Exam 1

Day 1

$\boxed{1}$ Find all positive integers $n \geq 2$ such that for all integers i, j that $0 \leq i, j \leq n$, $i + j$ and $\binom{n}{i} + \binom{n}{j}$ have same parity.

Proposed by Mr.Etesami

$\boxed{2}$ Consider ω is circumcircle of an acute triangle ABC. D is midpoint of arc BAC and I is incenter of triangle ABC. Let DI intersect BC in E and ω for second time in F. Let P be a point on line AF such that PE is parallel to AI. Prove that PE is bisector of angle BPC.

Proposed by Mr.Etesami

$\boxed{3}$ Let n be a positive integer. Let S be a subset of points on the plane with these conditions:

$i)$ There does not exist n lines in the plane such that every element of S be on at least one of them.

$ii)$ for all $X \in S$ there exists n lines in the plane such that every element of $S - X$ be on at least one of them.

Find maximum of $\mid S \mid$.

Proposed by Erfan Salavati

Day 2

$\boxed{1}$ Consider $m + 1$ horizontal and $n + 1$ vertical lines $(m, n \geq 4)$ in the plane forming an $m \times n$ table. Cosider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m - 1)(n - 1)$ interior vertices (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that

$$A = B - C + m + n - 1.$$

Proposed by Ali Khezeli

$\boxed{2}$ The function $f : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}^{\geq 0}$ satisfies the following properties for all $a, b \in \mathbb{R}^{\geq 0}$:

a) $f(a) = 0 \Leftrightarrow a = 0$

b) $f(ab) = f(a)f(b)$

c) $f(a + b) \leq 2\max\{f(a), f(b)\}$.

Prove that for all $a, b \in \mathbb{R}^{\geq 0}$ we have $f(a + b) \leq f(a) + f(b)$.

Proposed by Masoud Shafaei

$\boxed{3}$ The pentagon $ABCDE$ is inscirbed in a circle w. Suppose that w_a, w_b, w_c, w_d, w_e are reflections of w with respect to sides AB, BC, CD, DE, EA respectively. Let A' be the second intersection point of w_a, w_e and define B', C', D', E' similarly. Prove that

$$2 \leq \frac{S_{A'B'C'D'E'}}{S_{ABCDE}} \leq 3,$$

where S_X denotes the surface of figure X.

Proposed by Morteza Saghafian, Ali khezeli

Exam 2

Day 1

1. Is it possible to put $\binom{n}{2}$ consecutive natural numbers on the edges of a complete graph with n vertices in a way that for every path (or cycle) of length 3 where the numbers a, b and c are written on its edges (edge b is between edges c and a), b is divisible by the greatest common divisor of the numbers a and c?

Proposed by Morteza Saghafian

2. Let $g(x)$ be a polynomial of degree at least 2 with all of its coefficients positive. Find all functions $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$f(f(x) + g(x) + 2y) = f(x) + g(x) + 2f(y) \quad \forall x, y \in \mathbb{R}^+.$$

Proposed by Mohammad Jafari

3. Suppose $ABCD$ is a parallelogram. Consider circles w_1 and w_2 such that w_1 is tangent to segments AB and AD and w_2 is tangent to segments BC and CD. Suppose that there exists a circle which is tangent to lines AD and DC and externally tangent to w_1 and w_2. Prove that there exists a circle which is tangent to lines AB and BC and also externally tangent to circles w_1 and w_2.

Proposed by Ali Khezeli

Day 2

1. For positive reals a, b and c with $ab + bc + ca = 1$, show that

$$\sqrt{3}(\sqrt{a} + \sqrt{b} + \sqrt{c}) \leq \frac{a\sqrt{a}}{bc} + \frac{b\sqrt{b}}{ca} + \frac{c\sqrt{c}}{ab}.$$

Proposed by Morteza Saghafian

2. Points A and B are on a circle ω with center O such that $\frac{\pi}{3} < \angle AOB < \frac{2\pi}{3}$. Let C be the circumcenter of the triangle AOB. Let l be a line passing through C such that the angle between l and the segment OC is $\frac{\pi}{3}$. l cuts tangents in A and B to ω in M and N respectively. Suppose circumcircles of triangles CAM and CBN, cut ω again in Q and R respectively and theirselves in P (other than C). Prove that $OP \perp QR$.

Proposed by Mehdi E'tesami Fard, Ali Khezeli

$\boxed{3}$ We call a subset B of natural numbers *loyal* if there exists natural numbers $i \leq j$ such that $B = \{i, i+1, ..., j\}$. Let Q be the set of all *loyal* sets.

Now for every subset $A = \{a_1 < a_2 < ... < a_k\}$ of $\{1, 2, ..., n\}$ we set

$f(A) = \max_{1 \leq i \leq k-1} a_{i+1} - a_i$, and $g(A) = \max_{B \subseteq A, B \in Q} |B|$.

And we define

$F(n) = \sum_{A \subseteq \{1,2,...,n\}} f(A)$ and $G(n) = \sum_{A \subseteq \{1,2,...,n\}} g(A)$.

Prove that there exists $m \in \mathbb{N}$ such that for each natural number $n > m$, we have $F(n) > G(n)$.

(By $|A|$ we mean the number of elements of A and if $|A| \leq 1$, we define $f(A)$ to be zero).

Proposed by Javad Abedi

Exam 3

Day 1

$\boxed{1}$ Consider a regular 2^k-gon with center O and label its sides clockwise by $l_1, l_2, ..., l_{2^k}$. Reflect O with respect to l_1, then reflect the resulting point with respect to l_2 and do this process until the last side. Prove that the distance between the final point and O is less than the perimeter of the 2^k-gon.

Proposed by Hesam Rajabzade

$\boxed{2}$ Do there exist 2000 real numbers (not necessarily distinct) such that all of them are not zero and if we put any group containing 1000 of them as the roots of a monic polynomial of degree 1000, the coefficients of the resulting polynomial (except the coefficient of x^{1000}) be a permutation of the 1000 remaining numbers?

Proposed by Morteza Saghafian

$\boxed{3}$ Find all integer numbers x and y such that:

$$(y^3 + xy - 1)(x^2 + x - y) = (x^3 - xy + 1)(y^2 + x - y).$$

Proposed by Mahyar Sefidgaran

Day 2

$\boxed{1}$ Suppose p is an odd prime number. We call the polynomial $f(x) = \sum_{j=0}^{n} a_j x^j$ with integer coefficients i-remainder if $\sum_{p-1|j, j>0} a_j \equiv i \pmod{p}$. Prove that the set $\{f(0), f(1), ..., f(p-1)\}$ is a complete residue system modulo p if and only if polynomials $f(x), (f(x))^2, ..., (f(x))^{p-2}$ are 0-remainder and the polynomial $(f(x))^{p-1}$ is 1-remainder.

Proposed by Yahya Motevassel

$\boxed{2}$ Let n be a natural number. Suppose A and B are two sets, each containing n points in the plane, such that no three points of a set are collinear. Let $T(A)$ be the number of broken lines, each containing $n-1$ segments, and such that it doesn't intersect itself and its vertices are points of A. Define $T(B)$ similarly. If the points of B are vertices of a convex n-gon (are in *convex position*), but the points of A are not, prove that $T(B) < T(A)$.

Proposed by Ali Khezeli

3 Let O be the circumcenter of the acute triangle ABC. Suppose points A', B' and C' are on sides BC, CA and AB such that circumcircles of triangles $AB'C', BC'A'$ and $CA'B'$ pass through O. Let l_a be the radical axis of the circle with center B' and radius $B'C$ and circle with center C' and radius $C'B$. Define l_b and l_c similarly. Prove that lines l_a, l_b and l_c form a triangle such that it's orthocenter coincides with orthocenter of triangle ABC.

Proposed by Mehdi E'tesami Fard