Eighth Edition

CHAPTER

11

VECTOR MECHANICS FOR ENGINEERS:

DYNAMICS

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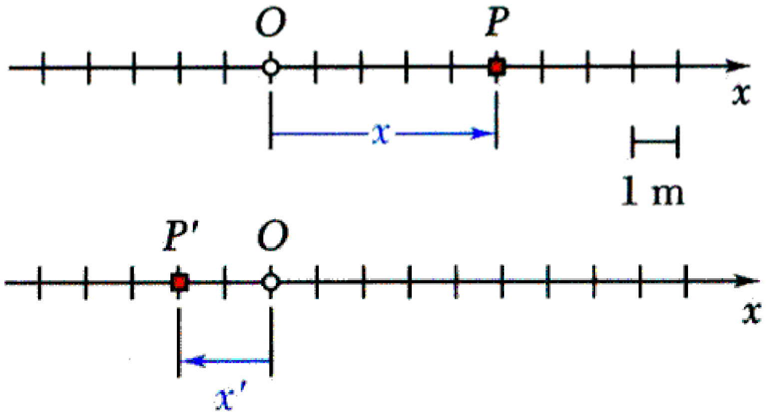
Kinematics of Particles

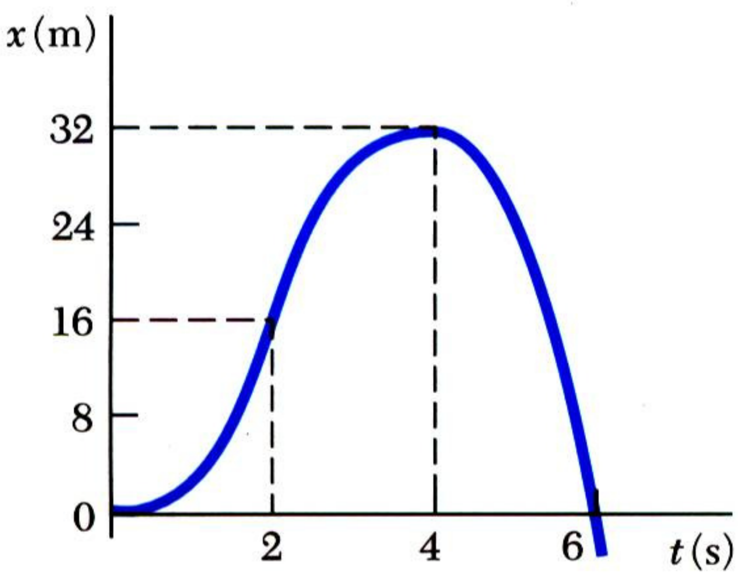
Contents

Introduction

- Dynamics includes:
  - *Kinematics*: study of the geometry of motion. Kinematics is used to relate displacement, velocity, acceleration, and time without reference to the cause of motion.
  - *Kinetics*: study of the relations existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.
- *Rectilinear* motion: position, velocity, and acceleration of a particle as it moves along a straight line.
- *Curvilinear* motion: position, velocity, and acceleration of a particle as it moves along a curved line in two or three dimensions.

Rectilinear Motion: Position, Velocity & Acceleration



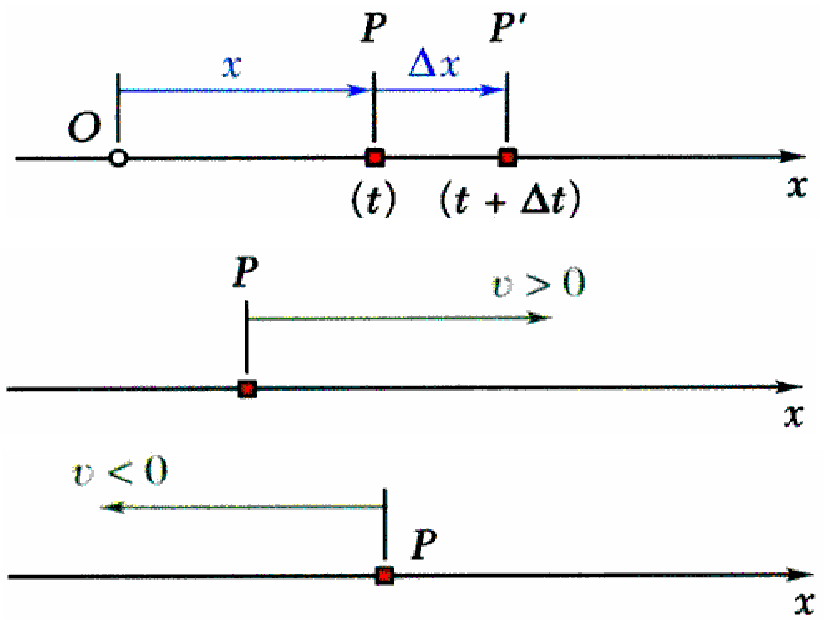


- Particle moving along a straight line is said to be in *rectilinear motion*.
- *Position coordinate* of a particle is defined by positive or negative distance of particle from a fixed origin on the line.
- The *motion* of a particle is known if the position coordinate for particle is known for every value of time *t*. Motion of the particle may be expressed in the form of a function, e.g.,

$$x = 6t^2 - t^3$$

or in the form of a graph *x* vs. *t*.

Rectilinear Motion: Position, Velocity & Acceleration



- Consider particle which occupies position *P* at time *t* and *P'* at *t+Δt*,

$$\textit{Average velocity} = \frac{\Delta x}{\Delta t}$$

$$\textit{Instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

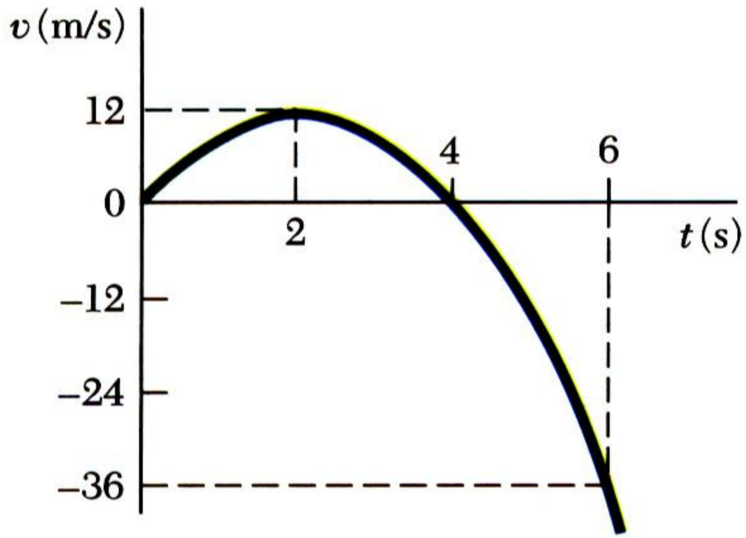
- Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as *particle speed*.

- From the definition of a derivative,

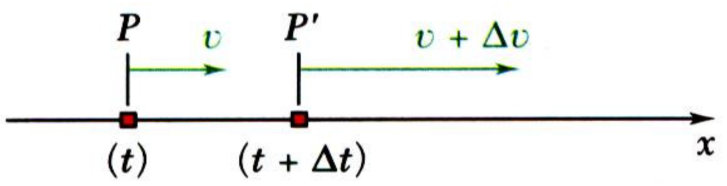
$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

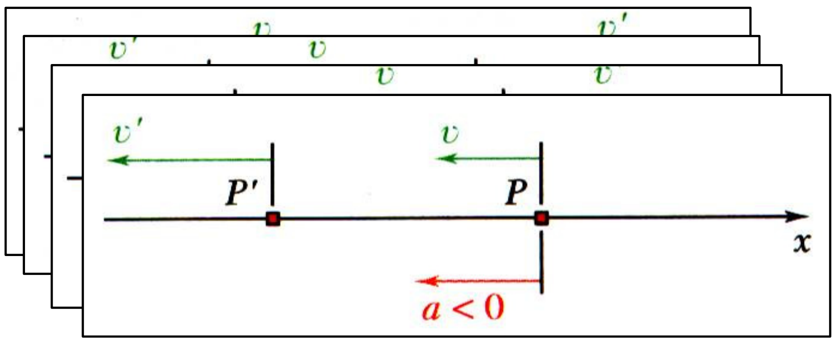
e.g., $x = 6t^2 - t^3$

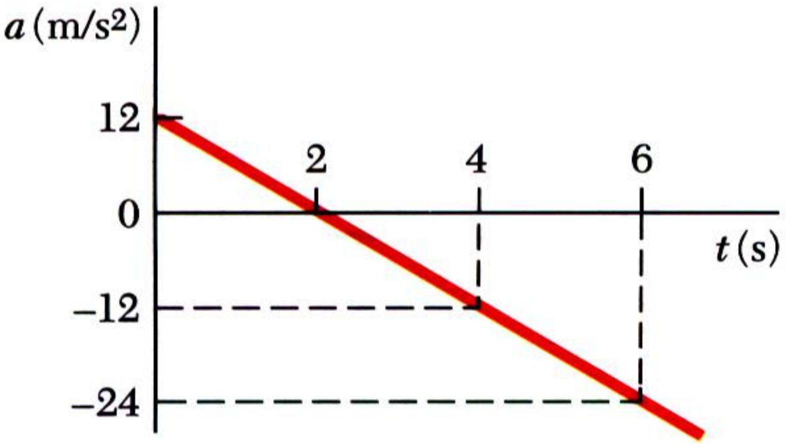
$$v = \frac{dx}{dt} = 12t - 3t^2$$



Rectilinear Motion: Position, Velocity & Acceleration







- Consider particle with velocity v at time t and v' at $t+\Delta t$,

$$\textit{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- Instantaneous acceleration may be:
  - positive: increasing positive velocity or decreasing negative velocity
  - negative: decreasing positive velocity or increasing negative velocity.

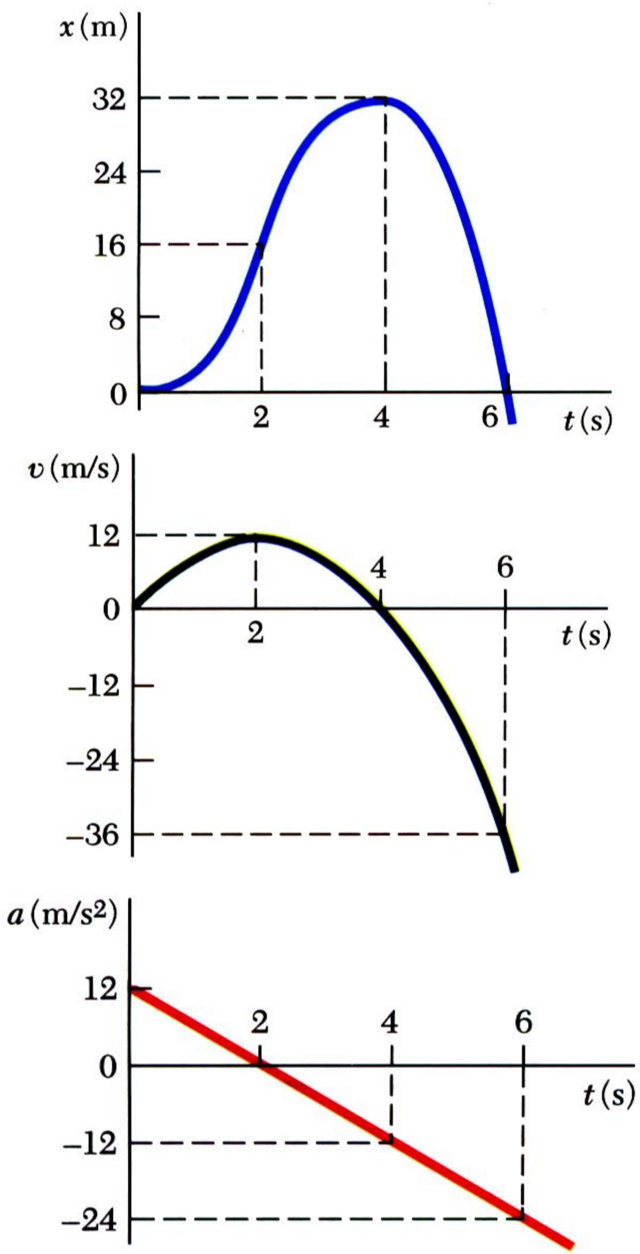
- From the definition of a derivative,

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

e.g. $v = 12t - 3t^2$

$$a = \frac{dv}{dt} = 12 - 6t$$

Rectilinear Motion: Position, Velocity & Acceleration



- Consider particle with motion given by

$$x = 6t^2 - t^3$$

$$v = \frac{dx}{dt} = 12t - 3t^2$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 12 - 6t$$

- at $t = 0$, $x = 0$, $v = 0$, $a = 12$ m/s^2
- at $t = 2$ s, $x = 16$ m, $v = v_{max} = 12$ m/s, $a = 0$
- at $t = 4$ s, $x = x_{max} = 32$ m, $v = 0$, $a = -12$ m/s^2
- at $t = 6$ s, $x = 0$, $v = -36$ m/s, $a = 24$ m/s^2

Determination of the Motion of a Particle

- Recall, *motion* of a particle is known if position is known for all time t.
- Typically, conditions of motion are specified by the type of acceleration experienced by the particle. Determination of velocity and position requires two successive integrations.
- Three classes of motion may be defined for:
  - acceleration given as a function of *time*, $a = f(t)$
  - acceleration given as a function of *position*, $a = f(x)$
  - acceleration given as a function of *velocity*, $a = f(v)$

Determination of the Motion of a Particle

- Acceleration given as a function of *time*, $a = f(t)$:

$$\frac{dv}{dt} = a = f(t) \qquad dv = f(t)\,dt \qquad \int_{v_0}^{v(t)} dv = \int_0^t f(t)\,dt \qquad v(t) - v_0 = \int_0^t f(t)\,dt$$

$$\frac{dx}{dt} = v(t) \qquad dx = v(t)\,dt \qquad \int_{x_0}^{x(t)} dx = \int_0^t v(t)\,dt \qquad x(t) - x_0 = \int_0^t v(t)\,dt$$

- Acceleration given as a function of *position*, $a = f(x)$:

$$v = \frac{dx}{dt} \text{ or } dt = \frac{dx}{v} \qquad a = \frac{dv}{dt} \text{ or } a = v\frac{dv}{dx} = f(x)$$

$$v\,dv = f(x)\,dx \qquad \int_{v_0}^{v(x)} v\,dv = \int_{x_0}^{x} f(x)\,dx \qquad \tfrac{1}{2}v(x)^2 - \tfrac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)\,dx$$

Determination of the Motion of a Particle

- Acceleration given as a function of velocity, $a = f(v)$:

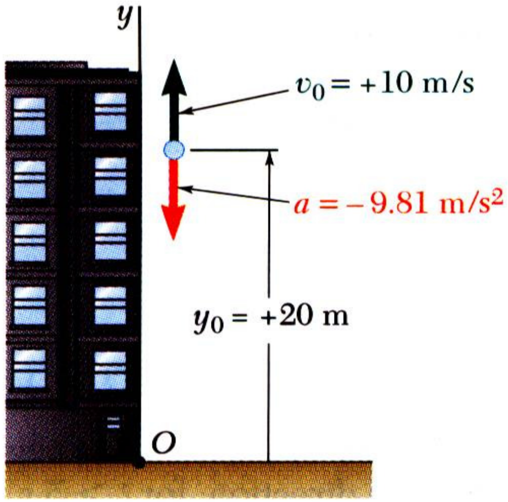
$$\frac{dv}{dt} = a = f(v) \qquad \frac{dv}{f(v)} = dt \qquad \int_{v_0}^{v(t)} \frac{dv}{f(v)} = \int_0^t dt$$

$$\int_{v_0}^{v(t)} \frac{dv}{f(v)} = t$$

$$v\frac{dv}{dx} = a = f(v) \qquad dx = \frac{v\,dv}{f(v)} \qquad \int_{x_0}^{x(t)} dx = \int_{v_0}^{v(t)} \frac{v\,dv}{f(v)}$$

$$x(t) - x_0 = \int_{v_0}^{v(t)} \frac{v\,dv}{f(v)}$$

Sample Problem 11.2



Ball tossed with 10 m/s vertical velocity from window 20 m above ground.

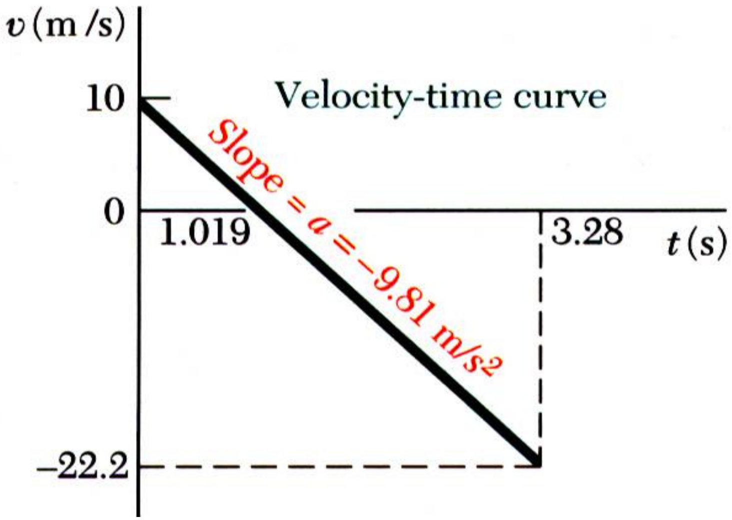
Determine:

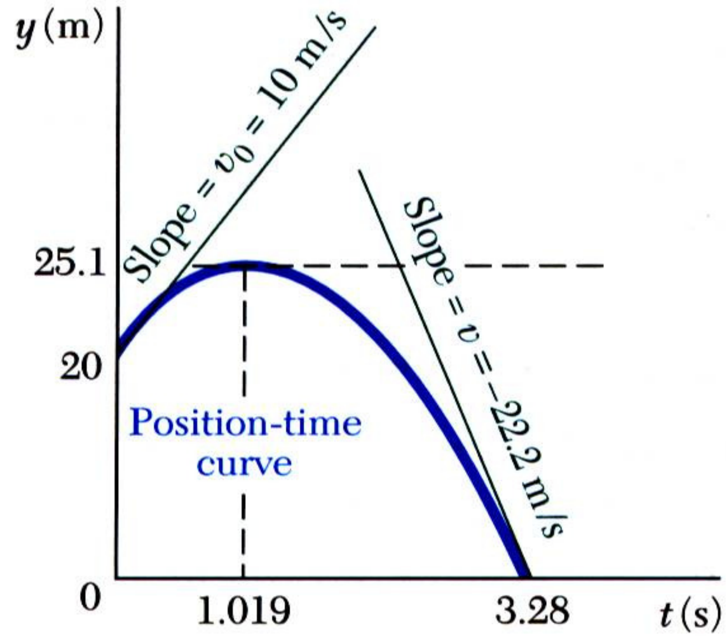
- velocity and elevation above ground at time *t*,
- highest elevation reached by ball and corresponding time, and
- time when ball will hit the ground and corresponding velocity.

SOLUTION:

- Integrate twice to find $v(t)$ and $y(t)$.
- Solve for t at which velocity equals zero (time for maximum elevation) and evaluate corresponding altitude.
- Solve for t at which altitude equals zero (time for ground impact) and evaluate corresponding velocity.

Sample Problem 11.2





SOLUTION:

- Integrate twice to find $v(t)$ and $y(t)$.

$$\frac{dv}{dt} = a = -9.81\,\mathrm{m/s^2}$$

$$\int_{v_0}^{v(t)} dv = -\int_0^t 9.81\,dt \qquad v(t) - v_0 = -9.81t$$

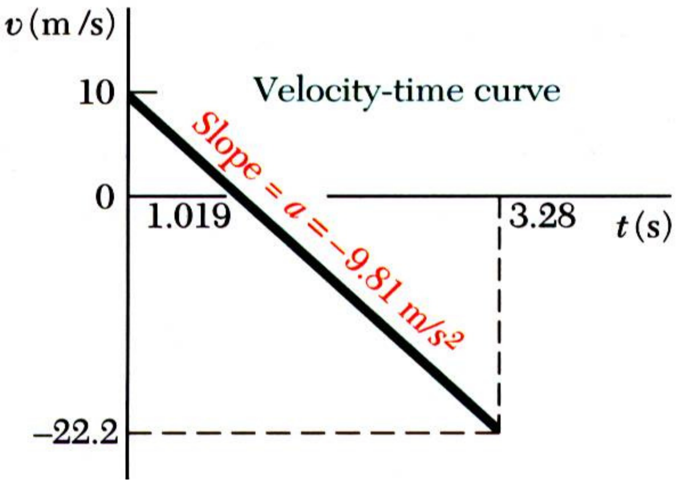
$$v(t) = 10\,\frac{\mathrm{m}}{\mathrm{s}} - \left(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t$$

$$\frac{dy}{dt} = v = 10 - 9.81t$$

$$\int_{y_0}^{y(t)} dy = \int_0^t (10 - 9.81t)\,dt \qquad y(t) - y_0 = 10t - \tfrac{1}{2}9.81t^2$$

$$y(t) = 20\,\mathrm{m} + \left(10\,\frac{\mathrm{m}}{\mathrm{s}}\right)t - \left(4.905\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2$$

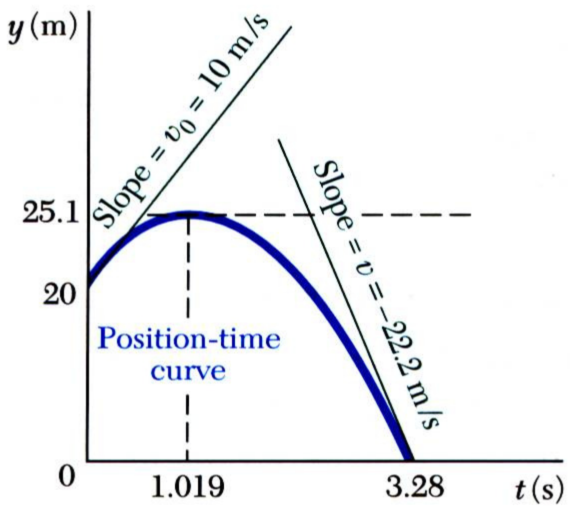
Sample Problem 11.2



- Solve for t at which velocity equals zero and evaluate corresponding altitude.

$$v(t) = 10\,\frac{\text{m}}{\text{s}} - \left(9.81\,\frac{\text{m}}{\text{s}^2}\right)t = 0$$

$$\boxed{t = 1.019\ \text{s}}$$



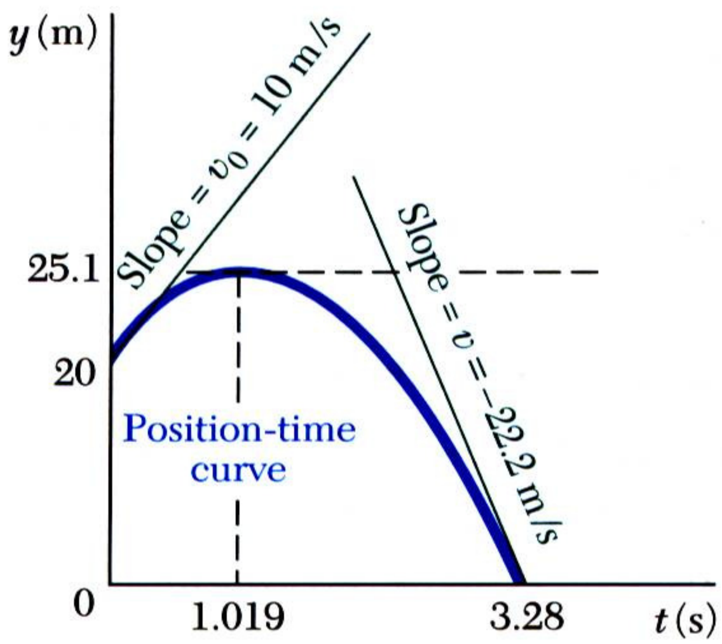
- Solve for t at which altitude equals zero and evaluate corresponding velocity.

$$y(t) = 20\ \text{m} + \left(10\,\frac{\text{m}}{\text{s}}\right)t - \left(4.905\,\frac{\text{m}}{\text{s}^2}\right)t^2$$

$$y = 20\ \text{m} + \left(10\,\frac{\text{m}}{\text{s}}\right)(1.019\ \text{s}) - \left(4.905\,\frac{\text{m}}{\text{s}^2}\right)(1.019\ \text{s})^2$$

$$\boxed{y = 25.1\ \text{m}}$$

Sample Problem 11.2



- Solve for t at which altitude equals zero and evaluate corresponding velocity.

$$y(t) = 20\,\text{m} + \left(10\,\frac{\text{m}}{\text{s}}\right)t - \left(4.905\,\frac{\text{m}}{\text{s}^2}\right)t^2 = 0$$

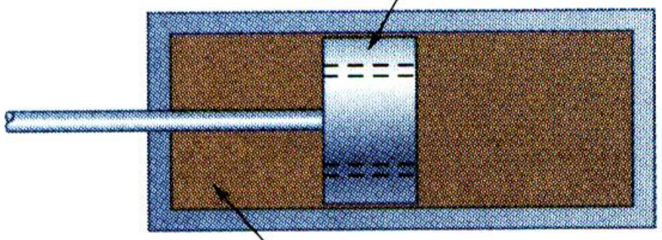
$$t = -1.243\,\text{s}\ \ (\text{meaningles s})$$
$$t = 3.28\,\text{s}$$

$$v(t) = 10\,\frac{\text{m}}{\text{s}} - \left(9.81\,\frac{\text{m}}{\text{s}^2}\right)t$$

$$v(3.28\,\text{s}) = 10\,\frac{\text{m}}{\text{s}} - \left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(3.28\,\text{s})$$

$$v = -22.2\,\frac{\text{m}}{\text{s}}$$

Sample Problem 11.3



$a = -kv$

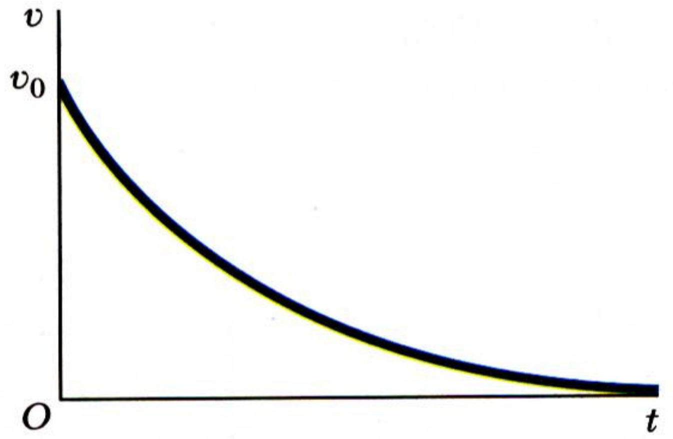
Brake mechanism used to reduce gun recoil consists of piston attached to barrel moving in fixed cylinder filled with oil. As barrel recoils with initial velocity v_0, piston moves and oil is forced through orifices in piston, causing piston and cylinder to decelerate at rate proportional to their velocity.

Determine $v(t)$, $x(t)$, and $v(x)$.

SOLUTION:

- Integrate $a = dv/dt = -kv$ to find $v(t)$.
- Integrate $v(t) = dx/dt$ to find $x(t)$.
- Integrate $a = v\, dv/dx = -kv$ to find $v(x)$.

Sample Problem 11.3

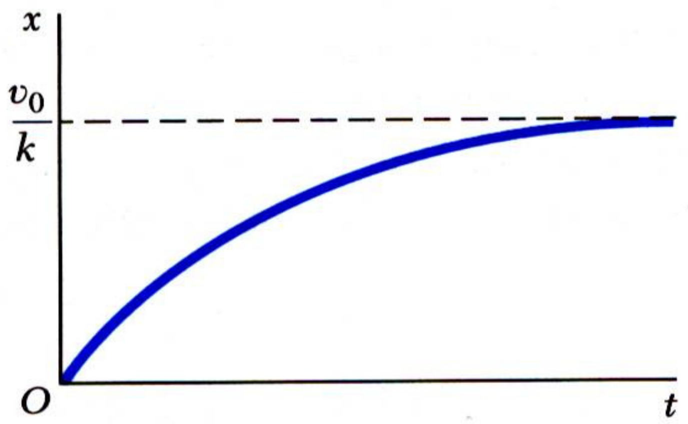


SOLUTION:

- Integrate $a = dv/dt = -kv$ to find $v(t)$.

$$a = \frac{dv}{dt} = -kv \qquad \int_{v_0}^{v(t)} \frac{dv}{v} = -k\int_0^t dt \qquad \ln\frac{v(t)}{v_0} = -kt$$

$$\boxed{v(t) = v_0 e^{-kt}}$$



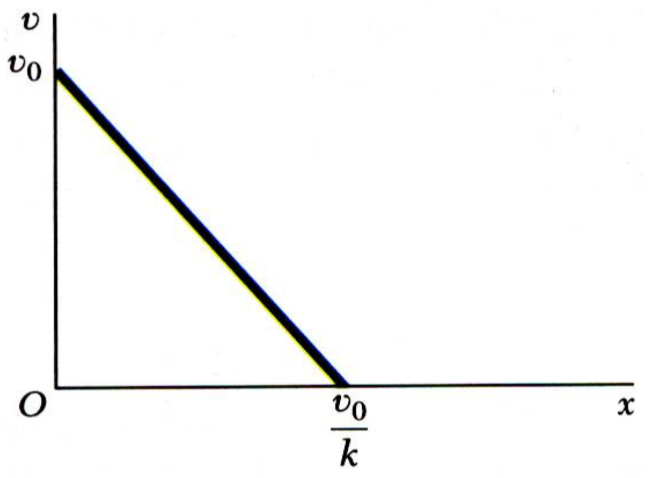
- Integrate $v(t) = dx/dt$ to find $x(t)$.

$$v(t) = \frac{dx}{dt} = v_0 e^{-kt}$$

$$\int_0^{x(t)} dx = v_0 \int_0^t e^{-kt}\, dt \qquad x(t) = v_0\left[-\frac{1}{k}e^{-kt}\right]_0^t$$

$$\boxed{x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)}$$

Sample Problem 11.3



- Integrate $a = v\,dv/dx = -kv$ to find $v(x)$.

$$a = v\frac{dv}{dx} = -kv \qquad dv = -k\,dx \qquad \int_{v_0}^{v} dv = -k\int_0^x dx$$

$$v - v_0 = -kx$$

$$\boxed{v = v_0 - kx}$$

- Alternatively,

with $\quad x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)$

and $\quad v(t) = v_0 e^{-kt} \quad$ or $\quad e^{-kt} = \frac{v(t)}{v_0}$

then $\quad x(t) = \frac{v_0}{k}\left(1 - \frac{v(t)}{v_0}\right)$

$$\boxed{v = v_0 - kx}$$

Uniform Rectilinear Motion

For particle in uniform rectilinear motion, the acceleration is zero and the velocity is constant.

$$\frac{dx}{dt} = v = \text{constant}$$

$$\int_{x_0}^{x} dx = v \int_{0}^{t} dt$$

$$x - x_0 = vt$$

$$x = x_0 + vt$$

Uniformly Accelerated Rectilinear Motion

For particle in uniformly accelerated rectilinear motion, the acceleration of the particle is constant.

$$\frac{dv}{dt} = a = \text{constant} \qquad \int_{v_0}^{v} dv = a\int_0^t dt \qquad v - v_0 = at$$

$$v = v_0 + at$$

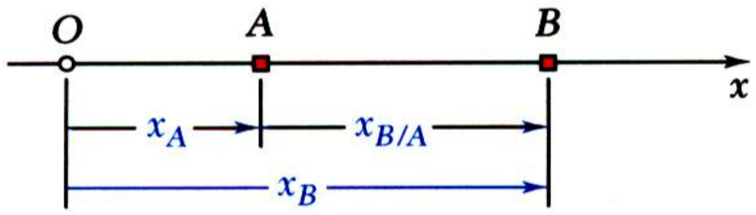
$$\frac{dx}{dt} = v_0 + at \qquad \int_{x_0}^{x} dx = \int_0^t (v_0 + at)dt \qquad x - x_0 = v_0 t + \tfrac{1}{2}at^2$$

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2$$

$$v\frac{dv}{dx} = a = \text{constant} \qquad \int_{v_0}^{v} v\,dv = a\int_{x_0}^{x} dx \qquad \tfrac{1}{2}\left(v^2 - v_0^2\right) = a(x - x_0)$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

Motion of Several Particles: Relative Motion



- For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.

$x_{B/A} = x_B - x_A =$ relative position of B with respect to A

$$x_B = x_A + x_{B/A}$$

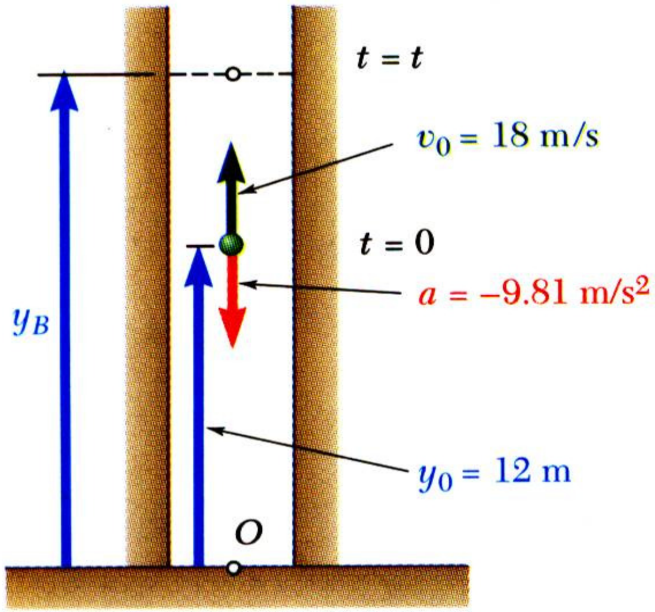
$v_{B/A} = v_B - v_A =$ relative velocity of B with respect to A

$$v_B = v_A + v_{B/A}$$

$a_{B/A} = a_B - a_A =$ relative acceleration of B with respect to A

$$a_B = a_A + a_{B/A}$$

Sample Problem 11.4



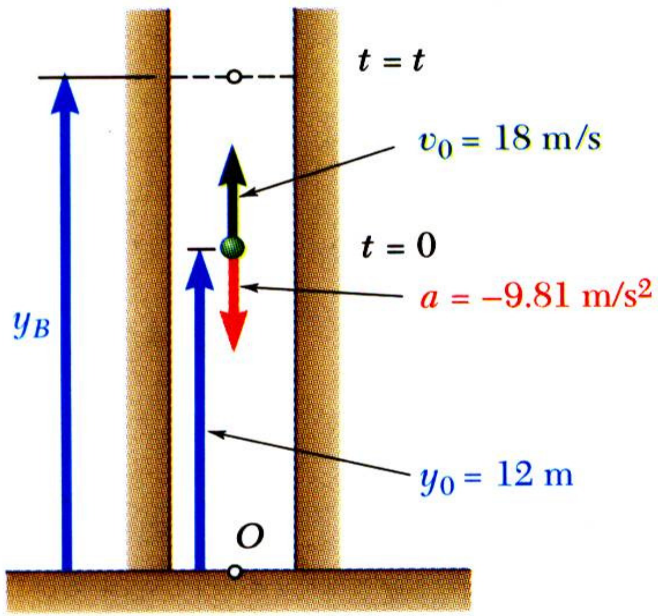
Ball thrown vertically from 12 m level in elevator shaft with initial velocity of 18 m/s. At same instant, open-platform elevator passes 5 m level moving upward at 2 m/s.

Determine *(a)* when and where ball hits elevator and *(b)* relative velocity of ball and elevator at contact.

SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.
- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.
- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.
- Substitute impact time into equation for position of elevator and relative velocity of ball with respect to elevator.

Sample Problem 11.4

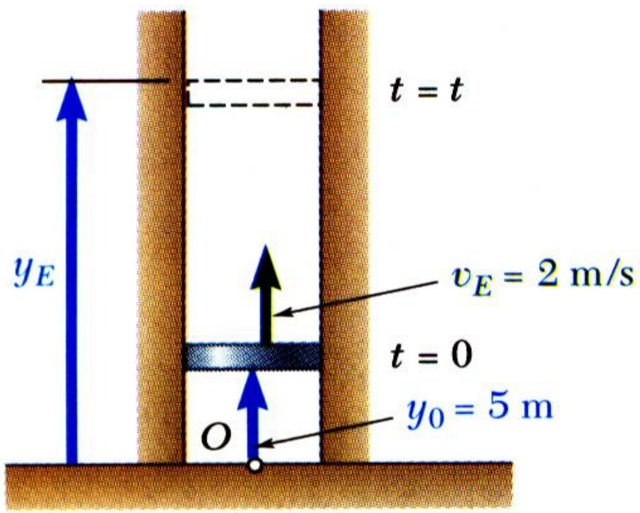


SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

$$v_B = v_0 + at = 18\,\frac{\text{m}}{\text{s}} - \left(9.81\,\frac{\text{m}}{\text{s}^2}\right)t$$

$$y_B = y_0 + v_0 t + \tfrac{1}{2}at^2 = 12\,\text{m} + \left(18\,\frac{\text{m}}{\text{s}}\right)t - \left(4.905\,\frac{\text{m}}{\text{s}^2}\right)t^2$$

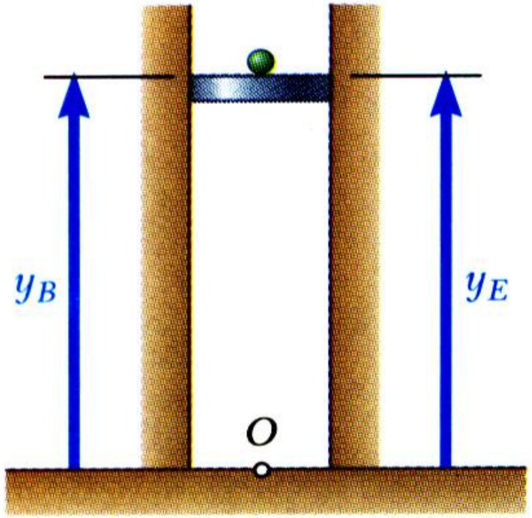


- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

$$v_E = 2\,\frac{\text{m}}{\text{s}}$$

$$y_E = y_0 + v_E t = 5\,\text{m} + \left(2\,\frac{\text{m}}{\text{s}}\right)t$$

Sample Problem 11.4



- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

$$y_{B/E} = \left(12 + 18t - 4.905t^2\right) - (5 + 2t) = 0$$

$$t = -0.39\text{ s} \quad (\text{meaningles s})$$
$$t = 3.65\text{ s}$$

- Substitute impact time into equations for position of elevator and relative velocity of ball with respect to elevator.

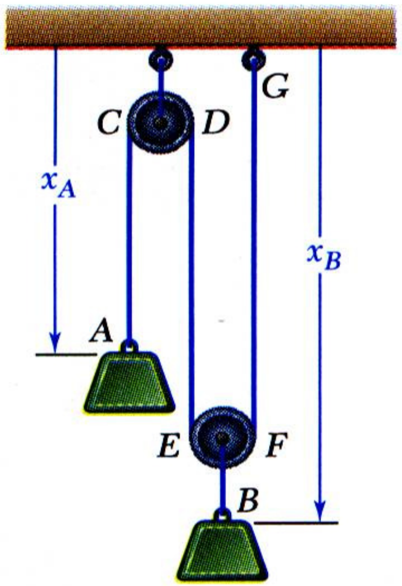
$$y_E = 5 + 2(3.65)$$

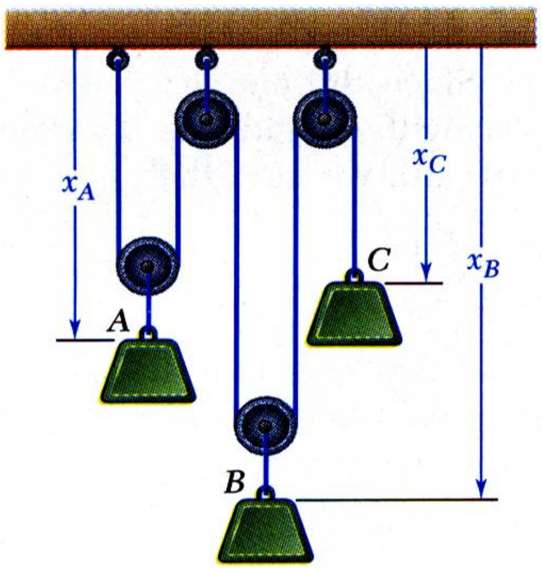
$$y_E = 12.3\text{ m}$$

$$v_{B/E} = (18 - 9.81t) - 2$$
$$= 16 - 9.81(3.65)$$

$$v_{B/E} = -19.81\ \frac{\text{m}}{\text{s}}$$

Motion of Several Particles: Dependent Motion





- Position of a particle may *depend* on position of one or more other particles.
- Position of block *B* depends on position of block *A*. Since rope is of constant length, it follows that sum of lengths of segments must be constant.

$$x_A + 2x_B = \text{constant} \quad \text{(one degree of freedom)}$$

- Positions of three blocks are dependent.

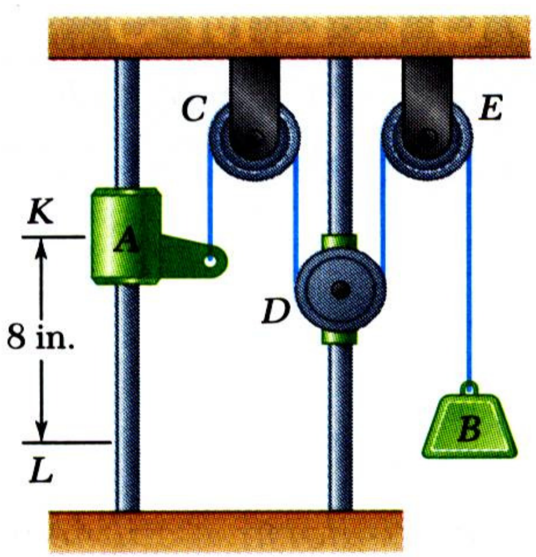
$$2x_A + 2x_B + x_C = \text{constant (two degrees of freedom)}$$

- For linearly related positions, similar relations hold between velocities and accelerations.

$$2\frac{dx_A}{dt} + 2\frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \quad \text{or} \quad 2v_A + 2v_B + v_C = 0$$

$$2\frac{dv_A}{dt} + 2\frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \quad \text{or} \quad 2a_A + 2a_B + a_C = 0$$

Sample Problem 11.5

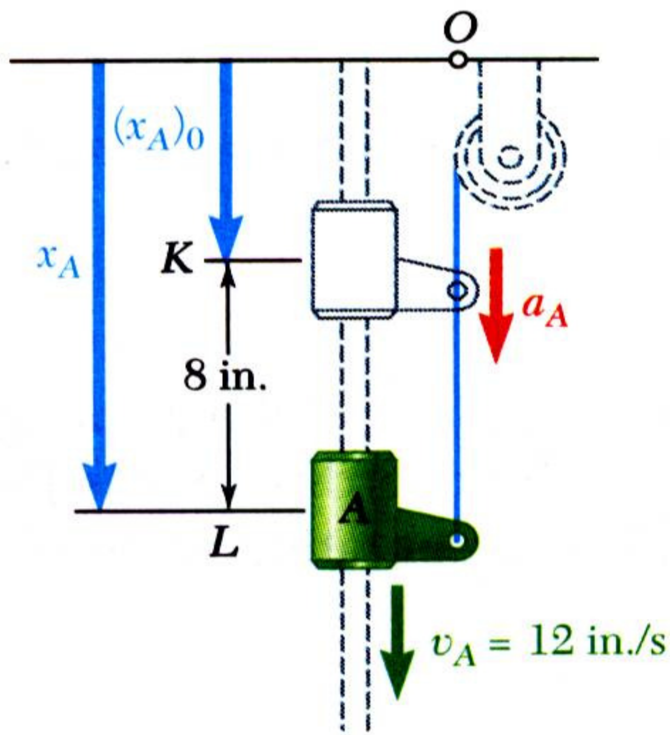


Pulley *D* is attached to a collar which is pulled down at 3 in./s. At $t = 0$, collar *A* starts moving down from *K* with constant acceleration and zero initial velocity. Knowing that velocity of collar *A* is 12 in./s as it passes *L*, determine the change in elevation, velocity, and acceleration of block *B* when block *A* is at *L*.

SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.
- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.
- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.
- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.
- Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.

Sample Problem 11.5



SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.

- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.

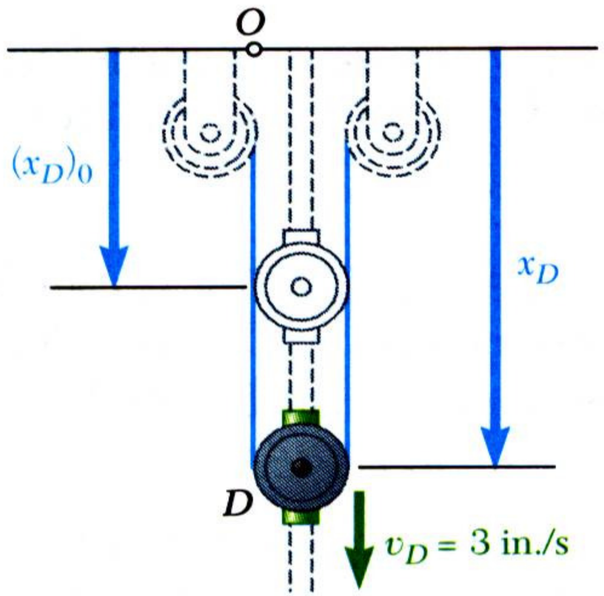
$$v_A^2 = (v_A)_0^2 + 2a_A\left[x_A - (x_A)_0\right]$$

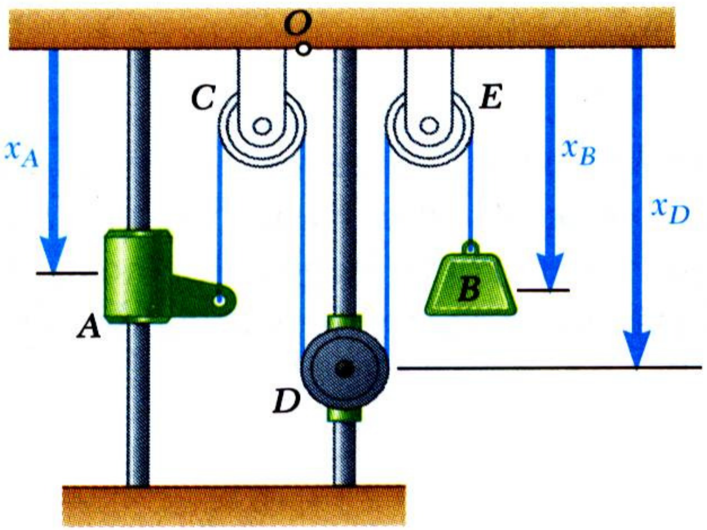
$$\left(12\,\frac{\text{in.}}{\text{s}}\right)^2 = 2a_A(8\,\text{in.}) \qquad a_A = 9\,\frac{\text{in.}}{\text{s}^2}$$

$$v_A = (v_A)_0 + a_A t$$

$$12\,\frac{\text{in.}}{\text{s}} = 9\,\frac{\text{in.}}{\text{s}^2}\,t \qquad t = 1.333\,\text{s}$$

Sample Problem 11.5





- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.

$$x_D = (x_D)_0 + v_D t$$

$$x_D - (x_D)_0 = \left(3\,\frac{\text{in.}}{\text{s}}\right)(1.333\text{ s}) = 4\text{ in.}$$

- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.

Total length of cable remains constant,

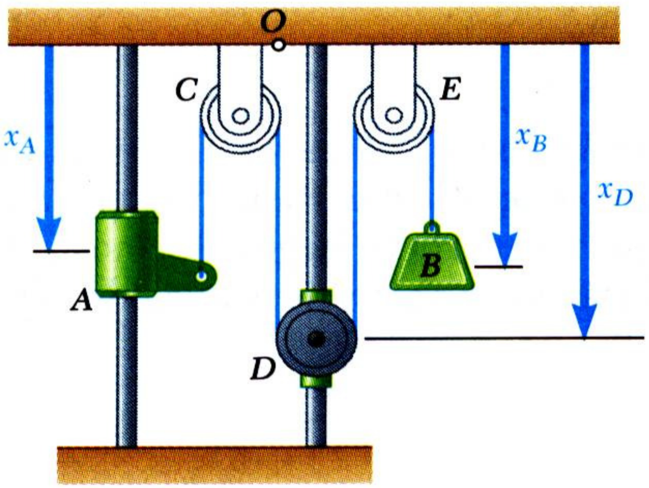
$$x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0$$

$$\left[x_A - (x_A)_0\right] + 2\left[x_D - (x_D)_0\right] + \left[x_B - (x_B)_0\right] = 0$$

$$(8\text{ in.}) + 2(4\text{ in.}) + \left[x_B - (x_B)_0\right] = 0$$

$$x_B - (x_B)_0 = -16\text{ in.}$$

Sample Problem 11.5



- Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.

$$x_A + 2x_D + x_B = \text{constant}$$

$$v_A + 2v_D + v_B = 0$$

$$\left(12\frac{\text{in.}}{\text{s}}\right) + 2\left(3\frac{\text{in.}}{\text{s}}\right) + v_B = 0$$

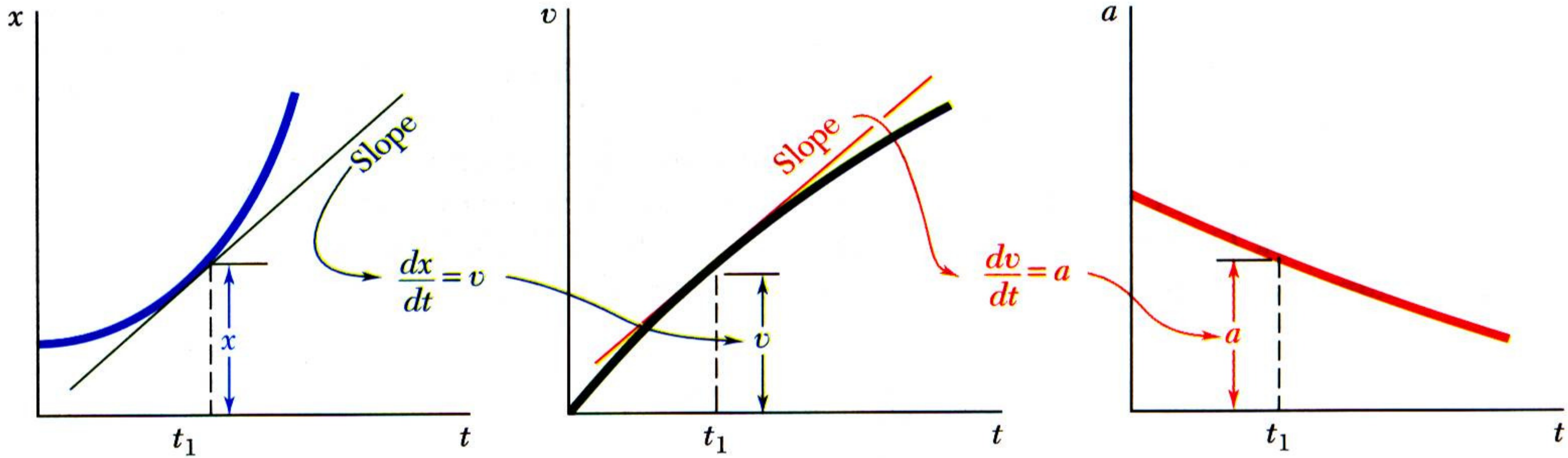
$$v_B = 18\frac{\text{in.}}{\text{s}}$$

$$a_A + 2a_D + a_B = 0$$

$$\left(9\frac{\text{in.}}{\text{s}^2}\right) + v_B = 0$$

$$a_B = -9\frac{\text{in.}}{\text{s}^2}$$

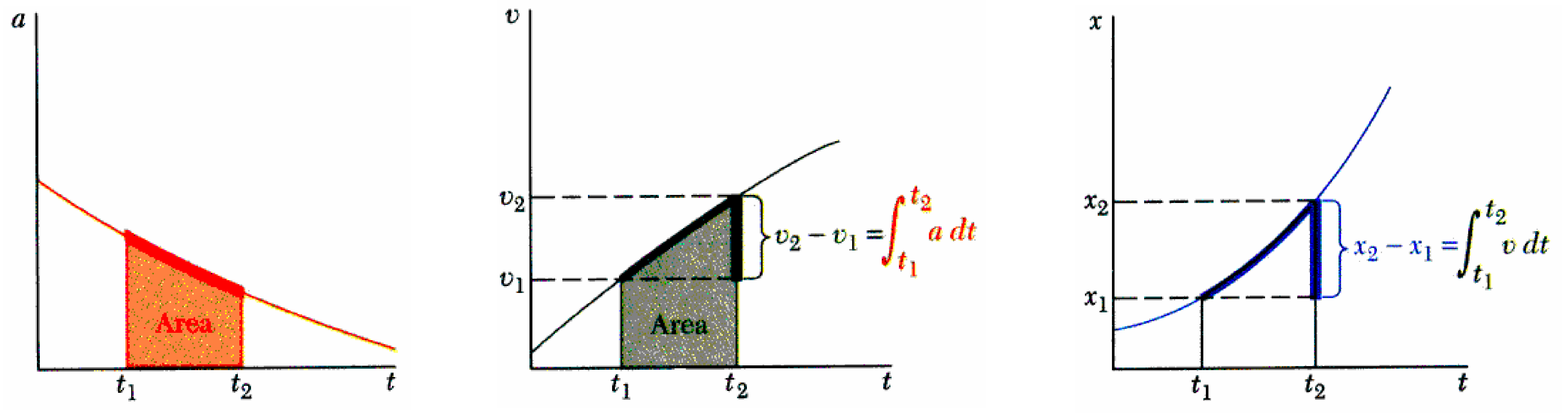
Graphical Solution of Rectilinear-Motion Problems



- Given the *x*-*t* curve, the *v*-*t* curve is equal to the *x*-*t* curve slope.

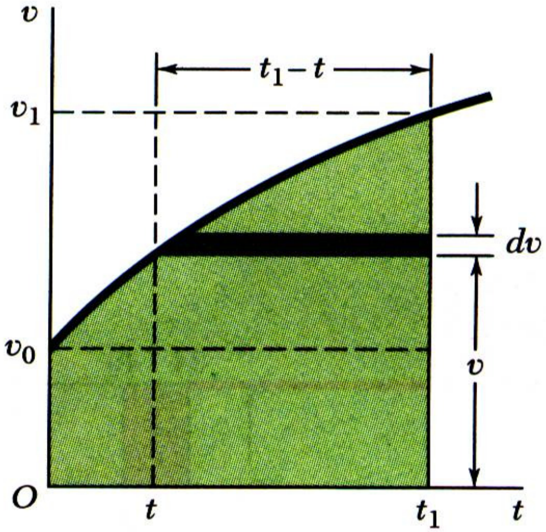
- Given the *v*-*t* curve, the *a*-*t* curve is equal to the *v*-*t* curve slope.

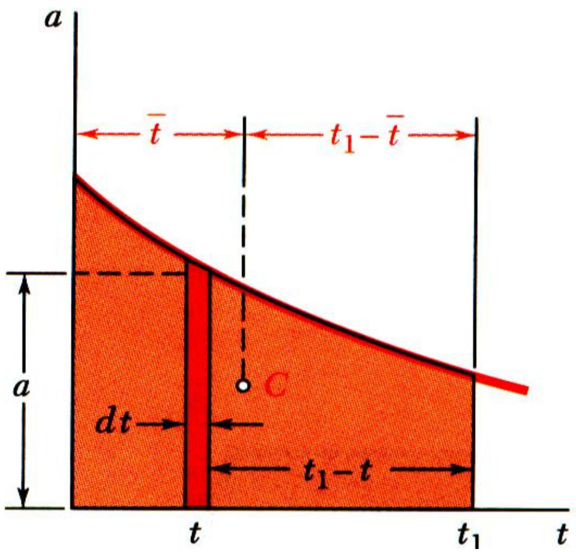
Graphical Solution of Rectilinear-Motion Problems



- Given the *a*-*t* curve, the change in velocity between t_1 and t_2 is equal to the area under the *a*-*t* curve between t_1 and t_2.
- Given the *v*-*t* curve, the change in position between t_1 and t_2 is equal to the area under the *v*-*t* curve between t_1 and t_2.

Other Graphical Methods





- *Moment-area method* to determine particle position at time t directly from the a-t curve:

$$x_1 - x_0 = \text{area under } v - t \text{ curve}$$

$$= v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t)\, dv$$

using $dv = a\,dt$,

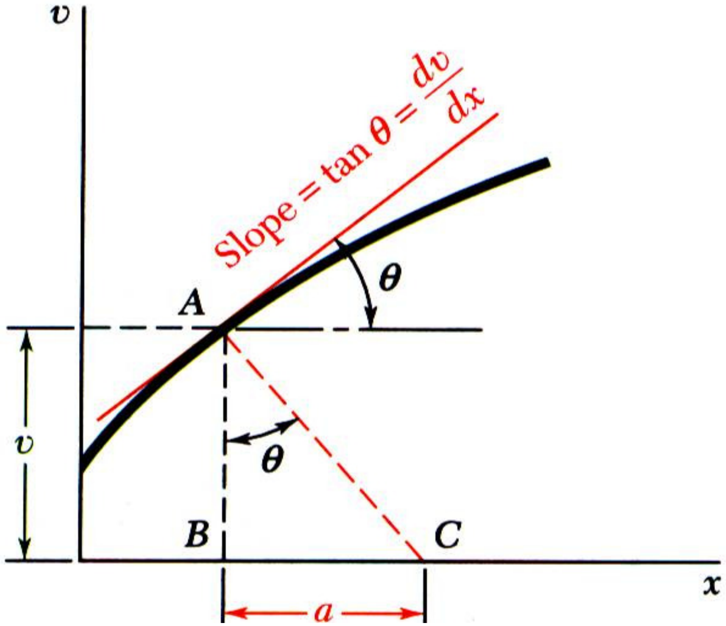
$$x_1 - x_0 = v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t)\, a\, dt$$

$$\int_{v_0}^{v_1} (t_1 - t)\, a\, dt = \text{first moment of area under } a\text{-}t \text{ curve with respect to } t = t_1 \text{ line.}$$

$$x_1 = x_0 + v_0 t_1 + (\text{area under } a\text{-}t \text{ curve})(t_1 - \bar{t})$$

$$\bar{t} = \text{abscissa of centroid } C$$

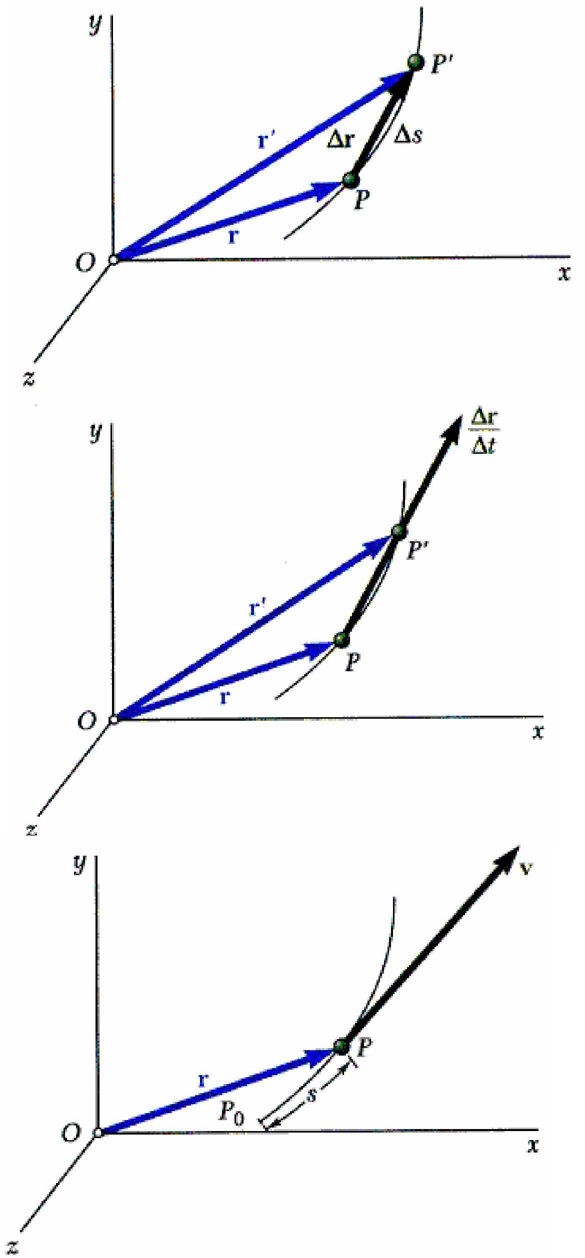
Other Graphical Methods



- Method to determine particle acceleration from *v-x* curve:

$$a = v\frac{dv}{dx}$$
$$= AB\tan\theta$$
$$= BC = \textit{subnormal} \text{ to } v\text{-}x \text{ curve}$$

Curvilinear Motion: Position, Velocity & Acceleration



- Particle moving along a curve other than a straight line is in *curvilinear motion*.

- *Position vector* of a particle at time *t* is defined by a vector between origin *O* of a fixed reference frame and the position occupied by particle.

- Consider particle which occupies position *P* defined by $\vec{r}$ at time *t* and *P'* defined by $\vec{r}'$ at $t + \Delta t$,

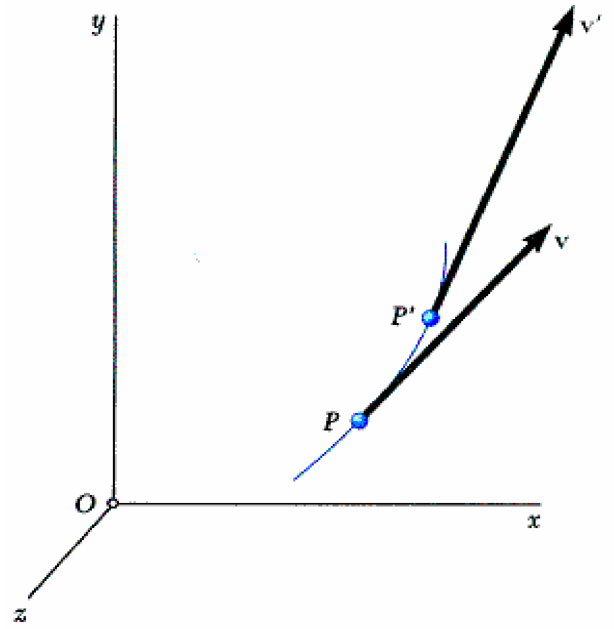
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

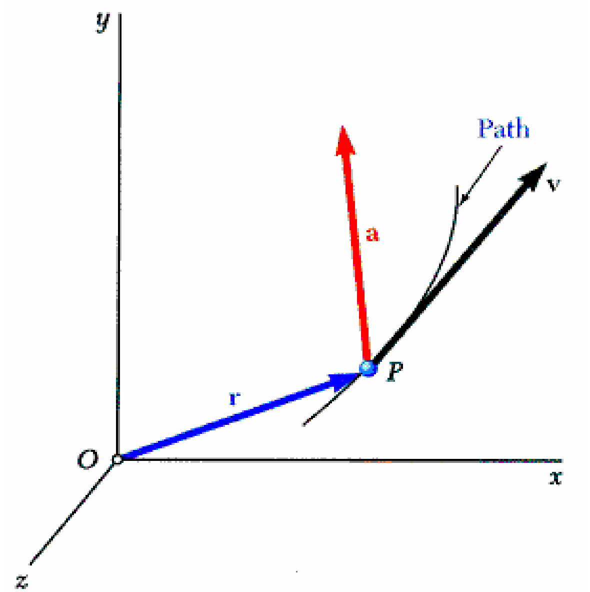
= instantaneous velocity (vector)

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

= instantaneous speed (scalar)

Curvilinear Motion: Position, Velocity & Acceleration





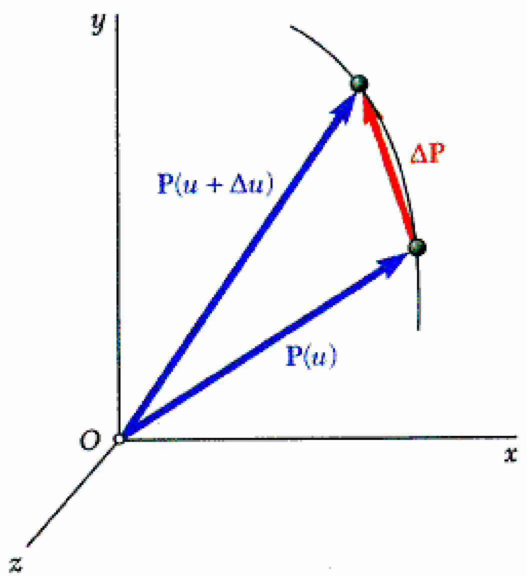
- Consider velocity $\vec{v}$ of particle at time t and velocity $\vec{v}'$ at $t + \Delta t$,

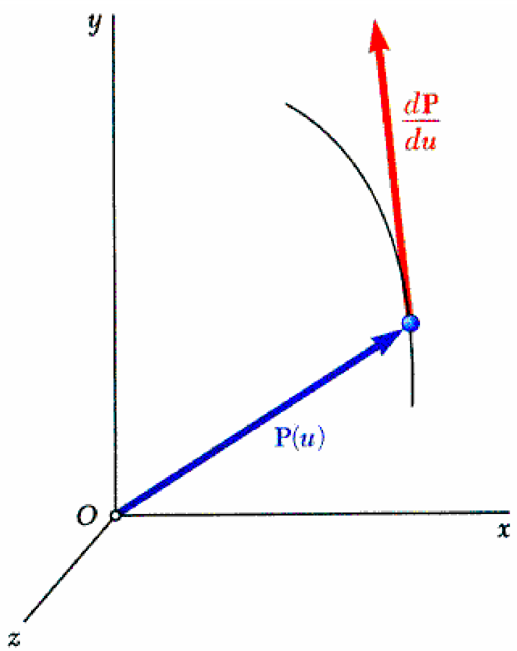
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$= \text{instantaneous acceleration (vector)}$$

- In general, acceleration vector is not tangent to particle path and velocity vector.

Derivatives of Vector Functions





- Let $\vec{P}(u)$ be a vector function of scalar variable u,

$$\frac{d\vec{P}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \vec{P}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{P}(u + \Delta u) - \vec{P}(u)}{\Delta u}$$

- Derivative of vector sum,

$$\frac{d\left(\vec{P} + \vec{Q}\right)}{du} = \frac{d\vec{P}}{du} + \frac{d\vec{Q}}{du}$$

- Derivative of product of scalar and vector functions,

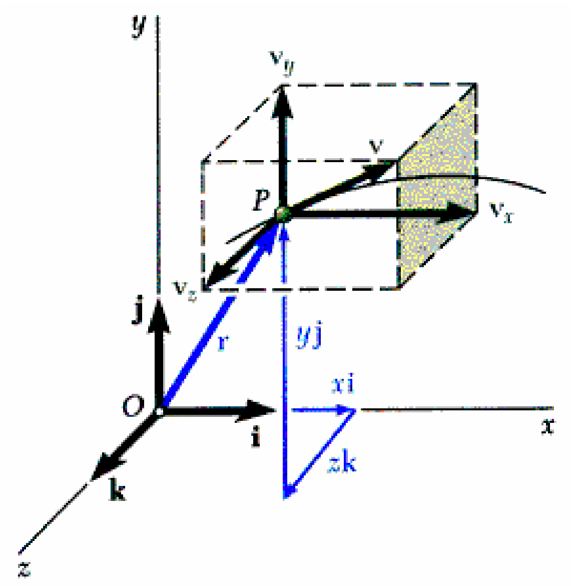
$$\frac{d\left(f\vec{P}\right)}{du} = \frac{df}{du}\vec{P} + f\frac{d\vec{P}}{du}$$

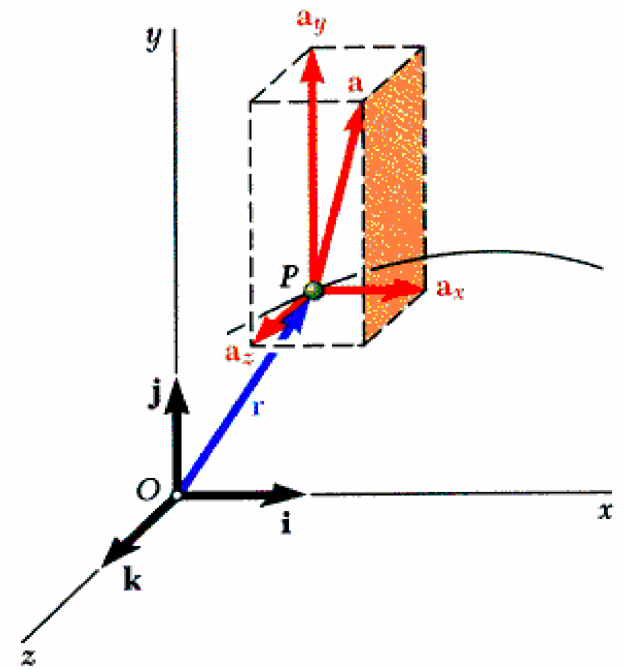
- Derivative of *scalar product* and *vector product*,

$$\frac{d\left(\vec{P} \bullet \vec{Q}\right)}{du} = \frac{d\vec{P}}{du} \bullet \vec{Q} + \vec{P} \bullet \frac{d\vec{Q}}{du}$$

$$\frac{d\left(\vec{P} \times \vec{Q}\right)}{du} = \frac{d\vec{P}}{du} \times \vec{Q} + \vec{P} \times \frac{d\vec{Q}}{du}$$

Rectangular Components of Velocity & Acceleration





- When position vector of particle *P* is given by its rectangular components,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

- Velocity vector,

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

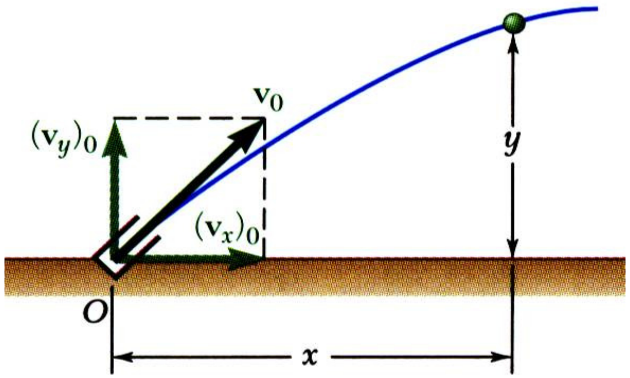
$$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

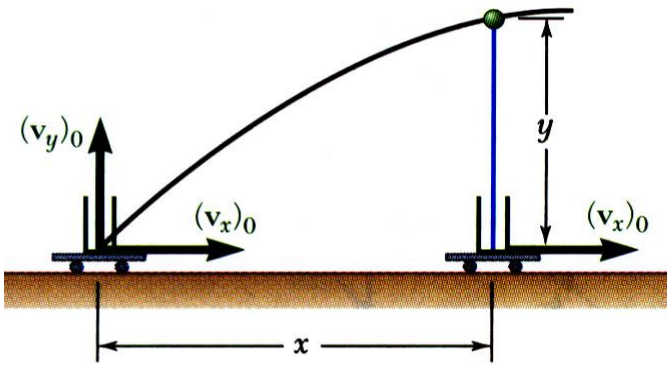
- Acceleration vector,

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

Rectangular Components of Velocity & Acceleration





- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$

with initial conditions,

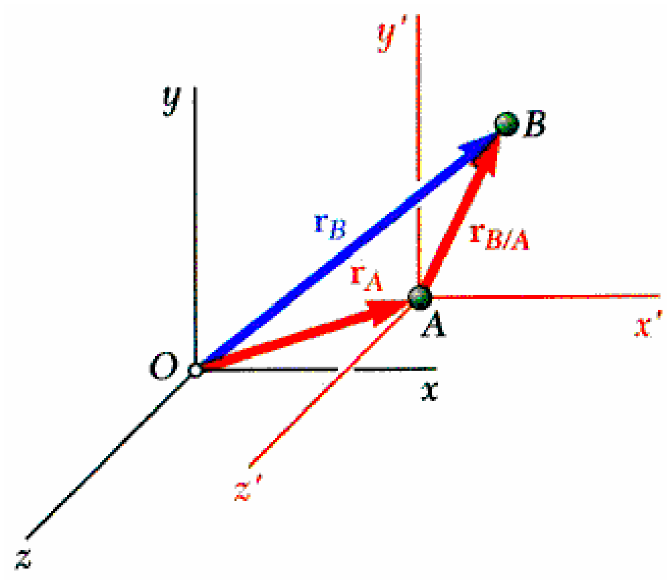
$$x_0 = y_0 = z_0 = 0 \qquad (v_x)_0, (v_y)_0, (v_z)_0 = 0$$

Integrating twice yields

$$v_x = (v_x)_0 \quad v_y = (v_y)_0 - gt \qquad v_z = 0$$
$$x = (v_x)_0 t \quad y = (v_y)_0 y - \tfrac{1}{2} gt^2 \quad z = 0$$

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.

Motion Relative to a Frame in Translation



- Designate one frame as the *fixed frame of reference*. All other frames not rigidly attached to the fixed reference frame are *moving frames of reference*.
- Position vectors for particles *A* and *B* with respect to the fixed frame of reference *Oxyz* are $\vec{r}_A$ and $\vec{r}_B$.
- Vector $\vec{r}_{B/A}$ joining *A* and *B* defines the position of *B* with respect to the moving frame *Ax'y'z'* and

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

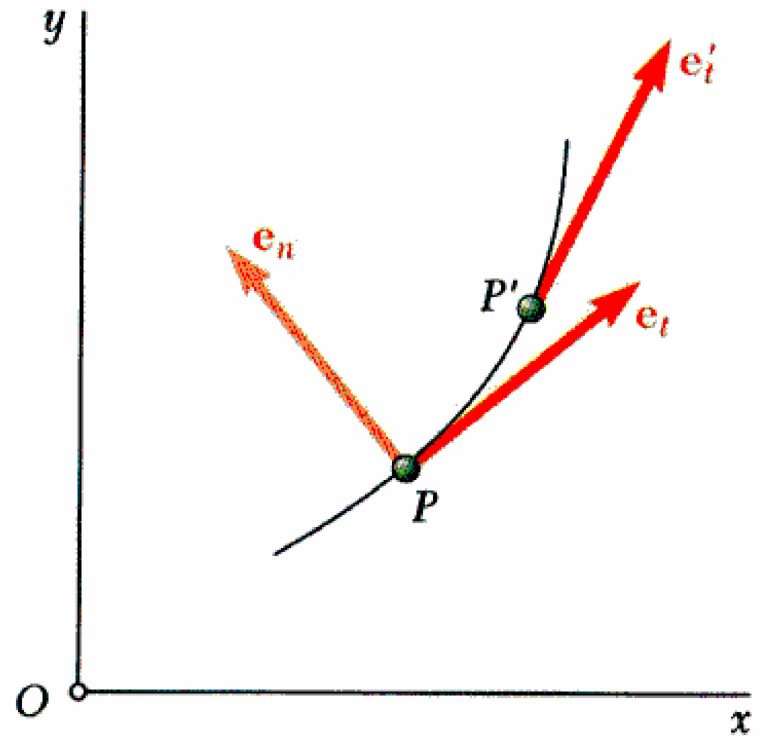
- Differentiating twice,

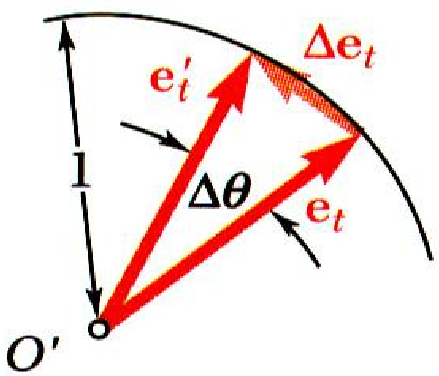
$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad \vec{v}_{B/A} = \text{velocity of } B \text{ relative to } A.$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \qquad \vec{a}_{B/A} = \text{acceleration of } B \text{ relative to } A.$$

- Absolute motion of *B* can be obtained by combining motion of *A* with relative motion of *B* with respect to moving reference frame attached to *A*.

Tangential and Normal Components





- Velocity vector of particle is tangent to path of particle. In general, acceleration vector is not. Wish to express acceleration vector in terms of tangential and normal components.

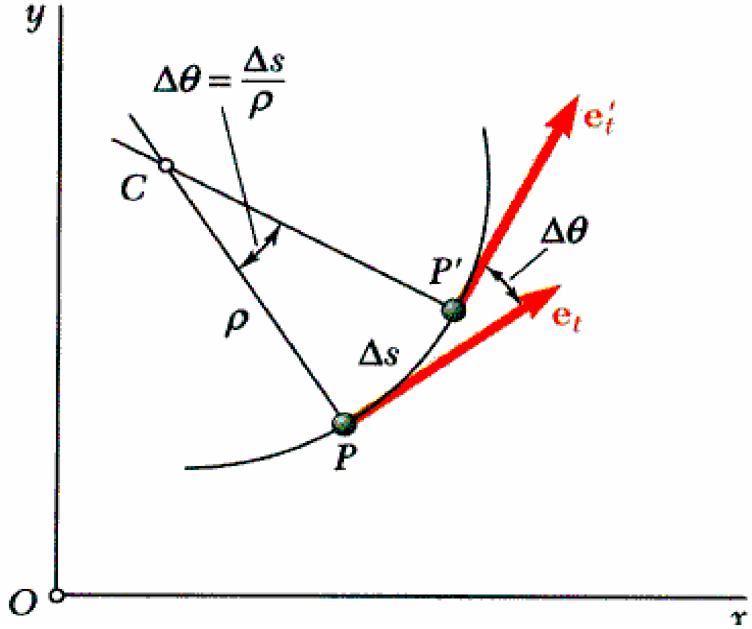
- $\vec{e}_t$ and $\vec{e}_t'$ are tangential unit vectors for the particle path at *P* and *P'*. When drawn with respect to the same origin, $\Delta\vec{e}_t = \vec{e}_t' - \vec{e}_t$ and $\Delta\theta$ is the angle between them.

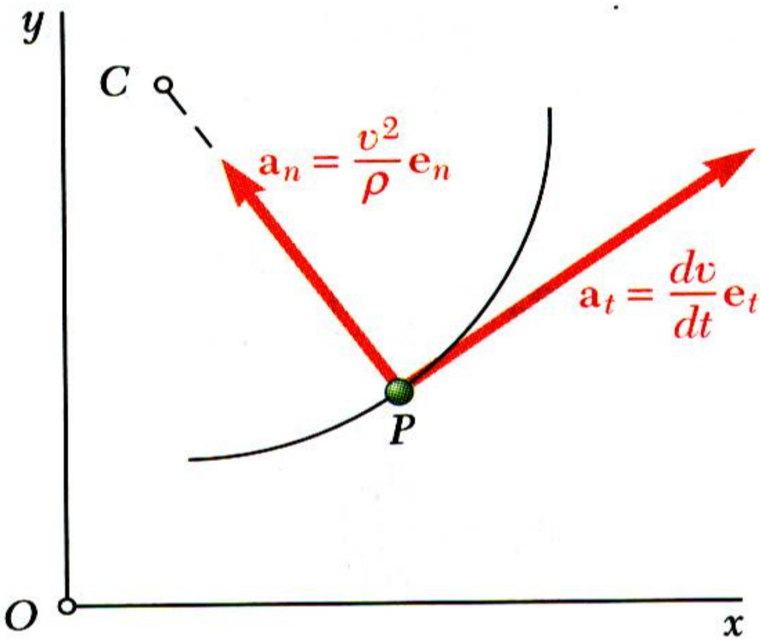
$$\Delta e_t = 2\sin(\Delta\theta/2)$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\vec{e}_t}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\vec{e}_n = \vec{e}_n$$

$$\vec{e}_n = \frac{d\vec{e}_t}{d\theta}$$

Tangential and Normal Components





- With the velocity vector expressed as $\vec{v} = v\vec{e}_t$ the particle acceleration may be written as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

but

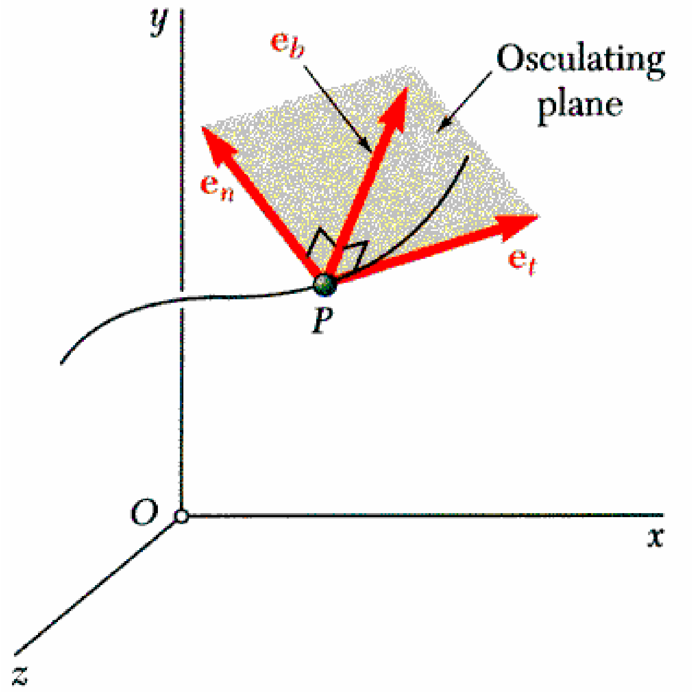
$$\frac{d\vec{e}_t}{d\theta} = \vec{e}_n \qquad \rho\, d\theta = ds \qquad \frac{ds}{dt} = v$$

After substituting,

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$

- Tangential component of acceleration reflects change of speed and normal component reflects change of direction.
- Tangential component may be positive or negative. Normal component always points toward center of path curvature.

Tangential and Normal Components



- Relations for tangential and normal acceleration also apply for particle moving along space curve.

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$

- Plane containing tangential and normal unit vectors is called the *osculating plane*.

- Normal to the osculating plane is found from

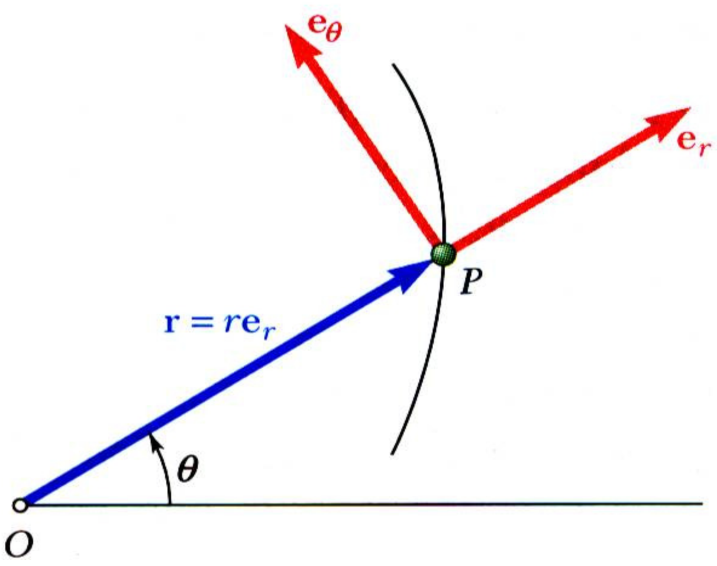
$$\vec{e}_b = \vec{e}_t \times \vec{e}_n$$

$$\vec{e}_n = principal \;\; normal$$

$$\vec{e}_b = binormal$$

- Acceleration has no component along binormal.

Radial and Transverse Components



$$\vec{r} = r\vec{e}_r$$

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad \frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$$

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \vec{e}_\theta \frac{d\theta}{dt}$$

$$\frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\vec{e}_r \frac{d\theta}{dt}$$

- When particle position is given in polar coordinates, it is convenient to express velocity and acceleration with components parallel and perpendicular to *OP*.

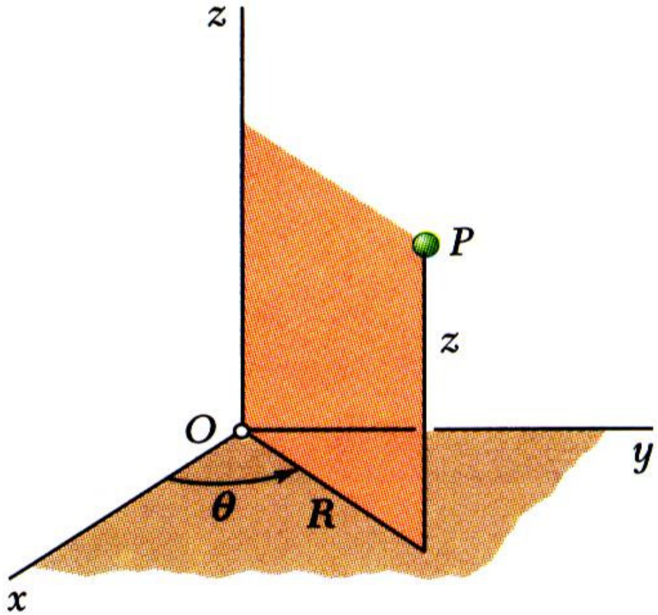
- The particle velocity vector is

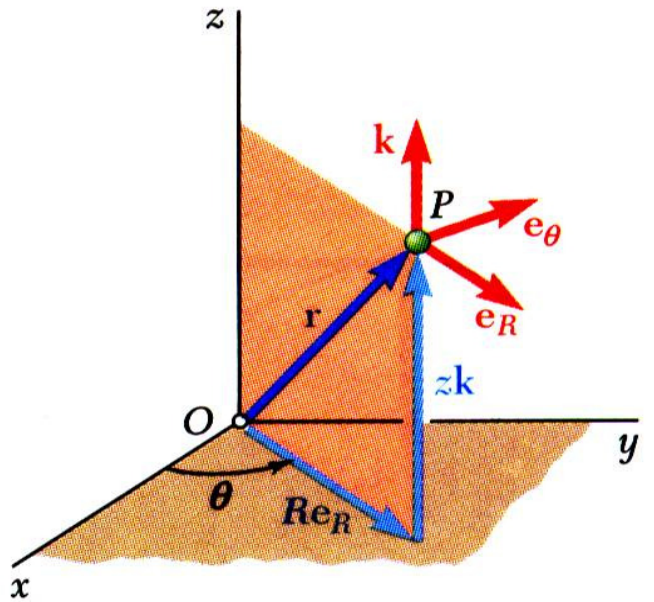
$$\vec{v} = \frac{d}{dt}\left(r\vec{e}_r\right) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$
$$= \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

- Similarly, the particle acceleration vector is

$$\vec{a} = \frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta\right)$$
$$= \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

Radial and Transverse Components





- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors $\vec{e}_R, \vec{e}_\theta$, and $\vec{k}$.

- Position vector,

$$\vec{r} = R\,\vec{e}_R + z\,\vec{k}$$

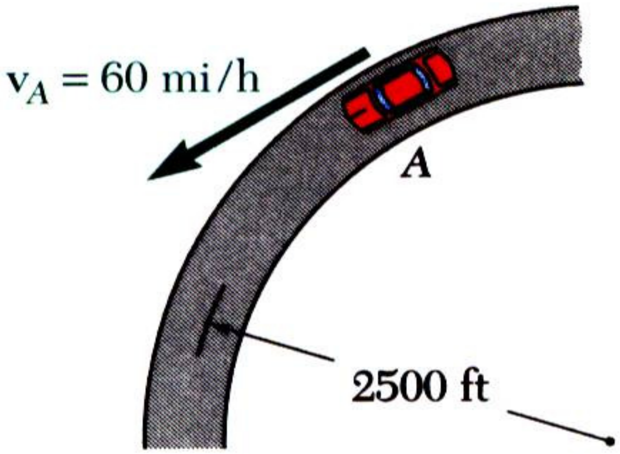
- Velocity vector,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\,\vec{e}_R + R\dot{\theta}\,\vec{e}_\theta + \dot{z}\,\vec{k}$$

- Acceleration vector,

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{R} - R\dot{\theta}^2\right)\vec{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\vec{e}_\theta + \ddot{z}\,\vec{k}$$

Sample Problem 11.10



A motorist is traveling on curved section of highway at 60 mph. The motorist applies brakes causing a constant deceleration rate.

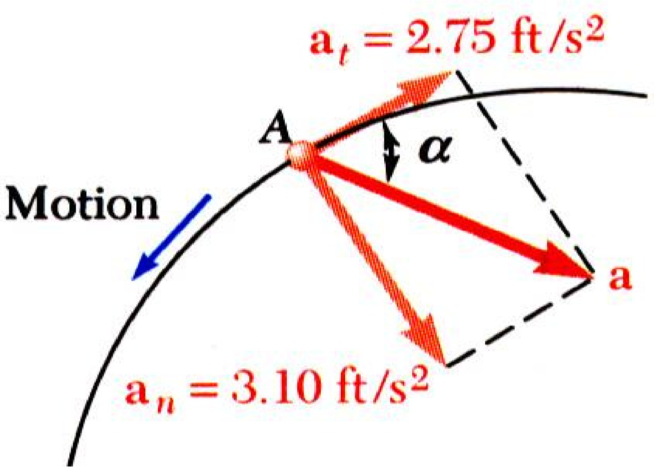
Knowing that after 8 s the speed has been reduced to 45 mph, determine the acceleration of the automobile immediately after the brakes are applied.

SOLUTION:

- Calculate tangential and normal components of acceleration.

- Determine acceleration magnitude and direction with respect to tangent to curve.

Sample Problem 11.10



$$60\,\text{mph} = 88\,\text{ft/s}$$
$$45\,\text{mph} = 66\,\text{ft/s}$$

SOLUTION:

- Calculate tangential and normal components of acceleration.

$$a_t = \frac{\Delta v}{\Delta t} = \frac{(66-88)\,\text{ft/s}}{8\,\text{s}} = -2.75\,\frac{\text{ft}}{\text{s}^2}$$

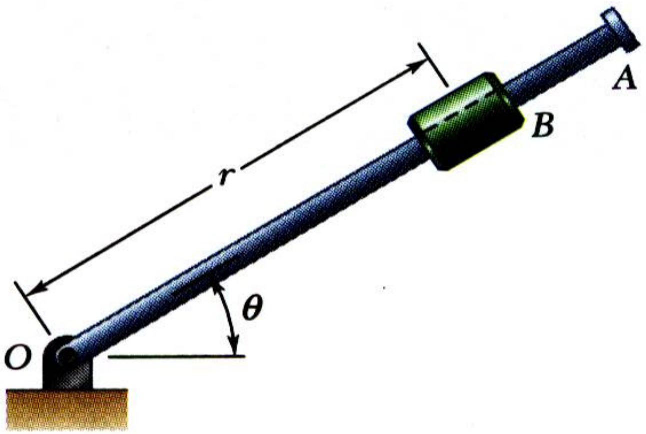
$$a_n = \frac{v^2}{\rho} = \frac{(88\,\text{ft/s})^2}{2500\,\text{ft}} = 3.10\,\frac{\text{ft}}{\text{s}^2}$$

- Determine acceleration magnitude and direction with respect to tangent to curve.

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(-2.75)^2 + 3.10^2} \qquad \boxed{a = 4.14\,\frac{\text{ft}}{\text{s}^2}}$$

$$\alpha = \tan^{-1}\frac{a_n}{a_t} = \tan^{-1}\frac{3.10}{2.75} \qquad \boxed{\alpha = 48.4^\circ}$$

Sample Problem 11.12



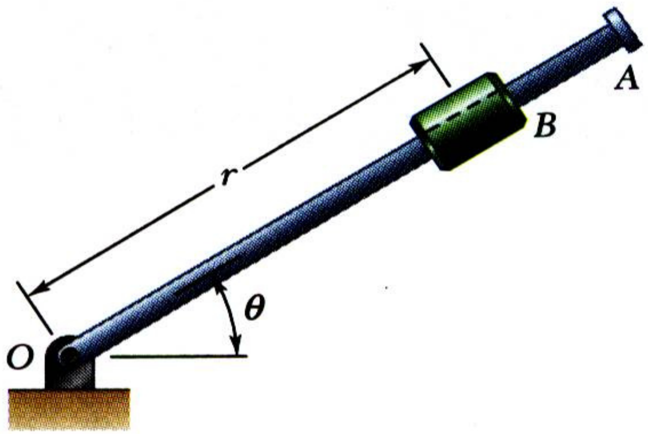
Rotation of the arm about O is defined by $\theta = 0.15t^2$ where θ is in radians and t in seconds. Collar B slides along the arm such that $r = 0.9 - 0.12t^2$ where r is in meters.

After the arm has rotated through 30°, determine *(a)* the total velocity of the collar, *(b)* the total acceleration of the collar, and *(c)* the relative acceleration of the collar with respect to the arm.

SOLUTION:

- Evaluate time t for $\theta = 30°$.
- Evaluate radial and angular positions, and first and second derivatives at time t.
- Calculate velocity and acceleration in cylindrical coordinates.
- Evaluate acceleration with respect to arm.

Sample Problem 11.12



SOLUTION:

- Evaluate time t for $\theta = 30^o$.

$$\theta = 0.15\,t^2$$

$$= 30° = 0.524\text{ rad} \qquad t = 1.869\text{ s}$$

- Evaluate radial and angular positions, and first and second derivatives at time t.

$$r = 0.9 - 0.12\,t^2 = 0.481\text{ m}$$

$$\dot{r} = -0.24\,t = -0.449\text{ m/s}$$

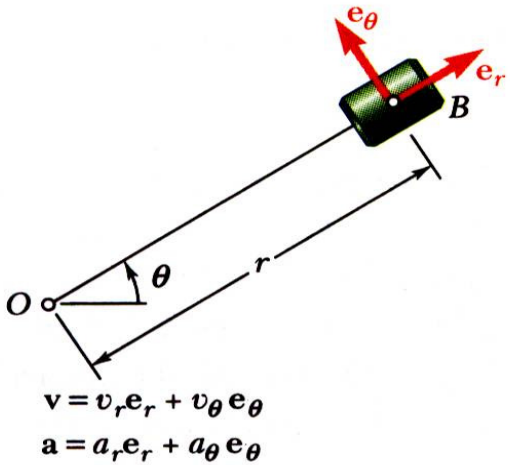
$$\ddot{r} = -0.24\text{ m/s}^2$$

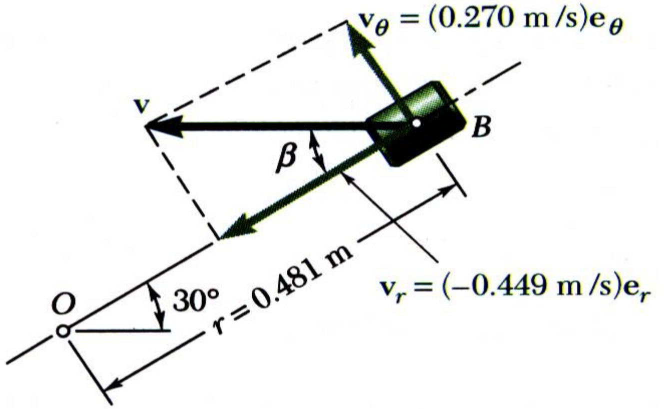
$$\theta = 0.15\,t^2 = 0.524\text{ rad}$$

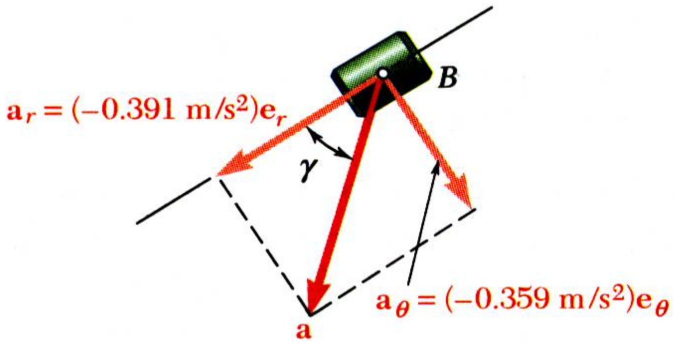
$$\dot{\theta} = 0.30\,t = 0.561\text{ rad/s}$$

$$\ddot{\theta} = 0.30\text{ rad/s}^2$$

Sample Problem 11.12







- Calculate velocity and acceleration.

$$v_r = \dot{r} = -0.449\ \text{m}/s$$

$$v_\theta = r\dot{\theta} = (0.481\ \text{m})(0.561\ \text{rad/s}) = 0.270\ \text{m/s}$$

$$v = \sqrt{v_r^2 + v_\theta^2} \qquad \beta = \tan^{-1}\frac{v_\theta}{v_r}$$

$$\boxed{v = 0.524\ \text{m/s} \qquad \beta = 31.0°}$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$= -0.240\ \text{m/s}^2 - (0.481\ \text{m})(0.561\ \text{rad/s})^2$$

$$= -0.391\ \text{m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

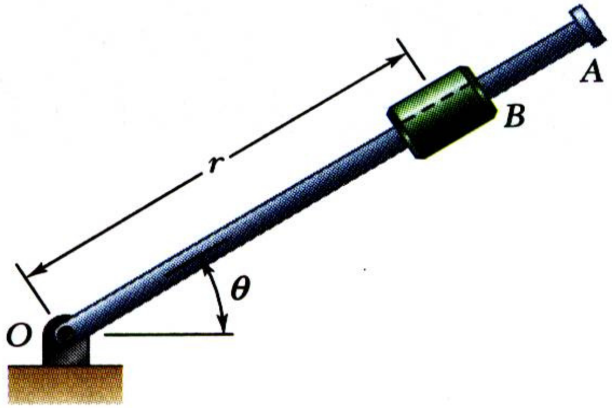
$$= (0.481\ \text{m})\left(0.3\ \text{rad/s}^2\right) + 2(-0.449\ \text{m/s})(0.561\ \text{rad/s})$$

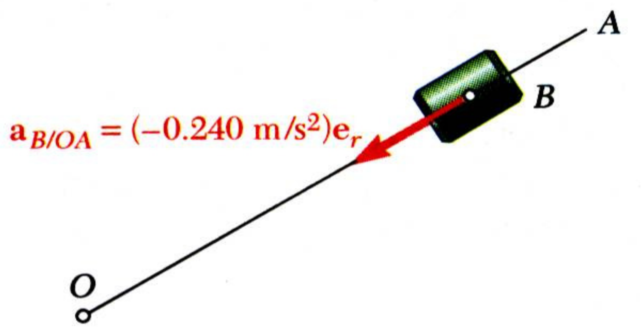
$$= -0.359\ \text{m/s}^2$$

$$a = \sqrt{a_r^2 + a_\theta^2} \qquad \gamma = \tan^{-1}\frac{a_\theta}{a_r}$$

$$\boxed{a = 0.531\ \text{m/s} \qquad \gamma = 42.6°}$$

Sample Problem 11.12





- Evaluate acceleration with respect to arm.

  Motion of collar with respect to arm is rectilinear and defined by coordinate *r*.

$$a_{B/OA} = \ddot{r} = -0.240\,\text{m/s}^2$$