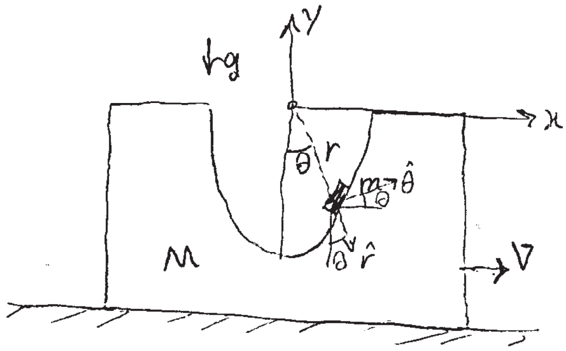
$$\vec{v}_m^{\,rel} = \dot r\hat r + r\dot\theta\hat\theta = r'\dot\theta(\sin\theta\,\hat x - \cos\theta\,\hat y) + r\dot\theta(\cos\theta\,\hat x + \sin\theta\,\hat y)$$

مسئله 1 - امتحان (۵) (1)

a)



پایستگی تکانه در راستای x :

$$MV + m[V + \dot\theta(r'\sin\theta + r\cos\theta)] = 0$$

$$\Rightarrow (m+M)V + m\dot\theta(r'\sin\theta + r\cos\theta) = 0$$

$$\Rightarrow V = \frac{-m}{m+M}(r'\sin\theta + r\cos\theta)\dot\theta \qquad (1)$$

پایستگی انرژی :

$$0 = -mgr\cos\theta + \frac{1}{2}MV^2 + \frac{1}{2}m\left[(r\sin\theta - r'\cos\theta)^2\dot\theta^2 + \left(V + (r'\sin\theta + r\cos\theta)\dot\theta\right)^2\right]$$

$$\Rightarrow 2mgr\cos\theta = (m+M)V^2 + 2mV\dot\theta(r'\sin\theta + r\cos\theta) + m\dot\theta^2(r^2+r'^2)$$

$$(1) \Rightarrow 2mgr\cos\theta = \frac{m^2}{m+M}\dot\theta^2(r'\sin\theta + r\cos\theta)^2 - 2\frac{m^2}{m+M}\dot\theta^2(r'\sin\theta + r\cos\theta)^2 + m(r^2+r'^2)\dot\theta^2$$

$$\Rightarrow 2gr\cos\theta = (r^2+r'^2)\dot\theta^2 - \frac{m}{m+M}(r'\sin\theta + r\cos\theta)^2\dot\theta^2 \Rightarrow \dot\theta^2 = \frac{2gr\cos\theta}{r^2+r'^2 - \frac{m}{m+M}(r\cos\theta + r'\sin\theta)^2} \qquad (2)$$

$$T_m = \frac{1}{2}m\left[(r\sin\theta - r'\cos\theta)^2\dot\theta^2 + \left(V + (r'\sin\theta + r\cos\theta)\dot\theta\right)^2\right] = \frac{1}{2}m\dot\theta^2\left[(r\sin\theta - r'\cos\theta)^2 + \frac{M^2}{(m+M)^2}(r'\sin\theta + r\cos\theta)^2\right]$$

$$\cancel{= \frac{1}{2}m\left[(r^2+r'^2)\dot\theta^2 + V^2 + 2V\dot\theta(r'\sin\theta + r\cos\theta)\right] =}$$

$$\cancel{= \frac{1}{2}m\dot\theta^2\left[r^2+r'^2 + \frac{m^2}{(m+M)^2}(r'\sin\theta + r\cos\theta)^2 - \frac{2m}{m+M}(r'\sin\theta + r\cos\theta)^2\right] = \frac{1}{2}m\dot\theta^2\left[r^2+r'^2 - \frac{m^2+2mM}{(m+M)^2}(r'\sin\theta + r\cos\theta)\right]}$$

$$T_M = \frac{1}{2}MV^2 = \frac{1}{2}m\dot\theta^2\left[\frac{mM}{(m+M)^2}(r'\sin\theta + r\cos\theta)^2\right]$$

$$\cancel{\Rightarrow \frac{T_m}{T_M} = \frac{m}{M} - 2\frac{m+M}{M} + \frac{r^2+r'^2}{\frac{mM}{(m+M)^2}(r'\sin\theta + r\cos\theta)^2} =}$$

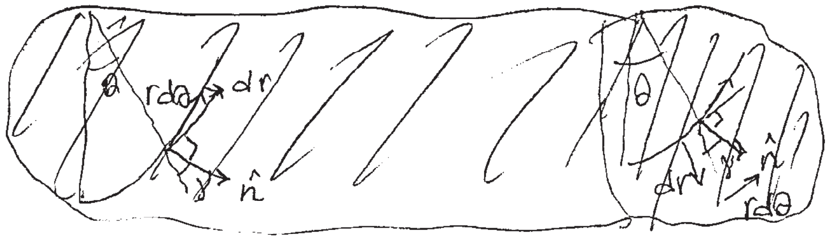
$$\Rightarrow \boxed{\frac{T_m}{T_M} = \frac{M}{m} + \frac{(m+M)^2}{mM}\cdot\left(\frac{r\sin\theta - r'\cos\theta}{r\cos\theta + r'\sin\theta}\right)^2}$$

b) $T = T_m + T_M = T_M\left(1 + \frac{T_m}{T_M}\right) \Rightarrow f(\theta) = \frac{T_M}{\frac{1}{2}m\dot\theta^2}\left(1 + \frac{T_m}{T_M}\right)$

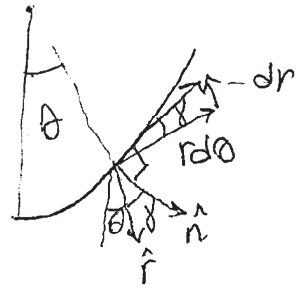
$$\Rightarrow \boxed{f(\theta) = (r\sin\theta - r\cos\theta)^2 + \left(\frac{M}{m+M}\right)^2(r'\sin\theta + r\cos\theta)^2}$$

$$mgr\cos\theta = \frac{1}{2}m f(\theta)\dot\theta^2 \Rightarrow \dot\theta^2 = \frac{2gr\cos\theta}{f(\theta)}$$

c)



$$\tan\gamma = \frac{-dr}{rd\theta} = \frac{-r'}{r}$$



$$F_x = N\sin(\theta+\gamma) = N(\sin\theta\cos\gamma + \cos\theta\sin\gamma)$$

$$= N\cos\gamma\left(\sin\theta - \frac{r'}{r}\cos\theta\right) = Ma_x$$

$$\therefore \cos\gamma = \frac{1}{\sqrt{1+\tan^2\gamma}} = \frac{r}{\sqrt{r^2+r'^2}}$$

$$\Rightarrow N = \frac{\sqrt{r^2+r'^2}}{r\sin\theta - r'\cos\theta}Ma_x$$

$$a_x = \frac{dV}{dt} = \frac{-m}{m+M}(r'\sin\theta + r\cos\theta)\ddot{\theta} - \frac{m}{m+M}(r'\cos\theta - r\sin\theta)\dot{\theta}^2 - \frac{m}{m+M}(r''\cos\theta + r''\sin\theta)\dot{\theta}^2$$

$$\dot{\theta}^2 = \frac{2gr\cos\theta}{f(\theta)} \Rightarrow \ddot{\theta} = \frac{d\dot{\theta}^2}{2d\theta} = g\frac{(r'\cos\theta - r\sin\theta)f(\theta) - r\cos\theta f'(\theta)}{f^2(\theta)}$$

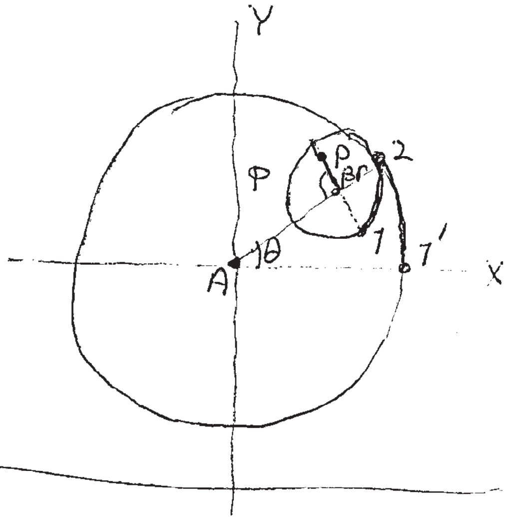
$$\Rightarrow a_x = \frac{-m}{m+M}(r\cos\theta + r'\sin\theta)\,g\,\frac{(r'\cos\theta - r\sin\theta)f(\theta) - r\cos\theta f'(\theta)}{f^2(\theta)} - \frac{m}{m+M}(r''\sin\theta + 2r'\cos\theta - r\sin\theta)\frac{2gr\cos\theta}{f(\theta)}$$

$$\Rightarrow N(\theta) = \frac{mMg}{m+M}\sqrt{r^2+r'^2}\left\{\frac{(r\cos\theta + r'\sin\theta)}{f(\theta)} - \frac{r\cos\theta(r\cos\theta + r'\sin\theta)f'(\theta)}{(r'\cos\theta - r\sin\theta)f^2(\theta)} + \frac{2r\cos\theta(r''\sin\theta + 2r'\cos\theta - r\sin\theta)}{(r'\cos\theta - r\sin\theta)f(\theta)}\right.$$

مسئله 2 - آزمون اول

الف) $\widehat{1'2} = \widehat{12}$

$$\Rightarrow R\theta = r\varphi \Rightarrow \boxed{\varphi = \alpha\theta}$$



ب)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (R-r)\cos\theta - \beta r\cos(\varphi-\theta) \\ (R-r)\sin\theta + \beta r\sin(\varphi-\theta) \end{pmatrix}$$

$$\boxed{= r\begin{pmatrix} (\alpha-1)\cos\theta - \beta\cos[(\alpha-1)\theta] \\ (\alpha-1)\sin\theta + \beta\sin[(\alpha-1)\theta] \end{pmatrix}}$$

$$\Rightarrow |PA| = \cancel{r\sqrt{(\alpha-1)^2+\beta^2}} = \boxed{r\sqrt{\beta^2+(\alpha-1)^2 - 2\beta(\alpha-1)\cos(\alpha\theta)}}$$

ج) $\dot\theta = \omega$

$$\Rightarrow \vec{v} = \begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = r(\alpha-1)\omega \boxed{\begin{pmatrix} -\sin\theta + \beta\sin[(\alpha-1)\theta] \\ \cos\theta + \beta\cos[(\alpha-1)\theta] \end{pmatrix}}$$

$$\Rightarrow |\vec v| = \cancel{r\omega(\alpha-1)\sqrt{1+\beta^2}} \quad \boxed{r\omega(\alpha-1)\sqrt{1+\beta^2+2\beta\cos(\alpha\theta)}}$$

د) $\alpha = 2,\ \beta = -1 \Rightarrow \boxed{\vec v = -2r\omega\sin\theta\ \hat{x}} \Rightarrow |\vec v| = \boxed{2r\omega|\sin\theta|}$

$$|\vec v| = 2r\omega|\sin\omega t| \Rightarrow S = \int_0^T v\,dt = 2r\omega\int_0^{2\pi/\omega}|\sin\omega t|\,dt = 4r\int_0^{\pi}\sin\theta\,d\theta = 8r\int_0^{\pi/2}\sin\theta\,d\theta = \boxed{8r}$$

ه) $\alpha = 2,\ \beta \neq 1, -1 \Rightarrow \begin{pmatrix} x/r \\ y/r \end{pmatrix} = \begin{pmatrix} \cos\theta - \beta\cos\theta \\ \sin\theta + \beta\sin\theta \end{pmatrix} = \begin{pmatrix} (1-\beta)\cos\theta \\ (1+\beta)\sin\theta \end{pmatrix}$

$$\Rightarrow \left(\frac{x}{r(1-\beta)}\right)^2 + \left(\frac{y}{r(1+\beta)}\right)^2 = 1 \Rightarrow \boxed{\text{بیضی است}}$$

ج) $\alpha=3,\ \beta=1$

$\Rightarrow v = 2r\omega\sqrt{2+2\cos(3\theta)} = 4r\omega\left|\cos\frac{3\theta}{2}\right| \Rightarrow S = \int_{\theta=0}^{2\pi} 4r\omega\left|\cos\frac{3\theta}{2}\right|dt = \frac{8r}{3}\int_{\theta=0}^{2\pi}\left|\cos\frac{3\theta}{2}\right|d\left(\frac{3\theta}{2}\right)$

$= \frac{8r}{3}\int_0^{3\pi}|\cos\gamma|\,d\gamma = 6\times\frac{8r}{3}\int_0^{\pi/2}\cos\gamma\,d\gamma = \boxed{16r}$

د)

$\boxed{r=3,\ \alpha=7/3,\ \beta=5/3}$

$\Rightarrow \vec{r}_p = 3\begin{pmatrix}\frac{4}{3}\cos\theta-\frac{5}{3}\cos(\frac{4}{3}\theta)\\ \frac{4}{3}\sin\theta+\frac{5}{3}\sin(\frac{4}{3}\theta)\end{pmatrix} = \begin{pmatrix}4\cos\theta-5\cos\frac{4}{3}\theta\\ 4\sin\theta+5\sin(\frac{4}{3}\theta)\end{pmatrix}$

$|r_{max}| = \sqrt{\beta^2+(\alpha-1)^2+2\beta(\alpha-1)}\ r = |\beta+\alpha-1|\,r$

$= 3r = 9 \Rightarrow$ $\boxed{\text{D شکل}}$

$\boxed{r=3,\ \alpha=7/3,\ \beta=1}$

$r_{max} = \frac{7}{3}r = 7 \Rightarrow$ $\boxed{\text{C شکل}}$

$|v| = r\omega(\alpha-1)\times 2\cos\frac{\alpha\theta}{2} \Rightarrow v(\theta=\frac{\pi}{\alpha})=0$

تیز شکل ✓

$\boxed{r=5,\ \alpha=7/5,\ \beta=2/5}$

$r_{max} = \frac{4}{5}r = 4 \Rightarrow$ $\boxed{\text{A شکل}}$

$\boxed{r=5,\ \alpha=7/5,\ \beta=3/5}$

$\Rightarrow r_{max} = r = 5 \Rightarrow$ $\boxed{\text{B شکل}}$

مسئله ③ امتحان اول

آ) صفر پتانسیل را ظرف 1 می‌گیریم *

$$P(\vec{r},\vec{v}) = C\left(\frac{m}{2\pi KT}\right)^{3/2} e^{\frac{-mv^2}{2KT}} \times e^{\frac{-U(\vec{r})}{KT}}$$

$$\int_{\varphi,\theta} dv_x dv_y dv_z = 4\pi v^2 dv \quad \Rightarrow \int P(\vec{r},\vec{v})\, dv_x dv_y dv_z = P(\vec{r}) = Ce^{\frac{-U(\vec{r})}{KT}}$$

$$\int P(\vec{r})\, dxdydz = 1 \Rightarrow \int_{V_1} Ce^{\frac{-U(\vec{r})}{KT}} dxdydz + \int_{V_2} Ce^{\frac{-U(\vec{r})}{KT}} dxdydz = 1$$

$$\Rightarrow \frac{1}{C} = V_1 + V_2 e^{\frac{-mgh}{KT}} \Rightarrow \boxed{C = \frac{1}{V_1 + V_2 e^{\frac{-mgh}{KT}}}} \Rightarrow P(\vec{r}) = \frac{e^{\frac{-U(\vec{r})}{KT}}}{V_1 + V_2 e^{\frac{-mgh}{KT}}}$$

ب) $dN_v = NP(\vec{r},\vec{v})\, d^3r\, d^3v \quad \Rightarrow dN = NP(\vec{r})\, d^3r$

$$\Rightarrow \begin{cases} N_1 = N\int_{V_1} P(\vec{r})\, d^3r = \dfrac{N}{V_1+V_2 e^{\frac{-mgh}{KT}}}\displaystyle\int_{V_1} d^3r = \boxed{\dfrac{V_1}{V_1+V_2 e^{\frac{-mgh}{KT}}} N} \\[2ex] N_2 = N\int_{V_2} P(\vec{r})\, d^3r = \boxed{\dfrac{V_2 e^{\frac{-mgh}{KT}}}{V_1+V_2 e^{\frac{-mgh}{KT}}} N} \end{cases}$$

$$\Rightarrow \begin{cases} P_1 = \dfrac{N_1 KT}{V_1} = \boxed{\dfrac{NKT}{V_1+V_2 e^{\frac{-mgh}{KT}}}} \\[2ex] P_2 = \dfrac{N_2 KT}{V_2} = \boxed{\dfrac{NKT\, e^{\frac{-mgh}{KT}}}{V_1+V_2 e^{\frac{-mgh}{KT}}}} \end{cases}$$

ج) $E_{in} = N_2 mgh + \frac{3}{2}N_1KT + \frac{3}{2}N_2KT = \boxed{\frac{3}{2}NKT + \frac{V_2 e^{\frac{-mgh}{KT}}}{V_1+V_2 e^{\frac{-mgh}{KT}}} Nmgh}$

$$C_V = \left(\frac{\partial E_{in}}{\partial T}\right)_{V_1,V_2} = \boxed{\frac{3}{2}NK + \frac{V_1V_2\, Nmgh}{\left(V_1+V_2 e^{\frac{-mgh}{KT}}\right)^2} \times \left(\frac{mgh}{KT^2}\right) e^{\frac{-mgh}{KT}}} =$$

$$= NK\left[\frac{3}{2} + \frac{V_1V_2 e^{\frac{-mgh}{KT}}}{\left(V_1+V_2 e^{\frac{-mgh}{KT}}\right)^2} \times \left(\frac{mgh}{KT}\right)^2\right]$$

ز)

I) $\frac{mgh}{KT} \gg 1 \Rightarrow \begin{cases} N_1 \simeq N \quad , \quad N_2 = 0 \\ P_1 = \frac{NKT}{V_1} \quad , \quad P_2 = 0 \\ E_{in} = \frac{3}{2}NKT \quad و \quad C_V = \frac{3}{2}NK \end{cases}$

مرتبه صفر :

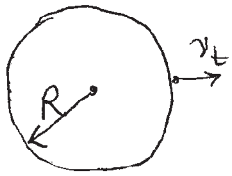
(اولین مرتبه غیرصفر)

II) $\frac{mgh}{KT} \ll 1 \Rightarrow \begin{cases} N_1 \doteq \frac{V_1}{V_1+V_2}N \quad و \quad N_2 = \frac{V_2}{V_1+V_2}N \\ P_1 = P_2 = \frac{NKT}{V_1+V_2} \\ E_{in} = NKT\left[\frac{3}{2} + \frac{V_2 e^{-\frac{mgh}{KT}}}{V_1 + V_2 e^{-\frac{mgh}{KT}}} \times \frac{mgh}{KT}\right] \doteq \frac{3}{2}NKT \quad و \quad C_V = \frac{3}{2}NK \end{cases}$

مسئله 4 آزمون ۶۶ :

آ) $p(v_r)\,dv_r = \sqrt{\frac{m}{2\pi kT}}\; e^{\frac{-mv_r^2}{2kT}}\, dv_r$

$$\xRightarrow{\text{ب)}} \alpha = \sqrt{\frac{m}{2\pi kT}} \int_{v_t}^{\infty} e^{\frac{-mv_r^2}{2kT}}\, dv_r = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{mv_t^2}{2kT}}}^{\infty} e^{-u^2}\, du = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\frac{mv_t^2}{2kT}}} e^{-u^2}\, du$$



سرعت فرار : $\frac{1}{2} v_t^2 = \frac{GM}{R} \Rightarrow v_t = \sqrt{\frac{2GM}{R}} \Rightarrow \begin{cases} v_t^{\text{ماه}} \simeq 2{,}37 \ \text{km/s} \\ v_t^{\text{زمین}} \simeq 11{,}2 \ \text{km/s} \end{cases}$

$\cancel{\alpha = \frac{1}{2} - \frac{1}{\sqrt{\pi}}(x_0 - x_0^3/3)}$

$$x_{0\,\text{ماه}}^{H_2} = \sqrt{\frac{2\times10^{-26}}{2\times 6.02} \times \frac{2{,}37^2 \times 10^6}{1.38\times10^{-23}\times 300}} \approx 1.5 \quad , \quad x_{0\,\text{ماه}}^{N_2} = 5.62$$

$$x_{0\,\text{زمین}}^{H_2} = 7.1 \quad , \quad x_{0\,\text{زمین}}^{N_2} \approx 26.5$$

تقریب : $(x_0 \gg 1)$

$$\alpha \approx \frac{1}{2} - \frac{1}{2}\left[1 - \frac{e^{-x_0^2}}{x_0\sqrt{\pi}}\left(1 - \frac{x_0^2}{2} + \dots\right)\right] = \frac{e^{-x_0^2}}{2x_0\sqrt{\pi}}\left(1 - \frac{x_0^2}{2} + \dots\right) \approx \frac{e^{-x_0^2}}{2x_0\sqrt{\pi}}$$

$$\Rightarrow \boxed{\alpha_{\text{ماه}}^{H_2} \approx 0{,}02} \ , \ \boxed{\alpha_{\text{ماه}}^{N_2} \approx 9{,}6 \times 10^{-16}} \ , \ \boxed{\alpha_{\text{زمین}}^{H_2} \approx 5.1 \times 10^{-24}} \ , \ \boxed{\alpha_{\text{زمین}}^{N_2} \approx 0 \,!}$$

ب)

$$p(v)dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{\frac{-mv^2}{2kT}} dv \Rightarrow \eta = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_{v_m}^{\infty} v^2 e^{\frac{-mv^2}{2kT}} dv = \frac{4}{\sqrt{\pi}}\int_{\sqrt{\frac{mv_m^2}{2kT}}}^{\infty} u^2 e^{-u^2} du$$

$$K_{max} = 13.6\,eV = \frac{1}{2} m_e V_m^2 \Rightarrow V_m^2 = \frac{2\times 13.6\times 1.6\times 10^{-19}}{m_e}$$

$$\Rightarrow m_e V^2 = 4.352\times 10^{-18}\ J$$

$$\Rightarrow x_0 = \sqrt{\frac{m_e V^2}{2kT}} = \sqrt{\frac{4.352\times 10^{-18}}{2\times 1.38\times 10^{-23}\times 2\times 10^{4}}} \simeq 2.81 \gg 1$$

$$\eta = \frac{4}{\sqrt{\pi}}\int_{x_0}^{\infty} u^2 e^{-u^2} du = \frac{4}{\sqrt{\pi}}\int_{x_0}^{\infty}\left[\frac{1}{2}e^{-u^2} - \frac{1}{2}\frac{d}{du}\left(ue^{-u^2}\right)\right]du = \frac{2}{\sqrt{\pi}}\int_{x_0}^{\infty} e^{-u^2} du - \frac{2}{\sqrt{\pi}}\, ue^{-u^2}\Big|_{x_0}^{\infty}$$

$$\Rightarrow \eta = \frac{2}{\sqrt{\pi}}\int_{x_0}^{\infty} e^{-u^2} du - \frac{2}{\sqrt{\pi}} x_0 e^{-x_0^2} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{x_0} e^{-u^2} du - \frac{2}{\sqrt{\pi}} x_0 e^{-x_0^2}$$

$$\Rightarrow \eta \approx 1 - \left[1 - \frac{e^{-x_0^2}}{x_0\sqrt{\pi}}\left(1 - \frac{x_0^2}{2} + \cdots\right)\right] - \frac{2x_0 e^{-x_0^2}}{\sqrt{\pi}} \approx \frac{e^{-x_0^2}}{\sqrt{\pi}}\left(\frac{1}{x_0} - 2x_0 - \frac{x_0}{2} + \cdots\right)$$

$$\Rightarrow \eta \approx \frac{e^{-x_0^2}}{x_0\sqrt{\pi}} = 7.47\times 10^{-5} = \boxed{0.007\ \%}$$

a)

مسئله ① آزمون دوم :

پایستگی تکانه : $MV + m(V + h\dot{\theta}(0)) = 0 \Rightarrow (m+M)V + mh\dot{\theta}(0) = 0$

سرعت m در $\theta = 0$

$$\Rightarrow V = \frac{-m}{m+M} h\dot{\theta}(0)$$

پایستگی انرژی : $\frac{1}{2}MV^2 + \frac{1}{2}m(V + h\dot{\theta}(0))^2 = mgh \Rightarrow 2mgh = \frac{Mm^2}{(m+M)^2}h^2\dot{\theta}^2(0) + \frac{mM^2}{(m+M)^2}h^2\dot{\theta}^2(0)$

$$\Rightarrow 2mgh = \frac{Mm}{M+m}h^2\dot{\theta}^2(0) \Rightarrow \dot{\theta}^2(0) = \frac{2(m+M)g}{Mh} \Rightarrow \dot{\theta}(0) = -\sqrt{\frac{2(m+M)g}{Mh}}$$

$$\Rightarrow V = \frac{m}{M+m} h \sqrt{\frac{2(m+M)g}{Mh}}$$

$$y = -r\cos\theta \Rightarrow \dot{y} = r\sin\theta\,\dot{\theta} - r'\dot{\theta}\cos\theta \Rightarrow \ddot{y} = (r\sin\theta - r'\cos\theta)\ddot{\theta} + (r'\sin\theta - r''\cos\theta)\dot{\theta}^2 + (r\cos\theta + r'\sin\theta)\dot{\theta}^2$$

$$\Rightarrow \ddot{y}_{\substack{(\theta=0)\\(r'=0)}} = -r''(0)\dot{\theta}^2(0) + r(0)\dot{\theta}^2(0) = (h - r''(0))\dot{\theta}^2(0)$$

$$m\ddot{y}(0) = N - mg \Rightarrow N = m\left[g + (h - r''_0)\frac{2(m+M)g}{Mh}\right] = \boxed{mg\left[1 + 2\frac{h - r''_0}{h}\left(\frac{m+M}{M}\right)\right]}$$

راه دوم : $\begin{cases} x = r\sin\theta \Rightarrow x' = r'\sin\theta + r\cos\theta \\ y = -r\cos\theta \Rightarrow y' = r\sin\theta - r'\cos\theta \Rightarrow y'' = r\cos\theta + 2r'\sin\theta - r''\cos\theta \end{cases} \Rightarrow R_0 = \frac{x_0'^2}{y_0''} = \frac{h^2}{h - r''_0}$

$$m\frac{(h^2\dot{\theta}_0^2)}{R} = N - mg \Rightarrow N = mg\left[1 + 2\frac{h - r''_0}{h}\left(\frac{m+M}{M}\right)\right]$$

---

b) $\begin{cases} \dot{x}_{rel} = (r'\sin\theta + r\cos\theta)\dot{\theta} \\ \dot{y} = (r\sin\theta - r'\cos\theta)\dot{\theta} \end{cases}$

فرض ← m از θ_0 رها می‌شود

پایستگی تکانه : $MV + m(V + \dot{x}_{rel}) = 0$

$$\Rightarrow V = \frac{-m}{m+M}\dot{x}_{rel} = \frac{-m}{m+M}(r'\sin\theta + r\cos\theta)\dot{\theta} \xrightarrow[\text{تقریب ۱}]{}$$

$$\begin{cases} \sin\theta = \theta \\ \cos\theta = 1 - \theta^2/2 \\ r(\theta) = h + \frac{1}{2}r''_0\theta^2 \Rightarrow r' = r''_0\theta \end{cases}$$

$$\Delta y = r\cos\theta - r(\theta_0)\cos\theta_0 = \left(h + \frac{1}{2}r''_0\theta^2\right)\left(1 - \frac{\theta^2}{2}\right) - \left(h + \frac{1}{2}r''_0\theta_0^2\right)\left(1 - \frac{\theta_0^2}{2}\right)$$

$$\Rightarrow \Delta y = \frac{1}{2}(r''_0 - h)(\theta^2 - \theta_0^2)$$

$$V = \frac{-m}{m+M}h\dot{\theta} \quad \Bigg\} \Rightarrow mg\frac{1}{2}(r''_0 - h)(\theta^2 - \theta_0^2) = \frac{1}{2}M\frac{m^2}{(m+M)^2}h^2\dot{\theta}^2 + \frac{1}{2}mv_m^2$$

$$v_m^2 = \dot{y}^2 + (V + \dot{x}_{rel})^2 \overset{\text{تقریب}}{\approx} (V + \dot{x}_{rel})^2 = \left(\frac{M}{m+M}h\dot{\theta}\right)^2$$

$$\Rightarrow mg(r''_0 - h)(\theta^2 - \theta_0^2) = \frac{Mm^2}{(m+M)^2}h^2\dot{\theta}^2 + \frac{M^2m}{(M+m)^2}h^2\dot{\theta}^2 = \frac{Mm}{M+m}h^2\dot{\theta}^2$$

$$\xrightarrow{\text{مشتق گیری}} 2mg(r''_0 - h)\theta\dot{\theta} = \frac{2Mm}{m+M}h^2\dot{\theta}\ddot{\theta} \Rightarrow 2g(r''_0 - h)\theta = \frac{M}{m+M}h^2\ddot{\theta} \Rightarrow \boxed{f = \frac{1}{2\pi}\sqrt{\frac{2(m+M)g(h - r''_0)}{Mh^2}}}$$

c) $r=(h-a)\cos\theta+a \Rightarrow r'=(a-h)\sin\theta \Rightarrow r''=(a-h)\cos\theta \Rightarrow r_0''=a-h$

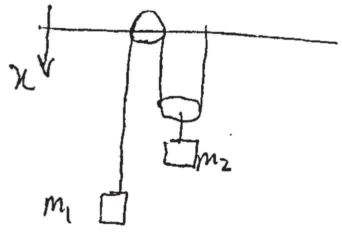
$$\Rightarrow N=mg\left[1+2\frac{h-r_0''}{h}\left(\frac{m+M}{M}\right)\right]=\boxed{mg\left[1+2\frac{2h-a}{h}\left(\frac{m+M}{M}\right)\right]}$$

$$\boxed{f=\frac{1}{2\pi}\sqrt{\frac{2(m+M)g(2h-a)}{h^2}}}$$

حل ② آزمون دوم :

a)

$x_1+2x_2=l \Rightarrow x_2=\frac{l-x_1}{2} \Rightarrow \dot{x}_2=-\frac{\dot{x}_1}{2}=-\frac{\dot{x}}{2}$



$\cancel{m_1 g x_0 \neq m_1 g x_1}$

$$-m_1 g x_0-\frac{m_2}{2}g(l-x_0)-\mu\left(\frac{x_0}{l}\right)g\left(\frac{x_0}{2}\right)-\mu\left(\frac{l-x_0}{2l}\right)g\left(\frac{l-x_0}{4}\right)\times 2$$
$$=-m_1 g x_1-\frac{m_2}{2}g(l-x_1)-\mu\left(\frac{x_1}{l}\right)g\left(\frac{x_1}{2}\right)-\mu\left(\frac{l-x_1}{2l}\right)g\left(\frac{l-x_1}{4}\right)\times 2+\frac{1}{2}m_1\dot{x}^2+\frac{1}{8}m_2\dot{x}^2+\frac{1}{2}\mu\left(\frac{l+x}{2l}\right)\dot{x}^2$$

$$\Rightarrow \dot{x}^2\left[\frac{1}{2}m_1+\frac{1}{8}m_2+\frac{l+x}{4l}\mu\right]=m_1 g(x-x_0)-m_2 g\left(\frac{x-x_0}{2}\right)+\mu g/2\left(\frac{x^2-x_0^2}{l}\right)+\frac{\mu g}{4l}\left((l-x)^2-(l-x_0)^2\right)$$

$$\boxed{\Rightarrow \dot{x}^2=\frac{m_1 g(x-x_0)-m_2 g/2\,(x-x_0)+\frac{3\mu g}{4l}(x^2-x_0^2)-\mu g/2\,(x-x_0)}{\frac{m_1}{2}+\frac{m_2}{8}+\frac{l+x}{4l}\mu}:=f(x)}$$

$$\Rightarrow \ddot{x}=\frac{d\dot{x}^2}{2dx}=\boxed{\frac{f'(x)}{2}}$$

---

b) تکانه $P=m_1\dot{x}-\frac{m_2}{2}\dot{x}+\frac{\mu}{l}\left(\frac{3x-l}{2}\right)\dot{x}$

$$\Rightarrow \frac{dP}{dt}=m_1\ddot{x}-\frac{m_2}{2}\ddot{x}+\frac{\mu}{l}\left(\frac{3x-l}{2}\right)\ddot{x}+\frac{3}{2}\frac{\mu}{l}\dot{x}^2=m_1 g+m_2 g+\mu g-F$$

$$\boxed{\Rightarrow F=(m_1+m_2+\mu)g-\left(m_1-\frac{m_2}{2}+\frac{\mu}{l}\left(\frac{3x-l}{2}\right)\right)\frac{f'(x)}{2}-\frac{3\mu}{2l}f(x)}$$

---

c) $\ddot{x}\big|_{x=x_0}=0 \quad : \forall x_0 \qquad \Rightarrow f'(x_0)=0$

$$f'(x)=\frac{\left(m_1 g-m_2 g/2+\frac{3}{2}\mu g x/l-\mu g/2\right)\left(\frac{m_1}{2}+\frac{m_2}{8}+\frac{l+x}{4l}\mu\right)-\frac{\mu}{4l}\left(f(x)\text{ صورت}\right)}{\left(\frac{m_1}{2}+\frac{m_2}{8}+\frac{l+x}{4l}\mu\right)^2}$$

$$f'(x_0)=0 \Rightarrow m_1-\frac{m_2}{2}+\frac{3\mu}{2}\cdot x_0/l-\mu/2=0 \Rightarrow x_0=\frac{2l}{3\mu}\left(\mu/2+m_2/2-m_1\right)$$

$$0<x_0<l \Rightarrow \boxed{\begin{cases}2m_1<\mu+m_2\\ m_2<2m_1+2\mu\end{cases}\quad \text{شرطها}}$$

d) $\begin{cases} 2m < 3m + {}^{2m}/_3 \\ 3m < 2m + ({}^{2m}/_3) \times 2 \end{cases}$ ✓

$$\Rightarrow x_0 = \frac{2l}{3\mu} \times 5/6\, m = \quad 5/6\, l$$

$$\Rightarrow f(0) = \frac{-m_1 g(\frac{5}{6}l) + m_2 g/_2 (\frac{5}{6}l) - 3\frac{\mu g}{4l}(\frac{5}{6}l)^2 + \mu g/_2(\frac{5}{6}l)}{\frac{m_1}{2} + \frac{m_2}{8} + \frac{\mu}{4}} =$$

$$= \frac{gl}{3} \; \frac{10m_2 - 20m_1 - 25/_2\,\mu + 10\mu}{m_2 + 4m_1 + 2\mu} = \frac{gl}{3} \times \frac{25m/_3}{25m/_3} = gl/_3 \Rightarrow \boxed{\dot{x}(0) = -\sqrt{gl/_3}}$$

$$f(x) = g\; \frac{(x - \frac{5}{6}l) - \frac{3}{2}(x - \frac{5}{6}l) + \frac{1}{2l}(x^2 - \frac{25}{36}l^2) - \frac{1}{3}(x - \frac{5}{6}l)}{\frac{1}{2} + \frac{3}{8} + \frac{1}{6}(1 + \frac{x}{l})}$$

$$\Rightarrow f(x) = gl\; \frac{\frac{1}{2}(\frac{x^2}{l^2} - \frac{25}{36}) - \frac{5}{6}(x/l - 5/6)}{7/8 + \frac{1}{6} + \frac{x}{6l}} = gl\; \frac{3(\frac{x^2}{l^2} - \frac{25}{36}) - 5(x/l - 5/6)}{\frac{25}{4} + x/l}$$

$$\Rightarrow \dot{x} = \frac{dx}{dt} = -\sqrt{gl}\; \sqrt{\frac{3(x^2/l^2 - \frac{25}{36}) - 5(x/l - 5/6)}{\frac{25}{4} + x/l}}$$

$$\Rightarrow -\sqrt{gl}\; T = \int_{5/12\, l}^{0} \frac{\sqrt{x/l + \frac{25}{4}}\; dx}{\sqrt{3(x^2/l^2 - \frac{25}{36}) - 5(x/l - \frac{5}{6})}}$$

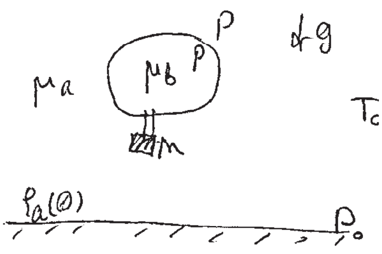
$$\Rightarrow T = \sqrt{l/g} \int_0^{5/12} \frac{\sqrt{x + 25/4}\; dx}{\sqrt{3(x^2 - 25/36) - 5(x - 5/6)}} = \sqrt{l/g} \int_0^{5/12} \frac{\sqrt{x + 25/4}\; dx}{\sqrt{3}\sqrt{x - 5/6}\sqrt{x + 5/6 - 5/3}}$$

$$\Rightarrow T = \sqrt{l/g} \int_0^{5/12} \frac{\sqrt{x + 25/4}}{\sqrt{3}\,(x - 5/6)}\, dx = \sqrt{\frac{l}{3g}} \int_0^{5/12} \frac{\sqrt{(x - 5/6) + \frac{85}{12}}\; dx}{(x - 5/6)}$$

$$\Rightarrow T = \sqrt{\frac{l}{3g}} \int_{-5/6}^{-5/12} \frac{\sqrt{u + \frac{85}{12}}\; du}{u} = 2\sqrt{\frac{l}{3g}} \left\{ \sqrt{\frac{85}{12} - \frac{5}{12}} - \sqrt{\frac{85}{12} - \frac{5}{6}} - \sqrt{\frac{85}{12}} \tanh^{-1}\sqrt{\frac{\frac{85}{12} - \frac{5}{12}}{85/12}} + \sqrt{\frac{85}{12}} \tanh^{-1}\sqrt{\frac{85/12 - 5/6}{85/12}} \right\}$$

$$\Rightarrow \boxed{T = \sqrt{\frac{4l}{3g}} \left\{ 2\sqrt{\frac{5}{3}} - \frac{5}{2} - \sqrt{\frac{85}{12}} \tanh^{-1}\sqrt{\frac{16}{17}} + \sqrt{\frac{85}{12}} \tanh^{-1}\sqrt{\frac{15}{17}} \right\}}$$

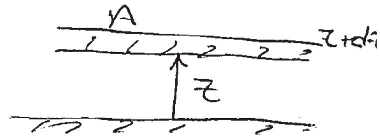
مسئله ③ آزمون ۹۰ :



الف) $\rho_a(0) V g = \rho_b(0) V g + Mg$

$$\Rightarrow \boxed{V_{min} = \frac{M}{\rho_a(0) - \rho_b(0)}}$$

ب)



$$P(z) A - P(z+dz) A = \rho(z) A\, dz\, g$$

$$\Rightarrow \left\{ \begin{array}{l} \frac{dP}{dz} = -\rho g \\ P\mu_a = \rho R T_0 \Rightarrow \rho = \frac{P\mu_a}{RT_0} \end{array} \right\} \Rightarrow \frac{dP}{P} = \frac{-\mu_a g\, dz}{RT_0} \Rightarrow \boxed{P = P_0 e^{-\frac{\mu_a g z}{RT_0}}}$$

$$, \quad \rho_a(z) = \rho_a(0) e^{-\frac{\mu_a g z}{RT_0}} = \frac{P_0 \mu_a}{RT_0} e^{-\frac{\mu_a g z}{RT_0}}$$

ج) $\rho_a(z) V g = \rho_b(z) V g + Mg \Rightarrow \rho_b(z) = \rho_a(z) - \frac{M}{V} = \frac{P(z)\mu_b}{RT(z)}$

$$\Rightarrow T(z) = \frac{\frac{\mu_b}{R} P(z)}{\rho_a(z) - \frac{M}{V}} = \frac{\frac{\mu_b P_0}{R} e^{-\frac{\mu_a g z}{RT_0}}}{\frac{P_0 \mu_a}{RT_0} e^{-\frac{\mu_a g z}{RT_0}} - \frac{M}{V}} = \boxed{\frac{T_0}{\frac{\mu_a}{\mu_b} - \frac{MRT_0}{P_0 \mu_b V} e^{\frac{\mu_a g z}{RT_0}}}}$$

د) $\frac{T_0}{T_{max}} = \frac{\mu_a}{\mu_b} - \frac{MRT_0}{\mu_b P_0 V} e^{\frac{\mu_a g z_m}{RT_0}} \Rightarrow \cancel{z_m = \frac{RT_0}{\mu_a g} \ln\left[\frac{MRT_0}{\mu_a P_0 V}(1 \right.}$

$$\Rightarrow \boxed{z_m = \frac{RT_0}{\mu_a g} \ln\left[\frac{\mu_a P_0 V}{MRT_0}\left(1 - \frac{\mu_b T_0}{\mu_a T_{max}}\right)\right]}$$

ه) $\rho_b(z) = \frac{P(z)\mu_b}{\alpha R\, T(z)} = \frac{\mu_b P_0 e^{-\frac{\mu_a g z}{RT_0}}}{\alpha R T_0} \left(\frac{\mu_a}{\mu_b} - \frac{MRT_0}{P_0 \mu_b V} e^{\frac{\mu_a g z}{RT_0}}\right) =$

$$= \frac{\mu_a P_0}{\alpha R T_0} e^{-\frac{\mu_a g z}{RT_0}} - \frac{M}{\alpha V} \Rightarrow (M + \rho_b(z) V)\ddot{z} = \rho_a(z) V g - Mg - \rho_b(z) V g$$

$$\Rightarrow \ddot{z} = g \frac{\left(1 - \frac{1}{\alpha}\right)\left(\frac{\mu_a P_0}{RT_0} e^{-\frac{\mu_a g z}{RT_0}} - \frac{M}{V}\right)}{\frac{1}{\alpha}\left(\frac{\mu_a P_0}{RT_0} e^{-\frac{\mu_a g z}{RT_0}} - \frac{M}{V}\right) + \frac{M}{V}} \Rightarrow \ddot{z} = g(\alpha - 1) - g(\alpha - 1) \frac{\alpha \frac{M}{V}}{\alpha \frac{M}{V} + \left(\frac{\mu_a P_0}{RT_0} e^{-\frac{\mu_a g z}{RT_0}} - \frac{M}{V}\right)}$$

ζ.) $$\ddot{z} = g(\alpha-1)\left[1 - \frac{\alpha M/V}{\alpha \frac{M}{V} + \left(\frac{\mu_a P_0}{RT_0} e^{-\frac{\mu_a g z}{RT_0}} - \frac{M}{V}\right)}\right] = \frac{dV^2}{2dz}$$

$$\Rightarrow V^2 = 2g(\alpha-1)\int_0^z \left\{1 - \frac{\alpha M/V}{\alpha \frac{M}{V} + \left(\frac{\mu_a P_0}{RT_0} e^{-\frac{\mu_a g z}{RT_0}} - \frac{M}{V}\right)}\right\} dz$$

$$\Rightarrow V^2 = 2g(\alpha-1)z - 2g\alpha\int_0^z \frac{(\alpha-1)\frac{M}{V}\,\frac{RT_0}{\mu_a P_0} e^{+\frac{\mu_a g z}{RT_0}}\,dz}{\frac{(\alpha-1)RT_0 M}{\mu_a P_0 V} e^{\frac{\mu_a g z}{RT_0}} + 1}$$

$$\Rightarrow V^2 = 2g(\alpha-1)z - 2g\alpha\cdot\frac{RT_0}{\mu_a g}\int_0^z \frac{d\left[1 + \frac{(\alpha-1)RMT_0}{\mu_a P_0 V} e^{\frac{\mu_a g z}{RT_0}}\right]}{\left[1 + \frac{(\alpha-1)RMT_0}{\mu_a P_0 V} e^{\frac{\mu_a g z}{RT_0}}\right]}$$

$$\Rightarrow V^2 = 2g(\alpha-1)z - \frac{2\alpha RT_0}{\mu_a}\ln\left[\frac{1 + \frac{(\alpha-1)MRT_0}{\mu_a P_0 V} e^{\frac{\mu_a g z}{RT_0}}}{1 + \frac{(\alpha-1)MRT_0}{\mu_a P_0 V}}\right]$$

$$\boxed{\Rightarrow V(z) = \sqrt{2g(\alpha-1)z - \frac{2\alpha RT_0}{\mu_a}\ln\left[\frac{1 + \frac{(\alpha-1)MRT_0}{\mu_a P_0 V} e^{\frac{\mu_a g z}{RT_0}}}{1 + \frac{(\alpha-1)MRT_0}{\mu_a P_0 V}}\right]}}$$

ζ5) $\alpha = 1+\epsilon$, $P = P_0(1+\beta z)$, $\beta = \frac{-\mu_a g}{RT_0}$ (βz کوچک است ؛ β)

$$\rho_a(z) = \frac{P_0\mu_a}{RT_0}(1+\beta z)$$

$$T(z) = \alpha\,\frac{P(z)\mu_b/R}{\rho_a(z) - M/V} = \frac{(1+\epsilon)\frac{\mu_b P_0}{RT_0}(1+\beta z)}{\frac{P_0\mu_a}{RT_0}(1+\beta z) - \frac{M}{V}}\,T_0$$

$$\Rightarrow \rho_b(z) = \frac{\mu_b P(z)}{RT(z)} = \frac{1}{(1+\epsilon)}\left(\frac{P_0\mu_a}{RT_0}(1+\beta z) - \frac{M}{V}\right)$$

$$\Rightarrow \left(\frac{M}{V} + \rho_b(z)\right)V\ddot{z} = \rho_a(z)Vg - \frac{M}{V}Vg - \rho_b(z)Vg \Rightarrow \ddot{z} = g\,\frac{\frac{\epsilon}{1+\epsilon}\left(\frac{P_0\mu_a}{RT_0}(1+\beta z) - \frac{M}{V}\right)}{\frac{\epsilon}{1+\epsilon}\frac{M}{V} + \frac{1}{1+\epsilon}\frac{P_0\mu_a}{RT_0}(1+\beta z)}$$

$$\Rightarrow \ddot{z} = g\epsilon\,\frac{\frac{P_0\mu_a}{RT_0}(1+\beta z) - \frac{M}{V}}{\frac{P_0\mu_a}{RT_0}(1+\beta z) + \epsilon\frac{M}{V}} = \frac{dV^2}{2dz} = g\epsilon\left[1 - (1+\epsilon)\frac{M/V}{\frac{P_0\mu_a}{RT_0}(1+\beta z) + \epsilon\frac{M}{V}}\right]$$

$$\Rightarrow V^2 = 2g\epsilon z - 2g\epsilon(1+\epsilon)\frac{M}{V}\times\frac{RT_0}{\beta P_0\mu_a}\int_0^z \frac{d\left[\epsilon\frac{M}{V} + \frac{P_0\mu_a}{RT_0}(1+\beta z)\right]}{\left[\epsilon\frac{M}{V} + \frac{P_0\mu_a}{RT_0}(1+\beta z)\right]}$$

$$\Rightarrow V^2 = 2g\epsilon z - 2g\epsilon(1+\epsilon)\frac{M}{V}\cdot\frac{RT_0}{\beta P_0\mu_a}\ln\left[\frac{\epsilon\frac{M}{V} + \frac{P_0\mu_a}{RT_0} + \frac{P_0\mu_a\beta}{RT_0}z}{\epsilon\frac{M}{V} + \frac{P_0\mu_a}{RT_0}}\right] = 2g\epsilon z - 2g\epsilon(1+\epsilon)\frac{M}{V}\cdot\frac{RT_0}{\beta P_0\mu_a}\times\frac{\frac{P_0\mu_a\beta z}{RT_0}}{\frac{P_0\mu_a}{RT_0} + \epsilon\frac{M}{V}}$$

$$\Rightarrow V^2 = 2g\epsilon z - 2g\epsilon\frac{M}{V}\times\frac{RT_0 z}{P_0\mu_a} = 2g\epsilon z\left(1 - \frac{MRT_0}{VP_0\mu_a}\right)$$

$$\Rightarrow \beta V^2 = 2g\epsilon\,(\beta z) - 2g\epsilon(1+\epsilon)\frac{M}{V}\int \frac{d(1+\beta z)}{\epsilon \frac{M}{V} + \frac{P_0 \mu_a}{RT_0}(1+\beta z)}$$

$$\Rightarrow \beta V^2 = 2g\epsilon(\beta z) - 2g\epsilon(1+\epsilon)\frac{M}{V} \times \frac{RT_0}{P_0 \mu_a} \ln\left[\frac{\epsilon \frac{M}{V} + \frac{P_0\mu_a}{RT_0}(1+\beta z)}{\epsilon \frac{M}{V} + \frac{P_0 \mu_a}{RT_0}}\right]$$

ε کوچک

$$\Longrightarrow \beta V^2 = 2g\epsilon(\beta z) - 2g\epsilon \cdot \frac{M}{V}\cdot \frac{RT_0}{P_0\mu_a}\ln(1+\beta z)$$

$$\Rightarrow \beta V^2 \simeq 2g\epsilon(\beta z)\left(1 - \frac{MRT_0}{P_0\mu_a V}\right) = \frac{-\mu_a g}{RT_0} V^2$$

$$\Rightarrow V^2 = \frac{2\epsilon M R^2 T_0^2}{P_0 \mu_a^2 V}\left(1 - \frac{P_0\mu_a V}{MRT_0}\right)\beta z$$

$$\Rightarrow V = \frac{dz}{dt} = \boxed{\frac{RT_0}{\mu_a}\sqrt{\frac{2M\epsilon}{P_0 V}\left(1 - \frac{\mu_a V P_0}{MRT_0}\right)\beta z}}$$

ز) $$V = \sqrt{2g\epsilon\left(1 - \frac{MRT_0}{VP_0\mu_a}\right)}\sqrt{z} = \frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{2\sqrt{z}} = dt\sqrt{\frac{g\epsilon}{2}\left(1 - \frac{MRT_0}{P_0 V \mu_a}\right)} = d\sqrt{z}$$

$$\Rightarrow \sqrt{z} = t\sqrt{\frac{g\epsilon}{2}\left(1 - \frac{MRT_0}{\mu_a P_0 V}\right)} \Rightarrow \boxed{z = \frac{g\epsilon}{2}\left(1 - \frac{MRT_0}{\mu_a P_0 V}\right)t^2}$$

ژ) $$T(z) = \frac{\frac{\mu_b}{R}P(z)}{\rho_a(z) - \frac{M}{V}} = \frac{\frac{\mu_b}{RT_0}P_0(1+\beta z)}{\frac{P_0\mu_a}{RT_0}(1+\beta z) - \frac{M}{V}}T_0 = T_{max} \Rightarrow \frac{T_0}{T_{max}} = \frac{\mu_a}{\mu_b} - \frac{MRT_0}{\mu_b P_0 V(1+\beta z)}$$

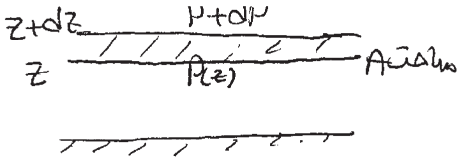
$$\Rightarrow (1+\beta z) = \frac{MRT_0}{\mu_b P_0 V\left(\frac{\mu_a}{\mu_b} - \frac{T_0}{T_m}\right)} = \frac{MRT_0}{\mu_a P_0 V\left(1 - \frac{\mu_b T_0}{\mu_a T_{max}}\right)}$$

$$\Rightarrow \beta\frac{g\epsilon}{2}\left(1 - \frac{MRT_0}{\mu_a P_0 V}\right)t^2 = \frac{MRT_0}{\mu_a P_0 V}\cdot\frac{1}{1 - \frac{\mu_b T_0}{\mu_a T_m}} - 1$$

$$\Rightarrow t = \boxed{\sqrt{\frac{2RT_0}{\mu_a \epsilon g^2\left(1 - \frac{MRT_0}{\mu_a P_0 V}\right)} - \frac{2MR^2T_0^2}{\mu_a^2 P_0 V \epsilon g^2\left(1 - \frac{\mu_b T_0}{\mu_a T_m}\right)\left(1 - \frac{MRT_0}{\mu_a P_0 V}\right)}}}$$

مسئله 4 آزمون دوم :

(الف



$$P(z)A - P(z+dz)A = \rho(z)\, g A\, dz \Rightarrow \boxed{\frac{dP}{dz} = -\rho(z)\, g}$$

ب) $n = \frac{m}{M} = \frac{\rho V}{M} \Rightarrow \frac{n}{V} = \frac{\rho}{M}$

$$\left(P + a\frac{n^2}{V^2}\right)\left(1 - \frac{n}{V}b\right) = \frac{n}{V}RT \Rightarrow \boxed{\left(P + a\frac{\rho^2}{M^2}\right)\left(1 - \frac{\rho}{M}b\right) = \frac{\rho RT}{M}}$$

ج) $P = \frac{\rho RT}{M - \rho b} - a\frac{\rho^2}{M^2} \Rightarrow \frac{dP}{dz} = \frac{d\rho}{dz}\left\{\frac{RT(M-b\rho)+b\rho RT}{(M-b\rho)^2} - 2a\frac{\rho}{M^2}\right\} = \left\{\frac{MRT}{(M-b\rho)^2} - \frac{2a\rho}{M^2}\right\}\frac{d\rho}{dz} = -\rho g$

$$\Rightarrow -g\,dz = \frac{MRT\, d\rho}{\rho(M-b\rho)^2} - \frac{2a}{M^2}d\rho \Rightarrow -gz = MRT\int_{\rho(0)}^{\rho}\frac{d\rho}{\rho(M-b\rho)^2} - \frac{2a}{M^2}(\rho - \rho_0)$$

$$\frac{1}{\rho(M/b-\rho)^2} = \frac{b}{M}\,\frac{(M/b-\rho)+\rho}{\rho(M/b-\rho)^2} = \frac{b}{M}\cdot\frac{1}{\rho(M/b-\rho)} + \frac{b}{M}\,\frac{1}{(M/b-\rho)^2} =$$

$$= \left(\frac{b}{M}\right)^2\frac{\rho + M/b - \rho}{\rho(M/b-\rho)} + \frac{b}{M}\cdot\frac{1}{(M/b-\rho)^2} = \left(\frac{b}{M}\right)^2\cdot\frac{1}{M/b-\rho} + \left(\frac{b}{M}\right)^2\frac{1}{\rho} + \left(\frac{b}{M}\right)\frac{1}{(M/b-\rho)^2}$$

$$\Rightarrow \int_{\rho_0}^{\rho}\frac{d\rho}{\rho(M-b\rho)^2} = \frac{1}{M^2}\ln\frac{\rho}{\rho_0} - \frac{1}{M^2}\ln\left(\frac{\rho - M/b}{\rho_0 - M/b}\right) + \frac{b}{Mb^2}\left(\frac{1}{M/b-\rho} - \frac{1}{M/b-\rho_0}\right)$$

$$\Rightarrow \boxed{-gz = -\frac{2a}{M^2}(\rho - \rho_0) + \frac{RT}{M}\ln\frac{\rho}{\rho_0} - \frac{RT}{M}\ln\left(\frac{b\rho - M}{b\rho_0 - M}\right) + \frac{RT}{M-b\rho} - \frac{RT}{M-b\rho_0}}$$

د) $b\rho_0 \ll M \Rightarrow \boxed{b \ll \frac{M}{\rho_0}}$ ← از جواب قبل

$$\frac{2a\rho_0}{M^2} \ll \frac{RT}{M} \Rightarrow \boxed{a \ll \frac{MRT}{\rho_0}}$$

~~ه) $U = \frac{3}{2}nRT - a\frac{n^2}{V} \Rightarrow dU = \frac{3}{2}nR\,dT + a\frac{n^2}{V^2}dV = -P\,dV = -\frac{\rho RT}{M-\rho b}dV + a\frac{\rho^2}{M^2}dV$~~

~~$\frac{dV}{V} = -\frac{d\rho}{\rho} \Rightarrow \frac{3}{2}nR\,dT = a\frac{n^2}{V}d\rho \Rightarrow \frac{3}{2}R\,dT = a\frac{\rho}{M}d\rho$ $\quad \frac{d\rho}{\rho} = -\frac{dV}{V} \Rightarrow$~~

~~$\Rightarrow \frac{3}{2}R\,dT - a\frac{\rho}{M}\cdot\frac{d\rho}{\rho} =$~~

h) $P = \frac{\rho RT}{M - b\rho} - a\frac{\rho^2}{M^2}$, $V = \frac{Mn}{\rho}$ $\Rightarrow dV = \frac{-Mn}{\rho^2} d\rho$

$U = \frac{3}{2} nRT - a\frac{n^2}{V} = \frac{3}{2} nRT - a\frac{n\rho}{M} \Rightarrow dU = \frac{3}{2} nRdT - a\frac{n}{M} d\rho = -PdV$

$\Rightarrow \frac{3}{2} RdT - \frac{a}{M} d\rho = \frac{M}{\rho^2} \cdot \frac{\rho RT}{M - b\rho} d\rho - \frac{M}{\rho^2} \cdot a\frac{\rho^2}{M^2} d\rho$

$\Rightarrow \frac{3}{2} RdT = \left(\frac{a}{M} + \frac{MRT}{\rho(M - b\rho)} - \frac{a}{M}\right) d\rho = \frac{MRT}{\rho(M - b\rho)} d\rho \Rightarrow \frac{3}{2}\frac{dT}{T} = \frac{d\rho}{\rho(1 - \frac{b}{M}\rho)} = \frac{\frac{b}{M}\rho + (1 - \frac{b}{M}\rho)}{\rho(1 - \frac{b}{M}\rho)} d\rho$

$\Rightarrow \frac{3}{2}\frac{dT}{T} = \frac{b}{M}\frac{d\rho}{1 - \frac{b}{M}\rho} + \frac{d\rho}{\rho} \Rightarrow \frac{3}{2}\ln\frac{T}{T_0} = \ln\frac{\rho}{\rho_0} - \ln\frac{\rho - M/b}{\rho_0 - M/b}$

$\Rightarrow \left(\frac{T}{T_0}\right)^{3/2} = \left(\frac{\rho}{\rho_0}\right)\left(\frac{b\rho_0 - M}{b\rho - M}\right) \Rightarrow \boxed{\frac{M - b\rho}{\rho} T^{3/2} = \text{const.}}$

---

g) $T = T_0 \left(\frac{\rho}{\rho_0}\right)^{2/3} \left(\frac{b\rho_0 - M}{b\rho - M}\right)^{2/3}$

$\Rightarrow \boxed{P = RT_0 \left(\frac{-b\rho_0 + M}{\rho_0}\right)^{2/3} \left(\frac{\rho}{-b\rho + M}\right)^{5/3} - a\frac{\rho^2}{M^2}}$

---

i) $\frac{dP}{dz} = -\rho g$ $\quad \frac{dP}{dz} = \frac{d\rho}{dz}\left\{ RT_0 \left(\frac{-b\rho_0 + M}{\rho_0}\right)^{2/3} \times \frac{5}{3}\left(\frac{\rho}{-b\rho + M}\right)^{2/3} \times \frac{+M}{(-b\rho + M)^2} - 2a\frac{\rho}{M^2} \right\} = -\rho g$

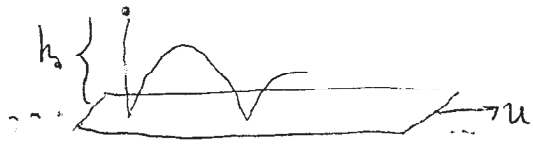
$\Rightarrow -g\,dz = +\frac{5}{3} MRT_0 \left(\frac{-b\rho_0 + M}{\rho_0}\right)^{2/3} \frac{d\rho}{\rho^{1/3}(b\rho - M)^{8/3}} - \frac{2a}{M^2} d\rho$

$\Rightarrow gz = \frac{2a}{M^2}(\rho - \rho_0) + -\frac{5}{3} MRT_0 \left(\frac{-b\rho_0 + M}{\rho_0}\right)^{2/3} \times \frac{3}{10 M^{8/3}} \times \rho^{2/3} \frac{(5 - 3\frac{b}{M}\rho)}{(1 - \frac{b}{M}\rho)^{5/3}}$

$\Rightarrow \boxed{-gz = \frac{2a}{M^2}(\rho_0 - \rho) + \frac{RT_0}{2\,M}\left(\frac{M - b\rho_0}{\rho_0}\right)^{2/3} \frac{\rho^{2/3}(5M - 3b\rho)}{(M - b\rho)^{5/3}}}$

مسئله ① آزمون سوم :

الف) $v_0 = \sqrt{2gh_0} \Rightarrow v_{y_1} = e\sqrt{2gh_0}$



علت پایستگی انرژی و ثابت بودن سرعت افقی در طول پرواز بین دو برخورد متوالی

سرعت عمودی قبل برخورد n+1 ، منفی مقدار آن بعد از برخورد n ام است

$$\Rightarrow v_{y_{n+1}} = e\,v_{y_n} \Rightarrow v_{y_n} = e^{n-1} v_{y_1} \Rightarrow \boxed{v_{y_n} = e^n\sqrt{2gh_0}}$$

ب) $v_{x_{n-1}} \leftarrow$ سرعت درست قبل برخورد n

$$\Delta P_x = m v_{x_n} - m v_{x_{n-1}} = \int_{\text{برخورد}} F_x\,dt = \mu\int N\,dt$$

$$\Delta P_y = m v_{y_n} - m(-v_{y_{n-1}}) = \int N\,dt \Rightarrow \frac{1}{m}\int N\,dt = v_{y_n} + v_{y_{n-1}} = e^{n-1}\sqrt{2gh_0}\,(1+e)$$

$$\Rightarrow v_{x_n} - v_{x_{n-1}} = \mu(1+e)\,e^{n-1}\sqrt{2gh_0}$$

$$\xrightarrow{v_{x_0}=0} v_{x_n} - v_{x_0} = \sum_{i=1}^{n}(v_{x_i} - v_{x_{i-1}}) = \mu(1+e)\sqrt{2gh_0}\sum_{i=1}^{n} e^{i-1} = \mu(1+e)\sqrt{2gh_0}\left(\frac{1-e^n}{1-e}\right)$$

$$\Rightarrow \boxed{v_{x_n} = \mu\sqrt{2gh_0}\times\left(\frac{1+e}{1-e}\right)(1-e^n)}$$

ج) $gT_n = v_{y_n} - (-v_{y_n}) = 2v_{y_n} \Rightarrow T_n = \frac{2\sqrt{2gh_0}}{g}e^n$

زمان برخورد n و n+1

$$\Rightarrow \Delta x_n = v_{x_n}T_n = \boxed{4\mu h_0\left(\frac{1+e}{1-e}\right)e^n(1-e^n)}$$

د) $x_n = \sum_{i=1}^{n-1}\Delta x_i = 4\mu h_0\left(\frac{1+e}{1-e}\right)\left(\sum_{i=1}^{n-1} e^i - \sum_{i=1}^{n-1}(e^2)^i\right) =$

$$= 4\mu h_0\left(\frac{1+e}{1-e}\right)\left(\frac{1-e^n}{1-e} - 1 - \frac{1-e^{2n}}{1-e^2} + 1\right) = 4\mu h_0\,\frac{1}{(1-e)^2}\left(1 - e^n + e - e^{n+1} - 1 + e^{2n}\right) =$$

$$= 4\mu h_0\,\frac{e}{(1-e)^2}\left[1 - e^{n-1} - e^n(1 - e^{n-1})\right] = \boxed{4\mu h_0\,e\,\frac{(1-e^{n-1})(1-e^n)}{(1-e)^2}}$$

ث) $v_{x_{N-1}} < u < v_{x_N}$

$$\Rightarrow \mu\sqrt{2gh_0}\left(\frac{1+e}{1-e}\right)(1-e^{N-1}) < u < \mu\sqrt{2gh_0}\left(\frac{1+e}{1-e}\right)(1-e^{N})$$

$$\Rightarrow 1-e^{N-1} < \frac{u}{\mu\sqrt{2gh_0}}\left(\frac{1-e}{1+e}\right) < 1-e^{N} \Rightarrow e^{N} < 1-\frac{u}{\mu\sqrt{2gh_0}}\left(\frac{1-e}{1+e}\right) < e^{N-1}$$

$$\cancel{\Rightarrow N < \log_e\left\{1-\frac{u}{\mu\sqrt{2gh_0}}\left(\frac{1-e}{1+e}\right)\right\} < N-1}$$

$$\Rightarrow N\ln e < \ln\left[1-\frac{u}{\mu\sqrt{2gh_0}}\left(\frac{1-e}{1+e}\right)\right] < (N-1)\ln e$$

$$\xrightarrow{\ln e<0} \boxed{N = 1 + \underbrace{\left[\frac{\ln\left\{1-\frac{u}{\mu\sqrt{2gh_0}}\left(\frac{1-e}{1+e}\right)\right\}}{\ln e}\right]}_{(\text{جزء صحیح})}}$$

ج) $x_N = 4\mu h_0 e\,\frac{(1-e^{N-1})(1-e^{N})}{(1-e)^2}$

$n>N$: $\quad \Delta x_n = uT_n = \frac{2u\sqrt{2gh_0}}{g}e^{n}$

$$\Rightarrow x_n = x_N + \sum_{i=N}^{n-1}\Delta x_i = x_N + \frac{2u\sqrt{2gh_0}}{g}\left(\sum_{i=1}^{n-1}e^{i} - \sum_{i=1}^{N-1}e^{i}\right) =$$

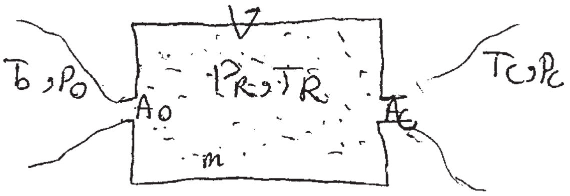
$$= x_N + \frac{2u\sqrt{2gh_0}}{g}\left(\frac{1-e^{n}}{1-e} - \frac{1-e^{N}}{1-e}\right)$$

$$\xrightarrow{n>N} \boxed{x_n = 4h_0\left\{\mu e\,\frac{(1-e^{N-1})(1-e^{N})}{(1-e)^2} + \sqrt{\frac{u^2}{2gh_0}}\left(\frac{e^{N}-e^{n}}{1-e}\right)\right\}}$$

مسئله ② آزمون سوم :

الف)

$$-dN^{v}_{R\to C} = \int_{r=0}^{v dt}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \frac{N}{V}P(v)\,dv \times \frac{A_C\cos\theta}{4\pi}\times \sin\theta\, d\theta\, d\varphi\, dr \quad , \quad P(v)=4\pi v^2\left(\frac{m}{2\pi K T_R}\right)^{3/2} e^{\frac{-mv^2}{2KT_R}}$$



$$\Rightarrow \frac{dN_{R\to C}}{dt} = \frac{-A_C\sqrt{K}}{V\sqrt{2\pi m}} N_R\sqrt{T_R}$$

$$\frac{N_R}{V}=\frac{P_R}{KT_R} \Rightarrow \frac{dN_{R\to C}}{dt} = \frac{-A_C}{\sqrt{2\pi m}}\cdot\frac{P_R}{\sqrt{KT_R}}$$

$$\Rightarrow \boxed{\frac{dN}{dt} = \frac{A_C}{\sqrt{2\pi m}}\cdot\frac{P_C}{\sqrt{KT_C}} + \frac{A_0}{\sqrt{2\pi m}}\cdot\frac{P_0}{\sqrt{KT_0}} - \frac{(A_0+A_C)}{\sqrt{2\pi m}}\cdot\frac{P_R}{\sqrt{KT_R}}}$$

ب)

$$-dE_{R\to C} = \int_{v=0}^{\infty}\int_{r=0}^{v dt}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \frac{N_R}{V}\cdot P(v)\,dv \times \frac{A_C\cos\theta}{4\pi}\times\left(\tfrac{1}{2}mv^2\right)\times \sin\theta\, d\theta\, d\varphi\, dr$$

$$\Rightarrow \frac{dE_{R\to C}}{dt} = \frac{-A_C K^{3/2}\sqrt{2}}{V\sqrt{\pi m}} N_R T_R^{3/2} = \frac{-\sqrt{2}A_C}{\sqrt{\pi m}} P_R\sqrt{KT_R}$$

$$\Rightarrow \bar{E}_{R\to C} = \frac{dE_{R\to C}}{dN_{R\to C}} = \boxed{2KT_R}$$

ج-۱)

$$\boxed{\frac{dE_R}{dt} = \frac{2A_C}{\sqrt{2\pi m}}P_C\sqrt{KT_C} + \frac{2A_0}{\sqrt{2\pi m}}P_0\sqrt{KT_0} - \frac{2(A_0+A_C)}{\sqrt{2\pi m}}P_R\sqrt{KT_R}}$$

ج-۲)

$$E_R=\frac{3}{2}N_R K T_R \Rightarrow \frac{dE_R}{dt} = \frac{3}{2}KT_R\frac{dN_R}{dt} + \frac{3}{2}N_R K\frac{dT_R}{dt} \quad , \quad N_R=\frac{P_R V}{KT_R}$$

$$\Rightarrow \frac{4A_C}{3\sqrt{2\pi m}}P_C\sqrt{KT_C} + \frac{4A_0}{3\sqrt{2\pi m}}P_0\sqrt{KT_0} - \frac{4(A_0+A_C)}{3\sqrt{2\pi m}}P_R\sqrt{KT_R}$$

$$= \frac{A_C}{\sqrt{2\pi m}}P_C\frac{KT_R}{\sqrt{KT_C}} + \frac{A_0}{\sqrt{2\pi m}}P_0\frac{KT_R}{\sqrt{KT_0}} - \frac{(A_0+A_C)}{\sqrt{2\pi m}}\sqrt{KT_R} + P_R V\frac{1}{T_R}\frac{dT_R}{dt}$$

$$\Rightarrow \boxed{\frac{dT_R}{dt} = \frac{A_C}{\sqrt{2\pi m}}\cdot\frac{P_C}{\sqrt{KT_C}}\cdot\frac{T_R}{P_R V}\cdot\left(\frac{4}{3}KT_C-KT_R\right) + \frac{A_0}{\sqrt{2\pi m}}\cdot\frac{P_0}{\sqrt{KT_0}}\cdot\frac{T_R}{P_R V}\left(\frac{4}{3}KT_0-KT_R\right) - \frac{(A_0+A_C)}{3\sqrt{2\pi m}}\cdot\frac{T_R\sqrt{KT_R}}{V}}$$

د)

$$E_R=\frac{3}{2}P_R V \Rightarrow \frac{dE_R}{dt}=\frac{3}{2}V\frac{dP_R}{dt}$$

$$\Rightarrow \boxed{\frac{dP_R}{dt} = \frac{4A_C}{3V\sqrt{2\pi m}}P_C\sqrt{KT_C} + \frac{4A_0}{3V\sqrt{2\pi m}}P_0\sqrt{KT_0} - \frac{4(A_0+A_C)}{3V\sqrt{2\pi m}}P_R\sqrt{KT_R}}$$

ث) $\left(\frac{dT_R}{dt}=0 \;,\; \frac{dP_R}{dt}=0\right)$ یا $\left(\frac{dN_R}{dt}=0 \;,\; \frac{dE_R}{dt}=0\right)$

$$\Rightarrow \begin{cases} \dfrac{A_C P_C}{\sqrt{T_C}} + \dfrac{A_0 P_0}{\sqrt{T_0}} = \dfrac{(A_0+A_C)\,P_R}{\sqrt{T_R}} \\[2ex] A_C P_C \sqrt{T_C} + A_0 P_0 \sqrt{T_0} = (A_0+A_C)\,P_R\sqrt{T_R} \end{cases} \xrightarrow{\text{تقسیم}} \boxed{T_R = \frac{A_C P_C \sqrt{T_C} + A_0 P_0 \sqrt{T_0}}{\dfrac{A_C P_C}{\sqrt{T_C}} + \dfrac{A_0 P_0}{\sqrt{T_0}}}}$$

$$\xRightarrow{\text{ضرب}} \boxed{P_R = \frac{\sqrt{\left(\dfrac{A_C P_C}{\sqrt{T_C}} + \dfrac{A_0 P_0}{\sqrt{T_0}}\right)\left(A_C P_C\sqrt{T_C} + A_0 P_0\sqrt{T_0}\right)}}{(A_0+A_C)}}$$

ج) $T_0 = 300\ K \;,\; T_C = 280\ K \;,\; A_0 = 0.2\ m^2 \;,\; A_C = 0.5\ m^2 \;,\; P_0 = 101.3\ kPa \;,\; P_C = 101.5\ kPa$

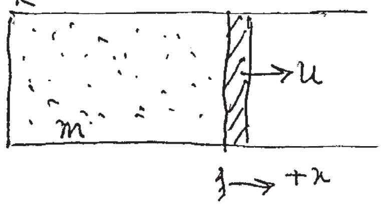
$\Rightarrow A_C P_C = 50.75\ N \;,\; A_0 P_0 = 20.26\ N$

$$\Rightarrow \begin{cases} \boxed{T_R = 288\ K} \\[2ex] \boxed{P_R = 101.47\ kPa} \end{cases}$$

سوال ۴ ③ آزمون سوم :

آ) $P(v_x)\,dv_x = \sqrt{\frac{m}{2\pi kT}}\; e^{-\frac{m v_x^2}{2kT}}\, dv_x$

$N = \frac{R}{k} n$



$$PA\,dt = \int_{v_x=u}^{+\infty} A(v_x-u)\,dt \times \frac{nR}{kV} \times 2m(v_x-u) \times P(v_x)\,dv_x$$

$$\left(v=\sqrt{\frac{m v_x^2}{2kT}}\right)$$

$$\Rightarrow P = \frac{2nmR}{kV}\sqrt{\frac{m}{2\pi kT}} \int_u^\infty (v_x-u)^2 e^{-\frac{m v_x^2}{2kT}}\, dv_x = \frac{2nmR}{kV}\sqrt{\frac{m}{2\pi kT}} \times \left(\frac{2kT}{m}\right)^{3/2} \times \int_{v=\sqrt{\frac{mu^2}{2kT}}}^{\infty} \left(v-\sqrt{\frac{mu^2}{2kT}}\right)^2 e^{-v^2}\, dv$$

$$\Rightarrow P = \frac{4nRT}{\sqrt{\pi}\, V} \int_{\sqrt{\frac{mu^2}{2kT}}}^{\infty} \left(v-\sqrt{\frac{mu^2}{2kT}}\right)^2 e^{-v^2}\, dv \qquad \epsilon := \sqrt{\frac{mu^2}{2kT}}$$

$$\Rightarrow P = \frac{4nRT}{\sqrt{\pi}\, V} \int_\epsilon^\infty (v-\epsilon)^2 e^{-v^2}\, dv = \frac{4nRT}{\sqrt{\pi}\, V}\left[\int_0^\infty (v-\epsilon)^2 e^{-v^2}\, dv - \int_0^\epsilon (v-\epsilon)^2 e^{-v^2}\, dv\right]$$

تا مرتبه دوم ϵ) $\int_0^\epsilon \underbrace{(v-\epsilon)^2}_{\text{مرتبه دوم}} e^{-v^2}\, dv \;=\!=\!=\; \int_0^\epsilon \underbrace{(v-\epsilon)^2}_{\text{مرتبه دو}} \underbrace{dv}_{\text{مرتبه ۱}} = 0$

$$\Rightarrow P^{(2)} = \frac{4nRT}{\sqrt{\pi}\, V}\left[\underbrace{\int_0^\infty v^2 e^{-v^2}\, dv}_{\frac{\sqrt{\pi}}{4}} - 2\epsilon \int_0^\infty v e^{-v^2}\, dv + \epsilon^2 \underbrace{\int_0^\infty e^{-v^2}\, dv}_{\sqrt{\pi}/2}\right]$$

$$\Rightarrow P = \frac{nRT}{V}\left[1 + 2\epsilon^2 - \frac{4\epsilon}{\sqrt{\pi}} \underbrace{\int_0^\infty e^{-v^2}\, d(v^2)}_{1}\right] = \boxed{\frac{nRT}{V}\left[1 - \frac{4}{\sqrt{\pi}}\sqrt{\frac{mu^2}{2kT}} + \frac{mu^2}{kT}\right]}$$

ب) $U = \frac{3}{2} nRT \Rightarrow dU = \frac{3}{2} nR\, dT = -P\, dV$

$$\Rightarrow \frac{3}{2}\, dT = -\frac{dV}{V}\left[T - \sqrt{\frac{8mu^2}{\pi k}}\sqrt{T} + \frac{mu^2}{k}\right]$$

$$\Rightarrow -\frac{2}{3}\frac{dV}{V} = \frac{dT}{T - \sqrt{\frac{8mu^2}{\pi k}}\sqrt{T} + \frac{mu^2}{k}} = \frac{2\sqrt{T}\, d\sqrt{T}}{\sqrt{T}^2 - \left(\sqrt{\frac{8mu^2}{\pi k}}\right)\sqrt{T} + \left(\frac{mu^2}{k}\right)} = \frac{d\left[\sqrt{T}^2 - \left(\sqrt{\frac{8mu^2}{\pi k}}\right)\sqrt{T} + \left(\frac{mu^2}{k}\right)\right] + \sqrt{\frac{8mu^2}{\pi k}}\, d\sqrt{T}}{\left[\sqrt{T}^2 - \left(\sqrt{\frac{8mu^2}{\pi k}}\right)\sqrt{T} + \left(\frac{mu^2}{k}\right)\right]}$$

$$\Rightarrow \ln\left(\frac{V}{V_0}\right)^{-2/3} = \ln\left(\frac{T - \sqrt{\frac{8mu^2}{\pi k}}\sqrt{T} + \frac{mu^2}{k}}{T_0 - \sqrt{\frac{8mu^2}{\pi k}}\sqrt{T_0} + \frac{mu^2}{k}}\right) + \sqrt{\frac{8mu^2}{\pi k}} \times \frac{1}{\sqrt{\frac{mu^2}{k}}\sqrt{1-\frac{2}{\pi}}} \times \left\{\tan^{-1}\frac{\sqrt{T} - \sqrt{\frac{2mu^2}{\pi k}}}{\sqrt{\frac{mu^2}{k}}\sqrt{1-\frac{2}{\pi}}}\,\Bigg|_{T_0}^{T}\right\}$$

$$\boxed{\Rightarrow \left(\frac{V}{V_0}\right)^{-2/3} = \left(\frac{T - \sqrt{\frac{8mu^2}{\pi k}}\sqrt{T} + \frac{mu^2}{k}}{T_0 - \sqrt{\frac{8mu^2}{\pi k}}\sqrt{T_0} + \frac{mu^2}{k}}\right) \times \exp\left\{\sqrt{\frac{8}{\pi-2}}\left(\tan^{-1}\frac{\sqrt{T} - \sqrt{\frac{2mu^2}{\pi k}}}{\sqrt{\frac{2mu^2}{\pi k}}\sqrt{\pi/2-1}} - \tan^{-1}\frac{\sqrt{T_0} - \sqrt{\frac{2mu^2}{\pi k}}}{\sqrt{\frac{2mu^2}{\pi k}}\sqrt{\pi/2-1}}\right)\right\}}$$

مسئلهٔ 4 آزمون سوم :

الف)
$$\vec{r}'(t_r) = a\cos(\omega t_r)\hat{x} + a\sin(\omega t_r)\hat{y}$$
$$\vec{r} = z\hat{z}$$
$$\Rightarrow \vec{r} - \vec{r}'(t_r) = z\hat{z} - a\cos(\omega t_r)\hat{x} - a\sin(\omega t_r)\hat{y} \Rightarrow |\vec{r} - \vec{r}'(t_r)| = \sqrt{z^2 + a^2} = c(t - t_r)$$
$$\Rightarrow \boxed{t_r(t) = t - \frac{\sqrt{a^2 + z^2}}{c}}$$

ب)
$$\vec{v}'(t_r) = a\omega\left(-\sin\omega t_r\,\hat{x} + \cos\omega t_r\,\hat{y}\right)$$
$$\Rightarrow \vec{R}\cdot\vec{v}'(t_r) = 0$$
$$\Rightarrow \Phi(0,0,z,t) = \frac{1}{4\pi\epsilon_0}\frac{Qq}{R} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Qq}{\sqrt{a^2+z^2}}}$$
$$\Rightarrow \boxed{\vec{A}(0,0,z,t) = \frac{Qq\,a\omega}{4\pi\epsilon_0 c^2\sqrt{a^2+z^2}}\left(-\sin\omega t_0\,\hat{x} + \cos\omega t_0\,\hat{y}\right)}$$

ج)
$$\left.\begin{cases}\nabla\Phi\cdot\hat{z} = \dfrac{\partial\Phi}{\partial z} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{Qqz}{[a^2+z^2]^{3/2}} \\ \dfrac{\partial\vec{A}}{\partial t}\cdot\hat{z} = 0\end{cases}\right\} \Rightarrow F_z = -\nabla\Phi = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Qqz}{[a^2+z^2]^{3/2}}}$$

د)
$$\vec{r} = (x,y,z) \Rightarrow \vec{r} - \vec{r}'(t_r) = (x - a\cos\omega t_r,\ y - a\sin\omega t_r,\ z) = \vec{R}$$
$$\Rightarrow |\vec{R}| = \sqrt{x^2+y^2+z^2+a^2-2ax\cos\omega t_r-2ay\sin\omega t_r} \overset{\text{تقریب}}{=} \sqrt{a^2+z^2}\left(1 - \frac{ax\cos\omega t_r + ay\sin\omega t_r}{a^2+z^2}\right) = c(t - t_r)$$
$$t_r = t_0 + t_1 \Rightarrow -ct_1 = -\sqrt{a^2+z^2}\,\frac{ax\cos\omega t_0 + ay\sin\omega t_0}{a^2+z^2} \Rightarrow t_1 = \frac{x\cos\omega t_0 + y\sin\omega t_0}{c\sqrt{a^2+z^2}}\,a$$
$$\Rightarrow \boxed{t_r = t_0 + \frac{a}{\sqrt{a^2+z^2}}\cdot\frac{x\cos\omega t_0 + y\sin\omega t_0}{c}}$$

ه)
$$\vec{R}\cdot\vec{v}'(t_r) = -a\omega x\sin\omega t_0 + a\omega y\cos\omega t_0$$
$$\Rightarrow \Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\cdot\frac{Qq}{\sqrt{a^2+z^2}}\cdot\frac{1}{1 - \dfrac{ax\cos\omega t_0 + ay\sin\omega t_0}{a^2+z^2} + \dfrac{a\omega x\sin\omega t_0 - a\omega y\cos\omega t_0}{c\sqrt{a^2+z^2}}}$$
$$\Rightarrow \boxed{\Phi(x,y,z,t) = \frac{Qq}{4\pi\epsilon_0\sqrt{a^2+z^2}}\left(1 + \frac{ax\cos\omega t_0 + ay\sin\omega t_0}{a^2+z^2} + \frac{-a\omega x\sin\omega t_0 + a\omega y\cos\omega t_0}{c\sqrt{a^2+z^2}}\right)}$$

$$\vec{v}\,'(t_1) = -a\omega \sin(\omega t_0+\omega t_1)\hat{x} + a\omega\cos(\omega t_0+\omega t_1)\hat{y} = (-a\omega\sin\omega t_0\,\hat{x} + a\omega\cos\omega t_0\,\hat{y}) - a\omega^2 t_1\cos\omega t_0\,\hat{x} - a\omega^2 t_1 \sin\omega t_0\,\hat{y}$$

$$\Rightarrow \vec{A}(x,y,z,t) = \frac{Qqa\omega}{4\pi\epsilon_0 c^2\sqrt{a^2+z^2}}\Big\{(-\sin\omega t_0\,\hat{x}+\cos\omega t_0\,\hat{y}) - \frac{a\omega\cos\omega t_0}{\sqrt{a^2+z^2}}\left(\frac{x\cos\omega t_0+y\sin\omega t_0}{c}\right)\hat{x} - \frac{a\omega\sin\omega t_0}{\sqrt{a^2+z^2}}\left(\frac{x\cos\omega t_0+y\sin\omega t_0}{c}\right)\hat{y}$$

$$+\frac{ax\cos\omega t_0+ay\sin\omega t_0}{a^2+z^2}(-\sin\omega t_0\,\hat{x}+\cos\omega t_0\,\hat{y}) + \frac{-a\omega x\sin\omega t_0+a\omega y\cos\omega t_0}{c\sqrt{a^2+z^2}}(-\sin\omega t_0\,\hat{x}+\cos\omega t_0\,\hat{y})\Big\}$$

$$\boxed{\Rightarrow \vec{A} = \frac{Qqa\omega}{4\pi\epsilon_0 c^2\sqrt{a^2+z^2}}\begin{pmatrix} -\sin\omega t_0 - \frac{a\omega x}{c\sqrt{a^2+z^2}}\cos(2\omega t_0) - \frac{a\omega y}{c\sqrt{a^2+z^2}}\sin(2\omega t_0) - \frac{ax}{2(a^2+z^2)}\sin(2\omega t_0) - \frac{ay}{2(a^2+z^2)}(1-\cos 2\omega t_0) \\ \cos\omega t_0 - \frac{a\omega x}{c\sqrt{a^2+z^2}}\sin(2\omega t_0) - \frac{a\omega y}{c\sqrt{a^2+z^2}}\cos(2\omega t_0) + \frac{ay}{2(a^2+z^2)}\sin(2\omega t_0) + \frac{ax}{2(a^2+z^2)}(1+\cos 2\omega t_0) \end{pmatrix}}$$

9)
$$\begin{cases} \dfrac{\partial\Phi}{\partial x} = \dfrac{Qq}{4\pi\epsilon_0\sqrt{a^2+z^2}}\left(\dfrac{a\cos\omega t_0}{a^2+z^2} - \dfrac{a\omega\sin\omega t_0}{c\sqrt{a^2+z^2}}\right) = \dfrac{Qqa}{4\pi\epsilon_0(a^2+z^2)}\left(\dfrac{\cos\omega t_0}{\sqrt{a^2+z^2}} - \dfrac{\omega}{c}\sin\omega t_0\right) \\ \dfrac{\partial\Phi}{\partial y} = \dfrac{Qq}{4\pi\epsilon_0\sqrt{a^2+z^2}}\left(\dfrac{a\sin\omega t_0}{a^2+z^2} + \dfrac{a\omega\cos\omega t_0}{c\sqrt{a^2+z^2}}\right) \end{cases}$$

$$\frac{\partial t_0}{\partial t} = 1 \quad \Rightarrow \left.\frac{\partial A}{\partial t}\right|_{x,y=0} = \frac{Qqa\omega^2}{4\pi\epsilon_0 c^2\sqrt{a^2+z^2}}\begin{pmatrix} -\cos\omega t_0 \\ -\sin\omega t_0 \end{pmatrix}$$

$$\boxed{\Rightarrow \vec{F} = \frac{Qqa}{4\pi\epsilon_0\sqrt{a^2+z^2}}\begin{pmatrix} \dfrac{\omega\sin\omega t_0}{c\sqrt{a^2+z^2}} - \dfrac{\cos\omega t_0}{a^2+z^2} + \dfrac{\omega^2\cos\omega t_0}{c^2} \\ -\dfrac{\omega\cos\omega t_0}{c\sqrt{a^2+z^2}} - \dfrac{\sin\omega t_0}{a^2+z^2} + \dfrac{\omega^2\sin\omega t_0}{c^2} \end{pmatrix}}$$

مسئله ۱ امتحان چهارم :

الف) $\frac{F}{A}=\mu\frac{\Delta v}{\Delta x} \Rightarrow [\mu]=\left[\frac{F\Delta x}{A\Delta v}\right]=\frac{MLT^{-2}\times L}{L^{2}\times LT^{-1}}=\boxed{ML^{-1}T^{-1}}$

ب) کمیت های موثر و ابعادشان : $[v]=LT^{-1}$ ، $[R]=L$ ، $[r]=L$ ، $[l]=L$ ، $[\rho]=ML^{-3}$ ، $[\mu]=ML^{-1}T^{-1}$ ، $[\Delta P]=ML^{-1}T^{-2}$

۷ کمیت ، ۳ بعد مختلف $\Leftarrow$ ۴ کمیت بدون بعد

v ، ΔP ، r ، l را مستقل می‌گیریم

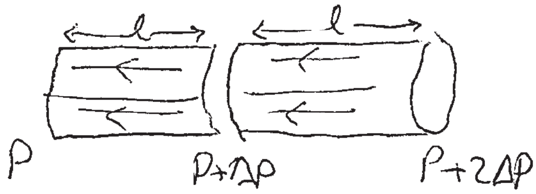
$$\Rightarrow \pi_1=\frac{R\rho v}{\mu} \quad , \quad \pi_2=\frac{\Delta P\,\rho\, l^{2}}{\mu^{2}} \quad , \quad \pi_3=\frac{r}{R} \quad , \quad \pi_4=\frac{l}{R}$$

$$\Rightarrow \pi_1=f_1(\pi_2,\pi_3,\pi_4) \Rightarrow v=\frac{\mu}{R\rho}f_1\left(\frac{\rho l^{2}\Delta P}{\mu^{2}},\frac{l}{R},\frac{r}{R}\right)$$

در معادلات ما، چون حرکت پایاست پس لختی ظاهر نمی‌شود. همچنین در دیگر نیروهای ما (ΔPA و نیروی گرانروی) هم جرم یا همان چگالی ظاهر نمی‌شود پس v به ρ ربطی ندارد

$$\Rightarrow v=\frac{\mu}{R\rho}\times\frac{\rho l^{2}\Delta P}{\mu^{2}}f_2\left(\frac{l}{R},\frac{r}{R}\right)=\frac{l^{2}\Delta P}{\mu R}f_2\left(\frac{l}{R},\frac{r}{R}\right)$$

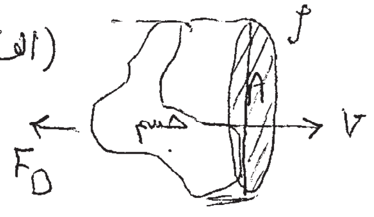
اگر دو لوله ی مشابه را به هم بچسبانیم، هیچ تغییری ایجاد نمی‌شود. پس کمیت موثر در v، $\frac{\Delta P}{l}$ است و این دو به صورت مجزا نباید ظاهر شوند.



$$\Rightarrow v=\frac{l^{2}\Delta P}{\mu R}\times\left(\frac{R}{l}\right)^{3}f_3\left(\frac{r}{R}\right)=\boxed{\frac{R^{2}\Delta P}{\mu l}f_3\left(\frac{r}{R}\right)}$$

ج) $r=0 \Rightarrow f_3\left(\frac{r}{R}\right)\Big|_{r=0}=C$ (ثابت) $\Rightarrow v_0=\frac{R^{2}\Delta P}{\mu l} \Rightarrow \frac{v_0'}{v_0}=\left(\frac{R'}{R}\right)^{2}=\boxed{4}$

مسئله ④ آزمون چهارم :



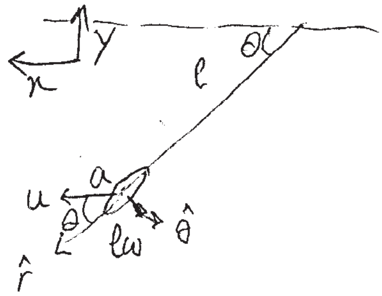
الف) $[\rho]=ML^{-3}$, $[v]=LT^{-1}$, $[A]=L^2$, $[F_D]=MLT^{-2}$

تعداد کمیت های بی بعد ← 1

$$\Rightarrow \pi = \frac{F_D}{\rho A v^2} = c \Rightarrow \boxed{F_D = c\rho A v^2}$$

$$\Rightarrow \left(\vec{F_D} = -c\rho A v^2\, \hat{v}\right)$$

ب) از ابعاد دست در برابر l صرف نظر کنیم



$$\vec{v} = u\cos\theta\,\hat{r} + (l\omega - u\sin\theta)\,\hat{\theta}$$

$$\vec{F_D} = -c\rho(\vec{A}\cdot\hat{v})v^2\hat{v} = -c\rho(\vec{A}\cdot\vec{v})\vec{v}$$

$$\vec{A} = a\hat{\theta} \Rightarrow \vec{A}\cdot\vec{v} = a(l\omega - u\sin\theta)$$

$$\Rightarrow \vec{F_D} = -c\rho a(l\omega - u\sin\theta)\vec{v}$$

نیروی جلو برنده
$$\Rightarrow F_D\cdot\hat{x} = -c\rho a(l\omega - u\sin\theta)(u - l\omega\sin\theta) = -c\rho a(l\omega u + l\omega u\sin^2\theta - u^2\sin\theta - l^2\omega^2\sin\theta)$$

$$\Rightarrow f = \boxed{c\rho a l^2\omega^2\left[\left(1+\frac{u^2}{l^2\omega^2}\right)\sin\theta - \frac{u}{l\omega}(1+\sin^2\theta)\right]}$$

پ) $$\bar{f}_{\text{دست و پا}} = 2c\rho a l^2\omega^2\left\{\left(1+\frac{u^2}{l^2\omega^2}\right)\frac{1}{\pi}\int_0^{\pi}\sin\theta\, d\theta - \frac{u}{l\omega} - \frac{u}{l\omega}\cdot\frac{1}{\pi}\int_0^{\pi}\sin^2\theta\, d\theta\right\}$$

$$\Rightarrow \boxed{\bar{f} = 2c\rho a l^2\omega^2\left[\frac{2}{\pi}\left(1+\frac{u^2}{l^2\omega^2}\right) - \frac{3u}{2l\omega}\right]}$$

ت) $$|f_{\text{مقاوم}}| = \bar{f} \Rightarrow c\rho a' u^2 = \bar{f} \Rightarrow \frac{a'}{a} = \frac{4}{\pi}\left(\frac{l^2\omega^2}{u^2}+1\right) - 3\frac{l\omega}{u}$$

$$\Rightarrow \left(\frac{l\omega}{u}\right)^2 - \frac{3\pi}{4}\left(\frac{l\omega}{u}\right) + 1 - \frac{\pi a'}{4a} = 0 \Rightarrow \frac{l\omega}{u} = \frac{3\pi}{8} \pm\sqrt{\left(\frac{3\pi}{8}\right)^2 + \frac{\pi a'}{4a} - 1}$$

$$\frac{\partial\left(\frac{l\omega}{u}\right)}{\partial a'} > 0 \Rightarrow + \text{ جواب صحیح است.} \Rightarrow \boxed{\omega = \frac{u}{l}\left[\frac{3\pi}{8} + \sqrt{\left(\frac{3\pi}{8}\right)^2 + \left(\frac{\pi a'}{4a}\right) - 1}\right]}$$

ث) $$v = \frac{200\,m}{102\,s} = \frac{100}{101}\,m/s \quad , \quad l \simeq \frac{1,9\,m}{3,1} \simeq 0,613\,m \Rightarrow \frac{u}{l} \simeq 1,615\ rad\,s^{-1}$$

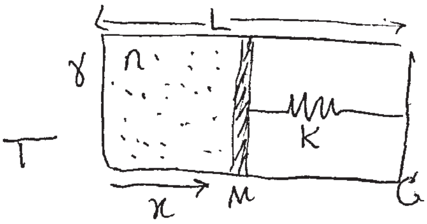
نسبت دست به قد

$a \simeq 170\ cm^2$

$a' \simeq 260\ cm^2$ ← [illegible] سر

$$\Rightarrow \omega = 2,44\,u/l \simeq 3,9\ rad/s \simeq \boxed{4\ rad/s} = \boxed{0,6\ rev/s}$$

مسئله ② آزمون چهارم :



الف)

$$\left.\begin{array}{l} PA = Kx_0 \\ PAx_0 = nRT \end{array}\right\} \Rightarrow Kx_0^2 = nRT$$

$$\Rightarrow x_0 = \sqrt{\frac{nRT}{K}} = \boxed{a}$$

ب) فرض می‌کنیم دمای گاز همیشه T است

$$M\ddot{x} = \frac{nRT}{x} - Kx \Rightarrow M\delta\ddot{x} = \frac{nRT}{a}\left(1 - \frac{\delta x}{a}\right) - Ka\left(1 + \frac{\delta x}{a}\right)$$

$$\Rightarrow M\delta\ddot{x} = -\frac{nRT}{a^2}\delta x - K\delta x \Rightarrow \delta\ddot{x} + \left(\frac{nRT}{Ma^2} + \frac{K}{M}\right)\delta x = 0$$

$$\Rightarrow \omega_1 = \sqrt{\frac{Ka^2}{Ma^2} + \frac{K}{M}} = \sqrt{2\frac{K}{m}} \Rightarrow \boxed{\omega_1 = \sqrt{2}\,\Omega}$$

ج)

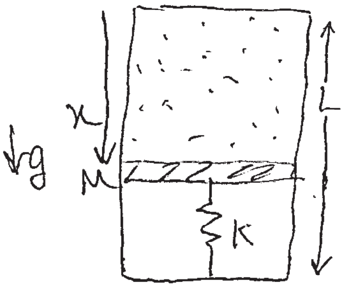
$$E_{in} = \frac{1}{2}Kx^2 + \underbrace{\cancel{\quad}}_{nC_vT} + CT$$

چون شبه‌تعادلی : $nRT = Kx^2 \Rightarrow E_{in} = \underbrace{\cancel{\quad}}_{n(C_v + R/2)T} + CT$

$$\Rightarrow dE_{in} = (nC_v + nR/2 + C)\,dT = dQ \Rightarrow C_1 = C + nR\left(\frac{1}{2} + \frac{1}{\gamma - 1}\right)$$

$$\Rightarrow \boxed{C_1 = C + \frac{\gamma + 1}{2(\gamma - 1)}nR}$$

د)



$$\frac{nRT}{x_0} + Mg = Kx_0$$

$$\Rightarrow x_0^2 - \frac{Mg}{K}x_0 - \frac{nRT}{K} = 0 \Rightarrow x_0^2 - bx_0 - a^2 = 0$$

$$\Rightarrow x_0 = \frac{b \pm \sqrt{b^2 + 4a^2}}{2} \xrightarrow{x_0 > 0} \boxed{x_0 = \frac{b + \sqrt{b^2 + 4a^2}}{2}}$$

هـ)

$$M\delta\ddot{x} = \frac{nRT}{x_0\left(1 + \frac{\delta x}{x_0}\right)} + Mg - Kx_0 - K\delta x$$

$$\Rightarrow M\delta\ddot{x} = -\frac{nRT}{x_0^2}\delta x - K\delta x \Rightarrow \omega_2^2 = \frac{K}{M} + \frac{nRT}{Mx_0^2} = \Omega^2\left(1 + \frac{a^2}{x_0^2}\right)$$

$$\Rightarrow \boxed{\omega_2 = \sqrt{1 + \frac{1}{\left(\frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} + 1}\right)^2}}\;\Omega}$$

g) $E_{in} = CT + \frac{nR}{\gamma-1}T + \frac{1}{2}kx^2 - Mgx$

$$\Rightarrow dE_{in} = dQ = C_2 dT = C dT + \frac{nR}{\gamma-1} dT + (kx - Mg)dx$$

$$\Rightarrow C_2 = C + \frac{nR}{\gamma-1} + (kx - Mg)\frac{dx}{dT}$$

$$x = \frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{nRT}{k}} \Rightarrow \frac{dx}{dT} = \frac{\frac{nR}{k}}{\sqrt{b^2 + 4\frac{nRT}{k}}}, \quad (kx - Mg) = k(x-b) = \frac{k}{2}\left(\sqrt{b^2+4\frac{nRT}{k}} - b\right)$$

$$\Rightarrow (kx - Mg)\frac{dx}{dT} = \frac{nR}{2} - \frac{nRb}{2\sqrt{b^2+4a^2}}$$

$$\boxed{\Rightarrow C_2 = C + \frac{nR(\gamma+1)}{2(\gamma-1)} - \frac{nRb}{2\sqrt{b^2+4a^2}}}$$

---

i) $C_1 = C + \frac{nR(\gamma+1)}{2(\gamma-1)} \Rightarrow C_1 - C_2 = \frac{nR}{2\sqrt{1+4a^2/b^2}}$ $\quad a^2 = \frac{nRT}{k}, \quad b = \frac{Mg}{k}$

$$\omega_1 = \sqrt{\frac{2k}{M}} \Rightarrow k = \frac{1}{2}M\omega_1^2 \Rightarrow a^2 = \frac{2nRT}{M\omega_1^2}, \quad b^2 = \frac{4g^2}{\omega_1^4} \Rightarrow 4a^2/b^2 = \frac{2nRT\omega_1^2}{Mg^2}$$

$$\Rightarrow C_1 - C_2 = \frac{nR}{2\sqrt{1+\frac{2nRT\omega_1^2}{Mg^2}}} \Rightarrow 1 + \frac{2nRT\omega_1^2}{Mg^2} = \frac{n^2R^2}{4(C_1-C_2)^2}$$

$$\boxed{\Rightarrow M = \frac{2nRT\omega_1^2}{g^2\left[\frac{n^2R^2}{4(C_1-C_2)^2} - 1\right]}} \Rightarrow \boxed{k = \frac{nRT\omega_1^4}{g^2\left[\frac{n^2R^2}{4(C_1-C_2)^2} - 1\right]}}$$

مسئلهٔ ③ آزمون چهارم :

الف) 

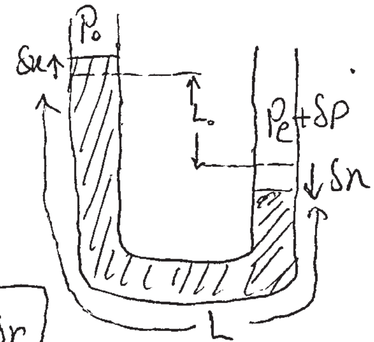
$$P_e\,dS - P_0\,dS = 2\frac{\tau}{r_0}dS \Rightarrow \boxed{P_e = P_0 + 2\tau/r_0}$$

$$P_e - \rho g L_0 = P_0 \Rightarrow \rho g L_0 = 2\tau/r_0 \Rightarrow \boxed{L_0 = \frac{2\tau}{\rho g r_0}}$$

ب) $r = r_0 + \delta r \quad , \quad T = T_0 + \delta T \quad , \quad P = P_e + \delta P$

$$\delta M = 0 \Rightarrow (P_e + \delta P)\,dS - P_0\,dS - \frac{2\tau}{r_0 + \delta r}dS = 0 \Rightarrow \delta P + 2\frac{\tau}{r_0} - 2\frac{\tau}{r_0 + \delta r} = 0$$

$$\Rightarrow \delta P + 2\frac{\tau}{r_0^2}\delta r = 0 \quad \Rightarrow \boxed{\delta P = -\frac{2\tau}{r_0^2}\delta r}$$

ج) 

$$(\delta P + P_e)A - P_0 A - \rho g A(L_0 + 2\delta x) = \rho A L\,\ddot{\delta x}$$

$$\Rightarrow \delta P + \frac{2\tau}{r_0} - 2\frac{\tau}{r_0} - 2\rho g\,\delta x = \rho L\,\ddot{\delta x}$$

$$\cancel{\Rightarrow \ddot{\delta x} = -2\rho g\,\delta x - \frac{2\tau}{r_0^2}\delta P} \Rightarrow \boxed{\ddot{\delta x} = -\frac{2g}{L}\delta x - \frac{2\tau}{\rho L r_0^2}\delta r}$$

$$\Rightarrow \boxed{\ddot{\delta x} = -2\xi\Omega\,\delta x - 2\Omega^2\,\delta r}$$

د) $dV = 4\pi r_0^2\,\delta r + A\,\delta x \qquad [\to \delta V = (4\pi r_0^2 - A)\delta r + A\,\delta r + A\,\delta x]$

$$\Rightarrow -\delta W = (P_e + \delta P)\,\delta V = P_e 4\pi r_0^2\,\delta r + P_e A\,\delta x$$

$$U = \frac{nR}{\gamma - 1}T \Rightarrow \frac{dU}{dt} = \frac{nR}{\gamma - 1}\dot{\delta T} = -P_e 4\pi r_0^2\,\dot{\delta r} - P_e A\,\dot{\delta x} - hA\,\delta T \qquad (1)$$

$$\cancel{(P_e + \delta P)(V_0 + \delta V) = nR(T_0 + \delta T) \Rightarrow P_e\delta V + V_0\delta P = nR\,\delta T \Rightarrow \delta T = \frac{P_e}{nR}4\pi r_0^2\delta r + \frac{P_e}{nR}A\delta x - \frac{2\tau V_0}{nR r_0^2}\delta r}$$

$$\cancel{\Rightarrow \left(\frac{nR}{\gamma - 1} + hA\right)\left(\frac{P_e}{nR}4\pi r_0^2\delta r - \frac{2\tau V_0}{nR r_0^2}\delta r + \frac{P_e}{nR}A\,\delta x\right) = -P_e A\delta x - P_e 4\pi r_0^2\delta r}$$

$$\cancel{\Rightarrow \left(\frac{nR}{\gamma - 1} + \frac{\tau 4\pi r_0^2}{r_0 T_0}\cdot\frac{\alpha v_0}{\beta}\right)\left(\frac{4\pi r_0^2\tau}{nR\beta r_0}\delta r - 2\tau\frac{4\pi r_0^2}{3\lambda r_0 nR}\delta r + \frac{4\pi r_0^2\tau}{nR\beta r_0}\alpha\,\delta x\right) = -\frac{\tau 4\pi r_0^2}{\beta r_0}\alpha\,\delta x - \frac{4\pi r_0^2\tau}{\beta r_0}\delta r}$$

$$\cancel{\Rightarrow \left(\frac{nR}{\gamma - 1} + \frac{4\pi r_0^2\tau}{r_0 T_0}\cdot\frac{\alpha v_0}{\beta}\right)\left(\frac{\delta r}{nR} + \frac{\alpha\,\delta x}{nR} - \frac{2\beta}{3\lambda nR}\delta r\right) = -\alpha\,\delta x - \delta r}$$

$$(P_e+\delta P)(V_0+\delta V)=nR(\delta T+T_0) \quad , \quad P_eV_0=nRT_0 \Rightarrow nR=\frac{\frac{\tau}{\beta r_0}\cdot\frac{4\pi r_0^3}{3\lambda}}{T_0}$$

$$\Rightarrow P_e\,\delta V+V_0\,\delta P=nR\,\delta T \Rightarrow nR\,\delta T=P_e 4\pi r_0^2\,\delta r+P_eA\,\delta x-\frac{2\tau V_0}{r_0^2}\,\delta r$$

(1) $$\Rightarrow \frac{P_e4\pi r_0^2}{\gamma-1}\delta\dot r+\frac{P_eA}{\gamma-1}\delta\dot x-\frac{2\tau V_0}{(\gamma-1)r_0^2}\delta\dot r=-P_e4\pi r_0^2\delta\dot r-P_eA\,\delta\dot x-hA\frac{4\pi r_0^2P_e}{nR}\delta r-hA\frac{P_eA}{nR}\delta x+\frac{2\tau V_0}{nRr_0^2}hA\,\delta r$$

$$\Rightarrow \frac{\gamma}{\gamma-1}P_e4\pi r_0^2\delta\dot r+\frac{\gamma}{\gamma-1}P_eA\,\delta\dot x-\frac{2\tau V_0}{r_0^2(\gamma-1)}\delta\dot r=-hA\frac{4\pi r_0^2P_e}{nR}\delta r-hA\frac{P_eA}{nR}\delta x+\frac{2\tau V_0}{nRr_0^2}hA\,\delta r$$

$$\Rightarrow \frac{\gamma}{\gamma-1}\cdot\frac{\tau}{\beta r_0}\delta\dot r+\frac{\gamma}{\gamma-1}\alpha\frac{\tau}{\beta r_0}\delta\dot x-\frac{2\tau}{3\lambda(\gamma-1)r_0}\delta\dot r=-\frac{\tau}{\beta r_0}\cdot\frac{3\alpha\lambda v_0}{r_0}\delta r-\frac{\tau}{\beta r_0}\cdot\frac{3\alpha\lambda v_0}{r_0}\alpha\,\delta x+2\frac{\tau\alpha v_0}{r_0^2}\delta r$$

$$\boxed{\Rightarrow\left(\frac{\gamma}{\gamma-1}-\frac{2\beta}{3\lambda(\gamma-1)}\right)\delta\dot r+\frac{\gamma}{\gamma-1}\alpha\,\delta\dot x=-\frac{3\alpha\lambda v_0}{r_0}\delta r-\frac{3\alpha^2\lambda v_0}{r_0}\delta x+2\beta\frac{\alpha v_0}{r_0}\delta r}$$

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۲) $$\delta\ddot x=-2\zeta\Omega\,\delta x-2\Omega^2\delta r\Rightarrow\delta r=\frac{-\zeta}{\Omega}\delta x-\frac{1}{2\Omega^2}\delta\ddot x\Rightarrow\delta\dot r=\frac{-\zeta}{\Omega}\delta\dot x-\frac{1}{2\Omega^2}\delta\dddot x$$

$$\Rightarrow\left(\frac{\gamma}{\gamma-1}-\frac{2\beta}{3\lambda(\gamma-1)}\right)\times\frac{-\zeta}{\Omega}\delta\ddot x-\frac{1}{2\Omega^2}\left(\frac{\gamma}{\gamma-1}-\frac{2\beta}{3\lambda(\gamma-1)}\right)\delta\dddot x+\frac{\gamma}{\gamma-1}\alpha\,\delta\dot x=-\frac{3\alpha^2\lambda v_0}{r_0}\delta x+\left(\frac{\zeta\delta\dot x}{\Omega}+\frac{\delta\ddot x}{2\Omega^2}\right)\left(\frac{3\alpha\lambda v_0}{r_0}-\frac{2\alpha\beta v_0}{r_0}\right)$$

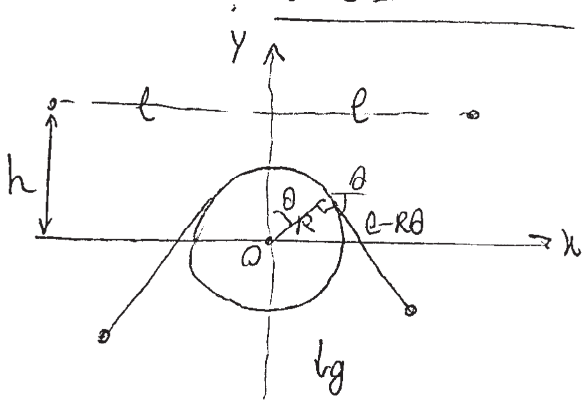
$$\boxed{\begin{aligned}\Rightarrow&\frac{1}{\Omega^2}\left(\frac{\gamma}{2(\gamma-1)}-\frac{\beta}{3\lambda(\gamma-1)}\right)\delta\dddot x+\frac{1}{\Omega^2}\left(\frac{3\alpha\lambda v_0}{2r_0}-\frac{\alpha\beta v_0}{r_0}\right)\delta\ddot x+\\&+\left(\frac{\gamma\zeta}{(\gamma-1)\Omega}-\frac{2\beta\zeta}{3\lambda\Omega(\gamma-1)}-\frac{\alpha\gamma}{\gamma-1}\right)\delta\dot x+\left(\frac{3\alpha\lambda v_0\zeta}{r_0\Omega}-\frac{2\alpha\beta v_0\zeta}{r_0\Omega}-\frac{3\alpha^2\lambda v_0}{r_0}\right)\delta x=0\end{aligned}}$$

---

۳) $$\frac{1}{\Omega}\approx 0\Rightarrow-\frac{\alpha\gamma}{\gamma-1}\delta\dot x-\frac{3\alpha^2\lambda v_0}{r_0}\delta x=0\Rightarrow\delta\dot x=-\underbrace{\frac{\gamma-1}{\gamma}\cdot\frac{3\alpha\lambda v_0}{r_0}}_{\text{مثبت باشد}}\delta x$$

$$\Rightarrow\boxed{\gamma>1}$$

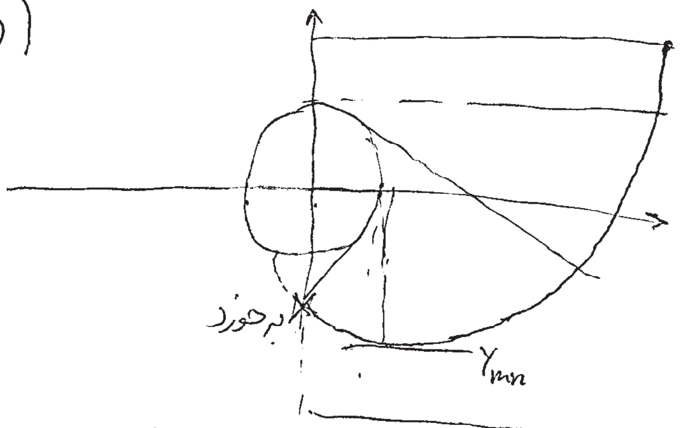
مسئله ① فاینال ۵۵



a)

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R\sin\theta + (l-R\theta)\cos\theta \\ R\cos\theta - (l-R\theta)\sin\theta \end{pmatrix}}$$

b)



$$\frac{dy}{d\theta} = 0 \Rightarrow -R\sin\theta - (l-R\theta)\cos\theta + R\sin\theta = 0$$

$$\xrightarrow{l-R\theta\neq 0} \cos\theta = 0 \Rightarrow \theta = \pi/2$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}_{\text{نقطه کمینه}} = \boxed{\begin{pmatrix} R \\ -(l-\frac{\pi}{2}R) \end{pmatrix}}$$

c)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \dot\theta\,(l-R\theta)\begin{pmatrix} -\sin\theta \\ -\cos\theta \end{pmatrix} \Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -(l-R\theta)\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\ddot\theta + R\dot\theta^2\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - (l-R\theta)\dot\theta^2\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}_{\text{نقطه کمینه}} = \begin{pmatrix} -(l-R\theta)\ddot\theta + R\dot\theta^2 \big|_{\theta=\pi/2} \\ (l-\frac{\pi}{2}R)\dot\theta^2 \end{pmatrix} \Rightarrow \ddot{Y}_m = (l-\tfrac{\pi}{2}R)\dot\theta_m^2$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = (l-R\theta)^2\dot\theta^2 \Rightarrow v_m^2 = (l-\tfrac{\pi}{2}R)^2\dot\theta_m^2$$

پایستگی انرژی: $mg(h-y_m) = \frac{1}{2}mv_m^2 \Rightarrow v_m^2 = 2g\left(h+(l-\tfrac{\pi}{2}R)\right) = (l-\tfrac{\pi}{2}R)^2\dot\theta_m^2$

$$\Rightarrow \ddot{Y}_m = (l-\tfrac{\pi}{2}R)\dot\theta_m^2 = 2g\left(1+\frac{h}{l-\frac{\pi}{2}R}\right)$$

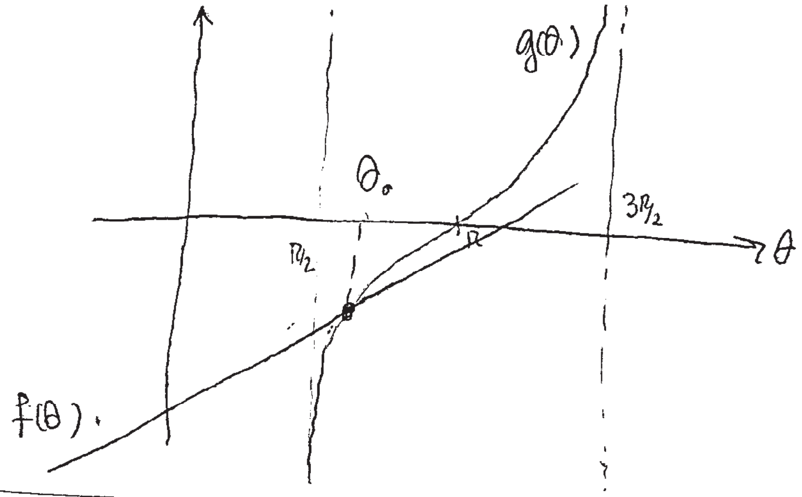
$$m\ddot{Y}_m = T_m - mg \Rightarrow \boxed{T_m = mg\left(3+\frac{2h}{l-\frac{\pi}{2}R}\right)}$$

d)

$$x(\theta_0) = 0 \Rightarrow \sin\theta_0 + \left(\frac{l}{R}-\theta_0\right)\cos\theta_0 = 0 \Rightarrow \boxed{\tan\theta_0 = \theta_0 - l/R}$$

روش ترسیمی :

$$\begin{cases} 1.\ f(\theta)=\theta-\ell/R \\ g(\theta)=\tan\theta \qquad \pi/2<\theta<\pi \end{cases}$$



e) $y(\theta_0) = R\cos\theta_0-(\ell-R\theta_0)\sin\theta_0 = R\cos\theta_0+R\tan\theta_0\sin\theta_0 = \boxed{\dfrac{R}{\cos\theta_0}} = R\sqrt{1+(\theta_0-\ell/R)^2}$

f) $\underline{\underline{\theta_0=3\pi/4}} \Rightarrow 3\pi/4-\ell/R=\tan 3\pi/4=-1 \Rightarrow \ell/R=3\pi/4+1 \Rightarrow \boxed{\ell=\left(1+\frac{3\pi}{4}\right)R}$

g) پس از برخورد $\begin{cases} v'_y=v_y \\ v'_x=-v_x \end{cases}$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -\left(\ell-\tfrac{3\pi}{4}R\right)\begin{pmatrix} \sin\frac{3\pi}{4} \\ \cos\frac{3\pi}{4} \end{pmatrix}\dot\theta = -R\begin{pmatrix} \sqrt2/2 \\ -\sqrt2/2 \end{pmatrix}\dot\theta = -\tfrac{\sqrt2}{2}R\begin{pmatrix} 1 \\ -1 \end{pmatrix}\dot\theta$$

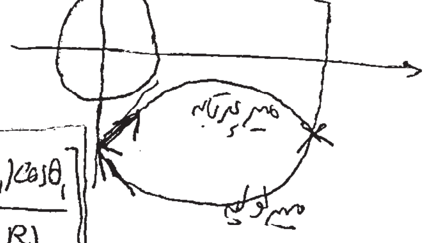
$$\Rightarrow v^2=v_x^2+v_y^2=R^2\dot\theta^2$$

$$\tfrac12 mv^2=mg(h-y) \Rightarrow R^2\dot\theta^2=2g(h+\sqrt2R) \Rightarrow R\dot\theta=\sqrt{2g(h+\sqrt2R)}$$

$$\Rightarrow \begin{pmatrix} v'_x \\ v'_y \end{pmatrix} = \tfrac{\sqrt2}{2}\sqrt{2g(h+\sqrt2R)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} v'_x=\sqrt{g(h+\sqrt2R)} \\ v'_y=\sqrt{g(h+\sqrt2R)} \end{cases}$$

مسیر پس از برخورد پرتابه است :
$$\left.\begin{cases} x=v'_x t \\ y=v'_y t-\tfrac12 gt^2 \end{cases}\right\} \Rightarrow y(x)=x-\frac{gx^2}{2v'^2_x} \Rightarrow \boxed{y(x)=x-\frac{x^2}{2(h+\sqrt2R)}}$$

h) حرکت تا جایی پرتابه است که مسیر پرتابه مسیر اولیه را قطع کند



$$\Rightarrow \boxed{R\cos\theta_1-(\ell-R\theta_1)\sin\theta_1=\left[R\sin\theta_1+(\ell-R\theta_1)\cos\theta_1\right]\times\left[1-\frac{R\sin\theta_1+(\ell-R\theta_1)\cos\theta_1}{2(h+\sqrt2R)}\right]}$$

جایگذاری x و y برای مسیر در معادله پرتابه

مسئله ② [illegible]

a) $p = \sqrt{2mT}$

$$\Rightarrow \begin{cases} \sqrt{2m_1 T} = \sqrt{2m_2 T_2}\,\cos\varphi \\ \sqrt{2m_1 T_1} = \sqrt{2m_2 T_2}\,\sin\varphi \\ T = T_1 + T_2 \end{cases} \xrightarrow[\text{مجذور}]{\text{جمع}} \cancel{}\; 2m_1(T+T_1) = 2m_2 T_2 \Rightarrow T_2 = \frac{m_1}{m_2}(T+T_1)$$

$$\Rightarrow T\left(1-\frac{m_1}{m_2}\right) = T_1\left(1+\frac{m_1}{m_2}\right) \Rightarrow \boxed{T_1 = \frac{m_2-m_1}{m_2+m_1}\,T} \qquad T_1 > 0 \xrightarrow[\text{وجود}]{\text{شرط}} \boxed{m_2 > m_1}$$

$$\tan\varphi = \sqrt{\frac{T_1}{T}} = \boxed{\sqrt{\frac{m_2-m_1}{m_2+m_1}}}$$

---

b)

$$\begin{cases} p = p_2\cos\varphi \\ p_1 = p_2\sin\varphi \\ \varepsilon + m_2c^2 = \varepsilon_1 + \varepsilon_2 \end{cases}$$

$$\Rightarrow p_2^2 = p^2 + p_1^2 \Rightarrow \varepsilon_2^2 - (m_2c^2)^2 = \varepsilon^2 + \varepsilon_1^2 - 2(m_1c^2)^2$$

$$\Rightarrow \varepsilon_2^2 = \varepsilon^2 + \varepsilon_1^2 + (m_2c^2)^2 + 2\varepsilon(m_2c^2) - 2\varepsilon_1(m_2c^2) - 2\varepsilon\varepsilon_1$$

$$\Rightarrow (m_1c^2)^2 = \varepsilon_1(m_2c^2) + \varepsilon\varepsilon_1 - \varepsilon(m_2c^2) \Rightarrow \boxed{\varepsilon_1 = \frac{(m_1c^2)^2 + (m_2c^2)\,\varepsilon}{\varepsilon + m_2c^2}}$$

$$\tan\varphi = \frac{p_1}{p} = \sqrt{\frac{\varepsilon_1^2 - (m_1c^2)^2}{\varepsilon^2 - (m_1c^2)^2}}$$

$$\varepsilon_1^2 - (m_1c^2)^2 = \frac{[(m_1c^2)^2 + (m_2c^2)\,\varepsilon]^2 - [m_1c^2\,\varepsilon + (m_1c^2)(m_2c^2)]^2}{[\varepsilon + m_2c^2]^2} =$$

$$= \frac{(m_1c^2)^2\left[(m_1c^2)^2 - (m_2c^2)^2\right] + \varepsilon^2\left[(m_2c^2)^2 - (m_1c^2)^2\right]}{[\varepsilon + m_2c^2]^2} = \frac{\varepsilon^2 - (m_1c^2)^2}{(\varepsilon + m_2c^2)^2}\left((m_2c^2)^2 - (m_1c^2)^2\right)$$

$$\Rightarrow \boxed{\tan\varphi = \frac{\sqrt{(m_2c^2)^2 - (m_1c^2)^2}}{\varepsilon + m_2c^2}} \qquad \tan\varphi \;\text{وجود دارد}\; \xrightarrow[\text{برقرار}]{\text{شرط}} \boxed{m_2 > m_1}$$

c) $T \ll m_1c^2$ $\qquad \epsilon := \dfrac{T}{m_1c^2} \Rightarrow T = \epsilon m_1c^2 = \mathcal{E} - m_1c^2 \Rightarrow \mathcal{E} = m_1c^2(1+\epsilon)$

$$\mathcal{E}_1 = \frac{(m_1c^2)^2 + (m_2c^2)\mathcal{E}}{\mathcal{E} + (m_2c^2)} = m_1c^2\,\frac{m_1c^2 + (1+\epsilon)m_2c^2}{m_2c^2 + m_1c^2 + \epsilon m_1c^2} = m_1c^2\,\frac{1+\epsilon\frac{m_2}{m_1+m_2}}{1+\epsilon\frac{m_1}{m_1+m_2}}$$

$$\cancel{\mathcal{E}_1 = m_1c^2\left[1+\frac{m_2-m_1}{m_2+m_1}\epsilon\right]} \Rightarrow \mathcal{E}_1 = m_1c^2\left(1+\epsilon\frac{m_2}{m_1+m_2}\right)\left(1-\epsilon\frac{m_1}{m_1+m_2}+\epsilon^2\frac{m_1^2}{(m_1+m_2)^2}\right)$$

$$\boxed{\Rightarrow T_1 = \mathcal{E}_1 - m_1c^2 = \underbrace{m_1c^2\,\epsilon\,\frac{m_2-m_1}{m_2+m_1}}_{\text{غیرنسبتی}} + \underbrace{m_1c^2\,\epsilon^2\,\frac{m_1(m_1-m_2)}{(m_1+m_2)^2}}_{\text{تصحیح نسبتی}}}$$

$$\tan\varphi = \frac{\sqrt{(m_2c^2)^2-(m_1c^2)^2}}{\mathcal{E}+m_2c^2} = \frac{\sqrt{(m_2c^2)^2-(m_1c^2)^2}}{m_1c^2+m_2c^2+\epsilon m_1c^2} = \sqrt{\frac{m_2-m_1}{m_2+m_1}} \times \frac{1}{1+\epsilon\frac{m_1}{m_1+m_2}} = \boxed{\sqrt{\frac{m_2-m_1}{m_2+m_1}}\left(1-\epsilon\frac{m_1}{m_1+m_2}\right)}$$

---

d) $m_2c^2 \gg \mathcal{E} > m_1c^2$:

$$\mathcal{E}_1 = \left\{\frac{\left(\frac{m_1}{m_2}\right)^2 + \frac{\mathcal{E}}{m_2c^2}}{\frac{\mathcal{E}}{m_2c^2}+1}\right\}(m_2c^2) \overset{\text{در تقریب اول}}{=} \frac{\mathcal{E}}{m_2c^2} \times m_2c^2 = \boxed{\mathcal{E}}$$

$$\tan\varphi = \frac{m_2c^2\sqrt{1-\left(\frac{m_1}{m_2}\right)^2}}{m_2c^2\left(1+\mathcal{E}/m_2c^2\right)} \overset{\text{در تقریب اول}}{=} 1 \Rightarrow \boxed{\varphi = 45^\circ}$$

$\mathcal{E} \gg m_1c^2, m_2c^2$:

$$\mathcal{E}_1 = \mathcal{E}\left\{\frac{\left(\frac{m_1c^2}{\mathcal{E}}\right)^2 + \frac{m_2c^2}{\mathcal{E}}}{1+\frac{m_2c^2}{\mathcal{E}}}\right\} = \mathcal{E} \times \frac{m_2c^2}{\mathcal{E}} = \boxed{m_2c^2}$$

$$\tan\varphi = \frac{\sqrt{\left(\frac{m_2c^2}{\mathcal{E}}\right)^2 - \left(\frac{m_1c^2}{\mathcal{E}}\right)^2}}{1+\frac{m_2c^2}{\mathcal{E}}} \overset{\text{در تقریب صفرم}}{=} 0 \Rightarrow \boxed{\varphi = 0}$$

---

e) $\mathcal{E} = m_1c^2 + 200\text{ MeV} = 1138.3\text{ MeV}$

$$\Rightarrow \mathcal{E}_1 = \frac{(938.3)^2 + 1138.3 \times 939.6}{1138.3 + 939.6}\text{ MeV} = 938.4\text{ MeV} \Rightarrow T_1 = \mathcal{E}_1 - 938.3\text{ MeV} = \underline{0/125\text{ MeV}}$$

$$\tan\varphi = \frac{\sqrt{939.6^2 - 938.3^2}}{1138.3 + 939.6} = 0/0238 \Rightarrow \varphi \approx \boxed{1/362^\circ}$$

مسئله ③ فاینال اول :

الف)
$$P = P_0 + \rho g (H-L), \quad PV = nRT_0 \Rightarrow \boxed{V = \frac{nRT_0}{P_0 + \rho_0 g (H-L)}}$$

ب)
$$\rho_0 g V = Mg \Rightarrow M_{max} = \rho_0 V = \boxed{\frac{nR\rho_0 T_0}{P_0 + \rho_0 g(H-L)}}$$

ج)
$$M\ddot{x} = Mg - \rho_0 g V(x) \Rightarrow \ddot{x} = g - \frac{\rho_0}{M} V(x) g$$

$$P(x) = P_0 + \rho_0 g x \Rightarrow V(x) = \frac{nRT_0}{P_0 + \rho_0 g x}$$

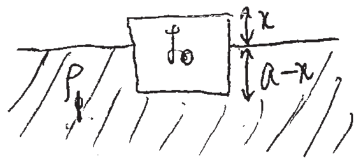
$$\boxed{\Rightarrow \ddot{x} = g - \frac{\rho_0 g}{M} \cdot \frac{nRT_0}{P_0 + \rho_0 g x}} \qquad \ddot{x} = \frac{d\dot{x}^2}{2dx} \Rightarrow \dot{x}^2(x) = \int_H^x 2\ddot{x}\, dx$$

$$\Rightarrow \dot{x}^2 = 2g(x-H) - \frac{nRT_0}{M} \ln \frac{P_0 + \rho_0 g x}{P_0 + \rho_0 g H}$$

$$\boxed{\Rightarrow \dot{x} = -\sqrt{\frac{nRT_0}{M} \ln\left(\frac{P_0 + \rho_0 g H}{P_0 + \rho_0 g x}\right) - 2g(H-x)}}$$

مسئله ④ فاینال اول :

الف)

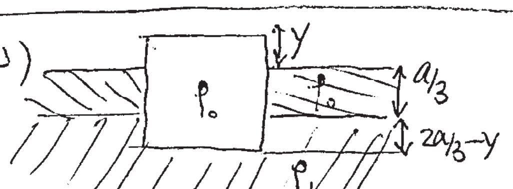


$$a^2 \rho_1 g (a-x) = \rho_0 g a^3$$

$$\Rightarrow \frac{a-x}{a} = \frac{\rho_0}{\rho_1} = 2/3 \Rightarrow x/a = 1/3 \Rightarrow x = a/3$$

$$\Rightarrow V_1 = a^2 x = \boxed{a^3/3}$$

ب)



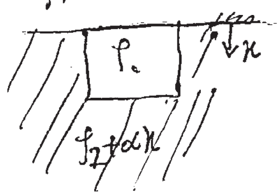
$$\rho_1 g a^2 (2a/3 - y) + \rho_0 g a^2 (a/3) = \rho_0 g a^3$$

$$\Rightarrow 2/3\, a \rho_0 = (2a/3 - y)\rho_1 \Rightarrow 2a/3 - y = 2/3\, a \times 2/3 = 4/9\, a$$

$$\Rightarrow y = 2a/3 - 4a/9 = 2/9\, a \Rightarrow \boxed{V_2 = 2/9\, a^3}$$

$$\Rightarrow 2a/3 - y = 4a/9 \Rightarrow \Delta y_{cm} = (a-x) - (2a/3 - y) = 2/3\, a - 4a/9 = \boxed{2/9\, a^3} \ (= y)$$

ج)



$$dP = \rho g dx = (\rho_2 + \alpha x) g\, dx \Rightarrow \Delta P_{,x} = \int_0^a dP = \rho_2 g a + \alpha g\, a^2/2$$

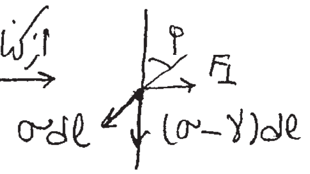
بالا و پایین مکعب

$$\Delta P a^2 = \rho_0 g a^3 \Rightarrow \rho_0 = \rho_2 + \frac{\alpha a}{2} \Rightarrow \boxed{\alpha = \frac{2(\rho_0 - \rho_2)}{a}}$$

مسئله ۴ (5) فاینال ۹۹ :

الف) به یک جزء طولی از مایع (مساحت و حجم صفر) در سطح آزاد نگاه می‌کنیم :

از کنار



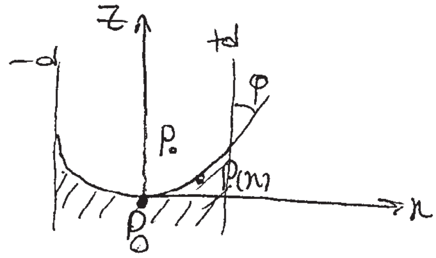
$$\sum F_{\parallel}=0 \Rightarrow \sigma dl\cos\varphi+(\sigma-\gamma)dl=0 \Rightarrow \cos\varphi=\frac{\gamma-\sigma}{\sigma} \Rightarrow \boxed{\varphi=\cos^{-1}\left(\frac{\gamma}{\sigma}-1\right)}$$

ب)

$$P_{(x)}=P_0-\rho g z \qquad z'(d)=\cot\varphi$$

$$P_\circ-P_0=\frac{\sigma}{R_\circ}=\sigma z''_\circ$$

$$P_\circ-P_{(x)}=\sigma\frac{z''}{(1+z'^2)^{3/2}}=\sigma z''_\circ+\rho g z$$



$$\Rightarrow \frac{dz'^2}{(1+z'^2)^{3/2}}=2z''_\circ dz+\frac{2\rho g}{\sigma}z\,dz \Rightarrow \left.\frac{-2}{\sqrt{1+z'^2}}\right|_{0=x}^{d}=2z''_\circ h+\frac{\rho g}{\sigma}h^2$$

$$\cancel{\Rightarrow 2\left(1-\frac{1}{\sqrt{1+\cot^2\varphi}}\right)=2z''_\circ d+\frac{\rho g}{\sigma}d^2 \Rightarrow 1-\sin\varphi=z''_\circ d+\frac{\rho g}{2\sigma}d^2}$$

$$\Rightarrow 2\left(1-\frac{1}{\sqrt{1+\cot^2\varphi}}\right)=2z''_\circ h+\frac{\rho g}{\sigma}h^2 \Rightarrow 1-\sin\varphi=z''_\circ h+\frac{\rho g}{2\sigma}h^2 \Rightarrow h^2+\frac{2\sigma z''_\circ}{\rho g}h-\frac{2\sigma}{\rho g}(1-\sin\varphi)=0$$

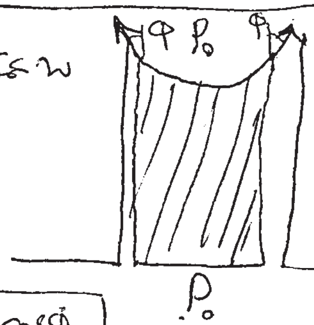
$$\xRightarrow{h>0} \boxed{h=\frac{-\sigma z''_\circ}{\rho g}+\sqrt{\left(\frac{\sigma z''_\circ}{\rho g}\right)^2+\frac{2\sigma}{\rho g}(1-\sin\varphi)}}$$

ج)

$$2dwH\rho g=2w\sigma\cos\varphi$$

$$\Rightarrow dH\rho g=\sigma\cos\varphi \Rightarrow H=\frac{\sigma\cos\varphi}{\rho g d}$$

w : عمق ظرف

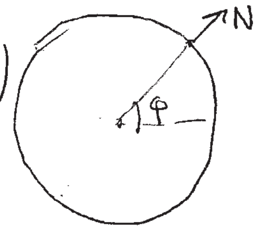


$$P_0=P_\circ-\rho g H \Rightarrow P_\circ-P_0=\rho g H=\sigma z''_\circ=\frac{\sigma\cos\varphi}{d} \Rightarrow \boxed{z''_\circ=\frac{\cos\varphi}{d}}$$

$$\Rightarrow h=\frac{-\sigma\cos\varphi}{\rho g d}+\sqrt{\left(\frac{\sigma\cos\varphi}{\rho g d}\right)^2+\frac{2\sigma}{\rho g}(1-\sin\varphi)}=\boxed{\frac{\sigma\cos\varphi}{\rho g d}\left\{\sqrt{1+\frac{2\rho g d^2}{\sigma\cos^2\varphi}(1-\sin\varphi)}-1\right\}}$$

مسئله ⑥ فاینال ترم :

آ)
$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + R\cos\varphi \\ y_1 + R\sin\varphi \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 - R\sin\varphi\,\dot{\varphi} \\ \dot{y}_2 + R\cos\varphi\,\dot{\varphi} \end{pmatrix} \quad (\mathrm{I})$$



(اگر نیروی وارد به حلقه در راستای شعاعی بود، آنرا مثبت می‌گیریم)

$$\begin{cases} M\begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{pmatrix} = N\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \\ m\begin{pmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{pmatrix} = -N\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} M\ddot{x}_1 + m\ddot{x}_2 = 0 \Rightarrow M\dot{x}_1 + m\dot{x}_2 = cte = 0 \overset{(\mathrm{I})}{\Rightarrow} (m+M)\dot{x}_1 = mR\sin\varphi\,\dot{\varphi} \\ M\ddot{y}_1 + m\ddot{y}_2 = 0 \Rightarrow M\dot{y}_1 + m\dot{y}_2 = cte = mv_0 \Rightarrow (m+M)\dot{y}_1 + mR\cos\varphi\,\dot{\varphi} = mv_0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \frac{m}{m+M}\begin{pmatrix} R\sin\varphi\,\dot{\varphi} \\ -R\cos\varphi\,\dot{\varphi} + v_0 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \frac{-M}{m+M}R\sin\varphi\,\dot{\varphi} \\ \frac{M}{m+M}R\cos\varphi\,\dot{\varphi} + \frac{m}{m+M}v_0 \end{pmatrix} \quad (\mathrm{II})$$

$$\Rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}\frac{mR}{m+M}\dot{\varphi}^2 + \begin{pmatrix} \sin\varphi \\ -\cos\varphi \end{pmatrix}\frac{mR}{m+M}\ddot{\varphi}$$

$$\begin{pmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}\frac{-MR}{m+M}\dot{\varphi}^2 + \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}\frac{MR}{m+M}\ddot{\varphi}$$

$$\Rightarrow \tan\varphi = \frac{\ddot{y}_1}{\ddot{x}_1} = \frac{-\ddot{\varphi}\cos\varphi + \dot{\varphi}^2\sin\varphi}{\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi} = \frac{\sin\varphi}{\cos\varphi} \Rightarrow -\ddot{\varphi}\cos^2\varphi + \dot{\varphi}^2\sin\varphi\cos\varphi = \ddot{\varphi}\sin^2\varphi + \dot{\varphi}^2\sin\varphi\cos\varphi$$

$$\Rightarrow \ddot{\varphi} = 0 \Rightarrow \dot{\varphi} = const = \dot{\varphi}(0) = \frac{v_0}{R} \Rightarrow \boxed{\varphi(t) = \frac{v_0}{R}t}$$

{از پایستگی انرژی هم می‌توان به نتیجه فوق رسید}

ب)
$$\cancel{\sqrt{\ddot{x}_1^2 + \ddot{y}^2} \neq \frac{mR}{m+M}} \quad M\begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{pmatrix} = \frac{MmR}{m+M}\left(\frac{v_0}{R}\right)^2\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = N\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \Rightarrow \boxed{N = \frac{mM}{m+M}\left(\frac{v_0^2}{R}\right)}$$

ج)
$$\begin{cases} \dot{x}_1 = \frac{m}{m+M}v_0 \sin\left(\frac{v_0}{R}t\right) \\ \dot{y}_1 = -\frac{mv_0}{m+M}\cos\left(\frac{v_0}{R}t\right) + v_0\frac{m}{m+M} \end{cases} \Rightarrow \boxed{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{mR}{m+M}\left\{1 - \cos\left(\frac{v_0}{R}t\right)\right\} \\ \frac{mv_0t}{m+M} - \frac{mR}{m+M}\sin\left(\frac{v_0}{R}t\right) \end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{mR}{m+M} \begin{pmatrix} 1-\cos(\frac{v_0}{R}t) \\ (\frac{v_0}{R}t) - \sin(\frac{v_0}{R}t) \end{pmatrix}$$

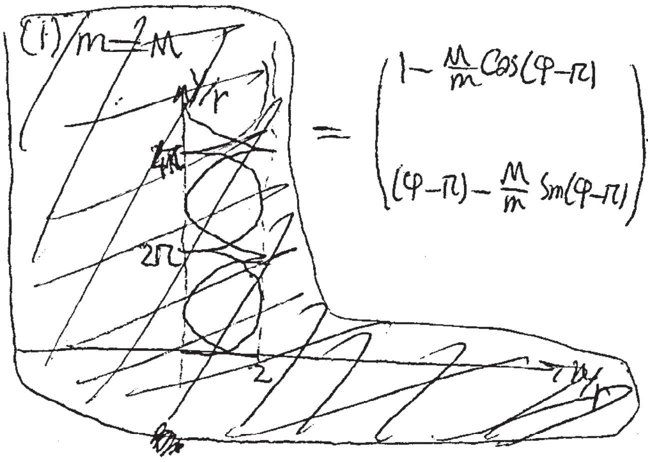
$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = R \begin{pmatrix} x_1/R + \cos\varphi \\ y_1/R + \sin\varphi \end{pmatrix} = \boxed{R \begin{pmatrix} \frac{m}{m+M} + \frac{M}{m+M}\cos(\frac{v_0}{R}t) \\ \frac{m}{m+M}(\frac{v_0}{R}t) + \frac{M}{m+M}\sin(\frac{v_0}{R}t) \end{pmatrix}}$$

ز) $\cancel{\begin{pmatrix} x_1/R \\ y_1/R \end{pmatrix} = \begin{pmatrix} \frac{m}{m+M}(1-\cos\varphi) \\ \varphi - \frac{m}{m+M}\sin\varphi \end{pmatrix}}$

$$\begin{pmatrix} \frac{x_1}{mR}\, m+M \\ \frac{y_1}{mR}\, m+M \end{pmatrix} = \begin{pmatrix} 1-\cos\varphi \\ \varphi - \sin\varphi \end{pmatrix} \leftarrow \text{سیکلوئید}$$

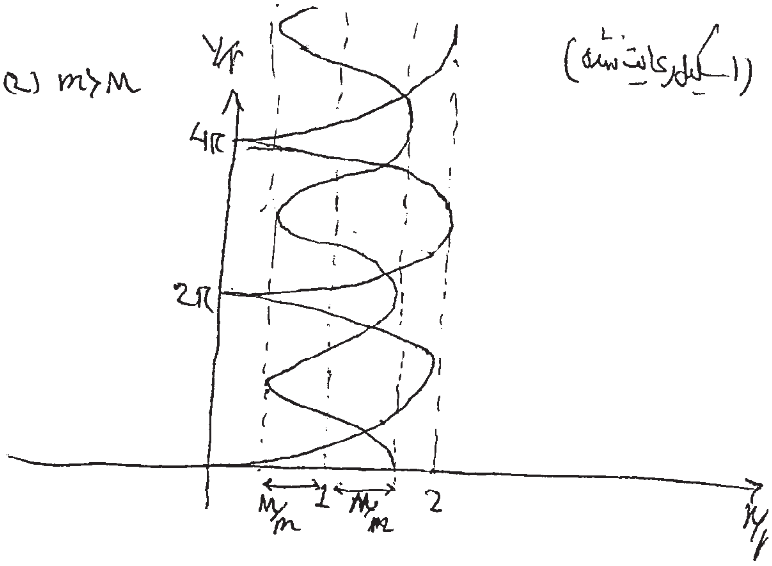
$$\begin{pmatrix} \frac{x_2}{mR}\cdot m+M \\ \frac{y_2}{mR}\cdot m+M \end{pmatrix} = \begin{pmatrix} 1+\frac{M}{m}\cos\varphi \\ \varphi + \frac{M}{m}\sin\varphi \end{pmatrix} \qquad r := \frac{m}{m+M}R$$

~~(1) m=2M~~

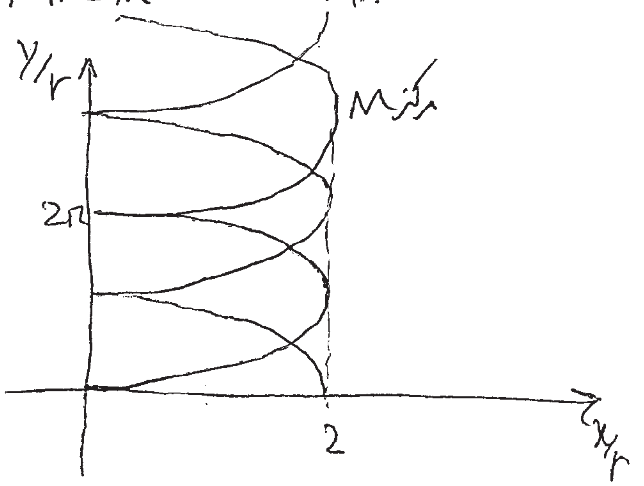


$$= \begin{pmatrix} 1-\frac{M}{m}\cos(\varphi-\pi) \\ (\varphi-\pi) - \frac{M}{m}\sin(\varphi-\pi) \end{pmatrix} + \begin{pmatrix} 0 \\ \pi \end{pmatrix}$$

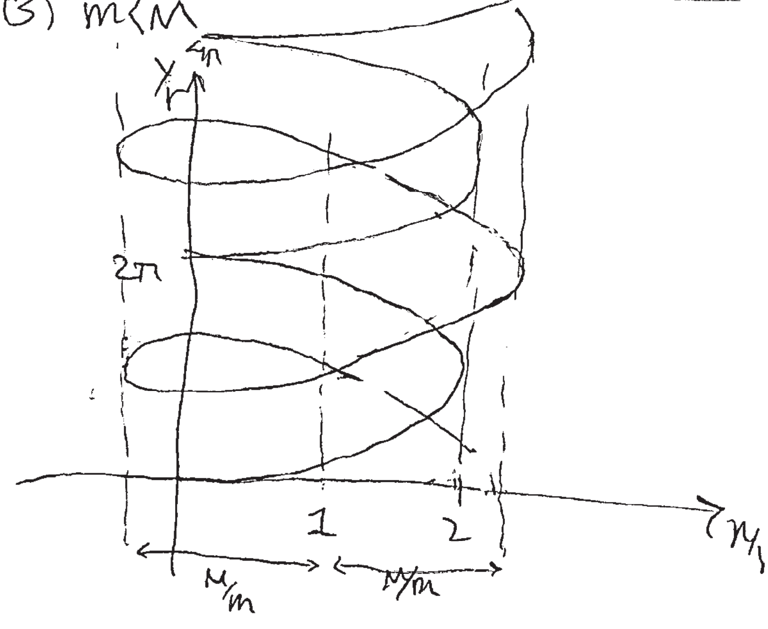
(2) $m>M$



(1) $m=M$



(3) $m<M$

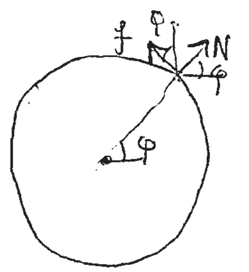


ت) $\frac{x_1}{r}=1-\cos\left(\frac{v_\circ}{R}t\right) \Rightarrow \boxed{T=2\pi\frac{R}{v_\circ}}$

$$\begin{pmatrix}\dot{x}_1\\ \dot{y}_1\end{pmatrix}=\frac{mv_\circ}{m+M}\begin{pmatrix}\sin\frac{v_\circ}{R}t\\ -\cos\frac{v_\circ}{R}t+1\end{pmatrix} \Rightarrow v_1=\sqrt{\dot{x}_1^2+\dot{y}_1^2}=\frac{mv_\circ}{m+M}\sqrt{2-2\cos\frac{v_\circ}{R}t}=\frac{2mv}{m+M}\left|\sin\frac{v_\circ t}{2R}\right|$$

$$\Rightarrow dS=\frac{2mv_\circ}{m+M}\left|\sin\frac{v}{2R}t\right|dt \Rightarrow S=\frac{2mv_\circ}{m+M}\int_0^{2\pi\frac{R}{v_\circ}}\left|\sin\frac{v}{2R}t\right|dt=4\frac{mv_\circ}{m+M}\int_0^{\pi\frac{R}{v_\circ}}\sin\left(\frac{v_\circ}{2R}t\right)dt$$

$$\Rightarrow S=\frac{4mv_\circ}{m+M}\times\frac{2R}{v_\circ}\int_0^{\pi/2}\sin\varphi\, d\varphi=\boxed{\frac{8mR}{m+M}}$$



ج)
$$\begin{cases}M\ddot{x}_1=N\cos\varphi-\mu|N|\sin\varphi\\ M\ddot{y}_1=N\sin\varphi+\mu|N|\cos\varphi\end{cases}$$

$$\begin{cases}m\ddot{x}_2=-N\cos\varphi+\mu|N|\sin\varphi\\ m\ddot{y}_2=-N\sin\varphi-\mu|N|\cos\varphi\end{cases}$$

پایستگی تکانه برقرار است

$$\Rightarrow \begin{pmatrix}\dot{x}_1\\ \dot{y}_1\end{pmatrix}=\begin{pmatrix}R\sin\varphi\,\dot{\varphi}\\ -R\cos\varphi\dot{\varphi}+v_\circ\end{pmatrix}\frac{m}{m+M} \quad , \quad \begin{pmatrix}\dot{x}_2\\ \dot{y}_2\end{pmatrix}=\begin{pmatrix}-\frac{M}{m+M}R\sin\varphi\,\dot{\varphi}\\ \frac{M}{m+M}R\cos\varphi\,\dot{\varphi}+\frac{m}{m+M}v_\circ\end{pmatrix}$$

(همانند بخش آ

$$\Rightarrow \begin{pmatrix}\ddot{x}_1\\ \ddot{y}_1\end{pmatrix}=\frac{mR\dot{\varphi}^2}{m+M}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}+\frac{mR\ddot{\varphi}}{m+M}\begin{pmatrix}\sin\varphi\\ -\cos\varphi\end{pmatrix}$$

$$\Rightarrow \begin{cases}\ddot{x}_1\cos\varphi+\ddot{y}_1\sin\varphi=\frac{mR\dot{\varphi}^2}{m+M}=\frac{N}{M} \Rightarrow N=\frac{mMR}{m+M}\dot{\varphi}^2\\ \ddot{y}_1\cos\varphi-\ddot{x}_1\sin\varphi=-\frac{mR\ddot{\varphi}}{m+M}=\mu\left|\frac{N}{M}\right| \Rightarrow |N|=\frac{-1}{\mu}\cdot\frac{mMR}{m+M}\ddot{\varphi}=\frac{mMR}{m+M}|\dot{\varphi}^2|\end{cases}$$

$$\Rightarrow \ddot{\varphi}=-\mu|\dot{\varphi}^2|=-\mu\dot{\varphi}^2=\frac{d\dot{\varphi}^2}{2d\varphi} \Rightarrow \frac{d\dot{\varphi}^2}{\dot{\varphi}^2}=-2\mu d\varphi \Rightarrow \dot{\varphi}^2=\dot{\varphi}_0^2e^{-2\mu\varphi}$$

$$\Rightarrow \dot{\varphi}=\frac{v_\circ}{R}e^{-\mu\varphi}=\frac{d\varphi}{dt} \Rightarrow \frac{v_\circ}{R}dt=e^{\mu\varphi}d\varphi \Rightarrow \frac{v_\circ}{R}t=\frac{e^{\mu\varphi}-1}{\mu}$$

$$\Rightarrow \boxed{\varphi=\frac{1}{\mu}\ln\left[1+\frac{\mu v_\circ}{R}t\right]} \qquad \Rightarrow \dot{\varphi}=\frac{v_\circ/R}{1+\frac{\mu v_\circ}{R}t}$$

ج ۲) $N=\frac{mMR}{m+M}\dot{\varphi}^2=$ [illegible] $=\boxed{\frac{mM}{m+M}\cdot\frac{(v_\circ^2/R)}{\left[1+\mu v_\circ/R\,t\right]^2}}$

ζ) $\varphi = \frac{1}{\mu}\ln\left[1+\mu\frac{v_0}{R}t\right] = \frac{1}{\mu}\left[\mu\frac{v_0}{R}t - \frac{\mu^2 v_0^2}{2R^2}t^2\right] = \boxed{\frac{v_0}{R}t\left(1-\frac{\mu v_0}{2R}t\right)}$

$N = \frac{mM}{m+M}\cdot\frac{(v_0^2/R)}{\left[1+\mu\frac{v_0}{R}t\right]^2} \approx \boxed{\frac{mM}{m+M}\left(\frac{v_0^2}{R}\right)\left(1-2\mu\frac{v_0}{R}t\right)}$

مسئله (7) _ فاینال دوم :

الف)

$$\begin{cases} t_1 = \dfrac{1}{\sqrt{1-V^2/c^2}}\left(t - \dfrac{vx}{c^2}\right) \\ x_1 = \dfrac{1}{\sqrt{1-V^2/c^2}}(x - vt) \\ y_1 = y \end{cases}$$

$$\boxed{\begin{cases} t_2 = \dfrac{1}{\sqrt{1-w^2/c^2}}\left(t_1 - \dfrac{w y_1}{c^2}\right) = \dfrac{1}{\sqrt{1-w^2/c^2}\sqrt{1-V^2/c^2}}\left(t - \dfrac{vx}{c^2}\right) - \dfrac{wy/c^2}{\sqrt{1-w^2/c^2}} \\ x_2 = x_1 = \text{———} = \dfrac{1}{\sqrt{1-V^2/c^2}}(x - vt) \\ y_2 = \dfrac{1}{\sqrt{1-w^2/c^2}}(y_1 - w t_1) = \dfrac{y}{\sqrt{1-w^2/c^2}} - \dfrac{w}{\sqrt{1-w^2/c^2}\sqrt{1-V^2/c^2}}\left(t - \dfrac{vx}{c^2}\right) \end{cases}}$$

ب)

$$\begin{cases} t_1' = \dfrac{1}{\sqrt{1-w^2/c^2}}\left(t - \dfrac{wy}{c^2}\right) \\ x_1' = x \\ y_1' = \dfrac{1}{\sqrt{1-w^2/c^2}}(y - wt) \end{cases}$$

$$\boxed{\begin{cases} t_2' = \dfrac{1}{\sqrt{1-V^2/c^2}}\left(t_1' - \dfrac{v x_1'}{c^2}\right) = \dfrac{1}{\sqrt{1-w^2/c^2}\sqrt{1-V^2/c^2}}\left(t - \dfrac{wy}{c^2}\right) - \dfrac{vx/c^2}{\sqrt{1-V^2/c^2}} \\ x_2' = \dfrac{1}{\sqrt{1-V^2/c^2}}(x_1' - v t_1') = \dfrac{x}{\sqrt{1-V^2/c^2}} - \dfrac{v}{\sqrt{1-w^2/c^2}\sqrt{1-V^2/c^2}}\left(t - \dfrac{wy}{c^2}\right) \\ y_2' = y_1' = \text{———} = \dfrac{1}{\sqrt{1-w^2/c^2}}(y - wt) \end{cases}}$$

ج)

تا مرتبه دوم

$$\begin{cases} t_2' - t_2 = \dfrac{\frac{v}{c}x - \frac{w}{c}y}{\sqrt{1-w^2/c^2}\ \sqrt{1-V^2/c^2}}\cdot\dfrac{1}{c} + \dfrac{\frac{w}{c}y}{c\sqrt{1-w^2/c^2}} - \dfrac{1}{c}\cdot\dfrac{\frac{v}{c}x}{\sqrt{1-v^2/c^2}} = 0 \\ x_2' - x_2 = \dfrac{vt}{\sqrt{1-v^2/c^2}}\left(1 - \dfrac{1}{\sqrt{1-w^2/c^2}}\right) + \dfrac{\frac{v}{c}\cdot\frac{w}{c}}{\sqrt{1-w^2/c^2}\sqrt{1-V^2/c^2}}\,y = \frac{v}{c}\cdot\frac{w}{c}\,y \\ y_2' - y_2 = \dfrac{wt}{\sqrt{1-w^2/c^2}}\left(\dfrac{1}{\sqrt{1-V^2/c^2}} - 1\right) - \dfrac{\frac{w}{c}\cdot\frac{v}{c}}{\sqrt{1-w^2/c^2}\sqrt{1-v^2/c^2}}\,x = -\frac{v}{c}\cdot\frac{w}{c}\,x \end{cases}$$

$$\Rightarrow \begin{cases} t_2' = t_2 \\ x_2' = x_2 + \frac{v}{c}\cdot\frac{w}{c}\,y_2 \\ y_2' = y_2 - \frac{v}{c}\cdot\frac{w}{c}\,x_2 \end{cases} \qquad \varepsilon = \frac{v}{c}\cdot\frac{w}{c} \Rightarrow \begin{cases} \sin\varepsilon = \frac{v}{c}\cdot\frac{w}{c} \\ \cos\varepsilon = 1 \end{cases} \Rightarrow \begin{pmatrix} t_2' \\ x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{pmatrix}\begin{pmatrix} t_2 \\ x_2 \\ y_2 \end{pmatrix}$$

{ تا تقریب ۰ $y = y_2$ ، $x = x_2$ }

تبدیل دوران حول z به اندازه $(-\varepsilon)$

د) $\begin{cases} t_1 = t \\ x_1 = x\cos\theta + y\sin\theta \\ Y_1 = -x\sin\theta + y\cos\theta \end{cases}$,

$$\boxed{\begin{cases} t_2 = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(t_1 - \dfrac{v x_1}{c^2}\right) = \dfrac{t}{\sqrt{1-v^2/c^2}} - \dfrac{1}{c}\cdot\dfrac{v}{c}\cdot\dfrac{x\cos\theta + y\sin\theta}{\sqrt{1-v^2/c^2}} \\ x_2 = \dfrac{1}{\sqrt{1-v^2/c^2}}(x_1 - v t_1) = \dfrac{x\cos\theta + y\sin\theta}{\sqrt{1-v^2/c^2}} - \dfrac{vt}{\sqrt{1-v^2/c^2}} \\ Y_2 = Y_1 = \text{------} = -x\sin\theta + y\cos\theta \end{cases}}$$

ه)

$$\begin{cases} t_1' = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(t - \dfrac{vx}{c^2}\right) \\ x_1' = \dfrac{1}{\sqrt{1-v^2/c^2}}(x - vt) \\ Y_1' = Y \end{cases} ,$$

$$\boxed{\begin{cases} t_2' = t_2 = \text{------} = \dfrac{t}{\sqrt{1-v^2/c^2}} - \dfrac{1}{c}\cdot\dfrac{v}{c}\,\dfrac{x}{\sqrt{1-v^2/c^2}} \\ x_2' = x_1'\cos\theta + y_1'\sin\theta = \dfrac{(x-vt)\cos\theta}{\sqrt{1-v^2/c^2}} + y\sin\theta \\ Y_2' = -x_1'\sin\theta + y_1'\cos\theta = \dfrac{-(x-vt)\sin\theta}{\sqrt{1-v^2/c^2}} + y\cos\theta \end{cases}}$$

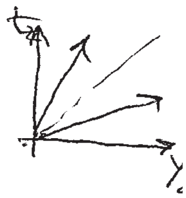
و)

$$\begin{cases} t_2' - t_2 = \dfrac{1}{c}\cdot\dfrac{v}{c}\cdot\dfrac{\overbrace{x(\cos\theta - 1)}^{\text{مرتبه دو}} + y\sin\theta}{\sqrt{1-v^2/c^2}} = \boxed{\dfrac{y}{c}\cdot\dfrac{v}{c}\theta} \\ x_2' - x_2 = y\sin\theta\left(\overbrace{\dfrac{-1}{\sqrt{1-v^2/c^2}} + 1}^{\text{مرتبه دوم}}\right) + \dfrac{vt}{\sqrt{1-v^2/c^2}}\overbrace{(1-\cos\theta)}^{\text{مرتبه دوم}} = \boxed{0} \\ y_2' - y_2 = x\sin\theta\left(\overbrace{1 - \dfrac{1}{\sqrt{1-v^2/c^2}}}^{\text{مرتبه دوم}}\right) + \dfrac{vt\sin\theta}{\sqrt{1-v^2/c^2}} = \boxed{ct\cdot\dfrac{v}{c}\theta} \end{cases}$$

$$\Rightarrow \begin{cases} ct_2' = ct_2 + \theta\dfrac{v}{c}\,y_2 \\ x_2' = x_2 \\ y_2' = y_2 + \theta\, v/c\; ct_2 \end{cases} \Rightarrow \begin{pmatrix} ct_2' \\ x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} \cos\varepsilon & 0 & \sin\varepsilon \\ 0 & 1 & 0 \\ \sin\varepsilon & 0 & \cos\varepsilon \end{pmatrix}\begin{pmatrix} ct_2 \\ x_2 \\ Y_2 \end{pmatrix} \quad \begin{cases} \varepsilon := \theta\, v/c \\ \sin\varepsilon = \theta\, v/c \\ \cos\varepsilon = 1 \end{cases}$$

(تا مرتبه صفرم $t = t_2$, $y = y_2$)

این تبدیل، غیر لورنتسی با سرعت θv است



مسئله 8 فاینال ۹۲ :

الف) $E=\gamma mc^2 \Rightarrow \left(\frac{mc^2}{E}\right)^2 = 1-\frac{v^2}{c^2} \Rightarrow v = c\sqrt{1-\left(\frac{mc^2}{E}\right)^2}$

$\Rightarrow v_A = c\sqrt{1-\left(\frac{m_A c^2}{E_A}\right)^2} \quad , \quad v_B = c\sqrt{1-\left(\frac{m_B c^2}{E_B}\right)^2}$

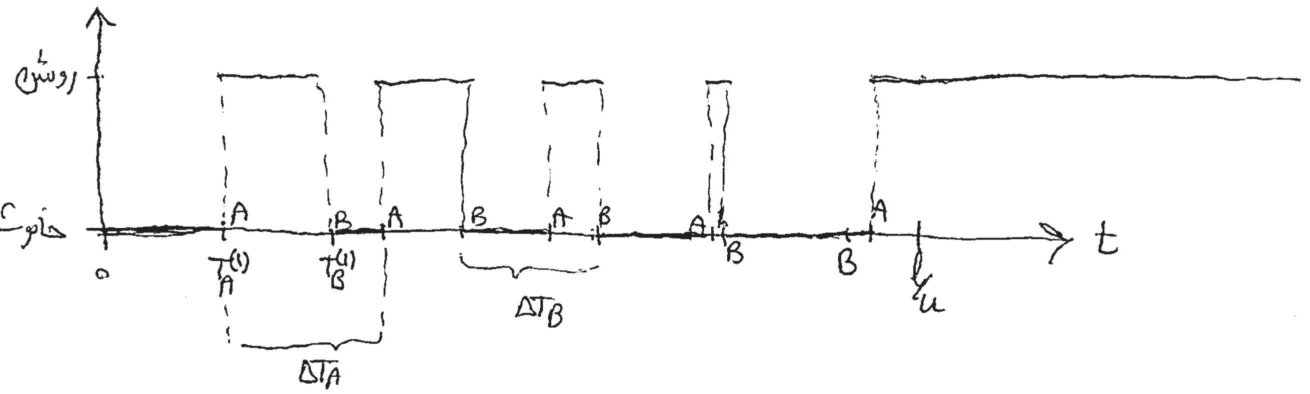
زمان رسیدن اولین ذره B ; $T_B^{(1)} = T + \frac{l-uT}{u+v_B} = \frac{l+v_BT}{u+v_B}$, $\Delta T_B = \frac{v_BT}{u+v_B}$ زمان رسیدن دو ذره متوالی B

زمان رسیدن n امین ذره B $\Rightarrow T_B^{(n)} = T_B^{(1)} + (n-1)\Delta T_B = \boxed{\frac{l+nv_BT}{u+v_B}}$

ب) $\cancel{T_A^{(1)} = T + \frac{uT}{u+v_B} = \frac{v_B+2u}{u+v_B}T \;,\; \Delta T_A = v_A}$

$T_A^{(1)} = T + \frac{uT}{v_A-u} = \frac{v_AT}{v_A-u}$, $\Delta T_A = \frac{v_AT}{v_A-u}$

$\Rightarrow T_A^{(n)} = T_A^{(1)} + (n-1)\Delta T_A = \frac{nv_AT}{v_A-u}$

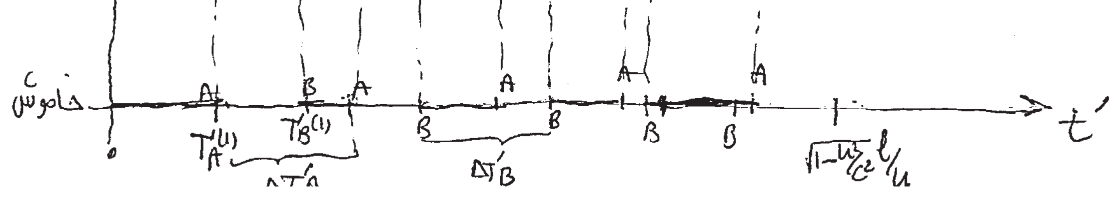


ج) $x'=0 \Rightarrow t = \gamma\left(t' + \frac{ux'}{c^2}\right) = \gamma t' \Rightarrow t' = \frac{t}{\gamma} = \sqrt{1-\frac{v^2}{c^2}}\, t$

$\Rightarrow \boxed{T_B'^{(n)} = \sqrt{1-\frac{v^2}{c^2}}\,\frac{l+nv_BT}{u+v_B}}$

د) $T_A'^{(n)} = \frac{1}{\gamma}T_A^{(n)} = \sqrt{1-\frac{v^2}{c^2}}\,\frac{nv_AT}{v_A-u}$

همان نمودار کشش ب که $t'=\frac{t}{\gamma}$



ت)

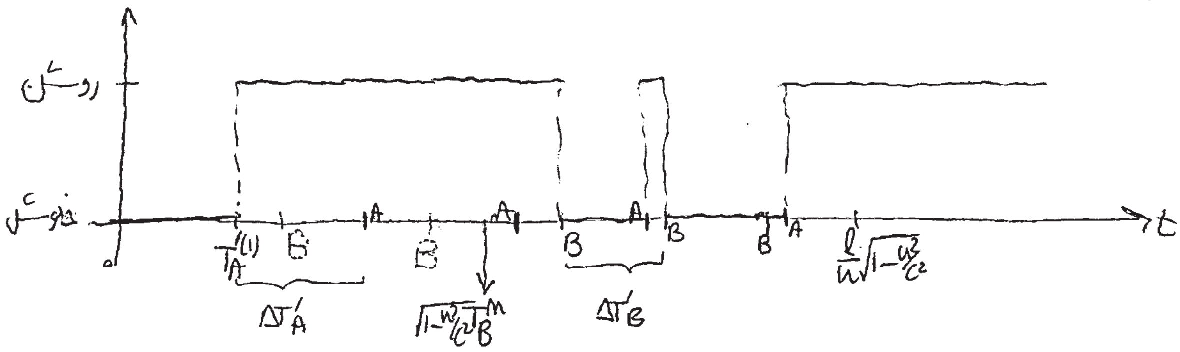
از دید ناظر O' : $\tau_A^O = \gamma_{v_A}\tau_A = \frac{E_A}{m_Ac^2}\tau_A$, $\tau_B^O = \frac{E_B}{m_Bc^2}\tau_B$

$$\Rightarrow x_A^{max} := v_A\tau_A^O = c\sqrt{1-\left(\frac{m_Ac^2}{E_A}\right)^2}\cdot\frac{E_A}{m_Ac^2}\tau_A = c\sqrt{\left(\frac{E_A}{m_Ac^2}\right)^2-1}\;\tau_A \quad , \quad x_B^{max} := v_B\tau_B^O$$

$T_A^M := \frac{x_A^{max}}{u}$, $T_B^M := \frac{l-x_B^{max}}{u}$ (در واقع نمودار را برای O رسم کرده ولی t را $\frac{1}{\gamma}$ برابر می‌کنیم!)

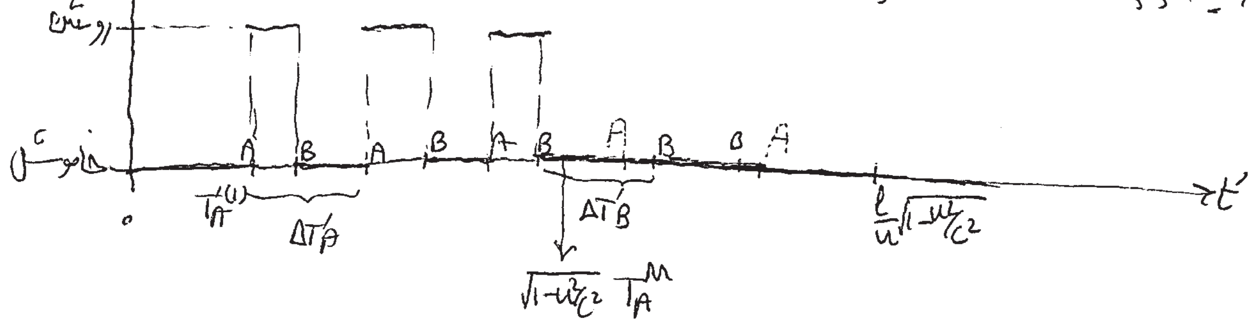
I) $x_A^{max}, x_B^{max} > l \Rightarrow$ همان نمودار بخش ت

II) $x_A^{max} > l$, $x_B^{max} < l$ {مکان A و B ها همان مکان آنها در نمودار (ت) است تنها از B $\frac{1}{\gamma}T_B^M$ به بعد وجود ندارد}

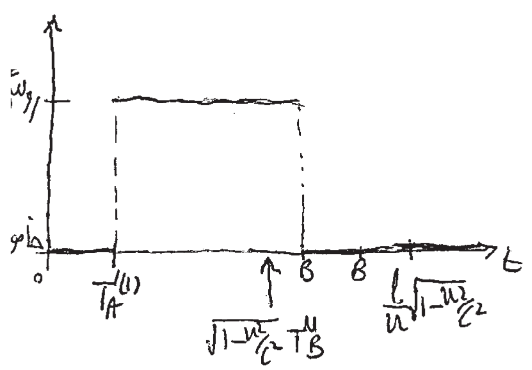


III) $x_A^{max} < l$, $x_B^{max} > l$ {مکان B ها همان نمودار "ت" و A ها همان نمودار "ت" است که از $\frac{1}{\gamma}T_A^M$ به بعد وجود ندارد}

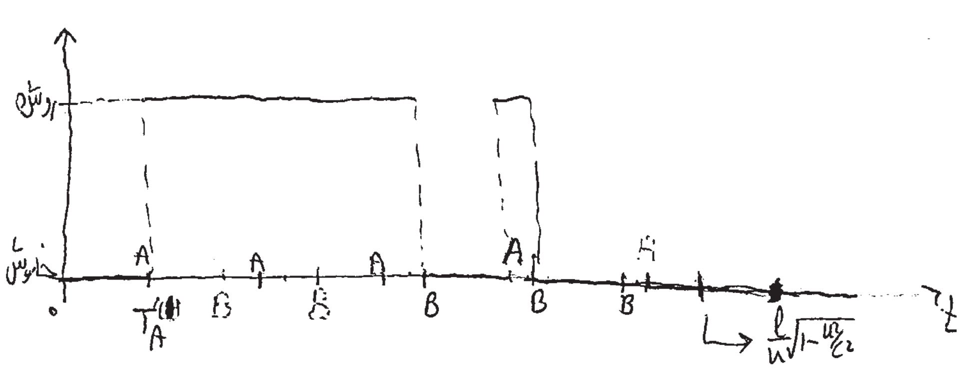
(البته اینکه چراغ تا کی خاموش باشد به پارامترها بستگی دارد)



IV) $x_A^{max} + x_B^{max} < l$



V) x_A^{max} و $x_B^{max} < l$, $x_A^{max} + x_B^{max} > l$



مسئله 9 ناظر دوم :

آ)

$$\begin{cases} K'_{ox} = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(K_o\cos\theta_o - \dfrac{v}{c}\cdot\dfrac{\omega_o}{c}\right) \\[2ex] K'_{oy} = K_o\sin\theta_o \qquad \longleftarrow K' \\[2ex] \dfrac{\omega'_o}{c} = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(\dfrac{\omega_o}{c} - \dfrac{v}{c}K_o\cos\theta_o\right) \end{cases}$$

$$\xrightarrow{\text{بعد بازتاب}} \begin{cases} K'_x = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(\dfrac{v}{c}\cdot\dfrac{\omega_o}{c} - K_o\cos\theta_o\right) \\[2ex] K'_y = K_o\sin\theta_o \qquad \longleftarrow K' \\[2ex] \dfrac{\omega'}{c} = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(\dfrac{\omega_o}{c} - \dfrac{v}{c}K_o\cos\theta_o\right) \end{cases}$$

$$K \Longrightarrow \begin{cases} K_x = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(K'_x + \dfrac{v}{c}\cdot\dfrac{\omega'}{c}\right) = \dfrac{1}{1-v^2/c^2}\left(\dfrac{v}{c}\cdot\dfrac{\omega_o}{c} - K_o\cos\theta_o + \dfrac{\omega_o}{c}\dfrac{v}{c} - \dfrac{v^2}{c^2}K_o\cos\theta_o\right) \\[2ex] K_y = K'_y = K_o\sin\theta_o \\[2ex] \dfrac{\omega}{c} = \dfrac{1}{\sqrt{1-v^2/c^2}}\left(\dfrac{\omega'}{c} + \dfrac{v}{c}K'_x\right) = \dfrac{1}{1-v^2/c^2}\left(\dfrac{\omega_o}{c} - \dfrac{v}{c}K_o\cos\theta_o + \dfrac{v^2}{c^2}\dfrac{\omega_o}{c} - \dfrac{v}{c}K_o\cos\theta_o\right) \end{cases}$$

$$\Longrightarrow \begin{cases} K_x = \dfrac{2v/c}{1-v^2/c^2}\cdot\dfrac{\omega_o}{c} - \dfrac{1+v^2/c^2}{1-v^2/c^2}K_o\cos\theta_o \\[2ex] K_y = K_o\sin\theta_o \\[2ex] \dfrac{\omega}{c} = \dfrac{1+v^2/c^2}{1-v^2/c^2}\dfrac{\omega_o}{c} - \dfrac{2v/c}{1-v^2/c^2}K_o\cos\theta_o \end{cases} \xrightarrow{K_o=\frac{\omega_o}{c}} \boxed{\omega = \omega_o\left[\dfrac{1+v^2/c^2 - 2v/c\cos\theta_o}{1-v^2/c^2}\right]}$$

$$\tan\theta = \frac{K_y}{-K_x} = \frac{\omega_o\sin\theta_o}{-\dfrac{2v/c}{1-v^2/c^2}\omega_o + \dfrac{1+v^2/c^2}{1-v^2/c^2}\omega_o\cos\theta_o} = \boxed{\frac{1-v^2/c}{1+v^2/c^2-2\dfrac{v}{c\cos\theta_o}}\tan\theta_o}$$

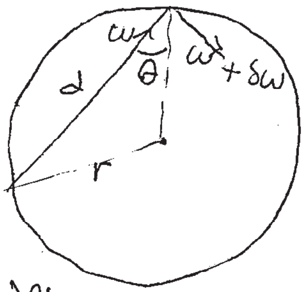
ب)

$$\omega = \omega_o\frac{1+\beta^2-2\beta\cos\theta_o}{1-\beta^2} = \boxed{\omega_o(1-2\beta\cos\theta_o)}$$

$$\tan\theta = \frac{1-\beta^2}{1+\beta^2-\dfrac{2\beta}{\cos\theta_o}}\tan\theta_o = \frac{\tan\theta_o}{1-\dfrac{2\beta}{\cos\theta_o}} = \left(1+\frac{2\beta}{\cos\theta_o}\right)\tan\theta_o = \tan(\theta_o+\delta\theta) = \tan\theta_o + \delta\theta\sec^2\theta_o$$

$$\Longrightarrow \delta\theta = 2\beta\sin\theta_o \quad \Longrightarrow \boxed{\theta = \theta_o + 2\beta\sin\theta_o}$$

الف) $\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \Rightarrow \frac{d\lambda}{\lambda} = \frac{-d\omega}{\omega}$



$d = 2r\cos\theta \Rightarrow \delta T = \frac{d}{c} = \frac{2r}{c}\cos\theta$

$\delta\omega = -2\omega\beta\cos\theta \Rightarrow \frac{-d\omega}{\omega} = 2\beta\cos\theta = \frac{d\lambda}{\lambda}$

$\Rightarrow d\lambda = 2\lambda\beta\cos\theta \Rightarrow \frac{d\lambda}{dt} = \frac{2\lambda\beta\cos\theta}{\frac{2r}{c}\cos\theta} = \frac{\lambda\beta c}{r} = \frac{\lambda v}{r}$

$\Rightarrow \boxed{\frac{d\lambda}{dt} = \frac{\lambda}{r}\frac{dr}{dt}}$

ب) $dU = d(uV) = -PdV = -\frac{1}{3}u\,dV \Rightarrow \frac{d(uV)}{uV} = -\frac{dV}{3V}$

$\Rightarrow \ln uV = C - \frac{1}{3}\ln V \Rightarrow uV = cV^{-1/3} \Rightarrow u^3V^4 = cte \Rightarrow uV^{4/3} = cte$

$\Rightarrow ur^4 = cte$

$\frac{d\lambda}{\lambda} = \frac{dr}{r} \Rightarrow \frac{\lambda}{r} = cte$

$\Rightarrow u\lambda^4 = cte \quad (1)$

در حالت کلی: $\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P = T\left(\frac{\partial P}{\partial T}\right)_V - P = \left(\frac{\partial (uV)}{\partial V}\right)_T \xrightarrow[\frac{\partial u}{\partial V}=0]{} u$

$\Rightarrow \frac{T}{3}\left(\frac{\partial u}{\partial T}\right)_V - \frac{u}{3} = u \Rightarrow T\frac{du}{dT} = 4u \Rightarrow \frac{du}{u} = 4\frac{dT}{T} \Rightarrow \frac{u}{T^4} = cte \quad (2)$

$(1), (2) \Rightarrow \lambda T = const \ (3) \quad \Rightarrow rT = cte$

$d(U_\lambda d\lambda) = -pdV = d(u_\lambda V d\lambda) \Rightarrow d(u_\lambda d\lambda)V + u_\lambda d\lambda dV = -\frac{1}{3}u_\lambda d\lambda dV$

$\Rightarrow \frac{d(u_\lambda d\lambda)}{u_\lambda d\lambda} = -\frac{4dV}{3V} = -4\frac{dr}{r} \Rightarrow u_\lambda d\lambda r^4 = cte$

برای [illegible]: $\frac{\lambda}{r} = \frac{\lambda_0}{r_0} \Rightarrow d\lambda = \frac{r}{r_0}d\lambda_0 \Rightarrow d\lambda \sim r$

$\Rightarrow u_\lambda r^5 = cte$

$\Rightarrow u_\lambda T^{-5} = const = f(\lambda T) \Rightarrow \boxed{u_\lambda = T^5 f(\lambda T)}$