All-Russian Olympiad 2014

– Grade level 9

Day 1

1 On a circle there are 99 natural numbers. If a, b are any two neighbouring numbers on the circle, then $a - b$ is equal to 1 or 2 or $\frac{a}{b} = 2$. Prove that there exists a natural number on the circle that is divisible by 3.

S. Berlov

2 Sergei chooses two different natural numbers a and b. He writes four numbers in a notebook: a, $a + 2$, b and $b + 2$. He then writes all six pairwise products of the numbers of notebook on the blackboard. Let S be the number of perfect squares on the blackboard. Find the maximum value of S.

S. Berlov

3 In a convex n-gon, several diagonals are drawn. Among these diagonals, a diagonal is called *good* if it intersects exactly one other diagonal drawn (in the interior of the n-gon). Find the maximum number of good diagonals.

4 Let M be the midpoint of the side AC of acute-angled triangle ABC with $AB > BC$. Let Ω be the circumcircle of ABC. The tangents to Ω at the points A and C meet at P, and BP and AC intersect at S. Let AD be the altitude of the triangle ABP and ω the circumcircle of the triangle CSD. Suppose ω and Ω intersect at $K \neq C$. Prove that $\angle CKM = 90°$.

V. Shmarov

Day 2

1 Define $m(n)$ to be the greatest proper natural divisor of $n \in \mathbb{N}$. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

N. Agakhanov

2 Let $ABCD$ be a trapezoid with $AB \parallel CD$ and Ω is a circle passing through A, B, C, D. Let ω be the circle passing through C, D and intersecting with CA, CB at A_1, B_1 respectively. A_2 and B_2 are the points symmetric to A_1 and B_1 respectively, with respect to the midpoints of CA and CB. Prove that the points A, B, A_2, B_2 are concyclic.

I. Bogdanov

3 In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

4 In a country of n cities, an express train runs both ways between any two cities. For any train, ticket prices either direction are equal, but for any different routes these prices are different. Prove that the traveler can select the starting city, leave it and go on, successively, $n-1$ trains, such that each fare is smaller than that of the previous fare. (A traveler can enter the same city several times.)

– Grade level 10

Day 1

1 Let a be *good* if the number of prime divisors of a is equal to 2. Do there exist 18 consecutive good natural numbers?

2 Given a function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x)^2 \le f(y)$ for all $x, y \in \mathbb{R}$, $x > y$, prove that $f(x) \in [0,1]$ for all $x \in \mathbb{R}$.

3 There are n cells with indices from 1 to n. Originally, in each cell, there is a card with the corresponding index on it. Vasya shifts the card such that in the i-th cell is now a card with the number a_i. Petya can swap any two cards with the numbers x and y, but he must pay $2|x-y|$ coins. Show that Petya can return all the cards to their original position, not paying more than $|a_1 - 1| + |a_2 - 2| + \ldots + |a_n - n|$ coins.

4 Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

M. Kungodjin

Day 2

1 Define $m(n)$ to be the greatest proper natural divisor of $n \in \mathbb{N}$. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

N. Agakhanov

2 Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.

F. Ivlev, F. Nilov

3 In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

4 Given are n pairwise intersecting convex k-gons on the plane. Any of them can be transferred to any other by a homothety with a positive coefficient. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these k-gons.

– Grade level 11

Day 1

1 Does there exist positive $a \in \mathbb{R}$, such that

$$|\cos x| + |\cos ax| > \sin x + \sin ax$$

for all $x \in \mathbb{R}$?

N. Agakhanov

2 Peter and Bob play a game on a $n \times n$ chessboard. At the beginning, all squares are white apart from one black corner square containing a rook. Players take turns to move the rook to a white square and recolour the square black. The player who can not move loses. Peter goes first. Who has a winning strategy?

3 Positive rational numbers a and b are written as decimal fractions and each consists of a minimum period of 30 digits. In the decimal representation of $a - b$, the period is at least 15. Find the minimum value of $k \in \mathbb{N}$ such that, in the decimal representation of $a + kb$, the length of period is at least 15.

A. Golovanov

4 Given a triangle ABC with $AB > BC$, Ω is circumcircle. Let M, N are lie on the sides AB, BC respectively, such that $AM = CN$. $K(.) = MN \cap AC$ and P is incenter of the triangle AMK, Q is K-excenter of the triangle CNK (opposite to K and tangents to CN). If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

M. Kungodjin

Day 2

1 Call a natural number n *good* if for any natural divisor a of n, we have that $a+1$ is also divisor of $n+1$. Find all good natural numbers.

S. Berlov

2 The sphere ω passes through the vertex S of the pyramid $SABC$ and intersects with the edges SA, SB, SC at A_1, B_1, C_1 other than S. The sphere Ω is the circumsphere of the pyramid $SABC$ and intersects with ω circumferential, lies on a plane which parallel to the plane (ABC).
Points A_2, B_2, C_2 are symmetry points of the points A_1, B_1, C_1 respect to midpoints of the edges SA, SB, SC respectively. Prove that the points A, B, C, A_2, B_2, and C_2 lie on a sphere.

3 If the polynomials $f(x)$ and $g(x)$ are written on a blackboard then we can also write down the polynomials $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ and $cf(x)$, where c is an arbitrary real constant. The polynomials $x^3 - 3x^2 + 5$ and $x^2 - 4x$ are written on the blackboard. Can we write a nonzero polynomial of form $x^n - 1$ after a finite number of steps?

4 Two players play a card game. They have a deck of n distinct cards. About any two cards from the deck know which of them has a different (in this case, if A beats B, and B beats C, then it may be that C beats A). The deck is split between players in an arbitrary manner. In each turn the players over the top card from his deck and one whose card has a card from another player takes both cards and puts them to the bottom of your deck in any order of their discretion. Prove that for any initial distribution of cards, the players can with knowing the location agree and act so that one of the players left without a card.

E. Lakschmanov