11^{th} Philippine Mathematical Olympiad

Questions, Answers, and Hints

Questions

Qualifying Stage, 18 October 2008

Part I. Each correct answer is worth two points.

1. Simplify: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{11}$.

(a) $\frac{940}{1155}$ (b) $\frac{941}{1155}$ (c) $\frac{942}{1155}$ (d) $\frac{943}{1155}$

2. If 30% of p is q, and 20% of q is 12, what is 50% of $p + q$?

(a) $\frac{1}{2}(q + 12)$ (b) 130 (c) $\frac{3}{10}p + \frac{1}{5}q$ (d) 100

3. Which of the following is equal to $\sqrt[3]{54} + \sqrt[6]{4}$?

(a) $\sqrt[3]{128}$ (b) $\sqrt[3]{56}$ (c) $\sqrt[9]{54 \cdot 4}$ (d) $\sqrt[6]{112}$

4. Let $ABCD$ be a square with each side of length 1 unit. If M is the intersection of its diagonals and P is the midpoint of MB, what is the square of the length of AP?

(a) $\frac{3}{4}$ (b) $\frac{5}{8}$ (c) $\frac{1}{2}$ (d) $\frac{3}{8}$

5. How many ordered pairs (x, y) of positive integers satisfy $2x + 5y = 100$?

(a) 8 (b) 9 (c) 10 (d) 11

6. Find the area of the circle that circumscribes a right triangle whose legs are of lengths 6 cm and 10 cm.

(a) 34π cm^2 (b) 68π cm^2 (c) 102π cm^2 (d) 136π cm^2

7. How many real roots does the equation $\log_{(x^2-3x)^3} 4 = \frac{2}{3}$ have?

(a) 0 (b) 2 (c) 3 (d) 4

8. Find the largest integer value of n such that $1 \times 3 \times 5 \times 7 \times \cdots \times 31 \times 33 \times 35$ is divisible by 3^n.

(a) 6 (b) 7 (c) 8 (d) 9

9. Let $0 < x < 1$. Which of the following has the largest value?

(a) x^3 (b) $x^2 + x$ (c) $x^2 + x^3$ (d) x^4

10. Find the sum of all the 4-digit positive numbers with no zero digit.

(a) 36644635 (b) 36644645 (c) 36445335 (d) 36446355

11. Find the number of real roots of the equation $x^5 - x^4 + x^3 - 4x^2 - 12x = 0$.

(a) 0 (b) 1 (c) 3 (d) 5

12. Let $ABCD$ be a trapezoid with $BC \| AD$ and $AB = BC = CD = \frac{1}{2}AD$. Determine $\angle ACD$.

(a) $75°$ (b) $85°$ (c) $90°$ (d) $135°$

13. How many 4-digit positive numbers, whose digits are from the set $\{1, 2, 3, 4\}$, are divisible by 4?

(a) 16 (b) 32 (c) 48 (d) 64

14. How many real numbers x satisfy the equation $x^x = x^2$?

(a) 1 (b) 2 (c) 4 (d) infinite

15. Let x be the solution of the equation

$$\frac{x+1}{1} + \frac{x+2}{2} + \frac{x+3}{3} + \cdots + \frac{x+2007}{2007} + \frac{x+2008}{2008} = 2008.$$

Which of the following is true?

(a) $x > 1$ (b) $x = 1$ (c) $0 < x < 1$ (d) $x \le 0$

Part II. Each correct answer is worth three points.

16. The roots of the quadratic equation $x^2 - 51x + k = 0$ differ by 75, where k is a real number. Determine the sum of the squares of the roots.

(a) 756 (b) 3825 (c) 4113 (d) 5625

17. Marco plans to give (not necessarily even) his eight marbles to his four friends. If each of his friends receives at least one marble, in how many ways can he apportion his marbles?

(a) 32 (b) 33 (c) 34 (d) 35

18. How many triangles (up to congruence) with perimeter 16 cm and whose lengths of its sides are integers?

(a) 3 (b) 4 (c) 5 (d) 6

19. Which of the following is not satisfied by any solution of the system

$$\begin{cases} x^2 - xy - 2y^2 & = & 4 \\ x^2 + 2xy + 3y^2 & = & 3\ ? \end{cases}$$

(a) $x = -2y$ (b) $9y = -x$ (c) $x = -4y$ (d) $y^2 = 1$

20. If a, b, c, d are real numbers with $a > b$ and $c > d$, which of the following is always true?

(a) $ac > bd$ (c) $ac + bd > ad + bc$

(b) $a^2 + c^2 > b^2 + d^2$ (d) $ad + bc > ac + bd$

21. Given that $0 < b < a$ and $a^2 + b^2 = 6ab$, what is the value of $\dfrac{a-b}{a+b}$?

(a) $\sqrt{2}$ (b) $1 + \sqrt{2}$ (c) $\frac{1}{2}\sqrt{2}$ (d) $-1 + \sqrt{2}$

22. How many values of n for which n and $\dfrac{n+3}{n-1}$ are both integers?

(a) 3 (b) 4 (c) 5 (d) 6

23. In an isosceles triangle ABC, where $AB = BC$, there exists a point P on the segment AC such that $AP = 6$, $PC = 4$, and $BP = 2$. Determine the perimeter of triangle ABC.

(a) $10 + 2\sqrt{7}$ (b) $10 + 4\sqrt{7}$ (c) $12 + 2\sqrt{7}$ (d) $12 + 4\sqrt{7}$

24. Each side of the square $ABCD$ is 12 meters long. The side AB is divided into three equal segments: AE, EF, and FB. Segments CE and DF intersect at point H. Find the area of triangle HCD.

(a) 48 m^2 (b) 54 m^2 (c) 60 m^2 (d) 72 m^2

25. An operation $*$ on the set of positive integers is defined by $a * b = (a + b)^{a-b}$. Evaluate

$$1024 * (512 * (256 * (128 * (64 * (32 * (16 * (8 * (4 * (2 * 1))))))))).$$

(a) 2027 (b) 2028 (c) 2047 (d) 2048

Part III. Each correct answer is worth six points.

26. If $a^3 + 12ab^2 = 679$ and $9a^2b + 12b^3 = 978$, find $a^2 - 4ab + 4b^2$.

(a) 1 (b) 9 (c) 25 (d) 49

27. How many ordered pairs (x, y) of positive integers satisfy the equation $\sqrt{y} = \sqrt{17} + \sqrt{x}$?

(a) none (b) 1 (c) 2 (d) infinite

28. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(a+b) = f(a)+f(b)$ and that $f(2008) = 3012$. What is $f(2009)$?

(a) 3012.5 (b) 3013 (c) 3013.5 (d) 3014

29. The length of each side of the squares $ABCD$ and $DEFG$ (both labeled in counter-clockwise direction) is 5 units. If AG is 8 units, how many units is BF?

(a) 11 (b) 12 (c) 13 (d) 14

30. A point M is chosen inside the square $ABCD$ in such a way that $\angle MAC = \angle MCD = x$. Determine $\angle ABM$.

(a) $90 - 2x$ (b) $180 - 3x$ (c) $90 - x$ (d) $2x$

Area Stage, 22 November 2008

Part I. No solution is needed. All answers must be in simplest form. Each correct answer merits two points.

1. For what integer a does the compound inequality $a < \sqrt{48} + \sqrt{140} < a + 1$ hold?

2. For what real numbers x is the equation $(x^2 + 6x + 10)^{x^2+2} = 1$ true?

3. Each element of the arithmetic sequence $101, 111, 121, \ldots, 201$ is multiplied to each element of the arithmetic sequence $212, 222, 232, \ldots, 302$. What is the sum when all these products are added?

4. Let x be a real number such that $\csc x + \cot x = 3$. Evaluate $\csc x - \cot x$.

5. The sum of the areas of two similar polygons is 65 square units. If their perimeters are 12 units and 18 units, respectively, find the area of the larger polygon.

6. Find all positive real values of x for which $x^{x^x} = (x^x)^x$.

7. In quadrilateral $ABCD$, it is given that $AB = BC = 4$ cm, $\angle ABC = 100°$, and $\angle CDA = 130°$. Find the length of BD.

8. How many polynomials are there of the form $x^3 - 10x^2 + cx + d$, where c and d are integers, such that the three roots are distinct positive integers?

9. The bases of a trapezoid are 9 in and 15 in, respectively. Its altitude is 7 in. Find the area of the quadrilateral formed by joining the midpoints of the adjacent sides of the trapezoid.

10. If $x \in \mathbb{R}$, what is the least possible value of the expression $|x + 1| + |x - 2| + |5 - x|$?

11. Chuckie was born before the year 2000. This year 2008, his age is exactly the sum of the digits of his year of birth. How old is Chuckie now?

12. Let $x, y, z \in \mathbb{R}$ such that $6x^2 + y^2 + z^2 = 2x(y + 2z)$. What is $x + y + z$?

13. In how many ways can the letters of the word OLYMPIAD be arranged if the vowels must be in alphabetical order?

14. Find the largest integer that divides all terms of the sequence $\{a_n\}$, where $a_n = n^5 - n$, $n \geq 1$.

15. A right triangle has sides of integral length and its area is equal to its perimeter. What is the least possible length of one of its legs?

16. Give the prime factorization of $3^{20} + 3^{19} - 12$.

17. How many integers between 2 and 10000 do not share a prime factor with 10000?

18. In isosceles triangle ABC, the base angles at B and C measure $40°$. The bisector of angle B intersects AC at point D, and BD is extended to point E so that $DE = AD$. Find $\angle BEC$.

19. At least how many 3-digit composite numbers should be chosen to ensure that at least two of the chosen numbers are not relatively prime?

20. Let AB and BC be two consecutive sides of a regular pentagon inscribed in a unit circle (that is, a circle of radius 1 unit). Find the value of $(AB \cdot AC)^2$.

Part II. Show the solution to each problem. A complete and correct solution merits ten points.

21. Consider the numbers $1, 10, 19, \ldots, 2008$, which form an arithmetic sequence. A number n is the sum of eleven distinct numbers from this sequence. How many (different) possible values of n are there?

22. Let a, b, and c be nonnegative real numbers such that $a + b + c = 1$. Prove that

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq \frac{1}{3}.$$

23. The bisector of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at a second point D. Let AD and BC intersect at point E, and F be the midpoint of segment BC. If $AB^2 + AC^2 = 2AD^2$, show that $EF = DF$.

National Stage, 24 January 2009

Oral Phase

15-Second Round. Each correct answer credits two points.

15.1 Define the operation "$\star$" by $a \star b = 4a - 3b + ab$ for all $a, b \in \mathbb{R}$. For what real numbers y does $12 = 3 \star y$?

15.2 Six numbers from a list of nine integers are 4, 9, 3, 6, 7, and 3. What is the largest possible value of the median of all nine integers in this list?

15.3 Triangle ABC has vertices with coordinates $A(1, 3)$, $B(4, 1)$, and $C(3, -5)$. A point D on AC is chosen so that the area of triangle ABD is equal to the area of triangle CBD. Find the length of segment BD.

15.4 How many positive integers less than 2009 are divisible by 28 but not by 12?

15.5 Find all real numbers x that satisfy the equation $(x^3 - x)^{x^3+x^2-2x} = 0$.

15.6 What is the maximum number of points of intersection of the graphs of two different fourth-degree polynomial functions $y = P(x)$ and $y = Q(x)$, each with leading coefficient 1?

15.7 Let f be a function satisfying $f(xy) = \dfrac{f(x)}{y}$ for all positive real numbers x and y. If $f(2008) = 1$, what is $f(2009)$?

15.8 Simplify the following expression

$$\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}.$$

15.9 If the letters of the word MATHEMATICS are repeatedly and consecutively written, what is the 2009th letter?

15.10 A rectangular piece of paper, 24 cm long by 18 cm wide, is folded once in such a way that two diagonally opposite corners coincide. What is the length of the crease?

15.11 A chemist has x liters of sugar solution that is $x\%$ sugar. How many liters of sugar must be added to this solution to yield a sugar solution that is $3x\%$ sugar?

15.12 In how many ways can 60 students be distributed into 6 buses if a bus can contain zero to 60 students?

15.13 If r_1, r_2, r_3, r_4 are the roots of the equation $4x^4 - 3x^3 - x^2 + 2x - 6 = 0$, what is

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\ ?$$

15.14 If $\sin x + \cos x = \sqrt{3} - 1$, what is $\sin 2x$?

15.15 The Philippines officially joined the International Mathematical Olympiad (IMO) in 1989. In what country was this IMO held?

30-Second Round. Each correct answer credits three points.

30.1 The first, fourth, and eighth terms of a nonconstant arithmetic sequence form a geometric sequence. If its twentieth term is 56, what is its tenth term?

30.2 Let α and β be acute angles such that $\tan\alpha = \frac{1}{7}$ and $\sin\beta = \frac{\sqrt{10}}{10}$. Find $\cos(\alpha + 2\beta)$.

30.3 A point P is outside a circle and is 15 cm from the center. A secant through P cuts the circle at Q and R so that the external segment PQ is 9 cm and QR is 8 cm. Find the radius of the circle.

30.4 What is the coefficient of x^5 in the polynomial expansion of $(2 - x + x^2)^4$?

30.5 All the students in a geometry class took a 100-point test. Six students scored 100, each student scored at least 50, and the mean score was 68. What is the smallest possible number of students in the class?

30.6 Let $a \geq b > 1$. What is the largest possible value of

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} \ ?$$

30.7 Let $P(x)$ be a polynomial that, when divided by $x - 19$, has the remainder 99, and, when divided by $x - 99$, has the remainder 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

30.8 In the plane, two concentric circles with radii 8 cm and 10 cm are given. The smaller circle divides a chord of the larger circle into three equal parts. Find the length of the chord.

30.9 Let f be a function such that $f(2 - 3x) = 4 - x$. Find the value of $\sum_{i=1}^{12} f(i)$.

30.10 The number 63 999 999 has exactly five prime factors. Find their sum.

60-Second Round. Each correct answer credits six points.

60.1 Equilateral triangle DEF is inscribed in equilateral triangle ABC with $DE \perp BC$. If the area of $\triangle DEF$ is 6 cm^2, what is the area of $\triangle ABC$?

60.2 For any positive integer n, define

$$f(n) = \begin{cases} \log_9 n & \text{if } \log_9 n \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

What is $\sum_{n=1}^{2009} f(n)$?

60.3 A point P is selected at random from the interior of the pentagon with vertices $A = (0, 2)$, $B = (4, 0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 4)$, and $E = (0, 4)$. What is the probability that $\angle APB$ is acute?

60.4 Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the positive integer n. Determine all two-digit numbers N that satisfy the equation

$$N = P(N) + 2S(N).$$

60.5 The vertices of a cube are each colored by either black or white. Two colorings of the cube are said to be *geometrically the same* if one can be obtained from the other by rotating the cube. In how many geometrically different ways can such coloring of the cube be done?

Written Phase

You are given three hours to solve all problems. Each item is worth eight points.

1. The sequence $\{a_0, a_1, a_2, \ldots\}$ of real numbers satisfies the recursive relation

$$n(n+1)a_{n+1} + (n-2)a_{n-1} = n(n-1)a_n$$

for every positive integer n, where $a_0 = a_1 = 1$. Calculate the sum

$$\frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{2008}}{a_{2009}}.$$

2. **(a)** Find all pairs (n, x) of positive integers that satisfy the equation $2^n + 1 = x^2$.

(b) Find all pairs (n, x) of positive integers that satisfy the equation $2^n = x^2 + 1$.

3. Each point of a circle is colored either red or blue.

(a) Prove that there always exists an isosceles triangle inscribed in this circle such that all its vertices are colored the same.

(b) Does there always exist an equilateral triangle inscribed in this circle such that all its vertices are colored the same?

4. Let k be a positive real number such that

$$\frac{1}{k+a} + \frac{1}{k+b} + \frac{1}{k+c} \leq 1$$

for any positive real numbers a, b, and c with $abc = 1$. Find the minimum value of k.

5. Segments AC and BD intersect at point P such that $PA = PD$ and $PB = PC$. Let E be the foot of the perpendicular from P to the line CD. Prove that the line PE and the perpendicular bisectors of the segments PA and PB are concurrent.

Answers and Hints

Qualifying Stage

1. b	**6.** a	**11.** c	**16.** c	**21.** c	**26.** b
2. b	**7.** b	**12.** c	**17.** d	**22.** d	**27.** d
3. a	**8.** d	**13.** d	**18.** c	**23.** b	**28.** c
4. b	**9.** b	**14.** b	**19.** c	**24.** b	**29.** d
5. b	**10.** d	**15.** d	**20.** c	**25.** c	**30.** a

Area Stage

1. 18	**11.** 23
2. -3	**12.** 0
3. 4268770	**13.** 6720
4. 1/3	**14.** 30
5. 45	**15.** 5
6. 1 and 2	**16.** $2^5 \cdot 3 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 757$
7. 4	**17.** 3999
8. 4	**18.** $80°$
9. 42	**19.** 12
10. 6	**20.** 5

21. Write $n = 11 + 9(a_1 + a_2 + \cdots + a_{11})$, where $a_1, a_2, \ldots, a_{11}$ are distinct elements of the set $\{0, 1, 2, \ldots, 223\}$. Let $X = a_1 + a_2 + \cdots + a_{11}$. By showing that the integers from 55 to 2398 are possible values of X, there are $2398 - 55 + 1 = 2344$ possible values of X (and also of n).

22. The desired inequality follows from the following inequalities:

(i) $2a\sqrt{bc} \le ab + ac$, $2b\sqrt{ac} \le ab + bc$, $2c\sqrt{ab} \le ac + bc$

(ii) $4a\sqrt{bc} \le a^2 + a^2 + b^2 + c^2$, $4b\sqrt{ac} \le a^2 + b^2 + b^2 + c^2$, $4c\sqrt{ab} \le a^2 + b^2 + c^2 + c^2$.

These inequalities follow from the AM-GM Inequality.

23. By symmetry, we assume that $AB < AC$. Use Ptolemy's Theorem and the Angle Bisector Theorem to show that

$$DF = EF = \frac{BC(AC - AB)}{2(AB + AC)}.$$

National Stage, Oral Phase

15.1. all real numbers

15.2. 7

15.3. $2\sqrt{2}$

15.4. 48

15.5. -1

15.6. 3

15.7. 2008/2009

15.8. 2

15.9. A

15.10. 22.5 cm

15.11. $2x^2/(100 - 3x)$

15.12. 6^{60}

15.13. 1/3

15.14. $3 - 2\sqrt{3}$

15.15. Germany

30.1. 36

30.2. $\sqrt{2}/2$

30.3. $6\sqrt{2}$ cm

30.4. -28

30.5. 17

30.6. 0

30.7. $-x + 118$

30.8. $9\sqrt{2}$ cm

30.9. 66

30.10. 577

60.1. 18 cm^2

60.2. 21/2

60.3. 11/16

60.4. 14, 36, and 77

60.5. 24

National Stage, Written Phase

1. It can be proved, by induction on n, that $a_{n-1}/a_n = n$ for every positive integer n. Thus, we obtain

$$\frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{2008}}{a_{2009}} = 1 + 2 + \cdots + 2009 = \frac{(2009)(2010)}{2} = 2019045.$$

2. **(a)** Rewrite the given equation into $2^n = (x-1)(x+1)$. With the power-of-two argument, the only pair that satisfies the equation is $(3, 3)$.

(b) If $n = 1$, then $x = 1$. So, $(1, 1)$ satisfies the equation. It is not possible that $n \geq 2$. Thus, $(1, 1)$ is the only pair that satisfies the equation.

3. **(a)** By the Pigeonhole Principle, there are three vertices of the same color that form an isosceles triangle.

(b) There is a coloring of the points of the circle such that no inscribed equilateral triangle has three vertices of the same color.

4. After considering three cases (when $k = 2$, when $k > 2$, and when $0 < k < 2$), one can conclude that the minimum value of k is 2.

5. Let the perpendicular bisectors of PA and PB meet at O. Let OP meet CD at F. It suffices to show that $OF \perp CD$.