Practical Applied Mathematics
Modelling, Analysis, Approximation

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Chapter 1

Introduction

Book born out of fascination with applied math as meeting place of physical world and mathematical structures.

have to be generalists, anything and everything potentially interesting to an applied mathematician

1.1 What is modelling/why model?

1.2 How to use this book

case studies as strands

must do exercises

1.3 acknowledgements

Have taken examples from many sources, old examples often the best. If you teach a course using other peoples' books and then write your own this is inevitable.

errors all my own

ACF, Fowkes/Mahoney, O2, green book, Hinch, ABT, study groups

Conventions. Let me introduce a couple of conventions that I use in this book. I use 'we', as in 'we can solve this by a Laplace transform', to signal the usual polite fiction that you, the reader, and I, the author, are engaged on a joint voyage of discovery. 'You' is mostly used to suggest that *you* should get your pen out and work though some of the 'we' stuff, a good idea in view of my fallible arithmetic. 'I' is associated with authorial opinions and can mostly be ignored if you like.

I have tried to draw together a lot of threads in this book, and in writing it I have constantly felt the need to sidestep in order to point out a connection with something else. On the other hand, I don't want you to lose track of the argument. As a compromise, I have used marginal notes and footnotes[1] with slightly different purposes.

Marginal notes are usually directly relevant to the current discussion, often being used to fill in details or point out a feature of a calculation.

[1]Footnotes are more digressional and can, in principle, be ignored.

Part I

Modelling techniques

Chapter 2

The basics of modelling

2.1 Introduction

This short introductory chapter is about mathematical modelling. Without trying to be too prescriptive, we discuss what we mean by the term modelling, why we might want to do it, and what kind of models are commonly used. Then, we look at some very standard models which you have almost certainly met before, and we see how their derivation is a blend of what are thought of as universal physical laws, such as conservation of mass, momentum and energy, with experimental observations and, perhaps, some ad hoc assumptions in lieu of more specific evidence.

One of the themes that run through this book is the applicability of all kinds of mathematical ideas to 'real-world' problems. Some of these arise in attempts to explain natural phenomena, for example models for water waves. We will see a number of these models as we go through the book. Other applications are found in industry, which is a source of many fascinating and non-standard mathematical problems, and a big 'end-user' of mathematics. You might be surprised to know how little is known of the detailed mechanics of most industrial processes, although when you see the operating conditions — ferocious temperatures, inaccessible or minute machinery, corrosive chemicals — you realise how expensive and difficult it would be to carry out detailed experimental investigations. In any case, many processes work just fine, having been designed by engineers who know their job. So where does mathematics come in? Some important uses are in quality control and cost control for existing processes, and simulation and design of new ones. We may want to understand why a certain type of defect occurs, or what is the 'rate-limiting' part of a process (the slowest ship, to be speeded up), or whether a novel idea is likely to work at all and if so, how to control it.

It is in the nature of real-world problems that they are large, messy and often rather vaguely stated. It is very rarely worth anybody's while producing a 'complete solution' to a problem which is complicated and whose desired outcome is not necessarily well specified (to a mathematician). Mathematics is usually most effective in analysing a relatively small 'clean' subproblem where more broad-brush approaches run into difficulty. Very often, the analysis complements a large numerical simulation which, although effective elsewhere, has trouble with this particular aspect of the problem. Its job is to provide understanding and insight to complement simulation, experiment and other approaches.

We begin with a chat about what models are and what they should do for us. Then we bring together some simple ideas about physical conservation laws, and how to use them together with experimental evidence about how materials behave to formulate closed systems of equations; this is illustrated with two canonical models for heat flow and fluid motion. There are many other models embedded elsewhere in the book, and we deal with these as we come to them.

2.2 What do we mean by a model?

There is no point in trying to be too precise in defining the term mathematical model: we all understand that it is some kind of mathematical statement about a problem that is originally posed in non-mathematical terms. Some models are *explicative*: that is, they explain a phenomenon in terms of simpler, more basic processes. A famous example is Newton's theory of planetary motion, whereby the whole complex motion of the solar system was shown to be a consequence of 'force equals mass times acceleration' and the inverse square law of gravitation. However, not all models aspire to explain. For example, the standard Black–Scholes model for the evolution of prices in stock markets, used by investment banks the world over, says that the percentage difference between tomorrow's stock price and today's is a normal random variable. Although this is a great simplification, in that it says that all we need to know are the mean and variance of this distribution, it says nothing about what will cause the price change.

All useful models, whether explicative or not, are *predictive*: they allow us to make quantitative predictions (whether deterministic or probabilistic) which can be used either to test and refine the model, should that be necessary, or for use in practice. The outer planets were found using Newtonian mechanics to analyse small discrepancies between observation and theory,[1]

[1] This is a very early example of an *inverse problem*: assuming a model and given

and the Moon missions would have been impossible without this model. Every day, banks make billions of dollars worth of trades based on the Black–Scholes model; in this case, since model predictions do not always match market prices, they may use the latter to refine the basic model (here there is no simple underlying mechanism to appeal to, so adding model features in a heuristic way is a reasonable way to proceed).

Most of the models we discuss in this book are based on differential equations, ordinary or partial: they are in the main deterministic models of continuous processes. Many of them should already be familiar to you, and they are all accessible with the standard tools of real and complex analysis, partial differential equations, basic linear algebra and so on. I would, however, like to mention some kinds of models that we don't have the space (and, in some cases I don't have the expertise) to cover.

- Statistical models.

Statistical models can be both explicative and predictive, in a probabilistic sense. They deal with the question of extracting information about cause and effect or making predictions in a random environment, and describing that randomness. Although we touch on probabilistic models, for a full treatment see a text such as [33].

- Discrete models of various kinds.

Many, many vitally important and useful models are intrinsically discrete: think, for example of the question of optimal scheduling of take-off slots from LHR, CDG or JFK. This is a vast area with a huge range of techniques, impinging on practically every other area of mathematics, computer science, economics and so on. Space (and my ignorance) simply don't allow me to say any more.

- 'Black box' models such as neural nets or genetic algorithms.

The term 'model' is often used for these techniques, in which, to paraphrase, a 'black box' is trained on observed data to predict the output of a system given the input. The user need never know what goes on inside the black box (usually some form of curve fitting and/or optimisation algorithm), so although these algorithms can have some predictive capacity they can rarely be explicative. Although often useful, this philosophy is more or less orthogonal to that behind the models in this book, and if you are interested see [15].

observations of the solution, determine certain model parameters, in this case the unknown positions of Uranus and Neptune. A more topical example is the problem of constructing an image of your insides from a scan or electrical measurements from electrodes on your skin. Unfortunately, such problems are beyond the scope of this book; see [10].

2.3 Principles of modelling: physical laws and constitutive relations

Work is heat and heat is work: the First Law of Thermodynamics, in mnemonic form.

Many models, especially ones based on mechanics or heat flow (which includes most of those in this book) are underpinned by physical principles such as conservation of mass, momentum, energy and electric charge. We may have to think about how we interpret these ideas, especially in the case of energy which can take so many forms (kinetic, potential, heat, chemical, ...) and be converted from one to another. Although they are in the end subject to experimental confirmation, the experimental evidence is so overwhelming that, with care in interpretation, we can take these conservation principles as assumptions.[2]

However, this only gets us so far. We can do very simple problems such as mechanics of point particles, and that's about it. Suppose, for example, that we want to derive the heat equation for heat flow in a homogeneous, isotropic, continuous solid. We can reasonably assume that at each point $\mathbf{x}$ and time t there is an energy density $E(\mathbf{x}, t)$ such that the internal (heat) energy inside any fixed volume V of the material is

$$\int_V E(\mathbf{x}, t)\, d\mathbf{x}.$$

We can also assume that there is a heat flux vector $\mathbf{q}(\mathbf{x}, t)$ such that the rate of heat flow across a plane with unit normal $\mathbf{n}$ is

$$\mathbf{q} \cdot \mathbf{n}$$

per unit area. Then we can write down conservation of energy for V in the form

$$\frac{d}{dt}\int_V E(\mathbf{x}, t)\, d\mathbf{x} + \int_{\partial V} \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n}\, dS = 0,$$

on the assumption that no heat is converted into other forms of energy. Next, we use Green's theorem on the surface integral and, as V is arbitrary, the 'usual argument' (see below) gives us

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{q} = 0. \tag{2.1}$$

At this point, we have to bring in some experimental evidence. We need to relate both E and $\mathbf{q}$ to the temperature $T(\mathbf{x}, t)$, by what are called *constitutive relations*. For many, but not all, materials, the internal energy is directly

[2] So we are making additional assumptions that we are not dealing with quantum effects, or matter on the scale of atoms, or relativistic effects. We deal only with models for human-scale systems.

proportional to the temperature,[3] written

$$E = \rho c T,$$

where ρ is the density and c is a constant called the *specific heat capacity*. Likewise, *Fourier's law* states that the heat flux is proportional to the temperature gradient,

$$\mathbf{q} = -k\nabla T.$$

Ask yourself why there is a minus sign. The Second Law of Thermodynamics in mnemonic form: heat cannot flow from a cooler body to a hotter one.

Putting these both into (2.1), we have

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T$$

as expected. The appearance of material properties such as c and k is a sure sign that we have introduced a constitutive relation, and it should be stressed that these relations between E, $\mathbf{q}$ and T are material-dependent and experimentally determined. There is no *a priori* reason for them to have the nice linear form given above, and indeed for some materials one or other may be strongly nonlinear.

Another set of models where constitutive relations pay a prominent role is models for solid and fluid mechanics.

2.3.1 Example: inviscid fluid mechanics

Let us first look at the familiar Euler equations for inviscid incompressible fluid motion,

'Oiler', not 'Yewler'.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p, \qquad \nabla\cdot\mathbf{u} = 0.$$

Here $\mathbf{u}$ is the fluid velocity and p the pressure, both functions of position $\mathbf{x}$ and time t, and ρ is the fluid density. The first of these equations is clearly 'mass $\times$ acceleration = force', bearing in mind that we have to calculate the acceleration following a fluid particle (that is, we use the convective derivative), and the second is mass conservation (now would be a good moment for you to do the first two exercises if this is not all very familiar material; a brief derivation is given in the next section).

The constitutive relation is rather less obvious in this case. When we work out the momentum balance for a small material volume V, we want

Remember a *material volume* is one whose boundary moves with the fluid velocity, that is, it is made up of fluid particles.

[3] It is an experimental fact that temperature changes in most materials are proportional to energy put in or taken out. However, both c and k may depend on temperature, especially if the material gradually melts or freezes, as for paraffin or some kinds of frozen fish. Such materials lead to nonlinear versions of the heat equation; fortunately, many common substances have nearly constant c and k and so are well modelled by the linear heat equation.

to encapsulate the physical law

$$\text{convective rate of change of momentum in } V = \text{forces on } V.$$

On the left, the (convective) rate of change of momentum in V is

$$\int_V \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) dV.$$

We then say that this is equal to the force on V, which is provided solely by the pressure and acts normally to ∂V. This is our constitutive assumption: that the internal forces in an inviscid fluid are completely described by a pressure field which acts isotropically (equally in all directions) at every point. Then, ignoring gravity, the force on V is

$$\int_{\partial V} -p\mathbf{n}\, dS = -\int_V \nabla p\, dV$$

by a standard vector identity, and for arbitrary V we do indeed retrieve the Euler equations.

2.3.2 Example: viscous fluids

Things are a little more complicated for a *viscous* fluid, namely one whose 'stickiness' generates internal forces which resist the motion. This model will be unfamiliar to you if you have never looked at viscous flow. If this is so, you can

(a) Just ignore it: you will then miss out on some nice models for thin fluid sheets and fibres in chapter **??**, but that's about all;

(b) Go with the flow: trust me that the equations are not only believable (an informal argument is given below, and in any case I am assuming you know about the inviscid part of the model) but indeed correct. As one so often has to in real-world problems, see what the mathematics has to say and let the intuition grow;

(c) Go away and learn about viscous flow; try the books by [28] or [2].

Viscosity is the property of a liquid that measures its resistance to shearing, which occurs when layers of fluid slide over one another. In the configuration of Figure 2.1, the force per unit area on either plate due to viscous drag is found for many liquids to be proportional to the shear rate U/h, and

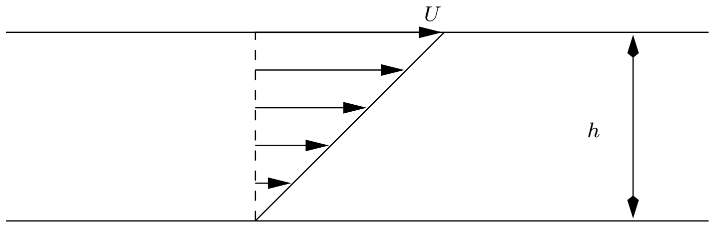

Figure 2.1: Drag on two parallel plates in shear, a configuration known as Couette flow. The arrows indicate the velocity profile.

is written $\mu U/h$ where the constant μ is called the *dynamic viscosity.* Such fluids are termed *Newtonian.*

Our strategy is again to consider a small element of fluid and on the left-hand side, work out the rate of change of momentum

$$\int_V \rho \frac{D\mathbf{u}}{Dt}\, dV,$$

while on the right-hand side we have

$$\int_{\partial V} \mathbf{F}\, dS,$$

the net force on its boundary. Then we use the divergence theorem to turn the surface integral into a volume integral and, as V is arbitrary, we are done.

Now for any continuous material, whether a Newtonian fluid or not, it can be shown (you will have to take this on trust: see [28] for a derivation) that there is a *stress tensor*, a matrix σ [NB want to get a bold greek font here, this one is not working] with entries σ_{ij} with the property that the force per unit area exerted by the fluid in direction i on a small surface element with normal n_j is $\sigma \cdot \mathbf{n} = \sigma_{ij} n_j$ (see Figure 2.2). It can also be shown that σ is symmetric: $\sigma_{ij} = \sigma_{ji}$. In an isotropic material (one with no built-in directionality), there are also some invariance requirements with respect to translations and rotations.

We are using the summation convention, that repeated indices are summed over from 1 to 3; thus for example

$$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}.$$

Is it clear that $\nabla \cdot \mathbf{u} = \partial u_i / \partial x_i$, and that

$$\nabla \cdot \sigma = \frac{\partial \sigma_{ij}}{\partial x_j}?$$

Thus far, our analysis could apply to any fluid. The force term in the equation of motion takes the form

$$\int_{\partial V} \sigma \cdot \mathbf{n}\, dS = \int_{\partial V} \sigma_{ij} n_j\, dS$$

which by the divergence theorem is equal to

$$\int_{\partial V} \nabla \cdot \sigma\, dS = \int_{\partial V} \frac{\partial \sigma_{ij}}{\partial x_j}\, dS,$$

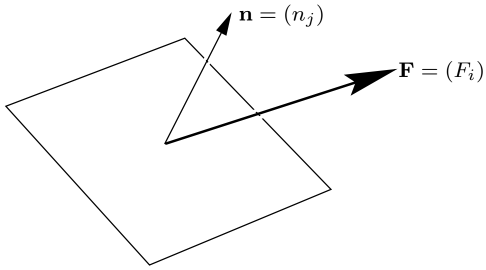

Figure 2.2: Force on a small surface element.

and so we have the equation of motion

$$\frac{D(\rho\mathbf{u})}{Dt} = \nabla \cdot \sigma. \tag{2.2}$$

We now have to say what kind of fluid we are dealing with. That is, we have to give a constitutive relation to specify σ in terms of the fluid velocity, pressure etc. For an inviscid fluid, the only internal forces are those due to pressure, which acts isotropically. The pressure force on our volume element is

$$\int_{\partial V} -p\mathbf{n}\,dS$$

with a corresponding stress tensor

$$\sigma_{ij} = -p\delta_{ij}$$

Which matrix has entries δ_{ij}? Interpret $\delta_{ij}v_j = v_i$ in matrix terms.

where δ_{ij} is the Kronecker delta. This clearly leads to the Euler momentum-conservation equation

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p.$$

When the fluid is viscous, we need to add on the contribution due to viscous shear forces. In view of the experiment of Figure **??**, it is very reasonable that the new term should be linear in the velocity gradients, and it can be shown, bearing in mind the invariance requirements mentioned above, that the appropriate form for σ_{ij} is

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For future reference we write out the components of σ in two dimensions:

$$\sigma_{ij} = \begin{pmatrix} -p + 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & -p + 2\mu\dfrac{\partial v}{\partial y} \end{pmatrix}. \tag{2.3}$$

Substituting this into the general equation of motion (2.2), and using the incompressibility condition $\nabla \cdot \mathbf{u} = \partial u_i/\partial x_i = 0$, it is a straightforward *exercise* to show that the equation of motion of a viscous fluid is

The emphasis mean you should do it.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0. \tag{2.4}$$

These equations are known as the *Navier–Stokes equations.* The first of them contains the corresponding inviscid terms, i.e.the Euler equations, with the new term $\mu\nabla^2\mathbf{u}$, which represents the additional influence of viscosity. As we shall see later, this term has profound effects.

2.4 Conservation laws

Perhaps we should elaborate on the 'usual argument' which, allegedly, leads to equation 2.1. Whenever we work in a continuous framework, and we have a quantity that is conserved, we offset changes in its *density*, which we call $P(\mathbf{x}, t)$ with equal and opposite changes in its *flux* $\mathbf{q}(\mathbf{x}, t)$. Taking a small volume V, and arguing as above, we have

$$\frac{d}{dt}\int_V P(\mathbf{x}, t)\, d\mathbf{x} + \int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS = 0,$$

the first term being the time-rate-of-change of the quantity inside V, and the second the net flux of it into V. Using Green's theorem on this latter integral,[4] we have

$$\int_V \frac{\partial P}{\partial t} + \nabla\cdot\mathbf{q}\, d\mathbf{x} = 0.$$

As V is arbitrary, we conclude that

$$\frac{\partial P}{\partial t} + \nabla\cdot\mathbf{q} = 0,$$

a statement which is often referred to as a *conservation law.*[5]

In the heat-flow example above, $P = \rho c T$ is the density of internal heat energy and $\mathbf{q} = -k\nabla T$ is the heat flux. Another familiar example is conservation of mass in a compressible fluid flow, for which the density is ρ and the mass flux is $\rho\mathbf{u}$, so that

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.$$

When the fluid is incompressible and of constant density, this reduces to

This is not as silly as it sounds: a fluid may be incompressible and have different densities in different places, the jargon being s*stratified.*

[4] Needless to say, this argument requires $\mathbf{q}$ to be sufficiently smooth, which can usually be verified *a posteriori*; in Chapter **??** we shall explore some cases where this smoothness is not present.

[5] Sometimes this term is reserved for cases in which $\mathbf{q}$ is a function of P alone.

$\nabla \cdot \mathbf{u} = 0$ as expected.

2.5 Conclusion

There are, of course, many widely used models that we have not described in this short chapter. Rather than give a long catalogue of examples, we'll move on, leaving other models to be derived as we come to them. We conclude with an important general point.

As stressed above, the construction of a model for a complicated process involves a blend of physical principles and (mathematical expressions of) experimental evidence; these may be supplemented by plausible ad hoc assumptions where direct experimental evidence is unavailable, or as a 'summary' model of a complicated system from which only a small number of outputs is needed. However, the initial construction of a model is only the first step in building a useful tool. The next task is to analyse it: does it make mathematical sense? Can we find solutions, whether explicit (in the form of a formula), approximate or numerical, and if so how? Then, crucially, what do these solutions (predictions) have to say about the original problem? This last step is often the cue for an iterative process in which discrepancies between predictions and observations prompt us to rethink the model. Perhaps, for example, certain terms or effects that we thought were small could not, in fact, safely be neglected. Perhaps some ad hoc assumption we made was not right. Perhaps, even, a fundamental mechanism in the original model does not work as we assumed (a negative result of this kind can often be surprisingly useful). We shall develop all of these themes as we go onwards.

Exercises

1. **Conservation of mass.** A uniform incompressible fluid flows with velocity $\mathbf{u}$. Take an arbitrary fixed volume V and show that the net mass flux across its boundary ∂V is

$$\int_{\partial V} \mathbf{u} \cdot \mathbf{n} \, dS.$$

Use Green's theorem to deduce that $\nabla \cdot \mathbf{u} = 0$. What would you do if the fluid were incompressible but of spatially-varying density (see §2.4)?

2. **The convective derivative.** Let $F(\mathbf{x}, t)$ be any quantity that varies with position and time, in a fluid with velocity $\mathbf{u}$. Let V be an arbitrary

material volume. Show that

$$\frac{D}{Dt}\int_V F\,dV = \int_V \frac{\partial F}{\partial t}\,dV + \int_{\partial V} F\mathbf{u}\cdot\mathbf{n}\,dS,$$

where the second term is there because the boundary of V moves. When the fluid is incompressible, use Green's theorem to deduce the convective derivative formula

Draw a picture of $V(t)$ and $V(t+\delta t)$ to see where it comes from.

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{u}\cdot\nabla F,$$

and verify that the left-hand side of the Euler momentum equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p$$

is the acceleration following a fluid particle.

3. **Potential flow has slip.** Suppose that a potential flow of an inviscid irrotational flow satisfies the no-slip condition $\mathbf{u} = \nabla\phi = \mathbf{0}$ at a fixed boundary. Show that the tangential derivatives of ϕ vanish at the surface so that ϕ is a constant (say zero) there. Show also that the normal derivative of ϕ vanishes at the surface and deduce from the Cauchy–Kowalevskii theorem (see [27]) that $\phi \equiv 0$ so the flow is static. (In two dimensions, you might prefer to show that $\partial\phi/\partial x - i\partial\phi/\partial y$ is analytic (= holomorphic), vanishes on the boundary curve, hence vanishes everywhere.)

4. **Waves on a membrane.** A membrane of density ρ per unit area is stretched to tension T. Take a small element A of it and use Green's theorem on the force balance

$$\iint_A \rho\frac{\partial^2 u}{\partial t^2}\,dA = \int_{\partial A} T\frac{\partial u}{\partial n}\,ds$$

to derive the equation of motion

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u,$$

where $c^2 = T/\rho$ is the wave speed.

Chapter 3

Units and dimensions

3.1 Introduction

This chapter and the next cover some simple ideas to do with dimensional analysis. They can be very helpful in understanding the basic physical mechanisms on which we will build mathematical models, but they are primarily the first step towards our main objective, to build up a systematic framework within which to assess such models of complex problems. Real-world situations, arising in industry or elsewhere, almost always involve many coupled physical processes. We may be able to write down models for each of them individually, and so for the whole, but faced with the resulting pages of equations, what then? Can we say anything about the 'structure' of the problem? What are the pivotal points? Are all the mechanisms we have put in equally significant? If not, how do we know, and which should we keep? Is it safe to put the equations on a computer?

We start with some basic material on dimensions and units; in the following chapter we move on to see how scaling reveals *dimensionless parameters* which, if small (or large) can point the way to useful approximation schemes. Along the way, we'll see gentle introductions to some of the models that we use repeatedly in later chapters. Almost all of these deal with reasonably familiar material and will not trouble you too much; the only possible exception is the material on electrostatics, and we don't have to do too much of that.

3.2 Units and dimensions

There is just one simple idea underpinning this section. If an equation models a physical process, then all the terms in it that are separated by $+$, $-$ or $=$

must have the same physical dimensions. If they did not, we would be saying something obviously ludicrous like

$$\text{apples} + \text{lawnmowers} = \text{light bulbs} + \text{whisky}.$$

This is the most basic of the many consistency (error-correcting) checks which you should build into your mathematics.

For example, is the answer real/positive/... when it obviously should be? Is it about the right size? If we expect the temperature to increase when we increase the input heat flux does our formula do what it should? Do the accelerations point in the same direction as the forces?

To quantify this idea, we'll use a fairly standard notation for the dimensions of quantities, denoted by square brackets: all units will be written in terms of the *primary quantities* mass $[\mathtt{M}]$, length $[\mathtt{L}]$, time $[\mathtt{T}]$, electric current $[\mathtt{I}]$ and temperature $[\Theta]$.[1] Once a specific set of unit has been chosen (we use the SI units here), these general quantities become specific; the SI units for our primaries are kg for kilogram, m for metre, s for second, A for ampere, K for kelvin (or we may use °C).[2]

Given the primary quantities, we can derive all other *secondary quantities* from them. Sometimes this is a matter of definition: for a velocity $\mathbf{u}$ we have

$$[\mathbf{u}] = [\mathtt{L}][\mathtt{T}]^{-1}.$$

In other cases we may use a physical law, as in

$$\text{force } F = \text{mass} \times \text{acceleration}, \quad \text{so} \quad [F] = [\mathtt{M}][\mathtt{L}][\mathtt{T}]^{-2};$$

the SI unit is the newton, N. Other instances of secondary quantities are

$$\text{pressure } P = \text{force per unit area}, \quad \text{so} \quad [P] = [\mathtt{M}][\mathtt{L}]^{-1}[\mathtt{T}]^{-2},$$

whose SI unit is the pascal, Pa;

$$\text{energy } E = \text{force} \times \text{distance moved}, \quad \text{so} \quad [E] = [\mathtt{M}][\mathtt{L}]^{2}[\mathtt{T}]^{-2},$$

the SI unit being the joule, J;

$$\text{power} = \text{energy per unit time},$$

giving the watt, W= J s^{-1}, and so on. The idea extends in an obvious way to physical parameters and properties of materials. For example,

$$\text{density } \rho = \text{mass per unit volume}, \quad \text{so} \quad [\rho] = [\mathtt{M}][\mathtt{L}]^{-3}.$$

[1] There are two more primary quantities, amount of a substance (SI unit the mole) and luminous intensity (the candela), but we don't need them in this book.

[2] You might imagine that it should not be necessary to stress the importance of choosing, and sticking to, a standard set of units for the primary quantities, and of stating what units are used. Examples such as the imperial/metric cock-up (one team using imperial units, another using metric ones) which led to the failure of the Mars Climate Orbiter mission in 1999 prove this wrong. How can *any* scientist seriously use feet and inches in this day and age?

3.2.1 Example: heat flow

We are going to see a lot of heat-flow problems in this book (I assume that you have already met the heat equation in an introductory PDE course). Let's begin by working out the basic dimensions of thermal conductivity k. By Fourier's law (an experimental fact), heat flux, which means the energy flow in a material per unit area per unit time, is proportional to temperature gradient:

$$\mathbf{q} = -k\nabla T.$$

Consistency check: why is there a minus sign?

Thus, noting that differentiation with respect to a spatial variable brings in a length scale on the bottom,

What does integration do?

$$\begin{aligned}[\mathbf{q}] &= [\text{energy}][\mathtt{L}]^{-2}[\mathtt{T}]^{-1} = [\mathtt{M}][\mathtt{T}]^{-3} \\ &= [k][\Theta][\mathtt{L}]^{-1},\end{aligned}$$

so that

$$[k] = [\mathtt{M}][\mathtt{L}][\mathtt{T}]^{-3}[\Theta]^{-1}.$$

The usual SI units of k, W m^{-1} K^{-1}, are chosen to be descriptive of what this parameter measures. It is an exercise now to check that the heat equation

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T, \tag{3.1}$$

in which c is the specific heat capacity, SI units J kg^{-1} K^{-1}, is dimensionally consistent.

Note also, for future reference, that the combination

$$\kappa = \frac{k}{\rho c},$$

known as the *thermal diffusivity*, has the dimensions $[\mathtt{L}]^2[\mathtt{T}]^{-1}$. The higher κ, the faster the material conducts heat: that is, heat put in is conducted more and absorbed less; you can see this because κ is the ratio of heat conduction (k) to absorption as internal energy (ρc). By way of examples, water with its large specific heat has $\kappa = 1.4 \times 10^{-7}$ m^2 s^{-1}, while for the much less dense air $\kappa = 2.2 \times 10^{-5}$. Amorphous solids such as glass ($\kappa = 3.4 \times 10^{-7}$ m^2 s^{-1}conduct less well than crystalline solids such as metals: for gold (an extreme and expensive example), $\kappa = 1.27 \times 10^{-4}$.

Given a length L, we can construct a time L^2/κ, which can be interpreted as the order of magnitude of the time it takes for you to notice an abrupt temperature change a distance L away. Conversely, during a specified time t, the abrupt temperature change propagates 'noticeably' a distance of order of magnitude $\sqrt{\kappa t}$.

Of course, the heat equation, being parabolic, has an infinite speed of propagation. What I mean by 'notice' is that the temperature change is not small. See the exercise 'similarity solution…' on page 36.

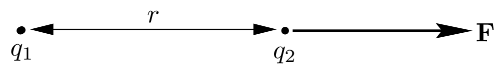

Figure 3.1: Force between two charges.

3.3 Electric fields and electrostatics

Several of the problems we look at in this book involve electromagnetic effects. Fortunately we only need a small subset of the wonderful edifice of electromagnetism, and most of what we use is a reminder of school physics, but written in more mathematical terms.

Models for electricity bring with them a stack of potentially confusing units. A good place to start is Coulomb's (experimental) observation that, in a vacuum, the force between two point charges q_1, q_2 is inversely proportional to the square of the distance r between them. We need a unit for charge, and as the relevant fundamental unit is the ampere, A, which measures the flow of electric charge per unit time down a wire, we find that it is one A s, known as the coulomb, C.[3] So, the force is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2},$$

with a sign convention consistent with 'like charges repel', as in Figure 3.1 in which q_1 and q_2 have the same sign. The constant ε_0 is known as the *(electric) permittivity of free space*, and the 4π is inserted to save a lot of occurrences of this factor in other formulae. Thus,

Notice that 4π, being a number, is omitted from this dimensional balance.

$$[\varepsilon_0] = \frac{([\mathtt{I}][\mathtt{T}])^2}{[\mathtt{L}]^2 \cdot [\mathtt{M}][\mathtt{L}][\mathtt{T}]^{-2}} = [\mathtt{M}]^{-1}[\mathtt{L}]^{-3}[\mathtt{T}]^4[\mathtt{I}]^2,$$

a combination which in SI is called one farad per metre (F m^{-1}) for a reason which will become clear if (when) you do the exercise 'capacitance' on page 31. The numerical value of ε_0 is approximately 8.85×10^{-12} F m^{-1}, from which we see that one coulomb is a colossal amount of charge. The attractive force between opposite charges of 1 C separated by 1 m is $(4\pi\varepsilon_0)^{-1}$; this is more than 10^8 N, and it would take two teams of 2,000 large elephants, each pulling their bodyweight, to drag them apart.

Suppose we regard charge 1 as fixed at the origin and charge 2 as a movable 'test charge' at the point $\mathbf{x}$. The force on it, now regarded as a vector, is

$$\mathbf{F} = q_2\mathbf{E}$$

Of course, $\hat{\mathbf{x}}$ is a unit vector along $\mathbf{x}$ and $r = |\mathbf{x}|$.

where

[3]See the exercises for the definition of the ampere.

$$\mathbf{E} = \frac{q_1 \hat{\mathbf{x}}}{4\pi\varepsilon_0 r^2} = \frac{q_1 \mathbf{x}}{4\pi\varepsilon_0 r^3} \tag{3.2}$$

is known as the *electric field* due to charge 1. Since $\nabla \wedge \mathbf{E} = \mathbf{0}$ for $\mathbf{r} \neq \mathbf{0}$, and $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ is simply connected, there is an *electric potential*

If you didn't know that $\nabla \wedge (\mathbf{x}/r^3) = \mathbf{0}$, check it using the formula

$$\nabla\wedge(\phi\mathbf{v}) = \nabla\phi\wedge\mathbf{v}+\phi\nabla\wedge\mathbf{v}$$

for scalar $\phi(\mathbf{x})$ and vector $\mathbf{v}(\mathbf{x})$.

$$\phi = \frac{q_1}{\varepsilon_0 r} \quad \text{with} \quad \mathbf{E} = -\nabla\phi$$

(the minus sign is conventional). Because $\nabla \cdot \mathbf{E} = 0$ away from $\mathbf{r} = \mathbf{0}$, we have

$$\nabla^2\phi = 0, \qquad \mathbf{x} \neq \mathbf{0}.$$

Instead of point charges, we may have a distributed charge density $\rho(\mathbf{x})$, which we can think of (in a loose way for now) as some sort of limit of a large number of point charges. Then we find that

Check this too: $\nabla \cdot (\phi\mathbf{v}) = \nabla\phi \cdot \mathbf{v} + \phi\nabla \cdot \mathbf{v}$.

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}.$$

We will see a justification for this equation in chapter **??** (see also the exercise 'Gauss' flux theorem' on page 31).[4]

We have strayed somewhat from our theme of units and dimensions. Returning to $\mathbf{E}$, we find from (3.2) that

$$[\mathbf{E}] = [\mathtt{M}][\mathtt{L}][\mathtt{T}]^{-3}[\mathtt{I}]^{-1};$$

it is measured in volts per metre, V m^{-1}, from which the units for ϕ are volts. Perhaps more usefully, since $q_2\mathbf{E}$ is a force, the formula

$$\text{work done} = \text{force} \times \text{distance moved}$$

tells us that the electric potential is the energy per unit charge expended in moving against the electric field:

$$q\,[\phi]_A^B = -\int_A^B q\mathbf{E} \cdot d\mathbf{x},$$

[4] In fact it is rather unusual to have $\rho \neq 0$, that is not to have charge neutrality, in the bulk of a material. The reason is that if the material is even slightly conducting, any excess charge moves (by mutual repulsion) to form a surface layer or, if it can escape elsewhere, it does so. If the material is a good insulator the charge cannot get into the interior anyway. In the next chapter we describe a situation where charge neutrality does not hold.

a formula which serves as a definition of ϕ.[5]

Thinking now of the rate at which work is done against the electric field (or just manipulating the definitions), we see that

$$1 \text{ volt} \times 1 \text{ ampere} = 1 \text{ watt},$$

and hence the dimensional correctness of the formula

$$P = VI$$

for the power dissipated when a current I flows across a potential difference V. When a current is carried by free electrons through a solid, the electric field forces the free electrons through the more-or-less fixed array of solid atoms, and the work done against this resistance is lost as heat at the rate VI. In many cases, the current is proportional to the voltage, giving the linear version of Ohm's law

$$V = IR,$$

from which the primary units of resistance R (SI unit the ohm, Ω) can easily be found. There are also many nonlinear resistors, for example diodes, in which R depends on I.

The only SI unit that is not a roman letter?

Sources and further reading

Barenblatt's book [4] has a lot of material about dimensional analysis, and is the source of the exercises on atom bombs and rowing. For electromagnetism I suggest the book by Robinson [35] if you can get hold of it, as his physical insight was unrivalled; failing this, try ....

Exercises

The first set of exercises is about electromagnetism. If you have never seen this topic before, do what you can, at least to get practice in working out units. But I hope that they will induce to learn more about this wonderful subject. Exercises on the rest of the chapter follow, on page 35.

[5]This is just the same idea as gravitational potential energy as a measure of the work done per unit mass against the gravitational field. If you have ever studied the Newtonian model for gravitation, which is also governed by the inverse square law, you will see the immediate analogy between electric field and gravitational force field, charge density and matter density, and electric and gravitational potentials. The major difference is of course that there are two varieties of charge, whereas matter apparently never repels other matter.

Electromagnetism

1. **Gauss' flux theorem.** Consider the electric field of a point charge (see Section 3.3). Integrate $\nabla \cdot \mathbf{E}$ over a spherical annulus $\epsilon < r < R$ and let $\epsilon \to 0$, $R \to \infty$, to show that

Note: ϵ is not to be confused with ε_0!

$$\lim_{\epsilon \to 0, R \to \infty} \iiint_{\epsilon < r < R} \nabla^2 \phi \, dV = -\frac{q_1}{\varepsilon_0};$$

note the absence of 4π from this formula. Generalise to a finite number of charges. Explain informally why the result is consistent with the continuous charge density equation $\nabla^2 \phi = -\rho/\varepsilon_0$.

2. **Capacitance.** A capacitor is a circuit device which stores charge. The archetypal capacitor consists of two parallel conducting plates, each of area A and separated by a distance d. If one of the plates is earthed and the other raised to a voltage V, it is found that there is a proportional charge Q on it (think of the current trying to get round the circuit and piling up). The constant of proportionality is called the *capacitance* C, so $C = Q/V$, measured in coulombs per volt, known as farads (F).

   Work out the dimensions of the farad in terms of primary quantities. Show that the formula

$$C = \frac{\varepsilon_0 A}{d}$$

   is dimensionally plausible. Check (for consistency) that it does what it should as A and d vary. Thinking of A as fixed and d as varying, explain why the units of ε_0 are F m^{-1}. (In fact $\varepsilon_0 \approx 8.85 \times 10^{-12}$ F m^{-1}.) How big is a 1 μF (quite a large value) capacitor if $d = 1$ mm? How big would a 1 F capacitor be? (In practice, capacitors are bulky objects which are made smaller by rolling them up, and by filling the space between the plates with a material of higher permittivity than ε_0.)

   Based solely on this dimensional analysis, make an order of magnitude guess at the capacitance of (a) an elephant (assumed conducting); (b) a homemade parallel-plate capacitor made from two ten-metre rolls of kitchen foil 30 cm wide separated by cling-film.

   If you walk across a nylon carpet you may become charged with static electricity, to a voltage of say 30 kV. (The charge appears on your shoes and is easily transported around you, because your body is quite a good conductor, to form a surface layer.) Estimate how much charge you accumulate. Given that air loses its insulating property and breaks down into an ionised gas at electric fields of around 3 MV m^{-1}, how far is your finger from the door handle when you discharge?

It is quite easy to work out the capacitance of a sphere of radius a. The electrostatic potential ϕ satisfies $\nabla^2\phi = 0$ for $r > a$, where r is distance from the centre of the sphere. If the sphere is raised to a voltage V relative to a potential of zero at infinity, we have $\phi = V$ on $r = a$ and $\phi \to 0$ as $r \to \infty$. We (you) can write down ϕ immediately. Now use Gauss' flux theorem, aka the divergence theorem, on a sphere $r = a+$ to show that the total charge on the sphere is

The notation $r = a+$ means do it for $r = a + \epsilon$ and let $\epsilon \downarrow 0$.

$$\varepsilon_0 \iint_{r=a} \frac{\partial \phi}{\partial r}\, dS$$

and deduce that the capacitance of the sphere is $4\pi\varepsilon_0 a$.

A capacitor with capacitance C is charged up to voltage V and discharged to earth (voltage 0) through a resistor of resistance R. If the charge on the capacitor is Q and the current to earth is I, explain why

$$Q = VC, \qquad I = \frac{dQ}{dt} \quad \text{and} \quad V = IR.$$

Find $I(t)$ and confirm that RC has the dimensions of time; interpret this time physically and explain why it increases with both R and C.

3. **Slow electrons.** The charge on an electron is approximately 1.6×10^{-19} C. In copper, there are about 8.5×10^{28} free electrons per cubic metre (this calculation is based on Avogadro's number, the density and atomic weight of copper, and one free electron per atom). What is the mean speed of the electrons carrying 1 A of current down a wire of diameter 1 mm? Does the answer surprise you?

4. **Forces between wires.** It is another experimental observation that the force F per unit length between infinitely long straight parallel wires in a vacuum, carrying currents I_1, I_2, is inversely proportional to the distance r between them, and directly proportional to each of the currents. This is written

$$F = \frac{\mu_0 I_1 I_2}{2\pi r}; \tag{3.3}$$

the factor 2π is again for convenience elsewhere. The constant μ_0 is known as the *permeability of free space*; what are its fundamental units? The SI units are henrys per metre, H m^{-1}.

Can you think why line currents get a factor 2π but point charges get a factor 4π?

Now recall that we have not yet defined the unit of current, the ampere. Because μ_0 and the currents in (3.3) are multiplied, there is a degree of indeterminacy in their scales (multiply the currents by α and divide μ_0 by α^2). We exploit this by arbitrarily (in fact it is a cunning choice from the practical point of view) setting

$$\mu_0 = 4\pi \times 10^{-7}\mathrm{H\ m}^{-1}$$

and then *defining* the ampere as the current that makes F exactly equal to 2×10^{-7} N m^{-1}.

We think of a current as generating a magnetic field, denoted by $\mathbf{B}$. The *Lorenz force law* states that the force on a charge q moving with velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is

Remember iron filings experiments to show the magnetic fields of bar magnets or wires? The filings line up in the direction of $\mathbf{B}$.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}).$$

Deduce the fundamental units of $\mathbf{B}$ (SI unit the tesla, T). Interpreting the currents as moving line charges, show that (3.3) is consistent with a magnetic field

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{e}_\theta$$

for a wire carrying current I along the z–axis of cylindrical polar coordinates (r, θ, z). How would iron filings on a plane normal to the wire line up in this case?

Show that, like the coulomb and farad, the tesla is an inconveniently large unit by working out the current required to give a field of 1 T at a distance of 1 m. How many 1 kW toasters would this current power at 250 V? (Ans: 1.25 million.) Why are electromagnets made of coils? The most powerful superconducting magnets, using coils to reinforce the field, have only recently broken the 10 T barrier.

5. **The speed of light.** Show that

$$c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$$

is a speed, and work out its numerical value. Do you recognise it?

6. **Electromagnetic waves.** OK, the result of the previous exercise is not a coincidence. We don't have the space to derive Maxwell's famous equations for $\mathbf{E}$ and $\mathbf{B}$, but here they are: in a vacuum, $\mathbf{E}$ and $\mathbf{B}$ satisfy

- $$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$
  This is Faraday's law of induction which says that time-varying magnetic fields generate electric fields.

- $$\frac{1}{\mu_0}\nabla \wedge \mathbf{B} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}.$$
  When there are currents present they appear as a source term $\mathbf{j}$, the current density, on the right-hand side of this equation, which is revealed as the model for generation of magnetic fields by currents. The term $\partial \mathbf{E}/\partial t$ is Maxwell's inspiration, the displacement current.

- $$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{E} = 0.$$
  The first of these says that there are no 'magnetic monopoles' (magnetic fields are only generated by currents, and magnetic lines of force have no ends), and the second is a special case of $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$, showing the generation of electric fields by charges.

Take these equations on trust and cross-differentiate them to show that $\mathbf{E}$ and $\mathbf{B}$ satisfy wave equations:

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B}, \qquad \frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}$$

where $c^2 = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$ as above. You may need the vector identity[6]

$$\nabla \wedge \nabla \wedge \mathbf{v} = \nabla\left(\nabla \cdot \mathbf{v}\right) - \nabla^2 \mathbf{v}.$$

7. **Planck's constant and the fine structure constant.** This book is not the place for an account of quantum mechanics. We can, however, note that underpinning it all is Schrödinger's equation

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m}\nabla^2 \psi = V\psi$$

for the wave function ψ of a particle of mass m moving in a potential V (ψ is complex-valued and $|\psi|^2$ is the probability density of the particle's

[6]Curl Curl is also a surf beach town near Sydney, lat.. long. ..

location). Find the dimensions of $\hbar$ (*Planck's constant is* $h = 2\pi\hbar$) and V. Show that the combination

$$\frac{e^2}{2\varepsilon_0 hc},$$

where e is the charge on an electron, is dimensionless. Such dimensionless ratios of fundamental constants are not coincidences, and this one, called the *fine structure constant*, plays an important role in quantum electrodynamics. It gets its name from its influence on the fine structure of the spectrum of light emitted by a glowing gas; crudely speaking it is the ratio of the speed an electron would have if it orbited a hydrogen nucleus in a circle (which it does not) to the speed of light. Its numerical value is very close to 1/137, a source of some fascination to numerologists. For more, see its own website `www.fine-structure-constant.org`.

Other exercises

1. **cgs units** An alternative system of units to SI is the cgs system, in which the unit of mass is the gramme (g) and the unit of length is the centimetre. Establish the following conversion table (which is really here for your reference), and construct the reverse table to turn SI into cgs.

	cgs	SI
Velocity	1 cm s^{-1}	10^{-2} m s^{-1}
Density	1 g cm^{-3}	10^{3} kg m^{-3}
Dynamic viscosity	1 poise	10^{-1} kg $m^{-1}s^{-1}$
Kinematic viscosity	1 cm^2 s^{-1}	10^{-4} m^2 s^{-1}
Pressure	1 dyne cm^{-2}	10^{-1} Pa
Energy	1 erg	10^{-7} J
Force	1 dyne	10^{-5} N
Surface tension	1 dyne cm^{-1}	10^{-3} N m^{-1}

2. **Imperial to metric.** Establish the quite useful relation

$$1 \text{ mph} \approx 0.447 \text{m s}^{-1}.$$

Using the Web or other sources for the definitions, show that

$$1 \text{ Btu} = 1 \text{ calorie},$$

a result which might be of use if you are interested in central heating.

Btu=British thermal unit, a measure of energy; kilocalories are worried about by dieters.

3. **Atom bombs.** An essentially instantaneous release of an amount E of energy from a very small volume (see the title of the exercise) creates a rapidly expanding high pressure fireball bounded by a very strong spherical shock wave across which the pressure drops abruptly to atmospheric. The pressure inside the fireball is so great that the ambient atmospheric pressure is negligible by comparison, and the only property of the air that determines the radius $r(t)$ of the fireball is its density ρ. Show dimensionally that

$$r(t) \propto E^{\frac{1}{5}} t^{\frac{2}{5}} \rho^{-\frac{1}{5}}.$$

   This result is due to GI Taylor, a colossus of British applied mathematics in the last century; whatever branch of fluid mechanics you look at, you will find that 'GI' wrote a seminal paper on it.[7] It can be used to deduce E from observations of $r(t)$; Taylor's publication of this observation [40] apparently caused considerable embarrassment in US military scientific circles where it was regarded as top secret.

4. **Rowing.** A boat carries N similar people, each of whom can put power P into propelling the boat. Assuming that they each require the same volume V of boat to accommodate them, show that the wetted area of the boat is $A \propto (NV)^{\frac{2}{3}}$ (here, as so often, the cox is ignored). Assuming inviscid flow, why might the drag force be proportional to $\rho U^2 A$, where U is the speed of the boat and ρ the density of water? (In saying this, we are ignoring drag due to waves created by the boat.) Deduce that the rate of energy dissipated by a boat travelling at speed U is proportional to $\rho U^3 A$, and put the pieces together to show that

$$U \propto N^{\frac{1}{9}} P^{\frac{1}{3}} \rho^{-\frac{1}{3}} V^{-\frac{2}{9}}.$$

   If we suppose, very crudely, that P and V are both proportional to body mass, is size an advantage to a rower?

   This example is based on a paper by McMahon [26], described in Barenblatt's book [4]; the theory agrees well with observed race times.

5. **Similarity solution to the heat equation.** Show that the problem

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \qquad x > 0, \quad t > 0,$$

   with

$$T(x,0) = 0, \qquad T(0,t) = T_0 > 0,$$

[7] Is it necessary to mention that the Taylor of Taylor's theorem was several hundred years earlier? One never knows these days.

which corresponds to instantaneous heating of a cold half-space from its boundary at $x = 0$, has a similarity solution

$$T = T_0 F\left(\frac{x}{\sqrt{\kappa t}}\right)$$

and find F in terms of the error function $\operatorname{erf}\xi = (2/\sqrt{\pi})\int_0^{\xi} e^{-s^2}\,ds$. Sketch F and interpret this solution in the light of the discussion at the end of Section 3.2.1.

Chapter 4

Dimensional analysis, scaling and non-dimensionalisation

4.1 Nondimensionalisation and dimensionless parameters

Like its predecessor, this chapter has one simple theme, but one with far-reaching repercussions. The key idea is this. Any equation we write down for a physical process models balances between physical mechanisms. Not all of these may be equally important, and we can begin to assess how important they are by scaling all the variables with 'typical' values — values of the size we expect to see, or dictated by the geometry, boundary conditions etc — so that the equation becomes *dimensionless.* Instead of a large number of physical parameters and variables, all with dimensional units, we are left with an equation written in *dimensionless variables.* All the physical parameters and typical values are collected together into a smaller number of *dimensionless parameters* (or *dimensionless groups*) which, when suitably interpreted, should tell us the relative importance of the various mechanisms.

All of this is much easier to see by working through an example than by waffly generalities. So here's a selection of three relatively simple physical situations where we can see the technique in action.

4.1.1 Example: advection-diffusion

We'll start with a combination of two very familiar models, heat conduction and fluid flow. When you stand in front of a fan to cool down, two mechanisms come into play: heat is conducted (diffuses) into the air, and is then carried away by it. The process of heat transfer via a moving fluid is called *advection*, as distinct from *convection*, which is hot-air-rises heat transfer due to density changes.[1] Both advection and convection are major mechanisms for heat transfer in systems such as the earth's atmosphere, oceans and molten core; almost any industrial process (think of cooling towers as a visible example); car engines; computers; you name it. Their analysis is of enormous practical importance.

It is often easier to analyse advection because we can usually decouple the question of finding the fluid flow from the heat-flow problem. In convection, the buoyancy force that drives the flow is strongly temperature-dependent — indeed, without it there would be no flow — and the problem correspondingly more difficult. For our first example, we'll consider two-dimensional flow of an incompressible liquid with a given (that is, we can calculate it separately) velocity $\mathbf{u}$ past a circular cylinder of radius a, with a free-stream velocity at large distances of $(U, 0)$. This is a simple model of, for example, the cooling of a hot pipe.

For the moment, it doesn't matter too much what we take for $\mathbf{u}$. Let's just use the standard inviscid flow model $\mathbf{u} = \nabla\phi$, where

$$\phi = U\left(r\cos\theta + \frac{a^2\cos\theta}{r}\right)$$

in plane polar coordinates. We need to generalise the heat conduction equation to include the advection. This needs us to recognise that when we write down conservation of energy in the form

$$\text{rate of change of energy of a particle} + \text{net heat flux into it} = 0,$$

we have to do this following a particle. Thus, the time derivative $\partial/\partial t$ in the usual heat equation (3.1) is replaced by the material (convective) derivative $\partial/\partial t + \mathbf{u}\cdot\nabla$, giving

If you don't quite believe this argument, do the exercise 'advection-diffusion' on page 54.

$$\rho c\left(\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T\right) = k\nabla^2 T. \tag{4.1}$$

Lastly we need some boundary conditions. The simplest ones are to have one constant temperature at infinity and another on the cylinder,[2] so

$$T \to T_\infty \quad \text{as} \quad r \to \infty, \qquad T = T_0 \quad \text{on} \quad r = a.$$

The problem is summarised in Figure 4.1.

[1] The usage is changing in the loose direction; convection is often used for both processes, subdivided where necessary into forced convection for advection, and natural convection for buoyancy-driven heat transport. A lot of the heat lost by a hot person in still air is by (natural) convection.

[2] The conditions on the cylinder are not especially realistic; a Newton condition of the form $-k\partial T/\partial n + h(T - T_0) = 0$ would be better; see the exercises on page 55.

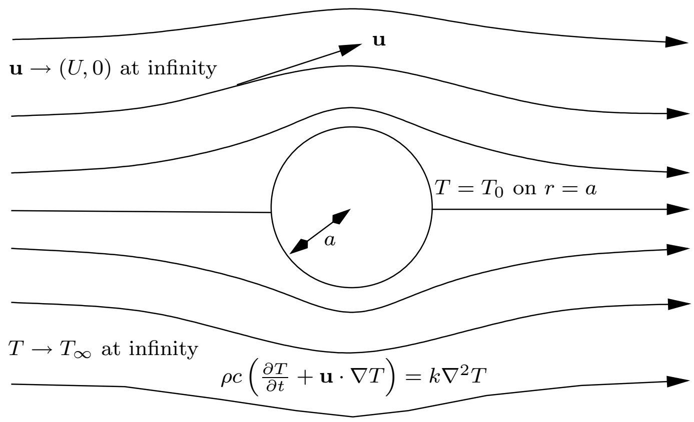

Figure 4.1: Advection-diffusion of heat from a cylinder.

In order to see the relative importance of advection (the left-hand side of (4.1)) and diffusion (the right-hand side), we scale all the variables with 'typical' values. The obvious candidate[3] for the length scale is a; then we can scale $\mathbf{u}$ with U and time with a/U, the order-of-magnitude residence time of a fluid particle near the cylinder. So, we write

$$\mathbf{x} = a\mathbf{x}', \quad \mathbf{u} = U\mathbf{u}', \quad t = (a/U)t'.$$

Only T has not yet been scaled. Clearly, we can write

$$T(\mathbf{x}, t) = T_\infty + (T_0 - T_\infty)\, T'(\mathbf{x}', t').$$

This gives

$$\frac{\rho c U}{a} \left(\frac{\partial T'}{\partial t'} + \mathbf{u}' \cdot \nabla' T' \right) = \frac{k}{a^2} \nabla'^2 T'. \tag{4.2}$$

Now comes a key point. All the terms in the original equation have the same physical dimensions. All our 'primed' quantities have no dimensions: they are just numbers. Thus, if we divide through (4.2) by one of the (still dimensional) quantities multiplying a 'primed' term, we will be left with a dimensionless term. Then, all the other terms in the equation must be dimensionless as well: and so, the physical parameters (a, ρ, etc) *must now occur in combinations which are dimensionless too.*

So, divide through (4.2) by k/a^2 to get

$$\frac{\rho c U a}{k} \left(\frac{\partial T'}{\partial t'} + \mathbf{u}' \cdot \nabla' T' \right) = \nabla'^2 T'.$$

[3]If the cross-section of the cylinder is another shape, we can use any measure of its 'diameter', although you can see an obvious difficulty here if it is, say, a long thin ellipse. We return to this point later.

We see that there is just one dimensionless combination in this problem,

$$\mathrm{Pe} = \frac{\rho c U a}{k} = \frac{U a}{\kappa},$$

known as the *Peclet number*. There is no dimensionless parameter in the boundary conditions because they scale linearly, to become

$$T' \to 0 \quad \text{as} \quad r' \to \infty, \qquad T' = 1 \quad \text{on} \quad r' = 1.$$

What would have happened if we had used the Newton condition?

There several things to say about this analysis. The first is the simple observation that, whereas the original problem has a large 'parameter space' consisting of the 7 parameters U, a, ρ, c, k, T_∞, T_0, the reduced problem contains the *single* dimensionless parameter Pe. That's quite a reduction; and if you think it is obvious, there are still plenty of mathematical subjects almost untouched by the idea (economics, for example).

It is worth noting that there may be more than one possible choice for some of the scales, and iteration may be needed to find the most appropriate one for a given problem. In our example, there are two other possible length scales, whose consequences are explored in the exercise 'Peclet numbers' on page 54. Usually, the obvious choice is the best.

Next, *all problems with the same value of* Pe *can be obtained by solving one canonical scaled problem for that value of* Pe. So, if we want to make an experimental analogue of a very large physical set-up, we can do it in a smaller setting as long as we achieve the same Peclet number.[4]

Next, and probably most important, the size of any dimensionless numbers in a problem tells us a great deal about the balance of the physical mechanisms involved, and about the behaviour of solutions. In our example, we can write

$$\begin{aligned}\mathrm{Pe} &= \frac{\rho c U a}{k} \\ &= \frac{\rho c U (T - T_\infty)}{k (T - T_\infty)/a} \\ &\approx \frac{\text{advective heat flow}}{\text{conductive heat flow}}.\end{aligned}$$

If Pe is large, advection dominates conduction, and vice versa if Pe is small. A 10 cm radius hot head in an air stream ($\rho \approx 1.3$ kg m^{-3}, $c \approx 993$ J kg^{-1} K^{-1}, $k \approx 0.24$ W m^{-1} K^{-1}) moving at 1 m s^{-1} from a fan has a Peclet number of about 500, undeniably large (large Peclet numbers are more common than small ones).

Any problem with a parameter equal to 500 must surely offer scope for judicious approximation: after all, $\frac{1}{500}$ is tiny, and we may hope to cross out the terms it multiplies without losing too much. If we do this in our equation

$$\frac{\partial T'}{\partial t'} + \mathbf{u}' \cdot \nabla' T' = \frac{1}{\mathrm{Pe}} \nabla'^2 T',$$

[4] We also have to ensure that the fluid velocity scales correctly. This may not be so easy given that it has its own equations of motion which may not behave properly. In our example, it is clear that the potential flow model does scale correctly, because it is linear.

we see that the convective derivative of T' is zero. That is, as we follow a particle its temperature does not change, and since all particles start far upstream at $x = -\infty$, T' everywhere has its upstream value of 0. Which is fine, until we realise that we can't satisfy $T' = 1$ on the cylinder. We shall have to wait until we have looked at asymptotic expansions before we can see how to get out of this jam.

A last remark is that one soon gets tired of writing primes on all scaled variables. As soon as the scalings have been introduced, it's usual to write the new dimensionless equations in the original notation, a pedantically incorrect but universal practice signalled by the phrase 'dropping the primes'.

4.1.2 Example: the damped pendulum

Sometimes the correct scales for one or more variables can only be deduced from the equations, as in the following example. The basic model for a linearly damped pendulum which is displaced an angle θ from the vertical (see Figure 4.2) is

$$l\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + g\sin\theta = 0,$$

and let us suppose that the initial angle and angular speed are prescribed:

$$\theta = \alpha_0, \quad \frac{d\theta}{dt} = \omega_0 \qquad \text{at} \quad t = 0.$$

Here k is the damping coefficient and g the acceleration due to gravity; their units are

$$[k] = [L][T]^{-1}, \qquad [g] = [L][T]^{-2}.$$

Check that these units are consistent with the equation; remember that θ is a number.

Combining the dimensional parameters l, k, g and ω_0, it is easy to see that there are *three* timescales built into the parameters of this problem:

$$t_1 = \sqrt{\frac{l}{g}}, \quad t_2 = \frac{l}{k}, \quad t_3 = \frac{1}{\omega_0}.$$

The first is the period of small undamped oscillations (linear theory). The second is the time over which the damping has an effect (solve $du/dt = -u/t_2$ and see that u decreases by a fraction $1/e$ in each time interval t_2). The third is prescribed by us: it tells us how long it takes the pendulum to cover one radian at its initial angular speed if no other forces act.

Let's scale time with $t_1 = \sqrt{l/g}$, which we do if we are expecting to see oscillatory behaviour. Then, writing

$$t = t_1 t',$$

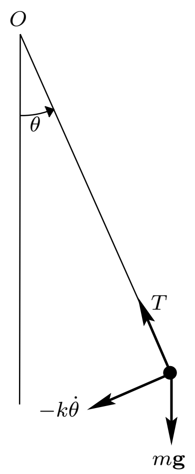

Figure 4.2: Motion of a simple pendulum.

we have the dimensionless model

$$\frac{d^2\theta}{dt'^2} + \frac{t_1}{t_2}\frac{d\theta}{dt'} + \sin\theta = 0,$$

with

$$\theta = \alpha_0, \quad \frac{d\theta}{dt'} = \frac{t_1}{t_3} \qquad \text{at} \quad t' = 0.$$

It contains two obviously dimensionless parameters,

$$\gamma = \frac{t_1}{t_2} = \sqrt{\frac{k^2}{gl}} \quad \text{and} \quad \beta_0 = \frac{t_1}{t_3} = \sqrt{\frac{\omega_0^2 l}{g}}.$$

The first of these, γ, is the ratio of the time over which the system responds to the physical mechanism of gravity (the period for small oscillations) and the timescale of damping. The second, β_0, is the ratio of the initial speed of the pendulum to the speed changes induced by gravity. There is a third dimensionless group, α_0, as angles are automatically dimensionless, and so the dimensionless model is

$$\frac{d^2\theta}{dt'^2} + \gamma\frac{d\theta}{dt'} + \sin\theta = 0,$$

with

$$\theta = \alpha_0, \quad \frac{d\theta}{dt'} = \beta_0 \qquad \text{at} \quad t' = 0.$$

We can make some immediate statements about the behaviour of the system just by looking at the sizes of our dimensionless parameters. For example, if γ is small, we expect the damping to have its effect over many cycles. If β and α_0 are both small, we hope that linearised theory will be valid; and so on. Later, in Chapter **??**, we'll see how to quantify some of these ideas.

4.1.3 Example: beams and strings

We all know that the motion of a string made of a material with density ρ (mass per unit volume, $[\rho] = [M][L]^{-3}$), of length L and cross-sectional area A, stretched to tension T, can be modelled by the equation[5]

$$\rho A \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < L,$$

where $y(x,t)$ is the amplitude of small transverse displacements.[6]

(a) What are the dimensions of T? (b) Show that $\sqrt{T/\rho A}$ is a speed. (c) Show that the wave speed is exactly $c = \sqrt{T/\rho A}$ (with no numerical prefactor) by substituting in a solution of the form $y(x - ct)$.

In the string model, the restoring force is provided by the component of the tension normal to the string, $T\,\partial y/\partial x$. If, on the other hand, we have a stiff beam or rod, the restoring force is caused by its resistance to bending, which can be shown to be proportional to $-\partial^3 y/\partial x^3$. If in addition there is a force perpendicular to the wire of magnitude F per unit length, we get the equation

$$\rho A \frac{\partial^2 y}{\partial t^2} + EAk^2 \frac{\partial^4 y}{\partial x^4} = F.$$

What is the wavespeed?

Here A is the cross-sectional area of the beam, while k is a constant with the dimensions of length known as the *radius of gyration* of the cross-section of the beam. In this model, k encapsulates the effect of the shape of the beam; for a fixed cross-sectional area, k is smallest for a circular cross-section, while for a standard I-beam it is large.[7] A derivation of this model is given in Section 5.1.

[5] You may have seen this in the form

$$\tilde{\rho} \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < L,$$

in which $\tilde{\rho}$ is the mass per unit length, or line density.

[6] Note the engineering rule of thumb

$$\text{wavespeed} = \sqrt{\frac{\text{stiffness}}{\text{inertia}}},$$

which applies very generally to non-dissipative linear systems.

[7] The definition of k is: in a cross-sectional plane, take coordinates (ξ, η) with origin at the centre of mass of the cross-section. Then,

$$Ak^2 = \iint_{\text{cross-section}} \xi^2 + \eta^2 \; d\xi \, d\eta.$$

The constant E is a property of the material from which the beam is made known as the *Young's modulus.* The larger it is, the more the material resists being bent, or sheared (bending leads to shearing). Lastly, ρ is again the underlying material density, which we use in preference to the line density because this model takes into account not just the cross-sectional area of the beam but also its shape).

If the force F is due to gravity we have

$$F = \rho A g.$$

But we might consider other forces, such as the drag from a fluid flowing past the beam, for example wind drag on a skyscraper, flagpole, car radio aerial or hair (see the next chapter); water drag on a reed bending in a stream; or the drag of gas escaping through a brush seal in a jet engine. For inviscid flows, the pressure in the liquid has typical magnitude $\rho_l U_l^2$, where ρ_l is the liquid density and U_l a measure of its speed (think of Bernoulli's equation $p + \frac{1}{2}\rho|\mathbf{u}|^2 =$ constant in steady irrotational flow). Because the flow about a cylinder (even a circular one) is not symmetrical in a real (as opposed to ideal) flow, there is a net pressure force on the cylinder, and its magnitude is roughly proportional to ρU^2.[8]

So, it is reasonable that the drag per unit length on an isolated cylinder in a flow with free stream velocity U_l can be well approximated by

$$\begin{aligned} F &= \text{geometric factor} \times \text{pressure} \times \text{perimeter} \\ &= c_d \times \rho_l U_l^2 \times k. \end{aligned}$$

Here the *drag coefficient* c_d depends on the Reynolds number and the shape and orientation of the cylinder (when we work out the drag force, we resolve the pressure, which acts normally to the surface, in the direction of the free stream and integrate over the perimeter; all this information is lumped into the drag coefficient), and we have used k as a measure of the length of the cross-sectional perimeter.

What other length might we have used? Why is k probably better?

To summarise, we have

$$\rho A \frac{\partial^2 y}{\partial t^2} + EAk^2 \frac{\partial^4 y}{\partial x^4} = c_d \rho_l U_l^2 k$$

as the equation for a beam (flagpole, reed) subject to a fluid drag force.

That is, Ak^2 is the moment of inertia of the cross-section.

[8] D'Alembert's paradox says that there is no drag on a cylinder in irrotational inviscid (potential) flow! In real life, even a very small viscosity has a profound effect, leading to completely different flows from the ideal ones. We'll get an idea why later on.

This model has a huge number of physical parameters, but we can get a lot of information from some simple scaling arguments. It is an exercise for you to do this:

1. Make the model dimensionless using the length of the beam L to scale x and to-be-determined scales y_0 and t_0 for y and t.
2. Verify that the units of E are $[M][L]^{-1}[T]^{-2}$.
3. Roughly how big is the steady-state displacement?
4. If the drag force is switched on suddenly at $t = 0$, over what timescale does the beam initially respond?
5. What is the timescale for free oscillations?

We will return to versions of this model at several places later in the book.

4.2 The Navier–Stokes equations and Reynolds numbers

Recall from Chapter **??** that the flow of an incompressible Newtonian viscous fluid is governed by the Navier–Stokes equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{4.3}$$

where $\mathbf{u}$ is the fluid velocity and p the pressure, both functions of position $\mathbf{x}$ and time t, while the physical parameters of the fluid are its density ρ and its *dynamic viscosity* μ.

Let us now look at how we should nondimensionalise the Navier–Stokes equations. We begin by noting that it is often useful to combine μ and ρ to form the *kinematic viscosity*

$$\nu = \frac{\mu}{\rho}.$$

Note the units of dynamic and kinematic viscosity: since, as in Figure **??**,

$$\text{force/area} = \mu U/h,$$

we have

$$[\mu] = [\mathtt{M}][\mathtt{L}]^{-1}[\mathtt{T}]^{-1}$$

(the SI unit is the pascal-second, Pa s), and so

Mnemonic: acres per annum; the acre is one of the old English units of area. Hectares per megasecond??

$$[\nu] = [\mathtt{L}]^2[\mathtt{T}]^{-1}.$$

Suppose we have flow past a body of typical size L, with a free-stream velocity $U\mathbf{e}_1$. As in the advection-diffusion problem, we scale all distances with L, time with L/U and velocities with U, writing

$$\mathbf{x} = L\mathbf{x}', \quad t = (L/U_\infty)t', \quad \mathbf{u} = U\mathbf{u}'.$$

Only p has not yet been scaled, and in the absence of any obvious exogenous scale we let the equations tell us what the possibilities are. For now, let's write

$$p = P_0 p'$$

and substitute all these into the momentum equation (clearly the mass conservation just becomes $\nabla' \cdot \mathbf{u}' = 0$). This gives

$$\frac{\rho U^2}{L}\left(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}'\right) = -\frac{P_0}{L}\nabla' p' + \frac{\mu U}{L^2}\nabla'^2 \mathbf{u}'. \tag{4.4}$$

Now we can divide through by one of the coefficients to leave a dimensionless term; because all the other terms must also be dimensionless, that will tell us the pressure scale. For example, divide through by $\rho U^2/L$, leaving the equation

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\frac{P_0}{\rho U^2}\nabla' p' + \frac{\mu}{\rho U L}\nabla'^2 \mathbf{u}'.$$

It is now clear that we can choose the 'inviscid' pressure scale

$$P_0 = \rho U^2$$

and when we do this we get the dimensionless equation in the form

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\nabla' p' + \frac{1}{\mathrm{Re}}\nabla'^2 \mathbf{u}', \tag{4.5}$$

where the dimensionless combination

$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

is known as the *Reynolds number* after the British hydrodynamicist Osborne Reynolds.

So what does this tell us? The most important conclusion is that *if viscous effects are all we have to worry about,*[9] *then all flows with the same*

[9]For example, we don't take account of temperature changes due to viscous dissipation, which may themselves affect the viscosity or density of the fluid.

Reynolds number are scaled versions of each other. This is the idea of the wind tunnel. We don't need to build full scale prototype aeroplanes or cars to test for lift and drag, we can use a scale model as long as we get the Reynolds number right. Furthermore, by forming dimensionless groups, we reduce the dimensionality of our parameter space as far as possible. In our example above, instead of the 4 physical parameters ρ, μ, L and U, we have the single combination Re. So for a given shape of body, in principle all we need to do is sweep through the Reynolds numbers from 0 to ∞ to find all the possible flows past a body of that shape.

Thinking now of the dimensionless parameters as encapsulating the competing (or balancing) physical mechanisms that led to our original equation, we can write the Reynolds number as

$$\begin{aligned}\mathrm{Re} &= \frac{\rho U L}{\mu} \\ &= \frac{\rho U^2}{\mu U/L}.\end{aligned}$$

The top of the last fraction is clearly a measure of the pressure force due to fluid inertia per unit area on a surface, while as we saw above, the bottom is a measure of viscous shear forces. So, the Reynolds number tells us the ratio of inertial forces to viscous ones. When it is large, the inertial forces dominate, while for small Re it is viscosity that wins.

In the former case, it is tempting to cross out the term multiplied by $1/\mathrm{Re}$ in the dimensionless equation (4.5); this leaves us with the *Euler equations*

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p, \qquad \nabla\cdot\mathbf{u} = 0$$

(we have dropped the primes), for which there is a large class of exact solutions when the flow is irrotational so that $\nabla \wedge \mathbf{u} = \mathbf{0}$. In this case, there is a velocity potential ϕ which satisfies Laplace's equation

$$\nabla^2\phi = 0$$

in the fluid. However, we must be very careful in making this approximation. One obvious reason is that for most viscous fluids we should apply the *no-slip condition* on rigid boundaries: this says that the fluid velocity at the boundary must equal the velocity of the boundary itself, so the fluid particles at the boundary stick to it. Most solutions of the Euler equations do not satisfy this condition, and the reconciliation of the two ideas led to boundary layer theory and the theory of matched asymptotic expansions, a triumph of twentieth-century applied mathematics which we look at briefly

See the exercise on page 23 where this is proved for potential flows.

in Chapter **??**. A second reason for proceeding with caution is the everyday observation that very fast (very large Reynolds number) flows are turbulent and so intrinsically unsteady. For these reasons one may worry that the inviscid model is one that exists in theory but is never seen in practice, but that would be unduly pessimistic. Boundary layer theory helps, and in many interesting flows either the Reynolds number is large but not enormous, or the flow takes place on a short timescale, so that turbulence does not have time to become a nuisance.

Returning to the theme of nondimensionalisation, what if Re is small, for slow viscous flow? Is is safe to say that since $\nabla'^2\mathbf{u}'$ is divided by Re, we simply set it equal to zero? No, it is not. If we do this, we are saying that pressure forces are not important, and it is common experience that they are. In such a situation, we should check whether there is an alternative scaling of the equations. It is not hard to see that there is a second possible pressure scale,

This is obvious from the definition of the (dimensionless) Re: why?

$$\tilde{P}_0 = \frac{\mu U}{L},$$

and it is an exercise to show that scaling p in this way leads to an alternative version of (4.5),[10]

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}'\right) = -\nabla p' + \nabla'^2 \mathbf{u}'.$$

If Re is small, maybe we can neglect the convective derivative (inertial) terms on the left to get the *Stokes Flow* model for slow flow:

$$\mathbf{0} = -\nabla p + \nabla^2 \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0.$$

As we continue, we shall see how we might justify dropping terms in this way (and why it might go wrong).

4.2.1 Water in the bathtub

We really should do this problem; even your aunt has heard of it. Is it true that water flows out of the bathtub with an anticlockwise swirl in the northern hemisphere and a clockwise swirl south of the equator?

Answer: only under very carefully controlled circumstances. Here's why. Remember the Coriolis theorem about transferring the equations of motion

There is nothing difficult about this: it is just the chain rule in disguise.

[10] Pedantically speaking, note that the p' in this equation is not the same as the p in the other dimensionless version of Navier–Stokes (4.5), as it has been scaled differently…

to a rotating coordinate system: if $\mathbf{v}$ is a vector, and we want its time derivative as measured in a frame rotating with angular velocity $\boldsymbol{\Omega}$, then

$$\left.\frac{d\mathbf{v}}{dt}\right|_{\text{fixed}} = \left.\frac{d\mathbf{v}}{dt}\right|_{\text{rotating}} + \boldsymbol{\Omega}\wedge\mathbf{v}.$$

The Navier–Stokes equations in a rotating frame are clearly

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\Omega}\wedge\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\left(\mathbf{g} - \boldsymbol{\Omega}\wedge\boldsymbol{\Omega}\wedge\mathbf{r}\right),$$

The origin is at the centre of the earth, and the $\boldsymbol{\Omega}\wedge\boldsymbol{\Omega}\wedge\mathbf{r}$ term is incorporated into the gravitational body force to give the 'apparent gravity'.

An exercise which you should carry out…

where in this case $\boldsymbol{\Omega}$ is the angular velocity of the earth, equal to 2π per 24 hours, about 7.3×10^{-5} radians per second, in the direction of the earth's axis of rotation. Now consider water moving at 1 m s^{-1} in a bath of size about 1 m. Clearly, if we scale the variables with representative values based on these figures (all of which are 1 in SI units), the ratio of the Coriolis term $\boldsymbol{\Omega}\wedge\mathbf{u}$ to the other acceleration terms is about the same as the numerical value of $|\boldsymbol{\Omega}|$ in SI units, i.e.less than 10^{-4}. That is, the rotation effect is tiny. In practice, other effects such as residual swirl from the way the water was put into the bath, or asymmetry in the plughole or the way the plug is pulled out, completely swamp the Coriolis effect unless the experiment is carried out under very carefully controlled conditions. On the other hand, If we look at rotating air masses on the scale of a hurricane or typhoon, the much greater length scale means that Coriolis effect is enormously important. As air masses leave the equator and travel north or south, they carry their angular momentum (whose direction is along the earth's axis of rotation) with them, and as they move round the curve of the earth it is transformed into rotatory motion in the tangent (locally horizontal) plane; but that is another story.

4.3 Buckingham's Pi-theorem

Let's take a short detour to state the only quasi-rigorous result in the area of dimensional analysis: the Buckingham Pi-theorem.

Suppose we have n independent physical variables and parameters $Q_1,\ldots,Q_n$ ($\mathbf{x}$, t, μ etc. above), and the solution of a mathematical model gives us one of these in terms of the others:

$$Q_1 = f(Q_2,\ldots,Q_n).$$

Suppose also that there are r independent basic physical dimensions ([M]*,* [L]*,* [T] *etc.).*

Then there are $n-r$ dimensionless[11] *combinations $\Pi_i(Q_j)$ and a function*

[11] And eponymous.

g such that

$$\Pi_1 = g(\Pi_2, \ldots, \Pi_{n-r}).$$

Example: the drag on a cylinder

Suppose a cylinder of length L and radius a is held in viscous fluid moving with far-field velocity U normal to the axis of the cylinder. How does the drag force depend on the parameters of the problem? What happens as $L \to \infty$?

There are 6 independent physical quantities in this problem:

- L and a, which are properties of the cylinder and both have dimensions $[\mathtt{L}]$;
- μ and ρ, which are properties of the fluid and have dimensions $[\mathtt{M}][\mathtt{L}]^{-1}[\mathtt{T}]^{-1}$, $[\mathtt{M}][\mathtt{L}]^{-3}$ respectively;
- The force F on the cylinder ($[\mathtt{M}][\mathtt{L}][\mathtt{T}]^{-2}$) and the free stream velocity U ($[\mathtt{L}][\mathtt{T}]^{-1}$).

In this case, $r = 3$ (for $[\mathtt{M}]$, $[\mathtt{L}]$ and $[\mathtt{T}]$), and so there must be $n - r = 3$ dimensionless quantities. One is obviously the aspect ratio L/a, and another is the Reynolds number $\mathrm{Re} = Ua/(\mu/\rho) = Ua/\nu$. For the third, a little experimentation shows that something of the form

In choosing a as the length to appear in Re, we are looking ahead to when we let $L \to \infty$.

$$\frac{F}{\rho U^2 [\mathtt{L}]^2}$$

will do, and we need to choose which lengths to use to replace $[\mathtt{L}]^2$. Here it helps to think what physical balance is expressed by this parameter. The top, F, is a force, while the bottom is the inviscid pressure scale ρU^2 multiplied by an area Hence it makes sense to use aL, which is a measure of the surface area of the cylinder, and our third dimensionless parameter is thus $F/(\rho U^2 aL)$.

Remember pressure = force per unit area.

Putting this all together, on dimensional grounds we have shown that the drag force is related to the other parameters by an equation of the form

$$F = \rho U^2 aL \times g(\mathrm{Re}, L/a)$$

for some function g.

If we further assume that our pipe is very long so that we have translational invariance along it, then instead of F and L as independent physical quantities, we only have the force per unit length F' (dimensions $[\mathtt{M}][\mathtt{T}]^{-2}$). Then, we get

$$F' = \rho U^2 a C_d(\mathrm{Re})$$

for some function C_d, which is just what we called a drag coefficient earlier in the chapter.

[? approx formula for C_d, flow parallel to generators?]

There is clearly some indeterminacy in the choice of the parameters and the representation of the drag coefficient. For example, we could have written the Reynolds number as UL/ν, or we could have introduced more convoluted parameters such as $UL^2/(\nu a)$, equal to L/a times our definition above, but this would not have had such clear physical implications. It helps to make the choices of parameters to correspond as closely as possible with the physical situation, although we can't always hope to get it right first time. Moreover, there are often genuine alternatives. In our example, we chose ρU^2 as our measure of the fluid pressures; this says that we expect inertia to be significant, and is the clearest way of writing the drag when the flow has large Reynolds number. However, as we saw earlier, we could have chosen $\mu U/a$ for the pressure scale, and this would have led to

$$F' = \mu U \widetilde{C}_d(\mathrm{Re}),$$

which might be more convenient if we are looking at slow flow. Of course, the drag coefficient is uniquely determined[12] as a function of the Reynolds number, so this is merely a relabelling exercise: it is easy to see that $\widetilde{C}_d(\mathrm{Re}) = \mathrm{Re}C_d(\mathrm{Re})$.

4.4 Onwards

We haven't done anything very technical in this chapter. This whole business of scaling is a combination of experience and plain common sense. The main point is that sensible scalings should reveal the primary balances between physical mechanisms in equations, leaving the remaining terms as smaller corrections, at least at first sight (it often happens that what we thought was a small correction rises up and hits us between the eyes: but that's all part of the experience). If we have the wrong scalings, it usually becomes apparent fairly soon. In later chapters we give an introduction to asymptotic analysis, a framework which allows us to make the idea of approximate solution more systematic.

[12] At large Reynolds number the flow is turbulent and so unsteady; the drag coefficient must be interpreted as a time average.

Sources and further reading

Acheson's book *From Calculus to Chaos* [1] has a lot more on the pendulum. The flagpole problem was lifted from the book of Fowkes & Mahoney [12], where many more details will be found. It is here partly as an exercise in scaling, but also as an introduction to the beam equation.

Exercises

1. **Advection-diffusion.** If $T(\mathbf{x}, t)$ is the temperature in an incompressible fluid which is moving with velocity $\mathbf{u}$, explain why the heat flux is

$$\rho c T \mathbf{u} - k \nabla T.$$

Take an arbitrary small volume V *fixed in space*, write energy conservation in the form

$$\frac{d}{dt} \int_V \rho c T \, dV + \int_{\partial V} (\rho c T \mathbf{u} - k \nabla T) \cdot \mathbf{n} \, dS = 0,$$

then use the divergence theorem and $\nabla \cdot \mathbf{u} = 0$ to derive (4.1). (Note that in the derivation on page 40 we used incompressibility to say that the density in the material volume remains constant. If the fluid is compressible, we have to worry about what we mean by the specific heat, because the density changes. That is, we have to think carefully about the thermodynamics of the problem. Fortunately, for most liquids the density change with temperature is small enough to be neglected in the convective derivative (although not necessarily in the buoyancy body force); in gas dynamics, two specific heats are considered, one at constant pressure and one at constant volume.)

2. **Peclet numbers.** Consider the advection-diffusion problem of Figure **??** on page **??**. Show that other possible length scales are

$$\frac{\kappa}{U}, \quad \frac{k(T_\infty - T_0)}{\rho U^3}.$$

If we use the first, what happens to the boundary $r = a$, and why might this be inconvenient? Explain why the denominator of the second is a kinetic energy flux and hence why it is an inappropriate length scale for this problem.

3. **The Boussinesq transformation.** Consider the steady-state dimensionless advection-diffusion problem

$$\mathrm{Pe}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\nabla^2 T,$$

in which the velocity is given by potential flow past a two-dimensional body (not necessarily a circular cylinder) with potential ϕ and streamfunction ψ:

$$u=\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y},\quad v=\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x},\qquad \nabla^2\phi=\nabla^2\psi=0.$$

Switch from (x,y) to (ϕ,ψ) as independent variables, so that

$$\frac{\partial}{\partial x}=\frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial\psi}=u\frac{\partial}{\partial\phi}-v\frac{\partial}{\partial\psi}$$

etcetera,[13] to show that the problem becomes

$$\mathrm{Pe}\frac{\partial T}{\partial\phi}=\frac{\partial^2 T}{\partial\phi^2}+\frac{\partial^2 T}{\partial\psi^2}$$

in the (ϕ,ψ) plane. If the flow is symmetric, what are to the boundary conditions in the new variables? (This problem can be solved by the Wiener–Hopf technique, but it is a complicated business.)

Note that the left-hand side is the directional derivative of T along streamlines, which are orthogonal; to the equipotentials (why?).

4. **The Kirchhoff transformation.** Suppose that the thermal conductivity of a material depends on the temperature. Show that the steady heat equation

$$\nabla\cdot(k(T)\nabla T))=0$$

can be transformed into Laplace's equation for the new variable $u=\int^T k(s)\,ds$.

5. **Newton's law of cooling and Biot numbers.** The process of cooling a hot object is a complicated one. In addition to conduction to the surroundings, it may involve both forced and natural convection if the body is immersed in a liquid or gas; there may be boiling, or thermal

[13] A shortcut: because $\phi+i\psi$ is an analytic (holomorphic) function $w(z)$ of $z=x+iy$, the Cauchy–Riemann equations let us simplify the Laplacian operator to

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\left|\frac{dw}{dz}\right|^2\left(\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial\psi^2}\right).$$

radiation. A very widely used model lumps all these effects into a single linear law, known as Newton's law of cooling: the heat flux per unit area from the body is given by

$$-k\left.\frac{\partial T}{\partial n}\right|_{\text{boundary}} = h(T - T_\infty),$$

where h is the *heat transfer coefficient and T_∞ is a measure of the ambient temperature*. What are the units of h? Explain in general terms why this law is reasonable (including the minus sign).

There are many empirical laws giving h in specific circumstances. For the specific example of black-body radiative transfer, it can be derived exactly. Recall that the *Stefan–Boltzmann law* says that the heat flux is

$$KT^4 - KT_\infty^4,$$

where T is the *absolute* temperature and K is a constant (what are its units?) Show that the Newton law is a good approximation if T is not too far from T_∞ and find h in this case.

If the body has typical temperature T_0 and size L, write the Newton law in dimensionless form as

$$-\frac{\partial T'}{\partial n'} = \mathrm{Bi}\, T'$$

where $\mathrm{Bi} = hL/k$ is known as a *Biot number*.

6. **Coffee.** Alphonse takes milk in his coffee and he has to carry the cup a long way from the machine to his desk. He wants the coffee to be as hot as possible when he gets there. Make a simple model to decide whether it is better to add the (cold) milk to the coffee at the machine or at his desk.

   Still on the subject of coffee, Bérénice takes sugar in hers. At time $t = 0$ she puts a lump in. If $V(t)$ is the volume and $A(t)$ the surface area of the undissolved lump, and the coffee is well-mixed, explain (on dimensional grounds) why a crude model for the evolution is

$$\frac{dV}{dt} \propto -A, \qquad A \propto V^{\frac{2}{3}}.$$

   Solve the model and show that V reaches zero in finite time.

   Now solve the differential equation

$$\frac{dV}{dt} = V^{\frac{2}{3}}, \quad t > 0; \qquad V(0) = 0.$$

I hope you found *all* the solutions:

$$V(t) = \begin{cases} 0, & 0 < t < t_0, \\ \left(\frac{1}{3}(t-t_0)\right)^3, & t \geq t_0, \end{cases}$$

for any $t_0 \geq 0$; the nonuniqueness arises because the right-hand side $V^{\frac{2}{3}}$ is not Lipschitz in V. (The solution $V \equiv 0$ of the differential equation is tangent to all the solutions $V = \left(\frac{1}{3}(t-t_0)\right)^3$.)

Your opportunity to review the Picard theorem on existence and uniqueness for first-order differential equations.

Hercule asks the question: if I observe the state of the sugar in Bérénice's coffee, can I deduce when she put it in? Show that he can do this if the lump is only partly dissolved but he can't if it is wholly gone. Hence give a physical interpretation of the non-uniqueness result immediately above. (Ivar Stakgold told me this example. I am happy to return the favour by recommending his book on Green's functions etc.; see page **??**.)

7. **Boiling an egg.** A spherical homogeneous (ie purely mathematical) egg of radius a is placed in cold water at temperature T_0, the egg being initially at this temperature too. Over a time t_0 the water temperature T_w is increased linearly to T_1, where it remains. The temperature T in the egg is modelled by the heat conduction equation

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T,$$

where ρ is the density, c the specific heat capacity and k the thermal conductivity of the egg, with the Newton boundary condition

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=a} = h(T - T_w).$$

Make the model dimensionless and identify the dimensionless parameters. What possible regimes might there be and how can you see this by looking at the size of your dimensionless parameters?

Is there any difference in your analysis if the egg is boiled by the traditional method of putting it into boiling water (assume the water temperature remains constant) and leaving it there while you sing your national anthem or some other suitable song (in England, the hymn Onward Christian Soldiers is traditional for this purpose)?

8. **Flagpole in an earthquake.** Suppose a flagpole is in still air, but that its base $y = 0$ is oscillated horizontally by an earthquake, so that the condition $y(0,t) = 0$ is replaced by

$$y(0,t) = a \cos \omega t, \tag{4.6}$$

the other boundary conditions remaining as

$$y_x(0,t) = 0, \quad y_{xx}(L,t) = y_{xxx}(L,t) = 0. \tag{4.7}$$

Nondimensionalize the unsteady unforced flagpole (beam) equation

$$\rho A y_{tt} + EAk^2 y_{xxxx} = 0 \tag{4.8}$$

using the timescale $1/\omega$ implicit in the boundary condition (4.6). What is the appropriate scale for y?

What is the radius of gyration of a circular cylinder of radius a?

A circular pole is 10 m high, and has a radius of 10 cm. It is made of steel for which $E_s = 2.0 \times 10^7$ kg m^{-1} s^{-2}, $\rho_s = 7.8 \times 10^3$ kg m^{-3}, and the oscillations are at a frequency of 1 Hz. What is ω? It is desired to simulate the behaviour of this pole using a wooden model of radius 1 cm and with the same value of ω. Given that $E_w \approx E_s/20$, $\rho_w \approx \rho_s/13$, how long should the model be?

9. **Flagpole under gravity.** Show from a vertical force balance that a vertical flagpole is subject to a compressive force $C(x)$ which satisfies

$$\frac{dC}{dx} = -A\rho g$$

(g is the acceleration due to gravity), with $C(L) = 0$. Hence find C; what is its value at $x = 0$, why?

It can be shown (see Section 5.1) that the effect of gravity is to modify the flagpole equation to

$$\rho A y_{tt} + (C y_x)_x + EAk^2 y_{xxxx} = 0$$

(the new term is just like the tension in the equation for waves on a string, but it is on the other side of the equation because C is a compression = negative tension). What is the dimensionless parameter that measures the relative importance of gravity for the pole of question 1? How big is it in that situation?

10. **Normal modes of strings and flagpoles.** A string with mass density ρ per unit length is stretched between $x = 0$ and $x = L$ to tension T. The end $x = L$ is held fixed, while the end $x = 0$ is oscillated transversely at frequency ω so that its displacement there is $y(0,t) = a\cos(\omega t)$. Find the time-periodic solution $y(x,t) = f(x)\cos(\omega t)$; does it exist for all ω and if not, what happens at the exceptional value(s)?

Show that the dimensionless unsteady flagpole equation of question 1 has solutions of the form $\cos / \sin \alpha x \times \cosh / \sinh \alpha x$ and find α. (Strictly voluntary, because rather hard work: find the time-periodic solution to question 1; you may want to use a symbolic manipulator such as Maple.)

11. **A layer of viscous fluid flowing on a surface.** A uniform layer of viscous fluid, of thickness h, flows down a plane inclined at an angle θ to the horizontal, so that the acceleration due to gravity down the plane is $g \sin \theta$. Show on dimensional grounds that the flux (per unit length 'into the page') is proportional to $h^3 g \sin \theta / \nu$, where ν is the kinematic viscosity.

    Explain (in terms of a force balance) why appropriate conditions at the free surface $y = h$ are $\sigma_{ij} n_j = 0$, where σ_{ij} is the stress tensor.

    Take coordinates x downhill along the plane and y normal to it. Show that there is a solution $\mathbf{u} = (u(y), 0, 0)$ and that the free surface conditions reduce to $p = 0$, $\partial u / \partial y = 0$. Find $u(y)$ and verify the dimensional analysis for $Q = \int_0^h u(y)\, dy$. Show also that this solution corresponds to one half of the Poiseuille flow of the previous question (see Figure 21.2 on page 256).

12. **Dimensional analysis of Poiseuille flow.** In a *Poiseuille flow* down a pipe, a Newtonian viscous fluid is forced down a circular tube of cross-sectional area A (or radius a) and length L by a pressure drop ΔP. Confirm that there are 6 independent physical quantities in this problem, and state their dimensions: L and a (or $\sqrt{A}$), which are properties of the pipe; μ and ρ, which are properties of the fluid; the input or output variables (specify one and find the other), ΔP and either a volume flux Q or an average velocity U. How is U related to Q and A?

    Use Buckingham to show that there are 3 independent dimensionless quantities and find them in their most useful forms. If this problem has a steady solution show that they are related by an equation of the form

    Obviously you can take products etc., but try to single out the best combinations.

    $$\Delta P = \rho U^2 F(\mathrm{Re}, L/a)$$

    Re being the Reynolds number Ua/ν.

    If we further assume that our pipe is very long so that we have translational invariance along it, then instead of ΔP and L as independent physical quantities, we only have the pressure gradient P'. Show that

$$P' = \frac{\rho U^2}{a}\phi(\mathrm{Re})$$

for some function ϕ.[14]

Using the information in the footnote, find relations between the volume flux and the pressure drop (a) for slow flow (b) for fast flow with $\mathrm{Re} < 2000$. How does the flux depend on the radius in each case?

Why does water come out of the tap (or a garden hose) in a thin but very fast jet when you put your finger over the end, but not when you take it away? If your WC is refilling, and you turn on a cold tap connected to the same water supply, why does the cistern stop making that sshh noise?

13. **Poiseuille flow: exact solution.** Consider the two-dimensional version of the flow of the previous exercise, in which a viscous liquid flows in the x–direction between two parallel plates at $y = \pm a$ under a pressure gradient P'. Assuming that

$$\frac{\partial}{\partial x}(\text{everything except } P) = 0, \qquad \frac{\partial p}{\partial x} = P',$$

show that there is an exact steady solution of the Navier–Stokes equations in which the velocity is $(u(y), 0)$ where

$$\mu\frac{\partial^2 u}{\partial y^2} = P'.$$

Applying the no-slip condition at $y = \pm a$, find $u(y)$ and hence the flux per unit length in the z–direction. Repeat the calculation for a circular pipe.

[14] As $\mathrm{Re} \to 0$, we have the analytical result (see the next exercise) that $\phi \sim (8Re)^{-1}$. Even though the flow from which this is derived is an exact solution of the Navier–Stokes equations for all Re, it is unstable. The effective drag for large Reynolds numbers is derived from measurements of time averages of much more complicated unsteady flows. This leads to empirical approximations such as $\phi \approx 32/\mathrm{Re}$ for $\mathrm{Re} < 2000$, and for $\mathrm{Re} > 3000$ ϕ is approximately half the root of

$$\frac{1}{\sqrt{\Phi}} = 2\log_{10}\left(\mathrm{Re}\sqrt{\Phi}\right) - 0.8.$$

Chapter 5

Case study: hair modelling and cable laying

In the next three chapters, we look at three 'real-world' problems, which all arose in industry. There are three reasons for presenting these case studies. One is simply to give some examples of modelling in action (the only way to get good at it is to do it). Another is to illustrate the techniques of the previous chapter in a less academic setting. Finally, we shall use these case studies, and others presented later in the book, to illustrate the techniques we develop later, although we do not have room to give full details of all that has been done on these problems (much of which is, ultimately, numerical). References to the literature are given at the end of the chapter.

You can skip these chapters and still read much of this book. Although you won't have wasted your money entirely, you will miss out on some nice applications of the methodology we describe later.

Both the models of this chapter are based on the Euler–Bernoulli beam equation for the bending of a slender elastic beam. This is such a common model (we have already seen it in the context of flagpoles) that it merits its own section, following which I have included a short section on the topical problem of modelling of hair. Then we turn to the rather harder problem of building a model for cable laying.

5.1 The Euler–Bernoulli model for a beam

We wrote down the Euler–Bernoulli model for the displacement of a slender nearly straight beam in Chapter **??**. In fact there is no requirement for the beam to be straight, but it must be slender for a crucial assumption in the following model to hold. Let us therefore consider a beam, or slender elastic

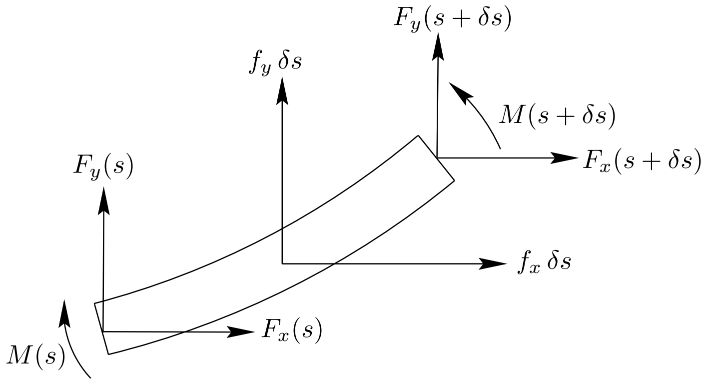

Figure 5.1: Forces and moments on an element of a beam.

rod, lying along the curve $\mathbf{r} = (x(s), y(s))$ in the plane (the equations are much more complicated in three dimensions). Here s is arclength, and if we let $\theta(s)$ be the angle between the curve and the x–axis, we have

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta, \quad \frac{d\theta}{ds} = \kappa,$$

where κ is the curvature.

If the beam is nearly straight and lies along the x–axis, you can think of F_x as a ten-sion/compression.

Now look at Figure 5.1, which shows a small element of the beam, of length δs. The forces acting on the element are the internal elastic forces acting on its ends and a body force with components (f_x, f_y) per unit length. We write the elastic forces at the ends as $(F_x(s), F_y(s))$ and $(F_x(s+\delta s), F_y(s+\delta s))$ respectively. In equilibrium the difference between these must cancel the body force $(f_x, f_y)\delta s$, and taking $\delta s \to 0$ we find the force balance equations

$$\frac{dF_x}{ds} + f_x = 0, \qquad \frac{dF_y}{ds} + f_y = 0.$$

Unlike a string, a beam resists being bent, by generating an internal bending moment $M(s)$ to balance the moment of the internal forces. Resolving the latter normal to the beam and taking moments about the left-hand end of our element, we find the equation

$$\frac{dM}{ds} - F_x \sin\theta + F_y \cos\theta = 0.$$

Let us pause for a moment and count equations and unknowns. The unknowns are θ (from which we can find x and y by integration), F_x, F_y and M, and we have three equations. However, we haven't yet said anything about the material the beam is made from. We need a constitutive relation to tell us something about how the forces and displacements are related. For a beam that started off straight and is bent into a curve, a good model is that

$$M = b\frac{d\theta}{ds},$$

that is, the bending moment is proportional to the curvature (stop and think why this is reasonable). A systematic derivation of this condition starting from the equations of linear elasticity is surprisingly difficult, and we will have to wait until Chapter **??** to see how to do it. We will also see there that the constant of proportionality b, known as the bending stiffness, is equal to EAk^2 where, as before, E is the Young's modulus, A the cross-sectional area of the beam, and k the radius of gyration of that cross-section. Our final beam equation is thus

$$\frac{dM}{ds} = EAk^2\frac{d\theta}{ds}.$$

It is easy to eliminate M, so we find the system

$$\frac{dF_x}{ds} + f_x = 0, \quad \frac{dF_y}{ds} + f_y = 0, \quad EAk^2\frac{d^2\theta}{ds^2} - F_x\sin\theta + F_y\cos\theta = 0 \quad (5.1)$$

for F_x, F_y and θ. It is a very straightforward exercise to show that when the beam is straight and nearly along the x–axis, so that $\theta \approx dy/dx$, we recover the system

$$EAk^2\frac{d^4y}{dx^4} - \frac{d}{dx}\left(F_x\frac{d\theta}{dx}\right) - f_y = 0, \quad \frac{dF_x}{dx} + f_x = 0$$

which is a generalisation of the flagpole equations to include body forces in both directions.

5.2 Hair modelling

One of the fastest growing customers for mathematical modelling is the entertainment industry. The main drivers are the demand for realistic real-time simulation in computer games, and the trend towards photo-realistic animated characters. Long hair and clothes are notoriously difficult to model; for example in the 2001 film *Final Fantasy*, about 20% of the production time was devoted to the 60,000 strands of lead character Aki's hair.[1] In this

[1]Water, with its longer mathematical pedigree, has been more successfully treated, a famous example being the ocean in *Titanic*, much of which was computer generated. It is said that a mathematician pointed out that the algorithm for waves did not conserve mass, and received the Hollywood mogul's reply, "I don't give a flying fish [actually, he used another word] if it loses mass, so long as it looks good". Ho hum.

short section we look at a very simple model for hair, in which each strand is treated individually and does not interact with its neighbours. This is only one of several possible models for hair, and at the time of writing this is a wide-open research area.

The idea is to treat the hair as an elastic rod of cross-sectional area A and density ρ, under gravity. Thus we just use the model of the previous section, with gravity providing the body force:

$$\frac{dF_x}{ds} = 0, \qquad \frac{dF_y}{ds} + \rho g A = 0, \quad b\frac{d^2\theta}{ds^2} - F_x \sin\theta + F_y \cos\theta = 0,$$

with the constitutive relation $M = b\,d\theta/ds$. Now the hair has a free end, at which $F_x = F_y = 0$, so measuring s from there we can easily find f_x and f_y, leaving the equation

$$b\frac{d^2\theta}{ds^2} + \rho g A s \cos\theta = 0.$$

for θ. Appropriate boundary conditions are quite easy in this case, as we can expect to prescribe θ where the hair enters the head (say normally), and we'll have $d\theta/ds = 0$ at $s = 0$ because that end of the hair is free. This is a relatively straightforward two-point boundary value problem to solve numerically using any of a variety of packages, although because this nonlinear system may have bifurcations the software must be able to handle these. (see Exercise ...). Solutions of this equation do indeed do more or less what real hair does, although the neglect of hair-hair interactions is a serious defect of the model. See the exercises for further properties of this problem.

5.3 Undersea cable-laying.

Cables and pipelines have been laid under the sea since the first electric telegraphs; nowadays they often hold optical fibres. Several factors compete in the design of cables: for example, strength and durability dictate large cables, while expense and speed of laying dictate thin ones. The process of laying is a dangerous time in the life of a cable, and very precise control of the operation is necessary to avoid damage while maximising the laying speed. In this case study, which recurs in Chapter **??**, we look at a model of the process of laying a cable from a ship, as shown in Figure 5.2. We consider the steady-state model, as a first step towards developing a dynamic model to enable real-time control of the operation.

As the ship moves forward the cable is unreeled from a large drum and passes through a 'tensioner', shown as a box above the stern of the ship. This has the effect of prescribing the angle at which the cable leaves the ship. The

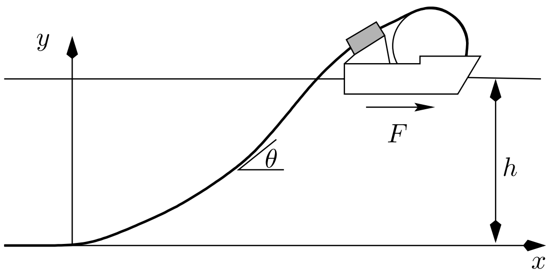

Figure 5.2: Cable laying from a ship.

ship exerts a force F on the cable, which also experiences buoyancy forces as it sinks to the sea bed. Our objective is to set up a model which allows us to calculate the shape of the cable (where does it feel the greatest stresses?) and the thrust needed from the ship.

5.4 Modelling and analysis

Let us now return to our cable-laying problem. Taking an origin at the point where the cable touches the sea bed, a distance h below the surface, we denote its position by $(x(s), y(s))$ for $0 < s < L$, where the wetted length L is as yet unknown. (For simplicity we are going to ignore the small length of cable in the air astern of the ship.) The angle between the cable and the horizontal is θ, as before, and the unit tangent and normal to the curve are $\mathbf{t} = (\cos\theta, \sin\theta)$ and $\mathbf{n} = (-\sin\theta, \cos\theta)$ respectively.

We are going to solve the beam system (5.1), and the main difficulty is in writing down the external forces f_x and f_y. There are three forces on the cable: one is its weight, a second is buoyancy, and a third is is drag from the water. The weight of the cable is easy, just contributing a term $\rho_c g A$ to the equation for F_y, where ρ_c is the density of the cable. We will focus on the buoyancy (the drag is dealt with in an exercise). Having completed the model, we then need to decide what boundary conditions to apply at $s = 0$ and $s = L$. The solution of the resulting two-point boundary value problem for a system of ordinary differential equations will almost inevitably be carried out numerically, again using a two-point boundary value problem solver, although in Chapter **??** we also look at an approximation for cables with low bending stiffness.

The buoyancy force per unit length on the cable is entirely due to hydrostatic pressure, and rather surprisingly it has two components. One is the ordinary Archimedes force, but we have to be careful in evaluating it. Consider a cylindrical element as in Figure 5.3. If the ends of the cylinder were

Old chestnut, claims new victims every year: you are in a boat on a lake, and you throw a brick over the side. Does the water level rise, fall or stay the same?

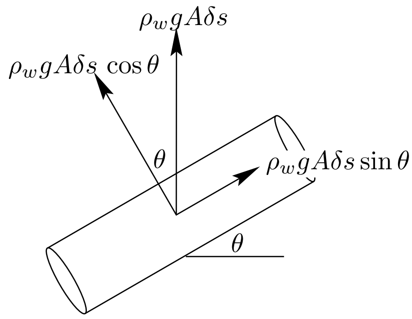

Figure 5.3: Hydrostatic force on a cylinder.

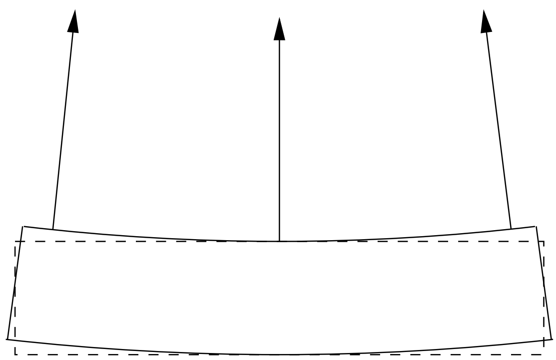

Figure 5.4: Normals on a slightly bent cylinder.

exposed to the water, the buoyancy force would be equal to the weight of the water displaced, namely $\rho_w gA$ per unit length, acting vertically upwards (see the exercises to prove this). Remembering that the pressure acts normally to the surface, and resolving along and normal to the cylinder, what remains after we have subtracted the contribution from the ends is a force per unit length of $\rho_w gA\cos\theta$ along the normal.

However, our cylinder is not quite straight. As shown in Figure 5.4, the surface area on the 'outside' of a bend is bigger than that on the 'inside', and so if a pressure p, here the hydrostatic pressure $\rho_w g(h-y)$, acts on a curved cylinder like this (but not on its endpoints), there is a net force in the normal direction. It is fairly clear that this extra force is proportional to the curvature — the area surplus/deficit is proportional to the rate of change of θ with s — and it can be shown (see the exercises) that the magnitude of this contribution to the buoyancy force is $pA\kappa$ per unit length, so the total buoyancy force is $\rho_w gA\cos\theta + pA\kappa$ along $\mathbf{n}$.

In summary, our model is

$$\frac{dF_x}{ds} + B_x = 0, \quad \frac{dF_y}{ds} + B_y - \rho_c g A = 0, \quad EAk^2\frac{d^2\theta}{ds^2} - F_x \sin\theta + F_y \cos\theta = 0, \tag{5.2}$$

where

$$(B_x, B_y) = \left(\rho_w g A \cos\theta + pA\frac{d\theta}{ds}\right)(-\sin\theta, \cos\theta). \tag{5.3}$$

5.4.1 Boundary conditions

We need four boundary conditions for this system in order to fix F_x, F_y and θ. Having done this, we will integrate $dx/ds = \cos\theta$, $dy/ds = \sin\theta$ from $s = 0$ to $s = L$, with the initial conditions $x(0) = y(0) = 0$. Then the condition $y(L) = h$ will tell us L and $x(L)$ will tell us the horizontal distance between touchdown of the cable and the ship. Of course, these integrations must be carried out numerically, and on the face of it the equations are even more nonlinear than the hair model; but read on.

Let us first think about the conditions at $s = 0$. We expect the cable to leave the sea bed smoothly, so we impose

$$\theta = 0, \quad \frac{d\theta}{ds} = 0 \tag{5.4}$$

at $s = 0$. The second of these conditions says that the bending moment is continuous at this point. As we shall see in Section 10.5.2, only a point force could cause a discontinuity in M.

Nothing else obvious can be applied at $s = 0$, so let us look at $s = L$. Here we know the angle at which the cable leaves the ship, and we know the horizontal force F_x:

$$\theta = \theta^*, \quad F_x = F \tag{5.5}$$

at $s = L$.

In the language of beam equations, we are imposing 'clamped' boundary conditions.

5.4.2 Effective forces and nondimensionalisation

Before we scale these equations, we note a potentially serious difficulty, and a neat extrication from it. Because p is hydrostatic, we have $p = \rho_w g(h - y)$. But this means that the p in (5.3) depends on y, and our scheme of only solving for y *after* finding θ looks doomed: it seems that the system is fully coupled. However, there is a *deus ex machina*. We first note that $dp/ds = -\rho_w g\, dy/ds = -\rho_w g \sin\theta$. Then, we define *effective horizontal and vertical forces* by

$$F_x^e = F_x + pA\cos\theta, \quad F_y^e = F_y + pA\sin\theta,$$

so that
$$\frac{dF_x^e}{ds} = \frac{dF_x}{ds} - pA\sin\theta\frac{d\theta}{ds} + A\cos\theta\frac{d\theta}{ds},$$
and similarly for dF_y^e/ds. When we substitute in (5.3), all the terms involving p vanish, and as $F_x^e = F_x$ at $y = 0$ where $p = 0$, there is no problem in applying the boundary condition (5.5). (The variables F_x and F_y are only steps on the way to θ, which is what we really need; so there is no loss in our not calculating them.)

Carrying out this simplification, we arrive at the system
$$\frac{dF_x^e}{ds} = 0, \quad \frac{dF_y}{ds} = \rho_c gA, \quad EAk^2\frac{d^2\theta}{ds^2} - F_x^e\sin\theta + F_y^e\cos\theta = 0,$$
with the boundary conditions
$$\theta = 0, \quad \frac{d\theta}{ds} = 0$$
at $s = 0$, and
$$\theta = \theta^*, \quad F_x^e = F$$
at $s = L$.

The scales for this system are clear. We scale x, y and s with h, and F_x^e, F_y^e with $\rho_c gAL$. Immediately dropping the primes, we have the dimensionless model

Consistency: check that the scale for F_x^e is indeed a force. It is probably slightly preferable to use this scale rather than F because F may be an unknown.

$$\frac{dF_x^e}{ds} = 0, \quad \frac{dF_y}{ds} = 1, \quad \epsilon\frac{d^2\theta}{ds^2} - F_x^e\sin\theta + F_y^e\cos\theta = 0, \tag{5.6}$$
with the boundary conditions
$$\theta = 0, \quad \frac{d\theta}{ds} = 0 \tag{5.7}$$
at $s = 0$, and
$$\theta = \theta^*, \quad F_x^e = F^* \tag{5.8}$$
at $s = \lambda$, where the three dimensionless parameters are
$$\epsilon = \frac{Ek^2}{\rho_c gh^3}, \quad F^* = \frac{F}{\rho_c gA}, \quad \lambda = L/h;$$
note that λ is unknown.

We can do a little better still: we can find F_x^e and F_y^e explicitly, and substituting into the equation for θ we have
$$\epsilon\frac{d^2\theta}{ds^2} - F^*\sin\theta + (F_0 + s)\cos\theta = 0, \tag{5.9}$$

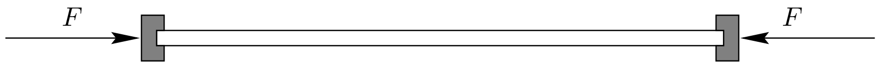

Figure 5.5: The Euler strut.

in which $F_0 = F_y^e(0)$ is an unknown constant. There, however, *three* boundary conditions for this equation, namely the relevant parts of (5.7), (5.8), and so we have an extra equation to tell us this unknown constant.

We return to this problem in Chapter **??**, where we show how to construct an approximate solution when ϵ is, as its name suggests, small, as is the case when the cable is heavy or the water is deep (it is clear that ϵ measures the relative importance of bending stiffness and cable weight). This kind of boundary value problem, with a small parameter multiplying the highest derivative, is often known as 'stiff' in a numerical context, and there are many specialised 'stiff solvers' to handle these problems.

Even though the beam is anything but stiff!

Sources and further reading

The cable-laying problem was proposed by (??); it is a simplifies version of more complicated three-dimensional 'upwinding' problems to do with the winding of wire onto a reel (the twist, or *torsion*, of the wire plays an important role in these situations). Ref to Love.

Exercises

1. **The Euler strut (i).**

   A thin rod of length L and bending stiffness b is clamped at each end and is compressed by a force F, as in Figure 5.5. Adapt the analysis of Section 5.1 to derive the dimensionless boundary value problem

   $$\frac{d^2\theta}{ds^2} + \alpha^2 \sin\theta = 0, \qquad \theta(0) = \theta(1) = 0,$$

   for the angle between the rod and the x axis, where $\alpha^2 = FL^2/b$. Show that $\theta = 0$ is always a solution; what does it represent?

   Now suppose that θ is small. Assuming that $\sin\theta \approx \theta$ (we will do this in more detail in Exercise ... of Chapter **??**), write down a linear two-point boundary value problem and show that its only solution is $\theta = 0$ unless $\alpha = n\pi$ for integral n. Deduce that as F is increased from zero, it is first possible to have a non-trivial solution ($\theta \neq 0$) when

$FL^2/b = \pi^2$ and sketch the resulting solution. What happens when $FL^2/b = 4\pi^2$

This appearance of a non-trivial solution as a parameter varies is known as a *bifurcation*. This one is easy to illustrate in practice with, say, a plastic ruler. On a larger scale, when putting up a modern tent with a carbon fibre pole, you have to bend the pole to fit it into its sockets. As you do so by bringing the ends together, starting with a straight pole, you initially go through the first buckling mode $\alpha = \pi$ (you may also see the second mode if the pole is long enough).

Buckling can also occur when the pole is held vertically, so the gravity supplies the compression, as the next example shows.

2. **Groan.** Take the hair model

$$b\frac{d^2\theta}{ds^2} + \rho g s \cos\theta = 0$$

with the boundary conditions $\theta = \theta_0$ (given) at $s = 0$ and $d\theta/ds = 0$ at $s = L$; explain what these conditions model. Look for a solution for a nearly vertical hair. That is, write $\theta = \pm\frac{\pi}{2} + \phi$ and derive two versions (related by $\xi \leftrightarrow -\xi$ of the *Airy equation*

That's 'orrible.

$$\frac{d^2\phi}{d\xi^2} \pm \xi\phi = 0, \qquad 0 < \xi < \xi_0 = L\sqrt{\rho g/b}$$

where ξ is a suitably scaled version of s. Which of $\pm$ is for upward-pointing hair and which for downward pointing?

Taking the $-$ sign for the standard definition, Airy's equation has linearly independent solutions $\mathrm{Ai}(\xi)$, $\mathrm{Bi}(\xi)$ which, for $\xi > 0$ both oscillate, while for $\xi < 0$ one grows exponentially and one decays, as in Figure **??**. What sort of solutions do you expect to see for (a) upwards (b) downwards pointing hair?

Compare $y'' + \lambda y = 0$ for $\lambda > 0$, $\lambda < 0$.

Show that as ξ_0 varies an upward-pointing hair can buckle via a bifurcation away from the vertical solution, and find the shortest length at which it does so in terms of Ai and Bi. Using the fact that for $x > 0$ Ai is decreasing and Bi is increasing, show that downward-pointing hairs cannot buckle away from the vertical solution.

3. **Waving hair and unsteady beams.** Consider an unsteady version of the Euler–Bernoulli beam model, in which the beam is parametrised as $(x(s,t), y(s,t))$. Justify the model

$$\frac{\partial F_x}{\partial s} + f_x = \rho A\frac{\partial^2 x}{\partial t^2}, \qquad \frac{\partial F_y}{\partial s} + f_y = \rho A\frac{\partial^2 y}{\partial t^2}, \quad \frac{dM}{ds} - F_x\sin\theta + F_y\cos\theta = 0,$$

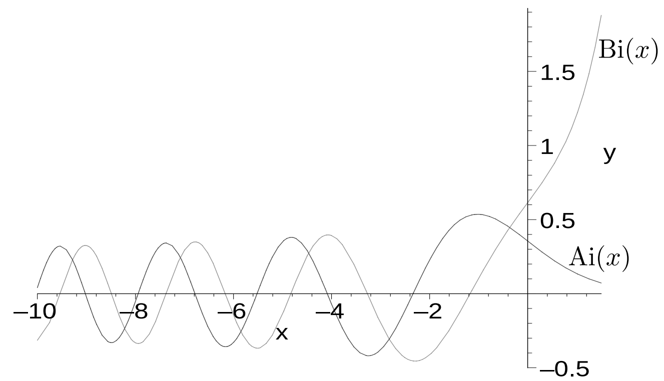

Figure 5.6: The Airy functions. $\mathrm{Ai}(0) = 3^{-\frac{2}{3}}/\Gamma(\frac{2}{3})$, $\mathrm{Bi}(0) = 3^{-\frac{1}{6}}/\Gamma(\frac{2}{3})$.

provided that the rate of change of angular momentum of the element can be ignored. Show that for a straight beam under constant tension the equation of motion for small displacements is

$$\rho A \frac{\partial^2 y}{\partial t^2} + b\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} = 0,$$

where T is the tension (that is, F_x) and gravity has been ignored.

4. **Eureka!** Dot both sides with a constant vector and use the divergence theorem to show the identity

$$\int_{\partial V} \Phi \mathbf{n}\, dS = \int_V \nabla\Phi\, dV$$

for sufficiently regular scalar functions Φ and volumes V. Put

$$\Phi = \rho_w g(\text{constant} - y)$$

to derive Archimedes' principle: the buoyant force on a body immersed in water of density ρ_w is equal to the weight of the water displaced. Well worth jumping out of the bath for.

Inspect the first term on the right-hand side of formula 5.3 for the buoyant force on an element of a submerged cable. What happens if $\theta = \frac{\pi}{2}$? Do you believe this? Would a cylindrical stick held vertically in water rise when let go? Resolve the apparent paradox.

5. **Hydrostatic force on a bent cylinder.** Consider the cylinder of figure 5.4, and suppose that arclength measured along the centreline is

s_0 at the left-hand end and $s_0+\delta s$ at the right-hand end. Suppose that the centreline has position $\mathbf{r}_0(s) = (x(s), y(s), 0)$ and that the cylinder is circular in each plane normal to this line, with radius ϵ. Suppose that a constant pressure p acts normally to the curved surface of the cylinder (but not the ends).

The correction due to hydrostatic variation in pressure over the element vanishes when $\delta s \to 0$.

Show that the tangent vector to the centerline is $\mathbf{t}(s) = (\cos\theta(s), \sin\theta(s), 0)$ and the normal (in the (x, y) plane) is $\mathbf{n} = (-\sin\theta, \cos\theta, 0)$. Show that $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b} = (0, 0, 1)$ are orthonormal. Show that a point on the surface can be written in the form

This will be familiar if you have done the Serret–Frenet formulae from differential geometry.

$$\mathbf{r} = \mathbf{r}_0(s) + \epsilon\cos\phi\,\mathbf{n}(s) + \epsilon\sin\phi\,\mathbf{b}(s), \qquad s_0 < s < s_0+\delta s, \quad 0 \le \phi < 2\pi.$$

Explain why

$$\frac{\partial \mathbf{r}}{\partial s} \wedge \frac{\partial \mathbf{r}}{\partial \phi}$$

is normal to the surface and why the unit normal to the surface and surface area element are

$$\mathbf{N} = \mathbf{n}\cos\phi + \mathbf{b}\sin\phi, \quad dS = \left|\frac{\partial \mathbf{r}}{\partial s} \wedge \frac{\partial \mathbf{r}}{\partial \phi}\right| d\phi ds = \epsilon(1 - \epsilon\kappa\cos\phi)\, d\phi\, ds,$$

where $\kappa = d\theta/ds$ is the curvature of the centreline. Deduce that

$$\int_S p\mathbf{N}\, dS = \pi\epsilon^2 \mathbf{n}\, \delta s,$$

and hence confirm formula (5.3).

6. **Water drag on a cylinder.** If a cylinder of radius a with axis along a line making an angle θ with the x–axis is placed in a fluid moving with far-field speed $(U, 0, 0)$, the drag per unit length on it is $\rho_w U^2 a(C_d^x(\theta), C_d^y(\theta), 0)$ (see Chapter 4). Incorporate this force into the cable-laying model when the ship moves forward with speed U, and identify the new dimensionless parameter which tells you the relative importance of drag and buoyancy.

Chapter 6

Case study: the thermistor 1

6.1 Heat and current flow in thermistors

A thermistor is a temperature-dependent resistor. A typical thermistor is a penny-shaped piece of a special ceramic material, about 1 mm thick and with a radius of 5 mm, and with metal contacts on the flat faces. The kind we are interested in becomes more resistive as it gets hotter, so it can be used as a fuse: if the current through the thermistor surges for any reason, the resulting Ohmic (I^2R) heating increases the resistance and so cuts the current. The beauty of this is that when the current goes away, the thermistor just cools down and normal operation can resume without anybody having to replace a fuse. Televisions have dozens of thermistors in them, and so do hairdryers as a protection against overheating, which is why they switch themselves off for a while if they get too hot.

There are various reasons for wanting to analyse the heat and current flow in thermistors. One is the obvious question of design: how do the characteristics, such as the switch-off time in response to a current surge, depend on the physical parameters? Another is an issue of quality control: some thermistors can crack because rapid thermal expansion caused by large temperature gradients stresses the material too much. The full analysis of cracking requires a model for thermoelasticity which is beyond the scope of this book; however, even an order of magnitude estimate of the temperature gradients could be used as an input to an 'engineering' rule of thumb for the likelihood of cracking.

6.1.1 A simple model

Let us begin with a simple model for a thermistor on its own, with a voltage V_0 applied across it at $t = 0$; then we extend this to a thermistor in a simple

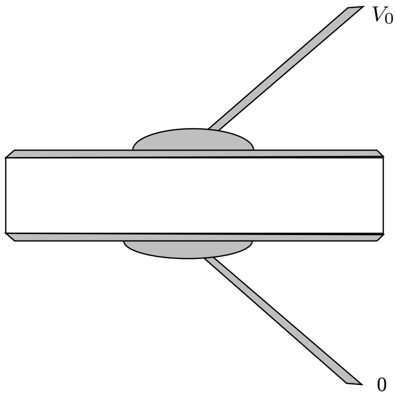

Figure 6.1: A thermistor: shaded regions are good (metallic) conductors.

circuit. We need first to think about how electric current flows through a solid. That is, we need a generalisation of Ohm's law $I = V/R$ for a resistor. This is straightforward. We assume that there is a local version of Ohm's law relating the current density $\mathbf{j}$ (units A m^{-2}) to the electric field E linearly:

$$\mathbf{j} = \sigma(T)\mathbf{E}$$

You may be more familiar with this as the *resistivity* $\rho(T) = 1/\sigma(T)$. What are its units?

where $\sigma(T)$ is a material property called the *conductivity*, whose dependence on temperature T, which is intrinsic to the proper working of the device, is shown explicitly. Now remember that there is an electric potential ϕ with $\mathbf{E} = -\nabla\phi$, and that current is conserved, so that $\nabla \cdot \mathbf{j} = 0$. Putting these together, we have

$$\mathbf{j} = -\sigma(T)\nabla\phi, \qquad \nabla \cdot (\sigma(T)\nabla\phi)) = 0.$$

We can also easily write down boundary conditions for the potential. If, as in Figure 6.1, the top and bottom of the thermistor are coated in an excellent conductor, the potential is very nearly constant on each, while there is no current through the sides. Thus, for $t > 0$, we have

$$\phi = V_0 \quad \text{on } z = H, \qquad \phi = 0 \quad \text{on } z = 0$$

and

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } r = a$$

where, clearly, (r, θ, z) are cylindrical polar coordinates with origin at the centre of the bottom face, while H is the thickness and r the radius of the cylinder.

Now we need to write down a model for the heat generation and conduction. That means we have to find a local version of the law for the power generated in a resistor, $VI = I^2R = V^2/R$. Clearly the local rate of heat production (volumetric heating) is

See the discussion of energy and work on page 29. You should check that the units are correct at W m^{-3}.

$$\mathbf{j} \cdot \mathbf{E} = \sigma |\nabla\phi|^2.$$

This appears as a source term in the heat equation for the temperature $T(\mathbf{x}, t)$,

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T + \sigma|\nabla\phi|^2.$$

Consistency: the heating term is positive so it acts to make T increase in time.

Boundary conditions for the heat equation are often problematical. The isothermal or perfectly insulated conditions beloved of exam question setters are rarely strictly applicable. It is safest to write down a Newton cooling law (see page 55)

$$-k\frac{\partial T}{\partial n} = h(T - T_a)$$

on the sides of the thermistor, where T_a is the ambient temperature and h the heat transfer coefficient. Taking some liberties, we may hope to model the cooling effect of the conducting top and bottom surfaces, together with the connecting wire and its solder, by a similar condition but with a larger value of h. Finally, because the heat equation is forward parabolic, we need an initial condition, for example

$$T(\mathbf{x}, 0) = T_a(\mathbf{x}).$$

6.2 Nondimensionalisation

This problem is not too hard to nondimensionalise. In the first instance, let us scale r and z with the thickness H, and time with the heat conduction scale H^2/κ, where $\kappa = \rho c/k$ is the thermal diffusivity and ρ, c and k are the density, specific heat capacity and thermal conductivity respectively. Let us now think about the temperature scale. The conductivity must change noticeably as the temperature varies, or the device would be pointless, and we can identify a temperature change ΔT over which it does so. Let us, therefore, use that as the temperature scale, writing $T - T_a = \Delta T\, u(\mathbf{x}', t')$. Lastly we'll use the external voltage V_0 as the scale for ϕ and the 'cold' value of the conductivity, σ_0, for $\sigma(T)$.

Scaling and immediately dropping the primes, we have the dimensionless equations

$$\nabla \cdot (\sigma\nabla\phi) = 0, \qquad \frac{\partial u}{\partial t} - \nabla^2 u = \gamma\sigma(u)|\nabla\phi|^2$$

for $0 < z < 1$, $0 \leq r < \alpha = a/H$. The boundary conditions are

$$\phi = 0, 1 \quad \text{on} \quad z = 0, 1 \quad \text{respectively}, \qquad \frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = \alpha,$$

and

$$\frac{\partial u}{\partial n} + \beta(\mathbf{x})u = 0$$

on the boundary, where the $\mathbf{x}$–dependence of β models the difference between the top/bottom and the side, β taking different values in the two cases.

There are now just three dimensionless parameters:

$$\alpha = \frac{a}{H}, \quad \beta = \frac{hH}{k} \quad \text{and} \quad \gamma = \frac{\sigma_0 V_0^2}{k\Delta T}.$$

Clearly, α measures the aspect ratio, β the heat transfer, and γ the competition between heat generation and conduction. When we put in typical physical values, namely

$$\begin{gathered}\rho = 5.6 \times 10^3\,\text{kg m}^{-3}, \quad c = 540\,\text{J kg}^{-1}\,\text{K}^{-1}, \quad k = 2\,\text{W m}^{-1}\,\text{K}^{-1},\\ \sigma_0 = 2\,\Omega^{-1}\,\text{m}^{-1}, \quad \Delta T = 100\,\text{K},\\ V_0 = 250\,\text{V}, \quad r = 5 \times 10^{-3}\,\text{m}, \quad H = 10^{-3}\,\text{m},\\ h = 10 \text{ (sides) to } 10^2 \text{ (top) W m}^{-2}\,\text{K}^{-1},\end{gathered}$$

we find that

$$\alpha = 5, \quad \beta = 10^{-2} \text{ (sides) to } 10^{-1} \text{ (top)}, \quad \gamma = 625.$$

Already we have learned a lot. We know that there are just three dimensionless parameters, and that two are large and one is small. The large aspect ratio α suggests that a one-dimensional model should perform well, and this notion is reinforced by the fact that β is especially small at the sides of the device: most of the heat generated will be lost through the top and bottom. The fact that γ is very large suggests that we may have chosen the wrong timescale, at least for the initial heating-up stage. However, the device does work, so the decrease of the conductivity as the temperature increases must eventually switch the heating term off, large though it appears to be. If we rescale time by writing $t = \gamma^{-1}\tau$, we find that

$$\frac{\partial u}{\partial \tau} = \sigma(u)|\nabla\phi|^2 + \frac{1}{\gamma}\nabla^2 u,$$

and with luck we can neglect the last term to simplify the problem considerably. However, it is not likely that we can explain spatial variations in the temperature without the last term, so there must be more to it than this. The full story is outlined in Chapter **??**.

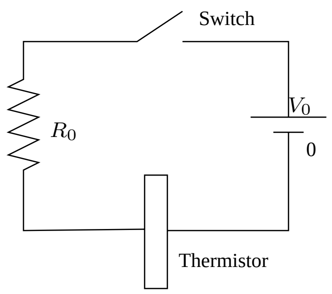

Figure 6.2: A thermistor in a circuit. The switch is closed at $t = 0$.

6.3 A thermistor in a circuit

In practice, our thermistor is likely to be part of a circuit, as shown in Figure 6.2, where the rest of the circuit is represented by a resistor of resistance R_0. This introduces some minor complications, as we no longer know the voltage drop across the thermistor, but instead we just have a relationship between this voltage and the current through the device. The model inside the thermistor is much as before, and there is no need to repeat the equations for T and ϕ. On the top and bottom of the thermistor, though, we have

$$\phi = 0, \quad z = 0, \qquad \phi = V(t), \quad z = H,$$

where $V(t)$ is not yet known. However, we can use Ohm's law for the resistor to say that the voltage drop across it is $I(t)R_0$, where $I(t)$ is the current in the circuit, and then we have

$$V_0 = I(t)R_0 + V(t)$$

by whichever of Kirchhoff's laws it is that says that the voltages round a closed circuit sum to zero.[1] We also have an expression for $I(t)$, as it is equal to the current flowing through the thermistor, namely

$$\iint_{z=H} \sigma(T)\nabla\phi \cdot \mathbf{n}\, dS,$$

which is just the current density integrated over the bottom face. Thus, in this case, the boundary condition for ϕ on $z = H$ is

Exercise: show from the equations that this is the same as the current density integrated over the top face.

$$\phi = V_0 - 2\pi R_0 \int_0^a \sigma(T) \left.\frac{\partial \phi}{\partial z}\right|_{z=H} r\, dr.$$

[1]See the exercise on page 79.

The effect of the external resistance in the dimensionless model is to bring in another parameter, from the boundary condition for ϕ. It is left to you to show that, with the same scales as above, the dimensionless model is

$$\nabla \cdot (\sigma \nabla \phi) = 0, \qquad \frac{\partial u}{\partial t} - \nabla^2 u = \gamma \sigma(u) |\nabla \phi|^2$$

for $0 < z < 1$, $0 \le r < \alpha = a/H$. The boundary conditions are

$$\phi = 0 \quad \text{on} \quad z = 0, \qquad \frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = \alpha,$$

and

$$\frac{\partial u}{\partial n} + \beta(\mathbf{x}) u = 0$$

Can you see why α^2 is separated out, and why we put the 2 in?

as before, with the new condition

$$\phi = 1 - \frac{2}{\delta \alpha^2} \int_0^\alpha \sigma(u) \frac{\partial \phi}{\partial z}\bigg|_{z=1} r\, dr,$$

where the 2 is for later convenience, and the new dimensionless parameter is

$$\delta = \frac{2}{\pi H R_0 \sigma_0 \alpha^2}.$$

A typical value for this parameter, given $R_0 = 400\,\Omega$, is 10^{-1}, which is quite small; note that formally setting $\delta = \infty$ we retrieve the problem with no external resistance.

6.3.1 The one-dimensional model

As we saw earlier, the large value of α and the small value of β on the sides of the thermistor suggest that a one-dimensional model should be a good approximation (we will have to wait until later in the book to see how to justify this). In such a model, ϕ and u are independent of r, and so we have the simpler problem

$$\frac{\partial}{\partial z}\left(\sigma \frac{\partial \phi}{\partial z}\right) = 0, \qquad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial z^2} = \gamma \sigma(u) \left|\frac{\partial \phi}{\partial z}\right|^2$$

for $0 < z < 1$, while the boundary conditions are

$$\phi = 0 \quad \text{on} \quad z = 0,$$

$$\phi = 1 - \frac{1}{\delta}\, \sigma(u) \frac{\partial \phi}{\partial z}\bigg|_{z=1},$$

and

$$\frac{\partial u}{\partial n} + \beta u = 0 \quad \text{on} \quad z = 0, 1.$$

Notice that we can integrate the equation for $\phi(z, t)$ once: can you see the physical interpretation of the result?

Some rather mathematical properties of this model are developed in the exercises on page 79. In the meantime, we move on to another case study with an electrical flavour.

Sources and further reading

The thermistor problem was brought to the Oxford Study Groups with industry by the British company STC, and has provoked a large mathematical literature for which [13] is a starting point.

Exercises

1. **Thermistor in a circuit; validity of Kirchhoff's law.** Strictly speaking, Kirchhoff's law of adding together voltages is not valid because the changing current generates a magnetic field which in turn generates a 'back emf', a voltage which opposes the current change. Show that the the back emf is small in our thermistor case study, as follows. (You may want to refer back to the exercises in Chapter 3.) If the circuit has typical length L, show that it is reasonable that a typical magnetic field strength is $B_0 = \mu_0 I_0/(2\pi L)$ and that the current has size $I_0 = V_0/R_0$. If $\mathbf{E}$ is the electric field strength, the typical back emf magnitude is L times the scale for $\mathbf{E}$. Show from Maxwell's equation

$$\nabla \wedge \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$$

   that the back emf scale works out as $\mu_0 I_0 L/(2\pi t_0)$, where t_0 is the timescale for changes in $I(t)$. If, say, $L = 10$ cm, $V_0 = 250$V and $R_0 = 500\ \Omega$, this is $10^{-8}/t_0$; verify that t_0 is a lot bigger than the timescale of $100/10^{-8}$ seconds necessary for the back emf to have magnitude 100 V. We can thus neglect $\partial\mathbf{B}/\partial t$, so that $\nabla \wedge \mathbf{E} \approx \mathbf{0}$; show that the change in voltage round the circuit is $\oint \mathbf{E} \cdot d\mathbf{s} \approx 0$.

2. **One-dimensional thermistors.** Consider the steady-state version of the one-dimensional thermistor problem, with the (not very realistic)

boundary conditions that $T = T_a$ on $z = 0, H$ and with no external resistance. Show that the dimensionless model is

$$\frac{\partial}{\partial z}\left(\sigma \frac{\partial \phi}{\partial z}\right) = 0, \qquad \frac{\partial^2 u}{\partial z^2} = -\gamma\sigma(u)\left|\frac{\partial \phi}{\partial z}\right|^2$$

for $0 < z < 1$, with

$$\phi = 0, 1, \quad u = 0 \quad \text{on} \quad z = 0, 1 \quad \text{respectively.}$$

Explain why $\phi = \frac{1}{2}$, $\partial u/\partial z = 0$ on $z = \frac{1}{2}$. Integrate the equation for ϕ once to show that

$$\sigma(u)\frac{\partial \phi}{\partial z} = I$$

where I is a constant (what is its physical interpretation?). Substitute for $\sigma\partial\phi/\partial z$ in the equation for u to show that

$$\frac{\partial u}{\partial z} = -\gamma I(\phi - \tfrac{1}{2}),$$

and then substitute for $(\partial\phi/\partial z)^2$ in the same equation to show that

$$\tfrac{1}{2}\gamma(\phi - \tfrac{1}{2})^2 = \int_0^{u_m} \frac{ds}{\sigma(s)},$$

where $u_m = u(\frac{1}{2})$ is the largest value of u, attained at $z = \frac{1}{2}$ (why?). Deduce that there can only be a steady solution if $\sigma(u)$ is such that

$$\int_0^{\infty} \frac{ds}{\sigma(s)} < \frac{\gamma}{8},$$

restate this inequality in dimensional terms, and interpret it physically. Give an example of a function $\sigma(u)$ for which it does not hold. What do you think happens to the solution of an initial-value problem if the inequality does not hold? Would we be more or less likely to have existence of a steady state with Newton cooling conditions for u?

3. **Thermistors and conformal mapping.** Consider the steady-state thermistor equations

$$\nabla \cdot (\sigma(u)\nabla\phi) = 0, \qquad \nabla^2 u = -\sigma(u)|\nabla\phi|^2$$

in two space dimensions but not necessarily in a rectangle. Suppose that the boundary of the thermistor has two conducting segments, on which $u = 0$ and $\phi = 0, 1$ respectively, separated by two insulating segments on which $\partial u/\partial n = 0$, $\partial\phi/\partial n = 0$.

Show that this system remains invariant under conformal maps $\xi + i\eta = f(x + iy)$ for analytic (holomorphic) f. Given that it is possible to map the thermistor region onto a certain rectangle with an obvious correspondence of boundary parts, show that the restriction on existence derived in the previous question holds irrespective of the geometry. (Note: the Riemann mapping theorem guarantees that the thermistor can be mapped onto any rectangle we choose, and that we can map *three* specified boundary points onto three specified points on the boundary of the rectangle. For example, we can map three of the 'changeover' points where the boundary conditions switch onto three corners of the rectangle. However, we can only map the fourth changeover point onto the fourth corner if the rectangle has a specific aspect ratio. For more on conformal mapping see [6, 9] or, for Matlab users, the Schwarz–Christoffel Toolbox of that package.)

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Chapter 7

Case study: electrostatic painting

7.1 Electrostatic painting

Many paints are based on organic solvents which, after application, evaporate and contribute to air pollution and global warming, and so they are coming under increasing regulatory pressure. A more environmentally friendly alternative for painting metal objects is to cover them with a layer of very small resin paint particles which, when the workpiece is put into an oven, melt and flow into into a smooth coating. (A similar process is used in 'flocking': here an object is coated in glue and then covered with short lengths of charged fibre. The charge makes the fibres stand on end, which is crucial to the final grass-like effect.) The particles are ejected from a gun which gives them an electric charge, and a potential difference is maintained between the gun and the workpiece, so the particles feel an electrostatic force which moves them towards the workpiece. On the other hand, they are also blown about by air currents, both imposed and generated by their own drag on the air.

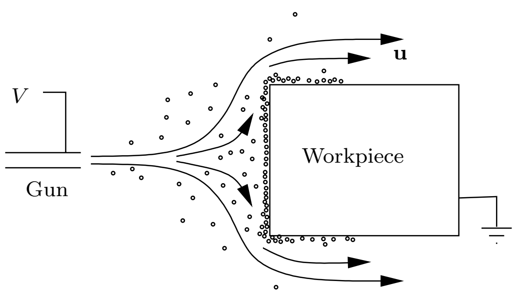

Figure 7.1: Electrostatic painting of an earthed metal workpiece.

We would like to know something about the controlling parameters of this process. In particular, we would like to get most of the particles to hit the workpiece and not to be carried away by the air flow (which must of course go round it). Do the particles influence the air flow, or are they passive? How thick is the final layer of resin? An attractive feature of this method of painting is that the electric field is strong, and so particles are especially attracted, at outward-pointing corners, which are hard to cover well with traditional methods. A less attractive feature is that it is very hard to see what is happening in the cloud of paint particles. A mathematical model may help to answer some of these questions.

7.2 Field equations

In this section, we begin to build a model of the painting process, by writing down 'field equations' to describe the fluid velocity. In this problem, it helps to start with some data, as that points to a reasonable model for the fluid/particle interaction. The workpiece has a typical size $L \approx 1$ m, and the observed air velocities have size $U_g \approx 1$ m s^{-1}. The air density and dynamic viscosity are $\rho_g \approx 1.3$, $\mu_g \approx 1.8 \times 10^{-5}$ in SI units (the kinematic viscosity ν_g is thus about 1.4×10^{-5}). The particles are tiny: they have radius $a \approx 10^{-5}$m and their mass is $m_p \approx 10^{-12}$ kg. There is an enormous number of them, a representative number density being $n_0 \approx 10^9$ per cubic metre of air. Lastly, turning to the electrical aspects, each particle carries a charge q_p of about 10^{-15}C, and the applied voltage is $V_0 \approx 10^5$V.

Now because the number density of particles is so large, the average particle separation, 10^{-3} m, is very small compared with the workpiece dimension L but very large compared with their mean radius a. It is reasonable to consider them as isolated particles when we work out the force on them, but when we work on larger length scales, we hope to get away with averaging their effects. We therefore consider the evolution of their local number density, which we think of as a continuous function $n(\mathbf{x}, t)$ representing the number of particles per unit volume measured over a small volume whose diameter is much bigger than the mean separation but much smaller than L. We proceed similarly in assuming that there is a local average particle velocity $\mathbf{v}_p(\mathbf{x}, t)$, and when we calculate the force exerted by the particles on the air: this is plausible if we can convince ourselves that neighbouring particles all feel very similar influences from the fluid, and all all have a similar effect on it.

We now start to write down some equations. The first is conservation of particles:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}_p) = 0. \tag{7.1}$$

Next, we think about the particle equation of motion. Unlike our flagpoles or cables, the particles feel a fairly slow flow past them. The 'local' Reynolds number for flow at 1 m s^{-1} past a ten-micron radius spherical particle is

$$\begin{aligned}\mathrm{Re}_{\text{particle}} &= \frac{U_g a}{\nu_g} \\ &\approx 0.7,\end{aligned}$$

and this is a considerable over-estimate since we should really use the relative (slip) velocity, which is likely to be smaller than 1 m s^{-1}. Now the force on a spherical particle in slow flow can be shown to be

See any good book on viscous flow.

$$-6\pi a\mu_g\left(\mathbf{v}_p - \mathbf{v}_g\right)$$

where $\mathbf{v}_g$ is the 'local' gas velocity, many particle radii away (but not so far as to be near neighbouring particles). Our particles are not spherical, but we'll still assume that they feel a force proportional to the slip velocity; we'll call it

$$-K\left(\mathbf{v}_p - \mathbf{v}_g\right)$$

where, on the basis of near-spherical particles, we expect that $K \approx 10^{-9}$ kg s^{-1}. Lastly we need to include the electrostatic force $q_p\mathbf{E}$, where $\mathbf{E}$ is the electric field, and the gravitational force $m_p\mathbf{g}$. Then, assuming that all neighbouring particles feel the same slip velocity, and have the same particle velocity, we can write down an equation of motion for the particles:

Note that D/Dt here is the convective (total) derivative.

$$m_p\frac{D\mathbf{v}_p}{Dt} = -K\left(\mathbf{v}_p - \mathbf{v}_g\right) + q_p\mathbf{E} + m_p\mathbf{g}. \tag{7.2}$$

Correspondingly, we have momentum and mass[1] conservation equations for the gas (let's keep things simple by writing down an inviscid model, leaving gravity out as it merely generates hydrostatic pressure):

$$\rho_g\frac{d\mathbf{v}_g}{dt} = -\nabla p + nK\left(\mathbf{v}_p - \mathbf{v}_g\right), \quad \nabla \cdot \mathbf{v}_g = 0. \tag{7.3}$$

The only potentially unfamiliar term here is the body force (force per unit volume) on the gas due to the particles. Whereas the particle equation of

[1]Another simplification I've slipped in is that, because the volume fraction of particles is so small, I've taken the gas volume fraction to equal 1. Technically, we should write down a two-phase model and a later exercise, on page 203, justifies our approximation.

motion is for individual particles, and thence for the averaged-out particles because of our assumption that all nearby particles behave similarly, the force by the particles on the gas is just the force on one representative particle, multiplied by the number density.

It only remains to write down Poisson's equation

$$\nabla \cdot (\varepsilon_g \mathbf{E}) = nq_p \tag{7.4}$$

for the averaged electric field (here $\varepsilon_g \approx 10^{-11}$ in SI units is the permittivity of air; it is very close to ε_0), and we have collected all the field equations of the model.[2]

7.3 Boundary conditions

For the sake of completeness, we should briefly discuss the boundary conditions for our model, although we aren't going to use them much. The main issue we should address is the question of how to deal with the thin layer of particles on the workpiece. There is little doubt that it is very thin — we don't want a centimetre-thick coating of paint! — and as far as the fluid is concerned we can assume that the workpiece forms a rigid boundary to complement whatever inflow conditions we impose at the gun. The particles satisfy the first-order equation (7.1), whose characteristics are the particle paths (see Chapter 8). Obviously we should impose an initial condition at the gun, and that is all we need.

Lastly, consider the electric potential. we can impose $\phi = V_0$ on the gun with a clear conscience, and likewise $\phi = 0$ at the workpiece, but we may worry that charge building up in the paint layer on the workpiece will alter the 'effective boundary condition' felt by ϕ. This depends to a large extent on the details of what happens in this layer. For example, if the charge can 'leak off' the particles to the workpiece (or electrons can move onto the particles if their charge is positive), the layer should be relatively passive and we can ignore it. On the other hand, if the charges remain in situ, we can still ignore the effect of this layer as long as the total charge it contains is small enough (see the exercise on page 89). Later in the book, we shall see how we can make this kind of ad hoc approximation more systematic.

[2] Caveat: we haven't dealt properly with the thin layer of paint on the workpiece, as shown in Figure 7.1. Clearly the particles cannot move freely in this layer and we need to treat it separately; see the next section and the exercises.

7.4 Nondimensionalisation

We have made great progress in producing a useful model. However, the result is undeniably complicated. Do we really need all the terms in these equations? Obviously we're going to have to solve it numerically, and for this reason if no other we should do what we can to check that the model is robust and suitable for a numerical attack.

Let's scale all the variables with typical values as before. Scale $\mathbf{x}$ with L, t with L/U_g, $\mathbf{v}_g$ and $\mathbf{v}_p$ with U_g, p with $\rho_g U_g^2$, and n with n_0. We have two choices for the scale for $\mathbf{E}$: one is the applied voltage V_0, while the other, $q_p n_0 L/\epsilon_g$, is derived from the Poisson equation (7.4) once all other scalings in it are fixed. It so happens that both are about the same size, 10^5 V m^{-1}, so let's save ink and use V_0.

Check this scaling.

Start with the particle equation of motion (7.2). Scaling and immediately dropping the primes, we get

It's obvious that the conservation of particle mass equation (7.1) isn't changed.

$$\frac{m_p U_g^2}{L}\frac{d\mathbf{v}_p}{dt} = -KU_g\left(\mathbf{v}_p - \mathbf{v}_g\right) + \frac{q_p V_0}{L}\mathbf{E} - m_p g\mathbf{k},$$

and, dividing by KU_g,

$$\frac{m_p U_g}{KL}\frac{d\mathbf{v}_p}{dt} = -\left(\mathbf{v}_p - \mathbf{v}_g\right) + \frac{q_p V_0}{KU_g L}\mathbf{E} - \frac{m_p g}{KU_g}\mathbf{k}. \qquad (7.5)$$

We see that the dimensionless quantity

$$\frac{m_p U_g}{KL} \approx 10^{-3}$$

is very small. With luck, we can neglect the term it multiplies, the particle acceleration: apart from an initial transient as they get up to speed near the gun, their inertia is dominated by viscous drag. The final dimensionless parameter,

$$\frac{m_p g}{KU_g} \approx 2 \times 10^{-2},$$

is also small. It decides the competition between gravity and viscous forces in favour of the latter: these particles fall slowly compared to the rate at which they are dragged about by the air. The second dimensionless parameter,

$$\frac{q_p V_0}{KU_g L} \approx 10^{-1}, \qquad (7.6)$$

compares the electrostatic forces with the drag forces. It too is small, though larger than the other two; we shall see some consequences of this in Chapter **??**.

This simple analysis has told us quite a lot. The scaled equation says that the particles follow the gas quite closely, with a small influence from the electrostatic force, and very minor contributions from gravity and inertia.[3] We can get a good approximation to the particle motion if we write

$$\mathbf{v}_p = \mathbf{v}_g + \frac{q_p V_0 L}{K U_g} \tilde{\mathbf{v}}_p, \tag{7.7}$$

Remember that this is a dimensionless equation: we are not directly equating a velocity to an electric field.

where, from (7.5),

$$\tilde{\mathbf{v}}_p = \mathbf{E} + \text{smaller terms.}$$

A preliminary conclusion from this analysis is that the device is not working terribly well: the particles are being swept along too much by the air.

Now let's look at the gas momentum equation. After scaling, this becomes

$$\frac{d\mathbf{v}_g}{dt} = -\nabla p + \frac{n_0 K L}{\rho_g U_g} n \left(\mathbf{v}_p - \mathbf{v}_g\right).$$

On the face of it, the dimensionless quantity

$$\frac{n_0 K L}{\rho_g U_g} \approx 1$$

is not small, indicating that the particles exert a body force on the air which is not small. However, remember that we decided above that the slip velocity $\mathbf{v}_p - \mathbf{v}_g$ (which this dimensionless parameter multiplies) *is* small. So, the particles do after all have a small effect on the gas, confirming that it would be a good idea to try to slow the air down to improve performance. We return to this problem in Chapter **??**.

Sources and further reading

Electrostatic painting was also a Study Group problem, from Courtaulds plc, and is documented further in [3].

[3] If the parameter in (7.6) had been *large*, not small, that would have told us that we had chosen a wrong scaling somewhere. There is no reason for $\mathbf{E}$ to be large — it has a perfectly good equation of its own in which there are no large parameters — and so we would have an equation with one large term in it and nothing to balance it. When the parameter (7.6) is small, we do have the a priori plausible approximation (7.7).

Exercises

1. **Paint layer.** Suppose that a thin layer of paint particles, deposited electrostatically as in the text, is growing on $y = 0$, and that its thickness is $y = h(x,t)$. Show that the normal velocity of the layer boundary is $(1 + h_x^2)^{\frac{1}{2}}\, h_t$, and relate this to $\mathbf{v}_p \cdot \mathbf{n}$ at the interface. How thick will the layer grow in 10 seconds? Justify the approximate boundary condition

   Subscripts indicate partial derivatives.

$$\frac{\partial h}{\partial t} = \mathbf{v}_p \cdot \mathbf{n}$$

   on the workpiece (we discuss this 'linearisation' of boundary conditions in Chapter **??**).

   Derive an order of magnitude estimate for the potential drop across the layer, assuming that the potential approximately satisfies

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\rho}{\varepsilon_0},$$

   where ρ is the density of charge in the layer. Assuming a reasonably close packing for the particles, express this order of magnitude in terms of the average particle radius a and charge q_p, and hence assess how thick the layer needs to be before the potential drop across it rivals the applied voltage.

Part II

Mathematical techniques

Chapter 8

Partial differential equations

This chapter is a short overview of partial diferential equations partial differential equations, with applications in view. The emphasis is largely on first-order quasilinear equations, for which many standard treatments don't provide much in the way of real-life examples; we'll see applications to e-mail and, in a case study, to traffic. We also take a brief look at the fully nonlinear case, with an eye to using it in Chapter **??** to see why we say that light travels in straight lines (and other problems). Last, we have a brief run through the standard theory of second order linear equations in two variables, for which the canonical examples of the wave equation, the heat equation and Laplace's equation, and their physical interpretations, are so well known that we don't need to give examples here.

8.1 First-order quasilinear partial differential equations: theory

We begin with a review of the elementary theory for the equation

$$a(x_1, x_2, u)\frac{\partial u}{\partial x_1} + b(x_1, x_2, u)\frac{\partial u}{\partial x_2} = c(x_1, x_2, u),$$

where a, b and c are given smooth functions, with initial values given in parametric form on a curve Γ, that is $u = u_0(s)$ on $x_1 = x_{10}(s)$, $x_2 = x_{20}(s)$. This should be familiar stuff: I am assuming that you have already studied this material. As illustrated in Figure 8.1, the partial differential equation written in the form

$$(a, b, c) \cdot \left(-\frac{\partial u}{\partial x_1}, -\frac{\partial u}{\partial x_2}, 1\right) = 0$$

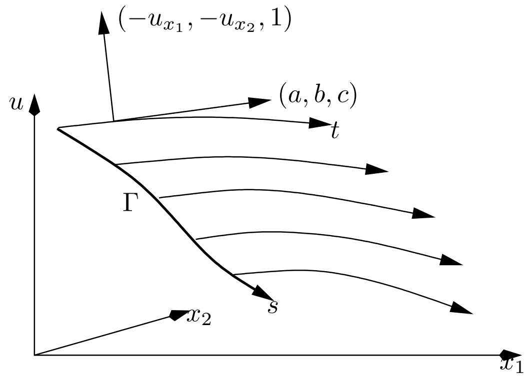

Figure 8.1: Solution of a first-order quasilinear equation by characteristics. Subscripts indicate partial derivatives.

shows that (a, b, c) is orthogonal to $(-\partial u/\partial x_1, -\partial u/\partial x_2, 1)$, which itself is normal to the solution surface $u - u(x_1, x_2) = 0$. It follows that if we solve the *characteristic equations*

Remember the c has to be on the right-hand side of the equation, or you will get an extraneous minus sign.

$$\frac{dx_1}{dt} = a(x_1, x_2, u),$$
$$\frac{dx_2}{dt} = b(x_1, x_2, u),$$
$$\frac{du}{dt} = c(x_1, x_2, u),$$

where t is a parameter along the characteristics, with the initial values

$$x_1(0) = x_{10}(s), \quad x_2(0) = x_{20}(s), \quad u(0) = u_0(s),$$

the curves so generated, known as *characteristics*, are tangent to the solution surface at each point. Glueing the characteristics together gives the solution surface, written in the parametric form

$$(x_1, x_2, u) = (x_1(s, t), x_2(s, t), u(s, t)),$$

and in principle we are done, at least near Γ. We see immediately the central role of characteristics (or their projections down onto the (x_1, x_2) plane, known as *characteristic projections*). They are curves along which informstion propagates: indeed, the left-hand side of the partial differential equation is just the directional derivative of u along the characteristic projections, so in this direction the partial differential equation reduces to an ordinary one.

There are several caveats to make about this procedure. One is that, as is clear from the geometrical point of view, the initial curve must not be tangent to a characteristic; for if it is, near the point of tangency we expect more than one value of u at each point (x_1, x_2); this is easily seen by showing that if there is one characteristic projection through such a point, in general there is another, carrying a different value of u. This insight is confirmed by a calculation in which we try to find all the partial derivatives of u at a point on Γ with the aim of constructing its Taylor sesries at that point. We know by differentiating $u_0(s)$ along Γ that

$$\frac{dx_{10}}{ds}\frac{\partial u}{\partial x_1} + \frac{dx_{20}}{ds}\frac{\partial u}{\partial x_2} = \frac{du_0}{ds},$$

and that

$$a(x_{10}, x_{20}, u_0)\frac{\partial u}{\partial x_1} + b(x_{10}, x_{20}, u_0)\frac{\partial u}{\partial x_2} = c(x_{10}, x_{20}, u_0);$$

regarding these as equations for $\partial u/\partial x_1$ and $\partial u/\partial x_2$, there is a unique solution provided that the determinant of coefficients

$$\begin{vmatrix} a & b \\ \dfrac{dx_{10}}{ds} & \dfrac{dx_{20}}{ds} \end{vmatrix} \neq 0,$$

and if it does vanish, so that there is no unique Taylor series, we get precisely the first two characteristic equations. Thus, if we require a unique solution u, Γ cannot be a characteristic.

Exercise: if it does vanish, show that the consistency condition for there to be a solution, albeit nonunique, is the third characteristic equation.

The second caveat concerns the region of existence of the solution. It is an obvious remark that if Γ has ends, we can only hope to find the solution in a region between the characteristics through those endpoints. Furthermore, by standard Picard theory, the characteristic equations have a unique solution for at least small t, that is in a small strip near Γ. How far we can go beyond this strip depends on two things. First, the local solution must not blow up, which it might well do for nonlinear equations. Secondly, and of more practical importance, we must (in principle) be able to find s and t uniquely from the solution of the first two characteristic equations in order to calculate u uniquely. That means that the Jacobian

$$\left|\frac{\partial(x_1, x_2)}{\partial(s, t)}\right|$$

must not vanish or become infinite. That in turn means that the characteristic projections are not allowed to cross, for if they did the different values of u propagated along the different characteristic projections would lead to many values of u at a single point. So even though we may have a perfectly good parametric solution of the form above, it does not correspond to a single-valued solution $u(x_1, x_2)$. In general, one can think of this coinciding with some kind of 'folding over' of the parametrised surface in x_1–x_2–u space. We return to the question of what to do if this happens in Section 8.3.

The third caveat is the degree of smoothness of the solution. At first sight, we expect the solution to have single-valued first partial derivatives, so that the partial differential equation makes sense. However, it is possible to extend our idea of solution by joining together smooth solution surfaces at curves at which the first partial derivatives have jumps; such a curve would look like a sheet slung over a rope and pulled out on either side, as in Figure 8.2. Such a discontinuity can only occur across a characteristic. We can see this by noting that when we showed that we can find the first partial derivatives of u uniquely unless they are given on a characteristic, we were in effect showing that jumps in these derivatives can only occur across a characteristic. That is, smooth solution surfaces intersect in characteristics. Alternatively, if $(x_1(t), x_2(t), u(t))$ is a curve in the solution surface along which u is continuous but there are jumps in it derivatives, from left to right, then we can differentiate the statement

$$[u] = 0$$

along the curve, where $[\cdot]$ means the jump across the curve, to get

$$\left[\frac{du}{dt}\right] = \left[\frac{\partial u}{\partial x_1}\right]\frac{dx_1}{dt} + \left[\frac{\partial u}{\partial x_2}\right]\frac{dx_2}{dt} = 0.$$

We also have the partial differential equation on either side, and taking the difference of this across the curve gives

Remember $[u] = 0$ so the coefficients a, b, c are continuous.

$$a\left[\frac{\partial u}{\partial x_1}\right] + b\left[\frac{\partial u}{\partial x_2}\right] = 0.$$

These two homogeneous equations for $[\partial u/\partial x_1]$ and $[\partial u/\partial x_2]$ only have a nonzero solution if the determinant of coefficients

$$\begin{vmatrix} \dfrac{dx_1}{dt} & \dfrac{dx_2}{dt} \\ a & b \end{vmatrix}$$

vanishes, and this is precisely the condition for $(x_1(t), x_2(t))$ to be a characteristic projection.

We see that characteristics are curves along which gradient discontinuities propagate from the initial curve. However, this does not let us deal with the blow-up described above, which can happen even with perfectly smooth initial data. In Section **??**, we shall see how to do that by introducing solutions that are themselves discontinuous, so-called *shocks*.

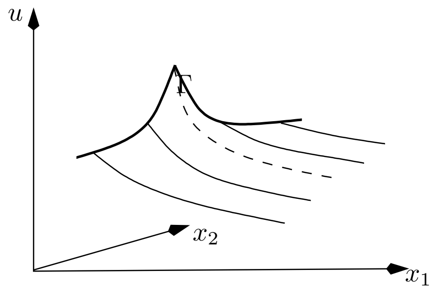

Figure 8.2: Solution of a first-order quasilinear equation with a gradient discontinuity.

8.2 Example: Poisson processes

First-order equations are often found in connection with various generating functions in probability. Many of these have their roots in the *Poisson process*, which is often used as a model for the number of occurrences in a given time of independent random events such as e-mails arriving in your inbox or calls coming to a telephone exchange. (I hesitate to propose the model for the arrival of London buses as there are good reasons for them to arrive in pairs or even threes.)

Suppose that we say that in a short time δt, there is a probability $\lambda\,\delta t$ that an e-mail arrives, a probability $1-\lambda\,\delta t$ that none arrive, and a negligible probability that two or more arrive at once. The constant λ, known as the *intensity*, measures your popularity (spam or otherwise). Starting with an empty inbox, and staying online, define the Poisson counter $N(t)$ as the number of e-mails you have at time t. Thus,

$$N(t+\delta t) = \begin{cases} N(t) & \text{with probability } 1-\lambda\,\delta t, \\ N(t)+1 & \text{with probability } \lambda\,\delta t. \end{cases}$$

What is the probability distribution of $N(t)$? Define

$$p_n(t) = P\left(N(t)=n\right).$$

There are two, and only two, ways that $N(t+\delta t)$ can equal n: either $N(t)=n$ and no new message arrives, or $N(t)=n-1$ and one message arrives. These events are disjoint and so we have

$$\begin{aligned} P\left(N(t+\delta t)=n\right) &= p_n(t+\delta t) \\ &= P\left(N(t)=n\right) P(\text{no new message}) \\ &\qquad + P\left(N(t)=n-1\right) P(1 \text{ new message}) \\ &= p_n(t)(1-\lambda\,\delta t) + p_{n-1}(t)\lambda\,\delta t. \end{aligned}$$

Expanding $p_n(t+\delta t)$ in a Taylor series,

$$p_n(t+\delta t) = p_n(t) + \delta t \frac{dp_n(t)}{dt} + \cdots$$

and taking $\delta t \to 0$, we get the system of differential equations

$$\frac{dp_n}{dt} = -\lambda \left(p_n - p_{n-1}\right), \quad n = 1, 2, 3, \ldots,$$

while separately

$$\frac{dp_0}{dt} = -\lambda p_0.$$

The initial conditions are $p_0(0) = 1$, $p_n(0) = 0$ for $n > 0$, and although one can, in principle, solve the equations sequentially, it's easier to define

$$G_N(x,t) = \sum_{n=0}^{\infty} x^n p_n(t).$$

Then, summing the differential equations,

$$\frac{\partial G_N}{\partial t} = -\lambda(1-x)G_N,$$

whence

$$G_N(x,t) = e^{-\lambda t} e^{\lambda t x} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} x^n.$$

so the probabilities are those of the Poisson distribution with mean λt.

This generating function only satisfies an ordinary differential equation, but now suppose that a virus is doing the rounds, spread by e-mail. Let $V(t)$ be the number of computers infected, and suppose that the probability of a new infection over the next δt is $\lambda V(t)\,\delta t$, that is, proportional to the number of infected computers. With $p_n(t) = P(V(t) = n)$ as before (but now for $n = 1, 2, \ldots$ and with $p_1(0) = 1$ to model one source of infection), we find

$$\frac{dp_n}{dt} = -\lambda \left(np_n - (n-1)p_{n-1}\right), \quad n = 1, 2, 3, \ldots$$

(here $p_0 = 0$), and then, following the calculation above, the generating function $G_V(x,t)$ satisfies

$$\frac{\partial G_V}{\partial t} + \lambda x(1-x)\frac{\partial G_V}{\partial x} = 0$$

with $G_V(x,0) = x$. The solution is easily found to be

Note the consistency check $G(1,t) = 1$ for both these examples: the probabilities sum to 1.

$$
\begin{aligned}
G_V(x,t) &= \frac{xe^{-\lambda t}}{1 - x\left(1 - e^{-\lambda t}\right)} \\
&= \sum_{n=1}^{\infty} x^n e^{-\lambda t} \left(1 - e^{-\lambda t}\right)^{n-1}.
\end{aligned}
$$

The mean of this distribution,

$$
\frac{\partial G_V}{\partial x}(1,t) = \sum_{n=1}^{\infty} n p_n(t),
$$

grows exponentially in t, as we would expect.

8.3 Shocks

We started our analysis of quasilinear equations such as the traffic model above by considering smooth solution surfaces with a unique normal at each point. Then, we realised that we can extend our class of solutions by allowing gradient discontinuities, as long as they occur across (propagate along) characteristics. However, blow-up still occurs when characteristic projections cross, because then we may get several values of u at the same place.

To see this in action consider the equation

$$
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.
$$

This equation is known as a *kinematic wave equation* and if we think of $u(x,t)$ as the speed of a particle moving along the x axis, it says that the speed of any given particle remains constant. It is easy to solve by characteristics with initial data $u(x,0) = u_0(x)$, say, corresponding to a snapshot at $t = 0$ of the speeds all along the x axis. The characteristic equations are

$$
\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = u, \quad \frac{du}{d\tau} = 0,
$$

so u remains constant along a characteristic whose projection has slope $dx/dt = u$. This simply repeats that the particles move along characteristics with constant speed u. So, to construct the solution, we simply draw all the characteristic projections through the innitial line $t = 0$, and read off the value of u at any point x and later time t. This procedure works fine if $u_0(x)$ is increasing, since then the characteristics spread out as in Figure 8.3.

But if $u_0(x)$ is dcreasing, we inevitably have a collision of characteristic projections — and particles — after a finite time, as in Figure 8.4. It has an

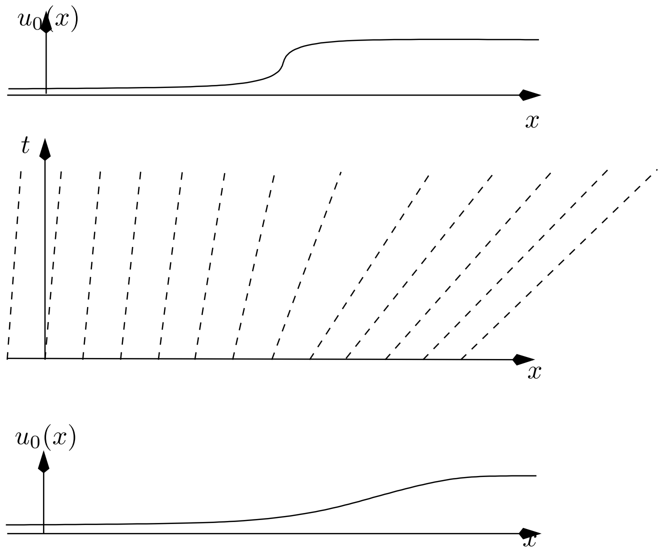

Figure 8.3: The kinematic wave equation with increasing initial data. The steep fall in $u_0(x)$ moves to the right and spreads out.

obvious physical interpretation that fast particles have caught up with slow ones and are trying to occupy the same space.

We could take the defeatist view that the solution ceases to exist at the moment when the characteristic projections first cross, and that is the end of the matter. On the other hand, we may try to extend our notion of what constitutes a solution, to allow not only gradient discontinuities but also *discontinuities in u itself.* That is, there may be a curve (or curves) $x = S(t)$ across which u has a jump. These jumps are called *shocks.* Famous physical examples include tidal bores[1] and the shock waves created by Concorde flying supersonically.

By 'physical example' we mean a physical situation that can be modelled by equations, here the shallow water equations and the equations of gas dynamics, that have shock solutions.

As one might expect, a drastic step like this is fraught with dangers. It does indeed turn out that we have generalised a bit too far, because we can have non-uniqueness of solutions with shocks. The ideas needed to deal with this are rather delicate and we refer to [?] for a fuller discussion. However, when the partial differential equation is a conservation law, we can give a heuristic derivation of a necessary condition at a shock, which in practice is sometimes sufficient as well.

[1]A tidal bore is an abrupt change in water level that propagates *upstream* from the river mouth, often appearing as a continually breaking wave. Prerequisites for them to form are a large tidal range and a slowly convergent estuary. Examples include the Severn bore and the Trent Aegir in the UK, and the Hooghly bore in India.

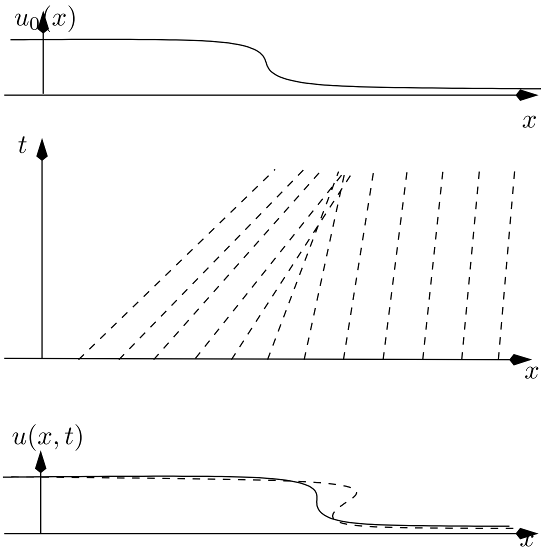

Figure 8.4: The kinematic wave equation with decreasing initial data. The steep fall in $u_0(x)$ gets steeper until, when the characteristic projections cross, it is vertical (solid line in the bottom graph). The dashed line in the bottom graph shows the multivalued solution we get if we continue to larger values of t.

8.3.1 The Rankine–Hugoniot conditions

Suppose that our quasilinear partial differential equation is in the conservation form (see Chapter 2)

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0,$$

where $P(x,t,u)$ is the density and $Q(x,t,u)$ the flux of a conserved quantity (usually they depend only on u). Now suppose that there is a curve $x = S(t)$ across which $u(x,t)$ has a discontinuity (jump). That means that P and Q also have jumps across this curve. However, we assume that overall conservation of the quantity, whatever it is (stuff, say). Let us do a simple-minded 'box' argument to see what this implies for the jumps in P and Q. Figure 8.5 sets the scene.

Compare this with the (in some ways more subtle) argument of Section 2.4.

In a small time δt, the shock moves by a small distance

$$\delta x = \frac{dS}{dt}\delta t.$$

The net flux into the region crossed by the shock is $Q_+ - Q_-$ in an obvious notation, + referring to $x > S(t)$. Hence the amount of stuff flowing into this small 'box' is $(Q_+ - Q_-)\delta t = [Q]_-^+ \delta t$. The amount of stuff in the box

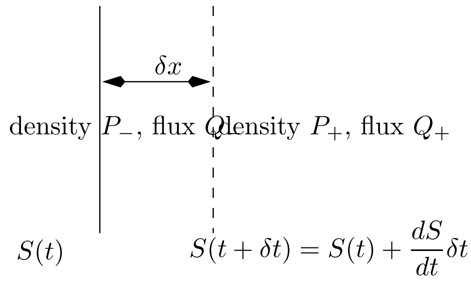

Figure 8.5: Derivation of the Rankine–Hugoniot condition.

before the shock arrives is $P_- \, \delta x$, and after it is $P_+ \, \delta x$. The difference must be accounted for by the net flux; thus,

$$\big[P\big]_-^+ \, \delta x = \big[Q\big]_-^+ \, \delta t.$$

Eliminating $\delta x / \delta t \approx dS/dt$, we arrive at

$$\frac{dS}{dt} = \frac{\big[Q\big]_-^+}{\big[P\big]_-^+}.$$

This relation between the two jumps is known as the *Rankine–Hugoniot condition* and it is necessary (but not sufficient) to give a unique solution with a shock.

8.4 Fully nonlinear equations and Charpit's method

8.4.1 Example: spray forming

Further reading

See [23] for an accessible introduction to Poisson processes.

Exercises

1. **Solution blow-up.** Consider the equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

Show that the general solution is given implicitly by $u(x,t) = f(x - tu(x,t))$ for arbitrary smooth f. Consider the initial value problem in which $u = u_0(x)$ on $t = 0$. Show that the parametric form of the solution is

$$t = \tau, \quad x = \tau u_0(s) + s, \quad u = f(s).$$

Deduce that u is constant on the characteristic projections $dx/dt = u$, which are thus straight lines.

Draw the characteristic projections and sketch the solution surface (if you can visualise it: see [27] Chapter 1) for the two kinds of 'ramp' initial data

$$u_{0\pm}(x) = \begin{cases} 0, & x < 0, \\ \pm x, & 0 \le x < 1, \\ \pm 1, & 1 \le x. \end{cases}$$

Which solution remains single-valued? (Note that the initial discontinuity in slope at $x = 0$ remains fixed on the characteristic projection $x = 0$, but the slope discontinuity at $x = 1$ is propagated along the characteristic projection $x = 1 \pm t$.)

2. **Waiting times for a Poisson process.** Suppose we start a Poisson process at time 0. What is the distribution of T, the time until the first event (clearly this is the same as the distribution of the interval between any two events)? We find it as follows. Let $F_T(t) = P(T < t)$. Explain why

$$F_T(t + \delta t) = (1 - \lambda \delta t) F_T(t)$$

and deduce that T has the negative exponential distribution with density $f_T(t) = \lambda e^{-\lambda t}$, $t > 0$. Figure 8.6 shows a histogram of the inter-arrival times of trades in the S&P 500 futures contract in Chicago (an open outcry market) with normal and log scales for the frequency; the latter is a good approximation to a straight line except for very short times between trades (thanks to Rashid Zuberi for these plots).

3. **Viral antidote.** Suppose that N computers are infected with a virus and, at time $t = 0$, I send them all an antidote which will cure the problem as soon as they open their inbox. Also assume that if n users are still infected at time t, then in the short time interval $(t, t + \delta t)$ one and only one user will log on, with probability $\mu n p_n(t) \delta t$. Why is this a reasonable model?

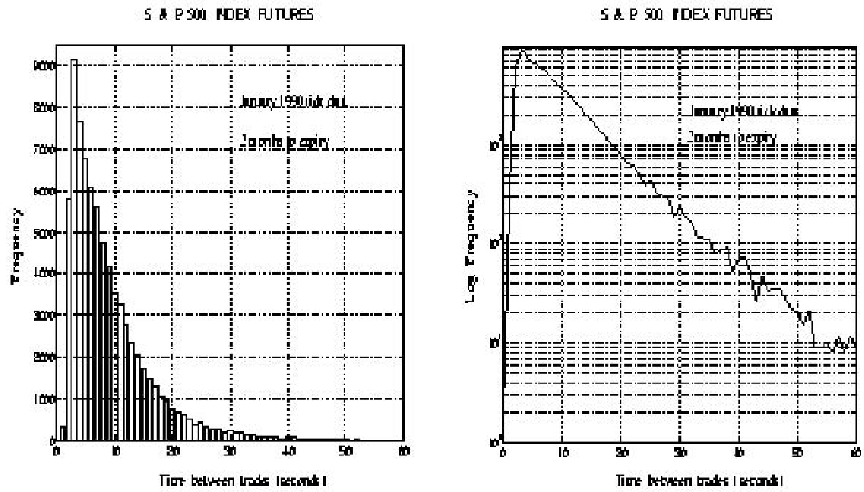

Figure 8.6: Standarda and log-linear plots of the inter-arrival time frequencies for a stock market.

If $p_n(t)$ is the probability that there are still n infected computers at time t, use the decomposition into disjoint events

$$\begin{aligned} P(n \text{ infected at } t+\delta t) = P(n+1 \text{ at } t \text{ and one logs on}) \\ + P(n \text{ at } t \text{ and none log on}) \end{aligned}$$

to show that

$$p_n(t+\delta t) = \mu(n+1)p_{n+1}(t)\delta t + (1-\mu n\,\delta t)p_n(t).$$

Letting $\delta t \to 0$, show that the generating function $G_A(x,t) = \sum_{n=0}^{\infty} p_n(t)x^n$ satisfies

$$\frac{\partial G_A}{\partial t} + \mu(x-1)\frac{\partial G_A}{\partial x} = 0.$$

Show that if there are N victims initially, the solution is

$$G_A(x,t) = \left(1+(x-1)e^{-\mu t}\right)^N.$$

What is the mean of this distribution?

Modify the argument to allow for both infection and cure.

Chapter 9

Case study: traffic modelling

9.1 Case study: traffic modelling

Mathematicians have long been interested in the problem of traffic, and the area is one of active research. A variety of models have been suggested with a view to understanding, for example, how and why traffic jams form, how to maximise carrying capacity of roads, or how best to use signals, speed limits and other controls to reduce journey times (the feedback effect whereby quicker journeys encourage more people to take to the roads is strangely absent from these analyses). Some models are based on discrete simulations of the movement of individual cars; as you may imagine, such models can be very large and complicated, and indeed they fall into the trendy area of Complex Systems. There is, however, a strand of traffic research which treats the cars as a continuum with a local number density and velocity which are more or less smooth functions of space and time, much as in the treatment of charged particles in the case study of Chapter 7. Models of this kind are unlikely ever to forecast the fine details of gridlock in New York City or even Oxford; but on the other hand they offer insights into the way in which traffic can behave, and they can to some extent be calibrated to (or at least compared with) observations. On the scale from parsimony (as few parameters and mechanisms as possible) to complexity, they are very much at the parsimonious end; the cost, a lack of realism, is balanced by a gain in understanding. They fit in well with my recommended philosophy of always trying to do the easiest problem first.

Let us, then, start with a toy model for cars travelling in one direction down a single-lane road (no overtaking) that is long and straight. Suppose that x measures distance along the road, and that we work on a large enough lengthscale, or we look from far enough away, that the cars can be treated as a continuum with number density $\rho(x, t)$ (cars per kilometer) and speed $u(x, t)$. Supposing further that no cars join or leave the road, we immediately write down 'conservation of cars' in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0,$$

as the flux of cars is clearly equal to ρu.

Given the continuum assumption, this equation is uncontroversial; but it is only one equation for two unknowns. We need some kind of 'constitutive relation' to close the system.

Blinkered drivers

One very simple model would be to say that, as they enter the road, drivers choose the constant speed they want to drive at, and then they drive at that speed no matter what happens. Of course, this is ludicrously unraelistic, but let's see what features it predicts. If the speed u of an individual car is constant, then the derivative of u following that car is zero:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

This equation is known as a *kinematic wave equation* and it is easy to solve by characteristics with initial data $u(x, 0) = u_0(x)$, say, corresponding to a snapshot at $t = 0$ of the speeds all along the road. The characteristic equations are

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = u, \quad \frac{du}{d\tau} = 0,$$

so u remains constant along a characteristic whose projection has slope $dx/dt = u$. This simply says that the cars move along characteristics with constant speed u. So, to construct the solution, we simply draw all the characteristic projections through the innitial line $t = 0$, and read off the value of u at any point x and later time t. This procedure works fine if $u_0(x)$ is increasing, since then the characteristics spread out as in Figure 9.1(a). But if $u_0(x)$ is dcreasing, we inevitably have a collision of characteristic projections — and cars — after a finite time, as in Figure 9.1(b). This is an example of the solution blow-up we discussed in Chapter 8, and here it has an obvious physical interpretation that fast cars have caught up with slow ones and are trying to occupy the same bit of road. That is, the model predicts that cars with different speeds will end up in the same place. Clearly, this model is inadequate as a description of how real traffic behaves. Its predictions are realistic within its severe limitations, but they are so far off the mark that we need to do something more sophisticated.

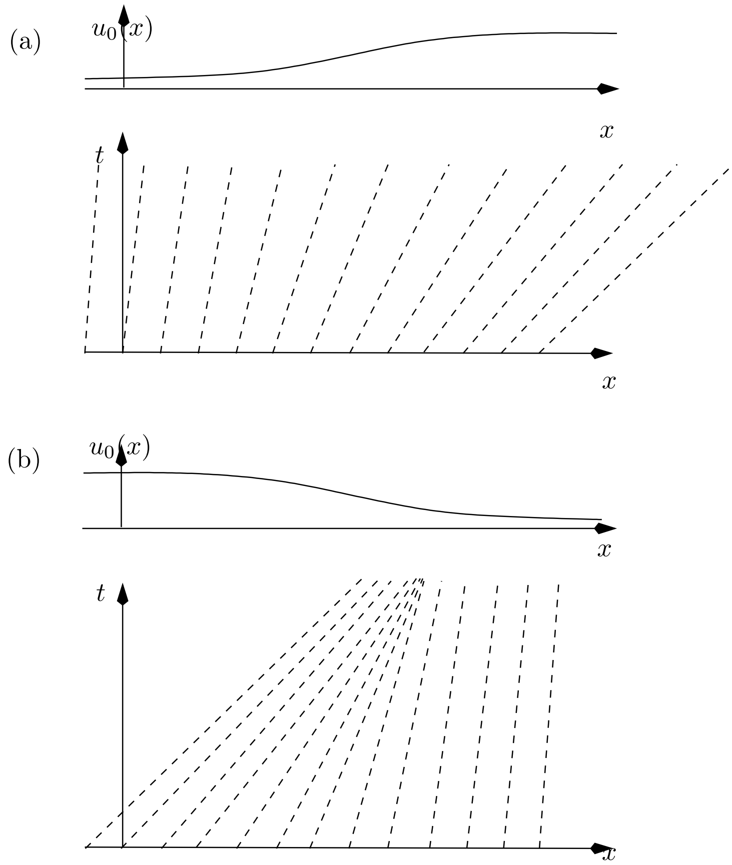

Figure 9.1: Initial data and characteristic projections for the kinematic wave equation.

9.1.1 Local speed-density laws

In our quest for greater realism, we should try to describe how drivers respond to the traffic around them. A simple way to do this is to propose a (constitutive) relation between the speed of cars at a point x and their density there. That is, we assume that

$$u = U(\rho)$$

for a suitable function U. This function should be determined experimentally from observations of local speed and density, or at least written down in a parametric form and the parameters calibrated (fitted) to observations of global features of the traffic flow (an example of an inverse problem). Before going too far down this road, let us see what happens when we put a simple U into the model. As heavy traffic generally moves more slowly than light traffic, we want $U(\rho)$ to be a decreasing function of ρ. We may assume a

Almost certainly $u_{\max}$ is greater than the speed limit...

maximum car speed $u_{\max}$, and it is reasonable to assume that cars drive at this speed on an empty road, when $\rho = 0$. Conversely, we can assume a maximum bumper-to-bumper density $\rho_{\max}$ at which the traffic comes to a complete halt, so $u = 0$. This suggests that the speed-density law

$$u = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

should be a reasonable qualitative description.

We can make an immediate and interesting observation. The flux of cars is

$$Q = u\rho = u_{\max}\rho_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)\frac{\rho}{\rho_{\max}},$$

It is an assumption of the model that all drivers behave in the same way, and that they all drive as fast as is consistent with the ambient traffic density.

and it is greatest when $\rho = \frac{1}{2}\rho_{\max}$, so that $u = \frac{1}{2}u_{\max}$. In this model the free-market individual desire of drivers to minimise their journey time by always driving as fast as possible does not necessarily deliver the maximum-flux solution for drivers as a whole.

Leaving this aside, let us see whether we still have blow-up. Making the trivial scalings $u = u_{\max}u'$, $\rho = \rho_{\max}\rho'$, with suitable scalings for x and t, we have the dimensionless equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho(1-\rho)\right) = \frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} = 0$$

(this is, of course, just a conservation law). The characteristic equations are

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = 1 - 2\rho, \quad \frac{d\rho}{d\tau} = 0,$$

so the characteristics are again straight as ρ is constant on them. However, bearing in mind that $0 < \rho < 1$, we see that we can easily prescribe initial data for ρ that will again lead to finite-time blow-up: the characteristic projections can have slopes of either sign and they can easily cross.

Clearly, we must either further tinker with the model so that blow-up is forbidden, or we must face up to the fact that it *will* happen in realistic models, and decide what to do about it.

9.2 Solutions with discontinuities: shocks and the Rankine–Hugoniot relations

We saw in Section 8.3 that the notion of a solution to a conservation law can be extended to allow jump discontinuities across

9.2.1 Traffic jams

9.2.2 Traffic lights

Exercises

1. **Blinkered cars.** Consider the kinematic wave equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

with $u(x,0) = u_0(x)$ a smooth decreasing function of x as in Figure 9.1(b). Find the solution in parametric form. Look at teh relevant Jacobian to show that the earliest time at which the characteristics cross is

$$t_{\min} = -\frac{1}{\min_{-\infty<x<\infty} u_0'(x)}.$$

Show that the rate at which cars get closer is $\partial u/\partial x$ and interpret the blow-up result above in this light.

2. **Traffic.** Consider the traffic model of Section **??**

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0,$$

where $Q = u\rho$ and $u = 1 - \rho$ for $0 \le \rho \le 1$. Show that u and ρ are constant on the characteristics

$$\frac{dx}{dt} = 1 - 2\rho,$$

and show that the Rankine–Hugoniot condition for the speed of a shock $x = S(t)$ is

$$\frac{dS}{dt} = \frac{[\rho(1-\rho)]_-^+}{[\rho]_-^+}.$$

(a) A tractor is travelling along the road at a quarter of the maximum speed and a very long queue of cars travelling at the same speed has built up behind it. At time $t = 0$ the tractor passes the origin $x = 0$ and immediately turns off the road. Sketch the characteristic diagram; show that there is an expansion fan for ρ centered at $x = 0$, $t = 0$ and find $\rho(x,t)$ for $t > 0$.

(b) A queue is building up at a traffic light $x = 1$ so that, when the light turns to green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0 & \text{for } x < 0 \text{ and } x > 1, \\ x & \text{for } 0 < x < 1. \end{cases}$$

Show that the characteristics, labelled by s and starting from $(s, 0)$, are given by $t = \tau$ and

$$\begin{aligned} x - s = \tau & \quad \text{in } x < \tau \text{ and } x > \tau + 1, \text{ on which } \rho = 0, \\ x - s = (1 - 2s)\tau & \quad \text{in } \tau < x < 1 - \tau, \text{ on which } \rho = s, \\ x - 1 = (1 - 2\rho_0)\tau & \quad \text{in } 1 - \tau < x < 1 + \tau, \text{ on which } \rho = \rho_0 = (\tau - x + 1)/(2\tau) \end{aligned}$$

(these last ones are an expansion fan starting from the light). Draw the characteristic projections in the (x, t) plane; show that all those starting with $0 < s < 1$ pass through one point and deduce that a collision first occurs at $x = 1/2$ at $t = 1/2$.

Harder: show that thereafter there is a shock $x = S(t)$ starting from $(\frac{1}{2}, \frac{1}{2})$ where

$$\frac{dS}{dt} = \frac{S + t - 1}{2t}.$$

Write $S(t) = 1 + \tilde{S}(t)$ to reduce this equation to one homogeneous in $\tilde{S}$ and t, and hence solve it.

3. **Two-lane traffic.** Explain why the one-lane model above might be extended to at two-lane model in the form

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}(\rho_1 u_1) = F_{12}(\rho_1, \rho_2, u_1, u_2),$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}(\rho_2 u_2) = -F_{12}(\rho_1, \rho_2, u_1, u_2),$$

and explain where F_{12} comes from. What general properties should F_{12} have, in your opinion? How would it differ for the American freeway in which overtaking is allowed on the inside lane, as compared to the British case in which (in principle if not in practice) it is not?

Chapter 10

The delta function and other distributions

10.1 Introduction

In this chapter we give a very informal introduction to *distributions*, also called *generalised functions*. We do two rather amazing things: we see how to differentiate a function with a jump discontinuity, and we develop a mathematical framework for point forces, masses, charges, sources etc. Furthermore, we find that these two ideas find their expression in the same mathematical object: the Dirac delta function.

When I learned proper real analysis for the first time, we spent ages agonising about continuity, left and right limits, one- and two-sided derivatives, and so on. The result was a lingering fear of pathological functions (zero except on the rationals, that sort of thing). It came as a great relief to find (much later on, alas) that one can make perfect sense of the derivative of the *Heaviside function*[1]

$$\mathcal{H}(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases}$$

This function and its derivative are ubiquitous in whole swathes of linear applied mathematics, not to mention continuous and discrete probability. I think distributions are invaluable in developing an intuitive framework for modelling and its interaction with mathematics. Don't be inhibited about using them: your mistakes are unlikely to do worse than lead to inconsisten-

[1] The value $\mathcal{H}(0) = 0$ has been assigned for consistency with probability, as we shall see; but for reasons that will shortly become clear it really doesn't matter what value we take.

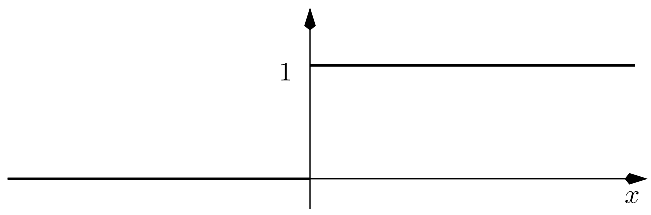

Figure 10.1: The Heaviside function $\mathcal{H}(x)$. Its derivative vanishes for all $x \neq 0$ but it still gets up from 0 to 1. How?

cies (which I hope you are constantly on the look out for) and plainly wrong answers, rather than the deadly 'plausible but fallacious' solution.

10.2 A point force on a stretched string; impulses

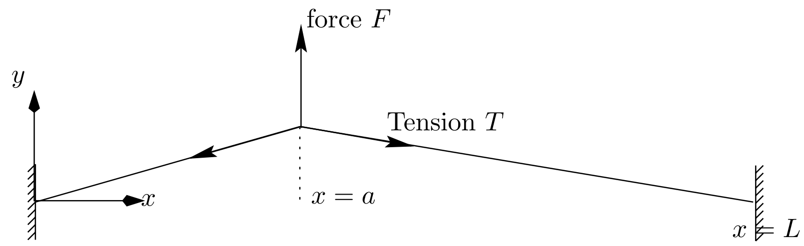

Figure 10.2: A string with a point force.

Let's start with a couple of motivating physical examples. We have all at some time worked out the displacement of a stretched string under the influence of a point force, as sketched in Figure 10.2. Under the standard assumptions that the string is effectively weightless, and that the force F (measured upwards, in the same direction as y) can be considered as acting at a point $x = a$ and only causes a small deflection, the equilibrium displacement $y(x)$ of the string satisfies

$$\frac{d^2y}{dx^2} = 0, \qquad 0 < x < a, \quad a < x < L, \tag{10.1}$$

Consistency check on the signs: $F > 0$ and dy/dx is negative to the right of a, positive to the left.

with the force balance condition

$$\left[T\frac{dy}{dx}\right]_{x=a-}^{x=a+} = -F. \tag{10.2}$$

Notice the implicit assumption that y itself is continuous at x_0 although its derivative is not.

Now we might ask, can we somehow put the force on the right-hand side of (10.1), and have the equilibrium conditions hold at $x = a$ as well? After all, if we have a distributed force per unit length $f(x)$ on the string, the usual force balance on a small element (see Figure 10.3) gives the equation[2]

$$T\frac{d^2y}{dx^2} = -f(x), \qquad 0 < x < L.$$

For example, when $f = -\rho g$, the gravitational force on a uniform wire of line density ρ, the displacement is a parabola (the small-displacement approximation to a catenary).

Can we devise some limiting process in which all the force becomes concentrated near $x = a$, with the total force $\int_0^L f(x)\,dx$ tending to F? A possible way to do this would be to take

Question expecting the answer yes.

$$f(x) = \begin{cases} F/2\epsilon & a-\epsilon < x < a+\epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

and then to let $\epsilon \to 0$. But would we get the same answer if we took the limit of some other concentrated force density, and in any case how, exactly, are we to interpret the result of this limiting process?

In a very similar vein, recall the concept of an impulse in mechanics. In one-dimensional motion, the velocity v of a particle under a force $f(t)$ satisfies Newton's equation

$$m\frac{dv}{dt} = f(t),$$

[2]You might wonder why there is a minus sign on the right. If we were to consider the unsteady motion of the string, Newton's Second Law in the form

$$\text{mass} \times \text{acceleration} = \text{force}$$

gives

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} + f,$$

leading to the minus sign in question. Many mathematicians, writing the wave equation as

$$\rho\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = f,$$

would write the equilibrium equation for the string as

$$-T\frac{d^2y}{dx^2} = f(x).$$

Note the absence of minus signs in the impulse example below.

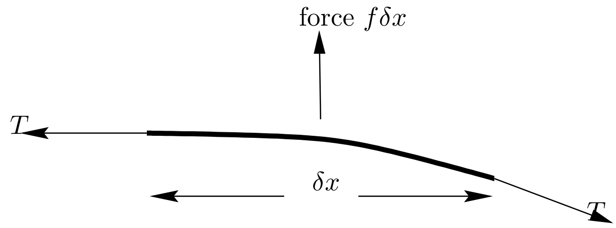

Figure 10.3: Force on an element of a string.

from which

$$v(t) = v(0) + \frac{1}{m}\int_0^t f(s)\,ds$$

If the force is very large but only lasts for a short time, say

$$f(t) = \begin{cases} I/\epsilon & 0 < t < \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then we can integrate the equation of motion from $t = 0$ to $t = \epsilon$ to find

$$v(\epsilon) = \frac{1}{m}\int_0^\epsilon \frac{I}{\epsilon}\,dt = \frac{I}{m}.$$

Notice that the wire slope has a jump discontinuity at a point force.

Letting $\epsilon \to 0$, we have the result of an *impulse* I: the velocity v changes discontinuously from 0 to I/m. Again, we can ask the question, can we put the limiting impulse directly into the equation of motion, rather than having to smooth it out and take a limit?

10.3 Informal definition of the delta and Heaviside functions

Obviously the answer to all our questions above is yes. The powerful and elegant theory of distributions allows us to model point forces and much more (dipoles, for example). However, the intuitive view of a point force (mass, charge,…) as the limit of a distributed force turns out to be technically very cumbersome, and nowadays a more concise and general, but physically less intuitive, treatment is preferred. This oblique approach requires some groundwork, and we defer a brief self-contained description until Chapter 11. You will survive if you don't read it, although I recommend that you do: it is not technically demanding or complex.

In this chapter we concentrate on the intuitive approach to the delta function. Although this is not how the theory is nowadays developed, *it*

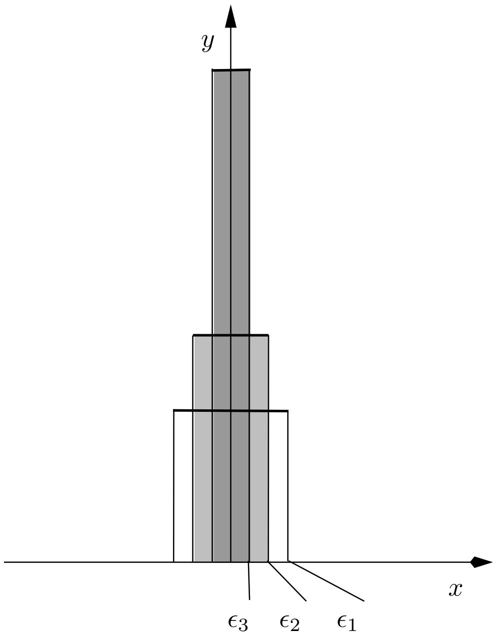

Figure 10.4: Three approximations to the delta function; $\epsilon_1 > \epsilon_2 > \epsilon_3 > 0$.

absolutely is how to visualise this central part of it. Taking the examples of the previous section and stripping away the physical background, consider the functions

$$f_\epsilon(x) = \begin{cases} 1/2\epsilon & -\epsilon < x < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

They are shown in Figure 10.4 for various values of ϵ. The following facts are obvious:

- $$\int_{-\infty}^{\infty} f_\epsilon(x)\, dx = 1 \qquad \text{for all } \epsilon > 0;$$
- for $x \neq 0$, $\quad f_\epsilon(x) \to 0 \quad$ as $\epsilon \to 0$.

Clearly the limiting 'function' is very strange indeed. It has a 'mass', or 'area under the graph', of 1, but that mass is all concentrated at $x = 0$. This is just what we need to model a point force, and even though we don't quite know how to interpret it rogorously, we provisionally christen the limit as the *delta function*, $\delta(x)$.

Two extremely useful properties of the delta function are now at least plausible. Firstly, as $\epsilon \to 0$,

$$\int_{-\infty}^{x} f_\epsilon(s)\, ds \to \begin{cases} 1 & x > 0, \\ 0 & x < 0, \end{cases}$$

For now, let's not worry what its value is at $x = 0$.

and the right-hand side is the Heaviside function $\mathcal{H}(x)$ with its jump discontinuity at $x = 0$. So, we should have

$$\int_{-\infty}^{x} \delta(s)\, ds = \mathcal{H}(x),$$

at least for $x \neq 0$. Furthermore, fingers crossed and appealing to the Fundamental Theorem of Calculus, we should conversely have

$$\frac{d}{dx}\mathcal{H}(x) = \delta(x).$$

That is, *delta functions let us differentiate functions with jump discontinuities.* The Heaviside function has a jump up of 1 at $x = 0$, and its derivative is $\delta(x)$, and by an obvious extension, the derivative of a function with a jump of A at $x = a$ contains a term $A\delta(x - a)$.

A proof is requested in the exercises

The second vital attribute of $\delta(x)$ is its 'sifting' property. It is intuitively clear that, for sufficiently smooth functions $\phi(x)$,

$$\int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\, dx \to \phi(0) \quad \text{as } \epsilon \to 0,$$

simply because all the mass of $f_\epsilon(x)$, and hence of the product $f_\epsilon(x)\phi(x)$, becomes concentrated at the origin. So, we conjecture that we can make sense of the statement

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\, dx = \phi(0) \tag{10.3}$$

and, by a simple change of variable,

$$\int_{-\infty}^{\infty} \delta(x - a)\phi(x)\, dx = \phi(a)$$

for any real a.

These assertions are eminently plausible. However, if you stop to think how you might make them mathematically acceptable, difficulties start to appear. Would we get the same results if we used a different approximating sequence $g_\epsilon(x)$? Do we need to worry about the value of $\mathcal{H}(0)$? Having differentiated $\mathcal{H}(x)$, can we define $d\delta/dx$? Clearly this last runs a big risk of being very dependent on the approximating sequence we use.

For example,

$$g_\epsilon(x) = \frac{1}{\epsilon\sqrt{2\pi}} e^{-\frac{1}{2}x^2/\epsilon^2}$$

as discussed in Section 10.4.2.

For all these reasons, and more, the theory is best developed slightly differently, without the 'epsilonology'.[3] The clue lies in the sifting property.

[3] See [27] page 97 for this neologism.

Using the fact that integration is a smoothing process, we can get away from the 'pointwise' view of functions which is so troublesome, and instead define distributions via *averaged* properties. An example is the integral (10.3), which leads to the definition of $\delta(x)$.[4] Before looking at this idea in more detail, we consider some examples.

10.4 Examples

10.4.1 A point force on a wire revisited

All our discussion suggests that we should model the point force F acting at $x = a$ by a term $F\delta(x-a)$ in the equilibrium equation for the displacement, and assume that the latter now holds for *all* x, so that

$$T\frac{d^2y}{dx^2} = -F\delta(x-a), \qquad 0 < x < L.$$

We now know that this means that the left-hand side is the derivative of a function which jumps by F at $x = a$. But the left-hand side is also the derivative of $T\,dy/dx$. Thus, putting the delta function into the equilibrium equation leads *automatically* to the force balance

Assuming we believe that differentiation still makes sense.

$$\left[T\frac{dy}{dx}\right]_{a-}^{a+} = -F,$$

and there is no need to state this separately.

10.4.2 Continuous and discrete probability.

We can interpret each of the approximations $f_\epsilon(x)$ of Figure **??** as the probability density of a random variable X_ϵ whose value is uniformly distributed on the interval $(-\epsilon, \epsilon)$. The mean of this distribution is 0 and its standard deviation is $\epsilon/\sqrt{3}$. As $\epsilon \to 0$, the random variable becomes equal to 1 with certainty, because its standard deviation tends to zero, and any random variable with zero standard deviation must be a constant. This suggests that we can interpret the delta function as the probability density 'function' of a

[4]The process of generalisation by looking at a weaker (smoother) definition using an integral, rather than a pointwise definition, is common in analysis. A famous example in applied mathematics is the definition of weak solutions to hyperbolic conservation laws, which leads to the Rankine–Hugoniot relations for a shock.

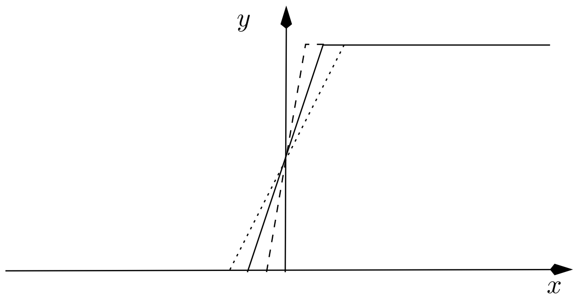

Figure 10.5: Cumulative density functions for the distributions of Figure ??.

variable whose probability of being equal to zero is 1. Likewise, the cumulative density function (distribution function) $F_{X_\epsilon}(x) = P(X_\epsilon < x)$ tends to the Heaviside function.[5]

In a similar vein, we can take approximations

$$g_\epsilon(x) = \frac{1}{\epsilon\sqrt{2\pi}} e^{-\frac{1}{2}x^2/\epsilon^2},$$

which are the density functions of normal random variables with mean zero and standard deviation ϵ. These also clearly tend to the delta function as $\epsilon \longrightarrow 0$.

Now suppose we have a coin-toss random variable X taking values ± 1 with equal probability $\frac{1}{2}$. As X can only equal 1 or -1, all its probability mass is concentrated at these values: its density function is zero for $x \neq \pm 1$. Clearly, the density of this random variable is

What is its distribution function?

$$f_X(x) = \tfrac{1}{2}\left(\delta(x+1) + \delta(x-1)\right).$$

In this way, we can unify continuous and discrete probability — at least when the number of discrete events is finite. The extension to infinitely many discrete events is much more difficult, and may require the tools of measure theory.

[5]In this case the strict inequality in the definition of F_{X_ϵ} suggests that we should take $\mathcal{H}(0) = 0$. Looking in the books on my shelf, I find that there is no consensus in the probability world whether to use $P(X < x)$ or $P(X \leq x)$ to define the distribution function (no wonder I can never remember). It is a matter of convention only, and would lead to corresponding conventional definitions of $\mathcal{H}(0)$. Another highly plausible definition is $\mathcal{H}(0) = \frac{1}{2}$, on the grounds that any Fourier series or inversion integral for a function with a jump converges to the average of the values on either side. This sort of hair splitting is one reason why the pointwise view of distributions is not really workable.

10.4.3 The fundamental solution of the heat equation

If we set $\epsilon = 2t$ in the functions g_ϵ of the previous section, we get the function

$$g(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}.$$

It is a straightforward exercise to show by direct differentiation that $g(x,t)$ satisfies the heat equation. As we saw above, as $t \downarrow 0$, $g(x,t) \to \delta(x)$. In summary, $g(x,t)$ satisfies the initial value problem

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2}, \qquad t > 0, \quad -\infty < x < \infty,$$
$$g(x,0) = \delta(x).$$

This solution represents the evolution of a 'hot spot', a unit amount of heat which at $t = 0$ is concentrated at $x = 0$.

Note the infinite propagation speed of the heat: $t = 0$ is a (double) characteristic of the heat equation. Note also the very rapid decay in the solution as $|x|$ increases.

With this solution, we can solve the more general initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad -\infty < x < \infty,$$
$$u(x,0) = u_0(x).$$

We first note that the initial data $u_0(x)$ can be written as

$$u_0(x) = \int_{-\infty}^{\infty} u_0(\xi)\delta(x-\xi)\, d\xi$$

by the picking-out property of the delta function. Now the evolution of a solution with initial data $\delta(x-\xi)$ is just $g(x-\xi,t)$ where g is as above. The integral over ξ amounts to superposing the initial data for these solutions, so that each point contributes a delta function weighted by $u_0(\xi)\, d\xi$. Because the heat equation is linear, we can superpose for $t > 0$ as well, so we have

Confirm that $u(x,t)$ satisfies the heat equation by differentiating under the integral sign.

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} u_0(\xi) g(x-\xi,t)\, d\xi \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} u_0(\xi) e^{-(x-\xi)^2/4t} d\xi. \end{aligned}$$

This solution has a physical interpretation as the superposition of elementary 'packets' of heat evolving independently.[6]

[6] There is also an interpretation in terms of random walkers following Brownian Motion: see the exercise 'Brownian Motion' on page 134.

10.5 'Balancing the most singular terms'

If we have an equation involving 'ordinary' functions, and there is a singularity on one side, there must be a balancing singularity somewhere else. For example, we could never find coefficients a_n such that

$$\frac{1}{\sin x} = a_0 + a_1 x + a_2 x^2 + \cdots$$

because the left-hand side clearly has a $1/x$ (simple pole) singularity at $x = 0$. On the other hand there *is* an expansion

This is just the Laurent expansion.

$$\frac{1}{\sin x} = \frac{a_{-1}}{x} + a_0 + a_1 x + a_2 x^2 + \cdots ,$$

and furthermore it is obvious that $a_{-1} = 1$ because $1/\sin x \sim 1/x$ as $x \to 0$. Thus, both sides have this singularity in their leading-order behaviour as $x \to 0$.

This is a simple but powerful idea, and it applies to distributions as well. In our naive approach, a delta function is obviously a 'function' with a particular singularity at $x = 0$. Thus, if part (for example the right-hand side) of an equation contains a delta function as its most singular term, there must be a balancing term somewhere else. For example, when we write

$$\frac{dv}{dt} = \frac{I}{m}\delta(t),$$

for the motion of a particle subject to a point force, there must be another singularity to balance the delta function. It can only be in dv/dt, so we know straightaway that v has a jump at $t = 0$; furthermore, we know that the magnitude of the jump is I/m, by 'comparing coefficients' of the delta functions. In this case it is trivial to find the balancing term, because there is only one candidate. Suppose, though, that the equation has a linear damping term:

Go back and look at the point force on a string in this light.

$$m\frac{dv}{dt} = -mkv + I\delta(t),$$

where $k > 0$ is the damping coefficient. The balancing singularity is still in the derivative dv/dt, simply because dv/dt always has worse singularities than v itself. Going back, we can check: if dv/dt has a delta, then v has a jump, which is indeed less singular.

Differentiation makes matters worse, integration makes them better.

10.5.1 The Rankine–Hugoniot conditions

In Chapter 8 we looked briefly at the Rankine–Hugoniot conditions for a first order conservation law

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

where, for example, P is the density ρ of traffic and Q the flux $u\rho$. We saw that we can construct solutions in which P and Q have jump discontinuities across a shock at $x = S(t)$, provided that

$$\frac{dS}{dt} = \frac{[Q]}{[P]}.$$

We can interpret this condition as a balance of delta functions. If P has a (time-dependent) jump of magnitude $[P](t)$ at $x = S(t)$, we can (very informally) write

$$P(x,t) = [P](t)\mathcal{H}\left(x - S(t)\right) + \text{smoother part},$$

and similarly for $Q(x,t)$. Differentiating, we find

$$\begin{aligned}\frac{\partial P}{\partial t} &= -[P](t)\delta\left(x - S(t)\right)\frac{dS}{dt} + \text{less singular terms},\\ \frac{\partial Q}{\partial x} &= [Q](t)\delta\left(x - S(t)\right) + \text{less singular terms}.\end{aligned}$$

Adding these and balancing the coefficients of the delta functions, the Rankine–Hugoniot condition drops out.

10.5.2 Case study: cable-laying

In Chapter 5, we wrote down the model

$$\frac{dF_x}{ds} = -B_x, \quad \frac{dF_y}{ds} = -B_y + \rho_c g A = 0, \quad EAk^2\frac{d^2\theta}{ds^2} - F_x\sin\theta + F_y\cos\theta = 0, \tag{10.4}$$

where

$$(B_x, B_y) = \left(\rho_w g A\cos\theta + pA\frac{d\theta}{ds}\right)(-\sin\theta, \cos\theta). \tag{10.5}$$

for a cable being laid on a sea bed, where θ is the angle between the cable and the horizontal. We stated, on a rather intuitive basis, that the boundary conditions at $s = 0$ are $\theta = 0$ (no worries about this one) and $d\theta/ds = 0$, namely continuity of θ and $d\theta/ds$, since $\theta = 0$ for $s < 0$. We can now see why this is necessary. If $d\theta/ds$ is not continuous, then $d^2\theta/ds^2$ has a delta function discontinuity at $s = 0$. But then there is no balancing term in the last equation of (10.4) since, loosely, (10.4) shows that both F_x and F_y are at least as continuous as B_x and B_y, and so from (10.5) they are no worse than $d\theta/ds$ with its assumed-for-a-contradiction jump discontinuity; we have duly obtained said contradiction.

Because there is a reaction force between the sea bed and the cable, and maybe some friction, we do not expect the right-hand sides of the first two of (10.4) to be continuous at $s = 0$.

10.6 Green's functions

10.6.1 Ordinary differential equations

We have all seen the definition of the Green's function for the self-adjoint two-point boundary value problem[7]

$$\mathcal{L}_x y(x) = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = f(x), \qquad 0 < x < 1, \tag{10.6}$$

$$y(0) = y(1) = 0. \tag{10.7}$$

Provided that the homogeneous problem ($f(x) \equiv 0$) has no non-trivial solutions, the Green's function is the function $G(x,\xi)$ that satisfies

$$\mathcal{L}_\xi G(x,\xi) = 0, \qquad 0 < \xi < x, \quad x < \xi < 1, \tag{10.8}$$

$$G(x,0) = G(x,1) = 0, \tag{10.9}$$

with some rather opaque (if you don't think about delta functions) conditions at $\xi = x$:

$$[G]_{\xi=x-}^{\xi=x+} = 0, \qquad \left[p(\xi)\frac{dG}{d\xi}\right]_{\xi=x-}^{\xi=x+} = 1 \quad \left(\text{or} \quad \left[\frac{dG}{d\xi}\right]_{\xi=x-}^{\xi=x+} = \frac{1}{p(x)}\right). \tag{10.10}$$

If we can solve this problem, then we have a representation for $y(x)$ as

$$y(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi.$$

Just like inverting a matrix $\mathbf{A}$ to solve $\mathbf{Ax} = \mathbf{b}$; see the exercise about this on page 133.

The point is, of course, that we need only calculate G once, and then we have the solution whatever we take for f.[8] In this way, we can think of

[7] The subscript to $\mathcal{L}$ tells you which variable to use. Strictly speaking, all the derivatives should be partial, but it seems to be conventional to stick to ordinary derivatives.

[8] A very common use of the Green's function is to turn a differential equation into an integral equation as a prelude to an iteration scheme to prove existence, uniqueness and regularity. Often the equation has a linear part and some nonlinearity as well, and we use the Green's function for the linear part. A simple example of this procedure is Picard's theorem for local existence and uniqueness of the solution to $d\mathbf{y}/dx = f(x,\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$ for a set of first-order equations, where the first step is to write

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_0^x f(\xi, \mathbf{y}(\xi))\, d\xi;$$

the only modification needed is to adapt the Green's function methodology to cater for initial value problems, as described in the exercise on that topic on page 130.

the operation of multiplying by the Green's function and integrating as the inverse of $\mathcal{L}$ and the boundary conditions.

This is all very well, but I don't think it gives a good intuitive feel for what the Green's function really *does*. Suppose, though, that we take the solution

$$y(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi \tag{10.11}$$

and apply $\mathcal{L}_x$ to it. Assuming that we can differentiate under the integral, we get

$$\begin{aligned} \mathcal{L}_x y(x) &= \int_0^1 \mathcal{L}_x G(x, \xi) f(\xi)\, d\xi \\ &= f(x). \end{aligned}$$

We recognise this: it is the sifting property. Whatever f we take, when we multiply $f(\xi)$ by $\mathcal{L}_x G(x, \xi)$ and integrate, we get $f(x)$. Thus, as a function of x, $G(x, \xi)$ satisfies

$$\mathcal{L}_x G(x, \xi) = \delta(x - \xi),$$

that is

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \delta(x - \xi),$$

Also, the boundary conditions $y(0) = y(1) = 0$ mean that we need to take

$$G(0, \xi) = G(1, \xi) = 0.$$

In summary, as a function of x, the Green's function satisfies the differential equation with a delta-function on the right-hand side, and with the homogeneous version of the original boundary conditions.

This calculation tells us several things. Thinking physically, it tells us that *the Green's function is the response of the system to a point stimulus (force, charge, ...) at* $x = \xi$. The solution (10.11) is then just the response to $f(x)$, regarded as a superposition of point stimuli (the delta function at $x = \xi$) weighted by $f(\xi)\, d\xi$.

Looking more mathematically, if we expand the differential operator $\mathcal{L}_x$ as

$$\mathcal{L}_x G(x, \xi) = p(x)\frac{d^2 G}{dx^2} + \text{lower order derivatives},$$

we see by balancing the most singular terms (the highest derivatives) that d^2G/dx^2 must have a delta function, scaled by $p(x)$, at $x = \xi$. That is,

$$[G]_{x=\xi^-}^{x=\xi+} = 0, \qquad \left[p(x)\frac{dG}{dx}\right]_{x=\xi^-}^{x=\xi+} = 1 \quad \left(\text{or} \quad \left[\frac{dG}{dx}\right]_{x=\xi^-}^{x=\xi+} = \frac{1}{p(\xi)}\right).$$

This should ring a bell. It is the same as the 'opaque' jump conditions (10.10), except that it refers to the x–dependence of $G(x,\xi)$ instead of the ξ–dependence. Indeed, comparing the original definition of G given in (10.8)–(10.10) and recalling that $G(0,\xi) = G(1,\xi) = 0$, we see that the two formulations are identical except that x and ξ are swapped. That is, we have established that

Note that $\delta(x-\xi)$ is obviously equal to $\delta(\xi - x)$.

$$G(x,\xi) = G(\xi,x)$$

and that

$$\mathcal{L}_\xi G(x,\xi) = \delta(\xi - x).$$

We are now in a position to tie together the x and ξ dependence of $G(x,\xi)$. Consider the integral

$$\int_0^1 y(\xi)\mathcal{L}_\xi G(x,\xi) - G(x,\xi)\mathcal{L}_\xi y(\xi)\, d\xi.$$

Inserting the right-hand sides of the differential equations for G and y, we get

$$\begin{aligned}\int_0^1 y(\xi)\mathcal{L}_\xi G(x,\xi) - G(x,\xi)\mathcal{L}_\xi y(\xi)\, d\xi &= \int_0^1 y(\xi)\delta(\xi - x) - G(x,\xi)f(\xi)\, d\xi \\ &= y(x) - \int_0^1 G(x,\xi)f(\xi)\, d\xi.\end{aligned}$$

On the other hand, integrating the same expression by parts, we get

$$\begin{aligned}\int_0^1 y(\xi)\mathcal{L}_\xi G(x,\xi) - G(x,\xi)\mathcal{L}_\xi y(\xi)\, d\xi &= \int_0^1 y(\xi)\left(\frac{d}{d\xi}\left(p(\xi)\frac{dG}{d\xi}\right) + q(\xi)G(x,\xi)\right) \\ &\quad - G(x,\xi)\left(\frac{d}{d\xi}\left(p(\xi)\frac{dy}{d\xi}\right) + q(\xi)y(\xi)\right)\, d\xi \\ &= \left[y(\xi)p(\xi)\frac{dG}{d\xi} - G(x,\xi)p(\xi)\frac{dy}{d\xi}\right]_0^1 \\ &\quad - \int_0^1 p(\xi)\frac{dy}{d\xi}\frac{dG}{d\xi} - p(\xi)\frac{dG}{d\xi}\frac{dy}{d\xi}\, d\xi \\ &= 0.\end{aligned}$$

Thus we retrieve our solution

$$y(x) = \int_0^1 G(x,\xi)f(\xi)\, d\xi.$$

As a function of x the differential operator for G is still $\mathcal{L}$ and the boundary conditions are the same as those for y.

This calculation is really the key to the whole procedure. It tells us that the differential operator and boundary conditions for G as a function of ξ must be such that we can integrate by parts and get zero (so in the second line of our calculation, we must have zero multiplying dy/dx, about which we know nothing at the endpoints). For a self-adjoint problem, such as the one we have here, G is symmetric and the two operators are the same. For more general problems, such as

$$\mathcal{L}_x y(x) = a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x),$$

we need to find the *adjoint operator* $\mathcal{L}^*$ which is such that

$$y(\xi)\mathcal{L}^*_\xi G(x,\xi) - G(x,\xi)\mathcal{L}_\xi y(\xi)$$

is an exact derivative and so can be integrated by parts. Provided that, as a function of ξ, $G(x,\xi)$ satisfies

$$\mathcal{L}^*_\xi G(x,\xi) = \delta(x-\xi),$$

with appropriate boundary conditions, we can integrate by parts as above to write $y(x)$ as an integral. For the general $\mathcal{L}$ just introduced, the adjoint operator is given by

$$\mathcal{L}^* v(x) = \frac{d^2}{dx^2}\left(a(x)v(x)\right) - \frac{d}{dx}\left(b(x)v(x)\right) + c(x)v(x),$$

as you will find out by doing the relevant exercise on page 131.

10.6.2 Partial differential equations

Much of the theory we have just seen can be generalised to linear partial differential equations. This is so much vaster a topic that it is only feasible to discuss one example in detail, the Green's function for Poisson's equation, which is probably the closest in spirit to the two-point boundary value problems we have been discussing so far. We then briefly mention two other canonical problems, for the heat equation and the wave equation.

We first have to generalise the delta function. In our informal style, this is easy: we just say that for $\mathbf{x} \in \mathbb{R}^n$, the delta function $\delta(\mathbf{x})$ is such that

$$\int_{\mathbb{R}^n} \delta(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} = \phi(\mathbf{0})$$

for all smooth functions $\phi(\mathbf{x})$. As before, we can think of this as a limiting process in which the delta function is the limit of a family of functions whose mass becomes more and more concentrated near the origin.[9] Thinking about how the integral is calculated, say in two dimensions with $d\mathbf{x} = dx\,dy$, we may also write

$$\delta(\mathbf{x}) = \delta(x)\delta(y),$$

and similarly in three or more variables.

Think of some physical interpretations for u, and then for the Green's function G.

Now suppose that we have to solve the problem

$$\mathcal{L}_{\mathbf{x}} u(\mathbf{x}) = \nabla^2 u(\mathbf{x}) = f(\mathbf{x})$$

in some region D, with the homogeneous Dirichlet boundary condition

$$u(\mathbf{x}) = 0 \qquad \text{on} \quad \partial D.$$

The Laplacian is self-adjoint ($\mathcal{L} = \mathcal{L}^*$) ...

We choose the Green's function to satisfy

$$\mathcal{L}_\xi G(\mathbf{x}, \xi) = \delta(\xi - \mathbf{x})$$

and look at the integral

$$\begin{aligned}\int_D u(\xi)\mathcal{L}_\xi G(\mathbf{x},\xi) - G(\mathbf{x},\xi)\mathcal{L}_\xi u(\mathbf{x})\; d\xi &= \int_D u(\xi)\nabla_\xi^2 G(\mathbf{x},\xi) - G(\mathbf{x},\xi)\nabla_\xi^2 u(\mathbf{x})\; d\xi \\ &= \int_D u(\xi)\delta(\xi - \mathbf{x}) - G(\mathbf{x},\xi) f(\xi)\; d\xi \\ &= u(\mathbf{x}) - \int_D G(\mathbf{x},\xi) f(\xi)\; d\xi.\end{aligned}$$

... because $u\nabla^2 G - G\nabla^2 u$ is a divergence and can be integrated (a generalisation of integration by parts).

On the other hand, using Green's theorem, we have

$$\int_D u(\xi)\nabla^2 G(\mathbf{x},\xi) - G(\mathbf{x},\xi)\nabla^2 u(\mathbf{x})\; d\xi \tag{10.12}$$

$$\begin{aligned}&= \int_{\partial D} u(\xi)\mathbf{n}\cdot\nabla_\xi G(\mathbf{x},\xi) - G(\mathbf{x},\xi)\mathbf{n}\cdot\nabla_\xi u(\mathbf{x})\; dS_\xi \\ &= 0,\end{aligned} \tag{10.13}$$

provided that we take $G(\mathbf{x}, \xi) = 0$ for $\xi \in \partial D$, where we do not know the normal derivative of u. Putting these together, we have

$$u(\mathbf{x}) = \int_D G(\mathbf{x},\xi) f(\xi)\, d\xi.$$

It is an easy generalisation to account for nonzero Dirichlet data $u(\mathbf{x}) = g(\mathbf{x})$ on ∂D: we just get an extra known term in (10.13).

[9] They might, but need not, be radially symmetric; we might, but won't, worry about how to define integrals in n dimensions.

Two more things should be said about this calculation. The first is that we have not yet said anything about the nature of the singularity of $G(\mathbf{x}, \xi)$ at $\mathbf{x} = \xi$ (in one space dimension, as we saw above, the first derivative of G has a jump and G itself is continuous). Knowing as we do that line (in two dimensions) or point (in three) charges generate electric fields which are solutions of Laplace's equation, we should not be surprised to see logs in two dimensions and inverse distances in three. This is confirmed by a simple version of the calculation we have just done.[10] In $\mathbb{R}^3$ for example, take $\xi = \mathbf{0}$ and suppose that

$$\nabla^2 G = \delta(\mathbf{x}) \tag{10.14}$$

Or line/point masses and their gravitational potentials, fluid sources and their velocity potentials, or heat sources and their steady-state temperature fields.

in the whole space. Clearly, then, G is radially symmetric: $G = G(r)$ where $r = |\mathbf{x}|$. That means that

$$G(r) = A + \frac{B}{r}$$

and if we want $G \to 0$ as $r \to \infty$, we take $A = 0$. Now use the divergence theorem on (10.14), integrating over a sphere of radius r centred at $\mathbf{x} = \mathbf{0}$. The left-hand side gives a surface integral equal to $-4\pi B/r$ and the volume integral of the delta function on the right is equal to 1. We conclude that the singular behaviour of $G(\mathbf{x}, \xi)$ near $\mathbf{x} = \xi$ is

The meaning of $\sim$ and $O(1)$ is explained in Chapter 13.

$$G(\mathbf{x}, \xi) \sim -\frac{1}{4\pi|\mathbf{x} - \xi|} + O(1),$$

and in two dimensions the corresponding result is

Can you now answer the marginal question after equation (3.3) on page 32?

$$G(\mathbf{x}, \xi) \sim \frac{1}{2\pi} \log|\mathbf{x} - \xi| + O(1).$$

The second point to make about the Green's function for the Laplacian is that it has a natural physical interpretation. The singular part we have just discussed gives us the electric potential due to a point charge (or whatever) with no boundaries. The remaining part, $G + 1/(4\pi|\mathbf{x} - \xi|)$, is known as the *regular part* of the Green's function and it gives the potential due to the image charge system induced by the boundary condition $G = 0$ on ∂D. Indeed, almost all the Green's functions for which explicit formulas are available are constructed by the method of images (possibly with the help of conformal maps).

[10] In the more classical treatment of Green's functions, you see essentially this calculation when you integrate $u\nabla^2 G - G\nabla^2 u$ over a region consisting of D with a sphere of radius ϵ around $\mathbf{x} = \xi$ removed. There, the singulat behaviour of G is prescribed (and looks mysterious: why this form?), whereas here it emerges naturally.

The heat and wave equations

You can safely ignore this section, but have a look if you have seen the classical treatments of these problems.

To round off, let's look quickly at two other equations that you are sure to have seen, the heat and wave equations in two space variables. We have already seen the fundamental solution to the initial-value problem for the heat equation on the whole line,

$$\mathcal{L}_{x,t}u = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = u_0(x).$$

It is no surprise that this is closely related to the Green's function. The adjoint to the forward heat equation is the backward heat equation, and as a function of ξ and τ (the analogue here of ξ above), $G(x,t;\xi,\tau)$ satisfies

$$\mathcal{L}^*_{\xi,\tau}G = \frac{\partial G}{\partial \tau} + \frac{\partial^2 G}{\partial \xi^2} = \delta(\xi - x)\delta(\tau - t),$$

Two minus signs from the exponent cancel.

and clearly (remembering the fundamental solution of the forward heat equation and reversing time),

$$G(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/4(t-\tau)}.$$

The usual integration in the form

$$\int_{-\infty}^{\infty}\int_0^t u\mathcal{L}_{\xi,\tau}G - G\mathcal{L}_{\xi,\tau}u \; d\tau \, d\xi$$

then yields precisely the solution we derived earlier. It is an exercise to generalise this result to the heat equation with a source term, $\mathcal{L}u = f(x,t)$; you will get a double integral involving the product of G and f which has the simple physical interpretation of being a superposition of solutions of initial value problems starting at different times. Do it and see.

For the inhomogeneous wave equation in the canonical form

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x \partial y} = f(x,y),$$

with Cauchy data u and $\partial u/\partial n$ given on a non-characteristic curve Γ, we proceed in the same spirit but differently in detail. One of the differences of detail is the the Green's function is now usually called a Riemann function,

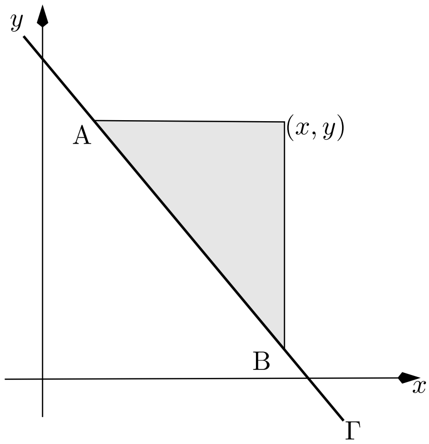

Figure 10.6: Domain of integration for the Riemann function for the wave equation.

and we denote it by $R(x, y; \xi, \eta)$. The differential operator $\partial^2/\partial x \partial y$ is self-adjoint, but we have to consider the direction of information flow carefully (see Figure **??**). When we solve

$$\mathcal{L}^* R = \frac{\partial^2 G}{\partial \xi \partial \eta} = \delta(\xi - x)\delta(\eta - y),$$

we look for a solution valid for $\xi < x$, $\eta < y$. The Riemann function for the wave operator is particularly simple:

Differentiate it and see.

$$R(x, y; \xi, \eta) = \mathcal{H}(x - \xi)\mathcal{H}(y - \eta),$$

i.e. it is equal to 1 in the quadrant $\xi < x$, $\eta < y$ and zero elsewhere. Then the 'usual' integral

$$\int u\mathcal{L}^* R - R\mathcal{L}u$$

is taken over the characteristic triangle shaded in Figure 10.6, and after use of Green's theorem yields the familiar D'Alembert solution (see [27]). Unfortunately, although it is not hard to prove that the Riemann function exists, only for a very few hyperbolic equations can it be found in closed form.

Sources and further reading

The material on Green's functions is just a small step into Sturm–Liouville and Hilbert–Schmidt theory and eigenfunction expansions/transform methods. If you want to explore further, [17] gives a straightforward account of the theory for ordinary differential equations, [27] present an informal introduction to the corresponding material for partial differential equations, and the excellent [38] contains a more thorough account.

Exercises

1. **Truncated random variables.** Suppose that X is a continuous random variable taking values in $(-\infty, \infty)$, for example Normal. The *truncated* variable Y is defined by
$$Y = \begin{cases} X & \text{if } X < a \\ a & \text{if } X \geq a. \end{cases}$$
What are its distribution and density functions?

2. **A useful identity.** Interchange the order of integration (draw a picture of the region of integration) to show that
$$\int_0^x \int_0^\xi f(s)\, ds\, d\xi = \int_0^x (x-\xi) f(\xi)\, d\xi.$$
Generalise to reduce an n–fold repeated integral of a function of a single variable to a single integral.

3. **Green's function for a stretched string.** Integrate twice to find the solution of the two-point boundary value problem
$$y'' = f(x), \quad 0 < x < 1, \qquad y(0) = y(1) = 0$$
in the form
$$y(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi.$$
Verify that if you differentiate twice under the integral sign and use the jump conditions at $\xi = x$ you recover the original problem.

4. **Green's function for an initial value problem.** Use the result of the ante-previous exercise to show that the solution of the initial value problem
$$y'' = f(x), \quad 0 < x < 1, \qquad y(0) = y'(0) = 0 \tag{10.15}$$
is
$$y(x) = \int_0^x (x-\xi) f(\xi)\, d\xi.$$

Now pick $X > x$ and write this answer in the form

$$y(x) = \int_0^X G(x,\xi) f(\xi)\, d\xi;$$

what is G? Show that G satisfies

$$\frac{d^2G}{d\xi^2} = \delta(x-\xi), \quad 0 < \xi < X, \qquad G = \frac{dG}{d\xi} = 0 \quad \text{at } x = X.$$

Verify by differentiating under the integral sign that your answer satisfies the original problem. What is the adjoint problem to the original problem (10.15)?

This kind of Green's function is the ordinary differential equation analogue of the Riemann function for a hyperbolic equation.

5. **Adjoint of a differential operator.** Suppose that

$$\mathcal{L}_x y = a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y.$$

Show that the adjoint is

$$\mathcal{L}_x^* v = \frac{d^2}{dx^2}\left(a(x)v\right) - \frac{d}{dx}\left(b(x)v\right) + c(x)v,$$

(a) by showing that $y\mathcal{L}_x^* v - v\mathcal{L}_x y$ can be integrated by parts as in the text;

The easy way if you know the answer.

(b) by writing

What you might do if you didn't know the answer and couldn't guess it.

$$\mathcal{L}_x^* v = A(x)\frac{d^2v}{dx^2} + B(x)\frac{dv}{dx} + C(x)v$$

and hacking away at the integration by parts (start by integrating the highest derivatives) until everything has been integrated.

Hence verify that self-adjoint operators are of the form

$$\mathcal{L}_x y = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y$$

for some functions $p(x)$ and $q(x)$.

Work though the calculations of Section 10.6.1 in the non-self-adjoint case.

Suppose now that $\mathcal{L}$ is self-adjoint but that the boundary conditions (known as *primary boundary conditions*) are $\alpha_0 y(0) + \beta_0 y'(0) = 0$, $\alpha_1 y(1) + \beta_1 y'(1) = 0$. What are the corresponding conditions satisfied by $G(x,\xi)$ (a) as a function of x, (b) as a function of ξ? What if $y(0) = 0$, $y(1) = y'(0)$?

6. **The Fredholm Alternative: linear algebra and two-point boundary value problems.** Suppose that $\mathbf{A}$ is an $n \times n$ matrix, and we want to solve the linear equations

$$\mathbf{A}\mathbf{y} = \mathbf{f}$$

for the vector $\mathbf{y}$ given $\mathbf{f}$. Show that, if y_1 and y_2 are two solutions, then their difference is an eigenvector of $\mathbf{A}$ with eigenvalue 0.

We all know that if the rank of $\mathbf{A}$ is n, then $\mathbf{A}$ is invertible, its determinant (equal to the product of the eigenvalues) is nonzero, and the solution $\mathbf{y}$ exists and is unique. Suppose now that the rank of $\mathbf{A}$ is $n-1$, so that the null space of $\mathbf{A}$ has dimension 1 and precisely one eigenvalue of $\mathbf{A}$ is zero. That is, there are vectors $\mathbf{v}$ and $\mathbf{w}$, unique up to multiplication by a scalar, such that

If $\mathbf{A}$ is symmetric, then $\mathbf{v} = \mathbf{w}$.

$$\mathbf{a}\mathbf{v} = \mathbf{0}, \qquad \mathbf{w}^\top \mathbf{a} = \mathbf{0}^\top;$$

they are the right and left eigenvectors of $\mathbf{A}$ with eigenvalue 0. Put another way, the corresponding homogeneous system $\mathbf{A}\mathbf{y} = \mathbf{0}$ has the nontrivial solution $c\mathbf{v}$ for any scalar c.

Premultiply $\mathbf{A}\mathbf{y} = \mathbf{f}$ by $\mathbf{w}^\top$ to show that

- **Either** $\mathbf{w}^\top\mathbf{f} = 0$, in which case the solution exists but is only unique up to addition of scalar multiples of $\mathbf{v}$;
- **Or** $\mathbf{w}^\top\mathbf{f} \neq 0$, in which case no solution exists at all.

Illustrate by finding the value of f_2 for which the equations

$$\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} 1 \\ f_2 \end{vmatrix}$$

have any solution at all; interpret geometrically.

This result is known as the *Fredholm Alternative.* It applied, *mutatis mutandis*, to two-point boundary value problems. For example, consider

$$\mathcal{L}y = \frac{d^2y}{dx^2} + \alpha^2 y = f(x), \quad 0 < x < 1, \qquad y(0) = y(1) = 0 \quad (10.16)$$

(the analogue of $\mathbf{A}\mathbf{y} = \mathbf{f}$). Show that the corresponding homogeneous problem $\mathcal{L}y = 0$ has only the trivial solution $y = 0$ unless $\alpha = m\pi$ for integral m (the analogue of $\mathbf{A}$ having zero for an eigenvalue). Find the corresponding eigenfunctions (analogous to $\mathbf{v}$ and $\mathbf{w}$, here equal as $\mathcal{L}$ is self-adjoint). Suppose that $\alpha = \pi$. Multiply (10.16) by the corresponding eigenfunction and integrate by parts to show that there is only a solution to (10.16) if

$$\int_0^1 f(x) \sin \pi x \, dx = 0.$$

Generalise to the case of any (not necessarily self-adjoint) second order differential operator.

Of course, this is not a coincidence. One could take a two-point boundary value problem and discretise it using finite difference approximations to the derivatives; the result would be a set of linear equations whose solvability or otherwise should, as $n \to \infty$, be the same as that of the original continuous problem.

7. **Matrix inversion.** In this question, we develop the matrix analogue of the calculation of Section 10.6.1 involving the Green's function for a two-point boundary value problem for an ordinary differential equation. For clarity, we use the summation convention (see page 19) throughout.

   Suppose that the matrix equation $\mathbf{A}\mathbf{y} = \mathbf{f}$ (in which $\mathbf{A}$ is not necessarily symmetric) is written in component form as

$$A_{ij} y_j = f_i \qquad \text{(identify this with } \mathcal{L}_x y = f).$$

   Let the inverse matrix $\mathbf{A}^{-1}$ have components $(A^{-1})_{ij} = G_{ij}$, so that from $\mathbf{y} = \mathbf{A}^{-1}\mathbf{f}$ we have

$$y_i = G_{ij} f_j \qquad \text{(identify with } y(x) = \int_0^1 G(x, \xi) f(\xi) \, d\xi).$$

   Let δ_{ij} be the Kronecker delta, the discrete analogue of the delta function. Show that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ are written

   That is, $\delta ij = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$. What is δ_{ii}?

$$G_{ij} A_{jk} = \delta_{ik} \qquad \text{(identify with } \mathcal{L}_x G = \delta(x - \xi)),$$
$$A_{ij} G_{jk} = \delta_{ik}.$$

   Take the transpose of the last equation to identify it with $\mathcal{L}_\xi^* G = \delta(\xi - x)$. Lastly, take the dot product with the vector (y_k) to show that

   Note that, just as $\delta(x - \xi) = \delta(\xi - x)$, so $\delta_{ij} = \delta_{ji}$.

$$0 = A_{ij} G_{jk} y_k - G_{ij} A_{jk} y_k = y_i - G_{ij} f_j;$$

   identify this with the calculation involving $\int y \mathcal{L}^* G - G \mathcal{L} y$.

8. **The fundamental solution of the heat equation.** Show that the heat equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
has similarity solutions of the form $u(x,t) = t^\alpha f(x/\sqrt{t})$ for all α and find the ordinary differential equation satisfied by f. Show that
$$\int_{-\infty}^{\infty} u(x,t)\,dx$$
is independent of t when $\alpha = -\frac{1}{2}$, use the result of the first exercise of this chapter to show that in this case $u(x,0) \propto \delta(x)$, and hence find the fundamental solution of the heat equation.

9. **Brownian Motion.** A particle performs the standard drunkard's random walk on the real line, in which in timestep i, of length δt, it moves by $X_i = \pm\delta x$ with equal probability $\frac{1}{2}$. It starts from the origin and the increments are independent. Define
$$W_n = \sum_1^n X_i.$$

This scaling is the simplest that allows proper time variation yet keeps the variance of the limit finite.

Show that $\mathbb{E}[W_n] = 0$, $\mathrm{var}[W_n] = n\delta X^2/\delta t$. Now let $n \to \infty$ with $n\delta t = t$ fixed and $\delta x = \sqrt{\delta t}$. Call the limiting process (assuming it exists!) W_t. Use the Central Limit Theorem to show that

- For each $t > 0$, W_t has the Normal distribution with mean zero and variance t.
- $W_0 = 0$.
- For each $0 \leq s < t$, $W_t - W_s$ is independent of W_s.

The resulting distribution is called *Brownian Motion* and it is central to modern analysis of financial markets. Give a heuristic argument that the sample paths (realisations) are continuous in t but not differentiable.

Now let $p(x,t)$ be the probability density function of many such reandom walks (as a function of position x for each t). Go back to the discrete random walk and, as in the discussion of Poisson processes in Chapter **??**, condition on one step to write down
$$p(x, t+\delta t) = \frac{1}{2}\left(p(x-\delta x, t) + p(x+\delta x, t)\right).$$

Expand the right-hand side in a Taylor series, use $\delta x = \sqrt{\delta t}$ to show that

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}.$$

Explain why $p(x, 0) = \delta(x)$ and hence find $p(x, t)$ from the previous exercise.

The $\frac{1}{2}$ in front of the second derivative in the heat equation is a diagnostic feature for a probabilist as distinct from a 'physical' applied mathematician.

10. **Regular part of the Green's function for the Laplacian.** A horizontal membrane stretched over a region D is stretched to tension T and a normal force f per unit area is then applied. The displacement (which, like the force, is measured vertically upwards) is zero on the boundary ∂D. Show that the displacement $u(x, y)$ of the membrane satisfies

$$T\nabla^2 u = -f \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D.$$

Suppose that $f(x, y) = \delta(\mathbf{x} - \xi)$ where $\mathbf{x} = (x, y)$ and $\xi = (\xi, \eta)$ is known. How is $u(x, y; \xi, \eta)$ related to the Green's function for the Laplacian in D?

Do not worry about the infinite displacement!

Now suppose that the force is due to a very heavy ball which is free to roll around, and that it is in equilibrium at ξ. Suppose that we model its effect by that of a point force. Take a small square centred on $\mathbf{x} = \xi$ and resolve forces in the x– and y–directions to show that the gradient of the regular part of G vanishes at $\mathbf{x} = \xi$. Do you think there is always just one such equilibrium point? If not, when might you have one and when more than one?

Can you find a dimensionless parameter to quantify this modelling assumption?

Chapter 11

Theory of distributions

The time has come to look at the theoretical underpinning of the delta function and its relatives. You may choose not to read this section, but I promise that it is not complex or technically demanding. We begin with a few (as few as we can get away with) necessary definitions.

11.1 Test functions

We noted earlier that the proper way to approach $\delta(x)$ was by thinking of the result of multiplying a suitably smooth function $\phi(x)$ and integrating to get $\phi(0)$. The first step in setting up a robust framework is to define a class of 'suitably smooth' functions, called *test functions*. We say that $\phi(x)$ is a test function if

- $\phi(x)$ is a C^∞ function. That is, it has derivatives of all orders at each point $x \in \mathbb{R}$.
- $\phi(x)$ has *compact support*: that is, it vanishes outside some interval (a, b).

Because every derivative of ϕ is itself differentiable, the derivatives are all continuous and bounded.

The first of these requirements makes these functions very smooth indeed.[1] This high degree of regularity guarantees a trouble-free ride for the theory, the reason being that if $\phi(x)$ is a test function, then so are all its derivatives.

We should note that test functions do exist (and that we never need to know much more than this: they are a background tool). The easiest way to see this is to construct one, using the famous example of a function which

[1] Roughly speaking, only real analytic functions (defined as equal to the sum of a convergent Taylor series) are smoother, and they can never be test functions because they cannot have compact support (why not?).

has derivatives of all orders, and hence a Taylor series, at $x = 0$, but which is not equal to the sum of its Taylor series. That is, look at

See the exercise on page 148. Perhaps those pathological real-analysis examples were more useful than I thought.

$$\Phi(x) = \begin{cases} 0 & x \leq 0, \\ e^{-1/x} & x > 0, \end{cases}$$

which vanishes for $x \leq 0$, is positive for $x > 0$, and is C^∞. The only thing wrong with this function is that it does not have compact support. To fix this up, just multiply by, say, $\Phi(1-x)$:

$$\phi(x) = \Phi(x)\Phi(1-x)$$

is a perfectly good test function with support on the interval $(0, 1)$.

We also need a definition of convergence for a sequence of test functions $\{\phi_n(x)\}$. We say that $\phi_n(x) \to 0$ as $n \to \infty$ if

- $\phi_n(x)$ and all its derivatives $\phi_n^{(m)}(x)$ tend to zero, uniformly in both x and m;
- There is an interval (a, b) containing the support of all the ϕ_n.

The first of these is an incredibly strong form of convergence: the ϕ_n have no room to wriggle at all. The second stops them from running away to infinity as n increases.

The only other thing to say about test functions is that we shall denote them by lower case greek letters, usually ϕ or ψ.

11.2 The action of a test function

Suppose that $f(x)$ is an integrable[2] function (we denote such functions by lower case roman letters f, g, etc.). We define the *action* of f on a test function $\phi(x)$ by

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx.$$

It's also a bit like an inner product: but note that f and ϕ lie in different spaces.

So, this action is a kind of weighted average of $f(x)$. If we know the action of f on all test functions, we should know all about f itself (a bit like recovering a probability distribution from its moments). It is obvious that the action, regarded as a map from the space of test functions to $\mathbb{R}$, satisfies the usual linearity properties, such as

$$\langle f, a\phi + b\psi \rangle = a\langle f, \phi \rangle + b\langle f, \psi \rangle,$$

[2] We sidestep the question of what we mean by this, exactly. Piecewise continuous will do for now.

for real constants a, b. Also, if $\phi_n(x) \to 0$ in the sense defined above, then $\langle f, \phi_n \rangle \to 0$ as a sequence of real numbers.

11.3 Definition of a distribution

In defining distributions, we use the very mathematical idea of taking things we already know about, here functions, and dropping some of their properties while retaining others in order to obtain something broader or more general. In this way, we see that distributions are indeed 'generalised functions', despite the inexplicable reluctance of some to use the term.

As foreshadowed above, the properties that we want to keep are those to do with the action of a function on a test function; that is, we keep the 'smoothing' idea of averaging while quietly dropping all worries about pointwise definition. We do this is such a way that all the properties of distributions are *consistent* with the corresponding properties of (say) piecewise continuous functions. Then, all such functions are subsumed within the larger class of distributions.

Measurable functions would be better, but that requires too much machinery.

The two properties that we keep are those we reached at the end of the previous section: linearity and continuity. We define: *a distribution $\mathcal{D}$ is a continuous linear map from the space of test functions to $\mathbb{R}$, denoted by*

$$\mathcal{D} : \phi \mapsto \langle \mathcal{D}, \phi \rangle \in \mathbb{R}.$$

The result of the map, $\langle \mathcal{D}, \phi \rangle$, is known as the *action* of $\mathcal{D}$ on ϕ. We say that two distributions are equal if their action is the same for all test functions.

The properties of linearity and continuity are as above:

$$\langle \mathcal{D}, a\phi + b\psi \rangle = a\langle \mathcal{D}, \phi \rangle + b\langle \mathcal{D}, \psi \rangle,$$

for real constants a, b, and

$$\text{if} \quad \phi_n(x) \to 0 \quad \text{as} \quad n \to \infty, \quad \text{then} \quad \langle \mathcal{D}, \phi_n \rangle \to 0.$$

Evidently any piecewise continuous function $f(x)$ corresponds to a distribution $\mathcal{D}_f$ with the obvious action $\langle \mathcal{D}_f, \phi \rangle = \langle f, \phi \rangle$. Indeed, we normally don't bother to write $\mathcal{D}_f$, but just use f itself. This is an example of the consistency referred to above.

We shall mostly use the letter style of $\mathcal{D}$, $\mathcal{H}$ to denote distributions, unless they already have a name. The set of test functions is often called [script D, need typeface for this] and the set of distributions is then written [script D prime]. Sometimes we write $\mathcal{D}(x)$ to emphasise the dependence on x; the dependence is of course in the test functions, but it's quite OK, and indeed a good idea, to think of distributions as depending on x as well.

Example: the delta function. There could be no better example than the delta distribution, δ or $\delta(x)$. It is defined as a distribution by its action on a test function $\phi(x)$:

$$\langle \delta, \phi \rangle = \phi(0).$$

We could also have written

$\langle \delta(x), \phi(x) \rangle = \phi(0)$

You should check carefully that this action does indeed define a distribution satisfying the properties above. Again, it is OK, and indeed a good idea, to think intuitively of the action of the delta function as

$$\langle \delta, \phi \rangle = \int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx.$$

However, you should always use the formal definition to prove anything about $\delta(x)$ or any other distribution.

11.4 Further properties of distributions

If our distributions are to be useful, we need to give them some more properties. We assume that, if $\mathcal{D}$ and $\mathcal{E}$ are distributions, a is a real constant, $\phi(x)$ is a test function and $\Phi(x)$ is a C^∞ function (not necessarily a test function), then there are new distributions $\mathcal{D} + \mathcal{E}$, $a\mathcal{D}$, $\mathcal{D}(x - a)$ and $\mathcal{D}(ax)$ such that

- $\langle \mathcal{D} + \mathcal{E}, \phi \rangle = \langle \mathcal{D}, \phi \rangle + \langle \mathcal{E}, \phi \rangle$;
- $\langle a\mathcal{D}, \phi \rangle = a\langle \mathcal{D}, \phi \rangle$;
- $\langle \mathcal{D}(x - a), \phi(x) \rangle = \langle \mathcal{D}(x), \phi(x + a) \rangle$;
- $\langle \mathcal{D}(ax), \phi(x) \rangle = \dfrac{1}{|a|}\langle \mathcal{D}, \phi(x/a) \rangle$.
- $\langle \Phi(x)\mathcal{D}(x), \phi(x) \rangle = \langle \mathcal{D}(x), \Phi(x)\phi(x) \rangle$.

Note how we slip in and out of stating the x–dependence explicitly.

Watch out for the modulus sign.

Note that $\Phi(x)\phi(x)$ is a test function even if $\Phi(x)$ is not.

You should check all these when $\mathcal{D}$ corresponds to an integrable function $f(x)$; it will give you intuition as to why the definitions have been made in this way. Note in particular that from the third definition, we have

$$\begin{aligned} \langle \delta(x - a), \phi(x) \rangle &= \langle \delta(x), \phi(x + a) \rangle \\ &= \phi(a). \end{aligned}$$

As expected, we have recovered the sifting property of the delta function.

11.5 The derivative of a distribution

One more idea completes our introduction to the distributional framework. If we want to make sense of ideas such as $d^2y/dx^2 = \delta(x-\xi)$, we had better have a definition of the derivative of a distribution. Again, consistency with ordinary functions provides the way in. If $f(x)$ is differentiable, with derivative $f'(x)$, then integrating by parts we calculate the action of $f'(x)$:

What properties of test functions do we use here?

$$\begin{aligned}\langle f'(x), \phi(x)\rangle &= \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx \\ &= f(x)\phi(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,dx \\ &= -\langle f(x), \phi'(x)\rangle.\end{aligned}$$

We define the *derivative* $\mathcal{D}'$ *of a distribution* $\mathcal{D}$ in terms of its action by

$$\langle \mathcal{D}', \phi\rangle = -\langle \mathcal{D}, \phi'\rangle$$

(note that $\phi'(x)$ is also a test function). The point is that although we do not know about $\mathcal{D}'$, we do know about $\mathcal{D}$, so we can calculate $\langle \mathcal{D}, \phi'\rangle$ and hence $\langle \mathcal{D}, \phi\rangle$.

For example, let us show that $\mathcal{H}'(x) = \delta(x)$. We define the Heaviside function $\mathcal{H}(x)$ by its action:

$$\langle \mathcal{H}, \phi\rangle = \int_0^{\infty} \phi(x)\,dx;$$

this is entirely consistent with our view of $\mathcal{H}(x)$ as the unit step function since

$$\mathcal{H}(x)\phi(x) = \begin{cases} 0, & x \le 0, \\ \phi(x), & x > 0. \end{cases}$$

Now consider the action of $\mathcal{H}'(x)$:

$$\begin{aligned}\langle \mathcal{H}'(x), \phi(x)\rangle &= -\langle \mathcal{H}(x), \phi'(x)\rangle \\ &= -\int_0^{\infty} \phi'(x)\,dx \\ &= \phi(0) \\ &= \langle \delta(x), \phi(x)\rangle.\end{aligned}$$

Since their actions are identical, we conclude that $\mathcal{H}'(x) = \delta(x)$ (as distributions).

We can extend this definition recursively, to give action of the m–th derivative of $\mathcal{D}$ as

$$\langle \mathcal{D}^{(m)}(x), \phi(x) \rangle = (-1)^m \langle \mathcal{D}, \phi^{(m)}(x) \rangle$$

for $m = 1, 2, 3, \ldots$. Because every derivative of a test function is a test function, we see that distributions have derivatives of all orders too, an example of the technical simplicity of this theory.

11.6 Extensions of the theory of distributions

We conclude with an overview (a glimpse, really) of two vital extensions of the theory just outlined.

11.6.1 More variables

It is a very straightforward business to define distributions in the context of functions of several variables. We first define test functions to have compact support and to be C^∞ in all their arguments. Then, we define distributions as continuous linear maps from this space of test functions to $\mathbb{R}$. In particular, the delta function satisfies

$$\langle \delta(\mathbf{x}), \phi(\mathbf{x}) \rangle = \phi(\mathbf{0}).$$

The partial derivatives of a distribution $\mathcal{D}(\mathbf{x})$ are defined recursively using the formula

$$\langle \frac{\partial \mathcal{D}}{\partial x_i}, \phi \rangle = -\langle \mathcal{D}, \frac{\partial \phi}{\partial x_i} \rangle.$$

Again, D has derivatives of all orders, and because the mixed partial derivatives of the test functions are always equal, so are the mixed partials of $\mathcal{D}$. Thus, identities such as $\nabla \wedge \nabla \mathcal{D} \equiv \mathbf{0}$ are automatically true for distributions. The whole theory is splendidly robust, and we need have no qualms at all about writing down equations such as $\nabla^2 G = \delta(\mathbf{x} - \xi)$.

11.6.2 Fourier transforms

Space does not permit a full description of the theory of Fourier transforms of distributions in one or more variables. Nevertheless, here is an outline. For technical reasons, we use a slightly different class of test functions, which are still C^∞ but no longer have compact support. Instead, they and all their derivatives decay faster than any power of x as $x \to \pm\infty$. In principle, this defines a different class of distributions (known as *tempered distributions* — the compact support ones are *Schwartz*[3] *distributions*), but we won't notice the difference.

The new test functions can be shown to have the nice property that if $\phi(x)$ is a test function then so is its Fourier transform; this is why we use this class of test functions. We write the transform as[4]

See the exercise on page 150 to see why this would not be so for compact support test functions.

$$\hat{\phi}(k) = \int_{-\infty}^{\infty} \phi(x) e^{ikx} dx.$$

This is just the usual Fourier transform; we write the inverse transform as

$$\check{\psi}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(k) e^{-ikx} dk,$$

and we recall the standard results

$$\widehat{\frac{d\phi}{dx}} = -ik\hat{\phi}, \qquad \widehat{x\phi} = -i\frac{d\hat{\phi}}{dk},$$

the first of which is established by integration by parts and the second by differentiation under the integral sign.

Let's see what the action of the Fourier transform of an ordinary function is on a test function. The Fourier transform of a tempered distribution $\mathcal{D}$ is then defined to be consistent with this; as ever, we look at its action and transfer the work to the test function. A formal calculation gives

You might want to write this out, swapping the dummy variables x and k in the second line.

$$\begin{aligned}\langle \hat{f}, \phi \rangle &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) e^{ikx} dx \right) \phi(k)\, dk \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi(k) e^{ikx} dk \right) f(x)\, dx \\ &= \langle f, \hat{\phi} \rangle.\end{aligned}$$

We therefore define

$$\langle \hat{\mathcal{D}}, \phi \rangle = \langle \mathcal{D}, \hat{\phi} \rangle,$$

and similarly we define the inverse by

Check this one for an ordinary function.

[3]Rather to my surprise, Schwartz, who invented the theory in 1944, died as recently as the time of writing.

[4]Beware: notations differ, both in the signs in the exponent and in the placement of the 2π which can appear in the exponent, or symmetrically as $1/\sqrt{2\pi}$ multiplying both the transform and its inverse. The definition here is probably the commonest among applied mathematicians.

$$\langle \check{\mathcal{D}}, \phi \rangle = \langle \mathcal{D}, \check{\phi} \rangle.$$

Notice how important it is that $\hat{\phi}$ should be a test function too. If it were not, we could not be confident that some of these actions are defined at all. Notice too that the factors of 2π don't appear here: they are all hidden in the inverse of ϕ.

Using these deceptively simple formulas, we can prove that the Fourier transform of the derivative $\mathcal{D}' = d\mathcal{D}/dx$ is $-ik\hat{\mathcal{D}}$:

Line 1 is the definition of the transform; line 2 is the distributional derivative; line 3 is a standard identity; in line 4 we swap x for k and shift it to the first argument of the action.

$$\begin{aligned}\langle \widehat{\mathcal{D}'}, \phi \rangle &= \langle \mathcal{D}', \hat{\phi} \rangle \\ &= -\langle \mathcal{D}, d\hat{\phi}/dk \rangle \\ &= -\langle \mathcal{D}, \widehat{ix\phi} \rangle \\ &= \langle -ik\hat{\mathcal{D}}, \phi \rangle\end{aligned}$$

as required. It is an exercise for you to prove that the transform of $x\mathcal{D}$ is $-ikd\hat{\mathcal{D}}/dk$.

We end this section by finding the transforms of $\delta(x)$ and 1. (Yes, 1 has a Fourier transform in this theory; so do x, $|x|$, etc.).[5] The transform of $\delta(x)$ must surely be 1: informally,

Very informally, because e^{ikx} is not a test function, although one could 'truncate' it by multiplying by a test function which is small for $|x| > R$ and taking $R \to \infty$.

$$\int_{-\infty}^{\infty} \delta(x) e^{ikx} dx = e^{ik0} = 1.$$

Formally,

$$\begin{aligned}\langle \hat{\delta}, \phi \rangle &= \langle \delta, \hat{\phi} \rangle \\ &= \hat{\phi}(0) \\ &= \int_{-\infty}^{\infty} \phi(x)\, dx \\ &= \langle 1, \phi \rangle\end{aligned}$$

so we do indeed have

$$\hat{\delta}(k) = 1.$$

For the inverse, we have

$$\begin{aligned}\check{\delta} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(k) e^{-ikx} dk \\ &= \frac{1}{2\pi},\end{aligned}$$

[5]The transforms of sums of delta functions are the characteristic functions of discrete random variables.

so taking the transform of both sides, remembering that $(\check{\delta})\hat{} = \delta$, we get

$$\hat{1}(k) = 2\pi\delta(k).$$

You may like to show this from the formal definitions alone, using the fact that for test functions $\langle 1, \check{\phi}\rangle = 2\pi\langle 1, \hat{\phi}\rangle$.

The heat equation

We conclude with an example: it's one we have seen before but we do it in a different way. Consider the initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = \delta(x).$$

This time we'll take a Fourier transform in x. The equation for $\hat{u}(k,t)$ is

$$\frac{\partial \hat{u}}{\partial t} = -k^2\hat{u}, \qquad -\infty < k < \infty, \quad t > 0,$$
$$u(x,0) = \hat{\delta}(k) = 1.$$

The solution is

$$\hat{u}(k,t) = e^{-k^2 t},$$

and inversion by any of a number of methods (see the exercise on page 151) yields the answer

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}.$$

Sources and further reading

The theory of distributions in its modern form was developed by Schwartz [37]; the epsilonological approach is exemplified by Lighthill's book [25]. My description of the modern theory is heavily based on the very approachable book by Richards & Youn [34] (my main quibble with that book is the intrusive 2π in the exponent of the Fourier Transform).

If the idea of extending our definition of functions to make sense of the result

$$\int_{-1}^{1} \frac{dx}{x^2} = -2$$

appeals to you then you should definitely read [34].

Exercises

1. **Constructing delta functions from continuous functions I: by the Lebesgue Dominated Convergence Theorem.** Suppose that

$f(x) \in L^1$ is continuous and $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Take a test function $\phi(x)$ and show that, as $\epsilon \to 0$,

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) \phi(x)\, dx \to \phi(0),$$

as follows. First show that

$$I_\epsilon = \int_{-\infty}^{\infty} f(s)\phi(\epsilon s)\, ds.$$

Next, show that

$$|f(s)\phi(\epsilon s)| < M\, |f(s)|$$

for some constant $M > 0$, that if $f(s) \in L^1$ then $f(s)\phi(\epsilon s) \in L^1$, and that, for each s, $f(s)\phi(\epsilon s) \to f(s)\phi(0)$ as $\epsilon \to 0$. Deduce from the Dominated Convergence Theorem that you can justify interchanging the limit and the integral:

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(s)\phi(\epsilon s)\, ds = \phi(0).$$

2. **Constructing delta functions from continuous functions II: by splitting the range of integration.** If you don't know about Lebesgue integration, derive the following slightly weaker result. Suppose that $f(x)$ is any continuous function with

$$\int_{-\infty}^{\infty} f(x)\, dx = 1, \qquad \int_{-\infty}^{\infty} |f(x)|\, dx < \infty, \quad \int_{-\infty}^{\infty} |xf(x)|\, dx < \infty.$$

Take a test function $\phi(x)$ and show that, as $\epsilon \to 0$,

$$I_\epsilon = \int_{-\infty}^{\infty} \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) \phi(x)\, dx \to \phi(0),$$

as follows. First write $x = \epsilon s$ in the integral and split the range of integration up to get

$$I_\epsilon = \int_{-\infty}^{-1/\sqrt{\epsilon}} + \int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} + \int_{1/\sqrt{\epsilon}}^{\infty} f(s)\phi(\epsilon s)\, ds.$$

Noting that $|\phi(x)|$ is bounded and using the idea that if $|h| < c$, $|\int gh| \leq \int |gh| \leq c \int |g|$, show that the first and third integrals tend to zero as $\epsilon \to 0$ because f is integrable. For the inner integral, expand $\phi(\epsilon s)$ using Taylor's theorem to get

$$\int_{-1/\sqrt{\epsilon}}^{1/\sqrt{\epsilon}} f(s)\Big(\phi(0) + \epsilon s \phi'\left(\xi(s)\right)\Big)\, ds$$

where $\xi(s)$ lies between 0 and s. Show that the first term in this integral tends to what we want and, noting that $|\phi'|$ is bounded, that the second tends to zero as $\epsilon \to 0$.

3. **Delta sequences.** Consider the functions

$$f_n = \frac{n}{\pi\left(1+n^2x^2\right)} \quad \text{and} \quad g_n = \frac{\sin nx}{\pi x}.$$

Sketch them and show that f_n tends to $\delta(x)$ as $n \to \infty$, in the distributional sense, so for any test function ϕ,

$$\langle f_n, \phi\rangle \to \phi(0)$$

as $n \to \infty$. Use the method of the previous question, but be careful when estimating the integrals as f_n does not satisfy all the conditions of that question. Repeat for g_n.

This might suggest that if $\delta_n(x)$ is a sequence tending to $\delta(x)$ then $\delta_n(0) \to \infty$. Construct a piecewise constant example to show that this is false.

4. **Discrete and continuous sources.** Suppose that $u(\mathbf{x})$ is a classical solution of $\nabla^2 u = f(\mathbf{x})$ in $\mathbb{R}^n$, $n \geq 2$, where $f(\mathbf{x})$ is smooth and has compact support, and appropriate growth conditions at infinity are assumed. Let $\phi(\mathbf{x})$ be a test function. Use Green's theorem in the form

$$\int_D v\nabla^2 w - w\nabla^2 v = \int_{\partial D} v\frac{\partial w}{\partial n} - w\frac{\partial v}{\partial n}, \tag{11.1}$$

where D is a region containing the support of f, to show that

$$\langle u, \nabla^2\phi\rangle = \langle f, \phi\rangle.$$

Now suppose that we approximate $f(\mathbf{x})$ by delta functions, defining the sequence of distributions

$$\mathcal{F}_n = \sum_1^n \alpha_n \delta(\mathbf{x} - \mathbf{x}_n)$$

and taking the limit $n \to \infty$ in such a way that all the weights α_n tend to zero but

$$\langle \mathcal{F}_n, \phi \rangle \to \langle f, \phi \rangle$$

for all test functions ϕ. Also let u_n be the solution of $\nabla^2 u_n = \mathcal{F}_n$. Show that

$$\langle u_n, \nabla^2 \phi \rangle = \langle \mathcal{F}_n, \phi \rangle,$$

and deduce that $u_n \to u$ (as a distribution). Interpret this result in terms of the gravitational potential due to a finite mass distribution (or in electrostatic terms).

5. **The function $e^{-1/x}$.** Consider

$$\Phi(x) = \begin{cases} 0 & x \le 0, \\ e^{-1/x} & x > 0. \end{cases}$$

Remember that $X^n e^{-X} \to 0$ as $X \to \infty$ for all N.

Show that for $x > 0$ its n–th derivative $\Phi^{(n)}(x)$ is a polynomial in $1/x$ times $e^{-1/x}$, and hence that $\lim_{x \downarrow 0} \Phi^{(n)}(x) = 0$. Deduce that the Taylor coefficients of this function are all zero. Does the complex function $e^{-1/z}$ have a Taylor series at $z = 0$? If not, what does it have?

6. **The distribution $\delta(ax)$.** Show from the interpretation as an integral that

$$\delta(ax) = \frac{1}{|a|} \delta(x).$$

7. **Derivatives of the delta function.** Show carefully, using the definition of a distributional derivative, that, if $\Psi(x)$ is a smooth (C^∞) function and $\mathcal{D}$ a distribution, then $(\mathcal{D}\Psi)' = \mathcal{D}'\Psi + \mathcal{D}\Psi'$ (Leibniz). Deduce that

$$x^n \delta^{(m)}(x) = \begin{cases} 0 & m < n, \\ \dfrac{(-1)^n m!}{(m-n)!} \delta^{(m-n)}(x) & m \ge n \end{cases}$$

($\delta^{(m)} = m$th derivative). What is $x\delta(x)$? Show that $\delta(x) = -x\delta'(x)$.

8. **Convergence of series of distributions.** We say that a sequence $\{\mathcal{D}_n\}$ of distributions converges to $\mathcal{D}$ if

$$\langle \mathcal{D}_n, \phi \rangle \to \langle \mathcal{D}, \phi \rangle$$

for all test functions $\phi(x)$. This is an incredibly tolerant form of convergence, because our definition of convergence of a sequence of test functions is so stringent: show that if $\mathcal{D}_n \to \mathcal{D}$, then the same applies to all the derivatives, so that $\mathcal{D}_n^{(m)} \to \mathcal{D}^{(m)}$. Show also that you can differentiate a convergent series of distributions term by term.

Find the Fourier series of the sawtooth function

$$f(x) = \begin{cases} \frac{1}{2} - \frac{x}{2\pi} & 0 < x < \pi, \\ -\frac{1}{2} - \frac{x}{2\pi} & -\pi < x < 0. \end{cases}$$

Now differentiate both sides, noting that the jumps of 1 in $f(x)$ at $x = 2n\pi$ contribute delta functions $\delta(x - 2n\pi)$, to establish the result

$$\sum_{n=-\infty}^{\infty} \delta(x - 2n\pi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \cos mx,$$

an identity which makes no classical sense but perfect distributional sense.

Note: it can be shown that every distribution $\mathcal{D}$ is the distributional limit of a sequence of test functions (which are C^∞). So the set of distributions is not unboundedly diverse.

9. **Derivative of a distribution.** Let $\mathcal{D}(x)$ be a distribution. Show (by considering its action) that

Remember that

$\langle \mathcal{D}(x+h), \phi(x) \rangle = \langle \mathcal{D}(x), \phi($

$$\mathcal{D}'(x) = \lim_{h \to 0} \frac{\mathcal{D}(x+h) - \mathcal{D}(x)}{h}.$$

Use the right-hand side of this equation to confirm (again by considering the action) that $\delta(x) = \mathcal{H}'(x)$.

10. **Dipoles.** The derivative of the delta function, $\delta'(x)$, is known as a (one-dimensional) *dipole*, which you can think of as the limit as $\epsilon \to 0$ of a positive delta function at $x = \epsilon$ and a negative one at $x = 0$ (see the previous exercise). What is its action on a test function $\psi(x)$?

In hydrodynamics, a mass dipole aligned with the x–axis is obtained as the limit of point (in two dimensions, line) sources of strength q at $(\pm\epsilon, 0, 0)$, keeping the product $m = 2\epsilon q$ constant as $\epsilon \to 0$. Explain why the velocity potential for inviscid irrotational flow with a point source at the origin satisfies

Notation clash! ϕ is not a test function here.

$$\nabla^2 \phi = q\delta(\mathbf{x})$$

and deduce that if there is a dipole as above at the origin, the potential satisfies

$$\nabla^2 \phi = m\frac{\partial \delta}{\partial x}.$$

(The right-hand side may also be written as $\delta'(x)\delta(y)\delta(z)$ in three dimensions, or $\delta'(x)\delta(y)$ in two.) Hence calculate the potential for a dipole and sketch the streamlines in two dimensions. Show that the potential $U(r\cos\theta + a^2\cos\theta/r)$ for flow past a cylinder consists of a uniform flow plus a dipole.

Interpret these results in terms of electric charges. (Whereas point charges generate electric fields, because there are no magnetic monopoles, the basic generator of magnetic fields is the infinitesimal current loop, giving a dipole field with lines of force similar to those of a bar magnet. Higher-order derivatives, called multipoles, are important in, for example, the analysis of the far field of radio transmitters.)

11. **Vector distributions.** Develop the following two ways of defining vector-valued distributions in $\mathbb{R}^3$. In both cases aim to establish the identities $\nabla\cdot\nabla\wedge\boldsymbol{\mathcal{D}} \equiv 0$, $\nabla\wedge\nabla\mathcal{D} \equiv \mathbf{0}$ for vector and scalar distributions $\boldsymbol{\mathcal{D}}$ and $\mathcal{D}$ respectively. You will need to establish variants of Green's theorem in order to define the action of the operators div and curl by integration by parts.

    (a) Take scalar test functions $\phi(\mathbf{x})$ and define their action on a vector function $\mathbf{v}(\mathbf{x})$ as the vector

    $$\langle \mathbf{v}, \phi\rangle = \int_{\mathbb{R}^{\nvDash}} \mathbf{v}(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x}.$$

    Then define a vector-valued distribution $\boldsymbol{\mathcal{D}}$ as a continuous linear map from the space of test functions to $\mathbb{R}^3$ consistently with this action.

    (b) Use vector test functions $\boldsymbol{\phi}$ and the action

    $$\langle \mathbf{v}, \boldsymbol{\phi}\rangle = \int_{\mathbb{R}^3} \mathbf{v}\cdot\boldsymbol{\phi}\,d\mathbf{x}.$$

12. **Open support test functions.** To get an idea why compact support test functions do not lead to a good theory for the distributional Fourier transform, work out the Fourier transform of

    It's not hard: just integrate.

    $$f(x) = \begin{cases} 1 & -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and observe that, unlike $f(x)$, $\hat{f}(k)$ does not have compact support. (Although $f(x)$ is not a test function, a similar result would hold if it were.) Now look at the definition of the Fourier transform to see why compact support test functions are not useful here.

13. **Commutation of the Fourier transform and its inverse.** Show directly from the definitions that if $\mathcal{D}$ is a distribution with Fourier transform $\hat{\mathcal{D}}$, then
$$(\hat{\mathcal{D}})\check{} = (\check{\mathcal{D}})\hat{} = \mathcal{D},$$
assuming that this holds for test functions.

14. **The inverse of e^{-k^2t}.**

    Find the inverse of $\hat{u}(k,t) = e^{-k^2t}$ in the following two ways.

    (a) Write down the inversion integral and complete the square in the exponent; then, thinking of the integral as a contour integral in the complex k–plane, move the integration contour to the line $\operatorname{Im} k = -x/2t$ (check that the endpoint contributions vanish) and evaluate a standard real integral, using the result $\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$.

    (b) Show that $\partial\hat{u}/\partial k = -2kt\hat{u}$, then use the standard identities for the transforms of $\partial u/\partial x$ and xu to obtain a similar ordinary differential equation for u; solve this and choose the 'constant of integration' (which is actually a function of t) to set $\int_{-\infty}^{\infty} u(x,t)\, dx = 1$ for all t (which is easy to show from the original problem).

15. **The pseudofunction $1/x$.** Obviously, $1/x$ is defined for $x \neq 0$ as an ordinary function. Its definition for all $x \in \mathbb{R}$ is achieved by defining its action on a test function $\phi(x)$:
$$\langle 1/x, \phi(x)\rangle = \lim_{\epsilon\to 0} \langle 1/x, \phi(x)\rangle_\epsilon,$$
where
$$\langle 1/x, \phi(x)\rangle_\epsilon = \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}\, dx;$$
the limiting integral, denoted by
$$⨍_{-\infty}^{\infty} \frac{\phi(x)}{x}\, dx,$$
is called a *Cauchy principal value integral.*

Show that the limit exists for all test functions $\phi(x)$. Show directly from the distributional definitions that

$$\frac{1}{x} = \frac{d}{dx}\log|x|;$$

that is, show that

$$\langle d\log|x|/dx, \phi(x)\rangle = -\langle 1/x, d\phi/dx\rangle$$

by considering the same statement with $\langle\cdot,\cdot\rangle$ replaced by $\langle\cdot,\cdot\rangle_\epsilon$ and letting $\epsilon \to 0$.

Show also (for future reference) that

$$\int_{-1}^{1} \frac{dx}{x} = 0. \tag{11.2}$$

16. **The Fourier transform of $\mathcal{H}(x)$.** A distribution $\mathcal{D}(x)$ is called *odd* if the result of its action gives $\mathcal{D}(-x) = -\mathcal{D}(x)$, and *even* if $\mathcal{D}(-x) = \mathcal{D}(x)$. Show that $\delta(x)$ is even. Show also that $x\delta(x) = 0$. If $\mathcal{H}(x)$ is the Heaviside function, show that $\widetilde{\mathcal{H}}(x) = \mathcal{H}(x) - \frac{1}{2}$ is odd.

    Show that the Fourier transform of a real-valued odd function is a purely imaginary odd function of k, and deduce (or assert) that the same applies to distributions.

    Since $\mathcal{H}'(x) = \delta(x)$, taking the Fourier transform gives

    $$-ik\widehat{\mathcal{H}} = \hat{\delta} = 1.$$

    However, before dividing through by k, we must realise that we can add $ck\delta(k)$ $(= 0)$ to the right-hand side, where c is an as yet unspecified complex constant. By considering instead the transform of the odd distribution $\widetilde{\mathcal{H}}(x)$, and recalling that $\hat{1} = 2\pi\delta(k)$, show that

    $$\widehat{\mathcal{H}}(k) = -\frac{1}{ik} + \pi\delta(k).$$

    Note that $\widehat{\mathcal{H}}$ requires the definition of $1/k$ introduced in the previous exercise.

17. **The Fourier transform of $\mathcal{H}(x)$ again.** Here are two more ways of calculating $\widehat{\mathcal{H}}(k)$.

(i) Consider

$$\int_0^{1/\epsilon} e^{ikx}dx = -\frac{1}{ik} + \frac{e^{ik/\epsilon}}{ik}.$$

The first part is already in the answer, so the second part must tend to $\pi\delta(k)$ as $\epsilon \to 0$. Write

$$\frac{e^{ik/\epsilon}}{ik} = \frac{\sin(k/\epsilon)}{k} - i\frac{\cos(k/\epsilon)}{k}$$

and note that the real part has been shown (in the exercise 'delta sequences') to give $\pi\delta(k)$. It remains to show that the principal value integral

$$⨍_{-\infty}^{\infty} \frac{\cos(k/\epsilon)\phi(k)}{k}\,dk \to 0$$

as $\epsilon \to 0$ for any test function ϕ. Write ϕ as the sum of its even and odd parts and note that we need only consider the odd part of ϕ as the integral of the even part vanishes by symmetry. Now proceed as in earlier exercises, splitting the range of integration into $|k| > \sqrt{\epsilon}$ and $|k| < \sqrt{\epsilon}$ and dealing with each separately. Alternatively, don't bother with the odd/even split, and just use (11.2) for the inner integral.

They are $\frac{1}{2}(\phi(k) \pm \phi(-k))$; show that both of these are test functions.

Use the decay properties of the test function to justify use of the Riemann–Lebesgue lemma for the outer integrals, and expand ϕ in a Taylor series for the inner one.

(ii) Consider

$$\mathcal{H}_\epsilon(x) = \mathcal{H}(x)e^{-\epsilon x},$$

which clearly has a Fourier transform for $\epsilon > 0$; show that it is

$$\widehat{\mathcal{H}}_\epsilon(k) = \frac{1}{\epsilon - ik}.$$

Writing

$$\frac{1}{\epsilon - ik} + \frac{1}{ik} = \frac{\epsilon}{\epsilon^2 + k^2} - \frac{i\epsilon^2}{k(\epsilon^2 + k^2)},$$

show that as $\epsilon \to 0$ the action of the right-hand side on a test function tends to that of $\pi\delta(k)$. (You will need to interpret the second term as a principal value integral; use the results of the earlier exercise 'delta sequences'.)

Note that this 'does the right sort of thing' as $\epsilon \to 0$: it tends to $-1/ik$ for $k \neq 0$, and to infinity for $k = 0$.

18. **More Fourier transforms.** What are the Fourier transforms of

$$x, \quad x^n, \quad |x|,$$

for integral $n > 0$?

Remember that $\widehat{(xf)} = -id\hat{f}/dk$.

Chapter 12

Case study: the pantograph

12.1 What is a pantograph?

In the late 1960s, British Rail was planning a new generation of high speed electric trains. One question was flagged as a potential problem area: could the waves generated in the overhead cable by the current-collecting device, which is called a *pantograph*,[1] build up and cause interruptions in the current flow, in particular when the train passes a support? At about the same time the US Air Force developed a facility in which a rocket slung from a taut cable was accelerated along the cable, to allow flight characteristics to be tested and to enable precise targeting for impact tests. In one such test, the rocket was accelerated to 1.04 times the wavespeed in the wire, with the dramatic result sketched in Figure 12.1.

The pantograph problem was one of the first to be discussed at an Oxford Study Group with Industry and has become a minor classic of mathematical modelling since the original paper [30]. As so often with industrial problems, it provoked a strand of theoretical research, into the so-called pantograph equation, which is still active.

[1] From the Greek, 'universal writer'. The original pantograph was a mechanical device whereby a series of linkages allowed a drawing to be copied exactly. The theory of linkages was at one time intensively studied because of their importance in machinery, the prototype problem being how to transform the up-and-down motion of a piston into circular motion of a wheel. It is related to Ptolemy's epicycle model of the heavens, in which the apparently irregular motion of the planets was accounted for by the assumption that they move around the earth in an arrangement of small circular orbits mounted on larger ones, much like various fairground rides (teacups, waltzer, cyclone). Alas, light pollution is such that fewer and fewer readers will have seen the planets except when they are very bright.

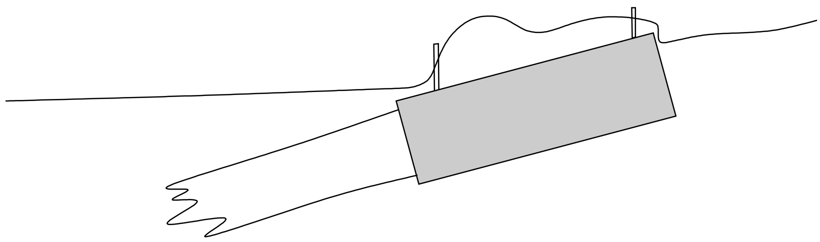

Figure 12.1: A rocket slung from a cable and accelerated to 1.04 times the wavespeed in the cable. Sketch based on an indistinct photograph in [36]; the cable is fairly accurately reproduced but the rest of the diagram is schematic.

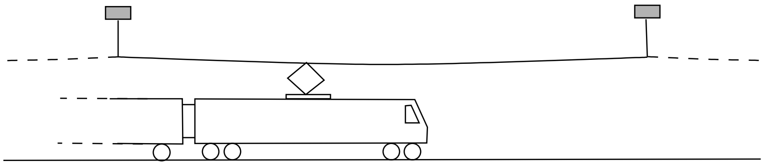

Figure 12.2: A locomotive and its power supply.

12.2 The model

Suppose, then, that a locomotive moves with constant speed U below a cable stretched to tension T between supports at $x = 0, \pm L, \pm 2L, \ldots$, as in Figure 12.2. We can distinguish three main parts of the system that we should model: the motion of the cable, the dynamics of the pantograph, and the dynamics of the supports. Some modelling assumptions are immediately reasonable:

- The cable can be modelled as a uniform string of line density ρ, because (based on typical values) its bending stiffness is small. Also its displacement from equilibrium is small. Thus, apart from the static displacement due to gravity, the vertical wire displacement $y(x,t)$ satisfies the wave equation away from the pantograph and the supports. (In a more sophisticated treatment, we may have to reconsider this assumption near to the pantograph and supports.)

- The contact area between the pantograph and the wire is small compared with L. This means that we can represent the effect of the pantograph by a point force at the position $x = Ut$, with a suitable choice of time origin.

We also make some simplifying assumptions, which can be relaxed at the cost of some complications:

- Although the supports can be surprisingly complex (see [30]), we assume that the wire is rigidly attached to them. This has the great advantage that what happens in one span of the cable does not affect what happens in the others. In our discussion below, we focus on the span $0 < x < L$, into which the train enters at $t = 0$.

- The pantograph itself can be modelled as a linear system, so that the force $F(t)$ that it exerts on the wire depends linearly on its vertical displacement $Y(t) = y(Ut, t)$. More specifically, we may expect

  – A spring force, intended to keep the pantograph in contact with the wire. For a linear spring this would contribute a term
$$F_0 - F_1 Y(t)$$
  to the force $F(t)$; here both F_0 and F_1 are positive and we expect that the combination $F_0 - F_1 Y$ is also positive for reasonable values of Y.

  – A damping force, which in the linear case has the form
$$-F_2 \frac{dY}{dt}.$$

  In the case of a rocket, a separate calculation of its dynamics adds a term proportional to d^2F/dt^2; see Exercise 8

Just as in our earlier examples, the point force is modelled by a delta function; the difference now is that it is moving. The motion of the wire is described by the inhomogenous wave equation

$$\rho \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = F(t)\delta(x - Ut) - \rho g,$$

where the last term models the gravitational force. With both spring and damping forces, the pantograph dynamics are modelled by

$$F(t) = F_0 - F_1 Y(t) - F_2 \frac{dY}{dt},$$

where $Y(t) = y(t, t)$. As for initial and boundary conditions, the wire starts at rest in its equilibrium shape $y_0(x) = -\rho g x(L-x)/2T$, and its displacement vanishes at $x = 0, L$.

Let us make this problem dimensionless. There are two velocities, U and the wavespeed $c = \sqrt{T/\rho}$, and we use the latter for scaling purposes, which

It hardly matters which we use. The choice of c is governed by some obscure aesthetic considerations of my own.

with the length scale L for x gives a timescale L/c; also, we write $U = cu$ (you might think of u as a Mach number for the train). In order to scale y, we can either use the maximum wire displacement under gravity, $y^*\rho g L^2/8T$, or we can use the displacement caused by a typical pantograph force. As we want to focus on the pantograph, let us use the latter and scale y with F^*L/T, where F^* is a typical size for the pantograph force (it might, for example, be equal to the constant force F_0). With these scalings, you should check that, the primes having been dropped, the dimensionless problem is

In doing the scalings, remember that $\delta(ax) = |a|^{-1}\delta(x)$.

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = f(t)\delta(x - ut) - \alpha, \quad 0 < x < 1, \tag{12.1}$$

where, retaining the notation $Y(t) = y(ut, t)$, the dimensionless force has the form

An exercise to work out the dimensionless coefficients f_0 etc in terms of their dimensional parents.

$$f(t) = f_0 - f_1 Y - f_2 \frac{dY}{dt} \tag{12.2}$$

and $\alpha = \rho g L/F^*$ is a dimensionless parameter measuring the ratio of the weight of the wire to the force exerted by the pantograph.

12.2.1 What happens at the contact point?

Looking ahead to calculating the displacement of the wire, we are clearly going to rely heavily on the general solution of the wave equation in the usual form. That means that we will need to join solutions of this type up across the train path $x = ut$, so we need to know what happens to the gradient of y across this line. We should proceed with caution when we see something unfamiliar like $\delta(x - ut)$: it is not immediately obvious what it means. One fairly safe way to proceed is to change coordinates to reduce it to rest. That is, we replace x by $\xi = x - ut$ and use ξ and tb as independent variables. When we do this, a straightforward chain rule calculation shows that (12.1) becomes

We started this chapter by writing down jump conditions on the (static) wire slope and went from there to the delta function; now the boot is on the other foot as we are confident that the delta function should be there but we don't know how to interpret it!

$$\frac{\partial^2 y}{\partial t^2} - 2u\frac{\partial^2 y}{\partial \xi \partial t} - (1 - u^2)\frac{\partial^2 y}{\partial \xi^2} = f(t)\delta(\xi) - \alpha.$$

Now we are in a position to balance the most singular terms. We know that y is continuous at $\xi = 0$ and assuming smoothness in t, $\partial^2 y/\partial t^2$ should also be continuous. The finger points at $\partial^2 y/\partial \xi^2$ as the most singular term, and we see that this has for its leading order singular behaviour a delta function of magnitude $-f(t)/(1 - u^2)$. That is, $\partial y/\partial \xi$, which is the same as $\partial y/\partial x$, has a jump of this magnitude:

Marginal note: as the right-hand side of this equation is a distribution, so is the left-hand side. However, the chain rule still applies for smooth coordinate changes.

$$\left[\frac{\partial y}{\partial x}\right]_{x=ut-}^{x=ut+} = -\frac{1}{1 - u^2} f(t), \tag{12.3}$$

At least for $U < c$; Figure 12.1 suggests otherwise for $U > c$!

Please check for

which is the time-dependent generalisation of the static condition (10.2) on page 112. A physical derivation of this condition is given in Exercise 1

12.3 Impulsive attachment for an undamped pantograph

The simplest situation to consider is one in which gravity is neglected ($\alpha = 0$) so that the cable is initially straight, and at $t = 0$ the train attaches to the cable impulsively at $x = 0$. In this case we expect disturbances to propagate ahead of and behind the train with (dimensionless) speed 1 so that the cable displacement is only nonzero for $-t < x < t$ as shown in the characteristic diagram of Figure 12.3.

Our strategy is to join together general solutions of the wave equation, of the form $g(t-x) + h(t+x)$, finding the arbitrary function involved from the conditions at the pantograph. Clearly the wire displacement is identically zero except in the regions 1, $ut < x < t$ and 2, $-t < x < ut$, shown in Figure 12.3. Otherwise, information would have to travel faster than the wavespeed. Across the characteristics $x = \pm t$, we expect to see a discontinuity in the derivatives of y, as we know that these can only propagate along charcteristics.

You may usually write $g(x-t)$; I do. It turns out that $g(t-x)$ is more convenient later, as we don't then get negative arguments for the function g_1.

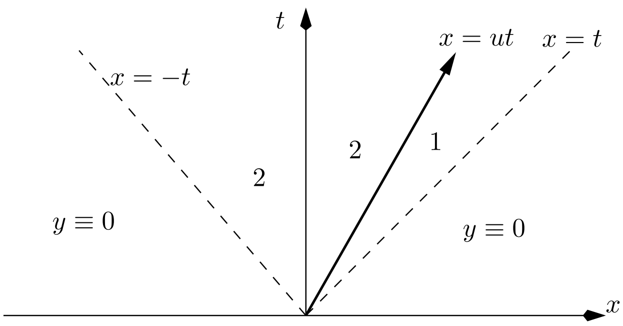

Figure 12.3: Characteristic diagram for impulsive attachment.

Bearing in mind that all the information comes from the train, the solution must have the form

$$y(x,t) = \begin{cases} g_1(t-x) & \text{in region 1} \\ h_2(t+x) & \text{in region 2,} \end{cases}$$

representing forward and backward travelling waves respectively. The functions g_1 and h_2 are as yet unknown, except that we can say that $g_1(0) = h_2(0) = 0$.

At the train $x = ut$, we first express the continuity of the cable:

$$g_1(t - ut) = h_2(t + ut). \tag{12.4}$$

Next, from (12.3), we have

$$-g_1'(t - ut) - h_2'(t + ut) = -\frac{1}{1 - u^2} f(t).$$

Using (12.2) to express $f(t)$ in terms of $Y(t) = g_1(t - ut)$ and eliminating $h_2(t + ut)$ by differentiating (12.4), we have

$$\begin{aligned} g_1'(t - ut) &= \frac{1}{2(1 - u)} f(t) \\ &= \frac{1}{2(1 - u)} \big(f_0 - f_1 g_1(t - ut) - (1 - u) f_2 g_1'(t - ut)\big). \end{aligned}$$

That is, $g_1(\xi)$ satisfies the ordinary differential equation

$$(1 - u)(2 + f_2)\frac{dg_1}{d\xi} + f_1 g_1 - f_0 = 0,$$

whose solution is easily found as a constant plus a decaying exponential. (The large-time behaviour has the pantograph displacement tending to the value f_0/f_1 at which the spring force vanishes; in practice this would be very large.)

12.4 Solution near a support

A rather more surprising thing happens when we look at what happens shortly after the train passes a rigid support. As the characteristic diagram 12.4 shows, there are again only two regions where the cable displacement is not zero. Let us again neglect the static displacement of the cable (this is even more realistic near a support where it is small). The difference between this configuration and impulsive attachment is that waves can be reflected off the rigid support, in region 2. Thus the cable displacement has the form

$$y(x, t) = \begin{cases} g_1(t - x) & \text{in region 1} \\ g_2(t - x) + h_2(t + x) & \text{in region 2.} \end{cases}$$

Continuity of the cable at $x = ut$ now gives

$$g_1(t - ut) = g_2(t - ut) + h_2(t + ut), \tag{12.5}$$

and the pantograph force balance is

$$-g_1'(t-ut)+g_2'(t-ut)-h_2'(t+ut) \tag{12.6}$$
$$= -\frac{1}{1-u^2}f(t)$$
$$= -\frac{1}{1-u^2}\left(f_0 - f_1 g_1(t-ut) - f_2(1-u)g_1'(t-ut)\right) \tag{12.7}$$

Lastly, we have the rigid support condition

$$g_2(t)+h_2(t)=0.$$

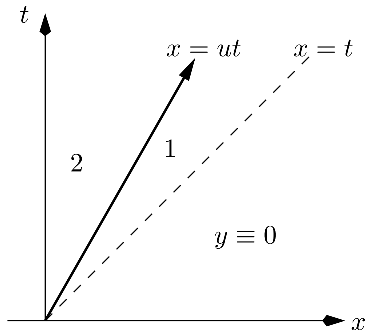

Figure 12.4: Characteristic diagram for the train passing a support.

With three equations for three unknown functions, we proceed confidently. We first eliminate h_2 to find

$$g_1(t-ut)=g_2(t-ut)-g_2(t+ut)$$

from (12.5), and

$$-g_1'(t-ut)+g_2'(t-ut)+g_2'(t+ut) = -\frac{1}{1-u^2}\left(f_0 - f_1 g_1(t-ut) - (1-u)f_2 g_1'(t-ut)\right)$$

from (12.7). Then we observe that we can eliminate $g_1(t-ut)$ throughout, to give (after tidying up)

$$(1+u)(2+f_2)g_2'(t+ut)+f_1 g_2(t+ut) = -f_0 + f_1 g_2(t-ut) + (1-u)f_2 g_2'(t-ut). \tag{12.8}$$

If we can solve this equation, we will have g_2 and hence g_1 and the force on the pantograph. The left-hand side of (12.8) is as expected, but the right-hand side is not. Because it contains the function g_2 evaluated at an *earlier* time than on the left-hand side, we have arrived not at an ordinary differential equation but a kind of *delay differential equation* for g_2. This kind of equation has come to be known as a *pantograph equation* and has given rise to a substantial literature in the last two decades.

Let us consider (12.8) in the special case $f_1 = 0$. It can then immediately be integrated once, to give

$$(2 + f_2)g_2(t + ut) = -f_0 t + f_2 g_2(t - ut).$$

Writing $\tau = t(1 + u)$ and

$$\mu = \frac{1 - u}{1 + u} < 1,$$

we have

$$(2 + f_2)g_2(\tau) = -\frac{f_0}{1 + u}\tau + f_2 g_2(\mu\tau).$$

We spot the particular solution $g_2(\tau) = a\tau$ where a is easily found, and we claim that this is the only solution. To show this, consider the difference between two solutions, which satisfies the homogeneous equation

$$g(\tau) = \frac{f_2}{2 + f_2} g(\mu\tau), \quad g(0) = 0;$$

note that the fraction on the right is less than 1. Suppose that for a fixed τ, $g(\tau) = g_0 \neq 0$. That means that $g(\mu\tau)$ is bigger in modulus than g_0 and, iterating, that

$$g(\mu^n\tau) = \left(\frac{2 + f_2}{2}\right)^n g_0.$$

But as $n \to \infty$, $\mu^n\tau \to 0$ and $|g(\mu^n\tau)| \to \infty$, a contradiction. Hence $g_0 = 0$ and the solution is unique.

It is possible to transform (12.8) to a delay differential equation with a constant time lag by making an exponential substitution; see Exercise 4.

12.5 Solution for a whole span

Let us look briefly at the cable motion in a whole span. This is the case we really need to analyse, because of the possibility that the solution will 'pile up' at the far end of the span $x = 1$. The characteristic diagram is now much more complicated, as indicated in Figure 12.5. The initial disturbance propagates as a gradient discontinuity across the characteristic $x = t$, as in the previous section, but then it reflects off the support at $x = 1$. We

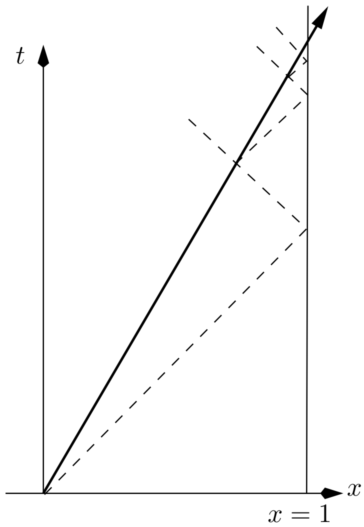

Figure 12.5: Characteristic diagram for a whole span. If, as here, $\frac{1}{3} < u < 1$, no reflections of characteristics from $x = 0$ reach the train.

therefore have a reflected characteristic, $x = 2 - t$, with another gradient discontinuity; and this in turn is reflected off the train path and so on, to generate an infinite series of characteristics separating regions in which the solution is smooth.

We have already looked at the regions near the first support, and we know from the previous section that when the pantograph force is a constant plus a linear damping term, the pantograph displacement $Y(t) = y(ut, t)$ is linear in t. This solution holds up until the time $t_1 = 2/(1+u)$ at which the reflection ($x = 2 - t$) of the leading characteristic ($x = t$) meets the train path. It is a reasonable guess that the pantograph displacement is a piecewise linear function of t at later times, and we can show that this is the case.

We adopt a slightly different, and more sophisticated approach to the wave equation than simply writing down its general solution. In Figure **??**, we see the train path with its images in the supports. We intend to extend the domain of definition from $0 < x < 1$ to the whole real line in the usual way so as to satisfy the conditions at the support; this means that the (fictitious) pantograph force from alternate, downward-sloping, parts of the image train path is minus that from the upward-sloping parts, and so is the pantograph displacement. The extended train path is shown in Figure 12.6, which also shows a characteristic triangle, which we call $\triangle$, for a point P above the reflection of the leading characteristic (recall that we already know the solution below this line).

Now suppose that we integrate the wave equation (12.1) (again with $\alpha = 0$ for simplicity) over the interior of the characteristic triangle $\triangle$, in the spirit

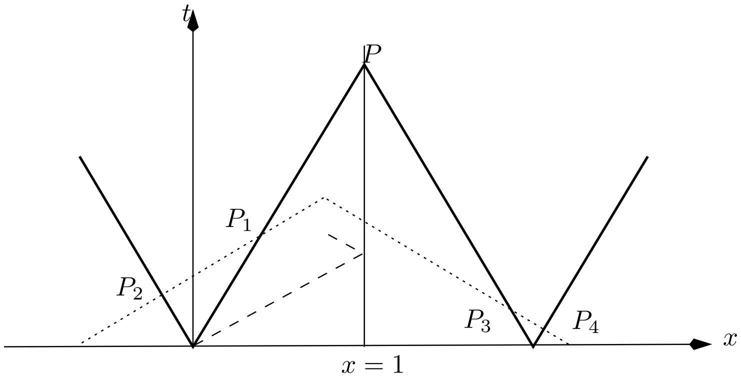

Figure 12.6: A characteristic triangle for $t > 2/(1+u)$. The dotted lines are characteristics; the dashed line is the leading characteristic and its reflection. The thick solid line is the train path with its images in the supports.

of Exercise 6. We has better be careful about the jumps across the train path, so for safety we integrate separately over each of the polygons that make up the triangle and consider integrals along both sides of the train path. Using Green's theorem, this gives

$$\int_\triangle \frac{\partial y}{\partial x}\,dt + \frac{\partial y}{\partial t}\,dx + \int_{\text{trainpath}} \left[\frac{\partial y}{\partial x}\right]\,dt + \left[\frac{\partial y}{\partial t}\right]\,dx = 0,$$

where as before the square brackets denote the jump in their contents. The first integral simply gives us $2y(P) = 2y(x,t)$. On the train path, we know that y itself has no jump, so

$$\left[\pm u\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t}\right] = 0,$$

with $+$ on the upward-sloping parts of the path and $-$ on the others. This lets us eliminate $[\partial y/\partial t]$, and then to use the fact that $dx = \pm u\,dt$ and the pantograph jump condition to show that

A factor of $1-u^2$ cancels.

$$2y(P) = \int_{\text{trainpath}} \pm f(s)\,ds,$$

where again the $\pm$ takes account of the image forces.

The next step is to let P approach the train path, and to calculate the t-coordinate of the points P_2, P_3 and P_4 (P_1 coincides with P). This is easily done, and results in

$$\begin{aligned} 2y(ut,t) &= 2Y(t) \\ &= -\int_0^{\mu t} + \int_0^{t} - \int_0^{t/\mu - 2/(1-u)} + \int_0^{t-2/(1+u)} f(s)\,ds, \end{aligned}$$

where $\mu = (1-u)/(1+u)$ is as above. If we differentiate this with respect to t, we can eliminate dY/dt on the left-hand side in favour of the pantograph force, using (12.2) (without the linear spring term: otherwise we get a genuine pantograph equation). This results in

$$\frac{2(f_0 - f(t))}{f_2} = -\mu f(\mu t) + f(t) - \frac{1}{\mu} f\left(\frac{t}{\mu} - \frac{2}{1-u}\right) + f\left(t - \frac{2}{1+u}\right), \quad (12.9)$$

again a delay equation, but now with *four* delays.

Fortunately, there is some structure to the solution. It is easy to show that the points at which the reflected characteristics in Figure 12.5 meet the train path are

$$t = t_n = \frac{1}{u}(1 - \mu^n).$$

Clearly, these are the only places at which any sort of discontinuity in $f(t)$ can occur. Furthermore, it is easy to see that, for $n > 1$,

$$t_n - \frac{2}{1+u} = \mu t_{n-1}, \quad \frac{t_n}{\mu} - \frac{2}{1-u} = t_{n-1}.$$

So, if t in (12.9) is equal to one of the t_n, then all the terms on the right-hand side occur in the same equation at $t = t_n$ or $t = t_{n-1}$. With all this going for us, we need only look for piecewise smooth functions between these points, and join them up across $t = t_n$, using (12.9) to relate the discontinuity at t_n to that at t_{n-1}. Simply writing (12.9) at $t = t_n\pm$ and subtracting the two,

$$\big[f(t)\big]_{t_n-}^{t_n+} = \frac{f_2}{\mu(2+f_2)} \big[f(t)\big]_{t_{n-1}-}^{t_{n-1}+},$$

and differentiating (12.9) gives the corresponding relation

$$\left[\frac{df}{dt}\right]_{t_n-}^{t_n+} = \frac{f_2}{\mu^2(2+f_2)} \left[\frac{df}{dt}\right]_{t_{n-1}-}^{t_{n-1}+}.$$

When you write out the right-hand side, you'll get f evaluated at $t_{n-1}\pm$, at $\mu t_n\pm$ and $\mu t_{n-1}\pm$. Only in exceptional cases could μt_n be equal to t_m for some $m < n$, so f is continuous ant these points and contributes nothing to the jump.

These are enough to determine a piecewise linear solution for values of t between the t_n, and it shows that the solution is well-behaved as $n \to \infty$ provided that $f_2/(\mu(2+f_2)) < 1$.

Sources and further reading

The description of the pantograph problem closely follows that of [39], but be careful: the notation is slightly different.

Exercises

1. **Conservation of momentum.** A string of line density ρ and tension T is pulled with speed $U < c = \sqrt{T/\rho}$ through a small frictionless ring as shown in Figure 12.7. Remembering that

$$\text{force} = \text{rate of change of momentum},$$

show that for small displacements the force on the ring is

$$F = -\rho(c^2 - U^2)\left[\frac{\partial y}{\partial x}\right]_-^+.$$

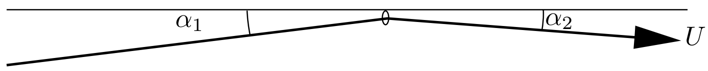

Figure 12.7: Conservation of momentum.

2. **Removing the static displacement.** Show that if we calculate the static displacement $y_s(x)$ of the cable and subtract it from $y(x,t)$, the difference $\bar{y}(x,t)$ satisfies the $\alpha = 0$ version of the problem but with an additional known time-dependent term in the force relation (12.2).

3. **Impulsive attachment of a point force.** A wire of density ρ is stretched to tension T. At time $t = 0$, the wire is atraight and motionless; a constant point force is implied impulsively at $x = 0$ and thereafter it moves with speed $U < c$. Draw a characteristic diagram, show that of the four regions in it, the wire displacement is only non-zero in region 1 ($Ut < x < ct$) and region 2 ($-ct < x < Ut$), and that the displacement there is of the form $g_1(t - x/c)$, $h_2(t + x/c)$ respectively. Apply the pantograph conditions (with a constant force) at $x = Ut$ to find these functions; sketch the wire displacement at a later time t. Repeat for $U > c$ and comment on the results.

   Repeat the exercise in the case that the wire is also subject to gravity, so that its initial (static) displacement is $y_s(x) = \frac{1}{2}\alpha x^2$. (Of course the wire is held up by distant supports.)

4. **Delay differential equations.** Consider the equation

$$y'(t) = \alpha_0 - \alpha_1 y(t - \tau), \quad t > 0,$$

where $\tau > 0$ is a constant, with the initial condition $y = 0$ for $-\tau \leq t < 0$ (this generalises the 'point' initial equation with no delay). Show that, with this initial condition, the Laplace transform of $y(t - \tau)$ is $e^{-p\tau}\bar{y}(p)$. Hence show that

$$\bar{y}(p) = \frac{\alpha_0}{p(p + \alpha_1 e^{-p\tau})}.$$

Put the right-hand side into partial fractions to deduce that the solution can be found as a constant plus a series of exponentials in t, involving the roots of $p + \alpha_1 e^{-p\tau} = 0$ (when $\alpha_1 > 0$ it can be shown that these have negative real parts so the associated exponentials decay). Confirm (for quality control) that you get the expected answer when $\tau = 0$.

Show that the substitution $t = e^s$ reduces the pantograph equation (12.8) to a more complicated constant-delay equation, defined on the whole real line.

5. **The solution for one span.** Complete the details of the solution of Section 12.5, finding the coefficients in the linear expression for $f(t)$ in each interval $t_n < t < t_{n+1}$. Include the static displacement of the wire (this generates a particular solution of the delay equation which you can subtract out).

6. **D'Alembert's solution to the wave equation.** Consider the initial value problem

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = G(x,t),$$

with initial conditions

$$y(x,0) = y_0(x), \quad \frac{\partial y}{\partial t}(x,0) = v_0(x), \qquad -\infty < x < \infty.$$

Write the left-hand side of the wave equation in divergence form and integrate over a characteristic triangle to derive the D'Alembert solution

$$y(x,t) = \tfrac{1}{2}\left(y_0(x-t) + y_0(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} v_0(s)\,ds + \tfrac{1}{2}\iint G(\xi,\tau)\,d\xi\,d\tau,$$

where the double integral is taken over the characteristic triangle.

Show that this solution is unique by considering the energy

$$E(t) = \tfrac{1}{2}\int_{-\infty}^{\infty} \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 dx.$$

7. **Pantograph with variable velocity.**

   (a) Suppose that a point force moves along the wire at $x = X(t)$, so that there is a jump in $\partial y/\partial x$ given by

$$\left[\frac{\partial y}{\partial x}\right] = -\frac{1}{1-(X'(t))^2}f(t).$$

   Bearing in mind that y is continuous, what is the corresponding jump in $\partial y/\partial t$? Modify the argument of Exercise 6 to show that if the wire is initially at rest,

$$y(x,t) = \frac{1}{2}\int f(\tau)\frac{1-\left(X'(\tau)\right)^2}{1-(X'(\tau))^2}\,d\tau,$$

   where the integral is along the part of $x = X(t)$ lying inside the characteristic triangle. (I know that the fraction is equal to 1. It's written like that because the two bits of it come from different places.)

   Why is this solution not valid if $X'(t) > 1$?

   (b) If $X = \frac{1}{2}at^2$ and $f(t) = 1$, show that for $0 < t < 1/a$, there is a region of the (x,t) plane in which

$$y(x,t) = t^*(x,t)/2,$$

   where $t^*(x,t)$ is the appropriate root of

$$x + (t - t^*) = \tfrac{1}{2}at^{*2}.$$

   (c) Define a distribution $\mathcal{D}(x,y) = \delta(x)f(y)$, where $f(y)$ is smooth. Explain why it is reasonable that

$$\int_0^1\int_0^1 \mathcal{D}(x,y)\,dx\,dy = \int_0^1 f(y)\,dy.$$

   How would you generalise this to $\mathcal{D}(x,y) = f(y)\delta(ax-by)$ for constant a and b, if the integral is over a general region? Use these ideas and Exercise 6 to derive the result of (a) from the equation of motion with a delta-function on the right-hand side.

8. **Dynamics of a rocket.** Consider a rocket of mass m slung from a long horizontal cable in the pantograph framework. Let its horizontal position be $x = X(t)$. Ignoring the static displacement of the cable, derive the dimensional model

$$\rho\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = F(t)\delta(x - X(t)), \quad 0 < x < \infty,$$

$$y(0,t) = 0, \quad y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0,$$

where

$$F(t) = -mg - m\frac{d^2Y}{dt^2}, \qquad Y(t) = y(t,t).$$

In the case $X(t) = Ut$ where U is constant, derive and solve an equation for the rocket displacement. When the rocket accelerates at a constant rate a, draw the characteristic diagram, indicating all the significant characteristics.

Part III

Asymptotic techniques

Chapter 13

Asymptotic expansions

13.1 Introduction

The rest of this book deals with systematic procedures to exploit small or large parameters in a dimensionless problem, a collection of ideas grouped together under the umbrella of *asymptotic analysis*. In this chapter, we open proceedings with the basics of what an asymptotic approximation is, following which we look at a selection of common techniques.

We start with a very simple example. Consider the quadratic

$$\epsilon x^2 + x - 1 = 0, \tag{13.1}$$

where ϵ is a fixed very small positive number, say 0.0000001. Forget for the moment that we know how to solve quadratics exactly: can we exploit the fact that ϵ is small to find approximate values for the roots? If $\epsilon = 0$, we have $x = 1$, and furthermore if we put $x = 1$ into the equation for small positive ϵ, the error, namely what remains on the left-hand side, is small; here it is ϵ. So, a natural first try is to write

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

where, obviously, $x_0 = 1$ (but it is reassuring to know that, as we see below, we can show this systematically). This kind of assumed form for an expansion is known as an *ansatz*; here we are assuming that ϵ crops up only in positive integral powers. We substitute this in:

$$\epsilon \left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right)^2 + x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots - 1 = 0,$$

then collect terms by powers of ϵ:

$$x_0 - 1 + \epsilon \left(x_1 + x_0^2\right) + \epsilon^2 \left(x_2 + 2x_0 x_1\right) + \cdots = 0.$$

Now we equate coefficients of successive powers of ϵ to zero. From ϵ^0,

$$x_0 - 1 = 0,$$

so $x_0 = 1$ as expected. From ϵ^1,

$$x_1 + x_0^2 = 0,$$

so $x_1 = -1$; from ϵ^2,

$$x_2 + 2x_0x_1 = 0,$$

so $x_2 = 2$, and so on. We have found x_0, x_1, x_2 recursively, giving us the approximation

$$x = 1 - \epsilon + 2\epsilon^2 + \cdots .$$

It is clear that we can carry on in this way to find as many terms as we like (and you can check the answer we get against the small–ϵ expansion of the quadratic formula).

But hold on! Don't quadratics have *two* roots? Where did we lose the other one? Well, one way to see is to look at what we did when we calculated x_0. In effect, we simply put $\epsilon = 0$ in (13.1), and so we lost the x^2 term which, of course, being the term of highest degree, tells us how many roots there are. Put another way, we said that the terms x and -1 must balance each other, leaving x^2 as a small correction, which we use to improve our solution iteratively. But is that the only possible balance? We can enumerate the other candidates:

- We might balance all three terms: this is obviously ridiculous, and in any case it is ruled out by our analysis above.

- We might balance ϵx^2 and -1, with x being smaller. This doesn't look so silly, but take it a bit further: if ϵx^2 balances -1, then the size of $|x|$ is $1/\sqrt{\epsilon}$ (which is large) plus a smaller correction. But then the term x, which was supposed to be small relative to ϵx^2 and -1, is in fact much bigger than either. It stands head and shoulders above the other two, with no counterbalance. We have not made the right choice.

- The only remaining possibility is to balance ϵx^2 and x.

If, then, ϵx^2 and x balance, we see that $|x|$ is of size $1/\epsilon$. So we *rescale*, writing

$$x = \frac{1}{\epsilon}X,$$

after which our quadratic (13.1) becomes

$$X^2 + X - \epsilon = 0,$$

confirming immediately that the third term (which was -1 before) is indeed small compared withy the other two. We expand X as above:

$$X = X_0 + \epsilon X_1 + \cdots ;$$

skipping the details (which you should work out), the lowest order terms clearly give

$$X_0^2 + X_0 = 0,$$

with the *two* roots

$$X_0 = -1 \quad \text{or} \quad X_0 = 0.$$

One root ($X_0 = 0$) simply reproduces the root we found earlier, while the root $X_0 = -1$ is the first term in the expansion of the one we did not find. It is left to you to calculate a couple more terms and verify that the expansions are correct by comparison with the exact solution.

Our example is mathematically trivial. However, it illustrates some important points about asymptotic approaches:

- When we equate coefficients of powers of ϵ to zero, we are in effect embedding our particular problem, with a given numerical value of ϵ, say 0.0000001, in a continuous set of problems for *all* ϵ in a small interval $[0, \epsilon^*)$ containing 0.0000001. If, as we hope, the dependence of the roots on ϵ has some smooth 'structure' as $\epsilon \to 0$, we should first be able to extract their general behaviour for all small ϵ, and only then reinstate our particular numerical value.

- Systematic approximation procedures start with the identification of the dominant balance(s) in an equation. Physical and mathematical intuition may both help in finding these balances as may iteration ideas (see the exercise on page 180); and once they are found, the remaining terms should be smaller corrections.

- It may be necessary to rescale some of the dependent or independent variables to achieve a balance.

Now it's time for some definitions.

13.2 Order notation

It is useful to have a way of writing down the idea that two functions are 'about the same size' near a point x_0 (usually 0 or ∞) or ϵ_0 (almost always 0, as ϵ is almost always used to denote a small parameter). We say:

$$f(x) = O(g(x)) \quad \text{as} \quad x \to x_0$$

if there is a constant A such that

$$|f(x)| \leq A\,|g(x)|$$

for all x sufficiently near x_0. So, for example,

$$3x + x^2 = O(x) \quad \text{as } x \to 0;$$

here any $A > 3$ will do. In our quadratic equation example, the roots $x_{(1)}$ and $x_{(2)}$ satisfy

$$x_{(1)}(\epsilon) = O(1) \quad \text{and} \quad x_{(2)}(\epsilon) = O(1/\epsilon)$$

as $\epsilon \to 0$. Successively more precise estimates for $x_{(1)}$ are

$$x_{(1)}(\epsilon) = 1 + O(\epsilon) \quad \text{and} \quad x_{(1)}(\epsilon) = 1 - \epsilon + O(\epsilon^2).$$

If we want a more specific estimate of the size of $f(x)$, we may try to find a function $g(x)$ whose 'leading order' behaviour is the same as that of $f(x)$. We write

$$f(x) \sim g(x) \quad \text{as} \quad x \to x_0$$

if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

So, $3x + x^2 \sim 3x$ as $x \to 0$, and

$$x_{(1)}(\epsilon) \sim 1 - \epsilon + 2\epsilon^2 \quad \text{as} \quad \epsilon \to 0.$$

Lastly there is a compact notation for the idea that one function is much smaller than another. We write

$$f(x) = o(g(x)) \quad \text{as} \quad x \to x_0$$

if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0;$$

this is often written $f(x) \ll g(x)$. So, for example,

$$e^{-x} = o(x^{-1}) \quad \text{as} \quad x \to \infty$$

and, for any n,

$$x^n \ll e^x \quad \text{as} \quad x \to \infty.$$

In our quadratic example,

$$x_{(1)}(\epsilon) = 1 - \epsilon + o(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

The order notation is most often used to quantify the error in an approximation, so that we know when we can safely use it. A good example is the remainder of the Taylor approximation (series). If we take $n+1$ terms of a Taylor series for $f(x)$ about x_0, the error is $o(x-x_0)^n$, and usually $O(x-x_0)^{n+1}$. For $n=0$,

$$f(x) = f(x_0) + O(x - x_0) \quad (\text{or } o(1));$$

for $n = 1$,

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + O(x - x_0)^2$$

(the error is also $o(x - x_0)$), and for $n = 2$,

$$f(x) \sim f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0)$$

with an error of $O(x - x_0)^3$ (or $o(x - x_0)^2$).

13.2.1 Asymptotic sequences and expansions

Suppose we are looking at a function of ϵ as $\epsilon \to 0$ (or any other limit point). We may aim to write its asymptotic behaviour in this limit in terms of simple functions of ϵ such as powers. A well-known example here is a power series in ϵ, if one exists, and we note that increasing powers of ϵ have an important property: each one is smaller than its predecessor, so that $\epsilon^{n+1} = O(\epsilon^n)$ as $\epsilon \to 0$. Less specifically, we may have non-integral powers, logs and so on, so we generalise this idea of using powers of ϵ by saying that a set of *gauge functions* $\{\phi_n(\epsilon)\}$, $n = 0, 1, 2, \ldots$, is an *asymptotic sequence* as $\epsilon \to 0$ if

$$\phi_{n+1}(\epsilon) = o\left(\phi_n(\epsilon)\right)$$

for all n. For example, $\{\epsilon^n\}$, $\left\{\epsilon^{\frac{n}{2}}\right\}$ are asymptotic sequences as $\epsilon \to 0$, while $\{e^{-nx}\}$ is an asymptotic sequence as $x \to \infty$. Making the right choice of asymptotic sequence for a specific problem is something of an art, albeit one in which common sense and simple iteration ideas play a large part.

Once we have an asymptotic sequence, we can expand functions. We say that $f(\epsilon)$ has an *asymptotic expansion* with respect to the asymptotic sequence $\{\phi_n(\epsilon)\}$ if there are constants a_k such that, for each n,

$$f(\epsilon) = \sum_{k=0}^{n} a_k \phi_k(\epsilon) + o\left(\phi_n(\epsilon)\right),$$

or

$$f(\epsilon) \sim \sum_{k=0}^{n} a_k \phi_k(\epsilon)$$

as $\epsilon \to 0$. In our quadratic equation example, we found the expansions

$$x_{(1)}(\epsilon) = 1 - \epsilon + 2\epsilon^2 + o(\epsilon^2)$$

with respect to the sequence $\{1, \epsilon, \epsilon^2, \ldots\}$, and

$$x_{(2)}(\epsilon) = \epsilon^{-1} + o(\epsilon^{-1})$$

with respect to the sequence $\{\epsilon^{-1}, 1, \epsilon, \epsilon^2, \ldots\}$; we were too lazy to calculate any more terms.

We very often have a function of several independent variables, here represented by a generic $\mathbf{x}$, and a small parameter ϵ. In such a case, we may look for an expansion in the form

$$f(\mathbf{x}; \epsilon) \sim \sum_{k=0}^{n} a_k(\mathbf{x}) \phi_k(\epsilon),$$

and we hope that the problem of calculating the a_k (sequentially) will be easier than finding $f(\mathbf{x}; \epsilon)$ all at one go. This the main reason for trying an asymptotic expansion in the first place.

13.3 Convergence and divergence

So what's the big deal: haven't we just found a straightforward generalisation of Taylor series? Well, no, not exactly. The point of a Taylor series is that, for a fixed value of ϵ, or whatever we've called our independent variable, as we take more and more terms the sum of the series gets closer and closer to the function it represents. That is, the series converges as the number of terms in the partial sums, n, tends to infinity. All Taylor series are thus *de facto* asymptotic expansions.

There are, however, *two* limiting processes going on when we write down an an asymptotic expansion, $n \to \infty$ and $\epsilon \to 0$, and they need not commute. When we do a Taylor (or Laurent) series expansion, we first take the limit as $n \to \infty$, and only then think what happens as ϵ varies. An asymptotic expansion, on the other hand, is designed to provide an accurate approximation as $\epsilon \to 0$ for each n, and many useful expansions don't converge at all as $n \to \infty$.

A famous example is the incomplete exponential integral:[1] evaluate

$$I(\epsilon) = \int_{1/\epsilon}^{\infty} x^{-1} e^{-x} dx.$$

Repeated integration by parts shows that

$$I(\epsilon) = \epsilon e^{-1/\epsilon} \left(1 - 1!\epsilon + 2!\epsilon^2 + \cdots + (-1)^n n!\epsilon^n\right) + R_n(\epsilon),$$

where $R_n(\epsilon)$ is easily shown to be asymptotically smaller than the last retained term; see the exercise on page 181.

It is quite clear that the series we have generated does not converge for *any* $\epsilon > 0$. That isn't the point, though. What *is* important is that as $\epsilon \to 0$ the series should give us an accurate description of the behaviour of the integral. That is, the smaller we take ϵ, the smaller should be the relative error of the approximation. In fact what happens is that, if we take a fixed value of ϵ, and take more and more terms in the expansion, at first successive terms get smaller and smaller (as they would for a convergent series); then, starting from values of n of $O(1/\epsilon)$, they increase again. The best approximation is given by cutting the series off at this optimal truncation point.[2] Even for $\epsilon = 1/4$, which is not particularly small, truncation of the series after 4 terms gives the reasonable approximation 0.00401... compared to 0.00378... from numerical integration. When $\epsilon = 1/8$, 8 terms of the series give $0.112434 \times e^{-1/8}$ compared with ... .

What are the gauge functions?

In most practically (as opposed to mathematically) generated asymptotic problems, we are unable to calculate enough terms to decide whether the asymptotic series is divergent or not. Indeed, it's usually next to impossible (or at least a rather strenuous exercise in mathematical weightlifting) trying to prove that the remainder after even one or two terms is small as it should be. We have to live with these lacunae: we proceed knowing that experience tells us that, mostly, things will work out.

For another asymptotic expansion which works well even when the relevant parameter, here n, is not small, try putting $n = 1$ in Stirling's formula, which says that $n! \sim n^{n+\frac{1}{2}} e^{-n}/\sqrt{2\pi}$ as $n \to \infty$.

[1]Approximation of integrals and special functions is a particularly happy hunting ground for asymptoticists, although we shan't be going that way much.

[2]The location of the optimal truncation point is determined by the value $n(\epsilon)$ at which successive terms have the same size. This series has a 'factorial-power' form for the terms in the expansion, a very general phenomenon, and it is easy to calculate that successive terms are closest in size when n is the integer part of $1/\epsilon$. The precise behaviour of the remaining error, and how to deal with it, is part of the trendy subject of *hyperasymptotics*, also known as *asymptotics beyond all orders* or *exponential asymptotics*.

Further reading

There are many excellent books on asymptotic expansions. A short but intensive introduction is the book of Hinch [18], whose first chapter probably put the quadratic example into my mind (if you think about it, $\epsilon x^2 + x - 1 = 0$ is the irreducible minimum of that kind of problem). If you are interested in ordinary differential equations try the book by O'Malley [32], or for a more wide-reaching treatment the books of Kevorkian & Cole [22] or Bender & Orszag [5]. Olver [31] is a good starting point for the analysis of special functions and, as ever, Carrier, Krook & Pearson [6] is very well worth reading.

Exercises

1. **Roots of a cubic.** Find expansions for the roots of

$$\epsilon x^3 + x - 1 = 0$$

as $\epsilon \to 0$ with at least two and preferably three nonzero terms in each expansion.

Draw graphs to see where the roots are.

Repeat for the real roots of $\epsilon x \tan x = 1$, and then for $x \tan x = \epsilon$. In the latter case you will have to consider the first root separately, as well as rescaling to get the large roots. In addition there is a range of roots you can't get approximations to; where is it?

2. **Iteration.**

Show that $x \log x \to 0$ as $x \to 0$; draw its graph. Suppose we want to find an asymptotic expansion for the solution to $x \log x = -\epsilon$, where $0 < \epsilon \ll 1$. In this case, it is not obvious what gauge functions we should use, so we find them by *iteration.* Write

$$x(\epsilon) \sim x_0(\epsilon) + x_1(\epsilon) + \cdots ,$$

where all we know is that $x_0 \gg x_1 \gg \cdots$. Take logarithms of the original equation (a key step, because it replaces multiplication of two small terms by addition of their (large) logarithms) and substitute this in to find

$$\log(x_0 + x_1 + \cdots) + \log(-\log(x_0 + x_1 + \cdots)) = \log \epsilon. \qquad (13.2)$$

Now ignore x_1 to show that $x_0 = \epsilon$. Put this back into (13.2) and expand the logarithms in powers of x_1/ϵ, which is small, to show that $x_1 = -\epsilon/|\log \epsilon|$ and calculate one more term in the expansion.

Repeat the calculation after making the simplifying initial scaling $x = \epsilon X$ (which you might not spot first time round).

3. **The exponential integral.** Show that our expression for the incomplete exponential integral is indeed an asymptotic expansion as follows. Consider
$$I(\epsilon) = \int_{1/\epsilon}^{\infty} x^{-1} e^{-x} dx.$$
Integrate by parts $n+1$ times to show that
$$I(\epsilon) = \epsilon e^{-1/\epsilon}\left(1 - 1!\epsilon + 2!\epsilon^2 + \cdots + (-1)^n n!\epsilon^n\right) + R_n(\epsilon),$$
and now integrate by parts once more to get
$$R_n = \epsilon e^{-1/\epsilon}(-1)^{n+1}(n+1)!\epsilon^{n+1} + R_n',$$
where a simple estimate using $e^{-x} \leq e^{-1/\epsilon}$ for $x \geq 1/\epsilon$ shows that R_n' is at most of the same order as the first term on the right-hand side of the expression for R_n. Conclude that, as $\epsilon \to 0$, $R_n = o(\epsilon^n e^{-1/\epsilon})$.

4. **Stockmarket crashes and six-sigma quality control.** The probability density function for the standard normal distribution $N(0,1)$ is
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$
Integrate by parts to find a one-term approximation for $P(X < x)$ as $x \to -\infty$ and show that it is asymptotically correct (see the previous exercise).

   'Six-sigma standards' in manufacturing demand that the probability that an individual component is defective is less than the probability of being 6 or more standard deviations away from the mean of a standard normal distribution. What is this probability, approximately? ($e^3 \approx 20$, $2\pi \approx \frac{25}{4}$.) If the manufacturers use this standard, what is the probability that none of the 10,000 components in a computer or the 1,000,000 components in an aeroplane is faulty?

   Hint: which well-known limit lets you work out an approximation to $(0.99)^{100}$?

   In the standard Black–Scholes model for financial markets, daily percentage changes in, say, the FTSE–100 or S&P–500 index are independent random variables which are approximately normal with very small mean and standard deviation of about 1%. What is the probability of a fall of 10% or more in one day? What is the probability of two such falls on consecutive days? (In October 1987 the UK stock market fell by more than 10% on both Black Monday and the day after. There have been several other changes of this magnitude in the (roughly) 25,000 days for which stock indices have been calculated.)

Chapter 14

Regular perturbations/expansions

14.1 Introduction

We begin our tour of asymptotic methods for simplifying complex problems with the most straightforward idea, that of a *regular asymptotic (or perturbation) expansion*. This is just the plain vanilla common-sense expansion you carry out when it seems that the dominant-balance terms in your model do indeed reflect the dominant physical mechanisms, and everything else is a small correction. For example, the expansion

$$x_{(1)}(\epsilon) = 1 - \epsilon + 2\epsilon^2 + o(\epsilon^2)$$

in our quadratic equation example of the previous chapter is beautifully regular. In some problems, we can characterise a regular expansion by saying that it is expected to be a *uniformly valid approximation* to the solution; for example, when we consider the standard model for waves on a string, we hope that, for all times and positions on the string, the wave equation is a good approximation to the fully nonlinear model we could write down for displacements that are not small. Having said this, there probably isn't a watertight definition of when an expansion is 'regular'; it may be safest just to leave it as 'any expansion that is not one of singular, boundary layer, multiple-scale, ...' and to let your sense of the meaning of the term grow with experience.

In the models we look at later on, we'll see a variety of scalings and transformations which help us to understand less straightforward situations; but after all these contortions, we end up with a regular expansion. When we've got to a regular expansion, nine times out of ten we've done as much simplifying as we can with asymptotic approximations. As a simple example, to find the other root $x_{(2)}(\epsilon)$ of the quadratic, we first had to introduce the singular (as $\epsilon \to 0$) scaling $x = X/\epsilon$, and only then find a regular expansion for X.

There is really not much more to say of a general nature. The rest of the chapter consists of a collection of examples of the regular perturbation technique in action. They are necessarily in order, because this is not a hypertext document, but they are not rigidly so: wander as you will.

14.2 Example: stability of a spacecraft in orbit

Many asymptotic techniques have their origin in astronomy. We begin with a simple one: the stability of circular planetary orbits. In the classical Newtonian model for motion of, say, a satellite or space station orbiting the earth, the satellite's plane polar coordinates $(r(t), \theta(t))$, with origin at the centre of the earth, satisfy

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}, \qquad \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,$$

where $\dot{} = d/dt$, G is the universal gravitational constant and M is the mass of the earth. Hence we retrieve Kepler's second (?) law

$$r^2\dot{\theta} = h,$$

a constant equal to the angular momentum (per unit mass, to be pedantic).

A circular orbit is an obvious solution with

$$r = a, \quad \dot{\theta} = \omega, \qquad \text{where} \qquad a^3\omega^2 = GM,$$

the relation between a and ω being Kepler's third (?) law. Suppose a booster rocket on the satellute gives it a small radial velocity ϵv (note that this does not change h). Is the orbit stable or will the satellite plunge to earth or fly off into the deeps of space?

Write

$$r(t) = a + \epsilon r_1(t) + \cdots, \quad \dot{\theta}(t) = \omega + \epsilon\dot{\theta}_1(t) + \cdots,$$

so that

$$\epsilon\ddot{r}_1 + \cdots - (a + \epsilon r_1 + \cdots)\left(\omega + \epsilon\dot{\theta}_1 + \cdots\right)^2 = -\frac{GM}{(a + \epsilon r_1 + \cdots)^2} \qquad (14.1)$$

and

$$(a + \epsilon r_1 + \cdots)^2 \left(\omega + \epsilon\dot{\theta}_1 + \cdots\right) = h = a^2\omega. \tag{14.2}$$

Expand the right-hand side of (14.1) by the binomial theorem, remember that $a^3\omega^2 = GM$, and the $O(\epsilon)$ terms in (14.1), (14.2) give

$$\ddot{r}_1 - \omega^2 r_1 - 2a\omega\dot{\theta}_1 = 0, \quad 2a\omega r_1 + a^2\dot{\theta}_1 = 0;$$

that is,

$$\ddot{r}_1 + \omega^2 r_1 = 0.$$

We see that r_1 oscillates without growing or decaying, so it looks as if the system is neutrally stable; this is not so surprising when we recall that the full system is conservative. The period of oscillation is equal to the original period of orbital rotation, and the perturbed orbit is slightly elliptical with the centre of the earth at one focus; the furthest and nearest distances from earth (apogee and perigee) occur $\frac{1}{4}$ and $\frac{3}{4}$ of an orbit after the initial thrust. Every half-orbit, the space station will return to the original location relative to earth: if the astronauts drop a spanner before applying the thrust, they will have two opportunities per orbit to reach out and grab it.

14.3 Linear stability

Linear stability analysis, of which the satellite problem is an example, is an archetypal example of a regular perturbation. We take a solution $\mathbf{u}_0$ of a system, often an equilibrium or a steady state such as a travelling wave, we perturb it to $\mathbf{u}_0 + \epsilon\mathbf{u}_1$, write down a regular perturbation expansion to determine $\mathbf{u}_1$ and see whether $\epsilon\mathbf{u}_1$ is small, or (in a time-dependent problem) remains small. The perturbation may be to the initial and/or boundary data of our problem, or to the geometry, or it may be structural via changes to the parameters or equations of the problem.

We say that a time-dependent system is *linearly stable* if a suitable norm (measure of the size) of the perturbation decays, *linearly unstable* if the norm grows, and (linearly) *neutrally stable* if it remains the same size.

In many systems the result of linearising about $\mathbf{u}_0$ is a linear evolution problem in the form

$$\frac{\partial \mathbf{u}_1}{\partial t} = \mathcal{L}\mathbf{u}_1$$

where $\mathcal{L}$ is a linear differential operator. If we are particularly lucky, $\mathcal{L}$ will have time-independent coefficients, and then the solution has the form

$$\mathbf{u}_1 = e^{\lambda t}\mathbf{U}_1$$

where $\mathbf{U}_1$, which is independent of t, is an eigensolution of $\mathcal{L}$ with eigenvalue λ:

$$\mathcal{L}\mathbf{U}_1 = \lambda \mathbf{U}_1$$

(we expect an eigenproblem because of the scaling invariance of a linear problem). This is all well illustrated by the very familiar phase-plane analysis we now briefly review.

14.3.1 Stability of critical points in a phase plane

You can do it in more than two dimensions, but two dimensions is much easier to analyse, because the dimension of the phase paths is one less than the dimension of the plane; in three or more dimensions the extra degree(s) of freedom make life much more difficult.

Phase-plane analysis of critical points is a classic example of linear stability analysis. Take a two-dimensional autonomous dynamical system

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2).$$

The *critical points* $\mathbf{x}_0$ are equilibrium points where $f(\mathbf{x}_0) = \mathbf{0}$. In order to analyse their stability, we first write down a regular expansion for $\mathbf{x}(t)$ about $\mathbf{x}_0$,

$$\mathbf{x} \sim \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \cdots ;$$

then, expanding $f(\mathbf{x})$ in a Taylor series about $\mathbf{x}_0$, at $O(\epsilon)$ we find a linear equation for $\mathbf{x}_1$:

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{J}\mathbf{x}_1,$$

It is de facto an asymptotic expansion provided that $|\mathbf{x} - \mathbf{x}_0| = O(1)$.

where $\mathbf{J}$, a constant matrix, is the Jacobian $\partial(f_1, f_2)/\partial(x_1, x_2)$ at $\mathbf{x}_0$. With a constant-coefficient equation, it is natural to look for a solution

$$\mathbf{x}_1 = e^{\lambda t}\mathbf{v}_1,$$

which reveals the eigenvalue equation

$$\mathbf{J}\mathbf{v}_1 = \lambda \mathbf{v}_1.$$

Here is a use for the canonical form reductions of linear algebra: we see a differential equation interpretation of the difference between algebraic and geometric multiplicity.

The eigenvalues are a conjugate pair: why?

The stability or otherwise of the fixed point is thus determined by the real parts of the eigenvalues of $\mathbf{J}$, as a positive real part for either will lead to exponential growth and hence instability. The details of the behaviour are surprisingly complicated, largely because of special cases when the eigenvalues are equal.[1] When they are distinct, things are easier and we have the familiar catalogue of possible behaviours: stable (unstable) nodes when both λ_1 and λ_2 are real and negative (positive); saddles for real eigenvalues of opposite signs; stable (unstable) spirals for complex eigenvalues with negative

[1] See [21].

(positive) real parts; and lastly centres when the eigenvalues are pure imaginary. Care is needed with the latter case which, unlike the rest, is clearly structurally unstable to small changes in the entries of **J**, as the following digressionary example shows.

14.3.2 Example (side track): a system which is neutrally stable but nonlinearly stable (or unstable)

Consider the two systems (one for $+$, one for $-$)

$$\dot{x} = y \pm x\left(x^2 + y^2\right), \tag{14.3}$$

$$\dot{y} = -x \pm y\left(x^2 + y^2\right). \tag{14.4}$$

If we look for solutions near the obvious equilibrium point $(0, 0)$, say with $x(0) = \epsilon\xi_0$, $y(0) = \epsilon\eta_0$, we can write $x = \epsilon X$, $y = \epsilon Y$, and then expand

$$X \sim X_0 + \epsilon^2 X_2 + \cdots, \quad Y \sim Y_0 + \epsilon^2 Y_2 + \cdots$$

(fairly clearly the $O(\epsilon)$ terms vanish). Then,

$$\dot{X}_0 = Y_0, \quad \dot{Y}_0 = -X_0,$$

and we have neutral stability since $X_0^2 + Y_0^2 = \xi_0^2 + \eta_0^2$ is constant. However, at $O(\epsilon^2)$ we find

$$\dot{X}_2 = \pm X_2\left(\xi_0^2 + \eta_0^2\right), \quad \dot{Y}_2 = \pm Y_2\left(\xi_0^2 + \eta_0^2\right),$$

and the $+$ system is plainly unstable at this order, while the $-$ system is stable. In fact this is in accordance with the exact result since $x \times (14.3) + y \times (14.4)$ gives

$$\frac{d}{dt}\left(x^2 + y^2\right) = \pm\left(x^2 + y^2\right)^2,$$

and it is an exercise to solve this equation and show that there is finite-time blow-up if we have the $+$ sign and existence for all t with the $-$ sign.

Clearly linear stability analysis is just the tip of the iceberg. Although it is easy to construct examples which are linearly stable and nonlinearly unstable, and vice versa, nonetheless as a general rule it is a good guide to the overall behaviour.

14.4 Example: the pendulum

Let us have a look at the pendulum model we introduced in Chapter **??**. Recall that the dimensionless pendulum model, without the primes on t, is

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \sin\theta = 0, \tag{14.5}$$

with

$$\theta = \alpha_0, \quad \frac{d\theta}{dt} = \beta_0 \qquad \text{at} \quad t = 0. \tag{14.6}$$

Of course, we can treat this equation via the phase plane (this relatively straightforward exercise is requested on page 195). However, the purpose of this chapter is to let you see the modus operandi of regular perturbations, so let's do this problem from scratch.

Suppose that α_0 is small, say $\alpha_0 = \epsilon a_0$ where $\epsilon \ll 1$, and $\beta_0 = 0$, so that we are releasing the pendulum from rest with only a small initial displacement.[2] Can we retrieve linear theory, and how big is the error?

Write

$$\theta \sim \theta_0 + \epsilon\theta_1 + \cdots . \tag{14.7}$$

Then it is obvious that $\theta_0 = 0$: it satisfies

$$\frac{d^2\theta_0}{dt^2} + \gamma\frac{d\theta_0}{dt} + \sin\theta_0 = 0,$$

with

$$\theta_0 = \frac{d\theta_0}{dt} = 0 \quad \text{at} \quad t = 0,$$

and the zero solution is unique by standard Picard theory.

With more experience, we would have seen this straightaway, and accounted for it by using the (regularly) scaled variable $\theta = \epsilon\tilde{\theta}$. However, let's press on. We now know that

$$\theta \sim \epsilon\theta_1 + o(\epsilon),$$

and so

$$\sin\theta \sim \epsilon\theta_1 + o(\epsilon).$$

Putting these two into (14.5), and retaining only the terms of $O(\epsilon)$, we find that

$$\frac{d^2\theta_1}{dt^2} + \gamma\frac{d\theta_1}{dt} + \theta_1 = 0,$$

[2] I could have chosen to expand in terms of α_0, instead of writing $\alpha_0 = \epsilon a_0$, and that might have looked less contrived. However, for continuity of exposition I want the small parameter to be called ϵ wherever possible.

while the $O(\epsilon)$ terms from the initial conditions (14.6) give

$$\theta_1 = a_0, \quad \frac{d\theta_1}{dt} = 0 \quad \text{at} \quad t = 0.$$

As promised, we have retrieved the linear theory.

As we noted in Chapter 13, what we have in effect done is to embed the problem for our particular value of α_0, say 0.001, in a family of problems parametrised by ϵ, and we are looking for an expansion valid for all ϵ in an interval containing $(0, 0.001)$. We hope that the solution depends smoothly on α_0 (and hence on ϵ) when α_0 is small, so that our procedure of expanding in powers of ϵ is justified. Indeed, if the solution is differentiable with respect to ϵ at $\epsilon = 0$, then we are just identifying a function by its Taylor series. Even if this is not the case, we hope that *the asymptotic expansion gives a good approximation to the solution as* $\epsilon \to 0$, the key requirement of such a representation.

14.5 Small perturbations of a boundary

In this section, we look at two problems in which the perturbation to a simple solution is induced by a small irregularity in the boundary of the domain in which we solve, rather than in the field equation itself.

14.5.1 Example: flow past a nearly circular cylinder

Suppose that we want to calculate potential flow past the slightly elliptical cylinder whose equation in plane polar coordinates is $r = a(1+\epsilon\cos\theta)$, where $\epsilon \ll 1$, and with velocity $(U, 0)$ at infinity. The velocity potential ϕ satisfies

A focus-directrix representation of an ellipse with small eccentricity ϵ is

$$a/r = 1 - \epsilon\cos\theta,$$

and expanding to $O(\epsilon)$ gives $r = a(1+\epsilon\cos\theta)$.

$$\nabla^2\phi = 0, \quad r > a(1+\epsilon\cos\theta), \tag{14.8}$$

with the boundary conditions

$$\frac{\partial\phi}{\partial n} = \mathbf{n}\cdot\nabla\phi = 0, \quad r = a(1+\epsilon\cos\theta) \qquad \phi \sim Ur\cos\theta + o(1), \quad r \to \infty. \tag{14.9}$$

We know the solution when $\epsilon = 0$, namely

$$\phi_0 = U\left(r\cos\theta + \frac{a^2}{r}\cos\theta\right), \tag{14.10}$$

and it seems very likely that the solution for $0 < \epsilon \ll 1$ is close to this. The only obstacle is that for $\epsilon > 0$ the boundary condition $\partial\phi/\partial n = 0$ is applied in an inconvenient place. We deal with this by *linearising* it onto $r = a$: we replace the exact boundary condition by an approximate one on the more convenient location. This entails two steps. First, we expand the condition $\mathbf{n} \cdot \nabla\phi = 0$ in powers of ϵ and discard small terms. Then, we use a second expansion to replace the resulting approximate condition at $r = a(1+\epsilon\cos\theta)$ by one on $r = a$. Again, we discard small terms, and as long as we do so consistently we should not degrade the accuracy of our approximation. I am going to go through this process in excruciating detail, because this is an important technique and one which many students do not get right first time.

Let us assume that we want to calculate the solution correct to $O(\epsilon)$. That means that as we go along we can discard any $O(\epsilon^2)$ terms (as long as we are confident they won't get divided by ϵ later). The unit normal to $r = a\,(1 + \epsilon f(\theta))$, for any smooth function $f(\theta)$, is[3]

In polars,

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial \theta}.$$

Note that a has dimensions of length but $\mathbf{n}$ is dimensionless: the a's cancel in the second line.

$$\begin{aligned}
\mathbf{n} &= \frac{\nabla(r - a\,(1+\epsilon f(\theta))}{|\nabla(r - a\,(1+\epsilon f(\theta))|} \\
&= \frac{\mathbf{e}_r - \epsilon f'(\theta)\mathbf{e}_\theta/[1+\epsilon f(\theta)]}{\left(1 + \epsilon^2 \frac{(f'(\theta))^2}{(1+\epsilon f(\theta))^2}\right)^{\frac{1}{2}}} \\
&= \mathbf{e}_r - \epsilon f'(\theta)\mathbf{e}_\theta + O(\epsilon^2).
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathbf{n} \cdot \nabla\phi|_{r=a(1+\epsilon f(\theta))} &= \left(\mathbf{e}_r \frac{\partial \phi}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial \phi}{\partial \theta}\right) \\
&= \left(\frac{\partial \phi}{\partial r} - \epsilon \frac{f'(\theta)}{a}\frac{\partial \phi}{\partial \theta}\right)\Bigg|_{r=a(1+\epsilon f(\theta))} + O(\epsilon^2). \qquad (14.11)
\end{aligned}$$

The next stage is to expand $\partial\phi/\partial r$ and $\partial\phi/\partial\theta$ in (14.11) in Taylor series about $r = a$. That is, we write

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a(1+\epsilon f(\theta))} = \left.\frac{\partial \phi}{\partial r}\right|_{r=a} + \epsilon a f(\theta) \left.\frac{\partial^2 \phi}{\partial r^2}\right|_{r=a} + O(\epsilon^2), \qquad (14.12)$$

and, as hindsight shows we only need one term for $\partial\phi/\partial\theta$,

$$\left.\frac{\partial \phi}{\partial \theta}\right|_{r=a(1+\epsilon f(\theta))} = \left.\frac{\partial \phi}{\partial \theta}\right|_{r=a} + O(\epsilon). \qquad (14.13)$$

We can now substitute for $\partial\phi/\partial r$ and $\partial\phi/\partial\theta$ from (14.12) and (14.13) in (14.11),

Do you now see why we only bother with one term for $\partial\phi/\partial\theta$?

to find that

$$\left.\frac{\partial\phi}{\partial n}\right|_{r=a(1+\epsilon f(\theta))} = \left.\left(\frac{\partial\phi}{\partial r} + \epsilon a f(\theta)\frac{\partial^2\phi}{\partial r^2} - \epsilon\frac{f'(\theta)}{a}\frac{\partial\phi}{\partial\theta}\right)\right|_{r=a} + O(\epsilon^2).$$

eck dimensions: the terms should e dimension /[L], which they

In our case, $f(\theta) = \cos\theta$, and so instead of the exact problem (14.8)–(14.9), we solve the approximate problem[4]

$$\nabla^2\phi = 0, \quad r > a,$$

with the boundary conditions

$$\frac{\partial\phi}{\partial r} + \epsilon a\cos\theta\frac{\partial^2\phi}{\partial r^2} + \epsilon\frac{\sin\theta}{a}\frac{\partial\phi}{\partial\theta} = 0, \quad r = a, \qquad \phi \sim Ur\cos\theta + o(1), \quad r \to \infty.$$

We have done the hard work. The approximate problem yields immediately to a regular expansion

$$\phi(r,\theta) \sim \phi_0(r,\theta) + \epsilon\phi_1(r,\theta) + O(\epsilon^2).$$

The leading order problem is just (as expected) the standard flow past a circular cylinder with solution ϕ_0 as given in (14.10). The problem for ϕ_1 is then

$$\nabla^2\phi_1 = 0, \quad r > a, \qquad \phi_1 = o(1), \quad r \to \infty,$$

with the approximate condition on $r = a$, More details for you to fill in.

$$\begin{aligned}\frac{\partial\phi_1}{\partial r} &= -a\cos\theta\frac{\partial^2\phi_0}{\partial r^2} - \sin\theta\frac{\partial\phi_0}{\partial\theta}\\ &= -2U\left(\cos^2\theta - \sin^2\theta\right)\\ &= -2U\cos 2\theta.\end{aligned}$$

We can look up ϕ_1 in our (mental) library of separable solutions of Laplace's equation, and our solution to $O(\epsilon)$ is

$$\phi(r,\theta) = U\left(r\cos\theta + \frac{a^2}{r}\cos\theta\right) + \epsilon Ua^3\frac{\cos 2\theta}{r^2} + O(\epsilon^2).$$

We can (should) run a couple of consistency checks on this solution. First, the correction ϕ_1 has the right dimensions, velocity $\times$ length. Second, look at the velocity correction on the x–axis. Our cylinder sticks out beyond the circle $r = a$ near the downstream end $\theta = 0$, and is inside near the upstream end $\theta = \pi$, and the flow has stagnation points on the boundary at Draw a picture.

$\theta = 0, \pi$. At the downstream stagnation point, the leading order horizontal velocity $\partial\phi_0/\partial r$ is small and positive because the leading order flow, which is overall left-to-right, has a stagnation point just to the left. Thus, to get zero horizontal velocity here, $\partial\phi_1/\partial r$ must be small and negative, which it is: we have got the right signs.

You can do the same argument at the upstream stagnation point, but you are more likely to lose a minus sign because you have to remember that $\partial/\partial r = -\partial/\partial x$ there.

14.5.2 Example: water waves

For our second example of a boundary perturbation, we look at the very classical problem of two-dimensional small-amplitude surface gravity waves on deep water. The new feature of the problem is that the boundary that is perturbed is itself unknown: it is called a *free boundary* or *free surface.* You may have done this problem in an ad hoc way, 'neglecting quadratic terms'. By now, you probably realise that this just means constructing an asymptotic expansion correct to $O(\epsilon)$, neglecting $O(\epsilon^2)$, and that is just what we do.

Let us build in the fact that the amplitude is small by writing the water surface as $y = \epsilon h(x,t)$, where $\epsilon \ll 1$ and $h = O(1)$ (and there is an implicit assumption that derivatives of h are not large either). Then the full problem to be solved for the velocity is

$$\nabla^2\phi = 0, \quad y < \epsilon h(x,t),$$

for which the free surface conditions are the kinematic condition

Particles in the surface stay there, so the material derivative of $y - \epsilon h(x,t)$ is zero. It also says that the normal velocity of the water is equal to the normal velocity of the interface.

$$\frac{D}{Dt}(y - \epsilon h(x,t)) = 0,$$

namely

$$\frac{\partial\phi}{\partial y} = \epsilon\left(\frac{\partial h}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x}\right), \quad y = \epsilon h(x,t), \tag{14.14}$$

and the Bernoulli condition

Woe to those who spell him Bernouilli.

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + gy = 0, \quad y = \epsilon h(x,t). \tag{14.15}$$

The unknown location of the surface makes this a formidably hard problem and even after decades of effort there are many open questions. The first step on the road, however, is easy.

We are aiming for an asymptotic expansion in powers of ϵ. A quick look at the kinematic condition (14.14) shows immediately that there is no $O(1)$ term in the velocity potential, and so its expansion has the form

Technically, I suppose we should expand $h(x,t)$ as well, but we only need one term so we don't bother.

[3] Note that the expansion may be invalid if $f'(\theta)$ is large.

[4] A pedant says: 'This function ϕ is different from the original one, so you should use a different notation for it.' I reply: 'Go away and leave me alone. There is too much unnecessary notation in the world without adding to it.'

$$\phi(x, y, t) \sim \epsilon\phi_1(x, y, t) + \cdots .$$

It is now clear that the leading order terms in the kinematic and dynamic boundary conditions are

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial h}{\partial t}, \quad \frac{\partial \phi_1}{\partial t} + gh = 0, \tag{14.16}$$

which apply on $y = \epsilon h(x, t)$ and then, by a trivial linearisation, on $y = 0$ without loss of accuracy to this order. The rest is history: taking a representative wave[5] $h(x, t) = e^{i(kx-\omega t)}$ with wavenumber k and frequency ω, we have from Laplace's equation and the first of (14.16) that

Waves with $k > 0$ travel to the right, $k < 0$ to the left; $|k|$ in ϕ ensures decay as $y \to -\infty$. Of course we take the real part of h and ϕ for the physical quantities.

$$\phi_1(x, y, t) = -\frac{i\omega}{|k|} e^{|k|y} e^{i(kx-\omega t)},$$

and then from the second of (14.16) we get the *dispersion relation*

$$\omega^2 = g|k|$$

giving us the phase speed $c = \omega/|k| = \sqrt{g/|k|}$ in terms of the wavenumber.

Check the dimensions.

This calculation can be viewed as a prototype of stability analyses of all sorts of free boundary problems, ranging from fluid flow to solidification of ice or steel. If we think of it in this light, it tells us that the surface of our water is neutrally stable, because the fact that ω is purely real tells us that small disturbances neither grow nor decay. If, on the other hand, we take $g < 0$, equivalent to having the water above the air, ω is purely imaginary and the linearised problem always has exponentially growing modes. Can you see why water falls out of a glass if you turn it upside down, even though the atmospheric pressure (about 10 m of water) is more than enough to hold it in?

Fill a glass to the brim, slide a piece of card across, and hold it in place while you invert the glass: it will stay there when you take your hand away. Why does this not work if there is an air gap before you put the card on?

14.6 Caveat expandator

Regular expansions don't always work. Sometimes the reasons for this are obvious: the procedure falls flat on its face early on. For example, consider the very easy differential equation

$$\epsilon\frac{dy}{dx} = y - 1, \quad x > 0, \qquad y(0) = 0,$$

[5]Representative because we can superpose these waves to solve any initial value problem, equivalent to taking a Fourier transform in x.

for small ϵ. A starightforward regular expansion in powers of ϵ gives $y \sim 1$, and all other term vanish. The regular expansion completely fails to satisfy the initial condition, and in this case inspection of the exact solution, $y = 1 - e{-}x/\epsilon$, shows that there is a *boundary layer* near $x = 0$, in which the solution changes too rapidly to be describable by a regular expansion. We look at problems of this type in Chapter 18. Another example where a regular expansion doesn't even get to first base is

$$\epsilon^2 \frac{d^2y}{dx^2} + y = 0,$$

whose solutions $e^{\pm ix/\epsilon}$ oscillate very rapidly. We look at these in Chapter 22.

There are, however, some problems where what goes wrong is more subtle. Let us return to the undamped small-displacement pendulum equation of Section 14.4,

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \qquad \theta(0) = \epsilon a_0, \frac{d\theta}{dt} = 0 \quad \text{at } t = 0.$$

We showed that we could recover linear theory as the first term in an expansion

$$\theta \sim \epsilon a_0 \cos t + \epsilon^2\theta_2 + \epsilon^3\theta_3 + O(\epsilon^4).$$

Exercise.

(I've kept two terms after θ_1 because it very soon becomes clear that $\theta_2 = 0$.). If, encouraged by this success, we continue, the problem for θ_3 is

As we all know, $\cos 3t = \cos^3 t - 4\cos t$.

$$\begin{aligned} \frac{d^2\theta_3}{dt^2} + \theta_3 &= \frac{a_0^3}{6}\cos^3 t \\ &= \frac{a_0^3}{24}\left(3\cos t + \cos 3t\right), \end{aligned}$$

with

$$\theta_3 = \frac{d\theta_3}{dt} = 0 \quad \text{at} \quad t = 0.$$

Why no $t\cos t$? Because the forcing function on the right-hand side is even, that's why: with a second derivative and an undifferentiated term, and zero initial derivative, we are bound to get an even solution. There's no point in making work for ourselves by putting in odd terms. For much the same reason there is no $\sin t$ term either (which makes fitting the initial zero value for θ_3 a

The solution is found after a small effort:

$$\theta_3 = \frac{a_0^3}{16} t\sin t - \frac{a_0^3}{192}\left(\cos 3t - \cos t\right).$$

There's just one problem with this solution. The term $t\sin t$ grows unboundedly as t increases. Eventually, when $\epsilon^3 t\sin t$ and $\epsilon\cos t$ are of the same size, that is when $t = O(1/\epsilon^2)$, the expansion is no longer valid, because successive terms are no longer decreasing in size. Terms like this are known as *secular terms* from the Latin for a century; the origin is in analysis of planetary orbits and in particular the effect of one planet's gravitational field on the motion of another, which is how the outer planets were found. (In the astronomical context, a century is a good-sized unit of time, long in comparison to our years but comparable to the years of the larger planets.)

The nonuniformity arises because we are trying to describe a periodic function of t whose period is not quite the 2π of the base solution θ_1. (Such nonuniformities are always lurking in problems with conserved quantities or similar structure if the functions we use to approximate are not quite compatible with the conserved quantities.) We take a brief look at problems of this kind in Chapter 17.

Exercises

1. **Space stations.** The space station is in a circular orbit about the earth at a distance a from the centre and with angular speed ω. Its tangential speed is increased from $a\omega$ to $a\omega + \epsilon v$ where $\epsilon \ll 1$. Carry out the linear stability analysis of the orbit (remember that the angular momentum has to be perturbed).

2. **Phase planes.** Referring back to Section 14.3.1, suppose that the Jacobian $\mathbf{J}$ is real and symmetric at a critical point. Show that the linearised equations can be reduced to

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

   where the axes of the coordinates X_i are along the eigenvectors of $\mathbf{J}$. Deduce that the orbits are locally given by the curves

$$|X_1|^{\lambda_1}\,|X_1|^{-\lambda_2} = \text{constant},$$

   and they look roughly like hyperbolae when $\lambda_1\lambda_2 < 0$.

   Show that when $\mathbf{J}$ is skew-symmetric, $\mathbf{J} = -\mathbf{J}^T$, the critical point is a centre.

3. **Pendulum phase planes.** Consider the damped pendulum equation

$$l\ddot{\theta} + k\dot{\theta} + g\sin\theta = 0, \qquad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \omega_0,$$

   made dimensionless with the timescale $t_0 = \sqrt{l/g}$, so that it becomes

$$\ddot{\theta} + \gamma\dot{\theta} + \sin\theta = 0, \qquad \theta(0) = \alpha_0, \quad \dot{\theta}(0) = \beta_0.$$

First suppose that $\gamma = 0$. Show that there are centres in the phase plane at $(\theta, \dot{\theta}) = (2n\pi, 0)$ and saddles at $((2n+1)\pi, 0)$. Sketch the phase plane, indicating the direction of the trajectories. Indicate the curves for which $\alpha_0 = 0$, $\beta_0 \ll 1$. Find a suitable scaling for θ to show that they are approximately circles.

Put one arrow on at, say, $(\alpha_0, 0)$ and the rest follow by continuity.

Still with $\gamma = 0$, indicate the curves for which $\beta_0 \gg 1$. What is the pendulum doing on one of these? Take $\alpha_0 = 0$ and rescale time by writing $t = \tilde{t}/\beta_0$; show that this gives

$$\frac{d^2\theta}{d\tilde{t}^2} + \frac{1}{\beta_0^2}\sin\theta = 0, \qquad \theta(0) = 0, \quad \frac{d\theta}{d\tilde{t}} = 1.$$

Now write

$$\theta = \theta_0 + \frac{1}{\beta_0^2}\theta_1 + \cdots,$$

and find θ_0 and θ_1 by equating terms of $O(1)$ and $O(1/\beta^2)$ to zero separately. Interpret these results.

Now suppose $\gamma > 0$. Show that the saddles remain saddles but the centres become stable spirals for $0 < \gamma < 2$. What happens for $\gamma > 2$? Sketch the phase plane when (a) $0 < \gamma \ll 1$, (b) $\gamma \gg 1$.

4. **Satellites.** Investigate the linear stability of a satellite orbit using the more general approach of Section 14.3.1 as follows. Write the equations for motion of a satellite,

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}, \qquad \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,$$

as a first-order system $\dot{x}_i = F_i(x_j)$ for $x_1 = r$, $x_2 = \dot{r}$ and $x_3 = \dot{\theta}$. Show that all points on the curve $x_1^3 x_3^2 = GM$, $x_2 = 0$, are equilibrium points. Taking a representative point $(a, 0, \omega)$ on this curve, show that the Jacobian $(\partial F_i/\partial x_j)$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 3\omega^2 & 0 & 2a\omega \\ 0 & -2\omega/a & 0 \end{pmatrix}.$$

Deduce from the trace and determinant of this matrix (which you can evaluate without detailed calculation) that at least one of the eigenvalues vanishes, that the remaining ones sum to zero, and so that just one vanishes and the other two are equal and opposite (consistency check). Confirm this by finding the eigenvalues as 0, $\pm i\omega$.

Note the zero eigenvalue, which is due to the existence of a non-isolated set of equilibrium points (why?). What would happen if you wrote the equations as a 4×4 system for r, $\dot{r}$, θ, $\dot{\theta}$?

5. **Motion under gravity near the earth.** Absolutely everybody does motion of projectiles early in their mathematical career: $m\ddot{\mathbf{r}} = -m\mathbf{g}$, $\mathbf{r}(0) = (0, 0, h)$, $\dot{\mathbf{r}}(0) = \mathbf{v}$. Expand in terms of $\epsilon = h/R_e$, where R_e is the radius of the earth, to reconcile this with the full Newtonian model in which the force on a particle is $m\nabla\phi$, in which the gravitational potential $\phi = GM_e/|\mathbf{R}|$, G being the universal gravitational constant, M_e the mass of the earth and $\mathbf{R}$ the position vector measured from the centre of the earth. What restriction on $\mathbf{v}$ is necessary for your approximation to be valid?

6. **Flagpoles again.**

   Look up your derivation of the dimensionless flagpole equation oscillated at the base, and write it in the form

$$\frac{\partial^2 y}{\partial t^2} + \alpha^4 \frac{\partial^4 y}{\partial y^4} = 0,$$

   with the boundary conditions

$$y_{xx} = y_{xxx} = 0 \quad \text{at } x = 1, \qquad y = \cos t, \quad y_x = 0 \quad \text{at } x = 0,$$

   and a condition of periodicity in time. Suppose that $\alpha \gg 1$, and write $\epsilon = 1/\alpha$. Find the solution correct to $O(\epsilon)$ by a regular perturbation method. What is happening physically in this regime?

7. **The Euler strut (ii).** Recall from Chapter **??** Exercise ... that for the Euler strut the angle with the x axis satisfies

$$\frac{d^2\theta}{ds^2} + \alpha^2 \sin\theta = 0, \qquad \theta(0) = \theta(1) = 0,$$

   where $\alpha^2 = FL^2/b$ is the bifurcation parameter. Show that if θ is small, the procedure of that exercise is equivalent to finding the first term in a regular expansion for θ.

   Now suppose that α is just above the critical value π so that $\alpha^2 = \pi^2 + \epsilon^2$ where ϵ is small. Seek a solution in which θ is small, so that $\theta = \delta\phi$, where $\delta \ll 1$ (so far, we do not know how big δ should be). Show that

$$\frac{d^2\phi}{ds^2} + \left(\pi^2 + \epsilon^2\right)\left(\phi - \delta^2\frac{\phi^3}{6} + \cdots\right) = 0, \qquad \phi(0) = \phi(1) = 0.$$

Conclude, provisionally, that a sensible choice for δ is $\delta = \epsilon$ (we return to this below).

Construct a regular expansion

$$\phi \sim \phi_0 + \epsilon^2 \phi_1 + O(\epsilon^4),$$

show that $\phi_0 = A \sin \pi s$ for an as yet unknown constant A, and write down the problem satisfied by ϕ_1. Multiply by ϕ_0 and integrate by parts to show that it only has a solution if

This is the Fredholm Alternative for a two-point boundary value problem; see page ...

$$\int_0^1 \left(\pi^2 \frac{\phi_0}{6} - \phi_0 \right) \sin \pi s \, ds = 0,$$

and conclude that ϕ_1 only exists if $A = 0, \pm 2\sqrt{2}/\pi$.

As α varies, define a measure $M_\theta(\alpha)$ of the size of the solution $\theta(s;\alpha)$ by

$$M_\theta(\alpha) = \max_{0 \le s \le 1} \theta(s;\alpha),$$

and ahow that for α near π, either $M_\theta(\alpha) = 0$ or

$$M_\theta(\alpha) = \pm 2\sqrt{2}\sqrt{\alpha^2 - \pi^2}/\pi.$$

Plot the *response diagram* $M_\theta(\alpha)$ against α and you will see why this bifurcation is called a *pitchfork bifurcation*.

Finally, go back and convince yourself that other choices for the magnitude of δ do not lead to sensible expansions. Also show (using the analysis above) that if α is slightly *below* the critical value, the only solution is $\phi = 0$.

8. **The forced logistic equation.** Explain why the equation

$$\frac{du}{dt} = ku(1-u)$$

is a crude model for population dynamics supported by a finite resource (what happens to u if it is small, or just above/below 1?). Which term in the equation corresponds to the size of the resource? Now suppose that the resource fluctuates seasonally so that the population equation is

$$\frac{du}{dt} = ku(1 + \epsilon \cos t - u)$$

Find a periodic solution $u = 1 + \epsilon u_1(t) + \cdots$ correct to $O(\epsilon^2)$.

Show that the equation can be solved exactly by putting $u = 1/v$. Does this help matters?

re (r, θ) are plane
ar coordinates.

9. **Electric potential of a nearly circular cylindrical annulus.** Find the electric potential ϕ, satisfying $\nabla^2\phi = 0$ between the two cylinders $r = a$, on which $\phi = 0$, and $r = b > a$, on which $\phi = V$. Suppose that the inner cylinder is perturbed to $r = a(1 + \epsilon \sin n\theta)$. Calculate ϕ correct to $O(\epsilon)$; to build up your arithmetical strength, calculate it correct to $O(\epsilon^2)$. What restriction on n is necessary for your expansion to be valid?

Chapter 15

Case study: electrostatic painting 2

15.1 Small parameters in the electropaint model

When we left this problem, we had a dimensionless model with a number of small parameters in it. Let's revisit it in the light of our discussion of regular expansions.

Recall that we have a number density n of particles, with velocity $\mathbf{v}_p$, an electric field $\mathbf{E}$, and gas velocity $\mathbf{v}_g$ and pressure p. There are several small dimensionless parameters in the model, and we'll leave them all out except the least small, which we call

$$\epsilon = \frac{q_p V_0 L}{K U_g}$$

(and apologise for the use of ε for electrical permittivity as well). The numerical value of ϵ is about 0.1. There is also one $O(1)$ parameter

$$A = \frac{n_0 K L}{\rho_g U_g}$$

whose numerical value is about 1.

The model consists of an equation of motion for the particles[1]

$$\mathbf{v}_p - \mathbf{v}_g = \epsilon \mathbf{E} \tag{15.1}$$

and conservation of particles,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_p) = 0; \tag{15.2}$$

[1]Now I really do need to apologise: how, to an applied mathematician of taste, could $\epsilon\mathbf{E}$ be anything but $\mathbf{D}$?

we have an equation for the electric field,

$$\nabla \cdot \mathbf{E} = n; \tag{15.3}$$

and lastly we have the equation of motion and conservation of mass for the gas,

$$\frac{d\mathbf{v}_g}{dt} = -\nabla p + An\left(\mathbf{v}_p - \mathbf{v}_g\right), \tag{15.4}$$

and

$$\nabla \cdot \mathbf{v}_g = 0. \tag{15.5}$$

If any of the following steps are not clear, take a minute to write them out.

Now expand

$$\mathbf{v}_p \sim \mathbf{v}_{p0} + \epsilon \mathbf{v}_{p1} + \cdots ,$$

with similar expansions for the other variables. It's clear from (15.1) that

$$\mathbf{v}_{0p} = \mathbf{v}_{0g},$$

which confirms that the particles follow the gas to leading order. If we use this on the right-hand side of (15.4), we see that $\mathbf{v}_{0g}$ just satisfies an ordinary fluid flow problem with no body force from the particles. Let's assume that we can solve this, and carry on.

The next thing to do is to calculate the evolution of the number density n. The leading order terms in (15.2) are

$$\frac{\partial n_0}{\partial t} + \nabla \cdot \left(n_0 \mathbf{v}_{p0}\right) = 0.$$

Bearing in mind that $\mathbf{v}_{p0} = \mathbf{v}_{g0}$ and $\nabla \cdot \mathbf{v}_{g0} = 0$, this simplifies to

$$\frac{\partial n_0}{\partial t} + \mathbf{v}_{g0} \cdot \nabla n_0 = 0,$$

a first-order hyperbolic equation[2] whose characteristics are, not surprisingly, the gas particle paths.

Having found n, the last task is to find the leading order electric field as the solution of

$$\nabla \cdot \mathbf{E}_0 = n_0.$$

We can now go round the cycle again, using the equations in the same order. First, (15.1) tells us that

$$\mathbf{v}_{p1} - \mathbf{v}_{g1} = \mathbf{E}_0, \tag{15.6}$$

[2] If you have only studied first-order partial differential equations in two independent variables, it is a relief to find that the extension to more independent variables is very straightforward; see [27], Chapter 1.

then (15.4) and (15.5) are a linear system for $\mathbf{v}_{g1}$. Thus, we know the correction to the particle velocity from (15.6); we we can also calculate n_1 from (15.2) and lastly $\mathbf{E}_1$ from (15.3). Notice how the asymptotic expansion suggests an order in which to solve the equations, which might also be a sensible basis for an iterative numerical scheme (at least in the steady case).

You can develop this problem further by doing the exercises on it.

Exercises

1. **Electrostatic painting I.** In the electrostatic painting model, we wrote down conservation of number of particles,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}_p) = 0$$

and conservation of mass for the gas (assumed incompressible)

$$\nabla \cdot \mathbf{v}_g = 0.$$

(Remember that this is dimensionless, and n is scaled with a typical number density $n_0 \approx 10^9$ m^{-3}; what is the average distance between the particles?)

In fact that isn't quite right, because this is a two-phase flow and gas may be displaced by particles and vice-versa. If we take a small (but large compared to the average particle separation) representative volume V, show that we can nevertheless justify it as follows:

(a) Show that the proportion of V that is occupied by particles is ϵn, where $\epsilon = 4\pi n_0 a^3/3$ if all the particles are spherical with the same radius a ($a \approx 10^{-5}$ m). Estimate the numerical size of ϵ.

(b) Deduce that the proportion of V occupied by gas is $1 - \epsilon n$.

(c) Using the general form

$$\frac{\partial(\text{density})}{\partial t} + \nabla \cdot (\text{flux}) = 0$$

for a conservation law, show that conservation of mass for the gas (remember it's incompressible so its density is constant) is

$$\frac{\partial(1-\epsilon n)}{\partial t} + \nabla \cdot ((1-\epsilon n)\mathbf{v}_g) = 0.$$

Show also that the conservation of particles equation given above is correct.

(d) Just to confirm, show that

$$\nabla \cdot (\epsilon n \mathbf{v}_p + (1 - \epsilon n)\mathbf{v}_g) = 0,$$

and interpret this flux in physical terms.

(e) Expand n and $\mathbf{v}_g$ in powers of ϵ, and show that the leading order equations are those given above.

2. **Electrostatic painting II.** Complete the derivation of the $O(\epsilon)$ equations for this problem, and verify that all the equations you obtain are linear in $\mathbf{v}_{g1}$, n_1 etc.

3. **Electrostatic painting III.** Consider steady-state solutions of the leading order ($O(1)$) equations for this problem. Show that $\mathbf{v}_{g0} \cdot \nabla n_0 = 0$. If the flow is two-dimensional, with stream function ψ, deduce that $n_0 = f(\psi)$ for some function f determined by the inlet conditions.

4. **Space charge.** A variant of the electropainting problem occurs when the charged particles are so small that they do not exert any significant body force on the gas. The sort of physical situation that can be modelled in this way is the motion of charged ions from a high-voltage DC power cable, or the electrostatic scrubbers used to clean power station emissions.

   If there is no imposed gas flow, briefly justify the model

$$\mathbf{v}_p = \mathbf{E}, \quad \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}_p) = 0, \quad \nabla \cdot \mathbf{E} = n.$$

   Given that we can write $\mathbf{E} = -\nabla\Phi$ (since from Maxwell's equations on this time scale we have $\nabla \wedge \mathbf{E} = \mathbf{0}$), show that the system becomes

$$\frac{\partial n}{\partial t} + n^2 - \nabla n \cdot \nabla\Phi = 0, \quad \nabla^2\Phi = -n.$$

   In the steady state, show that the characteristics of the first of these equations have tangent $-\nabla\Phi$. Deduce that they are orthogonal to the equipotentials, and parametrising them by τ, derive the ordinary differential equation

$$\frac{dn}{d\tau} + n^2 = 0$$

   along them. Show also how to model a point source of charged particles by allowing $n \to \infty$ as $\tau \to 0$.

   Now suppose that there is an imposed gas flow which is irrotational, so that there is a velocity potential ϕ with $\mathbf{v}_g = \nabla\phi$, where $\nabla^2\phi = 0$

(potential flow is often a very reasonable model upstream of an obstacle, less so downstream where the effects of boundary layers, separation and so on are felt). Show that there are parameter ranges where the model

$$\mathbf{v}_p = A\mathbf{v}_g + \mathbf{E}, \quad \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}_p) = 0, \quad \nabla \cdot \mathbf{E} = n.$$

is valid with A an $O(1)$ constant. Show that the results of the previous paragraph hold but with the characteristics derived from the modified potential $\Phi - A\phi$.

5. **Paint layer again.** Suppose that a thin layer of paint particles, deposited electrostatically as in the text, is growing on $y = 0$, and that its thickness is $y = h(x, t)$. If $\epsilon = H/L \ll 1$, where H and L are a typical thickness of the layer and length scale of the workpiece respectively, justify the approximate boundary condition

$$\frac{\partial h}{\partial t} = \mathbf{v}_p \cdot \mathbf{n}$$

on the workpiece (see the exercise on this topic in Chapter 7).

Chapter 16

Case study: piano tuning

16.1 The notes of a piano: the tonal system of Western music

This section contains a short description of the mathematical structure of the tonal system used for a piano. It can be omitted by those not interested.

The particular sound of a given note of a piano or other musical instrument is characterised reasonably well by its fundamental frequency and a variety of higher harmonics (damping rates also play a role). These harmonics are often (approximately — as we shall see, that is the point of this case study) integer multiples of the fundamental frequency f_1. On stringed instruments this is because the normal frequencies of a vibrating string are integer multiples of the fundamental, and wind instruments either have regular vibrating cavities (for example an organ tube) with the same integer harmonic ratios or, like a French horn, they are carefully (and expensively) made to sound this way.

See the exercise on page 215 for why a cymbal or gong sounds harsh.

When two or more notes are played together, their fundamentals and harmonics all interact. The tonal system of Western music has been strongly influenced by the features of this interaction; the mathematical construction we now outline goes back at least to the Pythagoreans of Ancient Greece. Suppose we play a note, called for example A, with fundamental frequency f_1^{A}; we hear frequencies f_1^{A}, $f_2^{\mathsf{A}} = 2f_1^{\mathsf{A}}$, $f_3^{\mathsf{A}} = 3f_1^{\mathsf{A}}$ and so on. We might expect the note A', with fundamental frequency $f_1^{\mathsf{A}'} = 2f_1^{\mathsf{A}}$ equal to twice that of A, to sound good with A', because its fundamental coincides with the first harmonic of A. It does indeed sound good, and the interval between the two, created by doubling the lower frequency, is called an *octave*. In a similar way, the note with fundamental frequency $3f_1^{\mathsf{A}}$, produces a harmonious blend with A, and so does the note an octave below it, whose frequency is $\frac{3}{2}f_1^{\mathsf{A}}$. This

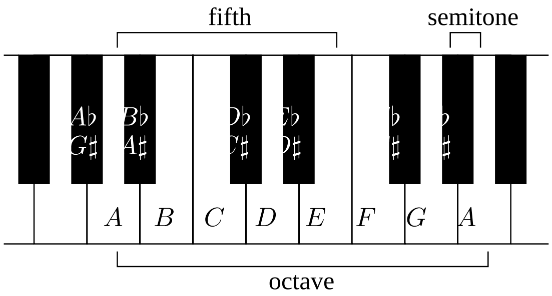

Figure 16.1: A section of a piano keyboard.

note is called **E**, and the interval corresponding to a frequency ratio of $\frac{3}{2}$ is called a *fifth*.

When a violinist tunes up by playing the **A** and **E** strings together and eliminating beats by turning a tuning peg, the beats that are eliminated are probably those between the second harmonic of the A string and the first harmonic of the E string. See below for a discussion of beats.

The next note to be constructed is the *fourth*, with frequency $\frac{4}{3}f_1^{\mathsf{A}}$, called **D**. Its frequency ratio is $\frac{4}{3}$, and we notice that since $\frac{3}{2} \times \frac{4}{3} = 2$, the interval from **E** to A' is also a fourth. Following this, we have the *major and minor thirds*, with ratios $\frac{5}{4}$ and $\frac{6}{5}$ respectively. These are the most important intervals and they make up, for example, the harmonious-sounding chords you hear at the ends of pieces of music.

It is apparent that we can continue this process of interval construction indefinitely, until we have notes with all rational multiples of f_1^{A}; this might be plausible in the context of a 'continuous' instrument like a violin or the human voice but it is clearly impractical for a piano. Moreover, given that the amplitude of the harmonics of a note decreases as we go to higher harmonics, it would be pointless because we could never hear the interactions. In practice, therefore, the process is truncated, and Western music is built around a tonal system consisting of 12 notes, separated by intervals called semitones. These notes contain the fifth (7 semitones), the fourth (5 semitones) and the major and minor thirds (4 and 3 semitones respectively). For reasons lost in history, only 7 letters are used to denote notes (these are the 'white notes' on a piano), the remaining ones being described with the help of two operators, ♯ (pronounced 'sharp') and ♭ ('flat') which, when placed after a note move it up by a semitone for a sharp and down for a flat.[1]

And, corresponding to the major and minor thirds, the minor and major sixths, with rations $\frac{8}{5}$ and $\frac{5}{3}$ respectively.

The sequence of notes can be written

$$\ldots, \mathsf{A}, \mathsf{A}\sharp = \mathsf{B}\flat, \mathsf{B}, \mathsf{C}, \mathsf{C}\sharp = \mathsf{D}\flat, \mathsf{D}, \mathsf{D}\sharp = \mathsf{E}\flat, \mathsf{E}, \mathsf{F}, \mathsf{F}\sharp = \mathsf{G}\flat, \mathsf{G}, \mathsf{G}\sharp = \mathsf{A}\flat, \mathsf{A}, \ldots$$

repeated up and down the piano in octaves.

If we look at this scheme more closely, we see that there is a contradiction in it. One manifestation of the inconsistency is that an octave should consist

[1] For musical reasons, other notations such as **C♭** (= **B**) or even **G♭♭** (= **F**) are possible, but they are irrelevant here.

of three consecutive major thirds of four semitones each, for example **A**–**C♯**–**F**–**A**. However, this gives a frequency ratio of $(\frac{5}{4})^3 = \frac{125}{64} < 2$, whereas it should give 2 exactly. Similarly the octave should be four consecutive minor thirds; but $(\frac{6}{5})^4 = \frac{1296}{625} > 2$. Another famous illustration of the inconsistency is obtained by constructing the 'circle of fifths', in which we go up by fifths, dropping down an octave as convenient:

$$\mathbf{A} \to \mathbf{E} \to \mathbf{B} \to \mathbf{F}\sharp \to \mathbf{C}\sharp = \mathbf{D}\flat \to \mathbf{A}\flat$$

(this is the 'furthest removed' note from **A**)

$$\to \mathbf{E}\flat \to \mathbf{B}\flat \to \mathbf{F} \to \mathbf{C} \to \mathbf{G} \to \mathbf{D} \to \mathbf{A}.$$

The frequency of our last **A**, $(\frac{3}{2})^{12}$, isn't a power of 2 as it should be. It's slightly sharp: $531441/4096 \approx 129.746 > 128$.

As a consequence of the inconsistency in construction, we can never tune an instrument so that all the intervals on it are perfectly in tune. For example, if we tune the fifths to be perfect, moving away from **A** in both directions, we get two different values for the furthest removed note, **A♭**. Going up, we get $(\frac{3}{2})^6$, going down we get $(\frac{2}{3})^6$, whose ratio is not a power of 2. Any other interval gives a similar result. How, then, are we to choose the fundamental frequencies of our twelve notes? The sound of two notes played together depends very strongly on the their interaction. Harmonics that are close together can give unpleasant sounding beats and sound out of tune, especially on an instrument like an organ in which the volume does not fall off. What compromise system should we use?

This question of *temperament* caused a great deal of trouble in the past, and I don't want to go into great detail about it here; literally hundreds of solutions have been proposed (see [20] for a popular history and [16] for a more technical derivation of some popular temperaments). The currently accepted solution[2] is to insist that each interval of a semitone corresponds to the same frequency ratio, which must terefore be $2^{\frac{1}{12}} \approx 1.0595$. With this compromise, called *equal temperament*, all notes and intervals are slightly wrong but at least no one note is more wrong than any other.

16.2 Tuning an ideal piano

The upshot of the previous section is that the goal of tuning a piano is to obtain certain frequency ratios between the fundamental frequencies of pairs of notes. Because the harmonics of an ideal piano string are integer multiples of the fundamental, they too are to be tuned in specified ratios.

[2]Some composers are returning to 'microtonal' music.

Moreover, these ratios are close to, but (apart from the octave) not exactly equal to, integer ratios. For example, the equal-temperament fifth has a ratio $2^{\frac{7}{12}} \approx 1.4983$.

The easiest intervals to tune are the octaves. If we use a tuning fork (mechanical or electronic) to tune one note of our piano, say the **A** above middle **C**, to its standard frequency of 440 Hz, then we can tune all the other **A**s on our instrument to frequencies of $2^{\pm k} \times 440$ Hz by eliminating beats between the fundamentals and the first harmonics of notes an octave apart. Then we can tune other notes by using intervals such as fifths, listening to the calculable (and measurable) beat rates on the appropriate harmonics (see Exercise 2).

Interlude: beats. How would we tune a note on a piano to be the same as a standard tone? The standard way is to play them together, and listen for the beats. Suppose they have the same amplitude a and phase (see the exercises for when they are not the same), but have slightly different frequencies ω and $\omega + \epsilon$ where ϵ is small. The sum of the signals is

$$a \cos \omega t + a \cos(\omega + \epsilon)t = 2a \cos(\omega + \tfrac{1}{2}\epsilon)t \cos \tfrac{1}{2}\epsilon t.$$

This is a modulated wave: it oscillates at the fast frequency $\omega + \frac{1}{2}\epsilon$, which is very close to ω, and its amplitude is modulated at the slow beat frequency ϵ. So the aim in tuning is to get the beat frequency to zero (or another specified rate) by tightening or loosening the piano strings (a very skilled business).[3]

Not $\frac{1}{2}\epsilon$: we hear two amplitude peaks for each cycle of $\cos \frac{1}{2}\epsilon t$.

16.3 A real piano

Now let's look at a real piano string. An ideal string satisfies the wave equation

$$\rho A \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < L,$$
$$y = 0 \quad \text{at} \quad x = 0, L.$$

and it's a piece of classical applied mathematics to show that the normal modes are

These are of course angular frequencies: the frequencies in Hz are $f_n = \omega_n/(2\pi) = nc/(2L)$, so that $1/f_n$ is the time taken for a signal travelling at the wave speed c to travel from one end of the string and back n times.

[3] Irrelevant digression: how loud do n instruments of an orchestra sound compared to one on its own? Answer: $\sqrt{n}$ times as loud, because the phases of the instruments are random. The sound signal from the whole orchestra is $\sum_i a_i \cos(\omega_i t + \phi_i)$ where a_i are the individual amplitudes, ω_i the frequencies and ϕ_i the phase shifts. Even if all the a_i are the same, the ϕ_i are in practice randomly distributed so (Central Limit Theorem) the root mean square amplitude (standard deviation) of the sum is $\sqrt{n}$ times an individual amplitude. This is one reason why the concerto can succeed as an art form (of course, skillful writing by composers may have something to do with it too.

$$y_n = e^{i\omega_n t} \sin \frac{n\pi x}{L}$$

where $\omega_n = n\pi c/L$ and $c^2 = T/(A\rho)$.

For later convenience, we give the dimensionless versions of these results. Scaling x with L and t with L/c and immediately dropping the primes, the equation is

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < 1,$$

with

$$y = 0 \quad \text{at} \quad x = 0, 1,$$

the normal modes are

$$y_n = e^{i\Omega_n t} \sin n\pi x$$

and the dimensionless frequencies are

$$\Omega_n = n\pi.$$

However, a real piano string has a small bending stiffness. A combination of the string model above and the beam models we used earlier (see Exercise 3 on page 70) gives us the dimensionless equation

$$\rho A \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} + EAk^2 \frac{\partial^4 y}{\partial x^4} = 0$$

for the string displacement. We can assess the size of the fourth-derivative term by scaling x and t as above, to get the dimensionless equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \epsilon \frac{\partial^4 y}{\partial x^4} = 0,$$

where

$$\epsilon = \frac{Ek^2}{\rho L^2 c^2} = \frac{EAk^2}{TL^2}.$$

Note the very sensitive dependence on the string thickness, as the fourth power of the radius since, roughly, $k \propto a$ and $A \propto a^2$.

Now for a circular string of radius a, $k^2 = \frac{1}{2}a^2$, so if $a = 1$ mm, $k^2 = \frac{1}{2} \times 10^{-6}$ m^2. If the string is made of steel, it has $E \approx 2 \times 10^{11}$ and $\rho = 7800$ in SI units. Suppose that the string is 1 m long and has a tension of 1 000 N (this is quite typical: the combined force of all the strings on a grand piano is several tonnes worth). Then

How many newtons in a tonne weight?

$$\epsilon = \frac{EAk^2}{TL^2} = \frac{2 \times 10^{11} \times \pi \times 10^{-6} \times \frac{1}{2} \times 10^{-6}}{10^3 \times 1^2} \approx 3.1 \times 10^{-4},$$

which is small indeed, but nevertheless has a noticeable effect, as we shall see. The frequency of this string is $c/(2L)$ is about 280 Hz, say the **C♯** above middle **C**.

Now let's calculate the normal modes of a string. In order to do this we need boundary conditions, two at each end. The simplest are that $y = 0$ (obviously) and that $\partial^2 y/\partial x^2 = 0$, so-called simply supported conditions, which are probably not a bad approximation to the truth as the string passes over a 'bridge' at each end. We shortcut the process of finding normal modes, which you would usually do by looking for separable solutions $y_n(x,t) = e^{i\Omega_n t} Y_n(x)$, by noting that with our choice of boundary conditions, there are solutions

$$y_n = e^{i\Omega_n t} \sin n\pi x$$

provided that

$$\Omega_n^2 = n^2\pi^2 + \epsilon n^4 \pi^4.$$

So, the normal frequencies are

$$\begin{aligned} \Omega_n &= n\pi \left(1 + \epsilon n^2 \pi^2\right)^{\frac{1}{2}} \\ &\sim n\pi \left(1 + \tfrac{1}{2}\epsilon n^2\pi^2 + o(\epsilon)\right). \end{aligned}$$

The fundamental frequency of our string is thus

$$\Omega_1 \sim \pi \left(1 + \tfrac{1}{2}\epsilon\pi^2\right)$$

and so the $(n-1)$th harmonic has frequency

$$\begin{aligned} \Omega_n &\sim n\pi \left(1 + \tfrac{1}{2}\epsilon n^2\pi^2\right) \\ &\sim n\Omega_1 \frac{1 + \frac{1}{2}\epsilon n^2\pi^2}{1 + \frac{1}{2}\epsilon\pi^2} \\ &\sim n\Omega_1 \left(1 + \tfrac{1}{2}\epsilon\pi^2 \left(n^2 - 1\right)\right), \end{aligned}$$

using the binomial expansion to simplify the fraction.

We see that the higher harmonics have slightly larger frequencies than the theoretical integer multiples of the fundamental, a property known as inharmonicity. So, if we tune the string **A′**, one octave above our **A**, by eliminating beats between its fundamental and the first harmonic of the lower **A** string, the fundamental frequency of the higher string will be $2(1 + \frac{3}{2}\epsilon\pi^2)$ times that of the lower one, not the theoretical twice. This phenomenon is known as *octave stretch*; over the 8 octaves of a piano, making the very crude assumption that the inharmonicities of the strings are all the same, the stretch is by a factor

$$\begin{aligned}(1 + \tfrac{3}{2}\epsilon\pi^2)^7 &\sim 1 + \frac{21}{2}\epsilon\pi^2 \\ &\approx 1.033.\end{aligned}$$

This may not look much, but it is more than half a semitone; in fact the inharmonicities on pianos can add up to as much as a whole semitone (the higher strings especially are very short and so have larger values of ϵ, and there are other effects due to the ends of the strings). It is not at all well known, even among pianists, that the treble strings of a piano are this much sharp of 'theoretical' values; fortunately there are no other instruments with a similar range that might accompany it. In the exercises you can work out how to deduce the inharmonicity by measuring beat rates, a first step in calculating the optimal tuning for a given instrument.

Sources and further reading

This case study describes joint work in progress with Paul Duggan, who tunes my piano while I do the calculations. There is a huge amount of fascinating stuff about musical instruments in [11].

Exercises

1. **Beats.** Suppose that we combine two signals

   $$a\cos(\omega t + \phi_1), \quad a\cos(\omega t + \epsilon + \phi_2).$$

   Show that the beats analysis is unaffected.

   Now combine signals with different amplitudes:

   $$a_1 \cos\omega t, \quad a_2\cos(\omega + \epsilon)t.$$

   Show that the output consists of a constant-amplitude signal at frequency ω, together with a signal that beats at frequency ϵ. To see this in detail, suppose $a_1 > a_2$ and write the combined signal as

   $$(a_1 - a_2)\cos\omega t + 2a_2\cos\left(\omega + \tfrac{1}{2}\epsilon\right)t\cos\tfrac{1}{2}\epsilon t.$$

   Then work out the average of the squared amplitude over a 'moving window' time interval which is large compared with the period of the fast oscillation at frequency ω, but small compared with the period of the slow modulation at frequency ϵ. [Making the substitution $\epsilon t = \tau$, you should get an integral something like

$$\int_{\tau}^{\tau+\delta} \left[(a_1 - a_2)\cos\left(\frac{\omega\tau'}{\epsilon}\right) + 2a_2 \cos\left(\frac{\omega}{\epsilon} + \frac{1}{2}\right)\tau' \cos\frac{1}{2}\tau' \right]^2 d\tau'$$

where $\epsilon \ll \delta \ll 1$. The squares of the first and second terms in the brackets average over many periods to a constant and a modulated amplitude respectively, and after a bit of diddling around the cross term is found to average to zero. Try it and see; use the Riemann–Lebesgue lemma if you want to be more rigorous.]

2. **Equal temperament.** Assume for this question that the harmonics of a string are integer multiples of the fundamental. A piano tuner tunes concert **A** at 440 Hz, and wishes to tune the **E** a fifth above, using equal temperament. This is to be done by counting the beat rate between the second harmonic of the **A** and the first harmonic of the **E** (in practice, it might be done with the sixth and fourth harmonics). Find a formula for the required beat rate and evaluate it numerically. Repeat for the sixth/fourth pair.

3. Pianos and harpsichords Suppose that you have a solution $y(x,t)$ of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L,$$

that is periodic in time with period T. Assuming sufficient smoothness, show that

$$\int_0^T \int_0^L c^2 \left(\frac{\partial y}{\partial x}\right)^2 dx\, dt = \int_0^T \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dt\, dx$$

and interpret this statement in terms of energy. If the solution to a general initial value problem for this string is expanded in a fourier series in x, in the form

$$y(x,t) = \sum_1^\infty (a_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)) \sin(n\pi x/L),$$

what is the ratio of the energy in each harmonic to that in the fundamental?

A piano string is set in motion by a hammer which imparts an instantaneous velocity V to the small segment $x_0 < x < x_0+h$, the remainder of the string string initially being at rest. Calculate the energy in each mode relative to the fundamental. Repeat for a harpsichord (or guitar) which is plucked by being let go from the piecewise linear static displacement you get when you displace the point x_0 by a distance a.

(The sound you hear is considerable modified by the soundboard and other parts of the instrument.)

4. **Waves on a circular membrane.** Recall from the exercises of Chapter **??** that waves on a circular membrane of radius a and density ρ per unit area, stretched to tension T, satisfy

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where $c^2 = T/\rho$ is the wave speed.

Show that there are solutions

$$u(r, \theta, t) = e^{-i\omega t} e^{im\theta} R(r)$$

where
$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) R = 0.$$

Putting $x = kr = \omega r/c$, reduce this to *Bessel's equation* of order m,

$$\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{m^2}{x^2}\right) R = 0.$$

Perform a local (Frobenius-style) analysis near $x = 0$ to show that there is only one solution that is bounded at $x = 0$; it is called $J_m(x)$. Deduce that the normal frequencies for a membrane clamped at its edges are $\omega_{m,n}$ where n labels the roots of $J_m(\omega_{m,n} a/c) = 0$. (It can be shown that there are infinitely many roots of $J_m(x)0$ and that they are asymptotic to $(n + \frac{1}{4})\pi$ as $n \to \infty$. However, the low harmonics are far from being integer multiples of the fundamental.) Sketch some nodal lines (lines where $R = 0$) for low values of m and n. Timpani (kettledrums) are much more complicated than this membrane because of the coupling with the air chamber.)

Or just look for a solution $R \sim x^\alpha$ as $x \to 0$ and find possible values of α by balancing the most singular terms.

5. **Cymbals and gongs.** A simple model for a cymbal or gong is to treat it as a circular elastic plate. It can be shown that the equation

of motion for small displacements $u(\mathbf{x}, t)$ of such a plate is

$$\rho\frac{\partial^2 u}{\partial t^2} + \frac{Eh^2}{12(1-\nu)^2}\nabla^4 u = 0,$$

where ρ is the density, E is the Young's modulus, h the thickness; the parameter ν, called Poisson's ratio, is a material property whose numerical value is often about $\frac{1}{3}$. Compare this equation with that of a beam and convince yourself that it is plausible (Poisson's ratio appears because the geometry of a plate is different from that of a beam).

To save real estate, define

$$c_L = \sqrt{\frac{E}{\rho(1-\nu)^2}},$$

which is the wavespeed for longitudinal waves in a plate. Show that time-periodic solutions $u(\mathbf{x}, t) = e^{-i\omega t}U(\mathbf{x})$ satisfy

$$\nabla^4 U - \frac{12\omega}{hc_L}U = \nabla^4 U - k^4 U = 0.$$

Check the dimensions.

Deduce that one-dimensional waves for which $u = e^{i(kx-\omega t)}$ are dispersive, with wavenumber and frequency related by

$$\omega = \frac{c_L h k^2}{\sqrt{12}}.$$

Now consider a circular plate and look for a solution

$$U(r, \theta, t) = R(r)\cos m\theta.$$

Noting that $\nabla^4 - k^4 = (\nabla^2 - k^2)(\nabla^2 + k^2)$, show that the general bounded solution for $R(r)$ is

$$R(r) = AJ_m(kr) + BI_m(kr)$$

where $I_m(kr) = i^{-m}J_m(ikr)$ is sometimes called a modified Bessel function of order m.

Write down (but do not attempt to solve) the normal frequency equation when the plate is clamped at its edges ($u = 0$ and $\partial u/\partial r = 0$). It is fairly clear that the roots are not in a harmonic progression, so the higher harmonics will clash with the fundamental. It is possible (but not recommended, on account of the heavy arithmetic) to find the normal frequencies for the more realistic case of free edges, with a similar lack of harmonicity.

This model is also useful in analysing flat-panel loudspeakers.

6. **Piano tuning.** Suppose we don't know the properties of our piano strings, but we believe that the frequencies of the harmonics (in Hz) of string k are given approximately by the formula

$$f_{k,n} = nf_{k,1}\left(1 + \varepsilon_k(n^2 - 1)\right),$$

where the inharmonicity coefficient ε_k may vary from string to string. A good piano tuner can hear the beats not just between the fundamental of one string, but also between pairs of harmonics. For example, if we have strings A, E, A', where A' is an octave above A and E is in between, the beats between $f_{A,3}$ and $f_{E,2}$ can be used to tune the E relative to the A, and then the beat rate between $f_{A,6}$ and $f_{E,4}$ can be measured. Show how to take measurements between pairs of harmonics (at most two per string) to determine the inharmonicity coefficients. (In practice, A and E are taken in the middle of the piano, and the beat rate between $f_{A,3}$ and $f_{E,2}$ is set to be 'narrow' by about 1 Hz in order to achieve equal temperament. That is, the frequency of the higher string is lowered from the beat-free $\frac{3}{2}$ times that of the lower string until beats occur at 1 per second.)

Chapter 17

Multiple scales and other methods for nonlinear oscillators

17.0.1 Poincaré–Linstedt for the pendulum

Let's go back to our original expansion (14.7) and write $\theta(t) = \epsilon\phi(t)$ as we should have done in the first place. Then, the first two terms in the expansion give

$$\frac{d^2\phi}{dt^2} + \phi = \frac{\epsilon^2}{6}\phi^3 + o(\epsilon^2),$$

or, writing $\delta = \epsilon^2/6$ to save arithmetic,

$$\frac{d^2\phi}{dt^2} + \phi = \delta\phi^3 + o(\delta),$$

with

$$\phi = a_0, \quad \frac{d\phi}{dt} = 0 \quad \text{at} \quad t = 0.$$

The trick here is to expand the period (or frequency) in powers of δ as well as expanding ϕ. So, we seek a solution $\phi(t)$ such that

$$\phi(t + 2\pi/\omega) = \phi(t)$$

for all t, where ϕ and ω both have expansions

$$\phi \sim \phi_0 + \delta\phi_1 + \cdots, \qquad \omega \sim \omega_0 + \delta\omega_1 + \cdots;$$

obviously $\omega_0 = 1$ but we'll derive this en route to more useful results.

We introduce a scaled time $\tau = \omega t$, so that we have 2π-periodicity in τ: $\phi(\tau + 2\pi) = \phi(\tau)$. This gives us

$$\omega^2 \frac{d^2\phi}{d\tau^2} + \phi = \delta\phi^3.$$

Substituting for the expansions for ω and ϕ and collecting terms of $O(1)$ and terms of $O(\delta)$, we get

$$\omega_0^2 \frac{d^2\phi_0}{d\tau^2} + \phi_0 = 0,$$

so the periodicity gives $\omega_0 = 1$ and then $\phi_0 = a_0 \cos\tau$: no surprises there. At $O(\delta)$, we find

$$\begin{aligned} \frac{d^2\phi_1}{d\tau^2} + \phi_1 &= \phi_0^3 - 2\omega_1 \frac{d^2\phi_0}{d\tau^2} \\ &= a_0^3 \cos^3\tau + 2\omega_1 \cos\tau \\ &= \tfrac{1}{4} a_0^3 \cos 3\tau + \left(\tfrac{3}{4} a_0^3 + 2\omega_1 a_0\right) \cos\tau. \end{aligned}$$

This time, we *can* eliminate the secular terms, which arise from the resonance between $\cos\tau$ on the right-hand side, with $\cos\tau$ which is also a solution of the homogeneous equation. We just set its coefficient equal to zero, and so we take

$$\omega_1 = -\frac{3a_0^2}{8}.$$

It's then straightforward to show that

$$\phi_1 = \frac{a_0^3}{32}\left(\cos\tau - \cos 3\tau\right).$$

If you want, you can verify this result by integrating the full pendulum equation exactly and then expanding the period for small initial amplitude.

Exercises

1. **Exact pendulum.**

   Multiply the undamped pendulum equation

   $$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

   by $d\theta/dt$ and integrate, using the initial conditions $\theta = \epsilon a_0$ and $d\theta/dt = 0$. Separate the variables in this first order equation to get an expression for half the period (if you want to look it up, it's an elliptic integral). Expand the integrand for small ϵ and integrate to confirm the Poincaré–Linstedt result.

2. **Precession of the perihelion of Mercury.** Recall that under Newtonian theory the planets move around the sun under the central force $-GM/r^2$ per unit mass, where M is the sun's mass and G is the universal gravitational constant (the forces due to other planets are ignored). Suppose that when it is nearest the sun (perihelion), Mercury is at a distance a from the sun and is travelling with speed v. Show that the equations of motion in plane polar coordinates,

$$\ddot{r} - r\dot{\theta}^2 = -GM/r^2, \quad r^2\dot{\theta} = av,$$

and the substitution $u = 1/r$, lead to

$$\frac{d^2u}{d\theta^2} + u = GM/a^2v^2, \quad \text{with} \quad u = 1/a, \quad \frac{du}{d\theta} = 0 \quad \text{at } \theta = 0,$$

and show that $u = A + B\cos\theta$ for some A and B which you should find. Sketch the orbit and note that it is 2π–periodic in θ.

Why is there no $\sin\theta$ term?

The theory of general relativity gives the modified equation

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{a^2v^2}\left(1 + \frac{3v^2a^2}{c^2}u^2\right),$$

where c is the speed of light. Writing $\epsilon = v^2/c^2$, find the solution up to $O(\epsilon)$ with the same initial conditions, and show that it is not 2π-periodic in θ (it may be periodic in t though). Show that the next perihelion (i.e the next value of θ at which $du/d\theta = 0$) occurs at $\theta \sim 2\pi + 2\pi \times 3(GM/av^2)^2\epsilon$. (Note that if $u(\theta;\epsilon) \sim u_0(\theta) + \epsilon u_1(\theta) + \cdots$, and $u_0'(\theta_0) = 0$, then the value of θ at which $u' = 0$ is found by writing it as $\theta \sim \theta_0 + \epsilon\theta_1 + \cdots$, and expanding the equation $u_0'(\theta_0 + \epsilon\theta_1 + \cdots) + \epsilon u_1'(\theta_0 + \cdots) = 0$ to $O(\epsilon)$. Here $\theta_0 = 2\pi$.)

Confirm your analysis by carrying out the Poincaré–Lindstedt expansion.

This result has been used as a test of general relativity. If $a = 46$ million km, $v \approx 60$ km/s, $G = 6.6710^{-11}$ N m^2 kg^{-1}, how big is the shift per (Mercury) year?

3. **Van der Pol and Rayleigh.** The Van der Pol equation is

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0, \quad \epsilon > 0.$$

It was written down as a model for a spontaneously oscillating valve circuit: by considering the damping term explain why this is plausible.

Where (for what values of x and $\dot{x}$) is energy taken out and where is it put in?

Rayleigh's equation

$$\ddot{x} + \epsilon(\dot{x}^2/3 - 1)\dot{x} + x = 0$$

was written down in connection with a model for a violin string. Show that it can be transformed into the Van der Pol equation by differentiation.

You may need one or other of the expressions $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$, $\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$.

Take $\epsilon \ll 1$ in the Van der Pol equation, and show that the periodic solution of the form $A\cos\tau + \epsilon u_1(\tau) + \cdots$, where $\tau = \omega t$ and $\omega \sim 1 + \epsilon\omega_1 + \cdots$, is only possible if $A = 2$ and $\omega_1 = 0$.

Harder: draw the phase plane, noting the existence of this periodic solution (known as a *limit cycle*).

Chapter 18

Boundary layers

18.1 Introduction

When might we not be able to construct a regular perturbation expansion for a function in terms of a parameter $\epsilon \to 0$? Or, if we have one, where might it not be valid? One thing that might go wrong is that either the function we are approximating, or the approximation itself, may have singularities. Another is that the approximation may slowly drift away from the true solution, as we saw for the second term of the small-amplitude regular expansion for the pendulum. A third possibility is that our function oscillates very rapidly, with a period of, say, $O(\epsilon)$: we look at this case in Chapter **??**. A fourth possibility is that the function changes rapidly in a very small layer, say of width $O(\epsilon)$, but is smooth elsewhere. Such a small layer is known as a *boundary layer* if attached to the boundary of the solution interval or domain, and an *interior layer* if it is internal; see Figure 18.1.

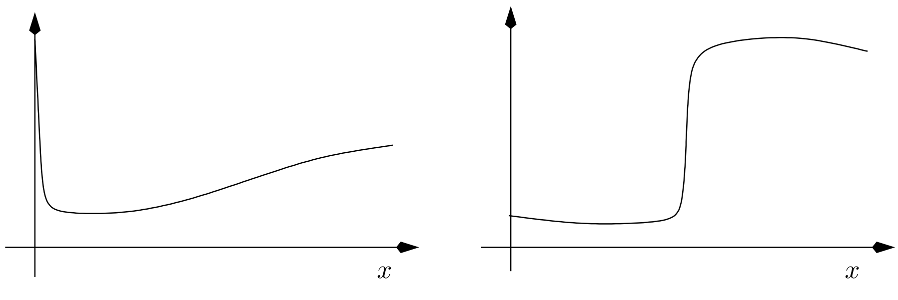

Figure 18.1: A function with a boundary layer at the origin, and one with an internal layer.

18.2 Functions with boundary layers; matching

Some functions come with built-in boundary layers. A prototype example, which crops up all over the place in applications, is

$$f(x;\epsilon) = e^{-x/\epsilon} \qquad \text{for} \quad 0 < x < 1, \quad \epsilon \to 0.$$

This function starts off with a value of 1 at $x = 0$, and becomes negligibly small, certainly smaller than any power of ϵ, by the time $x \gg O(\epsilon)$. All its effort is concentrated in a boundary layer of thickness $O(\epsilon)$ near the origin. This example is rather trivial, but it is fairly clear that if f is a bit more complicated, say

$$f(x;\epsilon) = e^{-x/\epsilon} g(x) + h(x),$$

where $g(x)$ and $h(x)$ are $O(1)$ functions, then we don't need to know all the details of g and h to have a pretty good idea of what f does. When $x = O(1)$, the term $e^{-x/\epsilon}$ is so small that we can forget about it, and we have the *outer expansion*

$$f(x;\epsilon) \sim h(x) + \text{exponentially small correction}$$

(the exponentially small correction often goes by the name of *transcendentally small terms*). On the other hand, when x is small, we expect $g(x)$ and $h(x)$ to be close to their initial values $g(0)$ and $h(0)$, so that

Notice that the limit as $x \to 0$ of the outer solution, $h(0)$, is not in general equal to $f(0;\epsilon)$. It is the job of the boundary layer to accommodate this discrepancy.

$$f(x;\epsilon) \sim g(0)e^{-x/\epsilon} + h(0),$$

although here it is not quite so obvious how big the error is.

The real point of this discussion is not to tell us how to expand functions which we already know. It is that we can often describe a function with a boundary layer with two expansions, one outer expansion valid away from the boundary layer, and one *inner expansion* valid in the boundary layer. In an application, the full function may be the solution of some horrendously difficult problem;[1] but if we can identify where the boundary layers are we may be able to formulate simpler problems for the inner and outer expansions, and thereby get a good description of the full solution without actually having to find it.

Before we plunge into a series of examples, we should first look a bit more closely at the question of how we 'join up', or *match*, the inner and outer expansions. We'll do this first assuming we know the full function,

[1] The Navier–Stokes equations spring to mind: the viscous boundary layer in high Reynolds number flow is an early and classic example of the technique in action.

so that we just verify that we can do it. Later, we use the matching to convey information between the two regions so as to complete the solution. For example, we may have undetermined constants as the result of solving a differential equation, and we fix these by matching.

18.2.1 Matching

There are various ways of joining together inner and outer expansions, and it is in the nature of the subject that none is universal: there are examples for which any method fails. However, the Van Dyke rule, which we now discuss, is as robust as any, and it certainly works for all the problems in this book.[2]

Let us return to the example we have just discussed, but with a slightly more complicated function

I strongly suggest that you work through the discussion; I have put some stepping stones of the calculation in the margin.

$$f(x;\epsilon) = e^{-x/\epsilon} g(x;\epsilon) + h(x;\epsilon)$$

where $g(x;\epsilon)$ and $h(x;\epsilon)$ have regular expansions

$$g(x;\epsilon) \sim g_0(x) + \epsilon g_1(x) + \cdots, \qquad h(x;\epsilon) \sim h_0(x) + \epsilon h_1(x) + \cdots,$$

valid in the whole domain, here the interval $[0,1]$. For example, take

$$g(x;\epsilon) = x + \epsilon x^2, \qquad h(x;\epsilon) = 1 + \epsilon e^x.$$

We can investigate this more closely by *rescaling* x in the boundary layer, writing $x = \epsilon X$. This gives

$$\begin{aligned} f(x;\epsilon) &= F(X;\epsilon) \\ &= g(\epsilon X)e^{-X} + h(\epsilon X). \end{aligned}$$

Now it should be safe to construct a regular expansion of $g(\epsilon X)$ and $h(\epsilon X)$, to give

$$\begin{aligned} F(X;\epsilon) &\sim e^{-X}\left(g(0) + \epsilon X g'(0) + \cdots\right) + h(0) + \epsilon X h'(0) + \cdots . \\ &\sim F_0(X) + \epsilon F_1(X) + \cdots \end{aligned}$$

Clearly the $O(1)$ term in this expansion, $F_0(X)$, agrees with the intuitive interpretation above. Moreover, as we go out of the boundary layer, that is

[2] A popular alternative is matching via an 'intermediate region' between the boundary layer and the outer solution; see the exercise on page 228. Much cruder is 'patching' in which we simply equate the values of the inner and outer expansions at a set value of (say) x: this cannot inform us about the structure of the problem but it can be a useful part of a numerical attack.

as X becomes very large (i.e. $X \gg \epsilon$), the term e^{-X} becomes exponentially small and can be neglected, so we have

$$\lim_{X\to\infty} F_0(X) = h(0)$$

which is the 'inner' limit of the outer solution.

$f(x) = e^{-x/\epsilon}h(x)$.

Simple ODE example such as

$$\epsilon y' + y = x, \quad y(0) = 1;$$

inner and outer expansions, matching, Van Dyke. Not too much detail.

Simple pde example:

$$\frac{\partial u}{\partial t} = \epsilon^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = 0, \quad u(0,t) = 1$$

Here there is a similarity solution.

Lastly

$$\frac{\partial T}{\partial x} = \epsilon^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

with $T \to 0$ as $|\mathbf{x}| \to \infty$ and $T = 1$ on $y = 0$, $x > 0$.

Then large Pe flow past cylinder? Boussinesq.

Travelling wave soln of burgers (traffic)?

18.3 Case study: cable laying

Recall that in our case study of laying an undersea cable (see Section 5.3), we wrote down a model in which the angle θ between the cable and the horizontal satisfies

$$\epsilon \frac{d^2\theta}{ds^2} - F^* \sin\theta + (F_0 + s)\cos\theta = 0,$$

in which F_0 is an unknown constant (equal to the dimensionless vertical force on the sea bed at the point where the cable touches down), F^* is a known dinensionless constant, and ϵ is a small dimensionless constant measuring the relative importance of cable rigidity and cable weight. The boundary conditions for the problem are

$$\theta = 0, \quad \frac{d\theta}{ds} = 0 \qquad \text{at} \quad s = 0,$$

and θ is prescribed at $s = \lambda$.

This problem is ideally suited to a boundary layer expansion, with a small parameter multiplying the highest derivative. The leading order outer solution $\theta_0(s)$ satisfies

$$\tan\theta_0 = \frac{s+F_0}{F*},$$

and it clearly does not satisfy the conditions at $s=0$. Before investing too much energy in it, let us look at the possibility of a boundary layer near $s=0$. Clearly θ is small in such a layer, and a little playing around, starting with the obvious guess that the boundary layer is for $s=O(\epsilon^{\frac{1}{2}})$, suggests the scalings

$$s=\epsilon^{\frac{1}{2}}\xi, \quad \theta=\epsilon^{\frac{1}{2}}\phi, \quad F_0=\epsilon^{\frac{1}{2}}f_0,$$

following which the leading order term in a regular expansion for ϕ satisfies

$$\frac{d^2\phi_0}{d\xi^2} - F^*\phi_0 + s + f_0 = 0.$$

Because there can be no exponentially growing term, the two boundary conditions at $\xi=0$ tell us both ϕ_0 and f_0:

$$\phi_0(\xi) = \frac{\xi}{F^*} - \frac{1}{(F^*)^{\frac{3}{2}}}\left(1-e^{-\xi(F^*)^{\frac{1}{2}}}\right), \quad f_0 = -\frac{1}{(F^*)^{\frac{1}{2}}}. \tag{18.1}$$

It is easy to see that this matches with the outer solution, since substituting for F_0 and writing $s=\epsilon^{\frac{1}{2}}\xi$ in our expression for θ_0, we see that the inner limit of the outer solution is

$$\epsilon^{\frac{1}{2}}\frac{\xi+f_0}{F^*},$$

which is just the same as the outer limit of the inner solution, obtained by neglecting the exponential term in (18.1).

We have also learned that F_0 is small, so away from the boundary layer the outer solution satisfies

$$\tan\theta_0 = \frac{s}{F^*}.$$

See the exercises for a demonstration that the solution of this equation is a catenary, as we might expect if bending stiffness is negligible.

Exercises

1. **A simple expansion near a singularity.** Consider the function

$$f(x;\epsilon) = \frac{1}{x+\epsilon}$$

as $\epsilon \to 0$. If $x = O(1)$, expand by the binomial theorem to show that

$$f(x;\epsilon) \sim \frac{1}{x} - \frac{\epsilon}{x^2} + \cdots .$$

Clearly this expansion is invalid near $x = 0$, as the first term is singular and the second term is larger than the first. Rescale $x = \epsilon X$ to find a valid approximation for small x. (This technique is useful for integrals of the form $\int_0^1 g(x)/(x+\epsilon)\,dx$.)

Of course, the ser in powers of x/ϵ does not converge $|x| < \epsilon$ (what is th correct series representation in this case?), but pretend we don't know this.

2. **Matching by intermediate regions.** The idea behind this matching principle is to choose a range of values of the independent variable(s) that is large compared to the boundary layer but small compared to the outer region. For example, in the problem described in Section 18.2.1, the intermediate region might be $x = O(\epsilon^{\frac{1}{2}})$. Then both inner and outer expansions are written in terms of an intermediate variable $x = \epsilon^{\frac{1}{2}}\xi$, re-expanded as asymptotic series in this new variable, and compared: they should be the same. Carry out this procedure for the example of Section 18.2.1.

   [MORE DETAILS HERE]

3. $y' + \epsilon y^2 = 0$.

4. **An artificial example.** Find an approximate solution to

$$\epsilon u'' + u' = \frac{u+u^3}{1+3u^2}, \qquad u(0) = 0, \quad u(1) = 1.$$

   First find the outer solution: which boundary condition will it satisfy, and why? Then do the boundary layer near $x = 0$ and carry out the matching. (The right-hand side of this example is selected (a) so that it gives an easy solution to the outer problem and (b) is uniformly Lipschitz in u, so there is no question of blow-up. I very much doubt that the full equation can be solved explicitly, but the approximation tells you all about the structure of the solution.)

You may need to convince yourself by drawing a graph that the equation $u + u^3 = a$ has a unique real root for each a.

5. **Singular expansion for a linear algebra problem.** Consider the problem

$$\begin{pmatrix} 1+\epsilon & \epsilon \\ 1-\epsilon & 2\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix},$$

   where $0 < \epsilon \ll 1$ and p, q are given. Draw the two lines whose intersection is the solution of these equations (a) when $p = q = 1$, (b) when $p = 1$, $q = 2$. What happens as $\epsilon \to 0$? Calculate the exact solution and verify that it is consistent with your graphical analysis, and that it is large when ϵ is small unless $p = q + O(\epsilon)$.

Write the problem as

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Recall the Fredholm Alternative theorem for the linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ (page 132):

- If $\mathbf{A}$ is invertible (in particular if none of its eigenvalues vanishes, so the homogeneous problem $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$), the solution is unique.
- Suppose on the other hand that $\mathbf{A}$ is not invertible, so that there is a vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \mathbf{0}$. That is, 0 is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ the corresponding (right) eigenvector; we'll assume for clarity that 0 is a simple eigenvalue. Then there is also a nontrivial (left eigen) vector $\mathbf{w}$ such that $\mathbf{A}^\top \mathbf{w} = \mathbf{0}$ and
  - If $\mathbf{w}^\top \mathbf{b} \neq 0$, then there is no solution to the original problem;
  - If on the other hand $\mathbf{w}^\top \mathbf{b} = 0$, there is a solution but it is not unique: the difference between any two solutions is a multiple of $\mathbf{v}$.

Now we find an asymptotic expansion for the solution of our problem. Write

$$\mathbf{A} = \mathbf{A}_0 + \epsilon \mathbf{A}_1, \qquad \mathbf{x} \sim \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \cdots .$$

Show that $\mathbf{A}_0$ has 0 for an eigenvalue and calculate the corresponding right and left eigenvectors $\mathbf{v}_0$ and $\mathbf{w}_0$. Deduce that the leading-order problem $\mathbf{A}_0\mathbf{x}_0 = \mathbf{p}$ only has a solution if $p - q = O(\epsilon)$.

From now on, take $p = q = 1$. Write down the general solution of $\mathbf{A}_0\mathbf{x}_0 = \mathbf{p}$ in the form 'particular solution + complementary solution', where the latter is a multiple α of $\mathbf{v}_0$ which is not determined at this order in the expansion. Write down the problem from the $O(\epsilon)$ terms in the expansion, and use the Fredholm alternative to show that it only has a solution if $\alpha = 2$. Noting that $\mathbf{x}_0$ is now uniquely determined, verify that it agrees with the small-ϵ expansion of the exact solution.

The analogy with linear ordinary differential equations suggested by this terminology is exact: see the exercise...(page 132).

6. **Cable laying with small bending stiffness.** In Section 18.3, we derived the equation

$$\tan\theta_0 = \frac{s}{F^*}.$$

for the leading order gradient of a cable with small bending stiffness. Remembering that $\tan\theta_0 = dy/dx = y'$ and that

$$\frac{ds}{dx} = \left(1 + (y')^2\right)^{\frac{1}{2}},$$

show that the solution consistent with $y(0) = 0$ and $y'(0) = 0$ (because $\theta_0(0) = 0$) is

$$y = F^* \left(\cosh(x/F^*) - 1\right).$$

Deduce that the ship's dimensionless position is at

$$x^* = F^* \cosh^{-1}(1 + 1/F^*)$$

and that the tensioner angle θ^* and dimensionless thrust F^* are related by

$$\tan^2\theta^* = \frac{1 + 2F^*}{(F^*)^2}.$$

Chapter 19

’Lubrication theory’ analysis: heat flow in long thin domains

19.1 ‘Lubrication theory’ approximations: slender geometries

We now turn to a class of approximation which derives its name from the classical theory of lubricated bearings in machinery, associated with Reynolds (the end of the 19th century was a great time to be a hydrodynamicist: there were indeed giants on the earth in those days). The distinguishing feature of problems to which it can be applied is that the physical domain is ‘long and thin’ in at least one direction, like a plate or rod. One might think of a lubrication solution as being ‘all boundary layer’; moreover, the geometry tells us where the boundary layer is. We scale the coordinate(s) in the ‘thin’ direction differently from the rest, and thereby hope to formulate a simpler problem by exploiting the smallness of the *slenderness parameter*

$$\epsilon = \frac{\text{typical thickness}}{\text{typical length}}.$$

Indeed, the full problem is usually very hard if not impossible to solve either explicitly or numerically, and even if we could solve it we would not necessarily gain understanding. As so often, it is usually very difficult even to prove that the lubrication approximation converges to the full solution in the appropriate limit (one variety of ‘rigorous asymptotics’).

This chapter is longer than most in the book. You can find excellent descriptions of most of the earlier material in other standard texts, but although lubrication expansions are common in practice and in research papers, they don’t feature prominently in textbooks. We’ll see applications to sheets and jets of fluids, as well as the original Reynolds problem, but we'll start with some simple problems in heat flow.

19.2 Heat flow in a bar of variable cross-section

We start with a very simple example: heat flow in a bar of variable cross-section, with insulated sides. Only in very rare cases can this problem be solved exactly, and a geometry of this kind does not lend itself very readily to a simple numerical discretisation. However, we can find a very good approximation to the solution with relatively little effort.

Consider steady heat flow in the domain $0 < x < L$, $-h(x) < y < h(x)$, where

$$\epsilon = \frac{H_0}{L} \ll 1,$$

in which H_0 is a 'typical size' for the bar thickness $h(x)$; this means that we can write

$$h(x) = H_0 H(x/L)$$

for some $O(1)$ function H. Let us also impose a temperature drop from $T = T_i$ at the inlet $x = 0$ to $x = L$, and have perfectly insulated sides. The temperature T satisfies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < L, \quad -h(x) < y < h(x),$$

with

$$T(0, y) = T_i, \qquad T(1, y) = 0,$$

and

$$\mathbf{n} \cdot \nabla T = \frac{\partial T}{\partial n} = 0 \qquad \text{on} \quad y = \pm h(x).$$

Let us first see what the answer is, by a physical argument. Then we'll derive it more mathematically. We argue as follows:

1. The heat flux is approximately unidirectional, along the bar, because no heat is lost through the sides. Thus, $T(x, y)$ is approximately independent of y (that is, it is approximately equal to its average across the bar; see the exercises), and we write $T(x, y) \approx T_0(x)$.

2. The heat flux $Q(x)$ across any line $x =$ constant is exactly equal to

$$\int_{-h(x)}^{h(x)} -k \frac{\partial T}{\partial x}\, dy.$$

Because $\partial T/\partial y \approx 0$, we have

$$Q(x) \approx -2hk\frac{\partial T}{\partial x}, \quad \text{that is} \quad .Q(x) \approx -2hk\frac{dT_0}{dx}.$$

3. Heat is conserved, so $dQ/dx = 0$, that is

$$\frac{d}{dx}\left(h(x)\frac{dT_0}{dx}\right) \approx 0,$$

and the solution of this ordinary differential equation, with $T_0(0) = T_i$, $T_0(L) = 0$, gives the 'leading order' behaviour of $T(x, y)$.

There is nothing at all wrong with this argument. However, we would like to be able to do a bit better. We would like to know how big the error is, when the approximation is valid and, most of all, how to attack more complicated long-thin problems where the physical argument is less clear-cut. This is what lubrication theory, in its general sense, does.

The crux of the lubrication approach is to exploit the slenderness by scaling x and y differently. We write

$$x = LX, \quad y = H_0Y;$$

that is, *we scale each variable with its own natural length scale.* This is the distinctive feature of the lubrication approach. Making the trivial scaling of T with T_i and dropping the primes, we find that

Recall that $h(x) = H_0H(X)$.

$$\epsilon^2\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = 0, \qquad 0 < X < 1, \quad -H(X) < Y < H(X),$$

with

$$T = 1 \quad \text{on} \quad X = 0, \qquad T = 0 \quad \text{on} \quad X = 1.$$

The conditions on $y = \pm h(x)$ take a little more work to scale. We have

$$\mathbf{n} = \frac{(\pm 1, h'(x))}{(1 + (h'(x))^2)^{\frac{1}{2}}} \quad \text{for } y = \pm h(x) \text{ respectively,}$$

and so $\mathbf{n} \cdot \nabla T = 0$, namely $\pm\,\partial T/\partial y - h'(x)\,\partial T/\partial x = 0$, becomes

$$\pm\frac{\partial T}{\partial Y} - \epsilon^2 H'(X)\frac{\partial T}{\partial X} = 0 \quad \text{on} \quad Y = \pm H(X).$$

Notice that the solution domain is now $O(1) \times O(1)$, but the small parameter ϵ has been moved into the field equation and boundary conditions.

Let us try writing

$$T(X,Y) \sim T_0(X,Y) + \epsilon^2 T_1(X,Y) + \cdots .$$

It is easy to see that the expansion is only in powers of ϵ^2. But you should write it all out to get a feel for how it works.

Then we find that

$$\frac{\partial^2 T_0}{\partial Y^2} = 0,$$

which together with the leading order approximation

$$\frac{\partial T_0}{\partial Y} = 0 \quad \text{on} \quad Y = \pm H(X)$$

means that

$$T_0 = T_0(X),$$

a function of X *which is as yet unknown.* That is all the information we get from the leading order equations and boundary conditions.

In order to find T_0, we have to look at the problem for T_1. This is

$$\frac{\partial^2 T_0}{\partial Y^2} + \epsilon^2 \frac{\partial^2 T_0}{\partial X^2} + \epsilon^2 \frac{\partial^2 T_1}{\partial Y^2} + \cdots = 0.$$

$$\frac{\partial^2 T_1}{\partial Y^2} = -\frac{\partial^2 T_0}{\partial X^2},$$

whose solution is clearly

$$T_1(X,Y) = -\tfrac{1}{2} Y^2 \frac{\partial^2 T_0}{\partial X^2} + \text{an arbitrary function of } X.$$

Aide-arithmétique:

$$\pm\left(\frac{\partial T_0}{\partial Y} + \epsilon^2 \frac{\partial T_1}{\partial Y}\right) - \epsilon^2 H' \frac{\partial T_0}{\partial X} + \cdots = 0.$$

Notice that the arbitrary function of X disappears; it is only found at $O(\epsilon^4)$.

The $O(\epsilon^2)$ terms in the boundary conditions are

$$\pm \frac{\partial T_1}{\partial Y} - H'(X) \frac{d^2 T_0}{dX^2} = 0 \quad \text{on} \quad Y = \pm H(X),$$

and putting these together we find that

$$-H(X) \frac{d^2 T_0}{dX^2} - H'(X) \frac{dT_0}{dX} = 0,$$

which is

$$\frac{d}{dX}\left(H(X) \frac{dT_0}{dX}\right) = 0$$

in confirmation of our intuitive argument. We have found out more, though: we now know that the error is $O(\epsilon^2)$ (and we could calculate it if we felt strong enough). We also know that the expansion will not work if any of the terms that we have assumed are $O(1)$ are large. In particular, it is not guaranteed to work if $H'(X)$ is large.

Remarks

Note the following features of the analysis, which are very common in this and other approximation schemes:

- The full problem (which here is an elliptic partial differential equation, Laplace's equation) has a unique solution.
- The leading-order approximate problem (here the ordinary differential equation $\partial^2 T_0/\partial Y^2 = 0$) does not have a unique solution.
- We eliminate the non-uniqueness by going to higher order, $O(\epsilon^2)$, in the expansion, and find a solvability condition which resolves the indeterminacy in the lowest order solution. This condition is essentially the Fredholm Alternative theorem. In the exercises you can see the the process of introducing indeterminacy at one order in an expansion, then resolving it at the next, for the very simple linear algebra problem

$$\begin{pmatrix} 1+\epsilon & \epsilon \\ 1-\epsilon & 2\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

19.3 Heat flow in a long thin domain with cooling

Let us more briefly look at a variation on this problem. Consider steady heat flow in the rectangular domain $0 < x < L$, $-H_0 < y < H_0$, where

$$\frac{H_0}{L} = \epsilon \ll 1,$$

with a temperature drop from T_i at $x = 0$ to 0 $x = L$, but now with Newton cooling at the sides, with a background temperature of 0 and heat transfer coefficient Γ. The temperature $T(x, y)$ satisfies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

with

$$T(0, y) = T_i, \qquad T(L, y) = 0,$$

and

$$\pm k \frac{\partial T}{\partial y} + \Gamma T = 0 \qquad \text{on} \quad y = \pm H_0.$$

Of course we can solve this problem by an eigenfunction expansion (see the exercises). But in a more complicated problem we might not be so clever, so let's see what the lubrication approach has to say.

We first find the answer by an elementary physical argument. *If* the heat flux is mostly in the x-direction (which is not quite so obvious as before), so that we can still work with the average of the temperature across the bar, and *if* the heat loss is proportional to this average temperature, then a straightforward 'box' argument shows that

$$\text{gradient of heat flux} = \text{rate of cooling},$$

or, again writing $T_0(x)$ for the approximate temperature,

$$-k\frac{d^2T_0}{dx^2}) \approx \Gamma T,$$

an ordinary differential equation for the approximate temperature, to be solved with $T_0 = T_i$ at $x = 0$ and $T_0 = 0$ at $x = L$. However, there is more prima facie doubt about this argument: for example, it requires heat to flow out of the bar, so $\partial T/\partial y$ cannot vanish, while maintaining that it is OK to work with the averaged value of T. Is this consistent? We know that it works when $\Gamma = 0$, the insulated case treated above, and it would be nice to know the other values of the heat transfer coefficient for which this approximation is valid, and to know the approximate temperature profile within the material (so we can verify that it is indeed nearly one-dimensional).

As above, we write

$$x = LX, \quad y = H_0 Y,$$

and scale $T(x, y)$ with T_i, to find that

$$\epsilon^2\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = 0, \qquad 0 < X < 1, \quad -1 < Y < 1,$$

with

$$T(0, Y) = 1, \qquad T(1, Y) = 0,$$

and

$$\frac{\partial T}{\partial Y} \pm \gamma T = 0 \quad \text{on} \quad Y = \pm 1;$$

here $\gamma = \epsilon L\Gamma/k$ is the dimensionless heat transfer coefficient, also called a Biot number (see page 55).

As the analysis below confirms, the most interesting case is when $\gamma = O(\epsilon^2)$, say $\gamma = \epsilon^2\alpha^2$, where $\alpha^2 = O(1)$ and the square is for later convenience, so we will proceed on that basis. If $\alpha^2 \ll O(1/\epsilon^2)$ then there is no heat loss through the sides $y = \pm H_0$ to leading order: that is, almost all the heat is conducted linearly from $X = 0$ to $X = 1$. (The small correction can be calculated by a regular perturbation expansion.) On the other hand, if $\alpha \gg O(1/\epsilon^2)$, then almost all the heat is lost in a small region near $X = 0$ (see the exercises).

As above, we expand

$$T(X,Y) \sim T_0(X,Y) + \epsilon^2 T_1(X,Y) + \cdots .$$

Then we find, as before,

$$T_0 = T_0(X),$$

as yet unknown. So, we move on to the problem for T_1,

$$\frac{\partial^2 T_1}{\partial Y^2} = -\frac{d^2 T_0}{dX^2}, \tag{19.1}$$

with

$$\frac{\partial T_1}{\partial Y} \pm \alpha^2 T_0(X) = 0 \quad \text{on} \quad Y = \pm 1. \tag{19.2}$$

The solution of (19.1) is clearly

$$T_1(X,Y) = -\tfrac{1}{2}\frac{d^2 T_0}{dX^2} Y^2 + \text{an arbitrary function of } X$$

and then from the boundary condition (19.2) we find an equation for $T_0(X)$:

$$-\frac{d^2 T_0}{dX^2} + \alpha^2 T_0 = 0,$$

which, after undoing the scalings, is exactly what we derived by a physical argument earlier. Incorporating the boundary conditions at $X = 0$ and $X = 1$, we have

$$T_0(X) = \frac{\sinh \alpha(1-X)}{\sinh \alpha},$$

and we can of course construct higher order corrections if we want.

19.4 Advection-diffusion in a long thin domain

Let's look at an extension of our previous examples, to include advection along the domain. Suppose that the material of our domain $0 < x < L$, $-H_0 < y < H_0$ is moving with speed U in the x-direction (think of modelling heat lost by hot water flowing through a radiator). The model for the steady temperature field is

$$\rho c U \frac{\partial T}{\partial x} = k \nabla^2 T,$$

and let's take boundary conditions

$$T = T_i \quad \text{on} \quad x = 0,$$

modelling a specified inlet temperature,

$$T = 0 \quad \text{on} \quad y = \pm H_0,$$

modelling excellent heat transfer to the surroundings, and

$$\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = L.$$

This last condition is not in fact an insulating boundary condition (remember the heat flux is $\rho c T(U, 0) - k\nabla T$), but rather a rough guess at a plausible outflow condition; it's always hard to know what to prescribe on an outflow boundary of this kind. But in any case the message of our analysis below is that it doesn't much matter what we do at this downstream end. We can even impose the condition $T = 0$, which is physically more or less impossible to realise, and the solution upstream won't be enormously affected (this case is dealt with in the exercises).

As in the previous examples, we scale x with L, y with $H_0 = \epsilon L$, and T with T_i, to get

$$\frac{\rho c U H_0^2}{kL} \frac{\partial T}{\partial X} = \frac{\partial^2 T}{\partial Y^2} + \epsilon^2 \frac{\partial^2 T}{\partial X^2}$$

with boundary conditions

$$T(0, Y) = 1, \quad T(X, \pm 1) = 0, \quad T(1, Y) = 0.$$

The dimensionless number

$$\mathrm{Pe} = \frac{\rho c U H_0^2}{kL}$$

is a Peclet number, measuring the relative effects of advection in the x-direction and conduction in the y-direction. We assume that it is $O(1)$ and, just for clarity, that it is equal to 1. As in the previous example, this is the only balance for which interesting action happens over all the length of our domain. Put another way, this is the balance for which the system can effectively transfer heat from the interior to the exterior.

We could of course again use an eigenfunction expansion (see the exercises). But again, this is messy. Instead, write

$$T(X,Y) \sim T_0(X,Y) + \epsilon^2 T_1(X,Y) + \cdots,$$

and it is soon clear that the leading order problem is

$$\frac{\partial T_0}{\partial X} = \frac{\partial^2 T_0}{\partial Y^2}, \quad 0 < X < 1, \tag{19.3}$$

with

$$T_0 = 0 \quad \text{on} \quad Y = \pm 1.$$

We now have to choose whether to impose $T_0 = 1$ at $X = 0$ or $\partial T_0/\partial X = 0$ at $X = 1$. We can't have both, as (19.3) is a parabolic equation with X as the 'timelike' direction. This gives us the clue: the equation is forward from $X = 0$ and backward from $X = 1$, and only the former gives us a well-posed problem. So, we take $T_0 = 1$ at $X = 0$.

The solution with $T_0(0,Y) = 1$ is easily found in the form

$$T_0(X,Y) = \sum_{n=0}^{\infty} \frac{2(-1)^n}{n+\frac{1}{2}} \cos\left((n+\tfrac{1}{2})\pi Y\right) e^{-(n+\frac{1}{2})^2\pi^2 X},$$

and of course it does not satisfy the condition at $X = 1$. We deal with this by introducing a boundary layer there. We want to rescale $X - 1$ so as to bring back the neglected term $\partial^2 T_0/\partial X^2$. So, we write $X - 1 = \delta\xi$, where $\xi < 0$, to give

$$\frac{1}{\delta}\frac{\partial T_0}{\partial \xi} = \frac{\epsilon^2}{\delta^2}\frac{\partial^2 T_0}{\partial \xi^2} + \frac{\partial^2 T_0}{\partial Y^2}.$$

Clearly the only plausible choice is to balance the first two terms, taking

$$\delta = \epsilon^2$$

(so this boundary layer is very small); then, writing T_b for the temperature in the boundary layer, we have

$$\frac{\partial T_b}{\partial \xi} = \frac{\partial^2 T_b}{\partial \xi^2} + \epsilon^2 \frac{\partial^2 T_b}{\partial Y^2}, \quad \xi < 0,$$

with

$$T_b(\xi, \pm 1) = 0, \quad \frac{\partial T_b}{\partial \xi}(0,Y) = 0.$$

The leading order term in a regular expansion in powers of ϵ^2 is easily found to be

The condition at $\xi = 0$ rules out the exponential solution of the differential equation.

$$T_{b0}(\xi, Y) = A(Y),$$

where all we know about the arbitrary function A is that $A(\pm 1) = 0$. So how do we find it?

We have not yet exploited the information coming into our boundary layer from the main flow. That is, we have to match with the 'outer' solution. At leading order, this is easy. We use the Van Dyke rule with one term in the inner and outer expansions. This tells us that

$$A(Y) = T_0(1, Y) = \sum_{n=0}^{\infty} \frac{2(-1)^n e^{-(n+\frac{1}{2})^2\pi^2}}{n+\frac{1}{2}} \cos\left((n+\tfrac{1}{2})\pi Y\right).$$

This is almost trivial, but perhaps counterintuitively it shows that the matching at leading order is between the values of the temperature and not its gradient, even though the boundary condition we have imposed is on the latter. We have to go to higher order to see this matching too; this is requested in the exercises, and here we just note that it is very plausible that the $O(\epsilon^2)$ term in the inner expansion, $\epsilon^2 T_b(\xi, Y)$ can match with an $O(1)$ outer temperature gradient because $\epsilon^2 \partial/\partial\xi = \partial/\partial X$.

Notice again that, where the full problem is elliptic, requiring boundary conditions all round the domain, the approximate problem is parabolic and we cannot impose a condition at $x = L$. The deficit is made up by the boundary layer, which allows the outer solution to accommodate to whatever we want at $x = L$ (because the boundary layer is very thin, it might not be easy to resolve numerically). Again, the approximate analysis tells us a lot about the structure of the problem, in both qualitative and quantitative terms.

Exercises

1. **Heat flow in a bar of variable cross-section: I.** In the problem of Section 19.2, define

$$\bar{T}(x) = \int_{-h(x)}^{h(x)} T(x, y)\, dy.$$

Show that $\bar{T}(x)$ exactly satisfies the equation

$$\frac{d}{dx}\left(h(x)\frac{d\bar{T}}{dx}\right) = 0$$

and interpret this physically.

2. **Heat flow in a bar of variable cross-section: II.** In this exercise we find an exact solution for steady heat flow in a long thin bar of variable cross-section, exploiting the fact that Laplace's equation is invariant under conformal maps.

   Consider the long thin rectangle $1 < \xi < e$, $-\epsilon < \eta < \epsilon$, $\epsilon \ll 1$. Show that its image under the conformal map $x + iy = \log(\xi + i\eta)$, with the branch cut out of the way on the negative real axis, is very close to the region between the curves $y = \pm\epsilon e^{-x}$, $0 < x < 1$. Show that the solution for steady heat flow in the rectangle, with $T = 0$ at $\xi = 1$, $T = 1$ at $\xi = e$, and insulated sides, is $T = (\xi - 1)/(e - 1)$. Writing this in terms of x and y, verify that this exact solution is consistent with the approximate solution derived in Section 19.2. Use other conformal maps to construct similar examples.

   How big is the error?

3. **Heat flow in a long thin domain.** Consider eigenfunction expansion solution to the problem

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < L, \quad -H < y < H,$$

   with

$$T(0, y) = 1, \qquad T(L, y) = 0,$$

   and

$$\pm k\frac{\partial T}{\partial y} + \Gamma T = 0 \qquad \text{on} \quad y = \pm H.$$

   Separate the variables to find eigenfunctions of the form

$$T_n(x, y) = \cos\alpha_n y \sinh\alpha_n(L - x)$$

   and show that the homogeneous boundary conditions on $x = L$, $y = \pm H$ are all satisfied provided that

$$\alpha_n \tan(\alpha_n H) = \Gamma/k$$

   (note that α_n is dimensional, 1/length). Verify that the eigenfunctions are orthogonal in y, and hence use the condition on $x = 0$ to calculate the coefficients in the expansion

$$T(x, y) = \sum_n a_n T_n(x, y).$$

   Verify that as $H/L \to 0$, this solution is accurately approximated by the 'lubrication' model of the text; consider all cases for the size of Γ.

4. **More heat flow in a long thin domain.** Suppose that the domain of the previous exercise is

$$0 < x < L, \quad -H_0\left(1 + f(x/L)\right) < y < H_0\left(1 + f(x/L)\right),$$

where $f(0) = 0$, $f(1) = 1$, and f is smooth. Suppose that the heat transfer boundary condition is

$$k\mathbf{n} \cdot \nabla T + \Gamma T = 0$$

on the lateral boundaries, where $\mathbf{n}$ is the unit normal. Would you be able to write down an eigenfunction expansion now? Show that the dimensionless model is

$$\epsilon^2 \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = 0, \qquad 0 < X < 1, \quad -1 - f(X) < Y < 1 + f(X),$$

with

$$T = 1 \quad \text{on} \quad X = 0, \qquad T = 0 \quad \text{on} \quad X = 1$$

and

$$\frac{\partial T}{\partial Y} \mp \epsilon f'(X) \frac{\partial T}{\partial X} \pm \epsilon^2 \alpha^2 \left(1 + \epsilon^2 f'^2(X)\right)^{\frac{1}{2}} T = 0 \quad \text{on} \quad Y = \pm\left(1 + f(X)\right)$$

(α is as defined in the text on page 236). Deduce that there is now a term of $O(\epsilon)$ in the expansion for T and find the ordinary differential equation for T_0.

5. **Still more heat flow in a long thin domain.** Consider the model above, but for a rectangular domain, namely

$$\epsilon^2 \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = 0, \quad 0 < X < 1, -1 < Y < 1,$$

with

$$T = 1 \quad \text{on } X = 0, \qquad T = 0 \quad \text{on } X = 1$$

and

$$\frac{\partial T}{\partial Y} \pm \epsilon^2 \alpha^2 T = 0 \quad \text{on } Y = \pm 1.$$

Suppose that $\epsilon^2 \alpha^2 = 1/\delta$, where $\delta \leq O(\epsilon)$ (by which I mean that $\delta = O(\epsilon)$ or $\delta = o(\epsilon)$), so that the heat transfer coefficient is large and hence $\epsilon^2 \alpha^2 \gg 1$. Show that scaling X with ϵ via $X = \epsilon \xi$ leads to the problem

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial Y^2} = 0, \qquad 0 < \xi < 1/\delta, \quad -1 < Y < 1,$$

with boundary conditions

$$T = 1 \quad \text{on} \quad X = 0, \qquad T = 0 \quad \text{on} \quad X = 1/\delta$$

and

$$T \pm \delta \frac{\partial T}{\partial Y} = 0 \quad \text{on} \quad Y = \pm 1.$$

Show that the leading order term in a regular expansion in powers of δ satisfies

$$\frac{\partial^2 T_0}{\partial \xi^2} + \frac{\partial^2 T_0}{\partial Y^2} = 0, \qquad 0 < \xi < \infty, \quad -1 < Y < 1,$$

with boundary conditions

$$T_0 = 1 \quad \text{on} \quad X = 0, \qquad T_0 \to 0 \quad \text{as} \quad X \to \infty$$

and

$$T_0 = 0 \quad \text{on} \quad Y = \pm 1.$$

Solve this problem by conformal mapping, an eigenfunction expansion, or a Fourier sine transform in ξ. Verify that the solution decays exponentially as $\xi \to \infty$, thereby justifying the replacement of the condition at $\xi = 1/\delta$ by one at $\xi = \infty$.

Show further that $\partial T/\partial Y$ becomes very large as $\xi \to 0$ on $Y = \pm 1$. Deduce that the expansion is not valid near the two corners $(0, \pm 1)$. Consider an inner expansion near $(0, -1)$: show that in coordinates

$$\xi = \delta \tilde{\xi}, \quad Y = -1 + \delta \tilde{Y}$$

and, with $T_0(\xi, Y) \sim \tilde{T}_0(\tilde{\xi}, \tilde{Y}) + \cdots$, the inner problem is to leading order

$$\frac{\partial^2 \tilde{T}_0}{\partial \tilde{\xi}^2} + \frac{\partial^2 \tilde{T}_0}{\partial \tilde{Y}^2} = 0, \qquad 0 < \tilde{\xi}, \tilde{Y} < \infty,$$

with

$$\tilde{T}_0 = 1 \quad \text{on} \quad \tilde{\xi} = 0, \qquad \frac{\partial \tilde{T}_0}{\partial \tilde{Y}} - \tilde{T}_0 = 0 \quad \text{on} \quad \tilde{Y} = 0$$

and the matching condition

$$\tilde{T}_0 \to \frac{2\theta}{\pi} \quad \text{as} \quad \tilde{\xi}^2 + \tilde{Y}^2 \to \infty,$$

where θ is the local polar angle. This problem is not easy to solve (the Mellin transform may be best).

6. **Heat flow with convection in a long thin domain.** Suppose that $T(x, y)$ satisfies

$$\rho c U \frac{\partial T}{\partial x} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + Q, \qquad 0 < x < L,\ -H < y < H,$$

where Q is a constant, with the boundary conditions

$$T = 0 \quad \text{on } x = 0,\ y = 0, h, \quad \frac{\partial T}{\partial x} = 0 \quad \text{on } x = L.$$

What is being modelled here? Now suppose that $H/L = \epsilon \ll 1$. Make the equation dimensionless by scaling x with L and y with H, and suppose that the Peclet number $\rho c U H^2/kL$ turns out to be equal to 1. What is the appropriate scale for T?

You should have arrived at the dimensionless equation

$$\frac{\partial T}{\partial X} = \epsilon^2 \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} + 1 \qquad 0 < X < 1,\ 0 < Y < 1,$$

with the boundary conditions

$$T = 0 \quad \text{on } X = 0,\ Y = 0, 1, \quad \frac{\partial T}{\partial X} = 0 \quad \text{on } X = 1.$$

Is this equation elliptic, parabolic or hyperbolic? Briefly indicate how you would find a separation-of-variables solution in the form

$$T(X, Y) = \sum a_n e^{\lambda_n X} \sin n\pi Y$$

where

$$\epsilon^2 \lambda_n^2 - \lambda_n - n^2 = 0.$$

For each $O(1)$ value of n, find expressions for the positive and negative roots of this equation as $\epsilon \to 0$. Find the leading order terms in an approximate solution to the original problem, and explain why the positive roots of the eigenvalue equation for λ_n correspond to the boundary layer contribution to the approximate solution.

7. **And still more on heat conduction in a long thin domain.** Find terms up to $O(\epsilon^2)$ in the outer and inner expansions of the solution of

$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial Y^2} + \epsilon^2 \frac{\partial^2 T}{\partial X^2}, \quad 0 < X < 1,$$

with boundary conditions

$$T(0,Y) = 1, \quad T(X,\pm 1) = 0, \quad T(1,Y) = 0, \quad \frac{\partial T}{\partial X}(1,Y) = 0$$

(you will have to go to $O(\epsilon^4)$ in the outer solution; save ink by writing b_n for the Fourier coefficients). Now carry out the matching to second order using Van Dyke's matching principle

2-term inner expansion of 2-term outer expansion =
2-term outer expansion of 2-term inner expansion.

That is, write the outer expansion in terms of the inner variable $\xi = (X-1)/\epsilon^2$. Expand the result, keeping terms of $O(\epsilon^2)$. Repeat this procedure for the inner expansion (notice how terms involving e^{ξ} are neglected in this expansion, being exponentially small). Compare the two expansions to identify all the unknown functions in the inner expansion.

Repeat the whole problem for the (physically unrealistic) condition $T = 0$ at $X = 1$ (if you are feeling tired by now, just do the $O(1)$ terms).

Can you see physically why we may have to do something different if we try to impose a zero-heat-flux condition at $x = L$?

Chapter 20

Case study: continuous casting of steel

20.1 Continuous casting of steel

One way of producing a continuous bar, or 'strand', of steel is to cast it continuously. Molten steel is poured into a large 'tundish' from which it emerges through a mould slot in the bottom. It is cooled by water pipes in the sides of the slot and, once it has emerged, by water sprays and jets (see Figure 20.1). When the steel emerges, it has a thin solid skin, which becomes thicker as the steel moves down. Nevertheless, the liquid steel extends far down the strand.

It is important to be able to control the location of the solid-liquid boundary, for safety reasons (the molten steel must not be allowed to spill out) and in order to get the metallurgy right. The former involves a very complicated situation near to the mould, which we do not attempt to model here. Instead, we write down a simple two-dimensional model for the latter (we don't tackle the problem of controlling the heat fluxes to achieve a desired solidification rate).

This kind of model can also be used for other solidification processes such as a Bridgeman crystal grower, in which a continuous strand of silicon is solidified very slowly (in order to minimise defects) by being passed along a conveyor belt and cooled from above and below.

Before proceeding, we remind ourselves of the Stefan model for solidification of a pure material. It is an experimental observation that a fixed amount of energy per unit mass is required to melt a pure[1] solid without changing its temperature, and the same amount of energy must be removed

[1]In this case study, we are going to ignore the complication that steel is an alloy.

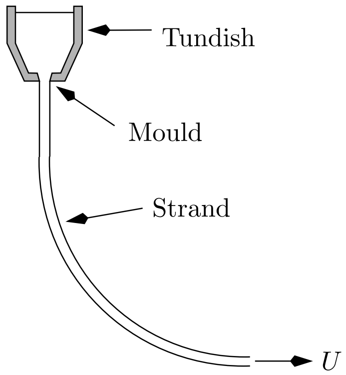

Figure 20.1: Continuous casting of steel

to solidify it. This heat is supplied or removed by the difference between the heat fluxes into and out of the solid-liquid interface. In one space dimension, we can carry out a 'box' argument for the configuration of Figure 20.2, in which solid is to the left of the interface.

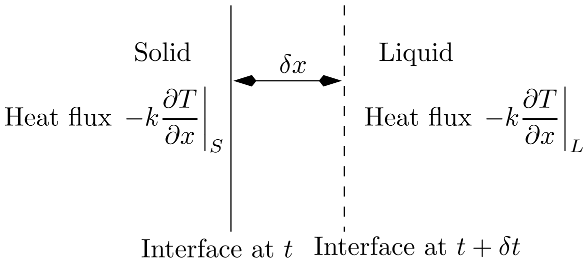

Figure 20.2: The Stefan problem

If the interface moves a distance δx in time δt, the latent heat absorbed (for melting, $\delta x < 0$) or released (for solidification, $\delta x > 0$) by that amount of material in changing phase is

$$\rho\lambda\,\delta x.$$

This must be balanced by the difference in heat fluxes over time δt,

$$\left[-k\frac{\partial T}{\partial x}\right]_S^L \delta t.$$

Hence we derive the Stefan condition for the speed of the interface,

$$\rho\lambda\frac{dx}{dt} = \left[-k\frac{\partial T}{\partial x}\right]_S^L.$$

The right-hand side of this condition is the net rate at which heat is supplied to the interface, while the left-hand side is the rate at which it used up or produced as the interface moves. In more dimensions, this argument is simply generalised so that for an interface with unit normal $\mathbf{n}$ from solid to liquid, and normal velocity V_n in that direction,

$$\rho\lambda V_n = [-k\mathbf{n}\cdot\nabla T]_S^L.$$

If this reminds you of the Rankine–Hugoniot condition (it should), see Exercise **??** for more details.

We are now in a position to write down a model for steady heat flow in the strand of steel. We straighten the strand out, modelling it by a rectangle, thus assuming that the effects of curvature are small, as can be verified later. We write U for the speed of the strand, and we immediately note that the Peclet number is very small, so the temperature approximately satisfies Laplace's equation. We make the further simplification that the liquid steel is exactly at its melting temperature, so we only have to find the temperature in the solid (a *one-phase problem*). We'll also take Newton cooling with a background temperature of T_∞ as a crude model for the effect of the water cooling.

It is important to notice that the liquid-solid interface is unknown: we have to find it as part of the solution. Let's write it as $y = \pm f(x)$ (see Figure 20.3). The solid temperature $T_S(x, y)$ satisfies

$$\frac{\partial^2 T_S}{\partial x^2} + \frac{\partial^2 T_S}{\partial y^2} = 0$$

in the solid region, with

$$\pm k\frac{\partial T_S}{\partial y} + \Gamma\left(T_S - T_\infty\right) = 0$$

on the edges $y = \pm H$.

On the liquid-solid interface $y = \pm f(x)$, we have

$$T_S = T_m,$$

the melting/solidification temperature, and the Stefan condition in the form

$$-k\frac{\partial T_S}{\partial n} = \rho\lambda U\cos\theta,$$

where θ is the angle between the normal to the interface and the x–axis, so that the normal velocity of the interface is $U\cos\theta$. This condition can also

Some details for checking.

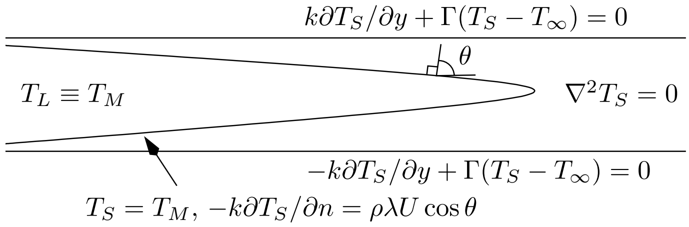

Figure 20.3: Model for continuous casting

be written

$$-k\left(\frac{\partial T_S}{\partial y} - \frac{df}{dx}\frac{\partial T_S}{\partial x}\right) = -\rho\lambda U\frac{df}{dx}.$$

For large x, we impose that $T_S \to T_\infty$, and we won't be too specific about the inlet conditions at this stage.

Now let's make the problem dimensionless. Obviously we'll scale y and f with H, but the length scale L for x is less obvious. We could of course use the length of the strand but a better idea is to *derive* the length scale from the balance between latent heat release and cooling. This also has the merit of telling us directly when our approximation is valid, and how long the molten region is expected to be. So, we write $x = LX$ and $y = HY$, where L is yet to be found but as usual $\epsilon = H/L \ll 1$, and we write $y = \pm f(x)$ as $Y = \pm F(X)$. We also need a scale for the temperature; this is built into the problem as

$$T_S = T_M + (T_M - T_\infty)\,T(X,Y).$$

So, dropping the primes, we have the dimensionless model

$$\frac{\partial^2 T}{\partial Y^2} + \epsilon^2\frac{\partial^2 T}{\partial X^2} = 0$$

in the solid, with the interface conditions

$$T = 0, \quad \frac{\partial T}{\partial Y} + \epsilon F'(X)\frac{\partial T}{\partial X} = \epsilon\tilde{\lambda}F'(X) \tag{20.1}$$

on $Y = \pm F(X)$, where

$$\tilde{\lambda} = \frac{\rho\lambda U}{((k(T_M - T_\infty)/H))}$$

is a dimensionless number which is written in this way to show that it measures the balance between latent heat release from an interface moving with speed U and conduction due to a temperature difference of $T_M - T_\infty$ across a distance of $O(H)$. The factor ϵ on the right-hand side of (20.1) arises because the interface only has a very small normal velocity.

Lastly the scaled cooling conditions are

$$\frac{\partial T}{\partial Y} \pm \alpha(T+1) = 0 \quad \text{on} \quad Y = \pm 1, \tag{20.2}$$

with $\alpha = \Gamma H\,(T_M - T_\infty)$. Bearing in mind the previous examples, we need the cooling rate to be small, so $\alpha \ll 1$, and we also need it to balance the rate of latent heat loss. We therefore determine L by making the choice

$$\epsilon = \alpha,$$

and check later that is it consistent.

Let's concentrate on the part of the strand where the liquid has not all solidified, and expand

$$T(X,Y) \sim T_0(X,Y) + \epsilon T_1(X,Y) + \cdots .$$

By symmetry, we need only focus on the top half of the strand. We easily have

$$T_0 = A_0(X) + B_0(X)Y,$$

Details which should be worked through.

where from the cooling condition (20.2) at lowest order $B(X) = 0$, and then from the melting temperature condition $A(X) = 0$ as well. So,

$$T(X,Y) \sim \epsilon T_1(X,Y) + \cdots ,$$

telling us that the temperature is everywhere within $O(\epsilon)$ of the melting temperature. Continuing, we have

$$T_1(X,Y) = C_1(X)\,(Y - F(X))\,,$$

Automatically incorporating the melting temperature, much more economical than grinding out $T_1 = A_1 Y + B_1$.

and now the '1' in the cooling condition (20.2) comes in to give

$$C_1(X) = -1.$$

Lastly we use the hitherto unexploited latent heat condition (20.1) to find that

$$F'(X) = -\frac{1}{\tilde{\lambda}},$$

so that, if the interface starts off from $Y = 1$ at $X = 0$,

$$F(X) = 1 - \frac{X}{\tilde{\lambda}}.$$

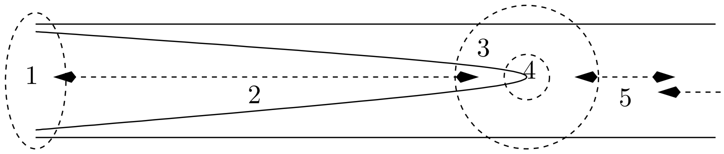

Figure 20.4: Regions for the continuous casting problem.

We have thus predicted the length of the liquid region ($L\tilde{\lambda}$) and the shape of the interface to lowest order (linear).

Clearly this analysis is not valid near the tip of the strand, where the upper and lower free surfaces meet, as the heat flow is obviously two-dimensional there. In fact one can carry out a more detailed analysis involving at least 6 regions (see Figure 20.4). Region 1 is an inlet region, from which all we need to know is a starting value for the interface. We have just analysed Region 2, which matches into Region 3, centred on the tip of the liquid region. This is essentially the problem of a half-line at temperature 0 with temperature of -1 on $Y = \pm 1$. Region 4 is necessary to resolve the singularity at the end of the half-line, and it shows that the tip of the liquid region is parabolic. Returning to Region 3, it matches into the intermediate Region 5, an intermediate region of length $O(H/\epsilon^{\frac{1}{2}})$, which enables the transition into Region 6 in which we finally have an eigenfunction expansion decaying exponentially as $x \to \infty$. Further details are given in the exercises.

Exercises

1.

Chapter 21

Lubrication theory for fluids

21.1 Thin fluid layers: classical lubrication theory

In this chapter, we describe the lubrication theiry analysis of a variety of thin fluid flows. Having done the heat conduction problems of Chapter 19, we shouldn't have too much trouble with the original (eponymous) lubrication theory model of flow of a viscous fluid in a thin bearing bounded by rigid surfaces. Then we'll generalise the approach to find equations for thin viscous sheets with free surfaces.

The simplest configuration is that of a *slider bearing*, in which one rigid surface slides over another as in Figure 21.1. These bearings are common in machinery ranging from the head floating over the hard disc of a computer[1] to enormous pumps and other engines. When the bearing is wrapped round into a circle, so that a rotating shaft can be supported, it is known as a *journal bearing*.

[1]Nowadays the gap between the head and the disc is so small that it is not clear that ordinary continuum models can safely be used for the fluid.

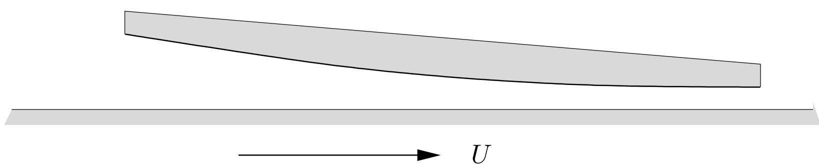

Figure 21.1: A slider bearing.

We'll look at two-dimensional flows only. Let us call the upper surface $y = H_0 H(x/L)$, where L is the length of the bearing and H_0 a representative value for the separation; as usual, $\epsilon = H_0/L \ll 1$. Let us also take axes in a frame in which the upper surface is stationary and the lower surface $y = 0$ moves to the right with velocity $(U, 0)$. The idea behind this bearing is to choose the shape $H(x/L)$ of the upper surface so that fluid dragged into the bearing (remember the no-slip condition on the lower surface) generates a high load-bearing pressure as it is forced through the converging part of the gap.

We'll start from the Navier–Stokes equations for the velocity $\mathbf{u} = (u, v)$ and the pressure p:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0,$$

with the no-slip boundary conditions

$$(u, v) = (U, 0) \quad \text{on} \quad y = 0,$$

$$(u, v) = (0, 0) \quad \text{on} \quad y = H_0 H(x/L).$$

Following our long-thin analysis above, and in contrast to our scaling when we last used these equations, we'll scale x and y differentially, x with L and y with H_0. When we come to the velocity $\mathbf{u} = (u, v)$, we have to scale its two components differentially as well, or we will not conserve mass. Obviously we want to scale u with U, and since (in unscaled variables)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

we need to scale v with ϵU. In the absence of any forced unsteady motion of the upper surface, the natural time scale is then L/U. Lastly, we need a scale P for the pressure p. In the absence of any obvious 'exogenous' scale, we'll work this out from the equations.

As in our analysis above, we use X and Y for the scaled coordinates, but we 'drop the primes' and stick with lower case letters for the dependent variables. It's a nasty hybrid notation, but capitals are so much harder to read, and we'll also be using dimensional equations later in the chapter so we want to be able to distinguish them at a glance.

The X–component of the scaled momentum equation is

$$\frac{\rho U^2}{L}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y}\right) = -\frac{P}{L}\frac{\partial p}{\partial X} + \frac{\mu U}{H_0^2}\left(\epsilon^2\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right).$$

If viscous shear forces are to do their job in generating pressure,[2] we have to choose P to balance terms on the right-hand side of this equation. Thus, $P = \mu UL/H_0^2$ We now have the back-of-the-envelope estimate $LP = \mu UL^3/H_0^2$ of the load per unit distance in the z-direction that this bearing can support.

Compare with the scaling $\mu U/L$ we used in deriving the slow flow equations earlier: here we have $(1/\epsilon^2)\mu U/L$, indicating the effectiveness of the long thin geometry in generating high pressures.

This scaling for p leaves one dimensionless parameter in the problem,

$$\mathrm{Re}' = \epsilon^2 \frac{UL}{\nu} = \epsilon^2 \mathrm{Re},$$

known as the *reduced Reynolds number*. We'll assume it is small; that is, the model is valid when it is small. This means in particular that all the inertial terms, some of which are nonlinear, are neglected.

Crossing off lots of small terms, our leading order model is then

Exercise...

$$\frac{\partial^2 u}{\partial Y^2} = \frac{\partial p}{\partial X}, \quad \frac{\partial p}{\partial Y} = 0, \quad \frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} = 0$$

for $0 < Y < H(X)$, with

$$u = 1, \quad v = 0 \quad \text{on} \quad Y = 0, \qquad u = v = 0 \quad \text{on} \quad Y = H(X).$$

It is straightforward to integrate these equations, firstly noting that $p = p(X)$, then finding

The flow is a combination of a Couette shear (the first two terms) and a Poiseuille flow with pressure gradient dp/dX, so with hindsight we could have written this down.

$$u = 1 - \frac{Y}{H(X)} - \frac{1}{2} Y \left(H(X) - Y\right) \frac{dp}{dX},$$

and lastly using the continuity equation integrated with respect to Y,

$$\frac{d}{dX} \int_0^{H(X)} u(X, Y)\, dY = 0,$$

to find *Reynolds' equation*

$$\frac{d}{dX}\left(H^3 \frac{dp}{dX}\right) = 6 \frac{dH}{dX}$$

for the pressure. Given $H(X)$, we solve this with ambient-pressure conditions at each end, and we can then calculate the load our bearing can support.

[2] At the other end of the viscosity range, one can make a model of an inviscid skimmer held up over a thin layer of water by inertial forces only.

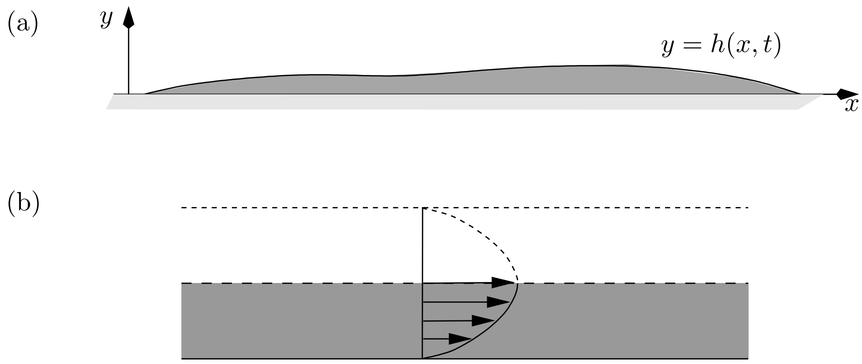

Figure 21.2: (a) Viscous layer spreading under gravity. (b) Velocity profile for Poiseuille flow.

21.2 Thin viscous fluid sheets on solid substrates

For our next application of the lubrication theory approach, we'll derive approximate equations for the evolution of thin sheets and fibres of a viscous fluid. These problems are a little more difficult, because the fluid has one or more *free surfaces*, whose locations have to be determined as part of the solution of the problem. We start with the case of a thin layer spreading out on a horizontal surface, a situation which arises in applications ranging in lateral scale from microns (layers of conductor applied in liquid form to a printed circuit board before being baked solid) through centimetres (paint on a wall, honey spilled on a table) to kilometres (magma flow from a volcano). We also briefly describe the corresponding model for flow on a vertical surface, before looking at free sheets, such as the glass sheets you would find when making a bottle by blowing, or a window by the float glass process. Lastly we look more briefly at manufacture of fibres, for example of glass (optical fibres) or polymer (artificial fabrics).

21.2.1 Viscous fluid spreading horizontally under gravity: intuitive argument

Imagine you spill a puddle of honey on a table. How does it spread out? Assume that the depth is much smaller than the spread, and for now take the two-dimensional situation sketched in Figure 21.2(a). Here is a physical argument, in four steps (all variables are dimensional).

(a) The flow is slow, so we use the Stokes equations (uncontroversial).

(b) The flow is driven by hydrostatic pressure and resisted by viscous shear forces (uncontroversial). The pressure is approximately hydrostatic, because vertical velocities are small enough that the viscous contribution to the forces in that direction is small (not so obvious; believe it for now). Thus

$$p(x, y, t) = \rho g \left(h(x, t) - y\right).$$

(c) The horizontal velocity u is much greater than the vertical velocity v and the free surface is almost horizontal. Moreover, on the free surface the shear stress, which is approximately $\mu\, \partial u/\partial y$, vanishes (uncontroversial, although we might want to check later). We can also regard this as a symmetry condition and thus, locally, the flow looks like the bottom half of flow between two parallel plates separated by $2h$ under a pressure gradient $\partial p/\partial x$ (see Figure 21.2(b)). The velocity profile is therefore parabolic:

$$u(x, y, t) = -\frac{1}{2\mu} y \left(2h(x, t) - y\right) \frac{\partial p}{\partial x},$$

and the horizontal flux is

$$\begin{aligned} Q(x, t) &= \int_0^{h(x,t)} u(x, y, t)\, dy \\ &= -\frac{1}{3\mu} h^3 \frac{\partial p}{\partial x} \\ &= -\frac{\rho g}{3\mu} h^3 \frac{\partial h}{\partial x}. \end{aligned}$$

(d) Mass conservation (uncontroversial) in the form

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0,$$

gives us a nonlinear diffusion equation for $h(x, t)$:

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} \frac{\partial}{\partial x} \left(h^3 \frac{\partial h}{\partial x} \right). \tag{21.1}$$

Note that this dimensional equation tells us the timescale for the spreading out. If x is scaled with L, h with a representative initial value H_0, then the timescale emerges immediately as $\mu L^2/(\rho g H_0^3)$. This looks reasonable: stickier fluids (larger μ) spread out more slowly, as do thin layers or fluids in regions of low g.

If, instead of gravity, surface tension at the interface drives the motion (as would be appropriate for thin layers of paint or conductor on a PCB), a very similar argument (see the exercises) shows that we get the *fourth-order* nonlinear diffusion equation

$$\frac{\partial h}{\partial t} + \frac{\gamma}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial^3 h}{\partial x^3}\right) = 0. \tag{21.2}$$

Not surprisingly, these equations with their evident structure have attracted a lot of theoretical analysis; natural questions to ask include 'if we start with a solution that is positive, does it remain so?' (yes for (21.1) and (21.2), but if h^3 in (21.2) had been h the answer would have been no) or 'if we have a dry patch where $h = 0$, what do we say at its edges?' (conserving mass is not too hard, but the extra condition for the fourth-order equation (21.2) is rather more problematic). Suggestions for further reading are given at the end of the chapter.

21.2.2 Viscous fluid spreading under gravity: systematic argument

You may be convinced by the derivation just given (I think I am). However, there are situations where a more precise approach is essential, so let's warm up for that by rederiving equation (21.1) by a systematic asymptotic approach.

Let's start with the slow flow equations

$$\nabla^2\mathbf{u} = \nabla p - \rho\mathbf{g}, \qquad \nabla\cdot\mathbf{u} = 0 \tag{21.3}$$

for the velocity $\mathbf{u} = (u, v)$, for which we have the no-slip condition

$$u = v = 0 \quad \text{on} \quad y = 0.$$

The big new feature in this problem is the free surface $y = h(x,t)$. It is unknown — we have to find it as part of the solution — and the boundary conditions applied on it are more complicated. The kinematic condition

$$v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{on} \quad y = h(x,t)$$

is easy enough,[3] and the other conditions, which say that no stresses act at the free surface, are written

$$\sigma_{ij}n_j = 0,$$

where

$$\mathbf{n} = (n_j) = \left(-\frac{\partial h}{\partial x}, 1\right) \Big/ \left(1 + \left(\frac{\partial h}{\partial x}\right)^2\right)^{\frac{1}{2}}$$

is the unit normal to the surface and

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is the stress tensor for a Newtonian viscous fluid. We recall from Chapter **??** that the components of σ are

$$\sigma_{ij} = \begin{pmatrix} -p + 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & -p + 2\mu\dfrac{\partial v}{\partial y} \end{pmatrix}.$$

Thus, the two components of the zero-stress condition,

$$\sigma_{11}n_1 + \sigma_{12}n_2 = 0, \quad \sigma_{12}n_1 + \sigma_{22}n_2 = 0,$$

are

$$-\frac{\partial h}{\partial x}\left(-p + 2\mu\frac{\partial u}{\partial x}\right) + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \tag{21.4}$$

$$-\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial h}{\partial x} - p + 2\mu\frac{\partial v}{\partial y} = 0 \tag{21.5}$$

Scale $x = LX$, $y = H_0Y = \epsilon LY$ as usual. Since we expect the flow to be driven by hydrostatic pressure, scale p with $\rho g H_0$. Then the horizontal component of the momentum, balancing $\mu\,\partial^2 u/\partial y^2$ with $\partial p/\partial x$, tells us the scale for u, namely $U_0 = \rho g H_0^3/\mu L$, and the scale for v is ϵU_0 so that mass conservation is not violated (it never is). Lastly our timescale is L/U_0 (or $H_0/\epsilon U_0$). You should check that we get the equations

Again in our hybrid notation in which independent variables are capitalised and dependent ones lower case.

[3]Either think of this as

$$\frac{D}{Dt}(y - h(x,t)) = 0$$

to express the fact that a particle in the surface remains there, or show that the normal to any curve $f(x, y, t) = 0$ is $\mathbf{n} = \nabla f/|\nabla f|$, the normal velocity of a point on the curve is $-(\partial f/\partial t)/|\nabla f|$, and equate this to $\mathbf{n} \cdot \mathbf{u}$.

$$\frac{\partial^2 u}{\partial Y^2} + \epsilon^2 \frac{\partial^2 u}{\partial X^2} = \frac{\partial p}{\partial X}, \tag{21.6}$$

$$\epsilon^2 \frac{\partial^2 v}{\partial Y^2} + \epsilon^4 \frac{\partial^2 v}{\partial X^2} = \frac{\partial p}{\partial Y} - 1, \tag{21.7}$$

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} = 0, \tag{21.8}$$

(the -1 in (21.7) is gravity) with the no-slip condition

$$u = v = 0 \quad \text{on} \quad Y = 0, \tag{21.9}$$

and, on $Y = h(X, T)$, the kinematic condition

$$v = \frac{\partial h}{\partial T} + u \frac{\partial h}{\partial X} \tag{21.10}$$

(note that unlike, say, water waves, our different scalings for u and v mean that we keep the term $u\, \partial h / \partial X$), and the stress-free conditions

$$-\frac{\partial h}{\partial X} \left(-p + 2\epsilon^2 \frac{\partial u}{\partial X} \right) + \frac{\partial u}{\partial Y} + \epsilon^2 \frac{\partial v}{\partial X} = 0, \tag{21.11}$$

$$-\epsilon^2 \left(\frac{\partial u}{\partial Y} + \epsilon^2 \frac{\partial v}{\partial X} \right) \frac{\partial h}{\partial X} - p + 2\epsilon^2 \frac{\partial v}{\partial Y} = 0. \tag{21.12}$$

We've done the hard work. Now we expand u, v, p in regular expansions[4]

$$u \sim u_0 + \epsilon^2 u_1 \quad \text{etcetera},$$

and we decide what order to solve the equations in. Clearly from (21.7) the pressure is hydrostatic to leading order, and from (21.12) it vanishes on $Y = h$, so we put a tick against those two equations, as we won't use them again at this order, and write down

$$p_0 = h(X, T) - Y.$$

Now we find u_0 using (21.6), with (21.9) and (21.11) for boundary conditions:

$$u_0(X, Y, T) = -\frac{1}{2} \frac{\partial h}{\partial X} Y (2h - Y).$$

Next integrate (21.11) with respect to Y and use the other part of (21.9) to find v_0, and substitution into (21.10) gives, as promised,

$$\frac{\partial h}{\partial T} = \frac{1}{3} \frac{\partial}{\partial X} \left(h^3 \frac{\partial h}{\partial X} \right).$$

Undoing the non-dimensionalisation leads immediately to the equation derived above.

A quicker way to do this is to note that the leading order flux is

$$Q_0(X, T) = \int_0^{h(X,T)} u_0(X, Y, T)\, dY$$

and to leading order mass conservation is

$$\frac{\partial h}{\partial T} + \frac{\partial Q_0}{\partial X} = 0.$$

[4] Strictly speaking, we should expand h as well, but as we only ever find the leading order terms we won't bother, sticking with $h(X, T)$.

21.2.3 A viscous fluid layer on a vertical wall

Suppose that our layer of fluid is on a vertical wall (or an inclined plane that is not almost horizontal). In this case gravity acts along the film, with the result that it balances the shear forces directly, rather than being transmitted through the pressure. The intuitive argument to derive the equation of motion is: Think paint.

(a) The flow is approximately unidirectional with velocity $u(x, y, t)$ in the x–direction, down the wall (y is measured out from the wall).

(b) The pressure is everywhere very small (because of zero-stress on the free surface $y = h(x, t)$). Instead, the body force ρg in the x–momentum equation drives the flow.

(c) Remembering that we have no-slip at $y = 0$, that is $u = 0$, and no-stress at $y = h(x, t)$, that is approximately $\partial u/\partial y = 0$, the flow is the same as half of a Poiseuille flow between $y = 0$ and $y = 2h$, driven by a pressure gradient ρg.

(d) The flux is therefore (standard calculation) $gh^3/(3\nu)$ and conservation of mass as above gives

$$\frac{\partial h}{\partial t} + \frac{g}{\nu} h^2 \frac{\partial h}{\partial x} = 0.$$

A systematic derivation of this equation by scaling techniques is asked for in the exercises. Notice that the new equation is *first-order*, not second order as for nearly horizontal flow.

21.3 Thin fluid sheets and fibres

For our last example in this series of models for of thin layers of a viscous fluid, we consider the evolution of a long thin viscous sheet stretched from its ends $x = 0$, L with a characteristic speed U. Now we have not one free surface but two, which adds some complexity to the analysis, as we shall see. This configuration is not very easy to realise, nor is it common in practice, although pizza makers come close (not that dough is anything close to a Newtonian fluid). The best example is probably the float glass process in which a layer of glass, which may be some hundreds of metres long and tens of metres wide but only a few millimetres thick, is floated on a bath of much less viscous molten tin. As it travels from one end to the other, it should reach a state

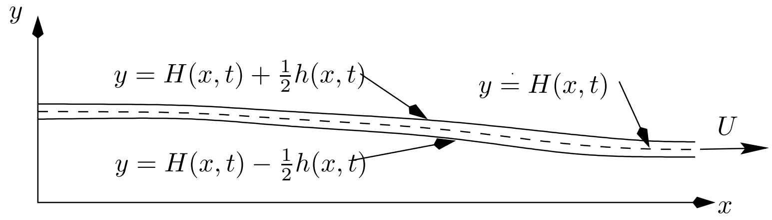

Figure 21.3: Drawing a thin sheet of viscous fluid.

of absolutely smooth pellucid perfection, so it is of vital importance to glass manufacturers to be able to control this process. However, the corresponding axisymmetric situation of a thread or fibre of fluid, which has only one free surface but still no fixed surfaces, is very common. Examples are manufacture of optical fibres from glass and artificial fabric fibres from polymers, both of which involve solidification of a liquid thread (so does making candy floss).[5]

There is a simple physical argument which leads to the correct answer (more or less). Much of it is familiar:

(a) The sheet is nearly flat and the velocity is approximately unidirectional, in the x–direction (along the sheet).

(b) The surfaces are stress-free so the x–velocity does not vary significantly across the sheet: it has the form $u(x, t)$.

This will be the σ_{11} stress component, the force per unit area in the x–direction across a plane with normal in the same direction.

(c) The stretching is resisted by viscous stresses (force per unit area) which, for a Newtonian fluid are proportional to the velocity gradient $\partial u/\partial x$. Thus the total force (per unit length perpendicular to the page) is proportional to $h\,\partial u/\partial x$, where $h(x, t)$ is the thickness of the sheet. As there are no external forces, this must be constant along the sheet:

$$\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) = 0. \tag{21.13}$$

(d) The second equation for h and u is mass conservation:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0,$$

and this completes the model. An identical argument holds for a thin fibre, with $h(x, t)$ replaced by the cross-sectional area $A(x, t)$.

[5]We should really include temperature-dependent viscosity in the model for both these fibres, and non-Newtonian fluid effects for the polymer. We'll keep things simple.

As ever, this analysis raises as many questions as it answers. In particular, it says nothing about the constant of proportionality in the relationship between the resistive stress and the velocity gradient. Clearly this constant is some sort of viscosity, and indeed it has a name, the *Trouton viscosity*, but how is it related to the usual dynamic viscosity μ? We don't need to know this if no forces (such as air drag) act at the surfaces of our sheet, because we cancel from both sides of the momentum balance equation (21.13), but it is crucial otherwise. In any case, if we integrate (21.13) we find

$$h\frac{\partial u}{\partial x} \propto T(t),$$

where $T(t)$ is the tension applied to our sheet, so we need the constant if we are to calculate the total tension needed to stretch the sheet. Only a more detailed analysis can tell us.

Or, replacing h by A, fibre.

21.3.1 The viscous sheet equations by a systematic argument

Let us call the surfaces of the sheet $y = \bar{h}(x,t) \pm \frac{1}{2}h(x,t)$, so that the centreline of the sheet is at $y = \bar{h}(x,t)$: we don't know a priori that it is symmetrical. The dimensional equations that we must solve are very similar to those of the previous section, but with gravity removed and the no-slip condition replaced by zero-stress conditions on both surfaces. We have the slow flow equations

$$\nabla^2\mathbf{u} = \nabla p, \qquad \nabla\cdot\mathbf{u} = 0$$

for the velocity $\mathbf{u} = (u, v)$, with the kinematic and dynamic (zero-stress) conditions

$$v = \frac{\partial}{\partial t}\left(\bar{h} \pm \tfrac{1}{2}h\right) + u\frac{\partial}{\partial x}\left(\bar{h} \pm \tfrac{1}{2}h\right), \qquad \sigma_{ij}n_j = 0$$

on $y = \bar{h}(x,t) \pm \frac{1}{2}h(x,t)$.

The scaling of x with L, y, $\bar{h}$ and h with a typical thickness H_0, and u, v with U, ϵU is much as before. It's not so easy to see a pressure scale here, so let's use the standard slow flow scale $\mu U/L$ as a first guess and let the equations tell us how (if at all) that should be corrected. The scaled equations are pretty much a cut-and-paste job too:

This scale is $O(\epsilon^2)$ smaller than the slider bearing scale because there are no solid surfaces to generate high pressures.

The ϵ's crop in different places because of our different pressure scale, so you may want to work through the details.

$$\frac{\partial^2 u}{\partial Y^2} + \epsilon^2 \frac{\partial^2 u}{\partial X^2} = \epsilon^2 \frac{\partial p}{\partial X}, \tag{21.14}$$

$$\frac{\partial^2 v}{\partial Y^2} + \epsilon^2 \frac{\partial^2 v}{\partial X^2} = \frac{\partial p}{\partial Y}, \tag{21.15}$$

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} = 0, \tag{21.16}$$

with the kinematic condition

$$v = \frac{\partial}{\partial T}\left(\bar{h} \pm \tfrac{1}{2}h\right) + u\frac{\partial}{\partial X}\left(\bar{h} \pm \tfrac{1}{2}h\right) \tag{21.17}$$

on $y = \bar{h} \pm \frac{1}{2}h$. The stress-free conditions become

$$-\epsilon^2\left(-p + 2\frac{\partial u}{\partial X}\right)\frac{\partial}{\partial X}\left(\bar{h} \pm \tfrac{1}{2}h\right) + \frac{\partial u}{\partial Y} + \epsilon^2\frac{\partial v}{\partial X} = 0, \tag{21.18}$$

$$-\left(\frac{\partial u}{\partial Y} + \epsilon^2\frac{\partial v}{\partial X}\right)\frac{\partial}{\partial X}\left(\bar{h} \pm \tfrac{1}{2}h\right) - p + 2\frac{\partial v}{\partial Y} = 0. \tag{21.19}$$

Now we expand

$$u \sim u_0 + \epsilon^2 u_1 + \cdots, \quad v \sim v_0 + \epsilon^2 v_1 + \cdots, \quad p \sim p_0 + \epsilon^2 p_1 + \cdots,$$
$$\bar{h} \sim \bar{h}_0 + \epsilon^2 \bar{h}_1 + \cdots, \quad h \sim h_0 + \epsilon^2 h_1 + \cdots,$$

take a deep breath and insert.[6]

At $O(1)$, equation (21.14) tells us that $\partial^2 u_0/\partial Y^2 = 0$, so

$$u_0 = u_0(X, T),$$

and so to leading order the flow is unidirectional (extensional) as promised. Looking through our equations, we see that (21.18) is also satisfied at this order, and so we tick it off and move on to the continuity equation (21.16), which tells us that

$$v_0 = -Y\frac{\partial u_0}{\partial X} + V_0(X, T),$$

where V_0 is found from the kinematic condition (21.17) as

$$V_0(X, T) = \frac{\partial}{\partial T}\left(\bar{h}_0 \pm \tfrac{1}{2}h_0\right) + \frac{\partial}{\partial X}\left(u_0\left(\bar{h}_0 \pm \tfrac{1}{2}h_0\right)\right).$$

[6] Warning: in this problem we are going to go to $O(\epsilon^2)$. If we are to be consistently accurate, at each stage we have to remember to expand the location where the free surface conditions are applied about the leading-order location $Y = \bar{h}_0 \pm \frac{1}{2}h_0$ (as in Section 14.5.1). Fortunately, we don't have to do that here.

This is true for both + and − signs and so, subtracting, we find

$$\frac{\partial h_0}{\partial T} + \frac{\partial (u_0 h_0)}{\partial X} = 0,$$

which is conservation of mass to leading order. Lastly we see from (21.15) that $\partial p_0/\partial Y = 0$, and so from (21.19) and the expression we have just found for v_0,

$$p_0(X, T) = -2\frac{\partial u_0}{\partial X}.$$

Let us pause and take stock. We have shown that, to leading order, the flow is extensional, and that mass is conserved, but this is only one equation for u_0, $\bar{h}_0$ and h_0. On the other hand, we have also shown that, in scaled terms,

$$(\sigma_{11})_0 = -p_0 + 2\frac{\partial u_0}{\partial X} = 4\frac{\partial u_0}{\partial X},$$

and so we expect the leading order tension to be

$$(h\sigma_{11}))_0 = 4h_0\frac{\partial u_0}{\partial X}.$$

Thus, we anticipate that

$$\frac{\partial}{\partial X}\left(4h_0\frac{\partial u_0}{\partial X}\right) = 0,$$

a second relation between h_0 and u_0, but not $\bar{h}_0$. In dimensional terms, this says that

$$\sigma_{11} \sim 4\mu\frac{\partial u}{\partial x}$$

and so the Trouton viscosity for a sheet is 4μ, a result we could never have guessed. (For a slender fibre it is an even less likely 3μ.)

This is encouraging, so let us press on to the $O(\epsilon^2)$ equations. We solve them in the same order as the $O(1)$ equations, namely (21.14) with (21.18), then (21.16) with (21.17), and lastly (21.15) with (21.19). We have reached the stage at which the arithmetical details become unedifying and are best dealt with in private; here is a sketch.

Hic opus, hic labor est.

From (21.14),

$$\frac{\partial^2 u_1}{\partial Y^2} = -3\frac{\partial^2 u_0}{\partial X^2},$$

which, with (21.18) on $Y = \bar{h}_0(X,T) \pm \frac{1}{2}h_0(X,T)$, gives

$$
\begin{aligned}
\left.\frac{\partial u_1}{\partial Y}\right|_{Y=\bar{h}_0-\frac{1}{2}h_0}^{Y=\bar{h}_0+\frac{1}{2}h_0} &= -3Y\left.\frac{\partial^2 u_0}{\partial X^2}\right|_{Y=\bar{h}_0-\frac{1}{2}h_0}^{Y=\bar{h}_0+\frac{1}{2}h_0} \\
&= \text{a lot of terms involving } u_0,\ \bar{h}_0,\ h_0.
\end{aligned}
$$

After simplification, we do indeed find that

$$
\frac{\partial}{\partial X}\left(4h_0\frac{\partial u_0}{\partial X}\right) = 0. \tag{21.20}
$$

The other equations are integrated in a similar way and lead, eventually, to the equation

$$
\frac{\partial}{\partial X}\left(4h_0\frac{\partial u_0}{\partial X}\frac{\partial \bar{h}_0}{\partial X}\right) = 0.
$$

Bearing in mind the equation (21.20) just found, this shows that

$$
\frac{\partial^2 \bar{h}_0}{\partial X^2} = 0,
$$

and so the sheet is, to lowest order, straight (the same applies to a fibre). This should not be taken to mean that all viscous sheets and fibres are straight, but rather that if they are being stretched on the timescale L/U of our analysis they must be straight. If the ends of a curved sheet are pulled apart, another model must be used, as they must when the sheet is being stretched so rapidly that the slow flow assumption does not hold.

I cannot leave this topic without pointing out that the nonlinear equations we have derived,

$$
\frac{\partial h_0}{\partial T} + \frac{\partial(u_0h_0)}{\partial X} = 0, \qquad \frac{\partial}{\partial X}\left(4h_0\frac{\partial u_0}{\partial X}\right) = 0,
$$

can be reduced to a linear system. You can find out about this by doing the exercise on page 275.

21.4 The beam equation (?)

Further reading

There is much more on the derivation of models for thin sheets and fibres in the papers [19], [8].

Exercises

(a) **Tilting pad bearings.** Calculate the pressure in a slider bearing of (dimensionless) thickness $H(X) = 1 + \alpha X$. Write down an integral for the load.

Now suppose that the upper surface of the bearing is pivoted freely at a point X_0 $(0 < X_0 < 1)$. Write down a moment condition for the bearing to be in equilibrium, and deduce a relation between the load and the angle α of the upper 'tilting pad'. (Don't try to simplify the integrals without a symbolic manipulator such as Maple.) This kind of bearing is self-adjusting: the pad tilts to accommodate whatever load is imposed.

(b) **Two-dimensional bearings.** Extend the analysis of the slider bearing to a rectangular upper surface above a flat lower surface, to derive a two-dimensional version of Reynolds' equation.

(c) **Squeeze films.** Suppose that the upper surface of a slider bearing is time-varying with characteristic frequency ω, for example by the imposition of a periodic load, so that the gap is $H(X, T)$ in dimensionless variables, in which t is scaled with $1/\omega$. Show that mass conservation is

$$\frac{\partial}{\partial X}\int_0^{H(X)} u(X,Y)\,dY + \sigma\frac{\partial H}{\partial T} = 0,$$

where $\sigma = \omega L/U$ is a dimensionless parameter known as the *bearing number*, and that Reynolds' equation becomes

$$\frac{\partial}{\partial X}\left(H^3\frac{\partial p}{\partial X}\right) = 6\frac{\partial H}{\partial X} + 12\sigma\frac{\partial H}{\partial T}.$$

Now suppose that $U = 0$ for the configuration of a slider bearing, but that the upper surface is oscillated up and down with frequency ω and amplitude a, thus forming a *squeeze film*. Scaling t with $1/\omega$ and v with $a\omega$, what are the appropriate scales for u and p? Show that the appropriate version of Reynolds' equation is

$$\frac{\partial}{\partial X}\left(H^3\frac{\partial p}{\partial X}\right) = 12\frac{\partial H}{\partial T}.$$

Show that by oscillating a suitably-shaped upper surface normal to the lower surface it is possible to generate a non-zero pressure (averaged over one cycle of oscillation) even if $U = 0$. This effect is used to move silicon chips around semiconductor plants on oscillating tracks with saw-tooth shaped surfaces (asymmetry in the surfaces generates a longitudinal pressure gradient which induces motion in that direction).

Show that if a constant load is applied normal to two flat parallel plates initially a distance H_0 apart, they take an infinite time to make contact. (In practice, no surface is absolutely flat, and small *asperities* in the surfaces make contact well before $t = \infty$. It is almost impossible to pull apart two optically flat surfaces that have been put together, and it can be surprisingly hard to lift a sheet of paper normal to a smooth surface. The trick is of course to slide the optically flat surfaces, and to lift the paper from the edge.)

(d) **Surface tension driven thin horizontal film.** Consider the evolution of a thin nearly flat horizontal fluid layer under the action of surface tension. Assume that the effect of surface tension is to give a jump in the normal stress across the fluid surface of

$$\gamma \times \text{curvature},$$

What are the dimensions of γ?

If s measures arclength along the curve $y = f(x)$ and ψ is the angle between the tangent and the x–axis, then the curvature is $\kappa = d\psi/ds$, which I hope you can show is equal to $f''/(1 + (f')^2)^{3/2}$.

where γ is the surface tension coefficient. Show that the curvature of a nearly flat interface $y = h(x, t)$ is approximately $\partial^2 h/\partial x^2$ and hence that the pressure in the flow is

$$p(x, y, t) \sim \gamma \frac{\partial^2 h}{\partial x^2}.$$

Deduce that the thickness satisfies

$$\frac{\partial h}{\partial t} + \frac{\gamma}{3\mu} \frac{\partial}{\partial x} \left(h^3 \frac{\partial^3 h}{\partial x^3} \right) = 0.$$

What is the timescale for the flow? What is the equation when we also consider variations in the third ('into the paper') direction?

Repeat the systematic asymptotic derivation in this case, imposing the stress conditions

$$\sigma_{ij} n_i n_j = \gamma \kappa, \quad \sigma_{ij} n_i t_j = 0,$$

where κ is the curvature with the appropriate sign, and $\mathbf{t} = (t_i)$ the unit tangent to the free surface.,

(e) **Similarity solution for thin fluid layer.** Show that the equation

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right)$$

(the dimensionless version of (21.1)) has similarity solutions of the form

$$h(x,t) = t^{-\alpha} f(x/t^{\alpha})$$

and find α and the ordinary differential equation satisfied by $f(\xi)$, where $\xi = x/t^{\alpha}$. Show further that this equation has solutions of the form

$$f(\xi) = \begin{cases} A(c^2-\xi)^{\beta}, & |\xi| < c, \\ 0, & |\xi| > c, \end{cases}$$

and find the constants A, c, β. (Although this solution, which has compact support, does not have continuous derivatives at $\xi = \pm c$, it represents the evolution of a blob of fluid whose extent is always finite; the mass flux at $x = \pm ct^{\alpha}$ is bounded. This property is generated by the nonlinearity and in particular the fact that the 'diffusion coefficient' (the h^3 multiplying $\partial^2 h/\partial x^2$) vanishes at $h = 0$. A linear diffusion equation, or one whose diffusion coefficient is bounded away from zero, could never produce such a solution. Notice also that this solution tends to $\delta(x)$ as $t \to 0$. Although the thin-film assumption is not valid if h is a delta function, one can still think of this solution as being the large-time asymptotic behaviour of any initial blob with compact support.)

(f) **Viscous liquid on an inclined plane** Give a careful asymptotic derivation of the equation

Note that this is dimensional.

$$\frac{\partial h}{\partial t} + \frac{g\sin\theta}{\nu}h^2\frac{\partial h}{\partial x} = 0$$

for the spreading down an inclined plane at an angle θ to the horizontal of a thin viscous fluid sheet. What model is appropriate if $\theta = O(\epsilon)$, where ϵ is the slenderness parameter of the sheet?

Harder: if the flow is over the surface $z = f(x,y)$, show that the generalisation is

$$\frac{\partial h}{\partial t} - \frac{g}{3\nu}F\nabla\cdot\left(h^3 F\nabla f\right) = 0,$$

where $F(x,y) = (1+|\nabla f(x,y)|)^{-\frac{1}{2}}$ and $h(x,y,t)$ is the layer thickness measured *normal* to the surface.

(g) **Liquid paint flow.** A thin layer of viscous paint flows down a vertical wall. Taking x downwards along the wall, write its thickness as $y = h(x,t)$, and work through the following alternative derivation of the equation for h. Because the layer is thin, its velocity may be taken to be approximately $u(x,y,t)$ in the x–direction. Gravity is resisted by the viscosity of the paint, resulting in a shearing force, which we assume to be approximately equal to $\mu\,\partial u/\partial y$, where μ is the viscosity of the fluid and $\partial u/\partial y$ is the velocity gradient. Use a force balance on a small fluid element to show that $\partial^2 u/\partial y^2 = -\rho g/\mu$. Using the boundary conditions $u = 0$ on $y = 0$ (no-slip) and $\partial u/\partial y = 0$ on $y = h(x,t)$ (no shear at the free surface), deduce that

$$u = \frac{\rho g}{2\mu} y(2h - y).$$

Show that mass conservation requires that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\,dy = 0,$$

that is,

$$\frac{\partial h}{\partial t} + \frac{\rho g}{\mu} h^2 \frac{\partial h}{\partial x} = 0.$$

Assume a length scale L for variations in h, and a typical thickness H. Make the equation dimensionless and write it in conservation form

$$\frac{\partial h}{\partial t} + \frac{\partial(\frac{1}{3}h^3)}{\partial x} = 0.$$

Write down the characteristic equations and draw the characteristic projections in the (x,t) plane (a) when $h(x,0) = h_0(x)$ is an increasing function of x, and (b) when it is decreasing. Interpret the results. Which flows quicker, a thick layer or a thin one?

You may want to refer back to the exercise on page 59 to remind yourself of the dimensional analysis of this problem.

Antonio, Bruno and Carlo are cooking. They have a very small amount of olive oil (a nice Newtonian fluid, unlike many liquids in the kitchen) in the bottom of one of those square bottles. Carlo turns the bottle upside down and holds it vertically while Bruno holds it flat side down at an angle to the horizontal. Assuming that the oil flows down the sides of the bottle rather than simply falling to the neck, who will wait longer, and why? Antonio, who has done the first part of this exercise, has a simple twist on Bruno's method which improves it significantly: what, and why?

OK, a cylinder of square cross-section…

Returning to the equation for $h(x,t)$ on a vertical wall, show that there is a similarity solution of the form $h(x,t) = t^{-\alpha} f(x/t^{\alpha})$ and find α (put this form into the equation and show that it works.) Show that the total mass of this solution is constant, and note that $h(x,0) = \delta(x)$. Find f if we assume that $h(0,t) = 0$ and only solve for $x > 0$. Show that if h is equal to this similarity solution for $0 < x < S(t)$, and is zero elsewhere so that there is a discontinuity (shock) at $x = S(t)$, then the Rankine–Hugoniot (shock) condition

$$\frac{dS}{dt} = \frac{\left[\frac{1}{3}h^3\right]}{[h]}$$

is satisfied provided that $S = At^{1/3}$ for constant A. How should this solution be interpreted?

Suppose now that the film thickness is nearly constant and look for small perturbations by finding solutions of the dimensionless equation in the form $h = 1 + \epsilon e^{i(kx+\omega t)}$ where $\epsilon \ll 1$ (like doing water waves). What is the relation between k and ω? Which direction do these waves travel in? (Note that there is only one direction of travel; water waves have two.) What is their dimensional speed? Show that the only smooth travelling wave solutions (i.e. solutions of the form $h(x,t) = g(x - Ut)$ for constant U) to the full equation are $h =$ constant. However, the linearised solution you have just found looks like a travelling wave. How do you reconcile these facts?

(h) **Linear stability of thin films on horizontal surfaces.** Investigate the linear stability of thin films under gravity or surface tension, by writing

$$h(x,t) \sim h_0 + \epsilon e^{ikx} e^{\lambda t}$$

in the dimensional equations and finding λ in terms of k. Note how the sign in front of the space derivatives changes when we go from second order to fourth order and relate to the linear stability result.

(i) **Marangoni effects in a thin layer.** Some flows are driven by variations in the surface tension coefficient γ. This may be due to temperature variations, or because there is a surfactant chemical in the fluid, or because some other effect such as evaporation of a solvent changes γ. The net effect is to induce a tangential (shear) stress at the interface which acts to drag the fluid from low surface

Paint drying, that riveting example.

tension regions to those where it is high. Foams are a particularly important practical example; the thin fluid sheets that form the bubble faces are stabilised by surfactants which counteract the tendency of the fluid to drain into the lower pressure regions where fluid sheets meet (known as Plateau borders; the pressure is lower there because of the curvature of the surface, as a sketch will show).

Suppose that we have a thin fluid layer as above and that the surface tension coefficient $\gamma(x)$ varies in a known way by an $O(1)$ amount (i.e. $\Delta\gamma/\gamma = O(1)$) over a horizontal distance L. Assuming that the Marangoni force translates into the (dimensional) boundary condition

$$\mu \left.\frac{\partial u}{\partial y}\right|_{y=h(x,t)} = \frac{d\gamma}{dx},$$

explain why the flow is locally equivalent to Couette flow with a linear velocity profile, and derive the equation

$$\frac{\partial h}{\partial t} + \frac{1}{2\mu}\frac{\partial}{\partial x}\left(h\frac{d\gamma}{dx}\right) = 0.$$

What is the timescale of the motion? How small would the surface tension variation with x have to be for the normal force (surface tension $\times$ curvature) to be significant? What is the equation for h in this case?

(j) **Tides.** Consider water waves in a basin $0 < x < L$, $-H < y < 0$. The velocity potential $\phi(x, y, t)$ and surface elevation $\eta(x, t)$ for small-amplitude waves satisfy

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad 0 < x < L,\ -H < y < 0,$$

with

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = -H, \qquad \frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}, \quad \frac{\partial\phi}{\partial t} + g\eta = 0 \quad \text{on } y = 0,$$

with suitable boundary conditions on $x = 0, L$. Make the problem nondimensional, scaling x with L and y with H, and using the timescale $\sqrt{L^2/gH}$, show that in the dimensionless version of the problem $\phi(x, y, t)$ satisfies the *elliptic* equation

$$\frac{\partial^2\phi}{\partial y^2} + \epsilon^2\frac{\partial^2\phi}{\partial x^2} = 0 \ \text{ in } \ -1 < y < 0,$$

with

$$\frac{\partial \phi}{\partial y}(x,-1,t) = 0, \quad \frac{\partial \phi}{\partial y}(x,0,t) + \epsilon^2 \frac{\partial^2 \phi}{\partial t^2}(x,0,t) = 0.$$

Show that an expansion in which

$$\phi \sim \phi_0(x,y,t) + \epsilon^2 \phi_1(x,y,t) + \cdots$$

satisfies the equation and boundary condition up to terms of $O(\epsilon^2)$ if $\phi_0 = \phi_0(x,t)$, where the $O(\epsilon^2)$ equation shows that ϕ_0 satisfies the *hyperbolic* equation

$$\frac{\partial^2 \phi_0}{\partial x^2} - \frac{\partial^2 \phi_0}{\partial t^2} = 0.$$

What (in dimensional terms) is the wavespeed?

This example, which is a *very* simple model for the tidal flows (it does not even have the daily periodicity built in, nor the rotation of the earth!), shows that the solution of an elliptic equation can sometimes be consistently approximated by that of a hyperbolic equation. The development of mathematical models for tide prediction preoccupied many famous minds — Newton and Laplace among them — and is described in [7]. One approach was to expand the water depth (as a function of time) as a series of harmonic terms, to reflect the periodic influence of the sun, moon etc. The summation of such a series by hand was a tedious and error-prone business, which was greatly facilitated by the invention by Lord Kelvin of a mechanical analogue based on pulleys. These machines were used until well into the twentieth century, and one of them can be seen in Liverpool Museum.

(k) **Shallow water equations.** There is no need only to consider sticky fluids in thin layers. In this question we derive the famous shallow water model for inviscid flow, starting with an intuitive derivation.

   i. Write down the two-dimensional Euler equations

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x},$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x} - \rho g,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

   for unsteady flow of an inviscid liquid under gravity.

ii. Assume that there is a base at $y = 0$ and a free surface at $y = h(x, t)$, that the layer is long and thin, and the flow fast enough that the velocity is approximately unidirectional and independent of depth, and hence of the form $(u(x, t), 0)$.

iii. Assume further that the pressure is approximately hydrostatic; show that

$$p(x, y, t) = \rho g \left(h(x, t) - y\right).$$

iv. Write down mass conservation.

v. Put these assumptions into the Euler equations to derive

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0, \qquad \frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0.$$

Linearise the system about the constant solution $u = 0$, $h = h_0$ and find the speed of propagation of small disturbances; compare with the previous exercise.

This hyperbolic system can describe all sorts of phenomena such as the Severn Bore or its less well known cousin the Trent Aegir (and a host of other bores around the world); they appear as shocks in the solutions. See [29] for lots more about the shallow water equations and their properties.

Now derive these equations by a lubrication scaling of the Euler equations, to justify the (very reasonable) assumptions made above. Scaling x with L, y with $\epsilon L = H_0$, t with L/U and p with $\epsilon \rho g L$, you should get at lowest order in ϵ

$$\begin{aligned}\frac{\partial u_0}{\partial T} + u_0\frac{\partial u_0}{\partial X} + v_0\frac{\partial u_0}{\partial Y} &= -\frac{1}{F^2}\frac{\partial p_0}{\partial X},\\ \frac{\partial p_0}{\partial Y} = -1, \qquad \frac{\partial u_0}{\partial X} + \frac{\partial v_0}{\partial Y} &= 0,\end{aligned}$$

together with the kinematic and dynamic free surface conditions on $Y = h(X, T)$. Here $F^2 = U^2/gH_0$ is a dimensionless parameter called the *Froude* (rhymes with crowd) number, measuring the inertia/gravity balance. Notice that the X-momentum equation is not quite the same as Now make the additional assumption that $\partial u_0/\partial Y = 0$ at the inlet or beginning of the flow. Calculate p_0 and deduce that $\partial u_0/\partial Y = 0$ throughout the flow. Write down the condition for irrotationality at leading order and compare; relate this to Kelvin's theorem in fully two-dimensional flow. Hence derive the shallow-water equations.

Harder: derive these equations starting from potential flow; you will not now have to assume irrotationality.

(l) **Boussinesq flow in a porous medium.** Suppose water flows in a porous rock (an *aquifer*) under the action of gravity. The French sewage engineer Darcy established the law

Consistency: why a minus sign?

$$\mathbf{u} = -K\nabla(p + \rho g y)$$

giving the fluid velocity $\mathbf{u}$ as proportional to the gradient of the pressure (after subtracting off the hydrostatic head); here y is vertically upwards and K is called the *mobility.* Assuming that water is essentially incompressible, show that this model is equivalent to potential flow with potential $\Phi = -K(p + \rho g y)$.

A thin layer of water (called a water mound in the trade) lies above a horizontal impermeable base at $y = 0$. Either intuitively, or by scaling, or both, derive the nonlinear diffusion equation

$$\frac{\partial h}{\partial t} = K\rho g \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right).$$

Find the similarity solution corresponding to a delta-function initial data.

(m) **Linearising the viscous sheet equations.** Take the viscous sheet equations in the form

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \qquad \frac{\partial}{\partial x}\left(4h\frac{\partial u}{\partial x}\right) = 0$$

and integrate to get

$$h\frac{\partial u}{\partial x} = f(t)$$

for some $f(t)$ (proportional to the tension). Define

$$\tau(t) = \int_0^t f(s)\,ds$$

and set $u(x,t) = f(t)v(x,\tau)$. Show that

$$h\frac{\partial v}{\partial x} = 1, \qquad \frac{Dh}{D\tau} = -1, \quad \text{where} \quad \frac{D}{D\tau} = \frac{\partial}{\partial \tau} + v\frac{\partial}{\partial x}.$$

Change the independent variables to h, τ (a *partial hodograph*

Differentiate $h = h(x,\tau)$ implicitly with respect to x and τ to show that $1 = h_x x_h$, $0 = h_x x_\tau + h_\tau$, then show that $v = x_\tau - x_h$, and differentiate again with respect to x to get v...

transformation) to show that $x(h, \tau)$ satisfies

$$\frac{\partial^2 x}{\partial h \partial \tau} - \frac{\partial^2 x}{\partial h^2} = \frac{1}{h}\frac{\partial x}{\partial h}.$$

By solving this linear equation for $\partial x/\partial h$, deduce that

$$\frac{\partial x}{\partial h} = (hF(h+\tau))^{-1}$$

for arbitrary F, and hence that

$$\frac{\partial h}{\partial x} = hF(h+\tau).$$

Notice that F can now be determined from the initial data for h, so the whole system can be solved explicitly.

Chapter 22

Ray theory and other ‘exponential’ approaches

22.1 Introduction

In this short final chapter we look at some problems which can be tackled using a different kind of singular scaling, via an exponent.

Exercises

(a) **String art.** Consider the family of lines generated by joining the point $(t,0)$ to $(0,1-t)$. Find its envelope ($(x-y)^2+1=2(x+y)$; why is this obviously a parabola?).

[There was a fortunately short lived phase of string art made by hammering lines of nails into a board and stretching highly-coloured shiny string between them, the endpoints of successive lengths of string being related as in the example above. The result was a complicated web with (a discrete approximation to) an envelope, often in the shape of a sailing boat or similar object. They can occasionally be seen in charity shops and holiday cottages even now and will doubtless at some point become highly collectable.)

(b) **Rays in an ellipse.** Consider the ray equations for $u_x^2+u_y^2=1$ inside the ellipse $x^2/a^2+y^2/b^2=1$, with $u=0$ on its boundary $(x_0(s),y_0(s))$. Show that the direction of a ray is $(p_0(s),q_0(s))$, and by differentiating the condition $u=0$ on the ellipse with respect to s, show that the rays are normal to the boundary. Use the parametric form $(a\cos s, b\sin s)$ for the boundary to show that the normals are

$$ax\sin s - by\cos s = (a^2 - b^2)\sin s\cos s,$$

deduce that there is a caustic on the envelope of these curves, namely

$$x = \frac{a^2 - b^2}{a}\cos^3 s, \qquad y = \frac{a^2 - b^2}{b}\sin^3 s;$$

sketch this curve.

(c) **The nephroid in a mug of milky tea.** Consider a semicircular reflector $x^2 + y^2 = 1$, $x < 0$, with a plane wave $-e^{ikx}$ incident from $x = +\infty$. Solve the ray equations with $u = -x$ on the circle (corresponding to a zero field on the reflector): show that the ray passing through $(\cos s, \sin s)$, $\pi/2 < s < 3\pi/2$, has direction (p_0, q_0), where $p_0 = \cos 2s$, $q_0 = \sin 2s$, and is therefore

$$x = \cos s + t\cos 2s, \qquad y = \sin s + t\sin 2s.$$

Eliminate t and find the envelope in the form

$$x = \sin s\sin 2s + \tfrac{1}{2}\cos s\cos 2s, \qquad y = -\sin s\cos 2s + \tfrac{1}{2}\sin 2s\cos s.$$

This curve is shown below: when you put a mug of milky coffee in the sun, you see a bright caustic in this shape. It is called a nephroid from its resemblance to a kidney. From the picture below you can see that there are 4 rays through each point outside the nephroid, just two inside.

Show that the ray above has the equation $x\sin 2s - y\cos 2s = \sin s$, and hence that it meets the circle again at the point $(\cos\alpha, \sin\alpha)$ where $\alpha = 3s - \pi$. Hence explain how to make a string art nephroid.

(d) **Sand.** Suppose I pile up dry sand on top of a horizontal flat rock until I can put no more on. Assume that the particles of sand on the surface are in limiting frictional equilibrium, that is the frictional force on them is equal to μ times the normal reaction where μ is the coefficient of friction. Assume also that all the particles of sand in the interior are also in equilibrium. Draw a force diagram for a surface particle and deduce that, with a suitable scaling, the height $u(x, y)$ of the sand satisfies

$$u_x^2 + u_y^2 = 1, \qquad u = 0 \quad \text{on the boundary.}$$

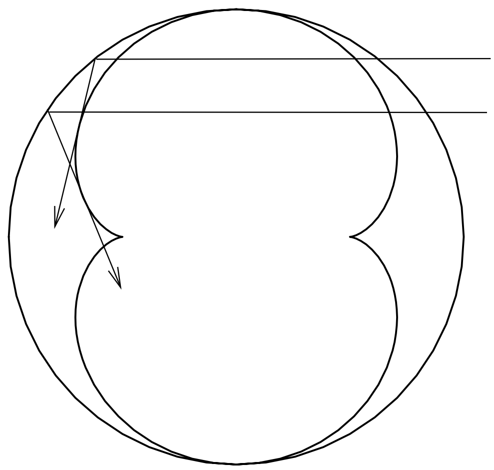

Figure 22.1: Rays and the caustic inside a cup of tea.

Write down the ray equations and show the rays are normal to the boundary. If the rock is circular, show that the rays all meet at the centre where $\partial(x, y)(s, t) = 0$; what is the surface? What do you think you get if the rock is square, rectangular, elliptic?

Note that in this problem the interpretation of u is quite different from that in geometric optics. Here, u must be single valued. There, it was quite acceptable to have more than one ray through a single point; this just said that the overall field was the superposition of several waves propagating through that point in different directions (of course some of them may have small amplitude). The generic singularity of geometric optics is a caustic, while for sandpiles it is a ridgeline.

Note also that if the sand is wet it may have an internal 'cohesion' i.e. the grains may stick together.This is much more difficult to model.

Chapter 23

Case study: the thermistor 2

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