a)} = \frac{f(x+a)-1}{f(x+a)+1}$ and after simplifying $f(x+a) = \frac{-1}{f(x-a)}$. Evaluating this last equation in $x + a$, we get that $f(x+2a) = \frac{-1}{f(x)}$, and then

$$f(x+4a) = \frac{-1}{f(x+2a)} = \frac{-1}{\frac{-1}{f(x)}} = f(x),$$

that is, f is periodic with period $4a$.

Example 6.2.8 (Belarus, 2005).

(a) *Consider a function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $f(n) = f(n + f(n))$ for all $n \in \mathbb{N}$. Prove that, if the image of f is finite, then f is periodic.*

(b) *Find a non-periodic function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = f(n + f(n))$, for all $n \in \mathbb{N}$.*

To prove (a) we do as follows. Since $f(n) = f(n + f(n))$, then $f(n + f(n) + f(n)) = f(n + f(n) + f(n + f(n))) = f(n + f(n)) = f(n)$. Therefore, $f(n) = f(n + 2f(n))$ and, by induction, $f(n) = f(n + kf(n))$ for all $k \in \mathbb{N}$.

Let $A = f(\mathbb{N}) = \{a_1, \dots, a_t\}$ and $T = a_1 \cdot \ \cdots \ \cdot a_t$. Let us see that T is a period of f. Since $f(n) = f(n + kf(n))$, for all k, it follows that

$$f(n) = f\left(n + \frac{T}{f(n)} f(n)\right) = f(n + T),$$

where $k = \frac{T}{f(n)}$. That is, T is a period of f.

(b) We would like to find a non-periodic function f that satisfies the equation. For $n = 2^k m$ with $k \in \mathbb{N} \cup \{0\}$ and m odd, we define $f(n) = 2^{k+1}$.
Now, let us see that the function satisfies the equation, $f(n + f(n)) = f(2^k m + 2^{k+1}) = f(2^k(m + 2)) = 2^{k+1} = f(n)$ and moreover, f is not periodic, otherwise its range would be finite, however, its image is the set of powers of 2.

6.2.4 Increasing and decreasing functions

We say that a function is **increasing** if for $x < y$ it follows that $f(x) < f(y)$. We say that a function is **non-decreasing** if for $x < y$ it follows that $f(x) \le f(y)$. Similarly, we say that a function is **decreasing** if for $x < y$ it follows that $f(x) > f(y)$, and it is **non-increasing** if for $x < y$ it happens that $f(x) \ge f(y)$.

Another way to guarantee the injectivity of a real function, whose domain is the real numbers, is to know if the function is increasing or decreasing.

Example 6.2.9 (IMO, 1992). *Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(x^2 + f(y)) = y + f(x)^2, \quad \text{for all } x, y \in \mathbb{R}.$$

Let $a = f(0)$. With $x = 0$, we have that $f(f(y)) = a^2 + y$. With $x = y = 0$, then $f(a) = a^2$, hence $f(x^2 + a) + a^2 = f(x)^2 + f(a)$. Applying f to both sides of the equation it follows that $f(f(x^2 + a) + a^2) = f(f(x)^2 + f(a))$. The left-hand side of the equality is

$$\begin{aligned} f(f(x^2 + a) + a^2) &= x^2 + a + f(a)^2 \\ &= x^2 + a + a^4, \end{aligned} \tag{6.5}$$

and the right-hand side of the equality is

$$\begin{aligned} f(f(x)^2 + f(a)) &= a + f(f(x))^2 \\ &= a + (a^2 + x)^2 = a + a^4 + 2a^2 x + x^2. \end{aligned} \tag{6.6}$$

Comparing both equations (6.5) and (6.6), we need to have that $2a^2x = 0$, hence $a = 0$. From here we conclude that $f(f(y)) = y$, for all $y \in \mathbb{R}$ and $f(x^2) = f(x)^2$, for all $x \in \mathbb{R}$.

The last equation guarantees that if $x \geq 0$, then $f(x) \geq 0$. Since $f(f(y)) = y$ we have that f is injective, hence $f(x) = 0$ if and only if $x = 0$. In this way when $x > 0$, then $f(x) > 0$.

Since $f(f(x)^2 + f(y)) = f(f(x))^2 + y = x^2 + y$, it follows that $f(x^2 + y) = f(f(f(x)^2 + f(y))) = f(x)^2 + f(y) = f(x^2) + f(y)$.
If $y < x$, then $x - y > 0$ and $x = x - y + y$. Applying the last equality to $\sqrt{x-y}$ and y, it follows that $f(x) = f(x - y + y) = f(x - y) + f(y) > f(y)$, that is, f is increasing. But f non-decreasing and $f(f(x)) = x$, guarantee that $f(x) = x$. In fact if $f(x) > x$, then $x = f(f(x)) > f(x)$ and if $f(x) < x$, then $x = f(f(x)) < f(x)$. Therefore, the only function that satisfies the functional equation is $f(x) = x$.

6.2.5 Bounded functions

We say that a function $f : A \subset \mathbb{R} \to \mathbb{R}$ is **bounded above** in A, if there exists $M \in \mathbb{R}$ such that $f(a) \leq M$, for all $a \in A$.

We say that a function $f : A \subset \mathbb{R} \to \mathbb{R}$ is **bounded below** in A, if there exists $N \in \mathbb{R}$ such that $f(a) \geq N$, for all $a \in A$.

We say that a function $f : A \subset \mathbb{R} \to \mathbb{R}$ is **bounded** in A, if there exists $M > 0$ such that $|f(x)| \leq M$, for all $x \in A$, or equivalently $-M \leq f(x) \leq M$, for all $x \in A$.

Example 6.2.10. *Let $m \geq 2$ be an integer number. Find all the bounded functions $f : [0, 1] \to \mathbb{R}$ such that, for $x \in [0, 1]$, it follows that*

$$f(x) = \frac{1}{m^2}\left\{f(0) + f\left(\frac{x}{m}\right) + f\left(\frac{2x}{m}\right) + \cdots + f\left(\frac{(m-1)\,x}{m}\right)\right\}.$$

If $|f(x)| \leq M$ for $x \in [0, 1]$, then, using the triangle inequality, it follows that

$$|f(x)| \leq \frac{1}{m^2}\sum_{j=0}^{m-1}\left|f\left(\frac{j}{m}\,x\right)\right| \leq \frac{mM}{m^2} = \frac{M}{m}.$$

Hence $|f(x)| \leq \frac{M}{m}$, for $x \in [0, 1]$. With this new bound, we can repeat the last argument to show that $|f(x)| \leq \frac{M}{m^2}$, for $x \in [0, 1]$. An inductive argument guarantees us that for all $n \in \mathbb{N}$, it follows that $|f(x)| \leq \frac{M}{m^n}$, for $x \in [0, 1]$, and when we let n go to infinity, we have that $f(x) = 0$, for all $x \in [0, 1]$, and this is the only function that satisfies the equation.

6.2.6 Continuity

When the values of a function $f(x)$ get closer to b as x tends to a, we say that b is the **limit** of $f(x)$ as x tends to a. In mathematical language this is usually written as follows:

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ such that } \ 0 < |x-a| < \delta \ \Rightarrow\ |f(x)-b| < \epsilon$$

and we write $\lim_{x\to a} f(x) = b$.

Also, we can consider $\lim_{x\to\infty} f(x) = b$, which is defined as: for all $\epsilon > 0$ there exists $M > 0$ such that if $x > M$ then $|f(x)-b| < \epsilon$.

One characterization of the limit concept when dealing with sequences is the following theorem.

Theorem 6.2.11. *Let f be a function, then $\lim_{x\to a} f(x) = b$ if and only if for every sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$, it follows that $\lim_{n\to\infty} f(a_n) = b$.*

The proof of this theorem will be given in Chapter 7.

Observation 6.2.12. *The previous theorem also states that $\lim_{x\to a} f(x)$ is not b, if there exists a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} f(a_n)$ is not b.*

Example 6.2.13 (IMO, 1983)**.** *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy:*

(a) *$f(xf(y)) = yf(x)$, for all positive real numbers x, y.*
(b) *$\lim_{x\to\infty} f(x) = 0$.*

If $x = 1$, then $f(f(y)) = yf(1)$, hence the function is bijective. In fact, if $f(x) = f(y)$, then $xf(1) = f(f(x)) = f(f(y)) = yf(1)$, and since $f(1) \neq 0$ we have that $x = y$, therefore f is injective; let us see now that f is surjective, let $c \in \mathbb{R}^+$ and take $y = \frac{c}{f(1)}$, then it follows that $f\left(f\left(\frac{c}{f(1)}\right)\right) = f(f(y)) = yf(1) = c$, hence f is surjective. Thus, f is bijective.

In particular, there is a y_0 such that $f(y_0) = 1$. Since $f(xf(y_0)) = y_0 f(x)$ for all $x > 0$, taking $x = 1$, we have that $f(1) = y_0 f(1)$, and then $y_0 = 1$, hence $x = 1$ is a fixed point.
If $x = y$, then $f(xf(x)) = xf(x)$, and $xf(x)$ is also a fixed point of f. If we show that the only fixed point of f is 1, we can conclude that $f(x) = \frac{1}{x}$.
From the equation $f\left(\frac{1}{a}f(a)\right) = af\left(\frac{1}{a}\right)$ and the fact that f is injective, we have that $f(a) = a$ if and only if $f\left(\frac{1}{a}\right) = \frac{1}{a}$. If there is a fixed point different from 1, hence there is a fixed point a greater than 1. But $f(a) = a$ implies, by induction, that $f(a^n) = a^n$, (for instance, $f(a^2) = f(af(a)) = af(a) = a^2$). Since $a^n \to \infty$ and $f(a^n) = a^n \to \infty$, this contradicts the fact that $\lim_{x\to\infty} f(x) = 0$. Therefore, the only fixed point is 1.
The function $f(x) = \frac{1}{x}$ satisfies the conditions of the problem.

We say that a function f is **continuous** at some point a if when we let x tend to a, $f(x)$ tends to $f(a)$, that is, $\lim_{x\to a} f(x) = f(a)$. Also, we say that f is **continuous on a set** A if f is continuous at every point $a \in A$.

One characterization of the continuity property is given by the following result.

Theorem 6.2.14. *A function f is continuous at a if and only if for every sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$, it follows that $\lim_{n\to\infty} f(a_n) = f(a)$.*

A set $D \subset \mathbb{R}$ is **dense** in the set of real numbers if every open interval in $\mathbb{R}$ has points in D.

Theorem 6.2.15. *The set of rational numbers is dense in the set of real numbers.*

Theorem 6.2.16. *If a function f is continuous on $\mathbb{R}$ and f is zero in a dense subset of the real numbers, then f is identically zero in $\mathbb{R}$.*

As a consequence, if the functions f and g are continuous and coincide on a dense subset of $\mathbb{R}$, then they coincide on all $\mathbb{R}$.

The proofs of these three theorems will be given in Section 7.4.

Example 6.2.17 (Nordic, 1998)**.** *Find all the continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation,*

$$f(x+y) + f(x-y) = 2(f(x) + f(y)), \text{ for all } x, y \in \mathbb{R}.$$

We have seen in Example 6.2.6 that $f(x) = f(1)x^2$, for $x \in \mathbb{Q}$ and, by Theorem 6.2.16, $f(x) = f(1)x^2$ for all $x \in \mathbb{R}$.

Exercise 6.8. *Let $f,\ g : \mathbb{R} \to \mathbb{R}$ be two functions that satisfy $f(g(x)) = g(f(x)) = -x$, for any real number x. Prove that f and g are odd functions.*

Exercise 6.9. *Find all surjective functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(f(x-y)) = f(x) - f(y), \text{ for all } x, y \in \mathbb{R}.$$

Exercise 6.10. *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(xf(y)) = xy, \text{ for all } x, y \in \mathbb{R}.$$

Exercise 6.11 (Belarus, 2005)**.** *Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy*

$$f(m-n+f(n)) = f(m) + f(n), \text{ for all } m, n \in \mathbb{N}.$$

Exercise 6.12 (IMO, 1990)**.** *Find a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ that satisfies the equation*

$$f(xf(y)) = \frac{f(x)}{y}, \text{ for all } x, y \in \mathbb{Q}^+.$$

Exercise 6.13. *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$xf(y) + yf(x) = (x+y)f(x)f(y), \text{ for all } x, y \in \mathbb{R}.$$

Exercise 6.14 (IMO, 1968). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property*

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2},$$

for all $x \in \mathbb{R}$ and a a fixed number.

(i) *Prove that f is periodic.*
(ii) *In case that $a = 1$, give an example of a function of this type.*

Exercise 6.15. *Let a, $b > 0$, find the values of m such that the equation*

$$|x-a| + |x-b| + |x+a| + |x+b| = m(a+b),$$

has at least one real solution.

6.3 Functional equations of Cauchy type

The functional equations of Cauchy type are:

$$\begin{aligned}
&(C_1) \qquad && f(x+y) = f(x) + f(y).\\
&(C_2) && f(x \cdot y) = f(x) + f(y).\\
&(C_3) && f(x+y) = f(x) \cdot f(y).\\
&(C_4) && f(x \cdot y) = f(x) \cdot f(y).
\end{aligned}$$

In order to establish what functions satisfy certain functional equations we should take into account the domain and the codomain where we want to solve the equation. For instance, if we take equation (C_2), and we want to solve it in all $\mathbb{R}$, considering $y = 0$, we obtain that the solution of the equation is $f(x) = 0$, for all x, which means the equation sought for was very simple. Therefore, it is more adequate for this functional equation to consider the set of real positive numbers as its domain.

6.3.1 The Cauchy equation $f(x+y) = f(x) + f(y)$

The first of the equations of Cauchy type is the most important. With this equation we will illustrate how some functional equations are solved. First, we will see how to determine some of the values that the functions take, and this will allow us to find, in a natural way, other values until we learn how the functions behave in the set of rational numbers.

Letting $x = y = 0$, we have that $f(0) = 2f(0)$, then $f(0) = 0$. If $y = -x$, we have that $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$, and then $f(-x) = -f(x)$, which tells us that the function f should be odd.

With $x = y$, we have that $f(2x) = 2f(x)$. Now, using induction, we can conclude that $f(nx) = nf(x)$, for any positive integer n. In fact, $f((n+1)x) = f(nx+x) = f(nx) + f(x) = nf(x) + f(x) = (n+1)f(x)$.

Recalling that $f(-x) = -f(x)$, we get $f(nx) = nf(x)$ for all $n \in \mathbb{Z}$. Now, since $f(x) = f(\frac{m}{m}x) = f(m\frac{x}{m}) = mf(\frac{x}{m})$, we have that $f(\frac{x}{m}) = \frac{1}{m}f(x)$, therefore $f(\frac{n}{m}x) = f(n\frac{x}{m}) = nf(\frac{x}{m}) = \frac{n}{m}f(x)$. Hence, $f(rx) = rf(x)$, for all $r \in \mathbb{Q}$ and all $x \in \mathbb{R}$.

Letting $c = f(1)$, we get $f(r) = cr$ for all $r \in \mathbb{Q}$.

We conclude that a function $f : \mathbb{Q} \to \mathbb{R}$ that satisfies equation (C_1) should have the form $f(r) = cr$, for all $r \in \mathbb{Q}$, with $c = f(1)$ a fixed constant. And a function of this type $f(x) = cx$ satisfies such a type of Cauchy equation, since $c(x+y) = cx + cy$, for any $x, y \in \mathbb{Q}$.

Additional hypothesis to the Cauchy equation $f(x+y) = f(x) + f(y)$

We would like to determine functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the first of the Cauchy type equations, which are also known as **additive** functions. We will see that with an additional hypothesis (we will analyze several of them), the conclusion is that f should be linear, that is, of the form $f(x) = ax$, for all $x \in \mathbb{R}$ and with $a = f(1)$.

(H_1) **The function is continuous in all $\mathbb{R}$.**

We know that $f(r) = cr$, for all $r \in \mathbb{Q}$. Since $f(x)$ and cx are continuous in $\mathbb{R}$ and coincide in $\mathbb{Q}$, we have, by Theorem 6.2.16, that the functions coincide in all $\mathbb{R}$. Hence $f(x) = cx$ for all $x \in \mathbb{R}$.

Example 6.3.1 (Jensen's equation)**.** *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation,*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \quad \textit{for all} \ \ x, y \in \mathbb{R}.$$

Note that by letting $x = y = 0$, we do not obtain information about what is the value of $f(0)$. Define $g(x) = f(x) - f(0)$, which is also continuous. A straightforward calculation shows that

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2},$$

but now the function g satisfies $g(0) = 0$. Taking $y = 0$ in the new equation, we have that

$$g\left(\frac{x}{2}\right) = \frac{g(x)}{2},$$

and after substituting in this last equation $x = u + v$, we get

$$g\left(\frac{u+v}{2}\right) = \frac{g(u+v)}{2}.$$

Hence, we can affirm that $g(u+v) = g(u) + g(v)$, for all $u, v \in \mathbb{R}$, that is, g is a continuous function that satisfies the first equation of Cauchy. Therefore, $g(x) = ax$, with $a = g(1)$, and then $f(x) = ax + b$, for all $x \in \mathbb{R}$ and $b = f(0)$.

(H_1') **The function is continuous only in $x = 0$.**

To reduce this to the previous case, it will be enough to show the following result.

Lemma 6.3.2. *If $f : \mathbb{R} \to \mathbb{R}$ is an additive function, that is, if it satisfies equation (C_1) and it is continuous at 0, then it is continuous at every real number a.*

Proof. Let $\{a_n\}$ be a sequence with $\lim_{n\to\infty} a_n = a$, then the sequence $\{a_n - a\}$ satisfies $\lim_{n\to\infty}(a_n - a) = 0$. Since f is continuous in 0, Theorem 6.2.14 guarantees that $\lim_{n\to\infty} f(a_n - a) = f(0) = 0$. But equation (C_1) implies that $f(a_n) = f(a_n - a) + f(a)$, then $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(a_n - a) + \lim_{n\to\infty} f(a) = f(a)$, which implies f is continuous in a. $\square$

(H_2) **The function is monotone.**

If the function f, besides being additive, is monotone (without loss of generality, we can assume that it is non-decreasing), then $f(x)$ should have the form $f(x) = cx$. To support this claim consider a real number x. Let $\{r_n\}$ and $\{s_n\}$ be sequences of rational numbers converging to x, with $r_n < x < s_n$ for all n.
By the monotonicity of the function f, $cr_n = f(r_n) \le f(x) \le f(s_n) = cs_n$. Taking the limit, we obtain $cx = \lim_{n\to\infty} cr_n \le f(x) \le \lim_{n\to\infty} cs_n = cx$. Thus, $f(x) = cx$.

The non-increasing case is similar. Moreover, we can change $\le$ to $<$ and reach the same conclusion, and similarly in the case $\ge$.

(H_3) **The function is positive (for positive numbers).**

If $f(x) > 0$ for $x > 0$ and if in addition it is additive, then it is increasing. In fact, if $x < y$ then $y - x > 0$, and $f(y - x) > 0$. Hence, $f(x) < f(x) + f(y - x) = f(x + (y - x)) = f(y)$. Similarly, we can consider the decreasing, non-decreasing and non-increasing cases, and in each one of them we can conclude that f is linear.

(H_4) **The function is bounded.**

If the additive function f is bounded in an interval of the form $[a, b]$, that is, if there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in [a, b]$, then the function must have the form $f(x) = cx$.

First note that $x \in [0, b-a]$ if and only if $x + a \in [a, b]$ and for $x \in [0, b-a]$ we have $|f(x)| = |f(x+a) - f(a)| \le |f(x+a)| + |f(a)| \le 2M$. This guarantees that f is bounded by $2M$ in $[0, b-a]$. Let $\alpha = b - a$, $c = \frac{f(\alpha)}{\alpha}$ and $g(x) = f(x) - cx$. We then have

(a) $g(x+y) = f(x+y) - c(x+y) = f(x) - cx + f(y) - cy = g(x) + g(y)$, that is, g is additive.

(b) $g(\alpha) = f(\alpha) - c\alpha = 0$.
(c) $g(x+\alpha) = g(x) + g(\alpha) = g(x)$, the function g is periodic with period α.
(d) For $x \in [0, \alpha]$, we have $|g(x)| = |f(x) - cx| \leq |f(x)| + |cx| \leq 2M + \left|\frac{f(\alpha)}{\alpha}\right| |\alpha| \leq 3M$, that is, g is bounded in the interval $[0, \alpha]$ and then, because it is periodic, it is bounded in all $\mathbb{R}$.

If $g(x_0) \neq 0$ for some real number x_0, then since $g(nx_0) = ng(x_0)$ for every integer number n, we can make $|g(nx_0)|$ as large as we wish, then g would not be bounded, which would be a contradiction to the part (d). Therefore, $g(x) = 0$ for any real number x and then $f(x) = cx$, for all x in $\mathbb{R}$.

(H_4') **The function is bounded on a neighborhood of 0.**

By (H_1'), it will be enough to show that f is continuous in 0.
Let $\{a_n\}$ be a sequence that converges to 0. We will use Theorem 6.2.14 to show that $f(a_n)$ converges to 0. Let $\epsilon > 0$, we will see that $|f(a_n)| < \epsilon$ for large n. If $M > 0$ is the bound for f in the interval $(-a, a)$, let us choose an integer N such that $\frac{M}{N} < \epsilon$.
Since $\lim_{n\to\infty} a_n = 0$, there exists n_0 such that $|a_n| < \frac{a}{N}$, for all $n \geq n_0$. Since $|Na_n| < a$, it follows that $|f(Na_n)| \leq M$ for $n \geq n_0$. But since $f(Na_n) = Nf(a_n)$, we have that $|f(a_n)| \leq \frac{M}{N} < \epsilon$, for $n \geq n_0$, as we wanted.

There are several conditions that can be added to the equation of Cauchy to make sure that the function that satisfies the equation is a linear function. Many of these conditions are the source of problems of the kind that appear in the mathematical olympiad. Let us see an example.

Example 6.3.3. *Find all the functions that satisfy the following equations:*

(a) $f(x+y) = f(x) + f(y)$, *for all* x, $y \in \mathbb{R}$.
(b) $f(xy) = f(x)f(y)$, *for all* x, $y \in \mathbb{R}$.

First note that if $x \geq 0$, then

$$f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) = (f(\sqrt{x}))^2 \geq 0.$$

But then, by (H_3), we have that f is linear, that is, it has the form $f(x) = cx$ with $c = f(1)$. Taking $x = y = 1$ in the equation (b), we get that $c = c^2$, hence $c = 0$ or $c = 1$. Then, $f(x) = 0$ or $f(x) = x$, are the only solutions to the problem.

6.3.2 The other Cauchy functional equations ⋆

The Cauchy equation $f(x \cdot y) = f(x) + f(y)$

We will find continuous solutions to this functional equation. If $y = 0$ belongs to the domain of f, then $f(x) = 0$. Now, suppose that the function is defined for $x \neq 0$. If we take $x = y = 1$ in the equation, we have that $f(1) = 0$. Also, considering $x = y = -1$, we get that $f(-1) = 0$. Now, taking $y = -1$, we obtain $f(-x) = f(x)$, that is, the function must be even and it will be determined by its behavior when x is positive. But if x, y are positive, there are u, $v \in \mathbb{R}$ such that $x = e^u$, $y = e^v$, and with them the equation[14] can be expressed as

$$f(e^u \cdot e^v) = f(e^u) + f(e^v).$$

If we let $g(u) = f(e^u)$, then $g(u + v) = g(u) + g(v)$, which is the first Cauchy equation, and we know that its solution is $g(u) = cu$, with $c = g(1) = f(e)$, then $f(x) = g(u) = f(e) \log x$ for $x > 0$, and $f(x) = f(e) \log |x|$ for $x \neq 0$.

The Cauchy equation $f(x + y) = f(x) \cdot f(y)$

First note that if for some y, $f(y) = 0$, then f is constant. This follows from $f(x) = f(x - y + y) = f(x - y)f(y) = 0$. If f is never zero, then it is positive since $f(x) = f(\frac{x}{2} + \frac{x}{2}) = (f(\frac{x}{2}))^2 > 0$. Hence, since f is always positive we can take logarithms on both sides of the equation in order to satisfy the functional equation,

$$\log f(x + y) = \log f(x) + \log f(y),$$

which is a functional equation of the first type, then $\log f(x) = cx$, with $c = \log f(1)$. Applying the exponential function, we have that $f(x) = e^{\log f(1)x} = f(1)^x$, for all $x \in \mathbb{R}$. Note that here we have found only the continuous solutions.

The Cauchy equation $f(x \cdot y) = f(x) \cdot f(y)$

As in the previous equation, if for some $y \neq 0$, $f(y) = 0$, then f is constant. This follows since $f(x) = f(\frac{x}{y} \cdot y) = f(\frac{x}{y})f(y) = 0$. If f is never zero, then for x positive, $f(x)$ is positive, since $f(x) = f(\sqrt{x} \cdot \sqrt{x}) = (f(\sqrt{x}))^2 > 0$. For $x = 1$, we have that $f(1) = (f(1))^2$, therefore $f(1) = 0$ or $f(1) = 1$. The first option has been studied before, and therefore $f(1) = 1$. Since $f(x^2) = (f(x))^2$, $f(-1) = \pm 1$ and taking $y = -1$ in the original equation, we have that $f(-x) = \pm f(x)$. Then it will be enough to see what happens with $x > 0$. After that we will have two options to extend to the negative real numbers, that is, whether making the function even or odd. Since the function f is positive for $x > 0$, we can apply the logarithmic function on both sides of the equality to get

$$\log f(x \cdot y) = \log f(x) + \log f(y).$$

Considering $g(x) = \log f(x)$, we have that g satisfies the second equation of Cauchy, then $g(x) = g(e) \log x$, hence $f(x) = x^{g(e)} = x^{\log f(e)}$. Thus the continuous solutions are $f(x) = 0$, $f(x) = \pm x^{\log f(e)}$. Observe that the point $x = 0$ remains outside the analysis we have made (when we take a logarithm), but at the end we include 0 and necessarily it has to happen that $f(0) = 0$, in order to have

[14] For the definition of the exponential function, e^x, and the function logarithm, $\log x$, see [21].

continuity there. However, there exists an exception, if $f(e) = 1$ there are two solutions. One of them is $f(x) = 1$, which is continuous, and the other solution is $f(x) = \operatorname{sign}(x)$, which is not continuous at $x = 0$.

Exercise 6.16. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(x^2) - f(y^2) = (x + y)(f(x) - f(y)), \text{ for all } x, y \in \mathbb{R}.$$

Exercise 6.17. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy*

$$f(x)f(y) - f(xy) = \frac{x}{y} + \frac{y}{x}, \text{ for all } x, y \in \mathbb{R}^+.$$

Exercise 6.18.

(i) *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the condition*

$$f(xf(y)) + f(yf(x)) = 2xy, \text{ for all } x, y \in \mathbb{R}^+.$$

(ii) (Short list IMO, 2002) *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(f(x) + y) = 2x + f(f(y) - x), \text{ for all } x, y \in \mathbb{R}.$$

Exercise 6.19. *Let f be a function such that for some number $a \in \mathbb{R}$ it satisfies*

$$f(x + a) = \frac{f(x) - 3}{f(x) - 2}, \text{ for all } x \in \mathbb{R}.$$

Prove that f is periodic.

Exercise 6.20. *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function such that the set $\{f(n) \mid n \in \mathbb{N}\}$ has an infinite number of elements. Prove that the period of f is an irrational number.*

Exercise 6.21 (Long list IMO, 1977)**.** *Determine all the real continuous functions $f(x)$ defined on the interval $(-1, 1)$, that satisfy the functional equation*

$$f(x + y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}, \quad \text{for } \ x + y, x, y \in (-1, 1). \tag{6.7}$$

Exercise 6.22. *Find all the continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) + f(y) = f\left(\frac{x + y}{1 - xy}\right), \text{ with } x, y \neq 1.$$

Exercise 6.23. *Find all the continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(x + y) = f(x) + f(y) + f(x)f(y), \text{ for all } x, y \in \mathbb{R}.$$

6.4 Recommendations to solve functional equations

Next we will present a series of recommendations of the things we need to do in order to find solutions of functional equations. Moreover, we will show some examples where we use these observations.

Substituting the variables for values. One of the first steps that we need to follow is to see if it is possible to determine some values of the function we are looking for, for instance $f(0)$, $f(1)$, etc. In some cases, the values can be found through direct substitution. But sometimes we may need to make a variable interchange. For instance, if we found something like $f(x+y)$, it is natural to make $y=-x$, to obtain $f(0)$.

Mathematical induction. We should have in mind that the principle of mathematical induction can help us. In these cases it is important to remember the induction basis. For instance, to know what is $f(1)$ or $f(j)$, and then later to be able to conjecture something more specific that could be a relation that allows us to go from n to $n+1$. Also, try to find expressions like $f\left(\frac{1}{n}\right)$, and afterwards expressions of the form $f(r)$, with $r \in \mathbb{Q}$. These situations, in general, can arise when dealing with equations with variables in $\mathbb{Q}$ or in $\mathbb{Z}$.

Basic properties of functions. It is important to know if the function is injective, surjective, bijective, periodic, even, odd, or with some kind of symmetry. This can help us to reduce the cases and to concentrate only on the set of numbers where the equation is valid.

Substitutions. Beside substitutions by specific values, we can try other more general substitutions, for instance, $\frac{1}{x}$, $x+1$, $x+y$, $x-y$.

Symmetry in the variables. If the equation has two (or more) variables, for instance x, y, we will always try to substitute the y by the x (and vice versa), and look always for symmetries in the variables.

Compare with the Cauchy equations. If our equation can be reduced or simplified to an equation of Cauchy type, then we have made good progress, since we already know the solutions to this type of equations.

Continuity, monotonicity. Investigate if the unknown function is monotone or continuous. This is very useful, since the problem could then be reduced to be solved on the rational numbers or on some dense subset of the real numbers.

Other numeral systems. In functional equations where the natural numbers are present, it can help us to work in another numeral system different from the 10-base system, for instance, moving to the binary system or the 3-base system.

Check. It is important to always check that the function we proposed as solving the equation, really does it. We should never forget this part.

Now, we will exhibit examples where these recommendations are put to use.

Example 6.4.1. *Find all functions $f : \mathbb{Q} \to \mathbb{R}$ that satisfy the following conditions,*

$$f(xy) = xf(y) + yf(x) \quad \textit{and} \quad f(x+y) = f(x^2) + f(y^2), \ \textit{for}\ x, y \in \mathbb{Q}.$$

If in the first equation we set $x = y = 0$, we obtain $f(0) = 0$, and taking $x = y = 1$, we have that $f(1) = 2f(1)$, that is, $f(1) = 0$.

On the other hand, taking $x = 0$ in the second equation, we get $f(y) = f(y^2)$, then the second equation becomes $f(x + y) = f(x) + f(y)$ and we know, by the Cauchy equation, that the function should be such that $f(x) = f(1)x$, for all $x \in \mathbb{Q}$. Moreover, since $f(1) = 0$ the only solution is $f(x) = 0$.

Example 6.4.2 (Short list IMO, 1988)**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a function that satisfies*

$$f(f(m) + f(n)) = m + n, \ \textit{for all}\ m, n.$$

Find all possible values of $f(1988)$.

In Example 6.2.1, we showed that the function is injective. Moreover, for $l < n$, we have that

$$f(f(m+l) + f(n-l)) = m + l + n - l = m + n = f(f(m) + f(n)). \tag{6.8}$$

The injectivity property tells us that

$$f(m+l) + f(n-l) = f(m) + f(n), \ \text{for}\ l, m, n \in \mathbb{N} \text{ and } l < n. \tag{6.9}$$

Now, by induction we will see that $f(n) = n$. First, for $n = 1$ let us see that $f(1) = 1$. If $b = f(1)$, then the following two equalities are true:

$$f(2b) = f(f(1) + f(1)) = 2 \quad \text{and} \quad f(b+2) = f(f(1) + f(2b)) = 1 + 2b.$$

Then, $b = 2$ is not possible, since, on the one hand we would have that $f(2 \cdot 2) = 2$ and on the other hand $f(2 + 2) = 1 + 2 \cdot 2 = 5$.

Neither is $b > 2$ possible, since using $f(2b) = 2$, $f(1) = b$ and equation (6.9), it follows that

$$\begin{aligned} b + 2 &= f(1) + f(2b) \\ &= f(1 + b - 2) + f(2b - (b-2)) \\ &= f(b-1) + f(b+2) \\ &= f(b-1) + 1 + 2b, \end{aligned}$$

then $f(b - 1) = 1 - b < 0$, which is not possible. Therefore, $b = f(1) = 1$. Suppose now that $f(n) = n$, then from the original equation and from the induction hypothesis, it follows that

$$n + 1 = f(f(n) + f(1)) = f(n+1).$$

Therefore, the only possible value of $f(1988)$ is 1988.

Example 6.4.3. *Find all increasing or decreasing functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x + f(y)) = f(x) + y, \textit{ for } x, y \in \mathbb{R}.$$

Letting $x = y = 0$, we get $f(f(0)) = f(0)$. Since f is increasing or decreasing, it follows that f is injective, then $f(0) = 0$.

Taking $x = 0$, we get $f(f(y)) = y$ for all $y \in \mathbb{R}$. Letting $a = f(y)$, it follows that $f(f(y)) = y = f(a)$, then

$$f(x + y) = f(x + f(a)) = f(x) + a = f(x) + f(y),$$

hence, f satisfies the additive Cauchy equation. Moreover, with the condition $f(f(y)) = y$, we have that the only solutions are $f(x) = x$ and $f(x) = -x$, which verify the functional equation.

Example 6.4.4. *Find all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *that satisfy the equation*

$$f(x + y) + f(x - y) = 2f(x), \textit{ for } x, y \in \mathbb{R}.$$

Letting $x = y$, it follows that $f(2x) = 2f(x) - f(0)$. Then, from the original equation we obtain $f(x+y)+f(x-y) = f(2x)+f(0)$; subtracting $2f(0)$ on both sides, we get $f(x + y) - f(0) + f(x - y) - f(0) = f(2x) - f(0)$.

Hence, $f(x) - f(0)$ is additive, since taking $u = x + y$, $v = x - y$ in the last equation, leads to $f(u) - f(0) + f(v) - f(0) = f(u + v) - f(0)$.

Since $f(x) - f(0)$ is continuous, it follows that $f(x) = f(0) + ax$, for all $x \in \mathbb{R}$.

Example 6.4.5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $f(x) = x$ *has no real solutions. Then* $f^n(x) = x$ *has no real solutions, where* f^n *is the composition of* f *with itself,* n *times, for any* $n \in \mathbb{N}$.

Since $f(x) = x$ has no real roots, that is, there is no $x \in \mathbb{R}$ such that $f(x) - x = 0$, then it is true that either $f(x) > x$ or $f(x) < x$, for all $x \in \mathbb{R}$. In fact, since $f(x) - x$ is continuous, and if f were positive on a point d and negative on another point e, then, by the intermediate value theorem[15], there would be a point x_0 between d and e such that $f(x_0) - x_0 = 0$, which is impossible. Therefore, $f(x) > x$ for all $x \in \mathbb{R}$ or $f(x) < x$ for all $x \in \mathbb{R}$.

If $f(x) > x$ for all $x \in \mathbb{R}$, then

$$x < f(x) < f(f(x)) < \cdots < f(f(\cdots f(x) \cdots)) < \cdots ,$$

and therefore $f(f(\cdots f(x) \cdots)) = x$ has no real solutions.

Similarly, if $f(x) < x$, we get that $f(f(\cdots f(x) \cdots)) = x$ has no real solutions.

Exercise 6.24. *Prove that there are no functions* $f : \mathbb{N} \to \mathbb{N}$ *that satisfy*

$$f(f(n)) = n + 1, \textit{ for all } n \in \mathbb{N}.$$

[15] See [21].

Exercise 6.25 (IMO, 1986). *Find functions* $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ *such that satisfy:*

(a) $f(xf(y))f(y) = f(x+y)$, *for* $x, y \geq 0$.
(b) $f(2) = 0$.
(c) $f(x) \neq 0$, *for all* x *such that* $0 \leq x < 2$.

Exercise 6.26. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x-y) = f(x+y)f(y), \text{ for all } x, y \in \mathbb{R}.$$

6.5 Difference equations. Iterations

In this section we will study two kinds of functional equations: those relating the values of $f(x)$ and $f(x+h)$ or, more generally, with those of $f(x+nh)$ for some $n \in \mathbb{N}$, which are called **difference equations**, and the functional equations relating $f(x)$ with its **iterations**, that is, with $f^2(x) = f(f(x)), \ldots, f^n(x) = \underbrace{f(f(\ldots f(x)\ldots))}_{n \text{ times}}$.

For the difference equations, we will use the **difference operator**, denoted by Δ, and which is defined, for a function $f : \mathbb{R} \to \mathbb{R}$, as

$$\Delta f(x) = f(x+h) - f(x), \tag{6.10}$$

for $x \in \mathbb{R}$, and where h is a fixed real number. Also, we will use the operator E defined by $Ef(x) = f(x+h)$, and the identity operator I defined by $If(x) = f(x)$, so that $\Delta = E - I$, that is, $\Delta f(x) = Ef(x) - If(x) = f(x+h) - f(x)$.

The following properties of the operators are seen to hold trivially.

Properties 6.5.1.

(a) $\Delta(af(x) + g(x)) = a\Delta f(x) + \Delta g(x)$, *for a fixed real number* a.
(b) $\Delta(f(x) \cdot g(x)) = Ef(x)\Delta g(x) + g(x)\Delta f(x)$.
(c) $\Delta \dfrac{f}{g}(x) = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)Eg(x)}$, *if* $g(x) \neq 0$.
(d) $\Delta^m f(x) = \Delta^{m-1}(\Delta f(x))$, *and also* $\Delta^m \Delta^n = \Delta^n \Delta^m = \Delta^{m+n}$.

Lemma 6.5.2. *For each integer number* $n \geq 1$, *it follows that* $\Delta^n x^n = h^n n!$ *and* $\Delta^m x^n = 0$ *for* $m > n$.

Proof. The proof is by induction on n.

If $n = 1$, we have that $\Delta x = (x+h) - x = h$ and $\Delta^2 x = \Delta h = 0$, then $\Delta^m x = 0$ for $m > 1$.

Suppose that the result is true for all $j < n$, then

$$\begin{aligned}
\Delta^n x^n &= \Delta^{n-1}(\Delta x^n) = \Delta^{n-1}((x+h)^n - x^n) \\
&= \Delta^{n-1}\left(\sum_{i=1}^{n-1}\binom{n}{i}x^{n-i}h^i\right) = \sum_{i=1}^{n-1}\binom{n}{i}h^i\Delta^{n-1}(x^{n-i}) \\
&= \binom{n}{1}h\Delta^{n-1}(x^{n-1}) = \binom{n}{1}hh^{n-1}(n-1)! = h^n n!,
\end{aligned}$$

and since Δ applied to a constant is zero, we have that $\Delta^m x^n = 0$ for $m > n$, so the induction step is true and the result holds. $\square$

Example 6.5.3. *If $P(x) = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial of degree n, it follows that $\Delta^n P(x) = a_n n! h^n$ and $\Delta^m P(x) = 0$ for $m > n$.*

In fact, by the previous lemma

$$\begin{aligned}
\Delta^n P(x) &= \Delta^n(a_0 + a_1x + \cdots + a_nx^n) \\
&= \Delta^n(a_0) + \Delta^n(a_1x) + \cdots + \Delta^n(a_nx^n) \\
&= a_n\Delta^n(x^n) \\
&= a_n n! h^n.
\end{aligned}$$

It is clear that if $m > n$,

$$\begin{aligned}
\Delta^m P(x) &= \Delta^{m-n}(\Delta^n P(x)) \\
&= \Delta^{m-n}(a^n n! h^n) = 0.
\end{aligned}$$

In general, we have the following theorem.

Theorem 6.5.4. *For any function f, it follows that*

$$\Delta^n f(x) = (E-I)^n f(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}f(x+(n-j)h).$$

Proof. The proof is by induction. The case $n = 1$ has been already validated. Suppose now the result holds true for n and let us see what happens for $n+1$.

$$\begin{aligned}
&\Delta^{n+1} f(x) = \Delta^n(f(x+h) - f(x)) \\
&= \sum_{j=0}^{n}(-1)^j\binom{n}{j}f\left(x+(n-j+1)h\right) - \sum_{j=0}^{n}(-1)^j\binom{n}{j}f\left(x+(n-j)h\right) \\
&= \sum_{j=0}^{n}(-1)^j\binom{n}{j}f\left(x+(n-j+1)h\right) \\
&\quad + \sum_{j=0}^{n}(-1)^{j+1}\binom{n}{j}f\left(x+(n+1-(j+1))h\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n}(-1)^j \binom{n}{j} f(x+(n+1-j)h) + \sum_{j=1}^{n}(-1)^j \binom{n}{j-1} f(x+(n+1-j)h) \\
&= \binom{n}{0} f(x+(n+1)h) + \sum_{j=1}^{n}(-1)^j \left[\binom{n}{j} + \binom{n}{j-1}\right] \\
&\qquad f(x+(n+1-j)h) + (-1)^{n+1}\binom{n}{n} f(x) \\
&= \sum_{j=0}^{n+1}(-1)^j \binom{n+1}{j} f(x+(n+1-j)h). \qquad \square
\end{aligned}
$$

When difference equations are applied to functions with variables among the non-negative integers and with $h = 1$, we get expressions of the form

$$\Delta f(0) = f(1) - f(0),\ \Delta f(1) = f(2) - f(1),\ \Delta f(2) = f(3) - f(2), \dots,$$

which are known as **sequences in finite differences** or **recurrent sequences**, notions that will be studied more carefully in Chapter 7.

Let us see an example dealing with iterations in the functional equation.

Example 6.5.5. *Find functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy*

$$f(f(n)) + f(n)^2 = n^2 + 3n + 3, \ \textit{for}\, n \in \mathbb{N}.$$

It is easy to verify that $f(n) = n + 1$ satisfies the equation. Let us see that this function is the only one that satisfies the equation.

If $f(n) > n+1$, then $f(n)^2 \geq (n+2)^2$, hence $f(f(n)) = n^2+3n+3-f(n)^2 \leq n^2 + 3n + 3 - (n + 2)^2 = -n - 1 < 0$ which is absurd. Therefore $f(n) \leq n + 1$.

If $f(n) < n+1$, we have that $f(n)^2 < (n+1)^2$, and then $f(f(n)) = n^2+3n+3-f(n)^2 > n^2+3n+3-(n+1)^2 = n+2 > f(n)+1$. Hence, $f(f(n)) > f(n)+1$, which is impossible as follows from the previous case. Therefore $f(n) < n + 1$ cannot hold and $f(n) = n + 1$ is the only solution.

Example 6.5.6. *Find continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following:*

$$\textit{For each } x \in \mathbb{R},\ \textit{there exists an integer}\, n \geq 1 \textit{ such that } f^n(x) = x. \tag{6.11}$$

First, let us see that the function is bijective. Suppose that for x, $y \in \mathbb{R}$, we have that $f(x) = f(y)$, by property (6.11), there exist n, $m \in \mathbb{N}$ with $f^n(x) = x$ and $f^m(y) = y$. It is clear that $f^{nm}(x) = x$ and $f^{nm}(y) = y$. But if $f(x) = f(y)$, then $f^{nm}(x) = f^{nm}(y)$, hence $x = y$. Therefore, f is injective.

Next, we will prove that the function is surjective. For each x, there exists $n \in \mathbb{N}$ with $x = f^n(x) = f(f^{n-1}(x))$, remember that $f^0(x) = x$.

Now, we will show that f is increasing or decreasing, which is highlighted in the following more general lemma, not just for the function in the example.

Lemma 6.5.7. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bijective, then f is increasing in all $\mathbb{R}$ or f is decreasing in all $\mathbb{R}$.*

Proof. Let us see that in every open interval the function is increasing or decreasing. Let a, $b \in \mathbb{R}$ with $a < b$. Since f is injective, then $f(a) < f(b)$ or $f(a) > f(b)$.

If $f(a) < f(b)$, we will prove that f is increasing in (a, b), consider x, y with $a < x < y < b$.

(a) It must happen that $f(a) < f(x) < f(b)$, otherwise $f(x) < f(a)$ or $f(b) < f(x)$. In the first case, since $f(x) < f(a) < f(b)$, by the intermediate value theorem there exists $x_1 \in (x, b)$ with $f(x_1) = f(a)$, which is a contradiction to the fact that f is injective. Similarly, if $f(a) < f(b) < f(x)$, by the intermediate value theorem there exists $x_2 \in (a, x)$ with $f(x_2) = f(b)$ which is impossible, since f is injective. Thus, $f(a) < f(x) < f(b)$.

(b) Similarly, we have that for y with $x < y < b$, it follows that $f(x) < f(y) < f(b)$. Therefore, f is increasing in (a, b).

The case when $f(a) > f(b)$ is similar, except that in this situation f will be decreasing. □

Let us come back to the example. We will show that if f is increasing, then $f(x) = x$ for all x. In fact, if for some x_0, we have that $f(x_0) \neq x_0$, then $f(x_0) > x_0$ or $f(x_0) < x_0$. But, since f is increasing we have that

$$x_0 < f(x_0) < f^2(x_0) < \cdots < f^n(x_0) < \ldots$$

or

$$x_0 > f(x_0) > f^2(x_0) > \cdots > f^n(x_0) > \ldots,$$

in any case, $f^n(x_0) \neq x_0$ for every $n \geq 1$, which contradicts (6.11).

But if f is decreasing, then $f^2(x) = x$ for all x, in fact, if f is decreasing, we have that f^2 is increasing and then $f^2(x) = x$.

Finally, we point out some properties that the function has: $f^2(x) = x$, for all $x \in \mathbb{R}$, and it can be proved that the number n in (6.11) is 1 or 2.

Exercise 6.27. *Find the sum*

$$\sum_{k=0}^{n} (-1)^k k^n \binom{n}{k}.$$

Exercise 6.28. *Find all functions $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ that satisfy*

$$f(f(f(n))) + f(f(n)) + f(n) = 3n, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Exercise 6.29. *Find all the continuous functions $f : [0, 1] \to [0, 1]$ that satisfy $f(0) = 0$, $f(1) = 1$ and $f^n(x) = x$, for all $x \in (0, 1)$ with $n \in \mathbb{N}$ fixed.*

Exercise 6.30. *Find all the continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that there is a natural number $n \geq 1$ with $f^n(x) = -x$, for all $x \in \mathbb{R}$.*

Chapter 7

Sequences and Series

7.1 Definition of sequence

A sequence of numbers $\{a_n\}$ can be thought of as a function f defined on the set of positive integers and whose images are a set of numbers A. This set can be: natural, integer, rational, real or complex numbers, that is,

$$\begin{aligned} f \ : \ \mathbb{N} &\to A \\ n &\mapsto f(n) = a_n. \end{aligned}$$

Sometimes it is useful to start the sequence with a_0. We call every element a_n of the sequence a **term** of the sequence. We can also think of a sequence as an infinite collection of ordered numbers.

In the mathematical olympiad, the problems related to sequences are of different kinds. In some of them it is asked to find specific terms of the sequence, in others to prove that the terms are related in some particular way or that they satisfy certain identities. Also, there are some problems that require one to find a closed formula of the nth term or to prove that the nth term satisfies some property. In the following examples, we present a variety of these situations, and in the next section we will give some properties and characteristics that will help us to achieve the goal we have just described.

The set of points a_n, with $n = 1, 2, \ldots$, is called the range of the sequence. The range of the sequence can be finite or infinite.

Example 7.1.1.

(a) *For the sequence* $\left\{a_n = \frac{1}{n}\right\}$, *the range is infinite;*

(b) *If* $\left\{a_n = n^2\right\}$, *then this sequence has infinite range;*

(c) *If* $\left\{a_n = 1 + \frac{(-1)^n}{n}\right\}$, *its range is infinite.*

(d) *The sequence* $\{a_n = (-1)^n\}$ *has finite range.*

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_7

In this first example, the sequences exhibit certain orders or patterns, but not all sequences are like these. In Chapter 2 we studied the arithmetic and geometric progressions, which are examples of sequences that have a pattern or a rule that can be given in an explicit way, but a sequence $\{a_n\}$ such that a_n is the nth digit in the decimal expression of π has no explicit rule.

Let us analyze several examples to get more familiar with sequences.

Example 7.1.2. *The sequence $a_0, a_1, a_2, \ldots,$ is defined as $a_0 = 0, a_1 = 1$ and, for $m \geq n \geq 0$, $a_{m+n} + a_{m-n} = \frac{a_{2m}+a_{2n}}{2}$. Find the value of a_{1000}.*

If $n = 0$, we have that $2a_m = \frac{a_{2m}+a_0}{2}$, then $a_{2m} = 4a_m$. If $m = 1$ and $n = 0$, $a_2 = 4a_1 = 4 = 2^2$. If $m = 2$ and $n = 1$, then $a_3 + a_1 = \frac{a_4+a_2}{2} = \frac{4a_2+4}{2} = 10$, hence $a_3 = 9 = 3^2$.

This suggests that $a_n = n^2$. We will use induction to prove this claim. In fact, it is only left to check the inductive step,

$$\begin{aligned} a_{n+1} + a_{n-1} &= \frac{a_{2n} + a_2}{2} = \frac{4a_n + a_2}{2} \\ a_{n+1} &= 2n^2 + 2 - (n-1)^2 = (n+1)^2. \end{aligned}$$

Thus, $a_{1000} = 1000^2$.

Example 7.1.3. *Define the sequence $\{a_n\}$ as $a_1 = a_2 = 1$ and, for $n \geq 1$, $a_{n+2} = a_{n+1}a_n + 1$. Which elements of the sequence are even and which ones are multiples of 4?*

By induction we can prove that a_n is a positive integer. If a_{n-1} and a_{n-2} are positive integers, then $a_n = a_{n-1}a_{n-2} + 1$ is also a positive integer. Let us see which terms are even. We have that $a_3 = a_2a_1 + 1 = 2$ is even, but a_4 and a_5 are not even; since by definition each of them is the sum of an even number and 1, then a_4 and a_5 are odd. However, a_6 is even and the formula for a_7 and a_8, tell us both are odd. Then, the sequence modulo 2 is 1, 1, 0, 1, 1, 0, ... The recursive relation $a_{n+2} = a_{n+1}a_n + 1$ generates an odd number if one of the factors is even, and an even number if both factors are odd. Then, the terms of the form a_{3k} are even.

If now we consider the sequence modulo 4, we see that it is given by 1, 1, 2, 3, 3, 2, 3, 3, ...; after the third term in the sequence, the numbers 2, 3, 3 are repeated, which shows that there are no multiples of 4.

Example 7.1.4. *The sequence $\{a_n\}$ is defined by $a_1 = a_2 = a_3 = 1$ and, for $n \geq 3$, by $a_{n+1} = \frac{1+a_na_{n-1}}{a_{n-2}}$. Then every element of the sequence is an integer.*

Observe that for $n \geq 3$, the elements of the sequence satisfy $a_{n+1}a_{n-2} = 1 + a_na_{n-1}$, therefore $a_{n+2}a_{n-1} = 1 + a_{n+1}a_n$.

Subtracting the first equation from the second, we get

$$a_{n+2}a_{n-1} - a_{n+1}a_{n-2} = a_{n+1}a_n - a_na_{n-1}.$$

After factoring and rearranging the terms, we obtain

$$(a_{n+2} + a_n)a_{n-1} = (a_n + a_{n-2})a_{n+1}$$
$$\frac{a_{n+2} + a_n}{a_{n+1}} = \frac{a_n + a_{n-2}}{a_{n-1}}.$$

If we define $b_n = \frac{a_n + a_{n-2}}{a_{n-1}}$, it follows that $b_{n+2} = b_n$. That is, it happens that the even terms of $\{b_n\}$ are all equal and the odd terms are also equal to each other. Then, since

$$b_3 = \frac{a_3 + a_1}{a_2} = 2 \quad \text{and} \quad b_4 = \frac{a_4 + a_2}{a_1} = \frac{\frac{1+a_3a_2}{a_1} + a_2}{a_1} = \frac{\frac{1+1}{1} + 1}{1} = 3,$$

we have that

$$a_n = \begin{cases} 3a_{n-1} - a_{n-2}, & \text{if } n \text{ is even} \\ 2a_{n-1} - a_{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

By induction, we can conclude that a_n is an integer number.

Example 7.1.5. *The sequence $\{a_n\}$ defined by $a_1 = 1$ and $a_{n+1} = a_n^2 + a_n$, for $n \geq 1$, satisfies that for any n, $\frac{1}{1+a_1} + \cdots + \frac{1}{1+a_n} < 1$.*

Since $a_{n+1} = a_n^2 + a_n$, it follows that $\frac{1}{a_{n+1}} = \frac{1}{a_n(a_n+1)} = \frac{1}{a_n} - \frac{1}{a_n+1}$, which is equivalent to $\frac{1}{a_n+1} = \frac{1}{a_n} - \frac{1}{a_{n+1}}$. Adding, we obtain

$$\sum_{j=1}^{n} \frac{1}{1+a_j} = \left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \cdots + \left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right) = 1 - \frac{1}{a_{n+1}} < 1.$$

Exercise 7.1 (Croatia, 2009)**.** *The sequence $\{a_n\}$ is defined by*

$$a_1 = 1, \quad a_2 = 3, \quad a_n = a_{n-1} + a_{n-2}, \quad \textit{for } n \geq 3.$$

Prove that $a_n < \left(\frac{7}{4}\right)^n$, for all n.

Exercise 7.2 (Croatia, 2009)**.** *The sequence $\{a_n\}$ is defined by*

$$a_1 = 1, \quad a_n = 3a_{n-1} + 2^{n-1}, \quad \textit{for } n \geq 2.$$

Find a formula for the general term a_n in terms of n.

Exercise 7.3. *The sequence $\{a_n\}$ is defined by*

$$a_1 = 1 \quad \textit{and} \quad a_{n+1} = 1 + a_1 a_2 \ldots a_n, \quad \textit{for } n \geq 1.$$

Prove that for all $n \geq 1$, $\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 2$.

Exercise 7.4. *The sequence* $\{a_n\}$ *is defined by*

$$a_1 = a_2 = 1 \quad and \quad a_{n+1} = \frac{a_n^2 + 1}{a_{n-1}}, \quad for \quad n \geq 2.$$

Prove that every term of the sequence is a positive integer.

Exercise 7.5 (MEMO, 2008). *Let* $\{a_n\}$ *be a sequence of positive integers such that* $a_n < a_{n+1}$ *for* $n \geq 1$. *Suppose that for all 4-tuples* (i, j, k, l) *of indices, such that* $1 \leq i < j \leq k < l$ *and* $i + l = j + k$, *it follows that* $a_i + a_l > a_j + a_k$. *Find the smallest possible value of* a_{2008}.

Exercise 7.6 (China, 2008). *A sequence of real numbers* $\{a_n\}$ *is defined by* $a_0 \neq 0, 1$, $a_1 = 1 - a_0$ *and* $a_{n+1} = 1 - a_n(1 - a_n)$, *for* $n = 1, 2, \ldots$. *Prove that for each positive integer* n,

$$(a_0 a_1 \ldots a_n)\left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) = 1.$$

Exercise 7.7. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences defined by the equations*

$$x_{n+1} = x_n^3 - 3x_n \quad and \quad y_{n+1} = y_n^3 - 3y_n.$$

If $x_0^2 = y_0 + 2$, *prove that* $x_n^2 = y_n + 2$, *for all* n.

Exercise 7.8. *The sequence* $\{a_n\}$ *is defined by*

$$a_1 = 1,\ a_2 = 12,\ a_3 = 20 \quad and \quad a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n, \ for \ \ n \geq 1.$$

Prove that $1 + 4a_n a_{n+1}$ *is a perfect square, for* $n \geq 1$.

7.2 Properties of sequences

In this section we study some properties of the sequences that are useful to find specific relations among terms of the sequences, find closed formulas, etc.

7.2.1 Bounded sequences

We say that a sequence $\{a_n\}$ is **bounded** if there exist $K > 0$ such that $|a_n| \leq K$, for all $n \in \mathbb{N}$. That is, we say that a sequence is bounded if its range is bounded.

For instance, it is clear that the sequences $\{a_n = \frac{1}{n}\}$ and $\{a_n = (-1)^n\}$ are bounded by 1. However, there are sequences for which the bound has to be found.

The most important example of a sequence that is not bounded is the following.

Example 7.2.1. *The sequence $\{a_n = n\}$ is not bounded, since the set of natural numbers is not bounded.*

Suppose that $\mathbb{N}$ is bounded above. Then, there exists $M > 0$ such that $n \leq M$, for all $n \in \mathbb{N}$. Take $\lfloor M \rfloor$ the greatest integer less than or equal than M, then the integer $\lfloor M \rfloor + 1$ satisfies that it is a positive integer with $M < \lfloor M \rfloor + 1$, hence M is not an upper bound for $\mathbb{N}$, which is a contradiction.

Example 7.2.2. *The sequence $\{a_n\}$ given by $0 < a_0 < a_0 + a_1 < 1$ and*

$$a_{n+1} + \frac{a_n - 1}{a_{n-1}} = 0, \quad \textit{for } n \geq 1,$$

is a bounded sequence.

Let us find a few terms of the sequence:

$$a_2 = \frac{1 - a_1}{a_0}, \quad a_3 = \frac{a_0 + a_1 - 1}{a_0 a_1}, \quad a_4 = \frac{1 - a_0}{a_1}, \quad a_5 = a_0, \quad a_6 = a_1.$$

Therefore, we see that the terms of the sequence are repeated every five terms, then it is bounded.

In the last example, we can observe that the term a_5 is equal to the term a_0, and in general we have that $a_{n+5} = a_n$, for all $n \in \mathbb{N}$. The sequences with this property have a special name, which will be studied next.

7.2.2 Periodic sequences

A sequence $\{a_n\}$ is **periodic**, with period $p \geq 1$, if it satisfies that $a_{n+p} = a_n$, for all $n \in \mathbb{N}$.

If a sequence is periodic with period p, then we can find all the values of the sequence if we know the values of the first p terms of the sequence. Actually, if $\{a_n\}$ is a sequence with period p and n is a positive integer, by Euclid's algorithm, we can express n as $n = pq + r$, with $0 \leq r < p$. Then $a_n = a_r$ if $r \neq 0$, and $a_n = a_p$ if $r = 0$. Also, observe that every periodic sequence is bounded, moreover, the sequences with finite rank are clearly bounded.

Example 7.2.3. *The sequence $\{a_n\}$ is defined by $a_{n+2} = \frac{1+a_{n+1}}{a_n}$. Find the value of a_{2013}.*

We analyze the first terms of the sequence

$$a_3 = \frac{1 + a_2}{a_1}, \quad a_4 = \frac{1 + a_3}{a_2} = \frac{1 + \frac{1+a_2}{a_1}}{a_2} = \frac{1 + a_1 + a_2}{a_1 a_2},$$

$$a_5 = \frac{1 + a_4}{a_3} = \frac{1 + \frac{1+a_1+a_2}{a_1 a_2}}{\frac{1+a_2}{a_1}} = \frac{(1 + a_1)(1 + a_2)}{\frac{a_1 a_2 (1+a_2)}{a_1}} = \frac{1 + a_1}{a_2},$$

$$a_6 = \frac{1 + a_5}{a_4} = \frac{1 + \frac{1+a_1}{a_2}}{\frac{1+a_1+a_2}{a_1 a_2}} = a_1, \quad a_7 = \frac{1 + a_6}{a_5} = \frac{1 + a_1}{\frac{1+a_1}{a_2}} = a_2.$$

Then, the sequence is periodic with period 5, that is, $a_{n+5} = a_n$ for all n. Therefore, $a_{2013} = a_3 = \frac{1+a_2}{a_1}$.

In some of the examples we have seen so far, we can notice that in order to define the general term of a sequence it is necessary to know some of the previous terms. We will come to this in the following section.

7.2.3 Recursive or recurrent sequences

Some of the sequences we have thus far studied satisfy the condition that the term a_{n+1} is a function of some of the previous terms, that is, $a_{n+1} = f(a_1, \dots, a_n)$. Sequences of this sort are known as **recurrent sequences or recursive sequences**. More precisely, we will say that $\{a_n\}$ satisfies the **recursive equation**

$$a_{n+1} = f(a_1, \dots, a_n), \tag{7.1}$$

if for every n, the terms of $\{a_n\}$ satisfy the last identity. Note that the function f is not the same for each n, for instance, if $f(a_1, \dots, a_n) = a_1 + a_2 + \cdots + a_n$ we have that $a_2 = f(a_1) = a_1$, $a_3 = f(a_1, a_2) = a_1 + a_2$, $\dots$.

The simplest examples of recursive sequences are the arithmetic progressions $a_n = a_1 + (n-1)d$ that satisfy the recurrent equation $a_{n+1} = a_n + d$ and the geometric progressions $a_n = r^{n-1}a_1$, that solve the recurrent equation $a_{n+1} = ra_n$.

Example 7.2.4. *A sequence that generalizes the arithmetic and geometric progressions is the sequence that solves the recursive equation, $a_{n+1} = r_n a_n + d_n$, where $\{r_n\}$ and $\{d_n\}$ are sequences independent of the terms a_n. Let us find a closed formula for a_n.*

It is evident that if for every integer n the equality $a_n = r_{n-1}a_{n-1} + d_{n-1}$ holds, then

$$\begin{aligned}
a_n &= r_{n-1}a_{n-1} + d_{n-1}\\
r_{n-1}a_{n-1} &= r_{n-1}r_{n-2}a_{n-2} + r_{n-1}d_{n-2}\\
r_{n-1}r_{n-2}a_{n-2} &= r_{n-1}r_{n-2}r_{n-3}a_{n-3} + r_{n-1}r_{n-2}d_{n-3}\\
&\ \vdots \qquad\qquad \vdots\\
r_{n-1}\cdots r_2 a_2 &= r_{n-1}\cdots r_2 r_1 a_1 + r_{n-1}\cdots r_2 d_1.
\end{aligned}$$

After adding and canceling terms, we get that

$$a_n = (r_{n-1}\cdots r_1)\, a_1 + \sum_{j=1}^{n-2} r_{n-1}\cdots r_{j+1}d_j + d_{n-1}.$$

In particular, for $a_{n+1} = ra_n + d$, it follows that $a_n = r^{n-1}a_1 + (1 + r + \cdots + r^{n-2})d$.

In Example (3.1.4) of the Hanoi's towers, we notice that the number of necessary movements h_n to move n disks from one stick to another, satisfies the recursive formula $h_{n+1} = 2h_n + 1$. This formula confirms the value that we found for h_n, because it implies that,

$$h_n = 2^{n-1} \cdot h_1 + (1 + 2 + \cdots + 2^{n-2}) \cdot 1 = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1.$$

We say that $\{a_n\}$ is a **recurrent linear sequence of order** $k \geq 1$, if it satisfies the recursive equation

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n,$$

where $c_1, \ldots, c_k$ are constant numbers.

For instance, the Fibonacci sequence $\{f_n\}$, defined as the sequence that satisfies the Fibonacci recursion formula $f_{n+1} = f_{n-1} + f_n$, with $f_1 = f_2 = 1$, is a recurrent linear sequence of order 2. A geometric progression is a recurrent linear sequence of order 1, since $a_{n+1} = ra_n$, and these sequences are the only ones of order 1, where r can be any number. An arithmetic progression satisfies $a_{n+1} = a_n + d$, which is not a linear recursion because of the constant term d. However, since $a_{n+2} = a_{n+1} + d$, it follows that $a_{n+2} - a_{n+1} = a_{n+1} - a_n$, so that $a_{n+2} = 2a_{n+1} - a_n$. Thus the arithmetic progressions are recurrent linear sequences of order 2.

Our next objective is to solve the linear recursions of second order, that is, we want to recognize the sequences that satisfy the recursive equation

$$a_{n+2} = ba_{n+1} + ca_n, \tag{7.2}$$

where b and c are fixed constants.

For instance, the recursive equation $a_{n+1} = 5a_n - 6a_{n-1}$ is solved by the sequences $\{2^n\}$ and $\{3^n\}$, which shows that there is not always only one sequence that solves the equation. Moreover, if $\{a_n\}$ and $\{b_n\}$ are sequences that solve a recursive linear equation, then also $\{Aa_n + Bb_n\}$ solves the equation, for any numbers A and B. Hence there could be several sequences solving equation (7.2).

However, if we have that the first two terms of each of two sequences that solve a linear recursion of order 2 are equal, then the two solutions coincide. This follows because, if the first two terms of each solution coincide, then the third terms coincide too and, by induction, all the terms of the two sequences coincide. An application of this remark can be found in the following example.

Example 7.2.5. *If $\{f_n\}$ is the Fibonacci sequence, then $f_{m+n} = f_m f_{n-1} + f_{m+1} f_n$, for $m \geq 0$ and $n \geq 1$, where $f_0 = 0$.*

Define the sequences $a_m = f_{m+n}$ and $b_m = f_m f_{n-1} + f_{m+1} f_n$, for $m \geq 0$ and with $n \geq 1$ fixed. It is easy to show that a_m and b_m satisfy the Fibonacci recursion formula. For instance, $a_{m+2} = f_{m+2+n} = f_{m+1+n} + f_{m+n} = a_{m+1} + a_m$ and

$$\begin{aligned} b_{m+2} &= f_{m+2} f_{n-1} + f_{m+3} f_n = (f_{m+1} + f_m) f_{n-1} + (f_{m+2} + f_{m+1}) f_n \\ &= (f_{m+1} f_{n-1} + f_{m+2} f_n) + (f_m f_{n-1} + f_{m+1} f_n) = b_{m+1} + b_m. \end{aligned}$$

On the other hand, $a_0 = f_n$, $a_1 = f_{n+1}$, $b_0 = f_0\ f_{n-1} + f_1 f_n = f_n$ and $b_1 = f_1 f_{n-1} + f_2 f_n = f_{n-1} + f_n = f_{n+1}$. Since both sequences satisfy the same linear recurrence of order 2 and coincide in the first two terms, we have that the sequences are equal. Therefore the identity holds.

Let us go back to see how to solve the linear recursions of order 2. Following the idea that the linear recursions of order 1 are solved by sequences of the form $a_n = A\lambda^n$, let us see what a sequence of the form $\{a_n = A\lambda^n\}$ should satisfy in order to be a solution of (7.2). Substituting in equation (7.2), we get

$$A\lambda^{n+2} = bA\lambda^{n+1} + cA\lambda^n.$$

If $A = 0$, it is clear that the constant sequence $a_n = 0$ satisfies (7.2). If $A \neq 0$, we can cancel A and after factoring we get $\lambda^n \left(\lambda^2 - b\lambda - c\right) = 0$.

Now, if for some integer n we have that $\lambda^n = 0$, that is, $\lambda = 0$ and, therefore $a_n = 0$, which we know solves the equation. Now, suppose $\lambda \neq 0$, hence

$$\lambda^2 - b\lambda - c = 0, \tag{7.3}$$

so that $\lambda = \frac{b \pm \sqrt{b^2+4bc}}{2}$ are the only possible values if the solution is of the form $a_n = A\lambda^n$.

Equation (7.3) is known as the **characteristic equation** of the recursion formula (7.2) and the polynomial on the left is known as the **characteristic polynomial**. To conclude we analyze two cases. The first corresponding to the roots of the characteristic equation being different and then the case when they are equal.

Case A. λ_1 and λ_2 are the solutions of equation (7.3), with $\lambda_1 \neq \lambda_2$.

In this case, we notice that $a_n = A\lambda_1^n + B\lambda_2^n$ solves equation (7.2). Now, let us see that, if $\{b_n\}$ is a sequence that satisfies the equation, then $b_n = A\lambda_1^n + B\lambda_2^n$ for some numbers A and B.

For this, we know that it is enough to see that $a_0 = b_0$ and $a_1 = b_1$. Then we have to solve

$$\begin{aligned} a_0 &= A + B \\ a_1 &= A\lambda_1 + B\lambda_2. \end{aligned}$$

But this system of two equations with two unknowns can be solved in a unique way for A and B. In fact,

$$A = \frac{a_0\lambda_2 - a_1}{\lambda_2 - \lambda_1} \quad \text{and} \quad B = \frac{a_1 - \lambda_1 a_0}{\lambda_2 - \lambda_1},$$

and this is the only solution of the system when $\lambda_2 - \lambda_1 \neq 0$.

Case B. The roots of the characteristic polynomial λ_1 and λ_2 coincide.

In this case $a_n = A\lambda_1^n + B\lambda_2^n$ is not a general solution anymore, since $a_n = (A+B)\lambda_1^n$, and it is not always possible to choose A and B such that $A+B = a_0$ and $(A+B)\lambda_1 = a_1$.

However, there is another solution of the recursion formula different from λ_1^n; this happens to be sequence $b_n = (n+1)\lambda_1^n$. In order to see that this sequence satisfies the recursion, first note that if $\lambda_1 = \lambda_2$ are the roots of $\lambda^2 - b\lambda - c = 0$, by Vieta, $b = 2\lambda_1$ and $c = -\lambda_1^2$. Then, the recursion is

$$a_{n+2} = 2\lambda_1 a_{n+1} - \lambda_1^2 a_n$$

and we can verify that

$$(n+3)\lambda_1^{n+2} = 2\lambda_1(n+2)\lambda_1^{n+1} - \lambda_1^2(n+1)\lambda_1^n,$$

which proves that $b_n = (n+1)\lambda_1^n$ solves the recursion formula.

Now, the two known solutions $b_n = \lambda_1^n$ and $c_n = (n+1)\lambda_1^n$ generate the general solution $a_n = A\lambda_1^n + (n+1)B\lambda_1^n$. In this case the initial conditions determine A and B, that is, there is only one pair of numbers A and B with

$$\begin{aligned} a_0 &= A + B \\ a_1 &= (A + 2B)\lambda_1. \end{aligned}$$

Actually, $A = \frac{2a_0\lambda_1 - a_1}{\lambda_1}$ and $B = \frac{a_1 - a_0\lambda_1}{\lambda_1}$. Then, in this case, the solution with initial conditions is unique.

We can summarize both cases in the following result.

Theorem 7.2.6.

(a) *If the roots of the equation $\lambda^2 - b\lambda - c = 0$ are different ($b^2 + 4c \neq 0$), then all the solutions $a_{n+2} = ba_{n+1} + ca_n$, of the recursion formula are of the form*

$$a_n = A\lambda_1^n + B\lambda_2^n,$$

where A and B are any real numbers.

(b) *If the equation $\lambda^2 - b\lambda - c = 0$ has only one double real root equal to $\lambda = \frac{b}{2}$, then all solutions of the recursion are of the form*

$$a_n = (A + (n+1)B)\lambda^n,$$

where A and B are any real numbers.

(c) *If a_0 and a_1 are given numbers, then A and B are determined by $a_0 = A+B$ and $a_1 = A\lambda_1 + B\lambda_2$ in case* (a)*, and by $a_0 = A + B$ and $a_1 = (A+2B)\lambda$ in case* (b).

For instance, the recursion $x_{n+2} = 2x_{n+1} - x_n$, has characteristic polynomial $\lambda^2 - 2\lambda + 1$, with $\lambda = 1$ as the only root. Then, the solutions are of the form $x_n = (c + dn)1^n = c + dn$, something we already know as arithmetic progressions.

Example 7.2.7. *Find the solutions of the Fibonacci recursion, $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$, for $n \geq 0$.*

The characteristic equation is given by $\lambda^2 - \lambda - 1 = 0$ and its roots are $\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{5}}{2}$, which are different. Then, the solutions of the recursion are of the form $f_n = A\lambda_1^n + B\lambda_2^n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$. Because the first terms are $f_0 = 0$ and $f_1 = 1$, then $A = \frac{1}{\sqrt{5}} = -B$. Hence the Fibonacci numbers f_n are

$$f_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

Example 7.2.8. *Find the solutions of the recursion defined by $a_0 = 0$, $a_1 = \sin\alpha$ and $a_{n+2} = 2\cos\alpha \cdot a_{n+1} - a_n$, for $n \geq 0$ and $\alpha \neq n\pi$.*

The characteristic polynomial of the given recursion is $\lambda^2 - 2\cos\alpha\,\lambda + 1 = 0$, which has solutions

$$\lambda_1, \lambda_2 = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} = \cos\alpha \pm i\sin\alpha.$$

Hence, the solutions of the recursion are of the form $a_n = A\lambda_1^n + B\lambda_2^n$. From the initial conditions we obtain that $a_0 = A + B = 0$ and $a_1 = A\lambda_1 + B\lambda_2 = \sin\alpha$. From the first equation we have that $B = -A$, so that $A(\lambda_1 - \lambda_2) = A(2i\sin\alpha) = \sin\alpha$, and then, using the fact that $\sin\alpha \neq 0$, $A = \frac{1}{2i}$. Therefore,

$$\begin{aligned} a_n &= \frac{1}{2i}\left\{(\cos\alpha + i\sin\alpha)^n - (\cos\alpha - i\sin\alpha)^n\right\} \\ &= \frac{1}{2i}(\cos n\alpha + i\sin n\alpha - \cos n\alpha + i\sin n\alpha) \\ &= \sin n\alpha. \end{aligned}$$

Example 7.2.9. *Analyze the non-linear recurrent equation $a_{n+1} = a_n^2 - 2$.*

It is clear that if $a_0 = a + \frac{1}{a}$, then $a_n = a^{2^n} + a^{-2^n}$ solves the recurrence, since $a_n^2 - 2 = \left(a^{2^n} + a^{-2^n}\right)^2 - 2 = a^{2^{n+1}} + a^{-2^{n+1}} = a_{n+1}$. If $|a_0| > 2$, then $a = \frac{a_0 + \sqrt{a_0^2 - 4}}{2}$ satisfies $a_0 = a + \frac{1}{a}$, if $|a_0| \leq 2$, so that we can take $a_0 = 2\cos\theta$, and therefore $a = e^{i\theta} = \cos\theta + i\sin\theta$.

7.2.4 Monotone sequences

A sequence $\{a_n\}$ of real numbers is **monotone increasing** if

$$a_n \leq a_{n+1}, \quad \text{for all } \; n \in \mathbb{N}.$$

The sequence is **increasing** if $a_n < a_{n+1}$, for all $n \in \mathbb{N}$.

Similarly, we say that a sequence $\{a_n\}$ of real numbers is **monotone decreasing** if

$$a_n \geq a_{n+1}, \quad \text{for all} \;\; n \in \mathbb{N}.$$

The sequence is **decreasing** if $a_n > a_{n+1}$, for all $n \in \mathbb{N}$.

Example 7.2.10.

(a) *Any arithmetic progression with difference* $d > 0$ *is an increasing sequence.*
(b) *Any geometric sequence with* $a_1 > 0$ *is monotone increasing if* $r \geq 1$.

In (a), if $\{a_n\}$ is the arithmetic progression with difference d, then $a_{n+1} - a_n = d > 0$, where clearly $a_n < a_{n+1}$.

In (b), if $\{a_n\}$ is a geometric progression with ratio $r \geq 1$, then is clear that $a_{n+1} = ra_n \geq a_n$.

Example 7.2.11. *The sequence defined by* $0 < a_1 < \frac{1}{2}$ *and* $a_{n+1} = 2a_n(1 - a_n)$ *is increasing.*

In order to see that it is increasing, we will need to show that $0 < a_n$ and $1 < \frac{a_{n+1}}{a_n} = 2(1 - a_n)$, that is, $0 < a_n < \frac{1}{2}$ for all n.

Let us proceed by induction. First note that if $0 < a_n < \frac{1}{2}$, then $a_{n+1} = 2a_n(1 - a_n) > 0$. Now using the geometric mean and the arithmetic mean inequality, we have that

$$a_{n+1} = 2a_n(1 - a_n) \leq 2\left(\frac{a_n + 1 - a_n}{2}\right)^2 = \frac{1}{2}.$$

The inequality is strict since the equality holds when $a_n = 1 - a_n = \frac{1}{2}$, which is not the case in the previous expression.

7.2.5 Totally complete sequences

A sequence $\{a_n\}$ of positive integers is called **totally complete** if every positive integer can be expressed as a sum of one or more different terms of the sequence.

Clearly the sequence of positive integers $\{1, 2, 3, \ldots, n, \ldots\}$ is totally complete.

Example 7.2.12. *The sequence of powers of* 2, $\{2^0, 2^1, 2^2, \ldots, 2^n, \ldots\}$ *is totally complete.*

We will give a proof using strong induction. The basis cases are evident, since $1 = 2^0$ and $2 = 2^1$. Suppose that any integer less than n can be written as a sum of different powers of 2. Let 2^m be the greatest power of 2 that is less than or equal to n, then $2^m \leq n < 2^{m+1}$. Set $d = n - 2^m$, which is clearly less than or equal to n, and also less than 2^m ($n < 2^{m+1}$ implies that $n - 2^m < 2^m$). By the induction hypothesis, d can be expressed as the sum of different powers of 2 and, since d is less than 2^m, this power is not included in the representation of d. Adding 2^m to the representation of d we get the representation of n.

Proposition 7.2.13. *A sequence of positive integers* $\{a_n\}$ *that satisfies* $a_1 = 1$ *and*

$$a_{n+1} \leq 1 + a_1 + a_2 + \cdots + a_n, \quad \textit{for all} n = 1, 2, \ldots,$$

is totally complete.

Proof. We will show, using induction on n, that every integer k less than or equal to $a_1 + a_2 + \cdots + a_n$, is the sum of different terms of $\{a_1, a_2, .., a_n\}$.
If $n = 1$, the only $k \leq a_1 = 1$ is $k = 1$ and clearly $k = a_1 = 1$.
If $n = 2$, the numbers k to consider are the ones that satisfy $k \leq a_2 \leq 1 + a_1$. Since $a_2 \leq 1 + a_1 = 2$, it follows that $a_2 = 1$ or 2. If $a_2 = 1$, $k = 1$ or 2, then $k = 1 = a_1$ and $k = 2 = a_1 + a_2$. If $a_2 = 2$, $k = 1$, 2 or 3, then $k = 1 = a_1$, $k = 2 = a_2$ and $k = 3 = a_1 + a_2$.
For the inductive step, suppose the statement true for n and let us prove it for $n + 1$. Consider a positive integer k that satisfies $k \leq a_1 + \cdots + a_n + a_{n+1}$. If $k \leq a_1 + \cdots + a_n$, by the induction hypothesis, such k is the sum of different elements of $\{a_1, a_2, \ldots, a_n\}$. Suppose then that $1 + a_1 + \cdots + a_n \leq k \leq a_1 + \cdots + a_n + a_{n+1}$. Then, since $a_{n+1} \leq 1 + a_1 + a_2 + \cdots + a_n$, it follows that $a_{n+1} \leq k \leq a_1 + \cdots + a_n + a_{n+1}$, hence $0 \leq k - a_{n+1} \leq a_1 + \cdots + a_n$. Now, if $k - a_{n+1} = 0$ the proof is finished, and if $0 < k - a_{n+1} \leq a_1 + \cdots + a_n$, then, by the induction hypothesis, $k - a_{n+1}$ is the sum of different elements of the set $\{a_1, a_2, \ldots, a_n\}$. Adding a_{n+1}, we have that k is the sum of different elements of $\{a_1, \ldots, a_n, a_{n+1}\}$, as we wanted to prove. $\square$

Example 7.2.14. *The Fibonacci sequence* $1, 1, 2, 3, 5, \ldots, f_n, \ldots$ *is totally complete.*

By the last proposition, it is enough to see that $f_{n+1} \leq 1 + f_1 + f_2 + \cdots + f_n$, for every $n \geq 1$. For $n = 1$, it is clear since $1 = f_2 \leq 1 + f_1 = 2$. For $n = 2$, the result follows because $2 = f_3 \leq 1 + f_1 + f_2 = 3$. For $n \geq 3$, it is enough to show that $1 + f_1 + f_2 + \cdots + f_n - f_{n+1} \geq 0$. But since $f_{n+1} = f_{n-1} + f_n$, it follows for $n \geq 3$ that

$$\begin{aligned} 1 + f_1 + f_2 + \cdots + f_n - f_{n+1} &= 1 + f_1 + f_2 + \cdots + f_n - (f_{n-1} + f_n) \\ &= 1 + f_1 + f_2 + \cdots + f_{n-2} \geq 0. \end{aligned}$$

7.2.6 Convergent sequences

A sequence $\{a_n\}$ **converges to or has limit** a if for all $\epsilon > 0$ there exists a natural number N such that, for all $n \geq N$, it follows that $|a_n - a| < \epsilon$. This can be written as,

$$\forall\, \epsilon > 0, \ \exists\, N \in \mathbb{N} \text{ such that, } \forall\, n \geq N, \ |a_n - a| < \epsilon.$$

We say that a is the **limit** of the sequence and write $\lim_{n\to\infty} a_n = a$.

A sequence diverges if it does not converge to any point a.

Example 7.2.15. *The sequence* $\{a_n = \frac{1}{n}\}$ *converges to* 0.

Given $\epsilon > 0$ we would like to show the existence of $N \in \mathbb{N}$ such that, for all $n \geq N$, it implies that $0 < \frac{1}{n} < \epsilon$. But if $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N}$, therefore it is enough to show the existence of N with $\frac{1}{N} < \epsilon$. If such N does not exist, that is if $\frac{1}{N} \geq \epsilon$, then $N \leq \frac{1}{\epsilon}$ for all $N \in \mathbb{N}$. This is a contradiction, because the natural numbers are not bounded.

Example 7.2.16. *The sequence* $\{a_n = a^n\}$ *converges to* 0 *if* $|a| < 1$.

If $|a| < 1$, then $\frac{1}{|a|} > 1$, hence $\frac{1}{|a|} = 1 + p$ with $p > 0$. Thus, $(1+p)^n = 1 + np + \frac{n(n-1)}{2}p^2 + \cdots \geq np$, and $|a|^n = \frac{1}{(1+p)^n} \leq \frac{1}{np}$. Choose $N \in \mathbb{N}$ such that if $n \geq N$, then $\frac{1}{n} < p\epsilon$, so that $|a|^n = \frac{1}{(1+p)^n} \leq \frac{1}{np} < \epsilon$. Thus $\{a^n\}$ converges to 0.

7.2.7 Subsequences

Given a sequence $\{a_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. The sequence $\{a_{n_k}\}$ is called a **subsequence** of $\{a_n\}$. Note that given a sequence, we can obtain an infinite number of subsequences from it.

Example 7.2.17. *The sequence* $\{a_n\}$ *defined by*

$$a_n = \begin{cases} \frac{1}{n}, & \textit{if } n \textit{ is even,} \\ n^2, & \textit{if } n \textit{ is odd,} \end{cases}$$

has subsequences that are convergent, not convergent, bounded, not bounded, increasing and decreasing.

First note that the sequence is not convergent, and neither bounded nor monotone, but we can find the following subsequences:

(a) If we take $\{a_{2n}\}$, the sequence is convergent, decreasing and bounded, since the terms are given by $\frac{1}{2n}$ for all $n \in \mathbb{N}$.
(b) If we take $\{a_{2n-1}\}$, the sequence is not convergent, increasing and not bounded, since the terms are given by $(2n-1)^2$, for all $n \in \mathbb{N}$.

Exercise 7.9. *Prove that the sequence* $\{a_n\}$ *defined by* $a_0 = 0$ *and for* $n \geq 0$, $a_{n+1} = \sqrt{4 + 3a_n}$, *is a bounded sequence.*

Exercise 7.10. *A sequence* $\{a_n\}$ *is defined by*

$$a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{a_n^2}, \quad \textit{for } n \geq 1.$$

Determine if the sequence is bounded or not and prove that $a_{9000} > 30$.

Exercise 7.11. *The terms of the sequence* $\{a_n\}$ *are positive and* $a_{n+1}^2 = a_n + 1$, *for all* n. *Prove that the sequence contains at least one irrational number.*

Exercise 7.12. *Find, in each case, the solutions of the recursive equation of degree* 3, $a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$, *if:*

(i) $a_1 = a_2 = a_3 = 1$.

(ii) $a_1 = 1$, $a_2 = 2$, $a_3 = 3$.

(iii) $a_1 = 1$, $a_2 = 4$, $a_3 = 9$.

Exercise 7.13. *The positive integers* $a_1, a_2, \ldots,$ *are bounded and form a sequence that satisfies the following condition: if* m *and* n *are positive integers, then* $a_m + a_n$ *is divisible by* a_{m+n}. *Prove that the sequence is periodic after some terms.*

Exercise 7.14. *Prove that* $a_n = n!$, *the number of permutations of* n *elements, and* d_n, *the number of permutations of* n *elements without fixed points, satisfy the recursive equation* $x_{n+1} = n(x_n + x_{n-1})$. *Why is* $a_n \neq d_n$ *for all* n*?*

Note: a permutation without fixed points is called a **derangement**.

Exercise 7.15.

(i) *Use the recursive formula for the number of derangements of a set of* n *elements, given by* $d_n = (n-1)(d_{n-1} + d_{n-2})$, *to prove that* $d_n = nd_{n-1} + (-1)^n$.

(ii) *Use Example* 7.2.4 *to justify the formula*

$$d_n = n!\left(1 + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \cdots + \frac{(-1)^n}{n!}\right).$$

Exercise 7.16. **Lucas' numbers** *are defined by the recurrence* $L_1 = 1$, $L_2 = 3$ *and* $L_{n+1} = L_n + L_{n-1}$, *for* $n \geq 2$. *Prove that*

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Exercise 7.17. *Solve the recursive equation* $b_{n+1} = \frac{b_n}{1+b_n}$, *for* $n = 0, 1, 2, \ldots,$ *where* b_0 *is a fixed positive number.*

Exercise 7.18. *Prove that the sequence defined by* $a_0 = a \neq 1$ *and* $a_{n+1} = \frac{1}{1-a_n}$, *is periodic.*

Exercise 7.19. *Solve the recursive equation* $a_{n+1} = 4 - \frac{4}{a_n}$. *Prove that* a_n *converges to* 2.

Exercise 7.20. *Prove that a sequence* $\{a_n\}$ *that satisfies* $a_1 = 1$ *and* $a_{n+1} \leq 2a_n$, *for* $n \geq 1$, *is totally complete.*

Exercise 7.21. *Prove that the sequence* $\{1, 2, 3, 5, 7, 11, 13, \ldots\}$ *of all prime numbers and the number* 1, *is totally complete.*

Exercise 7.22. *Two brothers inherit n golden pieces with total weight $2n$. Each piece has an integer weight and the heaviest of all the pieces does not weigh more than all the others together. Prove that if n is even, then the brothers can divide the golden pieces into two parts with equal weight.*

Exercise 7.23 (Romania, 2009). *A sequence $\{a_n\}$ is defined by*

$$a_1 = 2, \quad a_{n+1} = \sqrt{a_n + \frac{1}{n}}, \quad \text{for } n \geq 1.$$

Prove that the limit of the sequence exists and is 1.

7.3 Series

Given a sequence $\{a_n\}$, to denote the sum $a_p + a_{p+1} + \cdots + a_q$ with $p \leq q$, we use the notation

$$\sum_{n=p}^{q} a_n.$$

To a sequence $\{a_n\}$ we associate the sequence $\{s_n\}$, called the sequence of **partial sums**, given by

$$s_n = \sum_{k=1}^{n} a_k.$$

The infinite sum $a_1 + a_2 + a_3 + \cdots$ can be written in short form as

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n = \lim_{N \to \infty} s_N.$$

This last expression is called **infinite series** or, more simply, **series**. If $\{s_n\}$ converges to s, we say that the **series converges** to s and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the **sum** of the series.

We say that the **series diverges** if $\{s_n\}$ diverges.

Example 7.3.1. *Find the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

The partial sum is

$$\begin{aligned} s_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} \\ &= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right) = 1 - \frac{1}{n+1}. \end{aligned}$$

Since $\{\frac{1}{n+1}\} \to 0$, when $n \to \infty$, then $\{s_n\} \to 1$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Exercise 7.24. *If $\{a_n\}$ is an arithmetic progression with difference $d \neq 0$, prove that:*

(i) $\displaystyle\sum_{i=0}^{n} \frac{1}{a_i \cdot a_{i+1}} = \frac{1}{d}\left(\frac{1}{a_0} - \frac{1}{a_{n+1}}\right)$.

(ii) $\displaystyle\sum_{i=0}^{n} \frac{1}{a_i \cdot a_{i+1} \cdot a_{i+2}} = \frac{1}{2d}\left(\frac{1}{a_0\cdot a_1} - \frac{1}{a_{n+1}\cdot a_{n+2}}\right)$.

(iii) $\displaystyle\sum_{i=0}^{\infty} \frac{1}{a_i \cdot a_{i+1}} = \frac{1}{da_0}$. (iv) $\displaystyle\sum_{i=0}^{\infty} \frac{1}{a_i \cdot a_{i+1} \cdot a_{i+2}} = \frac{1}{2d\cdot a_0 \cdot a_1}$.

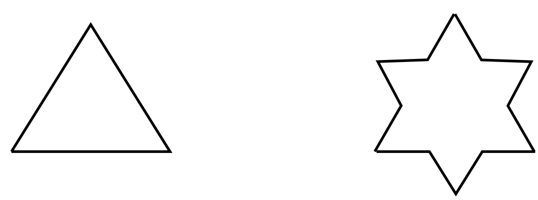
Exercise 7.25. *Let f_n be the Fibonacci sequence ($f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$). Find the sum of the following series:*

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}}$ (ii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}}$.

Exercise 7.26. *The* **Koch snowflake** *is obtained by means of the following process:*

(i) *In step* 0*, the curve is an equilateral triangle of side* 1.

(ii) *In step* $n+1$*, the curve is obtained from the curve in step* n*, dividing each one of the sides in* 3 *equal parts, and constructing externally in the middle part of the divided side, an equilateral triangle erasing the side in which it was constructed. In the following diagram steps* 0 *and* 1 *are shown.*



If P_n and A_n are the perimeter and the area, respectively, of the curve of step n, find:

(i) P_n (ii) A_n (iii) $\displaystyle\lim_{n\to\infty} P_n$ (iv) $\displaystyle\lim_{n\to\infty} A_n$.

Exercise 7.27. *Consider the harmonic progression* $\left\{\frac{1}{n}\right\}$. *The series* $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ *is known as the* **harmonic** *series.*

(i) *Prove that for* $n \geq 1$, $\dfrac{1}{2^n+1}+\dfrac{1}{2^n+2}+\cdots+\dfrac{1}{2^{n+1}} > \dfrac{1}{2}$.

(ii) *Prove that for* $n \geq 2$, $\dfrac{1}{n}+\dfrac{1}{n+1}+\cdots+\dfrac{1}{n^2} > 1$.

(iii) *Prove that for* $n \geq 2$, $\dfrac{1}{n-1}+\dfrac{1}{n}+\dfrac{1}{n+1} > \dfrac{3}{n}$.

(iv) *Use any of the previous inequalities to conclude that the harmonic series is divergent.*

7.3.1 Power series

The following expression is known as **formal power series** in the variable x with center at zero,

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots\,, \tag{7.4}$$

where $a_0, a_1, a_2, \ldots$ is an arbitrary sequence of numbers. The power series can be written as

$$f(x) = \sum_{n=0}^{\infty} a_nx^n.$$

Let us see some examples of how to calculate these series.

Example 7.3.2. *The sum of the* **geometric series** *is given by*

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}, \quad \textit{if} \quad |x| < 1.$$

Consider the partial sum

$$\begin{aligned} s_n &= a + ax + ax^2 + \cdots + ax^n \\ &= a(1 + x + \cdots + x^n) \\ &= a\left(\frac{1-x^{n+1}}{1-x}\right). \end{aligned}$$

Then, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{a(1-x^{n+1})}{1-x} = \frac{a}{1-x}$, since[16] $|x| < 1$. Therefore $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$.

[16]See Example 7.2.16.

Example 7.3.3. *The following series, known as the* **derivative series** *of the geometric series, converges to the given value*

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \quad \textit{for } |x| < 1.$$

Consider the partial sum $\sum_{k=0}^{n} kx^k = x + 2x^2 + 3x^3 + \cdots + nx^n$. This partial sum is the sum of the following partial sums:

$$\begin{aligned}
x + x^2 + x^3 + \cdots + x^n &= x\left(\frac{1-x^n}{1-x}\right) \\
x^2 + x^3 + \cdots + x^n &= x^2\left(\frac{1-x^{n-1}}{1-x}\right) \\
x^3 + \cdots + x^n &= x^3\left(\frac{1-x^{n-2}}{1-x}\right) \\
\vdots \quad & \quad \vdots \\
x^n &= x^n\left(\frac{1-x}{1-x}\right).
\end{aligned}$$

Adding these sums, we get

$$\begin{aligned}
x + 2x^2 + 3x^3 + \cdots + nx^n &= \frac{x(1-x^n) + x^2(1-x^{n-1}) + \cdots + x^n(1-x)}{1-x} \\
&= \frac{x + x^2 + \cdots + x^n - nx^{n+1}}{1-x} \\
&= \frac{x\left(\frac{1-x^n}{1-x}\right) - nx^{n+1}}{1-x} \\
&= \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}. \qquad (7.5)
\end{aligned}$$

The value of the infinite sum is

$$\sum_{n=1}^{\infty} nx^n = \lim_{n\to\infty}\left(\frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}\right) = \frac{x}{(1-x)^2}, \qquad (7.6)$$

since x^{n+1} and nx^{n+1} go to zero as $n \to \infty$, because $|x| < 1$. Therefore, canceling one x on both sides, we have the desired equality.

Example 7.3.4. *The sum of the following series is given by*

$$\sum_{n=0}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}, \quad \textit{for } |x| < 1.$$

Consider the partial sum

$$\begin{aligned} s_n = \sum_{k=0}^{n} k(k-1)x^k &= 2x^2 + 3\cdot 2x^3 + 4\cdot 3x^4 + \cdots + n(n-1)x^n \\ &= 2x\left(x + 3x^2 + 2\cdot 3x^3 + \cdots + \frac{n(n-1)}{2}x^{n-1}\right). \end{aligned}$$

We can write the factor in parentheses as

$$\begin{aligned} x + 3x^2 + 6x^3 + \cdots + \frac{n(n-1)}{2}x^{n-1} &= x + x^2 + x^3 \cdots + x^{n-1} \\ &\quad + 2x^2 + 2x^3 + \cdots + 2x^{n-1} \\ &\quad + 3x^3 + \cdots + 3x^{n-1} \\ &\quad \vdots \\ &\quad + (n-1)x^{n-1} \\ = x\left(\frac{1-x^{n-1}}{1-x}\right) + 2x^2\left(\frac{1-x^{n-2}}{1-x}\right) &+ \cdots + (n-1)x^{n-1}\left(\frac{1-x}{1-x}\right) \\ = \frac{[x + 2x^2 + \cdots + (n-1)x^{n-1}] - [x^n + 2x^n + \cdots + (n-1)x^n]}{1-x}&. \end{aligned}$$

By equation (7.5) and the Gauss sum[17], we have that this sum is equal to

$$\frac{\left[\frac{x(1-x^{n-1})}{(1-x)^2} - \frac{(n-1)x^n}{(1-x)}\right] - \left[x^n\left(\frac{n(n-1)}{2}\right)\right]}{1-x}. \tag{7.7}$$

Therefore the value of the series is

$$\sum_{n=0}^{\infty} n(n-1)x^n = \lim_{n\to\infty} 2x\left(\frac{\left[\frac{x(1-x^{n-1})}{(1-x)^2} - \frac{(n-1)x^n}{(1-x)}\right] - \left[x^n\left(\frac{n(n-1)}{2}\right)\right]}{1-x}\right) = \frac{2x^2}{(1-x)^3}.$$

7.3.2 Abel's summation formula

The sums calculated in some of the previous examples in this section can be simplified using what is known as the **Abel's summation formula**.

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two finite sequences of numbers. Then

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1} (a_i - a_{i+1})(b_1 + \cdots + b_i) + a_n(b_1 + b_2 + \cdots + b_n),$$

which can be proved simplifying the right-hand side of the identity.

[17] See Section 2.1.

Example 7.3.5. *Using the Abel's summation formula, find the value of the sum* $\sum_{k=1}^{n} kq^{k-1}$*, for* $q \neq 1$*.*

Using the Abel's summation formula, we obtain that

$$\begin{aligned}
\sum_{k=1}^{n} kq^{k-1} &= \sum_{k=1}^{n-1}(k-(k+1))(1+q+\cdots+q^{k-1}) + n(1+q+\cdots+q^{n-1}) \\
&= -\sum_{k=1}^{n-1} \frac{q^k-1}{q-1} + n\left(\frac{q^n-1}{q-1}\right) = -\frac{1}{q-1}\sum_{k=1}^{n-1} q^k + \frac{n-1}{q-1} + n\left(\frac{q^n-1}{q-1}\right) \\
&= \frac{nq^n}{q-1} - \frac{q^n-1}{(q-1)^2}.
\end{aligned}$$

Example 7.3.6 (Rearrangement inequality)**.** *If* $a_1 \leq a_2 \leq \cdots \leq a_n$ *and* $b_1 \leq b_2 \leq \cdots \leq b_n$ *are two collections of real numbers in increasing order and* $(b'_1, b'_2, \ldots, b'_n)$ *is a permutation of* $(b_1, b_2, \ldots, b_n)$*, then it follows that*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b'_1 + a_2b'_2 + \cdots + a_nb'_n.$$

Apply the Abel's summation formula to the difference of the sums

$$\begin{aligned}
\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_ib'_i &= \sum_{i=1}^{n} a_i(b_i - b'_i) \\
&= \sum_{i=1}^{n-1}(a_i - a_{i+1})\Bigg(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} b'_j\Bigg) + a_n\Bigg(\sum_{j=1}^{n} b_j - \sum_{j=1}^{n} b'_j\Bigg) \\
&= \sum_{i=1}^{n-1}(a_i - a_{i+1})\left(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} b'_j\right) \geq 0,
\end{aligned}$$

since for every $i = 1, \ldots, n-1$, $a_i \leq a_{i+1}$ and $\sum_{j=1}^{i} b_j \leq \sum_{j=1}^{i} b'_j$.

Exercise 7.28. *Find, using Abel's summation formula, the value of the sum*

$$\sum_{k=1}^{n} k^2 q^{k-1}, \text{ with } q \neq 1.$$

Exercise 7.29. *Prove that the following series converges to the given value:*

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}, \quad \text{for } |x| < 1.$$

Exercise 7.30. *Find the sum of the following series:*

(i) $\sum\limits_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$. (ii) $\sum\limits_{n=1}^{\infty} \left(\frac{-1}{3}\right)^n$. (iii) $\sum\limits_{n=1}^{\infty} \frac{2}{3^n}$. (iv) $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$. (v) $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$.

7.4 Convergence of sequences and series ⋆

In this section, we shall present the proofs of the convergence theorems and their properties. However, you can continue reading the book without studying this section.

Remember that a sequence $\{a_n\}$ **converges to or has a limit** a if

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ n \geq N \ \Rightarrow\ |a_n - a| < \epsilon.$$

If a is the **limit** of the sequence, we write $\lim_{n\to\infty} a_n = a$ or briefly $a_n \to a$. If for any $a \in \mathbb{R}$ the sequence $\{a_n\}$ does not converge to a, we will say that the sequence **diverges**.

Theorem 7.4.1. *If a sequence $\{a_n\}$ converges to a_1 and to a_2, then $a_1 = a_2$. That is, the limit is unique.*

Proof. If $a_1 \neq a_2$, take $\epsilon = \frac{1}{2}|a_1 - a_2| > 0$. Since $\{a_n\}$ converges to a_1 and to a_2, there exist N_1 and $N_2 \in \mathbb{N}$ such that

$$|a_n - a_1| < \epsilon, \text{ if } n \geq N_1 \quad \text{and} \quad |a_n - a_2| < \epsilon, \quad \text{if } n \geq N_2.$$

For $n \geq N = \max(N_1, N_2)$, it follows that

$$|a_1 - a_2| \leq |a_1 - a_n| + |a_2 - a_n| < 2\epsilon = |a_1 - a_2|,$$

which is a contradiction. Hence, $a_1 = a_2$. □

Next we present some properties of limits of sequences.

Theorem 7.4.2. *If $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and α is any real number, then:*

(a) $\lim\limits_{n\to\infty} (a_n + b_n) = a + b.$

(b) $\lim\limits_{n\to\infty} \alpha a_n = \alpha\, a.$

(c) $\lim\limits_{n\to\infty} a_n b_n = ab.$

(d) *If $b \neq 0$, then for $b_n \neq 0$ and n large enough it happens that*

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Proof. We will prove only parts (a) and (d); the rest is left as an exercise for the reader.

(a) Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then there exist N_1, $N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2}, \quad \text{for } n \geq N_1 \quad \text{and} \quad |b_n - b| < \frac{\epsilon}{2}, \quad \text{for } n \geq N_2.$$

If $N = \max(N_1, N_2)$, then for $n \geq N$ it follows that

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(d) Since $b \neq 0$, it follows that $\frac{|b|}{2} > 0$. Since $b_n \to b$, there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|b_n - b| < \frac{|b|}{2}$, then $|b| - |b_n| \leq |b_n - b| < \frac{|b|}{2}$ and $|b_n| > \frac{|b|}{2}$. Hence $\frac{1}{|b_n|} < \frac{2}{|b|}$, for all $n \geq N_1$.

Let $\varepsilon > 0$, since $b_n \to b$, considering $\frac{|b|^2\varepsilon}{2}$ there exists $N \geq N_1$ such that, for all $n \geq N$, $|b_n - b| < \frac{|b|^2}{2}\varepsilon$. Hence, for all $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b|\,|b_n|} \leq \frac{2\,|b_n - b|}{|b|^2} < \varepsilon. \qquad \square$$

Observations 7.4.3.

(a) *If $\{a_n\}$ converges to a, then any open interval that contains the number a has an infinite number of terms of the sequence. Moreover, outside this interval there are only a finite number of terms of the sequence.*

(b) *If a_n converges to a, then the sequence is bounded.*

Part (a) follows since there exists $\epsilon > 0$, with $I = (a - \epsilon, a + \epsilon)$, contained in the open interval, but in I there are an infinite number of terms of $\{a_n\}$.

In order to prove (b), observe that the interval $(a - \epsilon, a + \epsilon)$ contains all the terms a_n with $n \geq N$ for some N, then $|a_n| < |a| + \epsilon$. Therefore, if we define $K = \max\{|a| + \epsilon, |a_1|, \ldots, |a_n|\}$, it is clear that K is a bound for the sequence, that is, $|a_n| \leq K$ for all $n \in \mathbb{N}$.

Theorem 7.4.4. *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of real numbers. Suppose that there is $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ with $n \geq N$, $a_n \leq b_n \leq c_n$ holds. If $\{a_n\}$ and $\{c_n\}$ converge to the same limit a, then $\{b_n\}$ converges to a.*

Proof. Let $\epsilon > 0$, by hypothesis there exist N_1, $N_2 \in \mathbb{N}$ such that:

$$|a_n - a| < \epsilon, \quad \text{for } n \geq N_1 \quad \text{and} \quad |c_n - a| < \epsilon, \quad \text{for all } n \geq N_2.$$

If $N_0 = \max(N, N_1, N_2)$, then for all $n \geq N_0$, it follows that

$$-\epsilon < a_n - a \leq b_n - a \leq c_n - a < \epsilon,$$

which implies that $|b_n - a| < \epsilon$, that is, $\{b_n\}$ converges to a. $\square$

Theorem 7.4.5. *A sequence $\{a_n\}$ converges to a if and only if any subsequence of $\{a_n\}$ converges to a.*

Proof. If the sequence converges to a, then $\lim_{k\to\infty} a_k = a$, that is, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $k > N$, then $|a_k - a| < \epsilon$. Let n_k be an increasing sequence of positive integers and consider the subsequence $\{a_{n_k}\}$. Since $n_k \geq k$ and if $k > N$, then $|a_{n_k} - a| < \epsilon$, hence the subsequence converges.

Suppose now that the sequence does not converge to a, then there exists $\epsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $n > N$ such that $|a_n - a| > \epsilon$. For

$N = 1$, there is $n_1 > 1$ such that $|a_{n_1} - a| > \epsilon$. For n_1, there is $n_2 > n_1$ such that $|a_{n_2} - a| > \epsilon$. For n_2, there is $n_3 > n_2$ such that $|a_{n_3} - a| > \epsilon$. Proceeding in the same way, we construct a subsequence $\{a_{n_k}\}$ that does not converge to a, which is a contradiction. □

Theorem 7.4.6. *If $\{a_n\}$ is bounded, then there exists a subsequence of $\{a_n\}$ that converges.*

Proof. In order to prove this theorem, we need to construct a convergent subsequence. Since the sequence is bounded, we know that there is $M > 0$ such that $|a_n| \leq M$, for all n, that is, $-M \leq a_n \leq M$. Divide the closed interval[18] $[-M, M]$ into two intervals $[-M, 0]$ and $[0, M]$. Consider the interval where there are a infinite number of terms of the sequence; suppose that this interval is $[0, M]$. To construct the subsequence choose one of the terms in the interval $[0, M]$, say a_{n_1}. Again, divide the interval $[0, M]$ into two intervals $[0, \frac{M}{2}]$, $[\frac{M}{2}, M]$ and choose the interval that contains an infinite number of terms of the sequence. Without loss of generality, we can assume that the interval is $[\frac{M}{2}, M]$. Choose as a second element of the subsequence one term a_{n_2} such that $n_2 > n_1$ in the interval $[\frac{M}{2}, M]$. Continuing this process, we will get a subsequence $\{a_{n_k}\}$ and a collection of nested closed intervals of lengths $\frac{M}{2^n}$. The infinite intersection of these closed intervals is not empty, in fact, it is a unique point[19], say a. This point a is the limit of the subsequence, since $|a - a_{n_k}| < \frac{M}{2^k}$, moreover $|a - a_{n_l}| < \frac{M}{2^k}$, for $l \geq k$. □

Theorem 7.4.7. *Every upper bounded, increasing sequence of real numbers (monotonically increasing) is convergent. Similarly, every lower bounded decreasing sequence of real numbers (monotonically decreasing) is convergent.*

Proof. Let $\{a_n\}$ be an upper bounded increasing sequence. Since the sequence is bounded, Theorem 7.4.6 implies that there exists a subsequence $\{a_{n_k}\}$ that converges to a point a, that is, given $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that for all $k \geq K$, it follows that $|a_{n_k} - a| < \epsilon$. Let $N = n_K$, we would like to show that for all $n \geq N$ it follows that $|a_n - a| < \epsilon$. Since $n_k \leq n$ there is $j \geq k$ such that $n_K < \cdots < n_j \leq n < n_{j+1}$. Using that the sequence $\{a_n\}$ is increasing, we have that $a_{n_j} - a \leq a_n - a < a_{n_{j+1}} - a$, so that $|a_n - a| < \epsilon$. The other cases are similar. □

Theorem 7.4.8. *Let f be a function, then $\lim_{x\to a} f(x) = b$ if and only if for every sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = a$, it follows that $\lim_{n\to\infty} f(a_n) = b$.*

Proof. Suppose that $\lim_{n\to\infty} a_n = a$. Since $\lim_{x\to a} f(x) = b$, it follows that given $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - b| < \epsilon$. By the convergence of $\{a_n\}$ to a, for $\delta > 0$, there is $N \in \mathbb{N}$, such that if $n \geq N$,

---

[18] See Section 1.2, for the definition of interval.
[19] See [17].

$|a_n - a| < \delta$. Then, since $|a_n - a| < \delta$ if $n \geq N$, hence $|f(a_n) - b| < \epsilon$, that is, $f(a_n) \to b$.

Conversely, suppose that $\lim_{x\to a} f(x)$ is not b, that is, there exists $\epsilon > 0$, such that for every $\delta > 0$, there exists x with $|x - a| < \delta$ and $|f(x) - b| \geq \epsilon$.

In this way, for all $\delta = \frac{1}{n}$ with $n \in \mathbb{N}$, there exists a_n with $|a_n - a| < \frac{1}{n}$ and $|f(a_n) - b| \geq \epsilon$. Hence, $\{a_n\}$ is a sequence that converges to a and it suffices that $\{f(a_n)\}$ does not converge to b, which is a contradiction. $\square$

Theorem 7.4.9. *A function f is continuous at a if and only if for every sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} f(a_n) = f(a)$.*

The proof follows directly from the previous theorem, setting $b = f(a)$.

Theorem 7.4.10. *The set of rational numbers is dense in the set of real numbers.*

Proof. Let (a, b) be an open interval and let $\epsilon = b - a > 0$. As we proved in Example 7.2.15, there exists a positive integer n with

$$0 < \frac{1}{n} < \epsilon. \tag{7.8}$$

For some $m \in \mathbb{Z}$, it follows that $a < \frac{m}{n} < b$, otherwise a and b are between two consecutive numbers of the form $\frac{m}{n}$ and $\frac{m+1}{n}$, that is, $\frac{m}{n} \leq a < b \leq \frac{m+1}{n}$. Hence, $\epsilon = b - a \leq \frac{m+1}{n} - \frac{m}{n} = \frac{1}{n}$, which contradicts inequality (7.8). $\square$

Lemma 7.4.11. *If $D \subset \mathbb{R}$ is dense, then for every $x \in \mathbb{R}$ there exists a sequence $\{a_n\}$ in D with $\lim_{n\to\infty} a_n = x$.*

Proof. Let $x \in \mathbb{R}$, for every $n \in \mathbb{N}$ it follows, since D is dense, that there exists $a_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap D$. It is clear that if $|a_n - x| < \frac{1}{n}$ for all n, then $\lim_{n\to\infty} a_n = x$. $\square$

Theorem 7.4.12. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for all $x \in D$, where D is dense in $\mathbb{R}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. By the previous lemma, for $x \in \mathbb{R}$ there exists a sequence $\{a_n\}$ in D with $\lim_{n\to\infty} a_n = x$. Since f is continuous in x, by Theorem 7.4.8, it follows that $f(x) = \lim_{n\to\infty} f(a_n) = 0$. $\square$

Corollary 7.4.13. *If the functions f and g are continuous and coincide in a dense set, then they coincide in all points.*

The proof of the corollary follows from the last theorem, using $f - g$.

Chapter 8

Polynomials

8.1 Polynomials in one variable

A **polynomial** $P(x)$ in one variable x is an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where a_0, a_1, ..., a_n are constants and $n \in \mathbb{N}$. Every term of the polynomial is called a **monomial** or simply a **term**. The constants a_i are known as the **coefficients** of the polynomial. We will denote by $A[x]$ the set of polynomials with coefficients in A and variable x. Usually, the set A is $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. In this chapter we will study polynomials with complex coefficients, unless otherwise stated.

If $a_n \neq 0$, we say that the polynomial has **degree** n. In this sense, $a_n x^n$ is the most important term of the polynomial, because it defines the degree and it is called the **main term** . The number a_0 is the **constant term**. We write $\deg(P)$ to denote the degree of $P(x)$. A polynomial is constant if it has a unique term a_0. If the constant a_0 is different from 0 we say that the polynomial has degree zero. If $a_n = 1$, we say that the polynomial is **monic**.

There are some special names for polynomials whose degree is small. A polynomial is linear if it has degree 1. We have already studied the quadratic and cubic polynomials, which have degrees 2 and 3, respectively. If the polynomial has degree 4, it is called quartic.

In the same way we did for quadratic and cubic polynomials, we say that two polynomials are **equal** if its coefficients are equal term by term, that is, if the coefficients of the monomials of the same degree are equal.

A **zero** of the polynomial $P(x)$ is a number r such that $P(r) = 0$. When $P(r) = 0$, we also say that r is a **root** or a **solution** of the equation $P(x) = 0$.

As we did with the quadratic and cubic polynomials, we can add, subtract, multiply and divide polynomials.

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_8

Let

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$
$$Q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m,$$

be any two polynomials with $n \geq m$.
We define the sum as $(P+Q)(x) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\cdots+(a_m+b_m)x^m + a_{m+1}x^{m+1} + \cdots + a_nx^n$. The difference as $(P-Q)(x) = (a_0-b_0)+(a_1-b_1)x+(a_2-b_2)x^2+\cdots+(a_m-b_m)x^m + a_{m+1}x^{m+1}+\cdots+a_nx^n$. The product of a polynomial $P(x)$ and a constant c is $cP(x) = ca_0 + ca_1x + ca_2x^2 + \cdots + ca_nx^n$. The product of the two polynomials is

$$(PQ)(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots$$
$$+(a_0b_r + a_1b_{r-1} + \cdots + a_ib_{r-i} + \cdots + a_rb_0)x^r + \cdots + (a_nb_m)x^{n+m}.$$

Example 8.1.1. *In order to multiply two polynomials*[20] *we can use the previous definition or it is enough to multiply the coefficients. For instance, if we have the polynomials* $x^4 + 3x^3 + x^2 - 2x + 5$ *and* $3x^3 + 2x^2 + 6$*, its product can be obtained as follows:*

			1	3	1	−2	5
				3	2		6
			6	18	6	−12	30
	2	6	2	−4	10		
3	9	3	−6	15			
3	11	9	2	29	16	−12	30

The product polynomial is $3x^7 + 11x^6 + 9x^5 + 2x^4 + 29x^3 + 16x^2 - 12x + 30$.

Exercise 8.1. *Multiply the polynomials* $P(x) = 4x^3 + 2x^2 + 7x + 1$ *and* $Q(x) = 2x^2 + x + 8$*. Evaluate the two polynomials and their product at* $x = 2$.

Exercise 8.2. *Let* $P(x) = (1-x+x^2-\cdots+x^{100})(1+x+x^2+\cdots+x^{100})$*. Prove that after simplifying the product, the only terms left are those that have even powers of* x.

8.2 The division algorithm

Let

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \text{ with } a_n \neq 0,$$
$$Q(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0, \text{ with } b_m \neq 0$$

[20] See [4], p. 4

be polynomials of degree n and m, respectively, with $m \leq n$ and complex or real coefficients.

The **division algorithm** says that given polynomials $P(x)$ and $Q(x)$ there exist unique polynomials $H(x)$ and $R(x)$ with real or complex coefficients, according to the case, such that

$$P(x) = Q(x)H(x) + R(x), \quad \deg(R) < \deg(Q) \text{ or } R(x) = 0.$$

In order to show how to find $H(x)$ and $R(x)$, let us see an example. These polynomials are known as the **quotient** and the **remainder**, respectively.

Example 8.2.1. *Let $P(x) = x^5 + x^3 + 2x$ and $Q(x) = x^2 - x + 1$, divide*[21] *$P(x)$ by $Q(x)$ and find $H(x)$, $R(x)$.*

Dividing $P(x)$ by $Q(x)$ we get

$$\begin{array}{r|lllll}
 & \multicolumn{5}{l}{x^3 + x^2 + x} \\
\cline{2-6}
x^2 - x + 1 & x^5 & & +x^3 & & +2x \\
 & -x^5 & +x^4 & -x^3 & & \\
\cline{2-6}
 & & x^4 & & & +2x \\
 & & -x^4 & +x^3 & -x^2 & \\
\cline{3-6}
 & & & x^3 & -x^2 & +2x \\
 & & & -x^3 & +x^2 & -x \\
\cline{4-6}
 & & & & & x
\end{array}$$

In this case, $H(x) = x^3 + x^2 + x$ and $R(x) = x$.

If $R(x) = 0$, we say that $Q(x)$ **divides** $P(x)$ and we write $Q(x)|P(x)$. Note that the variable x must be the same in all polynomials; thus we can omit it sometimes.

The division of a polynomial $P(x)$ of degree n by a polynomial of the form $x - a$, gives

$$P(x) = (x - a)Q(x) + r, \quad \text{with} \quad r \in \mathbb{R} \quad \text{and} \quad \deg(Q) = n - 1.$$

Letting $x = a$, we get $r = P(a)$, and therefore

$$P(x) = (x - a)Q(x) + P(a) \;\text{ or }\; P(x) - P(a) = (x - a)Q(x). \tag{8.1}$$

It follows from equation (8.1) that

$$P(a) = 0 \;\text{ is equivalent to } P(x) = (x - a)Q(x), \tag{8.2}$$

for some polynomial $Q(x)$. Thus, we have proved the following theorem.

Theorem 8.2.2 (Factor theorem)**.** *The number a is a root of a polynomial $P(x)$ if and only if the polynomial $P(x)$ is divisible by $x - a$.*

[21] See [4] p. 58

A polynomial $H(x)$ is the **greatest common divisor** of $P(x)$ and $Q(x)$ if it satisfies:

(a) $H(x)$ divides $P(x)$ and $Q(x)$.

(b) If $K(x)$ is any other polynomial that divides $P(x)$ and $Q(x)$, then $K(x)$ divides $H(x)$.

It can be proved that $H(x)$ is unique, up to multiplication by a constant.

There is a method called **Euclid's algorithm**, that is used to find the greatest common divisor of two polynomials and which follows the same ideas of Euclid's algorithm to find the greatest common divisor of two integers. Let us see an example.

Example 8.2.3. *Find the greatest common divisor of the polynomials $x^4 - 2x^3 - x^2 + x - 1$ and $x^3 + 1$.*

We perform the following divisions of polynomials

$$
\begin{array}{r|l}
 & x - 2 \\
\hline
x^3 + 1 & x^4 - 2x^3 - x^2 + x - 1 \\
 & -x^4 \qquad\qquad\quad - x \\
\cline{2-2}
 & \quad -2x^3 - x^2 \qquad -1 \\
 & \quad +2x^3 \qquad\qquad +2 \\
\cline{2-2}
 & \qquad\quad - x^2 \qquad +1
\end{array}
$$

$$
\begin{array}{r|l}
 & -x \\
\hline
-x^2 + 1 & x^3 \qquad\quad +1 \\
 & -x^3 \quad +x \\
\cline{2-2}
 & \qquad\quad x \ +1
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & -x + 1 \\
\hline
x + 1 & -x^2 \qquad +1 \\
 & x^2 + x \\
\cline{2-2}
 & \qquad x + 1 \\
 & \quad -x - 1 \\
\cline{2-2}
 & \qquad\quad 0
\end{array}
$$

Then, as when dealing with integers, the greatest common divisor is $x + 1$, which is the remainder before reaching 0 as a remainder.

We can express the greatest common divisor above as a combination of the polynomials $x^4 - 2x^3 - x^2 + x - 1$ and $x^3 + 1$, following the inverse steps of the divisions as shown:

$$
\begin{aligned}
x + 1 &= (x^3 + 1) - (-x^2 + 1)(-x) = (x^3 + 1) + x(-x^2 + 1) \\
&= (x^3 + 1) + x\left[(x^4 - 2x^3 - x^2 + x + 1) - (x^3 + 1)(x - 2)\right] \\
&= x(x^4 - 2x^3 - x^2 + x + 1) + (1 - x(x - 2))(x^3 + 1) \\
&= x(x^4 - 2x^3 - x^2 + x + 1) + (-x^2 + 2x + 1)(x^3 + 1).
\end{aligned}
$$

If a_1 and a_2 are two different zeros of $P(x)$, then by the factor theorem, $P(x) = (x - a_1)Q_1(x)$, with $Q_1(x)$ a polynomial. Since $0 = P(a_2) = (a_2 - a_1)Q_1(a_2)$ and $a_1 \neq a_2$, then $Q_1(a_2) = 0$. Again, by the factor theorem $Q_1(x) = (x - a_2)Q_2(x)$, with $Q_2(x)$ a polynomial. Then,

$$P(x) = (x - a_1)(x - a_2)Q_2(x) \quad \text{with} \quad \deg(Q_2) = n - 2.$$

In general, if a_1, a_2, ..., a_m are different zeros of $P(x)$ we can write

$$P(x) = (x - a_1)(x - a_2)\dots(x - a_m)Q(x),$$

for some polynomial $Q(x)$, with $\deg(Q) = \deg(P) - m$.

If a is a zero of $P(x)$, then the factor theorem guarantees that there exists a polynomial $Q_1(x)$ with $P(x) = (x - a)Q_1(x)$. If $Q_1(a) \neq 0$, we say that a is a **zero of order 1**, but if $Q_1(a) = 0$ we say that a is a **zero of order greater than 1**. If there is $m \in \mathbb{N}$ and a polynomial $Q(x)$ such that,

$$P(x) = (x - a)^m Q(x), \text{ with } Q(a) \neq 0, \tag{8.3}$$

then a is a root or a zero of $P(x)$ of **multiplicity** m.

One of the important consequences of the factor theorem is the following result:

Theorem 8.2.4. *If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has $n + 1$ distinct roots, then the polynomial is identically zero.*

Proof. We proceed by induction on n. For $n = 1$, the result is clear, since a polynomial of degree 1, has only one root. Suppose that the result is true for $n - 1$; let us show that it is true for n. Suppose that $r_0, r_1, \dots, r_n$ are roots of the polynomial $P(x)$. By the factor theorem, $P(x) = (x - r_n)Q(x)$, where the polynomial $Q(x) = a_n x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_0$ has n distinct roots $r_0, r_1, \dots, r_{n-1}$. By induction, $Q(x)$ is identically zero, hence $P(x)$ also is identically zero. □

Observation 8.2.5. *The previous theorem guarantees that a polynomial of degree n must have at most n distinct roots.*

A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with $a_n \neq 0$ is called **reciprocal** if $a_i = a_{n-i}$, for all $i = 0, 1, \dots, n$.

Example 8.2.6. *The polynomials $x^n + 1$, $x^5 + 3x^3 + 3x^2 + 1$ and $6x^7 - 2x^6 + 4x^4 + 4x^3 - 2x + 6$ are reciprocal polynomials.*

Theorem 8.2.7. *A reciprocal polynomial $P(x)$ of degree $2n$, can be written as $P(x) = x^n Q(z)$, where $z = x + \frac{1}{x}$ and $Q(z)$ is a polynomial in z of degree n.*

Proof. Let $P(x) = a_0 x^{2n} + a_1 x^{2n-1} + \cdots + a_1 x + a_0$, then

$$P(x) = x^n \left(a_0 x^n + a_1 x^{n-1} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$
$$P(x) = x^n \left(a_0 \left(x^n + \frac{1}{x^n} \right) + a_1 \left(x^{n-1} + \frac{1}{x^{n-1}} \right) + \cdots + a_n \right).$$

Using the recursive formula (3.3),

$$x^{k+1} + \frac{1}{x^{k+1}} = \left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right) - \left(x^{k-1} + \frac{1}{x^{k-1}}\right),$$

it is clear that we can express each term $x^k + \frac{1}{x^k}$ as a polynomial in $z = x + \frac{1}{x}$. □

Exercise 8.3. *Divide $P(x) = x^8 - 5x^3 + 1$ by $Q(x) = x^3 + x^2 + 1$. Using the division algorithm, find the polynomials $H(x)$ and $R(x)$.*

Exercise 8.4. *Prove that, for $n \geq 1$, $(x-1)^2$ divides $nx^{n+1} - (n+1)x^n + 1$.*

Exercise 8.5. *Let n be a positive integer. Find the roots of the polynomial*

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \cdots + \frac{x(x+1)\ldots(x+n-1)}{n!}.$$

Exercise 8.6. *Determine the polynomials with real coefficients $P(x)$ that satisfy $P(0) = 0$ and $P(x^2 + x + 1) = P^2(x) + P(x) + 1$ for all $x \in \mathbb{R}$.*

Exercise 8.7. *Prove that any polynomial $P(x)$ of degree n, with $a_0 \neq 0$, is reciprocal if and only if*

$$x^n P\left(\frac{1}{x}\right) = P(x), \text{ for every } x \neq 0.$$

Exercise 8.8. *Prove that every reciprocal polynomial $P(x)$ of odd degree is divisible by $x + 1$ and its quotient is a reciprocal polynomial of even degree.*

Exercise 8.9. *If a is a root of a reciprocal polynomial $P(x)$, prove that $\frac{1}{a}$ is also a root of the polynomial.*

Exercise 8.10. *Determine for which integers n, the polynomial $1 + x^2 + x^4 + \cdots + x^{2n-2}$ is divisible by the polynomial $1 + x + x^2 + \cdots + x^{n-1}$.*

Exercise 8.11. *Prove that the greatest common divisor of $x^n - 1$ and $x^m - 1$ is $x^{(n,m)} - 1$, where (n,m) denotes the greatest common divisor of m and n.*

Exercise 8.12 (USA, 1977)**.** *Find all the pairs of positive integers (m,n) such that $1 + x + x^2 + \cdots + x^m$ divides $1 + x^n + x^{2n} + \cdots + x^{mn}$.*

Exercise 8.13 (Canada, 1971)**.** *Let $P(x)$ be a polynomial with integer coefficients. Prove that if $P(0)$ and $P(1)$ are odd, then $P(x) = 0$ has no integer solutions.*

8.3 Roots of a polynomial

8.3.1 Vieta's formulas

Vieta's formulas (4.1) and (4.2) can be generalized for polynomials of higher degree.

If a monic polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has n roots x_1, x_2, $\ldots$, x_n, then

$$\begin{aligned}
x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 &= (x-x_1)(x-x_2)\cdots(x-x_n)\\
= x^n - (x_1 + \cdots + x_n)x^{n-1} &+ (x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n)x^{n-2}\\
&+ \cdots + (-1)^n x_1\cdots x_n,
\end{aligned}$$

hence,

$$\begin{aligned}
a_{n-1} &= -(x_1 + \cdots + x_n)\\
a_{n-2} &= (x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n)\\
&\vdots\\
a_{n-j} &= (-1)^j \sum_{1\le i_1<\cdots<i_j\le n} x_{i_1}x_{i_2}\cdots x_{i_j}\\
&\vdots\\
a_0 &= (-1)^n x_1x_2\ldots x_n.
\end{aligned} \tag{8.4}$$

The formulas (8.4) are known as Vieta's formulas.

Example 8.3.1. *Consider the polynomial $P(x) = x^n - (x-1)^n$, where n is an odd positive integer. Calculate the sum and the product of its roots.*

The polynomial $P(x)$ can be written as

$$\begin{aligned}
P(x) &= x^n - \left(x^n - nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} - \cdots + (-1)^n\right)\\
&= x^n - \left(x^n - nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} - \cdots - 1\right)\\
&= nx^{n-1} - \frac{n(n-1)}{2}x^{n-2} + \cdots + 1.
\end{aligned}$$

Then, the sum of its roots is $\frac{n(n-1)}{2n} = \frac{n-1}{2}$ and the product of its roots is $\frac{1}{n}$.

8.3.2 Polynomials with integer coefficients

Consider the polynomial $P(x) = a_nx^n + \cdots + a_1x + a_0$ with integer coefficients. The difference $P(x) - P(y)$, can be written as

$$a_n(x^n - y^n) + \cdots + a_2(x^2 - y^2) + a_1(x - y),$$

where every term of the sum is a multiple of $x-y$. This leads us to the following arithmetic property of the polynomials in $\mathbb{Z}[x]$.

Theorem 8.3.2. *If $P(x)$ is a polynomial with integer coefficients, then $P(a)-P(b)$ is divisible by $a-b$, for any pair of different integers a and b.*

In particular, all integer roots of $P(x)$ divide $P(0)$.

There is a similar statement for the rational roots of polynomials $P(x)$ with integer coefficients.

Theorem 8.3.3 (Theorem of the rational root)**.** *Any rational root $\frac{p}{q}$, with $(p,q)=1$, of a polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ with integer coefficients, satisfies that p divides a_0 and q divides a_n.*

Proof. Let $\frac{p}{q}$ be a root of $P(x)$, then

$$q^nP\left(\frac{p}{q}\right)=a_np^n+a_{n-1}p^{n-1}q+\cdots+a_0q^n=0.$$

All the terms of the sum, except possibly the first, are multiples of q, and all the terms of the sum, except possibly the last, are multiples of p. Since p and q divide 0, it follows that $q|a_np^n$ and $p|a_0q^n$, and then the assertion follows, since $(p,q)=1$. $\square$

Corollary 8.3.4. *If $P(x)$ is a polynomial with integer coefficients that takes values $\{\pm1\}$ in three different integers, then $P(x)$ has no integer roots.*

Proof. Suppose that there are integers a, b, c and d such that $P(a)$, $P(b)$, $P(c)\in\{-1,1\}$ and $P(d)=0$. Then, since the integers a, b and c are different, $a-d$, $b-d$ and $c-d$ are also different and, by Theorem 8.3.2, all divide 1, which is impossible. $\square$

8.3.3 Irreducible polynomials

A polynomial $P(x)$ with integer coefficients is called **irreducible** in $\mathbb{Z}[x]$, if it cannot be written as a product of two non-constant polynomials with coefficients in $\mathbb{Z}$.

Example 8.3.5. *Any quadratic polynomial with at least one non-rational root is irreducible in $\mathbb{Z}[x]$. For instance, x^2-x-1 is irreducible in $\mathbb{Z}[x]$, since it has roots given by $\frac{1\pm\sqrt{5}}{2}$.*

Similarly, we define irreducibility over the set of polynomials with coefficients in $\mathbb{Q}$, $\mathbb{R}$. The next theorem claims that for polynomials with integer coefficients, the fact that the polynomial could be factored in $\mathbb{Q}[x]$ is equivalent to the fact that the polynomial could be factored in $\mathbb{Z}[x]$. Moreover, a polynomial with real coefficients always can be expressed as a product of linear polynomials and irreducible quadratic polynomials in $\mathbb{R}[x]$. In the case of a polynomial with complex coefficients, it can always be factored into linear factors over $\mathbb{C}[x]$.

Lemma 8.3.6 (Gauss' lemma). *If $P(x)$ has integer coefficients and $P(x)$ can be factored over the rational numbers, then $P(x)$ can be factored over the integers as well.*

Proof. Suppose that $P(x) = a_n x^n + \cdots + a_0$ has integer coefficients and that $P(x) = Q(x)R(x)$, where $Q(x)$ and $R(x)$ are non-constant polynomials with rational coefficients. Let q and r be the smallest natural numbers such that $qQ(x)$ and $rR(x)$ have integer coefficients. Then, if $d = qr$ it follows that $P_1(x) = dP(x) = qQ(x) \cdot rR(x) = Q_1(x)R_1(x)$ is a factorization of the polynomial $P_1(x)$ into two polynomials with integer coefficients $Q_1(x) = q_k x^k + \cdots + q_0$ and $R_1(x) = r_m x^m + \cdots + r_0$. Let a'_j, with $0 \le j \le n$, be the coefficients of $P_1(x)$. Based on this, we will construct the required factorization of $P(x)$.

Let p be a prime divisor of d. Then all coefficients of $P_1(x)$ are divisible by p. Now let us show that p divides all coefficients of $Q_1(x)$ or divides all coefficients of $R_1(x)$.

If p divides all coefficients of $Q_1(x)$, we are done. Otherwise, let i be such that $p | q_0, q_1, \ldots, q_{i-1}$, but $p \nmid q_i$. We have that $p | a'_i$ and $a'_i = q_0 r_i + \cdots + q_i r_0 \equiv q_i r_0 \mod p$, which implies that $p | r_0$. Moreover, $p | a'_{i+1}$ and $a'_{i+1} = q_0 r_{i+1} + \cdots + q_i r_1 + q_{i+1} r_0 \equiv q_i r_1 \mod p$, and then $p | r_1$. Proceeding in the same way, we can deduce that $p | r_j$, for all j. Then $\frac{R_1(x)}{p}$ has integer coefficients. Then we have a factorization of $\frac{d}{p} P(x)$ into two polynomials with integer coefficients. Taking all the prime divisors of d, we will eventually finish with a factorization of the polynomial $P(x)$ into two polynomials with integer coefficients. $\square$

Example 8.3.7. *If $a_1, a_2, \ldots, a_n$ are different integers, then the polynomial $P(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$ is irreducible over $\mathbb{Z}[x]$.*

Suppose that $P(x) = Q(x)R(x)$, for some non-constant polynomials $R(x)$ and $Q(x)$ with integer coefficients. Since $Q(a_i)R(a_i) = -1$ for $i = 1, \ldots, n$, then $Q(a_i) = 1$ and $R(a_i) = -1$ or $Q(a_i) = -1$ and $R(a_i) = 1$; in both cases, we have that $Q(a_i) + R(a_i) = 0$. Also, the polynomial $Q(x) + R(x)$ is not zero (because otherwise $P(x) = -Q(x)^2 \le 0$ for every real number x, but if x is very large, $P(x)$ is positive, a contradiction). Moreover, $Q(x) + R(x)$ has n zeros $a_1, \ldots, a_n$, which is impossible given that its degree is less than n.

A polynomial with integer coefficients is **primitive**, if its principal coefficient is positive and there is no integer number that divides all coefficients of the polynomial.

Theorem 8.3.8 (Eisenstein's irreducibility criterion). *Consider a polynomial*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

with integer coefficients. Let p be a prime number such that

(a) *p does not divide a_n,*
(b) *p divides every coefficient $a_0, a_1, \ldots, a_{n-1}$,*
(c) *p^2 does not divide a_0.*

Then $P(x)$ is irreducible over $\mathbb{Q}[x]$. Moreover, if $P(x)$ is primitive, then it is irreducible over $\mathbb{Z}[x]$.

Proof. Suppose that $P(x)$ is reducible over $\mathbb{Q}[x]$. By Gauss' lemma, $P(x) = Q(x)R(x)$, where $Q(x) = q_k x^k + \cdots + q_0$ and $R(x) = r_m x^m + \cdots + r_0$ are polynomials with integer coefficients. Since $a_0 = q_0 r_0$ is divisible by p but not by p^2, exactly one of q_0 or r_0 is a multiple of p. Suppose that $p|q_0$ and $p \nmid r_0$. Moreover, since $p|a_1 = q_0 r_1 + q_1 r_0$ it follows that $p|q_1 r_0$; then, $p|q_1$ and so on. We conclude that all coefficients $q_0, q_1, \ldots, q_k$ are divisible by p, but then $p|a_n$ since $a_n = q_k r_m$, which is a contradiction. $\square$

One of the most important applications of the Eisenstein criterion, is to show the irreducibility of the **cyclotomic polynomials**, $x^{p-1} + x^{p-2} + \cdots + x + 1$, with p a prime number. Note that the roots of this polynomial are the pth roots of unity, that is, the powers of $e^{\frac{2\pi i}{p}}$.

Example 8.3.9. *The polynomial $P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$, with p a prime number, is irreducible over $\mathbb{Q}[x]$.*

Note that $(x-1)P(x) = x^p - 1$. With the substitution $x = y+1$ in the last product we get

$$yP(y+1) = (y+1)^p - 1 = y^p + \binom{p}{1}y^{p-1} + \binom{p}{2}y^{p-2} + \cdots + \binom{p}{p-1}y.$$

Since $\binom{p}{i} = \frac{p(p-1)\cdots(p-i+1)}{i!}$, if $i < p$ then, since the prime p is not a factor of $i!$, $i!$ divides the product $(p-1)\cdots(p-i+1)$. This implies that $\binom{p}{i}$ is divisible by p. Dividing $yP(y+1)$ by y, it follows that $P(y+1)$ satisfies the conditions of the Eisenstein criterion and therefore it is an irreducible polynomial, hence $P(x)$ is also irreducible.

Let us see some applications in number theory of the previous example. Let p be an odd prime number and consider the polynomial $P(x) = x^{p-1} - 1$ with coefficients[22] in $\mathbb{Z}_p$. By Fermat's little theorem[23], each of the numbers 1, 2, ..., $p-1$ is a root of the polynomial, then

$$x^{p-1} - 1 = (x-1)(x-2)\cdots(x-p+1). \tag{8.5}$$

(a) Comparing the constant coefficients in the last identity we get Wilson's theorem:

$$(p-1)! \equiv -1 (\text{mod } p).$$

[22] $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ with the sum and product operations module p.
[23] See [8].

(b) If we expand the right-hand side of (8.5), the coefficient σ_j of x^{p-1-j} is the sum of all products of j elements of the set $\{1, 2, \dots, p-1\}$. Comparing coefficients, we get that p divides σ_j for $j = 1, 2, \dots, p-2$. Now assume that $p \geq 3$.

(c) (Wolstenholme) The numerator of the (reduced) fraction

$$\frac{m}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{p-1}$$

is divisible by p. In fact, $\frac{m}{n} = \frac{\sigma_{p-2}}{(p-1)!}$ and, since $p|\sigma_{p-2}$ and p is relatively prime to $(p-1)!$, it follows that $p|m$.

(d) Let m be as in the previous part. If $p \geq 5$, it follows that $p^2|m$.
Since $(x-1)\dots(x-p+1) = x^{p-1} - \sigma_1 x^{p-2} + \sigma_2 x^{p-3} + \cdots + \sigma_{p-1}$, it follows, after evaluating in $x = p$, that

$$(p-1)! = p^{p-1} - \sigma_1 p^{p-2} + \sigma_2 p^{p-3} + \cdots - \sigma_{p-2} p + \sigma_{p-1}.$$

Since $\sigma_{p-1} = (p-1)!$, we can reduce the last identity to

$$\sigma_{p-2} = p^{p-2} - \sigma_1 p^{p-3} + \cdots + \sigma_{p-3} p.$$

This shows that σ_{p-2} is divisible by p^2, since every σ_j is divisible by p and since p^2 and $(p-1)!$ are relatively prime, it follows that $p^2|m$.

Exercise 8.14. *Find the solutions of the system*

$$\begin{aligned} x + y + z &= w \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w}. \end{aligned}$$

Exercise 8.15. *Prove that the polynomial $x^4 - x^3 - 3x^2 + 5x + 1$ is irreducible over $\mathbb{Q}[x]$.*

Exercise 8.16. *Let $P(x)$ be a polynomial with integer coefficients. Prove that if $P^k(n) = n$, for some integer number $k \geq 1$, and for some integer number n, then for such integer n it follows that $P(P(n)) = n$.*

Exercise 8.17. *Find all polynomials of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with $a_j \in \{-1, 1\}$, such that all its roots are real numbers.*

Exercise 8.18 (USA, 1974)**.** *Let a, b, c be different integers and let $P(x)$ be a polynomial with integer coefficients. Prove that it is impossible that $P(a) = b$, $P(b) = c$ and $P(c) = a$.*

Exercise 8.19. *Prove that the polynomial with real coefficients $P(x) = x^n + 2nx^{n-1} + 2n^2 x^{n-2} + a_{n-3} x^{n-3} + \cdots + a_1 x + a_0$, cannot have all its roots real.*

8.4 The derivative and multiple roots ⋆

For a polynomial of degree n,

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n = \sum_{k=0}^{n} a_kx^k,$$

we define **the derivative** of $P(x)$ as the polynomial of degree $n-1$ given by $P'(x) = a_1 + 2a_2x + \cdots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1} = \sum_{k=1}^{n} ka_kx^{k-1}$.
It can be shown that if $P(x) = (x-a)Q(x)$, for some polynomial $Q(x)$, then

$$P'(x) = Q(x) + (x-a)Q'(x). \tag{8.6}$$

There is an important relationship between the roots of a polynomial $P(x)$ and the roots of its derivative $P'(x)$, given by the following theorem.

Theorem 8.4.1. *If for some positive integer m the polynomial $P(x)$ is divisible by $(x-a)^{m+1}$, then the polynomial $P'(x)$ is divisible by $(x-a)^m$. That is, if a is a zero of multiplicity $m+1$ of $P(x)$, then a is a zero of multiplicity m for $P'(x)$.*

Proof. For the proof we will use induction over m. For $m = 1$, if $P(x)$ is divisible by $(x-a)^2$, then $P(x) = (x-a)Q(x)$, where $Q(x)$ is a polynomial divisible by $x-a$, and $P'(x) = Q(x) + (x-a)Q'(x)$. Therefore, $P'(x)$ is divisible by $x-a$.
Suppose the result is true for $m-1$. Let $P(x)$ be divisible by $(x-a)^{m+1}$. Then $P(x) = (x-a)Q(x)$, where $Q(x)$ is divisible by $(x-a)^m$, and from $P'(x) = Q(x)+(x-a)Q'(x)$, it follows that $P'(x)$ is divisible by $(x-a)^m$, where it has been used that $Q'(x)$ is divisible by $(x-a)^{m-1}$, this being the induction hypothesis. □

Observe that if a is a zero of $P(x)$ with multiplicity one, then $P'(a) \neq 0$.

Example 8.4.2. *If m and n are integers, such that $0 < m < n$, then*

$$\sum_{k=1}^{n} (-1)^k k^m \binom{n}{k} = 0.$$

The problem is equivalent to showing that the polynomial

$$P_m(x) = \sum_{k=0}^{n} k^m \binom{n}{k} x^k,$$

has $x = -1$ as a root. Let us prove the following stronger result: $P_m(x)$ has a zero of multiplicity at least $n-m$ in $x = -1$. For this we will see that $(x+1)^{n-m}$ divides $P_m(x)$, for $0 \leq m < n$.

We proceed by induction on m. For $m = 0$, it follows that

$$P_0(x) = \sum_{k=0}^{n} k^0 \binom{n}{k} x^k = (1+x)^n,$$

then the statement is true.

Suppose that for m, with $0 < m < n$, $P_{m-1}(x)$ has -1 as a root of multiplicity $n-(m-1)$. By the previous theorem, $P'_{m-1}(x)$ has one root of multiplicity $n-m$ in -1. Then $P'_{m-1}(x) = (x+1)^{n-m}Q(x)$, for some polynomial $Q(x)$. But

$$P'_{m-1}(x) = \left(\sum_{k=0}^{n} k^{m-1}\binom{n}{k}x^k\right)' = \sum_{k=0}^{n} k^m \binom{n}{k} x^{k-1} = \frac{1}{x}P_m(x).$$

Hence, $P_m(x) = xP'_{m-1}(x) = x(x+1)^{n-m}Q(x)$, therefore $(x+1)^{n-m}$ divides $P_m(x)$.

Proposition 8.4.3. *If $P(x)$ is a polynomial such that $P(a) = P'(a) = \cdots = P^{(m-1)}(a) = 0$ and $P^{(m)}(a) \neq 0$, then a is a zero of $P(x)$ of multiplicity m. That is, $P(x) = (x-a)^m Q(x)$ for some polynomial $Q(x)$, with $Q(a) \neq 0$. Here $P^{(j+1)}(x)$ is the derivative of $P^{(j)}(x)$ and $P^{(1)}(x) = P'(x)$.*

Proof. Since $P(a) = 0$, then $P(x) = (x-a)Q_1(x)$ for some polynomial $Q_1(x)$. Since $P'(x) = Q_1(x) + (x-a)Q_1'(x)$, it follows that $P'(a) = Q_1(a) = 0$. Then $Q_1(x) = (x-a)Q_2(x)$ for some polynomial $Q_2(x)$, hence $P(x) = (x-a)^2Q_2(x)$. Proceeding in the same way we get that $P(x) = (x-a)^m Q_m(x)$, where $Q_m(x)$ is the polynomial $Q(x)$ we are looking for. □

Exercise 8.20. *Determine if the polynomial $P(x) = x^3 - x^2 - 8x + 12$ has multiple roots.*

Exercise 8.21. *Find all the triplets of real numbers (a,b,c) such that the polynomial $P(x) = x^3 + ax^2 + bx + c$ is divisible by $(x+1)^2$.*

8.5 The interpolation formula

Given two points in the Cartesian plane, there is a unique straight line that joins these two points. Then, for two pairs of real numbers $(\alpha_0, \beta_0), (\alpha_1, \beta_1)$, with $\alpha_0 \neq \alpha_1$, there is a unique polynomial $P(x)$ of degree at most 1, such that $P(\alpha_0) = \beta_0$ and $P(\alpha_1) = \beta_1$. This can be generalized as follows.

Theorem 8.5.1 (Lagrange interpolation formula)**.** *Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be different real numbers and let $\beta_0, \beta_1, \ldots, \beta_n$ be another $n+1$ set of real numbers. Then there exists a unique polynomial $P(x)$ of degree at most n such that $P(\alpha_i) = \beta_i$, for $0 \leq i \leq n$.*

Proof. First, let us find a polynomial that satisfies the conditions. Consider the polynomials

$$D_k(x) = \frac{(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{k-1})(x-\alpha_{k+1})\cdots(x-\alpha_n)}{(\alpha_k-\alpha_0)(\alpha_k-\alpha_1)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n)},$$

with $0 \leq k \leq n$, where the numerator and the denominator have n factors. It is clear that $D_k(x)$ has degree n, that $D_k(\alpha_k) = 1$ and that $D_k(\alpha_i) = 0$, if $i \neq k$. Then, the polynomial that satisfies the conditions is

$$P(x) = \sum_{k=0}^{n} \beta_k D_k(x).$$

To show the uniqueness of the polynomial, suppose that there are two polynomials $P_1(x)$ and $P_2(x)$ of degree at most n, such that

$$P_j(\alpha_i) = \beta_i, \text{ for } 0 \leq i \leq n, j = 1, 2.$$

Then the polynomial $P(x) = P_1(x) - P_2(x)$ has degree at most n and has $n+1$ distinct roots $\alpha_0, \alpha_1, \ldots, \alpha_n$. By Theorem 8.2.4, it follows that $P(x)$ is the zero polynomial, then $P_1(x) = P_2(x)$. □

Example 8.5.2. *For $1 < m < n$, the following identity holds:*

$$\sum_{k=1}^{n} (-1)^k k^m \binom{n}{k} = 0.$$

Construct a polynomial of degree at most n which takes the values $\beta_k = k^m$ in the points $\alpha_k = k$, with $k = 0, 1, 2, \ldots, n$, respectively. By the Lagrange interpolation formula it follows that the polynomial we are looking for is given by

$$P(x) = \sum_{k=0}^{n} k^m D_k(x),$$

where

$$D_k(x) = \frac{x(x-1)(x-2)\cdots(x-k+1)(x-k-1)\cdots(x-n)}{k(k-1)\cdots 1(-1)(-2)\cdots(k-n)}.$$

The coefficient of x^n, in the polynomial $D_k(x)$, is equal to

$$\frac{1}{k(k-1)\cdots 1(-1)(-2)\cdots(k-n)} = (-1)^{k-n}\frac{1}{k!(n-k)!} = \frac{(-1)^n}{n!}(-1)^k\binom{n}{k}$$

and therefore the coefficient of x^n, in the polynomial $P(x)$, turns out to be

$$\frac{(-1)^n}{n!}\sum_{k=0}^{n}(-1)^k k^m \binom{n}{k}.$$

On the other hand, the polynomial $Q(x) = x^m$ satisfies that $Q(k) = k^m$ for every $0 \leq k \leq n$. Then, by Theorem 8.5.1, $P(x)$ and $Q(x)$ are equal. Since $m < n$, the coefficient of x^n in the polynomial $P(x)$ is equal to 0.

Example 8.5.3. *Find a polynomial $P(x)$ such that $xP(x-1) = (x+1)P(x)$, for all $x \in \mathbb{R}$.*

For $x = 0$, the condition is equivalent to $P(0) = 0$. If $P(n) = 0$, then $P(n+1) = 0$ since $(n+1)P(n) = (n+2)P(n+1)$. Therefore $P(x)$ has an infinite number of zeros. Then, $P(x) \equiv 0$ is the only polynomial that satisfies the equation.

Exercise 8.22. *Let $P(x)$ be a polynomial of degree n such that $P(k) = 2^k$, for $k = 0, 1, \ldots, n$. Find $P(n+1)$.*

8.6 Other tools to find roots

8.6.1 Parameters

In order to study the roots of a polynomial of degree greater than two, sometimes it is useful to consider the independent terms as variables. These independent terms are called **parameters**. To illustrate this let us see the next example.

Example 8.6.1. *Find the solutions of the equation*

$$x^3(x+1) = 2(x+a)(x+2a),$$

where a is a real parameter.

Solving the equation is equivalent to solving the quartic equation

$$x^4 + x^3 - 2x^2 - 6ax - 4a^2 = 0.$$

This equation is difficult to solve. In some cases it is possible to factorize without any trouble the left-hand side, but in many cases it is not easy. However we can use some algebraic tricks; for instance, we can consider the number a as the variable and x as the parameter or constant. Then, we get the quadratic equation in a,

$$4a^2 + 6ax - x^4 - x^3 + 2x^2 = 0.$$

Using the formula to solve second-degree equations, the discriminant is given by

$$36x^2 + 16(x^4 + x^3 - 2x^2) = 4x^2(2x+1)^2,$$

which is a square number. Solving the equation for a, we obtain the two roots of the equation: $a_1 = -\frac{1}{2}x^2 - x$ and $a_2 = \frac{1}{2}x^2 - \frac{1}{2}x$. Then, we can factorize the equation as

$$x^4 + x^3 - 2x^2 - 6ax - 4a^2 = -4\left(a + \frac{1}{2}x^2 + x\right)\left(a - \frac{1}{2}x^2 + \frac{1}{2}x\right)$$
$$= (x^2 + 2x + 2a)(x^2 - x - 2a).$$

Finally, solving these second-degree equations we obtain the solutions $x_{1,2} = -1 \pm \sqrt{1-2a}$, $x_{3,4} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+8a}$, which are real if $a \in \left[-\frac{1}{8}, \frac{1}{2}\right]$.

Exercise 8.23.

(i) *Solve the equation* $\sqrt{5-x} = 5 - x^2$.
(ii) *Solve the equation* $\sqrt{a-x} = a - x^2$, *with* $a > 0$.

Exercise 8.24. *Solve the equation* $x = \sqrt{a - \sqrt{a+x}}$, *where* $a > 0$ *is a parameter.*

8.6.2 Conjugate

The idea we are about to consider is very simple: when in some algebraic expression there is a square root in the denominator of a fraction, sometimes it is useful to multiply the expression by some factor that cancels out the square root. Let us show this procedure with some simple examples.

Example 8.6.2. *If we want to eliminate the square root in the denominator of the expression*

$$\frac{1}{a+\sqrt{b}},$$

we should multiply it by $\frac{a-\sqrt{b}}{a-\sqrt{b}}$ *in order to obtain* $\frac{1}{a+\sqrt{b}} = \frac{a-\sqrt{b}}{a^2-b}$.

The number $a - \sqrt{b}$ is known as the conjugate of $a + \sqrt{b}$.

Example 8.6.3. *Solve the equation*

$$\sqrt{1+mx} = x + \sqrt{1-mx},$$

where m *is a real parameter.*

The equation is equivalent to $\sqrt{1+mx} - \sqrt{1-mx} = x$.

Multiplying and dividing by $\sqrt{1+mx} + \sqrt{1-mx}$, which is the conjugate of $\sqrt{1+mx} - \sqrt{1-mx}$, it follows that

$$\frac{2mx}{\sqrt{1+mx} + \sqrt{1-mx}} = x.$$

A solution is $x = 0$ and, if $x \neq 0$, then $2m = \sqrt{1+mx} + \sqrt{1-mx}$, hence m is positive. Squaring and simplifying, it follows that

$$2m^2 - 1 = \sqrt{1 - m^2x^2},$$

hence $2m^2 - 1 \geq 0$. Squaring again and solving for x we get

$$x = \pm 2\sqrt{1 - m^2}$$

and, since $1 - m^2 \geq 0$, it follows that $m \in \left[\frac{1}{\sqrt{2}}, 1\right]$.

Exercise 8.25. *Solve the equation*

$$\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}} = m\sqrt{\frac{x}{x+\sqrt{x}}},$$

where m is a real parameter.

Exercise 8.26 (Short list OIM, 2009)**.** *Find all triplets (x, y, z) of positive real numbers that satisfy the system of equations:*

$$\begin{aligned} x + \sqrt{y+11} &= \sqrt{y+76} \\ y + \sqrt{z+11} &= \sqrt{z+76} \\ z + \sqrt{x+11} &= \sqrt{x+76}. \end{aligned}$$

Exercise 8.27. *Let a, b, $c > 0$, solve the system:*

$$\begin{aligned} ax - by + \frac{1}{xy} &= c \\ bz - cx + \frac{1}{zx} &= a \\ cy - az + \frac{1}{yz} &= b. \end{aligned}$$

8.6.3 Descartes' rule of signs $\star$

The estimation of the number of positive roots of a polynomial $P(x)$, with real coefficients, can be achieved counting the number of changes of sign $C(P)$, in the sequence of non-zero coefficients of $P(x)$.

Example 8.6.4. *The polynomial*

$$P(x) = 3x^{12} + 4x^{10} - 2x^9 - 4x^8 - x^6 + 3x^5 - 2x^4 - 6x^2 + 11x,$$

has 4 changes of sign, that is, $C(P) = 4$.

Note that zero coefficients are neglected.

Theorem 8.6.5 (Descartes' rule of signs). *The number of positive real roots of a polynomial $P(x)$, with real coefficients, is less than or equal to the number of changes of sign, $C(P)$, that are produced among its coefficients (neglecting the zero coefficients and counting multiplicities of the roots). Similarly, the number of negative real roots of the polynomial is less than or equal to the changes of sign that appear among the coefficients of $P(-x)$.*
Moreover, if the number of positive roots is less than the number of changes of sign, then the number of positive roots differs from the number of changes of sign by an even number.

Proof. Suppose that the polynomial $P(x)$ has degree n, is monic and that the constant term is not zero, that is, $P(0) \neq 0$. Otherwise, we can factor one term of the form x^k, which does not contribute to the positive roots. Let us show first that the number of changes of sign and the number of positive roots have the same parity. The proof is by induction on n.
For $n = 1$, the polynomial has degree 1 and the result is clear, since $P(x) = x - a$, with $a > 0$, has one change of sign and the only positive root is $x = a$. If $P(x) = x + a$, there is no change of sign and the only solution is $x = -a$, which is negative.
Now suppose that $P(x)$ is a monic polynomial of degree $n > 1$, with $P(0) \neq 0$. There are two cases:

Case 1. If $P(0) < 0$, then the number of changes of sign must be odd since it starts with a positive number because the polynomial is monic, and it finishes with a negative number $P(0)$. Let us see that the number of positive roots of the polynomial is also odd.
Since the degree of $P(x)$ is n, the term x^n dominates for large values of x. Then for some large and positive value of x, say x_0, it follows that $P(x_0)$ is positive, then $P(x)$ must have a root[24] in the interval $(0, x_0)$, which is clearly positive.
Let k be such a root. Then $P(x) = (x-k)Q(x)$, with $Q(x)$ a polynomial of degree $n-1$ and $Q(0) = \frac{P(0)}{-k}$ positive. Hence, applying the induction hypothesis to $Q(x)$, we obtain that this polynomial has an even number of positive roots, therefore $P(x)$ has an odd number of positive zeros, the zeros of $Q(x)$ and k.

Case 2. If $P(0) > 0$ and the equation has no positive solutions, we are done, since zero is an even number. When the equation has some positive solution, we consider one of them, say k. As before, we have that $P(x) = (x - k)Q(x)$, with $Q(x)$ a polynomial of degree $n - 1$ and $Q(0) = \frac{P(0)}{-k}$ negative. Then $Q(x)$ has an odd number of changes of sign. Applying the induction hypothesis to $Q(x)$, we obtain that $Q(x)$ has an odd number of positive roots. Therefore, $P(x)$ has an even number of positive roots.

Until now we have shown that the number of changes of sign and the number of positive roots of a polynomial have the same parity. It is only left to prove that the number of sign changes is greater than or equal to the number of positive roots,

[24]This claim is guaranteed by the Intermediate Value Theorem, see [21].

that is, the number of sign changes is an upper bound for the number of positive roots. If there were more positive roots than sign changes in the coefficients of $P(x)$, then there must be at least two more positive roots than the number of sign changes, that is, there must be at least $C(P)+2$ positive roots.

On the other hand, $P'(x)$ has at least one change of sign[25] between every two roots of $P(x)$, hence there would be at least $C(P)+1$ roots of $P'(x)$. But $P'(x)$ has at least as many sign changes as $P(x)$, that is, $C(P)$, and moreover its degree is $n-1$. Under these conditions, the induction hypothesis tells us that such polynomial satisfies the rule of signs, that is, it has more sign changes than positive roots, which is a contradiction. Therefore, there are more changes of sign than positive roots. □

Example 8.6.6 (Poland, 2001). *Let $n \geq 3$ be an integer. Prove that a polynomial of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with $a_{n-1} = a_{n-2} = 0$ and at least one $a_k \neq 0$, cannot have all its roots real.*

Suppose that the polynomial satisfies that $a_0 \neq 0$ in order to assure that if it has a real root, it will be positive or negative. Applying Descartes' rule of signs, it follows that the number of real roots of the polynomial is at most $n-1$, hence at least one of them is a complex root.

If $a_0 = 0$, we can factorize the highest power of x that divides the polynomial and, in this case, work with the quotient polynomial as in the previous case.

8.7 Polynomials that commute

Two monic polynomials $P(x)$ and $Q(x)$, with real coefficients in one variable, commute if, for every real number x, it follows that

$$P(Q(x)) = Q(P(x)).$$

This means that they commute as polynomial functions. In this section we try to characterize all monic polynomials $Q(x)$ that commute with a given polynomial $P(x)$. Let us see a first example.

Example 8.7.1. *Find all monic polynomials of degree 3 that commute with the polynomial $P(x) = x^2 - \alpha$, for some α.*

Let $Q(x) = x^3 + ax^2 + bx + c$. The equality $P(Q(x)) = Q(P(x))$ can be written as

$$(x^3 + ax^2 + bx + c)^2 - \alpha = (x^2 - \alpha)^3 + a(x^2 - \alpha)^2 + b(x^2 - \alpha) + c.$$

Expanding these last expressions, we get that the right-hand side of the equality has only even powers of x, whereas on the left-hand side there is x^5 with coefficient

[25] See Rolle's theorem [21].

$2a$, forcing a to be zero. Hence the coefficient of x^3 in the left-hand side is $2c$, so that c is also zero. Therefore, $Q(x) = x^3 + bx$.

Canceling the parenthesis and equating coefficients of the corresponding powers of x, we get

$$2b = -3\alpha, \quad b^2 = 3\alpha^2 + b, \quad \alpha = \alpha^3 + b\alpha.$$

Letting $\alpha = 2\gamma$, it follows that $b = -3\gamma$. The second equation implies that $9\gamma^2 = 12\gamma^2 - 3\gamma$, that is, $\gamma^2 - \gamma = 0$, then $\gamma = 0$ or $\gamma = 1$. It is easy to verify that each of these values represents a solution of the system.

Summarizing, a polynomial $Q(x)$ of degree 3 that commutes with $P(x) = x^2 - \alpha$ exists only when $\alpha = 0$ or $\alpha = 2$. If $\alpha = 0$, $Q(x) = x^3$ and if $\alpha = 2$, $Q(x) = x^3 - 3x$.

Similarly, it can be shown that the only polynomial of degree 2 that commutes with $P(x) = x^2 - \alpha$ is $P(x)$ itself and that the only polynomial $Q(x)$ of degree 1 that commutes with $P(x)$ is $Q(x) = x$.

An important example of polynomials that commute are the so-called **Tchebyshev polynomials**, defined in the following way.

Let k be a non-negative integer, the Tchebyshev polynomial $T_k(x)$ is defined, for $-1 \leq x \leq 1$, in a recursive way as follows: $T_0(x) = 1$, $T_1(x) = x$ and, for $k \geq 2$,

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x). \tag{8.7}$$

The first Tchebyshev polynomials are $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$. The identity (8.7) and the induction principle guarantee us that T_k is a polynomial of degree k.

Lemma 8.7.2. *The Tchebyshev polynomials satisfy* $T_k(\cos t) = \cos(kt)$.

Proof. We proceed by induction. We have that $T_0(\cos t) = 1 = \cos 0 = \cos(0 \cdot t)$, moreover, $T_1(x) = x$ implies that $T_1(\cos t) = \cos t = \cos(1 \cdot t)$.

Suppose that $T_{k-1}(\cos t) = \cos[(k-1)t]$ and $T_{k-2}(\cos t) = \cos[(k-2)t]$.

Let us show the result for k. Since the equation (8.7) holds, then

$$\begin{aligned} T_k(\cos t) &= 2\cos t\, T_{k-1}(\cos t) - T_{k-2}(\cos t) \\ &= 2\cos t \cos[(k-1)t] - \cos[(k-2)t], \end{aligned} \tag{8.8}$$

where we have used the induction hypothesis.

Now, $\cos(kt) = \cos[(k-1)t + t] = \cos[(k-1)t] \cdot \cos t - \sin[(k-1)t] \cdot \sin t$. On the other hand, since $\sin[(k-1)t] = \sin[(k-2)t + t] = \sin[(k-2)t]\cos t + \sin t \cos[(k-2)t]$, it follows that,

$$\begin{aligned}\sin[(k-1)t]\,\sin t &= \sin[(k-2)t]\cos t\,\sin t + \cos[(k-2)t]\,\sin^2 t\\ &= \sin[(k-2)t]\cos t\,\sin t + \cos[(k-2)t](1-\cos^2 t)\\ &= \sin[(k-2)t]\cos t\,\sin t + \cos[(k-2)t] - \cos[(k-2)t]\cos^2 t\\ &= \cos t\,(\sin[(k-2)t]\,\sin t - \cos[(k-2)t]\cos t) + \cos[(k-2)t]\\ &= \cos t\,(-\cos[(k-1)t]) + \cos[(k-2)t].\end{aligned}$$

Therefore,

$$\begin{aligned}\cos kt &= \cos[(k-1)t]\cos t - \cos[(k-2)t] + \cos[(k-1)t]\cos t\\ &= 2\cos t\cos[(k-1)t] - \cos[(k-2)t],\end{aligned}$$

as desired. □

Also, it follows that the Tchebyshev polynomials commute,

$$T_k(T_m(x)) = T_{km}(x) = T_m(T_k(x)).$$

This follows from the simple fact that $\cos[k(mt)] = \cos[kmt] = \cos[m(kt)]$. However, T_k is not a monic polynomial, its principal coefficient is 2^{k-1}, which is an inconvenience. But this can be easily fixed if we define $P_k(x) = 2T_k(\frac{x}{2})$. This procedure keeps the commuting property.

Example 8.7.3 (IMO, 1976). *Let $P_1(x) = x^2 - 2$ and let $P_j(x) = P_1(P_{j-1}(x))$, for $j = 2, 3, \ldots$. Prove that for every positive integer n, the roots of the equation $P_n(x) = x$ are real and distinct.*

Let us write $x(t) = 2\cos t$. This function sends the interval $[0, \pi]$ to the interval $[-2, 2]$. Now, observe that

$$\begin{aligned}P_1(x) &= P_1(2\cos t) = 4(\cos t)^2 - 2 = 2\cos 2t,\\ P_2(x) &= P_1(P_1(x)) = 4(\cos 2t)^2 - 2 = 2\cos 4t,\\ &\vdots \qquad \vdots\\ P_n(x) &= 2\cos 2^n t.\end{aligned}$$

Equation $P_n(x) = x$ is equivalent to $2\cos 2^n t = 2\cos t$, with solutions $2^n t = \pm t + 2\pi k$, with $k = 0, 1, \ldots$. That is, the following 2^n values of t,

$$t = \frac{2k\pi}{2^n - 1} \quad \text{and} \quad t = \frac{2k\pi}{2^n + 1},$$

give 2^n real distinct values of $x(t) = 2\cos t$ that satisfy $P_n(x) = x$.

Let us see another example.

Example 8.7.4. *There exists an infinite sequence of polynomials $P_1(x)$, $P_2(x)$, …, $P_k(x)$, …, such that any two of them commute, the degree of $P_k(x)$ is k, and $P_1(x) = x$ and $P_2(x) = x^2 - 2$.*

One solution is immediate, considering the Tchebyshev polynomials that, as we already saw, commute. However, we will give a constructive proof. There is only one polynomial $P_k(x)$, of degree k, that commutes with $P_2(x) = x^2 - 2$ (see Exercise 8.28).

Let us write the first terms of the sequence we are looking for:

$$\begin{aligned}
P_1(x) &= x,\\
P_2(x) &= x^2 - 2,\\
P_3(x) &= x^3 - 3x,\\
P_4(x) &= x^4 - 4x^2 + 2,\\
P_5(x) &= x^5 - 5x^3 + 5x,\\
P_6(x) &= x^6 - 6x^4 + 9x^2 - 2.
\end{aligned}$$

Here, $P_4(x) = P_2(P_2(x))$, $P_6(x) = P_2(P_3(x))$. Observing the previous polynomials, we note they satisfy the relation $P_{k+1}(x) = xP_k(x) - P_{k-1}(x)$. This makes it natural to define the polynomials using this last recursion, for $k > 1$, and with $P_1(x) = x$, $P_2(x) = x^2 - 2$. We can prove, using induction, that these polynomials commute with $P_2(x)$ and, by Exercise 8.30, we obtain that any two of them commute.

Exercise 8.28. *Prove that there exists at most one polynomial of a given degree that commutes with a given polynomial of degree 2.*

Exercise 8.29. *Find all polynomials of degrees 4 and 8 that commute with a given polynomial of degree 2.*

Exercise 8.30. *Prove that if the polynomials $Q(x)$ and $R(x)$ both commute with the polynomial $P(x)$ of degree 2, then they commute with each other.*

Exercise 8.31. *Prove that two polynomials $P(x)$ and $Q(x)$, of degree 1, commute if and only if either both are monic or both have a common fixed point.*

Exercise 8.32. *Given a polynomial $R(x)$, define the polynomial $R_a(x) = R(x - a) + a$. Prove that if two polynomials $P(x)$ and $Q(x)$ commute, then $P_a(x)$ and $Q_a(x)$ commute as well.*

8.8 Polynomials of several variables

If x and y are the solutions of the quadratic equation $at^2 + bt + c = 0$, then $\frac{-b}{a} = x+y$ and $\frac{c}{a} = xy$. The expressions $x+y$ and xy are examples of polynomials in two variables x and y. In general, a **polynomial in two variables** x and y, is a sum of terms of the form cx^ky^m, where c is a constant, k and m are non-negative integers and we denote it by $P(x,y)$. The number $k+m$ is called the degree of the term, and the degree of the polynomial $P(x,y)$ is equal to the degree of the term with the largest degree. We can add, subtract and multiply polynomials of several variables in the same way as the polynomials in one variable. To simplify the polynomials, the terms of the same degree are grouped together. We will consider two types of polynomials with two variables: **symmetric polynomials**, that is, the ones that satisfy $P(x,y) = P(y,x)$, and **homogeneous polynomials** where all the terms have the same degree.

Similar definitions can be given for polynomials in three variables x, y and z. A polynomial, in three variables, is any finite sum of terms of the form $cx^ky^nz^m$, where k, n and m are non-negative integers. The degree of the polynomial is given by the maximum sum of the powers $k+m+n$. If all the terms have the same degree, we say that the polynomial is **homogeneous** and if it satisfies that $P(x,y,z) = P(z,x,y) = P(y,z,x) = P(x,z,y) = P(y,x,z) = P(z,y,x)$, then we say that $P(x,y,z)$ is **symmetric**.

Example 8.8.1.

(a) *The elementary symmetric polynomials* $\sigma_1 = x+y$ *and* $\sigma_2 = xy$ *are also homogeneous, the first one of degree* 1 *and the second one of degree* 2*.*

(b) *The polynomial* $x^2 + x + y + y^2$ *is symmetric but it is not homogeneous, meanwhile* $x^2y + 2y^3$ *is homogeneous but not symmetric.*

(c) *The sum of powers* $s_i = x^i + y^i$*, with* $i = 0, 1, 2, \ldots,$ *are symmetric.*

Theorem 8.8.2. *Any symmetric polynomial in* x *and* y *can be represented as a polynomial in* $\sigma_1 = x+y$ *and* $\sigma_2 = xy$*.*

Proof. In fact,

$$s_n = x^n + y^n = (x+y)(x^{n-1} + y^{n-1}) - xy(x^{n-2} + y^{n-2}) = \sigma_1 s_{n-1} - \sigma_2 s_{n-2},$$

where $s_i = x^i + y^i$. Then, we have the recursion

$$s_0 = 2, \quad s_1 = \sigma_1, \quad s_n = \sigma_1 s_{n-1} - \sigma_2 s_{n-2}, \text{ for } n \geq 2.$$

Now, the proof for any symmetric polynomial is simple. The terms of the form ax^ky^k do not cause any problem, since $ax^ky^k = a\sigma_2^k$. If the term bx^iy^k, with $i < k$, appears in the polynomial, then, by symmetry, the term bx^ky^i also has to be part of the polynomial. Grouping terms, it follows that

$$bx^iy^k + bx^ky^i = bx^iy^i(x^{k-i} + y^{k-i}) = b\sigma_2^i s_{k-i}.$$

But s_{k-i} can be expressed as a polynomial in terms of σ_1 and σ_2. □

The non-linear systems of symmetric equations in two variables x and y can be simplified using the substitution $\sigma_1 = x + y$ and $\sigma_2 = xy$. The degree of these equations will be reduced, since $\sigma_2 = xy$ is a second-degree term in x and y. As soon as we find σ_1 and σ_2, we can find the solutions x, y of the system of symmetric equations, either solving the quadratic equation $z^2 - \sigma_1 z + \sigma_2 = 0$ or solving the system $x + y = \sigma_1$, $xy = \sigma_2$.

Example 8.8.3. *Solve the system*

$$x^5 + y^5 = 31, \quad x + y = 1.$$

Take $\sigma_1 = x + y$, $\sigma_2 = xy$. Then

$$x^5 + y^5 = \sigma_1 s_4 - \sigma_2 s_3 = (x + y)(x^4 + y^4) - xy(x^3 + y^3) = 31. \tag{8.9}$$

Since $x^4 + y^4 = \sigma_1 s_3 - \sigma_2 s_2$, making the substitutions recursively, we obtain that equation (8.9) is transformed into $\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 = 31$. Then, the system we need to solve is

$$\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 = 31, \quad \sigma_1 = 1.$$

Substituting the value of σ_1 in the first equation, we get $\sigma_2^2 - \sigma_2 - 6 = 0$. This last equation has as solutions $\sigma_2 = 3$, -2. Hence, we need to solve $x + y = 1$, $xy = 3$, -2. Therefore, the solutions are:

$$\left(\frac{1 + i\sqrt{11}}{2}, \frac{1 - i\sqrt{11}}{2}\right), \left(\frac{1 - i\sqrt{11}}{2}, \frac{1 + i\sqrt{11}}{2}\right), (2, -1), (-1, 2).$$

The symmetric polynomials in three variables can be expressed in terms of the symmetric polynomials

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + yz + zx \text{ and } \sigma_3 = xyz.$$

The sum of the powers $s_i = x^i + y^i + z^i$, with $i = 0, 1, 2, \ldots$, can be expressed with σ_1, σ_2 and σ_3, in the following way:

$$\begin{aligned}
s_0 &= x^0 + y^0 + z^0 = 3,\\
s_1 &= x + y + z = \sigma_1,\\
s_2 &= x^2 + y^2 + z^2 = \sigma_1^2 - 2\sigma_2,\\
s_3 &= x^3 + y^3 + z^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3,
\end{aligned}$$

and, for $n \geq 3$, we have the recursive equation

$$s_n = \sigma_1 s_{n-1} - \sigma_2 s_{n-2} + \sigma_3 s_{n-3}.$$

The non-linear systems of symmetric equations in three variables x, y and z can be simplified using σ_1, σ_2 and σ_3. Once we have the equations in σ_1, σ_2 and

σ_3, we write the cubic equation $u^3 - \sigma_1 u^2 + \sigma_2 u - \sigma_3 = 0$, where σ_1, σ_2 and σ_3 are its coefficients. Then, the solutions (x_1, x_2, x_3) of this cubic equation are the solutions of the system. The other solutions can be obtained by permutation of the variables.

Example 8.8.4. *Solve the system*

$$x + y + z = 4, \quad x^2 + y^2 + z^2 = 14, \quad x^3 + y^3 + z^3 = 34.$$

Note that x, y, z are roots of the polynomial $P(u) = u^3 - (x + y + z)u^2 + (xy + yz + zx)u - xyz$. This becomes

$$P(u) = u^3 - \sigma_1 u^2 + \sigma_2 u - \sigma_3,$$

where $\sigma_1 = x + y + z$, $\sigma_2 = xy + yz + zx$ and $\sigma_3 = xyz$. Then

$$4 = s_1 = \sigma_1, \quad 14 = s_2 = \sigma_1^2 - 2\sigma_2 = 16 - 2\sigma_2, \text{ then } \sigma_2 = 1.$$

The numbers x, y and z are roots of $P(u)$, then

$$\begin{aligned} x^3 - 4x^2 + x - \sigma_3 &= 0, \\ y^3 - 4y^2 + y - \sigma_3 &= 0, \\ z^3 - 4z^2 + z - \sigma_3 &= 0. \end{aligned}$$

Adding the equations, we get that $\sigma_3 = -6$, and so the roots we need to find are the roots of the polynomial $P(u) = u^3 - 4u^2 + u + 6$. Observe that $u_1 = -1$ is a root, then we can factorize $P(u)$ as $P(u) = (u+1)(u^2 - 5u + 6)$. Solving $u^2 - 5u + 6 = 0$, we obtain the other roots, which in this case are $u_2 = 2$ and $u_3 = 3$. Hence, the solution of the system is $(x, y, z) = (-1, 2, 3)$ and all its permutations.

A technique used to generate integer roots for a family of quadratic polynomials or for a polynomial of several variables, is known as **Vieta's jumping**. This tool is used to find the roots of a quadratic polynomial in a recursive way. In general, when this technique is applied to polynomials of several variables only one variable is taken into account while the other variables are considered as constants. Let us see the following example.

Example 8.8.5. *If x, y are positive integers such that $\frac{x^2+y^2+1}{xy}$ is an integer, then such integer is equal to* 3*.*

Suppose that (x_0, y_0) is a pair of positive integers such that

$$\frac{x_0^2 + y_0^2 + 1}{x_0 y_0} = k,$$

with k an integer. Using this solution, we will find another solution (x_1, y_1) of positive integers such that $x_1 + y_1 < x_0 + y_0$. Suppose that $x_0 > y_0$ and let

$P(x) = x^2 - ky_0x + y_0^2 + 1$, then x_0 is a root of $P(x)$ and if x_1 is the other root, by Vieta's formulas, it follows that $x_0 + x_1 = ky_0$ and $x_0x_1 = y_0^2 + 1$. From the first equality, it follows that x_1 is an integer, and from the second equality, x_1 is positive since x_0 and $y_0^2 + 1$ are positive. Then $x_1 = \frac{y_0^2+1}{x_0}$ and, since x_0, y_0 are integers with $x_0 > y_0$, it follows that $x_0 \geq y_0 + 1$, so that $x_0^2 \geq (y_0 + 1)^2 > y_0^2 + 1$, and therefore $x_1 < x_0$. In this way, we have found another solution $(x_1, y_1) = (x_1, y_0)$ such that $x_1 + y_1 < x_0 + y_0$.

This process can be continued and we can find another solution of positive integers (x_1, y_1) such that the sum $x_1 + y_1$ is less than the sum of the elements of the previous solution. Since this process cannot be done infinitely, we must get a solution (x_0, y_0) such that $x_0 = y_0$. Then

$$k = \frac{y_0^2 + y_0^2 + 1}{y_0^2} = 2 + \frac{1}{y_0^2},$$

hence $2 < k \leq 3$, from where we deduce $k = 3$.

Exercise 8.33. *Solve the system*

$$x^5 + y^5 = 33, \qquad x + y = 3.$$

Exercise 8.34. *Find the solutions of the equation* $\sqrt[4]{97 - x} + \sqrt[4]{x} = 5$.

Exercise 8.35. *What is the relation between a, b and c if*

$$x + y = a, \quad x^2 + y^2 = b, \quad x^3 + y^3 = c?$$

Exercise 8.36 (IMO, 1961)**.** *What conditions must satisfy a and b in order that x, y, z are different positive real numbers and such that*

$$x + y + z = a, \quad x^2 + y^2 + z^2 = b^2, \quad xy = z^2?$$

Exercise 8.37. *Solve the system*

$$x + y + z = a, \quad x^2 + y^2 + z^2 = b^2, \quad x^3 + y^3 + z^3 = a^3.$$

Exercise 8.38. *Find the integer solutions of the equation*

$$(x + y^2)(x^2 + y) = (x + y)^3.$$

Exercise 8.39 (China, 2010)**.** *Find all integers k for which there are positive integers a, b, such that*

$$\frac{a+1}{b} + \frac{b+1}{a} = k.$$

Exercise 8.40. *Let x, y, z be non-zero integers, and such that*

$$\frac{x^2 + y^2 + z^2}{xyz}$$

is an integer. Prove that this integer is either 1 *or* 3.

Chapter 9

Problems

Problem 9.1. *Find the irrational numbers a such that $a^2 + 2a$ and $a^3 - 6a$ are rational numbers.*

Problem 9.2 (OMCC, 2010)**.** *Let p, q, r be rational numbers different from zero, such that*

$$\sqrt[3]{pq^2} + \sqrt[3]{qr^2} + \sqrt[3]{rp^2}$$

is a rational number different from zero. Prove that

$$\frac{1}{\sqrt[3]{pq^2}} + \frac{1}{\sqrt[3]{qr^2}} + \frac{1}{\sqrt[3]{rp^2}},$$

is also a rational number.

Problem 9.3 (APMO, 2005)**.** *Prove that for every irrational number a there exist irrational numbers b and b' such that $a + b$ and ab' are rational numbers and such that $a + b'$ and ab are irrational numbers.*

Problem 9.4. *Let $x_1, x_2, \ldots, x_n$ be positive integers less or equal than an integer number m. Prove that if the least common multiple of each pair of integers is greater than m, then the sum of the reciprocals of the n numbers is less than $\frac{3}{2}$.*

Problem 9.5 (APMO, 2013)**.** *Find all positive integers n such that $\frac{n^2+1}{\lfloor\sqrt{n}\rfloor^2+2}$ is an integer number.*

Problem 9.6 (IMO shortlist, 1998)**.** *Determine all the pairs (a, b) of real numbers such that $a\lfloor nb \rfloor = b\lfloor na \rfloor$, for every positive integer n.*

Problem 9.7. *If n and m are positive integers without common factors, prove that*

$$\left\{\frac{n}{m}\right\} + \left\{\frac{2n}{m}\right\} + \left\{\frac{3n}{m}\right\} + \cdots + \left\{\frac{(m-1)n}{m}\right\} = \frac{(m-1)}{2}.$$

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_9

Problem 9.8. *Find the real solutions of the system*

$$\begin{aligned} 9 &= \frac{1}{a} + \frac{1}{b} \\ 18 &= \left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}}\right)\left(1 + \frac{1}{\sqrt[3]{a}}\right)\left(1 + \frac{1}{\sqrt[3]{b}}\right). \end{aligned}$$

Problem 9.9. *The different real numbers a, b, c satisfy the identities $a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}$. Find the values that abc can get.*

Problem 9.10. *The positive real numbers a, b, c satisfy the identity $abc(a+b+c) = 1$. Find the minimum value of $(a+b)(a+c)$.*

Problem 9.11. *The real numbers a, b, c different from zero, satisfy the identity*

$$\left(1 + \frac{a}{bc}\right)\left(1 + \frac{b}{ca}\right)\left(1 + \frac{c}{ab}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right)^2.$$

Find the value of $a + b + c$.

Problem 9.12. *Let a, b, c be real numbers different from zero and such that $a + b + c = 0$. Find the value of*

$$\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right).$$

Problem 9.13. *Let a, b, c be non-zero and distinct real numbers, such that*

$$\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0.$$

Find the value of

$$\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2}.$$

Problem 9.14. *Let a, b, c be integers that satisfy the equality $ab + bc + ca = 1$. Prove that the number $(a^2+1)(b^2+1)(c^2+1)$ is a perfect square.*

Problem 9.15. *Let a, b, c be integers that satisfy the equality $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. Prove that the number $a^2 + b^2 + c^2$ is a perfect square.*

Problem 9.16. *The real numbers a, b, c satisfy the identity $(a+b+c)^2 = a^2+b^2+c^2$. Prove that:*

(i) $\displaystyle \frac{a^2}{a^2+2bc} + \frac{b^2}{b^2+2ca} + \frac{c^2}{c^2+2ab} = 1.$

(ii) $\displaystyle \frac{bc}{a^2+2bc} + \frac{ca}{b^2+2ca} + \frac{ab}{c^2+2ab} = 1.$

Problem 9.17 (Slovenia, 2005). *The real numbers a, b, c satisfy the identity $abc = 1$. Find the value of the expression*

$$\frac{a+1}{ab+a+1} + \frac{b+1}{bc+b+1} + \frac{c+1}{ca+c+1}.$$

Problem 9.18. *Let a, b, c be distinct, non-zero integers, such that $\frac{a-c}{b} + \frac{b+c}{a} = 2$. What is the value of $a - b$?*

Problem 9.19 (Czech-Slovak-Polish, 2005). *Let n be a positive integer. Find the real non-negative numbers $x_1, \ldots, x_n$ that solve the following system of equations:*

$$\begin{aligned} x_1 + x_2^2 + x_3^3 + \cdots + x_n^n &= n \\ x_1 + 2x_2 + 3x_3 + \cdots + nx_n &= \frac{n(n+1)}{2}. \end{aligned}$$

Problem 9.20 (Canada, 1971). *Let x, y be positive real numbers with $x + y = 1$. Prove that $\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9$.*

Problem 9.21 (Romania, 2007). *For real non-negative numbers x, y, z, prove that*

$$\frac{x^3+y^3+z^3}{3} \geq xyz + \frac{3}{4}\left|(x-y)(y-z)(z-x)\right|.$$

Problem 9.22 (Great Britain, 2010). *Let a, b, c be the lengths of the sides of a triangle, that satisfy $ab + bc + ca = 1$. Prove that $(a+1)(b+1)(c+1) < 4$.*

Problem 9.23 (OMCC, 2012). *Let a, b, c be real numbers such that $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} = 1$ and $ab + bc + ca > 0$. Prove that*

$$a + b + c - \frac{abc}{ab+bc+ca} \geq 4.$$

Problem 9.24 (IMO, 1964). *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c) \leq 3abc.$$

Problem 9.25 (IMO, 1975). *Consider two collections of numbers $x_1 \leq x_2 \leq \cdots \leq x_n$, $y_1 \leq y_2 \leq \cdots \leq y_n$ and a permutation $(z_1, z_2, \ldots, z_n)$ of $(y_1, y_2, \ldots, y_n)$. Prove that*

$$(x_1-y_1)^2 + \cdots + (x_n-y_n)^2 \leq (x_1-z_1)^2 + \cdots + (x_n-z_n)^2.$$

Problem 9.26 (IMO, 1978). *Let x_1, x_2, $\ldots$, x_n be distinct positive integers. Prove that*

$$\frac{x_1}{1^2} + \frac{x_2}{2^2} + \cdots + \frac{x_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Problem 9.27 (IMO shortlist, 2010). *Let* $x_1, \ldots, x_{100}$ *be non-negative real numbers such that* $x_i + x_{i+1} + x_{i+2} \leq 1$, *for every* $i = 1, \ldots, 100$ (*where* $x_{101} = x_1$, $x_{102} = x_2$). *Find the maximum possible value of the sum*

$$S = \sum_{i=1}^{100} x_i x_{i+2}.$$

Problem 9.28 (Czech-Slovak-Polish, 2010). *Let* a, b, x, y *be positive real numbers, with* $a \geq b$ *and* $ab \geq ax + by$. *Prove that:*

(i) $x + y \leq a$,
(ii) $\sqrt{a+b} \geq \sqrt{x} + \sqrt{y}$.

Problem 9.29 (Thailand, 2005). *Let* a, b, c *be real numbers, prove that*

$$\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2 \geq 5.$$

Problem 9.30. *Find all triplets of positive real numbers* a, b, c *such that they satisfy the following inequalities*

$$ab + 1 \leq 2c, \qquad bc + 1 \leq 2a, \qquad ca + 1 \leq 2b.$$

Problem 9.31 (Czech-Slovak-Polish, 2010). *Determine all triplets of positive integers* (a, b, c), *that satisfy the identities:*

$$\begin{aligned} a\sqrt{b} - c &= a \\ b\sqrt{c} - a &= b \\ c\sqrt{a} - b &= c. \end{aligned}$$

Problem 9.32 (IMO, 1968). *Let* a, b, c *be real numbers. Prove that the system of equations*

$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ \vdots \qquad & \quad \vdots \\ ax_{n-1}^2 + bx_{n-1} + c &= x_n \\ ax_n^2 + bx_n + c &= x_1, \end{aligned}$$

has a unique real solution if and only if $(b-1)^2 - 4ac = 0$.

Problem 9.33 (IMO shorlist, 1967). *Solve the following system:*

$$\begin{aligned} x^2 + x - 1 &= y \\ y^2 + y - 1 &= z \\ z^2 + z - 1 &= x. \end{aligned}$$

Problem 9.34 (OMM, 2011). *Solve the system,*

$$\begin{aligned} x_1^2 + x_1 - 1 &= x_2 \\ x_2^2 + x_2 - 1 &= x_3 \\ \vdots \qquad & \quad \vdots \\ x_{n-1}^2 + x_{n-1} - 1 &= x_n \\ x_n^2 + x_n - 1 &= x_1. \end{aligned}$$

Problem 9.35. *For $n \geq 3$, find all the positive solutions of the system*

$$\begin{aligned} x_1^2 &= x_2 + x_3 \\ x_2^2 &= x_3 + x_4 \\ \vdots \ &= \ \ \vdots \\ x_{n-1}^2 &= x_n + x_1 \\ x_n^2 &= x_1 + x_2. \end{aligned}$$

Problem 9.36. *Let x, y, z be real numbers such that $x+y+z = x^{-1}+y^{-1}+z^{-1} = 0$. Prove that*

$$\frac{x^6 + y^6 + z^6}{x^3 + y^3 + z^3} = xyz.$$

Problem 9.37 (Peru, 2009). *Let a, b, c, d be integers such that $a + b + c + d = 0$. Prove that $(bc - ad)(ac - bd)(ab - cd)$ is a perfect square.*

Problem 9.38. *Let a, b and c be real numbers different from zero, with $a+b+c = 0$. Prove that*

$$\frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} = \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}.$$

Problem 9.39. *Find all triplets of positive integers (a, b, c), such that $a^3 + b^3 + c^3 - 3abc = p$, where $p > 3$ is a prime number.*

Problem 9.40. *Find all positive integers that are solutions of the equation $x^3 - y^3 = xy + 61$.*

Problem 9.41 (Great Britain, 2008). *Find the minimum of* $x^2+y^2+z^2$*, where* x*,* y*,* z *are real numbers that satisfy* $x^3+y^3+z^3-3xyz=1$*.*

Problem 9.42 (Shortlist OMCC, 2011). *The positive real numbers* x*,* y*,* z*, satisfy that*

$$x+\frac{y}{z}=y+\frac{z}{x}=z+\frac{x}{y}=2.$$

Find the value of $x+y+z$*.*

Problem 9.43. *If* $\{a_n\}\subset\mathbb{R}^+$ *is an arithmetic progression, prove that*

$$\frac{1}{\sqrt{a_0}+\sqrt{a_1}}+\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n}{\sqrt{a_0}+\sqrt{a_n}}.$$

Problem 9.44. *The lengths of the sides of a right triangle are* $a<b<c$ *and they are in arithmetic progression. Prove that their difference* d *is equal to the radius of the incircle of the triangle.*

Problem 9.45. *Prove that in any partition of the set* $\{1,2,\dots,9\}$ *into two subsets, it is possible to find, in one of them, an arithmetic progression with three terms.*

Problem 9.46. *Let* a*,* b*,* c *be real numbers in arithmetic progression; prove that*

$$\frac{2}{9}(a+b+c)^3=a^2(b+c)+b^2(c+a)+c^2(a+b).$$

Problem 9.47. *Prove that in the arithmetic progression* $\{3,7,11,\dots,4k-1,\dots\}$*, there are an infinite number of primes.*

Problem 9.48. *Is it possible to divide the set of natural numbers in two subsets, such that none of them contains a non-constant arithmetic progression?*

Problem 9.49. *Is there a non-constant arithmetic progression where the terms of the progression are all prime numbers?*

Problem 9.50. *Prove that if an arithmetic progression of positive integers contains a perfect square then it has an infinite number of perfect squares.*

Problem 9.51 (OMM, 1999). *Prove that there do not exist* 1999 *prime numbers in arithmetic progression, all of them smaller than* 12345*.*

Problem 9.52 (OMM, 2005). *Let us say that a list of numbers* $a_1,a_2,\dots,a_m$ *contains an arithmetic triplet* a_i,a_j,a_k*, if* $i<j<k$ *and* $2a_j=a_i+a_k$*. For example,* 8, 1, 5, 2, 7 *has the following arithmetic triplet* $(8,5,2)$*, but* 8, 1, 2, 5, 7 *does not. Let* n *be a positive integer. Prove that the numbers* $1,2,\dots,n$ *can be rearranged in a list, such that this list does not contain an arithmetic triplet.*

Problem 9.53 (Czech-Slovak, 2010). *The four real solutions of the equation $ax^4 + bx^2 + a = 1$, form an increasing arithmetic progression. One of the solutions is also a solution of the equation $bx^2 + ax + a = 1$. Find all possible real values of a and b.*

Problem 9.54 (APMO, 2013). *For $2k$ real numbers $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$, define the sequence of numbers X_n by*

$$X_n = \sum_{i=1}^{k} \lfloor a_i n + b_i \rfloor, \quad (n = 1, 2, \ldots).$$

If the sequence X_n forms an arithmetic progression, prove that the sum $\sum_{i=1}^{k} a_i$ has to be an integer number.

Problem 9.55. *Prove that in any partition of $\{1, 2, 2^2, \ldots, 256\}$ into two subsets, it is possible to find in one of them three terms that are in geometric progression.*

Problem 9.56. *Let $n > 1$ and consider the collection of numbers $a_0, a_1, \ldots, a_n$ defined by*

$$a_0 = \frac{1}{2} \quad \text{and} \quad a_{k+1} = a_k + \frac{a_k}{n^2}, \quad \text{with} \quad k = 0, 1, \ldots, n-1.$$

Prove that $a_n < 1$.

Problem 9.57. *Let $a_1 < a_2 < \cdots < a_n < \cdots$ be an increasing sequence of positive integers such that:*

(i) *For every $n \geq 1$, $a_{2n} = a_n + n$.*
(ii) *If a_p is a prime number, then p is prime.*

Prove that $a_n = n$, for all $n \geq 1$.

Problem 9.58. *An arbitraty set of $m+n$ numbers is divided into two arbitrary sets $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n$. Order the numbers, in each set, in increasing form*

$$a_1 < a_2 < \cdots < a_m, \quad b_1 < b_2 < \cdots < b_n.$$

Then, the numbers in each set are divided again into two subsets $c_1, c_2, \ldots, c_m$ and $d_1, d_2, \ldots, d_n$ and let us arrange them in increasing order

$$c_1 < c_2 < \cdots < c_m, \quad d_1 < d_2 < \cdots < d_n.$$

Prove the equality

$$|a_1 - c_1| + |a_2 - c_2| + \cdots + |a_m - c_m| = |b_1 - d_1| + |b_2 - d_2| + \cdots + |b_n - d_n|.$$

Problem 9.59 (Poland, 1986). *Prove that, for every integer $n \geq 3$, the number $n!$ can be represented as the sum of n different divisors of $n!$.*

Problem 9.60. *For each prime number $p > 3$, prove that the number $\binom{2p-1}{p-1} - 1$ is divisible by p^3.*

Problem 9.61 (UK, 2004). *Let S be a set of rational numbers that satisfy:*

(i) $\frac{1}{2} \in S$.
(ii) *If $x \in S$, then $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.*

Prove that S contains all rational numbers in the interval $0 < x < 1$.

Problem 9.62 (IMO, 1988). *Prove that, if a, b are positive integers such that $ab+1$ divides $a^2 + b^2$, then $\frac{a^2+b^2}{ab+1}$ is a perfect square.*

Problem 9.63 (Ireland, 2007). *If a, b, c are roots of the polynomial $P(x) = x^3 - 2007x + 2002$, determine the value of*

$$\left(\frac{a-1}{a+1}\right)\left(\frac{b-1}{b+1}\right)\left(\frac{c-1}{c+1}\right).$$

Problem 9.64. *Determine all positive rational numbers a, b, c such that $a+b+c$, abc and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ are integers.*

Problem 9.65. *Let a, b, c be real numbers, not all zero. Prove that one of the equations $ax^2 + 2bx + c = 0$, $bx^2 + 2cx + a = 0$, $cx^2 + 2ax + b = 0$ has a real root.*

Problem 9.66 (Estonia, 2005). *Find all pairs of real numbers (a, b) such that the roots of the polynomials*

$$6x^2 - 24x - 4a \quad \textit{and} \quad x^3 + ax^2 + bx - 8$$

are all non-negative real numbers.

Problem 9.67. *Let $P(x)$ be a polynomial such that $|P(x)| \leq 1$, for $|x| \leq 1$.*

(i) (Short list IMO, 1986) *If $P(x) = ax^2 + bx + c$, find the maximum value of $|a| + |b| + |c|$.*
(ii) (Short list IMO, 1996) *If $P(x) = ax^3 + bx^2 + cx + d$, find the maximum value of $|a| + |b| + |c| + |d|$.*

Problem 9.68. *Let a, b, c, d be real numbers, with a and d distinct from zero. Prove that if the roots of the polynomials $ax^3 + bx^2 + cx + d$ and $dx^3 + cx^2 + bx + a$ are positive, then $\frac{bc}{ad} \geq 9$.*

Problem 9.69 (IMO, 1963). *Find all real solutions of the equation* $\sqrt{x^2-p}+2\sqrt{x^2-1}=x$, *where* p *is a real number.*

Problem 9.70 (China, 2008). *Let* $P(x)=ax^3+bx^2+cx+d$ *be a polynomial with real coefficients. If* $P(x)$ *has three positive real roots and* $P(0)<0$, *prove that* $2b^3+9a^2d-7abc\leq 0$.

Problem 9.71 (India, 2010). *Let* a, b, c *be integers with* b *even and* c *odd. Suppose that the equation* $x^3+ax^2+bx+c=0$ *has roots* α, β, γ, *with* $\alpha^2=\beta+\gamma$. *Prove that* α *is an integer and that* $\beta\neq\gamma$.

Problem 9.72. *Consider all quadratic equations* $x^2+px+q=0$, *where the coefficients* p, q *belong to the interval* $[-1,1]$. *Find all possible values of the solutions of those equations.*

Problem 9.73 (Hermite's identity). *Given a positive integer* n *and a real number* x, *prove that*

$$\lfloor nx\rfloor=\lfloor x\rfloor+\left\lfloor x+\frac{1}{n}\right\rfloor+\left\lfloor x+\frac{2}{n}\right\rfloor+\cdots+\left\lfloor x+\frac{n-1}{n}\right\rfloor.$$

Problem 9.74 (IMO, 1968). *For every positive integer* n, *prove that*

$$\left\lfloor\frac{n+1}{2}\right\rfloor+\left\lfloor\frac{n+2}{2^2}\right\rfloor+\cdots+\left\lfloor\frac{n+2^k}{2^{k+1}}\right\rfloor+\cdots=n.$$

Problem 9.75 (USA, 1981). *Prove that if* x *is a positive real number and* n *is a positive integer, then*

$$\lfloor nx\rfloor\geq\frac{\lfloor x\rfloor}{1}+\frac{\lfloor 2x\rfloor}{2}+\cdots+\frac{\lfloor nx\rfloor}{n}.$$

Problem 9.76. *The function* f *assigns to each non-negative integer* n, *the non-negative integer* $f(n)$, *such that:*

(i) $f(nm)=f(n)+f(m)$, *for* n, $m\geq 0$.
(ii) $f(n)=0$ *if the units digit of* n *is* 3.
(iii) $f(10)=0$.

Find the value of $f(1985)$.

Problem 9.77 (Czech-Slovak, 2010). *Find all functions* $f:\mathbb{R}^+\to\mathbb{R}^+$ *that satisfy*

$$f(x)f(y)=f(y)f(xf(y))+\frac{1}{xy},\quad \textit{with } x,y\in\mathbb{R}^+.$$

Problem 9.78 (Thailand, 2004). *Let $f : [0,1] \to \mathbb{R}$ be a function such that:*

(i) $f(0) = f(1) = 0$.
(ii) $|f(x) - f(y)| < |x - y|$, *for* x, $y \in [0,1]$ *with* $x \neq y$.

Prove that $|f(x) - f(y)| < \frac{1}{2}$, *for all* x, $y \in [0,1]$.

Problem 9.79 (IMO, 1978). *The set of all positive integers is the union of two disjoint subsets* $\{f(1), f(2), \ldots, f(n), \ldots\}$, $\{g(1), g(2), \ldots, g(n), \ldots\}$, *where*

$$f(1) < f(2) < \cdots < f(n) < \ldots,$$
$$g(1) < g(2) < \cdots < g(n) < \ldots,$$

and

$$g(n) = f(f(n)) + 1, \quad \text{for all} \quad n \geq 1.$$

Determine $f(240)$.

Problem 9.80 (IMO, 1981). *The function* $f(x,y)$ *satisfies*

$$f(0,y) = y + 1 \tag{9.1}$$
$$f(x+1,0) = f(x,1) \tag{9.2}$$
$$f(x+1,y+1) = f(x, f(x+1,y)), \tag{9.3}$$

for all non-negative integers x, y. *Determine the value of* $f(4, 1981)$.

Problem 9.81 (IMO, 1982). *The function* $f(n)$ *is defined for all positive integers* n *and takes non-negative values. Also, for all* m *and* n,

$$f(n+m) - f(m) - f(n) = 0 \quad \text{or} \quad 1,$$
$$f(2) = 0,\ f(3) > 0 \quad \text{and} \quad f(9999) = 3333.$$

Find $f(1982)$.

Problem 9.82 (IMO, 1983). *Find all functions* f *defined in the set of positive real numbers that satisfy the conditions:*

(i) $f(xf(y)) = yf(x)$, *for all positive numbers* x, y.
(ii) $f(x) \to 0$, *when* $x \to \infty$.

Problem 9.83 (IMO, 2010). *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that, for all* x, $y \in \mathbb{R}$, *the equality*

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

holds, where $\lfloor x \rfloor$ *denotes the greatest integer less than or equal to* x.

Problem 9.84 (APMO, 2011). *Find all upper bounded functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy*

$$f(xf(y)) + yf(x) = xf(y) + f(xy), \text{ for } x, y \in \mathbb{R}. \tag{9.4}$$

Problem 9.85 (OIM, 2009). *Let $\{a_n\}$ be a sequence defined by*

$$a_1 = 1, \quad a_{2n} = a_n + 1, \quad a_{2n+1} = \frac{1}{a_{2n}}, \quad \text{for } n \geq 1.$$

Prove that, for every rational number r, there is a unique positive integer n with $a_n = r$.

Problem 9.86. *Find the term a_{1000} of the sequence defined by*

$$a_0 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_n}{1 + na_n}, \quad \text{for} \quad n \geq 0.$$

Problem 9.87 (Short list IMO, 2010). *A sequence x_1, x_2, ..., is defined by $x_1 = 1$ and $x_{2k} = -x_k$, $x_{2k-1} = (-1)^{k+1}x_k$, for all $k \geq 1$. Prove that $x_1+x_2+\cdots+x_n \geq 0$, for all $n \geq 1$.*

Problem 9.88 (IMO, 2010). *Let a_1, a_2, ..., a_n, ... be a sequence of positive real numbers. Suppose that for some positive integer s, we have that*

$$a_n = \max\{a_k + a_{n-k} : \quad 1 \leq k \leq n-1\},$$

for all $n > s$. Prove that there exist positive integers l and N, with $l \leq s$, such that $a_n = a_{n-l} + a_l$, for all $n \geq N$.

Problem 9.89 (Bulgaria, 1987). *Let k be an integer greater than 1. Prove that there exist a prime number p and an increasing sequence of positive integers $a_1, a_2, \ldots, a_n, \ldots$, such that the terms of the sequence*

$$p + ka_1, p + ka_2, \ldots, p + ka_n, \ldots$$

are all prime numbers.

Problem 9.90 (Austria, 2005). *For real numbers a, b, c, set $s_n = a^n + b^n + c^n$, for $n \geq 0$. Suppose that $s_1 = 2$, $s_2 = 6$ and $s_3 = 14$. Prove that $\left|s_n^2 - s_{n-1}s_{n+1}\right| = 8$, for all $n \geq 2$.*

Problem 9.91 (IMO, 1982). *Consider an infinite sequence of positive real numbers $\{x_n\}$, such that $x_0 = 1$ and $x_{i+1} \leq x_i$, for all $i \geq 0$.*

(i) *Prove that, for any sequence that satisfies the given conditions, there exists an integer $n \geq 1$ such that*

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(ii) *Find a sequence with the given conditions such that*

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4, \quad \textit{for all} \ \ n \geq 1.$$

Problem 9.92 (Great Britain, 2009). *Find all sequences $\{a_n\}$ that satisfy the following conditions:*

(i) $a_{n+1} = 2a_n^2 - 1$, *for all* $n \geq 1$.
(ii) a_1 *is a rational number.*
(iii) $a_i = a_j$, *for some* i, j *with* $i \neq j$.

Problem 9.93. *If* $a_0 = 0$, $a_1 = 1$ *and* $a_n = 2a_{n-1} + a_{n-2}$, *prove that* $2^k | a_n$ *if and only if* $2^k | n$.

Problem 9.94 (Russia, 1989). *The sequence $\{a_n\}$ is such that*

$$|a_m + a_n - a_{m+n}| \leq \frac{1}{m+n}, \quad \textit{for all} \ \ m, n \geq 1.$$

Prove that $\{a_n\}$ is an arithmetic progression.

Problem 9.95 (Bulgaria, 1996). *The sequence $\{a_n\}$ is defined by*

$$a_1 = 1 \quad \textit{and} \quad a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}, \quad \textit{for} \ \ n \geq 1.$$

Prove that $\lfloor a_n^2 \rfloor = n$, *for all* $n \geq 4$.

Problem 9.96. *Let b_n be the units digit of the number* $1^1 + 2^2 + 3^3 + \cdots + n^n$. *Prove that the sequence $\{b_n\}$ is periodic with period* 100.

Problem 9.97. *Prove that* $\lfloor (1 + \sqrt{3})^{2n+1} \rfloor$ *is an even integer, for* $n \geq 0$.

Problem 9.98. *The sequence of integers $\{a_n\}$ is defined by* $a_1 = 3$, $a_2 = 5$ *and* $a_{n+2} = 3a_{n+1} - 2a_n$, *for* $n \geq 1$. *Prove that* $a_n = 2^n + 1$, *for all integers* $n \geq 1$.

Problem 9.99. *The sequence of integers $\{a_n\}$ is defined by* $a_1 = 1$, $a_2 = 2$ *and* $a_{n+2} = a_{n+1} - a_n$, *for* $n \geq 1$. *Prove that* $a_{n+6} = a_n$, *for all integers* $n \geq 1$.

Problem 9.100 (Short list IMO, 1986). *Let* $a_0 = a_1 = 1$ *and, for* $n \geq 0$, $a_{n+2} = 7a_{n+1} - a_n - 2$. *Prove that a_n is a perfect square, for every integer number* $n \geq 0$.

Problem 9.101 (IMO, 1976). *The sequence $\{u_n\}$ is defined by* $u_0 = 2$, $u_1 = \frac{5}{2}$ *and* $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$, *for* $n \geq 1$. *Prove that* $\lfloor u_n \rfloor = 2^{\frac{2^n - (-1)^n}{3}}$, *where* $\lfloor x \rfloor$ *is the integer part of* x.

Problem 9.102. *If* $a_n = \lfloor \sqrt{2n} \rfloor$, *for* $n \geq 1$, *what is the value of* a_{2020}*?*

Problem 9.103. *Let $P(x)$ be a polynomial of degree n with $P(j) = \frac{j}{j+1}$, for $j = 0, 1, \ldots, n$. Find $P(m)$, for $m > n$.*

Problem 9.104. *Let $P(x)$ be a polynomial of degree n with $P(j) = \frac{1}{j}$, for $j = 2^0, 2^1, \ldots, 2^n$. Find $P(0)$.*

Problem 9.105 (Short list IMO, 1981)**.** *Let $P(x)$ be a polynomial of degree n with $P(j) = \binom{n+1}{j}^{-1}$, for $j = 0, 1, \ldots, n$. Find $P(n+1)$.*

Problem 9.106 (IMO, 1993)**.** *Let $n > 1$ be an integer and let $P(x) = x^n + 5x^{n-1} + 3$. Prove that $P(x)$ is irreducible over $\mathbb{Z}[x]$.*

Problem 9.107 (Short list, 1997)**.** *Let p be a prime number and let $Q(x)$ be a polynomial of degree n with integer coefficients such that:*

(i) $Q(0) = 0$, $Q(1) = 1$,

(ii) *For any integer n, $Q(n)$ is congruent to 0 or 1 module p.*

Prove that $n \geq p - 1$.

Problem 9.108 (Short list IMO, 1997)**.** *Let $P(x)$ be a polynomial with real coefficients and $P(x) > 0$, for $x \geq 0$. Prove that there is a positive integer n such that the coefficients of the polynomial $(1 + x)^n P(x)$ are all positive.*

Problem 9.109 (Short list IMO, 2002)**.** *Let $P(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with integer coefficients a, b, c, d and $a \neq 0$. If $xP(x) = yP(y)$, for an infinite number of integers x, y, with $x \neq y$, prove that $P(x)$ has an integer root.*

Problem 9.110. *Consider the positive real numbers a, b, c. Solve the system of equations:*

$$xy = a, \qquad yz = b, \qquad zx = c.$$

Problem 9.111. *Consider the real numbers a, b, c. Solve the system of equations:*

$$x(y + z) = a, \qquad y(z + x) = b, \qquad z(x + y) = c.$$

Problem 9.112. *Consider the positive real numbers a, b, c. Solve the system of equations:*

$$\frac{xyz}{x + y} = c, \qquad \frac{xyz}{y + z} = a, \qquad \frac{xyz}{z + x} = b.$$

Problem 9.113 (Balkanic, 2002)**.** *Solve the system of equations:*

$$\begin{aligned} a^3 + 3ab^2 + 3ac^2 - 6abc &= 1, \\ b^3 + 3ba^2 + 3bc^2 - 6abc &= 1, \\ c^3 + 3cb^2 + 3ca^2 - 6abc &= 1. \end{aligned}$$

Problem 9.114. *Let $P(x)$ be a polynomial that takes integer values in the integers. Prove that there are integers c_0, c_1, ..., c_n such that*

$$P(x) = c_n\binom{x}{n} + c_{n-1}\binom{x}{n-1} + \cdots + c_0\binom{x}{0},$$

where $\binom{x}{j} = \frac{x(x-1)(x-2)\ldots(x-j+1)}{j!}$.

Problem 9.115. *Let p be an odd prime number and let $P(x) = x^p - x + p$. Prove that $P(x)$ is irreducible over $\mathbb{Z}[x]$.*

Problem 9.116 (IMO, 2004)**.** *Find all polynomials $P(x)$ with real coefficients that satisfy the equality*

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c),$$

for all real numbers a, b, c, with $ab + bc + ca = 0$.

Problem 9.117 (IMO, 2006)**.** *Let $P(x)$ be a polynomial of degree $n > 1$, with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, with k pairs of parenthesis. Prove that $Q(x)$ has at most n integer fixed points, that is, integers that satisfy the equation $Q(x) = x$.*

Problem 9.118. *Find all polynomials $P(x)$ such that $P(0) = 0$ and $P(x^2+1) = P(x)^2 + 1$, for all real numbers x.*

Problem 9.119. *Find all polynomials $P(x)$ such that $P(x)^2 - 2 = 2P(2x^2-1)$, for all real numbers x.*

Problem 9.120. *Let $P(x)$ and $Q(x)$ be monic polynomials such that $P(P(x)) = Q(Q(x))$, for all real numbers x. Prove that $P(x) = Q(x)$.*

Chapter 10

Solutions to Exercises and Problems

The first eight sections of this chapter contain all solutions of the exercises in the first eight chapters. In Section 9, you can find the solutions to the problems of Chapter 9. The difficulty of the problems in Chapter 9 is usually greater than the difficulty of the exercises that you find in the first eight chapters. However, solving the problems of this last chapter would be an excellent training in preparation for international mathematical competitions.

We recommend that the reader consult this last chapter just in case he or she cannot solve the exercises and problems alone.

10.1 Solutions to exercises of Chapter 1

Solution 1.1. (i) If $a < 0$, then $-a > 0$. Also use $(-a)(-b) = ab$. (ii) $(-a)b > 0$. (iii) $a < b \Leftrightarrow b - a > 0$, now use property (a). (iv) Use property (a). (v) $aa^{-1} = 1 > 0$. (vi) If $a < 0$, then $-a > 0$.

Solution 1.2. Observe that if $a^2 + b - (a + b^2) \in \mathbb{Q}$, then $(a - b)(a + b - 1) \in \mathbb{Q}$ and, since $a + b - 1 \in \mathbb{Q} \setminus \{0\}$, then $(a - b) \in \mathbb{Q}$. Therefore, if $a + b \in \mathbb{Q}$ and $a - b \in \mathbb{Q}$, then $2a$ and $2b$ are in $\mathbb{Q}$. Therefore, a and b are rational numbers.

Solution 1.3. If $a = 0$ or $b = 0$, then the result is clear. Now suppose $ab \neq 0$. Since $(a^2 + b^2)^2 - (a^4 + b^4) = 2a^2b^2$, we have that $a^2b^2 \in \mathbb{Q}$. Note that $a^6 + b^6 = (a^2 + b^2)^3 - 3a^2b^2(a^2 + b^2) \in \mathbb{Q}$, then $(a^3 + b^3)^2 - (a^6 + b^6) = 2a^3b^3 \in \mathbb{Q}$. Therefore,

$$ab = \frac{a^3b^3}{a^2b^2} \in \mathbb{Q} \quad \text{and} \quad a + b = \frac{a^3 + b^3}{a^2 + b^2 - ab} \in \mathbb{Q}.$$

Solution 1.4. (i) Suppose $\sqrt{p}$ is not an irrational number, that is, $\sqrt{p} = \frac{m}{n}$, where m, n are integers with $(m, n) = 1$, that is, m and n are relatively prime numbers. Squaring both sides of the equality, we get $pn^2 = m^2$, that is, p divides m^2, then p divides m. Therefore, $m = ps$ and $pn^2 = p^2s^2$ imply $n^2 = ps^2$, this guarantees that p divides n^2 and also divides n. Therefore, p divides m and n contradicting the fact that m and n are relatively prime.

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5_10

(ii) Suppose $\sqrt{m}$ is not an irrational number, that is, $\sqrt{m} = \frac{r}{s}$, where r, s are integers with $(r, s) = 1$. Squaring we have $ms^2 = r^2$. Since m is not a perfect square, it has a factor of the form p^α, where p is a prime number and α is a positive odd integer. Then p^α divides r^2, which means that the prime p appears an even number of times in the factor decomposition of r^2. Since r and s are relatively prime numbers, p does not divide s, hence p appears an odd number of times as a factor of ms^2, which is a contradiction.

Solution 1.5. If $a+b = ab = n$, then $b = n-a$ and $n = a(n-a)$. The last equation is equivalent to $a^2 - na + n = 0$; solving the equation we have

$$a = \frac{n \pm \sqrt{n^2 - 4n}}{2}, \quad \text{from where} \quad b = \frac{n \mp \sqrt{n^2 - 4n}}{2}.$$

For $n \geq 5$, we have $(n-3)^2 < n^2 - 4n < (n-2)^2$, therefore $\sqrt{n^2 - 4n}$ is an irrational number, and then a and b are irrational numbers.

Solution 1.6. Suppose $\frac{m}{n}$ is a root, with $(m, n) = 1$, then m and n cannot both be even. On the other hand, since $a\left(\frac{m}{n}\right)^2 + b\left(\frac{m}{n}\right) + c = 0$, we have that $am^2 + bmn + cn^2 = 0$. The right-hand side of the last equation is even and the left-hand side is odd. If m and n are odd numbers, the three terms of the left-hand side of the equation are odd. Now, if one term is even and the other is odd then two terms are even, the third odd and the sum is odd again. This contradiction implies that the equation cannot have rational roots.

Second Solution. The discriminant $b^2 - 4ac$ has to be a perfect square. But, since a, b and c are odd numbers, we can prove that $b^2 - 4ac \equiv 5 \mod 8$. However, the square of an odd number has remainder 1 modulo 8.

Solution 1.7. Let $u = a + \sqrt{b}$ and $v = a - \sqrt{b}$, then

$$\begin{aligned}
\sqrt{a+\sqrt{b}} = \sqrt{u} &= \frac{\sqrt{u}+\sqrt{v}}{2} + \frac{\sqrt{u}-\sqrt{v}}{2} \\
&= \sqrt{\frac{(\sqrt{u}+\sqrt{v})^2}{4}} + \sqrt{\frac{(\sqrt{u}-\sqrt{v})^2}{4}} \\
&= \sqrt{\frac{\frac{u+v}{2} + \sqrt{uv}}{2}} + \sqrt{\frac{\frac{u+v}{2} - \sqrt{uv}}{2}} \\
&= \sqrt{\frac{\frac{a+\sqrt{b}+a-\sqrt{b}}{2} + \sqrt{a^2-b}}{2}} + \sqrt{\frac{\frac{a+\sqrt{b}+a-\sqrt{b}}{2} - \sqrt{a^2-b}}{2}} \\
&= \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}},
\end{aligned}$$

as we wanted to prove.

Solution 1.8. (i) Let $x = \sqrt{a\sqrt{a\sqrt{a\sqrt{a\ldots}}}}$, then $x^2 = a\sqrt{a\sqrt{a\sqrt{a\sqrt{a\ldots}}}}$, therefore, $x^2 = ax$. Factorizing, $x(x-a) = 0$. Therefore, since a is positive the solution is $x = a$.

Second Solution. We can give another solution using series. We have

$$x = a^{\frac{1}{2}} a^{\frac{1}{4}} a^{\frac{1}{8}} \cdots = a^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots} = a,$$

since $\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, see Section 7.3.1.

(ii) Let $x = \sqrt{a\sqrt{b\sqrt{a\sqrt{b\ldots}}}}$, then $x^2 = a\sqrt{b\sqrt{a\sqrt{b\sqrt{a\ldots}}}}$, therefore $x^4 = a^2bx$. Since $x \neq 0$, $x^3 = a^2b$, then $x = \sqrt[3]{a^2b}$.

Second Solution. We can give another solution using series. We have

$$x = a^{\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots} b^{\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots} = a^{\frac{2}{3}} b^{\frac{1}{3}},$$

since $\sum_{j=1}^{\infty} \frac{1}{2^{2j}} = \frac{1}{3}$ and $\sum_{j=0}^{\infty} \frac{1}{2^{2j+1}} = \frac{2}{3}$, see Section 7.3.1.

Solution 1.9.

(i) If xy, yz and zx are in $\mathbb{Q}$, then $\frac{(xy)(zx)}{yz} = x^2 \in \mathbb{Q}$. Similarly, y^2, $z^2 \in \mathbb{Q}$. Therefore, $x^2 + y^2 + z^2 \in \mathbb{Q}$.

(ii) By (i) we have $(x^2)^2 + (xy)y^2 + (xz)z^2 = x(x^3 + y^3 + z^3) \in \mathbb{Q}$, then $x \in \mathbb{Q}$. Similarly, y, $z \in \mathbb{Q}$.

Solution 1.10. Since $a - \sqrt{ab} = a\left(1 - \frac{\sqrt{b}}{\sqrt{a}}\right)$, it is sufficient to prove that $1 - \frac{\sqrt{b}}{\sqrt{a}}$ is a rational number different from zero to claim that a is a rational number. But $\frac{b-\sqrt{ab}}{a-\sqrt{ab}} = \frac{\sqrt{b}(\sqrt{b}-\sqrt{a})}{\sqrt{a}(\sqrt{a}-\sqrt{b})} = -\frac{\sqrt{b}}{\sqrt{a}}$ is a rational number different from -1 (since $a \neq b$), therefore $1 - \frac{\sqrt{b}}{\sqrt{a}}$ is a rational number different from 0. Similarly, b is a rational number.

Solution 1.11. To solve (i), define $x = 0.111\ldots$, then $10x = 1.11\ldots$. Subtracting the first equation from the second, we get $9x = 1$, therefore $x = \frac{1}{9}$.

(ii) Let $x = 1.141414\ldots$, then $100x = 114.141414\ldots$. Subtracting the first equation from the second, we get $99x = 113$, therefore $x = \frac{113}{99}$.

Solution 1.12.

(i) First observe that $121_b = (1 \times b^2) + (2 \times b) + 1 = (b+1)^2$, then 121_b is a perfect square in any base $b \geq 2$.

(ii) Since $232_b = 2b^2 + 3b + 2$ has to be a square and since 3 is one of its digits, $b \geq 4$.

For $b = 4$, $232_4 = 46$, for $b = 5$, $232_5 = 67$, for $b = 6$, $232_6 = 92$ and for $b = 7$, $232_7 = 121$. Then, $b = 7$ is the smallest positive integer such that 232_b is a perfect square.

Solution 1.13. Suppose that $a > b$. Then for all integers $0 \le k \le n$, $x_n x_k a^n b^k \ge x_n x_k b^n a^k$, where the equality holds only when $k = n$ or $x_k = 0$. In particular, we have a strict inequality for $k = n - 1$. Adding, this becomes

$$x_n a^n \sum_{k=0}^{n} x_k b^k > x_n b^n \sum_{k=0}^{n} x_k a^k$$

or

$$\frac{x_n a^n}{A_n} > \frac{x_n b^n}{B_n}.$$

This implies that

$$\frac{A_{n-1}}{A_n} = 1 - \frac{x_n a^n}{A_n} < 1 - \frac{x_n b^n}{B_n} = \frac{B_{n-1}}{B_n}.$$

On the other hand, if $a = b$, then clearly $\frac{A_{n-1}}{A_n} = \frac{B_{n-1}}{B_n}$, and if $a < b$, using what we proved before, it follows that $\frac{A_{n-1}}{A_n} > \frac{B_{n-1}}{B_n}$. Therefore, $\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ if and only if $a > b$.

Solution 1.14. Note that $|a| = |a - b + b| \le |a - b| + |b|$, therefore we have $|a| - |b| \le |a - b|$. Similarly, following the same procedure, we have $|b| - |a| \le |b - a|$. From these two inequalities, we get $||a| - |b|| \le |a - b|$.

Solution 1.15.

(i) $|x - 1| - |x + 1| = 0$ is equivalent to $|x - 1| = |x + 1|$. Squaring both sides of the previous equation and solving $(x - 1)^2 = (x + 1)^2$, we have $4x = 0$, therefore the only solution is $x = 0$.

(ii) $|x - 1||x + 1| = 1$ is equivalent to $|x^2 - 1| = 1$, hence

$$\begin{aligned} x^2 - 1 &= 1 \quad &\text{or} \quad -(x^2 - 1) &= 1, \\ x^2 &= 2 \quad &\text{or} \quad x^2 &= 0, \\ x &= \pm\sqrt{2} \quad &\text{or} \quad x &= 0, \end{aligned}$$

therefore the solutions are $x = \pm\sqrt{2}$ and $x = 0$.

(iii) If $x > 1$ we get $|x + 1| = x + 1 > 2$, therefore there are no solutions.
If $x < -1$ we get $|x - 1| = -x + 1 > 2$, therefore there are no solutions.
If $-1 \le x \le 1$, then $x - 1 \le 0 \le x + 1$, therefore

$$|x - 1| + |x + 1| = (1 - x) + (x + 1) = 2.$$

Thus, the only values of x that satisfy the equality are $-1 \le x \le 1$.

Solution 1.16. From the first and third inequalities we have $z \geq |x+y| - 1 \geq 0$. Therefore, $z^2 \geq (|x+y|-1)^2$. Now, $2xy \geq z^2 + 1 \geq (|x+y|-1)^2 + 1 \geq 0$, then

$$2xy \geq x^2 + 2xy + y^2 - 2|x+y| + 2 \geq |x|^2 + 2xy + |y|^2 - 2|x| - 2|y| + 2,$$

cancelling out, $0 \geq |x|^2 + |y|^2 - 2|x| - 2|y| + 2 = (|x|-1)^2 + (|y|-1)^2$. Therefore $|x| = 1$ and $|y| = 1$. Since x and y have to be -1 or 1, but since $xy \geq 0$, both numbers have the same sign. For $x = y = 1$ or $x = y = -1$ we get, substituting in the original equations, that $2 - z^2 \geq 1$ and $z - 2 \geq -1$. Therefore, $z^2 \leq 1$ and $z \geq 1$. The only value of z that satisfies both inequalities is $z = 1$. Therefore, there are two solutions to the problem $x = y = z = 1$ and $x = y = -1$, $z = 1$.

Solution 1.17. Suppose that $a_1 < a_2 < \cdots < a_n$ is a collection with the largest quantity of integers that satisfy the property. It is clear that $a_i \geq i$, for all $i = 1, \ldots, n$.
If a and b are two integers from the collection $a > b$, since $|a-b| = a - b \geq \frac{ab}{100}$, we get $a\left(1 - \frac{b}{100}\right) \geq b$; therefore, if $100 - b > 0$, then $a \geq \frac{100b}{100-b}$.
Note that there are no two numbers a and b in the collection greater than 100; in fact if $a > b > 100$, then $a - b = |a-b| \geq \frac{ab}{100} > a$, which is false.
We also have that for integers a and b smaller than 100, we have $\frac{100a}{100-a} \geq \frac{100b}{100-b}$ if and only if $100a - ab \geq 100b - ab$ if and only if $a \geq b$.

It is clear that $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is a collection whose elements satisfy the property.
Now, $a_{11} \geq \frac{100a_{10}}{100-a_{10}} \geq \frac{100\cdot 10}{100-10} = \frac{100}{9} > 11$, which implies that $a_{11} \geq 12$.

$a_{12} \geq \frac{100a_{11}}{100-a_{11}} \geq \frac{100\cdot 12}{100-12} = \frac{1200}{88} > 13$, hence $a_{12} \geq 14$.

$a_{13} \geq \frac{100a_{12}}{100-a_{12}} \geq \frac{100\cdot 14}{100-14} = \frac{1400}{86} > 16$, hence $a_{13} \geq 17$.

$a_{14} \geq \frac{100a_{13}}{100-a_{13}} \geq \frac{100\cdot 17}{100-17} = \frac{1700}{83} > 20$, hence $a_{14} \geq 21$.

$a_{15} \geq \frac{100a_{14}}{100-a_{14}} \geq \frac{100\cdot 21}{100-21} = \frac{2100}{79} > 26$, hence $a_{15} \geq 27$.

$a_{16} \geq \frac{100a_{15}}{100-a_{15}} \geq \frac{100\cdot 27}{100-27} = \frac{2700}{73} > 36$, hence $a_{16} \geq 37$.

$a_{17} \geq \frac{100a_{16}}{100-a_{16}} \geq \frac{100\cdot 37}{100-37} = \frac{3700}{63} > 58$, hence $a_{17} \geq 59$.

$a_{18} \geq \frac{100a_{17}}{100-a_{17}} \geq \frac{100\cdot 59}{100-59} = \frac{5900}{41} > 143$, hence $a_{18} \geq 144$.

Moreover, as we have already observed, there are no two integers of the collection greater than 100, so the largest quantity is 18. A collection with 18 integers that satisfies the conditions is

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 17, 21, 27, 37, 59, 144\}.$$

Solution 1.18. By Example 1.3.2, $\lfloor 2a \rfloor = \lfloor a \rfloor + \lfloor a + \frac{1}{2} \rfloor$ and $\lfloor 2b \rfloor = \lfloor b \rfloor + \lfloor b + \frac{1}{2} \rfloor$, then the inequality that we have to prove is equivalent to

$$\lfloor a \rfloor + \left\lfloor a + \frac{1}{2} \right\rfloor + \lfloor b \rfloor + \left\lfloor b + \frac{1}{2} \right\rfloor \geq \lfloor a \rfloor + \lfloor b \rfloor + \lfloor a + b \rfloor,$$

then we only have to prove that $\lfloor a + \frac{1}{2} \rfloor + \lfloor b + \frac{1}{2} \rfloor \geq \lfloor a + b \rfloor$.

Let $a = n+y$, $b = m+x$, with $n, m \in \mathbb{Z}$ and $0 \leq x, y < 1$. Then $0 \leq x+y < 2$ and $a + b = n + m + x + y$. We have two cases:

(i) If $1 \leq x + y < 2$, then $\lfloor a + b \rfloor = n + m + 1$, and at least one of the numbers x or y is greater than or equal to $\frac{1}{2}$. Suppose that $x \geq \frac{1}{2}$, then $\lfloor b+\frac{1}{2} \rfloor = \lfloor m+x+\frac{1}{2} \rfloor = m+1$, therefore $\lfloor a+\frac{1}{2} \rfloor + \lfloor b+\frac{1}{2} \rfloor \geq m+n+1 = \lfloor a+b \rfloor$.

(ii) If $0 \leq x+y < 1$, then $\lfloor a+b \rfloor = n+m$ and $\lfloor a+\frac{1}{2} \rfloor + \lfloor b+\frac{1}{2} \rfloor \geq m+n = \lfloor a+b \rfloor$.

Solution 1.19. (i) We have that $\lfloor x \lfloor x \rfloor \rfloor = 1$ if and only if $1 \leq x \lfloor x \rfloor < 2$. If $x = m + y$, with $m \in \mathbb{Z}$ and $0 \leq y < 1$, then $1 \leq m^2 + my < 2$. Observe that $m = 0$ is impossible, as well as $m \geq 2$ or $m \leq -2$. Therefore, it only remains to be proved for $m = 1$ or $m = -1$.

If $m = 1$, then $1 \leq 1 + y < 2$, from which $0 \leq y < 1$ and then any x in the interval $[1, 2)$ satisfies the equation. If $m = -1$, then since $1 \leq m^2 + my < 2$, we have $1 \leq 1 - y < 2$, from which $0 \leq -y < 1$ and then $y = 0$ and $x = -1$. Therefore, the numbers that satisfy the equation are $x = -1$ and $x \in [1, 2)$.

(ii) Since $\lfloor x \rfloor \leq x \leq |x|$, it follows that $|x| - \lfloor x \rfloor \geq 0$, therefore $||x| - \lfloor x \rfloor| = |x| - \lfloor x \rfloor$. On the other hand, by property (c) in 1.3.1 we obtain $\lfloor |x| - \lfloor x \rfloor \rfloor = \lfloor |x| \rfloor - \lfloor x \rfloor$. Using the last equalities, the equation becomes $|x| - \lfloor x \rfloor = \lfloor |x| \rfloor - \lfloor x \rfloor$, which is equivalent to $|x| = \lfloor |x| \rfloor$; then $|x|$ is an integer number and the values of x that satisfy the equation are all the integers.

Solution 1.20. Add the three equations to obtain $2x + 2y + 2z = 6.6$, so that $x + y + z = 3.3$. If you subtract this equation from the original ones, we obtain

$$\begin{aligned} \{y\} + \lfloor z \rfloor &= 2.2, \\ \{x\} + \lfloor y \rfloor &= 1.1, \\ \{z\} + \lfloor x \rfloor &= 0. \end{aligned}$$

For the first equation, we obtain $\lfloor z \rfloor = 2$ and $\{y\} = 0.2$; the second equation becomes $\lfloor y \rfloor = 1$, $\{x\} = 0.1$, and the third $\lfloor x \rfloor = 0$ and $\{z\} = 0$. Therefore, the solution is $x = 0.1$, $y = 1.2$ and $z = 2$.

Solution 1.21. We have $\sqrt{n}+\sqrt{n+1} < \sqrt{4n+2}$ if and only if $2n+1+\sqrt{4n^2+4n} < 4n + 2$, which is equivalent to $\sqrt{4n^2+4n} < 2n + 1$. Squaring again, the last inequality is equivalent to $4n^2+4n < 4n^2+4n+1$. This proves that $\sqrt{n}+\sqrt{n+1} < \sqrt{4n+2}$, then $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor \leq \lfloor \sqrt{4n+2} \rfloor$.

Suppose that, for some positive integer n, $\lfloor\sqrt{n}+\sqrt{n+1}\rfloor \neq \lfloor\sqrt{4n+2}\rfloor$. Let $q=\lfloor\sqrt{4n+2}\rfloor$, then $\sqrt{n}+\sqrt{n+1} < q \leq \sqrt{4n+2}$. Squaring, we obtain $2n+1+\sqrt{4n^2+4n} < q^2 \leq 4n+2$, or equivalently, $\sqrt{4n^2+4n} < q^2-2n-1 \leq 2n+1$. Squaring again we find that $4n^2+4n < (q^2-2n-1)^2 \leq 4n^2+4n+1=(2n+1)^2$. Since there does not exist a square between two consecutive integers, we have $q^2-2n-1=2n+1$ or $q^2=4n+2$, which is equivalent to saying that $q^2 \equiv 2 \mod 4$. But this is a contradiction, since any square number is congruent to 0 or $1 \mod 4$. Therefore, we get the equality.

We now prove that $\lfloor\sqrt{4n+1}\rfloor = \lfloor\sqrt{4n+2}\rfloor = \lfloor\sqrt{4n+3}\rfloor$.

For the first equality, suppose that there exists n such that $m=\lfloor\sqrt{4n+1}\rfloor < m+1=\lfloor\sqrt{4n+2}\rfloor$, therefore $m \leq \sqrt{4n+1} < m+1 \leq \sqrt{4n+2}$, or $m^2 \leq 4n+1 < (m+1)^2 \leq 4n+2$. Therefore, since $4n+1$ and $4n+2$ are two consecutive integers, and since $(m+1)^2 > 4n+1$, therefore $(m+1)^2=4n+2$, and again we found a square number which has remainder 2 when we divide the number by 4, which is impossible. For the second equality, proceed in the same way.

Solution 1.22. For the first five parts use equations (1.2), (1.3) and (1.5). To prove (vi), use (iv) and (v).

Solution 1.23. For the first two parts (i) and (ii), use equations (1.2) and (1.3). To prove (iii), use (i) and (ii).

Solution 1.24. To prove (i) and (ii) just expand the left-hand side of the equations and rearrange the terms.
To prove (iii), (iv), (v) and (vi) make the operations on both sides of the equality and observe that they are equal.

Solution 1.25. To prove (i) and (ii) expand the right-hand side of the equations and simplify.

Solution 1.26. Use equations (1.2) and (1.3), and perform the operations on both sides of the equation.

Solution 1.27. Let $x=\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$, then

$$x-\sqrt[3]{2+\sqrt{5}}-\sqrt[3]{2-\sqrt{5}}=0.$$

By equation (1.7), if $a+b+c=0$, then $a^3+b^3+c^3=3abc$, therefore

$$x^3-\left(2+\sqrt{5}\right)-\left(2-\sqrt{5}\right)=3x\sqrt[3]{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)},$$

simplifying we have that $x^3+3x-4=0$. Clearly a root of the equation is $x=1$ and the other roots satisfy the equation $x^2+x+4=0$ which does not have real solutions. Since $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ is a real number, it follows that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1,$$

which is a rational number.

Solution 1.28. Observe that, if $x+y+z=0$, then it follows from equation (1.7) that $x^3+y^3+z^3=3xyz$. Since $(x-y)+(y-z)+(z-x)=0$, we obtain the factorization

$$(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x).$$

Solution 1.29. Observe that $(x+2y-3z)+(y+2z-3x)+(z+2x-3y)=0$, then it follows, from equation (1.7), that $(x+2y-3z)^3+(y+2z-3x)^3+(z+2x-3y)^3=3(x+2y-3z)(y+2z-3x)(z+2x-3y)$.

Solution 1.30. Let $a=\sqrt[3]{x-y}$, $b=\sqrt[3]{y-z}$, $c=\sqrt[3]{z-x}$, and suppose that $a+b+c=0$, then it follows, from equation (1.7), that $a^3+b^3+c^3=3abc$, but then $0=(x-y)+(y-z)+(z-x)=a^3+b^3+c^3=3abc=3\sqrt[3]{x-y}\sqrt[3]{y-z}\sqrt[3]{z-x}\neq 0$, which is absurd.

Solution 1.31. If we define $a=\sqrt[3]{r}$, $b=-\frac{1}{\sqrt[3]{r}}$ and $c=-1$, we have $a+b+c=0$, then $r-\frac{1}{r}-1=3\sqrt[3]{r}\left(-\frac{1}{\sqrt[3]{r}}\right)(-1)=3$, therefore $r-\frac{1}{r}=4$. Similarly, $r^3-\frac{1}{r^3}-4^3=3r\left(-\frac{1}{r}\right)(-4)=12$, therefore $r^3-\frac{1}{r^3}=76$.

Solution 1.32. It follows from

$$a^3+b^3+c^3-3abc=\begin{vmatrix} a & b & c\\ c & a & b\\ b & c & a\end{vmatrix}=\begin{vmatrix} 100b+10c+a & b & c\\ 100a+10b+c & a & b\\ 100c+10a+b & c & a\end{vmatrix}=\begin{vmatrix} bca & b & c\\ abc & a & b\\ cab & c & a\end{vmatrix}.$$

Solution 1.33. If we rewrite the equation as $m^3+n^3+(-33)^3-3mn(-33)=0$, and using equation (1.9), we get

$$(m+n-33)\left[(m-n)^2+(m+33)^2+(n+33)^2\right]=0.$$

The equation $m+n=33$ has 34 solutions with $mn\geq 0$ which are $(k,33-k)$, with $k=0,1,\ldots,33$, and the second factor is 0 only when $m=n=-33$, therefore there are 35 solutions.

Solution 1.34. If we rewrite the equation as $x^3+y^3+(-1)^3-3xy(-1)=0$, and using equation (1.9), we have

$$(x+y-1)\left[(x-y)^2+(y+1)^2+(-1-x)^2\right]=0.$$

Therefore, the points (x,y) satisfy $x+y=1$ or $x=y=-1$.

Solution 1.35. Substituting the equation of the hypothesis in equation (1.7), we get

$$\begin{aligned}(x+y+z)^3 - 3xyz &= x^3+y^3+z^3-3xyz \\ &= (x+y+z)(x^2+y^2+z^2-xy-yz-zx) \\ &= (x+y+z)((x+y+z)^2-3xy-3yz-3zx) \\ &= (x+y+z)^3 - 3(x+y+z)(xy+yz+zx),\end{aligned}$$

from where it is clear that $xyz = (x+y+z)(xy+yz+zx)$, therefore $(x+y)(y+z)(z+x) = 0$. Or use that $(x+y+z)^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x)$. Therefore, the solutions are $(x,-x,z)$, $(x,y,-y)$, $(x,y,-x)$, with x, y, z any real numbers.

Solution 1.36.

(i) Since $0 \leq b \leq 1$ and $1+a > 0$, it follows that $b(1+a) \leq 1+a$, then $0 \leq b-a \leq 1-ab$, therefore $0 \leq \dfrac{b-a}{1-ab} \leq 1$.

(ii) The inequality is clear. Since $1+a \leq 1+b$, we have $\frac{1}{1+b} \leq \frac{1}{1+a}$, then $\dfrac{a}{1+b} + \dfrac{b}{1+a} \leq \dfrac{a}{1+a} + \dfrac{b}{1+a} = \dfrac{a+b}{1+a} \leq 1$.

Solution 1.37. If we define $X = \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$, adding and substracting three times 1, leads to

$$\begin{aligned}X &= \frac{a}{b+c} + \frac{b+c}{b+c} + \frac{b}{a+c} + \frac{a+c}{a+c} + \frac{c}{a+b} + \frac{a+b}{a+b} - 3 \\ &= \frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} - 3 \\ &= (a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3 \\ &= \frac{1}{2}((a+b)+(b+c)+(a+c))\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3.\end{aligned}$$

Now, from the arithmetic and geometric mean inequality, we get $x+y+z \geq 3\sqrt[3]{xyz}$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3\sqrt[3]{\frac{1}{x}\frac{1}{y}\frac{1}{z}}$. Therefore, $X \geq \frac{1}{2} \cdot 3 \cdot 3 - 3 = \frac{3}{2}$.

Solution 1.38. Without loss of generality we can assume that $a \geq b \geq c$; the inequality is equivalent to $-a^3+b^3+c^3+3abc \geq 0$. But, by equation (1.9), $-a^3+b^3+c^3+3abc = \frac{1}{2}(-a+b+c)\left[(a+b)^2+(a+c)^2+(b-c)^2\right] \geq 0$, since, by the triangle inequality, $a < b+c$.

Solution 1.39. Observe that $\frac{1}{p} + \frac{1}{q} = 1$ implies $p+q = pq = s$. Now, $(p+q)^2 \geq 4pq$ implies $s \geq 4$.

To prove (i), observe that

$$\frac{1}{p(p+1)}+\frac{1}{q(q+1)}=\frac{1}{p}-\frac{1}{p+1}+\frac{1}{q}-\frac{1}{q+1}=1-\frac{p+q+2}{(p+1)(q+1)}$$
$$=1-\frac{s+2}{2s+1}=\frac{s-1}{2s+1}.$$

Therefore, we have to prove

$$\frac{1}{3}\leq\frac{s-1}{2s+1}\leq\frac{1}{2},$$

but $2s+1\leq 3s-3\Leftrightarrow 4\leq s$ and $2s-2\leq 2s+1\Leftrightarrow -2\leq 1$.

To prove (ii), show that

$$\frac{1}{p(p-1)}+\frac{1}{q(q-1)}=\frac{1}{p-1}-\frac{1}{p}+\frac{1}{q-1}-\frac{1}{q}=\frac{p+q-2}{(p-1)(q-1)}-1$$
$$=\frac{s-2}{s-s+1}-1=s-3\geq 1.$$

Solution 1.40. First, note that

$$\frac{a}{b}+\frac{a}{1-b}=\frac{a}{b(1-b)}\geq 4a,$$

since

$$b(1-b)\leq\left(\frac{b+(1-b)}{2}\right)^2=\frac{1}{4}.$$

Moreover, the equality holds if and only if $b=\frac{1}{2}$. Similarly,

$$\frac{b}{a}+\frac{b}{1-a}\geq 4b.$$

Therefore,

$$\frac{a}{b}+\frac{b}{a}+\frac{a}{1-b}+\frac{b}{1-a}\geq 4a+4b\geq 2\sqrt{4^2ab}=8\sqrt{k}.$$

With equality if and only if $a=b$. Then,

$$\frac{a}{b}+\frac{b}{a}+\frac{a}{1-b}+\frac{b}{1-a}\geq 8\sqrt{k}\geq 4$$

if and only if $k\geq\frac{1}{4}$, then the smallest number k is $\frac{1}{4}$.

Solution 1.41. Prove that $(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc=\frac{8}{9}(a+b+c)(ab+bc+ca)+\frac{1}{9}(a+b+c)(ab+bc+ca)-abc$, and from the arithmetic and geometric mean inequality, we have that $(a+b+c)(ab+bcb+ca)\geq\left(3\sqrt[3]{abc}\right)\left(3\sqrt[3]{(ab)(bc)(ca)}\right)=9abc$.

Solution 1.42. Using the arithmetic and geometric mean inequality, and the condition $(a+b)(b+c)(c+a)=1$, leads to

$$a+b+c \geq 3\sqrt[3]{\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right)} = \frac{3}{2},$$
$$abc = \sqrt{ab}\sqrt{bc}\sqrt{ca} \leq \left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right) = \frac{1}{8}.$$

Now, $1=(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc \geq \frac{3}{2}(ab+bc+ca)-\frac{1}{8}$, see Exercise 1.24 (iii).

Solution 1.43. By Exercise 1.24 (iii), it is enough to prove $ab+bc+ca+\frac{3}{a+b+c} \geq 4$. But

$$\begin{aligned} ab+bc+ca+\frac{3}{a+b+c} &= 3\left(\frac{ab+bc+ca}{3}\right)+\frac{3}{a+b+c} \\ &\geq 4\sqrt[4]{\left(\frac{ab+bc+ca}{3}\right)^3\left(\frac{3}{a+b+c}\right)}. \end{aligned}$$

Now use that $(ab+bc+ca)^2 \geq 3(ab\cdot bc+bc\cdot ca+ca\cdot ab)=3(a+b+c)$, and that $ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2}=3$.

Solution 1.44. Without loss of generality we can assume that $a \leq b \leq c$. Therefore $c^2 < c^2+a+b \leq c^2+2c < (c+1)^2$; this proves that c^2+a+b cannot be a perfect square.

Solution 1.45. To prove all the equalities of the exercise, just perform the operations and simplify.

Solution 1.46. To prove all the equalities of the exercise, just use the identity (1.7).

Solution 1.47. Expand both sides of the identities and compare.

Solution 1.48. We have

$$\begin{aligned} 0 &= x^2(y+z)-y^2(x+z) = xy(x-y)+(x^2-y^2)z \\ &= (x-y)(xy+xz+yz). \end{aligned}$$

Since $x \neq y$, we have $xy+xz+yz=0$. Multiplying by $x-z$ we obtain

$$\begin{aligned} 0 &= (x-z)(xy+xz+yz) = xz(x-z)+(x^2-z^2)y \\ &= x^2(y+z)-z^2(x+y), \end{aligned}$$

then $z^2(x+y)=x^2(y+z)=2$.

Solution 1.49. See that $(x+y+z)(xy+yz+zx) = xyz$ and by equation (1.23) we get $(x+y)(y+z)(z+x) = 0$. Therefore, the solutions (x, y, z, w) are of the form $(x, -x, z, z)$, $(x, y, -y, x)$ and $(x, y, -x, y)$, with x, y and z real numbers different from zero.

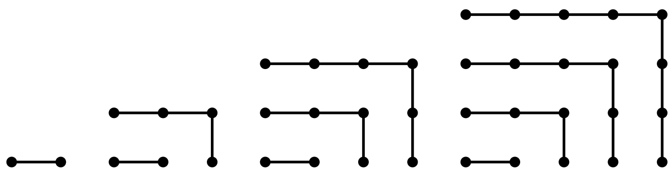
Solution 1.50. By equation (1.23) the condition is equivalent to $(x+y)(y+z)(z+x) = 0$. Therefore, one factor is zero, say $x+y = 0$. Then, since n is odd, $x^n + y^n = 0$, and also $\frac{1}{x^n} + \frac{1}{y^n} = 0$.

10.2 Solutions to exercises of Chapter 2

Solution 2.1. Call c_n the sum of the first n even numbers, then we have

$$\begin{array}{ccccccccc} c_n & = & 2 & + & 4 & + & \cdots & + & 2n \\ c_n & = & 2n & + & 2n-2 & + & \cdots & + & 2 \\ \hline 2c_n & = & (2n+2) & + & (2n+2) & + & \cdots & + & (2n+2) \end{array} \tag{10.1}$$

therefore, $2c_n = n(2n+2) = 2n(n+1)$, then $c_n = n(n+1)$. We can represent this sum as arrangements of points forming rectangles such as those below:



Solution 2.2.

(i) Let d and d' be the differences of the progressions $\{a_n\}$ and $\{b_n\}$, respectively. Then, $(a_{n+1} - a_n) \pm (b_{n+1} - b_n) = d \pm d'$. Rearranging the terms we have $(a_{n+1} \pm b_{n+1}) - (a_n \pm b_n) = d \pm d'$.

(ii) If d is the difference of the progression $\{a_n\}$, we obtain

$$\begin{aligned} b_{n+1} - b_n &= (a_{n+2}^2 - a_{n+1}^2) - (a_{n+1}^2 - a_n^2) \\ &= (a_{n+2} - a_{n+1})(a_{n+2} + a_{n+1}) - (a_{n+1} - a_n)(a_{n+1} + a_n) \\ &= d(a_{n+2} + a_{n+1} - a_{n+1} - a_n) \\ &= d(a_{n+2} - a_n) = 2d^2. \end{aligned}$$

Solution 2.3. If d is the difference of the progression $\{a_n\}$, we have

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{1}{a_j a_{j+1}} &= \frac{1}{d} \sum_{j=0}^{n-1} \frac{1}{a_j} - \frac{1}{a_{j+1}} = \frac{1}{d}\left(\frac{1}{a_0} - \frac{1}{a_n}\right) \\ &= \frac{1}{d}\left(\frac{a_n - a_0}{a_0\, a_n}\right) = \frac{1}{d}\left(\frac{a_0 + nd - a_0}{a_0\, a_n}\right) = \frac{n}{a_0\, a_n}. \end{aligned}$$

Solution 2.4. If $\{a_n\}$ is an arithmetic progression,

$$\begin{aligned} S_n &= a_0 + a_1 + \cdots + a_{n-1} = \frac{a_0 + a_{n-1}}{2} \cdot n \\ &= \frac{2a_0 + (n-1)d}{2} \cdot n = \frac{d}{2}n^2 + \left(a_0 - \frac{d}{2}\right) n, \end{aligned}$$

and with $A = \frac{d}{2}$ and $B = a_0 - \frac{d}{2}$ we get the result.

Suppose now that $S_n = a_0 + a_1 + \cdots + a_{n-1} = An^2 + Bn$, then $S_{n+1} = a_0 + a_1 + \cdots + a_{n-1} + a_n = A(n+1)^2 + B(n+1)$. Subtracting the first equation from the second, we have

$$\begin{aligned} a_n &= A(n+1)^2 + B(n+1) - An^2 - Bn \\ &= A(2n+1) + B = 2An + (A+B) \end{aligned}$$

and using Proposition 2.1.3, we get the expected result.

Solution 2.5. Suppose that $\{a_n\}$ is an arithmetic progression of order 2. Consider $S_n = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n - a_0$. By Exercise 2.4, we have $a_n - a_0 = An^2 + Bn$. Therefore, $a_n = An^2 + Bn + a_0 = P(n)$, where $P(x)$ is the polynomial of degree 2, given by $Ax^2 + Bx + a_0$.

Suppose now that each term of the progression a_n is equal to $P(n)$, where $P(x)$ is a polynomial of degree 2, that is, $a_n = An^2 + Bn + C$. It follows that

$$\begin{aligned} a_{n+1} - a_n &= A(n+1)^2 + B(n+1) + C - (An^2 + Bn + C) \\ &= 2An + (A+B). \end{aligned}$$

Therefore, by Proposition 2.1.3, $\{a_{n+1} - a_n\}$ is an arithmetic progression and thus $\{a_n\}$ is an arithmetic progression of order 2.

Solution 2.6. A consequence of the inequality between the geometric mean and the arithmetic mean, is $\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$. By Proposition 2.1.1, we get

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a_1 + a_n}{2} \cdot n \cdot \frac{1}{n} = \frac{a_1 + a_n}{2}.$$

To prove the left-hand side inequality, we use a similar version of the equality of Exercise 2.3

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}.$$

By the inequality between the harmonic mean and the geometric mean, we get

$$a_1 a_n = \frac{n-1}{\frac{1}{a_1 a_2} + \cdots + \frac{1}{a_{n-1} a_n}} \leq \sqrt[n-1]{(a_1 a_2)(a_2 a_3) \cdots (a_{n-1} a_n)}.$$

Then,

$$(a_1 a_n)^{n-1} \leq a_1 (a_2 \cdots a_{n-1})^2 a_n.$$

Therefore, $(a_1 a_n)^n \leq (a_1 a_2 \cdots a_n)^2$, that is, $\sqrt{a_1 a_n} \leq \sqrt[n]{a_1 a_2 \cdots a_n}$.

Solution 2.7. If p, $p+6$, $p+12$, $p+18$, $p+24$ are the 5 prime numbers, when we consider these numbers modulo 5, we have that they are congruent to p, $p+1$, $p+2$, $p+3$, $p+4$, but between five consecutive numbers we have always one that is divisible by 5, and since p is prime, then $p=5$ and hence the numbers are forced to be 5, 11, 17, 23 and 29.

Solution 2.8. The numbers m between 2^n and 2^{n+1} are part of an arithmetic progression with difference 1 starting in 2^n+1 and ending in $2^{n+1}-1$. Then, by Proposition 2.1.1,

$$S_n = \frac{2^n+1+2^{n+1}-1}{2}(2^{n+1}-1-2^n) = \frac{3\cdot 2^n}{2}(2^{n+1}-1-2^n),$$

and from this it is clear that 3 divides S_n.

Solution 2.9. Since a, b and c are in harmonic progression, in that order, we can suppose that $\frac{1}{a}=A-s$, $\frac{1}{b}=A$ and $\frac{1}{c}=A+s$, with $s\neq 0$. We have $\frac{1}{b-c}=\frac{A(A+s)}{s}$, $\frac{4}{c-a}=\frac{-2(A^2-s^2)}{s}$, $\frac{1}{a-b}=\frac{A(A-s)}{s}$ and $\frac{1}{c}-\frac{1}{a}=2s$. Therefore,

$$\frac{1}{b-c}+\frac{4}{c-a}+\frac{1}{a-b}=\frac{A^2+As-2A^2+2s^2+A^2-As}{s}=2s=\frac{1}{c}-\frac{1}{a}.$$

Solution 2.10. If a, b, c and d are in harmonic progression, then their inverses are in arithmetic progression. Suppose that $\frac{1}{b}=\frac{1}{a}+s$, $\frac{1}{c}=\frac{1}{a}+2s$ and $\frac{1}{d}=\frac{1}{a}+3s$, with $s\neq 0$, that is, we have $\frac{1}{b}=\frac{1+as}{a}$, $\frac{1}{c}=\frac{1+2as}{a}$ and $\frac{1}{d}=\frac{1+3as}{a}$. Then we have that, $b=\frac{a}{1+as}$, $c=\frac{a}{1+2as}$ and $d=\frac{a}{1+3as}$.

Then,

$$a+d=\frac{2a+3a^2s}{1+3as} \quad \text{and} \quad b+c=\frac{2a+3a^2s}{1+3as+2(as)^2}.$$

Since $1+3as<1+3as+2(as)^2$, we have that $a+d>b+c$, as desired.

Solution 2.11. Note that

$$\begin{aligned}
\frac{1}{c+a}-\frac{1}{b+c}&=\frac{b-a}{(b+c)(c+a)}=\frac{b^2-a^2}{(b+a)(b+c)(c+a)},\\
\frac{1}{a+b}-\frac{1}{c+a}&=\frac{c-b}{(a+b)(c+a)}=\frac{c^2-b^2}{(b+a)(b+c)(c+a)},
\end{aligned}$$

then $\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$ if and only if $b^2-a^2=c^2-b^2$. Then, $b+c$, $c+a$ and $a+b$ are in harmonic progression if and only if a^2, b^2 and c^2 are in arithmetic progression.

Solution 2.12. Suppose that a_0, a_1, … is the progression and that d is the difference of the progression, that is, $d=a_{n+1}-a_n$ for all $n\geq 0$, then $a_n=a_0+nd$.

By hypothesis, $a_0(a_0+d)$ and $a_0(a_0+2d)$ also belong to the progression, therefore $a_0(a_0+2d)-a_0(a_0+d)=n_0d$ for some integer number $n_0 \geq 1$, then $a_0d = n_0d$. Since the progression is increasing, it follows that $d > 0$ and therefore $a_0 = n_0$.

Change a_0 for any a_m in the previous argument and conclude that a_m is an integer number.

Solution 2.13.

(i) Observe that in the array, the left-hand side number of each row is as follows:

$$\begin{array}{ll}
1^\circ \text{ row} & 1 \\
2^\circ \text{ row} & 1+2=3 \\
3^\circ \text{ row} & 1+2+4=7 \\
4^\circ \text{ row} & 1+2+4+6=13 \\
5^\circ \text{ row} & 1+2+4+6+8=21 \\
\vdots & \vdots \\
100^\circ \text{ row} & 1+2+\cdots+2\cdot 99 = 1+2(1+\cdots+99) = 9901.
\end{array}$$

(ii) The sum of the numbers on the 100th row is,

$$\begin{aligned}
9901+9903+\cdots+10099 &= (9900+1)+(9900+3)+\cdots \\
&\quad + (9900+2\cdot 100-1) \\
&= 9900\cdot 100 + (1+3+\cdots+2\cdot 100-1) \\
&= 990000+100^2 = 10^6.
\end{aligned}$$

Solution 2.14. The array will be a square of size 100×100. Let S_1 be the sum of the numbers in the diagonal which goes from the top left corner to the bottom right corner and let S_2 be the sum of the numbers of the other main diagonal.

When we move one column to the right in the same row, the number increases by 1; if we move one column to the left in the same row, the number decreases by 1. When we move down in the same column, the number increases by 100.

From the top left corner to the bottom right corner through the diagonal, each number is one column to the right and one row below, that is, it is $1+100=101$ greater than the previous number of the diagonal. That is, the sum we want to calculate is the sum of the progression $1, 1+101, \ldots, 1+99\cdot 101$ that, by Proposition 2.1.1 (b), is

$$S_1 = \left(\frac{2\cdot 1 + 99\cdot 101}{2}\right) 100 = 500050.$$

From the top right corner, through the diagonal, each number is in the previous column, that is, one column to the left and one row below, that is, it is $-1+100 = 99$ greater than the previous number in the diagonal. That is, the sum we are looking for is the sum of the progression 100, $100 + 1 \cdot 99$, ..., $100 + 99 \cdot 99$ that, by Proposition 2.1.1 (b), is

$$S_2 = \left(\frac{2 \cdot 100 + 99 \cdot 99}{2} \right) 100 = 500050.$$

Therefore, $S_1 = S_2 = 500050$.

Solution 2.15. Call a_{ij}, where i denotes the number of the row and j denotes the number of the column, the corresponding position in the table.
Let x_0 and x_1 be the two numbers neighboring 0, x_0 on the right-hand side of 0 and x_1 on top of 0, then the last row can be filled up with 0, x_0, $2x_0$, $3x_0$ and $4x_0$, and the first column with $4x_1$, $3x_1$, $2x_1$, x_1 and 0.
If x is the number in the position a_{32}, the number that occupies the a_{42} position will be $\frac{1}{2}(x + x_0)$, but we also know that the number in the a_{42} position is $\frac{1}{2}(x_1 + 103)$. Therefore, $\frac{1}{2}(x + x_0) = \frac{1}{2}(x_1 + 103)$, solving for x we have that $x = x_1 + 103 - x_0$. Now, let y be the number occupying the a_{44} position, then we have $103 = \frac{1}{2}(y + \frac{1}{2}(x_1 + 103))$, solving for y we obtain $y = \frac{1}{2}(309 - x_1)$.

The number $\frac{1}{2}(309-x_1)-103 = \frac{1}{2}(103-x_1)$ is the difference of the progression in the fourth row, but this difference, added to the number in the a_{44} position, gives the number in position a_{45}, that is, $\frac{1}{2}(309 - x_1) + \frac{1}{2}(103 - x_1) = 206 - x_1$. However, we know that $206 - x_1 = \frac{1}{2}(186 + 4x_0)$, then

$$2x_0 + x_1 = 113. \tag{10.2}$$

Observe also that $\frac{1}{2}\left(74 + \frac{1}{2}(x_1 + 103)\right) = x_1 + 103 - x_0$. Simplifying we obtain

$$4x_0 - 3x_1 = 161. \tag{10.3}$$

Solving the system of equations (10.2) and (10.3), we get $x_0 = 50$ and $x_1 = 13$. With these values, now it is easy to complete the table, so the filled board is

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

Solution 2.16. If $\{a_n\}$ is a geometric progression with ratio r, we have that $a_n = a_0 r^n$. Similarly, if $\{b_n\}$ is a geometric progression with ratio s, then $b_n = b_0 s^n$. Therefore, since $b_n \neq 0$ for all n, we have $\frac{a_n}{b_n} = \frac{a_0}{b_0}\left(\frac{r}{s}\right)^n$.

Solution 2.17. Let $\{a_n\}$ be a geometric progression with ratio r and having the property that $a_{n+2} = a_{n+1} + a_n$. Since $a_n = a_0 r^n$, this property is equivalent to $a_0 r^{n+2} = a_0 r^{n+1} + a_0 r^n$. Since $a_0 \neq 0$ and $r \neq 0$, we certainly have that $r^2 = r+1$, which has as solutions $r = \frac{1\pm\sqrt{5}}{2}$. Then, the solutions are $\left\{a_n = a_0\left(\frac{1+\sqrt{5}}{2}\right)^n\right\}$ and $\left\{a_n = a_0\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$.

Solution 2.18.

(i) Since $P_n = a_0 \cdot a_1 \cdot \cdots \cdot a_{n-1}$ and $a_n = a_0 r^n$, for all n, we have

$$\begin{aligned} P_n &= a_0 \cdot a_1 \cdot \cdots \cdot a_{n-1} = a_0(a_0 r)(a_0 r^2)\cdots(a_0 r^{n-1}) \\ &= a_0^n r^{1+2+\cdots+(n-1)} = a_0^n r^{\frac{n(n-1)}{2}}. \end{aligned}$$

(ii) Since $P_n = a_0^n r^{\frac{n(n-1)}{2}}$, it is clear that

$$(P_n)^2 = \left(a_0^n r^{\frac{n(n-1)}{2}}\right)^2 = a_0^n a_0^n\, r^{n(n-1)} = a_0^n\left(a_0 r^{n-1}\right)^n = a_0^n a_{n-1}^n.$$

Solution 2.19. Since $a_{n+1} = a_n \cdot r$, then $b_{n+1} = \log a_{n+1} = \log(a_n \cdot r) = \log a_n + \log r = b_n + \log r$, and the result follows.

Solution 2.20. Factorizing we have

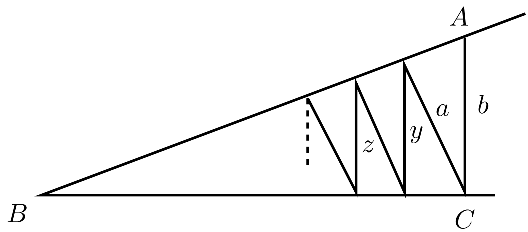
$$a^3b^3 + b^3c^3 + c^3a^3 - abc(a^3+b^3+c^3) = \left(-bc+a^2\right)\left(-ca+b^2\right)\left(-ab+c^2\right).$$

Solution 2.21. If the arithmetic progression is a, $a+d$, $a+2d$, $\ldots$, it is clear that $a + ad = a(1+d)$ is an element of the arithmetic progression, and also the integers $a(1+d)^n$, with $n \geq 1$, are part of the progression. But these terms form a geometric progression, then it is clear that these terms have to remain in the sequence and we have to eliminate the remaining terms of the original progression.

Solution 2.22. Consider $a < b < c$, then since the lengths of the sides are in geometric progression, we have $b = ar$ and $c = ar^2$, with r positive. Since the triangle is a right triangle, it follows that $a^2 + (ar)^2 = (ar^2)^2$. Simplifying the equation we get $1 + r^2 = r^4$, which can be solved for r^2. That is, $r^2 = \frac{1+\sqrt{5}}{2}$. Therefore, $r = \sqrt{\frac{1+\sqrt{5}}{2}}$.

Solution 2.23. Let d be the common difference of the progression. Then $a_2 = 1+d$, $a_5 = 1+4d$ and $a_{11} = 1+10d$. Since a_2, a_5, a_{11} form a geometric progression, we have $(1+4d)^2 = (1+d)(1+10d)$ or $6d^2 = 3d$. Since the arithmetic progression is not constant, we conclude that $d = \frac{1}{2}$ and the sum of the first 2009 terms is $2009 + \frac{2009\cdot 2008}{2}\cdot\frac{1}{2} = 2009\cdot 503$.

Solution 2.24. By the similarity of the triangles, we have that $\frac{b}{a} = \frac{a}{y}$, then $y = \frac{a^2}{b}$. Again, using the similarity of the triangles, we have $\frac{a}{y} = \frac{y}{z}$.



Then $z = \frac{y^2}{a}$, and substituting the value of y in this equation results in $z = \frac{a^4}{b^2 a} = \frac{a^3}{b^2}$.

(i) Carry on with this process and find that the sides of the polygonal measure b, a, $\frac{a^2}{b}$, $\frac{a^3}{b^2}$, $\frac{a^4}{b^3}$, …, which can be written as b, a, $a\left(\frac{a}{b}\right)$, $a\left(\frac{a}{b}\right)^2$, $a\left(\frac{a}{b}\right)^3$, …. Therefore, the nth segment measures $a\left(\frac{a}{b}\right)^{n-2}$.

(ii) The length of the n-sided polygonal line is then

$$\begin{aligned} b + a\left(\frac{a}{b}\right)^0 + a\left(\frac{a}{b}\right)^1 + \cdots + a\left(\frac{a}{b}\right)^{n-2} &= b + a\left(\frac{1-\left(\frac{a}{b}\right)^{n-1}}{1-\frac{a}{b}}\right) \\ &= b + ab\left(\frac{1-\left(\frac{a}{b}\right)^{n-1}}{b-a}\right). \end{aligned}$$

(iii) Since the number $\frac{a}{b}$ is less than one, then raising this number to the nth power and making n go to infinite leads to 0 as its limit. Then, the length of the polygonal line with an infinite number of sides is

$$\lim_{n\to\infty}\left(b + ab\left(\frac{1-\left(\frac{a}{b}\right)^{n-1}}{b-a}\right)\right) = b + \frac{ab}{b-a} = \frac{b^2}{b-a}.$$

Solution 2.25. The sum of each row is $R_n = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, then the sum of all the array is $S_T = n\left(\frac{n(n+1)(2n+1)}{6}\right)$. See now that the sum of each corridor is

$$C_k = 1^2 + 2^2 + \cdots + (k-1)^2 + k^3 = \frac{8\,k^3}{6} - \frac{k^2}{2} + \frac{k}{6}.$$

Then

$$\begin{aligned} S_T &= C_1 + C_2 + \cdots + C_n \\ &= \frac{8}{6}(1^3 + 2^3 + \cdots + n^3) - \frac{1}{2}(1^2 + 2^2 + \cdots + n^2) + \frac{1}{6}(1 + 2 + \cdots + n) \\ &= \frac{8}{6}(1^3 + 2^3 + \cdots + n^3) - \left(\frac{1}{2}\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{6}\left(\frac{n(n+1)}{2}\right). \end{aligned}$$

Therefore

$$\begin{aligned}\frac{8}{6}(1^3+2^3+\cdots+n^3) =& \frac{n^2(n+1)(2n+1)}{6}+\frac{n(n+1)(2n+1)}{12}-\frac{1}{6}\left(\frac{n(n+1)}{2}\right)\\ =&\frac{8}{6}\left(\frac{n(n+1)}{2}\right)^2.\end{aligned}$$

From this the sought for equality follows immediately.

Solution 2.26. Observe that the numbers $\{1, 4, 7, \ldots, 2998, 3001\}$ form an arithmetic progression $\{a_n\}$ with difference 3 and $a_1 = 1$. Each term of the sum can be seen as

$$\frac{1}{a_i a_{i+1}} = \frac{1}{a_{i+1}-a_i}\left(\frac{1}{a_i}-\frac{1}{a_{i+1}}\right) = \frac{1}{3a_i}-\frac{1}{3a_{i+1}}.$$

Therefore, we can calculate the sum as

$$\begin{aligned}\left(\frac{1}{3a_1}-\frac{1}{3a_2}\right)+\left(\frac{1}{3a_2}-\frac{1}{3a_3}\right)+\cdots+\left(\frac{1}{3\cdot 3000}-\frac{1}{3\cdot 3001}\right)\\ =\frac{1}{3}-\frac{1}{3\cdot 3001}=\frac{1000}{3001}.\end{aligned}$$

Solution 2.27.

(i) We have $\frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)$, then

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+2)} &= \sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)\\ &=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{n}-\frac{1}{n+2}\right)\\ &=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)\\ &=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}.\end{aligned}$$

(ii) We have $\frac{2k+1}{k^2(k+1)^2} = \frac{1}{k^2}-\frac{1}{(k+1)^2}$, then

$$\begin{aligned}\sum_{k=1}^{n}\frac{2k+1}{k^2(k+1)^2} &= \sum_{k=1}^{n}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)\\ &=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots+\frac{1}{n^2}-\frac{1}{(n+1)^2}\\ &=1-\frac{1}{(n+1)^2}.\end{aligned}$$

Solution 2.28.

(i) Since

$$\frac{k}{(k+1)!} = \frac{(k+1)-1}{(k+1)!} = \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}$$

the sum we have to calculate becomes

$$\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = 1 - \frac{1}{(n+1)!}.$$

(ii) The general term can be written as

$$\begin{aligned}\frac{k+1}{(k-1)!+k!+(k+1)!} &= \frac{k+1}{(k-1)![1+k+k(k+1)]}\\ &= \frac{k+1}{(k-1)!(k+1)^2}\\ &= \frac{1}{(k-1)!(k+1)} = \frac{k}{(k+1)!}.\end{aligned}$$

By the first part of the exercise, the sum is equal to $1 - \frac{1}{(n+1)!}$.

Solution 2.29. For any positive integer n, we have

$$1 + \frac{1}{n^2} + \frac{1}{(n+1)^2} = \frac{n^2(n+1)^2 + (n+1)^2 + n^2}{n^2(n+1)^2} = \frac{(n^2+n+1)^2}{n^2(n+1)^2}.$$

Then

$$\sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}} = \frac{n^2+n+1}{n^2+n} = 1 + \frac{1}{n(n+1)}.$$

Therefore, the sum is equal to

$$\sum_{n=1}^{2011}\left(1 + \frac{1}{n(n+1)}\right) = \sum_{n=1}^{2011}\left(1 + \frac{1}{n} - \frac{1}{n+1}\right) = 2012 - \frac{1}{2012}.$$

Solution 2.30. Define S as the product we have to calculate, that is,

$$S = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right)\cdots\left(1 + \frac{1}{2^{2^n}}\right).$$

Multiplying both sides of the equality by $\left(1 - \frac{1}{2}\right)$ and using that $\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right) = \left(1 - \frac{1}{2^2}\right)$, we get

$$\left(1 - \frac{1}{2}\right)S = \left(1 - \frac{1}{2^2}\right)\left(1 + \frac{1}{2^2}\right)\cdots\left(1 + \frac{1}{2^{2^n}}\right).$$

Proceeding in this way, we arrive at

$$\left(1-\frac{1}{2}\right)S=\left(1-\left(\frac{1}{2^{2^n}}\right)^2\right).$$

Therefore,

$$S=\frac{\left(1-\left(\frac{1}{2^{2^n}}\right)^2\right)}{\left(1-\frac{1}{2}\right)}=2\left(1-\left(\frac{1}{2^{2^n}}\right)^2\right).$$

10.3 Solutions to exercises of Chapter 3

Solution 3.1.

(i) If $n=1$, then $1=\frac{1-q}{1-q}=1$.

Suppose that $1+q+\cdots+q^{n-1}=\frac{1-q^n}{1-q}$. Then,

$$1+q+\cdots+q^{n-1}+q^n=\frac{1-q^n}{1-q}+q^n=\frac{1-q^{n+1}}{1-q},$$

as we wanted to prove.

Second Solution. Let $S=1+q+\cdots+q^{n-1}$, then $Sq=q+q^2+\cdots+q^n$. Substracting the first equality from the second leads to $Sq-S=q^n-1$, then $S=\frac{1-q^n}{1-q}$.

(ii) The proof is immediate from (i).

Solution 3.2. The result is valid for $n=1$, since $a_3=a_2+a_1=2$ and $a_3=1+a_1=2$.

Suppose that the result is valid for n, that is, $a_{n+2}=1+a_1+a_2+\cdots+a_n$. Then the formula is valid for $n+1$, since

$$a_{n+3}=a_{n+2}+a_{n+1}=1+a_1+a_2+\cdots+a_n+a_{n+1}.$$

Solution 3.3. For $n=0$ the statement is valid, because 3 divides $2+1=3$. The induction hypothesis for $n-1$ tells us that 3^n divides $2^{3^{n-1}}+1$.

We prove now the result for n. We start with

$$2^{3^n}+1=(2^{3^{n-1}}+1)\left[\left(2^{3^{n-1}}\right)^2-2^{3^{n-1}}+1\right].$$

By the induction hypothesis, the first factor is divisible by 3^n. The second factor is divisible by 3, since $2^{3^{n-1}}\equiv -1 \mod 3$. This proves the statement.

Solution 3.4. If $n=1$ we have three coins. Place in each plate of the weighing scale one coin, if the plates balance, the false coin is the third coin, which we did not place. If not, the plate that lifts is the one that has the false coin.

Suppose that it is true for 3^n coins. Consider 3^{n+1} coins. Divide the coins in three groups with 3^n coins in each. Put one of the groups in one plate of the balance and another group in the other plate. If the plates balance, the false coin is in the third group. If not, the false coin is in the group of the plate that raises. In both cases the problem will reduce to finding the false coin in a group with 3^n coins, but this can be done in n weighings and with the weighing already done we have $n+1$ weighings.

Solution 3.5. First note that for $n \geq 1$, it follows that $2^{n+3} \pm 1 = 2^n(2^3-1) + 2^n \pm 1 = 7 \cdot 2^n + (2^n \pm 1)$, then $7 \mid 2^n \pm 1$ if and only if $7 \mid 2^{n+3} \pm 1$. This equivalence shows an inductive step of the form $7 \mid 2^n \pm 1 \Rightarrow 7 \mid 2^{n+3} \pm 1$, and of the form $7 \nmid 2^n \pm 1 \Rightarrow 7 \nmid 2^{n+3} \pm 1$.

Now let us see the induction basis.

(i) For $n = 1, 2$ it follows that 7 does not divide $2^n - 1$. For $n = 3$, it follows that 7 divides $2^3 - 1 = 7$. Therefore, the integers n we are looking for are the multiples of 3.

(ii) For $n = 1, 2, 3$, it follows that 7 does not divide $2^n + 1$ (which are 3, 5 and 9). Thus 7 does not divide $2^n + 1$ with $n \geq 1$.

Solution 3.6. Observe that $a_1 + a_2 = 2^2$. Solving with respect to a_2, we show that $a_2 = 4 - a_1 = 4 - 3 = 3$. This suggests to us that $a_n = 2n - 1$. Suppose that $a_j = 2j-1$, for all $j < n$. It follows that $a_1+a_2+\cdots+a_n = 1+3+\cdots+(2n-3)+a_n = n^2$.

Since $1 + 3 + \cdots + (2n-3) = (n-1)^2$, it follows that $(n-1)^2 + a_n = n^2$. From this we conclude that $a_n = n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1$.

Solution 3.7. For $n = 1$, the identity is valid, the left-hand side is $\lfloor \frac{1}{2} \rfloor = 0$ and the right-hand side is $\lfloor \frac{1}{2} \rfloor \lfloor \frac{2}{2} \rfloor = 0$.

For $n = 2$, the identity is also valid; on the one hand we have $\lfloor \frac{1}{2} \rfloor + \lfloor \frac{2}{2} \rfloor = 1$ and on the other hand we have $\lfloor \frac{2}{2} \rfloor \lfloor \frac{3}{2} \rfloor = 1 \cdot 1 = 1$.

Now we suppose valid the identity for n and we prove that it is true for $n+2$. The left-hand side is

$$\begin{aligned}
&\left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{2} \right\rfloor \\
&\quad = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{2} \right\rfloor \\
&\quad = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} + 1 \right\rfloor \\
&\quad = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
&\quad = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left(\left\lfloor \frac{n+1}{2} \right\rfloor + 1 \right) = \left\lfloor \frac{n+2}{2} \right\rfloor \left\lfloor \frac{n+3}{2} \right\rfloor,
\end{aligned}$$

which is what we expected from the right-hand side.

Solution 3.8. Observe that $\sqrt{a_1} = \frac{1\cdot\sqrt{a_2}}{2}$. Solving this equation gives as a result $2 = \sqrt{a_2}$, from where $a_2 = 2^2$. This suggests $a_n = n^2$. Suppose that $a_j = j^2$, for all $j < n$. We then have

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{n-1}} = 1 + 2 + \cdots + n - 1 = \frac{(n-1)n}{2} = \frac{(n-1)\sqrt{a_n}}{2},$$

and therefore $n = \sqrt{a_n}$, that is, $a_n = n^2$.

Solution 3.9.

(i) For $n = 1$, we have $x^2 - 2x + 1 = (x-1)^2$.
For $n = 2$ we get $x^3 - 3x + 2 = (x-1)^2(x+2)$, since $(x-1)^2(x+2) = (x^2 - 2x + 1)(x+2) = x^3 - 3x + 2$.
Suppose that $x^{n+1} - (n+1)x + n = (x-1)^2(x^{n-1} + 2x^{n-2} + \cdots + n)$, then

$$\begin{aligned}
(x-1)^2&(x^n + 2x^{n-1} + \cdots + (n-1)x^2 + nx + (n+1)) \\
&= (x-1)^2 \left(x(x^{n-1} + 2x^{n-2} + \cdots + (n-1)x + n) + (n+1)\right) \\
&= x\left((x-1)^2(x^{n-1} + 2x^{n-2} + \cdots + (n-1)x + n\right) + (x-1)^2(n+1) \\
&= x\left(x^{n+1} - (n+1)x + n\right) + (x-1)^2(n+1) \\
&= x^{n+2} - (n+1)x^2 + nx + (n+1)x^2 - 2x(n+1) + (n+1) \\
&= x^{n+2} - (n+2)x + (n+1).
\end{aligned}$$

Therefore we have proved that $x^{n+1} - (n+1)x + n = (x-1)^2(x^{n-1} + 2x^{n-2} + \cdots + n)$. Now, if $x > 0$ the right-hand side of the above equality is greater tan or equal to zero, then $x^{n+1} - (n+1)x + n \geq 0$.

(ii) If $x = \frac{a}{b}$, with $a = \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1}$ and $b = \frac{x_1 + x_2 + \cdots + x_n}{n}$, we have that

$$\begin{aligned}
\frac{a^{n+1}}{b^{n+1}} - (n+1)\frac{a}{b} + n &\geq 0 \\
\frac{a^{n+1}}{b^{n+1}} &\geq (n+1)\left(\frac{\frac{x_1 + \cdots + x_{n+1}}{n+1}}{\frac{x_1 + \cdots + x_n}{n}}\right) - n \\
\frac{a^{n+1}}{b^{n+1}} &\geq \frac{n(x_1 + \cdots + x_{n+1})}{x_1 + \cdots + x_n} - n \\
\frac{a^{n+1}}{b^{n+1}} &\geq \frac{n(x_1 + \cdots + x_n)}{x_1 + \cdots + x_n} + \frac{nx_{n+1}}{x_1 + \cdots + x_n} - n \\
\frac{a^{n+1}}{b^{n+1}} &\geq \frac{nx_{n+1}}{x_1 + \cdots + x_n},
\end{aligned}$$

then

$$a^{n+1} \geq x_{n+1}\left(\frac{b^{n+1}}{\frac{x_1 + \cdots + x_n}{n}}\right) = x_{n+1}b^n,$$

which is what we wanted to prove.

(iii) Suppose that $\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n}$, then

$$\left(\frac{x_1+\cdots+x_{n+1}}{n+1}\right)^{n+1} \geq x_{n+1}\left(\frac{x_1+\cdots+x_n}{n}\right)^n \geq x_{n+1}(x_1\cdots x_n),$$

where the first inequality is due to the (ii) part and the second by the inductive step. Therefore, $\frac{x_1+\cdots+x_{n+1}}{n+1} \geq \sqrt[n+1]{x_1\cdots x_{n+1}}$, which is what we wanted to prove.

Solution 3.10. For $n = 1$ the result follows. Suppose that the result is true for $n-1$. Then $\sum_{i=1}^{n-1} e_ia_i$ has at least $\binom{n}{2}$ different values and the same holds for $\left(\sum_{i=1}^{n-1} e_ia_i\right) - a_n$. The greater value of these sums is $a_1 + a_2 + \cdots + a_{n-1} - a_n$, but it is clear that

$$\begin{aligned} a_1 + a_2 + \cdots + a_{n-1} - a_n &< a_1 + a_2 + \cdots + a_{n-2} - a_{n-1} + a_n \\ &< a_1 + \cdots + a_{n-3} - a_{n-2} + a_{n-1} + a_n \\ &\vdots \qquad\qquad \vdots \\ &< a_1 - a_2 + a_3 + \cdots + a_n \\ &< -a_1 + a_2 + a_3 + \cdots + a_n \\ &< a_1 + a_2 + a_3 + \cdots + a_n, \end{aligned}$$

then we have n additional different sums. Since $\binom{n}{2} + n = \binom{n+1}{2}$, the result is proved.

Solution 3.11. For $n = 1$ the statement is true, because in a sequence of three numbers (chosen from the set of numbers $\{0, 1\}$) there are two that are equal. Suppose the statement to be valid for $2n + 1$ numbers and consider a sequence with $2n + 3$ elements.

If for any i we have that $a_i = a_{i+1}$, then by the induction hypothesis, a_1, …, a_{i-1}, a_{i+2},…, a_{2n+3} has a subsequence even-balanced with $2n$ elements. To this subsequence add two equal elements and you will have a subsequence even-balanced with $2(n + 1)$ elements.

On the contrary, we have that $a_i \neq a_{i+1}$ for all $i \in \{1, 2, \ldots, n + 1\}$, therefore $a_i = a_{i+2}$ for all i. Since the sequence has an odd number of elements, the initial and last numbers have to be the same. Then, if we take out the element that occupies the central place of the sequence, we will have a subsequence with $2n$ elements that clearly is even-balanced, since the sequence is symmetric with respect to the central element.

Solution 3.12. Prove, by induction, that $a_k = k$. For $k = 1$ the statement is true, since $a_1^3 = a_1^2$ and $a_1 > 0$, hence $a_1 = 1$. Suppose the result is true for 1, 2, …, k and prove that it is valid for $k + 1$.

We then have that the equality

$$a_1^3 + a_2^3 + \cdots + a_n^3 = (a_1 + a_2 + \cdots + a_n)^2,$$

is satisfied by $n = 1, 2, \ldots k, k+1$. By the induction hypothesis, we have that $a_1 = 1$, $a_2 = 2$, …, $a_k = k$ and the condition for $n = k+1$ is

$$1^3 + 2^3 + \cdots + k^3 + a_{k+1}^3 = (1 + 2 + \cdots + k + a_{k+1})^2.$$

Expanding the right-hand side of the equality, we obtain

$$(1^3 + 2^3 + \cdots + k^3) + a_{k+1}^3 = (1 + 2 + \cdots + k)^2 + 2(1 + 2 + \cdots + k)a_{k+1} + a_{k+1}^2.$$

The first terms on both sides of the equality can be canceled out, and from there we get $a_{k+1}^3 = 2(1 + 2 + \cdots + k)a_{k+1} + a_{k+1}^2$; but this is equivalent to $a_{k+1}^3 = 2\frac{k(k+1)}{2}a_{k+1} + a_{k+1}^2$. Dividing by $a_{k+1} \neq 0$, we obtain $a_{k+1}^2 - a_{k+1} - k(k+1) = 0$. This is a quadratic equation in a_{k+1} with roots $-k$ and $k+1$. Since a_{k+1} is positive, we end up with $a_{k+1} = k+1$, which ends the proof.

Solution 3.13. We will do the proof using induction. For $n = 1$, since $a_1 \geq 1$, we have that $a_1^3 \geq a_1^2$, moreover, $a_1^3 = a_1^2$ if and only if $a_1 = 1$. Suppose the inequality is true for $n = k$ and consider $k+1$ positive integers such that $a_1 < a_2 < \cdots < a_k < a_{k+1}$. Then, we have $a_{k+1} \geq a_k + 1$, therefore

$$\frac{(a_{k+1} - 1)a_{k+1}}{2} \geq \frac{a_k(a_k + 1)}{2} = 1 + 2 + \cdots + a_k.$$

Note that the sum $1 + 2 + \cdots + a_k$ contains all positive integers less than or equal to a_k, then it is greater than or equal to $a_1 + a_2 + \cdots + a_k$, hence $\frac{(a_{k+1}-1)a_{k+1}}{2} \geq a_1 + a_2 + \cdots + a_k$, which multiplied by $2a_{k+1}$, is equivalent to $(a_{k+1}^2 - a_{k+1})a_{k+1} \geq 2(a_1 + a_2 + \cdots + a_k)a_{k+1}$, that is, $a_{k+1}^3 \geq 2(a_1 + a_2 + \cdots + a_k)a_{k+1} + a_{k+1}^2$.
On the other hand, the induction hypothesis implies that

$$a_1^3 + a_2^3 + \cdots + a_k^3 \geq (a_1 + a_2 + \cdots + a_k)^2.$$

Adding the last two inequalities, we obtain

$$a_1^3 + a_2^3 + \cdots + a_k^3 + a_{k+1}^3 \geq (a_1 + a_2 + \cdots + a_k + a_{k+1})^2,$$

from where the inequality is true for $n = k+1$.
It is not difficult to deduce, from the previous proof, that the equality follows if and only if $a_{k+1} = a_k = 1$ and $a_1^3 + a_2^3 + \cdots + a_k^3 = (a_1 + a_2 + \cdots + a_k)^2$. In the last identity, the induction hypothesis implies that if the equality holds, then $a_1 = 1$, $a_2 = 2$, …, $a_k = k$. Therefore, $a_{k+1} = a_k + 1$ implies that $a_{k+1} = k+1$. That is, the sequence is $a_i = i$, for $i = 1, 2, \ldots, k+1$. Reciprocally, we have equality for $a_1 = 1$, $a_2 = 2$, …, $a_k = k$, $a_{k+1} = k+1$ thanks to the classical formula.

Solution 3.14. (i) The original inequality for $n = 1$ can be verified directly. For $n \geq 2$, it is enough to prove by induction that

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}\right)\cdots\left(1+\frac{1}{2^n}\right) \leq \frac{5}{2}\left(1-\frac{1}{2^n}\right).$$

For $n = 2$ we have the equality.
The inductive step is reduced to verifying that

$$\frac{5}{2}\left(1-\frac{1}{2^n}\right)\left(1+\frac{1}{2^{n+1}}\right) \leq \frac{5}{2}\left(1-\frac{1}{2^{n+1}}\right).$$

To see this, observe that the previous inequality is equivalent to

$$\begin{aligned}
\left(1-\frac{1}{2^n}\right)\left(1+\frac{1}{2^{n+1}}\right) &\leq 1-\frac{1}{2^{n+1}}\\
1-\frac{1}{2^n}+\frac{1}{2^{n+1}}-\frac{1}{2^{2n+1}} &\leq 1-\frac{1}{2^{n+1}}\\
\frac{2}{2^{n+1}} &\leq \frac{1}{2^n}+\frac{1}{2^{2n+1}}\\
\frac{1}{2^n} &\leq \frac{1}{2^n}\left(1+\frac{1}{2^{n+1}}\right),
\end{aligned}$$

which is clearly true.

(ii) As in the previous part, it will be easier to see that

$$\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\cdots\left(1+\frac{1}{n^3}\right) < 3-\frac{1}{n}.$$

For the inductive step, it is necessary to verify that

$$\left(3-\frac{1}{n}\right)\left(1+\frac{1}{(n+1)^3}\right) \leq 3-\frac{1}{n+1}.$$

Performing all the operations and simplifying leads to

$$\begin{aligned}
3-\frac{1}{n}+\frac{3}{(n+1)^3}-\frac{1}{n(n+1)^3} &\leq 3-\frac{1}{n+1}\\
\frac{3}{(n+1)^3}+\frac{1}{n+1} &\leq \frac{1}{n}\left(1+\frac{1}{(n+1)^3}\right)\\
\frac{1}{n+1}\left(\frac{3}{(n+1)^2}+1\right) &\leq \frac{1}{n}\left(1+\frac{1}{(n+1)^3}\right)\\
n(n^2+2n+4) &\leq n^3+3n^2+3n+2\\
0 &\leq n^2-n+2,
\end{aligned}$$

and the last inequality follows.

Solution 3.15. Define $P(n)$ as the statement we want to prove. If $n = 1$ the equality follows and then $P(1)$ is true. To see that $P(2)$ is valid, consider the following set of equivalences:

$$\begin{aligned}
&\frac{1}{1+a_1}+\frac{1}{1+a_2}\geq\frac{2}{1+\sqrt{a_1a_2}}\\
&\Leftrightarrow\quad (2+a_1+a_2)(1+\sqrt{a_1a_2})\geq 2(1+a_1)(1+a_2)\\
&\Leftrightarrow\quad 2\sqrt{a_1a_2}+(a_1+a_2)\sqrt{a_1a_2}\geq a_1+a_2+2a_1a_2\\
&\Leftrightarrow\quad 2\sqrt{a_1a_2}(1-\sqrt{a_1a_2})+(a_1+a_2)(\sqrt{a_1a_2}-1)\geq 0\\
&\Leftrightarrow\quad (\sqrt{a_1a_2}-1)(a_1-2\sqrt{a_1a_2}+a_2)\geq 0.
\end{aligned}$$

But the last inequality is true, since $\sqrt{a_1a_2} \geq 1$ and $(\sqrt{a_1}-\sqrt{a_2})^2 \geq 0$.

We now see that $P(2^n) \Rightarrow P(2^{n+1})$.

$$\begin{aligned}
\sum_{i=1}^{2^{n+1}}\frac{1}{1+a_i}&=\sum_{i=1}^{2^n}\frac{1}{1+a_i}+\sum_{i=2^n+1}^{2^{n+1}}\frac{1}{1+a_i}\\
&\geq\frac{2^n}{1+\sqrt[2^n]{a_1\cdots a_{2^n}}}+\frac{2^n}{1+\sqrt[2^n]{a_{2^n+1}\cdots a_{2^{n+1}}}}\\
&\geq 2^n\left(\frac{2}{1+\sqrt{\sqrt[2^n]{a_1\cdots a_{2^n}}\sqrt[2^n]{a_{2^n+1}\cdots a_{2^{n+1}}}}}\right)\\
&=\frac{2^{n+1}}{1+\sqrt[2^{n+1}]{a_1\cdots a_{2^{n+1}}}}.
\end{aligned}$$

For the first inequality we used $P(2^n)$ twice and for the second we applied $P(2)$ to the numbers $\sqrt[2^n]{a_1\cdots a_{2^n}}$ and $\sqrt[2^n]{a_{2^n+1}\cdots a_{2^{n+1}}}$.

Now, let us see that $P(n+1) \Rightarrow P(n)$.

If we apply $P(n+1)$ to the numbers $a_1, \ldots, a_n, a_{n+1} = \sqrt[n]{a_1\cdots a_n}$, we have that

$$\begin{aligned}
\frac{1}{1+a_1}+\cdots+\frac{1}{1+a_n}+\frac{1}{1+a_{n+1}}&\geq\frac{n+1}{1+\sqrt[n+1]{a_1\cdots a_n\sqrt[n]{a_1\cdots a_n}}}\\
&=\frac{n+1}{1+\sqrt[n+1]{(a_1\cdots a_n)^{1+\frac{1}{n}}}}\\
&=\frac{n+1}{1+a_{n+1}}.
\end{aligned}$$

Hence

$$\frac{1}{1+a_1}+\cdots+\frac{1}{1+a_n}\geq\frac{n}{1+a_{n+1}}=\frac{n}{1+\sqrt[n]{a_1\cdots a_n}}.$$

Solution 3.16. To prove (i) and (ii) apply the binomial Theorem 3.2.3 to $(1+1)^n$ and $(1+2)^n$, respectively.

Solution 3.17. To prove (i) just apply formula (3.5) and for (ii) take $r = 1$ in (i).

Solution 3.18. (i) In equation $(1+x)^n(1+x)^n = (1+x)^{2n}$, the coefficient of the term x^n on the right-hand side of the equation is $\binom{2n}{n}$. If we expand the left-hand side of the equation we find that

$$\left[\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{j}x^j + \cdots + \binom{n}{n}x^n\right] \\ \times \left[\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n\right].$$

Therefore the term x^n will appear when we multiply $\binom{n}{j}x^j$ and $\binom{n}{k}x^k$, with $j+k = n$, and then

$$\sum_{\substack{0\le j,k\le n\\ j+k=n}} \binom{n}{j}\binom{n}{k} = \binom{2n}{n}.$$

But

$$\sum_{\substack{0\le j,k\le n\\ j+k=n}} \binom{n}{j}\binom{n}{k} = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{n-j} = \sum_{j=0}^{n} \binom{n}{j}^2,$$

since $\binom{n}{j} = \binom{n}{n-j}$.

(ii) In the same way as in (i), it is enough to compare the coefficient of x^r on both sides of the identity

$$(1+x)^n(1+x)^m = (1+x)^{n+m}$$

which turns out to be

$$\sum_{\substack{0\le j,k\le r\\ j+k=r}} \binom{n}{k}\binom{m}{j} = \binom{n+m}{r}.$$

Then,

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}.$$

(iii) Changing $\binom{m}{0}$ by $\binom{m+1}{0}$ on the left-hand side of the equality and using Pascal's formula (3.6), we have that

$$\binom{m+1}{0}+\binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} \\ = \binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n}$$

$$
\begin{aligned}
&= \binom{m+3}{2} + \binom{m+3}{3} + \cdots + \binom{m+n}{n} \\
&= \qquad \vdots \\
&= \binom{m+n}{n-1} + \binom{m+n}{n} \\
&= \binom{m+n+1}{n}.
\end{aligned}
$$

(iv) Changing $\binom{m}{m}$ by $\binom{m+1}{m+1}$ on the left-hand side of the equality and using Pascal's formula (3.6), we obtain

$$
\begin{aligned}
\binom{m+1}{m+1} + \binom{m+1}{m} &+ \binom{m+2}{m} + \cdots + \binom{n}{m} \\
&= \binom{m+2}{m+1} + \binom{m+2}{m} + \cdots + \binom{n}{m} \\
&= \binom{m+3}{m+1} + \binom{m+3}{m} + \cdots + \binom{n}{m} \\
&= \qquad \vdots \\
&= \binom{n}{m+1} + \binom{n}{m} = \binom{n+1}{m+1}.
\end{aligned}
$$

Solution 3.19. (i) Using part (ii) of Exercise 3.17, it follows that

$$j\binom{n}{j} = n\binom{n-1}{j-1},$$

then

$$
\begin{aligned}
\sum_{j=1}^{n} j\binom{n}{j} &= \sum_{j=1}^{n} n\binom{n-1}{j-1} = n\sum_{j=1}^{n}\binom{n-1}{j-1} \\
&= n\sum_{j=0}^{n-1}\binom{n-1}{j} = n\cdot 2^{n-1}.
\end{aligned}
$$

(ii) By part (i) of Exercise 3.17, it follows that

$$\binom{n+1}{j+1} = \frac{n+1}{j+1}\binom{n}{j},$$

from where

$$\frac{1}{j+1}\binom{n}{j} = \frac{1}{n+1}\binom{n+1}{j+1}$$

hence,

$$\begin{aligned}
\sum_{j=0}^{n} \frac{1}{j+1}\binom{n}{j} &= \sum_{j=0}^{n} \frac{1}{n+1}\binom{n+1}{j+1} \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j+1} \\
&= \frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j} \\
&= \frac{1}{n+1}\left[\left(\sum_{j=0}^{n+1}\binom{n+1}{j}\right) - \binom{n+1}{0}\right] \\
&= \frac{1}{n+1}[2^{n+1}-1].
\end{aligned}$$

Solution 3.20. (i) First, using repeatedly Pascal's formula (3.6) and part (ii) of Exercise 3.17, it follows that

$$\begin{aligned}
\frac{1}{j}\binom{n}{j} &= \frac{1}{j}\left[\binom{n-1}{j} + \binom{n-1}{j-1}\right] = \frac{1}{j}\binom{n-1}{j} + \frac{1}{n}\binom{n}{j} \\
&= \frac{1}{j}\left[\binom{n-2}{j} + \binom{n-2}{j-1}\right] + \frac{1}{n}\binom{n}{j} \\
&= \frac{1}{j}\binom{n-2}{j} + \frac{1}{n-1}\binom{n-1}{j} + \frac{1}{n}\binom{n}{j},
\end{aligned}$$

and carrying on in this way, we obtain

$$\frac{1}{j}\binom{n}{j} = \frac{1}{n}\binom{n}{j} + \frac{1}{n-1}\binom{n-1}{j} + \cdots + \frac{1}{j}\binom{j}{j}.$$

Therefore

$$\sum_{j=1}^{n}(-1)^{j+1}\frac{1}{j}\binom{n}{j} = \sum_{j=1}^{n}(-1)^{j+1}\left[\sum_{i=0}^{n-j}\frac{1}{n-i}\binom{n-i}{j}\right].$$

If we change the sum's order, our previous identity changes to

$$\sum_{i=0}^{n-1}\left[\sum_{j=1}^{n-i}(-1)^{j+1}\frac{1}{n-i}\binom{n-i}{j}\right].$$

Set $k = n-i$, then the right-hand side of the identity becomes

$$\sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^{j+1}\frac{1}{k}\binom{k}{j} = \sum_{k=1}^{n}\frac{1}{k}\left[\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\right] = \sum_{k=1}^{n}\frac{1}{k}.$$

(ii) Observe that the left-hand side of the equality is

$$\begin{aligned}
&\frac{1}{1\cdot 2}\binom{n}{1} - \frac{1}{2\cdot 3}\binom{n}{2} + \cdots + \frac{(-1)^{n+1}}{n(n+1)}\binom{n}{n} \\
&\quad = \left(1-\frac{1}{2}\right)\binom{n}{1} - \left(\frac{1}{2}-\frac{1}{3}\right)\binom{n}{2} + \cdots + (-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)\binom{n}{n} \\
&\quad = \left[\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \cdots + (-1)^{n+1}\frac{1}{n}\binom{n}{n}\right] - \left[\frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \cdots \right. \\
&\qquad \left. + \frac{(-1)^{n+1}}{n+1}\binom{n}{n}\right].
\end{aligned} \tag{10.4}$$

Let U be equal to the terms inside the first square bracket. By the previous statement we have that $U = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, and for the terms inside the second square bracket we use $\frac{1}{j+1}\binom{n}{j} = \frac{1}{n+1}\binom{n+1}{j+1}$. Then the above identity (10.4) takes the form

$$\begin{aligned}
&\frac{1}{1\cdot 2}\binom{n}{1} - \frac{1}{2\cdot 3}\binom{n}{2} + \cdots + \frac{(-1)^{n+1}}{n(n+1)}\binom{n}{n} \\
&\quad = U - \frac{1}{n+1}\left[\binom{n+1}{2} - \binom{n+1}{3} + \cdots + (-1)^{n+1}\binom{n+1}{n+1}\right] \\
&\quad = U - \frac{1}{n+1}\left[\binom{n+1}{0} - \binom{n+1}{1} + \binom{n+1}{2} + \cdots \right. \\
&\qquad\qquad \left. +(-1)^{n+1}\binom{n+1}{n+1} - (1-(n+1))\right] \\
&\quad = U - \frac{1}{n+1}[n] = U - \frac{n}{n+1} \\
&\quad = U - \frac{n+1-1}{n+1} = \frac{1}{2} + \cdots + \frac{1}{n+1}.
\end{aligned}$$

(iii) Call T the left-hand side of the equality, then

$$T = \frac{1}{1^2}\binom{n}{0} - \frac{1}{2^2}\binom{n}{1} + \cdots + \frac{(-1)^n}{(n+1)^2}\binom{n}{n}.$$

Multiplying by $n+1$, we have

$$(n+1)T = \frac{n+1}{1^2}\binom{n}{0} - \frac{n+1}{2^2}\binom{n}{1} + \cdots + (-1)^n\frac{n+1}{(n+1)^2}\binom{n}{n}.$$

Using the fact that $\frac{n+1}{j+1}\binom{n}{j} = \binom{n+1}{j+1}$, our identity becomes

$$\begin{aligned}
(n+1)T &= \frac{1}{1}\binom{n+1}{1} - \frac{1}{2}\binom{n+1}{2} + \cdots + (-1)^n\frac{1}{n+1}\binom{n+1}{n+1} \\
&= 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}.
\end{aligned}$$

Therefore, the sum we are looking for is $T = \frac{1}{n+1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right)$.

Solution 3.21. (i) Using the equality $\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}$ leads to the set of identities

$$\sum_{j=1}^{n}(-1)^j j\binom{n}{j} = \sum_{j=1}^{n}(-1)^j n\binom{n-1}{j-1} = n\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1}$$
$$= n\sum_{j=0}^{n-1}(-1)^{j+1}\binom{n-1}{j}$$

and using Example 3.2.4, we reach the result

$$n\sum_{j=0}^{n-1}(-1)^{j+1}\binom{n-1}{j} = n\cdot 0 = 0.$$

(ii) Again, using the equality $\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}$, it follows that

$$\sum_{j=1}^{n}(-1)^j j^2\binom{n}{j} = \sum_{j=1}^{n}(-1)^j n\cdot j\binom{n-1}{j-1} = n\sum_{j=1}^{n}(-1)^j j\binom{n-1}{j-1}$$
$$= n\sum_{j=1}^{n}(-1)^j (j-1)\binom{n-1}{j-1} + n\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1},$$
$$= n\sum_{j=2}^{n}(-1)^j (j-1)\binom{n-1}{j-1} + n\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1},$$

since in the above identity the first term of the first sum is zero.

Finally, by Example 3.2.4, and using the previous part of this exercise, we find

$$n\sum_{j=2}^{n}(-1)^j (j-1)\binom{n-1}{j-1} + n\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1} = n\cdot 0 + n\cdot 0 = 0.$$

Solution 3.22. Consider the following array, where in the mth row, for $m \geq 0$, we have the binomial coefficients $\binom{m}{j}$ modulo 2. On the left column is shown the number of the row written in base 2 and in the right column the number of odd binomial coefficients written in base 2.

$$\begin{array}{lccccccccccccccccl}
0 & & & & & & & & 1 & & & & & & & & & 2^0\\
1 & & & & & & & 1 & & 1 & & & & & & & & 2^1\\
2 & & & & & & 1 & & 0 & & 1 & & & & & & & 2^1\\
3 = 2 + 2^0 & & & & & 1 & & 1 & & 1 & & 1 & & & & & & 2^2\\
4 = 2^2 & & & & 1 & & 0 & & 0 & & 0 & & 1 & & & & & 2^1\\
5 = 2^2 + 2^0 & & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & & & 2^2\\
6 = 2^2 + 2^1 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & & 2^2\\
7 = 2^2 + 2^1 + 2^0 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 2^3
\end{array}$$

This array suggests to us that the number of odd binomial coefficients will be 2^k, with k the number of non-zero digits when we write n in base 2.

Note that if $n = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_r}$, with $\alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0$, then

$$\begin{aligned}(1+x)^n &= (1+x)^{2^{\alpha_1}} \cdots (1+x)^{2^{\alpha_r}} \\ &\equiv (1+x^{2^{\alpha_1}}) \cdots (1+x^{2^{\alpha_r}}) \mod 2.\end{aligned}$$

The previous identity follows because if we expand the binomial of the form $(1+x)^{2a}$ the first and the last binomial coefficients are 1 and the rest are even.

It is clear that if we expand $(1+x^{2^{\alpha_1}}) \cdots (1+x^{2^{\alpha_r}})$, there are 2^r terms.

Solution 3.23. The proof is based on the identity $\sum_{j=0}^{n} \binom{p}{j}^2 = \binom{2p}{p}$, as seen in Exercise 3.18 part (i).
Since $\binom{p}{j}$ is divisible by p for all $j = 1, 2, \ldots, p-1$, each term of the sum is divisible by p^2, with exception of the first and the last, which are equal to 1. Therefore, p^2 divides $\binom{2p}{p} - 2$. To finish the proof, observe that

$$\binom{2p-1}{p-1} - 1 = \frac{1}{2}\left(\binom{2p}{p} - 2\right).$$

Solution 3.24. For $p = 2$, we have that $2^1 = 1^3 + 1^3$ and for $p = 3$, we get $3^2 = 1^3 + 2^3$.
Let us see now that there is no prime number $p > 3$, for which there exist a, b and n such that $a^3 + b^3 = p^n$.

Suppose that we can find such numbers and that n is the smallest integer number that fulfills the conditions of the problem. Since $p \geq 5$, one of the numbers a or b is greater than 1, then $a^3 + b^3 \geq 5$. Since

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

and $a^2 - ab + b^2 = (a-b)^2 + ab \geq 2$, then p must divide $a+b$ and $a^2 - ab + b^2$. But then, p divides

$$(a+b)^2 - (a^2 - ab + b^2) = 3ab.$$

Since $p \geq 5$, p has to divide a or b, but since $p | a+b$, it follows that p divides a and it also divides b. Then, $a^3 + b^3 \geq 2p^3$, hence $n > 3$. Since

$$p^{n-3} = \frac{p^n}{p^3} = \frac{a^3+b^3}{p^3} = \left(\frac{a}{p}\right)^3 + \left(\frac{b}{p}\right)^3,$$

it follows that $n-3$ also satisfies the condition; then n is not a minimum.

Solution 3.25. Consider the equation $x^2 + y^2 + z^2 = 2xyz$. The left-hand side of the equation has exactly one even term or all three terms are even. If exactly one term is even, then the right-hand side of the equation is divisible by 4 and the left-hand side is divisible only by 2, so we have a contradiction. Then all terms are even, that is, $x = 2x_1$, $y = 2y_1$, $z = 2z_1$ and

$$x_1^2 + y_1^2 + z_1^2 = 4x_1y_1z_1. \tag{10.5}$$

From equation (10.5), following the same reasoning leads to $x_1 = 2x_2$, $y_1 = 2y_2$, $z_1 = 2z_2$ and

$$x_2^2 + y_2^2 + z_2^2 = 8x_2y_2z_2. \tag{10.6}$$

Again from equation (10.6), it follows that x_2, y_2, z_2 are even, and so on and so forth. Then

$$\begin{aligned}
x &= 2x_1 = 2^2x_2 = 2^3x_3 = \cdots = 2^nx_n = \cdots,\\
y &= 2y_1 = 2^2y_2 = 2^3y_3 = \cdots = 2^ny_n = \cdots,\\
z &= 2z_1 = 2^2z_2 = 2^3z_3 = \cdots = 2^nz_n = \cdots,
\end{aligned}$$

that is, if (x, y, z) is a solution, then x, y and z are divisible by 2^n for all n. This is impossible, unless $x = y = z = 0$.

Solution 3.26. The solutions are $a = b = 1$ or a and b consecutive square numbers.

We can write the divisibility condition as

$$k(ab + a + b) = a^2 + b^2 + 1, \tag{10.7}$$

for some integer number k. If $k = 1$, then the equation (10.7), is equivalent to $(a - b)^2 + (a - 1)^2 + (b - 1)^2 = 0$, from where $a = b = 1$. If $k = 2$, then equation (10.7) can be written as $4a = (b - a - 1)^2$, from where we deduce that a has to be a square number, that is, $a = d^2$. Then $b - d^2 - 1 = \pm 2d$, that is, $b = \pm(d \pm 1)^2$ and, a and b are consecutive square numbers.

Suppose now that $k \geq 3$, and let (a, b) be a solution with a the minimum and $a \leq b$. Write equation (10.7) as a quadratic equation in b,

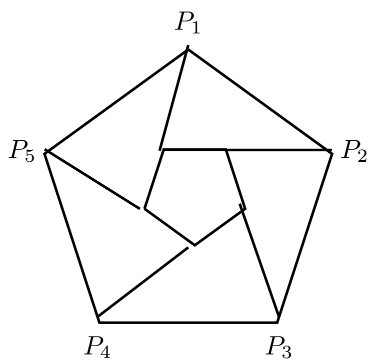
$$b^2 - k(a + 1)b + (a^2 - ka + 1) = 0.$$

Since root b is an integer, the other root r satisfies $b + r = k(a + 1)$ and it is also an integer. Since equation (10.7) has to be true if we substitute b by r, note that $k(ar + a + r) = a^2 + r^2 + 1 > 0$ implies $ar + a + r > 0$, and then we can conclude that $r > 0$. And since $a \leq b$ and the product of the roots $a^2 - ka + 1$ is less than a^2, we have $r < a$. But (r, a) is also a solution of (10.7), which contradicts a being a minimum.

Solution 3.27. First we prove that we cannot find an equilateral triangle such that the vertices are points with integer coordinates. Suppose we can find such a triangle. Let a be the length of the sides of the triangle such that the vertices are points with integer coordinates. The area of the triangle is $a^2\frac{\sqrt{3}}{4}$ which is an irrational number, since a^2 is an integer number. On the other hand, the area of any polygon whose vertices are points with integer coordinates is a rational number[26].

The vertices of a regular hexagon $P_1P_2P_3P_4P_5P_6$ cannot be points with integer coordinates, since $P_1P_3P_5$ would be an equilateral triangle whose vertices have integer coordinates.

Let $n \neq 3, 4, 6$. Suppose that $P_1P_2P_3 \ldots P_n$ is a regular n-gon with vertices having integer coordinates. Through the points P_1, P_2, $\ldots$, P_n draw the parallels to $\overrightarrow{P_2P_3}$, $\overrightarrow{P_3P_4}$, $\ldots$, $\overrightarrow{P_1P_2}$, respectively, as shown in the figure.



The points of intersection of the parallels are also points with integer coordinates and form a regular n-gon inside the first one. With the new n-gon we can proceed in the same form. This process can continue to generate an infinite number of n-gons. The square of the length of the sides of these polygons are integers that decrease in each step, but this is impossible.

Solution 3.28. Error. The given arguments assume that the set has at least 3 elements, and we use that a_1, a_n, a_{n+1} are different. We can say that the statement $P_1 \Rightarrow P_2$ is not valid.

Solution 3.29. Error. Statement $\mathcal{P}(n)$ is: for n coins among which one is false, it is enough to weigh 4 times to identify the false coin. When we take out one coin there are two cases: (a) the coin that we took out is genuine, (b) the coin we took out is false.

In the first case, the inductive step works, but in the second case it does not, because among the coins that remain we do not have a false coin.

Solution 3.30. Error. Statement $\mathcal{P}(0)$ implies $\mathcal{P}(1)$ is false.

[26] See [13].

10.4 Solutions to exercises of Chapter 4

Solution 4.1. The equation can be written as $\left(m - \frac{3}{2}\right)^2 - \frac{5}{4} = \left(n + \frac{1}{2}\right)^2 - \frac{5}{4}$ or $\left(m - \frac{3}{2}\right)^2 - \left(n + \frac{1}{2}\right)^2 = 0$, that is, $(m+n-1)(m-n-2) = 0$. Since m and n are positive integers, then $m + n - 1 > 0$, therefore m and n are solutions of the equation if and only if $m - n - 2 = 0$, that is, the solutions are $(m, n) = (a+2, a)$, where a is any positive integer.

Solution 4.2. Since $P(-1) = a - b + c$, $P(0) = c$ and $P(1) = a + b + c$ are integers, we have that $2a$, $2b$ y c are integers.

For $n = 2m$, with m an integer number, we have that $P(n) = P(2m) = a(4m^2)+b(2m)+c = (2m^2)(2a)+m(2b)+c$ is an integer, and for $n = 2m+1$, with m an integer number, we have that $P(n) = P(2m+1) = a(2m+1)^2+b(2m+1)+c = (2m^2 + 2m)(2a) + (2a) + m(2b) + (a + b + c)$ is also an integer.

Solution 4.3. If $\frac{p}{q}$ is a solution of $ax^2 + bx + c = 0$, it follows, multiplying by q^2, that $ap^2 + bpq + cq^2 = 0$. Then $p \mid cq^2$ and $q \mid ap^2$, but since $(p, q) = 1$, it follows that $p \mid c$ and $q \mid a$.

Solution 4.4. Note that $cx^2 + bx + a = x^2(c + b\frac{1}{x} + a\frac{1}{x^2})$, then the roots are the inverse of the roots of $ax^2 + bx + c$, therefore

$$(\alpha + \beta)(\alpha' + \beta') = (\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \geq 4.$$

Another way, by Vieta's formulas (4.1), $\alpha + \beta = -\frac{b}{a}$ and $\alpha' + \beta' = -\frac{b}{c}$, then $(\alpha + \beta)(\alpha' + \beta') = \frac{b^2}{ac}$. But $b^2 - 4ac \geq 0$, as they are real roots and since $a = \alpha\beta$, $c = \alpha'\beta'$ are positive.

Solution 4.5. Let x_1, x_2 be the zeros of $P(x)$. Then, by Vieta's formulas (4.1), we have $x_1 + x_2 = a - 2$ and $x_1x_2 = -a - 1$. Substitute in the identity $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2$ the values of the sum and the product of x_1 and x_2 to obtain $(a-2)^2 + 2(a+1) = a^2 - 2a + 6 = (a-1)^2 + 5 \geq 5$, with equality for $a = 1$. Then, $a = 1$ is the only number.

Solution 4.6. Observe that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{qr+pr+pq}{pqr}$. By Vieta's formula (4.2), $qr + pr + pq = 3$ and $pqr = -1$. Then, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = -3$.

Solution 4.7. Since p, q and r are roots of the given cubic equation, by Vieta's formula (4.2), it follows that $p + q + r = -b$ and $pq + qr + rp = c$.
Since $(p+q+r)^2 = p^2+q^2+r^2+2(pq+qr+rp)$, we obtain $(-b)^2 = p^2+q^2+r^2+2c$, and rearranging terms $b^2 - 2c = p^2 + q^2 + r^2$.
Therefore, a quadratic equation with the desired roots is

$$(x - (-b))(x - (b^2 - 2c)) = x^2 + (b - b^2 + 2c)x + (2bc - b^3) = 0.$$

Solution 4.8. Since x_1 and x_2 are solutions to equation $x^2 - \left(\frac{2m-1}{2}\right)x + \frac{m^2-3}{2} = 0$, it follows that $x_1 + x_2 = \frac{2m-1}{2}$ and $x_1 x_2 = \frac{m^2-3}{2}$. We need $x_1 = x_2 - \frac{1}{2}$, then it is necessary that $\left(x_2 - \frac{1}{2}\right) + x_2 = 2x_2 - \frac{1}{2} = \frac{2m-1}{2}$, that is, $x_2 = \frac{m}{2}$. Similarly, we want to have $\left(x_2 - \frac{1}{2}\right)x_2 = \frac{m^2-3}{2}$, then $x_2^2 - \frac{x_2}{2} = \frac{m^2}{2} - \frac{3}{2}$. Hence, substituting $x_2 = \frac{m}{2}$, we get $\frac{m^2}{4} - \frac{m}{4} = \frac{m^2}{2} - \frac{3}{2}$, that is $(m+3)(m-2) = 0$. Then, it follows that $m = -3$ or $m = 2$. Since m has to be positive, then $m = 2$. Therefore, $x_1 = \frac{1}{2}$ and $x_2 = 1$.

Solution 4.9. Since $P(x^2+1) = x^4 + 4x^2$, the polynomial $P(x)$ has to be of degree 2 and it is monic, that is, $P(x) = x^2 + bx + c$, then

$$(x^2+1)^2 + b(x^2+1) + c = x^4 + 4x^2.$$

Expand the equation to obtain $x^4 + 2x^2 + 1 + bx^2 + b + c = x^4 + 4x^2$, hence $2 + b = 4$ and $1 + b + c = 0$, then $b = 2$, $c = -3$. Substitution leads to $P(x) = x^2 + 2x - 3$, therefore $P(x^2-1) = (x^2-1)^2 + 2(x^2-1) - 3 = x^4 - 2x^2 + 1 + 2x^2 - 2 - 3 = x^4 - 4$.

Solution 4.10. Note that $a^3 + b^3 = c^3 + d^3$ if and only if $(a+b)(a^2 - ab + b^2) = (c+d)(c^2 - cd + d^2)$, but since $a + b = c + d \neq 0$, we can cancel these factors and obtain $a^2 - ab + b^2 = c^2 - cd + d^2$. On the other hand, from $a + b = c + d$ we get, squaring this equality, that $a^2 + 2ab + b^2 = c^2 + 2cd + d^2$. Subtracting these two last identities, it follows that $ab = cd$. Then, the two quadratic polynomials $x^2 - (a+b)x + ab$ and $x^2 - (c+d)x + cd$ coincide, in particular their roots are the same. But, by Vieta's formula (4.1) we know that the roots of the polynomials are $\{a, b\}$ and $\{c, d\}$, respectively. Therefore, $\{a, b\} = \{c, d\}$.

Solution 4.11. The polynomial has equal roots if the discriminant is zero, that is, $4 - 4\lambda\left(1 - \frac{1}{\lambda}\right) = 0$, then $\lambda\left(1 - \frac{1}{\lambda}\right) = 1$. Thus $\lambda = 2$ is the only possibility.

Solution 4.12. It is not possible. Otherwise, if the three polynomials had two real roots, the discriminants, $b^2 - 4ac$ and $c^2 - 4ab$, $a^2 - 4bc$ would be positive. Hence, $b^2 > 4ac$, $c^2 > 4ab$, $a^2 > 4bc$, and multiplying the inequalities we would have $a^2b^2c^2 > 64a^2b^2c^2$, which is false.

Solution 4.13. The solutions of the equation are given by

$$x = \frac{1 - 2k \pm \sqrt{(1-2k)^2 - 4k(k-2)}}{2k} = \frac{1 - 2k \pm \sqrt{1+4k}}{2k}.$$

The number x will be rational if $1 + 4k$ is a perfect square, that is, k has to be an integer of the form $k = \frac{n^2-1}{4}$, with n a positive integer. Since we want k to be an integer, $n^2 - 1$ has to be divisible by 4, but $n^2 - 1 = (n+1)(n-1)$ is divisible by 4 if and only if n is odd. Then, for $k = \frac{n^2-1}{4}$, with n odd, the roots of $kx^2 - (1-2k)x + k - 2 = 0$ are rational numbers.

Solution 4.14. If $a+b$ is a root of the polynomial $P(x) = x^2 + ax + b$, then $0 = (a+b)^2 + a(a+b) + b = b^2 + (3a+1)b + 2a^2$, and so b has to be a root of the polynomial $Q(x) = x^2 + (3a+1)x + 2a^2$. But $Q(x)$ will have an integer root if its discriminant $(3a+1)^2 - 4 \cdot 2a^2 = (a+3)^2 - 8$ is a perfect square. But two square numbers have difference 8 if and only if they are 1 and 9, then $(a+3)^2 = 9$, therefore $a = -6$ or $a = 0$. If $a = -6$, then $b = 8$ or 9 and if $a = 0$ then $b = 0$ or -1.

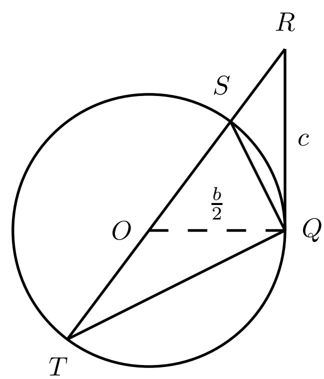
Hence, the only possible pairs (a, b) are: $(-6, 8)$, $(-6, 9)$, $(0, 0)$ and $(0, -1)$.

Solution 4.15. We want to solve equation $x^2 - 5x - 1 = n^2$, that is, $x^2 - 5x - (n^2+1) = 0$. The solutions of the equation are given by

$$x_{1,2} = \frac{5 \pm \sqrt{25 + 4n^2 + 4}}{2}. \tag{10.8}$$

For x to be an integer number it is necessary that $4n^2 + 29 = t^2$. Then, $t^2 - 4n^2 = (t-2n)(t+2n) = 29$, that is, $t + 2n = \pm 29$ and $t - 2n = \pm 1$ or $t + 2n = \pm 1$ and $t - 2n = \pm 29$. Solving these equations, it follows that $4n = \pm 28$, then $n = 7$ and $t = 15$ or $n = -7$ and $t = -15$. Substituting n in equation (10.8), we obtain $x_1 = 10$ and $x_2 = -5$.

Solution 4.16. We present the solution due to R. Descartes. Using Vieta's formulas (4.1), we find that if α and β are the polynomial roots, then $\alpha + \beta = -b$ and $\alpha\beta = -c^2$, then there is a negative root.



Consider the triangles RQS and RTQ, which are similar, since they share the angle of the vertex R and $\angle RTQ = \angle RQS$. Then, we have that $RT \cdot RS = RQ^2 = c^2$, therefore, $RS \cdot (-RT) = -c^2$. On the other hand, since $RT = RS + b$, we have $RS + (-RT) = -b$. Then, $-RT$ and RS satisfy Vieta's relations, therefore those numbers are the roots of the equation.

Solution 4.17. If $P(x) = x$ does not have real solutions, then $P(x) > x$, for all x or $P(x) < x$, for all x. Hence, $P(P(x)) > P(x) > x$ or $P(P(x)) < P(x) < x$, for all x, therefore it is impossible to have $P(P(x)) = x$.

Solution 4.18. If $P(P(P(x_0))) = P(x_0) = 0$ for some x_0, then

$$P(P(0)) = P(P(P(x_0))) = 0.$$

Hence, $P(0)$ is a zero of $P(x)$ and it is an integer because $P(x)$ is a polynomial with integer coefficients. Moreover, $P(P(P(P(0)))) = P(P(0)) = 0$, then $P(0)$ is also a root of $P(P(P(x)))$.

Solution 4.19. Note that $ax^2+bx+c > cx$ is equivalent to $P(x) = ax^2+(b-c)x+c > 0$; this guarantees that $a > 0$, $c = P(0) > 0$ and $(b-c)^2 - 4ac < 0$. On the other hand, to prove that $cx^2 - bx + a > cx - b$ is equivalent to proving that $cx^2 - (b+c)x + a + b > 0$, but since $c > 0$, it is enough that the discriminant be negative; such discriminant is $(b+c)^2 - 4c(a+b) = (b-c)^2 - 4ac$, but we have already proved that it is negative.

Solution 4.20. Suppose that $20(b-a)$ is an integer number. By symmetry we can also suppose that $b > a$, and then $20(b-a) \geq 1$. Since there are no real solutions, the discriminant of the polynomial $x^2 + 20bx + 10a$ is negative, therefore $10b^2 - a < 0$. Then, we have $10b^2 < a < b$ and $b < \frac{1}{10}$. Hence $0 < b - a < b < \frac{1}{10}$ and, then $20(b-a) < 2$, but if $20(b-a)$ is an integer number we have $20(b-a) = 1$ and then $b = a + \frac{1}{20}$. Thus, $10b^2 - a = 10(a + \frac{1}{20})^2 - a = 10a^2 + \frac{1}{40} > 0$, which is a contradiction. Therefore, $20(b-a)$ can never be an integer number.

Solution 4.21. If $P(x) = x^2 + bx + c$ satisfies $P(P(P(x_0))) = P(x_0) = 0$, for some x_0, then $P(P(0)) = P(P(P(x_0))) = 0$. Therefore,

$$0 = P(P(0)) = P(c) = c^2 + bc + c = c(c+b+1) = P(0)P(1).$$

Solution 4.22. Expand the following polynomial and use the relation $ab+ac+bc = de+df+ef$, to get that

$$\begin{aligned}
(x+a)(x+b)(x+c) - (x-d)(x-e)(x-f)& \\
&= x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc \\
&\qquad - x^3 + (d+e+f)x^2 - (de+df+ef)x + def \\
&= Nx^2 + abc + def.
\end{aligned}$$

Then, if we let $x = d$, the above expression becomes

$$(d+a)(d+b)(d+c) = Nd^2 + (abc + def).$$

Then, if N divides $abc+def$, then N divides $(d+a)(d+b)(d+c)$. Let p be a prime number such that p divides N, then p divides at least one of the factors $d+a$, $d+b$ or $d+c$. Then, $p \leq \max(d+a, d+b, d+c) < N$, that is, p is a factor of N and N is a composite number.

Solution 4.23. Since $P(x)$ and $Q(x)$ have integer coefficients, we can divide by the main coefficient and assume that $P(x) = (x-\alpha)(x-r)$ and $Q(x) = (x-\alpha)(x-s)$, where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, that is, $P(x)$ and $Q(x)$ are monic polynomials with rational coefficients. In a quadratic polynomial, if one root is an irrational number the other root is also irrational, since the sum of both roots has to be a rational number. Then, $r, s \in \mathbb{R}\backslash\mathbb{Q}$.

Note that $\alpha + r$, αr, $\alpha + s$ and αs are all rational numbers since they are the coefficients of $P(x)$ and $Q(x)$. Then

$$(\alpha + r) - (\alpha + s) = r - s \in \mathbb{Q}$$
$$\frac{\alpha r}{\alpha s} = \frac{r}{s} \in \mathbb{Q}.$$

Let $r - s = \frac{p}{q}$ and $\frac{r}{s} = \frac{m}{n}$. Solving for r from the second equation, it follows that $r = \frac{m}{n} \cdot s$, from where we get $\frac{m}{n} \cdot s - s = \frac{p}{q}$, that is, $s\left(\frac{m}{n} - 1\right) = \frac{p}{q}$. If $\frac{m}{n} - 1 \neq 0$, it follows that

$$s = \frac{\frac{p}{q}}{\frac{m}{n} - 1} \in \mathbb{Q}$$

which is a contradiction, then $\frac{m}{n} = 1$. Thus, $r = s$.

Solution 4.24. Multiplying both equations, it follows that

$$x^4 - (b_1 + b_2)x^3 + (c_1 + c_2 + b_1b_2)x^2 - (b_1c_2 + b_2c_1)x + c_1c_2 = 0.$$

On the other hand, since b_1, c_1, b_2 and c_2 are roots, we have by, Vieta's formulas (4.1.1) and equating the coefficients, that

$$\begin{aligned} b_1 + b_2 + c_1 + c_2 &= b_1 + b_2 \\ b_1b_2 + b_1c_1 + b_1c_2 + b_2c_1 + b_2c_2 + c_1c_2 &= c_1 + c_2 + b_1b_2 \\ b_1b_2c_1 + b_1b_2c_2 + b_1c_1c_2 + b_2c_1c_2 &= b_1c_2 + c_1b_2 \\ b_1b_2c_1c_2 &= c_1c_2. \end{aligned}$$

From the first equation, we obtain $c_1 = -c_2$, then from the second equation it follows that $c_1c_2 = c_1 + c_2 = 0$, from where we get $c_1 = c_2 = 0$, which contradicts the fact that c_1 and c_2 are different numbers. Hence, those polynomials do not exist.

Solution 4.25. Since $(a - b) + (b - c) + (c - a) = 0$, then some of the terms of the sum $a - b$, $b - c$ or $c - a$ are less than or equal to zero. Suppose, without loss of generality, that $a - b \leq 0$. Then, the discriminant of the third equation is $(c-a)^2 - 4(a-b) \geq 0$, that is, the third equation has a zero that is a real number.

Solution 4.26. Let $P(x) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc$ be the monic polynomial with zeros a, b and c. Let $A = ab + bc + ca$, $B = abc$ and

$T_n = a^n + b^n + c^n$. Then, $T_0 = 3$, $T_1 = 0$, $T_2 = (a+b+c)^2 - 2(ab+bc+ca) = -2A$. The equation $P(x) = 0$ is equivalent to $x^3 = -Ax + B$, from where $x^{n+3} = -Ax^{n+1} + Bx^n$. Then, it follows that $T_{n+3} = -AT_{n+1} + BT_n$. Now find T_3, T_4 and T_5.

Solution 4.27. Using $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$, we get $ab+bc+ca = 2$. Using the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$, we get $abc = -\frac{2}{3}$. Then, the cubic polynomial with roots a, b and c is $P(x) = x^3-3x^2+2x+\frac{2}{3}$; now, from $aP(a)+bP(b)+cP(c) = 0$, it follows that $a^4+b^4+c^4 = 9$.

Solution 4.28. Each of the equations $ax^2+bx+c = 0$ and $cx^2+bx+a = 0$ have two different real solutions, $a \neq 0$ and $c \neq 0$. Moreover, r is the root of $ax^2+bx+c = 0$ if and only if $\frac{1}{r}$ is a root of $cx^2 + bx + a = 0$. Therefore, $\{q_1, q_2\} = \left\{\frac{1}{p_1}, \frac{1}{p_2}\right\}$.

If p_1, q_1, p_2, q_2 are in arithmetic progression,

$$|p_1 - p_2| = |q_1 - q_2| = \left|\frac{1}{p_1} - \frac{1}{p_2}\right| = \frac{|p_1 - p_2|}{|p_1p_2|},$$

from where $|p_1p_2| = 1$.

Using Vieta's formula (4.1), we have $p_1p_2 = \frac{c}{a}$, so $|c| = |a|$ and then $a = \pm c$. If $a = c$, the two given quadratic equations are equal, and then $p_1 = q_1$, $p_2 = q_2$, which tell us that the difference of the progression is 0. Then, $p_1 = q_1 = p_2 = q_2$ which is a contradiction. Therefore, $a = -c$ or $a + c = 0$.

Solution 4.29. Let $T_n = a^n + b^n + c^n$ for each integer number n, then $T_0 = 3$, $T_1 = 0$ and $T_2 = (a + b + c)^2 - 2(ab + bc + ca) = -2(ab + bc + ca)$. Now, define $A = ab + bc + ca$ and $B = abc$; then, by Vieta's formulas (4.2), it follows that a, b and c are the roots of the equation $x^3 + Ax - B = 0$ and $T_2 = -2A$.
For $n \geq 0$, we substitute a, b and c in $x^{n+3} = -Ax^{n+1} + Bx^n$ and adding we obtain $T_{n+3} = -AT_{n+1} + BT_n$. Then

$$\begin{aligned} T_3 &= -AT_1 + BT_0 = 3B, \\ T_4 &= -AT_2 + BT_1 = 2A^2, \\ T_5 &= -AT_3 + BT_2 = -5AB. \end{aligned}$$

Hence, $\frac{T_5}{5} = -AB = \frac{T_3}{3} \cdot \frac{T_2}{2}$. Since $T_3 = T_5$, the last equality implies that $T_2 = \frac{6}{5}$.

Solution 4.30. Let $Q(x) = P(x) - 2$, since a, b and c are the roots of $Q(x)$, it is clear that $Q(x) = \alpha(x - a)(x - b)(x - c)$, for some integer number α. If for some integer number d we have $P(d) = 3$; then, since $1 = P(d) - 2 = Q(d) = \alpha(d-a)(d-b)(d-c)$, the factors on the right-hand side of the equation have to be -1 or 1, then two of $d-a$, $d-b$, $d-c$ are equal, so a, b, c are not different, which is a contradiction. This guarantees that there does not exist an integer number d with $P(d) = 3$.

10.5 Solutions to exercises of Chapter 5

Solution 5.1. For (i) and (ii) apply directly the definition. The proof of (iii) and (iv) is straightforward once you do the operations. To prove (v), after squaring both sides, observe that $z\bar{w} + w\bar{z} \leq 2|z\bar{w}|$. (vi) Do the operations using $|z|^2 = z\bar{z}$.

Solution 5.2. Let $z = x + iy$, then $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$. Hence, $\mathrm{Im}\big(z + \frac{1}{z}\big) = y + \frac{-y}{x^2+y^2} = 0$ is equivalent to $y(x^2 + y^2 - 1) = 0$, with solutions $y = 0$ or $x^2 + y^2 = 1$. The solution $y = 0$ means that the real axis satisfies the original equation and the solution $x^2 + y^2 = 1$ is the unit circle.

Second Solution. Write z in its polar form, that is, $z = r(\cos\theta + i\sin\theta)$. Then, $\frac{1}{z} = \frac{1}{r}(\cos(-\theta)+i\sin(-\theta)) = \frac{1}{r}(\cos\theta - i\sin\theta)$. Thus, $\mathrm{Im}\big(z + \frac{1}{z}\big) = (r - \frac{1}{r})\sin\theta = 0$, and then $r - \frac{1}{r} = 0$ or $\sin\theta = 0$. Equation $r - \frac{1}{r} = 0$ implies that $r = 1$, that is, the complex numbers that satisfy the equation are the ones on the unit circle; meanwhile the complex numbers z that satisfy $\sin\theta = 0$ are the complex numbers with argument 0 or π, that is, all the real numbers.

Third Solution. Observe that

$$\begin{aligned}
\mathrm{Im}\left(z + \frac{1}{z}\right) = 0 \quad &\Leftrightarrow \quad z + \frac{1}{z} = \bar{z} + \frac{1}{\bar{z}} \\
&\Leftrightarrow \quad (z\bar{z} - 1)(z - \bar{z}) = 0 \\
&\Leftrightarrow \quad |z| = 1 \;\text{ or }\; z = \bar{z} \\
&\Leftrightarrow \quad z \text{ is on the unit circle or the real axis.}
\end{aligned}$$

Solution 5.3. Use the fact that $z\bar{z} = |z|^2$ for every complex number z, and see that

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w, \\
|z - w|^2 &= (z - w)\overline{(z - w)} = z\bar{z} + w\bar{w} - z\bar{w} - \bar{z}w.
\end{aligned}$$

Then, $|z + w| = |z - w|$ if and only if $z\bar{w} + \bar{z}w = -z\bar{w} - \bar{z}w$, that is, $2(z\bar{w} + \bar{z}w) = 0$. Using the fact that $\mathrm{Re}\; z = \frac{z+\bar{z}}{2}$, we obtain $2(z\bar{w} + \bar{z}w) = 4\;\mathrm{Re}\;\bar{w}z = 0$. Since $\bar{w} = \frac{|w|^2}{w}$, then $\bar{w}z = \frac{|w|^2}{w}z$, hence $\mathrm{Re}\;\bar{w}z = \mathrm{Re}\;\frac{z}{w} = 0$. Therefore, $\frac{z}{w}$ is purely imaginary, that is, $\frac{z}{w} = ir$ for some $r \in \mathbb{R}$. Thus, $\frac{iz}{w} = -r$, which is a real number.

Second Solution. A geometric proof is the following. Since $w \neq 0$, we can divide the original equation by w, to obtain

$$\left|\frac{z}{w} + 1\right| = \left|\frac{z}{w} - 1\right|,$$

which means that $\frac{z}{w}$ is in the perpendicular bisector of the segment that joins 1 and -1, that is, the imaginary axis, then $\frac{z}{w}$ is purely imaginary and now we can conclude as above.

Solution 5.4. (i) Use the fact that $z\bar{z} = |z|^2$. The left-hand side of the identity is

$$\begin{aligned}|1-\bar{z}w|^2 - |z-w|^2 &= (1-\bar{z}w)(1-z\bar{w}) - (z-w)(\bar{z}-\bar{w}) \\ &= 1 + |z|^2|w|^2 - \bar{z}w - z\bar{w} - |z|^2 - |w|^2 + z\bar{w} + w\bar{z} \\ &= 1 + |z|^2|w|^2 - |z|^2 - |w|^2 .\end{aligned}$$

In the same way, the right-hand side of the identity is

$$\begin{aligned}(1+|zw|)^2 - (|z|+|w|)^2 &= 1 + 2|zw| + |z|^2|w|^2 - |z|^2 - |w|^2 - 2|z||w| \\ &= 1 + |z|^2|w|^2 - |z|^2 - |w|^2 .\end{aligned}$$

Therefore, using the above relations we reach the desired conclusions.

(ii) Proceed as before:

$$\begin{aligned}|1+\bar{z}w|^2 - |z+w|^2 &= (1+\bar{z}w)(1+z\bar{w}) - (z+w)(\bar{z}+\bar{w}) \\ &= 1 + |z|^2|w|^2 + \bar{z}w + z\bar{w} - |z|^2 - |w|^2 - z\bar{w} - w\bar{z} \\ &= 1 + |z|^2|w|^2 - |z|^2 - |w|^2 = (1-|z|^2)(1-|w|^2).\end{aligned}$$

Solution 5.5. It is clear that $z = 0$ if and only if $w = 0$, thus we can assume that both numbers are non-zero. Also suppose that $z \neq w$, then we need to show that $\bar{z}w = 1$. Simplifying the given identity, we obtain $z + zw\bar{w} - w - wz\bar{z} = 0$.
The last equality can be written as $z - w = zw(\bar{z} - \bar{w})$. Considering the norm on both sides, and using the fact that the norm of a complex number is equal to the norm of its conjugate, it follows that $|zw| = 1$, that is, $zw\bar{z}\bar{w} = 1$.
Multiplying the equality $z + zw\bar{w} - w - wz\bar{z} = 0$ by $\bar{z}$, we obtain $|z|^2 + 1 - \bar{z}w - w\bar{z}|z|^2 = 0$. Now, from this last equation we have that $(|z|^2+1)(1-\bar{z}w) = 0$, thus $\bar{z}w = 1$.

Solution 5.6. Observe that $z_1^2 + z_2^2 + z_3^2 = (z_1+z_2+z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$. Then, since $z_1+z_2+z_3 = 0$, it follows that $z_1^2+z_2^2+z_3^2 = -2(z_1z_2+z_2z_3+z_3z_1) = -2z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = -2z_1z_2z_3(\bar{z_1} + \bar{z_2} + \bar{z_3}) = 0$.

Solution 5.7. Since $z_1\bar{z}_1 = |z_1|^2 = 1$, then $\frac{1}{z_1} = \bar{z}_1$, and also $\frac{1}{z_2} = \bar{z}_2$. Then,

$$\frac{\bar{z}_1 + \bar{z}_2}{1 + \bar{z}_1\bar{z}_2} = \frac{\frac{1}{z_1} + \frac{1}{z_2}}{1 + \frac{1}{z_1}\frac{1}{z_2}} = \frac{z_1 + z_2}{1 + z_1z_2},$$

hence $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Solution 5.8. If some of a, b, c is zero, the result is clear. Then, suppose that a, b, $c \neq 0$.

It is enough to see that $(d-a)(d-b)(d-c) = 0$, or equivalently, that $d^3 - (a+b+c)d^2 + (ab+bc+ca)d - abc = 0$ and, by the hypothesis of the exercise, this is equivalent to showing that $(ab+bc+ca)d = abc$.

Since all numbers have the same norm, define $r = |a| = |b| = |c| = |d|$. On the other hand, it is known that $d = a + b + c$, then $\bar{d} = \overline{a+b+c} = \bar{a} + \bar{b} + \bar{c}$. Now, $\frac{\bar{d}d}{d} = \frac{\bar{a}a}{a} + \frac{\bar{b}b}{b} + \frac{\bar{c}c}{c} = r^2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$, and it follows that $\frac{1}{d} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. That is, $d(ab + bc + ca) = abc$, as we wanted to prove.

Solution 5.9. We show first that (i) is equivalent to (ii). If a, b, c are collinear, then $\arg(b - a) = \arg(c - a)$ or $\arg(b - a) = \arg(c - a) + \pi$, because both are on the same line, then it follows that $\arg\left(\frac{c-a}{b-a}\right) = \arg\left((c-a) \cdot \frac{1}{b-a}\right) = \arg(c - a) - \arg(b - a) = 0$ or π, which implies that $\frac{c-a}{b-a} \in \mathbb{R}$. Reciprocally, if $t = \frac{c-a}{b-a} \in \mathbb{R}$, then $c - a = t(b - a)$ or $c = (1 - t)a + tb$, which means that c is on the straight line that determines a and b, that is, a, b and c are collinear.
Now, we show that (iii) is equivalent to (iv). In order to do this, note that the determinant of the matrix in (iv) is equal to $b\bar{c} - c\bar{b} - a(\bar{c} - \bar{b}) + \bar{a}(c - b)$. Then $z = c\bar{b} - c\bar{a} - a\bar{b} \in \mathbb{R}$ is equivalent to $z = \bar{z}$, that is, $c\bar{b} - c\bar{a} - a\bar{b} = b\bar{c} - a\bar{c} - b\bar{a}$, which is the same as $b\bar{c} - c\bar{b} - a(\bar{c} - \bar{b}) + \bar{a}(c - b) = 0$.
Finally, observe that

$$\frac{c-a}{b-a} = \frac{c-a}{b-a} \cdot \frac{\bar{b}-\bar{a}}{\bar{b}-\bar{a}} = \frac{c\bar{b} - c\bar{a} - a\bar{b} + |a|^2}{|b-a|^2}.$$

Since $\frac{|a|^2}{|a-b|^2}$ is a real number, $\frac{c-a}{b-a} \in \mathbb{R}$ is equivalent to $c\bar{b} - c\bar{a} - a\bar{b} \in \mathbb{R}$, that is, (ii) is equivalent to (iii).

From the part (ii), it follows that the equation of the line through b and c is $\operatorname{Im}(\frac{z-c}{z-b}) = 0$.

Solution 5.10. By Exercise 5.9, z, i, iz are collinear if and only if

$$0 = \begin{vmatrix} 1 & i & -i \\ 1 & z & \bar{z} \\ 1 & iz & -i\bar{z} \end{vmatrix} = z\bar{z} - \frac{1+i}{2}\bar{z} - \frac{1-i}{2}z,$$

which is equivalent to $\left|z - \frac{1+i}{2}\right|^2 = \left|\frac{1+i}{2}\right|^2$. Then, the complex numbers that satisfy the condition of the exercise are the points in the circle with center $\frac{1+i}{2}$ and radius $\left|\frac{1+i}{2}\right| = \frac{\sqrt{2}}{2}$.

Solution 5.11. We will construct first the two squares which have as side length the segment determined by z and w. To do that, consider the points 0 and $z - w$ as two consecutive vertices of a square. Then one possible third vertex is $i(z - w)$, which is the rotation with angle 90° of the complex number $z - w$ (or it could be $-i(z - w)$ if we rotate $-90°$). Finally, the fourth vertex is the sum of the previous two, that is, $(z - w) + i(z - w)$ (or $(z - w) - i(z - w)$ in the other case). Then, to calculate the vertices of the squares that are formed with z and w as consecutive vertices, we should add w to the vertices of the squares found, that is, the vertices

we are looking for are z, w, $i(z-w)+w$ and $z+i(z-w)$ or z, w, $-i(z-w)+w$ and $z-i(z-w)$.
In the case where z and w are opposite vertices of the square, again translate one of the vertices to the origin, that is, now the opposite vertices are 0 and $z-w$. Now, in a square the diagonals intersect each other in a right angle in their midpoints, then we need to consider the two complex numbers orthogonal to $z-w$ and then translate them to the midpoint of $z-w$. That is, we need to consider the complex numbers $i\left(\frac{z-w}{2}\right)+\frac{z-w}{2}$ and $-i\left(\frac{z-w}{2}\right)+\frac{z-w}{2}$. Finally, adding w to all the previous vertices, we get the square with vertices w, $i\left(\frac{z-w}{2}\right)+\frac{z-w}{2}+w$, z, $-i\left(\frac{z-w}{2}\right)+\frac{z-w}{2}+w$.

Solution 5.12. Proceed by induction. The basis of induction is $n=2$. We know

$$(\cos\theta+i\sin\theta)^2=\cos^2\theta-\sin^2\theta+i2\cos\theta\sin\theta=\cos 2\theta+i\sin 2\theta,$$

where in the last equality we used the identity for the sum of angles for sine and cosine.
Suppose then that the identity is true for some $n=k$, that is, we have

$$(\cos\theta+i\sin\theta)^k=\cos k\theta+i\sin k\theta.$$

Then,

$$\begin{aligned}(\cos\theta+i\sin\theta)^{k+1}&=(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)^k\\&=(\cos\theta+i\sin\theta)(\cos k\theta+i\sin k\theta)\\&=\cos\theta\cos k\theta-\sin\theta\sin k\theta+i(\cos\theta\sin k\theta+\cos k\theta\sin\theta)\\&=\cos{(k+1)\theta}+i\sin{(k+1)\theta}.\end{aligned}$$

Solution 5.13. Equation $z+\frac{1}{z}=2\cos\theta$ can be rewritten as

$$z^2+1=2z\cos\theta \quad \text{or} \quad z^2-2z\cos\theta+1=0,$$

then $z=\cos\theta\pm\sqrt{\cos^2\theta-1}=\cos\theta\pm i\sin\theta$.
Using de Moivre's formula, it follows that $z^n=\cos n\theta\pm i\sin n\theta$, then

$$\frac{1}{z^n}=\frac{1}{\cos n\theta\pm i\sin n\theta}=\cos n\theta\mp i\sin n\theta.$$

Adding the last two identities leads to $z^n+\frac{1}{z^n}=2\cos n\theta$.

Solution 5.14. Equation $|z^2+\bar{z}^2|=1$ is equivalent to $|z^2+\bar{z}^2|^2=(z^2+\bar{z}^2)(\bar{z}^2+z^2)=1$, but using $|z|=1$, the last equation is equivalent to $(z^4+1)^2=z^4$, which can be factored as a difference of squares $(z^4-z^2+1)(z^4+z^2+1)=0$. These two quadratic equations can be solved using directly the general formula to obtain that the solutions are $z^2 = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and $z^2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, and from these equalities we can obtain the values of z.

Solution 5.15. (i) Let z_1, z_2 be the roots of the equation with $|z_1| = 1$. Since $z_1 z_2 = \frac{c}{a}$, we have that $|z_2| = \left|\frac{c}{a}\right| \frac{1}{|z_1|} = 1$. Now, since $z_1 + z_2 = -\frac{b}{a}$ and $|a| = |b|$, it follows that $|z_1 + z_2|^2 = 1$. This last equation is equivalent to

$$(z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = 1, \quad \text{or} \quad (z_1 + z_2)\left(\frac{1}{z_1} + \frac{1}{z_2}\right) = 1.$$

Hence, $(z_1 + z_2)^2 = z_1 z_2$, that is, $\left(-\frac{b}{a}\right)^2 = \frac{c}{a}$, which can be reduced to $b^2 = ac$.
(ii) It follows that $b^2 = ac$ and $c^2 = ab$ by the previous part. Multiplying both equalities we have that $b^2c^2 = a^2bc$, and then $a^2 = bc$. Therefore, $a^2 + b^2 + c^2 = ab + bc + ca$. This last identity is equivalent to $(a - c)^2 = (a - b)(b - c)$. Taking norms, $|c - a|^2 = |a - b| \cdot |b - c|$. Similarly, we can obtain $|b - c|^2 = |c - a| \cdot |a - b|$ and $|a - b|^2 = |b - c| \cdot |c - a|$. Adding the last equalities, we get $|b - c|^2 + |c - a|^2 + |a - b|^2 = |b - c| \cdot |c - a| + |c - a| \cdot |a - b| + |a - b| \cdot |b - c|$, which is equivalent to $(|b - c| - |c - a|)^2 + (|c - a| - |a - b|)^2 + (|a - b| - |b - c|)^2 = 0$, and the result follows.

Solution 5.16. Let $z^5 + az^4 + bz^3 + cz^2 + dz + e$ be the polynomial having as roots the numbers z_1, z_2, z_3, z_4, z_5. By Vieta, $a = \sum_{i=1}^{5} z_i = 0$ and $b = \sum_{i<j} z_i z_j = \frac{1}{2}\left(\sum_{i=1}^{5} z_i\right)^2 - \frac{1}{2}\sum_{i=1}^{5} z_i^2 = 0$. Also, we have

$$0 = \sum_{i=1}^{5} \overline{z_i} = \sum_{i=1}^{5} \frac{1}{z_i} = \frac{1}{z_1 z_2 z_3 z_4 z_5} \sum_{\text{cyclic}} z_1 z_2 z_3 z_4,$$

then $d = 0$ and $0 = \sum_{i<j} \overline{z_i z_j} = \frac{1}{z_1 z_2 z_3 z_4 z_5} \sum_{\text{cyclic}} z_1 z_2 z_3$, hence $c = 0$. Thus, the polynomial can be reduced to $z^5 + e$, which has as roots complex numbers that are the vertices of a regular pentagon.

Solution 5.17. If $a = b$, $|a - b + c|$ works; if $a = -b$, then $|a + b + c|$ works. Suppose that a is different from b and $-b$. Consider the numbers $a + b$, $a - b$, $-a + b$ and $-a - b$, which are the vertices of a rhombus of side 2. Taking as a center each of these vertices, construct disks of radius 1; these 4 disks cover the circle with center 0 and radius 1. In particular, the point c belongs to one of these disks, then the distance from the center of such a disk to c is less than or equal to 1.

Second Solution. The numbers a, b, c are the vertices of a triangle, with orthocenter in $a + b + c$. If the triangle is acute, then its orthocenter is inside the triangle and then $|a + b + c| \leq 1$. If the triangle is obtuse, without loss of generality, we can suppose that the obtuse angle is in a, then $-a$, b, c are the vertices of an acute triangle with orthocenter $-a + b + c$, which is inside the triangle, and in this case $|-a + b + c| \leq 1$.

Solution 5.18. If at least one of the numbers a, b, c is 0, the result follows. Consider $\alpha = \frac{a}{|a|}$, $\beta = \frac{b}{|b|}$, $\gamma = \frac{c}{|c|}$, then $|\alpha| = |\beta| = |\gamma| = 1$ and if $a\,|bc| + b\,|ca| + c\,|ab| = 0$ is divided by $|abc|$, then $\alpha + \beta + \gamma = 0$, hence α, β, γ are the vertices of an equilateral triangle. Thus, the angle between two of them is $\frac{\pi}{3}$. Using the cosine law, $|a-b|^2 = |a|^2 + |b|^2 + |a|\,|b| \geq 3\,|a|\,|b|$. Similarly, $|b-c|^2 \geq 3\,|b|\,|c|$ and $|c-a|^2 \geq 3\,|c|\,|a|$, then $|a-b|^2\,|b-c|^2\,|c-a|^2 \geq 3^3\,|a|^2\,|b|^2\,|c|^2$.

Solution 5.19. Since a, b, c have the same norm and $|abc| = 1$, it is clear that $|a| = |b| = |c| = 1$. Then, $1 = \overline{a} + \overline{b} + \overline{c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc} = ab + bc + ca$. The monic polynomial which has zeros a, b, c is,

$$\begin{aligned} P(z) &= (z-a)(z-b)(z-c) = z^3 - (a+b+c)z^2 + (ab+bc+ca)z - abc \\ &= z^3 - z^2 + z - 1 = (z-1)(z^2+1), \end{aligned}$$

hence, $\{a, b, c\} = \{1, i, -i\}$.

Solution 5.20. Multiplying both sides of the equation $x^4 + x^3 + x^2 + x + 1 = 0$ by $x-1$, we get $x^5 - 1 = 0$; then to find the roots of $x^4+x^3+x^2+x+1 = 0$ is equivalent to finding the roots of $x^5 - 1 = 0$, which are different from 1. These roots are the quintic roots of unity, given by w, w^2, w^3, w^4 where $w = \cos\left(\frac{2}{5}\pi\right) + i\sin\left(\frac{2}{5}\pi\right)$.

Second Solution. The equation can be solved dividing by x^2, and then making the substitution $y = x + \frac{1}{x}$, and finally using the general formula to solve a quadratic equation. That is,

$$\begin{aligned} x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 &= 0 \\ \left(x^2 + 2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 1 &= 0 \\ \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 &= 0 \\ y^2 + y - 1 &= 0. \end{aligned}$$

The roots of this last equation are $y_1 = \frac{-1+\sqrt{5}}{2}$, $y_2 = \frac{-1-\sqrt{5}}{2}$. It is left to find x solving the two equations

$$x + \frac{1}{x} = y_1 \quad \text{and} \quad x + \frac{1}{x} = y_2,$$

which are equivalent to $x^2 - y_1 x + 1 = 0$ and $x^2 - y_2 x + 1 = 0$. Solving these two equations we find the four roots we are looking for:

$$\begin{aligned} x_1 &= \frac{-1+\sqrt{5}}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}, & x_2 &= \frac{-1+\sqrt{5}}{4} - i\frac{\sqrt{10+2\sqrt{5}}}{4}, \\ x_3 &= \frac{-1-\sqrt{5}}{4} + i\frac{\sqrt{10-2\sqrt{5}}}{4}, & x_4 &= \frac{-1-\sqrt{5}}{4} - i\frac{\sqrt{10-2\sqrt{5}}}{4}. \end{aligned}$$

As a result of having two different methods for solving equation $x^4+x^3+x^2+x+1 = 0$, we can conclude that

$$\cos\left(\frac{2}{5}\pi\right) + i\sin\left(\frac{2}{5}\pi\right) = \frac{-1+\sqrt{5}}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}.$$

Solving for the real and the imaginary part, we get

$$\cos 72^\circ = \frac{-1+\sqrt{5}}{4}, \quad \sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

Solution 5.21. Note that the polynomial $x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1$ can be factored as

$$x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = (x+1)(x^5 + x^4 + x^3 + x^2 + x + 1).$$

In this way we have to find the roots of the equation

$$(x+1)(x^5 + x^4 + x^3 + x^2 + x + 1) = 0,$$

where it is clear that $x = -1$ is one of the roots. The other roots are complex numbers and can be calculated following the trick used in the previous problem. Multiply $x-1$ by $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ to get equation $x^6 - 1 = 0$ and find the roots distinct from 1. These roots are the 6th roots of unity, which can be calculated using de Moivre's formula (5.3).

Solution 5.22. If $n = 3m+2$ for some positive integer m, then the complex number $\cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right)$ is a solution with norm 1. Conversely, if z is a solution with norm 1, then $\bar{z} = \frac{1}{z}$ is also a solution. Then, $z^n + z + 1 = 0 = z^n + z^{n-1} + 1$, which implies that $z^{n-2} = 1$, $z^2 + z + 1 = 0$, $z^3 = 1$ with $z \neq 1$, hence $n = 3m+2$ for some positive integer m.

Second Solution. Let $P(z) = z^n + z + 1 = 0$. If $P(w) = 0$, with $|w| = 1$, then $w = \cos\theta + i\,\sin\theta$, and then, using de Moivre's formula (5.3), $w^n = \cos n\theta + i\sin n\theta$, it follows that $0 = (\cos n\theta + \cos\theta + 1) + i(\sin n\theta + \sin\theta)$. Then $\sin^2 n\theta = \sin^2\theta$ and $\cos^2 n\theta = \cos^2\theta + 2\cos\theta + 1$, and from this $\cos\theta = -\frac{1}{2}$. It follows that $w^3 = 1$ and $w^2 + w + 1 = 0$, therefore $w^n = w^2$, and then $n \equiv 2 \pmod 3$.

Conversely, if $n \equiv 2 \pmod 3$, for $w \neq 1$, with w a root of unity of order 3, $P(w) = 0$. Then $P(z) = z^n + z + 1 = (z^2 + z + 1)Q(z)$, for some polynomial $Q(z)$ with integer coefficients.

Solution 5.23. (i) Let $S = 1 + w + w^2 + \cdots + w^{n-1}$. Multiplying by w both sides of the equality, we get

$$Sw = w + w^2 + \cdots + w^{n-1} + w^n,$$

and subtracting from S the last equality, we obtain $S - Sw = 1 - w^n$. Therefore $S = \frac{1-w^n}{1-w} = 0$.

(ii) Let $S = 1 + 2w + 3w^2 + \cdots + nw^{n-1}$. Multiplying by w we get

$$Sw = w + 2w^2 + 3w^3 + \cdots + nw^n.$$

Then, $S - Sw = 1 + w + w^2 + w^3 + \cdots + w^{n-1} - nw^n = -nw^n$ (by the first part, $1 + w + w^2 + w^3 + \cdots + w^{n-1} = 0$). Therefore, $S = \frac{-nw^n}{1-w}$, hence $1 + 2w + 3w^2 + \cdots + nw^{n-1} = \frac{n}{w-1}$.

Solution 5.24. (i) Observe that if $w \neq 1$ is a nth root of unity, then $z^n - 1 = (z-1)(z-w)\dots(z-w^{n-1})$. Hence,

$$\frac{z^n - 1}{z - 1} = (z-w)(z-w^2)\dots(z-w^{n-1}) = z^{n-1} + z^{n-2} + \cdots + z + 1.$$

Now, let $z = 1$ in the previous equality in order to obtain

$$(1-w)(1-w^2)\dots(1-w^{n-1}) = n.$$

(ii) Consider the polynomial $P(z) = (z-w)(z-w^2)\dots(z-w^{n-1})$, and since $z^n - 1 = (z-1)(z-w)\dots(z-w^{n-1})$, we get $P(z) = \frac{z^n-1}{z-1} = z^{n-1} + z^{n-2} + \cdots + z + 1$. Note now that $\frac{P'(z)}{P(z)} = \frac{1}{z-w} + \frac{1}{z-w^2} + \cdots + \frac{1}{z-w^{n-1}}$, then it is enough to calculate $\frac{P'(1)}{P(1)}$.

Since $P'(z) = (n-1)z^{n-2} + (n-2)z^{n-3} + \cdots + 2z + 1$, we obtain $P'(1) = 1 + 2 + \cdots + (n-1) = \frac{(n-1)n}{2}$. Now, from $P(1) = n$ we can conclude that $\frac{P'(1)}{P(1)} = \frac{n-1}{2}$.

Solution 5.25. (i) Note that

$$\begin{aligned}
&(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) \\
&\quad = a^2 + ab\omega^2 + ac\omega + bc\omega^2 + ab\omega + b^2 + ac\omega^2 + ab\omega + c^2 \\
&\quad = a^2 + b^2 + c^2 + (ab + bc + ca)\omega + (ab + bc + ca)\omega^2 \\
&\quad = a^2 + b^2 + c^2 + (ab + bc + ca)(\omega + \omega^2) \\
&\quad = a^2 + b^2 + c^2 + (ab + bc + ca)(-1).
\end{aligned}$$

(ii) Substitute the equation obtained in (i).

Solution 5.26. The number of common vertices is given by the number of common roots of $z^{1982} - 1 = 0$ and $z^{2973} - 1 = 0$. Then, by Theorem 5.4.1, it follows that the number we are looking for is the greatest common divisor of $(1982, 2973) = 991$.

Solution 5.27. The roots of $x^2 + x + 1$ are $w = e^{i\frac{2\pi}{3}}$ and w^2. Using the relations $w^3 = 1$ and $1 + w + w^2 = 0$, we obtain

$$\begin{aligned}
n = 3k \quad &\Rightarrow \quad w^{6k} + w^{3k} + 1 = 1 + 1 + 1 = 3, \\
n = 3k+1 \quad &\Rightarrow \quad w^{6k+2} + w^{3k+1} + 1 = w^2 + w + 1 = 0, \\
n = 3k+2 \quad &\Rightarrow \quad w^{6k+4} + w^{3k+2} + 1 = w^4 + w^2 + 1 = w + w^2 + 1 = 0.
\end{aligned}$$

Therefore the answer is for all n that are not multiples of 3.

Solution 5.28. Use that x, y are of the form

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix},$$

in order to see that the product of both numbers is of the same form.

Second Solution. Use that

$$\begin{aligned} x = a^3 + b^3 + c^3 - 3abc &= (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) \\ &= P(1)P(\omega)P(\omega^2), \end{aligned}$$

for $P(z) = cz^2 + bz + a$ and ω a cubic root of unity.
Then, x belongs to S if and only if $x = P(1)P(\omega)P(\omega^2)$, hence, if

$$x = P(1)P(\omega)P(\omega^2) \quad \text{and} \quad y = Q(1)Q(\omega)Q(\omega^2)$$

for $Q(z)$ another polynomial of degree 2, we have that $xy = R(1)R(\omega)R(\omega^2)$ with $R(z) = P(z)Q(z)$. Note that $R(z)$ is of degree 4, and after dividing $R(z)$ by $z^3 - 1$, we get that $R(z) = (z^3-1)L(z) + R_1(z)$ with $R_1(z)$ of degree at most 2 and with $xy = R_1(1)R_1(\omega)R_1(\omega^2)$, then $xy \in S$.

Solution 5.29. Let $w = e^{2\pi i/5}$, then $w^5 = 1$. Evaluating in the original equation w, w^2, w^3, w^4, we obtain the following four equations:

$$\begin{aligned} P(1) + wQ(1) + w^2R(1) &= 0 \\ P(1) + w^2Q(1) + w^4R(1) &= 0 \\ P(1) + w^3Q(1) + wR(1) &= 0 \\ P(1) + w^4Q(1) + w^3R(1) &= 0. \end{aligned}$$

Now, if these equations are multiplied by $-w$, $-w^2$, $-w^3$, $-w^4$, respectively, we obtain:

$$\begin{aligned} -wP(1) - w^2Q(1) - w^3R(1) &= 0 \\ -w^2P(1) - w^4Q(1) - wR(1) &= 0 \\ -w^3P(1) - wQ(1) - w^4R(1) &= 0 \\ -w^4P(1) - w^3Q(1) - w^2R(1) &= 0. \end{aligned}$$

Using $1 + w + w^2 + w^3 + w^4 = 0$ and adding the equations, we get $5P(1) = 0$, that is, $x - 1$ divides $P(x)$.

Solution 5.30. If z is a root of $P(z)$, then z^2 is also a root. Hence, if $|z| > 1$ there will be an infinite number of roots, which is impossible since $P(z)$ is a polynomial. If $0 < |z| < 1$, the same will happen, and there will be an infinite number of roots. Then, all roots are 0 or they belong to the unit circle.

If $P(z)$ is a constant polynomial, then the constant polynomials $P(z) = 1$ and $P(z) = 0$ satisfy the equation.
If $P(z) = az + b$, with $a \neq 0$, substituting in the given equation, we get $(az + b)(-az + b) = az^2 + b$, that is, $az^2 + b = -a^2z^2 + b^2$. Since $a \neq 0$, it follows that $a = -1$ and $b^2 = b$, then $b = 0$ or $b = 1$, and in this case there are two polynomials, $P(z) = -z$ and $P(z) = 1 - z$.
If $P(z) = az^2 + bz + c$, with $a \neq 0$, then

$$P(z)P(-z) = (az^2 + bz + c)(az^2 - bz + c) = a^2z^4 + (2ac - b^2)z^2 + c^2.$$

Comparing with $P(z^2) = az^4 + bz^2 + c$, we obtain $a^2 = a$, $2ac - b^2 = b$ and $c^2 = c$. Since $a \neq 0$, it follows that $a = 1$; for $c^2 = c$ we have the solutions $c = 0$ and $c = 1$. For each of the values of c, we get two values for b: if $c = 0$, then $b = 0$ and $b = -1$; for $c = 1$, we get $b = 1$ and $b = -2$. Thus, in this case we have 4 polynomials that satisfy the given equation: $P(z) = z^2$, $P(z) = z^2 - z = -z(1-z)$, $P(z) = z^2 - 2z + 1 = (1 - z)^2$ and $P(z) = z^2 + z + 1$.

10.6 Solutions to exercises of Chapter 6

Solution 6.1. Observe that if $f(x) + f\left(\frac{1}{1-x}\right) = x$, then

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} \quad \text{and} \quad f\left(\frac{x-1}{x}\right) + f(x) = \frac{x-1}{x}.$$

Hence, $2f(x) = x + \frac{x-1}{x} - \frac{1}{1-x} = \frac{-x^3+x-1}{x(1-x)}$.

Solution 6.2. Prove by induction that $f(n) = n$.
Let $m = n = 1$ in order to see that $f(1) = 1$, and since $f(2) = 2$, by (ii) and using the induction hypothesis, it follows that $f(2k) = 2f(k) = 2k$ and $f(2k + 2) = f(2)f(k + 1) = 2(k + 1) = 2k + 2$.
Finally, by (iii), $2k = f(2k) < f(2k + 1) < f(2k + 2) = 2k + 2$, hence $f(2k + 1) = 2k + 1$. Then $f(n) = n$, for all $n \in \mathbb{N}$.

Solution 6.3. Taking $-x$ instead of x in the original equation, we obtain $-xf(-x) - 2xf(x) = -1$. Then

$$xf(x) + 2xf(-x) = -xf(-x) - 2xf(x),$$

hence, $3xf(x) = -3xf(-x)$. This we can substitute in the original equation to obtain $xf(x) = 1$.

Solution 6.4. Taking $-x$ instead of x in the original equation, we obtain $\frac{1}{-x}f(x) + f\left(\frac{1}{-x}\right) = -x$, then $f(x) - xf\left(\frac{-1}{x}\right) = x^2$. Now, taking x instead of $\frac{1}{x}$ in the original equation, we have that $xf\left(\frac{-1}{x}\right) + f(x) = \frac{1}{x}$. Adding the last two equalities leads to $2f(x) = x^2 + \frac{1}{x}$. Hence, the only function that satisfies the original equation is

$$f(x) = \frac{1}{2}\left(\frac{1}{x} + x^2\right).$$

Solution 6.5. Taking $y = -x$ in the original equation, we get $xf(x) + xf(-x) = 2xf(0)$, for all x. Then $f(x) + f(-x) = 2f(0)$ for all $x \neq 0$.

Taking $x+y$ and x in the functional equation, we get $(x+y)f(x+y) - xf(x) = yf(2x+y)$, and taking $2x+y$, $-x$, we obtain $(2x+y)f(2x+y) + xf(-x) = (3x+y)f(x+y)$.
Then, the last two equations can be rewritten as

$$x\left(f(x+y) - f(x)\right) = y\left(f(2x+y) - f(x+y)\right)$$
$$(2x+y)(f(2x+y) - f(x+y)) = x(f(x+y) - f(-x)).$$

Multiplying the first equation by $2x+y$ and the second one by y, and reducing we get $(2x+y)x(f(x+y) - f(x)) = yx(f(x+y) - f(-x))$. Canceling out x on both sides of the equation, simplifying and solving for $2xf(x+y)$, leads to $2xf(x+y) = (2x+y)f(x) - yf(-x)$. Substituting the value of $f(-x)$, gives us

$$\begin{aligned} 2xf(x+y) &= (2x+y)f(x) - y(2f(0) - f(x)) \\ &= 2(x+y)f(x) - 2yf(0). \end{aligned}$$

That is,

$$\begin{aligned} xf(x+y) &= (x+y)f(x) - yf(0) \\ xf(x+y) - xf(0) &= (x+y)f(x) - yf(0) - xf(0) \\ x(f(x+y) - f(0)) &= (x+y)(f(x) - f(0)). \end{aligned}$$

Now if $x = 1$, then $f(1+y) - f(0) = (1+y)(f(1) - f(0))$. Substituting $1+y = x$, we get $f(x) = f(0) + x(f(1) - f(0))$. Then, if we define the constants $m = f(1) - f(0)$, $b = f(0)$, the functions that satisfy the equation have the form $f(x) = mx + b$. Clearly, the functions of the form $f(x) = mx + b$ satisfy the original functional equation.

Solution 6.6. First, note that if $f(x+1) = f(x) + 1$, by induction it follows that $f(x+n) = f(x) + n$, for all $n \in \mathbb{N}$. Moreover, $f(x^2) = f(x)^2$ and $f(x+n) = f(x) + n$ imply that $f((x+n)^2) = f(x+n)^2 = (f(x)+n)^2$. Then $f(x^2 + 2xn + n^2) = f(x^2 + 2xn) + n^2 = f(x)^2 + 2f(x)n + n^2$, and then $f(x^2 + 2xn) = f(x)^2 + 2f(x)n$. Taking $x = 0$ and $n = 1$ in the last equation, we get $f(0)^2 + f(0) = 0$, then $f(0) = 0$. Moreover, taking $x = \frac{p}{q}$ and $n = q$ in the last equation, leads to $f\left(\frac{p^2}{q^2} + 2p\right) = f\left(\frac{p}{q}\right)^2 + 2qf(\frac{p}{q}) = f(\frac{p^2}{q^2}) + 2qf(\frac{p}{q})$ and, since $f\left(\frac{p^2}{q^2} + 2p\right) = f\left(\frac{p^2}{q^2}\right) + 2p$, then $f(\frac{p}{q}) = \frac{p}{q}$. That is, $f(x) = x$ for all $x \in \mathbb{Q}^+ \cup \{0\}$.

Solution 6.7. Suppose that x, y, z are different numbers. The equation can be rewritten as $x(f(y)-f(z))+yf(z) = f(x)(y-z)+zf(y)$; now subtracting on both sides $yf(y)$, we get $x(f(y) - f(z)) + y(f(z) - f(y)) = f(x)(y - z) + f(y)(z - y)$, hence

$$\frac{f(x) - f(y)}{x - y} = \frac{f(y) - f(z)}{y - z}.$$

Then, the slope between points $(x, f(x))$ and $(y, f(y))$ is equal to the slope between points $(y, f(y))$ and $(z, f(z))$, then every 3 points of the graph of f are collinear; hence the graph of f is a line and therefore $f(x) = mx + b$, for some real numbers m and b. In fact, m is the common slope and $b = f(0)$. Clearly, the affine functions $f(x) = mx + b$ satisfy the equation.

Second Solution. Taking $y = -1$, $z = 1$, we get $xf(-1) - f(1) + f(x) = -f(x) + f(-1) + xf(1)$, and solving for $f(x)$ the result is

$$f(x) = \frac{f(1) - f(-1)}{2}x + \frac{f(-1) + f(1)}{2}.$$

Solution 6.8. The equality $g(f(x)) = -x$, for any real number x, guarantees that $g(f(g(x))) = -g(x)$. If to the equality $f(g(x)) = -x$, we apply g to both sides, we obtain $g(f(g(x))) = g(-x)$. Then $g(-x) = -g(x)$, hence g is odd. Similarly we can prove that $f(-x) = -f(x)$.

Solution 6.9. Taking $y = 0$ in the original equation, we have $f(f(x)) = f(x)-f(0)$. But since f is surjective, given any real number y there exists x with $f(x) = y$, then $f(y) = y - f(0)$. Taking $y = 0$ in this last equation, we get $f(0) = 0$. Thus $f(x) = x$ for any real number x. It is clear that the function $f(x) = x$ satisfies the equation.

Solution 6.10. Taking $x = 0$ in the original equation gives us $f(0) = 0$. Now, if $x = 1$, we have $f(f(y)) = y$, then f is bijective. Taking $f(y)$ as y in the equation and using $f(f(y)) = y$, we get $f(xy) = xf(y)$. By symmetry in the variables x, y, also it is true that $f(xy) = yf(x)$, and then $xf(y) = yf(x)$. Hence for x, y different from 0, it follows that $\frac{f(x)}{x} = \frac{f(y)}{y}$, then $\frac{f(x)}{x}$ is constant and equal to $f(1)$, thus $f(x) = f(1)x$. Using $f(x) = f(1)x$ in the original equation, it follows that $xy = f(1)xf(y) = f(1)^2xy$ for x, $y \in \mathbb{R}$, then $f(1)^2 = 1$. Hence $f(x) = x$ or $f(x) = -x$ are the only continuous solutions of the equation.

Solution 6.11. Since $m - n + f(n) \geq 1$ holds for all $n \in \mathbb{N}$, then $f(n) \geq n$. Letting $F(n) = f(n) - n$, we can rewrite the functional equation as

$$F(m + F(n)) = F(m) + n, \quad \text{for all } \ m, n \in \mathbb{N}.$$

Taking $m = 1$ and adding 1 to both sides of the last equation, we have that $F(1 + F(n)) + 1 = F(1) + n + 1$, for $n \in \mathbb{N}$. If now we apply F on both sides and we use the new equation, we get $F(1)+1+F(n) = F(n+1)+1$, then

$$F(n+1) = F(n) + F(1), \quad \text{for all } \ n \in \mathbb{N}.$$

That is, $F(n) = F(1)n$ for all $n \in \mathbb{N}$, then $(m+nF(1))F(1) = mF(1)+n$, for all n, $m \in \mathbb{N}$. In this last equation, taking $n = m = 1$, leads to $F(1)(F(1)+1) = F(1)+1$, hence $F(1) = 1$, and then $f(n) = 2n$ for all $n \in \mathbb{N}$. Clearly $f(n) = 2n$ satisfies the functional equation.

Solution 6.12. In Example 6.2.4, we proved that the function is bijective.

With $y = 1$ and using the injectivity, it follows that $f(1) = 1$, and then $f(f(y)) = \frac{1}{y}$. Applying f to both sides, we get $\frac{1}{f(y)} = f\left(\frac{1}{y}\right)$.
For x, $y \in \mathbb{Q}^+$, take z such that $f(z) = y$, then

$$f(xy) = f(xf(z)) = \frac{f(x)}{z} = f(x)f(f(z)) = f(x)f(y).$$

In this way a function that satisfies the functional equation must satisfy the two equations

$$f(f(x)) = \frac{1}{x} \qquad \text{and} \qquad f(xy) = f(x)f(y).$$

One particular solution can be defined as follows. Let $p_1, p_2, \ldots$ be the ordered prime numbers and we define the function on the prime numbers as follows:

$$f(p_i) = \begin{cases} p_{i+1}, & \text{if } \ i \text{ is odd} \\ \frac{1}{p_{i-1}}, & \text{if} \, i \text{ is even} \end{cases}$$

and for a rational number $r = p_1^{n_1} \cdots p_k^{n_k}$, the function is defined as

$$f(r) = f(p_1)^{n_1} \cdots f(p_k)^{n_k},$$

where $n_k \in \mathbb{Z}$.

Solution 6.13. Taking $x = y$ in the original equation we get $xf(x) = x(f(x))^2$, then $x(f(x)^2 - f(x)) = 0$ for any real number x. Hence, for $x \neq 0$, we have $f(x)^2 = f(x)$, so that for every real number x it follows that $f(x) = 0$ or 1.

If for all $x \neq 0$ we have $f(x) = 0$, then by continuity it follows that $f(0) = 0$, and then f is identically zero on the real numbers.

If for some $x_0 \neq 0$ it happens that $f(x_0) = 1$, then taking x_0 in the original equation we obtain $x_0 f(y)+y = (x_0+y)f(y)$, hence $y = yf(y)$ for all real numbers y. Then, $f(y) = 1$ for all $y \neq 0$, and by continuity $f(0) = 1$, which guarantees that f is identically 1 on the real numbers.

Therefore, the only functions that satisfy the equation are the constant functions 0 and 1.

Solution 6.14. (i) We should expect that the period is related to a, then it is a good idea to iterate the function. By doing it we get

$$\begin{aligned} f(x+2a) &= \frac{1}{2} + \sqrt{f(x+a) - f(x+a)^2} \\ &= \frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{f(x) - f(x)^2} - \frac{1}{4} - \sqrt{f(x) - f(x)^2} - (f(x) - f(x)^2)} \\ &= \frac{1}{2} + \sqrt{\left(\frac{1}{2} - f(x)\right)^2} = \frac{1}{2} + \left| f(x) - \frac{1}{2} \right|. \end{aligned}$$

Since $f(x) \geq \frac{1}{2}$ for all x, then we have $f(x) = f(x+2a)$ for all x. Hence, f is periodic with period $2a$.

(ii) To find an example observe that $f(x) \geq \frac{1}{2}$ for all x and, on the other hand, the original equation guarantees that $\left(f(x+2) - \frac{1}{2}\right)^2 = f(x+1)(1 - f(x+1)) \leq \left(\frac{1}{2}\right)^2$, where the inequality follows from the geometric and the arithmetic mean inequality. Therefore, a possible example is, for $n \in \mathbb{Z}$, the function

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad 2n \leq x < 2n+1 \\ 1, & \text{if} \quad 2n+1 \leq x < 2n+2. \end{cases}$$

Solution 6.15. Suppose that the equation has at least one real solution x. Then

$$\begin{aligned} m(a+b) &= |x-a| + |x-b| + |x+a| + |x+b| \\ &\geq |(x-a) - (x+b)| + |(x-b) - (x+a)| = 2(a+b), \end{aligned}$$

and since $a + b > 0$, it follows that $m \geq 2$.

Conversely, suppose $m \geq 2$, then the equation has at least one real solution. In fact, if we define

$$f(x) = |x-a| + |x-b| + |x+a| + |x+b|,$$

observe that $f(0) = 2(a+b) \leq m(a+b)$ and $f(ma+mb) = 4m(a+b) > m(a+b)$. By the intermediate value theorem[27], there exists x such that $f(x) = m(a+b)$.

Solution 6.16. Without loss of generality we can assume that $f(0) = 0$, since the function $g(x) = f(x) - f(0)$ satisfies the equation and $g(0) = 0$.

Taking $y = 0$ in the equation, we have $f(x^2) = xf(x)$. Using this last equation and taking $y = 1$, we get $xf(x) - f(1) = (x+1)(f(x) - f(1))$, hence $f(x) = f(1)x$. This means all functions that satisfy the original equation are of the form $f(x) = f(1)x + f(0)$. It is easy to check that the affine functions $f(x) = mx + b$ satisfy the equation.

[27]See [21].

Solution 6.17. Taking $x = y = 1$ in the original equation, it follows that $f(1)^2 - f(1) = 2$, then $f(1) = 2$. Now, if we let $y = 1$, we get $f(x)f(1) - f(x) = x + \frac{1}{x}$, then $f(x) = x + \frac{1}{x}$. And it is clear that $f(x) = x + \frac{1}{x}$ satisfies the equation.

Solution 6.18. (i) In Example 6.2.2 we proved that these functions are injective. As we have seen in the example, taking $x = y$, leads to $f(xf(x)) = x^2$, in particular $f(f(1)) = 1$. Taking $x = f(1)$ in the last equation, gives us $f(1)^2 = f(f(1)f(f(1))) = f(f(1)) = 1$, hence $f(1) = 1$. Letting $y = 1$ in the original equation we obtain $f(x) + f(f(x)) = 2x$.

If $z > 0$, taking $x = zf(z)$ and $y = \frac{1}{z}$ in the functional equation, we get

$$f\left(zf(z)f\left(\frac{1}{z}\right)\right) + f\left(\frac{1}{z}f(zf(z))\right) = 2zf(z)\frac{1}{z}.$$

Then, using $f(zf(z)) = z^2$, it follows that

$$f\left(zf(z)f\left(\frac{1}{z}\right)\right) = f(z),$$

but since f is injective, it follows that $f(z)f\left(\frac{1}{z}\right) = 1$.

If now we take $x = z$, $y = \frac{1}{z}$ in the original functional equation, we get

$$f\left(zf\left(\frac{1}{z}\right)\right) + f\left(\frac{1}{z}f(z)\right) = 2,$$

but since $f\left(\frac{1}{z}\right) = \frac{1}{f(z)}$, then

$$f\left(\frac{z}{f(z)}\right) + f\left(\frac{f(z)}{z}\right) = 2.$$

Since $f(z)f\left(\frac{1}{z}\right) = 1$, it also follows that

$$f\left(\frac{z}{f(z)}\right) \cdot f\left(\frac{f(z)}{z}\right) = 1, \quad \text{then} \quad f\left(\frac{z}{f(z)}\right) = f\left(\frac{f(z)}{z}\right) = 1$$

and again the injectivity of f guarantees that $f(z) = z$.

(ii) In Example 6.2.3, we proved that these functions are surjective. Then, there exists a number x_0 such that $f(x_0) = 0$. Letting $x = x_0$ in the original equation, we obtain $f(y) = 2x_0 + f(f(y) - x_0)$, therefore if we make $z = f(y) - x_0$ we get $f(z) = z - x_0$.

Hence, the functions that satisfy the equation must have the form $f(z) = z + c$, for some constant c.

Solution 6.19. First note that 2 must not be in the image in order to consider the quotient $\frac{f(x)-3}{f(x)-2}$. If for some x, $f(x) = 1$ then $f(x + a) = \frac{-2}{-1} = 2$, hence 1 is not in the image either. Now, observe that

$$f(x+2a) = \frac{f(x+a)-3}{f(x+a)-2} = \frac{2f(x)-3}{f(x)-1}$$
$$f(x+3a) = f(x+2a+a) = \frac{f(x+2a)-3}{f(x+2a)-2} = f(x).$$

Then, $f(x)$ is periodic of period $3a$.

Solution 6.20. Let T be the period of f. Suppose that $T = \frac{p}{q}$, where p and q are relatively prime positive integers. Then, $qT = p$ is also a period of f. Let $n = kp + r$, where k and r are integers and $0 \leq r < p - 1$.
Then $f(n) = f(kp + r) = f(r)$, and $f(n) \in \{f(1), f(2), \ldots, f(p-1), f(p)\}$, for all positive integers n, which is a contradiction with the fact that $\{f(n) \mid n \in \mathbb{N}\}$ has an infinite number of elements.

Solution 6.21. Letting $x = y = 0$, we get

$$f(0) = \frac{2f(0)}{1 - f(0)^2},$$

which makes sense if $f(0) \neq \pm 1$. Then $f(0)^3 + f(0) = 0$, hence $f(0) = 0$.

Now take $g(x) = \arctan f(x)$, then $\tan g(x) = f(x)$ which is well defined for $x \in (-1, 1)$. Substituting in the equation (6.7) we obtain

$$\tan g(x+y) = \frac{\tan g(x) + \tan g(y)}{1 - \tan g(x) \tan g(y)} = \tan(g(x) + g(y)).$$

The last equality follows from the tangent formula for the sum of two angles. Now, apply the inverse tangent function on both sides of the equation to obtain

$$g(x+y) = g(x) + g(y) + k(x,y)\pi,$$

where $k(x,y)$ is a function that only takes integer values. On the other hand, since $f(0) = 0$ we have $g(0) = 0$ and then $k(0,0) = 0$. But since k is a continuous function, $k(x,y) = 0$ for all x, $y \in \mathbb{R}$, we get the equation

$$g(x+y) = g(x) + g(y),$$

which is the Cauchy equation whose continuous solution is $g(x) = \alpha x$. Hence, the solution of equation (6.7) is the function $f(x) = \tan \alpha x$.

Solution 6.22. Since

$$\frac{\tan u + \tan v}{1 - \tan u \tan v} = \tan(u+v),$$

we can take $x = \tan u$ and $y = \tan v$ with $xy \neq 1$, which is true if and only if $\tan u \tan v \neq 1$, that is, $u - v \neq \frac{\pi}{2}$. The equation becomes

$$f(\tan u) + f(\tan v) = f(\tan(u+v)),$$

then $f \circ \tan$ is additive and continuous, therefore $f(\tan u) = cu$, which implies that $f(x) = c \arctan x$.

Solution 6.23. The function $f(x) \equiv -1$ is a solution of the functional equation. If now we define $g(x) = f(x) + 1$, substituting we get

$$\begin{aligned} g(x+y) - 1 &= g(x) - 1 + g(y) - 1 + [g(x) - 1][g(y) - 1] \\ &= g(x) - 1 + g(y) - 1 + [g(x)g(y) - g(x) - g(y) + 1] \\ &= g(x)g(y) - 1. \end{aligned}$$

Therefore $g(x)$ satisfies the Cauchy equation $g(x+y) = g(x)g(y)$ and then $g(x) = a^x$, with $a \in \mathbb{R}^+$ and $f(x) = a^x - 1$.

Solution 6.24. Suppose that there exists a function that satisfies

$$f(f(n)) = n + 1. \tag{10.9}$$

Applying f to both sides of the equation, we obtain $f(n+1) = f(f(f(n))) = f(n) + 1$. Let us see by induction that $f(n+1) = f(1) + n$.

The case $n = 1$ is obvious, since $f(n+1) = f(n) + 1$ and after substituting $n = 1$, we get $f(1+1) = f(1) + 1$. Suppose the result true for $n - 1$ and prove it for n. Since it is true for $n - 1$, the following holds:

$$f(n+1) = f(n) + 1 = f((n-1)+1) + 1 = (f(1) + n - 1) + 1 = f(1) + n,$$

then $f(n+1) = f(1) + n$ for all $n \in \mathbb{N}$. From this last equation and by equation (10.9), we get

$$n + 1 = f(f(n)) = f(n) - 1 + f(1) = n - 1 + f(1) - 1 + f(1) = n - 2 + 2f(1).$$

Thus $f(1) = \frac{3}{2}$, which is a contradiction, since the image of f are the natural numbers. Therefore, no f exists that satisfies equation (10.9).

Solution 6.25. If $x \geq 2$, then $f(x) = f(x - 2 + 2) = f((x-2)f(2))f(2) = 0$, this together with (iii), implies that $f(x) = 0$ if and only if $x \geq 2$. For $0 \leq y < 2$, we have $f(y) \neq 0$ and $0 = f(2) = f(2 - y + y) = f((2-y)f(y))f(y)$, then $f((2-y)f(y)) = 0$, hence $(2-y)f(y) \geq 2$.
Taking $x = \frac{2}{f(y)}$, we get $f(x+y) = f(xf(y))f(y) = f(2)f(y) = 0$, then $x + y \geq 2$, but this implies $2 \geq (2-y)f(y)$. Since $(2-y)f(y) \geq 2$, then $(2-y)f(y) = 2$, that is, $f(u) = \frac{2}{2-y}$ for $0 \leq y < 2$.

Hence,

$$f(y) = \begin{cases} \frac{2}{2-y}, & \text{for } 0 \le y < 2 \\ 0, & \text{for } y \ge 2. \end{cases}$$

It is not difficult to see that f satisfies the conditions of the exercise.

Solution 6.26. It is clear that $f(x) = 0$ and $f(x) = 1$ are solutions; let us see that there are no other solutions.

Taking $x = y = 0$ in the functional equation, it follows that $f(0) = [f(0)]^2$, then either $f(0) = 0$ or $f(0) = 1$. Let us analyze the two cases:

(i) $f(0) = 0$. Letting $y = 0$ in the functional equation, we get $f(x) = f(x)f(0) = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

ii) $f(0) = 1$. First, we will see that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Letting $x = y$ in the original equation, we obtain $1 = f(0) = f(2x)f(x)$, then $f(x) \neq 0$ for all $x \in \mathbb{R}$.

Substituting x and y by $2u$ and u, respectively, we obtain $f(u) = f(3u)f(u)$, since $f(u) \neq 0$, then $f(3u) = 1$. Finally, letting $3u = x$, we get $f(x) = 1$ for all $x \in \mathbb{R}$. Therefore the only solutions are $f(x) = 0$ and $f(x) = 1$.

Solution 6.27. By Theorem 6.5.4, for a function f we have

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x + n - k) \quad \text{when } h = 1.$$

Moreover, by Example 6.5.3, if $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $\Delta^n P(x) = a_n n!$, when $h = 1$.

Consider $f(x) = P(x) = (n - x)^n$. The coefficient of x^n in this polynomial is $(-1)^n$, then $(-1)^n n! = \Delta^n P(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k - x)^n$. Therefore, letting $x = 0$ in the last equation, we get

$$\sum_{k=0}^{n} (-1)^k k^n \binom{n}{k} = (-1)^n n!.$$

Solution 6.28. We prove that the function f is injective. If $f(n) = f(m)$, then $f(f(f(n))) + f(f(n)) + f(n) = f(f(f(m))) + f(f(m)) + f(m)$, hence $3n = 3m$, that is, $n = m$.

Evaluating in $n = 0$, it follows that $f(f(f(0))) + f(f(0)) + f(0) = 0$, then $f(f(f(0))) = f(f(0)) = f(0) = 0$.

It is evident that $f(n) = n$ satisfies the equation; let us see that it is the only solution. By induction suppose that $f(k) = k$ for $0 \le k < n$. Since f is injective, $f(n)$ cannot take any of the values $0, 1, \ldots, n-1$, then $f(n) \ge n$ and also $f(f(n)) \ge n$ and $f(f(f(n))) \ge n$. Thus, $f(f(f(n))) + f(f(n)) + f(n) \ge 3n$.

By hypothesis, the equality must hold, then $f(f(f(n))) = f(f(n)) = f(n) = n$, which proves that $f(n) = n$, for all $n \in \mathbb{N} \cup \{0\}$.

Solution 6.29. Let us prove that $f(x) = x$. First, note that f is injective because if $f(x) = f(y)$, then

$$x = f^n(x) = f^n(y) = y,$$

hence $x = y$ and f is injective. Now, let us see that f is increasing. Suppose that there exist x_1 and x_2 such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$; since f is continuous in $[0, x_1]$, then by the intermediate value theorem[28] there is $c \in [0, x_1]$ such that $f(c) = f(x_2)$, which is a contradiction, since f is injective.

Now, assume that $x < f(x)$, then

$$f(x) < f(f(x)) = f^2(x) < \cdots < f^n(x) = x,$$

which is a contradiction. Similarly, if we suppose that $x > f(x)$ we reach another contradiction, therefore $f(x) = x$ is the only function that satisfies the conditions.

Solution 6.30. Note that f is injective. If for all $x, y \in \mathbb{R}$ we have that $f(x) = f(y)$, then $f^n(x) = f^n(y)$, hence $-x = -y$. Also, f is surjective because $-x = f(f^{n-1})(x)$.

Since $f(-x) = f(f^n(x)) = f^n(f(x)) = -f(x)$, we have that f is odd, therefore $f(0) = 0$.

But if f is bijective and continuous, then it is monotone. Let us prove that f cannot be increasing. If $x < y$ implies that always $f(x) < f(y)$, then $-x = f^n(x) < f^n(y) = -y$, and $y < x$, which is a contradiction. Thus, if f is decreasing, then x and $f(x)$ should have different signs for $x \neq 0$ (note that $x > 0$, implies that $f(x) < 0$, and $x < 0$, implies that $f(x) > 0$). Then, for $x \neq 0$, $xf(x) < 0$ and then x, $f(x)$, $f^2(x)$, $\ldots$, $f^n(x)$ alternate signs, but if $f^n(x) = -x$, then n is odd.

Let $x > 0$ and assume that $f(x) > -x$. Since f is decreasing and odd, we have

$$f(f(x)) < f(-x) = -f(x) < x,$$

again, since $f(x) < -f(f(x))$, being decreasing and odd

$$f(f(x)) > f(-f(f(x))) = -f(f(f(x))).$$

Continuing in this way, we get

$$x > -f(x) > f^2(x) > -f^3(x) > \cdots > -f^n(x) = x,$$

which is a contradiction, therefore $f(x) \leq -x$.

Similarly, we can show that for $x > 0$ it is not possible that $f(x) < -x$. Then $f(x) = -x$ for $x > 0$. Now, using that f is odd, we conclude that $f(x) = -x$, for all $x \in \mathbb{R}$. And $f(x) = -x$ satisfies the functional equation.

[28] See [21].

10.7 Solutions to exercises of Chapter 7

Solution 7.1. Proceed by induction. For $n = 1$, we have $a_1 = 1 < \frac{7}{4}$ and for $n = 2$, we have $a_2 = 3 = \frac{48}{16} < \frac{49}{16} = \left(\frac{7}{4}\right)^2$, which proves the induction basis.
For $n > 2$, suppose the result valid for $n-2$ and $n-1$, that is, $a_{n-2} < \left(\frac{7}{4}\right)^{n-2}$ and $a_{n-1} < \left(\frac{7}{4}\right)^{n-1}$. By direct calculation and using the induction hypothesis, it follows that

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \\ &< \left(\frac{7}{4}\right)^{n-1} + \left(\frac{7}{4}\right)^{n-2} = \left(\frac{7}{4}\right)^{n-2}\left(\frac{7}{4}+1\right) = \left(\frac{7}{4}\right)^{n-2}\left(\frac{11}{4}\right) \\ &< \left(\frac{7}{4}\right)^{n-2}\left(\frac{49}{16}\right) = \left(\frac{7}{4}\right)^{n}. \end{aligned}$$

Solution 7.2. Adding 2^n to both sides of $a_n = 3a_{n-1} + 2^{n-1}$, we get $a_n + 2^n = 3a_{n-1} + 3 \cdot (2^{n-1}) = 3(a_{n-1} + 2^{n-1})$, for all $n \geq 2$. Setting $b_n = a_n + 2^n$, we obtain $b_n = 3b_{n-1} = \cdots = 3^{n-1}b_1$. Since $b_1 = a_1 + 2 = 1 + 2 = 3$, it follows that $b_n = 3^{n-1} \cdot 3 = 3^n$, hence $a_n = 3^n - 2^n$.

Solution 7.3. Since $a_{n+1} = 1 + a_1a_2 \ldots a_n$, it follows that $a_1a_2 \ldots a_n = a_{n+1} - 1$, then $a_1a_2 \ldots a_{n-1} = a_n - 1$; hence $a_{n+1} - 1 = (a_n - 1)(a_n) > 0$, therefore

$$\frac{1}{a_{n+1}-1} = \frac{1}{(a_n-1)(a_n)} = \frac{1}{a_n-1} - \frac{1}{a_n}.$$

Finally, we get

$$\begin{aligned} \frac{1}{a_1} + \cdots + \frac{1}{a_n} &= 1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \\ &= 1 + \left(\frac{1}{a_2-1} - \frac{1}{a_3-1}\right) + \cdots + \left(\frac{1}{a_n-1} - \frac{1}{a_{n+1}-1}\right) \\ &= 1 + \frac{1}{a_2-1} - \frac{1}{a_{n+1}-1} = 2 - \frac{1}{a_{n+1}-1} < 2. \end{aligned}$$

Solution 7.4. By definition $a_{n+1}a_{n-1} = a_n^2 + 1$. Consider $a_{n+2}a_n = a_{n+1}^2 + 1$. If we subtract from this last equation the original identity, we get

$$a_{n+2}a_n - a_{n+1}a_{n-1} = a_{n+1}^2 - a_n^2,$$

which can be rewritten as $a_n(a_{n+2} + a_n) = a_{n+1}(a_{n+1} + a_{n-1})$. Therefore,

$$\frac{a_{n+2}+a_n}{a_{n+1}} = \frac{a_{n+1}+a_{n-1}}{a_n}.$$

Now, the sequence $b_n = \frac{a_{n+1}+a_{n-1}}{a_n}$ with $n \geq 2$ is constant[29] and since

$$b_2 = \frac{a_3 + a_1}{a_2} = \frac{\frac{a_2^2+1}{a_1} + a_1}{a_2} = 3,$$

it follows that $\frac{a_{n+1}+a_{n-1}}{a_n} = 3$, then $a_{n+1} = 3a_n - a_{n-1}$, for every $n \geq 2$. Now, the principle of mathematical induction helps us to conclude that every a_n is an integer and the original equation implies that each a_n is positive.

Solution 7.5. We have $a_2 - a_1 \geq 1$ and $a_{n+2} - a_{n+1} \geq (a_{n+1} - a_n) + 1$, where the second inequality follows by applying the condition to $(n, n+1, n+1, n+2)$ for all n. By induction, it is possible to show that $a_{n+1} - a_n \geq n$ for all $n \geq 1$. Therefore, $a_{n+1} \geq n + a_n$ and $a_1 \geq 1$, and again by induction we have $a_n \geq \frac{1}{2}(n^2 - n + 2)$. Since the sequence $a_n = \frac{1}{2}(n^2 - n + 2)$ satisfies the conditions of the problem (the condition $a_i + a_l > a_j + a_k$ in this case becomes $i^2 + l^2 > j^2 + k^2$, where we take $i = d - y$, $l = d + y$, $j = d - x$, $k = d + x$, with $0 \leq x < y$), the smallest value of a_{2008} is 2 015 029.

Solution 7.6. Note that $a_1 = 1 - a_0$ implies that $a_0 = 1 - a_1$.
Now, $a_2 = 1 - a_1(1 - a_1) = 1 - a_1 a_0$, and then by induction we have $a_n = 1 - a_0 a_1 \ldots a_{n-1}$. That is,

$$a_{n+1} = 1 - a_n(1 - a_n) = 1 - a_n(a_0 \ldots a_{n-1}) = 1 - a_0 \ldots a_{n-1} a_n.$$

The proof is finished using induction. For $n = 0, 1$ the identity follows immediately. Now suppose the statement holds for n and consider

$$\begin{aligned}
&(a_0 \ldots a_{n+1}) \left(\frac{1}{a_0} + \cdots + \frac{1}{a_{n+1}} \right) \\
&\quad = (a_0 \ldots a_n) \left(\frac{1}{a_0} + \cdots + \frac{1}{a_n} \right) a_{n+1} + (a_0 \ldots a_{n+1}) \frac{1}{a_{n+1}} \\
&\quad = a_{n+1} + a_0 \ldots a_n = 1.
\end{aligned}$$

Solution 7.7. For $n = 0$, we have $x_0^2 = y_0 + 2$. Now, use induction. Suppose that $x_k^2 = y_k + 2$ and prove that $x_{k+1}^2 = y_{k+1} + 2$.
Indeed, $x_{k+1}^2 = (x_k^3 - 3x_k)^2 = (x_k^2)^3 - 6(x_k^2)^2 + 9(x_k^2)$. Using the induction hypothesis, we have

$$x_{k+1}^2 = (y_k + 2)^3 - 6(y_k + 2)^2 + 9(y_k + 2) = y_k^3 - 3y_k + 2 = y_{k+1} + 2.$$

Solution 7.8. For $n = 1$, we have $1 + 4a_1 a_2 = 1 + 4(1)(12) = 49 = 7^2$. Now, we use induction to show that for $n \geq 2$, we have

$$1 + 4a_n a_{n+1} = (a_{n+1} + a_n - a_{n-1})^2.$$

That is, for $n = 2$ we get $1 + 4a_2 a_3 = 1 + 960 = 961 = 31^2$, and $(a_3 + a_2 - a_1)^2 = (20 + 12 - 1)^2 = 31^2$.

[29] See Example 7.1.4.

For the inductive step, suppose that $1 + 4a_n a_{n+1} = (a_{n+1} + a_n - a_{n-1})^2$. Note that

$$\begin{aligned}(a_{n+2} + a_{n+1} - a_n)^2 &= (2a_{n+1} + 2a_n - a_{n-1} + a_{n+1} - a_n)^2 \\ &= (2a_{n+1} + a_{n+1} + a_n - a_{n-1})^2 \\ &= 4a_{n+1}^2 + 4a_{n+1}(a_{n+1} + a_n - a_{n-1}) + (a_{n+1} + a_n - a_{n-1})^2 \\ &= 4a_{n+1}^2 + 4a_{n+1}(a_{n+1} + a_n - a_{n-1}) + (1 + 4a_n a_{n+1}) \\ &= 4a_{n+1}(2a_{n+1} + 2a_n - a_{n-1}) + 1 = 4a_{n+1}a_{n+2} + 1,\end{aligned}$$

as we wanted to prove.

Solution 7.9. Note that $a_1 = 2$, $a_2 < 4$, $a_3 < \sqrt{4 + 3 \cdot 4} = 4$. We show by induction that $a_n < 4$ for all $n \in \mathbb{N}$. For $n = 0$, 1, 2 and 3, it is clear. Suppose that $a_n < 4$, then $\sqrt{4+3a_n} < \sqrt{4+3\cdot 4} = 4$, and $a_{n+1} < 4$. Hence, the sequence is bounded by 4.

Solution 7.10. Since $a_{n+1} = a_n + \frac{1}{a_n^2}$, then $a_{n+1}^3 = a_n^3 + 3 + \frac{3}{a_n^3} + \frac{1}{a_n^6} > a_n^3 + 3$. Since $a_2^3 = 1 + 3 + 3 + 1 > 2 \cdot 3$, by induction it follows that $a_n^3 > 3n$. Therefore, $a_n > \sqrt[3]{3n}$ and the sequence is not bounded. Moreover, $a_{9000} > \sqrt[3]{27000} = 30$.

Solution 7.11. Suppose that all terms of the sequence are rational positive numbers, $a_n = \frac{p_n}{q_n}$, with $(p_n, q_n) = 1$. Then

$$\frac{p_{n+1}^2}{q_{n+1}^2} = a_{n+1}^2 = a_n + 1 = \frac{p_n}{q_n} + 1 = \frac{p_n + q_n}{q_n},$$

that is, $q_{n+1}^2(p_n + q_n) = q_n \cdot p_{n+1}^2$. Then, note that $q_n | q_{n+1}^2$ and $q_{n+1}^2 | q_n$, therefore $q_{n+1}^2 = q_n$ for all n.

Then, $q_{n+1} = (q_1)^{1/2^n}$ is a positive integer for all n. This happens only if $q_1 = 1$ and then $q_n = 1$ for all n, meaning that a_n is an integer for all n. Now, if $a_n = 1$, then $a_{n+1} = \sqrt{2}$, which is a contradiction. Then, $a_n > 1$ for all n. It follows that $a_{n+1}^2 - a_n^2 = a_n + 1 - a_n^2 = 1 + a_n(1 - a_n) < 0$ and $a_{n+1} < a_n$ for all n, that is, we have an infinite decreasing sequence of positive integers, which is a contradiction. Therefore, the sequence must contain irrational numbers.

Solution 7.12. It is not difficult to see that the constant sequences $\{a_n = A\}$, the linear sequences $\{a_n = Bn\}$ and the sequences of the form $\left\{a_n = Cn^2\right\}$, with A, B, C fixed numbers, solve the recurrence. Then, also the sequences

$$\left\{a_n = A + Bn + Cn^2\right\}$$

are solutions. Given the initial conditions, the solutions are $\{a_n = 1\}$, $\{a_n = n\}$ and $\left\{a_n = n^2\right\}$, respectively.

Solution 7.13. Since the sequence is bounded, some terms are repeated infinitely many times. Let K be the greatest number that is repeated infinitely many times in the sequence, and let N be a positive integer such that $a_i \leq K$ for $i \geq N$.

Choose $m \geq N$ such that $a_m = K$. We will prove that m is the period of the sequence, that is, $a_{i+m} = a_i$ for all $i \geq N$.

First, we suppose that $a_{i+m} = K$ for some i. Since $a_i + a_m$ is divisible by $a_{i+m} = K$, then $a_i = K = a_{i+m}$.

Now, if $a_{i+m} < K$, choose $j \geq N$ such that $a_{i+j+m} = K$, then it follows that $a_{i+m} + a_j < 2K$. Since $a_{i+m} + a_j$ is divisible by $a_{i+j+m} = K$, then $a_{i+m} + a_j = K$ and therefore $a_j < K$. Since $a_{i+j+m} = K$, the argument in the previous paragraph implies that $a_{i+j+m} = a_{i+j} = K$, and then K divides $a_i + a_j$. It follows that $a_i + a_j = K$, since $a_i \leq K$ and $a_j < K$. Therefore, $a_{i+m} = K - a_j = a_i$.

Solution 7.14. The sequence $a_n = n!$ satisfies the given recursion because $n(n! + (n-1)!) = n(n+1)(n-1)! = (n+1)!$.

The number of derangements of $n+1$ elements can be found as follows: Consider the permutations of $n+1$ elements without fixed points; the first element can be any of the n elements different from the first. Since there are n elements left, the d_{n+1} permutations can be divided into n groups according to which one was in the first place of the n elements different from the first. The groups have the same number of elements. Take one of the groups, say the one where the second element was in the first place.

The permutations are divided in two, when 1 goes to 2 and otherwise. In the first case, there are d_{n-1} derangements and in the second 1 is moved to any place different from 2 and the rest will move freely to a different place from the first, then there are d_n such permutations, hence $d_{n+1} = n(d_n + d_{n-1})$.

The sequences are different since the first terms are not equal, that is, $d_0 = 1$, $d_1 = 0$ and $a_0 = a_1 = 1$.

Solution 7.15. (i) Note that

$$\begin{aligned} d_n - nd_{n-1} &= -(d_{n-1} - (n-1)d_{n-2}) = (d_{n-2} - (n-2)d_{n-3}) = \cdots \\ &= (-1)^{n-2}(d_2 - 2d_1) = (-1)^{n-2}(1 - 2\cdot 0) = (-1)^n. \end{aligned}$$

(ii) A direct application of the formula in Example 7.2.4, leads to

$$\begin{aligned} d_n &= n(n-1)\cdots 2\cdot d_1 + \sum_{j=1}^{n-2} n(n-1)\cdots(j+2)(-1)^{j+1} + (-1)^n \\ &= (n!)d_1 + \sum_{j=1}^{n-2} \frac{n!}{(j+1)!}(-1)^{j+1} + (-1)^n\frac{n!}{n!} \\ &= n!\left(\frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \cdots + \frac{(-1)^n}{n!}\right) \\ &= n!\left(1 + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \cdots + \frac{(-1)^n}{n!}\right). \end{aligned}$$

Solution 7.16. The characteristic equation of the recursion is $x^2 - x - 1 = 0$ which has roots $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$. The solutions of the equation have the form $L_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Since $1 = L_1 = \frac{A+B}{2} + \frac{\sqrt{5}}{2}(A-B)$ and $3 = L_2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2$, then $A = B = 1$, hence $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$, for all n.

Solution 7.17. If $b_0 > 0$, then $b_1 = \frac{b_0}{1+b_0} > 0$ and by induction $b_n > 0$. Consider $a_n = \frac{1}{b_n}$; the recursive equation takes the form $a_{n+1} = a_n + 1$, which has as a solution the sequence $a_n = a_0 + n$. Then $b_n = \frac{b_0}{1+nb_0}$ is the solution of the equation.

Solution 7.18. Notice that

$$a_{n+3} = \frac{1}{1-a_{n+2}} = \frac{1}{1-\frac{1}{1-a_{n+1}}} = \frac{1}{1-\frac{1}{1-\frac{1}{1-a_n}}} = a_n.$$

Solution 7.19. Suppose that $a_n = \frac{b_{n+1}}{b_n}$, then the recursion takes the form $b_{n+2} = 4b_{n+1} - 4b_n$ which is linear of order 2. Its characteristic polynomial is $\lambda^2 - 4\lambda + 4 = 0$, which has as unique solution $\lambda = 2$. Then, $b_n = (A + nB)2^n$ for some numbers A and B. Hence $a_n = \frac{(A+(n+1)B)2^{n+1}}{(A+nB)2^n} = \frac{(A+(n+1)B)2}{(A+nB)}$ is the solution of the equation. Clearly a_n converges to 2.

Solution 7.20. By Proposition 7.2.13, it is enough to observe the following inequalities, where we use the fact that $a_{k+1} \le 2a_k$, for $k = 1, 2, \ldots, n$,

$$\begin{aligned} a_{n+1} &\le 2a_n = a_n + a_n \\ &\le a_n + 2a_{n-1} = a_n + a_{n-1} + a_{n-1} \\ &\vdots \\ &\le a_n + a_{n-1} + \cdots + a_2 + 2a_1 \\ &= a_n + a_{n-1} + \cdots + a_2 + a_1 + 1. \end{aligned}$$

Solution 7.21. Denote the sequence by $\{p_n\}$; prove that $p_{n+1} \le 2p_n$, for any $n \ge 1$. For $n = 1$ it is immediate, since $2 = p_2 = 2p_1 = 2$. For $n \ge 2$, we use **Bertrand's postulate**[30], which says that given an integer $m > 1$, there exists a prime number p such that $m < p < 2m$.
For p_k with $k \ge 2$, it follows, again by Bertrand's postulate, that there exists a prime number p with $p_k < p < 2p_k$. But this prime number is greater than or equal to the prime number after p_k, that is $p_{k+1} \le p$. Then, $p_{k+1} \le 2p_k$ and then, using the previous exercise, we have the result.

[30]See [15].

Solution 7.22. Let a_1, a_2, $\ldots$, a_n be the integer weights of each of the golden pieces and suppose that $a_1 \le a_2 \le \cdots \le a_n$. By hypothesis,

$$a_1 + a_2 + \cdots + a_n = 2n \quad \text{and} \quad a_n \le a_1 + a_2 + \cdots + a_{n-1}.$$

If $a_n = a_1 + a_2 + \cdots + a_{n-1}$, we are done. Case $a_1 \ge 2$ is clear since $a_i \ge 2$ for all i, and the condition $a_1 + a_2 + \cdots + a_n = 2n$, implies that $a_1 = a_2 = \cdots = a_n = 2$. Since n is even, we can perform the required partition.
Now, suppose $a_1 = 1$ and $a_n < a_1 + a_2 + \cdots + a_{n-1}$, then

$$a_n \le 1 + a_1 + a_2 + \cdots + a_{n-1}.$$

Then, it is enough to show that $a_{k+1} \le 1 + a_1 + a_2 + \cdots + a_k$, for $k = 1, 2, \ldots, n-2$. Suppose that the previous statement is not true, that is, let $a_{k+1} > 1 + a_1 + a_2 + \cdots + a_k$ for some $k \in \{1, 2, \ldots, n-2\}$. Then $a_{k+1} \ge k+2$, and $a_i \ge k+2$ for every $i = k+1, k+2, \ldots, n$. Moreover, we know that $a_i \ge 1$ for $i = 1, 2, \ldots, k$, then

$$2n = a_1 + a_2 + \cdots + a_n \ge k + (n-k)(k+2) = -k^2 + (n-1)k + 2n.$$

This implies that $k^2 - (n-1)k \ge 0$, therefore $k \le 0$ or $k \ge n-1$, and both contradict that $k \in \{1, 2, \ldots, n-2\}$, and so the result follows.

Now, to make the division follow the ideas of the proof of Proposition 7.2.13.

Solution 7.23. It is possible to show by induction that $1 \le a_n \le 2$ and that $a_n \ge a_{n+1}$, for all $n \ge 1$. Therefore, the limit of the sequence is equal to some number L (see Theorem 7.4.7) which satisfies $L = \sqrt{L}$, hence $L = 1$.

Solution 7.24. (i) Observe that

$$\begin{aligned}
\sum_{i=0}^{n} \left(\frac{1}{a_i} - \frac{1}{a_{i+1}} \right) &= \frac{1}{a_0} - \frac{1}{a_1} + \frac{1}{a_1} - \frac{1}{a_2} + \cdots + \frac{1}{a_n} - \frac{1}{a_{n+1}} \\
&= \frac{a_1 - a_0}{a_0\, a_1} + \frac{a_2 - a_1}{a_1\, a_2} + \cdots + \frac{a_{n+1} - a_n}{a_{n+1}\, a_n} \\
&= \frac{d}{a_0\, a_1} + \frac{d}{a_1\, a_2} + \cdots + \frac{d}{a_{n+1}\, a_n} \\
&= d \left(\sum_{i=0}^{n} \frac{1}{a_i\, a_{i+1}} \right).
\end{aligned}$$

On the other hand, the series $\sum_{i=0}^{n} (\frac{1}{a_i} - \frac{1}{a_{i+1}})$ is telescopic, hence

$$\sum_{i=0}^{n} \left(\frac{1}{a_i} - \frac{1}{a_{i+1}} \right) = \frac{1}{a_0} - \frac{1}{a_1} + \frac{1}{a_1} - \frac{1}{a_2} + \cdots + \frac{1}{a_n} - \frac{1}{a_{n+1}} = \frac{1}{a_0} - \frac{1}{a_{n+1}}.$$

Therefore, $\sum_{i=0}^{n} \frac{1}{a_i\, a_{i+1}} = \frac{1}{d} \left(\frac{1}{a_0} - \frac{1}{a_{n+1}} \right)$.

(ii) Observe that

$$\begin{aligned}
&\sum_{i=0}^{n}\left(\frac{1}{a_ia_{i+1}}-\frac{1}{a_{i+1}a_{i+2}}\right)\\
&=\frac{1}{a_0a_1}-\frac{1}{a_1a_2}+\frac{1}{a_1a_2}-\frac{1}{a_2a_3}+\cdots+\frac{1}{a_na_{n+1}}-\frac{1}{a_{n+1}a_{n+2}}\\
&=\frac{a_2-a_0}{a_0\,a_1\,a_2}+\frac{a_3-a_1}{a_1\,a_2\,a_3}+\cdots+\frac{a_{n+2}-a_n}{a_n\,a_{n+1}\,a_{n+2}}\\
&=\frac{2d}{a_0\,a_1\,a_2}+\frac{2d}{a_1\,a_2\,a_3}+\cdots+\frac{2d}{a_n\,a_{n+1}\,a_{n+2}}=2d\left(\sum_{i=0}^{n}\frac{1}{a_ia_{i+1}a_{i+2}}\right).
\end{aligned}$$

On the other hand, the series $\sum_{i=0}^{n}\left(\frac{1}{a_ia_{i+1}}-\frac{1}{a_{i+1}a_{i+2}}\right)$ is telescopic, then

$$\sum_{i=0}^{n}\left(\frac{1}{a_ia_{i+1}}-\frac{1}{a_{i+1}a_{i+2}}\right)=\frac{1}{a_0a_1}-\frac{1}{a_{n+1}a_{n+2}}.$$

Therefore, $\sum_{i=0}^{n}\frac{1}{a_i\,a_{i+1}\,a_{i+2}}=\frac{1}{2d}\left(\frac{1}{a_0a_1}-\frac{1}{a_{n+1}a_{n+2}}\right)$.

(iii) Since $a_1=a_0+d$, $a_2=a_1+d=a_0+2d$, $\ldots$, $a_{n+1}=a_0+(n+1)d$, it follows from part (i) that

$$\sum_{i=0}^{\infty}\frac{1}{a_ia_{i+1}}=\lim_{n\to\infty}\frac{1}{d}\left(\frac{1}{a_0}-\frac{1}{a_0+(n+1)d}\right)=\frac{1}{da_0}.$$

(iv) Since $a_1=a_0+d$, $a_2=a_1+d=a_0+2d$, $\ldots$, $a_{n+1}=a_0+(n+1)d$, we have from part (ii) that

$$\sum_{i=0}^{\infty}\frac{1}{a_ia_{i+1}a_{i+2}}=\lim_{n\to\infty}\frac{1}{2d}\left(\frac{1}{a_0a_1}-\frac{1}{(a_0+(n+1)d)(a_0+(n+2)d)}\right)=\frac{1}{2da_0a_1}.$$

Solution 7.25. Use the recurrence formula for the Fibonacci sequence in order to obtain the following equalities:

(i) $$\begin{aligned}
\sum_{n=2}^{\infty}\frac{f_n}{f_{n-1}f_{n+1}}&=\sum_{n=2}^{\infty}\frac{f_{n+1}-f_{n-1}}{f_{n-1}f_{n+1}}=\sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}}-\frac{1}{f_{n+1}}\right)\\
&=\lim_{N\to\infty}\left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f_N}-\frac{1}{f_{N+1}}\right)=\frac{1}{f_1}+\frac{1}{f_2}=2.
\end{aligned}$$

(ii) $$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}}=\sum_{n=2}^{\infty}\frac{f_n}{f_{n-1}f_nf_{n+1}}=\sum_{n=2}^{\infty}\frac{f_{n+1}-f_{n-1}}{f_{n-1}f_nf_{n+1}}\\
&=\sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n}-\frac{1}{f_nf_{n+1}}\right)=\lim_{N\to\infty}\left(\frac{1}{f_1f_2}-\frac{1}{f_Nf_{N+1}}\right)=\frac{1}{f_1f_2}=1.
\end{aligned}$$

Solution 7.26. In step 0, the equilateral triangle has perimeter 3. In step 1, the curve is formed by $4 \cdot 3$ segments of length $\frac{1}{3}$, then it has perimeter $\frac{1}{3} \cdot 4 \cdot 3$. In step 2, the curve is formed by $4^2 \cdot 3$ segments of length $\frac{1}{3^2}$. Then its perimeter is $\frac{1}{3^2} \cdot 4^2 \cdot 3$. In general, in step n the curve has perimeter $P_n = \frac{1}{3^n} \cdot 4^n \cdot 3 = \frac{4^n}{3^{n-1}}$. In this way, $\lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{4^n}{3^{n-1}} = \infty$, since P_n is a geometric progression with ratio $r = \frac{4}{3} > 1$.

The equilateral triangle of side 1 has area equal to $\frac{\sqrt{3}}{4}$. In step 1 the curve encloses an area equal to the sum of the area of the original triangle and the area of the three equilateral triangles of sides $\frac{1}{3}$, which are constructed on every side of the original triangle. That is, $A_1 = \frac{\sqrt{3}}{4} + 3\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{3^2}\right)$.

In step 2 another 4×3 equilateral triangles of side $\frac{1}{9}$ are added to the structure, then $A_2 = \frac{\sqrt{3}}{4} + 3\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{3^2}\right) + 4 \cdot 3\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{3^4}\right)$. In general, $A_n = \frac{\sqrt{3}}{4} + 3\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{3^2}\right) + 4 \cdot 3\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{3^4}\right) + \cdots + 4^{n-1} \cdot 3\left(\frac{\sqrt{3}}{4} \cdot \frac{1}{3^{2n}}\right)$.

Notice that we can write $A_n = \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\left(1 + \frac{4}{9} + \frac{4^2}{9^2} + \cdots + \frac{4^{n-1}}{9^{n-1}}\right)\right)$. In this way, we have that the sum inside the second parenthesis is the sum of a geometric progression with ratio $r = \frac{4}{9}$. Therefore, $\lim_{n\to\infty} A_n = \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\left(\frac{1}{1-\frac{4}{9}}\right)\right) = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\right) = \frac{2\sqrt{3}}{5}$.

Solution 7.27. (i) For $n \geq 1$, it follows that $\frac{1}{2^n+j} \geq \frac{1}{2^{n+1}}$, for all $j = 1, 2, \ldots, 2^n$. Since the inequality is strict for $j \neq 2^n$, we have

$$\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} > \frac{2^n}{2^{n+1}} = \frac{1}{2}.$$

(ii) The sum can be written as

$$\begin{aligned}\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} = \frac{1}{n} &+ \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right)\\ &+ \left(\frac{1}{2n+1} + \cdots + \frac{1}{3n}\right)\\ &+ \qquad \vdots\\ &+ \left(\frac{1}{(n-1)n+1} + \cdots + \frac{1}{n^2}\right).\end{aligned}$$

Since $\frac{1}{kn+1} + \cdots + \frac{1}{(k+1)n} \geq \frac{n}{(k+1)n} = \frac{1}{k+1}$, for each $k = 2, 3, \ldots, n-1$, we have

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} \geq \frac{1}{n} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

In order to reach the conclusion, observe that for $n \geq 2$, $\frac{1}{n} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.

(iii) Simplifying the inequality $\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}>\frac{3}{n}$, we see that it is equivalent to $n^2-1<n^2$, which is always true for $n\geq 1$.

(iv) From part (i), we get $1+\frac{1}{2}>\frac{1}{2}$, $\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$, $\frac{1}{5}+\cdots+\frac{1}{8}>\frac{1}{2}$, $\ldots$, $\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^{n+1}}>\frac{1}{2}$ then

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{n+1}}>\frac{n+1}{2},$$

hence we can conclude that the harmonic series is divergent.

Solution 7.28. Use Abel's summation formula to obtain

$$\begin{aligned}
\sum_{k=1}^{n}k^2q^{k-1} =&\sum_{k=1}^{n-1}(k^2-(k+1)^2)(1+q+\cdots+q^{k-1})+n^2(1+q+\cdots+q^{n-1})\\
=&-\sum_{k=1}^{n-1}(2k+1)\left(\frac{q^k-1}{q-1}\right)+n^2\left(\frac{q^n-1}{q-1}\right)\\
=&\sum_{k=1}^{n-1}\frac{q^k-1}{q-1}-2\sum_{k=1}^{n-1}(k+1)\left(\frac{q^k-1}{q-1}\right)+n^2\left(\frac{q^n-1}{q-1}\right)\\
=&\frac{1}{q-1}\left(\sum_{k=0}^{n-1}q^k-1\right)-\frac{2}{q-1}\left(\sum_{k=0}^{n-1}(k+1)(q^k-1)\right)\\
&+n^2\left(\frac{q^n-1}{q-1}\right)\\
=&\frac{q^n-1}{(q-1)^2}-\frac{n}{q-1}-\frac{2}{q-1}\left(\sum_{k=1}^{n}kq^{k-1}-\frac{n(n+1)}{2}\right)\\
&+n^2\left(\frac{q^n-1}{q-1}\right).
\end{aligned} \tag{10.10}$$

Using Example 7.3.5, we can see that equation (10.10) is equal to

$$\begin{aligned}
&\frac{q^n-1}{(q-1)^2}-\frac{2}{q-1}\left(\frac{nq^n}{q-1}-\frac{q^n-1}{(q-1)^2}\right)+\left(\frac{-n+n(n+1)+n^2(q^n-1)}{q-1}\right)\\
=&\frac{2(q^n-1)}{(q-1)^3}+\frac{(1-2n)q^n-1}{(q-1)^2}+\frac{n^2q^n}{q-1}.
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{n}k^2q^{k-1}=\frac{2(q^n-1)}{(q-1)^3}+\frac{(1-2n)q^n-1}{(q-1)^2}+\frac{n^2q^n}{q-1}.$$

Solution 7.29. Observe that

$$\sum_{n=0}^{\infty} n(n-1)x^n = \sum_{n=0}^{\infty} n^2x^n - \sum_{n=0}^{\infty} nx^n.$$

Now, use Examples 7.3.3 and 7.3.4, to obtain

$$\sum_{n=0}^{\infty} n^2x^n = \sum_{n=0}^{\infty} n(n-1)x^n + \sum_{n=0}^{\infty} nx^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}.$$

Solution 7.30. In order to prove (i), (ii) y (iii) use the fact that the series are geometric series with ratios $\frac{3}{4}$, $\frac{-1}{3}$ and $\frac{1}{3}$, respectively. Then, their sums are equal to $\frac{1}{1-r}$, where r is the ratio.

(iv) Use Example 7.3.3 to obtain $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

(v) Use Exercise 7.29 to prove that

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{2\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^3} + \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = \frac{8}{2} + \frac{4}{2} = 6.$$

10.8 Solutions to exercises of Chapter 8

Solution 8.1. The product of the coefficients of the polynomials is

		4	2	7	1
			2	1	8
		32	16	56	8
	4	2	7	1	
8	4	14	2		
8	8	48	25	57	8

Therefore, the product of the polynomials is $R(x) = 8x^5 + 8x^4 + 48x^3 + 25x^2 + 57x + 8$.

Evaluating in $x = 2$, we have $P(2) = 4 \cdot 2^3 + 2 \cdot 2^2 + 7 \cdot 2 + 1 = 55$, $Q(2) = 2 \cdot 2^2 + 2 + 8 = 18$ and $R(2) = 8 \cdot 2^5 + 8 \cdot 2^4 + 48 \cdot 2^3 + 25 \cdot 2^2 + 57 \cdot 2 + 8 = 990$.

Solution 8.2. Each factor of $P(x)$ is a geometric progression, then

$$\begin{aligned} P(x) &= (1 - x + x^2 - \cdots + x^{100})(1 + x + x^2 + \cdots + x^{100}) \\ &= \left(\frac{(-x)^{101} - 1}{-x - 1}\right)\left(\frac{x^{101} - 1}{x - 1}\right) = \left(\frac{x^{101} + 1}{x + 1}\right)\left(\frac{x^{101} - 1}{x - 1}\right) \\ &= \frac{(x^2)^{101} - 1}{x^2 - 1} = 1 + x^2 + \cdots + x^{200}. \end{aligned}$$

Solution 8.3. Apply the division algorithm to obtain $H(x)$ and $R(x)$, then

$$x^8 - 5x^3 + 1 = (x^5 - x^4 + x^3 - 2x^2 + 3x - 9)(x^3 + x^2 + 1) + 11x^2 - 3x + 10.$$

Hence $H(x) = x^5 - x^4 + x^3 - 2x^2 + 3x - 9$ and $R(x) = 11x^2 - 3x + 10$.

Solution 8.4. Let $P(x) = nx^{n+1} - (n+1)x^n + 1$, evaluating in $x = 1$, $P(1) = n - (n+1) + 1 = 0$, that is, $x = 1$ is a zero of $P(x)$, then $x - 1$ divides $P(x)$. In fact

$$P(x) = (x-1)(nx^n - x^{n-1} - \cdots - x - 1).$$

If $Q(x) = nx^n - x^{n-1} - x^{n-2} - \cdots - x - 1$, then $Q(1) = n - n = 0$ implies that $x - 1$ divides $Q(x)$, hence $(x-1)^2$ divides $P(x)$.

Solution 8.5. It is clear that $P_1(x) = 1 + x$ has as the only root -1 and $P_2(x)$ has roots -1 and -2. By induction, we will see that the roots of $P_n(x)$ are -1, -2, $\ldots$, $-n$. Suppose that $P_n(x)$ has roots -1, -2, $\ldots$, $-n$, then

$$P_n(x) = \frac{(x+1)(x+2)\cdots(x+n)}{n!}.$$

Hence,

$$\begin{aligned}
P_{n+1}(x) &= P_n(x) + \frac{x(x+1)(x+2)\cdots(x+n)}{(n+1)!}\\
&= \frac{(x+1)(x+2)\cdots(x+n)}{n!} + \frac{x(x+1)(x+2)\cdots(x+n)}{(n+1)!}\\
&= \frac{(x+1)(x+2)\cdots(x+n)(n+1) + x(x+1)(x+2)\cdots(x+n)}{(n+1)!}\\
&= \frac{(x+1)(x+2)\cdots(x+n)(x+n+1)}{(n+1)!},
\end{aligned}$$

which shows that the roots of $P_{n+1}(x)$ are -1, -2, $\ldots$, $-(n+1)$.

Solution 8.6. Since $P(0) = 0$, then $P(1) = P^2(0) + P(0) + 1 = 1$. Now, evaluate in $x = 1$, the identity $P(x^2+x+1) = P^2(x) + P(x) + 1$ to obtain $P(3) = 3$. Evaluating in $x = 3$, we get that $P(3^2 + 3 + 1) = P^2(3) + P(3) + 1 = 9 + 3 + 1 = 13$, then $P(13) = 13$.

Now, if we define $x_{n+1} = x_n^2 + x_n + 1$ and $P(x_n) = x_n$, then $P(x_{n+1}) = P(x_n^2 + x_n + 1) = P^2(x_n) + P(x_n) + 1 = x_n^2 + x_n + 1 = x_{n+1}$. This process constructs an infinite number of fixed points of $P(x)$ which are different, since $x_{n+1} - x_n = x_n^2 + 1 > 0$. But a polynomial cannot have an infinite number of fixed points unless $P(x) = x$.

Solution 8.7. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with $a_0 \neq 0$. Since

$$\begin{aligned} x^n P\left(\frac{1}{x}\right) &= x^n \left(a_n \left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + \cdots + a_1\left(\frac{1}{x}\right) + a_0\right) \\ &= a_n + a_{n-1}x^{n-1} + \cdots + a_1 x^{n-1} + a_0 x^n, \end{aligned}$$

we have $x^n P\left(\frac{1}{x}\right) = P(x)$ if and only if $a_n + a_{n-1}x^{n-1} + \cdots + a_1 x^{n-1} + a_0 x^n = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ if and only if the "complementary coefficients" are equal, that is, $a_i = a_{n-i}$, for all $i = 0, \ldots, n$.

Solution 8.8. Use the previous exercise to see that $P(-1) = 0$, then $P(x) = (x+1)Q(x)$, for a polynomial $Q(x)$ of degree $n-1$. Since

$$(x+1)Q(x) = P(x) = x^n P\left(\frac{1}{x}\right) = x^n \left(\frac{1}{x} + 1\right) Q\left(\frac{1}{x}\right),$$

it follows that $Q(x) = x^{n-1} Q\left(\frac{1}{x}\right)$. Hence, using again the previous exercise, it follows that $Q(x)$ is reciprocal.

Solution 8.9. A reciprocal polynomial does not have 0 as a root, since $a_n = a_0 \neq 0$, then $a \neq 0$. Since $a^n P\left(\frac{1}{a}\right) = P(a) = 0$, it follows that $P\left(\frac{1}{a}\right) = 0$, then $\frac{1}{a}$ is also a zero of $P(x)$.

Solution 8.10. If n is odd, it follows that $x^{2n-2} + x^{2n-4} + \cdots + x^4 + x^2 + 1$ can be factored as $(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1)(x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \cdots + 1)$, which proves that the polynomial in the left is divisible by $1 + x + \cdots + x^{n-1}$.

If n is even, -1 is a root of $1 + x + x^2 + \cdots + x^{n-1}$, but it is not a root of $1 + x^2 + x^4 + \cdots + x^{2n-2}$. Then $1 + x^2 + x^4 + \cdots + x^{2n-2}$ is not divisible by $1 + x + x^2 + \cdots + x^{n-1}$.

Solution 8.11. Suppose that $n \geq m$. Recall that Euclid's algorithm is used to find the greatest common divisor of (m, n), in the following way:

$$\begin{aligned} n &= m s_1 + r_1 \\ m &= r_1 s_2 + r_2 \\ &\vdots \\ r_{j-1} &= r_j s_{j+1} + 0, \end{aligned}$$

with $0 \leq r_i < r_{i-1}$ and $r_0 = m$, hence $(m, n) = r_j$.

Notice that

$$\begin{aligned} x^n - 1 &= x^{ms_1 + r_1} - 1 \\ &= x^{r_1}(x^{ms_1} - 1) + x^{r_1} - 1 \\ &= x^{r_1}\left(\frac{x^{ms_1} - 1}{x^m - 1}\right)(x^m - 1) + x^{r_1} - 1, \end{aligned}$$

then the division of x^n-1 by x^m-1, leaves remainder $x^{r_1}-1$, then $(x^n-1, x^m-1) = (x^m-1, x^{r_1}-1)$. Proceeding in the same way, we obtain $(x^n-1, x^m-1) = (x^m-1, x^{r_1}-1) = (x^{r_1}-1, x^{r_2}-1) = \cdots = (x^{r_{j-1}}-1, x^{r_j}-1) = x^{r_j}-1$. Therefore, $(x^n-1, x^m-1) = x^{(n,m)}-1$.

Solution 8.12. The problem is equivalent to finding all pairs (m, n) such that

$$\frac{(x^{mn+n}-1)(x-1)}{(x^{m+1}-1)(x^n-1)}$$

is a polynomial. Notice that x^n-1 and $x^{m+1}-1$ are divisors of $x^{mn+n}-1$. These factors can only have $x-1$ as a common factor, but by the previous exercise, $(x^{m+1}-1, x^n-1) = x^{(m+1,n)}-1$, therefore it will be enough to have $(m+1, n) = 1$.

Solution 8.13. If a is an integer with $P(a) = 0$, then $P(x) = (x-a)Q(x)$, for some polynomial $Q(x)$ with integer coefficients, and $P(0) = -aQ(0)$ and $P(1) = (1-a)Q(1)$. But if a is an integer, then either a or $1-a$ is even, and so one of $P(0)$ and $P(1)$ is even, which is a contradiction.

Solution 8.14. Consider the polynomial with roots x, y, z, that is, $P(u) = (u-x)(u-y)(u-z) = u^3 + au^2 + bu + c$. Then, by Vieta's formulas,

$$a = -x-y-z = -w, \quad b = xy + yz + zx = \frac{xyz}{w}, \quad c = -xyz,$$

therefore, $b = -\frac{c}{w} = \frac{c}{a}$. Hence, $P(u) = u^3 + au^2 + bu + ab = (u+a)(u^2+b)$, and $u = -a = w$ is a root. Without loss of generality, we can assume that it is x, that is, $x + y + z = x$, from where $y + z = 0$. This last equality implies that $y = -z$ and, then $b = -y^2$, hence the other roots are y and $-y$. That is, the roots are $(x, y, -y)$ and therefore the solutions of the system are the triplets $(x, y, -y)$ and its permutations, with $x, y \in \mathbb{R}$.

Solution 8.15. Since the coefficients of the polynomial are integers, it is enough to see that it is irreducible over $\mathbb{Z}[x]$. The polynomial has no integer roots, since a root of $x^4 - x^3 - 3x^2 + 5x + 1$ must divide 1, and then it must be 1 or -1, but $P(1) = 3$ and $P(-1) = -5$. Or, by Exercise 8.13, and because $P(0)$ and $P(1)$ are odd, $P(x) = 0$ has no integer solutions.

Hence, if $P(x)$ can be factored, it must be into two monic quadratic polynomials, as follows: $x^4 - x^3 - 3x^2 + 5x + 1 = (x^2 + bx + c)(x^2 + dx + e)$, with b, c, d, e integers.

Equating coefficients, it follows that:

$$\begin{aligned} b + d &= -1 \\ c + e + bd &= -3 \\ be + cd &= 5 \\ ce &= 1. \end{aligned}$$

The last identity implies $c = e = 1$ or $c = e = -1$. Now, the third identity takes the form $b+d = 5$ or $b+d = -5$. In any case, these equalities are in contradiction with the first identity.

Second Solution. It is enough to see that the polynomial is irreducible over $\mathbb{Z}_2[x]$. But in $\mathbb{Z}_2[x]$, the polynomial can be written as $x^4 + x^3 + x^2 + x + 1$. It is clear that 0 and 1 are not roots of this polynomial, then if it is reducible it should be decomposed as the product of two irreducible quadratic polynomials. But the only irreducible quadratic polynomial in $\mathbb{Z}_2[x]$ is x^2+x+1, and $(x^2+x+1)(x^2+x+1) = x^4 + x^2 + 1 \neq x^4 + x^3 + x^2 + x + 1$. This completes the proof.

Solution 8.16. Consider the integers $x_0 = n$ and $x_{k+1} = P(x_k)$, for $k \geq 0$. If $k = 1$, the result is immediate. Suppose now $x_k = x_0$, with $k \geq 2$, and define $d_i = x_{i+1} - x_i$. Since $d_i = x_{i+1} - x_i$ divides $P(x_{i+1}) - P(x_i) = x_{i+2} - x_{i+1} = d_{i+1}$, for all $i = 0, 1, \ldots, k-1$, and since $d_k = d_0 \neq 0$, it follows that $|d_0| = |d_1| = \cdots = |d_k|$.

Suppose that $d_0 = d_1$. In this case $d_2 = d_0$, otherwise $x_3 - x_2 = -(x_2 - x_1)$, then $x_3 = x_1$ and the sequence of iterates takes the form $x_0, x_1, x_2, x_1, x_2, \ldots$. Hence it should not exist x_k, with $k \geq 2$, that coincides with x_0. Similarly, if $d_j = d_0$, for every j, then $x_j = x_0 + jd_0 \neq x_0$, for every j, which is a contradiction. Hence $d_0 = -d_1$, that is, $x_1 - x_0 = -(x_2 - x_1)$. Therefore $x_0 = x_2$.

Solution 8.17. Suppose that $a_n = 1$. If $r_1, \ldots, r_n$ are the roots, by Vieta's formulas it follows that $r_1^2 r_2^2 \cdots r_n^2 = 1$ and

$$r_1^2 + r_2^2 + \cdots + r_n^2 = (r_1 + \cdots + r_n)^2 - 2 \sum_{1 \leq i < j \leq n} r_i r_j = a_{n-1}^2 - 2a_{n-2} \leq 3.$$

The inequality between the arithmetic and the geometric mean guarantees that

$$r_1^2 + r_2^2 + \cdots + r_n^2 \geq n \sqrt[n]{r_1^2 r_2^2 \cdots r_n^2} = n.$$

Both inequalities imply that $n \leq 3$.

In the case $n = 3$, from the eight polynomials of the form $x^3 \pm x^2 \pm x \pm 1$, the only ones that have three roots are $x^3 - x \pm (x^2 - 1) = (x^2 - 1)(x \pm 1)$. In the case $n = 2$, only the polynomials $x^2 \pm x - 1$ have its two roots real. In the case $n = 1$, the only polynomials are $x \pm 1$.

Solution 8.18. If $P(a) = b$, $P(b) = c$ and $P(c) = a$, then $P(P(P(a))) = P^3(a) = a$, by Exercise 8.16, and $P^2(a) = a$. But on the other hand, $P(P(a)) = P(b) = c$, then $c = a$, which is a contradiction.

Solution 8.19. Suppose that $r_1, \ldots, r_n$ are all the roots of $P(x)$ and that all are real; by Vieta's formulas it follows that

$$\sum_{i=1}^{n} r_i = -2n \quad \text{and} \quad \sum_{1 \leq i < j \leq n} r_i r_j = 2n^2. \tag{10.11}$$

On the other hand, the Cauchy–Schwarz inequality guarantees that $\left(\sum_{i=1}^n r_i\right)^2 \leq \sum_{i=1}^n r_i^2 \sum_{i=1}^n 1^2$, and then $-\sum_{i=1}^n r_i^2 \leq \frac{-1}{n}\left(\sum_{i=1}^n r_i\right)^2$. Hence

$$\begin{aligned}\sum_{1\leq i<j\leq n} r_i r_j &= \frac{1}{2}\left(\sum_{i=1}^n r_i\right)^2 - \frac{1}{2}\sum_{i=1}^n r_i^2\\ &\leq \left(\frac{1}{2}-\frac{1}{2n}\right)\left(\sum_{i=1}^n r_i\right)^2 = \frac{n-1}{2n}(-2n)^2 = 2n(n-1),\end{aligned}$$

contradicting equation (10.11).

Solution 8.20. Calculate the derivative of the polynomial $P(x)$, that is, $P'(x) = 3x^2 - 2x - 8$, which has roots $x = 2$ and $x = -\frac{4}{3}$. Since $P(2) = 0$, then 2 is a multiple root.

Solution 8.21. The fact that $P(x)$ is divisible by $(x+1)^2$ is equivalent to the fact that -1 must be a root of multiplicity at least 2 of $P(x)$, that is, $P(-1) = P'(-1) = 0$. This is equivalent to

$$\begin{aligned}-1 + a - b + c &= 0\\ 3 - 2a + b &= 0.\end{aligned}$$

Then, the solution's triplets are $(a, b, c) = (t, 2t-3, t-2)$, with $t \in \mathbb{R}$.

Solution 8.22. The polynomial of the Lagrange interpolation formula is

$$\begin{aligned}P(x) &= \sum_{k=0}^n (-1)^{n-k} 2^k \frac{x(x-1)(x-2)\cdots(x-k+1)(x-k-1)\cdots(x-n)}{k(k-1)\cdots(k-k+1)(k-k-1)\cdots(k-n)}\\ &= \sum_{k=0}^n (-1)^{n-k} 2^k \frac{x(x-1)(x-2)\cdots(x-k+1)(x-k-1)\cdots(x-n)}{k!\,(n-k)!}.\end{aligned}$$

Then,

$$\begin{aligned}&P(n+1)\\ &= \sum_{k=0}^n (-1)^{n-k} 2^k \frac{(n+1)n\cdots(n+1-(k-1))(n+1-k-1)\cdots(n+1-n)}{k!\,(n-k)!}\\ &= \sum_{k=0}^n (-1)^{n-k}\, 2^k\, \frac{(n+1)!}{k!\,(n-k)!(n+1-k)} = \sum_{k=0}^n (-1)^{n-k}\, 2^k\, \frac{(n+1)!}{k!\,(n+1-k)!}\\ &= \sum_{k=0}^n (-1)^{n-k}\, 2^k \binom{n+1}{k}.\end{aligned}$$

On the other hand,

$$\begin{aligned} 1 = (2-1)^{n+1} &= \sum_{k=0}^{n+1} \binom{n+1}{k} 2^k (-1)^{n+1-k} \\ &= -P(n+1) + 2^{n+1}. \end{aligned}$$

Therefore, $P(n+1) = 2^{n+1} - 1$.

Solution 8.23. (i) First observe that $x \in [-\sqrt{5}, \sqrt{5}]$, since the left-hand side is non-negative. Squaring both sides of the equation and rearranging, we get

$$x^4 - 10x^2 + x + 20 = 0,$$

which can be factorized as

$$(x^2 + x - 5)(x^2 - x - 4) = 0.$$

Then, $x^2 + x - 5 = 0$ or $x^2 - x - 4 = 0$. The solutions of these equations are, for the first $x_{1,2} = \frac{1}{2}\left(-1 \pm \sqrt{21}\right)$ and for the second $x_{3,4} = \frac{1}{2}\left(1 \pm \sqrt{17}\right)$. Only two of them are in the interval $[-\sqrt{5}, \sqrt{5}]$, and therefore the solutions of the equation are $\frac{1}{2}\left(-1 + \sqrt{21}\right)$ and $\frac{1}{2}\left(1 - \sqrt{17}\right)$.

(ii) As in the previous part, squaring both sides of the equation, it follows that

$$x^4 - 2ax^2 + x + a^2 - a = 0,$$

which is a quadratic equation in a,

$$a^2 - (2x^2 + 1)a + x^4 + x = 0.$$

The discriminant of the quadratic equation is $(2x-1)^2$, so that the roots of the equation are $a_1 = x^2 + x$ and $a_2 = x^2 - x + 1$. It follows that,

$$a^2 - (2x^2 + 1)a + x^4 + x = (a - x^2 - x)(a - x^2 + x - 1) = 0.$$

Now, solving the quadratic equations $x^2 + x - a = 0$ and $x^2 - x + 1 - a = 0$, give us the roots $x = \frac{-1 \pm \sqrt{1+4a}}{2}$ and $x = \frac{1 \pm \sqrt{1+4(a-1)}}{2}$. We can show that the root $\frac{-1+\sqrt{1+4a}}{2}$ is always in $[-\sqrt{a}, \sqrt{a}]$ while $\frac{-1-\sqrt{1+4a}}{2}$ is not. On the other hand, the root $\frac{1+\sqrt{1+4(a-1)}}{2}$ would be in $[-\sqrt{a}, \sqrt{a}]$, if $\frac{3}{4} \leq a \leq 1$ and the root $x = \frac{1-\sqrt{1+4(a-1)}}{2}$ belongs to $[-\sqrt{a}, \sqrt{a}]$ only when $a \geq 1$.

Solution 8.24. It is clear that $x \geq 0$, $a + x \geq 0$ and $a - \sqrt{a+x} \geq 0$. Squaring both sides of the equation leads to

$$x^2 = a - \sqrt{a+x} \quad \text{which is equivalent to} \quad \sqrt{a+x} = a - x^2.$$

Then, $a - x^2 \geq 0$, and after squaring, it follows that

$$a + x = (a - x^2)^2,$$

which is equivalent to the equation $P(x) = x^4 - 2ax^2 - x + a^2 - a = 0$. In order to solve this equation, consider a as the variable and x as the parameter, then

$$a^2 - (2x^2 + 1)a + x^4 - x = 0.$$

The discriminant of the equation is $(2x^2 + 1)^2 - 4(x^4 - x) = (2x + 1)^2$, then the roots of the equation are $a_1 = x^2 - x$ and $a_2 = x^2 + x + 1$. This implies that we can factorize $P(x)$ as

$$P(x) = (x^2 - x - a)(x^2 + x + 1 - a).$$

The positive roots of the equation $x^2 - x - a = 0$ do not satisfy the condition $a - x^2 \geq 0$, since $a - x^2 = -x$. The roots of $x^2 + x + 1 - a = 0$ are $x_1 = \frac{-1+\sqrt{4a-3}}{2}$ and $x_2 = \frac{-1-\sqrt{4a-3}}{2}$. Only x_1 can be non-negative, and this happens when $a \geq 1$.

Solution 8.25. First, observe that x is a positive number. Taking conjugates on the left-hand side of the equation leads to

$$\frac{2\sqrt{x}}{\sqrt{x + \sqrt{x}} + \sqrt{x - \sqrt{x}}} = \frac{m\sqrt{x}}{\sqrt{x + \sqrt{x}}},$$

then m is also a positive number. Dividing by $\sqrt{x}$ and simplifying the equation, gives us $(2 - m)\sqrt{x + \sqrt{x}} = m\sqrt{x - \sqrt{x}}$. Then $2 - m \geq 0$, and hence, after squaring, $(2-m)^2(x+\sqrt{x}) = m^2(x-\sqrt{x})$ which is equivalent to $(2-m)^2(\sqrt{x}+1) = m^2(\sqrt{x}-1)$. Solving for $\sqrt{x}$, we get $\sqrt{x} = \frac{m^2-2m+2}{2(m-1)}$, hence $m > 1$ and the solution to the equation is $x = \frac{(m^2-2m+2)^2}{4(m-1)^2}$, for $1 < m \leq 2$.

Solution 8.26. From the first equation, we get

$$x = \sqrt{y + 76} - \sqrt{y + 11} \text{ which is equivalent to } x(\sqrt{y + 76} + \sqrt{y + 11}) = 65.$$

Then,

$$x = \frac{65}{\sqrt{y + 76} + \sqrt{y + 11}}.$$

Similarly, we can obtain $y = \frac{65}{\sqrt{z+76}+\sqrt{z+11}}$ and $z = \frac{65}{\sqrt{x+76}+\sqrt{x+11}}$. Without loss of generality, we can assume that $x \leq y \leq z$. From these inequalities and since the function $\sqrt{x}$ is increasing, it follows that

$$\sqrt{x + 76} + \sqrt{x + 11} \leq \sqrt{y + 76} + \sqrt{y + 11} \leq \sqrt{z + 76} + \sqrt{z + 11}.$$

Then, taking inverses and multiplying by 65, it follows that $y \leq x \leq z$. Hence, $x = y$. From this equality, we obtain

$$x = \frac{65}{\sqrt{y+76}+\sqrt{y+11}} = \frac{65}{\sqrt{x+76}+\sqrt{x+11}} = z.$$

Thus $x = y = z$.

Then, in order to find the triplets, we need to find the solutions to the equation

$$x\left(\sqrt{x+76}+\sqrt{x+11}\right) = 65, \tag{10.12}$$

where x is a positive real number. Since the function $x\left(\sqrt{x+76}+\sqrt{x+11}\right)$ is monotone increasing for positive real numbers, there is at most one solution to the equation (10.12). Since 5 is a solution, the only triplet that solves the system is $(5, 5, 5)$.

Solution 8.27. Consider a, b and c as the unknowns, then we have a system of linear equations. Multiplying the first equation by x and then by y, next substituting cx in the second equation and cy in the last equation, we obtain

$$a(x^2+1) - b(xy+z) = \frac{1}{xz} - \frac{1}{y} \tag{10.13}$$

$$a(xy-z) - b(y^2+1) = -\frac{1}{x} - \frac{1}{yz}. \tag{10.14}$$

Now, multiply equation (10.13) by y^2+1 and equation (10.14) by $-(xy+z)$ and, add them. It follows that

$$a(x^2+y^2+z^2+1) = \frac{x^2+y^2+z^2+1}{xz},$$

hence $a = \frac{1}{xz}$. Similarly, we obtain $b = \frac{1}{yz}$ and $c = \frac{1}{xy}$. Since a, b, c are positive, x, y, z must have the same sign. Therefore, $abc = \frac{1}{(xyz)^2}$, hence $xyz = \pm\frac{1}{\sqrt{abc}}$. That is, the solutions to the system are

$$\left(\frac{b}{\sqrt{abc}}, \frac{a}{\sqrt{abc}}, \frac{c}{\sqrt{abc}}\right) \quad \text{and} \quad \left(-\frac{b}{\sqrt{abc}}, -\frac{a}{\sqrt{abc}}, -\frac{c}{\sqrt{abc}}\right).$$

Solution 8.28. Let $Q(x) = x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots + a_{k-1}x + a_k$ and $P(x) = x^2 + px + q$. Consider the equality

$$\begin{aligned}(x^k + a_1x^{k-1} + \cdots + a_k)^2 + p(x^k + a_1x^{k-1} + \cdots + a_k) + q \\ -(x^2+px+q)^k - a_1(x^2+px+q)^{k-1} - \cdots - a_k = 0.\end{aligned}$$

Calculating the coefficients of x^{2k}, x^{2k-1}, $\ldots$, x^1, x^0, we get a system of equations for a_1, a_2, $\ldots$, a_k, p, q. It is quite difficult to write this system, and to solve it even more. However, some useful observations can be made without solving the system.

When the polynomials are expanded, notice that the coefficients b_1, b_2, $\dots$, b_k of the powers of x^{2k-1}, x^{2k-2}, $\dots$, x^k, are

$$\begin{aligned}
b_1 &= 2a_1 + R_1(p,q) = 0, \\
b_2 &= 2a_2 + R_2(p,q,a_1) = 0, \\
\vdots \quad &\quad \vdots \\
b_k &= 2a_k + R_k(p,q,a_1,\dots,a_{k-1}) = 0,
\end{aligned}$$

where each R_i is some algebraic expression in terms of p, q, a_1, $\dots$, a_{k-1}. The first of these equations implies that a_1 can be expressed in terms of p and q; the second equation implies that a_2 can be expressed in terms of p, q and a_1, and therefore, only in terms of p and q. Similarly, we can conclude that all coefficients of the polynomial $Q(x)$ that commute with $P(x)$ can be expressed in a unique way in terms of p and q, which is what we wanted to prove.

Solution 8.29. First, let us see that the polynomial $Q(x) = P(P(x))$ commutes with $P(x)$. In fact, $Q(P(x)) = P(P(P(x))) = P(Q(x))$. This polynomial $Q(x)$ has degree 4 and, by the previous exercise, it is the only polynomial of degree 4 that commutes with $P(x)$. Similarly, it can be shown that the only polynomial of degree 8 that commutes with $P(x)$ is $R(x) = P(P(P(x)))$.

Solution 8.30. Let $S(x) = Q(R(x))$ and $T(x) = R(Q(x))$. Since $P(x)$ commutes with both $Q(x)$ and $R(x)$, it follows that $P(S(x)) = P(Q(R(x))) = Q(P(R(x))) = Q(R(P(x))) = S(P(x))$. Therefore, $P(x)$ commutes with $S(x)$. Similarly, it can be shown that $P(x)$ commutes with $T(x)$. Since $S(x)$ and $T(x)$ are monic polynomials of the same degree (if $Q(x)$ and $R(x)$ have degrees k and l, respectively, then $S(x)$ and $T(x)$ have degrees kl), by Exercise 8.28, it follows that $S(x) = T(x)$, that is, $Q(R(x)) = R(Q(x))$.

Solution 8.31. Let $P(x) = ax + b$ and $Q(x) = cx + d$. The condition $P(Q(x)) = Q(P(x))$ implies that $acx + ad + b = acx + bc + d$, that is, $d(a-1) = b(c-1)$, and from here we proceed by cases.
First, if $a = 1$, then $b(c-1) = 0$, and $b = 0$ or $c = 1$. If $b = 0$, it follows that $P(x) = x$ and $Q(x) = cx + d$ commute and they have the common fixed point $-\frac{d}{c-1}$. Now, if $c = 1$, then $P(x) = x + b$ and $Q(x) = x + d$, which clearly commute.
Second if $a \neq 1$, then $d = \frac{b(c-1)}{a-1}$ and we have the polynomials, $P(x) = ax + b$, with fixed point $-\frac{b}{a-1}$, and $Q(x) = cx + \frac{b(c-1)}{a-1}$ with fixed point $-\frac{b}{a-1}$, if $c \neq 1$. If $c = 1$, then $d = 0$, hence $P(x) = ax + b$ and $Q(x) = x$, a case that was already considered.

It is clear that if $P(x) = x + \alpha$ and $Q(x) = x + \beta$, then they commute. Now, if there is x_0 such that $P(x_0) = Q(x_0) = x_0$, then $P(x) = a(x - x_0) + x_0$ and $Q(x) = b(x - x_0) + x_0$. A direct calculation proves that the last two polynomials commute, which is the end of the proof.

Solution 8.32. Notice that

$$P_a(Q_a(x)) = P_a(Q(x-a)+a) = P(Q(x-a)+a-a)+a = P(Q(x-a))+a.$$

Similarly, $Q_a(P_a(x)) = Q(P(x-a))+a$. Then, $P_a(x)$ and $Q_a(x)$ commute if and only if $P(Q(x-a)) = Q(P(x-a))$, which is the case.

Solution 8.33. The system can be rewritten as

$$\begin{aligned} x^5+y^5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 &= 33 \\ \sigma_1 &= 3. \end{aligned}$$

Substituting the value of σ_1 in the last equation, we obtain the equation

$$15\sigma_2^2 - 5\cdot 27\sigma_2 + 9\cdot 27 = 33.$$

Simplifying the equation, we get $\sigma_2^2 - 9\sigma_2 + 14 = 0$.

The solutions are $\sigma_2 = 2$ and $\sigma_2 = 7$. Now, in order to obtain the values that we are looking for, we need to solve, for $\sigma_2 = 2$ and $\sigma_2 = 7$, the system

$$x+y = 3 \text{ and } xy = \sigma_2.$$

Solution 8.34. Define $y = \sqrt[4]{x}$ and $z = \sqrt[4]{97-x}$, then the equation can be rewritten as $y+z=5$. We have to solve the system of equations

$$\begin{aligned} y+z &= 5 \\ y^4+z^4 &= 97-x+x = 97. \end{aligned}$$

Now, $y^4+z^4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 = 97$. Substituting the value of σ_1, we have to solve the quadratic equation $2\sigma_2^2 - 100\sigma_2 + 5^4 = 97$. The solutions are $\sigma_2 = 44$ and $\sigma_2 = 6$.

To obtain the values of y and z, we must find the solutions to the systems of equations

$$\begin{aligned} y+z &= 5 \\ yz &= 44, \end{aligned} \qquad \begin{aligned} y+z &= 5 \\ yz &= 6. \end{aligned}$$

Solution 8.35. We have

$$x^3+y^3 = \sigma_1^3 - 3\sigma_1\sigma_2 = c. \tag{10.15}$$

From the second equation, we have $\sigma_1^2 - 2\sigma_2 = b$. Solving for σ_2, we get $\sigma_2 = \frac{a^2-b}{2}$, since $\sigma_1 = a$. Substituting in equation (10.15)

$$\begin{aligned} a^3 - 3a\left(\frac{a^2-b}{2}\right) &= c \\ a^3 - 3ab + 2c &= 0. \end{aligned}$$

Solution 8.36. Substituting the value of z^2 in the second equation, it follows that

$$x^2 + y^2 + xy = b^2.$$

From the first equation, we get

$$x + y - a = -z. \tag{10.16}$$

Squaring both sides leads to

$$\begin{aligned}(x+y)^2 - 2a(x+y) + a^2 &= z^2 \\ (x+y)^2 - 2a(x+y) + a^2 &= xy \qquad (10.17)\\ x^2 + xy + y^2 - 2a(x+y) &= -a^2.\end{aligned}$$

Since $x^2 + y^2 + z^2 = x^2 + y^2 + xy = b^2$, substituting in equation (10.17) and solving for $x + y$, results in $x + y = \frac{a^2+b^2}{2a}$. Substituting this value in (10.16), we get $z = \frac{a^2-b^2}{2a}$, that is, $xy = \frac{(a^2-b^2)^2}{4a^2}$ and $x + y = \frac{a^2+b^2}{2a}$. Observe, since z and a are positive, that $a^2 > b^2$. By Vieta's formulas, x and y are roots of the equation

$$w^2 - \left(\frac{a^2+b^2}{2a}\right) w + \frac{(a^2-b^2)^2}{4a^2} = 0. \tag{10.18}$$

But the solutions of equation (10.18) are

$$w_{1,2} = \frac{\frac{a^2+b^2}{2a} \pm \sqrt{\left(\frac{a^2+b^2}{2a}\right)^2 - 4\left(\frac{(a^2-b^2)^2}{4a^2}\right)}}{2}.$$

Since we are looking for real solutions, the discriminant must be positive, that is

$$\Delta = \left(\frac{a^2+b^2}{2a}\right)^2 - \frac{4(a^2-b^2)^2}{4a^2} = \frac{1}{4a^2}(3a^2-b^2)(3b^2-a^2) > 0.$$

Since $3a^2 > b^2$, then $3b^2 - a^2 > 0$. Hence, $3b^2 > a^2 > b^2$, which means $|b| < a < \sqrt{3}|b|$.

Solution 8.37. Take $\sigma_1 = x + y + z$, $\sigma_2 = xy + yz + zx$ and $\sigma_3 = xyz$. Then

$$\sigma_1 = a, \quad \sigma_2 = \frac{1}{2}(a^2 - b^2), \quad \sigma_3 = \frac{1}{2}a(a^2 - b^2),$$

where the third equation was obtained from the given factorization in equation (4.8).

Solve now,

$$\begin{aligned}u^3 - au^2 + \frac{1}{2}(a^2-b^2)u - \frac{1}{2}a(a^2-b^2) &= 0\\ (u-a)\left[u^2 + \frac{1}{2}(a^2-b^2)\right] &= 0.\end{aligned}$$

Its solutions are $u_1 = a$, $u_2 = \sqrt{\frac{b^2-a^2}{2}}$ and $u_3 = -\sqrt{\frac{b^2-a^2}{2}}$.

Solution 8.38. The equation is equivalent to $xy + x^2y^2 = 3xy^2 + 3x^2y$, which is equivalent to $xy(1 + xy - 3x - 3y) = 0$. If $(x, y) = (m, 0)$ or $(x, y) = (0, m)$, then they are solutions for any integer m.
If $xy \neq 0$, $1 + xy - 3x - 3y = 0$. Since $x = 3$ or $y = 3$ does not satisfy the equation, then we can divide by $x - 3$ or $y - 3$. Solving for y,

$$y = \frac{3x - 1}{x - 3} = 3 + \frac{8}{x - 3},$$

so that y is an integer if $x - 3$ divides 8. Then, $x - 3 = \pm 1$, $x - 3 = \pm 2$, $x - 3 = \pm 4$ and $x - 3 = \pm 8$. Therefore, $x \in \{-5, -1, 1, 2, 4, 5, 7, 11\}$, that is, the solutions are $(m, 0)$, $(0, m)$, for any integer m, and

$$(x, y) \in \{(-5, 2), (2, -5), (-1, 1), (1, -1), (4, 11), (11, 4), (5, 7), (7, 5)\}.$$

Solution 8.39. Let k be a fixed number and consider the solutions of the equation (a, b), with a, $b \in \mathbb{N}$, $a \geq b$, and from this set of solutions choose the one for which $a + b$ is minimum.

If we can show that $a = b$, then $1 + \frac{1}{a} = \frac{k}{2}$, hence $a = \frac{2}{k-2}$ and, since a is a positive integer, $k = 3$ or 4.

Let us see that $b = a$ in the following way. Suppose $a > b$ and notice that a is a solution of the equation $\frac{x+1}{b} + \frac{b+1}{x} = k$, or equivalently, it is a root of the quadratic equation $x^2 - (kb - 1)x + b^2 + b = 0$. If a_1 is the other root, by Vieta's formulas, it follows that $a + a_1 = kb - 1$ and $aa_1 = b^2 + b$. Hence, $a_1 + b = \frac{b^2+b}{a} + b$. Since $a > b$, it follows that $a \geq b + 1$ so that $b \geq \frac{b^2+b}{a} = a_1$. Therefore, the pair (b, a_1) is a solution of the original equation. Since (a, b) is the solution with minimum sum $a + b$, it follows that $a + b \leq b + a_1 = b + \frac{b^2+b}{a}$ so that $b^2 + b \geq a^2 \geq (b+1)^2 = b^2 + 2b + 1$, which implies $b + 1 \leq 0$. But this contradicts the fact that b is positive, therefore $a = b$.

Solution 8.40. Consider the case $k > 3$. Suppose that the integers a, b, c satisfy equation $a^2 + b^2 + c^2 = kabc$; then at least one of them is positive, and the other two are both positive or both negative. In the negative case, we can change the sign to both numbers to obtain a solution where all are positive. Then, without loss of generality, suppose that all three numbers are positive.

Now, let us prove that the three numbers are distinct. Suppose that the previous statement is false, for instance $a = b$. Then $2a^2 + c^2 = ka^2c$, that is, $c^2 = a^2(kc - 2)$. Therefore, $kc - 2$ is a perfect square and then there is an integer number $d \geq 1$ such that $kc = 2 + d^2$. Substituting the value of kc in the equation $2a^2 + c^2 = ka^2c$, it follows that $c^2 = d^2a^2$ or $c = da$. Now, $d^2 = kc - 2 = k(da) - 2$, hence $2 = d(ka - d)$ and this implies d divides 2, that is, $d = 1$ or 2. In both cases $ka = 3$, contradicting the fact that $k > 3$.

Then, suppose that $a > b > c \geq 1$. The triplet $(kbc - a, b, c)$ is also a solution of the equation $x^2 + y^2 + z^2 = kxyz$, with $kbc - a$ a positive integer, since $a(kbc - a) = b^2 + c^2$ and $a > 0$.

Consider now the polynomial $P(x) = x^2 - (kbc)x + b^2 + c^2$. The roots of this polynomial are a and $kbc - a$, moreover

$$P(b) = 2b^2 + c^2 - kb^2c \leq 2b^2 + c^2 - kb^2 < 3b^2 - kb^2 = (3-k)b^2 < 0.$$

The interval where $P(x)$ is negative is the interval with end points the roots of the polynomial. Then b is between a and $kbc - a$, and since $b < a$, it follows that $kbc - a < b$. Hence

$$\max\,(kbc - a, b, c) = b < a = \max\,(a, b, c).$$

Repeating this construction, we obtain a decreasing sequence of positive integers, which is something impossible, then there are no solutions of the original equation for $k > 3$.

Now, let $k = 2$. Suppose that $a^2 + b^2 + c^2 = 2abc$, where a, b, c are integers. Since $a^2 + b^2 + c^2$ is even, not all numbers a, b, c are odd. If exactly one of them is even, reducing modulo 4, we get $2 \equiv 0 \pmod 4$, a contradiction. Therefore, the three numbers are even, and they can be written as $a = 2a'$, $b = 2b'$ and $c = 2c'$, so that $a'^2 + b'^2 + c'^2 = 4a'b'c'$. The last equation is the case $k = 4$, which has no integer solutions except $(0, 0, 0)$, and from this we get $(a, b, c) = (2a', 2b', 2c') = (0, 0, 0)$.

For $k = 3$, a solution is $(1, 1, 1)$ and, for $k = 1$, consider multiples of 3 to reduce it to the case $k = 3$.

10.9 Solutions to problems of Chapter 9

Solution 9.1. Note that $b = a^2 + 2a + 1 = (a+1)^2 \in \mathbb{Q}$ and $b > 0$, then $a = -1 \pm \sqrt{b}$. On the other hand,

$$\begin{aligned} a^3 - 6a &= (-1 \pm \sqrt{b})^3 - 6(-1 \pm \sqrt{b}) \\ &= -1 \pm 3\sqrt{b} - 3b \pm b\sqrt{b} + 6 \mp 6\sqrt{b} \\ &= 5 - 3b \pm (b-3)\sqrt{b}. \end{aligned}$$

Since $a^3 - 6a$ and $5 - 3b$ are rational numbers, we have $(b-3)\sqrt{b} \in \mathbb{Q}$. If $b \neq 3$, $\sqrt{b} \in \mathbb{Q}$ and, then $a = -1 \pm \sqrt{b} \in \mathbb{Q}$ is a contradiction. Therefore $b = 3$ and then $a = -1 \pm \sqrt{3}$.
Moreover, it is clear that if $a = -1 \pm \sqrt{3}$, the numbers $a^2 + 2a = 2$ and $a^3 - 6a = -4$ are rational.

Solution 9.2. First, we make the following substitution to simplify the notation, $a = \sqrt[3]{pq^2}$, $b = \sqrt[3]{qr^2}$ and $c = \sqrt[3]{rp^2}$. We have to prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc}$

is a rational number. Since $abc = pqr$ is a rational number, it is enough to prove that $ab + bc + ca$ is a rational number. Notice that

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(ab+bc+ca)(a+b+c) - 3abc. \tag{10.19}$$

Since $a + b + c$ is a rational number, so is $(a + b + c)^3$ and clearly $a^3 = pq^2$, $b^3 = qr^2$ and $c^3 = rp^2$ are rational numbers. It is now straightforward, from equation (10.19) and since $a + b + c \neq 0$, that $ab + bc + ca$ is a rational number.

Solution 9.3. Since $a = a(2-a) - a(1-a)$, we have that the numbers $a(2-a)$ and $a(1-a)$ cannot be both rational numbers, then one of them is an irrational number; this one will define the number b, that is, b is $2-a$ or $1-a$. Notice that $a + (2-a) = 2$ and $a + (1-a) = 1$ are rational numbers, and $a(2-a)$ or $a(1-a)$ is an irrational number (if $-a(1-a)$ is an irrational number, then also $a(1-a)$ is an irrational number).

Similarly, since $\frac{1}{a} = \left(a + \frac{2}{a}\right) - \left(a + \frac{1}{a}\right)$, the numbers $a + \frac{2}{a}$ and $a + \frac{1}{a}$ cannot be both rational numbers; then one of them is an irrational number and b' is one of the numbers $\frac{2}{a}$ or $\frac{1}{a}$. Notice that, $ab' = 1$ or 2 is a rational number and $a + b' = a + \frac{2}{a}$ or $a + \frac{1}{a}$ is an irrational number.

Solution 9.4. For each i, there exist among $1, 2, \ldots, m$, $\lfloor m/x_i \rfloor$ multiples of x_i. None of them is a multiple of x_j for $j \neq i$, since the least common multiple of x_i and x_j is greater than m. Then, there exist $\lfloor m/x_1 \rfloor + \lfloor m/x_2 \rfloor + \cdots + \lfloor m/x_n \rfloor$ different numbers in $\{1, 2, \ldots, m\}$, and these numbers are divisible by some of the numbers x_1, x_2, $\ldots$, x_n. None of these last numbers can be 1 (unless $n = 1$, and in this case, the result is immediate). Therefore,

$$\left\lfloor \frac{m}{x_1} \right\rfloor + \left\lfloor \frac{m}{x_2} \right\rfloor + \cdots + \left\lfloor \frac{m}{x_n} \right\rfloor \leq m - 1.$$

Since $\frac{m}{x_i} < \lfloor \frac{m}{x_i} \rfloor + 1$ for each i, we have

$$m \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right) < m + n - 1.$$

If we prove that $n \leq (m+1)/2$, then

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 1 + \frac{n-1}{m} < \frac{3}{2}.$$

To prove that $n \leq (m+1)/2$, observe that the largest odd divisors of x_1, x_2, $\ldots$, x_n are all different, because otherwise, if two numbers have the same greatest odd divisor, one of them will be a multiple of the other, which will be a contradiction to the hypothesis. Therefore, n is less than or equal to the quantity of odd numbers among $1, 2, \ldots, m$, and the inequality follows.

Solution 9.5. Let $m = \lfloor \sqrt{n} \rfloor$ and $a = n - m^2$. We have $m \geq 1$, since $n \geq 1$. Now, from $n^2 + 1 = (m^2 + a)^2 + 1 \equiv (a-2)^2 + 1 \mod (m^2 + 2)$, it follows that the condition of the problem is equivalent to the fact that $(a-2)^2 + 1$ is divisible by $m^2 + 2$. Since

$$0 < (a-2)^2 + 1 \leq \max\{2^2, (2m-2)^2\} + 1 \leq 4m^2 + 1 < 4(m^2 + 2),$$

then $(a-2)^2 + 1 = k(m^2 + 2)$, for $k = 1, 2$ or 3. Now, we prove that none of these cases occur.

Case 1. When $k = 1$, we have $(a-2)^2 - m^2 = 1$, and this implies that $a - 2 = \pm 1$ and $m = 0$, but this contradicts the fact that $m \geq 1$.

Case 2. When $k = 2$, we have $(a-2)^2 + 1 = 2(m^2 + 2)$, but any perfect square is congruent to 0, 1, 4 modulo 8, and therefore $(a-2)^2 + 1 \equiv 1, 2, 5 \mod 8$, meanwhile $2(m^2 + 2) \equiv 4, 6 \mod 8$, then this case does not occur.

Case 3. When $k = 3$, we have $(a-2)^2 + 1 = 3(m^2 + 2)$. Since any perfect square is congruent to 0, 1 modulo 3, we have $(a-2)^2 + 1 \equiv 1, 2 \mod 3$, meanwhile $3(m^2 + 2) \equiv 0 \mod 3$, then this case is also impossible.

Solution 9.6. It is easy to prove that when $a = 0$ or $b = 0$ or $a = b$ or a and b are both integers, the identity follows.

Suppose now that a, b do not accomplish any of the above conditions. We have, for $n = 1$, that $\frac{a}{b} = \frac{\lfloor a \rfloor}{\lfloor b \rfloor}$, then $\frac{a}{b}$ is a rational number different from zero. Suppose that $\frac{a}{b} = \frac{p}{q}$, with $(p, q) = 1$.

If p is different from 1 and -1, then p divides $\lfloor na \rfloor$, for all n, in particular it divides $\lfloor a \rfloor$, therefore $a = kp + \epsilon$, for some $k \in \mathbb{N}$ and $0 \leq \epsilon < 1$. Notice that $\epsilon \neq 0$, otherwise $a = kp$ and $b = kq = \lfloor b \rfloor$ are integers.

Then, since there exists $n \in \mathbb{N}$, with $1 \leq n\epsilon < 2$, we have that $\lfloor na \rfloor = \lfloor knp + n\epsilon \rfloor = knp + 1$ is not divisible by p, which is a contradiction.

Similarly, it cannot happen for q to be different from 1 and -1. Therefore p, $q \in \{\pm 1\}$, but since $a \neq b$, we have that $b = -a$, then $\lfloor -a \rfloor = -\lfloor a \rfloor$, which is only possible if a is an integer number. Therefore, there are no more pairs of numbers (a, b) that satisfy the conditions the problem.

Solution 9.7. As we proved in Example 1.3.3, we have

$$\left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + \cdots + \left\lfloor \frac{(m-1)n}{m} \right\rfloor = \frac{(m-1)(n-1)}{2},$$

and we want to find the value of the sum

$$\left\{ \frac{n}{m} \right\} + \left\{ \frac{2n}{m} \right\} + \left\{ \frac{3n}{m} \right\} + \cdots + \left\{ \frac{(m-1)n}{m} \right\} = X.$$

Adding both equations, term by term, we get

$$\frac{n}{m} + \frac{2n}{m} + \frac{3n}{m} + \cdots + \frac{(m-1)n}{m} = X + \frac{(m-1)(n-1)}{2}, \tag{10.20}$$

factorizing $\frac{n}{m}$ on the left-hand side of the equation (10.20) and using Gauss addition formula, we have that the sum on the left-hand side is

$$\frac{n}{m}(1+2+\cdots+(m-1)) = \frac{n}{m}\left(\frac{(m-1)m}{2}\right) = \frac{n(m-1)}{2}.$$

Replacing this value in equation (10.20) and solving for X, we have

$$X = \frac{n(m-1)}{2} - \frac{(m-1)(n-1)}{2} = \frac{m-1}{2}(n-n+1) = \frac{(m-1)}{2}.$$

Solution 9.8. Since $(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$, it follows that

$$\begin{aligned}\left(\frac{1}{\sqrt[3]{a}}+\frac{1}{\sqrt[3]{b}}+1\right)^3 &= \frac{1}{a}+\frac{1}{b}+1+3\left(\frac{1}{\sqrt[3]{a}}+\frac{1}{\sqrt[3]{b}}\right)\left(\frac{1}{\sqrt[3]{b}}+1\right)\left(1+\frac{1}{\sqrt[3]{a}}\right)\\ &= 10+3(18) = 64,\end{aligned}$$

therefore $\frac{1}{\sqrt[3]{a}}+\frac{1}{\sqrt[3]{b}} = 3$, then the solutions are $(a,b) = \left(\frac{1}{8},1\right)$, $\left(1,\frac{1}{8}\right)$.

Solution 9.9. From the first identity, we get $a-b = \frac{1}{c}-\frac{1}{b} = \frac{b-c}{bc}$. Similarly, $b-c = \frac{c-a}{ca}$ and $c-a = \frac{a-b}{ab}$. Therefore $a-b = \frac{a-b}{(abc)^2}$ and, since a and b are different, we have $(abc)^2 = 1$, so that $abc = \pm 1$. It is clear that the numbers $(a,b,c) = (1,-\frac{1}{2},-2)$ and $(a,b,c) = (-1,\frac{1}{2},2)$ satisfy the identities and, with these triplets, we obtain the two possible values of abc.

Solution 9.10. Since $(a+b)(a+c) = a(a+b+c)+bc = \frac{1}{bc}+bc \geq 2$ and the equality holds when $bc = 1$, it follows that the minimum value is 2, and it is reached when $b = c = 1$ and $a = \sqrt{2}-1$.

Solution 9.11. The equation is equivalent to $(bc+a)(ca+b)(ab+c) = (ab+bc+ca-abc)^2$. We expand the equation and cancel out terms to obtain $abc(a^2+b^2+c^2+1) = 2abc(a+b+c)-2abc(ab+bc+ca)$, which is equivalent to $(a+b+c-1)^2 = 0$. Therefore, $a+b+c = 1$.

Solution 9.12. First prove that

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right) = -\frac{(b-c)(c-a)(a-b)}{abc}$$

and do the same for

$$\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right) = -9\frac{abc}{(b-c)(c-a)(a-b)}.$$

Therefore, the value we are looking for is 9.

Solution 9.13. Notice that

$$
\begin{aligned}
0 &= \left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)\\
&= \left(\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}\right)\\
&\quad+\left(\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}\right)
\end{aligned}
$$

and that

$$\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}=0.$$

Hence, the value we are looking for is 0.

Solution 9.14. Notice that $a^2+1=a^2+ab+bc+ca=(a+b)(a+c)$. Similarly, $b^2+1=(b+c)(b+a)$ and $c^2+1=(c+a)(c+b)$. Then $(a^2+1)(b^2+1)(c^2+1)=((a+b)(b+c)(c+a))^2$.

Solution 9.15. The condition $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, implies that $abc\neq 0$ and $ab+bc+ca=0$. Since $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$, it is clear that $a^2+b^2+c^2=(a+b+c)^2$.

Solution 9.16. Notice that the identity of the hypothesis implies that $ab+bc+ca=0$, so $\frac{a^2}{a^2+2bc}=\frac{a^2}{a^2-2(ab+ca)}=\frac{a}{3a-r}$, where $r=2(a+b+c)$. Then,

$$
\begin{aligned}
\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab} &= \frac{a}{3a-r}+\frac{b}{3b-r}+\frac{c}{3c-r}\\
&= \frac{27abc+\frac{r^3}{2}}{(3a-r)(3b-r)(3c-r)}=1,
\end{aligned}
$$

since $(3a-r)(3b-r)(3c-r)=27abc+\frac{r^3}{2}$.

From the equality $\frac{a^2}{a^2+2bc}+2\frac{bc}{a^2+2bc}=1$, we can also conclude that

$$\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{c^2+2ab}=1.$$

Solution 9.17. Notice that

$$
\begin{aligned}
&\frac{a+1}{ab+a+1}+\frac{b+1}{bc+b+1}+\frac{c+1}{ca+c+1}\\
&\quad=\frac{a(1+bc)}{a(b+1+bc)}+\frac{b+1}{bc+b+1}+\frac{c+1}{ca+c+1}=1+\frac{1}{bc+b+1}+\frac{c+1}{ca+c+1}\\
&\quad=1+\frac{1}{b(c+1+ca)}+\frac{c+1}{ca+c+1}=1+\frac{1+b(c+1)}{b(c+1+ca)}=2.
\end{aligned}
$$

Solution 9.18. The equation is equivalent to $a^2 - 2ab + b^2 - ac + bc = 0$. Defining $y = a - b$ and factorizing, we obtain $y^2 - cy = 0$. The roots of this quadratic equation are 0 and c, but 0 is not possible, since a and b are different. Hence, $a - b = c$.

Solution 9.19. If the numbers $x_1, \ldots, x_n$ solve the system, then

$$\begin{aligned} 0 &= x_1 + x_2^2 + \cdots + x_n^n - n - \left(x_1 + 2x_2 + \cdots + nx_n - \frac{1}{2}n(n+1)\right) \\ &= (x_2^2 - 2x_2 + 2 - 1) + (x_3^3 - 3x_3 + 3 - 1) + \cdots + (x_n^n - nx_n + n - 1). \end{aligned}$$

But, using the inequality between the geometric and the arithmetic mean, for each $k \geq 2$ and $x \geq 0$, we have

$$x^k + k - 1 = x^k + 1 + \cdots + 1 \geq k\sqrt[k]{x^k} = kx,$$

with equality if and only if $x = 1$.

Then, since each term of the sum $\left(x_k^k - kx_k + k - 1\right) \geq 0$ and the total sum is zero, we have that each term of the sum is zero, and this happens if each $x_k = 1$. Then, $x_2 = \cdots = x_n = 1$ and, recalling the first equation, we also have that $x_1 = 1$.

Solution 9.20. Observe that

$$\begin{aligned} \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9 \quad &\Leftrightarrow \quad (x+1)(y+1) \geq 9xy \\ &\Leftrightarrow \quad 2 \geq 8xy \\ &\Leftrightarrow \quad (x+y)^2 \geq 4xy \\ &\Leftrightarrow \quad (x-y)^2 \geq 0. \end{aligned}$$

Solution 9.21. The inequality is equivalent to

$$x^3 + y^3 + z^3 - 3xyz \geq \frac{9}{4}\left|(x-y)(y-z)(z-x)\right|.$$

But since

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right],$$

it is enough to prove that

$$\frac{1}{2}(x+y+z)\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right] \geq \frac{9}{4}\left|(x-y)(y-z)(z-x)\right|.$$

Let $p = |(x-y)(y-z)(z-x)|$, using the inequality between the geometric and the arithmetic mean, we have that

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 3\sqrt[3]{p^2}. \tag{10.21}$$

Now, since $|x-y| \le x+y$, $|y-z| \le y+z$, $|z-x| \le z+x$, we get

$$2(x+y+z) \ge |x-y| + |y-z| + |z-x|.$$

Applying again the inequality between the geometric and the arithmetic mean, we obtain

$$2(x+y+z) \ge 3\sqrt[3]{p}. \tag{10.22}$$

Hence, the result follows from inequalities (10.21) and (10.22).

Solution 9.22. If $a \ge 1$, since $b+c > a \ge 1$, we have that

$$ab+bc+ca = a(b+c)+bc > 1+bc > 1,$$

which is a contradiction to the fact that $ab+bc+ca = 1$, therefore $a < 1$. Similarly, we can prove that $b < 1$ and $c < 1$.
Thus, $(1-a)(1-b)(1-c) > 0$, then

$$1+ab+bc+ca > a+b+c+abc.$$

Adding 2 on both sides of the last inequality, and using the fact that $ab+bc+ca = 1$, we obtain $3+ab+bc+ca > 2+a+b+c+abc$, that is,

$$4 > 1+a+b+c+ab+bc+ca+abc = (a+1)(b+1)(c+1).$$

Solution 9.23. Notice that

$$\begin{aligned} a+b+c-\frac{abc}{ab+bc+ca} &= \frac{(a+b)(b+c)(c+a)}{ab+bc+ca} \\ &= \frac{a^2+b^2+c^2+3(ab+bc+ca)}{ab+bc+ca}, \end{aligned}$$

the last equality is valid, since the condition $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c} = 1$ is equivalent to $(a+b)(b+c)(c+a) = (a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b) = a^2+b^2+c^2+3(ab+bc+ca)$.
Then, the result to be proved is equivalent to $a^2+b^2+c^2 \ge ab+bc+ca$, which is valid using the inequality between the geometric mean and the arithmetic mean.

Solution 9.24. Since the expression is a symmetric function in a, b and c, we can assume, without loss of generality, that $c \le b \le a$. In such a case, $a\,(b+c-a) \le b\,(a+c-b) \le c\,(a+b-c)$.

For example, the first inequality can be justified as follows:

$$\begin{aligned} a\,(b+c-a) \le b\,(a+c-b) \quad &\Leftrightarrow \quad ab+ac-a^2 \le ab+bc-b^2 \\ &\Leftrightarrow \quad (a-b)\,c \le (a+b)\,(a-b) \\ &\Leftrightarrow \quad (a-b)\,(a+b-c) \ge 0. \end{aligned}$$

By the rearrangement inequality, see Example 7.3.6, we have

$$\begin{aligned}
&a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\\
&\quad\leq ba(b+c-a)+cb(c+a-b)+ac(a+b-c)\\
&a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\\
&\quad\leq ca(b+c-a)+ab(c+a-b)+bc(a+b-c).
\end{aligned}$$

Therefore, $2\left[a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\right]\leq 6abc$.

Solution 9.25. Squaring and rearranging the given inequality, it is equivalent to

$$\sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n x_iy_i + \sum_{i=1}^n y_i^2 \leq \sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n x_iz_i + \sum_{i=1}^n z_i^2.$$

But, since $\sum_{i=1}^n y_i^2 = \sum_{i=1}^n z_i^2$, the inequality we have to prove is equivalent to

$$\sum_{i=1}^n x_iz_i \leq \sum_{i=1}^n x_iy_i,$$

which is the rearrangement inequality (see Example 7.3.6).

Solution 9.26. Let $(a_1, a_2, \ldots, a_n)$ be a permutation of $(x_1, x_2, \ldots, x_n)$ with $a_1 \leq a_2 \leq \cdots \leq a_n$, and let $(b_1, b_2, \ldots, b_n) = \left(\frac{1}{n^2}, \frac{1}{(n-1)^2}, \cdots, \frac{1}{1^2}\right)$, that is, $b_i = \frac{1}{(n+1-i)^2}$, for the indices $i = 1, \ldots, n$.

Consider the permutation $(a_1', a_2', \ldots, a_n')$ of $(a_1, a_2, \ldots, a_n)$, defined by $a_i' = x_{n+1-i}$, for $i = 1, \ldots, n$.

Using the rearrangement inequality (see Example 7.3.6), we have

$$\begin{aligned}
\frac{x_1}{1^2}+\frac{x_2}{2^2}+\cdots+\frac{x_n}{n^2} &= a_1'b_1 + a_2'b_2 + \cdots + a_n'b_n\\
&\geq a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n\\
&= a_1b_n + a_2b_{n-1} + \cdots + a_nb_1\\
&= \frac{a_1}{1^2}+\frac{a_2}{2^2}+\cdots+\frac{a_n}{n^2}.
\end{aligned}$$

Since $1 \leq a_1$, $2 \leq a_2$, $\ldots$, $n \leq a_n$, we get

$$\frac{x_1}{1^2}+\frac{x_2}{2^2}+\cdots+\frac{x_n}{n^2} \geq \frac{a_1}{1^2}+\frac{a_2}{2^2}+\cdots+\frac{a_n}{n^2} \geq \frac{1}{1^2}+\frac{2}{2^2}+\cdots+\frac{n}{n^2} = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}.$$

Solution 9.27. First, define $x_{2i} = 0$, $x_{2i-1} = \frac{1}{2}$, for all $i = 1, \ldots, 50$. Then, we have $S = 50 \cdot \left(\frac{1}{2}\right)^2 = \frac{25}{2}$. Now we prove that always $S \leq \frac{25}{2}$.

Let $1 \leq i \leq 50$, under the conditions of the problem, we have $x_{2i-1} \leq 1 - x_{2i} - x_{2i+1}$ and $x_{2i+2} \leq 1 - x_{2i} - x_{2i+1}$. Using the inequality between the geometric mean and the arithmetic mean, we get

$$
\begin{aligned}
x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2} &\leq (1 - x_{2i} - x_{2i+1})x_{2i+1} + x_{2i}(1 - x_{2i} - x_{2i+1}) \\
&= (x_{2i} + x_{2i+1})(1 - x_{2i} - x_{2i+1}) \leq \left(\frac{(x_{2i} + x_{2i+1}) + (1 - x_{2i} - x_{2i+1})}{2}\right)^2 = \frac{1}{4}.
\end{aligned}
$$

Adding these inequalities, for $i = 1, 2, \ldots, 50$, we obtain

$$S = \sum_{i=1}^{50}(x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2}) \leq 50 \cdot \frac{1}{4} = \frac{25}{2}.$$

Solution 9.28. (i) From $bx + by \leq ax + by \leq ab$, it follows that $x + y \leq a$.

(ii) We have

$$
\begin{aligned}
\sqrt{x} + \sqrt{y} = \sqrt{\frac{ax}{a}} + \sqrt{\frac{by}{b}} &\leq \sqrt{ax + by}\,\sqrt{\frac{1}{a} + \frac{1}{b}} \\
&\leq \sqrt{ab\left(\frac{1}{a} + \frac{1}{b}\right)} = \sqrt{a + b}.
\end{aligned}
$$

The first inequality is given by the Cauchy–Schwarz inequality (see Example 4.2.3), and the second one follows from the hypothesis $ax + by \leq ab$.

Solution 9.29. Let $x = \frac{a}{a-b}$, $y = \frac{b}{b-c}$, $z = \frac{c}{c-a}$, then

$$
\begin{aligned}
(x-1)(y-1)(z-1) &= \left(\frac{b}{a-b}\right)\left(\frac{c}{b-c}\right)\left(\frac{a}{c-a}\right) \\
&= \left(\frac{a}{a-b}\right)\left(\frac{b}{b-c}\right)\left(\frac{c}{c-a}\right) \\
&= xyz.
\end{aligned}
$$

When we expand and cancel out terms, we obtain $x + y + z = xy + yz + zx + 1$. Then

$$
\begin{aligned}
&\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2 \\
&= (x+1)^2 + (y+1)^2 + (z+1)^2 \\
&= 3 + x^2 + y^2 + z^2 + 2(x + y + z) \\
&= 3 + x^2 + y^2 + z^2 + 2(xy + yz + zx + 1) \\
&= 5 + (x + y + z)^2 \geq 5.
\end{aligned}
$$

Solution 9.30. Suppose $a \leq b \leq c$, then $a^2 \leq bc$, therefore

$$a^2 + 1 \leq bc + 1 \leq 2a. \tag{10.23}$$

On the other hand, since $(a-1)^2 \geq 0$, we have $a^2 \geq 2a-1$. The last two inequalities imply that $a^2 = 2a - 1$, that is, $a = 1$.
The original second inequality can be rewritten, using that $a = 1$, as $bc \leq 1$. But since $1 \leq b \leq c$, we also have that $bc \geq 1$, therefore $bc = 1$.
The first and third inequalities becomes $b + 1 \leq 2c$ and $c + 1 \leq 2b$, therefore $(b+1)(c+1) \leq 4bc = 4$. If we expand and cancel out some terms, we get $b+c \leq 2$. The inequality between the geometric mean and the arithmetic mean, and the previous inequality, guarantee that $1 = bc \leq \left(\frac{b+c}{2}\right)^2 \leq 1$, then the equality holds, which is true only if $b = c$. Since $bc = 1$, then $b = c = 1$. Therefore, $a = b = c = 1$ is the only solution.

Solution 9.31. Without loss of generality, we can assume that $a = \max\{a, b, c\}$. Then $c(\sqrt{b} - 1) \leq a(\sqrt{b} - 1) = c$, hence $b \leq 4$.
We also have $b(\sqrt{c}-1) = a \geq b$, then $c \geq 4$. Now, $4 \leq c \leq c(\sqrt{c}-1) \leq c(\sqrt{a}-1) = b \leq 4$, then $b = c = 4$ and also $a = 4$. Therefore, there is a unique triplet that satisfies the equations, that is, $(4, 4, 4)$.

Solution 9.32. If $\{x_1, \dots, x_n\}$ is a real solution of the system, it is clear that also $\{x_2, x_3, \dots, x_n, x_1\}$ is a solution. But, by hypothesis, we only have one solution, then $x_1 = x_2 = \cdots = x_n$. The system reduces to only one equation $ax^2 + (b - 1)x + c = 0$, which has a unique solution if $(b - 1)^2 - 4ac = 0$.
Reciprocally, if $(b - 1)^2 - 4ac = 0$, the polynomial $P(x) = ax^2 + (b - 1)x + c = a\left(x + \frac{b-1}{2a}\right)^2$ has only one solution.

Adding the equations of the system, we have $\sum_{i=1}^{n} P(x_i) = 0$, but since either all the numbers have the same sign or are zero, then $P(x_i) = 0$ for every x_i. Hence $x_i = -\left(\frac{b-1}{2a}\right)$, and the system has as a unique solution $x_1 = x_2 = \cdots = x_n = -\left(\frac{b-1}{2a}\right)$.

Solution 9.33. If one of the variables x, y or z is equal to 1 or -1, then we obtain the solutions $(1, 1, 1)$ or $(-1, -1, -1)$, respectively. Now we will see that these are the only solutions of the system.

Let $f(t) = t^2 + t - 1$. If one of the variables x, y or z is greater than 1, for example $x > 1$, then we have $x < f(x) = y < f(y) = z < f(z) = x$, which is not possible. Therefore x, y, $z \leq 1$.

If one of the variables x, y or z is less than -1, choose $x < -1$. Since $f(t) = \left(t + \frac{1}{2}\right) - \frac{5}{4} \geq -\frac{5}{4}$, we then get $x = f(z) \in \left[-\frac{5}{4}, -1\right]$. But,

$$f\left(\left[-\frac{5}{4}, -1\right)\right) = \left(-1, \frac{-11}{16}\right) \subset (-1, 0) \quad \text{and} \quad f((-1, 0)) = \left[-\frac{5}{4}, -1\right),$$

then it follows that $y = f(x) \in (-1, 0)$, $z = f(y) \in \left[-\frac{5}{4}, -1\right)$ and $x = f(z) \in (-1, 0)$, which is a contradiction. Then, $-1 \le x, y, z \le 1$.

If $-1 < x, y, z < 1$, then $x > f(x) = y > f(y) = z > f(z) = x$, which is not possible. Therefore, there are no other solutions.

Solution 9.34. Add the n equations to obtain

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i - n = \sum_{i=1}^{n} x_i,$$

from where $\sum_{i=1}^{n} x_i^2 = n$.

On the other hand, rewrite the equations as follows:

$$\begin{aligned} x_1^2 + x_1 &= x_2 + 1 \\ x_2^2 + x_2 &= x_3 + 1 \\ \vdots \quad & \quad\ \vdots \\ x_{n-1}^2 + x_{n-1} &= x_n + 1 \\ x_n^2 + x_n &= x_1 + 1 \end{aligned}$$

and multiply the equations to obtain

$$\prod_{i=1}^{n} x_i(x_i + 1) = \prod_{i=1}^{n} (x_i + 1),$$

so that $\prod_{i=1}^{n} x_i = 1$ if $x_i \neq -1$, for all $1 \le i \le n$.

From the two equations, $\sum_{i=1}^{n} x_i^2 = n$ and $\prod_{i=1}^{n} x_i = 1$, we obtain, using the inequality between the geometric mean and the arithmetic mean, that

$$1 = \frac{n}{n} = \frac{\sum_{i=1}^{n} x_i^2}{n} \ge \sqrt[n]{\prod_{i=1}^{n} x_i^2} = 1,$$

and it follows that $x_1^2 = x_2^2 = \cdots = x_n^2 = 1$. Then, a possible solution is $(1, 1, \ldots, 1)$.

If some $x_i = -1$, by the symmetry of the equations, we can assume that $x_1 = -1$, and then it is easy to see that all $x_i = -1$. Hence the only other solution is $(-1, -1, \ldots, -1)$.

Solution 9.35. The only solution, with all the x_i's equal, is clearly $(2, 2, \ldots, 2)$. If there is another solution, let m and M be the minimum and maximum value of the x_i, respectively. Then $m < M$, and for some indexes j, k (taken modulo n), we would have $m^2 = x_j + x_{j+1} \ge 2m$ and $M^2 = x_k + x_{k+1} \le 2M$, from where $2 \le m < M \le 2$, which is absurd. Therefore $(2, 2, \ldots, 2)$ is the only solution.

Solution 9.36. The condition $x+y+z=0$ implies that $x^3+y^3+z^3=3xyz$. The condition $x^{-1}+y^{-1}+z^{-1}=0$ guarantees that $x^{-3}+y^{-3}+z^{-3}=3x^{-1}y^{-1}z^{-1}$. Now, since $x^6+y^6+z^6=(x^3+y^3+z^3)^2-2x^3y^3z^3(x^{-3}+y^{-3}+z^{-3})$, we have $x^6+y^6+z^6=9x^2y^2z^2-6x^2y^2z^2=3x^2y^2z^2$. Thus the result follows if we use again $x^3+y^3+z^3=3xyz$.

Solution 9.37. Since the sum of the numbers is zero, $d=-a-b-c$, therefore

$$\begin{aligned}
bc-ad&=bc+a(a+b+c)=a(a+b)+c(a+b)=(a+c)(a+b)\\
ac-bd&=ac+b(a+b+c)=b(a+b)+c(a+b)=(b+c)(a+b)\\
ab-cd&=ab+c(a+b+c)=c(c+a)+b(c+a)=(b+c)(c+a).
\end{aligned}$$

Then, $(bc-ad)(ac-bd)(ab-cd)=(a+b)^2(b+c)^2(c+a)^2$.

Solution 9.38. We have

$$\begin{aligned}
\sum_{\text{cyclic}}\frac{a^3}{bc}-\sum_{\text{cyclic}}\frac{a^2+b^2}{a+b}&=\sum_{\text{cyclic}}\frac{a^4}{abc}+\sum_{\text{cyclic}}\frac{a^2+b^2}{c}=\sum_{\text{cyclic}}\frac{a^4+ab(a^2+b^2)}{abc}\\
&=\frac{1}{abc}(a^4+b^4+c^4+ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2))\\
&=\frac{1}{abc}(a+b+c)(a^3+b^3+c^3)=0.
\end{aligned}$$

Solution 9.39. Since

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

and since $a+b+c>2$, then

$$a+b+c=p \ \text{ and } \ \left[(a-b)^2+(b-c)^2+(c-a)^2\right]=2.$$

Suppose that $a\geq b\geq c$. If $a>b>c$, then $a-b\geq 1$, $b-c\geq 1$ and $a-c\geq 2$, and this leads to $\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\geq 6>2$, which is absurd, therefore $a=b=c+1$ or $a-1=b=c$. Thus, we have that the prime number is of the form $p=3c+2$, in the first case or of the form $p=3c+1$, in the second case. Then the triplet is $(\frac{p+1}{3},\frac{p+1}{3},\frac{p-2}{3})$ in the first case, and $(\frac{p+2}{3},\frac{p-1}{3},\frac{p-1}{3})$ in the second case.

Solution 9.40. The equation is equivalent to

$$(3x)^3+(-3y)^3+(-1)^3-3(3x)(-3y)(-1)=1646,$$

which can be factorized as $(3x-3y-1)(9x^2+9y^2+1+9xy+3x-3y)=2\cdot 823$. Now, the first factor on the left-hand side is smaller than the second factor and, since 823 is a prime number and $3x-3y-1\equiv 2 \mod 3$, we get $3x-3y-1=2$ and $9x^2+9y^2+1+9xy+3x-3y=823$. Solving the system for positive real numbers leads to $x=6$ and $y=5$.

Solution 9.41. Using the identity (4.8), the condition $x^3 + y^3 + z^3 - 3xyz = 1$ is equivalent to

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 1. \tag{10.24}$$

Let $A = x^2 + y^2 + z^2$ and $B = x + y + z$. Observe that $B^2 - A = 2(xy + yz + zx)$. By identity (4.9) we have that $B > 0$. The equation (10.24) becomes

$$B\left(A - \frac{B^2 - A}{2}\right) = 1,$$

then $3A = B^2 + \frac{2}{B}$. Since $B > 0$, we apply the inequality between the geometric mean and the arithmetic mean to obtain $3A = B^2 + \frac{2}{B} = B^2 + \frac{1}{B} + \frac{1}{B} \geq 3$, that is, $A \geq 1$. The minimum $A = 1$ is reached, for example, with $(x, y, z) = (1, 0, 0)$.

Solution 9.42. Equations $x + \frac{y}{z} = 2$, $y + \frac{z}{x} = 2$, $z + \frac{x}{y} = 2$, imply that

$$zx + y = 2z, \qquad xy + z = 2x, \qquad yz + x = 2y$$

and that

$$xyz + y^2 = 2yz, \qquad xyz + z^2 = 2zx, \qquad xyz + x^2 = 2xy.$$

Therefore,

$$xy + yz + zx = x + y + z, \tag{10.25}$$

$$3xyz + (x^2 + y^2 + z^2) = 2(xy + yz + zx). \tag{10.26}$$

We also have that

$$\begin{aligned} 1 = \frac{y}{z}\frac{z}{x}\frac{x}{y} &= (2 - x)(2 - y)(2 - z) \\ &= 8 - 4(x + y + z) + 2(xy + yz + zx) - xyz. \end{aligned} \tag{10.27}$$

If we define $a = x + y + z$, we have by equation (10.25), that $xy + yz + zx = a$ and we also get that

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) = a^2 - 2a.$$

Now, from equation (10.26), it follows that $3xyz = -a^2 + 4a$. Finally, by equation (10.27), we can conclude that

$$1 = 8 - 4a + 2a - \frac{-a^2 + 4a}{3}.$$

Hence, we get the equation $a^2 - 10a + 21 = 0$, which has roots $a = 3$ and $a = 7$. Therefore, $x + y + z$ is equal to 3 or 7.
But, if $x+y+z = 7$, since $x + \frac{y}{z} + y + \frac{z}{x} + z + \frac{x}{y} = 6$, we have that $\frac{y}{z} + \frac{z}{x} + \frac{x}{y} = -1$, which is not possible for x, y, z positive numbers.

The sum $x + y + z = 3$, can be achieved with $x = y = z = 1$, which are also solutions of the equations, hence the only possible value for $x + y + z$ is 3.

Solution 9.43. Let d be the common difference of the progression $\{a_n\}$. Note that, for $j = 0, 1, \ldots, n-1$, we have that

$$\frac{1}{\sqrt{a_{j-1}} + \sqrt{a_j}} = \frac{1}{\sqrt{a_{j-1}} + \sqrt{a_j}} \cdot \frac{\sqrt{a_{j-1}} - \sqrt{a_j}}{\sqrt{a_{j-1}} - \sqrt{a_j}} = \frac{\sqrt{a_j} - \sqrt{a_{j-1}}}{a_j - a_{j-1}}.$$

Use the fact that $a_j - a_{j-1} = d$, for all j, in order to get

$$\sum_{j=1}^{n} \frac{1}{\sqrt{a_{j-1}} + \sqrt{a_j}} = \sum_{j=1}^{n} \frac{\sqrt{a_j} - \sqrt{a_{j-1}}}{d} = \frac{\sqrt{a_n} - \sqrt{a_0}}{d}.$$

Finally, observe that

$$\frac{\sqrt{a_n} - \sqrt{a_0}}{d} = \frac{a_n - a_0}{d(\sqrt{a_0} + \sqrt{a_n})} = \frac{a_0 + nd - a_0}{d(\sqrt{a_0} + \sqrt{a_n})} = \frac{n}{\sqrt{a_0} + \sqrt{a_n}}.$$

Solution 9.44. Let $d > 0$ be the common difference of the progression. Suppose that $a = b - d$, b and $c = b + d$ are the lengths of the sides of the triangle. Since $c^2 = a^2 + b^2$, we have $(b + d)^2 = (b - d)^2 + b^2$, therefore $b = 4d$. On the other hand, the area of the triangle is $\frac{a \cdot b}{2} = \frac{(b-d)4d}{2} = \frac{12d^2}{2} = 6d^2$, and it is also equal to the inradius r multiplied by the semiperimeter[31] $s = \frac{3d+4d+5d}{2} = 6d$. From this, $r \cdot 6d = 6d^2$, and then $r = d$.

Solution 9.45. Suppose that $\{1, 2, \ldots, 9\}$ has been divided into two subsets A and B such that neither of them contains an arithmetic progression. Suppose that $5 \in A$. It is clear that 1 and 9 cannot both be in A. Then, we have the following cases:

(i) If $1 \in A$ and $9 \in B$. Since $\{1, 5\} \subset A$, we have that $3 \in B$; $3, 9 \in B$ imply that $6 \in A$; $5, 6 \in A$ imply that $4, 7 \in B$; $3, 4 \in B$ imply that $2 \in A$; $7, 9 \in B$ imply that $8 \in A$. But, $\{2, 5, 8\} \subset A$ is an arithmetic progression, which is absurd.

(ii) If $9 \in A$ and $1 \in B$. This case is analogous to (i).

(iii) If $1, 9 \in B$. Then we have two subcases:

  (1) If $7 \in A$. In this case, $5, 7 \in A$ imply $6 \in B$ and $3 \in B$. Therefore, $\{3, 6, 9\} \subset B$, which is absurd.

  (2) If $7 \in B$. Since $7, 9 \in B$, we have that $8 \in A$; $1, 7 \in B$ imply $4 \in A$; $4, 5 \in A$ imply $3 \in B$; $1, 3 \in B$ imply that $2 \in A$. Therefore, again we have an arithmetic progression, that is, the progression $\{2, 5, 8\}$ is in A, which is a contradiction.

Solution 9.46. Observe that

$$\begin{aligned} &\frac{2}{9}(a + b + c)^3 - a^2(b + c) - b^2(c + a) - c^2(a + b) \\ &\qquad = \frac{1}{9}\left(a + b - 2c\right)\left(2a - b - c\right)\left(a - 2b + c\right). \end{aligned}$$

[31] See [5].

Solution 9.47. Suppose the number of prime numbers is not infinite. Let p be the greatest prime number of the progression. Consider the number $n = 4p! - 1$, which belongs to the progression. Since $n > p$, the number is a composite number and it does not have prime divisors of the form $4k - 1$ (the factors of this form belong to $p!$), then its prime divisors are of the form $4k + 1$. But the product of factors of the form $4k + 1$ is also a number of the form $4k + 1$, and then n must be of this form as well, which is a contradiction.

Solution 9.48. Divide the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ in the following way

$$\begin{array}{cccccccccccc} 1 & & & 4 & 5 & 6 & & & & & 11 & \dots \\ & 2 & 3 & & & & 7 & 8 & 9 & 10 & & \dots \end{array}$$

The sets we are looking for are $A = \{1, 4, 5, 6, 11, 12, 13, 14, 15, 22, \dots\}$ and $B = \{2, 3, 7, 8, 9, 10, 16, 17, 18, 19, 20, 21, 29, \dots\}$. In fact, each one of them has "gaps" between numbers as large as we want. Therefore, it is not possible to have in some of them an arithmetic progression, since the elements of the progression have a constant difference d, which will be overtaken by a proper gap.

Solution 9.49. The answer is no. If there is an arithmetic progression with difference d, we have that the d consecutive integers

$$(d + 1)! + 2, (d + 1)! + 3, \dots, (d + 1)! + (d + 1)$$

are composite numbers. But among them there has to be an element of the progression, because this progression has difference d, which is a contradiction.
Second Solution. Let $m > 1$ be a number in the progression. Then $m + md = m(d + 1)$ is also an element of the progression and is not a prime number.

Solution 9.50. Suppose that the arithmetic progression with difference d contains a perfect square, say a^2. Then the numbers a^2, $a^2 + d$, $a^2 + 2d, \dots$ are in the progression, and then the numbers $a^2 + (2a + d)d = (a + d)^2$ are also in the progression. Now, it is clear that the square numbers of the form $(a + kd)^2$, for every $k \in \mathbb{N}$, are also in the progression.

Solution 9.51. Suppose that there are 1999 prime numbers, smaller than 12345, in arithmetic progression. Let p be the first prime number in the progression and let r be the difference of the progression. Then the progression is p, $p + r$, $p + 2r$, $\dots$, $p + 1998r$.

The prime number p cannot be one of the prime numbers $2, 3, \dots, 1997$, because if it is one of them, then $p + pr$, which is in the progression, is not a prime number. Therefore, $p \geq 1999$.
Since p is an odd number and $p + r$ is prime, then r is even. All the even numbers are of the form $6n$, $6n + 2$ or $6n - 2$. Let us see now that r cannot be of the

form $6n+2$ nor can it take the form $6n-2$. In fact, since p is prime, it is of the form $6k+1$ or $6k-1$. In any of those four cases, there is in the progression a multiple of 3, $p+r=(6k+1)+(6n+2)$, $p+2r=(6k+1)+2(6n-2)$, $p+2r=(6k-1)+2(6n+2)$, $p+r=(6k-1)+(6n-2)$.
Therefore r is of the form $6n$ and then the progression is

$$p, p+6n, \ldots, p+1998(6n).$$

But $p \geq 1999$ and $n \geq 1$ imply that $p+1998(6n) \geq 1999+11988 = 13987 > 12345$. Hence, the numbers $p+jr$ cannot be all smaller than 12345.

Solution 9.52. For $n<3$, there does not exist a rearrangement with an arithmetic triplet. For $n=3$, the list 2, 1, 3 achieves the task. We will construct an example for n, using the examples of the previous values of n. On one side of the list, we put the even numbers between 1 and n, and on the other side the odd numbers. If the even numbers are j, we rearrange them using the example for the j numbers and then we have them multiplied by 2. If there are k odd numbers, to order them we use the example for the k multiplying those numbers by 2 and subtracting 1. In this way we obtain a rearrangement of the numbers from 1 to n. If on the even side, $2a$, $2b$ and $2c$ form an arithmetic triplet, then a, b and c is also an arithmetic triplet for the case j, which is absurd. If in the odd side, $2a-1$, $2b-a$ and $2c-1$ is an arithmetic triplet, then a, b and c is also an arithmetic triplet in the example for the case of k, which is also absurd. Finally, one term on the even side and one term on the odd side satisfy that their sum is odd, then there is not a third term in between them such that its double would be this sum. Then, the constructed rearrangement does not have arithmetic triplets.

Solution 9.53. Since there are 4 solutions for the first equation, $a \neq 0$. Let x_0 be the common solution to both equations.
Taking the difference of the equations, it follows that $ax_0^4 - ax_0 = 0$, which can also be written as $ax_0(x_0^3-1)=0$.
Then, the common solution is $x_0=0$ or $x_0=1$.
If $x_0=0$ in the first equation, we obtain $a=1$, therefore x_0 is a solution with multiplicity at least 2, but this is not possible because we know that there are four different roots.

Then the common solution is $x_0=1$. When we substitute in the first equation, we obtain $2a+b=1$, and then this equation can be rewritten as

$$ax^4+(1-2a)x^2+a-1=0,$$

which has 1 and -1 as solutions. Therefore

$$(x-1)(x+1)(ax^2-a+1)=0.$$

The quadratic equation $ax^2-a+1=0$ must have 2 different real solutions, say r and $-r$, with $r>0$.

There are two cases, $a > 1$ and $a < 0$.
If $a > 1$, then $0 < r < 1$ and the roots -1, $-r$, r, 1 are in arithmetic progression only when $r = \frac{1}{3}$. In such case $a = \frac{1}{1-r^2} = \frac{9}{8}$ and $b = 1 - 2a = -\frac{5}{4}$.
If $a < 0$, then $r > 1$, and the numbers $-r$, -1, 1, r are in arithmetic progression only if $r = 3$. In such case $a = \frac{1}{1-r^2} = -\frac{1}{8}$ and $b = \frac{5}{4}$.

Solution 9.54. Write $A = \sum_{i=1}^{k} a_i$ and $B = \sum_{i=1}^{k} b_i$. Now we add over i the corresponding terms in the inequalities

$$a_i n + b_i - 1 < \lfloor a_i n + b_i \rfloor \leq a_i n + b_i,$$

to obtain $An + B - k < X_n \leq An + B$. Now, suppose that $\{X_n\}$ is an arithmetic progression with common difference d, then $nd = X_{n+1} - X_1$ and $A + B - k < X_1 \leq A + B$. Combine the above inequalities to obtain $A(n+1) + B - k < nd + X_1 \leq A(n+1) + B$ or

$$An - k \leq An + (A + B - X_1) - k < nd < An + (A + B - X_1) < An + k,$$

from which we conclude that $|A - d| < \frac{k}{n}$, for any integer number n; then $A = d$. Since $\{X_n\}$ is a sequence of integers, d has to be also an integer number, therefore we conclude that A is an integer number.

Solution 9.55. Consider a partition of $\{1, \dots, 256\}$ into two subsets, A and B. Divide $\{1, \dots, 9\}$ in two subsets A_1 and B_1, in the following way: $k \in A_1$ (resp. B_1) if and only if $2^{k-1} \in A$ (resp. B). Clearly, $A_1 \cap B_1 = \emptyset$.
If $A_1 = \emptyset$ (or $B_1 = \emptyset$), then B_1 (or A_1) has three numbers in arithmetic progression a, b and c. Then 2^{a-1}, 2^{b-1} and 2^{c-1} is a geometric progression in B (or A).
If $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$, then $A_1 \cup B_1$ is a partition of $\{1, \dots, 9\}$. By Problem 9.45, one of the sets, say A_1, contains an arithmetic progression of three terms a, b and c. Then 2^{a-1}, 2^{b-1} and 2^{c-1} is a geometric progression in A.

Solution 9.56. Remember that the nth term in a geometric progression is $a_n = a_0 \cdot q^n$, where a_0 is the first term of the progression and q is the ratio.

Since $a_{k+1} = a_k \left(1 + \frac{1}{n^2}\right)$, the given collection of numbers coincide with the first terms of a geometric progression whose ratio is $1 + \frac{1}{n^2}$ and whose first term is given by $\frac{1}{2}$.

The nth term of the collection is given by

$$a_n = \frac{1}{2}\left(1 + \frac{1}{n^2}\right)^n.$$

We have to prove that $\left(1 + \frac{1}{n^2}\right)^n < 2$, for $n > 1$. By Newton's binomial theorem (Theorem 3.2.3), we have that

$$\left(1 + \frac{1}{n^2}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{n^2}\right)^i = \sum_{i=0}^{n} T_i, \quad \text{where} \quad T_i = \binom{n}{i} \left(\frac{1}{n^2}\right)^i.$$

Now, notice that

$$\frac{T_i}{T_{i+1}} = \frac{\frac{n!}{i!(n-i)!}}{\frac{n!}{(i+1)!(n-i-1)!}} \cdot \frac{\frac{1}{n^{2i}}}{\frac{1}{n^{2i+2}}} = \frac{(i+1)}{n-i} \cdot n^2 > 1,$$

hence the sequence T_i is decreasing, then $T_i < T_1 = \frac{1}{n}$. Therefore

$$\left(1+\frac{1}{n^2}\right)^n = 1 + T_1 + T_2 + T_3 + \cdots + T_n < 1 + nT_1 = 2.$$

Solution 9.57. Suppose that $a_1 = a$.

By induction we will see that $a_n = a+n-1$. Suppose that $a_n = a+n-1$ and prove that $a_{n+i} = a+n+i-1$, for $1 \leq i \leq n$. Since $a_{2n} = a_n + n$, the hypothesis leads us to $a_n = a+n-1$, then $a_{2n} = a+2n-1$. Now, since the sequence is increasing, it follows that

$$a+n-1 = a_n < a_{n+1} < \cdots < a_{n+i} < \cdots < a_{2n} = a+2n-1.$$

Hence

$$a_{n+i} = a+n+i-1 \quad \text{for} \quad 1 \leq i \leq n. \tag{10.28}$$

If $a_1 = 1$, then we get $a_n = n$, for all $n \geq 1$.
What remains to be proved is the induction basis, that is, $a_1 = 1$.
Suppose that $a_1 > 1$. Let p be the least prime number greater than $(a_1+1)! + a_1 + 1$. Of course, this prime number p is an element of the sequence, that is, $p = a_n = a_1 + (n-1)$, for some n. By equation (10.28), the a_n are consecutive numbers. Moreover, by property (ii), $n = p - a_1 + 1$ is also a prime number. Since $a_1 > 1$, $p - a_1 + 1 < p$, and by the way we have chosen p, we have that $p - a_1 + 1 \leq (a_1+1)! + a_1 + 1 < p$. Then $(a_1+1)!+2 \leq p - a_1 + 1 \leq (a_1+1)! + a_1 + 1$. But this is a contradiction, since among the numbers

$$(a_1+1)!+2, (a_1+1)!+3, \ldots, (a_1+1)!+a_1+1$$

there are no prime numbers, that is, all the numbers are composite numbers, since j divides $((a_1+1)!+j)$. The contradiction proves that $a_1 = 1$.

Solution 9.58. The proof is by induction over $n+m$. The statement is clear if $m+n = 2$. Suppose that the statement is true for $m+n < k$, and consider $m+n = k$ arbitrary numbers $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$. Define the sets

$$A = \{a_1, a_2, \ldots, a_m\}, \quad B = \{b_1, b_2, \ldots, b_n\},$$
$$C = \{c_1, c_2, \ldots, c_m\}, \quad D = \{d_1, d_2, \ldots, d_n\}.$$

We have two cases:

1. If $A \cap C \neq \emptyset$ or $B \cap D \neq \emptyset$. For example, suppose that $A \cap C \neq \emptyset$. This implies that $a_i = c_j$, for some indices $i, j \in \{1, 2, \ldots, m\}$. Without loss of

generality, let $i \leq j$. If $i = j$, one of the terms in the equality we want to prove is zero and, then we can apply directly the induction hypothesis. If $i < j$, then $c_i < \cdots < c_j = a_i < a_{i+1} < \cdots < a_j$, then

$$|a_i - c_i| + |a_{i+1} - c_{i+1}| + \cdots + |a_j - c_j| = (a_i - c_i) + (a_{i+1} - c_{i+1}) + \cdots + (a_j - c_j).$$

If we change the order of the terms, the value of the sum does not change, and this value is equal to

$$\begin{aligned}&(a_{i+1} - c_i) + (a_{i+2} - c_{i+1}) + \cdots + (a_j - c_{j-1}) + (a_i - c_j)\\ &\qquad = |a_{i+1} - c_i| + |a_{i+2} - c_{i+1}| + \cdots + |a_{j-1} - c_{j-2}| + |a_j - c_{j-1}|,\end{aligned}$$

since $a_i - c_j = 0$. Then the result follows from the induction hypothesis, for the $m + n - 1 = k - 1$ numbers

$$\begin{aligned}&a_1 < a_2 < \cdots < a_{i-1} < a_{i+1} < \cdots < a_m, \quad b_1 < b_2 < \cdots < b_n,\\ &c_1 < c_2 < \cdots < c_{j-1} < c_{j+1} < \cdots < c_m, \quad d_1 < d_2 < \cdots < d_n.\end{aligned}$$

2. If $A \cap C = B \cap D = \emptyset$. In this case, we have that a_1 is in D and b_1 is in C. Without loss of generality, suppose that $a_1 < b_1$. Then a_1 has to be equal to d_1. We will prove that b_1 has to be equal to c_1. If $b_1 = c_i$, for some $i > 1$, then $b_1 > c_1$ and, since $c_1 = b_j$, for some $j > 1$, we have that $b_1 > b_j$, which is a contradiction. Therefore, $b_1 = c_1$, and taking into account that $a_i = d_1$, this implies $|a_1 - c_1| = |b_1 - d_1|$. Now, we use the induction hypothesis for the numbers

$$\begin{aligned}&a_2 < a_3 < \cdots < a_m, \quad b_2 < b_3 < \cdots < b_n,\\ &c_2 < c_3 < \cdots < c_m, \quad d_2 < d_3 < \cdots < d_n.\end{aligned}$$

Solution 9.59. First, observe that for $n \geq 2$, we have

$$\sum_{j=1}^{n-1} \frac{j}{(j+1)!} = \sum_{j=1}^{n-1} \left(\frac{1}{j!} - \frac{1}{(j+1)!} \right) = 1 - \frac{1}{n!}. \tag{10.29}$$

Multiplying by $n!$ the equation (10.29), we obtain the desired representation

$$1 + \sum_{j=1}^{n-1} \frac{j \cdot n!}{(j+1)!} = n!.$$

Solution 9.60. Remember that every integer number relatively prime to p has a multiplicative inverse modulo p. Denote the inverse of x modulo p by x^{-1}. Observe that

$$\binom{2p}{p} - 2 = \sum_{k=1}^{p-1} \binom{p}{k}^2 = \sum_{k=1}^{p-1} \left(\frac{p}{k} \binom{p-1}{k-1} \right)^2.$$

Observe that $\frac{1}{k}\binom{p-1}{k-1}$ is an integer number, since it is equal to $\frac{1}{p}\binom{p}{k}$ and p divides $\binom{p}{k}$. Then the last sum is congruent, modulo p^3, to $p^2\sum_{k=1}^{p-1}\left(k^{-1}\binom{p-1}{k-1}\right)^2$. We prove that the sum is divisible by p. Observe that, modulo p, we have

$$\begin{aligned}\sum_{k=1}^{p-1}\left(k^{-1}\binom{p-1}{k-1}\right)^2 &\equiv \sum_{k=1}^{p-1}(k^{-1})^2\{(p-1)(p-2)\cdots(p-k+1)\\ &\quad\times[(k-1)(k-2)\cdots(p-k+1)]^{-1}\}^2\\ &\equiv \sum_{k=1}^{p-1}(k^{-1})^2\left[(-1)(1)^{-1}(-2)(2)^{-1}\cdots(-(k-1))(k-1)^{-1}\right]^2\\ &\equiv \sum_{k=1}^{p-1}(k^{-1})^2(-1)^{2k-2}.\end{aligned}$$

But the inverse numbers of 1, 2, ..., $p-1$, modulo p, are the same numbers in some other order. Then the sum is congruent to $\sum_{k=1}^{p-1}k^2$, which is equal to $(p-1)p(2p-1)/6$. This is divisible by p, since $p\neq 2,3$. Then $\binom{2p}{p}-2$ is divisible by p^3. Since

$$\binom{2p-1}{p-1}-1=\frac{1}{2}\left(\binom{2p}{p}-2\right),$$

it follows that $\binom{2p-1}{p-1}-1$ is divisible by p^3, as we wanted.

Solution 9.61. If $\frac{a}{b}\in S$, then by (ii), $\frac{1}{\frac{a}{b}+1}=\frac{b}{a+b}\in S$ and $\frac{\frac{a}{b}}{\frac{a}{b}+1}=\frac{a}{a+b}\in S$.
In particular, if $\frac{c}{d-c}\in S$, then $\frac{c}{d-c+c}=\frac{c}{d}\in S$ and if $\frac{d-c}{c}\in S$, then $\frac{c}{d-c+c}=\frac{c}{d}\in S$.
Consider a rational number $q_0=\frac{a_0}{b_0}$, with $(a_0,b_0)=1$ and $0<q_0<1$. Then it is enough to see that either $\frac{a_0}{b_0-a_0}$ or $\frac{b_0-a_0}{a_0}$ belongs to S.

If $q_0=\frac{1}{2}$ there is nothing to do. If $q_0<\frac{1}{2}$, then $a_0<b_0-a_0$. If $q_0>\frac{1}{2}$, then $b_0-a_0<a_0$.

Let

$$q_1=\begin{cases}\frac{a_0}{b_0-a_0}, & \text{if } q_0<\frac{1}{2}\\ \frac{b_0-a_0}{a_0}, & \text{if } q_0>\frac{1}{2},\end{cases}$$

then $0<q_1<1$ and if $q_1\in S$, then $q_0\in S$.
Now, if $q_1=\frac{a_1}{b_1}$, with $(a_1,b_1)=1$, and considering

$$q_2=\begin{cases}\frac{a_1}{b_1-a_1}, & \text{if } q_1<\frac{1}{2},\\ \frac{b_1-a_1}{a_1}, & \text{if } q_1>\frac{1}{2},\end{cases}$$

it follows that $0<q_2<1$ and if $q_2\in S$, then $q_1\in S$.
This process of going from q_k to q_{k+1} is possible if no q_k is equal to $\frac{1}{2}$; otherwise, the proof is complete ($q_k\in\frac{1}{2}\in S$ implies that $q_{k-1},\ldots,q_0\in S$).

The process cannot be infinite. If $q_0 < \frac{1}{2}$, then $\frac{a_1}{b_1} = q_1 = \frac{a_o}{b_0 - a_0}$, and $b_1 \mid b_0 - a_0$, hence $b_1 \leq b_0 - a_0 < b_0$. If $q_0 > \frac{1}{2}$, then $b_1 \mid a_0$ and $b_1 \leq a_0 < b_0$. Hence, in any case $b_1 < b_0$. Similarly, it follows that $b_{k+1} < b_k$, for all $k \geq 0$. Thus, $\{b_k\}$ is a decreasing infinite sequence of positive integers, which is impossible.

Solution 9.62. Suppose $\frac{a^2+b^2}{ab+1} = k$ and k is not a perfect square. Then $a^2 - kab + b^2 = k$. Suppose (a_0, b_0) is a solution of the equation.
By symmetry, we can assume that $a_0 \geq b_0 > 0$. We know that a_0 is a root of the quadratic equation $a^2 - kab_0 + b_0^2 - k = 0$. Let c_1 be the other root of the previous equation; the two roots satisfy $a_0 + c_1 = kb_0$ and $a_0c_1 = b_0^2 - k$, then $c_1 = kb_0 - a_0$ is an integer. Now, since k is not a square, $a_0c_1 \neq 0$, hence $c_1 \neq 0$.
If $c_1 < 0$, then $c_1^2 - kc_1b_0 + b_0^2 \geq c_1^2 + k + b_0^2 > k$, which is a contradiction, therefore $c_1 > 0$. Moreover, $c_1 = \frac{b_0^2 - k}{a_0} \leq \frac{b_0^2 - 1}{a_0} \leq \frac{a_0^2 - 1}{a_0} < a_0$. Thus (c_1, b_0) is a positive solution of $a^2 - kab + b^2 = k$, with $c_1 < a_0$. Proceed in the same way to construct a decreasing sequence of positive integers $a_0 > c_1 > c_2 > \cdots > 0$, but this cannot happen.

Solution 9.63. Since a, b, c are the roots of $P(x)$, by Vieta's formulas it follows that $P(x) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc = x^3 - 2007x + 2002$, and then $a+b+c = 0$, $ab+bc+ca = -2007$, $abc = -2002$.
Hence,

$$\begin{aligned}\left(\frac{a-1}{a+1}\right)\left(\frac{b-1}{b+1}\right)\left(\frac{c-1}{c+1}\right) &= \frac{abc - (ab+bc+ca) + a+b+c-1}{abc + ab+bc+ca + a+b+c+1} \\ &= \frac{-2002 - (-2007) - 1}{-2002 + (-2007) + 1} = \frac{4}{-4008} = -\frac{1}{1002}.\end{aligned}$$

Solution 9.64. If $x = a+b+c$, $y = abc$, $z = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ are integers, also $ab + bc + ca = abc(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) = yz$ is an integer. Moreover, a, b, c are the roots of the polynomial $w^3 - xw^2 + yzw - y = 0$. Since the coefficients of the polynomial are integers and the coefficient of w^3 is 1, by Gauss' lemma or by the rational root theorem, a, b, c are integers. Suppose that $1 \leq a \leq b \leq c$, since $1 \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{3}{a}$, it follows that $a = 1$, 2 or 3. If $a = 1$, then $\frac{1}{b} + \frac{1}{c} \leq 2$, and the only possibilities are $(b, c) = (1, 1)$ or $(2, 2)$. If $a = 2$, then $\frac{1}{2} \leq \frac{1}{b} + \frac{1}{c} \leq 1$, hence $(b, c) = (3, 6)$ or $(4, 4)$. If $a = 3$, $\frac{1}{b} + \frac{1}{c} = \frac{2}{3}$, and the only solution is $(b, c) = (3, 3)$. Thus, the solutions are $(a, b, c) = (1, 1, 1)$, $(1, 2, 2)$, $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$.

Solution 9.65. If one of a, b or c is zero, then one of the equations is linear and this one has a real solution. The discriminant of the equations are $4b^2 - 4ca$, $4c^2 - 4ab$, $4a^2 - 4bc$. Then, given that

$$(4b^2 - 4ca) + (4c^2 - 4ab) + (4a^2 - 4bc) = 2\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \geq 0,$$

one of the discriminants is non-negative and therefore the equation with that discriminant has a real solution.

Solution 9.66. Let u, v be the roots of the quadratic polynomial and let α, β, γ be the roots of the cubic polynomial, with all of these roots being non-negative. By Vieta's formulas,

$$\begin{gathered} u+v=4, \qquad uv=-\frac{2}{3}a, \\ \alpha+\beta+\gamma=-a, \quad \alpha\beta+\beta\gamma+\gamma\alpha=b, \quad \alpha\beta\gamma=8. \end{gathered}$$

Now, using the inequality between the geometric mean and the arithmetic mean,

$$\begin{gathered} 4=\left(\frac{4}{2}\right)^2=\left(\frac{u+v}{2}\right)^2 \geq uv = -\frac{2}{3}a, \\ -\frac{2}{3}a=\frac{2}{3}(\alpha+\beta+\gamma)\geq 2\sqrt[3]{\alpha\beta\gamma}=4. \end{gathered}$$

Hence, on both inequalities the equality holds, and then $u=v$, $\alpha=\beta=\gamma$ and $-\frac{2}{3}a=4$. Hence $a=-6$, $\alpha=\beta=\gamma=2$, therefore $b=12$. Thus, the only pair is $(a,b)=(-6,12)$.

Solution 9.67. We can choose appropriate signs in $\pm P(\pm x)$ in order to assume that a, $b\geq 0$.

(i) There are two cases:

(1) $c\geq 0$, $|a|+|b|+|c|=a+b+c=P(1)\leq 1$.
(2) $c<0$, $|a|+|b|+|c|=a+b-c=P(1)-2P(0)\leq 3$.

Then, 3 is the maximum that is attained with the polynomial $P(x)=2x^2-1$.

(ii) There are four cases:

(1) $c\geq 0$, $d\geq 0$, $|a|+|b|+|c|+|d|=a+b+c+d=P(1)\leq 1$.
(2) $c\geq 0$, $d<0$, $|a|+|b|+|c|+|d|=a+b+c-d=P(1)-2P(0)\leq 3$.
(3) $c<0$, $d\geq 0$, $|a|+|b|+|c|+|d|=a+b-c+d=\frac{4}{3}P(1)-\frac{1}{3}P(-1)+\frac{8}{3}P(\frac{1}{2})+\frac{8}{3}P(-\frac{1}{2})\leq 7$.
(4) $c,d<0$, $|a|+|b|+|c|+|d|=a+b-c-d=\frac{5}{3}P(1)-4P(\frac{1}{2})+\frac{4}{3}P(-\frac{1}{2})\leq 7$.

With the polynomial $P(x)=4x^3-3x$, the maximum 7 in this case, is attained.

Solution 9.68. Notice that $dx^3+cx^2+bx+a=x^3(d+c\frac{1}{x}+b\frac{1}{x^2}+a\frac{1}{x^3})$; then its roots are the inverse of the roots of ax^3+bx^2+cx+d. If $\{\alpha,\beta,\gamma\}$ are the roots of the polynomial ax^3+bx^2+cx+d, it follows, by Vieta's formulas, that $\alpha+\beta+\gamma=-\frac{b}{a}$ and $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=-\frac{c}{d}$. Using the inequality between the geometric mean and the arithmetic mean, we can conclude that $\frac{bc}{ad}=(\alpha+\beta+\gamma)(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma})\geq 9$.

Solution 9.69. Given that $2\sqrt{x^2-1} = x - \sqrt{x^2-p}$, and squaring both sides, we get that

$$\begin{aligned}
4(x^2-1) &= x^2 - 2x\sqrt{x^2-p} + x^2 - p\\
4x^2 - 4 &= 2x^2 - 2x\sqrt{x^2-p} - p\\
2x^2 + (p-4) &= -2x\sqrt{x^2-p},
\end{aligned}$$

then $2x^2 + (p-4) < 0$, since $x > 0$. Squaring the last equation and simplifying, it follows that

$$\begin{aligned}
4x^4 + 4x^2(p-4) + (p-4)^2 &= 4x^2(x^2-p)\\
8x^2(p-2) + (p-4)^2 &= 0\\
x &= \pm\frac{4-p}{\sqrt{8(2-p)}}.
\end{aligned}$$

Since x is a positive real number, then $p < 2$ and $x = \frac{4-p}{\sqrt{8(2-p)}}$.

Solution 9.70. Let α, β, γ be the roots of $P(x)$ that are positive. From Vieta's formulas, it follows that $\alpha+\beta+\gamma = -\frac{b}{a}$, $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$, $\alpha\beta\gamma = -\dfrac{d}{a}$. Since $d = P(0) < 0$ and $\alpha\beta\gamma > 0$, it follows that $a > 0$. Dividing by a^3, the inequality to be proved, it is enough to see that

$$2\left(\frac{b}{a}\right)^3 + 9\left(\frac{d}{a}\right) - 7\left(\frac{bc}{a^2}\right) \le 0,$$

that in terms of α, β, γ is

$$-2\left(\alpha+\beta+\gamma\right)^3 - 9\alpha\beta\gamma + 7\left(\alpha+\beta+\gamma\right)\left(\alpha\beta+\beta\gamma+\gamma\alpha\right) \le 0.$$

After simplifying, the left-hand side of the previous inequality is

$$\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2 \le 2\left(\alpha^3+\beta^3+\gamma^3\right).$$

This inequality follows from the rearrangement inequality applied in the following way,

$$\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha \le \alpha^3+\beta^3+\gamma^3,\ \beta^2\alpha + \gamma^2\beta + \alpha^2\gamma \le \alpha^3+\beta^3+\gamma^3.$$

Solution 9.71. By Vieta's formulas, it follows that $-a = \alpha+\beta+\gamma$, $b = \alpha\beta+\beta\gamma+\gamma\alpha$ and $-c = \alpha\beta\gamma$. Since $\alpha^2 = \beta+\gamma$, then $\alpha^2 = -\alpha-a$. Observe that $\alpha \ne 0$, otherwise $\alpha = 0$, and then $c = 0$, contradicting the fact that c is odd. Hence $\beta\gamma = -\frac{c}{\alpha}$ and $b = \alpha(\beta+\gamma) + \beta\gamma = \alpha^3 - \frac{c}{\alpha}$.

Therefore,

$$\begin{aligned}0 &= \alpha^4 - b\alpha - c = (\alpha+a)^2 - b\alpha - c = \alpha^2 + 2\alpha a + a^2 - b\alpha - c\\ &= -\alpha - a + 2\alpha a + a^2 - b\alpha - c = \alpha(2a-b-1) - (c - a^2 + a).\end{aligned}$$

Observe that $2a - b - 1 \neq 0$, otherwise $2a = b + 1$, and since b is even, $2a$ would be odd, a contradiction. Hence, from the previous equation, it follows that $\alpha = \frac{c-a(a-1)}{2a-b-1}$ is rational, and then it is an integer since the polynomial is monic and it has integer coefficients.
On the other hand, $-a = \alpha(\alpha+1)$ is even. If $\beta = \gamma$, then $2\beta = \beta + \gamma = -a - \alpha$. If β is rational, it must be an integer; then from equation $2\beta = -a - \alpha$, it follows that α is even, but then $c = -\alpha\beta\gamma$ is even, a contradiction since c is odd.

Solution 9.72. Let x_0 be a real solution of equation $x^2 + px + q = 0$. Then

$$x_0 = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \leq \frac{1 + \sqrt{1 + 4 \cdot 1}}{2} = \frac{1 + \sqrt{5}}{2} = s.$$

Notice that x_0 could be equal to s if $p = q = -1$.
If y is a real root of an equation of the form $t^2 + pt + q = 0$, with $p, q \in [-1, 1]$, then any number z with absolute value less than or equal to the absolute value of y, would be a root too. To see this, let $y^2 + py + q = 0$, for $p, q \in [-1, 1]$, and let $z = \alpha y$, where $|\alpha| \leq 1$. The equation $t^2 + \alpha pt + \alpha^2 q = 0$ has coefficients αp and $\alpha^2 q$ in the interval $[-1, 1]$, since $|\alpha| \leq 1$. Moreover, z is a root of this equation, since

$$z^2 + \alpha pz + \alpha^2 q = (\alpha y)^2 + \alpha p(\alpha y) + \alpha^2 q = \alpha^2(y^2 + py + q) = 0.$$

Therefore, all solutions of the quadratic equations belong to the interval $\left[-\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$.

Solution 9.73. Define $f : \mathbb{R} \to \mathbb{N}$ by

$$f(x) = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor - \lfloor nx \rfloor.$$

We have to show that $f(x) = 0$. Observe that

$$\begin{aligned}f\left(x + \frac{1}{n}\right) &= \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{1}{n} + \frac{1}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{1}{n} + \frac{n-1}{n} \right\rfloor - \left\lfloor n\left(x + \frac{1}{n}\right) \right\rfloor\\ &= \left\lfloor x + \frac{1}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor + \lfloor x + 1 \rfloor - \lfloor nx + 1 \rfloor,\end{aligned}$$

and since $\lfloor x + k \rfloor = \lfloor x \rfloor + k$, for every integer k, it follows that $f\left(x + \frac{1}{n}\right) = f(x)$, for every real number x. Hence, f is a periodic function with period $\frac{1}{n}$. In this way it is enough to study $f(x)$, for $0 \leq x < \frac{1}{n}$. But $f(x) = 0$ for all these values, then $f(x) = 0$ for every real number x.

Solution 9.74. The sought for identity can be rewritten as

$$\left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} + \frac{1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2^{k+1}} + \frac{1}{2} \right\rfloor + \cdots = n.$$

We now use a special case of the Hermite identity (see Problem 9.73 or Example 1.3.2), then for $n = 2$, $\left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor$. This implies that

$$\lfloor n \rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2^k} \right\rfloor - \left\lfloor \frac{n}{2^{k+1}} \right\rfloor + \cdots = n.$$

This last sum is telescopic and moreover $\left\lfloor \frac{n}{2^{k+1}} \right\rfloor = 0$, for k large enough.

Solution 9.75. This inequality can be proved using Hermite's identity, but here we present an idea for a shorter proof. Let S_n be the right-hand side of the inequality. Then, if we set $S_0 = 0$, we get

$$S_n - S_{n-1} = \frac{\lfloor nx \rfloor}{n}, \quad \text{for all} \quad n = 1, 2, \ldots.$$

Then

$$k(S_k - S_{k-1}) = \lfloor kx \rfloor, \quad \text{for} \quad k = 1, 2, \ldots, n+1.$$

Adding these $n+1$ equations, it follows that

$$-S_1 - S_2 - \cdots - S_n + (n+1)S_{n+1} = \lfloor (n+1)x \rfloor + \lfloor nx \rfloor + \cdots + \lfloor x \rfloor.$$

Now proceed by induction. The basis $n = 1$ is clear. Suppose that $S_k \leq \lfloor kx \rfloor$, for $1 \leq k \leq n$, then use the last identity for $(n+1)S_n$, hence

$$(n+1)S_{n+1} \leq \lfloor (n+1)x \rfloor + (\lfloor nx \rfloor + \lfloor x \rfloor) + \cdots + (\lfloor x \rfloor + \lfloor nx \rfloor).$$

Using n times the fact that $\lfloor u \rfloor + \lfloor v \rfloor \leq \lfloor u + v \rfloor$ for any real numbers u and v, it follows that $(n+1)S_{n+1} \leq (n+1)\lfloor (n+1)x \rfloor$, which ends the proof.

Solution 9.76. First observe that since $f(10) = 0$ and $f(10) = f(5) + f(2)$, then $f(5) + f(2) = 0$, but $f(n)$ is non-negative, then $f(5) = 0$ and $f(2) = 0$. On the other hand, $f(9) = f(3) + f(3) = 0$. Then, given that $1985 = 5 \cdot 397$, it follows that $f(1985) = 0$, since

$$\begin{aligned} f(1985) &= f(5) + f(397) \\ &= 0 + f(397) \\ &= f(9) + f(397) \\ &= f(9 \cdot 397) \\ &= f(3573) = 0. \end{aligned}$$

Solution 9.77. The function must satisfy that $f(y) > 0$, for $y > 0$. Then

$$f(xf(y)) = f(x) - \frac{1}{xyf(y)}. \tag{10.30}$$

Let $a = f(1) > 0$. Taking $x = 1$ and then $y = 1$ in (10.30), it follows respectively that

$$\begin{aligned} f(f(y)) &= f(1) - \frac{1}{yf(y)} = a - \frac{1}{yf(y)}, \quad \text{for } \ y \in \mathbb{R}^+ \\ f(xa) &= f(x) - \frac{1}{ax}, \quad \text{for } \ x \in \mathbb{R}^+. \end{aligned} \tag{10.31}$$

Taking $x = 1$ in the last equality, $f(a) = f(1) - \frac{1}{a} = a - \frac{1}{a}$.
Taking $x = a$ in the equation (10.30), it follows that

$$f(af(y)) = f(a) - \frac{1}{ayf(y)} = a - \frac{1}{a} - \frac{1}{ayf(y)}. \tag{10.32}$$

On the other hand, using equation (10.31),

$$f(af(y)) = f(f(y)) - \frac{1}{af(y)} = a - \frac{1}{yf(y)} - \frac{1}{af(y)}. \tag{10.33}$$

Combining equations (10.32) and (10.33), it follows that

$$\frac{1}{a} + \frac{1}{ayf(y)} = \frac{1}{yf(y)} + \frac{1}{af(y)}.$$

Hence $f(y) = 1 + \frac{a-1}{y}$, for $y \in \mathbb{R}^+$.
This is the only possible solution of the equation. Now substituting in the last equation, it follows that $(a-1)^2 = 1$, but since $a > 0$, the only choice is $a = 2$, and then $f(x) = 1 + \frac{1}{x}$ is the only solution.

Solution 9.78. For $x \in [0,1]$, $|f(x)| = |f(x) - f(0)| < |x - 0| = x$ and $|f(x)| = |f(x) - f(1)| < |x - 1| = 1 - x$. Then $|f(x)| < \min\{x, 1-x\}$, for $x \in [0,1]$.
If $|x - y| \le \frac{1}{2}$, then $|f(x) - f(y)| < |x - y| \le \frac{1}{2}$.
If $|x - y| > \frac{1}{2}$, without loss of generality, we can assume that $\frac{1}{2} \le x \le 1$ and that $y < \frac{1}{2}$. Since $|f(x)| < \min\{x, 1-x\} = 1 - x$ and also $|f(y)| < \min\{y, 1-y\} = y$, it follows that $|f(x) - f(y)| \le |f(x)| + |f(y)| < 1 - x + y = 1 - (x - y) < \frac{1}{2}$.

Solution 9.79. Let $F = \{f(n)\}$ and $G = \{g(n)\}$, for $n = 1, 2, \ldots$. Since $g(1) = f(f(1)) + 1 > 1$, then $f(1) = 1$ and $g(1) = 2$. Now we prove that if $f(n) = k$, then

$$f(k) = k + n - 1 \tag{10.34}$$

$$g(n) = k + n \tag{10.35}$$

$$f(k+1) = k + n + 1. \tag{10.36}$$

If we assume for the moment that these statements are true, they can be applied to $f(1) = 1$ to obtain $g(1) = 2$ and $f(2) = 3$. If we apply equation (10.34) to

$f(2) = 3$ and to the next numbers, we obtain a chain of results:

$$\begin{array}{cccc} f(3) = 4, & f(4) = 6, & f(6) = 9, & f(9) = 14, \\ f(14) = 22, & f(22) = 35, & f(35) = 56, & f(56) = 90, \\ f(90) = 145, & f(145) = 234, & f(234) = 378, & \ldots . \end{array}$$

But $f(240)$ is not in this chain. Observe that equation (10.36) generates larger numbers; for instance, if we apply it to $f(145) = 234$, we get $f(235) = 380$. Looking at the previous values of the chain, we can see that applying equation (10.36), $f(56) = 90$ and then $f(91) = 147$, $f(148) = 239$. Finally, $f(240) = 388$.

It only remains to prove equations (10.34), (10.35) and (10.36). Assuming that $f(n) = k$, it follows that the elements in the two disjoint subsets

$$\{f(1), f(2), \ldots, f(k)\} \quad \text{and} \quad \{g(1), g(2), \ldots, g(n)\}$$

cover all the natural numbers from 1 to $g(n)$, since $g(n) = f(f(n)) + 1 = f(k) + 1$. Counting the elements in the sets, it follows that $g(n) = k + n$ or $g(n) = f(n) + n$, which is equation (10.35). Equation (10.34) follows from $k + n = g(n) = f(k) + 1$. From equation $g(n) - 1 = f(f(n))$, notice that $g(n) - 1$ is an element of F, that is, two consecutive integers cannot be elements of G. Since $k + n$ is an element of G, it follows that both $k + n - 1$ and $k + n + 1$ are elements of F, moreover, they are two consecutive elements of F. Therefore, equation (10.34) implies that $k + n + 1 = f(k + 1)$.

Solution 9.80. Take $x = 0$ and $y = 1$ in equations (9.2) and (9.1) to get $f(1, 0) = f(0, 1) = 2$. Moreover, if we take $x = 0$ and $y = 0$, by equations (9.3) and (9.1), we get $f(1, 1) = f(0, f(1, 0)) = f(1, 0) + 1 = 3$. Now, if we take $x = 0$ and $y = 1$ in (9.3) and using the previous results, it follows that $f(1, 2) = f(0, f(1, 1)) = f(1, 1) + 1 = 4$. We claim that

$$f(1, y) = y + 2. \tag{10.37}$$

We already have verified this equation for $y = 0$, 1, 2. The inductive step follows from (9.3):

$$f(1, k) = f(0, f(1, k - 1)) = f(1, k - 1) + 1 = (k - 1) + 2 + 1 = k + 2.$$

Now, using induction, show that $f(2, y) = 2y + 3$. Observe that $f(2, 0) = f(1, 1) = 3$; now, from $f(2, y + 1) = f(1, f(2, y)) = f(2, y) + 2$, we obtain the inductive step. Also, it follows that $f(3, 0) = f(2, 1) = 5$ and $f(3, y + 1) = f(2, f(3, y)) = 2f(3, y) + 3$. Then, we get

$$\begin{aligned} f(3, y) &= 2f(3, y - 1) + 3 = 2(2f(3, y - 2) + 3) + 3 \\ &= \cdots = 2^y f(3, 0) + (2^{y-1} + \cdots + 2 + 1)3 \\ &= 2^y(2^3 - 3) + 3\frac{2^y - 1}{2 - 1} = 2^{y+3} - 3. \end{aligned}$$

Finally, once again by induction, we show that $f(4,y) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} - 3$, where the tower of numbers 2 has $y+3$ floors.

For this, notice that $f(4,0) = f(3,1) = 2^4 - 3 = 13$ and $f(4,y+1) = f(3, f(4,y)) = 2^{f(4,y)+3} - 3$, and the inductive step is completed.

Solution 9.81. Since $f(n+m) - f(m) - f(n) = 0$ or 1, it follows that $f(m+n) \geq f(m) + f(n)$.
If $m = n = 1$, then $f(2) \geq 2f(1)$, but $f(2) = 0$, that is, $0 \geq 2f(1)$, and since $f(n) \geq 0$, then $f(1) = 0$.
If $m = 2$ and $n = 1$, then $f(3) = f(2) + f(1) + \{0 \text{ or } 1\} = 0$ or 1. Since $f(3) > 0$, then $f(3) = 1$.
If $m = n = 3$, then $f(3+3) = f(2\cdot 3) = f(3) + f(3) + \{0 \text{ or } 1\}$, that is, $f(3+3) \geq 2 \cdot f(3) = 2$. Notice that

$$f(3+6) = f(3+2\cdot 3) \geq f(3) + f(2\cdot 3) \geq f(3) + 2\cdot f(3) = 3f(3).$$

Hence, $f(3n) = f(3+(n-1)3) \geq f(3) + (n-1)f(3) = n \cdot f(3) = n$.

Therefore $f(3n) \geq n$, for all n. If for some n_0 the inequality is strict, then for all $n \geq n_0$ the inequality is also strict.

Since $f(9999) = f(3\cdot 3333) = 3333$, it follows that $f(3n) = n$, for $3 \leq n \leq 3333$, in particular $f(3\cdot 1982) = 1982$. Now,

$$1982 = f(3\cdot 1982) = f(2\cdot 1982 + 1982) \geq f(2\cdot 1982) + f(1982) \geq 3\cdot f(1982),$$

then $1982 \geq 3 \cdot f(1982)$. Thus, $f(1982) \leq \frac{1982}{3} < 661$.
On the other hand, $f(1982) = f(1980+2) \geq f(1980) + f(2) = f(3\cdot 660) = 660$.
Therefore, $660 \leq f(1982) < 661$, that is, $f(1982) = 660$.

Solution 9.82. First, we show that 1 is in the image of f. For $x_0 > 0$, let

$$y_0 = \frac{1}{f(x_0)}.$$

Then condition (i) states that $f(x_0 f(y_0)) = y_0 f(x_0) = 1$, that is, 1 is in the image of f. Then there is a value of y such that $f(y) = 1$. This, together with $x = 1$ in (i), imply that $f(1\cdot 1) = f(1) = yf(1)$. Since $f(1) > 0$ by hypothesis, then $y = 1$ and hence $f(1) = 1$.

Taking $y = x$ in (i), we obtain that $f(xf(x)) = xf(x)$, for all $x > 0$. Then $xf(x)$ is a fixed point of f. Now, if a and b are fixed points of f, then using (i) with $x = a$, $y = b$, we guarantee that $f(ab) = ba$, hence ab is also a fixed point of f. Thus, the set of fixed points of f is closed under multiplication. In particular, if a is a fixed point, all non-negative integer powers of a are fixed points, that is,

a^n is a fixed point for all non-negative integers n. Then, by (ii), there are no fixed points greater than 1. Since $xf(x)$ is a fixed point, it follows that

$$xf(x) \le 1 \quad \text{or} \quad f(x) \le \frac{1}{x}, \quad \text{for all } x. \tag{10.38}$$

Let $a = xf(x)$, then $f(a) = a$. Now, use $x = \frac{1}{a}$ and $y = a$ in (i), to obtain

$$f\left(\frac{1}{a}f(a)\right) = f(1) = 1 = af\left(\frac{1}{a}\right),$$

then

$$f\left(\frac{1}{a}\right) = \frac{1}{a} \quad \text{or} \quad f\left(\frac{1}{xf(x)}\right) = \frac{1}{xf(x)}.$$

This shows that $\frac{1}{xf(x)}$ is also a fixed point of f for all $x > 0$. Then $f(x) \ge \frac{1}{x}$. This and (10.38) imply that $f(x) = \frac{1}{x}$. This function clearly satisfies the conditions of the problem.

Solution 9.83. Suppose that $\lfloor f(y) \rfloor = 0$ for some y, then the substitution $x = 1$ implies that $f(y) = f(1)\lfloor f(y) \rfloor = 0$. Then if $\lfloor f(y) \rfloor = 0$ for all y, it follows that $f(y) = 0$ for all y. This function obviously satisfies the conditions of the problem. Now, we have to consider the case when $\lfloor f(a) \rfloor \ne 0$, for some a. Then it follows from $f(\lfloor x \rfloor a) = f(x)\lfloor f(a) \rfloor$, that

$$f(x) = \frac{f(\lfloor x \rfloor a)}{\lfloor f(a) \rfloor}. \tag{10.39}$$

This means that $f(x_1) = f(x_2)$ if $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$, then $f(x) = f(\lfloor x \rfloor)$. Hence we can assume that a is an integer.
Now, we have

$$f(a) = f\left(2a \cdot \frac{1}{2}\right) = f(2a)\left\lfloor f\left(\frac{1}{2}\right)\right\rfloor = f(2a)\lfloor f(0) \rfloor,$$

this implies that $\lfloor f(0) \rfloor \ne 0$, then we can assume that $a = 0$. Therefore, equation (10.39) implies that $f(x) = \frac{f(0)}{\lfloor f(0) \rfloor} = C \ne 0$, for every x. Now, the condition of the problem is equivalent to equation $C = C\lfloor C \rfloor$, which is true exactly when $\lfloor C \rfloor = 1$. Then, the only functions that satisfy the conditions of the problem are $f(x) = 0$ and $f(x) = C$, with $\lfloor C \rfloor = 1$.

Solution 9.84. For $x = y = 1$, we have $f(f(1)) = f(1)$. Now, if we take $x = 1$ and $y = f(1)$, and since $f(f(1)) = f(1)$, it follows that $[f(1)]^2 = f(1)$. Then there are two possibilities, either $f(1) = 1$ or $f(1) = 0$.

If $f(1) = 1$, substituting $y = 1$ in the functional equation, we get $f(xf(1)) + f(x) = xf(1) + f(x)$, then $f(x) = x$, for all $x \in \mathbb{R}$. But this function is not bounded, hence $f(1) = 1$ is not true.

If $f(1) = 0$, taking $x = 1$, we get $f(f(y)) + yf(1) = f(y) + f(y)$, then $f(f(y)) = 2f(y)$. If $f(y) \in \operatorname{Img} f$, then $2f(y) \in \operatorname{Img} f$, and by induction, $f^n(y) = 2^n f(y) \in \operatorname{Img} f$. We conclude that $f(y) \leq 0$, because if $f(y) > 0$, it will follow that $2^n f(y) \in \operatorname{Img} f$, for all n, which is impossible since the function f is upper bounded.

Substituting x by $\frac{x}{2}$ and y by $f(y)$ and, noticing that $f(f(y)) = 2f(y)$, we obtain $f(xf(y)) + f(y)f\left(\frac{x}{2}\right) = xf(y) + f\left(\frac{x}{2}f(y)\right)$. Then $xf(y) - f(xf(y)) = f(y)f\left(\frac{x}{2}\right) - f\left(\frac{x}{2}f(y)\right) \geq 0$, since $f(x) \leq 0$, for all x. All these results together and equation (9.4) imply that $yf(x) \geq f(xy)$.

Considering that $yf(x) \geq f(xy)$, and taking $x > 0$, $y = \frac{1}{x}$, we obtain $f(x) \geq 0$. Since $f(x) \leq 0$, then $f(x) = 0$. Clearly $f(x) = 0$, for all $x \in \mathbb{R}$, satisfies the functional equation and it is bounded.

Suppose that f is not identically zero, then there exists $x_0 < 0$ such that $f(x_0) < 0$. Let $y_0 = f(x_0)$, then $f(y_0) = f(f(x_0)) = 2f(x_0) = 2y_0$. For any $x < 0$, it follows that $y_0 x > 0$, then $f(y_0 x) = f(2y_0 x) = 0$. Hence, after substituting y by y_0 in (9.4), we get $f(2y_0 x) + y_0 f(x) = 2y_0 x + f(xy_0)$, thus $f(x) = 2x$, for all $x < 0$. Hence, the other solution is $f(x) = 0$ for $x \geq 0$, and $f(x) = 2x$, for $x < 0$.

Therefore, the only solutions are

$$f(x) = 0 \quad \text{and} \quad f(x) = \begin{cases} 0, & \text{if} \quad x \geq 0, \\ 2x, & \text{if} \quad x < 0. \end{cases}$$

It is easy to verify that these functions satisfy the conditions of the problem.

Solution 9.85. First notice that $a_{2n} > 1$ and $a_{2n+1} < 1$, for all $n \geq 1$.
The proof of the following statement will be by induction for $k \geq 2$.

P_k : For every pair of positive integers a, b with $(a, b) = 1$ and $a + b \leq k$, there exists an integer n such that $a_n = \frac{a}{b}$.
If $k = 2$, the only positive integers a, b that satisfy $(a, b) = 1$ and $a + b \leq 2$, are $a = b = 1$, and in such a case a_1 satisfies $\frac{a}{b} = a_1 = 1$.
Now, suppose that the statement is true for $k \geq 2$; we will prove that it is valid for $k + 1$. Let a, b be positive integers that satisfy $(a, b) = 1$ and $a + b \leq k + 1$.
If $a > b$, then $a - b$ and b satisfy that $(a - b, b) = 1$ and $(a - b) + b = a \leq k$, then by the induction hypothesis, there is an integer n with $a_n = \frac{a-b}{b}$. Then

$$a_{2n} = a_n + 1 = \frac{a - b}{b} + 1 = \frac{a}{b}.$$

If $a < b$, then $b - a$ and a satisfy that $(b - a, a) = 1$ and $(b - a) + a = b \leq k$; hence by the induction hypothesis, there is an integer n with $a_n = \frac{b-a}{a}$. Then

$$a_{2n+1} = \frac{1}{a_{2n}} = \frac{1}{a_n + 1} = \frac{1}{\frac{b-a}{a} + 1} = \frac{a}{b}.$$

Let us now see the uniqueness of the representation.
If $a_n = a_m > 1$, then m and n are even. However, if $a_n = a_m < 1$, then m and n are odd. In the first case, $n = 2n'$ and $m = 2m'$, and then $a_n = a_m$ implies that $a_{n'} = a_{m'}$. In the second case, $n = 2n'+1$ and $m = 2m'+1$, then $a_n = a_m$ implies that $\frac{1}{a_{2n'}} = \frac{1}{a_{2m'}}$, and then $a_{n'} = a_{m'}$. Thus, in any case the equality $a_n = a_m$ leads to another equality $a_{n'} = a_{m'}$, where the subindices are smaller. This process can only be applied a finite number of steps to conclude that $n' = m' = 1$ or $n' = 1$ or $m' = 1$. In the first case, it follows that $n = m$, and in the other cases we conclude that $a_{n'} = a_1 = 1$ or $a_{m'} = a_1 = 1$, but this is not possible, since at the beginning we pointed out that the a_n's are not equal to 1, when $n > 1$.

Solution 9.86. Since $a_{n+1} = \frac{a_n}{1+na_n}$, for $n \geq 0$, it follows that $\frac{1}{a_{n+1}} = \frac{1+na_n}{a_n} = \frac{1}{a_n} + n$. Then

$$\begin{aligned}
\frac{1}{a_{1000}} &= \frac{1}{a_{999}} + 999 \\
&= \frac{1}{a_{998}} + 999 + 998 \\
&= \qquad \vdots \\
&= \frac{1}{a_1} + 999 + 998 + \cdots + 1 \\
&= 1 + \frac{999 \cdot 1000}{2} = 499501.
\end{aligned}$$

Therefore $a_{1000} = \frac{1}{499501}$.

Solution 9.87. From the definition of x_i it follows that for every integer k,

$$x_{4k-3} = x_{2k-1} = -x_{4k-2} \quad \text{and} \quad x_{4k-1} = x_{4k} = -x_{2k} = x_k. \tag{10.40}$$

Then, if we set $S_n = \sum_{i=1}^{n} x_i$, it follows that

$$S_{4k} = \sum_{i=1}^{k}((x_{4k-3} + x_{4k-2}) + (x_{4k-1} + x_{4k})) = \sum_{i=1}^{k}(0 + 2x_k) = 2S_k, \tag{10.41}$$

$$S_{4k+2} = S_{4k} + (x_{4k+1} + x_{4k+2}) = S_{4k}. \tag{10.42}$$

Also, observe that $S_n = \sum_{i=1}^{n} x_i \equiv \sum_{i=1}^{n} 1 = n \mod 2$. We will show, by induction on k, that $S_i \geq 0$, for all $i \leq 4k$. The basis is true since $x_1 = x_3 = x_4 = 1$, $x_2 = -1$. For the inductive step, suppose that $S_i \geq 0$, for all $i \leq 4k$. Using relations (10.40), (10.41) and (10.42), we obtain that

$$S_{4k+4} = 2S_{k+1} \geq 0, \qquad S_{4k+2} = S_{4k} \geq 0,$$

$$S_{4k+3} = S_{4k+2} + x_{4k+3} = \frac{S_{4k+2} + S_{4k+4}}{2} \geq 0.$$

Then, it will be enough to show that $S_{4k+1} \geq 0$. If k is odd, then $S_{4k} = 2S_k \geq 0$; since k is odd, S_k is also odd, and then $S_{4k} \geq 2$. Therefore $S_{4k+1} = S_{4k} + x_{4k+1} \geq 1$. Reciprocally, if k is even, then $x_{4k+1} = x_{2k+1} = x_{k+1}$. Thus $S_{4k+1} = S_{4k} + x_{4k+1} = 2S_k + x_{k+1} = S_k + S_{k+1} \geq 0$ and the induction is complete.

Solution 9.88. First, from the problem conditions, it follows that each a_n $(n > s)$ can be expressed as $a_n = a_{j_1} + a_{j_2}$ with $j_1, j_2 < n$, $j_1 + j_2 = n$. If, say, $j_1 > s$, then we can proceed in the same way with a_{j_1}, and so on. Finally, we represent a_n as

$$a_n = a_{i_1} + \cdots + a_{i_k}, \tag{10.43}$$
$$1 \leq i_j \leq s, \quad i_1 + \cdots + i_k = n. \tag{10.44}$$

Moreover, if a_{i_1} and a_{i_2} are the numbers in (10.43), obtained on the last step, then $i_1 + i_2 > s$. Hence we can adjust (10.44) as

$$1 \leq i_j \leq s, \quad i_1 + \cdots + i_k = n, \quad i_1 + i_2 > s. \tag{10.45}$$

On the other hand, suppose that the indices $i_1, \ldots, i_k$ satisfy conditions (10.45). Then, writing $s_j = i_1 + \cdots + i_j$, from the problem's condition we have

$$a_n = a_{s_k} \geq a_{s_{k-1}} + a_{i_k} \geq a_{s_{k-2}} + a_{i_{k-1}} + a_{i_k} \geq \cdots \geq a_{i_1} + \cdots + a_{i_k}.$$

Summarizing these observations, we get the following lemma.

Lemma 10.9.1. *For every $n > r$, we have $a_n = \max\{a_{i_1} + \cdots + a_{i_k}\}$, where the collection $(i_1, \ldots, i_k)$ satisfies (10.45).*

Now we write $r = \max\left\{\frac{a_i}{i} \mid 1 \leq i \leq s\right\}$, and fix some index $l \leq s$, such that $s = \frac{a_l}{l}$.
Consider some integer $n \geq s^2 l + 2s$ and choose an expansion of a_n in the form (10.43), (10.45). Then we have $n = i_1 + \cdots + i_k \leq sk$, so $k \geq n/s \geq sl + 2$. Suppose that none of the numbers $i_3, \ldots, i_k$ equals l. Then by the pigeonhole principle there is an index $1 \leq j \leq s$ which appears among $i_3, \ldots, i_k$ at least l times, and surely $j \neq l$. Let us delete these l occurrences of j from $(i_1, \ldots, i_k)$, and add j occurrences of l instead, obtaining a sequence $(i_1, i_2, i'_3, \ldots, i'_{k'})$ also satisfying (10.45). Using the lemma, we have

$$a_{i_1} + \cdots + a_{i_k} = a_n \geq a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}},$$

or, after removing the common terms, $la_j \geq ja_l$, then $\frac{a_l}{l} \leq \frac{a_j}{j}$. The definition of l leads to $la_j = ja_l$, hence

$$a_n = a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}}.$$

Thus, for every $n \geq s^2 l + 2s$, we have found a representation of the form (10.43), (10.45) with $i_j = l$, for some $j \geq 3$. Rearranging the indices we may assume that $i_k = l$.

Finally observe that in this representation the indices $(i_1, \ldots, i_{k-1})$ satisfy the conditions (10.45), with n replaced by $n-l$. Thus, from the lemma, we get $a_{n-l} + a_l \geq (a_{i_1} + \cdots + a_{i_{k-1}}) + a_l = a_n$, which, by the problem's condition, implies $a_n = a_{n-l} + a_l$ for each $n \geq s^2 l + 2s$, as desired.

Solution 9.89. For each $i = 1, 2, \ldots, k-1$, let P_i be the set of all prime numbers congruent to i modulo k. Each prime number (except possibly k) is contained in exactly one of the sets $P_1, P_2, \ldots, P_{k-1}$. Since there are an infinite number of prime numbers, at least one of these sets is infinite, say P_i. Let $p = x_1 < x_2 < \cdots < x_n < \cdots$ be its elements arranged in increasing order, and define $a_n = \frac{x_{n+1}-p}{k}$, for $n = 1, 2, \ldots$.

Then the sequence $p + ka_n$ contains all members of P_i, starting with x_2. The numbers a_n are positive integers, p is prime and the sequence $\{a_n\}$ is increasing, hence it satisfies the conditions of the problem.

Solution 9.90. The hypothesis implies $a+b+c = 2$, $ab+bc+ca = -1$ and $abc = 0$. Then a, b, c are the roots of the cubic polynomial $x^3 - 2x^2 - x = 0$, which are 0, $1 \pm \sqrt{2}$. Then we can assume that $a = 1 + \sqrt{2}$, $b = 1 - \sqrt{2}$ and $c = 0$. Hence, $a^2 + b^2 = 6$ and $ab = -1$.

Now, we have that $s_{n-1}s_{n+1} = (a^{n-1} + b^{n-1})(a^{n+1} + b^{n+1}) = a^{2n} + b^{2n} + a^{n-1}b^{n-1}(a^2 + b^2) = s_n^2 - 2a^n b^n + (ab)^{n-1}(a^2 + b^2) = s_n^2 - 2(ab)^n + 6(ab)^{n-1} = s_n^2 - 8(-1)^n$, then $\left|s_n^2 - s_{n-1}s_{n+1}\right| = |8(-1)^n| = 8$.

Solution 9.91. (i) We show that the series

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots, \quad \text{with } 1 = x_0 \geq x_1 \geq \cdots > 0, \tag{10.46}$$

has sum greater than or equal to 4. This clearly implies that some partial sum of the series is greater than or equal to 3.999.

Let L be the infimum (the greatest lower bound) of the sum of all series of the form (10.46). Clearly $L \geq 1$, since the first term $\frac{1}{x_1} \geq 1$. For all $\epsilon > 0$ we can find a sequence $\{x_n\}$ such that

$$L + \epsilon > \frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots. \tag{10.47}$$

Setting $y_n = \frac{x_{n+1}}{x_1}$, with $n \geq 0$, it follows that $1 = y_0 \geq y_1 \geq y_2 \geq \cdots > 0$. The series on the right-hand side of the inequality (10.47) can be written as

$$\frac{1}{x_1} + x_1 \left(\frac{y_0^2}{y_1} + \frac{y_1^2}{y_2} + \frac{y_2^2}{y_3} + \cdots \right).$$

By definition of L, the series inside the parenthesis has a sum greater than or equal to $\geq L$. Hence, by (10.47), we have that $L + \epsilon > \frac{1}{x_1} + x_1 L$.

Applying the inequality between the arithmetic mean and the geometric mean on the right-hand side of the inequality we get $L + \epsilon > 2\sqrt{L}$. Since this is true for all $\epsilon > 0$, it follows that $L \geq 2\sqrt{L}$. Thus, $L^2 \geq 4L$, and since $L > 0$, this implies $L \geq 4$.

(ii) Let $x_n = \frac{1}{2^n}$, then

$$\sum_{n=0}^{\infty} \frac{x_n^2}{x_{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} = 4,$$

and the partial sums of this series are less than 4.

Solution 9.92. By (i) and (ii), it follows that all the elements in the sequence are rational numbers. Suppose that $a_k = \frac{p}{q}$, with p and q integers such that $(p, q) = 1$, then $a_{k+1} = \frac{(2p^2-q^2)}{q^2}$. Since $(p, q) = 1$, it follows that $(2p^2 - q^2, q^2) = (2p^2, q^2) = (2, q^2)$, which is equal to either 1 or 2.
If $q > 2$, then the denominator of a_{k+1} is greater than the denominator of a_k. Hence, the sequence of denominators will be increasing and therefore it cannot be an equality among the terms of the sequence.
Moreover, if $|a_k| > 1$, then writing $|a_k| = 1+e$, it follows that $a_{k+1} = 1+4e+2e^2 > |a_k|$, which gives an increasing sequence.
Therefore, in order that the terms of the sequence repeat themselves, the first term must have denominator at most 2 and must be between -1 and 1, this gives us only 5 possible values:

- $a_0 = -1$, which gives the sequence $-1, 1, 1, 1, \ldots$.
- $a_0 = -\frac{1}{2}$, which gives the sequence $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots$.
- $a_0 = 0$, which gives the sequence $0, -1, 1, 1, 1, \ldots$.
- $a_0 = \frac{1}{2}$, which gives the sequence $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots$.
- $a_0 = 1$, which gives the sequence $1, 1, 1, 1, \ldots$.

Solution 9.93. The first terms of the sequence are $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 12$, $a_5 = 29$, $a_6 = 70$, $a_7 = 169$. Observe that $a_{n+1} = 2a_n + a_{n-1} = a_2a_n + a_1a_{n-1}$ and $a_{n+2} = 2a_{n+1} + a_n = 2(2a_n + a_{n-1}) + a_n = 5a_n + 2a_{n-1} = a_3a_n + a_2a_{n-1}$. Then we can conjecture that

$$a_{n+m} = a_{m+1}a_n + a_ma_{n-1}. \tag{10.48}$$

To show it we use induction on m. For $m = 1$, 2 it was already proved. Now, suppose that the equality is true for $m = k - 1$ and for $m = k$, and show that the relation holds for $k + 1$. Observe that

$$\begin{aligned} a_{n+k+1} &= 2a_{n+k} + a_{n+k-1} = 2(a_{k+1}a_n + a_ka_{n-1}) + a_ka_n + a_{k-1}a_{n-1} \\ &= (2a_{k+1} + a_k)a_n + (2a_k + a_{k-1})a_{n-1} = a_{k+2}a_n + a_{k+1}a_{n-1}, \end{aligned}$$

which finishes the induction.

If we let $n = m$ in (10.48), we obtain the equality $a_{2n} = a_n(a_{n+1} + a_{n-1})$.

Also note that if n is even, then a_n is even, which is easy to deduce using the formula of the sequence $a_n = 2a_{n-1} + a_{n-2}$. Now, if n is odd, then $a_n \equiv 1 \mod 4$. In order to see this, use induction again. For $n = 1$, the result follows since $a_1 = 1$, then assume that $a_{2n-1} \equiv 1 \mod 4$. Then, since $a_{2n+1} = 2a_{2n} + a_{2n-1}$, we only need to observe that a_{2n} is even, which finishes the induction.

Summarizing, if n is odd, that is, if there is no power of 2 dividing n, then $a_n \equiv 1 \mod 4$, hence there is no power of 2 dividing a_n.

If n is even, then $n-1$ and $n+1$ are odd, hence $a_{n+1} + a_{n-1} \equiv 2 \mod 4$, that is, in the equality $a_{2n} = a_n(a_{n+1} + a_{n-1})$, only one extra factor of 2 appears inside the parenthesis, and this ends the proof.

Solution 9.94. Since

$$|a_{m+1} + a_n - a_{m+n+1}| \le \frac{1}{m+n+1}, \quad |a_m + a_{n+1} - a_{m+n+1}| \le \frac{1}{m+n+1},$$

from the two inequalities it follows that

$$|(a_{m+1} - a_m) - (a_{n+1} - a_n)| \le \frac{2}{m+n+1} < \frac{2}{n}.$$

Now, using twice the previous inequality, we get

$$\begin{aligned}
&|(a_{m+1} - a_m) - (a_{n+1} - a_n)| \\
&\quad \le |(a_{m+1} - a_m) - (a_{k+1} - a_k) + (a_{k+1} - a_k) - (a_{n+1} - a_n)| \\
&\quad \le |(a_{m+1} - a_m) - (a_{k+1} - a_k)| + |(a_{k+1} - a_k) - (a_{n+1} - a_n)| \le \frac{2}{k} + \frac{2}{k} = \frac{4}{k}.
\end{aligned}$$

Since k is arbitrary, $|(a_{m+1} - a_m) - (a_{n+1} - a_n)|$ is equal to 0, then $a_{n+1} - a_n$ is constant.

Solution 9.95. The function $f(x) = \frac{x}{n} + \frac{n}{x}$ is decreasing in $(0, n]$ (it can be seen that $f'(x) < 0$, or that the graph is decreasing with the identity $f(x) - f(y) = \frac{(x-y)(xy-n^2)}{xyn}$). First we see, using induction in n, that $\sqrt{n} \le a_n \le \frac{n}{\sqrt{n-1}}$, for $n \ge 3$.

For $n = 3$ is clear, since $\sqrt{3} \le a_3 = 2 \le \frac{3}{\sqrt{2}}$. Suppose the result true for n, $a_{n+1} = f(a_n) \le f(\sqrt{n}) = \frac{n+1}{\sqrt{n}}$. On the other hand, $a_{n+1} = f(a_n) \ge f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}} > \sqrt{n+1}$, then the result follows.

Now, let us see that $a_n < \sqrt{n+1}$. If $a_{n+1} = f(a_n) \ge \frac{n}{\sqrt{n-1}}$ is true for $n \ge 3$, then $a_n \ge \frac{n-1}{\sqrt{n-2}}$ for $n \ge 4$. Since f is decreasing, then

$$a_{n+1} = f(a_n) \le f\left(\frac{n-1}{\sqrt{n-2}}\right) = \frac{(n-1)^2 + n^2(n-2)}{(n-1)n\sqrt{n-2}} < \sqrt{n+2},$$

for $n \ge 4$. Hence $\lfloor a_n^2 \rfloor = n$, for $n \ge 5$. The case $n = 4$ follows easily after noticing that $a_4 = \frac{13}{6}$.

Solution 9.96. Denote by $l(n)$ the last digit of a positive integer n, that is, the units digit. The sequence $\{l(n)\}$ is periodic with period 10. Now, for a fixed positive integer a, the sequence $\{l(a^n)\}$ is periodic with period equal to 1 if a ends in 0, 1, 5, 6; the period is equal to 2 if a ends in 4 or 9; and the period is 4 if a ends in 2, 3, 7, 8.
Since the least common multiple of 10 and 4 is 20, and if

$$m = (n+1)^{n+1} + (n+2)^{n+2} + \cdots + (n+20)^{n+20},$$

then $l(m)$ does not depend on n. We now calculate the last digit of $1^1 + 2^2 + 3^3 + \cdots + 20^{20}$. By the periodicity of the sequence of the form $\{l(a^n)\}$, the last digit of this number is the same as the one in

$$1 + 2^2 + 3^3 + 4^4 + 5 + 6 + 7^3 + 8^4 + 9 + 1 + 2^4 + 3 + 4^2 + 5 + 6 + 7 + 8^2 + 9.$$

The units digit of this last number is 4. Therefore, the last digit of a sum of the form $(n+1)^{n+1} + (n+2)^{n+2} + \cdots + (n+100)^{n+100}$ is equal to $l(4 \cdot 5) = l(20) = 0$. Thus, b_n is periodic with period 100.

Solution 9.97. Notice that $(1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1}$ is an even integer. In order to see this, simplify the binomials to obtain

$$\begin{aligned}
&(1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1} \\
&\quad = \sum_{j=0}^{2n+1} \binom{2n+1}{j} \left(\sqrt{3}\right)^j + \sum_{j=0}^{2n+1} \binom{2n+1}{j} \left(-\sqrt{3}\right)^j \\
&\quad = 2\sum_{j=0}^{n} \binom{2n+1}{2j} \left(\sqrt{3}\right)^{2j} = 2\sum_{j=0}^{n} \binom{2n+1}{2j} 3^j.
\end{aligned}$$

Now, since $-1 < 1-\sqrt{3} < 0$, it follows that $-1 < (1-\sqrt{3})^{2n+1} < 0$, therefore $\left\lfloor (1+\sqrt{3})^{2n+1} \right\rfloor = \left\lfloor (1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1} \right\rfloor = (1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1}$ is an even integer.

It can be shown that 2^{n+1} divides $\left\lfloor (1+\sqrt{3})^{2n+1} \right\rfloor$. To see this, observe that

$$\begin{aligned}
\left(1+\sqrt{3}\right)^{2n+1} + \left(1-\sqrt{3}\right)^{2n+1} &= (1+\sqrt{3})(4+2\sqrt{3})^n + (1-\sqrt{3})(4-2\sqrt{3})^n \\
&= 2^n(1+\sqrt{3})(2+\sqrt{3})^n + 2^n(1-\sqrt{3})(2-\sqrt{3})^n.
\end{aligned}$$

It is only left to see that 2 divides $(1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n$, and for that use again Newton's binomial theorem.

Solution 9.98. The characteristic polynomial of the recursion $a_{n+2} = 3a_{n+1} - 2a_n$ is $\lambda^2 - 3\lambda + 2$, which has as roots $\lambda = 2, 1$. Then $a_n = A\lambda^n + B$ for some real numbers A and B which are determined by $3 = a_1 = A \cdot 2 + B$ and $5 = a_2 = A \cdot 2^2 + B$. The previous system has solutions $A = B = 1$, then $a_n = 2^n + 1$, for $n \geq 1$.

Solution 9.99. The relation $a_{n+6} = a_n$ follows from the relation $a_{n+2} = a_{n+1} - a_n$, since $a_{n+6} = a_{n+5} - a_{n+4} = a_{n+4} - a_{n+3} - a_{n+4} = -(a_{n+2} - a_{n+1}) = -(a_{n+1} - a_n - a_{n+1}) = a_n$.

Solution 9.100. Since $a_2 = 2^2$, $a_3 = 5^2$, $a_4 = 13^2$, we can conjecture that $a_n = F_{2n-1}^2$. Use strong induction in order to show the result. A relation between 3 Fibonacci numbers that helps is $F_{2n+1} = 3F_{2n-1} - F_{2n-3}$.

Solution 9.101. Calculate the first terms,

$$\begin{aligned}
u_1 &= \frac{5}{2} = 2 + \frac{1}{2}\\
u_2 &= u_1(u_0^2 - 2) - u_1 = \left(2 + \frac{1}{2}\right)(2^2 - 2) - \left(2 + \frac{1}{2}\right) = 2 + \frac{1}{2}\\
u_3 &= u_2(u_1^2 - 2) - u_1 = \left(2 + \frac{1}{2}\right)\left(\left(2 + \frac{1}{2}\right)^2 - 2\right) - \left(2 + \frac{1}{2}\right)\\
&= \left(2 + \frac{1}{2}\right)\left(2^2 + \frac{1}{2^2}\right) - \left(2 + \frac{1}{2}\right) = \left(2 + \frac{1}{2}\right)\left(2^2 - 1 + \frac{1}{2^2}\right) = 2^3 + \frac{1}{2^3}\\
u_4 &= \left(2^3 + \frac{1}{2^3}\right)\left(\left(2 + \frac{1}{2}\right)^2 - 2\right) - \left(2 + \frac{1}{2}\right)\\
&= \left(2^3 + \frac{1}{2^3}\right)\left(2^2 + \frac{1}{2^2}\right) - \left(2 + \frac{1}{2}\right) = 2^5 + \frac{1}{2^5}\\
u_5 &= \left(2^5 + \frac{1}{2^5}\right)\left(\left(2^3 + \frac{1}{2^3}\right)^2 - 2\right) - \left(2 + \frac{1}{2}\right) = 2^{11} + \frac{1}{2^{11}}.
\end{aligned}$$

This allows us to conjecture that $u_n = 2^{r_n} + \frac{1}{2^{r_n}}$ for some numbers r_n which must be determined. If the numbers r_n are integers, then $\lfloor u_n \rfloor = 2^{r_n}$.

If $u_n = 2^{r_n} + \frac{1}{2^{r_n}}$, then from the original equation it follows that

$$\begin{aligned}
u_{n+1} &= \left(2^{r_n} + \frac{1}{2^{r_n}}\right)\left(\left(2^{r_{n-1}} + \frac{1}{2^{r_{n-1}}}\right)^2 - 2\right) - \left(2 + \frac{1}{2}\right)\\
&= \left(2^{r_n} + 2^{-r_n}\right)\left(2^{2r_{n-1}} + 2^{-2r_{n-1}}\right) - \left(2 + \frac{1}{2}\right)\\
&= 2^{r_n + 2r_{n-1}} + 2^{-r_n - 2r_{n-1}} + 2^{r_n - 2r_{n-1}} + 2^{-r_n + 2r_{n-1}} - \left(2 + \frac{1}{2}\right).
\end{aligned}$$

Therefore, if we can find a sequence $\{r_n\}$ that satisfies $r_{n+1} = r_n + 2r_{n-1}$ and $r_n - 2r_{n-1} = (-1)^n$, the proof is complete.

The characteristic equation of the recursion $r_{n+1} = r_n + 2r_{n-1}$ is $\lambda^2 - \lambda - 2 = 0$, and its roots are $\lambda = 2, -1$, and since $r_0 = 0$ and $r_1 = 1$, it follows that $r_n = \frac{2^n - (-1)^n}{3}$ is the solution, and it is also solution of the other recursion.

Hence, $\lfloor u_n \rfloor = 2^{\frac{2^n - (-1)^n}{3}}$, for all $n \in \mathbb{N}$.

Solution 9.102. First notice that the integers n that satisfy $\lfloor\sqrt{2n}\rfloor = m$ are the ones that satisfy $m \le \sqrt{2n} < m+1$. Then $m^2 \le 2n < (m+1)^2$, but even numbers between m^2 (inclusive) and $(m+1)^2$ are m, if m is odd, and $m+1$ if m is even. Hence the sequence a_n is

$$1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots,$$

where the integer k appears k times in the sequence if k is odd or $k+1$ times if k is even.

When we write all the numbers 63, we arrive to the term a_n with $n = 1+3+3+5+5+\cdots+63+63 = 2(1+3+\cdots+63)-1 = 2(32)^2-1 = 2047$. This last term and the 62 previous ones are equal to 63, in particular $a_{2020} = 63$.

Solution 9.103. Since $P(0) = 0$, it follows that $P(x) = xQ(x)$, for some polynomial $Q(x)$ of degree $n-1$ that satisfies $Q(j) = \frac{1}{j+1}$, for $j = 1, 2, \ldots, n$. If $R(x) = (x+1)Q(x) - 1$, then the degree of $R(x)$ is n and $R(j) = 0$, for $j = 1, 2, \ldots, n$. Hence, $R(x) = a_0(x-1)(x-2)\ldots(x-n)$. If $m > n$,

$$Q(m) = \frac{a_0(m-1)(m-2)\ldots(m-n)+1}{m+1}.$$

Evaluating $R(x)$ in $x = -1$, it follows that $-1 = a_0(-2)(-3)\ldots(-n-1)$, then $a_0 = \frac{(-1)^{n+1}}{(n+1)!}$. Hence,

$$P(m) = mQ(m) = \frac{(-1)^{n+1}}{(n+1)!}\frac{m(m-1)(m-2)\ldots(m-n)}{m+1} + \frac{m}{m+1}.$$

Solution 9.104. Consider $Q(x) = xP(x) - 1 = c(x-1)(x-2)\ldots(x-2^n)$. For $x \ne 1, 2, \ldots, 2^n$, it follows that

$$\frac{Q'(x)}{Q(x)} = \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-2^n}.$$

Since $Q(0) = -1$ and $Q'(x) = P(x) + xP'(x)$, then

$$P(0) = Q'(0) = -\left(-\frac{1}{1} - \frac{1}{2} - \cdots - \frac{1}{2^n}\right) = \frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = 2 - \frac{1}{2^n}.$$

Solution 9.105.
Lemma. *If $P(x)$ is a polynomial of degree less than or equal to n, then*

$$\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}P(i) = 0.$$

Proof of the lemma. Proceed by induction on n. For $n = 0$, $\binom{1}{0}P(0) - \binom{1}{1}P(1) = P_0 - P_0 = 0$.

Suppose valid the result for all $n \leq k$ and consider the polynomial $P(x)$ of degree $k+1$. The polynomial $P(x) - P(x+1)$ has degree less than or equal to k, then

$$
\begin{aligned}
0 &= \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (P(i) - P(i+1)) \\
&= \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} P(i) + \sum_{i=1}^{k+2} (-1)^i \binom{k+1}{i-1} P(i) \\
&= P(0) + \sum_{i=1}^{k+1} (-1)^i \left[\binom{k+1}{i} + \binom{k+1}{i-1} \right] P(i) + (-1)^{k+2} P(k+2) \\
&= \sum_{i=0}^{k+2} (-1)^i \binom{k+2}{i} P(i).
\end{aligned}
$$

Hence, for the polynomial $P(x)$ the result holds and therefore the proof of the lemma is complete. $\square$

Now, apply the lemma to the polynomial $P(x)$ of the problem,

$$
\begin{aligned}
0 &= \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} \binom{n+1}{i}^{-1} + (-1)^{n+1} P(n+1) \\
&= \sum_{i=0}^{n} (-1)^i + (-1)^{n+1} P(n+1),
\end{aligned}
$$

hence $P(n+1)$ is 1 if n is even and it is 0 if n is odd.

Solution 9.106. Suppose that there are polynomials $Q(x)$ and $R(x)$ with integer coefficients such that $P(x) = Q(x)R(x)$. Since $P(0) = 3$, we can assume, without loss of generality, that $|Q(0)| = 3$. If $Q(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0$, with $a_0 = \pm 3$, $R(x) = x^l + b_{l-1}x^{l-1} + \cdots + b_0$ and $P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$, it follows that $c_j = a_j b_0 + a_{j-1} b_1 + \cdots$.

Let j be the smallest index such that 3 does not divide a_j, then 3 neither divides c_j, since $3 \nmid b_0$. Hence, $j \geq n-1$, and then $k \geq n-1$ and $l \leq 1$. Thus the polynomial $R(x)$ has the form $\pm x \pm 1$, but neither 1 or -1 are roots of $P(x)$. Therefore, $P(x)$ is irreducible over $\mathbb{Z}[x]$.

Solution 9.107. Suppose that the degree of $Q(x)$ is $n \leq p-2$. Using the lemma in Problem 9.105,

$$
0 = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} Q(i) \equiv \sum_{i=0}^{p-1} Q(i) \mod p,
$$

since $\binom{p-1}{i} \equiv (-1)^i \mod p$. But this is not possible if $Q(0) = 0$, $Q(1) = 1$ and $Q(i) \equiv 0, 1 \mod p$.

Solution 9.108. Since $P(x) > 0$, for $x \geq 0$, the polynomial can be decomposed in the following way

$$P(x) = a_0(x + a_1)\ldots(x + a_n)(x^2 - b_1x + c_1)\ldots(x^2 - b_mx + c_m),$$

with $a_i > 0$, for $0 \leq i \leq n$, and each quadratic polynomial $x^2 - b_jx + c_j$ has no real roots.

Since the product of polynomials with positive coefficients is a polynomial with positive coefficients, and since the factors $(x+a_i)$ already have positive coefficients, it will be enough to analyze the quadratic factors.

Let $x^2 - bx + c$ be a quadratic polynomial, with $b^2 - 4c < 0$. Then

$$\begin{aligned}(1 + x)^n(x^2 - bx + c) &= \sum_{i=0}^{n} \binom{n}{i} x^i(x^2 - bx + c)\\ &= \sum_{i=0}^{n+2} \left[\binom{n}{i-2} - b\binom{n}{i-1} + c\binom{n}{i}\right] x^i\\ &= \sum_{i=0}^{n+2} C_i x^i,\end{aligned}$$

where

$$\begin{aligned}C_i &= \left[\binom{n}{i-2} - b\binom{n}{i-1} + c\binom{n}{i}\right]\\ &= \frac{n!\left[(b + c + 1)i^2 - ((b + 2c)n + (2b + 3c + 1))i + c(n^2 + 3n + 2)\right]}{i!(n - i + 2)!}.\end{aligned}$$

Now, C_i will be negative if its discriminant is negative (depends of i). The discriminant is

$$\begin{aligned}D &= ((b + 2c)n + (2b + 3c + 1))^2 - 4(b + c + 1)(c(n^2 + 3n + 2))\\ &= (b^2 - 4c)n^2 - 2Un + V,\end{aligned}$$

where $U = 2b^2 + bc + b - 4c$ and $V = (2b + c + 1)^2 - 4c$. But since $b^2 - 4c < 0$, it will be sufficient to take n large enough, and then we will do this with every quadratic factor.

Solution 9.109. Notice that

$$\begin{aligned}xP(x) &= yP(y)\\ &\Leftrightarrow \quad a(x^4 - y^4) + b(x^3 - y^3) + c(x^2 - y^2) + d(x - y) = 0\\ &\Leftrightarrow \quad a(x^3 + x^2y + xy^2 + y^3) + b(x^2 + xy + y^2) + c(x + y) + d = 0.\end{aligned}$$

If $u = x + y$ and $v = x^2 + y^2$, then $x^2 + xy + y^2 = \frac{u^2+v}{2}$ and the previous equation becomes $auv + \frac{b}{2}(u^2+v) + cu + d = 0$, or equivalently, $(2au+b)v = -(bu^2+2cu+2d)$.

Since $v \geq \frac{u^2}{2}$, it is clear that $u^2\,|2au+b| \leq 2\left|bu^2 + 2cu + 2d\right|$, which is true only for a finite number of values of u. Since there are an infinite number of pairs of integers (x, y) with $xP(x) = yP(y)$, there is an integer u such that $xP(x) = (u - x)P(u - x)$ for an infinite number of integers x, but since $P(x)$ is a polynomial, the latter is true for every real number x.
If $u \neq 0$, then u is a root of $P(x)$.
If $u = 0$, then $P(x) = -P(-x)$; this implies $b = d = 0$. Then $P(x) = ax^3 + cx = x(ax^2 + c)$, hence $u = 0$ is a root of $P(x)$.

Solution 9.110. It is clear that $(xyz)^2 = abc$, then $xyz = \pm\sqrt{abc}$, hence $x = \frac{\pm\sqrt{abc}}{b}$, $y = \frac{\pm\sqrt{abc}}{c}$, $z = \frac{\pm\sqrt{abc}}{a}$ solve the system.

Solution 9.111. It is clear that $xy + yz + zx = \frac{1}{2}(a + b + c)$, $xy = \frac{1}{2}(a + b - c)$, $yz = \frac{1}{2}(b + c - a)$, $zx = \frac{1}{2}(c + a - b)$. Hence $x = \pm\sqrt{\frac{(c+a-b)(a+b-c)}{2(b+c-a)}}$, $y = \pm\sqrt{\frac{(b+c-a)(a+b-c)}{2(c+a-b)}}$, $z = \pm\sqrt{\frac{(b+c-a)(c+a-b)}{2(a+b-c)}}$ solve the system.

Solution 9.112. The system is equivalent to the following system:

$$\frac{1}{xy} + \frac{1}{xz} = \frac{1}{a}, \qquad \frac{1}{yz} + \frac{1}{yx} = \frac{1}{b}, \qquad \frac{1}{zx} + \frac{1}{zy} = \frac{1}{c}.$$

By the previous problem,

$$\frac{1}{x} = \pm\sqrt{\frac{(\frac{1}{c} + \frac{1}{a} - \frac{1}{b})(\frac{1}{a} + \frac{1}{b} - \frac{1}{c})}{2(\frac{1}{b} + \frac{1}{c} - \frac{1}{a})}} = \pm\sqrt{\frac{(ab + bc - ca)(bc + ca - ab)}{2abc(ca + ab - bc)}}$$

and similarly for the other variables.

Solution 9.113. If A, B, C are the expressions on the left-hand sides of the equations, it follows that $-A + B + C = (-a + b + c)^3$, $A - B + C = (a - b + c)^3$, $A + B - C = (a + b - c)^3$ and $-A + B + C = A - B + C = A + B - C = 1$. The system is equivalent to $-a + b + c = 1$, $a - b + c = 1$, $a + b - c = 1$, which has the unique solution $(a, b, c) = (1, 1, 1)$.

Solution 9.114. Proceed by induction on n. The case $n = 1$ is trivial. Suppose that it is true for $n > 1$. The polynomial $Q(x) = P(x + 1) - P(x)$ has degree $n - 1$ and takes integer values in the integers, then by the induction hypothesis, there are integers $a_0, \ldots, a_{n-1}$ such that

$$Q(x) = a_{n-1}\binom{x}{n-1} + \cdots + a_0\binom{x}{0}.$$

For any integer $x > 0$, it follows that $P(x) = P(0) + Q(0) + Q(1) + \cdots + Q(x-1)$. Using the identity $\binom{0}{k} + \binom{1}{k} + \cdots + \binom{x-1}{k} = \binom{x}{k+1}$, for any integer k, we obtain the desired representation of $P(x)$,

$$P(x) = a_{n-1}\binom{x}{n} + \cdots + a_0\binom{x}{1} + P(0).$$

Solution 9.115. Let r be a zero of $P(x)$. Then $|r|^p - |r| \leq |r^p - r| = p$. If $|r| \geq p^{\frac{1}{p-1}}$, then

$$|r|^p - |r| = |r|(|r|^{p-1} - 1) \geq p^{\frac{1}{p-1}}(p-1) > p,$$

which is a contradiction. Here we have used $p^{\frac{1}{p-1}} > \frac{p}{p-1}$, which follows from $p^{p-1} = ((p-1)+1)^{p-1}$. Therefore, $|r| < p^{\frac{1}{p-1}}$.
Suppose that $P(x)$ is the product of two non-constant polynomials $Q(x)$ and $R(x)$ with integer coefficients. One of these polynomials, say $Q(x)$, has constant term equal to $\pm p$. On the other hand, the zeros $r_1, r_2, \ldots, r_k$ of $Q(x)$ satisfy $|r_1|, \ldots, |r_k| < p^{\frac{1}{p-1}}$, and moreover $r_1 \cdots r_k = \pm p$, hence $k \geq p$, which is impossible.

Solution 9.116. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. For every x, the triplet $(a, b, c) = (6x, 3x, -2x)$ satisfies the condition $ab + bc + ca = 0$. The condition in $P(x)$ implies that $P(3x) + P(5x) + P(-8x) = 2P(7x)$, for all x. Now comparing coefficients on both sides of the equality, it follows that the number $K(i) = (3^i + 5^i + (-8)^i - 2 \cdot 7^i) = 0$, if $a_i \neq 0$. Since $K(i)$ is negative for i odd and positive for $i = 0$ or for $i \geq 6$ even, then $a_i = 0$, for $i \neq 2$, 4. Therefore, $P(x) = a_2 x^2 + a_4 x^4$, for any real numbers a_2 and a_4. It is easy to see that all polynomials of the previous form satisfy the conditions set for the problem.

Solution 9.117. We have shown that if for some integer t, $Q(t) = t$, then $P(P(t)) = t$ (see Exercise 8.16). If such t also satisfies $P(t) = t$, the number of solutions is clearly at most the degree of $P(x)$, which is equal to n.

Let $P(t_1) = t_2$, $P(t_2) = t_1$, $P(t_3) = t_4$ and $P(t_4) = t_3$, where $t_1 \neq t_i$, for $i = 2, 3, 4$. Then we have that $t_3 - t_1$ divides $t_4 - t_2$ and vice versa; therefore $t_3 - t_1 = \pm(t_4 - t_2)$. Similarly, we have that $t_3 - t_2 = \pm(t_4 - t_1)$.

Suppose that we have positive signs in both equalities: $t_3 - t_2 = t_4 - t_1$ and $t_3 - t_1 = t_4 - t_2$. Substracting these equalities, we find $t_1 - t_2 = t_2 - t_1$, which is a contradiction. Then, at least one of the equalities has negative sign. For each one of those cases, this means that $t_3 + t_4 = t_1 + t_2$, or equivalently, $t_1 + t_2 - t_3 - P(t_3) = 0$. Let $C = t_1 + t_2$, then it has been proved that each integer number that is a fixed point of $Q(x)$, different from t_1 and t_2, is a root of the polynomial $F(x) = C - x - P(x)$. This is also valid for t_1 and t_2, and since $P(x)$ has degree $n > 1$, the polynomial $F(x)$ has the same degree, therefore it has no more than n roots. Hence, we have reached the result.

Solution 9.118. For the first positive integers, it follows that

$$\begin{aligned} P(1) &= P(0^2+1) = P(0)^2+1 = 1 \\ P(2) &= P(1^2+1) = P(1)^2+1 = 2 \\ P(5) &= P(2^2+1) = P(2)^2+1 = 5 \\ P(26) &= P(5^2+1) = P(5)^2+1 = 26. \end{aligned}$$

Then, for $x_0 = 0$ and for $n \geq 1$, it follows that $x_n = x_{n-1}^2 + 1$. Then $P(x_n) = P(x_{n-1}^2+1) = P(x_{n-1}^2)+1 = x_{n-1}^2+1 = x_n$. Hence, $P(x)$ has an infinite number of fixed points, therefore $P(x) = x$.

Solution 9.119. Let $P(1) = a$, then it follows that $a^2 - 2a - 2 = 0$. Since $P(x) = (x-1)P_1(x) + a$, for some polynomial $P_1(x)$, substituting in the original equation and simplifying leads to $(x-1)P_1(x)^2 + 2aP_1(x) = 4(x+1)P_1(2x^2-1)$. For $x = 1$, it follows that $2aP_1(1) = 8P_1(1)$, and together with $a \neq 4$, implies that $P_1(1) = 0$. Hence, $P_1(x) = (x-1)P_2(x)$, for some polynomial $P_2(x)$. Then $P(x) = (x-1)^2P_2(x) + a$.
Suppose that $P(x) = (x-1)^nQ(x)+a$, where $Q(x)$ is a polynomial with $Q(1) \neq 0$. Again, substituting in the original equation and simplifying, we get $(x-1)^nQ(x)^2 + 2aQ(x) = 2(2x+2)^nQ(2x^2-1)$, which implies that $Q(1) = 0$, which is a contradiction. Therefore $P(x) = a$.

Solution 9.120. It is clear that $P(x)$ and $Q(x)$ have the same degree, say n. The cases $n = 0$ and $n = 1$ are clear. Suppose that $R(x) = P(x) - Q(x) \neq 0$ and that $0 < k \leq n-1$ is the degree of $R(x)$, then

$$P(P(x)) - Q(Q(x)) = [Q(P(x)) - Q(Q(x))] + R(P(x)).$$

Writing $Q(x) = x^n + \cdots + a_1x + a_0$, we obtain

$$Q(P(x)) - Q(Q(x)) = [P(x)^n - Q(x)^n] + \cdots + a_1[P(x) - Q(x)].$$

The main coefficient of the polynomial $Q(P(x)) - Q(Q(x))$ is n and it is equal to the coefficient of the term x^{n^2-n+k}. On the other hand, the degree of the polynomial $R(P(x))$ is equal to $kn < n^2 - n + k$. Therefore the main coefficient of $P(P(x)) - Q(Q(x))$ is n, which is a contradiction with the fact that the polynomial is zero.
It is left to prove the case $R(x)$ equal to some constant c. Then the condition $P(P(x)) = Q(Q(x))$ implies that $Q(Q(x)+c) = Q(Q(x)) - c$, hence the equality $Q(y+c) = Q(y) - c$ follows for an infinite number of values of y. Thus $Q(y+c) \equiv Q(y) - c$, which is only possible for $c = 0$, which can be proved comparing the coefficients and using that $Q(x)$ is monic.

Notation

The following notation is standard:

$\mathbb{N}$	the positive integers or the natural numbers
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rational numbers
$\mathbb{Q}^+$	the positive rational numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}^+$	the positive real numbers
$\mathbb{I}$	the irrational numbers
$\mathbb{C}$	the complex numbers
$\mathbb{Z}_p$	is $\{0, 1, \ldots, p-1\}$ with the sum and product modulo p.
$\Leftrightarrow$	if and only if
$\Rightarrow$	imply
$a \in A$	the element a belongs to the set A
$A \subset B$	A is subset of B
$\|x\|$	the absolute value of the number x
$\|z\|$	the module of the complex number z
$\{x\}$	the fractional part of the number x
$\lfloor x \rfloor$	the integer part of the number x
$[a, b]$	the set of real numbers x such that $a \leq x \leq b$
(a, b)	the set of real numbers x such that $a < x < b$
$P(x)$	the polynomial P in the variable x
$\deg(P)$	the degree of the polynomial $P(x)$
$f : [a, b] \to \mathbb{R}$	the function f defined in $[a, b]$ with values in $\mathbb{R}$
$f'(x)$	the derivative of the function $f(x)$
$f''(x)$	the second derivative of the function $f(x)$
$f^{(n)}(x)$	the nth derivative of the function $f(x)$
$f(x)^n$	the nth power of a function $f(x)$

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5

$f^n(x)$	the nth iteration of a function $f(x)$
$\Delta f(x)$	the diference operator of $f(x)$
$\det A$	the determinant of a matrix A
$\sum_{i=1}^n a_i$	the sum $a_1 + a_2 + \cdots + a_n$
$\prod_{i=1}^n a_i$	the product $a_1 \cdot a_2 \cdot \ldots \cdot a_n$
$\prod_{i\neq j} a_i$	the product for all $a_1, a_2, \ldots, a_n$ except a_j
$\max\{a, b, \ldots\}$	the maximum value among $a, b, \ldots$
$\min\{a, b, \ldots\}$	the minimum value among $a, b, \ldots$
$\sqrt{x}$	the square root of x
$\sqrt[n]{x}$	the nth root of the real number x
$\exp x = e^x$	the exponential function
$\sum_{\text{cyclic}} f(a, b, \ldots)$	represents the sum of the function f evaluated in all the cyclic permutations of the variables $a, b, \ldots$

We use the following notation for the source of the problems:

AMC	American Mathematical Competition
APMO	Asian Pacific Mathematical Olympiad
IMO	International Mathematical Olympiad
MEMO	Middle European Mathematical Olympiad
OMCC	Mathematical Olympiad of Central America and the Caribean
OIM	Iberoamerican Mathematical Olympiad
OMM	Mexican Mathematical Olympiad
(country, year)	problem corresponding to the mathematical olympiad celebrated in that country, in that year, in some stage

Bibliography

[1] Andreescu T., Andrica D., *Complex numbers from A to ... Z*, Birkhäuser, 2005.

[2] Andreescu T., Gelca R., *Mathematical Olympiad Challenges*, Birkhäuser, 2000.

[3] Andreescu T., Enescu B., *Mathematical Olympiad Treasures*, Birkhäuser, 2006.

[4] Barbeau E.J., *Polynomials*, Springer-Verlag, 1989.

[5] Bulajich Manfrino R., Gómez Ortega J.A., *Geometría*, Cuadernos de Olimpiadas, Instituto de Matemáticas de la Universidad Nacional Autónoma de México, Sociedad Matemática Mexicana, 2012.

[6] Bulajich Manfrino R., Gómez Ortega J.A., Valdez Delgado R., *Desigualdades*, Cuadernos de Olimpiadas, Instituto de Matemáticas de la Universidad Nacional Autónoma de México, Sociedad Matemática Mexicana, 2010.

[7] Bulajich Manfrino R., Gómez Ortega J.A., Valdez Delgado R., *Inequalities: A Mathematical Olympiad Approach*, Birkhäuser, 2009.

[8] Cárdenas H., Lluis E., Raggi F., Tomás F. *Álgebra Superior*, Editorial Trillas, 1973.

[9] Djukić D., Janković V., Matić I., Petrović N., *The IMO Compendium*, Springer, 2006.

[10] Engel A., *Problem-Solving Strategies*, Springer, 1998.

[11] Fine B., Resenberger G., *The Fundamental Theorem of Algebra*, Springer, 1997.

[12] Goldberg S., *Introduction to Difference Equations*, Dover Publications, 1958.

[13] Gómez Ortega J.A., Valdez Delgado R., Vázquez Padilla, *Principio de las Casillas*, Cuadernos de Olimpiadas, Instituto de Matemáticas de la Universidad Nacional Autónoma de México, Sociedad Matemática Mexicana, 2011.

[14] Honsberger R., *Ingenuity in Mathematics*, vol. 23 in New Mathematical Library series, 1962.

[15] Niven I., Montgomery H., Zuckerman H., *An Introduction to the Theory of Numbers*, Wiley, 5th edition, 1991.

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5

[16] Remmert R., *Theory of Complex Functions*, Springer, 1999.

[17] Rudin W., *Principles of Mathematical Analysis*, McGraw-Hill, 1976.

[18] Savchev S., Andreescu T., *Mathematical Miniatures*, The Mathematical Association of America, 2003.

[19] Small C.G., *Functional Equations and How to Solve Them*, Springer, 2007.

[20] Soberón P., *Problem-Solving Methods in Combinatorics: An Approach to Olympiad Problems*, Birkhäuser, 2013.

[21] Spivak M., *Calculus*, Editorial Reverté, 1993.

[22] Tabachnikov, S. (editor), *Kvant Selecta: Algebra and Analysis II*, American Mathematical Society, 1999.

[23] Venkatachala B.J., *Functional Equations. A Problem Solving Approach*, Prism Books Pvt Ltd, 2002.

Index

R.B. Manfrino et al, *Topics in Algebra and Analysis*,
DOI 10.1007/978-3-319-11946-5