بسم الله الرحمن الرحیم

Network Security

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Content of this Chapter

- **The RSA Cryptosystem**
- Implementation aspects
- Finding Large Primes
- Attacks and Countermeasures
- Lessons Learned

The RSA Cryptosystem

- Martin Hellman and Whitfield Diffie published their landmark public-key paper in 1976
- Ronald Rivest, Adi Shamir and Leonard Adleman proposed the asymmetric RSA cryptosystem in1977
- Until now, RSA is the most widely use asymmetric cryptosystem although elliptic curve cryptography (ECC) becomes increasingly popular
- RSA is mainly used for two applications
    - Transport of (i.e., symmetric) keys (cf. Chptr 13 of *Understanding Cryptography*)
    - Digital signatures (cf. Chptr 10 of *Understanding Cryptography*)

Encryption and Decryption

- RSA operations are done over the integer ring Z_n (i.e., arithmetic modulo n), where $n = p * q$, with p, q being large primes
- Encryption and decryption are simply exponentiations in the ring

Definition

Given the public key $(n,e) = k_{pub}$ and the private key $d = k_{pr}$ we write

$$y = e_{k_{pub}}(x) \equiv x^e \bmod n$$

$$x = d_{k_{pr}}(y) \equiv y^d \bmod n$$

where $x, y \in Z_n$.

We call $e_{k_{pub}}()$ the encryption and $d_{k_{pr}}()$ the decryption operation.

- In practice x, y, n and d are very long integer numbers (≥ 1024 bits)
- The security of the scheme relies on the fact that it is hard to derive the „private exponent" d given the public-key (n, e)

Key Generation

- Like all asymmetric schemes, RSA has set-up phase during which the private and public keys are computed

Algorithm: RSA Key Generation

Output: public key: $k_{pub} = (n, e)$ and private key $k_{pr} = d$

1. Choose two large primes p, q
2. Compute $n = p * q$
3. Compute $\Phi(n) = (p-1) * (q-1)$
4. Select the public exponent $e \in \{1, 2, \ldots, \Phi(n)-1\}$ such that gcd$(e, \Phi(n)) = 1$
5. Compute the private key d such that $d * e \equiv 1 \bmod \Phi(n)$
6. **RETURN** $k_{pub} = (n, e)$, $k_{pr} = d$

Remarks:

- Choosing two large, distinct primes p, q (in Step 1) is non-trivial
- gcd$(e, \Phi(n)) = 1$ ensures that e has an inverse and, thus, that there is always a private key d

Example: RSA with small numbers

ALICE

Message $\color{red}{\boldsymbol{x = 4}}$

BOB

1. Choose $p = 3$ and $q = 11$
2. Compute $n = p * q = 33$
3. $\Phi(n) = (3\text{-}1) * (11\text{-}1) = 20$
4. Choose $e = 3$
5. $d \equiv e^{-1} \equiv 7 \bmod 20$

$K_{pub} = (33,3)$ ←

$y = x^e \equiv 4^3 \equiv 31 \bmod 33$

$y = 31$ →

$y^d = 31^7 \equiv \color{red}{\boldsymbol{4 = x}} \bmod 33$

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- The RSA Cryptosystem
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Implementation aspects

- The RSA cryptosystem uses only one arithmetic operation (modular exponentiation) which makes it conceptually a simple asymmetric scheme
- Even though conceptually simple, due to the use of very long numbers, RSA is orders of magnitude slower than symmetric schemes, e.g., DES, AES
- When implementing RSA (esp. on a constrained device such as smartcards or cell phones) close attention has to be paid to the correct choice of arithmetic algorithms
- The square-and-multiply algorithm allows fast exponentiation, even with very long numbers…

Square-and-Multiply

- **Basic principle**: Scan exponent bits from left to right and square/multiply operand accordingly

Algorithm: Square-and-Multiply for x^H *mod n*

Input: Exponent H, base element x, Modulus n

Output: $y = x^H \bmod n$

1. Determine binary representation $H = (h_t, h_{t-1}, ..., h_0)_2$
2. **FOR** $i = t-1$ **TO** 0
3. $\quad y = y^2 \bmod n$
4. $\quad$ **IF** $h_i = 1$ **THEN**
5. $\qquad y = y * x \bmod n$
6. **RETURN** y

- Rule: Square in every iteration (Step 3) and multiply current result by x if the exponent bit $h_i = 1$ *(Step 5)*
- Modulo reduction after each step keeps the operand y small

Example: Square-and-Multiply

- Computes x^{26} without modulo reduction
- Binary representation of exponent: $26 = (1,1,0,1,0)_2 = (h_4,h_3,h_2,h_1,h_0)_2$

Step		Binary exponent	Op	Comment
1	$x = x^1$	$(1)_2$		Initial setting, h_4 processed
1a	$(x^1)^2 = x^2$	$(10)_2$	SQ	Processing h_3
1b	$x^2 * x = x^3$	$(11)_2$	MUL	$h_3 = 1$
2a	$(x^3)^2 = x^6$	$(110)_2$	SQ	Processing h_2
2b	-	$(110)_2$	-	$h_0 = 0$
3a	$(x^6)^2 = x^{12}$	$(1100)_2$	SQ	Processing h_1
3b	$x^{12} * x = x^{13}$	$(1101)_2$	MUL	$h_1=1$
4a	$(x^{13})^2 = x^{26}$	$(11010)_2$	SQ	Processing h_0
4b	-	$(11010)_2$	-	$h_0 = 0$

- Observe how the exponent evolves into $x^{26} = x^{11010}$

Complexity of Square-and-Multiply Alg.

- The square-and-multiply algorithm has a logarithmic complexity, i.e., its run time is proportional to the bit length (rather than the absolute value) of the exponent
- Given an exponent with t+1 bits

  $$H = (h_t, h_{t-1}, \ldots, h_0)_2$$

  with $h_t = 1$, we need the following operations
  - *# Squarings* $= t$
  - Average # multiplications $= 0.5\ t$
  - Total complexity: *#SQ + #MUL* $= 1.5\ t$
- Exponents are often randomly chosen, so *1.5 t* is a good estimate for the average number of operations
- Note that each squaring and each multiplication is an operation with very long numbers, e.g., 2048 bit integers.

Speed-Up Techniques

- Modular exponentiation is computationally intensive
- Even with the square-and-multiply algorithm, RSA can be quite slow on constrained devices such as smart cards
- Some important tricks:
  - Short public exponent *e*
  - Chinese Remainder Theorem (CRT)
  - Exponentiation with pre-computation *(not covered here)*

Fast encryption with small public exponent

- Choosing a small public exponent *e* does not weaken the security of RSA
- A small public exponent improves the speed of the RSA encryption significantly

Public Key	***e* as binary string**	**#MUL + #SQ**
$2^1+1 = 3$	$(11)_2$	1 + 1 = 2
$2^4+1 = 17$	$(1\ 0001)_2$	4 + 1 = 5
$2^{16} + 1$	$(1\ 0000\ 0000\ 0000\ 0001)_2$	16 + 1 = 17

- This is a commonly used trick (e.g., SSL/TLS, etc.) and makes RSA the fastest asymmetric scheme with regard to encryption!

Fast decryption with CRT

- Choosing a small private key *d* results in security weaknesses!
  - In fact, d must have at least *0.3t* bits, where *t* is the bit length of the modulus *n*
- However, the Chinese Remainder Theorem (CRT) can be used to (somewhat) accelerate exponentiation with the private key *d*
- Based on the CRT we can replace the computation of

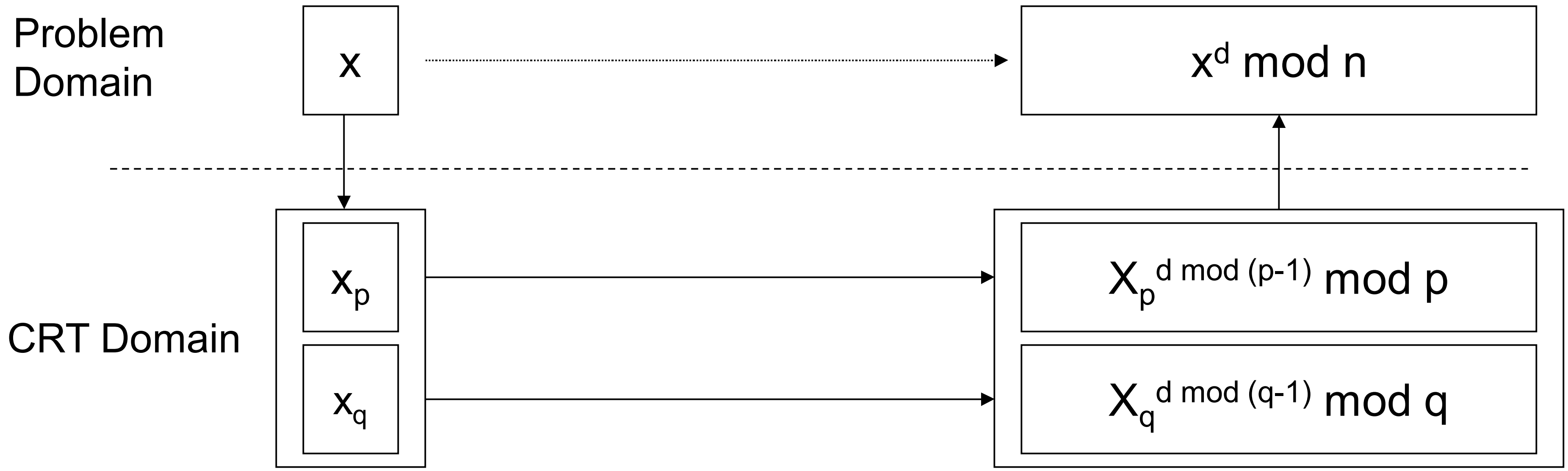
$$x^{d \bmod \Phi(n)} \bmod n$$

by two computations

$$x^{d \bmod (p-1)} \bmod p \quad \text{and} \quad x^{d \bmod (q-1)} \bmod q$$

where *q* and *p* are „small" compared to *n*

Basic principle of CRT-based exponentiation



- CRT involves three distinct steps

  (1) Transformation of operand into the CRT domain

  (2) Modular exponentiation in the CRT domain

  (3) Inverse transformation into the problem domain

- These steps are equivalent to one modular exponentiation in the problem domain

CRT: Step 1 – Transformation

- Transformation into the CRT domain requires the knowledge of *p* and *q*
- *p* and *q* are only known to the owner of the private key, hence CRT cannot be applied to speed up encryption
- The transformation computes (x_p, x_q) which is the representation of *x* in the CRT domain. They can be found easily by computing

$$x_p \equiv x \bmod p \quad \text{and} \quad x_q \equiv x \bmod q$$

CRT: Step 2 – Exponentiation

- Given d_p and d_q such that

$$d_p \equiv d \bmod (p\text{-}1) \quad \text{and} \quad d_q \equiv d \bmod (q\text{-}1)$$

one exponentiation in the problem domain requires two exponentiations in the CRT domain

$$y_p \equiv x_p^{d_p} \bmod p \quad \text{and} \quad y_q \equiv x_q^{d_q} \bmod q$$

- In practice, p and q are chosen to have half the bit length of n, i.e., $|p| \approx |q| \approx |n|/2$

CRT: Step 3 – Inverse Transformation

- Inverse transformation requires modular inversion twice, which is computationally expensive

$$c_p \equiv q^{-1} \bmod p \qquad \text{and} \qquad c_q \equiv p^{-1} \bmod q$$

- Inverse transformation assembles y_p, y_q to the final result $y \bmod n$ in the problem domain

$$y \equiv [\, q * c_p \,] * y_p + [\, p * c_q \,] * y_q \bmod n$$

- The primes p and q typically change infrequently, therefore the cost of inversion can be neglected because the two expresssions

  $[\, q * c_p \,]$ and $[\, p * c_q \,]$

  can be precomputed and stored

Complexity of CRT

- We ignore the transformation and inverse transformation steps since their costs can be neglected under reasonable assumptions
- Assuming that *n* has $t+1$ bits, both *p* and *q* are about $t/2$ bits long
- The complexity is determined by the two exponentiations in the CRT domain. The operands are only $t/2$ bits long. For the exponentiations we use the square-and-multiply algorithm:
  - # squarings (one exp.): $\#SQ = 0.5\ t$
  - # aver. multiplications (one exp.): #MUL = 0.25t
  - Total complexity: $2 * (\#MUL + \#SQ) = 1.5t$
- This looks the same as regular exponentations, but since the operands have half the bit length compared to regular exponent., each operation (i.e., multipl. and squaring) is 4 times faster!
- Hence CRT is **4 times** faster than straightforward exponentiation

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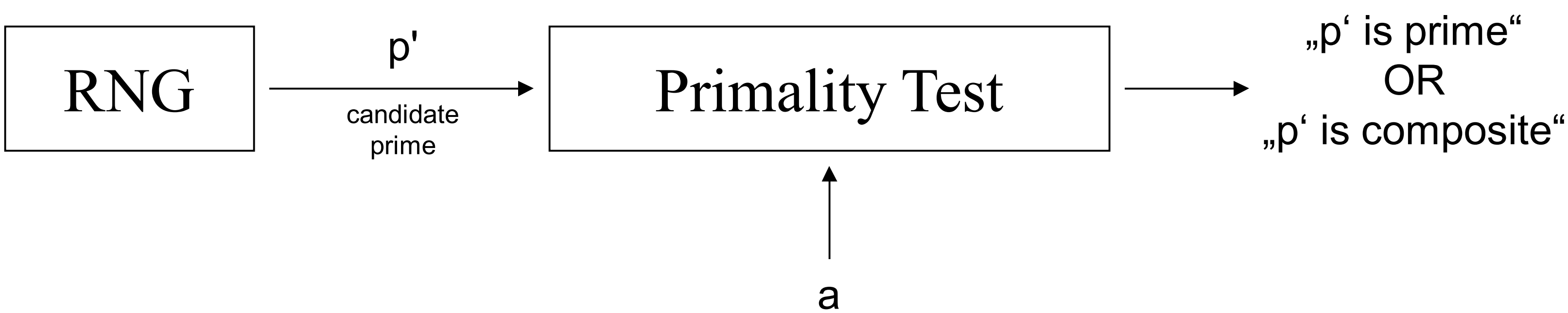
Finding Large Primes

- Generating keys for RSA requires finding two large primes *p* and *q* such that $n = p * q$ is sufficiently large
- The size of *p* and *q* is typically half the size of the desired size of *n*
- To find primes, random integers are generated and tested for primality:



- The random number generator (RNG) should be non-predictable otherwise an attacker could guess the factorization of *n*

Primality Tests

- Factoring *p* and *q* to test for primality is typically not feasible
- However, we are not interested in the factorization, we only want to know whether *p* and *q* are composite
- Typical primality tests are probabilistic, i.e., they are not 100% accurate but their output is correct with very high probability
- A probabilistic test has two outputs:
  - „p‘ is composite“ – always true
  - „p‘ is a prime“ – only true with a certain probability
- Among the well-known primality tests are the following
  - Fermat Primality-Test
  - Miller-Rabin Primality-Test

Fermat Primality-Test

- Basic idea: Fermat‘s Little Theorem holds for all primes, i.e., if a number *p‘* is found for which $a^{p'-1} \not\equiv 1 \bmod p'$, it is not a prime

Algorithm: Fermat Primality-Test

Input: Prime candidate *p‘*, security parameter *s*

Output: „*p‘* is composite“ or „*p‘* is likely a prime“

1. **FOR** $i = 1$ **TO** s
2. choose random $a \in \{2,3, \ldots, p'-2\}$
3. **IF** $a^{p'-1} \not\equiv 1 \bmod p'$ **THEN**
4. **RETURN** „*p‘* is composite“
5. **RETURN** „*p‘* is likely a prime“

- For certain numbers („Carchimchael numbers“) this test returns „*p‘* is likely a prime“ often – although these numbers are composite
- Therefore, the Miller-Rabin Test is preferred

Theorem for Miller-Rabin's test

- The more powerful Miller-Rabin Test is based on the following theorem

Theorem

Given the decomposition of an odd prime candidate *p'*

$$p' - 1 = 2^u * r$$

where *r* is odd. If we can find an integer *a* such that

$$a^r \not\equiv 1 \bmod p' \qquad \text{and} \qquad a^{r2^j} \not\equiv p' - 1 \bmod p'$$

For all $j = \{0, 1, \ldots, u-1\}$, then *p'* is composite.

Otherwise it is probably a prime.

- This theorem can be turned into an algorithm

Miller-Rabin Primality-Test

Algorithm: Miller-Rabin Primality-Test

Input: Prime candidate p' with $p'-1 = 2^{u} * r$ security parameter s

Output: „p' is composite" or „p' is likely a prime"

1. **FOR** $i = 1$ **TO** s
2. choose random $a \in \{2,3, ..., p'-2\}$
3. $z \equiv a^r \bmod p'$
4. **IF** $z \neq 1$ **AND** $z \neq p'-1$ **THEN**
5. **FOR** $j = 1$ **TO** $u-1$
6. $z \equiv z^2 \bmod p'$
7. **IF** $z = 1$ **THEN**
8. **RETURN** „p' is composite"
9. **IF** $z \neq p'-1$ **THEN**
10. **RETURN** „p' is composite"
11. **RETURN** „p' is likely a prime"

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Attacks and Countermeasures 1/3

- There are two distinct types of attacks on cryptosystems
  - **Analytical attacks** try to break the mathematical structure of the underlying problem of RSA
  - **Implementation attacks** try to attack a real-world implementation by exploiting inherent weaknesses in the way RSA is realized in software or hardware

Attacks and Countermeasures 2/3

RSA is typically exposed to these analytical attack vectors

- **Mathematical attacks**
  - The best known attack is factoring of n in order to obtain $\Phi(n)$
  - Can be prevented using a sufficiently large modulus n
  - The current factoring record is 664 bits. Thus, it is recommended that n should have a bit length between 1024 and 3072 bits
- **Protocol attacks**
  - Exploit the malleability of RSA, i.e., the property that a ciphertext can be transformed into another ciphertext which decrypts to a related plaintext – without knowing the private key
  - Can be prevented by proper padding

Attacks and Countermeasures 3/3

- Implementation attacks can be one of the following
  - **Side-channel analysis**
    - Exploit physical leakage of RSA implementation (e.g., power consumption, EM emanation, etc.)
  - **Fault-injection attacks**
    - Inducing faults in the device while CRT is executed can lead to a complete leakage of the private key

More on all attacks can be found in Section 7.8 of *Understanding Cryptography*

Attacks and Countermeasures 2/2

- RSA is typically exposed to these analytical attack vectors *(cont'd)*
  - **Protocol attacks**
    - Exploit the malleability of RSA
    - Can be prevented by proper padding
- Implementation attacks can be one of the following
  - **Side-channel analysis**
    - Exploit physical leakage of RSA implementation (e.g., power consumption, EM emanation, etc.)
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Lessons Learned

- RSA is the most widely used public-key cryptosystem
- RSA is mainly used for key transport and digital signatures
- The public key *e* can be a short integer, the private key *d* needs to have the full length of the modulus *n*
- RSA relies on the fact that it is hard to factorize *n*
- Currently 1024-bit cannot be factored, but progress in factorization could bring this into reach within 10-15 years. Hence, RSA with a 2048 or 3076 bit modulus should be used for long-term security
- A naïve implementation of RSA allows several attacks, and in practice RSA should be used together with padding

Content of this Chapter

- Diffie–Hellman Key Exchange
- The Discrete Logarithm Problem
- Security of the Diffie–Hellman Key Exchange
- The Elgamal Encryption Scheme

Diffie–Hellman Key Exchange: Overview

- Proposed in 1976 by **Whitfield Diffie and Martin Hellman**
- **Widely used**, e.g. in Secure Shell (SSH), Transport Layer Security (TLS), and Internet Protocol Security (IPSec)
- The Diffie–Hellman Key Exchange (DHKE) is a key exchange protocol and **not** used for encryption

  (For the purpose of encryption based on the DHKE, ElGamal can be used.)

Diffie–Hellman Key Exchange: Set-up

1. Choose a large prime p.
2. Choose an integer $\alpha \in \{2, 3, \ldots, p-2\}$.
3. Publish p *and* α.

Diffie–Hellman Key Exchange

Alice		Bob
Choose random private key $k_{prA}=a \in \{1,2,\ldots,p\text{-}1\}$		Choose random private key $k_{prB}=b \in \{1,2,\ldots,p\text{-}1\}$
Compute corresponding public key $k_{pubA}= A = \alpha^a \bmod p$	$\xrightarrow{\;A\;}$	
	$\xleftarrow{\;B\;}$	Compute correspondig public key $k_{pubB}= B = \alpha^b \bmod p$
Compute common secret $k_{AB} = B^a = (\alpha^a)^b \bmod p$		Compute common secret $k_{AB} = A^b = (\alpha^b)^a \bmod p$

We can now use the joint key k_{AB} for encryption, e.g., with AES

$$y = AES_{kAB}(x) \xrightarrow{\quad y \quad} x = AES^{-1}_{kAB}(y)$$

Diffie–Hellman Key Exchange: Example

Domain parameters p=29, α=2

Alice

Choose random private key
$k_{prA} = a = 5$

Compute corresponding public key
$k_{pubA} = A = 2^5 = 3 \bmod 29$

$A \longrightarrow$

Bob

Choose random private key
$k_{prB} = b = 12$

Compute correspondig public key
$k_{pubB} = B = 2^{12} = 7 \bmod 29$

$\longleftarrow B$

Alice:

Compute common secret
$k_{AB} = B^a = 7^5 = 16 \bmod 29$

Bob:

Compute common secret
$k_{AB} = A^b = 3^{12} = 16 \bmod 29$

Proof of correctness:

Alice computes: $B^a = (\alpha^b)^a \bmod p$
Bob computes: $A^b = (\alpha^a)^b \bmod p$

i.e., Alice and Bob compute the same key k_{AB}!

The Discrete Logarithm Problem

Discrete Logarithm Problem (DLP) in $Z_p{}^*$

- Given is the finite cyclic group $Z_p{}^*$ of order $p-1$ and a primitive element $\alpha \in Z_p{}^*$ and another element $\beta \in Z_p{}^*$.
- The DLP is the problem of determining the integer $1 \le x \le p-1$ such that $\alpha^x \equiv \beta \bmod p$
- This computation is called the **discrete logarithm problem (DLP)**

$$x = log_\alpha \beta \bmod p$$

- Example: Compute x for $5^x \equiv 41 \bmod 47$

Remark: For the coverage of groups and cylcic groups, we refer to Chapter 8 of *Understanding Cryptography*

The Generalized Discrete Logarithm Problem

- Given is a finite cyclic group G with the group operation $\circ$ and cardinality n.
- We consider a primitive element $\alpha \in G$ and another element $\beta \in G$.
- The discrete logarithm problem is finding the integer x, where $1 \leq x \leq n$, such that:

$$\beta = \underbrace{\alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha}_{x \text{ times}} = \alpha^x$$

The Generalized Discrete Logarithm Problem

The following discrete logarithm problems have been proposed for use in cryptography

1. The multiplicative group of the prime field Z_p *or a subgroup of it. For instance,* the classical DHKE uses this group (cf. previous slides), but also Elgamal encryption or the Digital Signature Algorithm (DSA).
2. The cyclic group formed by an elliptic curve (see Chapter 9)
3. The multiplicative group of a Galois field $GF(2^m)$ or a subgroup of it. *S*chemes such as the DHKE can be realized with them.
4. Hyperelliptic curves or algebraic varieties, which can be viewed as generalization of elliptic curves.

Remark: The groups 1. and 2. are most often used in practice.

Attacks against the Discrete Logarithm Problem

- Security of many asymmetric primitives is based on the difficulty of computing the DLP in cyclic groups, i.e.,

  Compute x for a given α and β such that $\beta = \alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha = \alpha^x$

- The following algorithms for computing discrete logarithms exist
  - Generic algorithms: Work in any cyclic group
    - Brute-Force Search
    - Shanks' Baby-Step-Giant-Step Method
    - Pollard's Rho Method
    - Pohlig-Hellman Method
  - Non-generic Algorithms: Work only in specific groups, in particular in Z_p
    - The Index Calculus Method
- Remark: Elliptic curves can only be attacked with generic algorithms which are weaker than non-generic algorithms. Hence, elliptic curves are secure with shorter key lengths than the DLP in prime fields Z_p

Attacks against the Discrete Logarithm Problem

Summary of records for computing discrete logarithms in Z_p^*

Decimal digits	Bit length	Date
58	193	1991
68	216	1996
85	282	1998
100	332	1999
120	399	2001
135	448	2006
160	532	2007

In order to prevent attacks that compute the DLP, it is recommended to use primes with a length of at least 1024 bits for schemes such as Diffie-Hellman in Z_p^*

Security of the classical Diffie–Hellman Key Exchange

- Which information does Oscar have?
  - α, p
  - $k_{pubA} = A = \alpha^a \bmod p$
  - $k_{pubB} = B = \alpha^b \bmod p$
- Which information does Oscar want to have?
  - $k_{AB} = \alpha^{ba} = \alpha^{ab} = \bmod p$
  - This is kown as Diffie-Hellman Problem (DHP)
- The only known way to solve the DHP is to solve the DLP, i.e.
  1. Compute $a = log_{\alpha} A \bmod p$
  2. *Compute* $k_{AB} = B^a = \alpha^{ba} = \bmod p$

  It is conjectured that the DHP and the DLP are equivalent, i.e., solving the DHP implies solving the DLP.
- To prevent attacks, i.e., to prevent that the DLP can be solved, choose $p > 2^{1024}$

The Elgamal Encryption Scheme: Overview

- Proposed by Taher Elgamal in 1985
- Can be viewed as an extension of the DHKE protocol
- Based on the intractability of the discrete logarithm problem and the Diffie–Hellman problem

The Elgamal Encryption Scheme: Principle

Alice		Bob
		choose $d = k_{prB} \in \{2,\ldots,p\text{-}2\}$
	$\xleftarrow{\ \beta\ }$	compute $\beta = k_{pubB} = \alpha^d \bmod p$
choose $i = k_{prA} \in \{2,\ldots,p\text{-}2\}$		
compute ephemeral key $k_E = k_{pubA} = \alpha^i \bmod p$	$\xrightarrow{\ k_E\ }$	
		compute $k_M = k_E^{\,d} \bmod p$
compute $k_M = \beta^i \bmod p$		
encrypt message $x \in Z_p{}^*$: $y = x \cdot k_M \bmod p$	$\xrightarrow{\ y\ }$	
		decrypt $x = y \cdot k_M^{-1} \bmod p$

This looks very similar to the DHKE! The actual Elgamal protocol re-orders the computations which helps to save one communication (cf. next slide)

The Elgamal Encryption Protocol

Alice		Bob
		choose large prime p
		choose primitive element $\alpha \in Z_p{}^*$ or in a subgroup of $Z_p{}^*$
		choose $d = k_{prB} \in \{2,\ldots,p\text{-}2\}$
	$\xleftarrow{k_{pubB} = (p, \alpha, \beta)}$	compute $\beta = k_{pubB} = \alpha^d \bmod p$
choose i = $k_{prA} \in \{2,\ldots,p\text{-}2\}$		
compute $k_E = k_{pubA} = \alpha^i \bmod p$		
compute masking key $k_M = \beta^i \bmod p$		
encrypt message $x \in Z_p{}^*$: $y = x \cdot k_M \bmod p$	$\xrightarrow{(k_E,\, y)}$	
		compute masking key $k_M = k_E{}^d \bmod p$
		decrypt $x = y \cdot k_M{}^{-1} \bmod p$

Computational Aspects

- Key Generation
  - Generation of prime p
  - p has to of size of at least 1024 bits
  - cf. Section 7.6 in *Understanding Cryptography* for prime-finding algorithms
- Encryption
  - Requires two modular exponentiations and a modular multiplictation
  - All operands have a bitlength of $\log_2 p$
  - Efficient execution requires methods such as the square-and-multiply algorithm (cf. Chapter 7)
- Decryption
  - Requires one modular exponentiation and one modulare inversion
  - As shown *in Understanding Cryptography*, the inversion can be computed from the ephemeral key

Security

- Passive attacks
  - Attacker eavesdrops p, α, $\beta = \alpha^d$, $k_E = \alpha^i$, $y = x \cdot \beta^i$ and wants to recover x
  - Problem relies on the DLP

- Active attacks
  - If the public keys are not authentic, an attacker could send an incorrect public key (cf. Chapter 13)
  - An Attack is also possible if the secret exponent i is being used more than once (cf. *Understanding Cryptography* for more details on the attack)

Lessons Learned

- The Diffie–Hellman protocol is a widely used method for key exchange. It is based on cyclic groups.
- The discrete logarithm problem is one of the most important one-way functions in modern asymmetric cryptography. Many public-key algorithms are based on it.
- For the Diffie–Hellman protocol in $Z_p{}^*$, *the prime p should be at least 1024 bits* long. This provides a security roughly equivalent to an 80-bit symmetric cipher.
- For a better long-term security, a prime of length 2048 bits should be chosen.
- The Elgamal scheme is an extension of the DHKE where the derived session key is used as a multiplicative masked to encrypt a message.
- Elgamal is a probabilistic encryption scheme, i.e., encrypting two identical messages does not yield two identical ciphertexts.