A generalization of α-dominating set and its complexity

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Abstract

Let $G = (V, E)$ be a simple and undirected graph. For some real number α with $0 < \alpha \le 1$, a set $D \subseteq V$ is called an *α-dominating set* in G if every vertex v outside D has at least $\alpha \cdot d_v$ neighbor(s) in S where d_v is the degree of v. The cardinality of a minimum α-dominating set in a graph G is called the *α-domination number* of G and denoted by $\gamma_\alpha(G)$. In this paper, we introduce a generalization of α-dominating set, that we call it *f_{deg}-dominating set*. Given a function f_{deg} where f_{deg} is as $f_{deg} : \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$, and f_{deg} may not be an integer-value function. A set $D \subseteq V$ is called an f_{deg}-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \ge f_{deg}(d_v)$. In this paper, for this new concept, we will present some results on the its NP-completeness, APX-completeness and inapproximability.

1 Introduction

Let $G = (V, E)$ be an undirected and simple graph. A set $D \subseteq V$ is called a *dominating* set if every vertex outside D has at least one neighbor in D. The cardinality of a minimum dominating set is called the *domination number* of G denoted by $\gamma(G)$. In 2000, Dunbar et al. [5], introduced the concept of α-domination. Let α be a real number with $0 < \alpha \le 1$. A set $D \subseteq V$ is called an α-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \ge \alpha \times d_v$ where $N(v)$ is the set of all neighbors of v in G, and $d_v := |N(v)|$ is the degree of v. Also, let k be a real number with $k \ge 1$. A set $D \subseteq V$ is called a *k-dominating* set in G if for every vertex v outside D, $|N(v) \cap D| \ge k$.

Now consider the definition of α-dominating. One generalization of this concept is that instead of having at least $\alpha \times d_v$ neighbors in D for each vertex $v \notin D$, we have at least $f(d_v)$ neighbors in D, for some special function f. By selecting $f(x) = \alpha x$, the definition match the α-dominating. It seems that this generalization is much near to the reality. Hence, in this paper, we define the *f_{deg}-dominating* set. Given a function f_{deg} where f_{deg} is as $f_{deg} : \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$, and f_{deg} may not be an integer-value function. A set $D \subseteq V$ is called an f_{deg}-dominating set in G if for every vertex v outside D, $|N(v) \cap D| \ge f_{deg}(d_v)$. In this paper, we consider the graphs with no isolated vertices. We can easily extend the results for the graphs with isolated vertices. In this paper, we prove the NP-completeness of the following problem: given a graph G and a positive integer k, decide whether G has an f_{deg}-dominating set S with $|S| \le k$. Moreover, we prove that the problem of finding a minimum f_{deg}-dominating set when $f_{deg}(x) = k$ (in the other words, the k-dominating set) for any integer $k \ge 1$ is APX-complete (there is no PTAS). Also, we present some inapproximability result for the problem of finding a minimum f_{deg}-dominating set for constant function $f_{deg}(x) = k$.

2 NP-completeness result

In this section, we will prove that the problem of finding the f_{deg}-domination number of a graph is NP-complete, for every given function f_{deg} with some special properties. It is well known that the following decision problem, denoted by 3-REGULAR DOMINATION (3RDM), is NP-complete [6]: given a 3-regular graph $G = (V, E)$ and a positive integer k, does G has a dominating set S with $|S| \le k$? Now, consider the following decision problem, denoted by f-DOMINATION (fDM): given a graph $G = (V, E)$ without isolated vertices and a positive integer k, does G has an f_{deg}-dominating set S with $|S| \le k$?

We will show that fDM is NP-complete for some special functions. We will extend the proof of the result in which that α-domination is NP-complete (see [5]).

Theorem 2.1.. *If an increasing function f_{deg} with domain $\mathbb{N}$ satisfies*

a. $\forall x \in \mathbb{N}, 0 < f_{deg}(x) \le x$,

b. $\exists x_0 > 0$ *such that* $\forall x \ge x_0,\ x + 1 \ge f_{deg}(x + 3)$.

c. *For every two integers x and y, $f_{deg}(y + x) \le f_{deg}(y) + f_{deg}(x)$,*

d. *For a given $x \in \mathbb{N}$, there is $y \in \mathbb{N}$, such that $y > x$ and $f_{deg}(y) \le x$,*

then, the problem fDM is an NP-complete problem.

Sketch of Proof. Let f_{deg} be an arbitrary function that has the conditions of the theorem. We fix the function f. We can easily see that $fDM \in NP$. Now, we proof the completeness. We make a transformation from 3RDM to fDM. Suppose that x is the smallest integer such that $(x+1) \ge f_{deg}(x+3)$, and y is the largest integer with $y > x$ and $x \ge f_{deg}(y)$. Consider the complete graph K_{y+1} and assume that $U = \{v_1, v_2, \ldots, v_x\}$ is a subset of vertices of K_{y+1} with x elements. We call the vertex set of K_{y+1} by W.

We transform a 3-regular graph G to a graph denoted by $\hat{G}$ by joining each vertex of set U to all vertices of G. Assume that S is a dominating set in G such that $|S| \le k$. Consider the set $D = S \cup U$. Using the conditions **b** and **d**, it is easy to see that D is an f_{deg}-dominating set in $\hat{G}$ with $|D| \le x + k$.

Now, we assume that D is an f_{deg}-dominating set in $\hat{G}$ with $|D| \le x+k$. Among all f_{deg}-dominating set in $\hat{G}$ with $|D| \le x + k$, we suppose that D is the one with maximum $|D \cap U|$. Also, without loss of generality we can suppose that there is a vertex in $W - U$ that is outside D. Using conditions **a, b, c,** and **d**, it is not hard to prove that the set $D \cap V(G)$ is a dominating set in G with $|D \cap V(G)| \le k$. Because 3RDM is NP-complete [6], fDM is also NP-complete for the function f that satisfies the conditions of Theorem 2.1. □

There are many functions that satisfy the conditions of Theorem 2.1, such as $\sqrt{x}$, $\ln x$ and $\frac{x}{2}$.

3 APX-completeness result

In this section, we prove that the problem of finding a minimum f_{deg}-dominating set of a graph with maximum degree $k + 2$ and $f_{deg}(x) = k$ for any $k \ge 1$ is APX-complete (there is no PTAS). We denote the problem of finding a minimum f_{deg}-dominating set of a graph where $f_{deg}(x) = k$ by MIN k-DOM SET, and when the problem is restricted to the graphs with maximum degree $k + 2$, we call it MIN k-DOM SET-$(k + 2)$.

At first, we recall the L-reduction.

Definition 3.1. (L-reduction)[2]. Given two NP optimization problems F and G and a polynomial transformation f from instances of F to instances of G, we say that f is an L-reduction if there are two positive constants α and β such that for every instance x of F

1. $opt_G(f(x)) \le \alpha opt_F(x)$
2. for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ we can, in polynomial time, find a solution y' of x with $m_F(f(x), y') = c_1$ such that $|opt_F(x) - c_1| \le \beta |opt_G(f(x)) - c_2|$.

To prove that a problem F is APX-complete, it is sufficient to prove that F ∈APX and there is an L-reduction from some APX-complete problem to problem F.

Theorem 3.2 ([?]). *For a graph G,* MIN k-DOM SET *can be approximated in polynomial time by a factor of* $\ln(2\Delta(G)) + 1$ *where $\Delta(G)$ is the maximum degree of G.*

Theorem 3.3. MIN k-DOM SET-$(k + 2)$ *is an APX-complete problem for any $k \ge 1$.*

Sketch of Proof. The case $k = 1$ proved in [1]. Consider $k > 1$. Clearly, by Theorem 3.2, if the degree of vertices of the graph is bounded by a constant then the approximation ratio is constant. Thus the problem MIN k-DOM SET-$(k + 2)$ is in APX. Suppose that $G = (V, E)$ is a graph of bounded degree 3. Construct a graph $G_k = (V_k, E_k)$ of bounded degree $k + 2$ as follows. Create a set S_v of $k - 1$ new vertices for each vertex v. Join each vertex $v \in V$ to $k-1$ vertices of S_v. Given a k-dominating set D_k of $G_k = f_k(G)$ (f_k is a transformation from G to G_k. Recall Definition 3.1), we can find a dominating set D in G as $D = D_k - \left(\bigcup_{v \in V(G)} S_v\right)$. So $\gamma(G) \le |D| = |D_k| - (k - 1)n$, where $n = |V|$. Also, given a dominating set D of G, clearly the set $D_k = \left(\bigcup_{v \in V(G)} S_v\right) \cup D$ is a k-dominating set in G_k. So $\gamma_k(G_k) \le |D_k| = |D| + (k - 1)n$. Hence, we can easily conclude that $\gamma_k(G_k) = \gamma(G) + (k - 1)n$.

Finally, using the above argument, we can find an L-reduction with parameters $\alpha = 4k - 3$ and $\beta = 1$. So, the problem MIN k-DOM SET-$(k + 2)$ is APX-complete. □

4 Inapproximability result on MIN k-DOM SET

In this section, we presents some inapproxmabillity result for MIN k-DOM SET.

Theorem 4.1 ([3]). *For any constant $\epsilon > 0$ there is no polynomial time algorithm approximating* MIN 1-DOM SET *within a factor of $(1 - \epsilon) \ln n$ unless NP ⊆ DTIME($n^{O(\log \log n)}$). The same result holds for bipartite graphs.*

Theorem 4.2. *For every $k \ge 1$ and every $\epsilon > 0$, there is no polynomial time algorithm approximating sc Min k-*DOM SET *for bipartite graphs within a factor of $(1 - \epsilon) \ln n$, unless NP ⊆ DTIME($n^{O(\log \log n)}$).*

Sketch of Proof. It is sufficient that, we make some modifications in the proof of Theorem 4.1. We make a reduction from domination on a bipartite graph G with n vertices such that $n + 2k - 2 \le n^{1+\epsilon}$ and $\gamma(G). \ge \frac{2(k-1)(1+2\epsilon)}{\epsilon^2}$. Then we transform the bipartite graph $G = (V_1, V_2, E)$ into a bipartite graph G' by adding to it two sets K_1 and K_2 each have $k - 1$ new vertices inducing a graph with no edges. Join each vertex of V_1 to each vertex of K_2 and join each vertex of V_2 to each vertex of K_1. We can easily prove that $\gamma_k(G') \le \gamma(G)+2k-2$. Now, suppose that there is a polynomial time approximation algorithm that computes a k-dominating set D' for G' such that $|D'| \le (1 - \epsilon) \ln(|V(G')|)\gamma_k(G')$. It is easy t.o see that $D := D' \cap V(G)$ is a dominating set in G. So,

$$
\begin{aligned}
|D| &\le |D'| \\
&\le (1 - \epsilon)(\ln |V(G')|)\gamma_k(G') \text{ (suppose that } n := |V(G')|) \\
&\le (1 - \epsilon)(\ln n)(1 + \epsilon + \epsilon^2)\gamma(G) \\
&= (1 - \epsilon')(\ln n)\gamma(G),
\end{aligned}
$$

. where $\epsilon' = \epsilon^3 > 0$. Therefore, the set D approximates a minimum dominating set in G within factor $(1 - \epsilon') \ln n$. But this contradicts Theorem 4.1. This completes the proof. □

5 Figure and Table

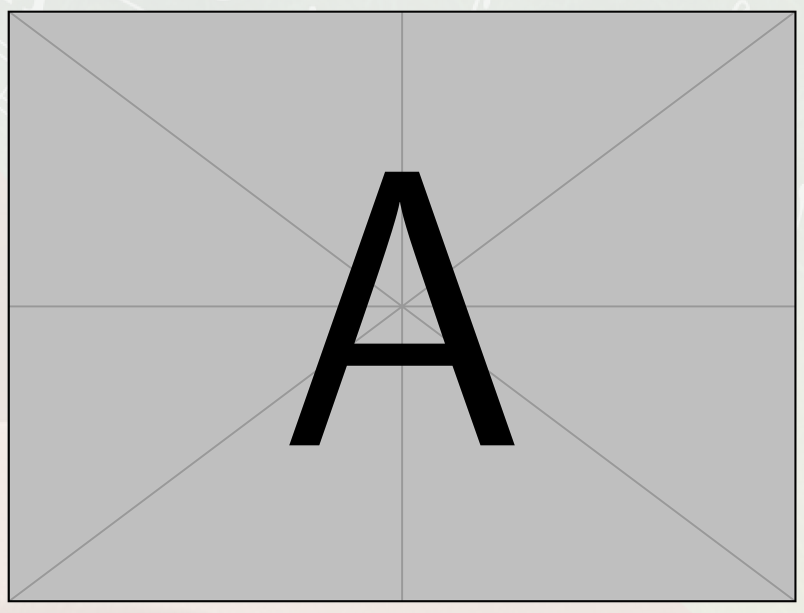

Figure 1: A sample figure caption

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First column head	Second column head	Third column head
N/A	$x^2 + 1$	6
−20	y	11
−12	$x + y$	7

Table 1: A sample table caption

References

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