بسم الله الرحمن الرحیم

Reducibility

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What we are going to discuss?

- Undecidable problems from language theory
  - Reductions via computation histories
- Mapping reducibility
  - Computable functions
  - Formal definition of mapping reducibility
- Post correspondence problem, or PCP

$$ALL_{CFG} = \{\langle G \rangle | G \text{ is a CFG and } L(G) = \Sigma^*\}$$

Undecidable

Theorem 5.13

Do not forget EQ_{CFG} is undecidable (Exercise 5.1).

Post Correspondence Problem, or PCP

≠ Probabilistically Checkable Proof

$$\left\{ \left[\frac{\mathrm{b}}{\mathrm{ca}}\right], \left[\frac{\mathrm{a}}{\mathrm{ab}}\right], \left[\frac{\mathrm{ca}}{\mathrm{a}}\right], \left[\frac{\mathrm{abc}}{\mathrm{c}}\right] \right\}$$

$$\left[\frac{\mathrm{a}}{\mathrm{ab}}\right]\left[\frac{\mathrm{b}}{\mathrm{ca}}\right]\left[\frac{\mathrm{ca}}{\mathrm{a}}\right]\left[\frac{\mathrm{a}}{\mathrm{ab}}\right]\left[\frac{\mathrm{abc}}{\mathrm{c}}\right]$$

matching

$$\begin{array}{|c|c|cc|c|ccc|} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \end{array}$$

$$\begin{array}{|cc|cc|c|cc|c|} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \end{array}$$

$$PCP = \{\langle P\rangle \mid P \text{ is an instance of PCP with a match}\}$$

Undecidable

Theorem 5.15

$PCP = \{\langle P\rangle | P \text{ is an instance of PCP with a match}\}$

- We reduce it from A_{TM} with computation histories
  - Given a TM M and an input w, we construct an instance of PCP P where M accepts w if there is a match in P.
- How can we construct P so that a match is an accepting computation history for M on w?
  - Each domino links a position or positions in one configuration with the corresponding ones in the next configuration.
  - Some simplifications:

MPCP

1. TM M on input w never tries to move its head off the left-hand end of tape
2. If $w = \varepsilon$, the string $\sqcup$ is used in place of w in the construction
3. PCP is modified such that a match must start with the first domino $\left[\frac{t_1}{b_1}\right]$.

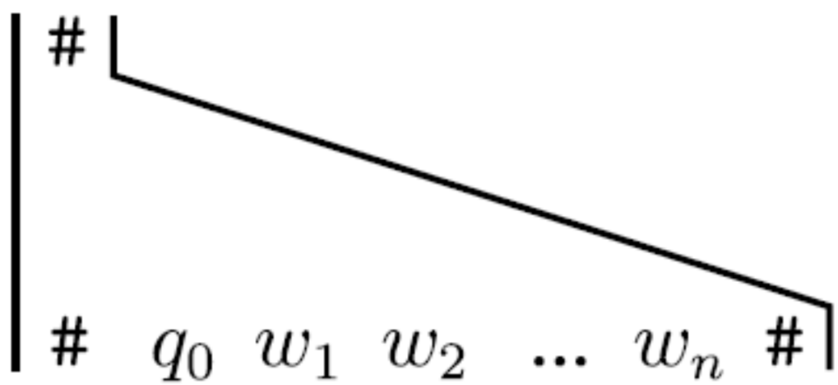
$PCP = \{\langle P\rangle | P \text{ is an instance of PCP with a match}\}$

- Assume TM R decides PCP, then we construct a TM S which decides A_{TM}.
  - Let $M = \left(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}}\right)$
  - TM S needs to construct an instance of PCP P that has a match iff M accepts w.
    - At first, S constructs and instance of MPCP P'

MPCP P'

1. The first domino is

$$\left[\frac{t_1}{b_1}\right] = \left[\frac{\#}{\#q_0 w_1 w_2 \ldots w_n \#}\right]$$



MPCP P'

2. For every $a, b \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{\text{reject}}$, if $\delta(q, a) = (r, b, \boldsymbol{R})$, put $\left[\frac{qa}{br}\right]$ into P'.
3. For every $a, b \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{\text{reject}}$, if $\delta(q, a) = (r, b, \boldsymbol{L})$, put $\left[\frac{cqa}{rcb}\right]$ into P'.
4. For every $a \in \Gamma$, put $\left[\frac{a}{a}\right]$ into P'.
5. Put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P'.

MPCP P'

6. For every $a \in \Gamma$, put $\left[\frac{a q_{\text{accept}}}{q_{\text{accept}}}\right]$ and $\left[\frac{q_{\text{accept}} a}{q_{\text{accept}}}\right]$ into P'.
7. Finally, add the following domino into P':

$$\frac{q_{\text{accept}}\#\#}{\#}$$

MPCP P'

- The MPCP instance P' constructed so far has a match iff M accepts w.
- However, if we treat it as a PCP instance, it always has a match …
  - What's that match?
- We need to convert MPCP P' into an instance of PCP P where a match in P exists iff M accepts w.
  - How?

Our trick

- For string $u = u_1 u_2 \dots u_n$, we define:
  - $\star u = * u_1 * u_2 * \cdots * u_n$
  - $\star u \star = * u_1 * u_2 * \cdots * u_n *$
  - $u \star = u_1 * u_2 * \cdots * u_n *$
- For MPCP instance $P' = \left\{\frac{t_1}{b_1}, \frac{t_2}{b_2}, \frac{t_3}{b_3}, \dots, \frac{t_k}{b_k}\right\}$, we construct the following PCP instance

$$P = \left\{\frac{\star t_1}{\star b_1 \star}, \frac{\star t_1}{b_1 \star}, \frac{\star t_2}{b_2 \star}, \frac{\star t_3}{b_3 \star}, \dots, \frac{\star t_k}{b_k \star}, \frac{* \bowtie}{\bowtie}\right\}.$$

$PCP = \{\langle P \rangle | P \text{ is an instance of PCP with a match}\}$

Undecidable

Theorem 5.15