جواب سؤال 1

a.

$$\vec{V} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} + \vec{\omega} \times (-R\hat{z})$$

$$= \begin{pmatrix} v_x - \omega_y R \\ v_y + \omega_x R \end{pmatrix}$$

b. $$\vec{f} = -\mu m g \hat{v} = -\mu m g \frac{\begin{pmatrix} v_x - \omega_y R \\ v_y + \omega_x R \end{pmatrix}}{\sqrt{(v_x - \omega_y R)^2 + (v_y + \omega_x R)^2}}$$

c. $$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = -\mu g \frac{\begin{pmatrix} v_x - \omega_y R \\ v_y + \omega_x R \end{pmatrix}}{\sqrt{(v_x - \omega_y R)^2 + (v_y + \omega_x R)^2}}$$

d.

$$\frac{2}{5} m R^2 \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \end{pmatrix} = (-R\hat{z}) \times \vec{f}$$

$$= \mu m g R \frac{\begin{pmatrix} -v_y - \omega_x R \\ v_x - \omega_y R \end{pmatrix}}{\sqrt{(v_x - \omega_y R)^2 + (v_y + \omega_x R)^2}}$$

$$\Rightarrow \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \end{pmatrix} = \frac{5}{2} \mu \frac{g}{R} \frac{\begin{pmatrix} -v_y - \omega_x R \\ v_x - \omega_y R \end{pmatrix}}{\sqrt{(v_x - \omega_y R)^2 + (v_y + \omega_x R)^2}}$$

e.

$$\dot{\omega}_x = \frac{5}{2}\frac{\dot{v}_y}{R}$$

$$\dot{\omega}_y = \frac{-5}{2}\frac{\dot{v}_x}{R}$$

f.

$$\Rightarrow \omega_x = \frac{5}{2}\left(\frac{v_y}{R} - \frac{v_0}{R}\right) \quad , \quad \omega_y = \frac{-5}{2} v_x/_R + \omega_0$$

$$\Rightarrow \frac{\dot{v}_x}{\dot{v}_y} = \frac{v_x - \omega_y R}{v_y + \omega_x R} = \frac{v_x + \frac{5}{2}v_x - \omega_0 R}{v_y + \frac{5}{2}v_y - \frac{5}{2}v_0}$$

$$\Rightarrow \frac{\dot{v}_x}{\dot{v}_y} = \frac{v_x - \frac{2}{7}\omega_0 R}{v_y - \frac{5}{7}v_0}$$

g.

تغییر متغیر $u_x = v_x - \frac{2}{7}\omega_0 R \quad , \quad u_y = v_y - \frac{5}{7}v_0$

$$\Rightarrow \frac{du_x}{u_x} - \frac{du_y}{u_y} = 0 \Rightarrow \frac{u_x}{u_y} = c = \frac{\dot{v}_x}{\dot{v}_y}$$

$t=0$ در $\frac{\dot{v}_x}{\dot{v}_y} = \frac{-2/7\,\omega_0 R}{2/7\,v_0} = -\frac{\omega_0 R}{v_0}$

h.

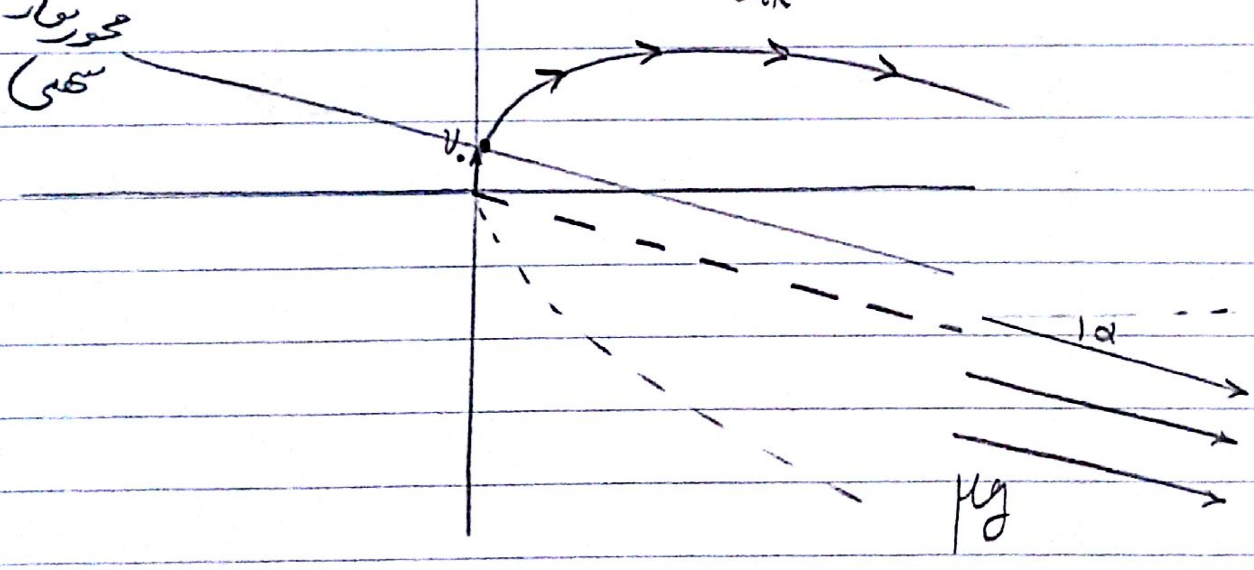
$$a_x = \mu g \frac{\omega_0 R}{\sqrt{v_0^2+\omega_0^2 R^2}} \qquad a_y = -\mu g \frac{v_0}{\sqrt{v_0^2+\omega_0^2 R^2}}$$

$$x = \frac{1}{2} a_x t^2 \,,\; y = \frac{1}{2} a_y t^2 + v_{0y} t$$

$$\Rightarrow y = \frac{-v_0}{\omega_0 R} x + v_0 \sqrt{\frac{2x\sqrt{v_0^2+\omega_0^2R^2}}{\mu g \omega_0 R}}$$

و یا با توجه به ثابت بودن اندازه و جهت نیروی می‌توان گفت مسیر حتماً سهمی است

که نسبت به محور x به اندازه‌ی $\alpha = \tan^{-1} \frac{v_0}{\omega_0 R}$ منحرف شده است.



$$\tan\alpha = \frac{v_0}{\omega_0 R}$$

جواب سوال ۲

a. $T_0 = m_1 r_0 \omega_0^2 \quad , \quad T_0 \cos\varphi_0 = m_2 g$

$$T_0 \sin\varphi_0 = m_2 (L - r_0)\, \Omega_0^2 \sin\varphi_0$$

$\Rightarrow$

$$T_0 = \frac{m_2 g}{\cos\varphi_0} \quad , \quad \omega_0^2 = \frac{m_2 g}{m_1 r_0 \cos\varphi_0} \quad , \quad \Omega_0^2 = \frac{g}{(L - r_0)\cos\varphi_0}$$

---

b.

جرم ۱ : $m_1 \begin{pmatrix} \ddot{r} - r\omega^2 \\ 2\dot{r}\omega + r\dot{\omega} \end{pmatrix} = \begin{pmatrix} -T - \mu m_1 g \dfrac{\dot{r}}{\sqrt{\dot{r}^2 + r^2\omega^2}} \\ -\mu m_1 g \dfrac{r\omega}{\sqrt{\dot{r}^2 + r^2\omega^2}} \end{pmatrix}$

جرم ۲ : $m_2 \begin{pmatrix} \ddot{\rho} - \rho\Omega^2 \\ 2\dot{\rho}\Omega + \rho\dot{\Omega} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} -T \sin\varphi \\ 0 \\ T\cos\varphi - m_2 g \end{pmatrix}$

$\rho = (L - r)\sin\varphi$

$\dot{\rho} = (L - r)\cos\varphi\,\dot{\varphi} - \dot{r}\sin\varphi$

$\ddot{\rho} = (L - r)\cos\varphi\,\ddot{\varphi} - (L - r)\sin\varphi\,\dot{\varphi}^2 - 2\dot{r}\cos\varphi\,\dot{\varphi} - \ddot{r}\sin\varphi$

$z = -(L - r)\cos\varphi \quad , \quad \dot{z} \quad , \quad \ddot{z}$ ✓

C, D.

$$T = T_0 + T_1 \ , \ \omega = \omega_0 + \omega_1 \ , \ \varphi = \varphi_0 + \varphi_1 \ , \ r = r_0 + r_1$$

فقط جملات مرتبه‌ی یک از معادلات را نگه می‌داریم: (تصحیح مسئله شد!)

$$m_1(\ddot{r}_1 - r_1\omega_0^2 - r_0(2\omega_0\omega_1)) = -T_1 \quad (I)$$

$$m_1(2\dot{r}_1\omega_0 + r_0\dot{\omega}_1) = -m_1\mu g \quad (II)$$

معادلات برای ذره m_1

$$\rho = (L-r)\sin\varphi = (L-r_0-r_1)(\sin\varphi_0 + \cos\varphi_0\,\varphi_1)$$

$$\Rightarrow \rho_1 = (L-r_0)\cos\varphi_0\,\varphi_1 - r_1\sin\varphi_0$$

$$z = -(L-r_0-r_1)(\cos\varphi_0 - \sin\varphi_0\,\varphi_1)$$

$$\Rightarrow z_1 = r_1\cos\varphi_0 + (L-r_0)\sin\varphi_0\,\varphi_1$$

$$(III)\ m_2\left\{(L-r_0)\cos\varphi_0\,\ddot{\varphi}_1 - \ddot{r}_1\sin\varphi_0 - \left\{(L-r_0)\cos\varphi_0\,\varphi_1 - r_1\sin\varphi_0\right\}\Omega_0^2 - (L-r_0)\sin\varphi_0\,(2\Omega_0\Omega_1)\right\} = -T_1\sin\varphi_0 - T_0\cos\varphi_0\,\varphi_1$$

برای ذره‌ی m_2

(IV) $\rho^2\Omega = \rho_0^2\Omega_0 \Rightarrow 2\rho_0\rho_1\Omega_0 + \rho_0^2\Omega_1 = 0$

$$\Rightarrow 2(L-r_0)\sin\varphi_0\{(L-r_0)\cos\varphi_0\,\varphi_1 - r_1\sin\varphi_0\}\Omega_0$$
$$+ (L-r_0)^2\sin^2\varphi_0\,\Omega_1 = 0$$

(V) $(\ddot{r}_1\cos\varphi_0 + (L-r_0)\sin\varphi_0\,\ddot{\varphi}_1)m_2 = T_1\cos\varphi_0 - T_0\sin\varphi_0\,\varphi_1$

ω_1 را برحسب r_1 از معادله II بدست می‌آوریم و از آنجا T_1 را برحسب r

از معادله I ، از IV ، Ω_1 را برحسب r و φ بدست می‌آوریم و با جایگذاری

در معادلات V و III دو معادله برحسب r و φ بدست می‌آوریم ...

$$\text{II} \xrightarrow{\text{انتگرال‌گیری}} -\mu g t = 2r_1\omega_0 + r_0\omega_1$$

$$\Rightarrow \omega_1 = -\frac{\mu g}{r_0}t - 2\frac{r_1}{r_0}\omega_0$$

$$(\text{I}) \Rightarrow T_1 = m_1\left\{-2r_0\omega_0\left(2\frac{r_1}{r_0}\omega_0 + \frac{\mu g}{r_0}t\right) + r_1\omega_0^2 - \ddot{r}_1\right\}$$

$$= m_1\{-3r_1\omega_0^2 - \ddot{r}_1 - 2\omega_0\mu g t\}$$

$$(\mathrm{IV}) \Rightarrow \Omega_1 = \frac{-2}{(L-r_0)\sin\varphi_0}\left\{(L-r_0)\cos\varphi_0\,\varphi_1 - r_1\sin\varphi_0\right\}$$

$$(\mathrm{III}): \frac{-g}{\cos\varphi_0}\cos\varphi_0\,\varphi_1 - \frac{m_1}{m_2}\left\{-3r_1\omega_0^2 - \ddot{r}_1 - 2\omega_0\mu g t\right\}\sin\varphi_0$$

$$= (L-r_0)\cos\varphi_0\,\ddot{\varphi}_1 - \ddot{r}_1\sin\varphi_0 - \Omega_0^2\left([L-r_0]\cos\varphi_0\,\varphi_1 - r_1\sin\varphi_0\right)$$

$$-2(L-r_0)\sin\varphi_0\,\Omega_0^2\left(\frac{-2}{(L-r_0)\sin\varphi_0}\left\{(L-r_0)\cos\varphi_0\,\varphi_1 - r_1\sin\varphi_0\right\}\right)$$

$$\Rightarrow \quad r_1\left(\frac{3m_1}{m_2}\omega_0^2\sin\varphi_0 - \Omega_0^2\sin\varphi_0 + 4\Omega_0^2\sin\varphi_0\right)$$

$$+ 2\omega_0\frac{m_1}{m_2}\mu g t\sin\varphi_0 + \ddot{r}_1\left(\frac{m_1}{m_2}\sin\varphi_0 + \sin\varphi_0\right)$$

$$+\varphi_1\left(-g + \Omega_0^2(L-r_0)\cos\varphi_0 - 4\Omega_0^2(L-r_0)\cos\varphi_0\right)$$

$$+\ddot{\varphi}_1\left(-(L-r_0)\cos\varphi_0\right) = 0$$

$$\Rightarrow \ddot{r}_1\left(1+\frac{m_1}{m_2}\right)\sin\varphi_0 - \ddot{\varphi}_1\left((L-r_0)\cos\varphi_0\right)$$

$$+ r_1\left(\frac{m_1}{m_2}\omega_0^2 + \Omega_0^2\right)(3\sin\varphi_0) - \varphi_1\left(g + 3\Omega_0^2(L-r_0)\cos\varphi_0\right)$$

$$+ 2\omega_0\frac{m_1}{m_2}\mu g t\sin\varphi_0 = 0$$

$$\Rightarrow \ddot{r}_1 - \frac{m_2 \cot\varphi_0}{m_1+m_2}(L-r_0)\ddot{\varphi}_1 + 3\left(\frac{m_1\omega_0^2 + m_2\Omega_0^2}{m_1+m_2}\right) r_1 - \left(\frac{m_2 g + 3\Omega_0^2 (L-r_0)\cos\varphi_0\, m_2}{(m_1+m_2)\sin\varphi_0}\right)\varphi_1 + \frac{2\omega_0 m_1 \mu g}{m_1+m_2}\, t = 0$$

از معادله دیگر نیز داریم :

$$\ddot{r}_1 \cos\varphi_0 + (L-r_0)\sin\varphi_0\, \ddot{\varphi}_1 = -\frac{m_1}{m_2}\left(\ddot{r}_1 + 3\omega_0^2 r_1 + 2\omega_0 \mu g t\right)\cos\varphi_0 - \frac{g}{\cos\varphi_0}\sin\varphi_0\, \varphi_1$$

$$\Rightarrow \ddot{r}_1\left(1+\frac{m_1}{m_2}\right)\cos\varphi_0 + (L-r_0)\sin\varphi_0\, \ddot{\varphi}_1 + 3\frac{m_1}{m_2}\omega_0^2 \cos\varphi_0\, r_1 + g\tan\varphi_0\, \varphi_1 + 2\frac{m_1}{m_2}\omega_0 \mu g \cos\varphi_0\, t = 0$$

$$\Rightarrow \ddot{r}_1 + \frac{m_2 \tan\varphi_0}{m_1+m_2}(L-r_0)\ddot{\varphi}_1 + \frac{3 m_1 \omega_0^2}{m_1+m_2} r_1 + \frac{m_2 g \tan\varphi_0}{(m_1+m_2)\cos\varphi_0}\varphi_1 + \frac{2m_1}{m_1+m_2}\omega_0 \mu g\, t = 0$$

حال با توجه به راهنمایی مسئله : (با جایگذاری جواب در معادلات)

$$a_2 A t + a_3 B t + a_4 t = 0$$

$$a_2' A t + a_3' B t + a_4' t = 0$$

$$a_2 A + a_3 B = -a_4$$
$$a_2' A + a_3' B = -a_4'$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{\begin{pmatrix} a_4' a_3 - a_3' a_4 \\ a_4 a_2' - a_4' a_2 \end{pmatrix}}{a_2 a_3' - a_3 a_2'}$$

$$a_2 = \frac{3(m_1 \omega_0^2 + m_2 \Omega_0^2)}{m_1 + m_2} \qquad a_2' = \frac{3 m_1 \omega_0^2}{m_1 + m_2}$$

$$a_3 = \frac{-m_2}{m_1 + m_2} \left(g \csc\varphi_0 + 3\Omega_0^2 (L - r_0) \cot\varphi_0 \right)$$

$$a_3' = \frac{m_2 g \sin\varphi_0}{(m_1 + m_2)\cos^2\varphi_0}$$

$$a_4 = a_4' = \frac{2m_1}{m_1 + m_2} \omega_0 \mu g$$

$$\Rightarrow A = 2m_1\omega_0\mu g \; \frac{-m_2 g\csc\varphi_0 - 3m_2\Omega_0^2(L-r_0)\cot\varphi_0 - m_2 g\dfrac{\sin\varphi_0}{\cos^2\varphi_0}}{3(m_1\omega_0^2+m_2\Omega_0^2)\,m_2 g\dfrac{\sin\varphi_0}{\cos^2\varphi_0} + \cdots}$$

$$\frac{\cdots}{\cdots + m_2\left(g\csc\varphi_0 + 3\Omega_0^2(L-r_0)\cot\varphi_0\right)\left(3m_1\omega_0^2\right)}$$

بنابراین حرکت خطی r (A) برابر است با :

$$\boxed{A = -\frac{2}{3}\mu m_1\omega_0 g \; \frac{g\csc\varphi_0 + 3\Omega_0^2(L-r_0)\cot\varphi_0 + g\tan\varphi_0\sec\varphi_0}{g\tan\varphi_0\sec\varphi_0\,(m_1\omega_0^2+m_2\Omega_0^2) + m_1\omega_0^2\left(g\csc\varphi_0 + 3\Omega_0^2(L-r_0)\cot\varphi_0\right)}}$$

به همین صورت برای B داریم : (حرکت خطی مربوط به φ)

$$B = 2m_1\omega_0\mu g \; \frac{3m_1\omega_0^2 - 3(m_1\omega_0^2+m_2\Omega_0^2)}{3(m_1\omega_0^2+m_2\Omega_0^2)\,m_2 g\dfrac{\sin\varphi_0}{\cos^2\varphi_0} + m_2\left(g\csc\varphi_0+3\Omega_0^2(L-r_0)\cot\varphi_0\right)\left(3m_1\omega_0^2\right)}$$

$$\boxed{B = -2\mu m_1\omega_0 g \; \frac{\Omega_0^2}{g\tan\varphi_0\sec\varphi_0\,(m_1\omega_0^2+m_2\Omega_0^2) + m_1\omega_0^2\left(g\csc\varphi_0 + 3\Omega_0^2(L-r_0)\cot\varphi_0\right)}}$$

$$f.\ \omega_1 = 0 \xrightarrow{\mathrm{II}} \frac{-\mu g t}{2\omega_0} = r_1$$

$$\Rightarrow A = \frac{-\mu g}{2\omega_0}$$

درست است
از رابطه بالا

$$\Rightarrow \left(\tfrac{4}{3} m_1 \omega_0^2\right)\left(g\{\csc\varphi_0 + \tan\varphi_0 \sec\varphi_0\} + 3\Omega_0^2 (L - r_0)\cot\varphi_0\right)$$

$$= g\{\tan\varphi_0 \sec\varphi_0 + \csc\varphi_0\} m_1 \omega_0^2 + 3\Omega_0^2 (L - r_0)\cot\varphi_0\, m_1 \omega_0^2$$

$$+ g \tan\varphi_0 \sec\varphi_0\, m_2 \Omega_0^2$$

$$\Rightarrow m_1 \omega_0^2 g\{\tan\varphi_0 \sec\varphi_0 + \csc\varphi_0\} = 5\Omega_0^2 (L - r_0)\cot\varphi_0\, m_1 \omega_0^2$$

$$+ 3g \tan\varphi_0 \sec\varphi_0\, m_2 \Omega_0^2$$

$$\Rightarrow \tan\varphi_0 \sec\varphi_0 + \csc\varphi_0 = 5\frac{\Omega_0^2 (L - r_0)}{g}\cot\varphi_0$$

$$+ 3\frac{m_2}{m_1}\frac{\Omega_0^2}{\omega_0^2}\tan\varphi_0 \sec\varphi_0$$

با جایگذاری مقادیر ω_0 و Ω_0 برحسب g و φ_0 :

$$\tan\varphi_0 \sec\varphi_0 + \csc\varphi_0 = 5\csc\varphi_0 + 3\frac{r_0}{L - r_0}\tan\varphi_0 \sec\varphi_0$$

$$\Rightarrow \tan\varphi_0 \sec\varphi_0 \left(\frac{L - r_0 - 3r_0}{L - r_0}\right) = 4\csc\varphi_0$$

$$\Rightarrow \frac{L - 4r_0}{L - r_0} = 4\cot^2\varphi_0 \Rightarrow \varphi_0 = \tan^{-1}\left(2\sqrt{\frac{L - r_0}{L - 4r_0}}\right)$$

نکته‌ی مهم :

$$\mathrm{II} \rightarrow \omega_1 = -\frac{\mu g}{r_0} t - \frac{2\omega_0}{r_0} r_1$$

و $r_1 = At + A'\cos\gamma t + A''\sin\gamma t$

$$\Rightarrow \omega_1 = -\frac{2\omega_0}{r_0}\left(A'\cos\gamma t + A''\sin\gamma t\right) + \left(-\frac{\mu g}{r_0} - \frac{2\omega_0}{r_0}A\right)t$$

با شرایط اولیه‌ی برای r_1 داریم :

$$A' = 0$$
$$A''\gamma = -A$$

$$\Rightarrow \omega_1 = \left(-\frac{2\omega_0}{r_0}A - \frac{\mu g}{r_0}\right)t + \frac{2A}{\gamma}\frac{\omega_0}{r_0}\sin\gamma t$$

$A \neq 0$ است بنابراین حتی اگر شرط بدست آمده برقرار باشد

یعنی $\varphi_0 = \tan^{-1}\left(2\sqrt{\frac{L-r_0}{L-4r_0}}\right)$ باز هم ω_1 تا مرتبه‌ی ۱ ناصفر است.

در واقع منظور سؤال از $\omega_1 = 0$، قسمت خطی ω_1 می‌باشد که برابر

با صفر باشد.

جواب سوال ۳

a.

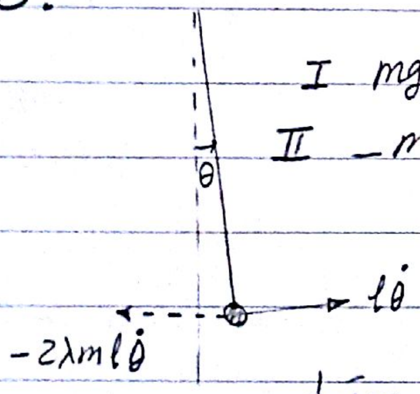
$$\frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta) = E_0$$

$$\Rightarrow m l^2 \ddot{\theta} + mgl \sin\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\theta \ll 1 \Rightarrow \sin\theta \simeq \theta \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$$

b.



$$\text{I} \quad mg\cos\theta - T = -ml\dot{\theta}^2$$

$$\text{II} \quad -mg\sin\theta - 2\lambda m l\dot{\theta} = ml\ddot{\theta}$$

I $\xrightarrow{\text{تا مرتبه ۱}}$ $T = mg$

II $\xrightarrow{\text{تا مرتبه ۱}}$ $\ddot{\theta} + 2\lambda\dot{\theta} + \frac{g}{l}\theta = 0$

$$\Rightarrow \text{if } \theta = e^{st} \Rightarrow s = -\lambda \pm \sqrt{\lambda^2 - g/l}$$

$$\lambda^2 < g/l \Rightarrow s = -\lambda \pm i\sqrt{\frac{g}{l} - \lambda^2}$$

$$\Rightarrow \theta(t) = e^{-\lambda t}\left(A\sin\sqrt{\frac{g}{l}-\lambda^2}\,t + B\cos\sqrt{\frac{g}{l}-\lambda^2}\,t\right)$$

بالنسبة لشرائط البدينة

$$\theta(t) = \theta_0 e^{-\lambda t}\left(\cos\sqrt{\frac{g}{l}-\lambda^2}\,t + \frac{\lambda}{\sqrt{\frac{g}{l}-\lambda^2}}\sin\sqrt{\frac{g}{l}-\lambda^2}\,t\right)$$

c. $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6}\right) = 0$$

if $\theta = \theta_0 \sin\omega t$

$$\Rightarrow -\omega^2\theta_0 \sin\omega t + \frac{g}{l}\left(\theta_0 \sin\omega t - \frac{\theta_0^3}{6}\left[\frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t\right]\right)$$

$= 0$

$$\Rightarrow -\omega^2\theta_0 + \frac{g}{l}\theta_0 - \frac{g}{l}\frac{\theta_0^3}{8} = 0$$

$$\Rightarrow \omega^2 = \frac{g}{l} - \frac{g}{l}\frac{\theta_0^2}{8} \Rightarrow \omega = \sqrt{\frac{g}{l}}\left(1 - \frac{\theta_0^2}{16}\right)$$

$$\Rightarrow T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16}\right)$$

روش دیگر برای بدست آوردن T

با پایستگی انرژی :

$$\frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

$$\Rightarrow \quad \dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0)$$

با استفاده از Silverman relationship :

$$|\dot{\theta}| = \sqrt{\frac{2g}{l}}(\cos\theta - \cos\theta_0)^{1/2}$$

$\dot{\theta} < 0$

با لحاظ جهت $(-)$ حرکت :

$$\Rightarrow \quad \frac{-d\theta}{dt} = \sqrt{\frac{2g}{l}}(\cos\theta - \cos\theta_0)^{1/2}$$

$$\Rightarrow \quad dt = \sqrt{\frac{l}{2g}} \frac{-d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\Rightarrow \int_0^{T/4} dt = \int_{\theta_0}^{0} \sqrt{\frac{l}{2g}} \frac{-d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\Rightarrow \quad T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

تا اولین مرتبه غیر صفر :

$$\Rightarrow T \approx 4\sqrt{\frac{\ell}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\frac{\theta_0^2}{2} - \frac{\theta^2}{2}}}$$

$$= 4\sqrt{\frac{\ell}{2g}}\,\sqrt{2}\,\sin^{-1}\left(\frac{\theta}{\theta_0}\right)\Big|_0^{\theta_0} = 4\sqrt{\frac{\ell}{2g}}\,\sqrt{2}\,\frac{\pi}{2} = 2\pi\sqrt{\frac{\ell}{g}}$$

مراتب بالاتر :

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$T = 2\sqrt{\frac{\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}$$

تغییر متغیر

$$\sin z = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \Longrightarrow \sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin z = \epsilon \sin z$$

$$\Rightarrow \frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \epsilon\cos z\,dz$$

$$\Rightarrow T = 2\sqrt{\frac{\ell}{g}} \int_0^1 \frac{2\epsilon\,dz\cos z}{\sqrt{1-\epsilon^2 z^2}\sqrt{\epsilon^2 - \epsilon^2\sin^2 z}}$$

$$= 4\sqrt{\frac{\ell}{g}} \int_0^{\pi/2} \frac{dz}{\sqrt{1-\epsilon^2\sin^2 z}}$$

$$\Rightarrow T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}\epsilon^2 \sin^2 z + \dots\right) dz$$

$$= 4\sqrt{\frac{l}{g}} \left(\frac{\pi}{2} + \frac{\epsilon^2}{2}\frac{\pi}{4} + \dots\right)$$

$$= 4\sqrt{\frac{l}{g}} \left(\frac{\pi}{2} + \frac{\sin^2 \theta_0/2}{2}\frac{\pi}{4} + \dots\right)$$

$$T \simeq 4\sqrt{\frac{l}{g}} \left(\frac{\pi}{2} + \frac{\theta_0^2}{32}\pi\right) = 2\pi\sqrt{\frac{l}{g}} \left(1 + \frac{\theta_0^2}{16}\right)$$

روش های دیگری نیز وجود دارد ...