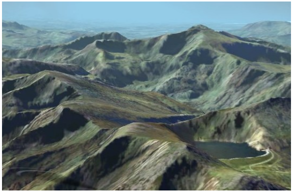
Delaunay Triangulations

Computational Geometry

Lecture 12: Delaunay Triangulations

Motivation: Terrains

- a terrain is the graph of a function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$
- we know only height values for a set of measurement points
- how can we interpolate the height at other points?
- using a triangulation



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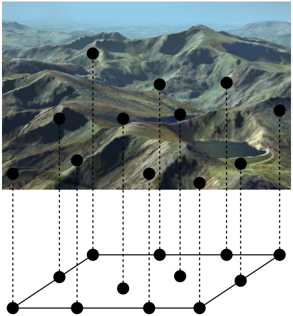

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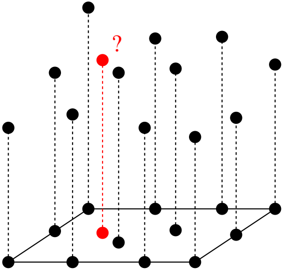

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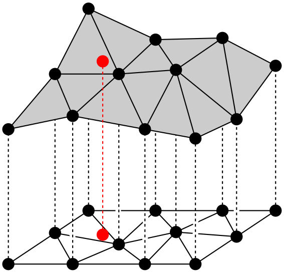

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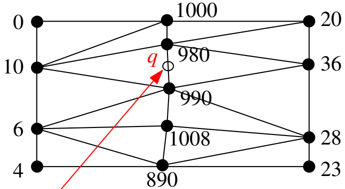
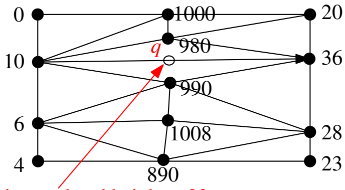
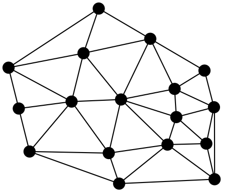
  – but which?





Triangulation

Let $P = \{p_1, \ldots, p_n\}$ be a point set. A triangulation of P is a maximal planar subdivision with vertex set P.



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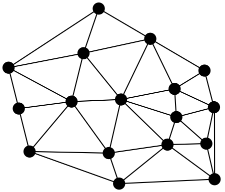
Complexity:

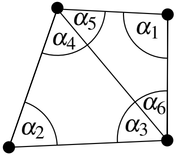
- $2n - 2 - k$ triangles
- $3n - 3 - k$ edges

where k is the number of points in P on the convex hull of P.



Angle Vector of a Triangulation

- Let $\mathcal{T}$ be a triangulation of P with m triangles and $3m$ vertices. Its angle vector is $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ where $\alpha_1, \ldots, \alpha_{3m}$ are the angles of $\mathcal{T}$ sorted by increasing value.
- Let $\mathcal{T}'$ be another triangulation of P. We define $A(\mathcal{T}) > A(\mathcal{T}')$ if $A(\mathcal{T})$ is lexicographically larger than $A(\mathcal{T}')$.
- $\mathcal{T}$ is angle optimal if $A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of P.



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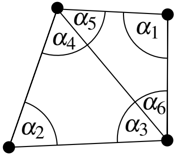

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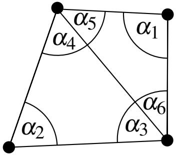

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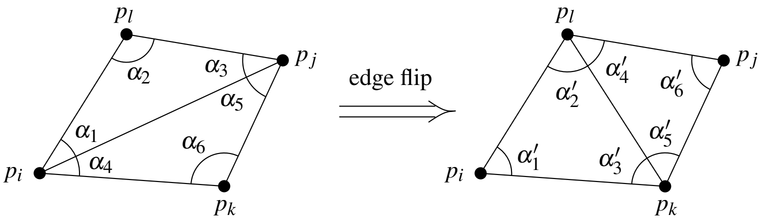
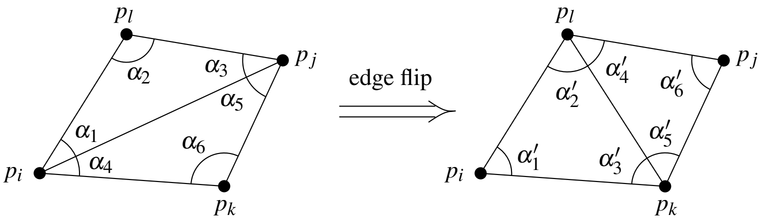
Edge Flipping



- Change in angle vector:
  $\alpha_1, \ldots, \alpha_6$ are replaced by $\alpha'_1, \ldots, \alpha'_6$.
- The edge $e = \overline{p_i p_j}$ is illegal if $\min_{1 \leq i \leq 6} \alpha_i < \min_{1 \leq i \leq 6} \alpha'_i$.
- Flipping an illegal edge increases the angle vector.

Edge Flipping



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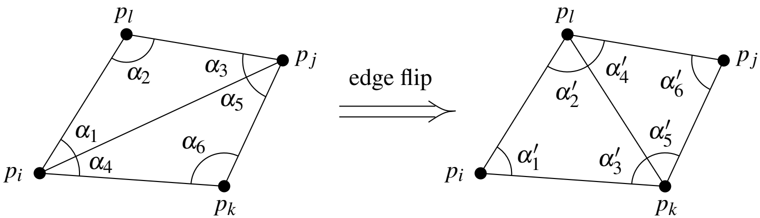

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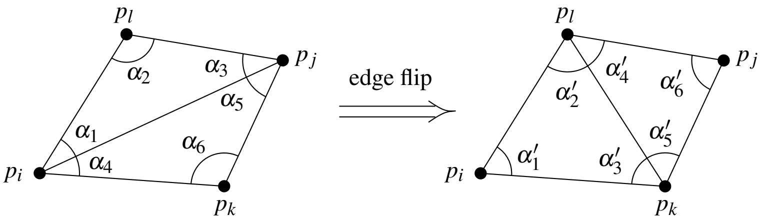

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Characterisation of Illegal Edges

How do we determine if an edge is illegal?

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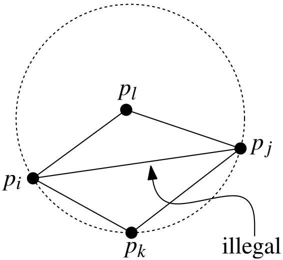
Lemma: The edge $\overline{p_i p_j}$ is illegal if and only if p_l lies in the interior of the circle C.

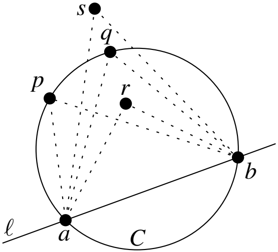


Thales Theorem

Theorem: Let C be a circle, ℓ a line intersecting C in points a and b, and p, q, r, s points lying on the same side of ℓ. Suppose that p, q lie on C, r lies inside C, and s lies outside C. Then

$$\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb,$$

where $\measuredangle abc$ denotes the smaller angle defined by three points a, b, c.



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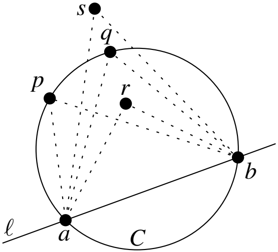

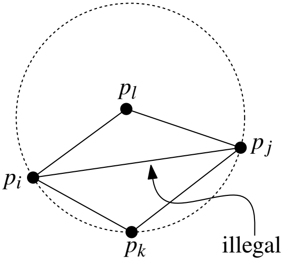

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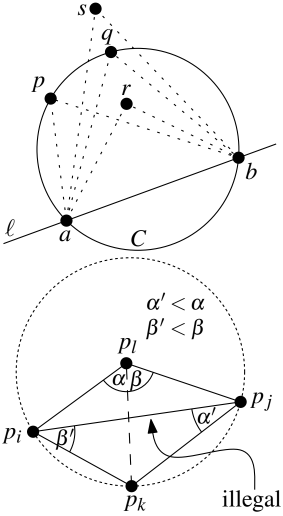

Legal Triangulations

A legal triangulation is a triangulation that does not contain any illegal edge.

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Algorithm LEGALTRIANGULATION($\mathcal{T}$)
Input. A triangulation $\mathcal{T}$ of a point set P.
Output. A legal triangulation of P.
1. **while** $\mathcal{T}$ contains an illegal edge $\overline{p_i p_j}$
2. **do** (* Flip $\overline{p_i p_j}$ *)
3. Let $p_i p_j p_k$ and $p_i p_j p_l$ be the two triangles adjacent to $\overline{p_i p_j}$.
4. Remove $\overline{p_i p_j}$ from $\mathcal{T}$, and add $\overline{p_k p_l}$ instead.
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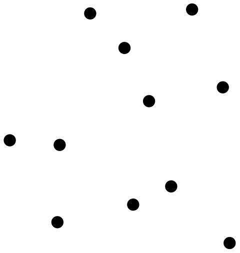
Question: Why does this algorithm terminate?

Voronoi Diagram and Delaunay Graph

- Let P be a set of n points in the plane.
- The Voronoi diagram $\mathrm{Vor}(P)$ is the subdivision of the plane into Voronoi cells $\mathcal{V}(p)$ for all $p \in P$.
- Let $\mathcal{G}$ be the *dual graph* of $\mathrm{Vor}(P)$.
- The Delaunay graph $\mathcal{DG}(P)$ is the *straight line embedding* of $\mathcal{G}$.
- **Question:** How can we compute the Delaunay graph?



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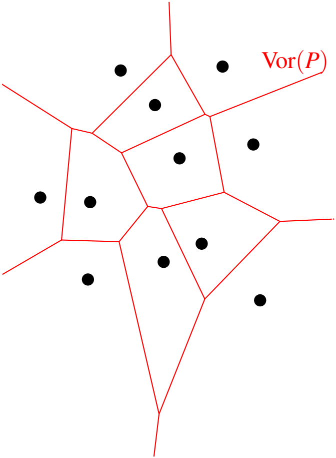

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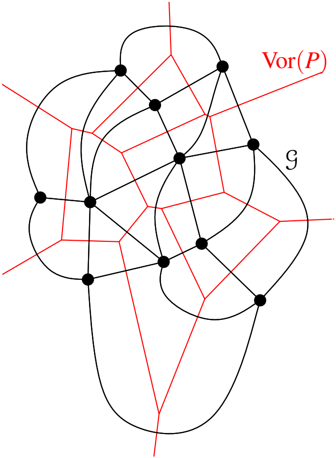

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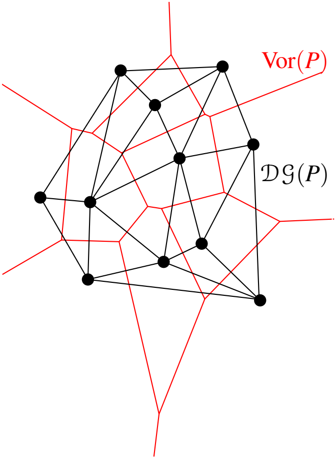

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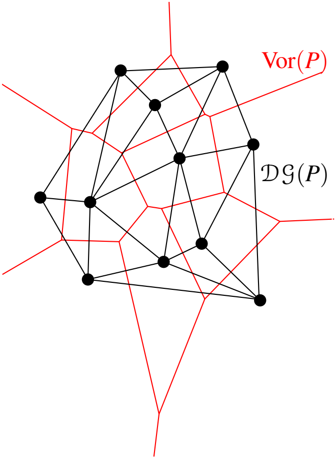
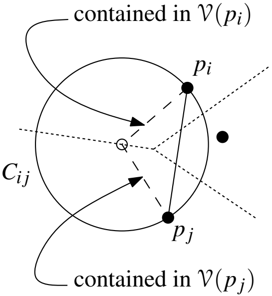


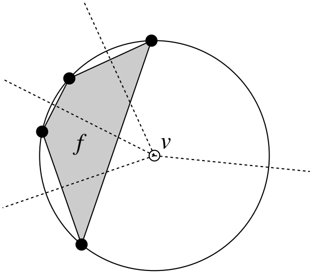
Planarity of the Delaunay Graph

Theorem: The Delaunay graph of a planar point set is a plane graph.



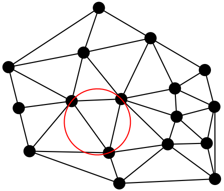
Delaunay Triangulation

If the point set P is in *general position* then the Delaunay graph is a triangulation.



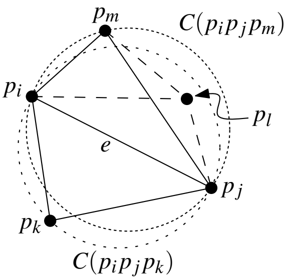
Empty Circle Property

Theorem: Let P be a set of points in the plane, and let $\mathcal{T}$ be a triangulation of P. Then $\mathcal{T}$ is a Delaunay triangulation of P if and only if the circumcircle of any triangle of $\mathcal{T}$ does not contain a point of P in its interior.



Delaunay Triangulations and Legal Triangulations

Theorem: Let P be a set of points in the plane. A triangulation $\mathcal{T}$ of P is legal if and only if $\mathcal{T}$ is a Delaunay triangulation.



Angle Optimality and Delaunay Triangulations

Theorem: Let P be a set of points in the plane. Any angle-optimal triangulation of P is a Delaunay triangulation of P. Furthermore, any Delaunay triangulation of P maximizes the minimum angle over all triangulations of P.

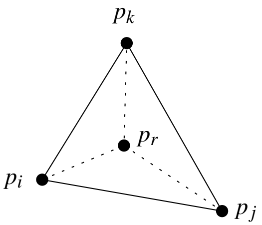
Randomized Incremental Construction

Algorithm DELAUNAYTRIANGULATION(P)
Input. A set P of $n+1$ points in the plane.
Output. A Delaunay triangulation of P.

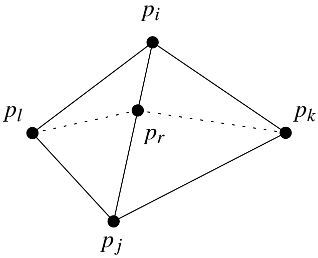
1. Initialize $\mathcal{T}$ as the triangulation consisting of an outer triangle $p_0 p_{-1} p_{-2}$ containing points of P, where p_0 is the lexicographically highest point of P.
2. Compute a random permutation $p_1, p_2, \ldots, p_n$ of $P \setminus \{p_0\}$.
3. **for** $r \leftarrow 1$ **to** n
4.   **do**
5.   LOCATE($p_r, \mathcal{T}$)
6.   INSERT($p_r, \mathcal{T}$)
7. Discard p_{-1} and p_{-2} with all their incident edges from $\mathcal{T}$.
8. **return** $\mathcal{T}$

Randomized Incremental Construction

p_r lies in the interior of a triangle



p_r falls on an edge



Randomized Incremental Construction

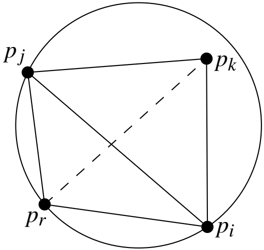
INSERT($p_r, \mathcal{T}$)

1. **if** p_r lies in the interior of the triangle $p_i p_j p_k$
2. **then** Add edges from p_r to the three vertices of $p_i p_j p_k$, thereby splitting $p_i p_j p_k$ into three triangles.
3. LEGALIZEEDGE($p_r, \overline{p_i p_j}, \mathcal{T}$)
4. LEGALIZEEDGE($p_r, \overline{p_j p_k}, \mathcal{T}$)
5. LEGALIZEEDGE($p_r, \overline{p_k p_i}, \mathcal{T}$)
6. **else** ($*$ p_r lies on an edge of $p_i p_j p_k$, say the edge $\overline{p_i p_j}$ $*$)
7. Add edges from p_r to p_k and to the third vertex p_l of the other triangle that is incident to $\overline{p_i p_j}$, thereby splitting the two triangles incident to $\overline{p_i p_j}$ into four triangles.
8. LEGALIZEEDGE($p_r, \overline{p_i p_l}, \mathcal{T}$)
9. LEGALIZEEDGE($p_r, \overline{p_l p_j}, \mathcal{T}$)
10. LEGALIZEEDGE($p_r, \overline{p_j p_k}, \mathcal{T}$)
11. LEGALIZEEDGE($p_r, \overline{p_k p_i}, \mathcal{T}$)

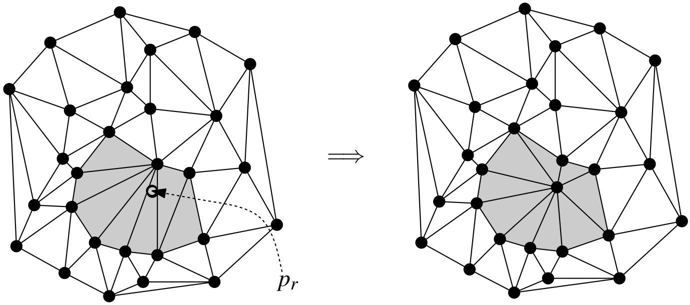
Randomized Incremental Construction

LegalizeEdge(p_r, $\overline{p_i p_j}$, $\mathcal{T}$)

1. (∗ The point being inserted is p_r, and $\overline{p_i p_j}$ is the edge of $\mathcal{T}$ that may need to be flipped. ∗)
2. **if** $\overline{p_i p_j}$ is illegal
3. **then** Let $p_i p_j p_k$ be the triangle adjacent to $p_r p_i p_j$ along $\overline{p_i p_j}$.
4. (∗ Flip $\overline{p_i p_j}$: ∗) Replace $\overline{p_i p_j}$ with $\overline{p_r p_k}$.
5. LegalizeEdge(p_r, $\overline{p_i p_k}$, $\mathcal{T}$)
6. LegalizeEdge(p_r, $\overline{p_k p_j}$, $\mathcal{T}$)

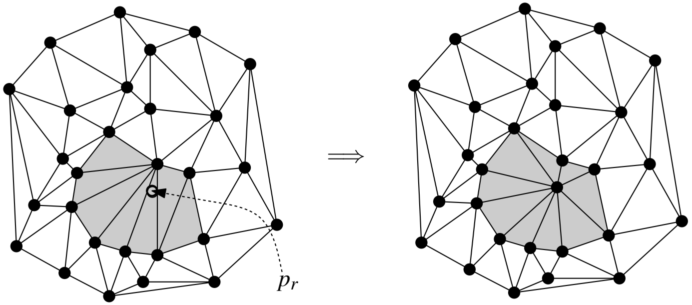


Randomized Incremental Construction



All edges created are incident to p_r.

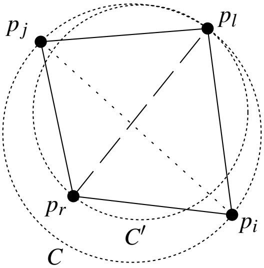
Randomized Incremental Construction



All edges created are incident to p_r.

Correctness: Are new edges legal?

Randomized Incremental Construction



Correctness:

For any new edge there is an empty circle through endpoints.

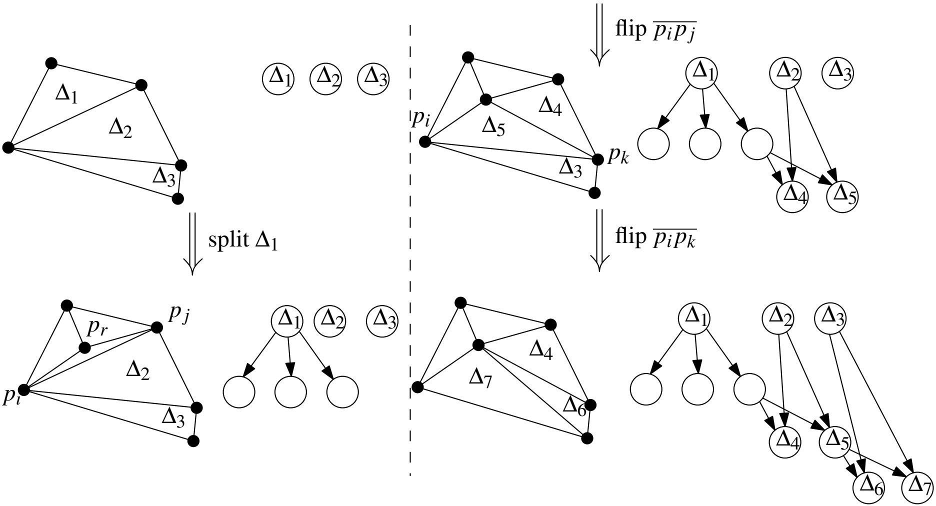
New edges are legal.

Randomized Incremental Construction

Initializing triangulation: treat p_{-1} and p_{-2} symbolically.
No actual coordinates.
Modify tests for point location and illegal edges to work as if far away.

Point location: search data structure.
Point visits triangles of previous triangulations that contain it.

Randomized Incremental Construction



Analysis

1. Expected total number of triangles created in $O(n)$
2. Expected total number of triangles visited while search point location data structure: $O(n \log n)$

We will only consider the first (see book for second)

Analysis

Lemma: Total number of triangles created is at most $9n+1$.

- How many triangles are created when inserting p_r?
- Backwards analysis: Any point of $p_1, \ldots, p_r$ has the same probability $1/r$ to be p_r.
- Expected degree of $p_r \leq 6$.
- Number of triangles created $\leq 2\text{degree}(p_r) - 3$ (Why? Count flips.)
- $2 \cdot 6 - 3 = 9$
- $+$ outer triangle

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Theorem: The Delaunay triangulation of n points can be computed in $O(n \log n)$ expected time.