All-Russian Olympiad 2013

– Grade level 9

Day 1

1 Given three distinct real numbers a, b, and c, show that at least two of the three following equations

$$(x-a)(x-b)=x-c$$
$$(x-c)(x-b)=x-a$$
$$(x-c)(x-a)=x-b$$

have real solutions.

2 Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocentre of triangle ADE is the midpoint of BC.

3 100 distinct natural numbers $a_1, a_2, a_3, \ldots, a_{100}$ are written on the board. Then, under each number a_i, someone wrote a number b_i, such that b_i is the sum of a_i and the greatest common factor of the other 99 numbers. What is the least possible number of distinct natural numbers that can be among $b_1, b_2, b_3, \ldots, b_{100}$?

4 N lines lie on a plane, no two of which are parallel and no three of which are concurrent. Prove that there exists a non-self-intersecting broken line $A_0A_1A_2A_3...A_N$ with N parts, such that on each of the N lines lies exactly one of the N segments of the line.

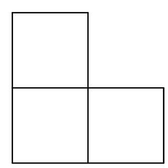
Day 2

1 $2n$ real numbers with a positive sum are aligned in a circle. For each of the numbers, we can see there are two sets of n numbers such that this number is on the end. Prove that at least one of the numbers has a positive sum for both of these two sets.

2 Peter and Basil together thought of ten quadratic trinomials. Then, Basil began calling consecutive natural numbers starting with some natural number. After each called number, Peter chose one of the ten polynomials at random and plugged in the called number. The results were recorded on the board. They eventually form a sequence. After they finished, their sequence was arithmetic. What is the greatest number of numbers that Basil could have called out?

3 Squares $CAKL$ and $CBMN$ are constructed on the sides of acute-angled triangle ABC, outside of the triangle. Line CN intersects line segment AK at X, while line CL intersects line segment BM at Y. Point P, lying inside triangle ABC, is an intersection of the circumcircles of triangles KXN and LYM. Point S is the midpoint of AB. Prove that angle $\angle ACS = \angle BCP$.

4 On a 55×55 square grid, 500 unit squares were cut out as well as 400 L-shaped pieces consisting of 3 unit squares (each piece can be oriented in any way) [refer to the figure]. Prove that at least two of the cut out pieces bordered each other before they were cut out.



– Grade level 10

Day 1

1 Given three distinct real numbers a, b, and c, show that at least two of the three following equations

$$(x-a)(x-b) = x-c$$

$$(x-c)(x-b) = x-a$$

$$(x-c)(x-a) = x-b$$

have real solutions.

2 Circle is divided into n arcs by n marked points on the circle. After that circle rotate an angle $2\pi k/n$ (for some positive integer k), marked points moved to n *new points* , dividing the circle into n *new arcs*. Prove that there is a new arc that lies entirely in the one of the old ars.
(It is believed that the endpoints of arcs belong to it.)

I. Mitrophanov

3 Find all positive integers k such that for the first k prime numbers $2, 3, \ldots, p_k$ there exist positive integers a and $n > 1$, such that $2 \cdot 3 \cdot \ldots \cdot p_k - 1 = a^n$.

V. Senderov

4 Inside the inscribed quadrilateral $ABCD$ are marked points P and Q, such that $\angle PDC + \angle PCB$, $\angle PAB + \angle PBC$, $\angle QCD + \angle QDA$ and $\angle QBA + \angle QAD$ are all equal to 90°. Prove that the line PQ has equal angles with lines AD and BC.

A. Pastor

Day 2

1 Does exist natural n, such that for any non-zero digits a and b

$$\overline{ab} \mid \overline{anb} \text{ ?}$$

(Here by $\overline{x\ldots y}$ denotes the number obtained by concatenation decimal digits x, $\ldots$, y.)

V. Senderov

2 Peter and Vasil together thought of ten 5-degree polynomials. Then, Vasil began calling consecutive natural numbers starting with some natural number. After each called number, Peter chose one of the ten polynomials at random and plugged in the called number. The results were recorded on the board. They eventually form a sequence. After they finished, their sequence was arithmetic. What is the greatest number of numbers that Vasil could have called out?

A. Golovanov

3 The incircle of triangle ABC has centre I and touches the sides BC, CA, AB at points A_1, B_1, C_1, respectively. Let I_a, I_b, I_c be excentres of triangle ABC, touching the sides BC, CA, AB respectively. The segments I_aB_1 and I_bA_1 intersect at C_2. Similarly, segments I_bC_1 and I_cB_1 intersect at A_2, and the segments I_cA_1 and I_aC_1 at B_2. Prove that I is the center of the circumcircle of the triangle $A_2B_2C_2$.

L. Emelyanov, A. Polyansky

4 A square with horizontal and vertical sides is drawn on the plane. It held several segments parallel to the sides, and there are no two segments which lie on one line or intersect at an interior point for both segments. It turned out that the segments cuts square into rectangles, and any vertical line intersecting the square and not containing segments of the partition intersects exactly k rectangles of the partition, and any horizontal line intersecting the square and not containing segments of the partition intersects exactly ℓ rectangles. How much the number of rectangles can be?

I. Bogdanov, D. Fon-Der-Flaass

– Grade level 11

Day 1

1 Let $P(x)$ and $Q(x)$ be (monic) polynomials with real coefficients (the first coefficient being equal to 1), and $\deg P(x) = \deg Q(x) = 10$. Prove that if the equation $P(x) = Q(x)$ has no real solutions, then $P(x+1) = Q(x-1)$ has a real solution.

2 The inscribed and exscribed sphere of a triangular pyramid $ABCD$ touch her face BCD at different points X and Y. Prove that the triangle AXY is obtuse triangle.

3 Find all positive k such that product of the first k odd prime numbers, reduced by 1 is exactly degree of natural number (which more than one).

4 On each of the cards written in 2013 by number, all of these 2013 numbers are different. The cards are turned down by numbers. In a single move is allowed to point out the ten cards and in return will report one of the numbers written on them (do not know what). For what most w guaranteed to be able to find w cards for which we know what numbers are written on each of them?

Day 2

1 101 distinct numbers are chosen among the integers between 0 and 1000. Prove that, among the absolute values of their pairwise differences, there are ten different numbers not exceeding 100.

2 Let a, b, c, d be positive real numbers such that $2(a+b+c+d) \geq abcd$. Prove that

$$a^2 + b^2 + c^2 + d^2 \geq abcd.$$

3 The head of the Mint wants to release 12 coins denominations (each - a natural number rubles) so that any amount from 1 to 6543 rubles could be paid without having to pass, using no more than 8 coins. Can he do it? (If the payment amount you can use a few coins of the same denomination.)

4 Let ω be the incircle of the triangle ABC and with centre I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.