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THE DISTRIBUTION OF PRODUCTS OF BETA, GAMMA AND GAUSSIAN RANDOM VARIABLES*

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1. Summary. The probability density functions of products of independent beta, gamma and central Gaussian random variables are shown to be Meijer G-functions. The density function of products of random beta variables is a Meijer G-function which is expressible in closed form when the parameters are integers. Recursion formulas are developed for the evaluation of the Meijer G-functions representing products of random gamma variables and products of random Gaussian variables $N(0, \sigma_i)$. These results include earlier results obtained by Springer and Thompson [1], [2] and Lomnicki [3], [4] as special cases.

2. Introduction. Unlike the distribution of sums of independent random variables, the distribution of products of more than two has received relatively little attention. In a previous paper [1], the authors have developed fundamental methods for the derivation of the distribution of products of independent random variables, which were used to obtain density functions in explicit form for products of Cauchy and central Gàussian variables in [1], and some special cases of beta variables in [1]. These methods are employed here to analyze products and quotients of beta and gamma variables. A recursion formula is also derived which is particularly convenient for use with electronic computers in the evaluation of the Meijer G-functions representing the density functions of products of independent gamma and Gaussian random variables.

In this paper the density functions of products of independent random variables are obtained directly through the use of the Mellin integral transform. The Mellin integral transform of $f(x)$, defined only for $x \geqq 0$, is

$$M\{f(x)|s\} = E[x^{s-1}] = \int_0^\infty x^{s-1}f(x)\,dx. \tag{1}$$

Under suitable restrictions [5, pp. 46–47], satisfied by all density functions considered in this paper, there is an inverse formula

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}M\{f(x)|s\}\,ds \tag{2}$$

for which the identity relation

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\left\{\int_0^\infty y^{s-1}f(y)\,dy\right\}ds \tag{3}$$

is valid almost everywhere. The path of integration is any line parallel to the imaginary axis and lying within the strip of analyticity of $M\{f(x)|s\}$. The Mellin integral transform of the density function $g(z)$ of the product $z = x_1x_2\cdots x_N$

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of N independent random variables x_i with density functions $f_i(x_i)$ is [1]

$$M\{g(z)|s\} = \prod_{i=1}^{N} M\{f_i(x_i)|s\}, \tag{4}$$

and the density function $g(z)$ is

$$g(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s} \prod_{i=1}^{N} M\{f_i(x_i)|s\}\, ds. \tag{5}$$

It is convenient to introduce the following notation in connection with the Euler psi function [6, p. 258], the Riemann zeta function [6, p. 807], and Euler's constant, respectively:

$$\psi(w + a) = \frac{d}{dw}\ln\Gamma(w + a), \tag{6}$$

$$\zeta(\beta, a) = \sum_{\alpha=0}^{\infty} (a + \alpha)^{-\beta}, \tag{7}$$

$$C = \psi(1) = \zeta(1, 1). \tag{8}$$

The Meijer G-function, to which reference will be made later, is defined by the contour integral:

$$G_{pq}^{mn}\left(z\left|\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q\end{matrix}\right.\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s} \frac{\prod_{j=1}^{m}\Gamma(s + b_j)\cdot\prod_{j=1}^{n}\Gamma(1 - a_j - s)}{\prod_{j=n+1}^{p}\Gamma(s + a_j)\prod_{j=m+1}^{q}\Gamma(1 - b_j - s)}\, ds, \tag{9}$$

where c is a real constant defining a Bromwich path separating the poles of $\Gamma(s + b_j)$ from those of $\Gamma(1 - a_k - s)$ and where the empty product $\prod_{j=r}^{r-1}$ is defined to be unity. A more detailed discussion of the G-function and examples are given in [7, pp. 374–379].

3. The product of independent gamma variables. Given N independent gamma variables x_i having density functions

$$f_i(x_i) = \frac{1}{\Gamma(b_i)} x_i^{b_i - 1}\exp(-x_i), \qquad b_i > 0, \quad 0 \leqq x_i < \infty, \tag{10}$$

the following theorem can be proved.

THEOREM 1. *The probability density function $g(z)$ of the product $z = x_1 x_2 \cdots x_N$ of N independent gamma variables is a Meijer G-function multiplied by a normalizing constant K, i.e.,*

$$g(z) = K G_{0N}^{N0}(z|b_1 - 1, b_2 - 1, \cdots, b_N - 1), \tag{11}$$

where

$$\frac{1}{K} = \sum_{i=1}^{N}\Gamma(b_i).$$

Proof. Since [7, Formula (1), p. 312] the Mellin integral transform of $\exp(-x)$ is $\Gamma(s)$ and

$$M\{x^b f(x)|s\} = M\{f(x)|s + b\}, \tag{12}$$

it follows that the density function of the gamma variable (10) has the Mellin transform

$$M\{f_i(x_i)|s\} = \frac{\Gamma(s + b_i - 1)}{\Gamma(b_i)}. \tag{13}$$

Then the Mellin integral transform of the probability density function $g(z)$ of the product of N independent random gamma variables is

$$M\{g(z)|s\} = \prod_{i=1}^{N} \frac{\Gamma(s + b_i - 1)}{\Gamma(b_i)}, \tag{14}$$

and

$$g(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s} \prod_{i=1}^{N} \frac{\Gamma(s + b_i - 1)}{\Gamma(b_i)} \, ds, \tag{15}$$

which is the Meijer G-function

$$G_{0N}^{N0}(z|b_1 - 1, b_2 - 1, \cdots, b_N - 1), \qquad \operatorname{Re} z \geqq 0, \tag{16}$$

multiplied by the constant

$$K = 1/\prod_{i=1}^{N} \Gamma(b_i). \tag{17}$$

The Meijer G-function

$$G(z|b_1, \cdots, b_N) \tag{18}$$

will now be expressed in a form which allows a direct numerical evaluation. To represent (18) in such a form it will be assumed in this application, without loss of generality, that the b_i are real with $b_N \geqq b_{N-1} \geqq \cdots \geqq b_1 > -1$. Each b_i can then be written in a unique way as

$$b_i = d_k + p_{ik}, \tag{19}$$

where $0 \geqq d_k > -1$ and p_{ik} is a nonnegative integer. The constants $d_1, d_2, \cdots, d_n$ are distinct and n is the number of different values among the $b_1, b_2, \cdots, b_N$ which do not differ by an integer. Finally it is seen that there can be found unique integers $r_{0k}, r_{1k}, \cdots, r_{pk}; m_k; p_k \geqq 0; \sum_{k=1}^{n} m_k = N$ such that

$$\prod_{i=1}^{N} \Gamma(s + b_i) = \prod_{k=1}^{n} (s + d_k)^{r_{0k}}(s + d_k + 1)^{r_{1k}} \cdots (s + d_k + p_k)^{r_{pk}}\Gamma^{m_k}(s + d_k), \tag{20}$$

where here and in the following r_{pk} denotes $r_{p_k k}$ and thus avoids writing the double subscript which is implied. That this rather elaborate notational machinery is truly convenient in accounting for the poles of the integrand function in (9) will be seen presently.

THEOREM 2. *The Meijer G-function* $G_{0N}^{N0}(z|b_1 - 1, b_2 - 1, \cdots, b_N - 1)$ *with real parameters* $b_N \geqq b_{N-1} \geqq \cdots \geqq b_1 > 0$ *has a representation*

$$(21) \qquad G_{0N}^{N0}(z|b_1 - 1, \cdots, b_N - 1) = \sum_{j=0}^{\infty} \sum_{k=1}^{n} R_{jk}(z, -d_k - j), \qquad \operatorname{Re} z > 0,$$

where the $R_{jk}(z, -d_k - j)$ *are obtained as explicit functions of z which can be evaluated by numerical methods.*

Proof. Upon substituting (20) in the definition (9) the integral representation of $G_{0N}^{N0}(z|b_1 - 1, \cdots, b_N - 1)$ is

$$(22) \qquad G_{0N}^{N0}(z|b_1 - 1, \cdots, b_N - 1) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} \prod_{k=1}^{n} \Gamma^{m_k}(s + d_k) \cdot \prod_{t=0}^{p_k} (s + d_k + t)^{r_{tk}} \, ds,$$

where $c > 1$ is arbitrary and d_k, $-1 < d_k \leqq 0$, along with the integers m_k, p_k, r_{pk}, are defined uniquely by (19) and (20). Since the conditions of Jordan's lemma are satisfied, one obtains, by applying the residue theorem [8] to (22),

$$(23) \qquad G_{0N}^{N0}(z|b_1 - 1, \cdots, b_N - 1) = \sum_{j=0}^{\infty} \sum_{k=1}^{n} R_{jk}(z, -d_k - j), \qquad \operatorname{Re} z > 0.$$

The relation

$$(24) \qquad R_{jk}(z, s) = \frac{1}{(m_k - 1)!} \frac{d^{m_k - 1}}{ds^{m_k - 1}} z^{-s} U^{(0)}(s, k),$$

where

$$(25) \qquad U^{(0)}(s, k) = (s + d_k + j)^{m_k} \prod_{a=1}^{n} \left\{ \Gamma^{m_a}(s + d_a) \prod_{t=0}^{p_a} (s + d_a + t)^{r_{ta}} \right\}$$

defines the residue at $s = -d_k - j$, $k = 1, 2, \cdots, n$, $j = 0, 1, 2, \cdots$, which for j sufficiently large is a pole of the integrand function of order m_k. Utilizing Leibniz's rule for the differentiation of a product, one can express the residue (24) in the form

$$(26) \qquad R_{jk}(z, s) = \frac{z^{-s}}{(m_k - 1)!} \sum_{i=0}^{m_k - 1} \binom{m_k - 1}{i} \left(\ln \frac{1}{z} \right)^i U^{(m_k - 1 - i)}(s, k).$$

To complete the representation of $G_{0N}^{N0}(z)$ it is necessary to produce an explicit algorithm for computing the successive derivatives

$$(27) \qquad U^{(q)}(s, k) = \frac{d^q}{ds^q} U^0(s, k), \qquad k = 1, 2, \cdots, n, \quad q = 0, 1, 2, \cdots.$$

To derive such an algorithm, note that (25) can be written in the form

$$(28) \qquad U^{(0)}(s, k) = \left\{ \frac{\Gamma(s + d_k + j + 1)}{(s + d_k) \cdots (s + d_k + j - 1)} \right\}^{m_k} \prod_{t=0}^{p_k} (s + d_k + t)^{r_{tk}} \prod_{\substack{a=1 \\ a \neq k}}^{n} \Gamma^{m_a}(s + d_a) \left\{ \prod_{t=0}^{P_a} (s + d_a + t)^{r_{ta}} \right\}.$$

If one now evaluates $d \ln U^{(0)}/ds$, recalling that $\psi(s+a) = d \ln \Gamma(s+a)/ds$, it follows that

$$U^{(1)}(s,k) = U^{(0)}(s,k)V^{(0)}(s,k), \tag{29}$$

where

$$\begin{aligned} V^{(0)}(s,k) &= m_k\psi(s+d_k+j+1) - m_k \sum_{t=0}^{j-1} \frac{1}{s+d_k+t} + \sum_{t=0}^{p_k} r_{tk}(s+d_k+t)^{-1} \\ &\quad + \sum_{\substack{a=1 \\ a\neq k}}^{n} m_a\psi(s+d_a) + \sum_{a=1}^{n}\sum_{t=0}^{p_a} r_{ta}(s+d_a+t)^{-1}. \end{aligned} \tag{30}$$

Application of Leibniz's rule to (29) gives

$$U^{(q+1)}(s,k) = \sum_{m=0}^{q} \binom{q}{w} U^{(q-m)}(s,k)V^{(m)}(s,k), \qquad q = 0,1,2,\cdots, \tag{31}$$

which relation provides the algorithm for computing the successive derivatives of $U^{(0)}(s,k)$, since the $V^{(m)}(s,k)$ can be obtained directly from (30) as

$$\begin{aligned} V^{(m)}(s,k) &= m_k\psi^{(m)}(s+d_k+j+1) + (-1)^{m+1}(m+1)!(s+d_k+t)^{-m-1} \\ &\quad + \sum_{t=0}^{p_k} (-1)^m m!(s+d_k+t)^{-(m+1)} + \sum_{a=1}^{n} m_a\psi^{(m)}(s+d_a) \\ &\quad + \sum_{a=1}^{n}\sum_{t=0}^{p_a} (-1)^m m!(s+d_a+t)^{-(m+1)}. \end{aligned} \tag{32}$$

The Euler psi function ψ and its derivatives $\psi^{(m)}$ as required in (32) are easily evaluated to any desired degree of accuracy for real values of the argument by using available tables and/or convenient recurrence relations, specifically,

$$\begin{aligned} \psi(1) &= -C, \\ C &= \text{Euler's constant}, \\ \psi(x+1) &= \frac{1}{x} + \psi(x), \quad x > 0, \\ \psi^{(k)}(x) &= (-1)^{k+1}k!\zeta(k+1,x) \\ &= (-1)^{k+1}k! \sum_{a=0}^{\infty} \frac{1}{(x+a)^{k+1}}. \end{aligned}$$

Substitution into (31) of $U^{(0)}(s,k)|_{s=-(d_k+j)}$ and $V^{(m)}(s,k)|_{s=-(d_k+j)}$, as obtained from (28) and (32) respectively, permits the evaluation of the residues $R_{jk}(z, -d_k - j)$ in (26), $k = 1, 2, \cdots, n$, $j = 0, 1, 2, \cdots$. Summing R_{jk} over j and k then yields (23).

To summarize, $G_{0N}^{N0}(z|b_1 - 1, \cdots, b_N - 1)$ has a representation (23) which can be evaluated using (24), (26), (28), (29), (30) and the recurrence relation (31). The process is well adapted for an electronic computer but is tedious for hand computation if several terms in the expansion (23) are required.

The following corollary and Theorem 3 relate to prior results of Lomnicki [3].

COROLLARY. *The probability density function of the product $z = x_1 x_2 \cdots x_N$ of N identically distributed independent gamma variables is the Meijer G-function*

$$g(z) = \frac{1}{\Gamma^N(b)} G_{0N}^{N0}(z|b-1). \tag{33}$$

The corollary follows immediately from (16) by setting $b_1 = b_2 = \cdots = b_N = b$.

THEOREM 3. *The Meijer G-function $G_{0N}^{N0}(z|b-1)$ with integer parameter b has the representation*

$$G_{0N}^{N0}(z|b-1) = \sum_{j=b-1}^{\infty} \sum_{k=0}^{N-1} \sum_{m=0}^{k-1} (-1)^{k-m} \cdot \frac{(-\ln z)^{N-1-k} N z^j \left\{ \zeta(k-m,1) - \sum_{i=0}^{j-1} \frac{1}{(j-i)^{k-m}} \right\} U^{(m)}(s)}{(N-1-k)!m!(k-1-m)!} \Bigg|_{s=-j}, \tag{34}$$

where

$$U^{(k)}(s)\Big|_{s=-j} = \frac{d^k}{ds^k} \left(\frac{\Gamma(s+b-1+j)}{(s+b-1)(s+b)\cdots(s+b-2+j)} \right)^N \Bigg|_{s=-j}$$

and can be expressed recursively as

$$U^{(k)}(s)\Big|_{s=-j} = N \sum_{m=0}^{k-1} \frac{(-1)^{k-1}(k-1)!}{m!(k-1-m)!} \left\{ \zeta(k-m,1) - \sum_{i=0}^{j-1} \frac{1}{(j-i)^{k-m}} \right\} U^{(m)}(s)\Big|_{s=-j}, \quad j = b-1, b-2, \cdots. \tag{35}$$

Proof. From (14), the Mellin transform of the density function $g(z)$ of the product of N identically distributed random gamma variables with parameter $b-1$ is

$$M\{g(z)|s\} = \frac{1}{\Gamma^N(b)} \Gamma^N(s+b-1),$$

while from (15) and the corollary to Theorem 2,

$$G_{0N}^{N0}(z|b-1) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} \Gamma^N(s+b-1)\, ds. \tag{36}$$

Since the conditions of Jordan's lemma hold, an expansion of (36) by residues yields

$$G_{0N}^{N0}(z) = \sum_{j=b-1}^{\infty} R(z,N,j), \tag{37}$$

where

$$R(z,N,j) = \frac{1}{(N-1)!} \frac{d^{N-1}}{ds^{N-1}} \{(s+j)^N z^{-s} \Gamma^N(s+b-1)\} \Big|_{s=-j}, \quad j = b-1, b-2, \cdots. \tag{38}$$

The problem then is to evaluate the residue $R(z, N, j)$ which is simplified by the application of Leibniz's rule for differentiation of products:

$$R(z,N,j) = z^{-s}\sum_{k=0}^{N-1}\frac{(-\ln z)^{N-1-k}}{(N-1-k)!k!}\frac{d^k}{ds^k} \cdot [(s+b-1+j)^N\Gamma^N(s+b-1)]\Big|_{s=-j}. \tag{39}$$

The problem is now reduced to that of obtaining a recursive formula for

$$\frac{d^k}{ds^k}[(s+b-1+j)\Gamma(s+b-1)]^N.$$

To simplify the analysis, note that

$$(s+b-1+j)\Gamma(s+b-1) = \frac{\Gamma(s+b+j)}{(s+b-1)(s+b)\cdots(s+b-2+j)} \tag{40}$$

and let

$$U(s) = \left[\frac{\Gamma(s+b+j)}{(s+b-1)(s+b)\cdots(s+b-2+j)}\right]^N.$$

Then

$$\ln U(s) = N\left[\ln\Gamma(s+b+j) - \sum_{i=0}^{j-1}\ln(s+b-1+i)\right]. \tag{41}$$

and differentiation of (41) gives

$$U^{(1)}(s) = NU(s)\left\{\psi(s+b+j) - \sum_{i=0}^{j-1}\frac{1}{s+b-1+i}\right\}. \tag{42}$$

Applying Leibniz's rule for differentiation of products to (42) yields the recursive relationship for the kth derivative in terms of the preceding $k-1$:

$$U^{(k)}(s)\Big|_{s=-j} = N\sum_{m=0}^{k-1}(-1)^{k-m}\frac{(k-1)!}{m!(k-1-m)!} \cdot\left\{\zeta(k-m,1) - \sum_{i=0}^{j-1}\frac{1}{(j-i)^{k-m}}\right\}U^{(m)}(s)\Big|_{s=-j}, \tag{43}$$

where $U^{(0)}(s)$ denotes $U(s)$. Substitution of this expression for $U^{(k)}(s)$ into (39) and (37) yields (34), which establishes the theorem. The representation of $G_{0N}^{N0}(z|b-1)$ as given by Theorem 3 is particularly convenient for adaptation to electronic computers. Lomnicki [3] has expressed the representation in a different form reducible to that of (37) but less conducive to computer evaluation since it does not express $U^{(n)}(s)$ explicitly in terms of $U^{(j)}(s)$, $j = 0, 1, 2, \cdots, n-1$. This could have been done by differentiating $G'(s)$ by Leibniz's rule, where in Lomnicki's notation

$$G'(s) = A(s)G(s),$$

$A(s)$ and $G(s)$ being defined by (10) and (11) of [3].

THEOREM 4. *The quotient of two independent random gamma variables* (10) *is a beta variable of the second kind having as density function the Meijer G-function* $G_{00}^{11}(z|_{b_1-1}^{-b_2})$ *multiplied by the constant* $(\Gamma(b_1)\,\Gamma(b_2))^{-1}$.

Proof. Let $h(z)$ denote the probability density function of the quotient

$$z = x_1/x_2$$

of two gamma random variables x_1 and x_2 with probability density function (10). It is well known that the Mellin transform of $f_i(x_i)$, $i = 1, 2$, is

$$M\{f(x_i)|s\} = \frac{\Gamma(b_i + s)}{\Gamma(b_i)}$$

and that the probability density function $g_i(1/x_i)$ of the random variable $1/x_i$ has the Mellin transform

$$M(g_i(1/x_i)|s) = M(f_i(x_i)| -s + 2).$$

Now $z = x_1/x_2$, so that

$$M(h(z)|s) = \frac{\Gamma(s + b_1 - 1)\Gamma(b_2 + 1 - s)}{\Gamma(b_1)\Gamma(b_2)}. \tag{44}$$

But this is precisely the Mellin transform of

$$h(z) = \frac{z^{b_1 - 1}}{B(b_1, b_2)(1 + z)^{b_1 + b_2}},$$

the probability density function of a beta variable of the second kind. Since a probability density function is uniquely determined by its Mellin transform, it follows that the probability density function of the quotient of two gamma random variables is identical with that of a beta variable of the second kind. Furthermore, the Mellin inversion integral corresponding to the Mellin transform (44) and yielding $h(z)$ is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{z^{-s}\Gamma(s + b_1 - 1)\,\Gamma(b_2 + 1 - s)}{\Gamma(b_1)\,\Gamma(b_2)}\,ds = \frac{1}{\Gamma(b_1)\,\Gamma(b_2)} G_{00}^{11}(z|_{b_1-1}^{-b_2}),$$

which in conjunction with the preceding statement establishes Theorem 4.

COROLLARY. *The probability density function of a beta variable of the second kind is a Meijer G-function multiplied by a normalizing constant.*

4. The product of independent Gaussian variables. It will now be shown that the probability density function of the product of independent Gaussian random variables is a Meijer G-function, whose representation in series form will also be derived.

THEOREM 5. *The probability density function of the product* $z = \prod_{i=1}^{N} x_i$ *of* N *independent Gaussian random variables* $N(0, \sigma_i)$, $i = 1, 2, \cdots, N$, *is a Meijer G-function multiplied by a normalizing constant* H, *i.e.*,

$$g(z) = HG_{N0}^{N0}\left(z^2 \prod_{i=1}^{N} \frac{1}{2\sigma_i}\middle|0\right), \tag{45}$$

where

$$H = \left[(2\pi)^{N/2} \prod_{i=1}^{N} \sigma_i\right]^{-1}.$$

Proof. Since the Mellin transform of $g(z)$ is

$$M\{g(z)|s\} = \left(\frac{1}{\pi}\right)^{N/2} 2^{N(s-1)/2}\left(\prod_{i=1}^{N} \sigma_i^{s-1}\right)\left[\Gamma\left(\frac{s}{2}\right)\right]^N,$$

the density function $g(z)$ is obtained by evaluating the inversion integral

$$g(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}M\{g(z)|s\}\, ds \tag{46}$$

along a line parallel to the imaginary axis and lying to the right of the origin [1]. If one makes the substitution

$$w = z^2 \prod_{i=1}^{N} \frac{1}{2\sigma_i^2}$$

in (46), he obtains the equivalent density function

$$g(z)\, dz = h(w)\, dw,$$

where

$$h(w) = \left(\frac{1}{\prod_{i=1}^{N} (2\pi\sigma_i^2)^{1/2}}\right)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-s}\Gamma^N(s)\, ds. \tag{47}$$

But by the definition (9) of the Meijer G-function it follows that

$$h(w) = HG_{0N}^{N0}(w|0).$$

Then the result (45) follows directly from (46).

THEOREM 6. *The Meijer G-function $G_{0N}^{N0}(w|0)$ has the representation*

$$G_{0N}^{N0}(w|0) = \sum_{j=0}^{\infty} R(w, N, j),$$

where

$$R(w, N, j) = w^j \sum_{k=0}^{N-1} \frac{(-\ln w)^{N-1-k}}{(N-1-k)!k!} U^{(k)}(s)$$

and $U^{(k)}(s)$ is obtained from the lower derivatives by the recursion formula

$$U^{(k)}(s) = N \sum_{m=0}^{k-1} (-1)^{k-m}\frac{(k-1)!}{m!}\left\{\zeta(k-m, 1) - \sum_{i=0}^{j-1} \frac{1}{(j-i)^{k-m}}\right\}U^{(m)}\Bigg|_{s=-j}.$$

Proof. From (9) and (47), the Meijer G-function to be evaluated is

$$G_{0N}^{N0}(w|0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-s}\Gamma^N(s)\, ds. \tag{48}$$

Since the conditions of Jordan's lemma are met, it follows from the residue theorem that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-s}\Gamma^N(s)\,ds = \sum_{j=0}^{\infty} R(w,N,j), \tag{49}$$

where

$$R(w,N,j) = \frac{1}{(N-1)!}\frac{d^{N-1}}{ds^{N-1}}\{(s+j)^N w^{-s}\Gamma^N(s+j)\}\bigg|_{s=-j}, \quad j=0,1,2,\cdots. \tag{50}$$

The problem is to evaluate the derivatives in (50) at the Nth order poles occurring at negative integral values of s. The evaluation is considerably simplified by applying Leibniz's rule for the differentiation of products, which gives

$$\begin{aligned} &\frac{d^{N-1}}{ds^{N-1}}\{(s+j)^N w^{-s}\Gamma^{(N)}(s+j)\}\bigg|_{s=-j} \\ &\qquad = w^{-s}\sum_{k=0}^{N-1}\frac{(-\ln w)^{N-1-k}}{(N-1-k)!k!}\frac{d^k}{ds^k}(s+j)^N\Gamma^N(s)\bigg|_{s=-j}. \end{aligned} \tag{51}$$

To simplify the notation, let

$$U(s) = [(s+j)\Gamma(s)]^N,$$

$$U^{(k)}(s) = \frac{d^k}{dU^k}U(s), \qquad k \geqq 1,$$

$$U^{(0)}(s) = U(s).$$

To further expedite the analysis, note that $U(s)$ can be written in the form

$$U(s) = \left[\frac{\Gamma(s+j+1)}{s(s+1)(s+2)\cdots(s+j-1)}\right]^N \tag{52}$$

from which

$$\ln U(s) = N\left\{\ln\Gamma(s+j+1) - \sum_{i=0}^{j-1}\ln(s+i)\right\}. \tag{53}$$

Differentiation of (53) leads immediately to the result

$$U^{(1)}(s) = NU(s)\left\{\psi(s+j+1) - \sum_{i=0}^{j-1}\frac{1}{s+i}\right\}. \tag{54}$$

Application of Leibniz's rule to (54) yields the recursion formula

$$\begin{aligned} U^{(k)}(s)\bigg|_{s=-j} &= N\sum_{m=0}^{k-1}\frac{(-1)^{k-m}(k-1)!}{m!(k-1-m)!} \\ &\quad\cdot\left\{\zeta(k-m,1) - \sum_{i=0}^{j-1}\frac{1}{(j-i)^{k-m}}\right\}U^{(m)}(s)\bigg|_{s=-j}, \quad j=0,1,2,\cdots, \end{aligned} \tag{55}$$

which when used in conjunction with (54), (52), (51), (50), (49) and (48) establishes Theorem 6.

5. The product of independent beta variables. The theorems of this section establish the fact that the density function of the product of independent beta variables is a Meijer G-function, the representation of which is derived in series form. For the special case of identically distributed beta variables, the result reduces to that given by Lomnicki [4].

THEOREM 7. *The probability density function of the product $z = \prod x_i$ of N independent beta random variables is a Meijer G-function multiplied by a normalizing constant K, i.e.,*

$$g(z) = KG_{NO}^{NO}\left(z \middle| \begin{matrix} a_1 + b_1 - 1, \cdots, a_N + b_N - 1 \\ a_1 - 1, \cdots, a_N - 1 \end{matrix}\right),$$

where

$$K = \prod_{i=1}^{N} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)}.$$

Proof. Consider N independent beta random variables x_i with integral parameters a_i, b_i, namely,

$$f_i(x_i) = \frac{1}{B(a_i, b_i)} x_i^{a_i - 1}(1 - x_i)^{b_i - 1}, \qquad i = 1, 2, \cdots, N. \tag{56}$$

Since the Mellin transform of (56) is (see [7, Formula (31), p. 311])

$$M\{f_i(x_i)|s\} = \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \frac{\Gamma(a_i - 1 + s)}{\Gamma(a_i + b_i - 1 + s)}, \tag{57}$$

the density function $g(z)$ is obtained by evaluating the Mellin inversion integral

$$g(z) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} z^{-s} \prod_{i=1}^{N} M\{f_i(x_i)|s\}\, ds.$$

It follows immediately from (57) and the definition of the Meijer G-function that

$$g(z) = KG_{NO}^{NO}\left(z \middle| \begin{matrix} a_1 + b_1 - 1, \cdots, a_N + b_N - 1 \\ a_1 - 1, a_2 - 1, \cdots, a_N - 1 \end{matrix}\right), \tag{58}$$

where

$$K = \prod_{i=1}^{N} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)}.$$

This establishes Theorem 7.

THEOREM 8. *The Meijer G-function*

$$G_{NO}^{NO}\left(z \middle| \begin{matrix} a_1 + b_1 - 1, \cdots, a_N + b_N - 1 \\ a_1 - 1, a_2 - 1, \cdots, a_N - 1 \end{matrix}\right)$$

with integral parameters has the closed form representation

$$G_{NO}^{NO}\left(z \middle| \begin{matrix} a_1 + b_1 - 1, \cdots, a_N + b_N - 1 \\ a_1 - 1, a_2 - 1, \cdots, a_N - 1 \end{matrix}\right) = \sum_{k=1}^{m} \sum_{j=0}^{e_k - 1} K_{kj} z^{d_k - 1}(-\ln z)^{e_k - j - 1}, \tag{59}$$

where d_k denotes the m different integers which occur with multiplicity e_k among the $a_i - 1, a_i - 2, \cdots, a_i + b_i - 2$ for $i = 1, 2, \cdots, N$.

Proof. Consider N beta variables of the first kind with integral parameters a_i, b_i with p.d.f. (56). Let $g(z)$ denote the p.d.f. of $z = \prod_{i=1}^{N} x_i$. Since the Mellin integral transform of (56) is

$$M(f_i(x)|s) = \frac{B(s + a_1 - 1, b_i)}{B(a_i, b_i)}$$

$$= \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \frac{\Gamma(s + a_i - 1)}{\Gamma(s + a_i - 1 + b_i)},$$

it follows that

$$(60) \qquad M(g(z)|s) = \prod_{i=1}^{N} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \frac{1}{(s + a_i - 1)(s + a_i) \cdots (s + a_i - 2 + b_i)}$$

and

$$(61) \qquad g(z) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} z^{-s} M\{g(z)|s\}\, ds.$$

The problem lies in evaluating (61) in closed form. To simplify the evaluation one can, without loss of generality, introduce new parameters d_k, e_k in (60) and write

$$(62) \qquad M(g(z)|s) = \prod_{i=1}^{N} \frac{(a_i + b_i - 1)!}{(a_i - 1)!} \frac{1}{(s + d_1 - 1)^{e_1}(s + d_2 - 1)^{e_2} \cdots (s + d_m - 1)^{e_m}},$$

where $d_k \neq d_j$, $k \neq j$ and where there occur m distinct integers $d_1 < d_2 < \cdots < d_m$ of multiplicity $e_1, e_2, \cdots, e_m$, respectively, among the $a_i - 1, a_i, \cdots, a_i + b_i - 2$ for $i = 1, 2, \cdots, N$. If $M(g(z)|s)$ as given by (62) is now substituted into (61), the resultant inversion integral may be evaluated by the methods of residues, since the conditions of Jordan's lemma are satisfied. Specifically,

$$g(z) = \left(\prod_{i=1}^{N} \frac{(a_i + b_i - 1)!}{(a_i - 1)!}\right) \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{z^{-s}\, ds}{(s + d_1 - 1)^{e_1} \cdots (s + d_m - 1)^{e_m}}$$

$$= \left(\prod_{i=1}^{N} \frac{(a_i + b_i - 1)!}{(a_i - 1)!}\right) \sum_{k=1}^{m} R_k,$$

where R_k is the residue at the kth pole of order e_k, $k = 1, 2, \cdots, m$, namely

$$(63) \qquad R_k = \frac{1}{(e_k - 1)!} \frac{d^{e_k - 1}}{ds^{e_k - 1}} (s + d_k - 1)^{e_k} z^{-s} \prod_{q=1}^{m} (s + d_q - 1)^{-e_q} \Bigg|_{s = -(d_k - 1)}.$$

Using Leibniz's rule for the differentiation of products, one can write (63) in the form

$$(64) \qquad R_k = z^{-s} \sum_{j=0}^{e_k - 1} \frac{(-\ln z)^{e_k - 1 - j}}{(e_k - 1 - j)!\, j!} W^{(j)}(s) \Bigg|_{s = -(d_k - 1)},$$

where

$$W(s) = \prod_{\substack{q=1 \\ q \neq k}}^{m} (s + d_q - 1)^{-e_q}.$$

Differentiation of ln $W(s)$ yields

$$W^{(1)}(s) = -W(s) \sum_{\substack{q=1 \\ q \neq k}}^{m} \frac{e_q}{s + d_q - 1}$$

to which Leibniz's rule for differentiation of products may be applied to obtain $W^{(j)}(s)|_{s=-(d_k-1)} = K_{kj}$, where

$$K_{kj} = \sum_{r=0}^{j} \sum_{\substack{q=1 \\ q \neq k}}^{m} (-1)^{r+1} \binom{j}{r} \frac{r! e_q}{(d_q - d_k)^{r+1}}, \tag{65}$$

and where $W^{(j)}(s)$ denotes the jth derivative of $W(s)$. Thus,

$$g(z) = \left(\prod_{i=1}^{N} \frac{(a_i + b_i - 1)!}{(a_i - 1)!} \right) \sum_{k=1}^{m} \sum_{j=0}^{e_k - 1} \frac{K_{kj} z^{d_k - 1} (-\ln z)^{e_k - 1 - j}}{(e_k - 1 - j)! j!}, \tag{66}$$

where the coefficients K_{kj} are given by (65). From (66) and (58), it follows that

$$G_{N0}^{N0}\left(z \middle| \begin{matrix} a_1 + b_1 - 1, \cdots, a_N + b_N - 1 \\ a_1 - 1, a_2 - 1, \cdots, a_N - 1 \end{matrix} \right) = \sum_{k=1}^{m} \sum_{j=0}^{e_k - 1} K_{kj} z^{d_k - 1} (-\ln z)^{e_k - 1 - j}. \tag{67}$$

COROLLARY. *The p.d.f. of the product $z = \prod_1^N x_i$ of N independent identically distributed beta random variables with the probability density function* (56) *and integer parameters a, b has the closed form representation*

$$g(z) = \left(\frac{(a + b - 1)!}{(a - 1)!} \right)^N N \sum_{k=1}^{b+1} \sum_{j=0}^{N-1} K_{kj} z^{a+k-2} (-\ln z)^{N-1-j}, \tag{68}$$

where

$$K_{kj} = \sum_{\substack{q=1 \\ q \neq k}}^{b} \sum_{r=0}^{j} (-1)^{r+1} \binom{j}{r} \frac{r!}{(s + a + q - 2)^{r+1}} W(s)^{(j-r)} \Bigg|_{s=-(a+k-2)}.$$

Proof. Since all the poles are of order N and $a_i = a, b_i = b, i = 1, 2, \cdots, N$, the result follows directly from Theorem 8 by putting $m = b + 1$, $e_k = N$, $k = 1, 2, \cdots, m; d_q = a + q - 2, q = 1, 2, \cdots, m$.

6. Mixed products of independent beta and gamma variables. The final result established in this paper concerns the product of M beta variables and $N - M$ gamma variables, and is stated in the following theorem.

THEOREM 9. *The probability density function $g(z)$ of the product*

$$z = x_1 x_2 \cdots x_M y_{M+1} y_{M+2} \cdots y_N$$

of M independent beta variables and N − M independent gamma variables is a Meijer G-function multiplied by a normalizing constant K, i.e.,

$$g(z) = KG_{MN}^{NO}\left(z\left|\begin{matrix} a_1 + b_1 - 1, a_2 + b_2 - 1, \cdots, a_M + b_M - 1 \\ a_1 - 1, a_2 - 1, \cdots, a_M - 1, c_{M+1} - 1, \cdots, c_N - 1\end{matrix}\right.\right),$$

where

$$K = \prod_{i=1}^{M} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \prod_{j=M+1}^{N} \frac{1}{\Gamma(c_j)}. \tag{69}$$

Proof. Let y_i denote the gamma random variable with probability density function

$$f_i(y_i) = \frac{1}{\Gamma(c_i)} y_1^{ci-1} \exp(-y_i), \qquad i = 1, 2, \cdots, M.$$

Then, since the random variables are independent, it follows from (13) and (57) that

$$M\{g(z)|s\} = \left\{\prod_{i=1}^{M} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)} \prod_{j=M+1}^{N} \frac{1}{\Gamma(c_j)}\right\} \frac{\prod_{i=1}^{M} \Gamma(a_i - 1 + s) \prod_{j=M+1}^{N} \Gamma(s + c_j - 1)}{\prod_{i=1}^{M} \Gamma(a_i + b_i - 1 + s)}.$$

The density function $g(z)$ is then given by the inversion integral

$$g(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} M\{g(z)|s\}\, ds$$

which is precisely

$$G_{ON}^{NO}\left(z\left|\begin{matrix} a_1 + b_1 - 1, a_2 + b_2 - 1, \cdots, a_M + b_M - 1 \\ a_1 - 1, a_2 - 1, \cdots, a_M - 1, c_{M+1} - 1, \cdots, c_N - 1\end{matrix}\right.\right)$$

multiplied by a constant. That is,

$$g(z) = KG_{ON}^{NO}\left(z\left|\begin{matrix} a_1 + b_1 - 1, a_2 + b_2 - 1, \cdots, a_M + b_M - 1 \\ a_1 - 1, a_2 - 1, \cdots, a_M - 1, c_{M+1} - 1, \cdots, c_N - 1\end{matrix}\right.\right),$$

where K is given by (69).

7. Examples. In this section, two examples are given, consisting of the probability density functions of products of three independent beta random variables and six independent Gaussian random variables. Both the graph and the analytical form of the density function are given.

Example 1. *Product of beta random variables.* Figure 1 consists of a graph of the probability density function $g(z)$ of the product $z = x_1x_2x_3$, where $x_i, i = 1, 2, 3$, are independent beta random variables whose probability density functions are

$$f(x_i) = \frac{1}{B(a_i, b_i)} x_i^{a_i - 1}(1 - x_i)^{b_i - 1}, \qquad 0 \leqq x_i \leqq 1, \quad i = 1, 2, 3,$$

$$a_1 = 9, \quad b_1 = 3; \quad a_2 = 8, \quad b_2 = 3; \quad a_3 = 4, \quad b_3 = 2.$$

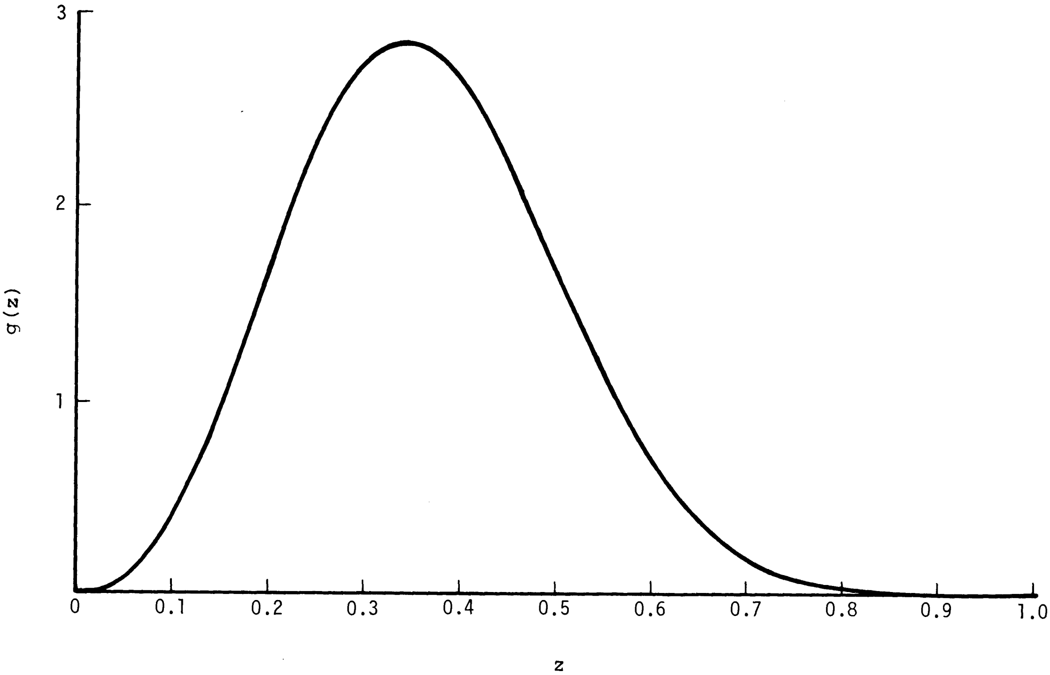

FIG. 1. *Probability density function of product of 3 beta random variables*

The density and distribution functions are, respectively,

$$g(z) = \frac{3960}{7}z^3 - 1980z^4 + 99{,}000z^7 + (374{,}220 + 356{,}400 \ln z)z^8 - (443{,}520 - 237{,}600 \ln z)z^9 - \frac{198{,}000}{7}z^{10}$$

and

$$G(z) = \frac{990}{7}z^4 - 396z^5 + 12{,}375z^8 + (37{,}180 + 39{,}600 \ln z)z^9 - (46{,}728 - 23{,}760 \ln z)z^{10} - \frac{18{,}000}{7}z^{11}.$$

Example 2. *Product of Gaussian random variables.* The density function $g(z)$ of the product $z = \prod_{i=1}^{6} x_i$ of six identically distributed independent Gaussian random variables

$$f(x_i) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_i^2}{2}\right), \qquad -\infty < x_i < \infty, \quad i = 1, 2, 3, 4, 5, 6,$$

was tabulated in a previous paper [1] and is presented graphically in Fig. 2. The density function is

$$h(z) = \sum_{j=0}^{\infty} R_j,$$

where

$$R_j = z^j \sum_{k=0}^{5} \frac{1}{(5-k)!k!}(-\ln z)^{5-k} \frac{d^k}{ds^k}\left(\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)}\right)^6\Bigg|_{s=-j},$$

$$j = 0, 1, 2, \cdots .$$

The number of residues R_j which must be evaluated depends upon the required accuracy of the density function. An excellent indication of the accuracy attained

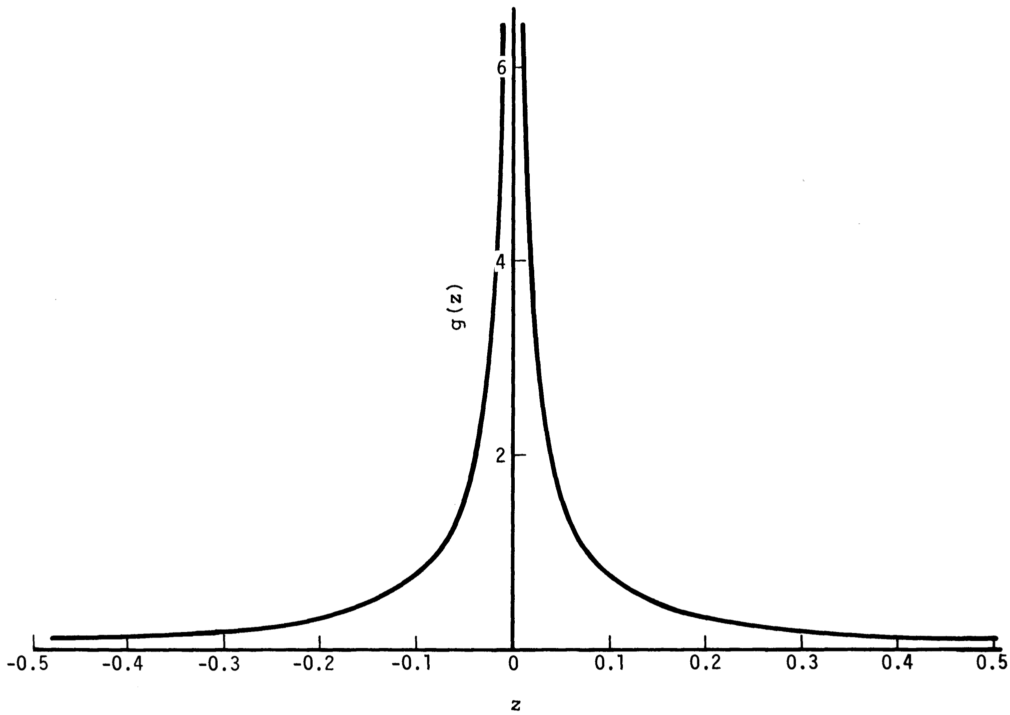

FIG. 2. *Probability density function of product of* 6 *Gaussian random variables*

in the derived probability density function can be obtained by computing the first and second absolute moments $E[|z|]$ and $E[z^2]$ since it is known that

$$E[|z|^m] = \prod_{i=1}^{N} E[|x_i|^m], \qquad m = 0, 1, 2, \cdots .$$

The first and second absolute moments of the probability density function shown graphically in Fig. 2 are correct to five decimal places.

REFERENCES

[1] M. D. SPRINGER AND W. E. THOMPSON, *The distribution of products of independent random variables*, this Journal, 14 (1966), pp. 511–526.

[2] ———, *Bayesian confidence limits for the products of N binomial parameters*, Biometrika, 53 (1966), pp. 611–613.

[3] Z. A. LOMNICKI, *On the distribution of products of random variables*, J. Roy. Statist. Soc. Ser. B, 29 (1967), pp. 513–524.

[4] ———, *On the distribution of products of independent beta variables*, Tech. Rep. 50, Laboratory of Statistical Research, Department of Mathematics, University of Washington, Seattle, 1967.

[5] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, 1937.

[6] *Handbook of Mathematical Functions*, Applied Mathematics Series 55, National Bureau of Standards, Washington, D.C., 1964.

[7] H. BATEMAN, *Tables of Integral Transforms*, vol. I, McGraw-Hill, New York, 1954.

[8] E. T. WHITTAKER AND G. N. WATSON, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1940.