EUCLID, fl. 300 BCE[1]

The name Euclid is known to almost every high school student as the author of *The Elements*, the long studied treatise on geometry and number theory. No other book except the *Bible* has been so widely translated and circulated. From the time it was written it was regarded as an extraordinary work and was studied by all mathematicians, even the greatest mathematician of antiquity — Archimedes, and so it has been through the 23 centuries that have followed. It is unquestionably the best mathematics text ever written and is likely to remain so into the distant future.

This miniature found in a manuscript of the Roman surveyors in Wolfenbüttel, 6th century CE is purportedly an image of Euclid.

1 Euclid, the mathematician

Little is known about **Euclid**, fl. 300BC, the author of **The Elements**. He taught and wrote in Egypt at the Museum and Library at Alexandria, which was founded in about 300 BCE by Ptolemy I Soter, who [2]

Almost everything about him comes from Proclus' **Commentary**, 4th cent AD. He writes that Euclid collected Eudoxus' theorems, perfected many of Theaetetus', and completed fragmentary works left by others. His synthesis of these materials was so masterful that scarcely any mathematician today is unfamiliar with this work.

Euclid is said to have said to the first Ptolemy who inquired if there was a shorter way to learn geometry than the Elements:

> ...there is no royal road to geometry

Another anecdote relates that a student after learning the very first proposition in geometry, wanted to know what he would gain by knowing such proposition, whereupon Euclid called his slave and said, ”Give him threepence since he must needs make gain by what he learns.”

There are also remarks in the Islamic literature that attributes names to Euclid's father and grandfather, that gives his birthplace as Tyre, and provides a very few other details about Euclid, including the admonition placed on the doors of many Greek schools forbidding anyone from entering who has not first learned the elements of Euclid.

Of the character of Euclid there is only a remark by Pappus that Euclid was unassuming, not boasting of his work and honest and fair to the contributions of others. These comments seem to have come as a pointed contrast to Apollonius[3]

$\star$ He , who we will discuss later. This, 700 years after Euclid's death, can scarcely be considered authoritative. Indeed, by this time Euclid was more legend than person.

2 Sources of *The Elements*

Before Euclid there was geometry. The latest compiler before Euclid was Theudius, whose textbook was used in the Academy. It was was

[2] Ptolemy I was a Macedonian general in the army of Alexander the Great. He became ruler of Egypt in 323 BCE upon Alexander's death and reigned to 285/283 BCE.

[3] Apollonius was known as the “great geometer” because of his work on conics. He seems to have felt himself a rival of Archimedes, twenty five years his senior. His accomplishments in proving tangencies without coordinates is singularly remarkable, and he is considered one of the greatest of the ancients of the Helenistic period.

probably the one used by Aristotle. But soon after *The Elements* appeared, all others were forgotten. If the greatness of a masterpiece can be measured by the number of people that study it, *The Elements* must rank second of all written works, with only the *The Bible* preceding it. Judging by the number of references, it must have been a classic almost from the time of publication. The most accomplished mathematicians of antiquity studied *The Elements*, and several of them wrote commentaries on it. Among them are Heron, Proclus, Pappus, Theon of Alexandria, and Simplicius. Some authors added books (chapters) and other improved or modified the theorems or proofs. In fact, considerable effort has been expended to determine what the original work contained. This is difficult in that it was written about 2300 years ago, and no copies are extant. Only a few potsherds dating from 225 BC contain notes about some propositions, Many new editions were issued. The most significant was prepared by Theon of Alexandria, 4^{th} century, CE. Theon's scholarly recension was for centuries the basis of all known translations. Another version was found in the Vatican by Peyrard (early 19^{th} century) with the customary attributions to Theon absent. From this, it was possible to determine an earlier, root version of *The Elements* closer to the original. However, it was not until the Danish scholar J. L. Heiberg in 1883-1888, working with the Peyrard manuscript and the best of the Theonine manuscripts together with commentaries by Heron and others, that a new and definitive text was constructed. This version is widely regarded as closest of all to the original, both in organization and constitution.

When the Greek world crumbled in the 5^{th} century, it was the Islamics that inherited the remains. At first disdaining any regard for ancient work and indeed destroying what they found, substantially on religious bases, they later embraced the Greek learning through as many ancient texts as could be recovered. They actively sought out the remaining Greek editions, even by making lavish purchases, and translated them to Arabic. We will discuss Islamic mathematical contributions to our mathematical heritage in more detail later. For now it suffices to say that it was the Arabic translations that provided the primary source materials for the Latin translations that were to emanate from Moorish Spain in the 12^{th} and 13^{th} centuries.

Three Arabic translations were made during the Islamic period of enlightenment. One was produced by al-Hajjaj ibn Yusuf ibn Matar, first for the Abbassid caliph Harun ar-Rashid (ruled 786-809) and again for the caliph al-Ma'Mun (ruled 813-833); The second was made by Hunayn ibn Ishaq (ruled 808-873), in Baghdad. His translation was revised by Thabit ibn Qurrah[4] The third was made by Nasir ad-Din at-Tusi in the 13th century.

Of the Latin translations, the first of these was produced by the Englishman Adelard of Bath (1075 - 1164) in about 1120. Adelard obtained a copy of an Arabic version in Spain, where he travelled while disguised as a Muslim student. There is, however, some evidence that *The Elements* was known in England even two centuries earlier. Adelard's translation, which was an abriged version with commentary, was followed by a version offered by the Italian Gherard of Cremona (1114 - 1187) who was said to have translated the '15 books' of *The Elements*. Certainly this was one of the numerous editions This version was written in Spain. Because it contains a number of Greek words such as *rhombus* where Adelard's version contains the Arabic translations, it is likely independent of Adelard's version. Moreover, Gherard no doubt used Greek sources as well. Gherard's manuscript was thought lost but was discovered in 1904 in France. It is a clearer translation that Adelard's, without abbreviations and without editing, being a word for word translation containing the revised and critical edition of Thabit's version. A third translation from the Arabic was produced by Johannes Campanus of Novara (1205 - 1296) that came in the late 13th century. The Campanus translation is similar to the Adelard version but it is clearer and the order of theorem and proof is as now, with the proof following the proposition statement.

The first direct translation from the Greek without the Arabic intermediate versions was made by Bartolomeo Zamberti in 1505. The *editio princeps* of the Greek text was published at Basel in 1533 by Simon Grynaeus. The first edition of the complete works of Euclid was the Oxford edition of 1703, in Greek and Latin, by David Gregory. All texts, including the one we quote from, are now superceded by *Euclidis Opera Omnia* (8 volumes and a supplement, 1883-1916), which were edited by J.L. Heiberg and H. Menge.

The earliest known copy of *The Elements* dates from 888AD and is currently located in Oxford.

[4] Abu'l-Hasan Thabit ibn Qurra (826 - 901) was court astronomer in Baghdad, though he was a native of Harran. Thabit generalized Pythagoras's theorem to an arbitrary triangle. He was regarded as Arabic equivalent of Pappus, the commentator on higher mathematics. He was also founder of the school that translated works by Euclid, Archimedes, Ptolemy, and Eutocius. Without his efforts many more of the ancient books would have been lost.

Note. There is an important web site at www.perseus.tufts.edu which details many facets of the ancient Greek world. It also contains the statements of the propositions in *The Elements*.

3 Euclid's Other Works

Five works by Euclid have survived to our day:

1. *The Elements*
2. *Data* — a companion volume to the first six books of the *Elements* written for beginners. It includes geometric methods for the solution of quadratics.
3. *Division of Figures* — a collection of thirty-six propositions concerning the division of plane configurations. It survived only by Arabic translations.
4. *Phaenomena* — on spherical geometry, it is similar to the work by Autolycus
5. *Optics* — an early work on perspective including optics, catoptrics, and dioptrics.

All these are in the TAMU library.

Three works by Euclid have not survived:

1. *Porisms* — possibly an ancient version of analytic geometry.
2. *Surface Loci* — ?
3. *Pseudaria* — ?

4 *The Elements*

The Elements was one of the first books printed with the Gutenberg press, though not by Gutenberg personally. It was first published in Venice by Erhard Ratdolt. This book had $2\frac{1}{2}$ inch margins in which were placed the figures. It was the first mathematical book of importance printed. The reason this or other mathematical texts were published so late was the technical difficulty of printing the figures. There is a remarkable similarity with the contemporary difficulty of producing mathematical typography for Web-based course. That is substantially the reason why these materials are in Acrobat pdf format. Our source for the results in *The Elements* are from the Sir Thomas L. Heath translation into English of Heiberg's Greek version. The general style of *The Elements* contrasts dramatically with a modern mathematics textbook. Indeed, these days only research monographs have a similar style. Namely, there is no examples, no motivations, no calculation, no witty remarks, no introduction, no preamble. The expensive method of manuscript reproduction, hand transcriptions, probably dictated this economy of scale. However, original commentary and the like may have been lost through the many new editions and translations.

4.1 *The Elements* — Structure: Thirteen Books

It comes as a surprise to many that *The Elements* contains so much mathematics, including number theory and aspects of series and limits. *The Elements* can be topically divided into four sections.

- Books I-VI — Plane geometry
- Books VII-IX — Theory of Numbers
- Book X — Incommensurables
- Book XI-XIII — Solid Geometry

Each of the books was organized in the following order.

- Definitions
- Axioms or common notions — general statements obvious to all
- Postulates — particular to the subject at hand
- Theorems

Present here is the considerable influence of Aristotle, who outlined the logical requirements of an argument. The axioms were general statements, so primitive and so true that there could be no hope of any sort of proof. A typical example: *If equals be added to equals, the wholes are equal.* This axiom, used repeatedly in almost every area of mathematics is completely fundamental. Axioms have bearing throughout all of reason. Postulates are the primitive basis of the subject at hand, and in *The Elements* form the set of constructs that are possible. In Book I there are five postulates. Here is one: *To describe a circle with any center and distance.* This means the Euclid states without proof that a circle of any diameter and radius may be constructed. This postulate, just barely more that defining what a circle is, allows circles to be constructed as needed. Of course, the theorems constitute the main content of the material at hand. This organization, which is the standard today, is remarkable in that it was developed concurrently with the materials themselves. It is reasonable to conclude that the theorems of *The Elements* assumed through many forms and were proved many ways before Euclid locked them into his timeless masterpiece.

4.2 *The Elements* — Book I

- Definitions — 23

1. A *point* is that which has no part
2. A *line* is breadthless length.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.

⋮

15. A *circle* is a plane figure contained by one line such that all the straight lines meeting it from one point among those lying within the figure are equal to one another.
16. And the point is called the center of the circle

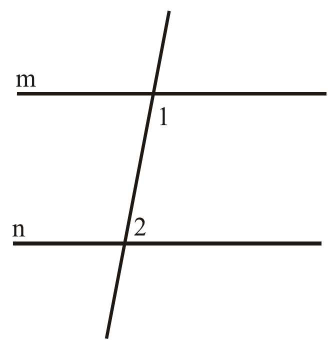
⋮

17. A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

⋮

23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions do not meet one another in either direction.

- Postulates — 5
  1. To draw a straight line from any point to any point.
  2. To produce a finite straight line continuously in a straight line.
  3. To describe a circle with any center and distance.
  4. That all right angles are equal to one another.
  5. That, if a straight line falling on two straight lines make the interior angles on the same side less than to right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the to right angles.



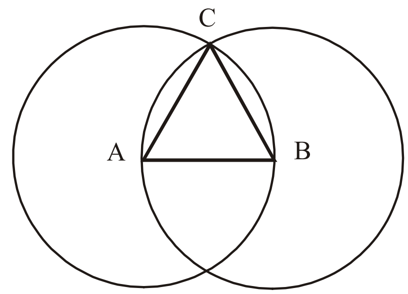
- Axioms — 5
  1. Things which are equal to the same thing are also equal to one another.
  2. If equals be added to equals, the wholes are equal.
  3. If equals be subtracted from equals, the remainders are equal.
  4. Things which coincide with one another are equal to one another.
  5. The whole is greater that the part.

Some Logic

- *A syllogism*: "a syllogism in discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so." Example: If all monkeys are primates and all primates are mammals, then it follows that all monkeys are mammals.
- *modus ponens*: If p, then q. p. Therefore q.
- *modus tolens*: If p, then q. Not q. Therefore, not p.

The 48 propositions of Book I comprise much of the standard one year high school geometry course. The most famous of all them is Proposition I-47, the Pythagorean Theorem, which was discussed in the chapter on Pythagoras. Here we shall consider a few of the results with their proofs as samples of the work.

Proposition I-1. On a given finite straight line to construct an equilateral triangle.



Proof. To prove this construct circles at A and B of radius AB. Argue that the intersection point C is equidistant from A and B, and since it lies on the circles, the distance is AB. ■

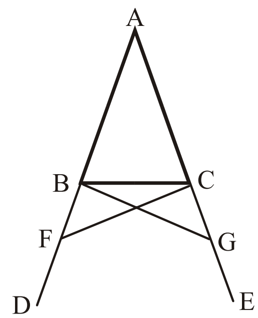
Note that in Proposition I-1, Euclid can appeal only to the definitions and postulates. But he doesn't use the Aristotelian syllogisms, rather he uses *modus ponens*. Note also that there is a subtle assumption of the continuous nature of the plane made in the *visual* assumption that the circles intersect. Flaws of this type went essentially unresolved up until modern times.

4.3 *The Elements* — Book I

Proposition I-4. (SAS) If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal sides also equal, then the two triangles are congruent.

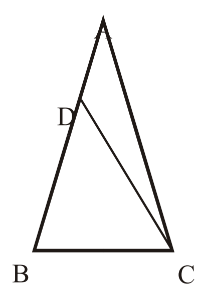
Note: The modern term *congruent* is used here, replacing Euclid's assertion that "each part of one triangle is equal to the corresponding part of the other." Euclid assumes that rigid translation or rotation leaves figures invariant and this is the final step, though never take, of every congruence proof. The one figure can be placed upon the other, with all sides and angles in correspondence, means they are congruent.

Proposition I-5. In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines are produced further, the angles under the base will be equal to one another.



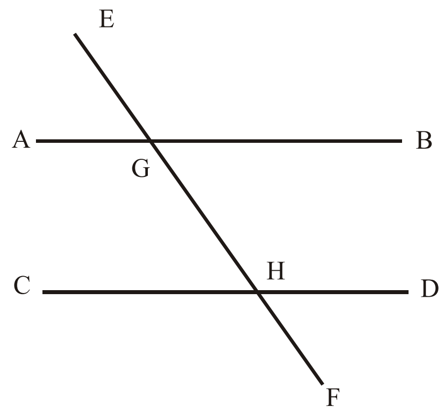
Proof. Extend AB to D and AC to E. Mark off equal distances BF and CG on their respective segments. Now argue that since AF and AG are equal and AC and AB are equal and the triangles ACF and ABG share the included angle at A, they must be congruent. This means than the sides FC and GB are equal. Hence, triangles FCB and GCB are (SAS) congruent. Therefore, the angles $\angle FBC$ and $\angle GCB$ are equal, from which the conclusion follows. ■

Proposition I-6. If in a triangle two angles are equal to one another, then the opposite sides are also equal.



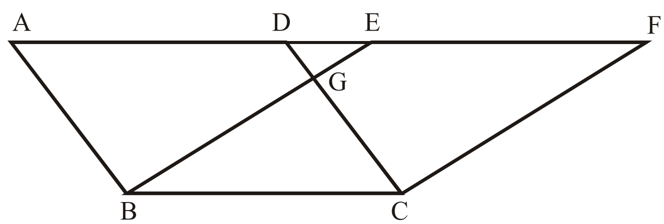
Proof. We are given that $\angle ABC = \angle ACB$. Assume $AB \neq AC$. Assume $AB > AC$. Make D so that $DC = AB$. Now argue that triangles ABC and DBC are congruent. Thus $\triangle DBC$, the part is equal to $\triangle ABC$, the whole. This cannot be.

Proposition I-29. A straight line intersecting two parallel straight line makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.



Proof. Assume $\angle AGH > \angle GHD$. Then the sum of $\angle AGH$ and $\angle BGH$ is greater than the sum of $\angle BGH$ and $\angle GHD$. But the first sum is two right angles. (Proposition I-13.) Thus the second sum is less than two right angles and thus the line are not parallel.

Proposition I-35. Parallelograms which are on the same base and in the same parallels are equal to one another.



Proof. The proof follow directly once the triangles BAE and 'CDF are shown to be congruent. And this step is argued via SAS congruence.

With I-35 established, it is shown in Proposition I-37 that *triangles which are on the same base and in the same parallels are equal to one another*, and in Proposition I-41 that if a triangle and parallel have the same base and are between the same parallels, then the triangle is half the parallelogram (in area). These together with Proposition I-46 on the constructibility of a square on any segment are the main tools in the proof of the Pythagorean theorem. The formal statement is

Proposition I-47. In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

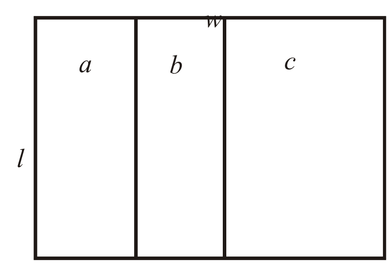
See more details and diagrams in the chapter on Pythagoras and the Pythagoreans.

4.4 *The Elements* — Book II

Book II, with 14 Theorems, is differs from Book I in that it deals with rectangles and squares. It can be termed **geometric algebra**. There is some debate among Euclid scholars as to whether it was extracted directly from Babylonian mathematics. In any event, it is definitely more difficult to read than Book I material.

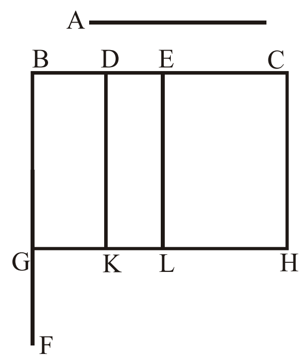
Definition. Any rectangle is said to be contained by the two straight lines forming the right angle. Euclid never multiplies the length and width to obtain area. There is no such process. He does multiply numbers (integers) times length.

Proposition II-1. If there are two straight lines, and one of them is cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the uncut straight line and each of the segments.



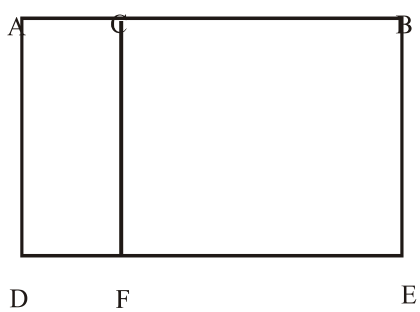
$lw = l(a+b+c) = la+lb+lc$

It should be apparent that this is the distributive law for multiplication through addition. Yet, it is expressed purely in terms of geometry.

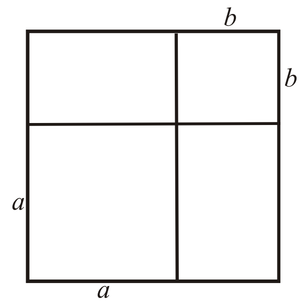


Proof. Let A and BC be the two lines. Make the random cuts at D and E. Let BF be drawn perpendicular to BC and cut at G so that BG is the same as A. Complete the diagram as shown. Then BH is equal to BK, DL, EH Now argue that the whole is the sum of the parts.

Proposition II-2. If a straight line be cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole.



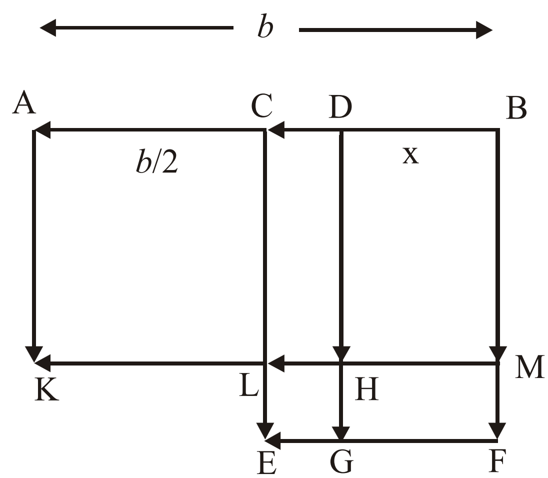
Proposition II-4. If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.



$$(a+b)^2 = a^2 + b^2 + 2ab$$

Note the simplicity of visualization and understanding for the binomial theorem for $n = 2$. Many propositions give geometric solutions to quadratic equations.

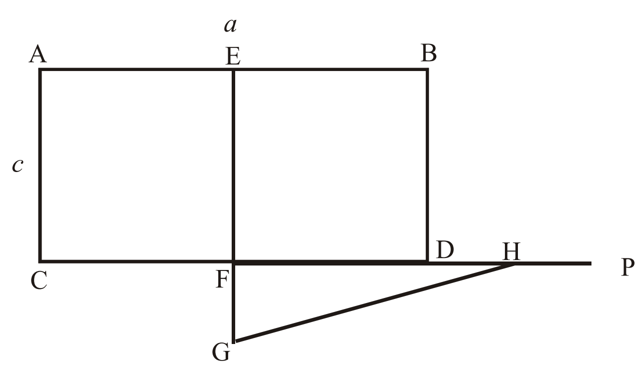
Proposition II-5. If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.



This proposition translates into the quadratic equation

$$(b-x)x + (b/2-x)^2 = (b/2)^2.$$

Proposition II-14. To construct a square equal to a given rectilinear figure.



Proof. Assume $a > c$. Solve $x^2 = ac$. Construct at the midpoint of AB, and produce the line EG of length $(a+c)/2$. Therefore length of the segment FG is $(a-c)/2$. Extend the line CD to P and construct the line GH of length $(a+c)/2$ (H is on this line.). By the Pythagorean theorem the length of the line FH has square given by

$$\left(\frac{a+c}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2 = ac$$

4.5 *The Elements* — Book III

Book III concerns circles, begins with 11 definitions about circles. For example, the definition of the equality of circles is given (circles are equal if they have the same diameter). Tangency is interesting in that it relies considerably on visual intuition:

Definition 2. A straight line is said to *touch a circle* which, meeting the circle and being produced, does not cut the circle.

Definition 3. A *segment of a circle* is the figure contained by a straight line and a circumference of a circle.

Other concepts are segments, angles of segments, and similarity of segments of circles are given. Euclid begins with the basics.

Proposition III-1. To find the center of a given circle.

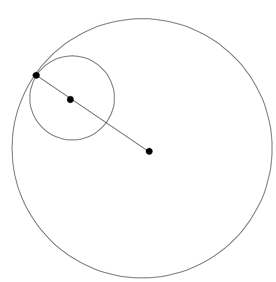
Proposition III-2. If on the circumference of a circle two points be take at random, the straight line joining the points will fall within the circle.

Proposition III-5. If two circles cut (touch) one another, they will not have the same center.

The inverse problem: III-9. If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the center of the circle.

The Elements — Book III

III-11. If two circles touch one another internally, and their centers be taken, the straight line joining their centers, if it be also produced, will fall on the point of contact.



III-16. The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; ... .

Proposition III-31. (Thales Theorem) In a circle the angle in the semicircle is a right angle, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.

4.6 *The Elements* — Book IV

— 16 theorems Construction of regular polygons was a preoccupation of the Greeks. Clearly equilateral triangles and squares can be constructed, that is, inscribed in a circle. Bisection allows any number of doublings, e.g. hexagons and octagons. The inscribed pentagon is a more challenging construction. This book is devoted to the circumscribing and inscribing regular and irregular polygons into circles. As ususal, Euclid begins with appropriate definitions. For example, a very general notion of *inscribed figure* is given.

Definition 1. A rectilineal figure is said to be inscribed in a rectilineal figure when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

Definition 2. Similarly a figure is said to be circumscribed about a figure when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

Definitions 3 and 4 give the meaning of inscribed in and circumscribed about a circle; in the former case the *angles* are required to lie on the circumference in contrast with the sides. In all there are seven definitions. Even the most basic result is considered by Euclid as we see in the opening proposition.

Proposition IV-1. Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

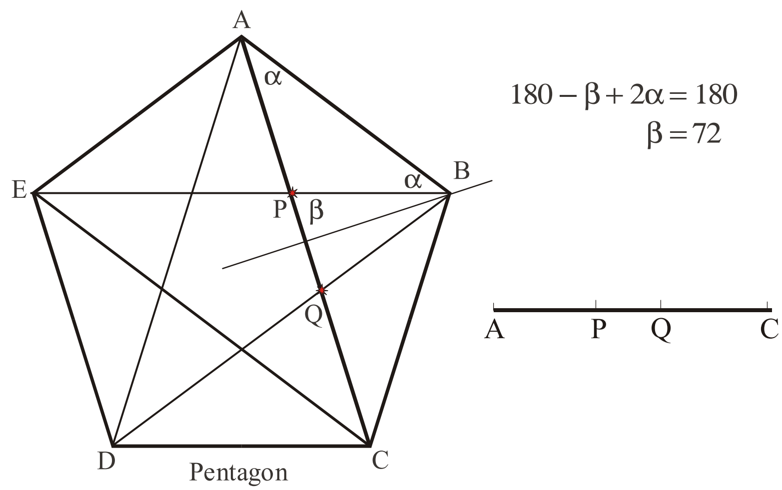
For example,

Proposition IV-5. About a given triangle to circumscribe a circle.

Proposition IV-10. To construct an isosceles triangle having each of the angles at the base double of the remaining one.

Proposition IV-10 is the key to proving the celebrated

Proposition IV-11. In a given circle to inscribe an equilateral and equiangular pentagon.



4.6.1 More regular figures.

The next regular figure to be inscribed in a circle was the 17-gon. But this is not in *The Elements* . Requiring more than 2100 years to find it, the key was understanding which polygon it should be. For this the spark of a young genius in the form of no less a mathematician than Carl Frederich Gauss (1777 - 1855) was needed. He discovered the 17-gon in 1796, at age 18.

In fact, when he was a student at Göttingen, he began work on his major publication *Disquisitiones Arithmeticae*, one of the great classics of the mathematical literature. Toward the end of this work, he included this result about the 17-gon but more!!! He proved that the only regular polygons that can be inscribed in a circle have

$$N = 2^m p_1 p_2 \dots p_r$$

sides, where m is a integer and the $p's$ are Fermat primes.

⋆ **Fermat numbers** are of the form

$$2^{2^n} + 1.$$

where n is an integer. For the first few integers they are prime and are called Fermat primes.

We have the following table of polygons that can be inscribed in a circle:

n	$2^{2^n}+1$	discoverer
0	3	ancients
1	5	ancients
2	17	Gauss
3	65	Gauss
4	65,537	Gauss

For many years, it was an open question as to whether all such numbers, $2^{2^n} + 1$, primes? In about 1730 another young genius, Leonhard Euler (1707 - 1783) factored the next one as $2^{2^5}+1 = 4,294,967,296 = 641 \cdot 6,700,417$ The Fermat numbers were not all primes. Indeed, no others are known as primes. A contemporary of Gauss, Fernidand Eisenstein (1823-1852) conjectured the following subset of the Fermat numbers consists only of primes:

$$2^2 + 1,\ 2^{2^2} + 1,\ 2^{2^{2^2}} + 1,\ 2^{2^{2^{2^2}}} + 1, \ldots$$

This conjecture has not been verified. The first three are the Fermat primes, 5, 17, 65,537. The next number has 19,729 digits. Even though prime numbers are now known with millions of digits, this number, with not even 20,000 digits, is almost intractable. It is not a Mersenne number (i.e. a number of the form $2^p - 1$, where p is a prime), so the Lucas-Lehmer test does not apply[5]. This limits the tests that can be applied. The most primitive test, that of attempting to divide all primes with 10,000 or fewer digits would require vastly more than the storage capacity of all the computers on earth to hold them and far more than the computational power of them all to perform the calculations.[6] So, another special test must be determined if the primality of such numbers is to be tested.

[5] See the chapter on Pythagoras and the Pythagoreans

[6] Just to convince you of this, simply suppose that everyone on earth has a computer with 1000 gigabites of storage and that the governments has 100 times that collective amount. This gives less than $10^{14+10} = 10^{24}$ bytes of storage. Compare this with just the number of primes, not even the storage requirements, in the required range range, which exceeds 10^{9995} bytes. Furthermore, if each of these computers operated at 1 *teraflop* (1 trillion floating point operations per second), only about 10^{37} computations could be carried out in the next century.

4.7 The Elements — Book V — 25 theorems

Book V treats ratio and proportion. Euclid begins with 18 definitions about magnitudes beginning with a part, multiple, ratio, be in the same ratio, and many others. **Definition 1.** A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.

This means that it divides the greater with no remainder.

Definition 4. Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, exceeding on another.

This is essentially the Archimedian Axiom: *If* $a < b$*, then there is an integer* n *such that* $na > b$*.*

In the modern theory of partially ordered spaces, a special role is played by those spaces which have the so-called *Archimedian Property*. Consider Definition 5 on same ratios devised by Eudoxus to reckon with incommensurables.

Definition 5. Magnitudes are said to *be in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

In modern notation, we say the magnitudes, a, b, c, d are *in the same ratio* $a : b = c : d$ if for all positive integers m and n

$$ma > mc \quad \text{then} \quad nb > nd,$$

and similarly for $<$ and $=$. Subtly, this definition requires an infinity of tests to verify two sets of numbers are in the same ratio.

Proposition V-1. If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will all be of all.

In modern notation, let the magnitudes be $a_1, a_2, \cdots, a_n$ and let m be the multiple. Then,

$$ma_1 + ma_2 + \cdots + ma_n = m(a_1 + a_2 + \cdots + a_n).$$

Proposition V-8. Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

In modern term, let $a > b$, and c is given. Then

$$a/c > b/c,$$

and

$$c/b > c/a.$$

The Elements — Book VI — 33 theorems

Book VI is on **similarity** of figures. It begins with three definitions.

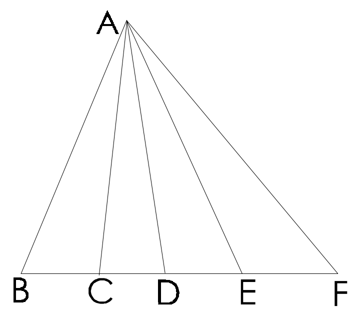
Definition 1. *Similar rectilineal figures* are such as have their angles severally equal and the sides about the equal angles proportional.

Definition 2. A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment, so is the greater to the less.

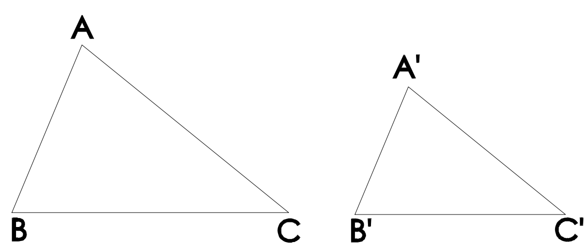
Definition 3. The height of any figure is the perpendicular drawn from the vertex to the base.

4.8 The Elements — Book VI

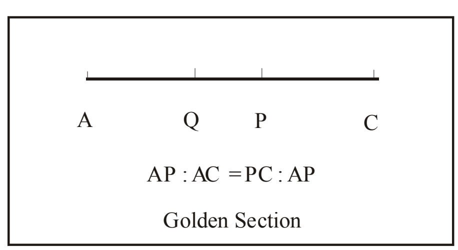
Proposition VI-1. Triangles and parallelograms which are under the same height are to one another as their bases.

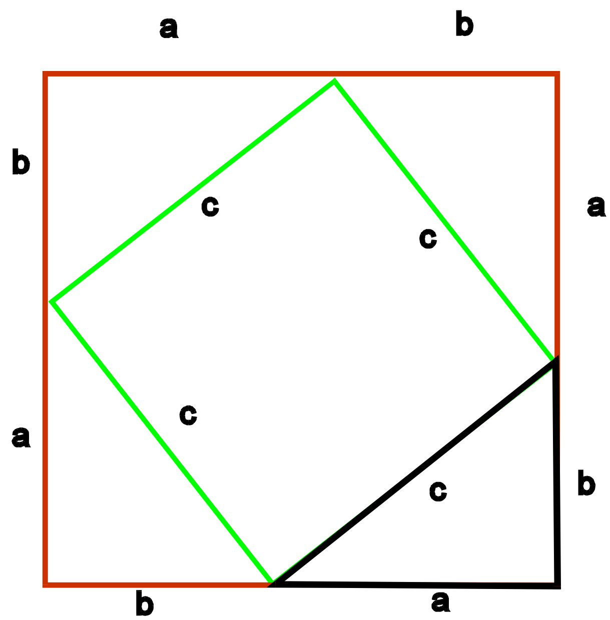


Proposition VI-5. If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.



Proposition VI-30. To cut a given finite straight line in extreme and mean ratio.





Proof of Pythagorean Theorem using similarity

The picture says....

$$\begin{aligned} (a+b)^2 &= c^2 + 4(\frac{1}{2}ab) \\ a^2 + 2ab + b^2 &= c^2 + 2ab \\ a^2 + b^2 = c^2 & \end{aligned}$$

Of course, you must prove all the similarity rigorously.

4.9 *The Elements* — Book VII — 39 theorems

Book VII is the first book of three on number theory. Euclid begins with definitions of unit, number, parts of, multiple of, odd number, even number, prime and composite numbers, etc.

Definition 11. A *prime* number is that which is measured by the unit alone.

Definition 12. Numbers *prime to one another* are those which are measured by the unit alone as a common measure.

Proposition VII-21. Numbers prime to one another are the least of those which have the same ratio with them.

Proposition VII-23. If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number.

Proposition VII-26. If two numbers be prime to two numbers, both to each, their products also will be prime to one another.

Proposition VII-31. Any composite number is measured by some prime number.

Proposition VII-32. Any number either is prime or is measured by some prime number.

4.10 *The Elements* — Book VIII — 27 theorems

Book VIII focuses on what we now call geometric progressions, but were called **continued proportions** by the ancients. Much of this is no doubt due to Archytas of Tarentum, a Pythagorean. Numbers are in continued proportion if

$$a_1 : a_2 = a_2 : a_3 = \dots .$$

We would write this as

$$a_1 = a,\ a_2 = ar,\ a_3 = ar^2,\ a_4 = ar^3, \dots .$$

which is of course the same thing.

Proposition VII-1. If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them.

Consider 5:3 and 8:6 and 10:6 and 16:12.

Proposition VIII-8. If between two numbers there are numbers in continued proportion with them, then, however any numbers are between them in continued proportion, so many will also be in continued proportion between numbers which are in the same ratio as the original numbers.

Euclid concerns himself in several other propositions of Book VIII with determining the conditions for inserting mean proportional numbers between given numbers of various types. For example,

Proposition VIII-20. If one mean proportional number falls between two numbers, the numbers will be similar plane numbers.

In modern parlance, suppose $a : x = x : b$, then $x^2 = ab$.

4.11 *The Elements* — Book IX — 36 theorems

The final book on number theory, Book IX, contains more familiar type number theory results.

Proposition IX-20. Prime numbers are more than any assigned multitude of prime numbers.

Proof. Let $p_1, \ldots, p_n$ be all the primes. Define $N = p_1 p_2 \cdots p_n + 1$. Then, since N must be composite, one of the primes, say $p_1 \mid N$. But this is absurd!

1 Proposition IX-35. If as many numbers as we please are in continued proportion, and there is subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it.

We are saying let the numbers be $a, ar, ar^2, \ldots, ar^n$, The the differences are $a(r-1)$ and $a(r^n-1)$. Then, the theorem asserts that

$$a/a(r-1) = (a + ar + \cdots + ar^{n-1})/a(r^n - 1).$$

Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

say that there are more prime numbers than *A, B, C.*

Proposition 36

If as many numbers as we please beginning from an unit be set out continuously If double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

For let as many numbers as we please, *A, B, C, D,* beginning from an unit be set out in double proportion, until the sum of all becomes prime,

let *E* be equal to the sum, and let *E* by multiplying *D* make *FG;*
I say that *FG* is perfect.

For, however many *A, B C, D* are in multitude, let so many *E, RK, L, M* be taken in double proportion beginning from *E;*

therefore, ex *qequali* as *A* is to *D,* so is *E* to *M.* [vii. 14]
Therefore the product of *E, D* is equal to the product of *A, M* [vii. **19]**

and the product of E, D *1)* is *FG;*

4.12 *The Elements* — Book X — 115 theorems

Many historians consider this the most important of the thirteen books. It is the longest and probably the best organized. The purpose is the classification of the incommensurables. The fact that the anathema to the Pythagoreans, the incommensurable is placed in Book X, the number of greatest significance to them, may be more than a coincidence. Perhaps a slight toward the Pythagoreans; perhaps a sense of humor — if not, the irony is almost as remarkable.

Definition 1. Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.

Note in the following definition how Euclid distinguishes magnitudes and lengths/areas.

Definition 1. Straight lines are *commensurable in square* when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure.

The first proposition is fundamental. It is Eudoxus' *method of exhaustion*.

Proposition X-I. Two unequal magnitudes being given, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, there will be left some magnitude less that the lesser of the given magnitudes.

This proposition allows an approximating process of arbitrary length.

Proposition X-36. If two rational straight lines commensurable in square only be added together, the whole is irrational.

4.13 *The Elements* — Book X1-XIII

The final three chapters of The **Elements** are on solid geometry and the use of a limiting process in the resolution of area and volume problems. For example,

Proposition XII-2. Circles are to one another as the squares on the diameters.

$\star$ You will note there is no "formula" expressed.

Proposition XII-7. An pyramid is a third part of the prism which has the same base with it an equal height.

Proposition XII-18. Spheres are to one another in the triplicate ratio of their respective diameters.