An Introduction to
Objective Bayesian Statistics

José M. Bernardo

Universitat de València, Spain

<jose.m.bernardo@uv.es>
http://www.uv.es/~bernardo

Université de Neuchâtel, Switzerland

March 10th–March 18th, 2004

Summary

1. *Concept of Probability*

   Introduction. Notation. Statistical models.
   Intrinsic discrepancy. Intrinsic convergence of distributions.
   Foundations. Probability as a rational degree of belief.

2. *Basics of Bayesian Analysis*

   Parametric inference. The learning process.
   Reference analysis. No relevant initial information.
   Inference summaries. Point and interval estimation.
   Prediction. Regression.
   Hierarchical models. Exchangeability.

3. *Decision Making*

   Structure of a decision problem. Intrinsic Loss functions.
   Formal point estimation. Intrinsic estimation.
   Hypothesis testing. Bayesian reference criterion (BRC).

1. Concept of Probability

1.1. Introduction

- Tentatively accept a *formal* statistical model
  - Typically suggested by informal descriptive evaluation
  - Conclusions conditional on the assumption that model is correct
- Bayesian approach firmly based on *axiomatic foundations*
  - Mathematical need to describe by probabilities all uncertainties
  - Parameters *must* have a (*prior*) distribution describing available information about their values
  - *Not* a description of their variability (*fixed unknown* quantities), but a description of the *uncertainty* about their true values.
- Important particular case: no relevant (or subjective) initial information
  - *Prior* only based on model assumptions and well-documented data
  - *Objective Bayesian Statistics*:
    Scientific and industrial reporting, public decision making

- *Notation*

□ Under conditions C, $p(\boldsymbol{x}\,|\,C)$, $\pi(\boldsymbol{\theta}\,|\,C)$ are, respectively, *probability* densities (or mass) functions of *observables* $\boldsymbol{x}$ and *parameters* $\boldsymbol{\theta}$

$$p(\boldsymbol{x}\,|\,C) \geq 0,\ \int_{\mathcal{X}} p(\boldsymbol{x}\,|\,C)\,d\boldsymbol{x} = 1,\ \ \mathrm{E}[\boldsymbol{x}\,|\,C] = \int_{\mathcal{X}} \boldsymbol{x}\,p(\boldsymbol{x}\,|\,C)\,d\boldsymbol{x},$$

$$\pi(\boldsymbol{\theta}\,|\,C) \geq 0,\ \int_{\Theta} \pi(\boldsymbol{\theta}\,|\,C)\,d\boldsymbol{\theta} = 1,\ \ \mathrm{E}[\boldsymbol{\theta}\,|\,C] = \int_{\Theta} \boldsymbol{\theta}\,\pi(\boldsymbol{\theta}\,|\,C)\,d\boldsymbol{\theta}.$$

□ Special densities (or mass) functions use specific notation, as $\mathrm{N}(x\,|\,\mu, \sigma^2)$, $\mathrm{Bi}(x\,|\,n, \theta)$, or $\mathrm{Pn}(x\,|\,\lambda)$. Other examples:

Beta	$\{\mathrm{Be}(x\,|\,\alpha, \beta),\quad 0 < x < 1,\quad \alpha > 0,\ \beta > 0\}$ $\mathrm{Be}(x\,|\,\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{\alpha-1}(1-x)^{\beta-1}$
Gamma	$\{\mathrm{Ga}(x\,|\,\alpha, \beta),\quad x > 0,\quad \alpha > 0,\ \beta > 0\}$ $\mathrm{Ga}(x\,|\,\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\,x^{\alpha-1}e^{-\beta x}$
Student	$\{\mathrm{St}(x\,|\,\mu, \sigma^2, \alpha),\quad x \in \Re,\quad \mu \in \Re,\ \sigma > 0,\ \alpha > 0\}$ $\mathrm{St}(x\,|\,\mu, \sigma^2, \alpha) = \frac{\Gamma\{(\alpha+1)/2)\}}{\Gamma(\alpha/2)}\frac{1}{\sigma\sqrt{\alpha\pi}}\left[1 + \frac{1}{\alpha}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-(\alpha+1)/2}$

- *Statistical Models*

□ *Statistical model* generating $\boldsymbol{x} \in \mathcal{X}$, $\{p(\boldsymbol{x} \,|\, \boldsymbol{\theta}), \boldsymbol{x} \in \mathcal{X}, \boldsymbol{\theta} \in \Theta\}$

Parameter vector $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\} \in \Theta$. *Parameter space* $\Theta \subset \Re^k$.
Data set $\boldsymbol{x} \in \mathcal{X}$. *Sampling space* $\mathcal{X}$, of arbitrary structure.

□ *Likelihood function* of $\boldsymbol{x}$, $l(\boldsymbol{\theta} \,|\, \boldsymbol{x})$.

$l(\boldsymbol{\theta} \,|\, \boldsymbol{x}) = p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$, as a function of $\boldsymbol{\theta} \in \Theta$.

□ *Maximum likelihood estimator (mle)* of $\boldsymbol{\theta}$

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\boldsymbol{x}) = \arg\sup_{\boldsymbol{\theta} \in \Theta} \; l(\boldsymbol{\theta} \,|\, \boldsymbol{x})$$

□ Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ *random sample* (iid) from model if

$p(\boldsymbol{x} \,|\, \boldsymbol{\theta}) = \prod_{j=1}^{n} p(x_j \,|\, \boldsymbol{\theta}), \; x_j \in \mathcal{X}, \quad \mathcal{X} = \mathcal{X}^n$

□ Behaviour under repeated sampling (general, not iid data)

Considering $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots\}$, a (possibly infinite) sequence
of possible replications of the *complete* data set $\boldsymbol{x}$.

Denote by $\boldsymbol{x}^{(m)} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$ a finite set of m such replications.

□ Asymptotic results obtained as $m \to \infty$

1.2. Intrinsic Divergence

- *Logarithmic divergences*

  - The logarithmic divergence (Kullback-Leibler) $k\{\hat{p}\,|\,p\}$ of a density $\hat{p}(\boldsymbol{x})$ from its true density $p(\boldsymbol{x})$, is

    $$k\{\hat{p}\,|\,p\} = \int_{\mathcal{X}} p(\boldsymbol{x}) \log \frac{p(\boldsymbol{x})}{\hat{p}(\boldsymbol{x})}\, d\boldsymbol{x}, \;\; \text{(provided this exists)}$$

    The functional $k\{\hat{p}\,|\,p\}$ is non-negative, (zero iff, $\hat{p}(\boldsymbol{x}) = p(\boldsymbol{x})$ a.e.) and *invariant* under one-to-one transformations of $\boldsymbol{x}$.

  - But $k\{p_1\,|\,p_2\}$ is *not symmetric* and diverges if, strictly, $\mathcal{X}_2 \subset \mathcal{X}_1$.

- *Intrinsic discrepancy between distributions*

  - $\delta\{p, q\} = \min\left\{\int_{\mathcal{X}} p(\boldsymbol{x}) \log \frac{p(\boldsymbol{x})}{q(\boldsymbol{x})}\, d\boldsymbol{x}, \int_{\mathcal{X}} q(\boldsymbol{x}) \log \frac{q(\boldsymbol{x})}{p(\boldsymbol{x})}\, d\boldsymbol{x}\right\}$

    The *intrinsic discrepancy* $\delta\{p, q\}\}$ is non-negative, (zero iff, $\hat{p} = p$ a.e.) *invariant* under one-to-one transformations of $\boldsymbol{x}$,

  - Defined if $\mathcal{X}_2 \subset \mathcal{X}_1$ or $\mathcal{X}_1 \subset \mathcal{X}_2$, operative interpretation as the minimum amount of information (in *nits*) required to discriminate.

- *Interpretation and calibration of the intrinsic discrepancy*

□ Let $\{p_1(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1), \boldsymbol{\theta}_1 \in \Theta_1\}$ or $\{p_2(\boldsymbol{x}\,|\,\boldsymbol{\theta}_2), \boldsymbol{\theta}_2 \in \Theta_2\}$ be two alternative statistical models for $\boldsymbol{x} \in X$, one of which is assumed to be true. The intrinsic divergence $\delta\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\} = \delta\{p_1, p_2\}$ is then *minimum expected log-likelihood ratio in favour of the true model*.

Indeed, if $p_1(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1)$ true model, the expected log-likelihood ratio in its favour is $\mathrm{E}_1[\log\{p_1(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1)/p_2(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1)\}] = k\{p_2\,|\,p_1\}$. If the true model is $p_2(\boldsymbol{x}\,|\,\boldsymbol{\theta}_2)$, the expected log-likelihood ratio in favour of the true model is $k\{p_2\,|\,p_1\}$. But $\delta\{p_2\,|\,p_1\} = \min[k\{p_2\,|\,p_1\}, k\{p_1\,|\,p_2\}]$.

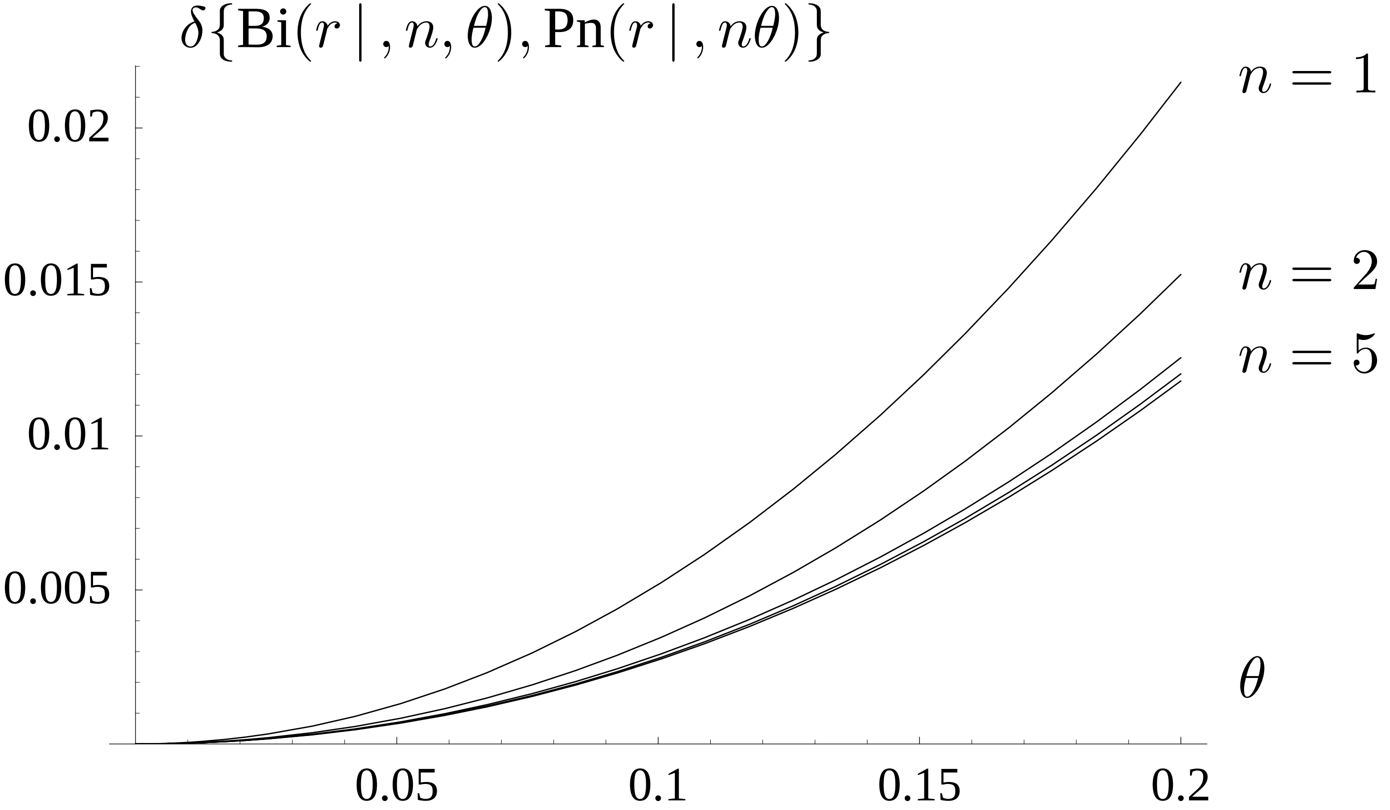
□ *Calibration.* $\delta = \log[100] \approx 4.6$ *nits*, likelihood ratios for the true model larger than 100 making *discrimination very easy*.

$\delta = \log(1+\varepsilon) \approx \varepsilon$ *nits*, likelihood ratios for the true model may about $1+\epsilon$ making *discrimination very hard*.

Intrinsic Discrepancy δ	0.01	0.69	2.3	4.6	6.9
Average Likelihood Ratio for **true** model $\exp[\delta]$	1.01	2	10	100	1000

- *Example.* Conventional Poisson approximation $\mathrm{Pn}(r \,|\, n\theta)$ of Binomial probabilities $\mathrm{Bi}(r \,|\, n, \theta)$

$$\delta(\mathrm{Bi}, \mathrm{Pn}) = \delta(n, \theta) = k(\mathrm{Pn} \,|\, \mathrm{Bi}) = \sum_{r=0}^{n} \mathrm{Bi}(r \,|\, n, \theta) \log \frac{\mathrm{Bi}(r \,|\, n, \theta)}{\mathrm{Pn}(r \,|\, n\theta)}$$

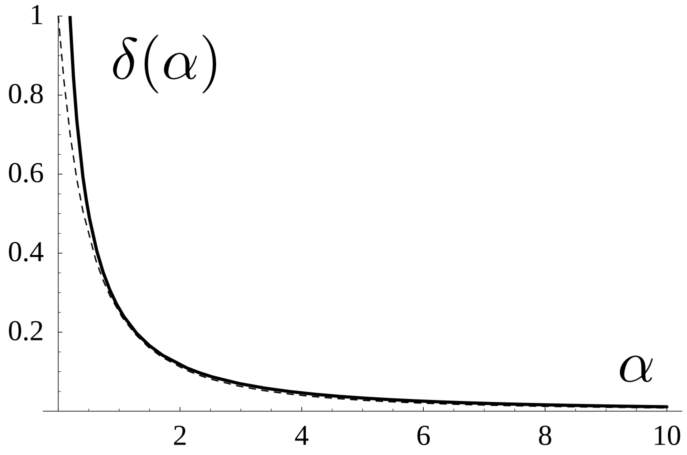


- *Intrinsic Convergence of Distributions*

  □ *Intrinsic Convergence.* A sequence of probability densities (or mass) functions $\{p_i(\boldsymbol{x})\}_{i=1}^{\infty}$ converges *intrinsically* to $p(\boldsymbol{x})$ if (and only if) the intrinsic divergence between $p_i(x)$ and $p(x)$ converges to zero. *i.e.*, iff $\lim_{i\to\infty} \delta(p_i, p) = 0$.

  □ *Example.* Normal approximation to a Student distribution.

$$\begin{aligned}\delta(\alpha) &= \delta\{\mathrm{St}(x \,|\, 0, 1, \alpha), \mathrm{N}(x \,|\, 0, 1)\} \\ &= \int_{-\infty}^{\infty} \mathrm{N}(x \,|\, 0, 1) \log \frac{\mathrm{N}(x \,|\, 0, 1)}{\mathrm{St}(x \,|\, 0, 1, \alpha)}\, dx \approx \frac{1}{(1+\alpha)^2}\end{aligned}$$



The function $\delta(\alpha)$ converges rapidly to zero. $\delta(18) = 0.004$.

1.3. Foundations

- *Foundations of Statistics*

□ Axiomatic foundations on rational description of uncertainty imply that the uncertainty about all unknown quantities should be measured with *probability* distributions $\{\pi(\boldsymbol{\theta}\,|\,C), \boldsymbol{\theta} \in \Theta\}$ describing the plausibility of their given available conditions C.

□ Axioms have a strong intuitive appeal; examples include

- *Transitivity of plausibility*.
  If $E_1 > E_2\,|\,C$, and $E_2 > E_3\,|\,C$, then $E_1 > E_3\,|\,C$
- *The sure-thing principle*.
  If $E_1 > E_2\,|\,A, C$ and $E_1 > E_2\,|\,\overline{A}, C$, then $E_1 > E_2\,|\,C$).

□ Axioms are not a *description* of actual human activity, but a *normative* set of principles for those aspiring to rational behaviour.

□ "Absolute" probabilities do not exist. Typical applications produce $\Pr(E\,|\,\boldsymbol{x}, A, K)$, a measure of rational belief in the occurrence of the *event* E, given data $\boldsymbol{x}$, assumptions A and available knowledge K.

- *Probability as a Measure of Conditional Uncertainty*

  □ Axiomatic foundations imply that $\mathrm{Pr}(E \,|\, C)$, the *probability* of an event E given C is *always* a conditional measure of the (presumably rational) uncertainty, on a $[0,1]$ scale, about the occurrence of E in conditions C.

  - *Probabilistic diagnosis*.V is the event that a person carries a virus and $+$ a positive test result. *All* related probabilities, *e.g.*,
    $\mathrm{Pr}(+ \,|\, V) = 0.98$, $\mathrm{Pr}(+ \,|\, \overline{V}) = 0.01$, $\mathrm{Pr}(V \,|\, K) = 0.002$,
    $\mathrm{Pr}(+ \,|\, K) = \mathrm{Pr}(+ \,|\, V)\mathrm{Pr}(V \,|\, K) + \mathrm{Pr}(+ \,|\, \overline{V})\mathrm{Pr}(\overline{V} \,|\, K) = 0.012$
    $\mathrm{Pr}(V \,|\, +, A, K) = \dfrac{\mathrm{Pr}(+ \,|\, V)\mathrm{Pr}(V \,|\, K)}{\mathrm{Pr}(+ \,|\, K)} = 0.164$ (Bayes' Theorem)
    are conditional uncertainty measures (and proportion estimates).

  - *Estimation of a proportion*.Survey conducted to estimate the proportion θ of positive individuals in a population.
    Random sample of size n with r positive.
    $\mathrm{Pr}(a < \theta < b \,|\, r, n, A, K)$, a conditional measure of the uncertainty about the event that θ belongs to $[a, b]$ *given* assumptions A, initial knowledge K and data $\{r, n\}$.

- *Measurement of a physical constant*.Measuring the unknown value of physical constant μ, with data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, considered to be measurements of μ subject to error. Desired to find $\Pr(a < \mu < b \,|\, x_1, \ldots, x_n, A, K)$, the *probability* that the unknown value of μ (fixed in nature, but unknown to the scientists) belongs to $[a, b]$ given the information provided by the data $\boldsymbol{x}$, assumptions A made, and available knowledge K.

- The statistical model may include *nuisance* parameters, unknown quantities , which have to be eliminated in the statement of the final results.

  For instance, the precision of the measurements described by unknown standard deviation σ in a $\mathrm{N}(x \,|\, \mu, \sigma)$ normal model

- Relevant scientific information may impose *restrictions* on the admissible values of the quantities of interest. These must be taken into account.

  For instance, in measuring the value of the gravitational field g in a laboratory, it is known that it must lie between 9.7803 m/sec^2 (average value at the Equator) and 9.8322 m/sec^2 (average value at the poles).

- *Future discrete observations*.Experiment counting the number r of times that an event E takes place in each of n replications. Desired to forecast the number of times r that E will take place in a future, similar situation, $\Pr(r \,|\, r_1, \ldots, r_n, A, K)$.
  For instance, no accidents in each of $n = 10$ consecutive months may yield $\Pr(r = 0 \,|\, \boldsymbol{x}, A, K) = 0.953$.
- *Future continuous observations*.Data $\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$. Desired to forecast the value of a future observation $\boldsymbol{y}$, $p(\boldsymbol{y} \,|\, \boldsymbol{x}, A, K)$.
  For instance, from breaking strengths $\boldsymbol{x} = \{y_1, \ldots, y_n\}$ of n randomly chosen safety belt webbings, the engineer may find $\Pr(y > y^* \,|\, \boldsymbol{x}, A, K) = 0.9987$.
- *Regression*.Data set consists of pairs $\boldsymbol{x} = \{(\boldsymbol{y}_1, \boldsymbol{v}_1), \ldots, (\boldsymbol{y}_n, \boldsymbol{v}_n)\}$ of quantity $\boldsymbol{y}_j$ observed in conditions $\boldsymbol{v}_j$.
  Desired to forecast the value of $\boldsymbol{y}$ in conditions $\boldsymbol{v}$, $p(\boldsymbol{y} \,|\, \boldsymbol{v}, \boldsymbol{x}, A, K)$.
  For instance, y contamination levels, v wind speed from source; environment authorities interested in $\Pr(y > y^* \,|\, v, \boldsymbol{x}, A, K)$

2. Basics of Bayesian Analysis

2.1. Parametric Inference

- *Bayes' Theorem*

  □ Let $M = \{p(\boldsymbol{x} \,|\, \boldsymbol{\theta}), \boldsymbol{x} \in \mathcal{X}, \boldsymbol{\theta} \in \Theta\}$ be an statistical model, let $\pi(\boldsymbol{\theta} \,|\, K)$ be a probability density for $\boldsymbol{\theta}$ given prior knowledge K and let $\boldsymbol{x}$ be some available data.

  $$\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}, M, K) = \frac{p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, K)}{\int_\Theta p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, K)\, d\boldsymbol{\theta}},$$

  encapsulates all information about $\boldsymbol{\theta}$ given data and prior knowledge.

  □ Simplifying notation, Bayes' theorem may be expressed as

  $$\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta}):$$

  The posterior is proportional to the likelihood times the prior. The missing proportionality constant $[\int_\Theta p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}]^{-1}$ may be deduced from the fact that $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})$ must integrate to one. To identify a posterior distribution it suffices to identify a *kernel* $k(\boldsymbol{\theta}, \boldsymbol{x})$ such that $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}) = c(\boldsymbol{x})\, k(\boldsymbol{\theta}, \boldsymbol{x})$. This is a very common technique.

- *Bayesian Inference with a Finite Parameter Space*

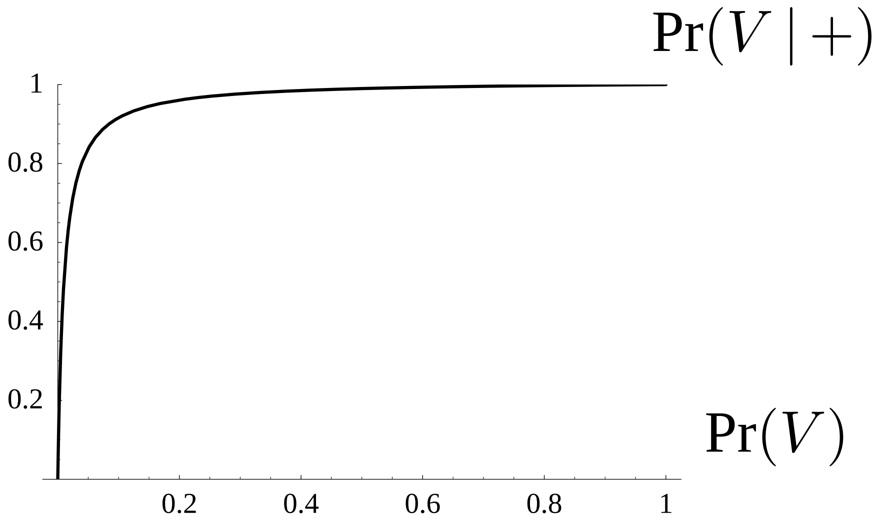
- Model $\{p(\boldsymbol{x} \,|\, \theta_i), \boldsymbol{x} \in \mathcal{X}, \theta_\rangle \in \times\}$, with $\Theta = \{\theta_1, \ldots, \theta_m\}$, so that θ may only take a *finite* number m of different values. Using the finite form of Bayes' theorem,

$$\Pr(\theta_i \,|\, \boldsymbol{x}) = \frac{p(\boldsymbol{x} \,|\, \theta_i) \Pr(\theta_i)}{\sum_{j=1}^{m} p(\boldsymbol{x} \,|\, \theta_j) \Pr(\theta_j)}\,, \quad i = 1, \ldots, m.$$

- *Example: Probabilistic diagnosis.* A test to detect a virus, is known from laboratory research to give a positive result in 98% of the infected people and in 1% of the non-infected. The posterior probability that a person who tested positive is infected is

$$\Pr(V \,|\, +) = \frac{0.98\, p}{0.98\, p + 0.01\, (1-p)}$$

as a function of $p = \Pr(V)$.



- Notice sensitivity of posterior $\Pr(V \,|\, +)$ to changes in the prior $p = \Pr(V)$.

- *Example: Inference about a binomial parameter*

□ Let data $\boldsymbol{x}$ be n Bernoulli observations with parameter θ which contain r positives, so that $p(\boldsymbol{x} \,|\, \theta, n) = \theta^r (1-\theta)^{n-r}$.

□ If $\pi(\theta) = \mathrm{Be}(\theta \,|\, \alpha, \beta)$, then

$$\pi(\theta \,|\, \boldsymbol{x}) \propto \theta^{r+\alpha-1}(1-\theta)^{n-r+\beta-1}$$

kernel of $\mathrm{Be}(\theta \,|\, r+\alpha, n-r+\beta)$.

□ Prior information (K)

$P(0.4 < \theta < 0.6) = 0.95$,

and symmetric, yields $\alpha = \beta = 47$;

□ No prior information $\alpha = \beta = 1/2$

□ $n = 1500, r = 720$

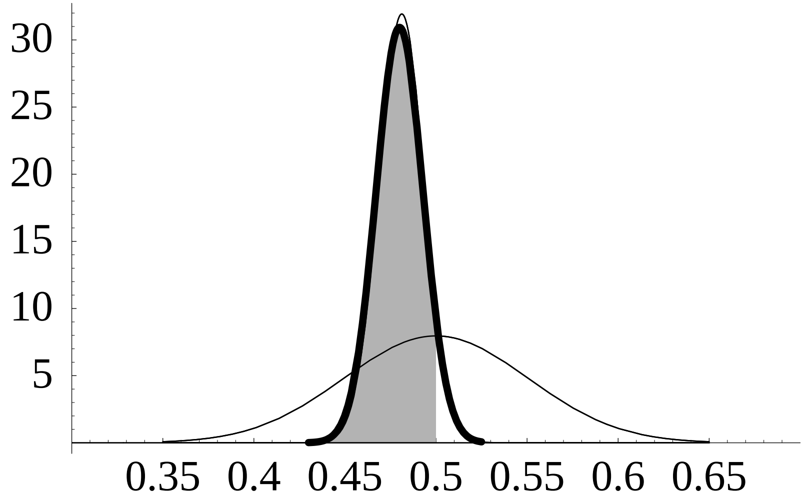
$P(\theta < 0.5 \,|\, \boldsymbol{x}, K) = 0.933$

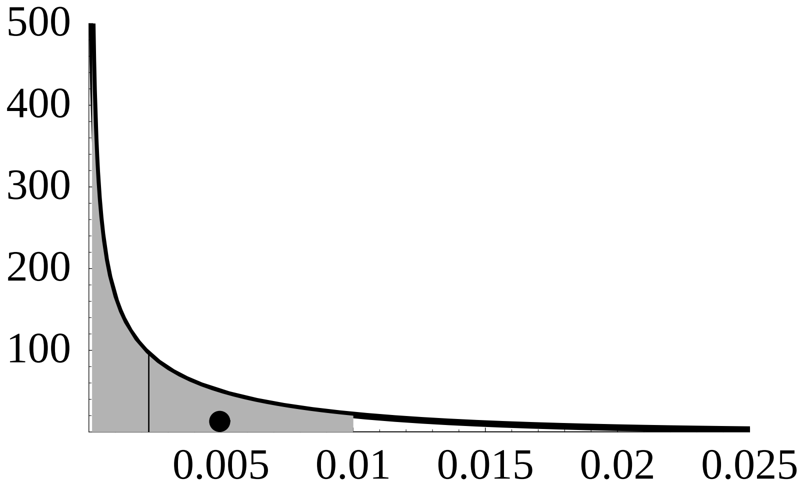
$P(\theta < 0.5 \,|\, \boldsymbol{x}) = 0.934$

□ $n = 100, r = 0$

$P(\theta < 0.01 \,|\, \boldsymbol{x}) = 0.844$

Notice: $\hat{\theta} = 0$, but $\mathrm{Me}[\theta \,|\, \boldsymbol{x}] = 0.0023$





- *Sufficiency*

□ Given a model $p(\boldsymbol{x} \mid \boldsymbol{\theta})$, a function of the data $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x})$, is a *sufficient* statistic if it encapsulates all information about $\boldsymbol{\theta}$ available in $\boldsymbol{x}$.

□ Formally, $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x})$ is *sufficient* if (and only if), for any prior $\pi(\boldsymbol{\theta})$ $\pi(\boldsymbol{\theta} \mid \boldsymbol{x}) = \pi(\boldsymbol{\theta} \mid \boldsymbol{t})$. Hence, $\pi(\boldsymbol{\theta} \mid \boldsymbol{x}) = \pi(\boldsymbol{\theta} \mid \boldsymbol{t}) \propto p(\boldsymbol{t} \mid \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})$.

□ This is equivalent to the frequentist definition; thus $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x})$ is sufficient iff $p(\boldsymbol{x} \mid \boldsymbol{\theta}) = f(\boldsymbol{\theta}, \boldsymbol{t}) g(\boldsymbol{x})$.

□ A sufficient statistic always exists, for $\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{x}$ is obviously sufficient

A much simpler sufficient statistic, with fixed dimensionality independent of the sample size, often exists.

This is case whenever the statistical model belongs to the *generalized exponential family*, which includes many of the more frequently used statistical models.

□ In contrast to frequentist statistics, Bayesian methods are independent on the possible existence of a sufficient statistic of fixed dimensionality.

For instance, if data come from an Student distribution, there is *no sufficient statistic* of fixed dimensionality: *all data are needed*.

- *Example: Inference from Cauchy observations*

□ Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ random from $\mathrm{Ca}(x \,|\, \mu, 1) = \mathrm{St}(x \,|\, \mu, 1, 1)$.

□ Objective reference prior for the location parameter μ is $\pi(\mu) = 1$.

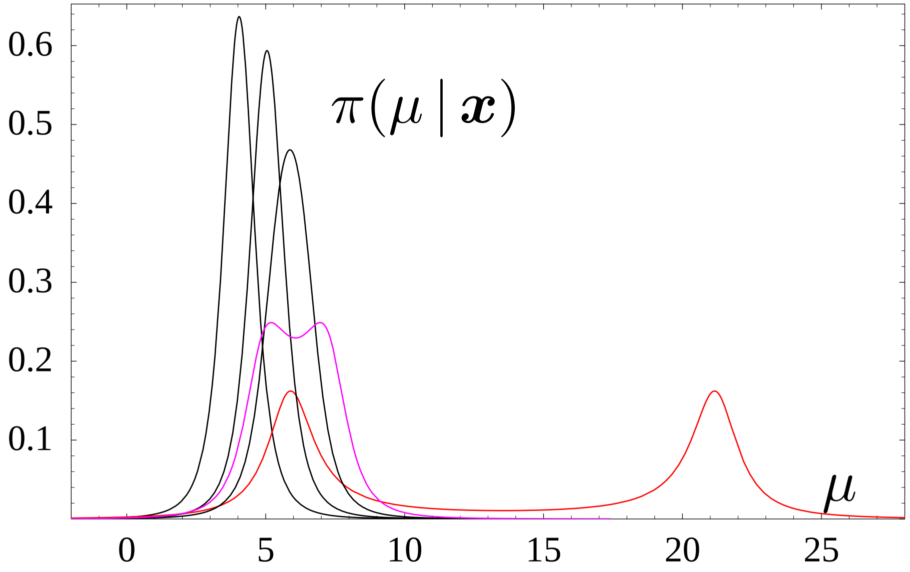
□ By Bayes' theorem,

$$\pi(\mu \,|\, \boldsymbol{x}) \propto \prod_{j=1}^{n} \mathrm{Ca}(x_j \,|\, \mu, 1)\pi(\mu) \propto \prod_{j=1}^{n} \frac{1}{1 + (x_j - \mu)^2}\,.$$

Proportionality constant easily obtained by numerical integration.

□ Five samples of size $n = 2$ simulated from $\mathrm{Ca}(x \,|\, 5, 1)$.

x_1	x_2
4.034	4.054
21.220	5.831
5.272	6.475
4.776	5.317
7.409	4.743



- *Improper prior functions*

  □ Objective Bayesian methods often use functions which play the role of prior distributions but are *not* probability distributions.

  □ An *improper prior function* is an non-negative function $\pi(\boldsymbol{\theta})$ such that $\int_\Theta \pi(\boldsymbol{\theta})\, d\boldsymbol{\theta}$ is not finite.

  The Cauchy example uses the improper prior function $\pi(\mu) = 1$, $\mu \in \Re$.

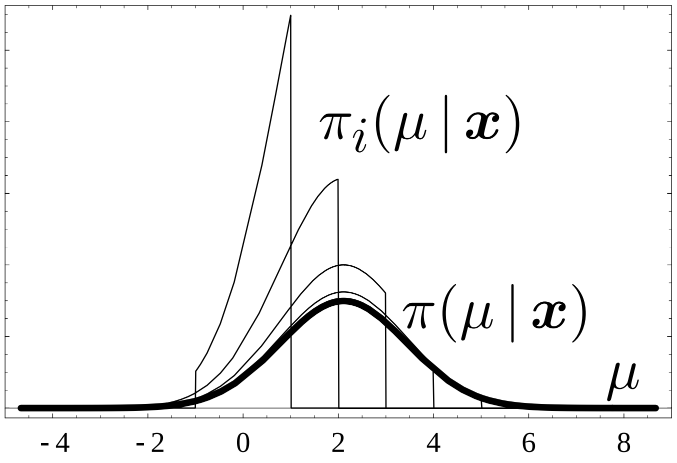
  □ $\pi(\boldsymbol{\theta})$ is an improper prior function, $\{\Theta_i\}_{i=1}^{\infty}$ an increasing sequence approximating Θ, such that $\int_{\Theta_i} \pi(\boldsymbol{\theta}) < \infty$, and $\{\pi_i(\boldsymbol{\theta})\}_{i=1}^{\infty}$ the proper priors obtained by *renormalizing* $\pi(\boldsymbol{\theta})$ within the Θ_i's.

  □ For any data $\boldsymbol{x}$ with likelihood $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$, the sequence of posteriors $\pi_i(\boldsymbol{\theta} \,|\, \boldsymbol{x})$ converges intrinsically to $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})$.

  □ Normal data, σ known, $\pi(\mu) = 1$.

$$\pi(\mu \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \mu, \sigma)\pi(\mu)$$
$$\propto \exp[-\frac{n}{2\sigma^2}(\overline{x} - \mu)^2]$$
$$\pi(\mu \,|\, \boldsymbol{x}) = \mathrm{N}(\mu \,|\, \overline{x}, \sigma^2/n)$$

Example: $n = 9,\ \overline{x} = 2.11,\ \sigma = 4$



- *Sequential updating*

  - Prior and posterior are terms *relative* to a set of data.
  - If data $\boldsymbol{x} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ are sequentially presented, the final result will be the same whether data are globally or sequentially processed.

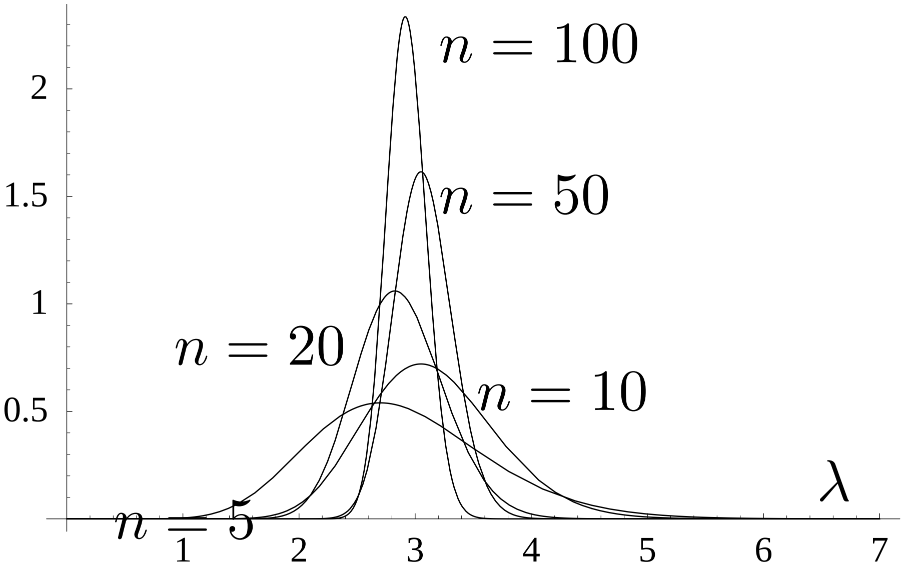
$$\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i+1}) \propto p(\boldsymbol{x}_{i+1} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_i).$$

  The "posterior" at a given stage becomes the "prior" at the next.

  - Typically (but not always), the new posterior, $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i+1})$, is more concentrated around the true value than $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_i)$.
  - Posteriors $\pi(\lambda \,|\, x_1, \ldots, x_i)$ from increasingly large simulated data from Poisson $\mathrm{Pn}(x \,|\, \lambda)$, with $\lambda = 3$

$$\pi(\lambda \,|\, x_1, \ldots, x_i) = \mathrm{Ga}(\lambda \,|\, r_i + 1/2, i)$$

$$r_i = \textstyle\sum_{j=1}^{i} x_j$$



- *Nuisance parameters*

  □ In general the *vector of interest* is not the whole parameter vector $\boldsymbol{\theta}$, but some function $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\theta})$ of possibly lower dimension.

  □ By Bayes' theorem $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta})$. Let $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{\theta}) \in \Omega$ be another function of $\boldsymbol{\theta}$ such that $\boldsymbol{\psi} = \{\boldsymbol{\phi}, \boldsymbol{\omega}\}$ is a bijection of $\boldsymbol{\theta}$, and let $J(\boldsymbol{\psi}) = (\partial\boldsymbol{\theta}/\partial\boldsymbol{\psi})$ be the Jacobian of the inverse function $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{\theta})$.

  From probability theory, $\pi(\boldsymbol{\psi} \,|\, \boldsymbol{x}) = |J(\boldsymbol{\psi})| [\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})]_{\boldsymbol{\theta}=\boldsymbol{\theta}(\boldsymbol{\psi})}$

  and $\pi(\boldsymbol{\phi} \,|\, \boldsymbol{x}) = \int_\Omega \pi(\boldsymbol{\phi}, \boldsymbol{\omega} \,|\, \boldsymbol{x})\, d\boldsymbol{\omega}$.

  □ Any valid conclusion on $\boldsymbol{\phi}$ will be contained in $\pi(\boldsymbol{\phi} \,|\, \boldsymbol{x})$.

  □ Particular case: *marginal posteriors*

  Often model directly expressed in terms of vector of interest $\boldsymbol{\phi}$, and vector of nuisance parameters $\boldsymbol{\omega}$, $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}) = p(\boldsymbol{x} \,|\, \boldsymbol{\phi}, \boldsymbol{\omega})$.

  Specify the prior $\quad \pi(\boldsymbol{\theta}) = \pi(\boldsymbol{\phi})\, \pi(\boldsymbol{\omega} \,|\, \boldsymbol{\phi})$

  Get the joint posterior $\quad \pi(\boldsymbol{\phi}, \boldsymbol{\omega} \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \boldsymbol{\phi}, \boldsymbol{\omega})\, \pi(\boldsymbol{\omega} \,|\, \boldsymbol{\phi})\, \pi(\boldsymbol{\phi})$

  Integrate out ω, $\quad \pi(\boldsymbol{\phi} \,|\, \boldsymbol{x}) \propto \pi(\boldsymbol{\phi}) \int_\Omega p(\boldsymbol{x} \,|\, \boldsymbol{\phi}, \boldsymbol{\omega})\, \pi(\boldsymbol{\omega} \,|\, \boldsymbol{\phi})\, d\boldsymbol{\omega}$

- *Example: Inferences about a Normal mean*

□ Data $\boldsymbol{x} = \{x_1, \ldots\ x_n\}$ random from $N(x \,|\, \mu, \sigma^2)$. Likelihood function $p(\boldsymbol{x} \,|\, \mu, \sigma) \propto \sigma^{-n} \exp[-n\{s^2 + (\overline{x} - \mu)^2\}/(2\sigma^2)]$, with $n\overline{x} = \sum_i x_i$, and $ns^2 = \sum_i (x_i - \overline{x})^2$.

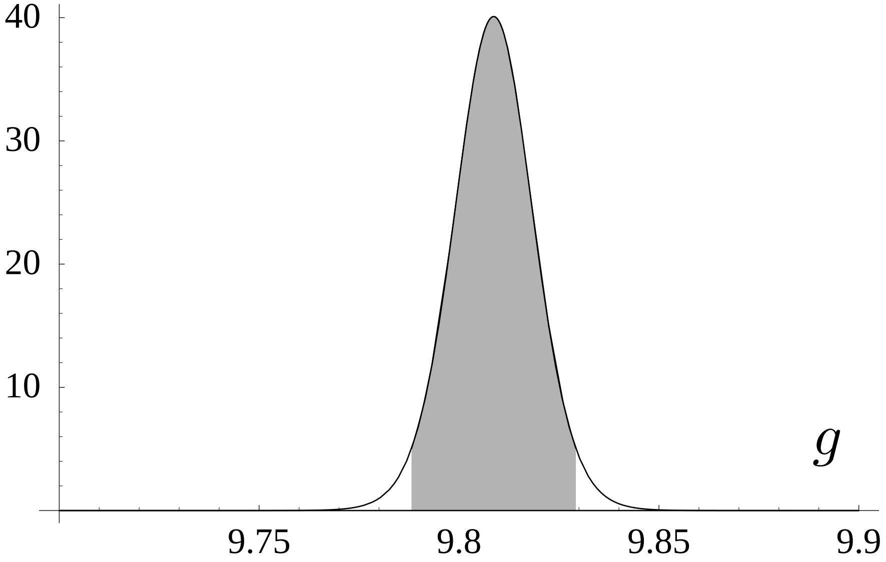
□ Objective prior is uniform in both μ and $\log(\sigma)$, ***i.e.***, $\pi(\mu, \sigma) = \sigma^{-1}$. Joint posterior $\pi(\mu, \sigma \,|\, \boldsymbol{x}) \propto \sigma^{-(n+1)} \exp[-n\{s^2 + (\overline{x} - \mu)^2\}/(2\sigma^2)]$.

□ Marginal posterior $\pi(\mu \,|\, \boldsymbol{x}) \propto \int_0^\infty \pi(\mu, \sigma \,|\, \boldsymbol{x})\, d\sigma \propto [s^2 + (\overline{x} - \mu)^2]^{-n/2}$, kernel of the Student density $\mathrm{St}(\mu \,|\, \overline{x}, s^2/(n-1), n-1)$

□ Classroom experiment to measure gravity g yields $\overline{x} = 9.8087$, $s = 0.0428$ with $n = 20$ measures.

$\pi(g \,|\, \overline{x}, s, n)$
$= \mathrm{St}(g \,|\, 9.9087, 0.0001^2, 19)$

$\Pr(9.788 < g < 9.829 \,|\, \boldsymbol{x})$
$= 0.95$ (shaded area)



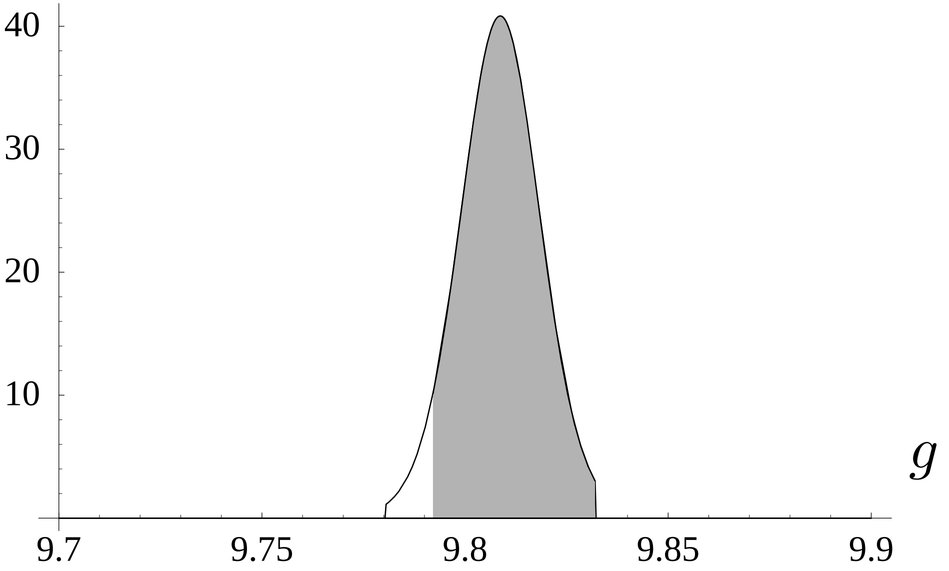
- *Restricted parameter space*

□ Range of values of $\boldsymbol{\theta}$ restricted by contextual considerations.
If $\boldsymbol{\theta}$ known to belong to $\Theta_c \subset \Theta$, $\pi(\boldsymbol{\theta}) > 0$ iff $\boldsymbol{\theta} \in \Theta_c$
By Bayes' theorem,

$$\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}, \boldsymbol{\theta} \in \Theta_c) = \begin{cases} \dfrac{\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})}{\int_{\Omega_c} \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \, d\boldsymbol{\theta}}, & \text{if} \quad \boldsymbol{\theta} \in \Theta_c \\ 0 & \text{otherwise} \end{cases}$$

□ To incorporate a restriction, it suffices to *renormalize* the unrestricted posterior distribution to the set $\Theta_c \subset \Theta$ of admissible parameter values.

□ Classroom experiment to measure gravity g with restriction to lie between
$g_0 = 9.7803$ (equator)
$g_1 = 9.8322$ (poles).
$\Pr(9.7803 < g < 9.8322 \,|\, \boldsymbol{x})$
$= 0.95$ (shaded area)



- *Asymptotic behaviour, discrete case*

  - If the parameter space $\Theta = \{\theta_1, \theta_2, \ldots\}$ is *countable* and

    The true parameter value θ_t is *distinguishable* from the others,*i.e.*,

    $\delta\{p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_t), p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_i)) > 0$, $i \neq t$,

    $$\lim_{n\to\infty} \pi(\theta_t \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1$$
    $$\lim_{n\to\infty} \pi(\theta_i \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 0, \quad i \neq t$$

  - To prove this, take logarithms is Bayes' theorem,

    define $z_i = \log[p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_i)/p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_t)]$,

    and use the strong law of large numbers on the n

    i.i.d. random variables $z_1, \ldots, z_n$.

  - For instance, in probabilistic diagnosis the posterior probability of the true disease converges to one as new relevant information accumulates, *provided* the model distinguishes the probabilistic behaviour of data under the true disease from its behaviour under the other alternatives.

- *Asymptotic behaviour, continuous case*

□ If the parameter θ is *one-dimensional and continuous*, so that $\Theta \subset \Re$, and the model $\{p(\boldsymbol{x} \,|\, \theta),\ \boldsymbol{x} \in \mathcal{X}\}$ is *regular*: basically,
$\mathcal{X}$ does not depend on θ,
$p(\boldsymbol{x} \,|\, \theta)$ is twice differentiable with respect to θ

□ Then, as $n \to \infty$, $\pi(\theta \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ converges intrinsically to a *normal* distribution with mean at the mle estimator $\hat{\theta}$, and with variance $v(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \hat{\theta})$, where
$v^{-1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \hat{\theta}) = -\sum_{j=1}^{n} \frac{\partial^2}{\partial\theta^2} \log[p(\boldsymbol{x}_j \,|\, \theta]$

□ To prove this, express is Bayes' theorem as
$\pi(\theta \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \propto \exp[\log \pi(\theta) + \sum_{j=1}^{n} \log p(\boldsymbol{x}_j \,|\, \theta)]$,
and expand $\sum_{j=1}^{n} \log p(\boldsymbol{x}_j \,|\, \theta)]$ about its maximum, the mle $\hat{\theta}$

□ The result is easily generalized to the case $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\}$, to obtain a limiting multivariate Normal $\mathrm{N}_k(\boldsymbol{\theta} \,|\, \hat{\boldsymbol{\theta}}, \boldsymbol{V}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \hat{\boldsymbol{\theta}})\}$.

- *Asymptotic behaviour, continuous case. Simpler form*

□ Using the strong law of large numbers on the sums above a simpler, less precise approximation is obtained:

□ If the parameter $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\}$ is continuous, so that $\Theta \subset \Re^k$ and the model $\{p(\boldsymbol{x} \,|\, \theta),\ \boldsymbol{x} \in \mathcal{X}\}$ is *regular*; basically:

$\mathcal{X}$ does not depend on $\boldsymbol{\theta}$

$p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$ is twice differentiable with respect to each of the θ_i's

□ As $n \to \infty$, $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ converges intrinsically to a *multivariate normal* distribution $\mathrm{N}_k\{\boldsymbol{\theta} \,|\, \hat{\boldsymbol{\theta}}, n^{-1}\boldsymbol{F}^{-1}(\hat{\boldsymbol{\theta}})\}$ with mean the mle $\hat{\boldsymbol{\theta}}$ and precision (inverse of variance) matrix $n\,\boldsymbol{F}(\hat{\boldsymbol{\theta}})$, where $\boldsymbol{F}$ is Fisher's information matrix, of general element

$$\boldsymbol{F}_{ij}(\boldsymbol{\theta}) = -\mathrm{E}_{\boldsymbol{x} \,|\, \boldsymbol{\theta}}\left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log p(\boldsymbol{x} \,|\, \boldsymbol{\theta})\right]$$

□ From this result, the properties of the multivariate Normal immediately yield the asymptotic forms for the *marginal* and the *conditional* posterior distributions of any subgroup of the θ_j's.

- *Example: Asymptotic approximation with Poisson data*

□ Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ random from $\mathrm{Pn}(x \,|\, \lambda) \propto e^{-\lambda}\lambda^x$
hence, $p(\boldsymbol{x} \,|\, \lambda) \propto e^{-n\lambda}\lambda^r$, $r = \Sigma_j\, x_j$, and $\hat{\lambda} = r/n$.

Fisher's function is $F(\lambda) = -\mathrm{E}_{x \,|\, \lambda}\left[\dfrac{\partial^2}{\partial\lambda^2}\log \mathrm{Pn}(x \,|\, \lambda)\right] = \dfrac{1}{\lambda}$

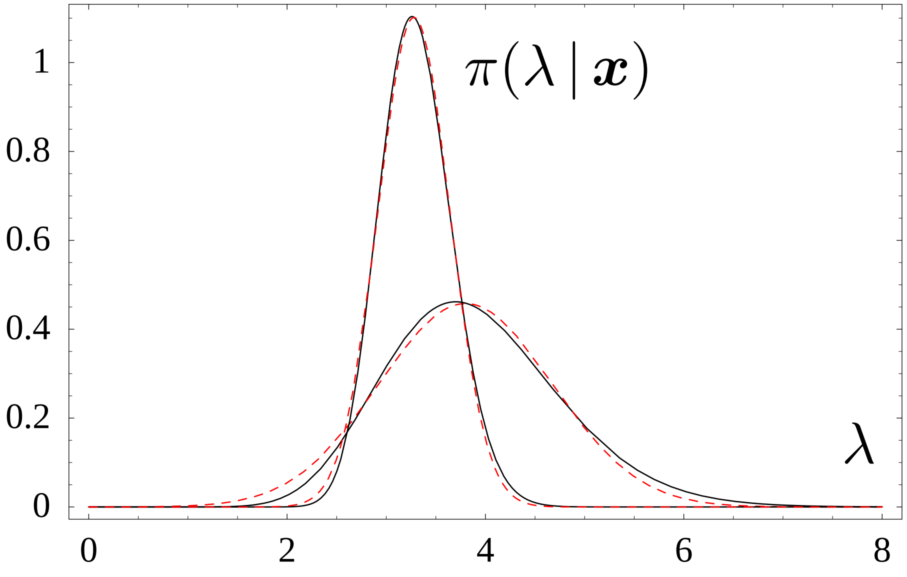
□ The objective prior function is $\pi(\lambda) = F(\lambda)^{1/2} = \lambda^{-1/2}$

Hence $\pi(\lambda \,|\, \boldsymbol{x}) \propto e^{-n\lambda}\lambda^{r-1/2}$

the of $\mathrm{Ga}(\lambda \,|\, r + \frac{1}{2}, n)$

□ The Normal approximation is
$\pi(\lambda \,|\, \boldsymbol{x}) \approx \mathrm{N}\{\lambda \,|\, \hat{\lambda}, n^{-1}F^{-1}(\hat{\lambda})\}$
$= \mathrm{N}\{\lambda \,|\, r/n, r/n^2\}$

□ Samples $n = 5$ and $n = 25$
simulated from Poisson $\lambda = 3$
yielded $r = 19$ and $r = 82$



2.2. Reference Analysis

- *No Relevant Initial Information*

  □ Identify the mathematical form of a “noninformative” prior. One with *minimal effect, relative to the data, on the posterior distribution of the quantity of interest*.

  □ Intuitive basis:

  Use *information theory* to measure the amount on information about the quantity of interest to be expected from data. This depends on prior knowledge: the more it is known, the less the amount of information the data may be expected to provide.

  Define the *missing information* about the quantity of interest as that which infinite independent replications of the experiment could possible provide.

  Define the *reference prior* as that which *maximizes the missing information about the quantity if interest*.

- *Expected information from the data*

  □ Given model $\{p(\boldsymbol{x} \,|\, \theta), \boldsymbol{x} \in \mathcal{X}, \theta \in \Theta\}$, the *amount of information* $I^{\theta}\{\mathcal{X}, \pi(\theta)\}$ which may be expected to be provided by $\boldsymbol{x}$, about the value of θ is defined by
  $I^{\theta}\{\mathcal{X}, \pi(\theta)\} = \mathrm{E}_{\boldsymbol{x}}[\, \int_{\Theta} \pi(\theta \,|\, \boldsymbol{x}) \log \frac{\pi(\theta \,|\, \boldsymbol{x})}{\pi(\theta)} \, d\theta],$
  the expected logarithmic divergence between prior and posterior.

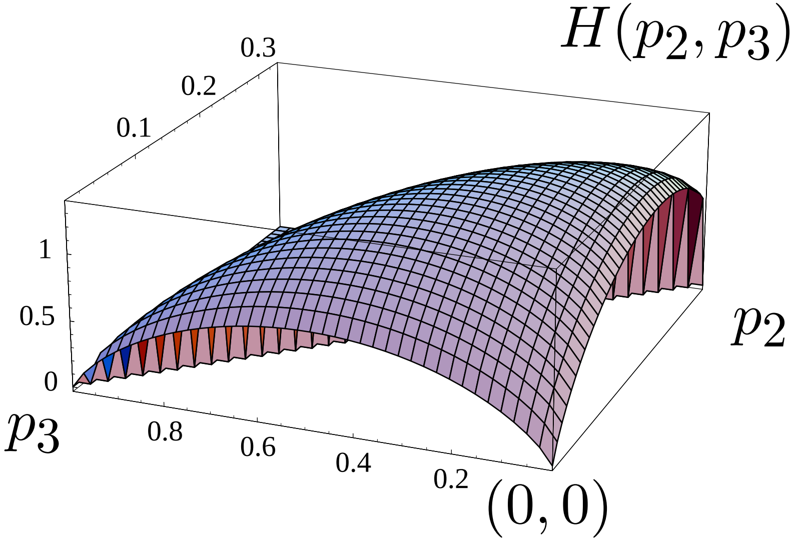
  □ Consider $I^{\theta}\{\mathcal{X}^{k}, \pi(\theta)\}$ the information about θ which may be expected from k conditionally independent replications of the original setup. As $k \to \infty$, this would provide any *missing information* about θ. Hence, as $k \to \infty$, the functional $I^{\theta}\{\mathcal{X}^{k}, \pi(\theta)\}$ will approach the missing information about θ associated with the prior $\pi(\theta)$.

  □ Let $\pi_k(\theta)$ be the prior which maximizes $I^{\theta}\{\mathcal{X}^{k}, \pi(\theta)\}$ in the class $\mathcal{P}$ of strictly positive prior distributions compatible with accepted assumptions on the value of θ (which be the class of *all* strictly positive priors).

  The *reference prior* $\pi^{*}(\theta)$ is the limit as $k \to \infty$ (in a sense to be made precise) of the sequence of priors $\{\pi_k(\theta), k = 1, 2, \ldots\}$.

- *Reference priors in the finite case*
  - If θ may only take a *finite* number m of different values $\{\theta_1, \ldots, \theta_m\}$ and $\pi(\theta) = \{p_1, \ldots, p_m\}$, with $p_i = \Pr(\theta = \theta_i)$, then $\lim_{k\to\infty} I^\theta\{\mathcal{X}^k, \pi(\theta)\} = H(p_1, \ldots, p_m) = -\sum_{i=1}^m p_i \log(p_i)$, that is, the *entropy* of the prior distribution $\{p_1, \ldots, p_m\}$.
  - In the finite case, the reference prior is that with *maximum entropy* within the class $\mathcal{P}$ of priors compatible with accepted assumptions. (cf. Statistical Physics)
  - If, in particular, $\mathcal{P}$ contains *all* priors over $\{\theta_1, \ldots, \theta_m\}$, the reference prior is the *uniform* prior, $\pi(\theta) = \{1/m, \ldots, 1/m\}$. (cf. Bayes-Laplace postulate of insufficient reason)
  - Prior $\{p_1, p_2, p_3, p_4\}$ in genetics problem where $p_1 = 2p_2$.

    Reference prior is $\{0.324, 0.162, 0.257, 0.257\}$



- *Reference priors in one-dimensional continuous case*

  - Let $\pi_k(\theta)$ be the prior which maximizes $I^{\theta}\{\mathcal{X}^k, \pi(\theta)\}$ in the class $\mathcal{P}$ of acceptable priors.

    For any data $\boldsymbol{x} \in \mathcal{X}$, let $\pi_k(\theta \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \theta)\, \pi_k(\theta)$ be the corresponding posterior.

  - The *reference posterior density* $\pi^*(\theta \,|\, \boldsymbol{x})$ is defined to be the intrinsic limit of the sequence $\{\pi_k(\theta \,|\, \boldsymbol{x}), k = 1, 2, \ldots\}$

    A *reference prior function* $\pi^*(\theta)$ is any positive function such that, for all $\boldsymbol{x} \in \mathcal{X}$, $\pi^*(\theta \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \theta)\, \pi^*(\theta)$.
    This is defined up to an (irrelevant) arbitrary constant.

  - Let $\boldsymbol{x}^{(k)} \in \mathcal{X}^k$ be the result of k independent replications of $\boldsymbol{x} \in \mathcal{X}$.
    With calculus of variations, the exact expression for $\pi_k(\theta)$ is found to be

    $$\pi_k(\theta) = \exp\left[\mathrm{E}_{x^{(k)} \,|\, \theta}\left\{\log \pi_k(\theta \,|\, x^{(k)})\right\}\right]$$

    For large k, this allows a *numerical derivation* of the reference prior by repeated simulation from $p(\boldsymbol{x} \,|\, \theta)$ for different θ values.

- *Reference priors under regularity conditions*

□ Let $\tilde{\theta}_k = \tilde{\theta}(x^{(k)})$ be a consistent, asymptotically sufficient estimator of θ. In regular problems this is often the case with the mle estimator $\hat{\theta}$.

The exact expression for $\pi_k(\theta)$ then becomes, for large k,

□ $\pi_k(\theta) \approx \exp[\mathrm{E}_{\tilde{\theta}_k \,|\, \theta}\{\log \pi_k(\theta \,|\, \tilde{\theta}_k)\}]$

As $k \to \infty$ this converges to $\pi_k(\theta \,|\, \tilde{\theta}_k)|_{\tilde{\theta}_k = \theta}$

□ Let $\tilde{\theta}_k = \tilde{\theta}(x^{(k)})$ be a consistent, asymptotically sufficient estimator of θ. Let $\pi(\theta \,|\, \tilde{\theta}_k)$ be any asymptotic approximation to $\pi(\theta \,|\, x^{(k)})$, the posterior distribution of θ.

Hence, $\pi^*(\theta) = \pi(\theta \,|\, \tilde{\theta}_k)|_{\tilde{\theta}_k = \theta}$

□ Under regularity conditions, the posterior distribution of θ is asymptotically Normal, $\mathrm{N}(\theta \,|\, \hat{\theta}, n^{-1} F^{-1}(\hat{\theta}))$, where $F(\theta) = -\mathrm{E}[\partial^2 \log p(\boldsymbol{x} \,|\, \theta)/\partial\theta^2]$ is Fisher's information function.

Hence, $\pi^*(\theta) = F(\theta)^{1/2}$ (cf. Jeffreys' rule).

- *One nuisance parameter*

  - *Two parameters*: reduce the problem to a *sequential* application of the one parameter case. Probability model is $\{p(\boldsymbol{x} \,|\, \theta, \lambda, \theta \in \Theta, \lambda \in \Lambda\}$ and a θ-reference prior $\pi_\theta^*(\theta, \lambda)$ is required. Two steps:

    (i) Conditional on θ, $p(\boldsymbol{x} \,|\, \theta, \lambda)$ only depends on λ, and it is possible to obtain the *conditional* reference prior $\pi^*(\lambda \,|\, \theta)$.

    (ii) If $\pi^*(\lambda \,|\, \theta)$ is proper, integrate out λ to get the one-parameter model $p(\boldsymbol{x} \,|\, \theta) = \int_\Lambda p(\boldsymbol{x} \,|\, \theta, \lambda)\, \pi^*(\lambda \,|\, \theta)\, d\lambda$, and use the one-parameter solution to obtain $\pi^*(\theta)$.

    The θ-reference prior is then $\pi_\theta^*(\theta, \lambda) = \pi^*(\lambda \,|\, \theta)\, \pi^*(\theta)$.
    The required reference posterior is $\pi^*(\theta \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \theta)\pi^*(\theta)$.

  - If $\pi^*(\lambda \,|\, \theta)$ is an *improper* prior function, proceed within an increasing sequence $\{\Lambda_i\}$ over which $\pi^*(\lambda \,|\, \theta)$ is integrable and, for given data $\boldsymbol{x}$, obtain the corresponding sequence of reference posteriors $\{\pi_i^*(\theta \,|\, \boldsymbol{x}\}$.

    The required reference posterior $\pi^*(\theta \,|\, \boldsymbol{x})$ is their intrinsic limit.

    A θ-reference prior is any positive function such that, for any data $\boldsymbol{x}$, $\pi^*(\theta \,|\, \boldsymbol{x}) \propto \int_\Lambda p(\boldsymbol{x} \,|\, \theta, \lambda)\, \pi_\theta^*(\theta, \lambda)\, d\lambda$.

- *The regular two-parameter continuous case*

  □ Model $p(\boldsymbol{x} \mid \theta, \lambda)$. If the joint posterior of (θ, λ) is asymptotically normal, the θ-reference prior may be derived in terms of the corresponding Fisher's information matrix, $\boldsymbol{F}(\theta, \lambda)$.

  $$\boldsymbol{F}(\theta, \lambda) = \begin{pmatrix} F_{\theta\theta}(\theta, \lambda) & F_{\theta\lambda}(\theta, \lambda) \\ F_{\theta\lambda}(\theta, \lambda) & F_{\lambda\lambda}(\theta, \lambda) \end{pmatrix}, \quad \boldsymbol{S}(\theta, \lambda) = \boldsymbol{F}^{-1}(\theta, \lambda),$$

  The θ-reference prior is $\pi^*_\theta(\theta, \lambda) = \pi^*(\lambda \mid \theta)\, \pi^*(\theta)$, where

  $\pi^*(\lambda \mid \theta) \propto F^{1/2}_{\lambda\lambda}(\theta, \lambda),\ \lambda \in \Lambda$, and, if $\pi^*(\lambda \mid \theta)$ is proper,

  $\pi^*(\theta) \propto \exp\{\int_\Lambda \pi^*(\lambda \mid \theta)\, \log[S^{-1/2}_{\theta\theta}(\theta, \lambda)]\, d\lambda\},\ \theta \in \Theta.$

  □ If $\pi^*(\lambda \mid \theta)$ is not proper, integrations are performed within an approximating sequence $\{\Lambda_i\}$ to obtain a sequence $\{\pi^*_i(\lambda \mid \theta)\, \pi^*_i(\theta)\}$, and the θ-reference prior $\pi^*_\theta(\theta, \lambda)$ is defined as its intrinsic limit.

  □ Even if $\pi^*(\lambda \mid \theta)$ is improper, if θ and λ are variation independent,

  $S^{-1/2}_{\theta\theta}(\theta, \lambda) \propto f_\theta(\theta)\, g_\theta(\lambda)$, and $F^{1/2}_{\lambda\lambda}(\theta, \lambda) \propto f_\lambda(\theta)\, g_\lambda(\lambda)$,

  Then $\pi^*_\theta(\theta, \lambda) = f_\theta(\theta)\, g_\lambda(\lambda)$.

- *Examples: Inference on normal parameters*

  □ The information matrix for the normal model $\mathrm{N}(x \,|\, \mu, \sigma)$ is

  $$\boldsymbol{F}(\mu, \sigma) = \begin{pmatrix} \sigma^{-2} & 0 \\ 0 & 2\sigma^{-2} \end{pmatrix}, \quad \boldsymbol{S}(\mu, \sigma) = \begin{pmatrix} \sigma^{2} & 0 \\ 0 & \sigma^{2}/2 \end{pmatrix};$$

  Since μ and σ are variation independent, and both $F_{\sigma\sigma}$ and $S_{\mu\mu}$ factorize
  $\pi^*(\sigma \,|\, \mu) \propto F_{\sigma\sigma}^{1/2} \propto \sigma^{-1}$, $\pi^*(\mu) \propto S_{\mu\mu}^{-1/2} \propto 1$.
  The μ-reference prior, as anticipated, is
  $\pi^*_\mu(\mu, \sigma) = \pi^*(\sigma \,|\, \mu)\, \pi^*(\mu) = \sigma^{-1}$,
  i.e., uniform on both μ and $\log \sigma$

  □ Since $\boldsymbol{F}(\mu, \sigma)$ is diagonal the σ-reference prior is
  $\pi^*_\sigma(\mu, \sigma) = \pi^*(\mu \,|\, \sigma)\pi^*(\sigma) = \sigma^{-1}$, the same as $\pi^*_\mu(\mu, \sigma) = \pi^*_\sigma(\mu, \sigma)$.

  □ In fact, it may be shown that, for location-scale models,
  $p(x \,|\, \mu, \sigma) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$,
  the reference prior for the location and scale parameters are always
  $\pi^*_\mu(\mu, \sigma) = \pi^*_\sigma(\mu, \sigma) = \sigma^{-1}$.

- Within any given model $p(\boldsymbol{x} \mid \boldsymbol{\theta})$ the ϕ-reference prior $\pi^*_\phi(\boldsymbol{\theta})$ maximizes the missing information about $\phi = \phi(\boldsymbol{\theta})$ and, in multiparameter problems, that prior *may change with the quantity of interest* ϕ.

- For instance, within a normal $\mathrm{N}(x \mid \mu, \sigma)$ model, let the *standardized mean* $\phi = \mu/\sigma$. be the quantity of interest.

  Fisher's information matrix in terms of the parameters ϕ and σ is $\boldsymbol{F}(\phi,\sigma) = J^t\, \boldsymbol{F}(\mu,\sigma)\, J$, where $J = (\partial(\mu,\sigma)/\partial(\phi,\sigma))$ is the Jacobian of the inverse transformation; this yields

  $$\boldsymbol{F}(\phi,\sigma) = \begin{pmatrix} 1 & \phi\sigma^{-1} \\ \phi\sigma^{-1} & \sigma^{-2}(2+\phi^2) \end{pmatrix},$$

  with $F^{1/2}_{\sigma\sigma} \propto \sigma^{-1}$, and $S^{-1/2}_{\phi\phi} \propto (1+\phi^2/2)^{-1/2}$.

- The ϕ-reference prior is, $\pi^*_\phi(\phi,\sigma) = (1+\phi^2/2)^{-1/2}\sigma^{-1}$. Or, in the original parametrization, $\pi^*_\phi(\mu,\sigma) = (1+(\mu/\sigma)^2/2)^{-1/2}\sigma^{-2}$, which is different from $\pi^*_\mu(\mu,\sigma) = \pi^*_\sigma(\mu,\sigma)$.
  This prior is shown to lead to a reference posterior for ϕ with *consistent marginalization properties*.

- *Many parameters*

  □ The reference algorithm generalizes to any number of parameters. If the model is $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}) = p(\boldsymbol{x} \,|\, \theta_1, \ldots, \theta_m)$, a joint reference prior $\pi^*(\phi_m \,|\, \phi_{m-1}, \ldots, \phi_1) \times \ldots \times \pi^*(\phi_2 \,|\, \phi_1) \times \pi^*(\phi_1)$ may sequentially be obtained for each *ordered parametrization*, $\{\phi_1(\boldsymbol{\theta}), \ldots, \phi_m(\boldsymbol{\theta})\}$.

  Reference priors are *invariant* under reparametrization of the $\phi_i(\boldsymbol{\theta})$'s.

  □ The choice of the ordered parametrization $\{\phi_1, \ldots, \phi_m\}$ describes the particular prior required, namely that which *sequentially* maximizes the missing information about each of the ϕ_i's, conditional on $\{\phi_1, \ldots, \phi_{i-1}\}$, for $i = m, m-1, \ldots, 1$.

  □ Example: *Stein's paradox*. Data random from a m-variate normal $\mathrm{N}_m(\boldsymbol{x} \,|\, \boldsymbol{\mu}, \boldsymbol{I})$. The reference prior function for any permutation of the μ_i's is uniform, and leads to appropriate posterior distributions for any of the μ_i's, but cannot be used if the quantity of interest is $\theta = \sum_i \mu_i^2$, the distance of $\boldsymbol{\mu}$ to the origin.

  The reference prior for $\{\theta, \lambda_1, \ldots, \lambda_{m-1}\}$ produces, for any choice of the λ_i's, an appropriate the reference posterior for θ.

2.3. Inference Summaries

- *Summarizing the posterior distribution*

  □ *The* Bayesian final *outcome* of a problem of inference about any unknown quantity $\boldsymbol{\theta}$ *is* precisely the *posterior density* $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x}, C)$.

  □ Bayesian inference may be described as the problem of stating a probability distribution for the quantity of interest encapsulating all available information about its value.

  □ In one or two dimensions, a *graph of the posterior probability density* of the quantity of interest conveys an intuitive summary of the main conclusions. This is greatly appreciated by users, and is an important asset of Bayesian methods.

  □ However, graphical methods not easily extend to more than two dimensions and elementary *quantitative* conclusions are often required.

  The simplest forms to *summarize* the information contained in the posterior distribution are closely related to the conventional concepts of point estimation and interval estimation.

- *Point Estimation: Posterior mean and posterior mode*

□ It is often required to provide point estimates of relevant quantities. Bayesian point estimation is best described as a *decision problem* where one has to *choose* a particular value $\tilde{\boldsymbol{\theta}}$ as an approximate proxy for the actual, unknown value of $\boldsymbol{\theta}$.

□ Intuitively, any location measure of the posterior density $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})$
may be used as a point estimator. When they exist, either
$\mathrm{E}[\boldsymbol{\theta} \,|\, \boldsymbol{x}] = \int_{\Theta} \boldsymbol{\theta}\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta}$ *(posterior mean)*, or
$\mathrm{Mo}[\boldsymbol{\theta} \,|\, \boldsymbol{x}] = \arg\sup_{\boldsymbol{\theta} \in \Theta} \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})$ *(posterior mode)*
are often regarded as natural choices.

□ *Lack of invariance*. Neither the posterior mean not the posterior mode are invariant under reparametrization. The point estimator $\tilde{\boldsymbol{\psi}}$ of a bijection $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$ will generally not be equal to $\boldsymbol{\psi}(\tilde{\boldsymbol{\theta}})$.

In pure “inferential” applications, where one is requested to provide a point estimate of the vector of interest without an specific application in mind, it is difficult to justify a non-invariant solution.

• *Point Estimation: Posterior median*

□ A summary of a multivariate density $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})$, where $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_k\}$, should contain summaries of:
(i) each of the marginal densities $\pi(\theta_i \,|\, \boldsymbol{x})$,
(ii) the densities $\pi(\phi \,|\, \boldsymbol{x})$ of other functions of interest $\phi = \phi(\boldsymbol{\theta})$.

□ In *one-dimensional continuous* problems the *posterior median*, is easily defined and computed as
$\mathrm{Me}[\theta \,|\, \boldsymbol{x}] = q\,; \quad \int_{\{\theta \leq q\}} \pi(\theta \,|\, \boldsymbol{x})\, d\theta = 1/2$

The one-dimensional posterior median has many attractive properties:
(i) it is *invariant* under bijections, $\mathrm{Me}[\phi(\theta) \,|\, \boldsymbol{x}] = \phi(\mathrm{Me}[\theta \,|\, \boldsymbol{x}])$.
(ii) it *exists* and it is *unique* under very wide conditions
(iii) it is rather *robust* under moderate perturbations of the data.

□ The posterior median is often considered to be the best 'automatic' Bayesian point estimator in one-dimensional continuous problems.

□ The posterior median is not easily used to a multivariate setting. The natural extension of its definition produces *surfaces* (not points).

General invariant multivariate definitions of point estimators is possible using Bayesian *decision theory*

- *General Credible Regions*

  - To describe $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})$ it is often convenient to quote regions $\Theta_p \subset \Theta$ of given probability content p under $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})$. This is the intuitive basis of graphical representations like boxplots.

  - A subset Θ_p of the parameter space Θ such that
    $\int_{\Theta_p} \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta} = p, \quad \text{so that } \Pr(\boldsymbol{\theta} \in \Theta_p \,|\, \boldsymbol{x}) = p,$
    is a *posterior p-credible region* for $\boldsymbol{\theta}$.

  - A credible region is invariant under reparametrization:
    If Θ_p is p-credible for $\boldsymbol{\theta}$, $\boldsymbol{\phi}(\Theta_p)$ is a p-credible for $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\theta})$.

  - For any given p there are generally infinitely many credible regions. Credible regions may be selected to have minimum size (length, area, volume), resulting in *highest probability density* (HPD) regions, where all points in the region have larger probability density than all points outside.

  - HPD regions are *not invariant* : the image $\boldsymbol{\phi}(\Theta_p)$ of an HPD region Θ_p will be a credible region for $\boldsymbol{\phi}$, but will not generally be HPD.
    There is no reason to restrict attention to HPD credible regions.

- *Credible Intervals*

  - In *one-dimensional continuous* problems, posterior quantiles are often used to derive credible intervals.
  - If $\theta_q = Q_q[\theta \,|\, \boldsymbol{x}]$ is the q-quantile of the posterior distribution of θ,
the interval $\Theta_p = \{\theta;\ \theta \leq \theta_p\}$ is a p-credible region,
and it is invariant under reparametrization.
  - *Equal-tailed* p-credible intervals of the form
$\Theta_p = \{\theta;\ \ \theta_{(1-p)/2} \leq \theta \leq \theta_{(1+p)/2}\}$
are typically unique, and they invariant under reparametrization.
  - Example: Model $\mathrm{N}(x \,|\, \mu, \sigma)$. *Credible intervals for the normal mean*.
The reference posterior for μ is $\pi(\mu \,|\, \boldsymbol{x}) = \mathrm{St}(\mu \,|\, \overline{x}, s^2/(n-1), n-1)$.
Hence the reference *posterior* distribution of $\tau = \sqrt{n-1}(\mu - \overline{x})/s$,
a function of μ, is $\pi(\tau \,|\, \overline{x}, s, n) = \mathrm{St}(\tau \,|\, 0, 1, n-1)$.

    Thus, the equal-tailed p-credible intervals for μ are
$\{\mu;\ \mu \in \overline{x} \pm q_{n-1}^{(1-p)/2}\, s/\sqrt{n-1}\}$,
where $q_{n-1}^{(1-p)/2}$ is the $(1-p)/2$ quantile of a standard Student density
with $n-1$ degrees of freedom.

- *Calibration*

  □ In the normal example above , the expression $t = \sqrt{n-1}(\mu - \overline{x})/s$ may *also* be analyzed, for fixed μ, as a *function of the data*.

  The fact that the *sampling* distribution of the statistic $t = t(\overline{x}, s \,|\, \mu, n)$ is *also* an standard Student $p(t \,|\, \mu, n) = \mathrm{St}(t \,|\, 0, 1, n-1)$ with the same degrees of freedom implies that, in this example, objective Bayesian credible intervals are *also* be *exact* frequentist confidence intervals.

  □ *Exact numerical agreement* between Bayesian credible intervals and frequentist confidence intervals is the *exception, not the norm*.

  □ For *large samples*, convergence to normality implies *approximate numerical agreement*. This provides a frequentist *calibration* to objective Bayesian methods.

  □ Exact numerical *agreement* is obviously *impossible when the data are discrete*: Precise (non randomized) frequentist confidence intervals do not exist in that case for most confidence levels.

  The computation of Bayesian credible regions for continuous parameters is however *precisely the same* whether the data are *discrete or continuous*.

2.4. Prediction

- *Posterior predictive distributions*

  □ Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, $x_i \in \mathcal{X}$, set of “homogeneous” observations. Desired to predict the value of a future observation $x \in \mathcal{X}$ generated by the same mechanism.

  □ From the foundations arguments the solution *must* be a probability distribution $p(x \,|\, \boldsymbol{x}, K)$ describing the uncertainty on the value that x will take, given data $\boldsymbol{x}$ and any other available knowledge K. This is called the (posterior) *predictive density* of x.

  □ To derive $p(x \,|\, \boldsymbol{x}, K)$ it is necessary to specify the *precise sense* in which the x_i’s are judged to be *homogeneous*.

  □ It is often directly assumed that the data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ consist of a *random sample* from some specified model, $\{p(x \,|\, \boldsymbol{\theta}), x \in \mathcal{X}, \boldsymbol{\theta} \in \Theta\}$, so that $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}) = p(x_1, \ldots, x_n \,|\, \boldsymbol{\theta}) = \prod_{j=1}^{n} p(x_j \,|\, \boldsymbol{\theta})$.

  If this is the case, the solution to the prediction problem is immediate once a prior distribution $\pi(\boldsymbol{\theta})$ has been specified.

- *Posterior predictive distributions from random samples*

  - Let $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, $x_i \in \mathcal{X}$ a random sample of size n from the statistical model $\{p(x \,|\, \boldsymbol{\theta}), x \in \mathcal{X}, \boldsymbol{\theta} \in \Theta\}$
    Let $\pi(\boldsymbol{\theta})$ a prior distribution describing available knowledge (in any) about the value of the parameter vector $\boldsymbol{\theta}$.
    The *posterior predictive distribution* is

    $$p(x \,|\, \boldsymbol{x}) = p(x \,|\, x_1, \ldots, x_n) = \int_\Theta p(x \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta}$$

    This encapsulates all available information about the outcome of any future observation $x \in \mathcal{X}$ from the same model.

  - To prove this, make use the total probability theorem, to have
    $p(x \,|\, \boldsymbol{x}) = \int_\Theta p(x \,|\, \boldsymbol{\theta}, \boldsymbol{x})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta}$
    and notice the new observation x has been assumed to be conditionally independent of the observed data $\boldsymbol{x}$, so that $p(x \,|\, \boldsymbol{\theta}, \boldsymbol{x}) = p(x \,|\, \boldsymbol{\theta})$.

  - The observable values $x \in \mathcal{X}$ may be either *discrete* or *continuous* random quantities. In the discrete the predictive distribution will be described by its probability *mass* function; if the continuous case, by its probability *density* function. Both are denoted $p(x \,|\, \boldsymbol{x})$.

- *Prediction in a Poisson process*

  □ Data $\boldsymbol{x} = \{r_1, \ldots, r_n\}$ random from $\mathrm{Pn}(r \mid \lambda)$. The reference posterior density of λ is $\pi^*(\lambda \mid \boldsymbol{x}) = \mathrm{Ga}(\lambda \mid , t + 1/2, n)$, where $t = \Sigma_j \, r_j$.

  The (reference) posterior predictive distribution is

  $$p(r \mid \boldsymbol{x}) = \Pr[r \mid t, n] = \int_0^\infty \mathrm{Pn}(r \mid \lambda) \, \mathrm{Ga}(\lambda \mid , t + \tfrac{1}{2}, n) \, d\lambda$$

  $$= \frac{n^{t+1/2}}{\Gamma(t+1/2)} \, \frac{1}{r!} \, \frac{\Gamma(r+t+1/2)}{(1+n)^{r+t+1/2}} \, ,$$

  an example of a Poisson-Gamma probability mass function.

  □ For example, no flash floods have been recorded on a particular location in 10 consecutive years. Local authorities are interested in forecasting possible future flash floods. Using a Poisson model, and assuming that meteorological conditions remain similar, the probabilities that r flash floods will occur next year in that location are given by the Poisson-Gamma mass function above, with $t = 0$ and $n = 10$. This yields, $\Pr[0 \mid t, n] = 0.953$, $\Pr[1 \mid t, n] = 0.043$, and $\Pr[2 \mid t, n] = 0.003$.

  Many other situations may be described with the same model.

- *Prediction of Normal measurements*

  - Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ random from $N(x \,|\, \mu, \sigma^2)$. Reference prior $\pi^*(\mu, \sigma) = \sigma^{-1}$ or, in terms of the precision $\lambda = \sigma^{-2}$, $\pi^*(\mu, \lambda) = \lambda^{-1}$.

    The *joint* reference posterior, $\pi^*(\mu, \lambda \,|\, \boldsymbol{x}) \propto p(\boldsymbol{x} \,|\, \mu, \lambda)\, \pi^*(\mu, \lambda)$, is $\pi^*(\mu, \lambda \,|\, \boldsymbol{x}) = N(\mu \,|\, \overline{x}, (n\lambda)^{-1})\, \mathrm{Ga}(\lambda \,|\, (n-1)/2, ns^2/2)$.

  - The predictive distribution is

    $$\pi^*(x \,|\, \boldsymbol{x}) = \int_0^\infty \int_{-\infty}^\infty \mathrm{N}(x \,|\, \mu, \lambda^{-1})\, \pi^*(\mu, \lambda \,|\, \boldsymbol{x})\, d\mu\, d\lambda$$
    $$= \{(1+n)s^2 + (\mu - \overline{x})^2\}^{-n/2},$$

    which is a kernel of the *Student* density

    $p(x \,|\, \boldsymbol{x}) = \mathrm{St}(x \,|\, \overline{x}, s^2 \frac{n+1}{n-1}, n-1)$.

  - *Example*. Production of safety belts. Observed breaking strengths of 10 randomly chosen webbings have mean $\overline{x} = 28.011$ kN and standard deviation $s = 0.443$ kN. Specification requires $x > 26$ kN.

    Reference posterior predictive $p(x \,|\, \boldsymbol{x}) = \mathrm{St}(x \,|\, 28.011, 0.490, 9)$.

    $\Pr(x > 26 \,|\, \boldsymbol{x}) = \int_{26}^\infty \mathrm{St}(x \,|\, 28.011, 0.240, 9)\, dx = 0.9987$.

- *Regression*
  - Often *additional information* from relevant covariates. Data structure, set of pairs $\boldsymbol{x} = \{(\boldsymbol{y}_1, \boldsymbol{v}_1), \ldots (\boldsymbol{y}_n, \boldsymbol{v}_n)\}$; $\boldsymbol{y}_i$, $\boldsymbol{v}_i$, both vectors. Given a new observation, with $\boldsymbol{v}$ known, predict the corresponding value of $\boldsymbol{y}$. Formally, compute $p\{\boldsymbol{y} \,|\, \boldsymbol{v}, (\boldsymbol{y}_1, \boldsymbol{v}_1), \ldots (\boldsymbol{y}_n, \boldsymbol{v}_n)\}$.
  - Need a model $\{p(\boldsymbol{y} \,|\, \boldsymbol{v}, \boldsymbol{\theta}), \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{\theta} \in \Theta\}$ which makes precise the probabilistic relationship between $\boldsymbol{y}$ and $\boldsymbol{v}$. The simplest option assumes a *linear dependency* of the form $p(\boldsymbol{y} \,|\, \boldsymbol{v}, \boldsymbol{\theta}) = \mathrm{N}(\boldsymbol{y} \,|\, \boldsymbol{V}\boldsymbol{\beta}, \Sigma)$, but far more complex structures are common in applications.
  - *Univariate linear regression on k covariates*. $Y \subset \Re$, $\boldsymbol{v} = \{v_1, \ldots, v_k\}$. $p(y \,|\, \boldsymbol{v}, \boldsymbol{\beta}, \sigma) = \mathrm{N}(y \,|\, \boldsymbol{v}\boldsymbol{\beta}, \sigma^2)$, $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_k\}^t$. Data $\boldsymbol{x} = \{\boldsymbol{y}, \boldsymbol{V}\}$, $\boldsymbol{y} = \{y_1, \ldots, y_n\}^t$, and $\boldsymbol{V}$ is the $n \times k$ matrix with the $\boldsymbol{v}_i$'s as rows. $p(\boldsymbol{y} \,|\, \boldsymbol{V}, \boldsymbol{\beta}, \sigma) = \mathrm{N}_n(\boldsymbol{y} \,|\, \boldsymbol{V}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I}_n)$; reference prior $\pi^*(\boldsymbol{\beta}, \sigma) = \sigma^{-1}$.

    Predictive posterior is the Student density

    $$p(y \,|\, \boldsymbol{v}, \boldsymbol{y}, \boldsymbol{V}) = \mathrm{St}(y \,|\, \boldsymbol{v}\hat{\boldsymbol{\beta}}, f(\boldsymbol{v}, \boldsymbol{V}) \tfrac{ns^2}{n-k}, n-k)$$

    $$\hat{\boldsymbol{\beta}} = (\boldsymbol{V}^t \boldsymbol{V})^{-1} \boldsymbol{V}^t \boldsymbol{y}, \quad ns^2 = (\boldsymbol{y} - \boldsymbol{v}\hat{\boldsymbol{\beta}})^t (\boldsymbol{y} - \boldsymbol{v}\hat{\boldsymbol{\beta}})$$

    $$f(\boldsymbol{v}, \boldsymbol{V}) = 1 + \boldsymbol{v}(\boldsymbol{V}^t \boldsymbol{V})^{-1} \boldsymbol{v}^t$$

- *Example: Simple linear regression*

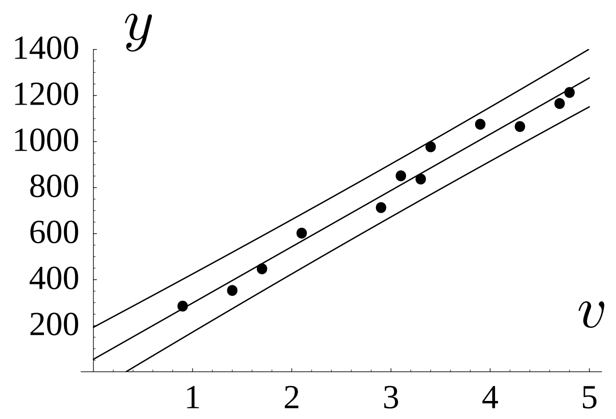
  - One covariate and a constant term; $p(y \mid v, \boldsymbol{\beta}, \sigma) = \mathrm{N}(y \mid \beta_1 + \beta_2 v, \sigma^2)$
    Sufficient statistic is $\boldsymbol{t} = \{\overline{v}, \overline{y}, s_{vy}, s_{vv}\}$, with $n\overline{v} = \Sigma v_j$, $n\overline{y} = \Sigma y_j$, $s_{yv} = \Sigma v_j y_j / n - \overline{v}\,\overline{y}$, $s_{vv} = \Sigma v_j^2 / n - \overline{v}^2$.

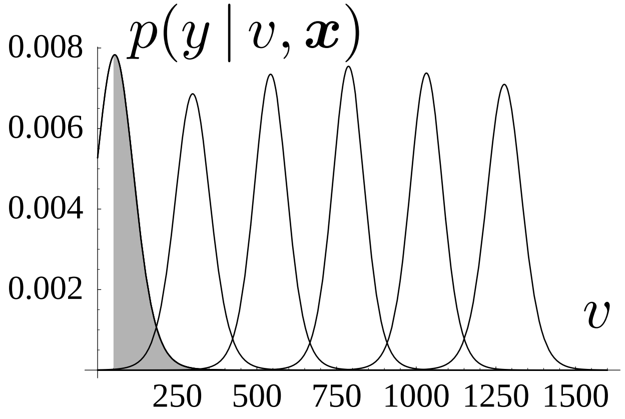
    $$p(y \mid v, \boldsymbol{t}) = \mathrm{St}(y \mid \hat{\beta}_1 + \hat{\beta}_2 v, f(v, \boldsymbol{t})\,\tfrac{ns^2}{n-2}, n-2)$$

    $$\hat{\beta}_1 = \overline{y} - \hat{\beta}_2 \overline{v}, \qquad \hat{\beta}_2 = \frac{s_{vy}}{s_{vv}},$$

    $$ns^2 = \textstyle\sum_{j=1}^{n} (y_j - \hat{\beta}_1 - \hat{\beta}_2 x_j)^2$$

    $$f(v, \boldsymbol{t}) = 1 + \frac{1}{n} \frac{(v - \overline{v})^2 + s_{vv}}{s_{vv}}$$





  - Pollution density ($\mu gr/m^3$), and wind speed from source (m/s).

y_j	1212	836	850	446	1164	601
v_j	4.8	3.3	3.1	1.7	4.7	2.1
y_j	1074	284	352	1064	712	976
v_j	3.9	0.9	1.4	4.3	2.9	3.4

$$\Pr[y > 50 \mid v = 0, \boldsymbol{x}] = 0.66$$

2.4. Hierarchical Models

- *Exchangeability*

  - □ Random quantities are often “homogeneous” in the precise sense that only their *values* matter, not the *order* in which they appear. Formally, this is captured by the notion of *exchangeability*. The set of random vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is exchangeable if their joint distribution is invariant under permutations. An infinite sequence $\{\boldsymbol{x}_j\}$ of random vectors is exchangeable if all its finite subsequences are exchangeable.

  - □ *Any random sample from any model is exchangeable*. The *representation theorem* establishes that if observations $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ are exchangeable, they are a *a random sample* from some model $\{p(\boldsymbol{x} \,|\, \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta\}$, labeled by a *parameter vector* $\boldsymbol{\theta}$, *defined* as the limit (as $n \to \infty$) of some function of the $\boldsymbol{x}_i$’s. Information about $\boldsymbol{\theta}$ in prevailing conditions C is *necessarily* described by *some* probability distribution $\pi(\boldsymbol{\theta} \,|\, C)$.

  - □ Formally, the joint density of any finite set of exchangeable observations $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ has an *integral representation* of the form
    $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \,|\, C) = \int_\Theta \prod_{i=1}^n p(\boldsymbol{x}_i \,|\, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, C)\, d\boldsymbol{\theta}$.

- *Structured Models*
  - Complex data structures may often be usefully described by partial exchangeability assumptions.
  - *Example: Public opinion.* Sample k different regions in the country. Sample n_i citizens in region i and record whether or not ($y_{ij} = 1$ or $y_{ij} = 0$) citizen j would vote A. Assuming exchangeable citizens within each region implies
$p(y_{i1}, \ldots, y_{in_i}) = \prod_{j=1}^{n_i} p(y_{ij} \,|\, \theta_i) = \theta_i^{r_i}(1-\theta_i)^{n_i - r_i}$,
where θ_i is the (unknown) proportion of citizens in region i voting A and $r_i = \Sigma_j y_{ij}$ the number of citizens voting A in region i.

    Assuming regions exchangeable within the country similarly leads to
$p(\theta_1, \ldots, \theta_k) = \prod_{i=1}^{k} \pi(\theta_i \,|\, \boldsymbol{\phi})$
for some probability distribution $\pi(\theta \,|\, \boldsymbol{\phi})$ describing the political variation within the regions. Often choose $\pi(\theta \,|\, \boldsymbol{\phi}) = \mathrm{Be}(\theta \,|\, \alpha, \beta)$.
  - The resulting *two-stages hierarchical Binomial-Beta model*
$\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\}$, $\boldsymbol{y}_i = \{y_{i1}, \ldots, y_{in_i}\}$, random from $\mathrm{Bi}(y \,|\, \theta_i)$,
$\{\theta_1, \ldots, \theta_k\}$, random from $\mathrm{Be}(\theta \,|\, \alpha, \beta)$
provides a far richer model than (unrealistic) simple binomial sampling.

- *Example: Biological response.* Sample k different animals of the same species in specific environment. Control n_i times animal i and record his responses $\{\boldsymbol{y}_{i1}, \ldots, \boldsymbol{y}_{in_i}\}$ to prevailing conditions. Assuming exchangeable observations within each animal implies
  $p(\boldsymbol{y}_{i1}, \ldots, \boldsymbol{y}_{in_i}) = \prod_{j=1}^{n_i} p(\boldsymbol{y}_{ij} \,|\, \boldsymbol{\theta}_i)$.
  Often choose $p(\boldsymbol{y}_{ij} \,|\, \boldsymbol{\theta}_i) = \mathrm{N}_r(\boldsymbol{y} \,|\, \boldsymbol{\mu}_i, \Sigma_1)$, where r is the number of biological responses measured.

  Assuming exchangeable animals within the environment leads to
  $p(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k) = \prod_{i=1}^{k} \pi(\boldsymbol{\mu}_i \,|\, \boldsymbol{\phi})$
  for some probability distribution $\pi(\boldsymbol{\mu} \,|\, \boldsymbol{\phi})$ describing the biological variation within the species. Often choose $\pi(\boldsymbol{\mu} \,|\, \boldsymbol{\phi}) = \mathrm{N}_r(\boldsymbol{\mu} \,|\, \boldsymbol{\mu}_0, \Sigma_2)$.
- The *two-stages hierarchical multivariate Normal-Normal model*
  $\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\}$, $\boldsymbol{y}_i = \{\boldsymbol{y}_{i1}, \ldots, \boldsymbol{y}_{in_i}\}$, random from $\mathrm{N}_r(\boldsymbol{y} \,|\, \boldsymbol{\mu}_i, \Sigma_1)$,
  $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k\}$, random from $\mathrm{N}_r(\boldsymbol{\mu} \,|\, \boldsymbol{\mu}_0, \Sigma_2)$
  provides a far richer model than (unrealistic) simple multivariate normal sampling.
- Finer subdivisions, *e.g.*, subspecies within each species, similarly lead to hierarchical models with more stages.

- *Bayesian analysis of hierarchical models*

  □ A *two-stages hierarchical model* has the general form
  $\boldsymbol{x} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k\}$, $\boldsymbol{y}_i = \{\boldsymbol{z}_{i1}, \ldots, \boldsymbol{z}_{in_i}\}$
  $\boldsymbol{y}_i$ random sample of size n_i from $p(\boldsymbol{z} \,|\, \boldsymbol{\theta}_i)$, $\boldsymbol{\theta}_i \in \Theta$,
  $\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_k\}$, random of size k from $\pi(\boldsymbol{\theta} \,|\, \boldsymbol{\phi})$, $\boldsymbol{\phi} \in \Phi$.

  □ Specify a *prior distribution* (or a reference prior function)
  $\pi(\boldsymbol{\phi})$ for the *hyperparameter vector* $\boldsymbol{\phi}$.

  □ Use *standard probability theory* to compute all desired
  posterior distributions:
  $\pi(\boldsymbol{\phi} \,|\, \boldsymbol{x})$ for inferences about the hyperparameters,
  $\pi(\boldsymbol{\theta}_i \,|\, \boldsymbol{x})$ for inferences about the parameters,
  $\pi(\psi \,|\, \boldsymbol{x})$ for inferences about the any function $\psi = \psi(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_k)$
  of the parameters,
  $\pi(\boldsymbol{y} \,|\, \boldsymbol{x})$ for predictions on future observations,
  $\pi(t \,|\, \boldsymbol{x})$ for predictions on any function $t = t(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)$
  of m future observations

  □ *Markov Chain Monte Carlo* based *software* available for the necessary computations.

3. Decision Making

3.1 Structure of a Decision Problem

- *Alternatives, consequences, relevant events*

  □ A decision problem if two or more possible courses of action; $\mathcal{A}$ is the class of possible *actions*.

  □ For each $a \in \mathcal{A}$, Θ_a is the set of *relevant events*, those may affect the result of choosing a.

  □ Each pair $\{a, \boldsymbol{\theta}\}$, $\boldsymbol{\theta} \in \Theta_a$, produces a consequence $c(a, \boldsymbol{\theta}) \in \mathcal{C}_a$. In this context, $\boldsymbol{\theta}$ if often referred to as the *parameter of interest*.

  □ The class of pairs $\{(\Theta_a, \mathcal{C}_a), a \in \mathcal{A}\}$ describes the *structure* of the decision problem. Without loss of generality, it may be assumed that the possible actions are mutually exclusive, for otherwise the appropriate Cartesian product may be used.

  □ In many problems the class of relevant events Θ_a is the same for all $a \in \mathcal{A}$. Even if this is not the case, a comprehensive *parameter space* Θ may be defined as the union of all the Θ_a.

- *Foundations of decision theory*

□ Different sets of principles capture a minimum collection of logical rules required for “rational” decision-making.

These are axioms with strong intuitive appeal.
Their basic structure consists of:

- The *Transitivity* of preferences:
  If $a_1 > a_2$ given C, and $a_2 > a_3$ given C,
  then $a_1 > a_3$ given C.
- The *Sure-thing principle*:
  If $a_1 > a_2$ given C and E, and $a_1 > a_2$ given C and not E
  then $a_1 > a_2$ given C.
- The existence of *Standard events*:
  There are events of known plausibility.
  These may be used as a unit of measurement, and
  have the properties of a probability measure

□ These axioms are not a description of human decision-making, but a *normative* set of principles defining *coherent* decision-making.

- *Decision making*

□ Many different axiom sets.
All lead basically to the same set of conclusions, namely:

• The consequences of wrong actions should be evaluated in terms of a real-valued *loss* function $L(a, \boldsymbol{\theta})$ which specifies, on a numerical scale, their undesirability.

• The uncertainty about the parameter of interest $\boldsymbol{\theta}$ should be measured with a *probability distribution* $\pi(\boldsymbol{\theta} \,|\, C)$

$$\pi(\boldsymbol{\theta} \,|\, C) \geq 0, \quad \boldsymbol{\theta} \in \Theta, \qquad \int_\Theta \pi(\boldsymbol{\theta} \,|\, C)\, d\boldsymbol{\theta} = 1,$$

describing all available knowledge about its value, given the conditions C under which the decision must be taken.

• The relative undesirability of available actions $a \in \mathcal{A}$ is measured by their *expected loss*

$$\overline{L}(a \,|\, C) = \int_\Theta L(a, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, C)\, d\boldsymbol{\theta}, \quad a \in \mathcal{A}.$$

- *Intrinsic loss functions: Intrinsic discrepancy*

  □ The loss function is typically *context dependent*.

  □ In mathematical statistics, *intrinsic* loss functions are used to measure the distance between between statistical models.

  They measure the *divergence between the models* $\{p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1), \boldsymbol{x} \in \mathcal{X}\}$ and $\{p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2), \boldsymbol{x} \in \mathcal{X}\}$ as some *non-negative* function of the form $L[p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1), p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2)]$ which is zero if (and only if) the two distributions are equal almost everywhere.

  □ The *intrinsic discrepancy* between two statistical models is simply the intrinsic discrepancy between their sampling distributions, *i.e.*,

$$\begin{aligned}\delta\{p_1, p_2\} &= \delta\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\} \\ &= \min\left\{\int_{\mathcal{X}} p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1) \log \frac{p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1)}{p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2)}\, d\boldsymbol{x}, \right. \\ &\qquad\qquad \left. \int_{\mathcal{X}} p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2) \log \frac{p_2(\boldsymbol{x} \,|\, \boldsymbol{\theta}_2)}{p_1(\boldsymbol{x} \,|\, \boldsymbol{\theta}_1)}\, d\boldsymbol{x} \right\}\end{aligned}$$

  □ The intrinsic discrepancy is an *information-based, symmetric, invariant intrinsic loss*.

3.2 Formal Point Estimation

- *The decision problem*

  - Given statistical model $\{p(\boldsymbol{x}\,|\,\boldsymbol{\omega}), \boldsymbol{x} \in \mathcal{X}, \boldsymbol{\omega} \in \Omega\}$, quantity of interest $\boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{\omega}) \in \Theta$. A *point estimator* $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}(\boldsymbol{x})$ of $\boldsymbol{\theta}$ is some function of the data to be regarded as a proxy for the unknown value of $\boldsymbol{\theta}$.
  - To choose a point estimate for $\boldsymbol{\theta}$ is a *decision problem*, where the action space is $\mathcal{A} = \Theta$.
  - Given a *loss function* $l(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta})$, the posterior expected loss is

    $$\overline{L}[\tilde{\boldsymbol{\theta}}\,|\,\boldsymbol{x}] = \int_{\Theta} L(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta}\,|\,\boldsymbol{x})\, d\boldsymbol{\theta},$$

    The corresponding *Bayes estimator* is that function of the data,

    $$\boldsymbol{\theta}^* = \boldsymbol{\theta}^*(\boldsymbol{x}) = \arg \inf_{\tilde{\boldsymbol{\theta}} \in \Theta} \overline{L}[\tilde{\boldsymbol{\theta}}\,|\,\boldsymbol{x}],$$

    which minimizes that expectation.

• *Conventional estimators*

□ The *posterior mean* and the *posterior mode* are the Bayes estimators which respectively correspond to a *quadratic* an a *zero-one* loss functions.

• If $L(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) = (\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta})^t(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta})$, then, assuming that the mean exists, the Bayes estimator is the *posterior mean* $E[\boldsymbol{\theta} \,|\, \boldsymbol{x}]$.

□ • If the loss function is a zero-one function, so that $L(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) = 0$ if $\tilde{\boldsymbol{\theta}}$ belongs to a ball of radius ε centered in $\boldsymbol{\theta}$ and $L(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) = 1$ otherwise then, , assuming that a unique mode exists, the Bayes estimator converges to the *posterior mode* $\mathrm{Mo}[\boldsymbol{\theta} \,|\, \boldsymbol{x}]$ as the ball radius ε tends to zero.

□ If θ is *univariate and continuous*, and the loss function is *linear*,

$$L(\tilde{\theta}, \theta) = \begin{cases} c_1(\tilde{\theta} - \theta) & \text{if} \quad \tilde{\theta} \geq \theta \\ c_2(\theta - \tilde{\theta}) & \text{if} \quad \tilde{\theta} < \theta \end{cases}$$

then the Bayes estimator is the *posterior quantile* of order $c_2/(c_1 + c_2)$, so that $\Pr[\theta < \theta^*] = c_2/(c_1 + c_2)$.

In particular, if $c_1 = c_2$, the Bayes estimator is the *posterior median*.

□ Any θ value may be optimal: it all depends on the loss function.

- *Intrinsic estimation*

□ Given the statistical model $\{p(\boldsymbol{x}\,|\,\boldsymbol{\theta}), \boldsymbol{x} \in \mathcal{X}, \boldsymbol{\theta} \in \Theta\}$ the intrinsic discrepancy $\delta(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ between two parameter values $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ is the intrinsic discrepancy $\delta\{p(\boldsymbol{x}\,|\,\boldsymbol{\theta}_1), p(\boldsymbol{x}\,|\,\boldsymbol{\theta}_2)\}$ between the corresponding probability models.

This is symmetric, non-negative (and zero iff $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$), invariant under reparametrization and invariant under bijections of $\boldsymbol{x}$.

□ The intrinsic estimator is the *reference* Bayes estimator which corresponds to the loss defined by the *intrinsic discrepancy*:

- The expected loss with respect to the reference posterior distribution

$$d(\tilde{\boldsymbol{\theta}}\,|\,\boldsymbol{x}) = \int_\Theta \delta\{\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}\}\, \pi^*(\boldsymbol{\theta}\,|\,\boldsymbol{x})\, d\boldsymbol{\theta}$$

is an objective measure, in information units, of the *expected* discrepancy between the model $p(\boldsymbol{x}\,|\,\tilde{\boldsymbol{\theta}})$ and the true (unknown) model $p(\boldsymbol{x}\,|\,\boldsymbol{\theta})$.

- The *intrinsic estimator* $\boldsymbol{\theta}^* = \boldsymbol{\theta}^*(\boldsymbol{x})$ is the value which minimizes such expected discrepancy,

$$\boldsymbol{\theta}^* = \arg \inf_{\tilde{\boldsymbol{\theta}} \in \Theta} d(\tilde{\boldsymbol{\theta}}\,|\,\boldsymbol{x}).$$

- *Example: Intrinsic estimation of the Binomial parameter*

□ Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, random from $p(x \mid \theta) = \theta^x (1-\theta)^{1-x}$, $r = \Sigma x_j$. Intrinsic discrepancy $\delta(\tilde{\theta}, \theta) = n \min\{k(\tilde{\theta} \mid \theta), k(\theta \mid \tilde{\theta})\}$, $k(\theta_1 \mid \theta_2) = \theta_2 \log \frac{\theta_2}{\theta_1} + (1-\theta_2) \log \frac{1-\theta_2}{1-\theta_1}$, $\quad \pi^*(\theta) = \mathrm{Be}(\theta \mid \frac{1}{2}, \frac{1}{2})$ $\pi^*(\theta \mid r, n) = \mathrm{Be}(\theta \mid r + \frac{1}{2}, n - r + \frac{1}{2})$.

□ Expected reference discrepancy

$$d(\tilde{\theta}, r, n) = \int_0^1 \delta(\tilde{\theta}, \theta)\, \pi^*(\theta \mid r, n)\, d\theta$$

□ Intrinsic estimator

$$\theta^*(r, n) = \arg\min_{0<\tilde{\theta}<1} d(\tilde{\theta}, r, n)$$

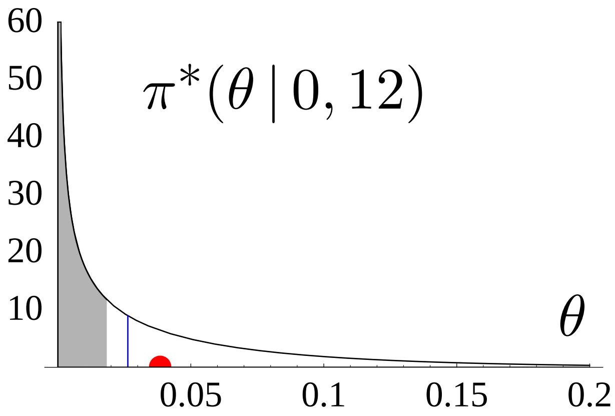
From invariance, for any bijection $\phi = \phi(\theta)$, $\phi^* = \phi(\theta^*)$.

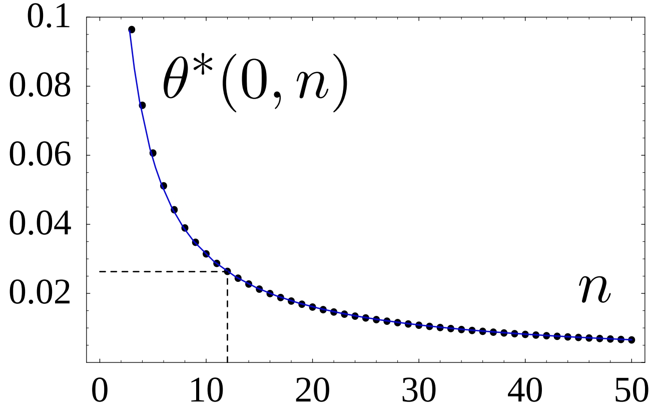
□ Analytic approximation

$$\theta^*(r, n) \approx \frac{r+1/3}{n+2/3}, \qquad n > 2$$

□ $n = 12, r = 0, \quad \theta^*(0, 12) = 0.026$

$\mathrm{Me}[\theta \mid \boldsymbol{x}] = 0.018$, $\mathrm{E}[\theta \mid \boldsymbol{x}] = 0.038$





3.3 Hypothesis Testing

- *Precise hypothesis testing as a decision problem*

  □ The posterior $\pi(\boldsymbol{\theta} \mid D)$ conveys intuitive information on the values of $\boldsymbol{\theta}$ which are *compatible* with the observed data $\boldsymbol{x}$: those with a *relatively high probability density*.

  □ Often a particular value $\boldsymbol{\theta}_0$ is suggested for special consideration:

  - Because $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ would greatly simplify the model
  - Because there are context specific arguments suggesting that $\boldsymbol{\theta} = \boldsymbol{\theta}_0$

  More generally, one may analyze the *restriction* of parameter space Θ to a subset Θ_0 which may contain more than one value.

  □ Formally, testing the hypothesis $H_0 \equiv \{\boldsymbol{\theta} = \boldsymbol{\theta}_0\}$ is a *decision problem* with just two possible actions:

  - a_0: to *accept* H_0 and work with $p(\boldsymbol{x} \mid \boldsymbol{\theta}_0)$.
  - a_1: to *reject* H_0 and keep the general model $p(\boldsymbol{x} \mid \boldsymbol{\theta})$.

  □ To proceed, a *loss* function $L(a_i, \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$, describing the possible consequences of both actions, must be specified.

- *Structure of the loss function*

  □ Given data $\boldsymbol{x}$, optimal action is to reject H_0 (action a_1) *iff* the expected posterior loss of accepting, $\int_\Theta L(a_0, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta}$, is *larger* than the expected posterior loss of rejecting, $\int_\Theta L(a_1, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta}$, *i.e.*, iff

  $$\int_\Theta [L(a_0, \boldsymbol{\theta}) - L(a_1, \boldsymbol{\theta})]\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta} = \int_\Theta \Delta L(\boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta} > 0.$$

  Therefore, only the loss difference $\Delta L(\boldsymbol{\theta}) = L(a_0, \boldsymbol{\theta}) - L(a_1, \boldsymbol{\theta})$, which measures the *advantage* of rejecting H_0 as a function of $\boldsymbol{\theta}$, has to be specified: The hypothesis should be rejected whenever the *expected* advantage of rejecting is positive.

  □ The advantage $\Delta L(\boldsymbol{\theta})$ of rejecting H_0 as a function of $\boldsymbol{\theta}$ should be of the form $\Delta L(\boldsymbol{\theta}) = l(\boldsymbol{\theta}_0, \boldsymbol{\theta}) - l^*$, for some $l^* > 0$, where

  • $l(\boldsymbol{\theta}_0, \boldsymbol{\theta})$ measures the *discrepancy* between $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$ and $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$,

  • l^* is a positive *utility constant* which measures the advantage working with the simpler model when it is true.

  □ The Bayes criterion will then be: *Reject* H_0 if (and only if)

  $\int_\Theta l(\boldsymbol{\theta}_0, \boldsymbol{\theta})\, \pi(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta} > l^*$, that is if (and only if)

  the *expected discrepancy* between $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$ and $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$ is *too large.*

- *Bayesian Reference Criterion*

□ An good choice for the function $l(\boldsymbol{\theta}_0, \boldsymbol{\theta})$ is the *intrinsic discrepancy*,
$\delta(\boldsymbol{\theta}_0, \boldsymbol{\theta}) = \min\{k(\boldsymbol{\theta}_0 \,|\, \boldsymbol{\theta}),\, k(\boldsymbol{\theta} \,|\, \boldsymbol{\theta}_0)\}$,
where $k(\boldsymbol{\theta}_0 \,|\, \boldsymbol{\theta}) = \int_{\mathcal{X}} p(\boldsymbol{x} \,|\, \boldsymbol{\theta}) \log\{p(\boldsymbol{x} \,|\, \boldsymbol{\theta})/p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)\} d\boldsymbol{x}$.
If $\boldsymbol{x} = \{x_1, \ldots, x_n\} \in \mathcal{X}^n$ is a random sample from $p(x \,|\, \boldsymbol{\theta})$, then

$$k(\boldsymbol{\theta}_0 \,|\, \boldsymbol{\theta}) = n \int_{\mathcal{X}} p(x \,|\, \boldsymbol{\theta}) \log \frac{p(x \,|\, \boldsymbol{\theta})}{p(x \,|\, \boldsymbol{\theta}_0)}\, dx.$$

□ For objective results, exclusively based on model assumptions and data, the *reference* posterior distribution $\pi^*(\boldsymbol{\theta} \,|\, \boldsymbol{x})$ should be used.

□ Hence, *reject if (and only if) the expected reference posterior intrinsic discrepancy $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ is too large*,

$d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x}) = \int_{\Theta} \delta(\boldsymbol{\theta}_0, \boldsymbol{\theta})\, \pi^*(\boldsymbol{\theta} \,|\, \boldsymbol{x})\, d\boldsymbol{\theta} > d^*$, for some $d^* > 0$.

This is the *Bayesian reference criterion (BRC)*.

□ The *reference test statistic* $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ is nonnegative, it is invariant both under reparametrization and under sufficient transformation of the data, and it is a measure, in natural information units (nits) of the expected discrepancy between $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$ and the true model.

- *Calibration of the BRC*

□ The reference test statistic $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ is the posterior expected discrepancy of the intrinsic discrepancy between $p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$ and $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})$. Hence,

• A reference test statistic value $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ of, say, $\log(10) = 2.303$ nits implies that, given data $\boldsymbol{x}$, the *average* value of the likelihood ratio *against* the hypothesis, $p(\boldsymbol{x} \,|\, \boldsymbol{\theta})/p(\boldsymbol{x} \,|\, \boldsymbol{\theta}_0)$, is expected to be about 10, suggesting some *mild evidence* against $\boldsymbol{\theta}_0$.

• Similarly, a value $d(\boldsymbol{\theta}_0 \,|\, \boldsymbol{x})$ of $\log(100) = 4.605$ indicates an average value of the likelihood ratio against $\boldsymbol{\theta}_0$ of about 100, indicating rather *strong evidence* against the hypothesis, and $\log(1000) = 6.908$, a rather conclusive likelihood ratio against the hypothesis of about 1000.

□ As expected, there are strong connections between the BRC criterion for precise hypothesis testing and intrinsic estimation:

• The *intrinsic estimator* is the value of $\boldsymbol{\theta}$ with minimizes the reference test statistic: $\boldsymbol{\theta}^* = \arg\inf_{\boldsymbol{\theta}\in\Theta} d(\boldsymbol{\theta} \,|\, \boldsymbol{x})$.

• The regions defined by $\{\boldsymbol{\theta};\ d(\boldsymbol{\theta} \,|\, \boldsymbol{x}) \leq d^*\}$ are invariant *reference posterior $q(d^*)$-credible regions* for $\boldsymbol{\theta}$. For regular problems and large samples, $q(\log(10)) \approx 0.95$ and $q(\log(100)) \approx 0.995$.

- *A canonical example: Testing a value for the Normal mean*

□ In the simplest case where the variance σ^2 is known,

$$\delta(\mu_0, \mu) = n(\mu - \mu_0)^2/(2\sigma^2), \qquad \pi^*(\mu \,|\, \boldsymbol{x}) = \mathrm{N}(\mu \,|\, \overline{x}, \sigma^2/n),$$

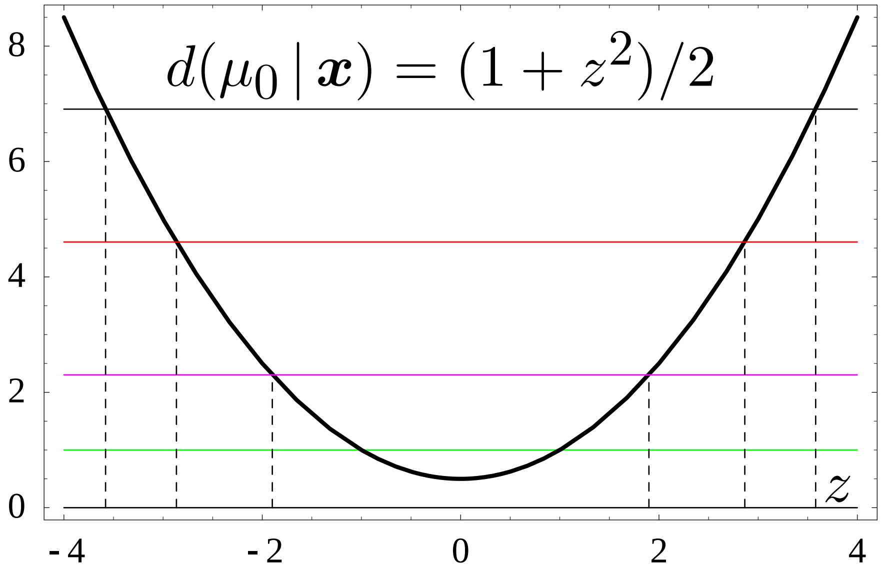
$$d(\mu_0 \,|\, \boldsymbol{x}) = \tfrac{1}{2}(1 + z^2), \qquad z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$$

Thus rejecting $\mu = \mu_0$ if $d(\mu_0 \,|\, \boldsymbol{x}) > d^*$ is equivalent to rejecting if $|z| > \sqrt{2d^* - 1}$ and, hence, to a conventional two-sided frequentist test with significance level $\alpha = 2(1 - \Phi(|z|))$.

d^*	$\lvert z \rvert$	α
$\log(10)$	1.8987	0.0576
$\log(100)$	2.8654	0.0042
$\log(1000)$	3.5799	0.0003

□ The expected value of $d(\mu_0 \,|\, \boldsymbol{x})$ if the hypothesis is true is

$$\int_{-\infty}^{\infty} \tfrac{1}{2}(1 + z^2)\mathrm{N}(z \,|\, 0, 1)\, dz = 1$$



- *Fisher's tasting tea lady*

□ Data $\boldsymbol{x} = \{x_1, \ldots, x_n\}$, random from $p(x \mid \theta) = \theta^x (1-\theta)^{1-x}$, $r = \Sigma x_j$. Intrinsic discrepancy $\delta(\theta_0, \theta) = n\ \min\{k(\theta_0 \mid \theta), k(\theta \mid \theta_0)\}$, $k(\theta_1 \mid \theta_2) = \theta_2 \log \frac{\theta_2}{\theta_1} + (1-\theta_2) \log \frac{1-\theta_2}{1-\theta_1}$, $\quad \pi^*(\theta) = \text{Be}(\theta \mid \frac{1}{2}, \frac{1}{2})$

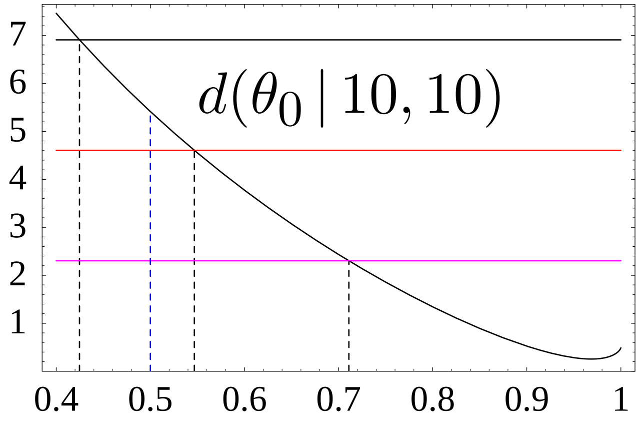
Intrinsic test statistic

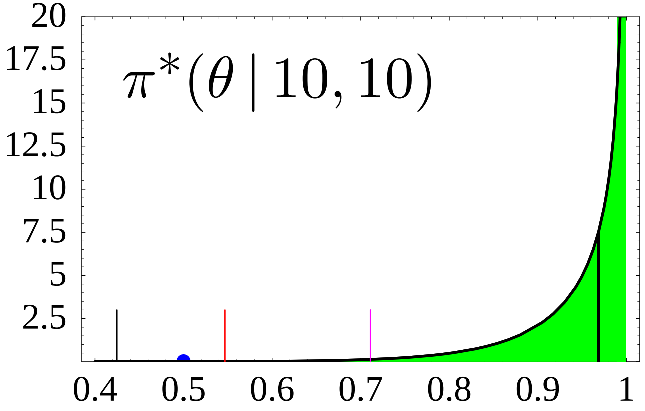
$$d(\theta_0 \mid \boldsymbol{x}) = \int_0^1 \delta(\tilde{\theta}, \theta)\, \pi^*(\theta \mid \boldsymbol{x})\, d\theta$$

□ Fisher's example: $\boldsymbol{x} = \{10, 10\}$
Test $\theta_0 = 1/2$, $\theta^*(\boldsymbol{x}) = 0.9686$

$d(\theta^* \mid \boldsymbol{x})$	θ^*	$\Pr[\theta < \theta^* \mid \boldsymbol{x}]$
log[10]	0.711	0.99185
log[100]	0.547	0.99957
log[1000]	0.425	0.99997

Using $d^* = \log[100] = 4.61$,
the value $\theta = 0.5$ is rejected.
$\Pr[\theta < 0.5 \mid \boldsymbol{x}] = 0.00016$

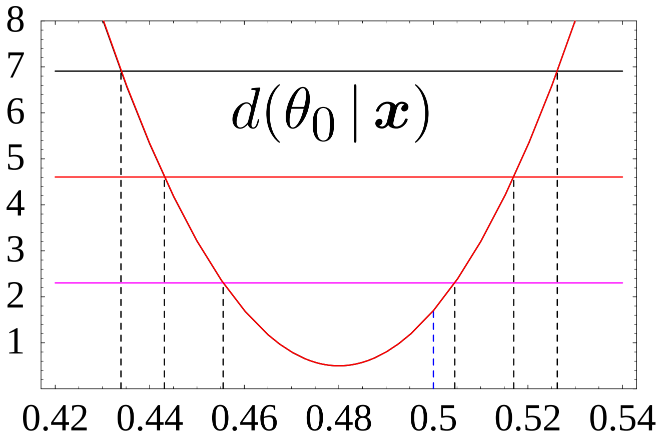




- *Asymptotic approximation*

□ For large samples, the posterior approaches $\mathrm{N}(\theta \,|\, \hat{\theta}, n^{-1}F^{-1}(\hat{\theta}))$, where $F(\theta)$ is Fisher's information function. Changing variables, the posterior distribution of $\phi = \phi(\theta) = \int F^{1/2}(\theta)\, d\theta = 2 \,\mathrm{arc\,sin}\, \sqrt{\theta})$ is approximately normal $\mathrm{N}(\phi \,|\, \hat{\phi}, n^{-1})$. Since $d(\theta, \boldsymbol{x})$ is invariant,

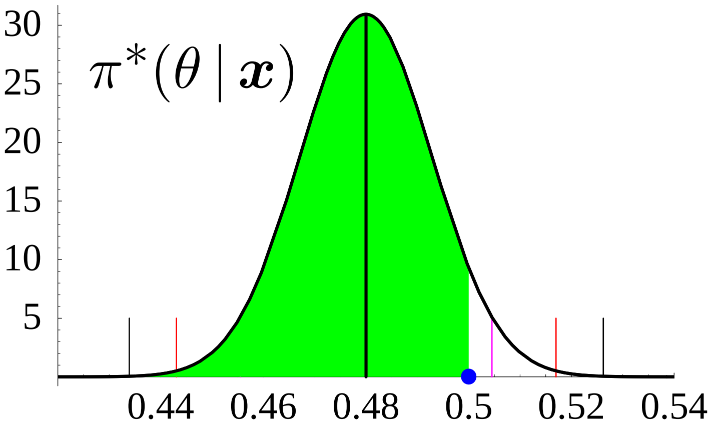
$$d(\theta_0, \boldsymbol{x}) \approx \tfrac{1}{2}[1 + n\{\phi(\theta_0) - \phi(\hat{\theta})\}^2].$$



- *Testing for a majority ($\theta_0 = 0.5$)*

$$\boldsymbol{x} = \{720, 1500\}, \quad \theta^*(\boldsymbol{x}) = 0.4800$$

$d(\theta^* \,\vert\, \boldsymbol{x})$	$R = (\theta_0^*, \theta_1^*)$	$\Pr[\theta \in R \,\vert\, \boldsymbol{x}]$
log[10]	(0.456, 0.505)	0.9427
log[100]	(0.443, 0.517)	0.9959
log[1000]	(0.434, 0.526)	0.9997



Very mild evidence against $\theta = 0.5$:
$d(0.5 \,|\, 720, 1500) = 1.67$
$\Pr(\theta < 0.5 \,|\, 720, 1500) = 0.9393$

Basic References

Bernardo, J. M. (2003). Bayesian Statistics.
Encyclopedia of Life Support Systems (EOLSS). Paris: UNESCO. (in press)
On line: **http://www.uv.es/˜bernardo/**

Gelman, A., Carlin, J. B., Stern, H. and Rubin, D. B. (1995).
Bayesian Data Analysis. London: Chapman and Hall.

Bernardo, J. M. and Smith, A. F. M. (1994).
Bayesian Theory. Chichester: Wiley.

Bernardo, J. M. and Ramón, J. M. (1998).
An introduction to Bayesian reference analysis: Inference on the ratio of multinomial parameters. *The Statistician* **47**, 1–35.

Bernardo, J. M. and Rueda, R. (2002). Bayesian hypothesis testing: A reference approach. *Internat. Statist. Rev.* **70**, 351–372.

Bernardo, J. M. and Juárez, M. (2003). Intrinsic estimation. *Bayesian Statistics 7* (J. M. Bernardo, M. J. Bayarri, J. O. Berger, A. P. Dawid, D. Heckerman, A. F. M. Smith and M. West, eds.). Oxford: University Press, 465-476.