40^{th} International Mathematical Olympiad

Bucharest

Day I

July 16, 1999

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

   for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

2. Let n be a fixed integer, with $n \geq 2$.

   (a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

   holds for all real numbers $x_1, \cdots, x_n \geq 0$.

   (b) For this constant C, determine when equality holds.

3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

   N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

   Determine the smallest possible value of N.

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4. Determine all pairs (n, p) of positive integers such that

   p is a prime,
   n not exceeded $2p$, and
   $(p-1)^n + 1$ is divisible by n^{p-1}.

5. Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively.

   Prove that CD is tangent to G_2.

6. Determine all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

   $$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

   for all real numbers x, y.